Alberto Barchielli
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Quantum Trajectories and Measurements in Continuous Time

The Diffusive Case

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To Mary, Cecilia, Caterina

Preface

Quantum trajectory theory is largely employed in theoretical quantum optics and quantum open system theory and is closely related to the conceptual formalism of quantum mechanics (quantum measurement theory). However, even research articles show that not all the features of the theory are well known or completely exploited. We wrote this monograph mainly for researchers in theoretical quantum optics and related fields with the aim of giving a self-contained and solid presentation of a part of quantum trajectory theory (the diffusive case) together with some significant applications (mainly with purposes of illustration of the theory, but which in part have been recently developed). Another aim of the monograph is to introduce to this subject post-graduate or PhD students. To help them, in the most mathematical and conceptual chapters, summaries are given to fix ideas. Moreover, as stochastic calculus is usually not in the background of the studies in physics, we added Appendix A to introduce these concepts. The book is written also for mathematicians with interests in quantum theories. Quantum trajectory theory is a piece of modern theoretical physics which needs an interplay of various mathematical subjects, such as functional analysis and probability theory (stochastic calculus), and offers to mathematicians a beautiful field for applications, giving suggestions for new mathematical developments. Appendix B presents the modern formalism of quantum mechanics and has the double role of collecting notions and results used throughout the book and of introducing to this subject peoples without a background in the axiomatic of quantum mechanics.

The so-called stochastic Scrödinger equation and stochastic master equation, which are the key equations of quantum trajectory theory, have been introduced also in different contexts, such as dynamical reduction theories. So, we developed the theory of such equations (existence of solutions, properties, etc.) also independently of the theory of measurements continuous in time.

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Milano, Italy
December 2008

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Matteo Gregoratti

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Part II Physical Applications

Chapter 1
Introduction

1.1 Quantum Open Systems

Quantum mechanics started as a theory of closed systems: the state of the system is a vector of norm one in a Hilbert space and it evolves in time according to the Schrödinger equation (B.11). In order to describe also a possible uncertainty on the initial state, a "statistical" formulation of quantum mechanics has been developed: the states are represented by statistical operators (Sect. B.3.1), also called density matrices, and their evolution is given by the von Neumann equation (B.18). This statistical formulation revealed to be well suited also for open systems. General evolution equations for density operators appeared under the names of master equations and quantum dynamical semigroups [1–5] (Sect. B.3.3); these concepts were generalised and gave rise to the theory of quantum Markov semigroups [6, 7].

The contemporary successes of stochastic processes and stochastic differential equations (SDEs) for classical systems gave a strong motivation to try something similar in quantum open system theory and two classes of SDEs were introduced: quantum SDEs [8–11], driven by non-commuting noises, and classical SDEs [12–15], driven by ordinary, commuting noises. All these descriptions of quantum open systems are connected and one can shift from one approach to the other [4, 5, 16–18, 37].

In this book we want to present the approach to quantum open system theory based on classical SDEs, with particular emphasis on continuous measurements.

We speak of continuous measurements when a quantum system is monitored with continuity in time. Traditional presentations of quantum mechanics consider only instantaneous measurements, but continuous measurements on quantum systems are a common experimental practice; typical cases are the various forms of photon detection. The statements of a quantum theory about an observable are of probabilistic nature; so, it is natural that a quantum theory of continuous measurements gives rise to stochastic processes. Moreover, a continuously observed system is certainly open. All these arguments show that the development of a quantum theory of continuous measurements needs quantum measurement theory, open system theory, operator theory, quantum probability, quantum and classical stochastic

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processes, etc. The first consistent paper treating continuous measurements was published in 1969 [20]. It concerns counting of quanta, but some ideas on quantum counting formulae for photons had already been introduced before [21]. The main field of applications was quantum optics, but applications of continuous measurement theory are now becoming important also in quantum information and quantum control.

1.2 Approaches to Continuous Measurements

There are essentially three approaches to continuous measurements [22–24]. The first and the third one (chronologically) are "morally" equivalent and both give the probability distribution of the output process together with the stochastic evolution of the system, due to the interaction with the measuring apparatus and to the information acquirement. The second approach to continuous measurements is set at a higher level, as it introduces also a quantum description of the measuring apparatus itself and of its interaction with the system. These approaches have received various degrees of development, any one of them has its own merits and its range of applicability; the three approaches are consistent and one can go from one to the other and this feature is certainly at the bases of the flexibility and interest of the theory.

The first approach [1, 25–29] is the *operational* one, which is based on positive operator-valued measures or (generalised) observables and operation-valued measures or *instruments* [1, 22, 30–32]; very general results have been obtained inside this approach by using the Fourier transforms of the operator measures and quantum analogs of Markov semigroups [26, 33–38]. A variant of the first approach is based on the Feynman integral [25, 38, 39].

The second approach [40–43] is based on *quantum stochastic calculus* and quantum SDEs [8–11, 44]; it is connected to quantum Langevin equations and to the notion of input and output fields in quantum optics [3, 11, 16, 45]. For a review of this approach to continuous measurements see [46]. The book [47] explains how quantum stochastic differential equations arise as suitable limits of more fundamental "Hamiltonian" descriptions.

The last approach [48–50] is based on classical SDEs and the notion of *a posteriori states* [51]. It was originated by Belavkin's work on quantum filtering [43, 52–55] and it is related to notions appeared in quantum optics, such as quantum-state diffusion models, quantum trajectories, Monte–Carlo wave function method, unravelling of master equations [4, 17, 56–58].

The central aim of this book is to introduce the reader to this last formulation of continuous measurement theory and to show how this approach is connected to the operational one, where the general formulation of quantum mechanics explicitly appears. We do not present the connections with the approach based on quantum stochastic calculus, which can be found in [16, 19, 22, 59].

1.3 Classical SDEs in Continuous Measurement Theory

In the theory of continuous measurements one meets four kinds of SDEs with two kinds of noises.

The four kinds of equations are: (1) linear for non-normalised vectors in a Hilbert space, (2) nonlinear for normalised vectors in a Hilbert space (the stochastic Schrödinger equation), (3) linear for positive trace class operators, and (4) nonlinear for density matrices (the stochastic master equation). A change of probability measure underlies the passage from the linear case to the nonlinear one.

The two kinds of noises characterize the diffusive and the jump case. In the diffusive case, we meet a Wiener processes W in the linear equations and a Wiener process $\widehat{W}$ in the nonlinear equations (W and $\widehat{W}$ are connected by a Girsanov transformation). In the jump case, we have n Poisson process in the linear equations and a counting process in the nonlinear equations (the Poisson and the counting processes are the same process under two different probability laws).

Let us stress the role of the stochastic equations in Hilbert spaces. Here the open systems are no more treated by equations for statistical operators, but we go back to a stochastic version of the Schrödinger equation. Moreover, from the numerical point of view it is easier to work in Hilbert spaces rather than in matrix or operator spaces and SDEs in Hilbert space give an efficient starting point for simulations of time evolutions of open systems [5, 56, 57, 60, 61].

Here the focus is on open system theory and continuous measurements, but the same type of SDEs arose in quantum mechanics also for other purposes. We already quoted numerical simulations of master equations and quantum filtering, but they were also introduced as modifications of quantum mechanics to give dynamical descriptions of the von Neumann reduction postulate [14] and in the framework of the so-called dynamical reduction theories [62–66].

This presentation is restricted to the cases in which the driving noises are Wiener processes. For the cases with jumps see [16, 49, 50, 67–70]. Moreover, to take things simpler we work in a finite dimensional Hilbert space. This avoids analytical complications, but it contains the essential structure of the theory and is enough for the simplest applications.

1.4 The Plan of the Book

The book is divided in two parts, one for the presentation of the theoretical structure of the theory and the other for applications. The two appendices are intended to be a primary in stochastic differential equations (Appendix A) and in quantum mechanics (measurement theory and open systems – Appendix B).

Chapter 2 is devoted to the Hilbert-space formulation of the theory and it is centred on the presentation of the *stochastic Schrödinger equation*. Chapters 3 and 5 present the formulation in terms of statistical operators; now the key concept is that

of the *stochastic master equation*. While the observables of the theory, represented by *positive operator-valued measures*, have already been introduced in Chap. 2, the full connection of quantum trajectory theory with the general axiomatic structure of quantum mechanics is given in Chap. 4. Here also the moments and the spectrum of the output of the measurement are studied. Chapter 6 connects quantum trajectory theory with quantum information. Measures of information, such as mutual entropies, are introduced; they quantify the information extracted from the observed quantum system by the continuous measurement.

Chapter 7 gives some ideas on how to construct concrete physical models, mainly in quantum optics, and how to use the theory developed in the first part in order to describe two types of photodetection: *heterodyne* and *homodyne* detection. A concrete model for a two-level atom stimulated by a monochromatic laser is given in Chap. 8 and its heterodyne and homodyne spectra are studied in Chap. 9. Chapter 10 is devoted to the effects produced on the homodyne spectrum by feedback and control; the first part of this chapter presents an interesting scheme, due to Wiseman and Milburn [71], which allows to introduce feedback loops in the theory. Chapters 9 and 10 present also many physical effects typical of different quantum systems, such as squeezing, line narrowing, thermal and dephasing broadening, etc.

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Part I
General Theory

Chapter 2
The Stochastic Schrödinger Equation

2.1 Introduction

In this chapter, we introduce the theory of measurements in continuous time (diffusive case) starting from the particular but important case of complete observation. This allows to present the Hilbert space formulation of the theory, where the state of the observed quantum system is described by a vector in the Hilbert space $\mathcal{H}$ of the system. Even if this is a special case of the more general theory presented in Chaps. 3, 4 and 5, it deserves a separate treatment for different reasons: it is instructive, it uses only the Hilbert space formulation of quantum mechanics, it is of interest on its own because the stochastic Scrödinger equation presented in this chapter has also been used in different contexts [1–6], some mathematical results of the following chapter will relay anyhow on the theory presented here, and Hilbert space SDEs are the key starting point for efficient numerical simulations of the dynamics of open quantum systems [1, 7].

First, we introduce the class of SDEs in Hilbert spaces which we are interested in and we present their mathematical properties. After that, we discuss their physical interpretation and start to develop the theory of continuous measurements.

Given the initial (pure) state $\psi_0 \in \mathcal{H}$ of the measured quantum system, the aim is to get two stochastic processes together with the probability distribution of their trajectories:

- the output $W(t)$ of the continuous measurement;
- the system state $\widehat{\psi}(t)$, whose evolution includes the continuous measurement and which is continuously conditioned on the observed output;
- the physical probability distribution of the processes $W(t)$ and $\widehat{\psi}(t)$.

The system state $\widehat{\psi}(t)$ is called a posteriori state, as it depends on the trajectory observed for $W(s)$ in the time interval $0 \leq s \leq t$. The knowledge of the physical probability distribution of $W(t)$ and $\widehat{\psi}(t)$ allows to consider and to compute mean values at a given time, just as correlations and multi-time moments.

There are two possible ways to develop the theory: to start from the nonlinear evolution equation of the a posteriori states $\widehat{\psi}(t)$ or from the linear evolution equation of the so-called non-normalised a posteriori states $\psi(t)$. We prefer to begin with

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this second approach which is the direct generalisation of the traditional description of an instantaneous measurement.

When a quantum system undergoes a "von Neumann measurement" of an observable represented by a self-adjoint operator X with discrete eigenvalues x_k and eigen-projections E_k, one usually fixes the space $\Omega = \{x_1, x_2, \ldots\}$ of the possible outcomes and, for every $x_k \in \Omega$, uses the corresponding projection E_k to introduce the linear state transformation (von Neumann reduction postulate):

$$\psi_0 \mapsto \psi_1(x_k) := E_k \psi_0.$$

Then, $\psi_1(x_k)$ gives both the physical probability distribution for the outcome X and the a posteriori state $\widehat{\psi}_1$: if ψ_0 is the initial system state, then

- $\|\psi_1(x_k)\|^2$ is the probability of observing $X = x_k$;
- $\widehat{\psi}_1(x_k) = \psi_1(x_k)/\|\psi_1(x_k)\|$ is the a posteriori state when $X = x_k$.

In order to generalise consistently such a representation of a measurement to the continuous time case, we use the powerful mathematical tools of stochastic calculus and thus we prefer to begin with their presentation.

Section 2.2 is devoted to the theory of homogeneous linear SDEs. To read this section, one needs the notions of filtration, stochastic process, martingale, stochastic integral with respect to a Wiener process and strong solution of an SDE; moreover, familiarity with the Itô formula is essential. All these topics of stochastic calculus are recalled in Sects. A.2, A.3 and A.4. In Sect. 2.3, the subclass of linear SDEs of our concern is presented and studied. Here the notions of exponential martingale, change of probability measure and Girsanov transformation are needed; they are recalled in Sect. A.5.

The SDE approach to the quantum theory of open systems and of continuous measurements is given in the rest of the chapter, starting from Sect. 2.4. In this chapter, only the Hilbert space formulation of quantum mechanics is needed, as it is presented in Sect. B.2. The key notion is "positive operator-valued measure", a mathematical object which represents a general quantum mechanical observable.

As already said in Sect. 1.3, we work in a finite dimensional Hilbert space, which is enough to give the main ideas of the stochastic approach to open systems and continuous measurements and to develop the simplest applications. For results and examples in infinite dimensional Hilbert spaces, see [7–27].

2.2 Linear Stochastic Differential Equations

Assumption 2.1. The Hilbert space of the quantum system is $\mathcal{H} = \mathbb{C}^n$.

The SDEs we consider are driven by white noise, the derivative of the Wiener process. So, let us introduce such a stochastic process and fix the framework needed for SDEs.

Assumption 2.2. We fix a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$ satisfying usual conditions (Sect. A.2.2) and a continuous d-dimensional Wiener process $W = \{W(t), t \geq 0\}$, with increments independent of the past (Definition A.21). We assume

$$\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t \,. \tag{2.1}$$

The symbol $\mathbb{E}_\mathbb{Q}$ indicates the expectation with respect to the probability $\mathbb{Q}$.

2.2.1 An Homogeneous Linear SDE in Hilbert Space

Let us start by considering a generic homogeneous linear SDE with "multiplicative noise" for an $\mathcal{H}$-valued process $\psi = \{\psi(t),\ t \geq 0\}$:

$$\begin{cases} \mathrm{d}\psi(t) = K(t)\psi(t)\,\mathrm{d}t + \displaystyle\sum_{j=1}^{d} R_j(t)\psi(t)\,\mathrm{d}W_j(t)\,, \\ \psi(0) = \psi_0\,, \qquad\qquad \psi_0 \in \mathcal{H}. \end{cases} \tag{2.2}$$

Assumption 2.3. The initial condition ψ_0 is non random. The coefficients $R_j(t)$, $K(t)$ are (non-random) linear operators on $\mathcal{H}$. The functions $t \mapsto K(t)$ and $t \mapsto R_j(t)$ are measurable and such that $\forall T \in (0, +\infty)$

$$\sup_{t\in[0,T]} \|K(t)\| < +\infty, \qquad \sup_{t\in[0,T]} \left\| \sum_j R_j(t)^* R_j(t) \right\| < +\infty. \tag{2.3}$$

Theorem 2.4. *Under Assumption 2.3, the linear SDE* (2.2) *admits strong solutions in* $[0, +\infty)$. *Pathwise uniqueness and uniqueness in law hold. Moreover, for any* $p \geq 2$ *and* $T > 0$, *there exists a constant* $C(p, T)$ *such that*

$$\mathbb{E}_\mathbb{Q}\left[\sup_{t\in[0,T]} \|\psi(t)\|^p \right] \leq C(p, T)\left(1 + \|\psi_0\|^p\right). \tag{2.4}$$

Proof. Let us make the identifications $b(x, t) = K(t)x$, $\sigma_j(x, t) = R_j(t)x$. We have the estimates

$$\|b(x, t)\| = \|K(t)x\| \leq \|K(t)\|\,\|x\|\,,$$

$$\sum_j \big\|\sigma_j(x, t)\big\|^2 = \sum_j \big\|R_j(t)x\big\|^2 = \big\langle x \big| \textstyle\sum_j R_j(t)^* R_j(t)x \big\rangle \leq \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \|x\|^2 .$$

Obviously, we also have $\sum_j \big\|\sigma_j(x, t) - \sigma_j(y, t)\big\|^2 = \sum_j \big\|\sigma_j(x - y, t)\big\|^2$ and $\|b(x, t) - b(y, t)\| = \|b(x - y, t)\|$. Then, Hypotheses A.25, A.32 and A.34 hold

with $L(T) = 2\max\left\{\sup_{t\in[0,T]}\|K(t)\|^2,\ \sup_{t\in[0,T]}\left\|\sum_j R_j(t)^* R_j(t)\right\|\right\}$, $M(T) = \sqrt{2L(T)}$, and Theorems A.36 and A.38 give the statements. □

Let us recall that the existence of strong solutions means that (2.2) admits a solution for every choice of the probability space, of the filtration and of the Wiener process (see Definition A.27). For the notions of uniqueness see Definitions A.28 and A.29.

In our construction, the stochastic basis and the Wiener process are fixed by Assumption 2.2. Then, by ψ we denote the continuous, adapted process (Itô process – see Sect. A.3.4) satisfying

$$\psi(t) = \psi_0 + \int_0^t K(s)\psi(s)\,\mathrm{d}s + \sum_{j=1}^{d}\int_0^t R_j(s)\psi(s)\,\mathrm{d}W_j(s)\,; \tag{2.5}$$

such a process is unique up to indistinguishableness (Sect. A.4.1).

Remark 2.5. In the following, the natural filtration of the increments of the Wiener process and its augmented version will be important: for $0 \le s \le t$, we define

$$\mathcal{D}_t^s := \sigma\{W(r) - W(s),\ r \in [s,t]\}, \qquad \overline{\mathcal{D}}_t^s := \mathcal{D}_t^s \vee \mathcal{N}; \tag{2.6}$$

$\mathcal{N}$ is the class of the $\mathbb{Q}$-null sets in $\mathcal{F}$.

Because of the properties of a Wiener process, the filtration $\{\overline{\mathcal{D}}_t^s,\ t \in [s,+\infty)\}$ satisfies the usual conditions: $\overline{\mathcal{D}}_t^s$ is independent of $\mathcal{F}_s$ and $\overline{\mathcal{D}}_t^s \subset \overline{\mathcal{D}}_t^0 \subset \mathcal{F}_t \subset \mathcal{F}$, for $0 \le s \le t$.

Because of the existence of strong solutions and of the fact that the initial condition is non-random, the continuous $(\mathcal{F}_t)$-adapted process ψ is also $\left(\overline{\mathcal{D}}_t^0\right)$-adapted.

2.2.2 The Stochastic Evolution Operator

Equation (2.2) being a linear equation, we can introduce a stochastic process of operators $A_t^0(\omega)$ giving the application $\psi_0 \mapsto \psi(t,\omega)$. Indeed, let us consider the operator-valued processes A_t^s, with $t \ge s \ge 0$, defined by the SDE

$$\begin{cases} \mathrm{d}A_t^s = K(t)A_t^s\,\mathrm{d}t + \displaystyle\sum_{j=1}^{d} R_j(t)A_t^s\,\mathrm{d}W_j(t)\,, \\ A_s^s = \mathbb{1}\,. \end{cases} \tag{2.7}$$

This is a linear SDE for an $n \times n$-dimensional complex process; so, exactly as for (2.2), in $\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\right)$ there is a pathwise unique, continuous, adapted solution.

Proposition 2.6. *Under Assumption 2.3, the linear SDE* (2.7) *admits strong solutions in* $[s, +\infty)$, $\forall s \geq 0$. *Pathwise uniqueness and uniqueness in law hold. Moreover, for any* $p \geq 2$ *and* $T > s$, *there exists a constant* $C(p, T)$ *such that*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[s,T]} \left\|A_t^s\right\|_2^{\,p}\right] \leq C(p,T)\left(1+n^{p/2}\right). \tag{2.8}$$

Proof. Let us make the identifications $b(a,t) = K(t)a$, $\sigma_j(a,t) = R_j(t)a$, $a \in M_n$. Now a, b, σ_j are vectors whose components are labelled by a couple of indices; then, the relevant norm is the Hilbert–Schmidt one (B.3). We have the estimates

$$\begin{aligned}\|b(a,t)\|^2 &= \|K(t)a\|_2^{\,2} = \mathrm{Tr}\left\{a^*K(t)^*K(t)a\right\} = \mathrm{Tr}\left\{K(t)^*K(t)aa^*\right\}\\ &\leq \left\|K(t)^*K(t)\right\| \left\|aa^*\right\|_1 = \|K(t)\|^2\,\mathrm{Tr}\left\{aa^*\right\} = \|K(t)\|^2\,\|a\|_2^{\,2},\end{aligned}$$

$$\begin{aligned}\sum_j \left\|\sigma_j(a,t)\right\|^2 &= \sum_j \left\|R_j(t)a\right\|_2^{\,2} = \sum_j \mathrm{Tr}\left\{a^*R_j(t)^*R_j(t)a\right\}\\ &= \mathrm{Tr}\left\{\textstyle\sum_j R_j(t)^*R_j(t)aa^*\right\} \leq \left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\| \,\|aa^*\|_1\\ &= \left\|\sum_j R_j(t)^*R_j(t)\right\| \,\|a\|_2^{\,2}.\end{aligned}$$

Then, the proof goes on as in Theorem 2.4, exactly with the same constants. Note that $\|\mathbb{1}\|_2^{\,2} = \mathrm{Tr}\{\mathbb{1}\} = n$. □

Because of the properties stated in the following proposition, A_t^s is called *stochastic evolution operator*. In mathematical terms, A_t^0 is the *fundamental matrix* of the linear equation (2.2), while in the physical literature the term *propagator* is more used.

Proposition 2.7. *For* $0 \leq s \leq t$, A_t^s *is* $\mathbb{Q}$*-independent of* $\mathcal{F}_s$ *and* $\overline{\mathcal{D}}_t^s$*-measurable. Moreover, for every given* $0 \leq r \leq s$, *almost surely (a.s.) we have*

$$A_t^s A_s^r = A_t^r\,, \qquad \forall t \geq s, \tag{2.9}$$

$$\psi(t) = A_t^0\psi_0\,, \qquad \forall t \geq 0\,. \tag{2.10}$$

More explicitly, the continuous processes $t \mapsto A_t^s A_s^r$ and $t \mapsto A_t^r$ are indistinguishable; the same holds for the processes $t \mapsto \psi(t)$ and $t \mapsto A_t^0\psi_0$.

Proof. Because of the existence of strong solutions and pathwise uniqueness, the random variable A_t^s is $\overline{\mathcal{D}}_t^s$-measurable; then, the statement about the independence follows from the independent increment property of the Wiener process.

Let us fix $s \geq r \geq 0$ and set

$$B_t := \begin{cases} A_t^r, & \text{if } r \leq t < s, \\ A_t^s A_s^r, & \text{if } t \geq s. \end{cases}$$

Then, by (2.7) we have for $t \geq s$

$$\begin{aligned} B_t = A_t^s A_s^r &= A_s^r + \int_s^t K(u) A_u^s A_s^r \, \mathrm{d}u + \sum_j \int_s^t R_j(u) A_u^s A_s^r \, \mathrm{d}W_j(u) \\ &= \mathbb{1} + \int_r^s K(u) A_u^r \, \mathrm{d}u + \sum_j \int_r^s R_j(u) A_u^r \, \mathrm{d}W_j(u) + \int_s^t K(u) B_u \, \mathrm{d}u \\ &\quad + \sum_j \int_s^t R_j(u) B_u \, \mathrm{d}W_j(u) = \mathbb{1} + \int_r^t K(u) B_u \, \mathrm{d}u + \sum_j \int_r^t R_j(u) B_u \, \mathrm{d}W_j(u); \end{aligned}$$

by the definition of B, the same equation holds also for $t < s$. Therefore, B_t and A_t^r satisfy the same equation and, by uniqueness, they are indistinguishable. This proves (2.9). Similarly $\psi(t)$ and $A_t^0 \psi_0$ satisfy the same equation and, so, (2.10) holds. □

Also the adjoint A_t^{s*} of the stochastic evolution operator is a continuous, adapted process and for $t \geq s$ it satisfies

$$\begin{cases} \mathrm{d}A_t^{s*} = A_t^{s*} K(t)^* \, \mathrm{d}t + \sum_{j=1}^d A_t^{s*} R_j(t)^* \, \mathrm{d}W_j(t) \,, \\ A_s^{s*} = \mathbb{1} \,. \end{cases} \tag{2.11}$$

2.2.2.1 The Stochastic Liouville Formula

It is important to prove other properties of the stochastic evolution operator and in particular that the matrix A_t^s is a.s. invertible.

Proposition 2.8. *For every given initial time $s \geq 0$, the Wronskian determinant $D_t^s := \det A_t^s$ is given by the stochastic Liouville formula*

$$D_t^s = \exp\left(\int_s^t \mathrm{Tr}\left\{ K(r) - \frac{1}{2} \sum_j R_j(r)^2 \right\} \mathrm{d}r + \sum_j \int_s^t \mathrm{Tr}\left\{ R_j(r) \right\} \mathrm{d}W_j(r) \right). \tag{2.12}$$

This equality holds a.s. for every $t \geq s$ and, so, $\mathbb{Q}(D_t^s > 0, \ \forall t \geq s) = 1$. Then, the operator A_t^s is a.s. invertible and the process $(A_t^s)^{-1}$ satisfies the SDE

$$\mathrm{d}(A_t^s)^{-1} = (A_t^s)^{-1} \Big[\textstyle\sum_j R_j(t)^2 - K(t) \Big] \mathrm{d}t - \sum_{j=1}^d (A_t^s)^{-1} R_j(t) \mathrm{d}W_j(t) \,. \tag{2.13}$$

Finally, for every $0 \leq s \leq t$*, the following representation holds a.s.:*

$$A_t^s = A_t^0 (A_s^0)^{-1}. \tag{2.14}$$

Proof. Let $s \geq 0$ be a given initial time. By differentiating the explicit expression of the determinant, which is a polynomial in the matrix elements of A_t^s, and by using the Itô formula for products, in the proof of Theorem 2.2 in [28] the following formula for the stochastic differential of D_t^s is obtained:

$$\begin{aligned} \mathrm{d}D_t^s = &\left[\mathrm{Tr}\left\{ K(t) - \frac{1}{2} \sum_j R_j(t)^2 \right\} + \frac{1}{2} \sum_j \left(\mathrm{Tr}\left\{ R_j(t) \right\} \right)^2 \right] D_t^s \, \mathrm{d}t \\ &+ \sum_j \mathrm{Tr}\left\{ R_j(t) \right\} D_t^s \, \mathrm{d}W_j(t) \,. \end{aligned} \tag{2.15}$$

But this is a one-dimensional linear SDE with initial condition $D_s^s = \mathbb{1}$. Again the solution is pathwise unique and it is an exercise in stochastic calculus to verify that (2.12) solves this linear SDE. Thus, $\mathbb{Q}(D_t^s > 0, \ \forall t \geq s) = 1$ and A_t^s is a.s. invertible for every $t \geq s$.

To prove (2.13), let us consider the equation

$$\mathrm{d}Z_t^s = Z_t^s \Big[\textstyle\sum_j R_j(t)^2 - K(t) \Big] \mathrm{d}t - \displaystyle\sum_{j=1}^{d} Z_t^s R_j(t) \mathrm{d}W_j(t) \,, \qquad Z_s^s = \mathbb{1} \,. \tag{2.16}$$

Once more the solution is unique. By Itô formula for products one gets $\mathrm{d}\left(Z_t^s A_t^s \right) = 0$. Together with $Z_s^s A_s^s = \mathbb{1}$ and continuity in t, this gives $Z_t^s A_t^s = \mathbb{1}$ for every $t \geq s$. By multiplying on the right by $\left(A_t^s \right)^{-1}$, which exists, we get $Z_t^s = \left(A_t^s \right)^{-1}$ for every $t \geq s$ and (2.13) is proved.

By using (2.9), we have $A_t^0 (A_s^0)^{-1} = A_t^s A_s^0 (A_s^0)^{-1} = A_t^s$ and (2.14) is proved.

□

2.2.3 The Square Norm of the Solution

Let us now study the behaviour of the norm of $\psi(t)$, which will be a key object in the whole construction.

Proposition 2.9. *We have*

$$\begin{aligned} \|\psi(t)\|^2 = \|\psi_0\|^2 &+ \int_0^t \Big\langle \psi(s) \Big| \left(K(s) + K(s)^* + \textstyle\sum_j R_j(s)^* R_j(s) \right) \psi(s) \Big\rangle \mathrm{d}s \\ &+ \sum_{j=1}^{d} \int_0^t \langle \psi(s) | (R_j(s) + R_j(s)^*) \psi(s) \rangle \mathrm{d}W_j(s) \,. \end{aligned} \tag{2.17}$$

Moreover, $\forall T \geq 0$*,*

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \sum_j \left|\langle\psi(t)|(R_j(t)+R_j(t)^*)\psi(t)\rangle\right|^2 \mathrm{d}t\right] < +\infty, \tag{2.18}$$

and the stochastic integral in (2.17) *is a square-integrable continuous martingale.*

Proof. By Itô formula, we get

$$\begin{aligned}
\mathrm{d}\,\|\psi(t)\|^2 &= \langle\psi(t)|\mathrm{d}\psi(t)\rangle + \langle\mathrm{d}\psi(t)|\psi(t)\rangle + \langle\mathrm{d}\psi(t)|\mathrm{d}\psi(t)\rangle \\
&= \langle\psi(t)|K(t)\psi(t)\rangle\mathrm{d}t + \sum_j \langle\psi(t)|R_j(t)\psi(t)\rangle\mathrm{d}W_j(t) + \langle K(t)\psi(t)|\psi(t)\rangle\mathrm{d}t \\
&\quad + \sum_j \langle R_j(t)\psi(t)|\psi(t)\rangle\mathrm{d}W_j(t) + \sum_j \left\|R_j(t)\psi(t)\right\|^2 \mathrm{d}t \\
&= \left\langle\psi(t)\right| \left(K(t)+K(t)^* + \textstyle\sum_j R_j(t)^*R_j(t)\right)\psi(t)\big\rangle\mathrm{d}t \\
&\quad + \sum_j \langle\psi(t)|(R_j(t)+R_j(t)^*)\psi(t)\rangle\mathrm{d}W_j(t)\,,
\end{aligned}$$

which gives (2.17).

For every $x \in \mathcal{H}$, let P_x be the one-dimensional orthogonal projection on the Hilbert ray containing x and recall that $R_j(t)^*P_xR_j(t) \geq 0$ and $R_j(t)^*(\mathbb{1}-P_x)R_j(t) \geq 0$. Then, we have

$$\begin{aligned}
\sum_j \langle x|(R_j(t)+R_j(t)^*)x\rangle^2 &\leq 4\sum_j \left|\langle x|R_j(t)x\rangle\right|^2 = 4\,\|x\|^2 \sum_j \langle x|R_j(t)^*P_xR_j(t)x\rangle \\
&\leq 4\,\|x\|^2 \sum_j \langle x|R_j(t)^*R_j(t)x\rangle \leq 4\,\|x\|^4 \left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|;
\end{aligned}$$

so, the following estimate holds: $\forall x \in \mathcal{H}$,

$$\sum_j \langle x|(R_j(t)+R_j(t)^*)x\rangle^2 \leq 4\,\|x\|^4 \left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|. \tag{2.19}$$

By using this inequality and the L^p estimate (2.4) given in Theorem 2.4, we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \sum_j \langle\psi(t)|(R_j(t)+R_j(t)^*)\psi(t)\rangle^2\mathrm{d}t\right] \\
&\leq 4 \sup_{t\in[0,T]} \left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\| T\,\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\leq t\leq T}\|\psi(t)\|^4\right] \\
&\leq 4 \sup_{t\in[0,T]} \left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\| TC(4,T)\left(1+\|\psi_0\|^4\right) < +\infty,
\end{aligned}$$

and (2.18) is proved. Then, the integrand process $\langle\psi(t)|(R_j(t) + R_j(t)^*)\psi(t)\rangle$ belongs to the space $\mathcal{M}^2$ for every j (Sect. A.3.1), and the stochastic integral in (2.17) is a square-integrable continuous martingale (Sect. A.3.3). □

2.3 The Linear Stochastic Schrödinger Equation

For the physical interpretation anticipated in Section 2.1 and discussed in Sect. 2.4, we are not interested in (2.2) in general, but only when $\|\psi(t)\|^2$ is a martingale of mean one and can be interpreted as a probability density with respect to $\mathbb{Q}$.

2.3.1 A Key Restriction

In order to reduce $\|\psi(t)\|^2$ to a martingale, we need the vanishing of the integrand in the time integral in (2.17) for every initial condition, i.e.

$$K(t) + K(t)^* + \sum_j R_j(t)^* R_j(t) = 0,$$

which is equivalent to the following assumption.

Assumption 2.10. The operator $K(t)$ has the structure

$$K(t) = -\mathrm{i}H(t) - \frac{1}{2}\sum_{j=1}^{d} R_j(t)^* R_j(t)\,, \tag{2.20}$$

where $H(t)$ is a self-adjoint operator on $\mathcal{H}$, called *effective Hamiltonian* of the system.

By Assumptions 2.3 and 2.10, the function $t \mapsto H(t)$ is measurable and

$$\forall T \in (0, +\infty), \qquad \sup_{t\in[0,T]} \|H(t)\| < +\infty\,. \tag{2.21}$$

Proposition 2.8 gives $\|\psi(t)\| > 0$ and we can define the continuous processes

$$\widehat{\psi}(t) := \|\psi(t)\|^{-1}\ \psi(t)\,, \tag{2.22}$$

$$m_j(t) := \big\langle\widehat{\psi}(t)\big|\big(R_j(t) + R_j(t)^*\big)\widehat{\psi}(t)\big\rangle = 2\,\mathrm{Re}\big\langle\widehat{\psi}(t)\big|R_j(t)\widehat{\psi}(t)\big\rangle. \tag{2.23}$$

Theorem 2.11. *Under Assumptions 2.2 and 2.10, the square norm* $\|\psi(t)\|^2$ *of the solution of the SDE* (2.2) *is a positive, continuous martingale and*

$$\|\psi(t)\|^2 = \|\psi_0\|^2 \exp\left\{\sum_j \left[\int_0^t m_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t m_j(s)^2\mathrm{d}s\right]\right\}. \tag{2.24}$$

Moreover, $\forall p \geq 1$,

$$\sup_{0\leq t\leq T} \mathbb{E}_{\mathbb{Q}}\left[\|\psi(t)\|^{2p}\right] \leq \mathbb{E}_{\mathbb{Q}}\left[\sup_{0\leq t\leq T} \|\psi(t)\|^{2p}\right] < +\infty. \tag{2.25}$$

Proof. Being an Itô process, ψ is continuous and this holds for its square norm.

By Assumption 2.10 and the definitions (2.22), (2.23), equation (2.17) reduces to

$$\|\psi(t)\|^2 = \|\psi_0\|^2 + \sum_j \int_0^t m_j(s)\,\|\psi(s)\|^2\,\mathrm{d}W_j(s). \tag{2.26}$$

By Proposition 2.9, the positive continuous process $\|\psi(t)\|^2$ is a square-integrable martingale. By taking m as given, (2.26) is a Doléans equation whose solution is unique and given by (2.24) (cf. Proposition A.41 and (Eqs. (A.23), (A.24), (A.25), (A.26)).

By inequality (2.19), we have

$$\begin{aligned} \sum_j m_j(t)^2 &\leq 4\left\|\textstyle\sum_j R_j(t)^* R_j(t)\right\|, \\ \int_0^T \sum_j m_j(t)^2\mathrm{d}t &\leq 4 \sup_{t\in[0,T]}\left\|\textstyle\sum_j R_j(t)^* R_j(t)\right\| T. \end{aligned} \tag{2.27}$$

Then, the last statement follows from Proposition A.42. □

In the following, we shall call *linear stochastic Schrödinger equation* the original SDE (2.2) for an $\mathcal{H}$-valued process ψ under all Assumptions 2.1, 2.2, 2.3 and 2.10, i.e.

$$\begin{cases} \mathrm{d}\psi(t) = \left(-\mathrm{i}H(t) - \dfrac{1}{2}\displaystyle\sum_{j=1}^d R_j(t)^* R_j(t)\right)\psi(t)\,\mathrm{d}t + \sum_{j=1}^d R_j(t)\psi(t)\,\mathrm{d}W_j(t)\,, \\ \psi(0) = \psi_0\,, \qquad\qquad \psi_0 \in \mathcal{H}. \end{cases} \tag{2.28}$$

Of course, the solution is the continuous, adapted stochastic process $\psi(t) = A_t^0\psi_0$, where the stochastic evolution operator A_t^s and its adjoint A_t^{s*} still satisfy the SDEs (2.7) and (2.11) with $K(t) = -\mathrm{i}H(t) - \dfrac{1}{2}\displaystyle\sum_{j=1}^d R_j(t)^* R_j(t)$ and $H(t) = H(t)^*$.

2.3.2 A Change of Probability

Assumption 2.12. The initial condition is normalised: $\|\psi_0\| = 1$.

$\|\psi(t)\|^2$ being a positive martingale with $\mathbb{E}_{\mathbb{Q}}\left[\|\psi(t)\|^2\right] = 1$ by the discussion in Sect. A.5.3 and Remark A.46, we have the following.

Remark 2.13. For any $T > 0$ the equation

$$\widehat{\mathbb{P}}^T_{\psi_0}(F) := \int_F \|\psi(T,\omega)\|^2\, \mathbb{Q}(\mathrm{d}\omega) \equiv \mathbb{E}_{\mathbb{Q}}\left[1_F \|\psi(T)\|^2\right], \qquad F \in \mathcal{F}_T, \tag{2.29}$$

defines a new probability law $\widehat{\mathbb{P}}^T_{\psi_0}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{Q}_T$, the restriction of $\mathbb{Q}$ to $\mathcal{F}_T$. Let us denote by $\widehat{\mathbb{E}}^T_{\psi_0}$ the expectation with respect to $\widehat{\mathbb{P}}^T_{\psi_0}$.

Moreover, $\left\{\widehat{\mathbb{P}}^T_{\psi_0},\ T > 0\right\}$ is a consistent family of probabilities, in the sense that

$$0 < S < T,\ F \in \mathcal{F}_S \ \Rightarrow\ \widehat{\mathbb{P}}^T_{\psi_0}(F) = \widehat{\mathbb{P}}^S_{\psi_0}(F)\,. \tag{2.30}$$

Then, Girsanov theorem (Theorem A.45 and Proposition A.47) gives the following fundamental result. The class of integrand processes $\mathcal{L}^2$ is defined in Sect. A.3.1.

Theorem 2.14. *Under the law $\widehat{\mathbb{P}}^T_{\psi_0}$ defined by* (2.29)*, the continuous processes*

$$\widehat{W}_j(t) := W_j(t) - \int_0^t m_j(s)\mathrm{d}s, \qquad j = 1, \ldots, d, \quad t \in [0, T], \tag{2.31}$$

are independent, standard Wiener processes with respect to the filtration $(\mathcal{F}_t)$.

Given d stochastically integrable processes $G_j(t)$, i.e. $G_j \in \mathcal{L}^2$, the Itô integrals $\sum_j \int_0^t G_j(s)\mathrm{d}\widehat{W}_j(s)$ and $\sum_j \int_0^t G_j(s)\mathrm{d}W_j(s)$ are defined for every $t \in [0, T]$, each one under its corresponding probability law, and we have $\mathbb{Q}$-a.s. and $\widehat{\mathbb{P}}^T_{\psi_0}$-a.s.

$$\sum_{j=1}^d \int_0^t G_j(s)\mathrm{d}\widehat{W}_j(s) = \sum_{j=1}^d \int_0^t G_j(s)\mathrm{d}W_j(s) - \sum_{j=1}^d \int_0^t G_j(s)m_j(s)\mathrm{d}s, \quad \forall t \in [0, T]\,. \tag{2.32}$$

Proposition 2.15. *The processes $\widehat{\psi}$, m, $\widehat{W}$ are $(\overline{\mathcal{D}}^0_t)$-adapted.*

Proof. The statement follows immediately from the definitions (2.22), (2.23), (2.31) and Proposition 2.7. □

2.4 The Physical Interpretation

Let us begin with a list of the mathematical objects involved by the linear stochastic Schrödinger equation (2.28) and their heuristic interpretation in the theory of continuous measurements, in analogy with the traditional representation of an instantaneous discrete measurement (Sect. 2.1).

- ψ_0 is the initial state of the quantum system;
- $(\Omega, \mathcal{F})$ is the measurable space of the possible outcomes of the experiment;
- $\mathcal{F}_t$ is the collection of events verifiable already at time t;
- the d stochastic processes $W_j(t)$ are the output of the continuous measurement and their derivatives $\dot{W}_j(t)$ can be interpreted as instantaneous imprecise measurements of the quantum observables $R_j(t) + R_j(t)^*$ performed at time t;
- $\overline{\mathcal{D}}_t^0$ is the collection of events verifiable already at time t which effectively regard the continuous measurement;
- the stochastic linear state transformation $\psi_0 \mapsto \psi(t) = A_t^0 \psi_0$ gives both the probability of the events, which could occur up to time t, and the state of the quantum system conditioned on the observation in the time interval $[0, t]$:
  - $\widehat{\mathbb{P}}_{\psi_0}^T$ is the physical probability law of the events which could occur in $[0, T]$;
  - $\widehat{\psi}(t, \omega)$ is the state of the system at time t, conditioned on having observed the trajectory $s \mapsto W(s, \omega)$ up to time t.

When the canonical realisation of the Wiener process is used, i.e. when the only output of the experiment is the diffusive process W, the outcome ω itself can be identified with the trajectory of the output; indeed in this case we have $W(s, \omega) = \omega(s)$ (see Remark A.23). Then $\psi(t, \omega)$, $\|\psi(t, \omega)\|$ and $\widehat{\psi}(t, \omega)$ depend only on $\omega(s)$ for $0 \le s \le t$. In particular, $\|\psi(t, \omega)\|^2$ is the density of probability (with respect to the Wiener measure) of observing $W(s) = \omega(s)$ in the time interval $0 \le s \le t$.

When $\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\right)$ is bigger than the canonical realisation of the Wiener process, still $\psi(t, \omega)$, $\|\psi(t, \omega)\|$ and $\widehat{\psi}(t, \omega)$ depend only on $W(s)$ for $0 \le s \le t$ because the stochastic processes $\psi(t)$, $\|\psi(t)\|$ and $\widehat{\psi}(t)$ are adapted to $(\overline{\mathcal{D}}_t^0)$ and thus $\psi(t, \omega) = \psi(t, \omega')$ if $W(s, \omega) = W(s, \omega')$ for $0 \le s \le t$ (maybe except for a set of null probability).

Therefore, even if from a mathematical point of view it can be convenient to work with a Wiener process W with increments independent of the past in an arbitrary filtration $(\mathcal{F}_t)$, from a physical point of view the relevant filtration is always $(\overline{\mathcal{D}}_t^0)$: it contains all the events regarding the output W of the measurement and, moreover, only these events really condition the system state $\widehat{\psi}$.

What we have to do now is to show that this interpretation is consistent with the general formulation of quantum mechanics. However, let us first add two further remarks on the physical interpretation.

The use itself of linear SDEs to assign the evolution of $\psi(t)$ implies a Markovian hypothesis about the observed quantum system and the measurement process: for every $0 \le s \le t$, in spite of all the information available at time s (the initial state ψ_0 and all the events in $\mathcal{F}_s$), the conditioned state $\widehat{\psi}(s)$ at time s is sufficient to evaluate the conditional state $\widehat{\psi}(t)$ at time t (together with the output W in $[s, t]$, of course).

There are two typical but physically different interpretations of the linear stochastic Schrödinger equation (2.28). Sometimes it is obtained by starting from a free closed evolution of the quantum system and introducing the continuous measurement as a perturbation, by adding a stochastic term in the evolution equation for

every continuously monitored quantum observable $R_j(t) + R_j(t)^*$. In this case, one can think of possibly switching off the measurement ($R_j(t) \equiv 0$), and the linear stochastic Schrödinger equation (2.28) reduces to an ordinary Schrödinger equation $\mathrm{d}\psi(t) = -\mathrm{i}H(t)\psi(t)\,\mathrm{d}t$. Other times, the linear stochastic Schrödinger equation (2.28) is obtained by starting from an open evolution of the quantum system $\mathcal{H}$ and introducing continuous measurements which acquire information on the system without introducing extra perturbations (e.g. the continuous monitoring of an atom by the detection of its fluorescence light). In this case, the "mean" evolution of the quantum system is not modified by the continuous measurement, but it is "unravelled" in many different trajectories according to the observed output W.

2.4.1 The POM of the Output and the Physical Probabilities

First, we introduce properly the positive operator-valued measure (see Definition B.1) associated with the continuous measurement in the time interval $[0, T]$. Taking the stochastic evolution operator A_T^0 associated with the linear stochastic Schrödinger equation (2.28), we can define

$$\widehat{E}_T(F) := \int_F A_T^0(\omega)^* A_T^0(\omega)\mathbb{Q}(\mathrm{d}\omega) \equiv \mathbb{E}_{\mathbb{Q}}\left[1_F A_T^{0*} A_T^0\right], \qquad F \in \mathcal{F}_T\,. \tag{2.33}$$

Then, $\widehat{E}_T$ is a positive operator-valued measure (POM) on the value space $(\Omega, \mathcal{F}_T)$. Indeed, it is positive and σ-additive by construction and, moreover,

$$\langle\psi_0|\widehat{E}_T(\Omega)\psi_0\rangle = \mathbb{E}_{\mathbb{Q}}\left[\langle\psi_0|A_T^{0*}A_T^0\psi_0\rangle\right] = \mathbb{E}_{\mathbb{Q}}\left[\|\psi(T)\|^2\right] = \|\psi_0\|^2, \qquad \forall\psi_0 \in \mathcal{H},$$

which implies $\widehat{E}_T(\Omega) = \mathbb{1}$ by the normalisation of ψ_0.

The POM $\widehat{E}_T$ assigns to each event in $\mathcal{F}_T$, according to the axioms of Sect. B.2.1, just the probability $\widehat{\mathbb{P}}^T_{\psi_0}$ that we called physical probability. Indeed, by (2.10) and (2.33) we get

$$\langle\psi_0|\widehat{E}_T(F)\psi_0\rangle = \mathbb{E}_{\mathbb{Q}}\left[\langle\psi_0|A_T^{0*}A_T^0\psi_0\rangle 1_F\right] = \int_F \|\psi(T,\omega)\|^2\,\mathbb{Q}(\mathrm{d}\omega) \tag{2.34}$$

and, by (2.29),

$$\langle\psi_0|\widehat{E}_T(F)\psi_0\rangle = \widehat{\mathbb{P}}^T_{\psi_0}(F), \qquad \forall F \in \mathcal{F}_T, \tag{2.35}$$

which is the standard formula for probabilities in the Hilbert space formulation of quantum mechanics.

Moreover, $\left\{\widehat{E}_T,\ T > 0\right\}$ is a consistent family of POMs, in the sense that

$$0 < S < T,\ F \in \mathcal{F}_S \ \Rightarrow\ \widehat{E}_T(F) = \widehat{E}_S(F)\,. \tag{2.36}$$

Indeed, for every $0 < S < T$, $F \in \mathcal{F}_S$, A_T^S being independent of $\mathcal{F}_S$, one gets

$$\begin{aligned}\widehat{E}_T(F) &= \mathbb{E}_{\mathbb{Q}}\left[1_F A_T^{0*} A_T^0\right] = \mathbb{E}_{\mathbb{Q}}\left[1_F A_S^{0*} A_T^{S*} A_T^S A_S^0\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[1_F A_S^{0*}\, \mathbb{E}_{\mathbb{Q}}\left[A_T^{S*} A_T^S | \mathcal{F}_S\right] A_S^0\right] = \mathbb{E}_{\mathbb{Q}}\left[1_F A_S^{0*}\, \mathbb{E}_{\mathbb{Q}}\left[A_T^{S*} A_T^S\right] A_S^0\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[1_F A_S^{0*} A_S^0\right] = \widehat{E}_S(F).\end{aligned}$$

Another way to look at (2.33) is to say that $A_t^{0*} A_t^0$ is the density (or Radon–Nikodym derivative) of the POM $\widehat{E}_t$ with respect to the probability measure

$$\mathbb{Q}_t := \mathbb{Q}\big|_{\mathcal{F}_t}. \tag{2.37}$$

By recalling that $A_t^{0*} A_t^0$ is $\mathcal{F}_t$-measurable, we can write

$$\frac{\widehat{E}_t(\mathrm{d}\omega)}{\mathbb{Q}_t(\mathrm{d}\omega)} = A_t^0(\omega)^* A_t^0(\omega). \tag{2.38}$$

We already discussed the fact that the filtration $(\mathcal{F}_t)$ could be unnecessarily large: the natural value space, when the output of the continuous measurement is the process W in the time interval $[0, t]$, is $(\Omega, \overline{\mathcal{D}}_t^0)$. Moreover, we could perform the measurement only in the time interval $[s, t]$. As in the evolution equations only the increments of W appear (through the $\mathrm{d}W$ term), the natural candidate to be the output in the time interval $[s, t]$ is the process $W(r) - W(s)$, $r \in [s, t]$, which generates the set of events $\overline{\mathcal{D}}_t^s$. Thus, analogous to (2.33), we define a POM $\widehat{E}_t^s$ on the value space $(\Omega, \overline{\mathcal{D}}_t^s)$ by

$$\widehat{E}_t^s(F) := \int_F A_t^s(\omega)^* A_t^s(\omega)\, \mathbb{Q}(\mathrm{d}\omega) \equiv \mathbb{E}_{\mathbb{Q}}\left[1_F A_t^{s*} A_t^s\right], \qquad F \in \overline{\mathcal{D}}_t^s. \tag{2.39}$$

By this definition, we have that $\widehat{E}_t^0$ is the restriction of $\widehat{E}_t$ to $\overline{\mathcal{D}}_t^0$. Also the new POMs (2.39) are consistent with respect to t. In order to use $\widehat{E}_t^s$ for an arbitrary $s > 0$ one needs to know the system state at time s.

By noticing that the positive operator-valued random variable $A_t^{s*} A_t^s$ is $\overline{\mathcal{D}}_t^s$-measurable, we get that the analog of (2.38) is

$$\frac{\widehat{E}_t^s(\mathrm{d}\omega)}{\mathbb{Q}(\mathrm{d}\omega)\big|_{\overline{\mathcal{D}}_t^s}} = A_t^{s*}(\omega) A_t^s(\omega). \tag{2.40}$$

Summing up, the POM representing the output of the continuous measurement in the time interval $[s, t]$ is $\widehat{E}_t^s$. Even if $s = 0$, the relevant POM is $\widehat{E}_t^0$, not $\widehat{E}_t$. We can also say that the physical probability, the probability of the events determined by the output in the time interval $[0, T]$, is $\widehat{\mathbb{P}}_{\psi_0}^T\big|_{\overline{\mathcal{D}}_t^0}$.

The probability $\widehat{\mathbb{P}}^T_{\psi_0}$ of events in the (augmented) natural filtration of W is obtained from a POM, as prescribed by quantum mechanics, whose value space is $\left(\Omega, \overline{\mathcal{D}}^0_T\right)$. We interpret W as the output of a continuous measurement performed on the quantum system $\mathcal{H}$ in the time interval $[0, T]$ and $\widehat{\mathbb{P}}^T_{\psi_0}$ as the corresponding physical probability. Moreover, from Girsanov formula (2.31) we have

$$W(t) = \widehat{W}(t) + \int_0^t m(s)\mathrm{d}s, \qquad t \in [0, T],$$

which says that the output process $W(t)$ decomposes to the sum of a Wiener process $\widehat{W}(t)$ and a process $\int_0^t m(s)\mathrm{d}s$ with trajectories of bounded variation. Let us remark that, even if it could be suggestive to interpret the two addenda as noise and signal, the two processes are typically not independent.

Remark 2.16. Here it is worthwhile to be more precise on the notion of output of the measurement. As already said, the choice of the two-time σ-algebras $\mathcal{D}^s_t$ or $\overline{\mathcal{D}}^s_t$, which are determined by the increments of W, reflects the fact that we consider as events which can be observed in the time interval $[s, t]$ only the events related to the increments of W with extreme times inside $[s, t]$, not the ones determined by $W(r)$ with $r \in [s, t]$. So, in this time interval, we observe the increments $W(r) - W(u)$, $s \leq u < r \leq t$, or functionals of these increments. "Morally" the output is the singular process $\dot{W}(r)$, $r \in [s, t]$. In the whole book we always understand this interpretation, even when we write that the output is W.

2.4.2 The A Posteriori States

Now we would like to justify the interpretation of $\widehat{\psi}(t)$ as the conditional state of the system at time t, i.e. as *a posteriori state* at time t (cf. Sect. B.4.3.2). We shall do this properly in Sect. 4.1.1, where, in a more general setup, we shall introduce explicitly the instruments. In the present paragraph, we only show that this interpretation is consistent with the present construction.

Let us consider an event F regarding the output in the time interval $[s, t]$, that is $F \in \overline{\mathcal{D}}^s_t$. If we evaluate its probability at the beginning of the experiment, when we only know that the initial state of the system is ψ_0, then we get $\widehat{\mathbb{P}}^T_{\psi_0}(F)$. On the other hand, if we reconsider the same event F at time s, when we have gathered all the information coming from the measurement in the time interval $[0, s]$, then its probability can be updated and it is given by $\widehat{\mathbb{P}}^T_{\psi_0}(F|\mathcal{F}_s) = \widehat{\mathbb{E}}^T_{\psi_0}[1_F|\mathcal{F}_s]$ (Sect. A.1.2.2). This is an $\mathcal{F}_s$-measurable random variable, as it depends on what is observed up to time s. The following proposition states that it can be computed using the POM $\widehat{E}^s_t$ defined by (2.39) and just $\widehat{\psi}(s)$ as the conditional state of the system at time s.

Proposition 2.17. *For all $F \in \overline{\mathcal{D}}^s_t$, $0 \leq s < t \leq T$, we have*

$$\widehat{\mathbb{P}}^T_{\psi_0}(F|\mathcal{F}_s) = \left\langle \widehat{\psi}(s) \middle| \widehat{E}^s_t(F)\widehat{\psi}(s) \right\rangle = \widehat{\mathbb{P}}^T_{\psi_0}(F|\overline{\mathcal{D}}^0_s). \tag{2.41}$$

Proof. For all $\mathcal{F}_s$-measurable bounded random variables Y, we get

$$\begin{aligned}\widehat{\mathbb{E}}^T_{\psi_0}[1_F Y] &= \mathbb{E}_{\mathbb{Q}}\left[\|\psi(t)\|^2 1_F Y\right] = \mathbb{E}_{\mathbb{Q}}\left[\langle\psi_0|A^{0*}_s A^{s*}_t A^s_t A^0_s\psi_0\rangle 1_F Y\right]\\ &= \mathbb{E}_{\mathbb{Q}}\left[\langle\psi_0|A^{0*}_s\,\mathbb{E}_{\mathbb{Q}}\left[1_F A^{s*}_t A^s_t|\mathcal{F}_s\right]A^0_s\psi_0\rangle Y\right]\\ &= \mathbb{E}_{\mathbb{Q}}\left[\langle\psi(s)|\,\mathbb{E}_{\mathbb{Q}}\left[1_F A^{s*}_t A^s_t\right]\psi(s)\rangle Y\right]\\ &= \widehat{\mathbb{E}}^T_{\psi_0}\left[\langle\widehat{\psi}(s)|\,\mathbb{E}_{\mathbb{Q}}\left[1_F A^{s*}_t A^s_t\right]\widehat{\psi}(s)\rangle Y\right];\end{aligned}$$

we have used the equality $\psi(t) = A^s_t A^0_s\psi_0$ and the independence of $1_F A^{s*}_t A^s_t$ from $\mathcal{F}_s$, which follows from Proposition 2.7. This computation proves that $\widehat{\mathbb{E}}^T_{\psi_0}[1_F|\mathcal{F}_s] = \left\langle\widehat{\psi}(s)\right|\mathbb{E}_{\mathbb{Q}}\left[1_F A^{s*}_t A^s_t\right]\widehat{\psi}(s)\rangle$. By using the definition of $\widehat{E}^s_t$ we have $\widehat{\mathbb{E}}^T_{\psi_0}[1_F|\mathcal{F}_s] = \left\langle\widehat{\psi}(s)\right|\widehat{E}^s_t(F)\widehat{\psi}(s)\rangle$. By the fact that $\overline{\mathcal{D}}^0_s \subset \mathcal{F}_s$ and that $\left\langle\widehat{\psi}(s)\right|\widehat{E}^s_t(F)\widehat{\psi}(s)\rangle$ is $\overline{\mathcal{D}}^0_s$-measurable, we have

$$\begin{aligned}\widehat{\mathbb{E}}^T_{\psi_0}\left[1_F\middle|\overline{\mathcal{D}}^0_s\right] &= \widehat{\mathbb{E}}^T_{\psi_0}\left[\widehat{\mathbb{E}}^T_{\psi_0}[1_F|\mathcal{F}_s]\middle|\overline{\mathcal{D}}^0_s\right]\\ &= \widehat{\mathbb{E}}^T_{\psi_0}\left[\left\langle\widehat{\psi}(s)\right|\widehat{E}^s_t(F)\widehat{\psi}(s)\rangle\middle|\overline{\mathcal{D}}^0_s\right] = \left\langle\widehat{\psi}(s)\right|\widehat{E}^s_t(F)\widehat{\psi}(s)\rangle.\end{aligned}$$ □

Remark 2.18. As suggested in the presentation before the proposition, by comparing (2.41) with (2.35), we see that we can interpret the state $\widehat{\psi}(s)$ as the conditional state of the system at time s; we call $\widehat{\psi}(t)$ the *a posteriori state* at time t. Considering also (2.34), we call $\psi(t)$ the *non-normalised a posteriori state* at time t.

With this interpretation in mind, we consider again the output and, thanks to the representation

$$W_j(t) = \widehat{W}_j(t) + \int_0^t \left\langle\widehat{\psi}(s)\right|\left(R_j(s) + R_j(s)^*\right)\widehat{\psi}(s)\rangle\mathrm{d}s, \qquad t \in [0,T], \tag{2.42}$$

we say that $\dot{W}_j(t)$ is an imprecise measurement of the quantum observable $R_j(t) + R_j(t)^*$. We shall consider again this interpretation in Sect. 4.3.

Remark 2.19 (A phase change). Let us consider now the normalised random vector $\widehat{\phi}(t,\omega) = \mathrm{e}^{\mathrm{i}\alpha(t,\omega)}\widehat{\psi}(t,\omega)$, where $\{\alpha(t),\ t \geq 0\}$ is an arbitrary $(\overline{\mathcal{D}}^0_t)$-adapted real process. By substituting $\widehat{\phi}(t)$ to $\widehat{\psi}(t)$ in (2.41), this formula continues to hold true. This means that $\widehat{\phi}(t)$ has the same right of $\widehat{\psi}(t)$ to the name of "a posteriori state". But this is nothing more than the stochastic version of the usual statement in quantum mechanics that a phase change of the state vector does not alter any physical quantity.

2.4.3 Infinite Time Horizon

Given the initial state ψ_0, we have a consistent set of probabilities $\widehat{\mathbb{P}}^T_{\psi_0}$, $T > 0$. As stated by Theorem 2.14, each $\widehat{\mathbb{P}}^T_{\psi_0}$ modifies the properties of the stochastic process

W, in the corresponding time interval $[0, T]$. A natural question is whether it is possible to have a unique probability for $T \to +\infty$. This would be useful, for instance, to study the long-time behaviour of W under the physical probability. By the discussion in Sect. A.5.5, we know that this is possible when the consistent measures are defined on standard Borel spaces. Therefore, if we consider the new probabilities restricted to $\mathcal{D}_t^0$, the natural (not augmented) filtration of W, we have a consistent set of probabilities on standard Borel spaces and we get that there exists a unique probability $\widehat{\mathbb{P}}_{\psi_0}^{\infty}$ on $\mathcal{D}_\infty^0 := \bigvee_{t>0} \mathcal{D}_t^0$ such that for every $T > 0$

$$\widehat{\mathbb{P}}_{\psi_0}^{\infty}(F) = \widehat{\mathbb{P}}_{\psi_0}^{T}(F), \qquad \forall F \in \mathcal{D}_T^0 \,. \tag{2.43}$$

Nevertheless, even if we choose $\mathcal{F} = \mathcal{D}_\infty^0$, each augmented σ-algebra $\overline{\mathcal{D}}_T^0$ is strictly greater than $\mathcal{D}_T^0$ and the limit probability $\widehat{\mathbb{P}}_{\psi_0}^{\infty}$ typically does not agree with $\widehat{\mathbb{P}}_{\psi_0}^{T}$ on the whole $\overline{\mathcal{D}}_T^0 \subset \mathcal{D}_\infty^0$. In order to work inside the filtration $(\mathcal{D}_t^0)$, it is enough to consider $(\mathcal{D}_t^0)$-adapted versions of the processes A_t^0, $\psi(t)$, $\widehat{\psi}(t)$, $m(t)$, $\widehat{W}(t)$. What we lose is that we are no more sure to have continuity in time for every ω.

Just to have an example of the differences, let us consider the POMs. By restricting $\widehat{E}_t^s$ to $\mathcal{D}_t^s$, we get from (2.40)

$$\widehat{E}_t^s(\mathrm{d}\omega)\big|_{\mathcal{D}_t^s} = \mathbb{E}_{\mathbb{Q}}\left[A_t^{s*} A_t^s | \mathcal{D}_t^s\right](\omega)\mathbb{Q}(\mathrm{d}\omega)\big|_{\mathcal{D}_t^s} \,, \tag{2.44}$$

but $\mathbb{E}_{\mathbb{Q}}\left[A_t^{s*} A_t^s | \mathcal{D}_t^s\right](\omega) = A_t^{s*}(\omega) A_t^s(\omega)$, $\mathbb{Q}$-a.s.

2.4.4 The Conservative Case

A very particular case is when the operators $R_j(t)$ are anti-selfadjoint [23]:

$$R_j(t) = -\mathrm{i} V_j(t) \,, \qquad V_j(t)^* = V_j(t) \,. \tag{2.45}$$

Equations (2.23), (2.26), (2.45) give $m_j(t) = 0$ and (for $\|\psi_0\| = 1$) $\|\psi(t)\| = 1$, $\forall t$. This implies $\widehat{\mathbb{P}}_{\psi_0}^{T} = \mathbb{Q}_T$, $\forall T > 0$, so that the randomness does not depend on the quantum system: the W_j are pure noises and there is no true measurement on the system.

The linear stochastic Shrödinger equation becomes

$$\mathrm{d}\psi(t) = -\mathrm{i}\left[H(t)\mathrm{d}t + \sum_j V_j(t)\mathrm{d}W_j(t)\right]\psi(t) - \frac{1}{2}\sum_j V_j(t)^2\psi(t)\mathrm{d}t \,, \tag{2.46}$$

and one can check that $(A_t^s)^*$ and $(A_t^s)^{-1}$ satisfy the same SDE: $(A_t^s)^* A_t^s \equiv \mathbb{1}$. Thus the system undergoes a stochastic unitary evolution: the quantum system has a unitary evolution in a random environment which determines the stochastic potential acting on the system. Even if W is observed, the measurement does not acquire any

information on the quantum system itself, but it only detects which unitary evolution occurs among the possible ones.

This class of stochastic Shrödinger equations was introduced as a model of dissipative evolution, with W not observed. In this case, all the physical quantities are obtained with a mean with respect to W. For an example of this approach to quantum open systems, see [2].

2.5 The Stochastic Schrödinger Equation

A key point of the theory is to show that the a posteriori states satisfy an SDE, closed in $\widehat{\psi}(t)$ itself. The structure of such an equation is not of usual type, but it is possible, after some work, to arrive at a theorem giving existence and uniqueness of the solutions. Different approaches to the existence and uniqueness problem, in finite and infinite dimensional Hilbert spaces, are given in [24, 27, 29, 30].

2.5.1 The Stochastic Differential of the A Posteriori State

Let us compute the stochastic differential of the a posteriori state $\widehat{\psi}(t) = \|\psi(t)\|^{-1}\ \psi(t)$ under the physical probability $\widehat{\mathbb{P}}^T_{\psi_0}$ and in terms of the new Wiener process $\widehat{W}$. To put in full evidence the dependence of the differential on $\widehat{\psi}(t)$ itself, it is useful to introduce the quantities

$$n_j(t,x) := \langle x|R_j(t)x\rangle\,, \qquad t\in[0,+\infty), \quad x\in\mathcal{H}\,. \tag{2.47}$$

Note that

$$m_j(t) = 2\,\mathrm{Re}\,n_j\left(t,\widehat{\psi}(t)\right). \tag{2.48}$$

Proposition 2.20. *Under the probability $\widehat{\mathbb{P}}^T_{\psi_0}$, the stochastic differential of $\widehat{\psi}(t)$, $0\le t<T$, is*

$$\begin{aligned}\mathrm{d}\widehat{\psi}(t) = &\sum_j\left[R_j(t)-\mathrm{Re}\,n_j\left(t,\widehat{\psi}(t)\right)\right]\widehat{\psi}(t)\,\mathrm{d}\widehat{W}_j(t)\\ &+\left[K(t)+\sum_j\left(\mathrm{Re}\,n_j\left(t,\widehat{\psi}(t)\right)\right)R_j(t)-\frac{1}{2}\sum_j\left(\mathrm{Re}\,n_j\left(t,\widehat{\psi}(t)\right)\right)^2\right]\widehat{\psi}(t)\,\mathrm{d}t\,.\end{aligned} \tag{2.49}$$

Proof. It is enough to apply Itô rules to $\widehat{\psi}(t) = \|\psi(t)\|^{-1}\ \psi(t)$ under the probability $\widehat{\mathbb{P}}^T_{\psi_0}$. By using (2.42) we can transform (2.2) into

$$\mathrm{d}\psi(t) = \sum_j R_j(t)\psi(t)\,\mathrm{d}\widehat{W}_j(t) + \left(K(t) + \sum_j m_j(t)R_j(t)\right)\psi(t)\,\mathrm{d}t\,.$$

By using (2.24), (2.42) and the fact that $\|\psi(t)\| > 0$ with probability one, we get

$$\begin{aligned}\|\psi(t)\|^{-1} &= \exp\left\{-\frac{1}{2}\sum_j\left[\int_0^t m_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t m_j(s)^2\mathrm{d}s\right]\right\}\\ &= \exp\left\{-\frac{1}{2}\sum_j\left[\int_0^t m_j(s)\mathrm{d}\widehat{W}_j(s) + \frac{1}{2}\int_0^t m_j(s)^2\mathrm{d}s\right]\right\};\end{aligned}$$

by Itô formula, this gives

$$\begin{aligned}\mathrm{d}\,\|\psi(t)\|^{-1} &= \|\psi(t)\|^{-1}\left\{-\frac{1}{2}\sum_j\left[m_j(t)\mathrm{d}\widehat{W}_j(t) + \frac{1}{2}m_j(t)^2\mathrm{d}t\right] + \frac{1}{8}\sum_j m_j(t)^2\mathrm{d}t\right\}\\ &= -\frac{1}{2}\,\|\psi(t)\|^{-1}\sum_j\left[m_j(t)\mathrm{d}\widehat{W}_j(t) + \frac{1}{4}m_j(t)^2\mathrm{d}t\right].\end{aligned}$$

Finally, by using the Itô rules for the differential of a product, we obtain

$$\begin{aligned}\mathrm{d}\widehat{\psi}(t) &= \sum_j R_j(t)\widehat{\psi}(t)\,\mathrm{d}\widehat{W}_j(t) + \left(K(t) + \sum_j m_j(t)R_j(t)\right)\widehat{\psi}(t)\,\mathrm{d}t\\ &\quad -\frac{1}{2}\sum_j m_j(t)\widehat{\psi}(t)\,\mathrm{d}\widehat{W}_j(t) - \frac{1}{8}\sum_j m_j(t)^2\widehat{\psi}(t)\,\mathrm{d}t - \frac{1}{2}\sum_j m_j(t)R_j(t)\widehat{\psi}(t)\,\mathrm{d}t\\ &= \sum_j\left[R_j(t) - \frac{m_j(t)}{2}\right]\widehat{\psi}(t)\,\mathrm{d}\widehat{W}_j(t)\\ &\quad + \left[K(t) + \sum_j\frac{m_j(t)}{2}R_j(t) - \sum_j\frac{m_j(t)^2}{8}\right]\widehat{\psi}(t)\,\mathrm{d}t\,.\end{aligned}$$

By using the notation $n_j(t,x)$ introduced in Definition (2.47), we get (2.49). □

2.5.1.1 A Stochastic Phase Change

Let us stress that no physical consequence depends on the phase of $\widehat{\psi}(t)$: consider the presentation of quantum mechanics given in Appendix B, the POM (2.33), the probabilities (2.35), the processes m_j (2.23), the output W (2.42), etc. So, we are allowed to make any change of phase on $\widehat{\psi}(t)$, even a stochastic one. In this order of ideas, we introduce the new normalised vectors

$$\widehat{\phi}(t) := \exp\left\{ -\mathrm{i}\sum_j \int_0^t \mathrm{Re}\, n_j\left(s, \widehat{\psi}(s)\right) \mathrm{Im}\, n_j\left(s, \widehat{\psi}(s)\right)\, \mathrm{d}s \right.$$
$$\left. -\mathrm{i}\sum_j \int_0^t \mathrm{Im}\, n_j\left(s, \widehat{\psi}(s)\right)\, \mathrm{d}\widehat{W}_j(s) \right\} \widehat{\psi}(t)\,. \tag{2.50}$$

The vectors $\widehat{\phi}(t)$ have the same right to the name of "a posteriori states" as the vectors $\widehat{\psi}(t)$. By applying Itô formula to (2.50) and by using the differential (2.49) and the fact that $n_j\left(t, \widehat{\psi}(t)\right) \equiv n_j\left(t, \widehat{\phi}(t)\right)$, we get the stochastic differential of $\widehat{\phi}(t)$:

$$\mathrm{d}\widehat{\phi}(t) = \sum_j \left[R_j(t) - n_j\left(t, \widehat{\phi}(t)\right)\right] \widehat{\phi}(t)\, \mathrm{d}\widehat{W}_j(t)$$
$$+ \left[K(t) + \sum_j \overline{n_j\left(t, \widehat{\phi}(t)\right)}\, R_j(t) - \frac{1}{2}\sum_j \left|n_j\left(t, \widehat{\phi}(t)\right)\right|^2 \right] \widehat{\phi}(t)\, \mathrm{d}t. \tag{2.51}$$

Thus the choice (2.50) gives a simple expression for $\mathrm{d}\widehat{\phi}(t)$ which is commonly used in the literature, just as (2.49).

2.5.2 Four Stochastic Schrödinger Equations

Both equalities (2.49) and (2.51) are closed equations, in the stochastic processes $\widehat{\psi}$ and $\widehat{\phi}$ respectively, and both are known under the name of *stochastic Schrödinger equation* [9]. However, we got them for normalised vector processes and thus if we want to interpret them as SDEs for $\mathcal{H}$-vector processes, we need to extend them also to non-normalised vectors. There is not a unique way to do such an extension and we present for each of them two extensions, the most natural ones.

Equalities (2.49) and (2.51) involve the quantities $n_j(t, x)$ for normalised x. The first type of extension is to allow for a non-normalised x in the quadratic form (2.47) defining n_j; in this way polynomial coefficients are obtained. The second type of extension is to write $n_j(t, x)/\|x\|^2$ everywhere n_j appears in the differentials of normalised states and then to extend the resulting expressions in the natural way to non-normalised x; in this way we obtain coefficients with at most linear growth.

Thus, we obtain four nonlinear stochastic Schrödinger equations ($\ell = 1, 2, 3, 4$)

$$\begin{cases} \mathrm{d}X^\ell(t) = \sum_j L_j^\ell\left(t, X^\ell(t)\right) X^\ell(t)\, \mathrm{d}\widehat{W}_j(t) + K^\ell\left(t, X^\ell(t)\right) X^\ell(t)\, \mathrm{d}t\,, \\ X^\ell(0) = x_0\,, \quad x_0 \in \mathcal{H}\,, \end{cases} \tag{2.52}$$

where the quantities $L_j^\ell(t, x)$ and $K^\ell(t, x)$ are defined in the following; they are introduced in order to write the four SDEs always in a compact form.

The extension of the stochastic Schrödinger equation for $\widehat{\psi}$ with polynomial coefficients is obtained by taking

$$L_j^1(t,x) := R_j(t) - \operatorname{Re} n_j(t,x), \tag{2.53a}$$

$$\begin{aligned} K^1(t,x) &:= K(t) + \sum_j \big(\operatorname{Re} n_j(t,x)\big) R_j(t) - \frac{1}{2}\sum_j \big(\operatorname{Re} n_j(t,x)\big)^2 \\ &\equiv -\mathrm{i}\left[H(t) + \frac{\mathrm{i}}{2}\sum_j \big(\operatorname{Re} n_j(t,x)\big)\big(R_j(t) - R_j(t)^*\big)\right] \\ &\quad -\frac{1}{2}\sum_j L_j^1(t,x)^* L_j^1(t,x). \end{aligned} \tag{2.53b}$$

The extension of the stochastic Schrödinger equation for $\widehat{\psi}$ with linearly growing coefficients is given by the choice

$$L_j^2(t,x) := R_j(t) - \operatorname{Re}\frac{n_j(t,x)}{\|x\|^2}\,, \tag{2.54a}$$

$$\begin{aligned} K^2(t,x) &:= K(t) + \sum_j \left(\operatorname{Re}\frac{n_j(t,x)}{\|x\|^2}\right) R_j(t) - \frac{1}{2}\sum_j \left(\operatorname{Re}\frac{n_j(t,x)}{\|x\|^2}\right)^2 \\ &\equiv -\mathrm{i}\left[H(t) + \frac{\mathrm{i}}{2}\sum_j \left(\operatorname{Re}\frac{n_j(t,x)}{\|x\|^2}\right)\big(R_j(t) - R_j(t)^*\big)\right] \\ &\quad -\frac{1}{2}\sum_j L_j^2(t,x)^* L_j^2(t,x). \end{aligned} \tag{2.54b}$$

The extension of the stochastic Schrödinger equation for $\widehat{\phi}$ with polynomial coefficients is the one with

$$L_j^3(t,x) := R_j(t) - n_j(t,x), \tag{2.55a}$$

$$\begin{aligned} K^3(t,x) &:= K(t) + \sum_j \overline{n_j(t,x)}\, R_j(t) - \frac{1}{2}\sum_j \big|n_j(t,x)\big|^2 \\ &\equiv -\mathrm{i}\left[H(t) + \frac{\mathrm{i}}{2}\sum_j \big(\overline{n_j(t,x)}\, R_j(t) - n_j(t,x) R_j(t)^*\big)\right] \\ &\quad -\frac{1}{2}\sum_j L_j^3(t,x)^* L_j^3(t,x). \end{aligned} \tag{2.55b}$$

Finally, the extension of the stochastic Schrödinger equation for $\widehat{\phi}$ with linearly growing coefficients is obtained by taking

$$L_j^4(t,x) := R_j(t) - \frac{n_j(t,x)}{\|x\|^2}\,, \tag{2.56a}$$

$$\begin{aligned} K^4(t,x) &:= K(t) + \sum_j \frac{\overline{n_j(t,x)}}{\|x\|^2}\, R_j(t) - \sum_j \frac{\left|n_j(t,x)\right|^2}{2\,\|x\|^4} \\ &\equiv -\mathrm{i}\left[H(t) + \frac{\mathrm{i}}{2}\sum_j \left(\frac{\overline{n_j(t,x)}}{\|x\|^2}\, R_j(t) - \frac{n_j(t,x)}{\|x\|^2}\, R_j(t)^* \right) \right] \\ &\quad -\frac{1}{2}\sum_j L_j^4(t,x)^* L_j^4(t,x). \end{aligned} \tag{2.56b}$$

We are using the convention that

$$\frac{n_j(t,x)}{\|x\|^2} = 0 \qquad \text{for } x = 0. \tag{2.57}$$

When $\|x\| = 1$ we have $L^1(t,x) = L^2(t,x)$ and $K^1(t,x) = K^2(t,x)$ and the SDE (2.52) for $\ell = 1, 2$ reduces to (2.49) when $x_0 = \psi_0$, if one proves that the solution stays normalised for all t. However, the two equations are different when the initial condition has no norm one. Similarly, for $\|x\| = 1$ we have $L^3(t,x) = L^4(t,x)$ and $K^3(t,x) = K^4(t,x)$ and the SDE (2.52) for $\ell = 3, 4$ reduces to (2.51) when $x_0 = \phi_0$, if one proves that the solution stays normalised for all t.

2.5.2.1 The Conservative Case

This is the case $R_j(t)^* = -R_j(t)$ of Section 2.4.4, corresponding to dissipation, but no effective measurement. By setting $R_j(t) = -\mathrm{i}V_j(t)$, with $V_j(t)^* = V_j(t)$, we find

$$\mathrm{Re}\, n_j(t,x) = 0\,, \tag{2.58}$$

$$L_j^1(t,x) = L_j^2(t,x) = -\mathrm{i}V_j(t)\,, \tag{2.59a}$$

$$K^1(t,x) = K^2(t,x) = K(t) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j V_j(t)^2\,, \tag{2.59b}$$

$$L_j^3(t,x) = -\mathrm{i}\left[V_j(t) - \langle x|V_j(t)x\rangle\right], \tag{2.60a}$$

$$L_j^4(t,x) = -\mathrm{i}\left[V_j(t) - \|x\|^{-2}\,\langle x|V_j(t)x\rangle\right], \tag{2.60b}$$

$$K^3(t,x) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j L_j^3(t,x)^* L_j^3(t,x)\,, \tag{2.60c}$$

$$K^4(t,x) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j L_j^4(t,x)^* L_j^4(t,x)\,. \tag{2.60d}$$

Then, for $\ell = 1, 2$ the stochastic Schrödinger equations (2.52) are linear and they coincide with the corresponding linear stochastic Schrödinger equation (2.46), while for $\ell = 3, 4$ they are nonlinear, but only due to a non-influent phase factor.

2.5.2.2 A Peculiar Case of Continuous Measurement

In the literature, when the case is considered of usual observables followed with continuity in time, a common choice is to take $R_j(t)^* = R_j(t)$ and to identify the continuously measured observables with $2R_j(t)$. In this case, the four stochastic Schrödinger equations (2.52) reduces to two, with a particularly simple form:

$$\operatorname{Im} n_j(t,x) = 0\,, \tag{2.61}$$

$$L_j^1(t,x) = L_j^3(t,x) = R_j(t) - n_j(t,x)\,, \tag{2.62a}$$

$$K^1(t,x) = K^3(t,x) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j \big[R_j(t) - n_j(t,x)\big]^2\,, \tag{2.62b}$$

$$L_j^2(t,x) = L_j^4(t,x) = R_j(t) - \|x\|^{-2}\, n_j(t,x)\,, \tag{2.62c}$$

$$K^2(t,x) = K^4(t,x) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j \big[R_j(t) - \|x\|^{-2}\, n_j(t,x)\big]^2\,. \tag{2.62d}$$

2.5.3 Existence and Uniqueness of the Solution

We have introduced four nonlinear SDEs (2.52) of the type of (A.14) with drift coefficients $b(x,t) = K^\ell(t,x)x$ and diffusion coefficients $\sigma_j(x,t) = L_j^\ell(t,x)x$, $\ell = 1, \dots, 4$, $j = 1, \dots, d$, given by (2.53), (2.54), (2.55) and (2.56).

Remark 2.21. For every finite time horizon $T > 0$ the following statements hold.

- The drift and the diffusion coefficients of the four SDEs (2.52) satisfy Hypothesis A.25 (measurability condition).
- The expression $\langle x|b(x,t)\rangle + \frac{1}{2}\sum_j \big\|\sigma_j(x,t)\big\|^2$ goes into

$$\begin{aligned}
&\langle x|K^\ell(t,x)x\rangle + \frac{1}{2}\sum_j \big\|L_j^\ell(t,x)x\big\|^2 \\
&\quad = \begin{cases}
-\mathrm{i}\langle x|H(t)x\rangle + \mathrm{i}\displaystyle\sum_j \big(\operatorname{Re} n_j(t,x)\big)\big(\operatorname{Im} n_j(t,x)\big), & \ell = 1, \\
-\mathrm{i}\langle x|H(t)x\rangle + \dfrac{\mathrm{i}}{\|x\|^2}\displaystyle\sum_j \big(\operatorname{Re} n_j(t,x)\big)\big(\operatorname{Im} n_j(t,x)\big), & \ell = 2, \\
-\mathrm{i}\langle x|H(t)x\rangle, & \ell = 3, 4.
\end{cases}
\end{aligned} \tag{2.63}$$

Therefore, the four sets of coefficients satisfy also the monotone condition (Hypothesis A.35) with $C(T) = 0$.
- By construction, the coefficients of the SDEs (2.52) with $\ell = 2, 4$ satisfy also the linear growth condition (Hypothesis A.34).

Lemma 2.22. *Let T be any finite time horizon. Then*

- *the coefficients of the SDEs* (2.52) *with $\ell = 1, 3$ satisfy the local Lipschitz condition (Hypothesis A.33);*
- *the coefficients of the SDEs* (2.52) *with $\ell = 2, 4$ satisfy the global Lipschitz condition (Hypothesis A.32).*

Proof. The coefficients of the SDEs (2.52) with $\ell = 1, 3$ are polynomials in the components of x; together with the boundedness Assumption 2.3, this gives by standard arguments that the local Lipschitz condition (A.17) holds.

Let us now consider the case $\ell = 2, 4$. Given two vectors x, y in $\mathcal{H}$, let us set

$$\hat{x} := \frac{x}{\|x\|}, \qquad \hat{y} := \frac{y}{\|y\|}, \qquad P_x := |\hat{x}\rangle\langle\hat{x}|, \qquad P_y := |\hat{y}\rangle\langle\hat{y}|, \tag{2.64a}$$

$$\hat{x}_\perp := \frac{(\mathbb{1} - P_y)x}{\big\|(\mathbb{1} - P_y)x\big\|}, \qquad \hat{y}_\perp := \frac{(\mathbb{1} - P_x)y}{\|(\mathbb{1} - P_x)y\|}. \tag{2.64b}$$

With these notations we can write

$$L_j^4(t,x)x = (\mathbb{1} - P_x)R_j(t)x\,, \tag{2.65a}$$

$$L_j^2(t,x)x = L_j^4(t,x)x + \mathrm{i}\big[\mathrm{Im}\, n_j\,(t,\hat{x})\big]x\,, \tag{2.65b}$$

$$K^4(t,x)x = K(t)x + g(t,x) - \frac{1}{2}\,P_x g(t,x)\,, \tag{2.65c}$$

$$g(t,x) := \sum_j R_j(t)P_x R_j(t)^* x\,, \tag{2.65d}$$

$$K^2(t,x) = K^4(t,x) + \mathrm{i}\sum_j \big[\mathrm{Im}\, n_j\,(t,\hat{x})\big]\,R_j(t)x + \frac{1}{2}\sum_j \big[\mathrm{Im}\, n_j\,(t,\hat{x})\big]^2 x\,. \tag{2.65e}$$

By using

$$\|x-y\|^2 = \big\|(\mathbb{1} - P_y)x\big\|^2 + \big\|P_y x - y\big\|^2 = \|(\mathbb{1} - P_x)y\|^2 + \|P_x y - x\|^2\,,$$
$$\|y\| = \|y - x + x\| \le \|y - x\| + \|x\|\,, \qquad \|x\| \le \|x - y\| + \|y\|\,,$$

we get

$$\big\|(\mathbb{1} - P_y)x\big\| \le \|y - x\|\,, \qquad \big\|(\mathbb{1} - P_y)\hat{x}\big\| \le 1, \qquad |\langle\hat{x}|\hat{y}\rangle| \le 1,$$
$$\|y\|\,\big\|(\mathbb{1} - P_y)\hat{x}\big\| \le \|y - x\|\,\big\|(\mathbb{1} - P_y)\hat{x}\big\| + \big\|(\mathbb{1} - P_y)x\big\| \le 2\,\|y - x\|\,,$$
$$\|(\mathbb{1} - P_x)y\| \le \|y - x\|\,, \qquad \|(\mathbb{1} - P_x)\hat{y}\| \le 1, \qquad \|x\|\,\|(\mathbb{1} - P_x)\hat{y}\| \le 2\,\|y - x\|\,,$$

$$\begin{aligned}\left||\langle\hat{y}|\hat{x}\rangle|^2\,\langle\hat{y}|x\rangle - \langle\hat{y}|y\rangle\right| &= \left|\langle\hat{y}|P_xP_yx\rangle - \langle\hat{y}|y\rangle\right| \\ &= \left|\langle\hat{y}|P_xP_yx\rangle - \langle\hat{y}|P_xy\rangle - \langle\hat{y}|(\mathbb{1}-P_x)y\rangle\right| \\ &\le \left|\langle\hat{y}|P_xP_y(x-y)\rangle\right| + |\langle\hat{y}|(\mathbb{1}-P_x)y\rangle| \\ &\le \|x-y\| + \|(\mathbb{1}-P_x)y\| \le 2\,\|x-y\|\,,\end{aligned}$$

$$\|R_i(t)\|^2 = \left\|R_i(t)^*\right\|\,\|R_i(t)\| = \left\|R_i(t)^*R_i(t)\right\| \le \left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|.$$

Let us check the global Lipschitz condition for the various coefficients. Consider first L^4:

$$\begin{aligned}\sum_j \left\|L_j^4(t,x)x - L_j^4(t,y)y\right\|^2 &= \sum_j \left\|(\mathbb{1}-P_x)R_j(t)x - (\mathbb{1}-P_y)R_j(t)y\right\|^2 \\ &= \sum_j \left\|(\mathbb{1}-P_x)\left[R_j(t)x - (\mathbb{1}-P_y)R_j(t)y\right]\right\|^2 \\ &\quad + \sum_j \left\|P_x(\mathbb{1}-P_y)R_j(t)y\right\|^2 \\ &= \sum_j \left\|(\mathbb{1}-P_x)\left[(\mathbb{1}-P_y)R_j(t)(x-y) + P_yR_j(t)x\right]\right\|^2 \\ &\quad + \sum_j \left|\langle(\mathbb{1}-P_y)\hat{x}|R_j(t)y\rangle\right|^2;\end{aligned}$$

we have

$$\begin{aligned}\sum_j \left|\langle(\mathbb{1}-P_y)\hat{x}|R_j(t)y\rangle\right|^2 &\le \sum_j \left\|(\mathbb{1}-P_y)\hat{x}\right\|^2\,\left\|R_j(t)y\right\|^2 \\ &= \sum_j \left\|(\mathbb{1}-P_y)\hat{x}\right\|^2\,\langle y|R_j(t)^*R_j(t)y\rangle \\ \le \left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|\,\left\|(\mathbb{1}-P_y)\hat{x}\right\|^2\,\|y\|^2 &\le 4\left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|\,\|y-x\|^2\end{aligned}$$

and

$$\begin{aligned}&\sum_j \left\|(\mathbb{1}-P_x)\left[(\mathbb{1}-P_y)R_j(t)(x-y) + P_yR_j(t)x\right]\right\|^2 \\ &\quad\le \sum_j \left(\left\|(\mathbb{1}-P_x)(\mathbb{1}-P_y)R_j(t)(x-y)\right\| + \left\|(\mathbb{1}-P_x)P_yR_j(t)x\right\|\right)^2 \\ &\quad\le \sum_j \left(\left\|R_j(t)(x-y)\right\| + \|(\mathbb{1}-P_x)\hat{y}\|\,\left|\langle\hat{y}|R_j(t)x\rangle\right|\right)^2\end{aligned}$$

$$
\begin{aligned}
&\le 2\sum_j \left(\left\| R_j(t)(x-y) \right\|^2 + \|(\mathbb{1}-P_x)\hat{y}\|^2 \left\| R_j(t)x \right\|^2 \right) \\
&\le 2 \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \left(\|x-y\|^2 + \|(\mathbb{1}-P_x)\hat{y}\|^2 \, \|x\|^2 \right) \\
&\le 10 \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \|x-y\|^2 ,
\end{aligned}
$$

and so

$$
\sum_j \left\| L_j^4(t,x)x - L_j^4(t,y)y \right\|^2 \le 14 \sup_{t\in[0,T]} \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \|x-y\|^2 .
$$

Therefore, $L_\bullet^4(t,x)x$ is globally Lipschitz.

We now consider K^4: we have

$$
\begin{aligned}
\|g(t,x)-g(t,y)\| &= \left\| \textstyle\sum_j R_j(t)\big(P_x R_j(t)^* x - P_y R_j(t)^* y\big) \right\| \\
&\le \left\| \textstyle\sum_j R_j(t)(\mathbb{1}-P_y)P_x R_j(t)^* x \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t) P_y P_x (\mathbb{1}-P_y) R_j(t)^* x \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t) P_y P_x P_y R_j(t)^* (\mathbb{1}-P_y) x \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t)\big(P_y P_x P_y R_j(t)^* P_y x - P_y R_j(t)^* y\big) \right\| \\
&= \Big\{ \left\| \textstyle\sum_j R_j(t)|\hat{x}_\perp\rangle\langle\hat{x}|R_j(t)^*\hat{x}\rangle \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t)|\hat{y}\rangle\langle\hat{x}_\perp|R_j(t)^*\hat{x}\rangle \right\| |\langle\hat{y}|\hat{x}\rangle| \\
&\quad + \left\| \textstyle\sum_j R_j(t)|\hat{y}\rangle\langle\hat{y}|R_j(t)^*\hat{x}_\perp\rangle \right\| |\langle\hat{y}|\hat{x}\rangle|^2 \Big\} \left\| (\mathbb{1}-P_y)x \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t)|\hat{y}\rangle\langle\hat{y}|R_j(t)^*\hat{y}\rangle \right\| \left| |\langle\hat{y}|\hat{x}\rangle|^2 \langle\hat{y}|x\rangle - \langle\hat{y}|y\rangle \right| \\
&\le \Big\{ \left\| \textstyle\sum_j R_j(t)|\hat{x}_\perp\rangle\langle\hat{x}|R_j(t)^*\hat{x}\rangle \right\| + \left\| \textstyle\sum_j R_j(t)|\hat{y}\rangle\langle\hat{x}_\perp|R_j(t)^*\hat{x}\rangle \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t)|\hat{y}\rangle\langle\hat{y}|R_j(t)^*\hat{x}_\perp\rangle \right\| \\
&\quad + 2 \left\| \textstyle\sum_j R_j(t)|\hat{y}\rangle\langle\hat{y}|R_j(t)^*\hat{y}\rangle \right\| \Big\} \|x-y\| ,
\end{aligned}
$$

$$
\begin{aligned}
\|g(t,x)-g(t,y)\| &\le 5\sum_i \|R_i(t)\|^2 \|x-y\| \\
&\le 5d \sup_{t\in[0,T]} \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \|x-y\| .
\end{aligned}
$$

Moreover,

$$\begin{aligned}
&\big\| P_x g(t,x) - P_y g(t,y) \big\| \\
&\quad = \big\| P_y P_x g(t,x) + (\mathbb{1} - P_y) P_x g(t,x) - P_y P_x g(t,y) - P_y (\mathbb{1} - P_x) g(t,y) \big\| \\
&\quad \le \big\| (\mathbb{1} - P_y) P_x g(t,x) \big\| + \big\| P_y (\mathbb{1} - P_x) g(t,y) \big\| + \big\| P_y P_x \big(g(t,x) - g(t,y) \big) \big\| \\
&\quad = \big\| (\mathbb{1} - P_y) P_x g(t,x) \big\| + |\langle \hat{y} | (\mathbb{1} - P_x) g(t,y) \rangle| + \big\| P_y P_x \big(g(t,x) - g(t,y) \big) \big\| \\
&\quad \le \big\| (\mathbb{1} - P_y) x \big\| \, |\langle \hat{x} | g(t,\hat{x}) \rangle| + \| (\mathbb{1} - P_x) y \| \, \| g(t,\hat{y}) \| + \| g(t,x) - g(t,y) \| \\
&\quad \le 7d \sup_{t \in [0,T]} \Big\| \textstyle\sum_j R_j(t)^* R_j(t) \Big\| \, \| x - y \| \, .
\end{aligned}$$

Therefore, $K^4(t,x)x$ is globally Lipschitz.

We now consider L^2 and K^2, which are related to the previous coefficients by (2.65). The differences with respect to the terms with $\ell = 4$ have similar structures; it is enough to check one of such differences:

$$\begin{aligned}
\big\| n_j\big(t,\hat{x}\big) x - n_j\big(t,\hat{y}\big) y \big\| &\le \big| n_j\big(t,\hat{x}\big) \big| \, \| x - y \| + \big| n_j\big(t,\hat{x}\big) - n_j\big(t,\hat{y}\big) \big| \, \| y \| \\
&\le 2 \, \big\| R_j(t) \big\| \, \| x - y \| + \big| n_j\big(t,\hat{x}\big) \, \| y \| - \langle \hat{y} | R_j(t) x \rangle \big| \\
&\le 2 \, \big\| R_j(t) \big\| \, \| x - y \| + \Big\| \, \| y \| \, \hat{x} - \| x \| \, \hat{y} \Big\| \, \big\| R_j(t) \big\| \le 3 \, \big\| R_j(t) \big\| \, \| x - y \| \, .
\end{aligned}$$

Therefore, $L^2(t,x)x$ and $K^2(t,x)x$ are globally Lipschitz. □

Theorem 2.23. *Every one of the four SDEs* (2.52) *admits a strong solution in the time interval* $[0, +\infty)$. *Pathwise uniqueness and uniqueness in law hold. Moreover, the norm of the solutions of the equations with* $\ell = 2, 4$ *is conserved,*

$$\big\| X^2(t) \big\|^2 = \big\| X^2(0) \big\|^2 , \qquad \big\| X^4(t) \big\|^2 = \big\| X^4(0) \big\|^2 , \tag{2.66}$$

while for $\ell = 1, 3$ *we have*

$$\begin{aligned}
1 - \big\| X^\ell(t) \big\|^2 &= \Big(1 - \big\| X^\ell(0) \big\|^2 \Big) \\
&\quad \times \exp\Big\{ -2 \sum_j \int_0^t \mathrm{Re}\, n_j \left(s, X^\ell(s) \right) \big[\mathrm{d}\widehat{W}_j(s) + \mathrm{Re}\, n_j \left(s, X^\ell(s) \right) \mathrm{d}s \big] \Big\} .
\end{aligned} \tag{2.67}$$

Proof. Uniqueness and existence of solutions is by Remark 2.21, Lemma 2.22 and Theorem A.36.

By computations similar to those in (2.17), one gets

$$\begin{aligned}
\mathrm{d} \big\| X^\ell(t) \big\|^2 &= 2 \Big(1 - \big\| X^\ell(t) \big\|^2 \Big) \sum_j \mathrm{Re}\, n_j \left(t, X^\ell(t) \right) \mathrm{d}\widehat{W}_j(t) \, , \ \text{for } \ell = 1, 3, \\
\mathrm{d} \big\| X^\ell(t) \big\|^2 &= 0 \, , \ \text{for } \ell = 2, 4 \, .
\end{aligned}$$

Then, the statements about the norm follow from Proposition A.41 applied to the stochastic processes $Z(t) = 1 - \left\| X^\ell(t) \right\|^2$. □

Thus, in the case of polynomial coefficients ($\ell = 1, 3$), the solutions X^ℓ of the stochastic Schrödinger equation move inside the unit ball if $\left\| X^\ell(0) \right\| < 1$, on the unit sphere if $\left\| X^\ell(0) \right\| = 1$ and outside the unit ball if $\left\| X^\ell(0) \right\| > 1$. In the case of linearly growing coefficients ($\ell = 2, 4$), the solutions X^ℓ of the stochastic Schrödinger equation move on the corresponding spheres of radius $\left\| X^\ell(0) \right\|$.

If we take the four equations with the same normalised initial condition, by uniqueness, we have that the solutions of the equations of number 1 and 2 coincide and the same holds for the solutions of numbers 3 and 4. Moreover, the solutions of 1 or 2 and of 3 or 4 are connected by (2.50).

Of course, when the stochastic Schrödinger equation (2.52) is considered in the probability space $\left(\Omega, \mathcal{F}_T, \widehat{\mathbb{P}}_\psi^T \right)$ for $\ell = 1, 2$ and normalised initial condition, its solution $\widehat{\psi}$ is the normalisation (2.22) of the solution ψ of the linear stochastic Schrödinger equation (2.28) in $(\Omega, \mathcal{F}_T, \mathbb{Q})$.

2.5.4 The Stochastic Schrödinger Equation as a Starting Point

By the results of the previous subsection, we have that both the SDEs for a posteriori states (2.49) and (2.51) with initial condition $\widehat{\psi}(0) = \widehat{\phi}(0) = \psi_0$, $\|\psi_0\| = 1$, have a unique (pathwise and in law) strong solution with $\left\| \widehat{\psi}(t) \right\| = \left\| \widehat{\phi}(t) \right\| = 1$. The solutions of the two equations are connected by the relation (2.50).

This point is very important because it gives the possibility of starting the whole theory from the nonlinear stochastic Schrödinger equation; we sketch this construction just below. For the theory of continuous measurements, this is only an alternative possibility, but conceptually this is needed when the nonlinear SDE is postulated for some reason, as for a modification of quantum mechanics [3, 6, 15, 31], or it is used for stochastic simulations of quantum dynamical semigroups as explained in Sect. 3.2.3.2. The problem of strong solutions, in the more general context of infinite dimensional Hilbert spaces and equations involving unbounded operators as coefficients, was already studied in [32].

Every one of the four stochastic Schrödinger equations (2.52) can be taken as starting point; let us choose the SDE with $\ell = 2$. Let us fix a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, +\infty)}, \mathbb{P} \right)$ in usual hypotheses and let $\widehat{B}$ be a continuous Wiener process in this basis with increments independent of the past. Let $\widehat{\psi}$ be a solution of (2.52), $\ell = 2$, with the Wiener process $\widehat{B}$ and initial condition $\psi_0 \in \mathcal{H}$, $\|\psi_0\| = 1$. By Theorem 2.23, the solution is unique and its norm is conserved: $\left\| \widehat{\psi}(t) \right\| = 1$, $\forall t \in [0, +\infty)$. Due to the normalisation for every time, the stochastic differential of $\widehat{\psi}(t)$ reduces to (2.49), i.e.

$$\mathrm{d}\widehat{\psi}(t) = \sum_j \left[R_j(t) - \frac{1}{2} m_j(t) \right] \widehat{\psi}(t)\, \mathrm{d}\widehat{B}_j(t) + \left[K(t) + \frac{1}{2} \sum_j m_j(t) R_j(t) - \frac{1}{8} \sum_j m_j(t)^2 \right] \widehat{\psi}(t)\, \mathrm{d}t, \quad (2.68a)$$

$$m_j(t) = 2\,\mathrm{Re}\langle \widehat{\psi}(t) | R_j(t) \widehat{\psi}(t) \rangle. \quad (2.68b)$$

In the case of a continuous measurement, besides the stochastic evolution of the state $\widehat{\psi}(t)$, we have to introduce also the stochastic output and its relation with $\widehat{\psi}(t)$. The output is the stochastic process with components

$$B_j(t) = \widehat{B}_j(t) + \int_0^t m_j(s) \mathrm{d}s. \quad (2.69)$$

The physical probability is $\mathbb{P}$. Notice that, having chosen the nonlinear stochastic Schrödinger equation as a starting point, the system state $\widehat{\psi}(t)$ at time t depends on $\widehat{B}(s)$, $0 \le s \le t$, which is not the observed output. Anyway we are still allowed to interpret $\widehat{\psi}(t)$ as the system state at time t conditioned by the observation of the output $B(s)$ for $0 \le s \le t$ because the knowledge of $B(s)$, $0 \le s \le t$, is equivalent to the knowledge of $\widehat{B}(s)$, $0 \le s \le t$. Heuristically one can think that the knowledge of the trajectory of $B(s)$ in $[0, t]$ determines the corresponding trajectory of $\widehat{B}(s)$ and thus the value of $\widehat{\psi}(t)$. The correct mathematical statement is that the two processes generate the same augmented filtration:

$$\sigma\left\{ B(s), s \in [0, t] \right\} \vee \mathcal{N} = \sigma\left\{ \widehat{B}(s), s \in [0, t] \right\} \vee \mathcal{N}. \quad (2.70)$$

Indeed, the inclusion $\subset$ is obvious because of (2.69) and because the process $\widehat{\psi}$ is adapted to the augmented natural filtration of $\widehat{B}$ thanks to Theorem 2.23. The opposite inclusion $\supset$ follows from the possibility of recovering the linear stochastic Schrödinger equation and by its theorem of existence and uniqueness of strong solutions. Let us show this fact.

Given the initial state ψ_0 of the system, consider the positive continuous process

$$q(t) = \exp\left\{ -\frac{1}{2} \sum_j \left[\int_0^t m_j(s) \mathrm{d}\widehat{B}_j(s) + \frac{1}{2} \int_0^t m_j(s)^2 \mathrm{d}s \right] \right\}. \quad (2.71)$$

Its square $q(t)^2$ is a positive $\mathbb{P}$-martingale and

$$\mathbb{Q}^t_{\psi_0}(\mathrm{d}\omega) = q(t, \omega)^2\, \mathbb{P}(\mathrm{d}\omega) \quad (2.72)$$

defines a new probability on $(\Omega, \mathcal{F}_t)$; the probabilities $\mathbb{Q}^t_{\psi_0}$, $t \geq 0$, are consistent. By Girsanov theorem, under the law $\mathbb{Q}^T_{\psi_0}$ the process $B(t)$, $t \in [0, T]$, with components (2.69) is a multidimensional standard Wiener process.

Let us define

$$\psi(t) = q(t)^{-1}\, \widehat{\psi}(t)\,; \tag{2.73}$$

by Itô calculus we get, under $\mathbb{Q}^T_{\psi_0}$, the linear stochastic Schrödinger equation (2.28):

$$\mathrm{d}\psi(t) = \sum_j R_j(t)\psi(t)\mathrm{d}B_j(t) + K(t)\psi(t)\mathrm{d}t. \tag{2.74}$$

Thus, Theorem 2.4 guarantees that $\psi(t)$ is adapted to the augmented filtration of $B(t)$ and then the same is true for $q(t) = \|\psi(t)\|^{-1}$, $\widehat{\psi}(t) = q(t)\psi(t)$ and $\widehat{B}(t)$. This completes the proof of (2.70).

Finally, the uniqueness in law of the solutions of all the equations involved guarantees that, for every finite interval of time $[0, T]$, the law of B under $\mathbb{P}$ and the law of W under $\widehat{\mathbb{P}}^T_{\psi_0}$ coincide. So, the two approaches, the one starting from the linear stochastic Schrödinger equation and the one starting from the nonlinear one, are completely equivalent.

2.6 The Linear Approach Versus the Nonlinear One

As the theory can be formulated by starting either from the linear stochastic Schrödinger equation, or from the nonlinear one, let us give here just some hints of comparison between the two approaches.

- Advantages of the linear approach:
  - Direct generalisation of the traditional description of an instantaneous measurement.
  - Clear analytical relation between the a posteriori state and the observed output: if the canonical realisation of the Wiener process is used, then the a posteriori states $\psi(t)$ and $\widehat{\psi}(t)$ are explicitly functions of the trajectory of the output $W(s)$ for $0 \leq s \leq t$.
- Characteristic features of the linear approach:
  - The output process W is a fixed function from the sample space Ω to $C^d_0(0, \infty)$, the space of all $\mathbb{R}^d$-valued continuous functions of a positive variable. Its physical properties depend on the physical probability $\widehat{\mathbb{P}}^T_{\psi_0}$, which changes on $(\Omega, \mathcal{F})$ according to the choice of the initial system state ψ_0.

- Disadvantages of the linear approach:
  - The linear stochastic Schrödinger equation is not suitable for numerical simulations as the norm of the non-normalised a posteriori state $\psi(t)$ can become very small.
- Advantages of the nonlinear approach:
  - The stochastic Schrödinger equation directly gives the a posteriori state $\widehat{\psi}(t)$.
  - The stochastic Schrödinger equation is suitable for numerical simulations [7, 32–35].
- Characteristic features of the nonlinear approach:
  - The probability $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$ is fixed. The output B is a function from the sample space Ω to $C_0^d(0,\infty)$, the space of all $\mathbb{R}^d$-valued continuous functions of a positive variable, which changes according to the choice of the initial system state ψ_0 (thus modifying its physical properties).
- Disadvantages of the nonlinear approach:
  - Non-transparent relation between the a posteriori state $\widehat{\psi}(t)$ and the output $B(t)$.

2.7 Tricks to Simplify the Equations

In special cases, some peculiar time dependencies can be eliminated and/or more compact forms of the stochastic Schrödinger equation can be obtained. Let us see how.

2.7.1 Time-Dependent Coefficients and Unitary Transformations

A particularly interesting case is when the time dependence of the coefficients in the linear stochastic Schrödinger equation (2.28) can be eliminated by using a unitary transformation. Let us assume that there exists a self-adjoint operator H_0 such that

$$\mathrm{e}^{\mathrm{i}H_0 t} R_j(t) \mathrm{e}^{-\mathrm{i}H_0 t} = R_j(0)\,, \qquad \mathrm{e}^{\mathrm{i}H_0 t} H(t) \mathrm{e}^{-\mathrm{i}H_0 t} = H(0)\,. \tag{2.75}$$

In the physical literature, this transformation is known as the use of a (suitable) *interaction picture*. We define the "interaction Hamiltonian" $H_I := H(0) - H_0$ and

$$R_j^0 := R_j(0)\,, \qquad K^0 := K(0) - \mathrm{i}H_0 \equiv -\mathrm{i}H_I - \frac{1}{2}\sum_j R_j^{0*} R_j^0\,. \tag{2.76}$$

By setting

$$\Phi(t) := \mathrm{e}^{\mathrm{i}H_0 t}\,\psi(t)\,, \tag{2.77}$$

we get $\mathrm{d}\Phi(t) = \mathrm{i}H_0\Phi(t)\mathrm{d}t + \mathrm{e}^{\mathrm{i}H_0 t}K(t)\psi(t)\mathrm{d}t + \mathrm{e}^{\mathrm{i}H_0 t}\sum_j R_j(t)\psi(t)\mathrm{d}W_j(t)$. By inserting before $\psi(t)$ the identity $\mathbb{1} = \mathrm{e}^{-\mathrm{i}H_0 t}\mathrm{e}^{\mathrm{i}H_0 t}$, we obtain the linear SDE with time-independent coefficients

$$\mathrm{d}\Phi(t) = K^0\Phi(t)\mathrm{d}t + \sum_j R_j^0\Phi(t)\mathrm{d}W_j(t)\,. \tag{2.78}$$

We can now redo the whole construction of probabilities and a posteriori states by starting from this equation instead of from (2.2). We have $\|\psi(t)\|^2 = \|\Phi(t)\|^2$, $m_j(t) = 2\,\mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle = 2\,\mathrm{Re}\langle\widehat{\Phi}(t)|R_j^0\widehat{\Phi}(t)\rangle$ and nothing changes for what concerns the physical probabilities. We have only to recall that the a posteriori states are given by $\widehat{\psi}(t) = \mathrm{e}^{-\mathrm{i}H_0 t}\widehat{\Phi}(t)$.

In the example of Section 8.1, we use just this trick in order to simplify the time dependence of the coefficients.

2.7.2 Complex Noise

When one of the coefficients $R_j(t)$ in the linear stochastic Schrödinger equation (2.28) differs from another one only by a multiplicative factor i (imaginary unit), the equations assume a simpler form by introducing complex Wiener processes [5, 33, 36–38]. Let us illustrate this fact in the case $d = 2$.

Assume that we have

$$R_1(t) = \frac{1}{\sqrt{2}}\,R(t), \qquad R_2(t) = \frac{\mathrm{i}}{\sqrt{2}}\,R(t). \tag{2.79}$$

Then, we define the complex Wiener process

$$W(t) = \frac{1}{\sqrt{2}}\,W_1(t) + \frac{\mathrm{i}}{\sqrt{2}}\,W_2(t), \tag{2.80}$$

for which the Itô rules turn out to be $\mathrm{d}W(t)^2 = 0$, $\mathrm{d}\overline{W(t)}\mathrm{d}W(t) = \mathrm{d}t$. With these notations the linear SDE (2.28) becomes

$$\mathrm{d}\psi(t) = R(t)\psi(t)\mathrm{d}W(t) + K(t)\psi(t)\mathrm{d}t\,, \qquad K(t) = -\mathrm{i}H(t) - \frac{1}{2}\,R(t)^*R(t). \tag{2.81}$$

Also the nonlinear stochastic Schrödinger equation assumes a simpler form in this case, especially if we consider the a posteriori states $\widehat{\phi}(t)$ with a changed phase:

$$\mathrm{d}\widehat{\phi}(t) = \left[R(t) - \langle\widehat{\phi}(t)|R(t)\widehat{\phi}(t)\rangle\right]\widehat{\phi}(t)\,\mathrm{d}\widehat{W}(t) + \left[K(t) + \langle\widehat{\phi}(t)|R(t)^*\widehat{\phi}(t)\rangle\,R(t) - \frac{1}{2}\left|\langle\widehat{\phi}(t)|R(t)\widehat{\phi}(t)\rangle\right|^2\right]\widehat{\phi}(t)\,\mathrm{d}t, \quad (2.82)$$

$$\widehat{W}(t) = \frac{1}{\sqrt{2}}\,\widehat{W}_1(t) + \frac{\mathrm{i}}{\sqrt{2}}\,\widehat{W}_2(t) = W(t) - \int_0^t \langle\widehat{\phi}(s)|R(s)^*\widehat{\phi}(s)\rangle\mathrm{d}s. \quad (2.83)$$

2.8 Summary: The Stochastic Schrödinger Equation

2.8.1 The Linear Stochastic Schrödinger Equation

2.8.1.1 Hilbert Space and System Operators

Assumptions 2.1, 2.3, 2.10.

- The Hilbert space of the quantum system under consideration is $\mathcal{H} = \mathbb{C}^n$.
- The effective Hamiltonian $H(t)$ and the system operators $R_j(t)$, $j = 1, \ldots, d$, (dissipative terms) are non-random linear operators on $\mathcal{H}$; $H(t)$ is self-adjoint: $H(t)^* = H(t)$.
- The functions $t \mapsto H(t)$ and $t \mapsto R_j(t)$ are measurable and, for every $T \in (0, +\infty)$,

$$\sup_{t\in[0,T]} \|H(t)\| < +\infty, \qquad \sup_{t\in[0,T]} \left\| \sum_j R_j(t)^* R_j(t) \right\| < +\infty.$$

- We use the shorthand notation: $K(t) := -\mathrm{i}H(t) - \frac{1}{2}\sum_{j=1}^{d} R_j(t)^* R_j(t)$.

2.8.1.2 Reference Probability Space and Filtrations

Assumption 2.2, Remark 2.5.

- $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$ is a stochastic basis satisfying the usual conditions, which means that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space, $(\mathcal{F}_t)$ is a filtration of sub-σ-algebras of $\mathcal{F}$, $\mathcal{F}_t = \bigcap_{s:s>t} \mathcal{F}_s$, $\mathbb{Q}(A) = 0 \Rightarrow A \in \mathcal{F}_t$, $\forall t \geq 0$.
- $\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$, $\qquad \mathcal{N} := \{B \in \mathcal{F} : \mathbb{Q}(B) = 0\}$.
- The symbol $\mathbb{E}_\mathbb{Q}$ indicates the expectation with respect to $\mathbb{Q}$.
- W is a continuous d-dimensional Wiener process defined in $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$. In particular, the process W has increments independent of the past with respect to the filtration $(\mathcal{F}_t)$.
- The natural filtration of the increments of W: $\mathcal{D}_t^s := \sigma\{W(u) - W(s),\ u \in [s, t]\}$.

- The augmented natural filtration of the increments of W: $\overline{\mathcal{D}}_t^s := \mathcal{D}_t^s \vee \mathcal{N}$.
- The filtration $\{\overline{\mathcal{D}}_t^s,\ t \in [s, +\infty)\}$ satisfies the usual conditions: $\overline{\mathcal{D}}_t^s$ is independent of $\mathcal{F}_s$ and $\overline{\mathcal{D}}_t^s \subset \overline{\mathcal{D}}_t^0 \subset \mathcal{F}_t \subset \mathcal{F}$, for $0 \le s \le t$.

2.8.1.3 The Linear Stochastic Schrödinger Equation

Assumptions 2.2, 2.12, equations (2.2), (2.7), (2.11), Propositions 2.6, 2.7, 2.8, Theorem 2.11.

- The linear stochastic Schrödinger equation (2.28):

$$\mathrm{d}\psi(t) = K(t)\psi(t)\,\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\,\mathrm{d}W_j(t)\,.$$

- Initial condition: a non-random $\psi_0 \in \mathcal{H}$, $\quad \|\psi_0\| = 1$.
- The solution is an $\mathcal{H}$-valued process ψ, which is continuous and $\left(\overline{\mathcal{D}}_t^0\right)$-adapted.
- $\|\psi(t)\|^2$ is a mean one, continuous martingale.
- The stochastic evolution operator, or propagator, A_t^s is a continuous process in $t \ge s$, which is $\left(\overline{\mathcal{D}}_t^s\right)$-adapted and independent of $\mathcal{F}_s$. It satisfies

$$\mathrm{d}A_t^s = K(t)A_t^s\,\mathrm{d}t + \sum_{j=1}^{d} R_j(t)A_t^s\,\mathrm{d}W_j(t)\,, \qquad A_s^s = \mathbb{1}.$$

- The adjoint operator $(A_t^s)^*$ satisfies

$$\mathrm{d}(A_t^s)^* = (A_t^s)^*K(t)^*\mathrm{d}t + \sum_{j=1}^{d}(A_t^s)^*R_j(t)^*\mathrm{d}W_j(t)\,, \qquad (A_s^s)^* = \mathbb{1}.$$

- $\psi(t) = A_t^0\psi_0, \qquad A_t^r = A_t^s A_s^r$ for $0 \le r \le s \le t$.
- $\det A_t^s > 0, \qquad A_t^s = A_t^r\left(A_s^r\right)^{-1}$ for $0 \le r \le s \le t$.
- The inverse operator $(A_t^s)^{-1}$ satisfies

$$\mathrm{d}(A_t^s)^{-1} = (A_t^s)^{-1}\left[\sum_j R_j(t)^2 - K(t)\right]\mathrm{d}t - \sum_{j=1}^{d}(A_t^s)^{-1}R_j(t)\mathrm{d}W_j(t)\,,$$

with $(A_s^s)^* = \mathbb{1}$.

2.8.1.4 The Physical Probability and the A Posteriori States

Equations (2.22), (2.23), (2.33), (2.39), (2.43), Theorems 2.11, 2.14, Proposition 2.17, Remarks 2.13, 2.16, 2.18.

- $\psi(t)$ is the non-normalised a posteriori state at time t.
- $\|\psi(t)\| > 0, \quad \widehat{\psi}(t) := \|\psi(t)\|^{-1}\ \psi(t)$.
- $\widehat{\psi}(t)$ is the a posteriori state at time t.
- $m_j(t) := \big\langle \widehat{\psi}(t) \big| \big(R_j(t) + R_j(t)^*\big)\widehat{\psi}(t)\big\rangle = 2\,\mathrm{Re}\big\langle \widehat{\psi}(t)\big|R_j(t)\widehat{\psi}(t)\big\rangle$.
- $\|\psi(t)\|^2 = \exp\Big\{\sum_j \Big[\int_0^t m_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t m_j(s)^2\mathrm{d}s\Big]\Big\}$.
- The expression $\widehat{\mathbb{P}}^T_{\psi_0}(\mathrm{d}\omega) = \|\psi(T,\omega)\|^2\,\mathbb{Q}(\mathrm{d}\omega)\Big|_{\mathcal{F}_T}$ defines the "physical" probability on $(\Omega, \mathcal{F}_T)$. The expectation with respect to $\widehat{\mathbb{P}}^T_{\psi_0}(\mathrm{d}\omega)$ is denoted by $\widehat{\mathbb{E}}^T_{\psi_0}$. The physical probability for the events regarding the output W up to time T is $\widehat{\mathbb{P}}^T_{\psi_0}\big|_{\overline{\mathcal{D}}^0_T}$.
- The family of probabilities $\Big\{\widehat{\mathbb{P}}^T_{\psi_0},\ T > 0\Big\}$ is consistent, which means that for any choice of $T > t \geq 0$ we have $\widehat{\mathbb{P}}^T_{\psi_0}(F) = \widehat{\mathbb{P}}^t_{\psi_0}(F), \forall F \in \mathcal{F}_t$.
- There exists a unique probability $\widehat{\mathbb{P}}^\infty_{\psi_0}$ on $\mathcal{D}^0_\infty := \bigvee_{t>0}\mathcal{D}^0_t$ such that

$$\widehat{\mathbb{P}}^\infty_{\psi_0}(F) = \widehat{\mathbb{P}}^T_{\psi_0}(F), \qquad \forall T > 0, \quad \forall F \in \mathcal{D}^0_T\,.$$

- Under the physical law $\widehat{\mathbb{P}}^T_{\psi_0}$, the process with components

$$\widehat{W}_j(t) := W_j(t) - \int_0^t m_j(s)\mathrm{d}s, \qquad t \in [0,T],$$

  is a continuous Wiener processes with increments independent of the past. It is $(\overline{\mathcal{D}}^0_t)$-adapted.
- The stochastic integrals with respect to W and $\widehat{W}$ are linked by (2.32).
- POMs and probabilities:

$$\widehat{E}^s_t(F) := \int_F A^s_t(\omega)^* A^s_t(\omega)\mathbb{Q}(\mathrm{d}\omega), \qquad \forall F \in \overline{\mathcal{D}}^s_t\,,$$
$$\widehat{\mathbb{P}}^T_{\psi_0}(F) = \langle\psi_0|\widehat{E}^0_T(F)\psi_0\rangle, \qquad \forall F \in \overline{\mathcal{D}}^0_T\,.$$

- Consistency of the POMs:

$$0 \leq r < s < t, \quad F \in \overline{\mathcal{D}}^r_s \quad \Rightarrow \quad \widehat{E}^r_t(F) = \widehat{E}^r_s(F).$$

- For all $F \in \overline{\mathcal{D}}^s_t$, $0 \leq s < t \leq T$, we have

$$\widehat{\mathbb{P}}^T_{\psi_0}(F|\mathcal{F}_s) = \big\langle \widehat{\psi}(s)\big|\widehat{E}^s_t(F)\widehat{\psi}(s)\big\rangle = \widehat{\mathbb{P}}^T_{\psi_0}(F|\overline{\mathcal{D}}^0_s).$$

2.8.2 The Nonlinear Stochastic Schrödinger Equation

- Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,+\infty)}, \mathbb{P}\big)$ be a stochastic basis in the usual hypotheses and $\widehat{B}$ be a continuous Wiener process in this basis with increments independent of the past.
- Both the nonlinear SDEs

$$\begin{aligned}\mathrm{d}\widehat{\psi}(t) = &\sum_j \big[R_j(t) - \mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle\big]\,\widehat{\psi}(t)\,\mathrm{d}\widehat{B}_j(t) + K(t)\widehat{\psi}(t)\,\mathrm{d}t\\ &+\sum_j\left[\big(\mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle\big)\,R_j(t) - \frac{1}{2}\big(\mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle\big)^2\right]\widehat{\psi}(t)\,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\widehat{\phi}(t) = &\sum_j \big[R_j(t) - \langle\widehat{\phi}(t)|R_j(t)\widehat{\phi}(t)\rangle\big]\,\widehat{\phi}(t)\,\mathrm{d}\widehat{B}_j(t) + K(t)\widehat{\phi}(t)\,\mathrm{d}t\\ &+\sum_j\left[\overline{\langle\widehat{\phi}(t)|R_j(t)\widehat{\phi}(t)\rangle}\,R_j(t) - \frac{1}{2}\big|\langle\widehat{\phi}(t)|R_j(t)\widehat{\phi}(t)\rangle\big|^2\right]\widehat{\phi}(t)\,\mathrm{d}t,\end{aligned}$$

with initial condition $\widehat{\psi}(0) = \widehat{\phi}(0) = \psi_0$, $\|\psi_0\| = 1$, have a unique (pathwise and in law) strong solution with $\big\|\widehat{\psi}(t)\big\| = \big\|\widehat{\phi}(t)\big\| = 1$. The solutions of the two equations are connected by the relation

$$\begin{aligned}\widehat{\phi}(t) = \exp\Big\{&-\mathrm{i}\sum_j\int_0^t \mathrm{Re}\langle\widehat{\psi}(s)|R_j(s)\widehat{\psi}(s)\rangle\,\mathrm{Im}\langle\widehat{\psi}(s)|R_j(s)\widehat{\psi}(s)\rangle\,\mathrm{d}s\\ &-\mathrm{i}\sum_j\int_0^t \mathrm{Im}\langle\widehat{\psi}(s)|R_j(s)\widehat{\psi}(s)\rangle\,\mathrm{d}\widehat{B}_j(s)\Big\}\widehat{\psi}(t).\end{aligned}$$

- The output of the measurement is the process $B(t)$, $t \geq 0$, under the law $\mathbb{P}$, with components

$$B_j(t) = \widehat{B}_j(t) + \int_0^t m_j(s)\mathrm{d}s\,,$$

where

$$m_j(t) = 2\,\mathrm{Re}\langle\widehat{\phi}(t)|R_j(t)\widehat{\phi}(t)\rangle = 2\,\mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle.$$

More precisely, the output is the collection of the increments of B in the interval of observation; heuristically, the output is the time derivative of B.

- The square of

$$q(t) = \exp\left\{-\frac{1}{2}\sum_j \left[\int_0^t m_j(s)\mathrm{d}\widehat{B}_j(s) + \frac{1}{2}\int_0^t m_j(s)^2\mathrm{d}s\right]\right\}$$

  is a positive $\mathbb{P}$-martingale, and

$$\mathbb{Q}^t_{\psi_0}(\mathrm{d}\omega) = q(t,\omega)^2\,\mathbb{P}(\mathrm{d}\omega)$$

  defines a new probability on $(\Omega, \mathcal{F}_t)$; the probabilities $\mathbb{Q}^t_{\psi_0}$, $t \geq 0$, are consistent.
- Under the law $\mathbb{Q}^T_{\psi_0}$ the process $B(t)$, $t \in [0, T]$, is a multidimensional standard Wiener process.
- The random vector

$$\psi(t) = q(t)^{-1}\,\widehat{\psi}(t),$$

  under the law $\mathbb{Q}^T_{\psi_0}$, satisfies the linear SDE

$$\mathrm{d}\psi(t) = \sum_j R_j(t)\psi(t)\mathrm{d}B_j(t) + K(t)\psi(t)\mathrm{d}t.$$

- In particular cases, the SDEs involved in the theory can be simplified by some tricks, for instance by using unitary transformations (in the case in Sect. 2.7.1) or complex Wiener processes (in the case in Sect. 2.7.2).

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Chapter 3
The Stochastic Master Equation: Part I

A satisfactory theory of continuous measurements has to be developed according to the axioms of quantum mechanics, that is by introducing, more or less explicitly, the associated instruments (Sect. B.4). This approach requires the statistical formulation of quantum mechanics (see Sect. B.3). This chapter generalises to this framework the theory developed in Chap. 2 and it extends the results to the case of incomplete measurements. Now, the key notions are "statistical operator", "stochastic master equation", "master equation" and "quantum dynamical (or Markov) semigroup".

3.1 From Hilbert Space to Statistical Formulation

Let us start by considering a continuous measurement described by the linear stochastic Schrödinger equation (2.28) performed on a quantum system with a (possibly) mixed initial state. The situation could be described by using a random pure initial condition (cf. Sect. A.4.4), but it is more instructive to use statistical operators, which represent the proper starting point to develop a measurement theory.

3.1.1 A Mixed Initial State

As done in Sect. B.3.1 for the general case, we introduce the statistical formulation for the continuous measurements, step by step, by starting from the Hilbert space formulation with uncertainty on the initial state and by using a Bayesian reasoning.

Let the initial state be ψ_α with probability $p(\alpha)$; we have $\|\psi_\alpha\| = 1$, $p(\alpha) > 0$, $\sum_\alpha p(\alpha) = 1$. We fix a time interval $[0, T]$ and all the times introduced in this section are intended to be in this interval.

Let us think of the index α as the value of some classical random variable observed at time zero. If we observe the value α, the initial state is ψ_α and, according to the theory of Chapter 2, the density of the physical probability up to time t, given α, is

$$p_t(\omega|\alpha) = \|\psi_\alpha(t, \omega)\|^2 ,$$

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where

$$\psi_\alpha(t,\omega) = A_t^0(\omega)\psi_\alpha\,; \tag{3.1}$$

see (2.10) and (2.29). Then, the joint density is

$$p_t(\alpha,\omega) = p_t(\omega|\alpha)p(\alpha)\,,$$

and, by the formula of total probabilities, the marginal density for the continuous output is

$$p_t(\omega) = \sum_\alpha p_t(\alpha,\omega) = \sum_\alpha p(\alpha)\,\|\psi_\alpha(t,\omega)\|^2\,. \tag{3.2}$$

This is the density of the physical probability if α is unknown. As $\|\psi_\alpha(t)\|^2$ is a $\mathbb{Q}$-martingale (see Theorem 2.11 and Remark 2.13), also the linear combination p_t has this property and it forms a consistent family of probability densities when t varies. Finally, the probability of α conditional on the output up to time t is

$$p_t(\alpha|\omega) = \frac{p_t(\omega,\alpha)}{p_t(\omega)} = \frac{p(\alpha)\,\|\psi_\alpha(t,\omega)\|^2}{\sum_\beta p(\beta)\,\big\|\psi_\beta(t,\omega)\big\|^2}\,. \tag{3.3}$$

Let us now consider the a posteriori states. Conditionally on α and the continuous output up to t, the system state in $\mathcal{H}$ is $\widehat{\psi}_\alpha(t) = \frac{\psi_\alpha(t)}{\|\psi_\alpha(t)\|}$. In the language of the statistical operators (recall that the set of statistical operators is $\mathcal{S}(\mathcal{H})$ defined by (B.14)), such a state is represented by the one-dimensional projection (pure state) $|\widehat{\psi}_\alpha(t,\omega)\rangle\langle\widehat{\psi}_\alpha(t,\omega)|$. If we condition only on the continuous output, we have to weight such a state with the conditional probabilities (3.3) and we get the mixed a posteriori state

$$\widetilde{\rho}(t,\omega) = \sum_\alpha p_t(\alpha|\omega)|\widehat{\psi}_\alpha(t,\omega)\rangle\langle\widehat{\psi}_\alpha(t,\omega)|, \tag{3.4}$$

which is the analog of (B.36). Note that $\widetilde{\rho}(0)$ is independent of ω and it is given by

$$\widetilde{\rho}(0) = \sum_\alpha p(\alpha)|\psi_\alpha\rangle\langle\psi_\alpha| =: \rho_0 \in \mathcal{S}(\mathcal{H}). \tag{3.5}$$

If we do not observe the initial random variable with values α, we need only the probability density p_t and the a posteriori states $\widetilde{\rho}(t)$. The important point now is that these two quantities depend only on ρ_0 and not on its decomposition (3.5) in terms of $p(\alpha)$ and ψ_α. Indeed, from (3.3) and (3.4) we get

$$\widetilde{\rho}(t) = \sum_\alpha \frac{p(\alpha)\,\|\psi_\alpha(t)\|^2}{\sum_\beta p(\beta)\,\big\|\psi_\beta(t)\big\|^2}\,|\widehat{\psi}_\alpha(t)\rangle\langle\widehat{\psi}_\alpha(t)| = \frac{\sum_\alpha p(\alpha)|\psi_\alpha(t)\rangle\langle\psi_\alpha(t)|}{\sum_\beta p(\beta)\,\big\|\psi_\beta(t)\big\|^2}\,. \tag{3.6}$$

Then, if we define

$$\widetilde{\sigma}(t) = A_t^0 \rho_0 A_t^{0*}\,, \tag{3.7a}$$

equations (3.1), (3.2), (3.6) give

$$p_t = \mathrm{Tr}\{\widetilde{\sigma}(t)\} \tag{3.7b}$$

and

$$\widetilde{\rho}(t) = \mathrm{Tr}\,\{\widetilde{\sigma}(t)\}^{-1}\,\widetilde{\sigma}(t). \tag{3.7c}$$

We stress again that (3.7) shows that all the physical quantities, probability densities and a posteriori states depend only on ρ_0 not on its decomposition. Therefore, ρ_0 acquires the meaning of initial state of the quantum system, in agreement with the statistical formulation of quantum mechanics given in Sect. B.3.1. Summarising, we have that the construction of Chap. 2 can be reformulated in the language of statistical operators and mixed initial states are allowed by substituting Assumption 2.12 with the following one.

Assumption 3.1. The initial state is a statistical operator $\rho_0 \in \mathcal{S}(\mathcal{H})$.

Of course, Assumption 2.12 corresponds to the pure case $\rho_0 = |\psi_0\rangle\langle\psi_0|$.

Then, given the initial state ρ_0, the continuous adapted stochastic process $\widetilde{\sigma}(t)$ (3.7a) is defined. The map $\rho_0 \mapsto \widetilde{\sigma}(t)$ is automatically linear and completely positive. The trace $\mathrm{Tr}\{\widetilde{\sigma}(t)\}$ is a positive $\mathbb{Q}$-martingale of mean one which is interpreted as the density of the physical probability.

By using this density, the physical probability distribution of the outcomes of the continuous measurements is given by: $\forall F \in \mathcal{F}_t$, $t \geq 0$,

$$\mathbb{P}^t_{\rho_0}(F) = \mathbb{E}_{\mathbb{Q}}\left[1_F\,\mathrm{Tr}\,\{\widetilde{\sigma}(t)\}\right]. \tag{3.8}$$

The family of probabilities $\left(\mathbb{P}^t_{\rho_0},\ t \geq 0\right)$ is consistent. Of course the physical probability can be obtained also by using the initial state ρ_0 and the POM $\widehat{E}_t$ defined by (2.33). Indeed, by using (3.7a) and the cyclic property of the trace we get

$$\mathbb{P}^t_{\rho_0}(F) = \mathrm{Tr}\left\{\widehat{E}_t(F)\rho_0\right\}, \qquad \forall F \in \mathcal{F}_t\,, \tag{3.9}$$

which is the analog of (B.16).

Finally, the a posteriori states at time t are given now by the random variable $\widetilde{\rho}(t) : \Omega \to \mathcal{S}(\mathcal{H})$, defined by the normalisation of $\widetilde{\sigma}(t)$ (3.7c).

To complete the proof that the theory of continuous measurements is consistent with the modern axiomatic formulation of quantum mechanics of Sect. B.4, we have just to show that, actually, the map $\rho_0 \mapsto \left\{\mathbb{P}^t_{\rho_0}, \widetilde{\rho}(t)\right\}$ defines an instrument. Then $\widetilde{\rho}(t)$ will be the a posteriori state of that instrument.

But that is not all. We want to show that, when t varies in $[0, T]$, we get a continuous family of instruments which describes a continuous measurement, we want to replace the linear stochastic Schrödinger equation (2.28) with a new starting point coherent with the statistical formulation, and, of course, we want to study the general properties of such a continuous measurement. Moreover, we want to develop this program in the more general framework of an incomplete measurement, as when not all components of W are really observed. This will take the present and the following two chapters. We begin by studying the mathematical properties of the process $\widetilde{\sigma}$ and of the process σ which will be introduced by considering incomplete observation.

3.1.2 The Linear Stochastic Master Equation – I

First of all, let us compute the stochastic differential of $\widetilde{\sigma}$. By Itô formula, the definition (3.7a) and the SDEs (2.7) and (2.11), we get

$$\mathrm{d}\widetilde{\sigma}(t) = \mathcal{L}(t)\big[\widetilde{\sigma}(t)\big]\mathrm{d}t + \sum_{j=1}^{d} \mathcal{R}_j(t)\big[\widetilde{\sigma}(t)\big]\mathrm{d}W_j(t), \tag{3.10}$$

where $\mathcal{R}_j(t)$ and $\mathcal{L}(t)$ are the linear maps on M_n, the space of $n \times n$ complex matrices τ, defined by

$$\mathcal{R}_j(t)[\tau] := R_j(t)\tau + \tau R_j(t)^*, \tag{3.11}$$

$$\mathcal{L}(t)[\tau] := -\mathrm{i}[H, \tau] + \frac{1}{2}\sum_{j=1}^{d} \left([R_j\tau, R_j^*] + [R_j, \tau R_j^*]\right). \tag{3.12}$$

The map $\mathcal{L}(t)$ is a *Liouville operator* and (3.10), which turns out to be a closed linear SDE in $\widetilde{\sigma}$, is called *linear stochastic master equation*.

3.2 The Master Equation

Before introducing the case of incomplete observations and studying the linear stochastic master equation, we want to analyse the relation between the stochastic Schrödinger equation (2.28) or (2.49) and the master equation associated to the Liouvillian (3.12).

3.2.1 The Mean Statistical Operator

For every $t \geq 0$, (3.7a) defines a random positive operator $\widetilde{\sigma}(t)$ and, independently of the interpretation in terms of continuous measurements, it is significant to consider its mean with respect to the probability $\mathbb{Q}$,

$$\eta(t) := \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\sigma}(t)\right] = \int_{\Omega} \widetilde{\sigma}(t,\omega)\,\mathbb{Q}(\mathrm{d}\omega). \tag{3.13}$$

By construction, $\eta(t)$ is a statistical operator. As a direct consequence of (3.8) and (3.7c), we get also

$$\eta(t) = \mathbb{E}^T_{\rho_0}\left[\widetilde{\rho}(t)\right] = \int_{\Omega} \widetilde{\rho}(t,\omega)\,\mathbb{P}^T_{\rho_0}(\mathrm{d}\omega), \tag{3.14}$$

so that $\left\{\mathbb{P}^T_{\rho_0}, \widetilde{\rho}(t)\right\}$ is a demixture of $\eta(t)$ (Sect. B.3.1).

Since, in the case of continuous measurements, $\widetilde{\rho}(t)$ is the a posteriori state at time t, by (3.14) $\eta(t)$ has the physical meaning of unconditioned state of the system at time t, which is the state of the system at time t if the continuous measurement is performed without any filtering on the basis of the output W. Note that, even in the case of pure initial state, the statistical formulation is needed in order to introduce the notion of unconditioned state.

3.2.2 The Mean Dynamics

The study of the mean dynamics is interesting even without a physical interpretation of $\eta(t)$ in terms of continuous measurements.

3.2.2.1 The Master Equation

Since the stochastic integral in (3.10) is a matrix-valued martingale, with zero mean, we get

$$\eta(t) = \rho_0 + \int_0^t \mathcal{L}(s)[\eta(s)]\mathrm{d}s\,, \tag{3.15}$$

and, at least almost everywhere in time, we obtain the *master equation* (cf. (B.21))

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta(t) = \mathcal{L}(t)[\eta(t)], \qquad \eta(0) = \rho_0. \tag{3.16}$$

Therefore, the evolution of $\eta(t)$ is memoryless and it is governed by the same Liouvillian $\mathcal{L}(t)$ which appears in the linear stochastic master equation (3.10).

Equation (3.16) is linear and, so, we can define its evolution map. Let $\mathcal{T}(t,s)$, $0 \le s \le t$, be the two parameters family of maps on M_n defined by the evolution equation

$$\mathcal{T}(t,s) = \mathrm{Id}_n + \int_s^t \mathcal{L}(r) \circ \mathcal{T}(r,s)\mathrm{d}r\,, \tag{3.17}$$

where Id_n is the identity map on M_n. Then, by the memoryless structure of this equation, we have

$$\mathcal{T}(t,r) = \mathcal{T}(t,s) \circ \mathcal{T}(s,r)\,, \qquad 0 \le r \le s \le t\,, \tag{3.18}$$

$$\eta(t) = \mathcal{T}(t,0)[\rho_0]. \tag{3.19}$$

The proof of (3.18) is simply to show that both its sides satisfy the same differential equation with respect to $t \ge s$ with the same initial condition.

3.2.2.2 Stochastic Representation of the Mean Dynamics

The dynamics $\mathcal{T}(t,s)$ admits the stochastic representation

$$\mathcal{T}(t,s)[\tau] = \mathbb{E}_{\mathbb{Q}}\left[A_t^s \tau A_t^{s*}\right], \qquad \forall \tau \in M_n\,, \tag{3.20}$$

as one sees again from the fact that both sides of this equation satisfy the same differential equation with the same initial condition. This representation immediately shows that the evolution $\mathcal{T}$ is completely positive and, since we know that (2.33) defines a POM, that the evolution $\mathcal{T}$ is trace preserving:

$$\mathrm{Tr}\left\{\mathcal{T}(t,s)[\tau]\right\} = \mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}\left[A_t^s \tau A_t^{s*}\right]\right\} = \mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}\left[A_t^{s*} A_t^s\right]\tau\right\} = \mathrm{Tr}\left\{\tau\right\}.$$

The composition law (3.18) can be seen also as consequence of the stochastic representation (3.20) and of the composition law (2.9) of the operators A_t^s:

$$\begin{aligned}
\mathcal{T}(t,r)[\tau] &= \mathbb{E}_{\mathbb{Q}}\left[A_t^r \tau A_t^{r*}\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[A_t^s A_s^r \tau A_s^{r*} A_t^{s*} \middle| \overline{\mathcal{D}}_t^s\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[A_t^s\, \mathbb{E}_{\mathbb{Q}}\left[A_s^r \tau A_s^{r*} \middle| \overline{\mathcal{D}}_t^s\right] A_t^{s*}\right] = \mathbb{E}_{\mathbb{Q}}\left[A_t^s\, \mathbb{E}_{\mathbb{Q}}\left[A_s^r \tau A_s^{r*}\right] A_t^{s*}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[A_t^s \mathcal{T}(s,r)[\tau] A_t^{s*}\right] = \mathcal{T}(t,s)\left[\mathcal{T}(s,r)[\tau]\right].
\end{aligned}$$

Thus, from (3.7a), (3.13), we can represent the solution of the master equation (3.16) as

$$\eta(t) = \mathbb{E}_{\mathbb{Q}}\left[A_t^0 \rho_0 A_t^{0*}\right] = \sum_\alpha p(\alpha)\, \mathbb{E}_{\mathbb{Q}}\left[|\psi_\alpha(t)\rangle\langle\psi_\alpha(t)|\right], \tag{3.21}$$

where A_t^0 satisfies the SDE (2.7) and $\psi_\alpha(t)$ the linear stochastic Schrödinger equation (2.28). By using the a posteriori states

$$\widehat{\psi}_\alpha(t) = \|\psi_\alpha(t)\|^{-1}\ \psi_\alpha(t)\,,$$

we get also

$$\eta(t) = \sum_{\alpha} p(\alpha)\, \widehat{\mathbb{E}}^{T}_{\psi_\alpha} \left[|\widehat{\psi}_\alpha(t)\rangle\langle\widehat{\psi}_\alpha(t)| \right], \tag{3.22}$$

where $\widehat{\mathbb{E}}^{T}_{\psi_\alpha}$ is the expectation with respect to the probability generated by the consistent densities $\|\psi_\alpha(t)\|^2$ (see the definition of $\widehat{\mathbb{E}}^{T}_{\psi_0}$ in Sect. 2.3.2).

Both (3.21) and (3.22) are stochastic representations of the dynamics with Liouvillian $\mathcal{L}(t)$. Sometimes people say that the stochastic Schrödinger equation (2.49) gives an *unravelling* of the master equation (3.16).

Let us stress that the operators $H(t)$ and $R_j(t)$ determine $\mathcal{L}(t)$, but not vice versa: given a unique master equation, many different unravellings are possible with SDEs of diffusive type and many more if also SDEs with jumps are allowed.

3.2.3 Quantum Dynamical Semigroups

3.2.3.1 The Autonomous Case

When the operators H, R_j are time independent, also $\mathcal{L}$ is time independent, the solution of the evolution equation (3.17) is

$$\mathcal{T}(t, s) = \mathrm{e}^{\mathcal{L}(t-s)}, \tag{3.23}$$

and the solution of the master equation (3.16) is

$$\eta(t) = \mathrm{e}^{\mathcal{L}t}[\rho_0]. \tag{3.24}$$

By comparing the expression of $\mathcal{L}$ given by (3.12) with (B.20), we see that $\mathcal{L}$ is the generic generator of a quantum dynamical semigroup. Therefore, the differential equation (3.16) satisfied by the mean statistical operator $\eta(t)$ is a generic time-independent quantum master equation.

The connection between linear SDEs and semigroups of operators $\mathrm{e}^{\mathcal{L}t}$ was already noticed in [1] (independently of quantum mechanics and without our key restriction (2.20)), while in [2, 3] our case was studied exactly (there the aim was "dilations of quantum dynamical semigroups" and not continuous measurements).

3.2.3.2 Stochastic Representation of Quantum Dynamical Semigroups

In the autonomous case, the (3.21) and (3.22) give two stochastic representations of a generic quantum dynamical semigroup.

Representations of the first kind,

$$\mathrm{e}^{\mathcal{L}t}[\rho_0] = \mathbb{E}_{\mathbb{Q}} \left[A^0_t \rho_0 A^{0*}_t \right] = \sum_{\alpha} p(\alpha)\, \mathbb{E}_{\mathbb{Q}} \left[|\psi_\alpha(t)\rangle\langle\psi_\alpha(t)| \right], \tag{3.25}$$

where A_t^0 satisfies the SDE (2.7) and $\psi_\alpha(t)$ the linear stochastic Schrödinger equation (2.28), have been used to study quantum dynamical semigroups with unbounded generators in infinite dimensional Hilbert spaces [4, 5].

Representations of the second kind,

$$\mathrm{e}^{\mathcal{L}t}[\rho_0] = \sum_\alpha p(\alpha)\, \widehat{\mathbb{E}}^T_{\psi_\alpha}\left[|\widehat{\psi}_\alpha(t)\rangle\langle\widehat{\psi}_\alpha(t)|\right], \tag{3.26}$$

where $\widehat{\psi}_\alpha(t)$ satisfies the stochastic Schrödinger equation (2.52), are at the basis of stochastic simulations of solutions of quantum master equations [6]. The master equation (3.16) is a system of n^2 deterministic linear differential equations, while a nonlinear stochastic Schrödinger equation (2.52) involves only n components. From the numerical point of view, it can be more cumbersome to solve n^2 ordinary equations than to simulate n SDEs many times $\big($in order to have an estimate of the mean $\sum_\alpha p(\alpha)\, \widehat{\mathbb{E}}^T_{\psi_\alpha}\big)$.

3.3 An Incomplete Observation

In the statistical formulation, it is possible not only to consider mixed initial states (Sect. 3.1.1), not only to decide wether to observe or do not the whole output W (Sect. 3.2.1), but also to observe only some components of W, say the first m components, $1 \le m \le d$. Let us say that we have *m observed channels* and *d–m unobserved channels*. What we have to do is to give the physical probabilities restricted to this incomplete output and the associated a posteriori states.

To give the physical probability and the associated POMs, it is of course sufficient to restrict the old ones to the new filtration. Let us introduce the natural filtration of the first m components of the Wiener process:

$$\mathcal{G}_t^s := \sigma\left\{W_j(r) - W_j(s),\ r \in [s,t],\ j = 1,\dots,m\right\}; \tag{3.27a}$$

we define also

$$\mathcal{G} := \bigvee_{t\ge 0} \mathcal{G}_t^0\,, \qquad \mathcal{N}_\mathcal{G} := \{A \in \mathcal{G} : \mathbb{Q}(A) = 0\}\,, \tag{3.27b}$$

$$\overline{\mathcal{G}}_t^s := \mathcal{G}_t^s \vee \mathcal{N}_\mathcal{G}\,. \tag{3.27c}$$

By comparing with the filtration $\mathcal{D}$ generated by all the components of the Wiener process introduced in (2.6), we see that the following inclusions hold:

$$\mathcal{G}_t^s \subset \mathcal{D}_t^s \subset \mathcal{F}_t\,, \qquad \overline{\mathcal{G}}_t^s \subset \overline{\mathcal{D}}_t^s \subset \mathcal{F}_t\,.$$

We can now restrict the POM (2.39) to the observed events $\overline{\mathcal{G}}_t^s$:

$$E_t^s(G) := \widehat{E}_t^s(G) = \mathbb{E}_{\mathbb{Q}}\left[1_G A_t^{s*} A_t^s\right], \qquad G \in \overline{\mathcal{G}}_t^s. \tag{3.28}$$

Let us note that, for $G \in \overline{\mathcal{G}}_t^0$, one has $\widehat{E}_t^0(G) = \widehat{E}_t(G)$; then, the restriction of the physical probability (2.33) to the observed events can be expressed as

$$\mathbb{P}_{\rho_0}^t(G) = \mathrm{Tr}\left\{E_t^0(G)\rho_0\right\}, \qquad G \in \overline{\mathcal{G}}_t^0. \tag{3.29}$$

Regarding the a posteriori states, $\widetilde{\rho}(t)$ (3.7c) is the state of the system conditioned on the observation of the whole output $W_1(s), \ldots, W_d(s)$, $0 \le s \le t$, so that the natural definition of a posteriori state at time t, conditioned on the observation only of $W_1(s), \ldots, W_m(s)$, $0 \le s \le t$, is

$$\rho(t) := \mathbb{E}_{\rho_0}^T\left[\widetilde{\rho}(t)\middle|\overline{\mathcal{G}}_t^0\right]. \tag{3.30}$$

Note that, by the measurability properties of A_t^0 given in Proposition 2.7, $\mathbb{E}_{\rho_0}^T\left[\widetilde{\rho}(t)\middle|\overline{\mathcal{D}}_t^0\right]$ and $\widetilde{\rho}(t)$ are a.s. equal; so, $\rho(t)$ reduces to $\widetilde{\rho}(t)$ when $m = d$ and their interpretations are consistent.

From $\widetilde{\rho}(t)$ we got $\rho(t)$ by conditioning; analogously, we define

$$\sigma(t) := \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\sigma}(t)\middle|\overline{\mathcal{G}}_t^0\right]. \tag{3.31}$$

The map $\rho_0 \mapsto \sigma(t)$ is linear and completely positive, as $\sigma(t) = \mathbb{E}_{\mathbb{Q}}[A_t^0 \rho_0 A_t^{0*} | \overline{\mathcal{G}}_t^0]$. Moreover, by (3.8), (3.29), (3.31) one has

$$\mathbb{P}_{\rho_0}^t(G) = \mathbb{E}_{\mathbb{Q}}\left[1_G \,\mathrm{Tr}\left\{\sigma(t)\right\}\right], \qquad \forall G \in \overline{\mathcal{G}}_t^0. \tag{3.32}$$

By positivity, we have $\mathrm{Tr}\{\sigma(t)\} = \|\sigma(t)\|_1$; again $\mathrm{Tr}\{\sigma(t)\}$ is a $\mathbb{Q}$-martingale and $\mathbb{P}_{\rho_0}^t(G)$ gives rise to a consistent set of probabilities with respect to $t \ge 0$. Obviously, we have

$$\mathbb{E}_{\rho_0}^T\left[\rho(t)\right] = \mathbb{E}_{\rho_0}^T\left[\widetilde{\rho}(t)\right] = \mathbb{E}_{\mathbb{Q}}[\sigma(t)] = \mathbb{E}_{\mathbb{Q}}[\widetilde{\sigma}(t)] = \eta(t). \tag{3.33}$$

Other properties of the various quantities we have introduced are collected in the following proposition.

Proposition 3.2. *Let $\rho(t)$ and $\sigma(t)$ be defined by* (3.30) *and* (3.31)*, respectively. Then, one has*

$$\sigma(t) = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\sigma}(t)\middle|\mathcal{G}\right], \tag{3.34}$$

$$\rho(t) = \left(\mathrm{Tr}\{\sigma(t)\}\right)^{-1} \sigma(t), \tag{3.35}$$

$$\mathbb{P}^t_{\rho_0}\big(G|\overline{\mathcal{G}}^0_s\big) = \mathrm{Tr}\left\{E^s_t(G)\rho(s)\right\}, \quad \forall G \in \overline{\mathcal{G}}^s_t, \quad 0 \le s < t. \tag{3.36}$$

Proof. Conditional expectations enjoy the following property [7, Exercise E3.2, pp. 60, 285]:

- if X is a random variable in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_1$ and $\mathcal{F}_2$ are two sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_2$ is independent of $\sigma(X) \vee \mathcal{F}_1$, then we have $\mathbb{E}[X|\mathcal{F}_1 \vee \mathcal{F}_2] = \mathbb{E}[X|\mathcal{F}_1]$.

If we take $\widetilde{\sigma}(t)$ for X and $\mathcal{F}_1 = \overline{\mathcal{G}}^0_t$, $\mathcal{F}_2 = \bigvee_{T:T>t} \overline{\mathcal{G}}^t_T$, the conditions of the previous property are satisfied. Indeed, $\widetilde{\sigma}(t)$ is $\overline{\mathcal{D}}^0_t$-measurable, which gives $\sigma\left(\widetilde{\sigma}(t)\right) \subset \overline{\mathcal{D}}^0_t$, and we have $\overline{\mathcal{G}}^0_t \subset \overline{\mathcal{D}}^0_t$, which gives $\sigma\left(\widetilde{\sigma}(t)\right) \vee \mathcal{F}_1 \subset \overline{\mathcal{D}}^0_t$. Moreover, the increments of the Wiener process are independent of the past, which gives that $\overline{\mathcal{D}}^0_t$ and $\mathcal{F}_2$ are independent. Then, we have $\mathcal{F}_1 \vee \mathcal{F}_2 = \mathcal{G}$ and the equality of the two conditional expectations (3.31) and (3.34) follows.

Equation (3.35) holds because, for all bounded $\overline{\mathcal{G}}^0_t$-measurable random variables Y, we have

$$\begin{aligned}\mathbb{E}^t_{\rho_0}\left[Y\rho(t)\right] &= \mathbb{E}^t_{\rho_0}\left[Y\,\mathbb{E}^t_{\rho_0}\left[\widetilde{\rho}(t)\middle|\overline{\mathcal{G}}^0_t\right]\right] = \mathbb{E}^t_{\rho_0}\left[Y\,\widetilde{\rho}(t)\right] = \mathbb{E}_{\mathbb{Q}}\left[Y\widetilde{\sigma}(t)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[Y\sigma(t)\right] = \mathbb{E}^t_{\rho_0}\left[Y\frac{\sigma(t)}{\mathrm{Tr}\{\sigma(t)\}}\right].\end{aligned}$$

Equation (3.36) is proved by the following equalities: for all $\overline{\mathcal{G}}^0_s$-measurable bounded random variables Y

$$\begin{aligned}&\mathbb{E}^t_{\rho_0}\left[1_G Y\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\{\sigma(t)\}1_G Y\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\{\widetilde{\sigma}(t)\}1_G Y\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{A^s_t\widetilde{\sigma}(s)A^{s*}_t\right\}1_G Y\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{1_G A^{s*}_t A^s_t\widetilde{\sigma}(s)\right\}Y\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}[1_G A^{s*}_t A^s_t|\mathcal{F}_s]\widetilde{\sigma}(s)\right\}Y\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}[1_G A^{s*}_t A^s_t]\widetilde{\sigma}(s)\right\}Y\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{E^s_t(G)\widetilde{\sigma}(s)\right\}Y\right] = \mathbb{E}^t_{\rho_0}\left[\mathrm{Tr}\left\{E^s_t(G)\widetilde{\rho}(s)\right\}Y\right] = \mathbb{E}^t_{\rho_0}\left[\mathrm{Tr}\left\{E^s_t(G)\rho(s)\right\}Y\right].\end{aligned}$$

□

Equation (3.36) reinforces the interpretation of $\rho(t)$ as the a posteriori state at time t of the continuous measurement under Assumption 3.1 and in the case of incomplete observation. Equations (3.7c), (3.30), (3.31), (3.35) show that to get the a posteriori states we can normalise and then to take the conditional expectation with respect to the physical probability or to take the conditional expectation with respect to the reference probability and then to normalise.

Let us compute the stochastic differential of σ. By linearity and the independence of the components of the Wiener process, from the linear stochastic master equation (3.10) we get

$$\begin{cases} \mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t)[\sigma(t)]\mathrm{d}W_j(t), \\ \sigma(0) = \rho_0\,, \end{cases} \tag{3.37}$$

where the linear maps $\mathcal{R}_j(t)$ and $\mathcal{L}(t)$ are just those defined by (3.11) and (3.12). Thus we have found a closed linear SDE for σ which generalises (3.10). Again, we call it *linear stochastic master equation*. Now we want to show that the SDE (3.37) can indeed be chosen as a starting point for the mathematical description of continuous measurements.

3.4 The Statistical Formulation

Starting from the Hilbert space formulation of continuous measurement theory, extending the evolution model from pure to mixed states and introducing incomplete observations, we have obtained the linear stochastic master equation (3.37). In particular, the measured but unobserved outputs $W_{m+1}, \ldots, W_d$ affect the system evolution only through their incoherent contribution to the Liouvillian $\mathcal{L}(t)$ and thus appear as generic dissipations into the external world. Therefore, in order to model also continuous measurements in the presence of dissipative effects not related to the measurement itself, but always in a Markovian regime, now we show that the linear stochastic master equation itself can be chosen as a starting point. We change the point of view and set the whole construction without direct reference to the Hilbert space equations. We can say that we are presenting the *quantum trajectory theory* (or the continuous measurement theory) in the statistical formulation [8–10].

3.4.1 The Linear Stochastic Master Equation – II

The starting point is the linear stochastic master equation for an operator-valued process σ:

$$\begin{cases} \mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\,\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t)[\sigma(t)]\,\mathrm{d}W_j(t), \\ \sigma(0) = \rho_0 \in \mathcal{S}(\mathcal{H}). \end{cases} \tag{3.38}$$

As in Assumption 2.2, W is a continuous m-dimensional Wiener process. The *Liouvillian* $\mathcal{L}(t)$ and the maps $\mathcal{R}_j(t)$ have already been introduced, but let us collect here all the hypotheses on the coefficients appearing in (3.38).

Assumption 3.3. The process W is a continuous m-dimensional Wiener process in a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$ satisfying usual conditions (Sect. A.2.2) and $\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$; W has increments independent of the past (Definition A.21).

The maps $\mathcal{R}_j(t)$, $\mathcal{L}(t)$ are linear operators over the space M_n of $n \times n$ complex matrices τ with the structure

$$\mathcal{R}_j(t)[\tau] = R_j(t)\tau + \tau R_j(t)^*, \tag{3.39}$$

$$\mathcal{L}(t) = \mathcal{L}_0(t) + \mathcal{L}_1(t), \tag{3.40a}$$

where

$$\mathcal{L}_1(t)[\tau] = \sum_{j=1}^{m} \left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{ R_j(t)^* R_j(t), \tau \right\} \right), \tag{3.40b}$$

$$\mathcal{L}_0(t)[\tau] = -\mathrm{i}[H(t), \tau] + \sum_{j=m+1}^{d} \left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{ R_j(t)^* R_j(t), \tau \right\} \right). \tag{3.40c}$$

The coefficients $R_j(t)$, $H(t)$ are (non-random) linear operators on $\mathcal{H} \equiv \mathbb{C}^n$ and $H(t) = H(t)^*$. The functions $t \mapsto H(t)$ and $t \mapsto R_j(t)$ are measurable and such that $\forall T \in (0, +\infty)$

$$\sup_{t\in[0,T]} \|H(t)\| < +\infty, \qquad \sup_{t\in[0,T]} \left\| \sum_{j=1}^{d} R_j(t)^* R_j(t) \right\| < +\infty. \tag{3.41}$$

The mathematical and physical properties of the SDE (3.38) depend on the operators $\mathcal{L}_0(t)$ and $R_j(t)$, $j = 1, \ldots, m$. The decomposition of $\mathcal{L}_0(t)$ given in (3.40c) is not unique and can be chosen at one's convenience.

There are two typical but different physical interpretations of the linear stochastic master equation (3.38). Sometimes it is obtained starting from the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t) = \mathcal{L}_0(t)[\eta(t)]$$

and introducing the continuous measurement as a perturbation, by adding in the evolution equation the stochastic terms associated to the outputs $W_1, \ldots, W_m$ and the corresponding $\mathcal{L}_1(t)$. In this case the Liouvillian $\mathcal{L}_0(t)$ is supposed to contain the Hamiltonian part of the dynamics and any dissipative effect not related to the observation and one can think about switching off the measurement ($R_j(t) \equiv 0, \ \forall j = 1, \ldots, m$) so that the linear stochastic master equation (3.38) reduces to the original ordinary master equation with Liouvillian $\mathcal{L}_0(t)$. Other times the linear stochastic master equation (3.38) is obtained starting from the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t) = \mathcal{L}(t)[\eta(t)]$$

and introducing a continuous measurement which acquires information on the system without introducing extra perturbations (e.g. the continuous monitoring of an atom by the detection of its fluorescence light). In this case the mean dynamics is not modified by the continuous measurement, but it is "unravelled" in many different trajectories according to the observed output W.

In any case, we say that the operator $H(t)$ has a role of *effective Hamiltonian* of the system and that the operators $R_j(t)$ with indexes $j = m+1, \dots, d$ characterise the *unobserved channels*. On the other side we say that the operators $R_j(t)$ with indexes $j = 1, \dots, m$, appearing as coefficients in the diffusive part of SDE (3.38) and in $\mathcal{L}_1(t)$, characterise the *observed channels*. If $m < d$, that is if the Liouvillian $\mathcal{L}_0(t)$ is not simply Hamiltonian, we say that the measurement is incomplete.

3.4.1.1 The Stochastic Evolution Map

We consider also the fundamental solution $\mathcal{A}(t, s)$ of the linear SDE (3.38), defined by

$$\begin{cases} \mathrm{d}\mathcal{A}(t,s) = \mathcal{L}(t) \circ \mathcal{A}(t,s)\,\mathrm{d}t + \displaystyle\sum_{j=1}^{m} \mathcal{R}_j(t) \circ \mathcal{A}(t,s)\,\mathrm{d}W_j(t), \\ \mathcal{A}(s,s) = \mathrm{Id}_n. \end{cases} \tag{3.42}$$

We call $\mathcal{A}(t, s)$ the *stochastic evolution map* or, borrowing a terminology used in theoretical physics, the *propagator* associated to the linear SDE.

3.4.1.2 The Natural Filtrations of W

We introduce the natural two-times filtrations of W:

$$\mathcal{G}_t^s := \sigma\left\{W_j(r) - W_j(s),\ r \in [s,t],\ j = 1, \dots, m\right\}, \tag{3.43a}$$

$$\mathcal{G} := \bigvee_{t \geq 0} \mathcal{G}_t^0\,, \qquad \mathcal{N}_{\mathcal{G}} := \{A \in \mathcal{G} : \mathbb{Q}(A) = 0\}\,, \tag{3.43b}$$

$$\overline{\mathcal{G}}_t^s := \mathcal{G}_t^s \vee \mathcal{N}_{\mathcal{G}}\,. \tag{3.43c}$$

Let us note that

$$\mathcal{G} \subset \mathcal{F}, \qquad \mathcal{N}_{\mathcal{G}} \subset \mathcal{N} \equiv \{A \in \mathcal{F} : \mathbb{Q}(A) = 0\}\,, \qquad \mathcal{G}_t^s \subset \overline{\mathcal{G}}_t^s \subset \mathcal{F}_t.$$

Moreover, $\left(\Omega, \mathcal{G}, (\overline{\mathcal{G}}_t^0), \mathbb{Q}\right)$ is a stochastic basis satisfying usual conditions.

Theorem 3.4. *Under Assumptions 3.3, the linear stochastic master equation* (3.38) *admits continuous strong solutions in* $[0, +\infty)$*. Pathwise uniqueness and uniqueness in law hold. The solution* $\sigma(t)$ *of* (3.38)*, with initial condition* $\sigma(0) = \rho_0 \in \mathcal{S}(\mathcal{H})$*, is non-negative. Moreover,* $p(t) := \mathrm{Tr}\{\sigma(t)\}$ *is a mean one* $\mathbb{Q}$*-martingale, it is a.s. strictly positive and it can be written as*

$$p(t) = \mathrm{Tr}\{\sigma(t)\} = \exp\left\{\sum_{j=1}^{m}\left[\int_0^t v_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t v_j(s)^2\mathrm{d}s\right]\right\}, \tag{3.44}$$

where

$$v_j(t) := \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\rho(t)\right\} = 2\,\mathrm{Re}\,\mathrm{Tr}\left\{R_j(t)\rho(t)\right\}, \tag{3.45}$$

$$\rho(t) := p(t)^{-1}\,\sigma(t)\,. \tag{3.46}$$

The linear SDE (3.42) *admits strong solutions in* $(s, +\infty)$*, for every* $s \geq 0$*. Pathwise uniqueness and uniqueness in law hold.* $\mathcal{A}(t, s)$ *is* $\mathbb{Q}$*-independent of* $\mathcal{F}_s$*,* $\overline{\mathcal{G}}_t^s$*-measurable, completely positive and continuous in* t*. Moreover, for* $0 \leq r \leq s \leq t$ *one has a.s.*

$$\mathcal{A}(t, s) \circ \mathcal{A}(s, r) = \mathcal{A}(t, r), \qquad \sigma(t) = \mathcal{A}(t, 0)[\rho_0]. \tag{3.47}$$

The master equation

$$\mathcal{T}(t, s) = \mathrm{Id}_n + \int_s^t \mathcal{L}(r) \circ \mathcal{T}(r, s)\,\mathrm{d}r \tag{3.48}$$

admits a unique solution in $[s, +\infty)$*, for every* $s \geq 0$*. Moreover, the solution admits the representation*

$$\mathcal{T}(t, s) = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{A}(t, s)\right], \tag{3.49}$$

it is continuous in t*, completely positive, trace preserving, and it satisfies the composition law* (3.18)*,*

$$\mathcal{T}(t, r) = \mathcal{T}(t, s) \circ \mathcal{T}(s, r), \qquad 0 \leq r \leq s \leq t\,. \tag{3.50}$$

Proof. Equation (3.38) is for a $(n \times n)$-dimensional process and (3.42) for a $(n^2 \times n^2)$-dimensional one; in both cases we have finite dimensional processes. The bounds (3.41) and the linearity give that the global Lipschitz condition A.32 and the linear growth condition A.34 hold. Then, as the measurability condition A.25 trivially holds, Theorem A.36 gives the existence of strong solutions and the uniqueness statements for both SDEs. By completeness, let us check in detail Hypotheses A.32 and A.34 for $t \in [0, T]$. Here below we use all the properties of the matrix norms given by (B.2)–(B.9).

First of all, let us note that

$$\left\|\tau^*\tau\right\|_1 = \left\|\tau\tau^*\right\|_1 = \|\tau\|_2^2\,,$$

which follows from the definitions of the two norms and from the positivity of $\tau^*\tau$ and $\tau\tau^*$. Moreover, for any matrix A we have

$$\|A\tau\|_2^2 = \mathrm{Tr}\left\{\tau^* A^* A \tau\right\} = \mathrm{Tr}\left\{A^* A \tau \tau^*\right\} \leq \left\|A^* A\right\| \left\|\tau\tau^*\right\|_1 = \|A\|^2 \|\tau\|_2^2 ,$$

so that

$$\begin{aligned}\left\|A\tau A^*\right\|_2^2 &= \mathrm{Tr}\left\{A\tau^* A^* A \tau A^*\right\} = \mathrm{Tr}\left\{A^* A \tau^* A^* A \tau\right\} \\ &\leq \left\|A^* A \tau^*\right\|_2 \left\|A^* A \tau\right\|_2 \leq \left\|A^* A\right\| \left\|\tau^*\right\|_2 \left\|A^* A\right\| \|\tau\|_2 = \left\|A^* A\right\|^2 \|\tau\|_2^2 .\end{aligned}$$

We also set

$$\ell_T := \max\left(\sup_{0\leq t\leq T} \|H(t)\| , \sup_{0\leq t\leq T} \left\|\sum_{j=1}^d R_j(t)^* R_j(t)\right\|\right) ; \tag{3.51}$$

by (3.41), $\ell_T < +\infty$. Since $\sum_j R_j(t)^* R_j(t)$ is a sum of positive operators, we have also

$$\left\|R_j(t)\right\|^2 = \left\|R_j(t)^* R_j(t)\right\| \leq \ell_T . \tag{3.52}$$

In the case of (3.38) the relevant norm, needed in Hypotheses A.32 and A.34, is the Hilbert–Schmidt norm. We have

$$\begin{aligned}\|\mathcal{L}(t)[\tau]\|_2 &\leq 2\|H(t)\tau\|_2 + \sum_{j=1}^d \left\|R_j(t)\tau R_j(t)^*\right\|_2 + \left\|\sum_{j=1}^d R_j(t)^* R_j(t)\tau\right\|_2 \\ &\leq 2\|H(t)\| \|\tau\|_2 + \sum_{j=1}^d \left\|R_j(t)^* R_j(t)\right\| \|\tau\|_2 + \left\|\sum_{j=1}^d R_j(t)^* R_j(t)\right\| \|\tau\|_2 \\ &\leq (3+d)\ell_T \|\tau\|_2 ,\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^m \left\|\mathcal{R}_j(t)[\tau]\right\|_2^2 &\leq 2\sum_{j=1}^m \left(\left\|R_j(t)\tau\right\|_2^2 + \left\|\tau R_j(t)^*\right\|_2^2\right) \\ &\leq 4\sum_{j=1}^m \left\|R_j(t)\right\|^2 \|\tau\|_2^2 \leq 4m\ell_T \|\tau\|_2^2 .\end{aligned} \tag{3.53}$$

These two estimates imply both Hypothesis A.32 and Hypothesis A.34.

The proof of existence and uniqueness of the solution of SDE (3.42) is completely similar and it is based on the estimates

$$\sum_{k,l=1}^n \|\mathcal{L}(t)\circ\mathcal{A}(t;s)[|k\rangle\langle l|]\|_2^2 \leq (3+d)^2 \ell_T^2 \sum_{k,l=1}^n \|\mathcal{A}(t;s)[|k\rangle\langle l|]\|_2^2 ,$$

$$\sum_{k,l=1}^{n}\sum_{j=1}^{m}\left\|\mathcal{R}_j(t)\circ\mathcal{A}(t;s)[|k\rangle\langle l|]\right\|_2^2 \leq 4m\ell_T\sum_{k,l=1}^{n}\|\mathcal{A}(t;s)[|k\rangle\langle l|]\|_2^2\,,$$

where $\{|k\rangle\}_{k=1}^n$ is a basis in $\mathcal{H}$.

The continuity in t of $\sigma(t)$ and $\mathcal{A}(t,s)$ comes from the fact that we are working in a stochastic basis in usual hypotheses and it is included in Definition A.27 of strong solution.

Because of the existence of strong solutions and pathwise uniqueness, the random variable $\mathcal{A}(t,s)$ is $\overline{\mathcal{G}}_t^s$-measurable; then, the statement about the $\mathbb{Q}$-independence of $\mathcal{F}_s$ follows from the independent-increment property of the Wiener process. Moreover, the two sides of the composition law in (3.47) satisfy the same SDE (3.42) for $t \leq s$ and so they are equal by the uniqueness statement of Theorem A.37. Analogously, $\sigma(t)$ and $\mathcal{A}(t,0)[\rho_0]$ are a.s. equal because they satisfy the same SDE (3.38) with the same initial condition.

Once the operators $H(t)$ and $R_j(t)$ appearing in (3.40) have been fixed and the stochastic evolution operator A_t^s solution of the SDE (2.7) with $K(t)$ given by (2.20) has been constructed, one can check that the map $\mathbb{E}_{\mathbb{Q}}\left[A_t^s\bullet A_t^{s*}\middle|\overline{\mathcal{G}}_t^s\right]$ satisfies the same SDE as $\mathcal{A}(t,s)$. Since a conditional expectation is a completely positive map and the same is true for a map of the type $\rho \mapsto A\rho A^*$, the map $\mathbb{E}_{\mathbb{Q}}\left[A_t^s\bullet A_t^{s*}\middle|\overline{\mathcal{G}}_t^s\right]$ is completely positive. Therefore, by the uniqueness in law of the solution of the SDE (3.42), $\mathcal{A}(t,s)$ is completely positive, too. Let us stress that this part of the proof is not direct but goes through the Hilbert space representation.

We can see the master equation (3.48) as a particular case of SDE; by the properties of the coefficients and Theorem A.36, for every $s \geq 0$ the solution is unique. Moreover, by Theorem A.37, the mean of $\mathcal{A}(t,s)$ exists and the stochastic integral in (3.42) has mean zero, so that $\mathbb{E}_{\mathbb{Q}}[\mathcal{A}(t,s)]$ is well defined and satisfies (3.48). For what concerns the composition law, the two sides of (3.50) satisfy the same equation with the same initial condition; they are equal by uniqueness of the solution. The continuity in t follows from the integral representation (3.48), the complete positivity from the same property of $\mathcal{A}(t,s)$ and the trace preserving property from the structure of any Liouville operator, which guarantees $\mathrm{Tr}\{\mathcal{L}(r)[\tau]\} = 0$ for every operator τ.

The complete positivity of $\mathcal{A}(t,0)$ and $\rho \geq 0$, imply $\sigma(t) \geq 0$. By taking the trace of the linear stochastic master equation (3.38), we get

$$\mathrm{Tr}\{\sigma(t)\} = 1 + \sum_{j=1}^{m}\int_0^t 2\,\mathrm{Re}\left(\mathrm{Tr}\{R_j(s)\sigma(s)\}\right)\mathrm{d}W_j(s). \tag{3.54}$$

By the bound (3.41) and the estimate of Theorem A.37 for the process σ, we have that the integrand in the equation above is in the class $\mathcal{M}^2$; therefore, the stochastic integral is a $\mathbb{Q}$-martingale. Let $\rho_\star$ be a fixed statistical operator and let us define

$$\rho(t) = \begin{cases} (\mathrm{Tr}\{\sigma(t)\})^{-1}\ \sigma(t), & \text{if } \mathrm{Tr}\{\sigma(t)\} > 0, \\ \rho_\star, & \text{if } \mathrm{Tr}\{\sigma(t)\} = 0. \end{cases}$$

Then, (3.54) can be written as

$$\mathrm{Tr}\{\sigma(t)\} = 1 + \sum_{j=1}^{m} \int_0^t \mathrm{Tr}\{\sigma(s)\} v_j(s)\, \mathrm{d}W_j(s),$$

where v_j is given by (3.45). The solution of this Doléans equation is unique and it is given by (3.44). Being of exponential form, it is strictly positive with probability one. □

Remark 3.5. Let us recall that the *adjoint* $\mathcal{O}^*$ of a linear map $\mathcal{O}$ on M_n is defined by $\mathrm{Tr}\{\tau \mathcal{O}^*[a]\} = \mathrm{Tr}\{\mathcal{O}[\tau]a\}$, $\forall \tau,\ a \in M_n$. If the map $\mathcal{O}$ is completely positive, also $\mathcal{O}^*$ is such. Note that $\mathrm{Tr}\{\tau \mathcal{O}^*[\mathbb{1}]\} = \mathrm{Tr}\{\mathcal{O}[\tau]\}$. See Sect. B.1.2, (B.7), and Section B.4.1.

Remark 3.6. By the previous remark, the maps $\mathcal{A}(t,s)^*$ and $\mathcal{T}(t,s)^*$ are completely positive and the trace preserving property of $\mathcal{T}(t,s)$ is equivalent to

$$\mathcal{T}(t,s)^*[\mathbb{1}] = \mathbb{1}. \tag{3.55}$$

Remark 3.7. Another important and intuitive property is

$$\mathbb{E}_{\mathbb{Q}}\left[\sigma(t)\middle|\overline{\mathcal{G}}_t^s\right] = \mathcal{A}(t,s)[\eta(s)]. \tag{3.56}$$

Indeed, by (3.47) and the fact that $\mathcal{A}(t,s)$ is $\overline{\mathcal{G}}_t^s$-measurable, we have $\mathbb{E}_{\mathbb{Q}}[\sigma(t)|\overline{\mathcal{G}}_t^s] = \mathcal{A}(t,s)\big[\mathbb{E}_{\mathbb{Q}}[\sigma(s)|\overline{\mathcal{G}}_t^s]\big]$. By the fact that the noises have independent increments, we have that $\sigma(s)$ is independent from $\overline{\mathcal{G}}_t^s$ and $\mathbb{E}_{\mathbb{Q}}[\sigma(s)|\overline{\mathcal{G}}_t^s] = \mathbb{E}_{\mathbb{Q}}[\sigma(s)] = \eta(s)$. This gives (3.56).

3.4.2 The Physical Probabilities, the A Posteriori and the A Priori States

Starting from the linear stochastic master equation (3.38) we introduce the physical probabilities, the a posteriori states and the a priori states. Let us recall that the initial state at time zero is the statistical operator $\rho_0 \in \mathcal{S}(\mathcal{H})$.

Definition 3.8. *We define the quantities*

$$E_t^s(G) := \int_G \mathcal{A}(t,s;\omega)^*[\mathbb{1}]\mathbb{Q}(\mathrm{d}\omega) \equiv \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,s)^*[\mathbb{1}]\right], \qquad G \in \overline{\mathcal{G}}_t^s, \tag{3.57}$$

$$\mathbb{P}^t_{\rho_0}(G) := \mathrm{Tr}\left\{E^0_t(G)\rho_0\right\} = \mathbb{E}_{\mathbb{Q}}\left[1_G \,\mathrm{Tr}\left\{\sigma(t)\right\}\right], \qquad G \in \overline{\mathcal{G}}^0_t . \tag{3.58}$$

Proposition 3.9. *E^s_t is a positive operator-valued measure on the value space $\left(\Omega, \overline{\mathcal{G}}^s_t\right)$ and $\mathbb{P}^t_{\rho_0}$ is a probability measure on the value space $\left(\Omega, \overline{\mathcal{G}}^0_t\right)$. The family of probability measures $\left\{\mathbb{P}^t_{\rho_0}\,,\ t > 0\right\}$ is consistent, i.e. $\mathbb{P}^t_{\rho_0}(G) = \mathbb{P}^s_{\rho_0}(G)$ for any $G \in \overline{\mathcal{G}}^0_s$ with $0 < s < t$. Analogously, we have the consistency of the POMs:*

$$0 \le s < t < T, \quad G \in \overline{\mathcal{G}}^s_t \qquad \Rightarrow \qquad E^s_t(G) = E^s_T(G). \tag{3.59}$$

Let T be an arbitrary positive time. Under the probability $\mathbb{P}^T_{\rho_0}$, the processes

$$\widehat{W}_j(t) := W_j(t) - \int_0^t v_j(s)\mathrm{d}s, \qquad j = 1, \ldots, m, \quad t \in [0, T], \tag{3.60}$$

are independent, $(\overline{\mathcal{G}}^0_t)$-adapted, standard Wiener processes.

Proof. By the properties of $\mathcal{A}(t, s)$ we have

$$0 \le \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t, s)^*[\mathbb{1}]\right] \le \mathbb{E}_{\mathbb{Q}}\left[\mathcal{A}(t, s)^*[\mathbb{1}]\right] = \mathcal{T}(t, s)^*[\mathbb{1}] = \mathbb{1}. \tag{3.61}$$

Then, from the Definition (3.57) one can check that all the properties characterising a POM hold. The consistency of the POMs follows from the composition property of the propagator $\mathcal{A}$, the independence of $1_G\mathcal{A}(t, s)^*$ and $\mathcal{A}(T, t)^*$ and $\mathcal{T}(T, t)^*[\mathbb{1}] = \mathbb{1}$:

$$\begin{aligned} E^s_T(G) &= \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(T, s)^*[\mathbb{1}]\right] = \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t, s)^* \circ \mathcal{A}(T, t)^*[\mathbb{1}]\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t, s)^* \circ \mathcal{T}(T, t)^*[\mathbb{1}]\right] = \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t, s)^*[\mathbb{1}]\right] = E^s_t(G). \end{aligned}$$

E^0_t being a POM and ρ_0 a state, (3.58) defines a probability measure. Consistency follows from the fact that $\mathrm{Tr}\{\sigma(t)\}$ is a martingale or from the consistency of the POMs.

The statement on $\widehat{W}(t)$ is Girsanov theorem. □

The interpretation is the one we have announced in the previous sections.

The observables of the theory are represented by the POMs E^0_t and the pre-measurement state by ρ_0. Then, the physical probabilities are defined by (3.58) and their structure in terms of a POM and a state guarantees that the usual axioms of quantum mechanics are not violated. Moreover, we can write

$$\mathbb{P}^t_{\rho_0}(\mathrm{d}\omega) = \mathrm{Tr}\left\{\sigma(t, \omega)\right\} \mathbb{Q}(\mathrm{d}\omega)\Big|_{\overline{\mathcal{G}}^0_t} . \tag{3.62}$$

The value space of the POM E^0_t is $\left(\Omega, \overline{\mathcal{G}}^0_t\right)$, but $\overline{\mathcal{G}}^0_t$ is generated by W and this allows to identify the m-dimensional process W with the output. The output of the

measurement has to be considered under the physical probability $\mathbb{P}^T_{\rho_0}$. By Girsanov formula (3.60), we can see the output as the sum of a Wiener process and of a process with trajectories of bounded variation,

$$W_j(t) = \widehat{W}_j(t) + \int_0^t v_j(s)\mathrm{d}s\,, \qquad j = 1, \ldots, m\,. \tag{3.63}$$

Apart from trivial cases, the two terms are not independent.

Furthermore, as in Section 2.4.3, one has that it exists a unique probability $\mathbb{P}_{\rho_0}$ on $\mathcal{G}$ such that $\mathbb{P}_{\rho_0}\Big|_{\mathcal{G}^0_t} = \mathbb{P}^t_{\rho_0}\Big|_{\mathcal{G}^0_t}$, $\forall t > 0$.

The random statistical operator $\rho(t)$ defined in (3.46) can be consistently interpreted as the state of the system at time t conditional on the output observed up to time t: for every $0 \le s \le t \le T$, the conditional probability $\widehat{\mathbb{P}}^T_{\rho_0}(G|\overline{\mathcal{G}}^0_s)$ of an event $G \in \overline{\mathcal{G}}^s_t$ (Sect. A.1.2.2) can be computed using the POM E^s_t defined by (3.57) and just $\rho(s)$ as the conditional state of the system at time s. Indeed, taken $G \in \overline{\mathcal{G}}^s_t$, for all $\overline{\mathcal{G}}^0_s$-measurable random variables Y we have

$$\begin{aligned}
&\mathbb{E}^T_{\rho_0}[1_G Y] = \mathbb{E}_{\mathbb{Q}}[\mathrm{Tr}\{\sigma(t)\}1_G Y] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}[1_G \mathcal{A}(t,s)|\overline{\mathcal{G}}^0_s]\sigma(s)\right\} Y\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}[1_G \mathcal{A}(t,s)]\sigma(s)\right\} Y\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\mathbb{E}_{\mathbb{Q}}[1_G \mathcal{A}(t,s)^*[\mathbb{1}]]\sigma(s)\right\} Y\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{E^s_t(G)\sigma(s)\right\} Y\right] = \mathbb{E}^T_{\rho_0}\left[\mathrm{Tr}\left\{\rho(s)E^s_t(G)\right\} Y\right].
\end{aligned}$$

This proves that $\mathrm{Tr}\left\{\rho(s)E^s_t(G)\right\}$ is the conditional expectation of 1_G given $\overline{\mathcal{G}}^0_s$. So, we have: $\forall G \in \overline{\mathcal{G}}^s_t$, $0 \le s \le t \le T$,

$$\mathbb{P}^T_{\rho_0}(G|\overline{\mathcal{G}}^0_s) = \mathrm{Tr}\left\{\rho(s)E^s_t(G)\right\}. \tag{3.64}$$

The random state $\rho(t)$ is called *a posteriori state*.

With this interpretation in mind, we consider again the output W and, thanks to the representation (3.63), (3.45) we say that $\dot{W}_j(t)$ is an imprecise measurement of the quantum observable $R_j(t) + R_j(t)^*$ at time t. We shall consider again this interpretation in Section 4.3.

The mean of the a posteriori state

$$\eta(t) := \mathbb{E}^T_{\rho_0}[\rho(t)] = \mathbb{E}_{\mathbb{Q}}[\sigma(t)] \tag{3.65}$$

is the state to be assigned to the system when the result of the observation is not known or not taken into account; it is known as *a priori state*. We have

$$\eta(0) = \rho_0, \qquad \eta(t) = \mathcal{T}(t,0)[\rho_0]. \tag{3.66}$$

Thus the non-selective dynamics associated to the continuous measurement (3.38) is governed by the ordinary master equation of Liouvillian $\mathcal{L}(t)$.

By using the composition property $\mathcal{A}(t,0) = \mathcal{A}(t,s) \circ \mathcal{A}(s,0)$ and the fact that $\mathcal{A}(t,s)$ and $\mathcal{A}(s,0)$ are $\mathbb{Q}$-independent, we obtain for $0 \leq s < t \leq T$

$$\mathbb{E}_{\mathbb{Q}}[\sigma(t)|\mathcal{F}_s] = \mathcal{T}(t,s)[\sigma(s)], \qquad \mathbb{E}^T_{\rho_0}[\rho(t)|\mathcal{F}_s] = \mathcal{T}(t,s)[\rho(s)]. \tag{3.67}$$

3.4.3 Back to the Hilbert Space Formulation

Even when the continuous measurement is represented by the SDE (3.38) with coefficients (3.39) and (3.40), where $\mathcal{L}_0$ contains dissipative terms not necessarily linked to the measurement itself, from a mathematical point of view it is always possible to associate such terms to the unobserved output of some additional measurements and to go back to the Hilbert space formulation. Of course, there is no uniqueness in this representation and, eventually, the stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$ needs to be enlarged.

Given the linear stochastic master equation (3.38), one fixes a representation for $\mathcal{L}_0(t)$ as (3.40c), thus getting the operators $H(t)$ and $R_j(t)$, $j = m+1, \ldots, d$. Then, one introduces a further $(d–m)$-dimensional Wiener process, independent of the previous m-dimensional one and forms a d-dimensional Wiener process W. If the original stochastic basis is too small to accommodate the new components of the Wiener process, one has to enlarge the filtration $(\mathcal{F}_t)$ or even the sample space Ω.

Finally, one associates also $W_{m+1}, \ldots, W_d$ to a continuous measurement on the quantum system by postulating the stochastic Schrödinger equation (2.28) for its evolution. This is the *unravelling* of the stochastic master equation.

By this procedure, one constructs also the stochastic evolution operators A^s_t in the Hilbert space $\mathcal{H}$ and, by the strong existence and the pathwise uniqueness of the solutions of the linear stochastic master equation (3.38), one gets

$$\sigma(t) = \mathcal{A}(t,0)[\rho_0] = \mathbb{E}_{\mathbb{Q}}\left[A^0_t \rho_0 A^{0*}_t \middle| \overline{\mathcal{G}}^0_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{\sigma}(t)\middle|\overline{\mathcal{G}}^0_t\right], \tag{3.68a}$$

$$\rho(t) = \mathbb{E}^T_{\rho_0}\left[\widetilde{\rho}(t)\middle|\overline{\mathcal{G}}^0_t\right]. \tag{3.68b}$$

3.5 The Stochastic Master Equation

By differentiating (3.46) we get a stochastic evolution equation for the a posteriori states, known in the physical literature as *stochastic master equation*.

Proposition 3.10. *Under the physical probability $\mathbb{P}^T_{\rho_0}$, the a posteriori states satisfy the nonlinear SDE*

$$\mathrm{d}\rho(t) = \mathcal{L}(t)[\rho(t)]\mathrm{d}t + \sum_{j=1}^{m}\big[R_j(t)\rho(t) + \rho(t)R_j(t)^* - v_j(t)\rho(t)\big]\mathrm{d}\widehat{W}_j(t) \tag{3.69}$$

with initial condition $\rho(0) = \rho_0 \in \mathcal{S}(\mathcal{H})$.

The quantities $v_j(t)$ are real random variables which depend on $\rho(t)$ and are given by (3.45).

Proof. By using (3.38) and (3.63) we can express the stochastic differential of $\sigma(t)$ in terms of the new noise $\widehat{W}$:

$$\begin{aligned} \mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\mathrm{d}t + \sum_{j=1}^{m}\big(R_j(t)\sigma(t) + \sigma(t)R_j(t)^*\big)\mathrm{d}\widehat{W}_j(t) \\ + \sum_{j=1}^{m}\big(R_j(t)\sigma(t) + \sigma(t)R_j(t)^*\big)v_j(t)\,\mathrm{d}t\,. \end{aligned}$$

From formula (3.44) we have immediately

$$\begin{aligned} (\mathrm{Tr}\{\sigma(t)\})^{-1} &= \exp\left\{-\sum_{j=1}^{m}\left[\int_0^t v_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t v_j(s)^2\mathrm{d}s\right]\right\} \\ &= \exp\left\{-\sum_{j=1}^{m}\left[\int_0^t v_j(s)\mathrm{d}\widehat{W}_j(s) + \frac{1}{2}\int_0^t v_j(s)^2\mathrm{d}s\right]\right\}; \end{aligned}$$

by Itô formula we get

$$\mathrm{d}\,(\mathrm{Tr}\{\sigma(t)\})^{-1} = -\,(\mathrm{Tr}\{\sigma(t)\})^{-1}\sum_{j=1}^{m} v_j(t)\mathrm{d}\widehat{W}_j(t).$$

Finally, by Itô formula for products we get (3.69). □

Let us stress that our starting point was the linear SDE (3.38) for $\sigma(t)$ in the stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$. Then, we constructed the a posteriori states $\rho(t)$ by (3.46) and the stochastic basis $\Big(\Omega, \mathcal{G}, \big(\overline{\mathcal{G}}_t^0\big), \mathbb{P}_{\rho_0}^T\Big)$. Finally, we showed that, in this new stochastic basis, $\rho(t)$ satisfies (3.69). So, we have by construction that the nonlinear SDE (3.69) has a solution in a particular stochastic basis: according to Definition A.26 we have shown that (3.69) has a weak solution. In Chap. 5 we shall show that the SDE (3.69) can be extended from $\mathcal{S}(\mathcal{H})$ to the whole M_n, has strong solutions, and can be taken as starting point of the whole theory.

3.6 Summary of Linear Quantum Trajectories

3.6.1 The Master Equation

- The Hilbert space of the quantum system under consideration is $\mathcal{H} = \mathbb{C}^n$. The set of statistical operators is $\mathcal{S}(\mathcal{H}) = \big\{\rho \in M_n : \rho \geq 0, \mathrm{Tr}\{\rho\} = 1\big\}$.
- The effective Hamiltonian $H(t)$ and the system operators $R_j(t)$, $j = 1, \ldots, d$, (dissipative terms) are non-random linear operators on $\mathcal{H}$; $H(t)$ is self-adjoint:

$H(t)^* = H(t)$. The functions $t \mapsto H(t)$ and $t \mapsto R_j(t)$ are measurable and, $\forall T \in (0, +\infty)$,

$$\sup_{t\in[0,T]} \|H(t)\| < +\infty, \qquad \sup_{t\in[0,T]} \left\| \sum_j R_j(t)^* R_j(t) \right\| < +\infty.$$

- Let us take $1 \le m \le d$; we introduce the Liouville operator as $\mathcal{L}(t) = \mathcal{L}_0(t) + \mathcal{L}_1(t)$, where, $\forall \tau \in M_n$,

$$\mathcal{L}_1(t)[\tau] = \sum_{j=1}^{m} \left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{ R_j(t)^* R_j(t), \tau \right\} \right),$$

$$\mathcal{L}_0(t)[\tau] = -\mathrm{i}[H(t), \tau] + \sum_{j=m+1}^{d} \left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{ R_j(t)^* R_j(t), \tau \right\} \right).$$

- The master equation and the composition law:

$$\mathcal{T}(t, s) = \mathrm{Id}_n + \int_s^t \mathcal{L}(r) \circ \mathcal{T}(r, s)\, \mathrm{d}r\,,$$
$$\mathcal{T}(t, r) = \mathcal{T}(t, s) \circ \mathcal{T}(s, r)\,, \qquad 0 \le r \le s \le t\,;$$

$\mathcal{T}(t, s)$ is continuous in t, completely positive and trace preserving.

3.6.2 Reference Probability Space and Filtrations

- $\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\right)$ is a stochastic basis satisfying the usual conditions. We take also $\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t\ge 0} \mathcal{F}_t$. The symbol $\mathbb{E}_\mathbb{Q}$ indicates the expectation with respect to $\mathbb{Q}$ and we set $\mathcal{N} := \{B \in \mathcal{F} : \mathbb{Q}(B) = 0\}$.
- $W = \{W(t),\ t \ge 0\}$ is a continuous m-dimensional Wiener process defined in $\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\right)$. The process W has increments independent of the past with respect to the filtration $(\mathcal{F}_t)$.
- The natural filtration of the increments of W is $\mathcal{G}_t^s = \sigma\left\{ W_j(r) - W_j(s), r \in [s, t], j = 1, \ldots, m \right\}$. We set also $\mathcal{G} := \bigvee_{t\ge 0} \mathcal{G}_t^0$ and $\mathcal{N}_\mathcal{G} = \{A \in \mathcal{G} : \mathbb{Q}(A) = 0\}$. The augmented natural filtration of W in $\left(\Omega, \mathcal{G}, \mathbb{Q}\right)$ is $\overline{\mathcal{G}}_t^s := \mathcal{G}_t^s \vee \mathcal{N}_\mathcal{G}$.
- The filtration $\{\overline{\mathcal{G}}_t^s,\ t \in [s, +\infty)\}$ satisfies the usual conditions; $\overline{\mathcal{G}}_t^s$ is independent of $\mathcal{F}_s$ and $\overline{\mathcal{G}}_t^s \subset \overline{\mathcal{G}}_t^0 \subset \mathcal{F}_t \subset \mathcal{F}$, for $0 \le s \le t$.

3.6.3 The Linear Stochastic Master Equation for Non-normalised States

- The linear stochastic master equation:

$$\begin{cases} \mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t)[\sigma(t)]\mathrm{d}W_j(t), \\ \sigma(0) = \rho_0 \in \mathcal{S}(\mathcal{H}), \end{cases}$$

$$\mathcal{R}_j(t)[\tau] := R_j(t)\tau + \tau R_j(t)^*, \qquad \forall \tau \in M_n .$$

- $p(t) = \mathrm{Tr}\{\sigma(t)\}$ is a mean one $\mathbb{Q}$-martingale, is a.s. strictly positive and can be written as

$$p(t) = \mathrm{Tr}\{\sigma(t)\} = \exp\left\{ \sum_{j=1}^{m} \left[\int_0^t v_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t v_j(s)^2 \mathrm{d}s \right] \right\},$$

$$v_j(t) := \mathrm{Tr}\left\{ \left(R_j(t) + R_j(t)^* \right) \rho(t) \right\} = 2\,\mathrm{Re}\,\mathrm{Tr}\left\{ R_j(t)\rho(t) \right\}.$$

- A posteriori states: $\rho(t) := (\mathrm{Tr}\{\sigma(t)\})^{-1}\ \sigma(t)$.
- The stochastic evolution map and its properties:

$$\mathrm{d}\mathcal{A}(t,s) = \mathcal{L}(t) \circ \mathcal{A}(t,s)\,\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t) \circ \mathcal{A}(t,s)\,\mathrm{d}W_j(t), \qquad \mathcal{A}(s,s) = \mathrm{Id}_n.$$

  $\mathcal{A}(t,s)$ is $\mathbb{Q}$-independent of $\mathcal{F}_s$, $\overline{\mathcal{G}}_t^s$-measurable, completely positive and continuous in t. For $0 \le r \le s \le t$ one has a.s.

$$\mathcal{A}(t,s) \circ \mathcal{A}(s,r) = \mathcal{A}(t,r), \qquad \sigma(t) = \mathcal{A}(t,0)[\rho_0].$$

- Mean dynamics and a priori states:

$$\mathcal{T}(t,s) = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{A}(t,s)\right], \qquad \eta(t) = \mathbb{E}_{\mathbb{Q}}[\sigma(t)] = \mathcal{T}(t,0)[\rho_0].$$

- Master equation: $\dfrac{\mathrm{d}}{\mathrm{d}t}\,\eta(t) = \mathcal{L}(t)[\eta(t)].$
- Conditioning: $0 \le s < t$

$$\mathbb{E}_{\mathbb{Q}}[\sigma(t)|\mathcal{F}_s] = \mathcal{T}(t,s)[\sigma(s)], \qquad \mathbb{E}_{\mathbb{Q}}\left[\sigma(t)\middle|\overline{\mathcal{G}}_t^s\right] = \mathcal{A}(t,s)[\eta(s)].$$

3.6.4 The Physical Probability and the POMs

- The POM on the value space $\left(\Omega, \overline{\mathcal{G}}_t^s\right)$:

$$E_t^s(G) = \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,s)^*[\mathbb{1}]\right], \qquad G \in \overline{\mathcal{G}}_t^s .$$

  The POMs are consistent: $0 \le s < t < T, G \in \overline{\mathcal{G}}_t^s \;\Rightarrow\; E_t^s(G) = E_T^s(G)$.
- The physical probability on the value space $\left(\Omega, \overline{\mathcal{G}}_t^0\right)$:

$$\mathbb{P}_{\rho_0}^t(G) = \mathbb{E}_{\mathbb{Q}}\left[1_G p(t)\right] = \mathrm{Tr}\left\{E_t^0(G)\rho\right\}, \qquad G \in \overline{\mathcal{G}}_t^0 .$$

  The physical probabilities are consistent: $0 < s < t, G \in \overline{\mathcal{G}}_s^0 \;\Rightarrow\; \mathbb{P}_{\rho_0}^t(G) = \mathbb{P}_{\rho_0}^s(G)$.
- The new Wiener process and the output: under the law $\mathbb{P}_{\rho_0}^T$, the processes

$$\widehat{W}_j(t) := W_j(t) - \int_0^t v_j(s)\mathrm{d}s, \qquad j = 1, \ldots, m, \quad t \in [0, T],$$

  are independent, $(\overline{\mathcal{G}}_t^0)$-adapted, standard Wiener processes. The output is the process $W(t)$, $0 \le t \le T$, under the physical probability $\mathbb{P}_{\rho_0}^T$. More precisely, the output is the set of the increments of W as explained in Remark 2.16.
- Conditioning: $0 \le s < t \le T \Rightarrow \mathbb{E}_{\rho_0}^T[\rho(t)|\mathcal{F}_s] = \mathcal{T}(t,s)[\rho(s)]$.

3.6.5 The Nonlinear Stochastic Master Equation for Normalised States

- The a posteriori states $\rho(t)$ satisfy the stochastic master equation

$$\mathrm{d}\rho(t) = \mathcal{L}(t)[\rho(t)]\mathrm{d}t + \sum_{j=1}^{m}\left[R_j(t)\rho(t) + \rho(t)R_j(t)^* - v_j(t)\rho(t)\right]\mathrm{d}\widehat{W}_j(t),$$

  with $\rho(0) = \rho_0 \in \mathcal{S}(\mathcal{H})$.

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Chapter 4
Continuous Measurements and Instruments

In this chapter we complete the task of showing that the SDE approach can be reduced to the usual formulation of quantum mechanics. The last notion which we need is the one of "instrument"; its definition, meaning and properties are presented in Sect. B.4.

We give also the important concept of characteristic operator, a kind of Fourier transform of the instruments, we show that it satisfies an evolution equation, in some sense similar to a master equation, and we show how to obtain explicit formulae for the moments of the output by means of this characteristic operator.

Finally, we introduce the notion of spectrum of a stochastic process and we study the spectrum of the output of the continuous measurement.

Recall that $\mathcal{H} = \mathbb{C}^n$ (Assumption 2.1).

4.1 The Instruments

In this section we show how the theory of continuous observations can be reformulated in terms of instruments.

4.1.1 The Construction of the Instruments

Let us start from the formulation based on the linear stochastic master equation, presented in Sect. 3.4.1. The main equation is the SDE (3.38) for $\sigma(t)$, its fundamental solution $\mathcal{A}(t,s)$ satisfies Eq. (3.42), all the coefficients are given in Assumption 3.3 and equations (3.39), (3.40) and the natural filtration $\mathcal{G}_t^s$ of the m-dimensional Wiener process $W(t)$ and its augmentation $\overline{\mathcal{G}}_t^s$ are given by (3.43).

Definition 4.1. *By using the stochastic evolution map $\mathcal{A}(t,s)$, $t > s \geq 0$, we define the map $\mathcal{I}_t^s(G)$, $G \in \overline{\mathcal{G}}_t^s$, by*

$$\mathcal{I}_t^s(G)[\tau] := \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,s)[\tau]\right], \qquad \forall \tau \in M_n\,. \tag{4.1}$$

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Proposition 4.2. *Equation* (4.1) *defines an instrument* $\mathcal{I}_t^s$ *on the value space* $(\Omega, \overline{\mathcal{G}}_t^s)$. *For* $t > s > r \geq 0$, *the following composition law holds:*

$$\mathcal{I}_t^r(G_1 \cap G_2) = \mathcal{I}_t^s(G_2) \circ \mathcal{I}_s^r(G_1), \qquad \forall G_1 \in \overline{\mathcal{G}}_s^r, \quad \forall G_2 \in \overline{\mathcal{G}}_t^s. \tag{4.2}$$

Proof. To see that $\mathcal{I}_t^s$ is an instrument is easy. By inequality (3.61) and by the linearity and the complete positivity of $\mathcal{A}(t,s)$ and of the expectation, $\mathcal{I}_t^s(G)$ is an operation. The normalisation is equivalent to the trace preserving property of the mean dynamics (3.49). The σ-additivity property comes from the properties of the indicator functions 1_G and the continuity of expectations.

By the properties of independence and measurability of $\mathcal{A}(t,s)$ given in Theorem 3.4 and by the composition property (3.47), we get

$$\begin{aligned}
\mathcal{I}_t^r(G_1 \cap G_2)[\tau] &= \mathbb{E}_{\mathbb{Q}}\left[1_{G_1 \cap G_2}\mathcal{A}(t,r)[\tau]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[1_{G_1}1_{G_2}\mathcal{A}(t,s) \circ \mathcal{A}(s,r)[\tau] \,\middle|\, \overline{\mathcal{G}}_t^s\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[1_{G_2}\mathcal{A}(t,s)\left[\mathbb{E}_{\mathbb{Q}}\left[1_{G_1}\mathcal{A}(s,r)[\tau] \,\middle|\, \overline{\mathcal{G}}_t^s\right]\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[1_{G_2}\mathcal{A}(t,s)\left[\mathbb{E}_{\mathbb{Q}}\left[1_{G_1}\mathcal{A}(s,r)[\tau]\right]\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[1_{G_2}\mathcal{A}(t,s)\left[\mathcal{I}_s^r(G_1)[\tau]\right]\right] = \mathcal{I}_t^s(G_2)\left[\mathcal{I}_s^r(G_1)[\tau]\right].
\end{aligned}$$

□

Remark 4.3.

1. By comparing (4.1) with Eq. (3.57), we see that

$$\mathcal{I}_t^s(G)^*[\mathbb{1}] = E_t^s(G); \tag{4.3}$$

by (B.30) it means that the positive operator-valued measure (the generalised observable) associated with the instrument $\mathcal{I}_t^s$ is the POM E_t^s discussed in Proposition 3.9.

2. By comparing (4.1) with Eq. (3.49) we get the connection of the instrument with the mean dynamics:

$$\mathcal{I}_t^s(\Omega) = \mathcal{T}(t,s). \tag{4.4}$$

3. From Eqs. (4.2) and (4.4), we get, for $0 \leq r \leq s < t \leq u$ and $G \in \overline{\mathcal{G}}_t^s$,

$$\mathcal{I}_u^r(G) = \mathcal{I}_u^t(\Omega) \circ \mathcal{I}_t^s(G) \circ \mathcal{I}_s^r(\Omega) = \mathcal{T}(u,t) \circ \mathcal{I}_t^s(G) \circ \mathcal{T}(s,r). \tag{4.5}$$

4. From the previous points, we get a generalised consistency property for the POMs:

$$0 \leq r \leq s < t \leq u, \quad G \in \overline{\mathcal{G}}_t^s \qquad \Rightarrow \qquad E_u^r(G) = \mathcal{T}(s,r)^*\left[E_t^s(G)\right]. \tag{4.6}$$

5. Let us consider now the initial state at time zero $\rho_0 \in \mathcal{S}(\mathcal{H})$ and the physical probabilities (3.58): we can write for all $G \in \overline{\mathcal{G}}_t^0$

$$\mathcal{I}_t^0(G)[\rho_0] = \mathbb{E}_{\mathbb{Q}}[1_G\, \sigma(t)], \tag{4.7}$$

$$\mathbb{P}_{\rho_0}^t(G) = \mathbb{E}_{\mathbb{Q}}\left[1_G \operatorname{Tr}\{\sigma(t)\}\right] = \operatorname{Tr}\left\{E_t^0(G)\rho_0\right\} = \operatorname{Tr}\left\{\mathcal{I}_t^0(G)[\rho_0]\right\}. \tag{4.8}$$

Remark 4.4. When the equation for matrices are obtained from the Hilbert space SDEs of Chap. 2, as done in Sections 3.1–3.3, by the uniqueness properties of the solutions of Eqs. (3.38), (3.42), we have

$$\mathcal{I}_t^s(G)[\tau] = \mathbb{E}_{\mathbb{Q}}\left[1_G A_t^s \tau A_t^{s*}\right], \qquad \forall G \in \overline{\mathcal{G}}_t^s, \quad \forall \tau \in M_n.$$

4.1.1.1 A Priori States

By Eqs. (3.66) and (4.4), we get the identification

$$\eta(t) = \mathcal{T}(t,0)[\rho_0] = \mathcal{I}_t^0(\Omega)[\rho_0], \tag{4.9}$$

which says that the a priori state $\eta(t)$, already introduced in Sect. 3.4.2, is indeed the a priori state for the instrument $\mathcal{I}_t^0$ with pre-measurement state ρ_0, according to the general definition (B.33).

4.1.1.2 A Posteriori States

By Eqs. (3.46), (4.8), (4.7), we get $\forall G \in \overline{\mathcal{G}}_t^0$

$$\int_G \rho(t,\omega)\mathbb{P}_{\rho_0}^t(\mathrm{d}\omega) = \mathbb{E}_{\rho_0}^t\left[1_G\, \rho(t)\right] = \mathbb{E}_{\mathbb{Q}}[1_G\, \sigma(t)] = \mathcal{I}_t^0(G)[\rho_0], \tag{4.10}$$

which says that the a posteriori state $\rho(t)$, already introduced in Eq. (3.46) and discussed in Sect. 3.4.2, is indeed the a posteriori state for the instrument $\mathcal{I}_t^0$ with pre-measurement state ρ_0, according to the general definition (B.34).

4.1.2 Interpretation of the Instruments and the Output

4.1.2.1 The Composition Law

Let us consider n events $G_j \in \overline{\mathcal{G}}_{t_j}^{t_{j-1}}$, $0 \le t_0 < t_1 < \cdots < t_n \le T$. Then, the composition law (4.2) gives

$$\mathcal{I}_T^0(G_1 \cap G_2 \cap \cdots \cap G_n) = \mathcal{T}(T,t_n) \circ \mathcal{I}_{t_n}^{t_{n-1}}(G_n) \circ \cdots \circ \mathcal{I}_{t_1}^{t_0}(G_1) \circ \mathcal{T}(t_0,0), \tag{4.11}$$

$$\mathbb{P}^T_{\rho_0}(G_1 \cap G_2 \cap \cdots \cap G_n) = \mathrm{Tr}\left\{\mathcal{I}^0_T(G_1 \cap G_2 \cap \cdots \cap G_n)\,[\rho_0]\right\}$$
$$= \mathrm{Tr}\left\{\mathcal{I}^{t_{n-1}}_{t_n}(G_n) \circ \cdots \circ \mathcal{I}^{t_0}_{t_1}(G_1)\,[\eta(t_0)]\right\}. \qquad (4.12)$$

By comparing this with Eq. (B.41), we can say that, when we have events related to disjoint time intervals, the continuous measurement can be seen as a time-ordered sequence of measurements (cf. Sect. B.4.3). Indeed, the class of continuous measurements described in this book were originally obtained just as a limit of repeated measurements at discrete times [1–3].

Equation (4.11) is a kind of "independence property" at the operator level, but this does not imply that the physical probabilities (4.12) factorise nor that the output process is Markovian.

By Eq. (B.42), we have also that the conditional state $\rho_t(G)$ at time t, given the result $G \in \overline{\mathcal{G}}^s_t$, $0 \le s \le t$, is

$$\rho_t(G) = \frac{\mathcal{I}^0_t(G)\,[\rho_0]}{\mathbb{P}^t_{\rho_0}(G)}. \qquad (4.13)$$

By Eq. (4.10), the connection between the conditional state $\rho_t(G)$ and the a posteriori state $\rho(t)$ is

$$\rho_t(G) = \frac{\int_G \rho(t,\omega)\mathbb{P}^t_{\rho_0}(\mathrm{d}\omega)}{\mathbb{P}^t_{\rho_0}(G)}. \qquad (4.14)$$

If $G \in \overline{\mathcal{G}}^r_s$, $0 \le r < s \le t$, Eqs. (4.9), (4.11), (4.13) give

$$\rho_t(G) = \frac{\mathcal{T}(t,s) \circ \mathcal{I}^r_s(G)\,[\eta(r)]}{\mathbb{P}^s_{\rho_0}(G)}. \qquad (4.15)$$

4.1.2.2 The Observables of the Continuous Measurement

Let us consider the instrument $\mathcal{I}^s_t$ and the POM E^s_t; they are measures on the value space $(\Omega, \overline{\mathcal{G}}^s_t)$. The σ-algebra $\overline{\mathcal{G}}^s_t$ is generated by the increments of the process W with times inside $[s,t]$, not by the process itself. From the results of the measurement represented by $\mathcal{I}^s_t$ we cannot reconstruct any information on $W(r)$, $r \in [s,t]$, unless $s = 0$ just because we decided that $W(0) = 0$. As already discussed in Remark 2.16, the output of the continuous measurement is not exactly the process W, but the set of all its increments. At a heuristic level, this means that, in an infinitesimal time interval around t, the output is its time derivative $\dot{W}(t)$. However, $\dot{W}(t)$ at a fixed time t is not well defined, because the typical trajectories of the Wiener process are not differentiable. In this section we show that $\dot{W}$ has meaning as generalised stochastic process. A *generalised stochastic process* [4, 5] is a linear random functional on a suitable test function space, as Schwartz space. We can

say that generalised stochastic processes are the random analog of distributions; as for distributions, a generalised notion of derivative can be introduced for such processes. Our process $\dot{W}$ is the first (generalised) derivative of a well behaving process (the Wiener process under $\mathbb{Q}$), and we know that we can integrate L^2-functions; so, we can use as test functions a space larger than Schwartz space.

Test Functions

Let us introduce $L^2_{\mathrm{loc}} := L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^m)$, the space of the locally square integrable functions k from $\mathbb{R}_+$ into $\mathbb{R}^m$; $k \in L^2_{\mathrm{loc}}$ means that, for all finite $t > 0$, we have $1_{(0,t)}k \in L^2((0,t);\mathbb{R}^m) =: L^2_t$. From now on a *test function* is any element of $L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^m)$.

Observables

Let us take $k \in L^2_{\mathrm{loc}}$ and $0 \le s < t \le T$. We define the random variables

$$X^s_t(k) := \sum_{j=1}^m \int_s^t k_j(r)\,\mathrm{d}W_j(r). \tag{4.16}$$

By Proposition A.47, the integral above is well defined either under the probability $\mathbb{Q}$ or under $\mathbb{P}^T_{\rho_0}$. Let us note that $X^s_t(k)$ is $\overline{\mathcal{G}}^s_t$-measurable and that

$$X^s_t(k) = X^0_T(1_{(s,t)}k).$$

Either under $\mathbb{Q}$ or under $\mathbb{P}^T_{\rho_0}$, $X^s_t(\bullet)$ is a linear random functional on L^2_{loc} and, so, it is a generalised stochastic process.

Remark 4.5. Heuristically, (4.16) can be written as $X^s_t(k) = \sum_{j=1}^m \int_s^t k_j(r)\dot{W}_j(r)\mathrm{d}r$ and the generalised process $X^s_t(\bullet)$ is what gives a rigourous meaning to the process $\dot{W}$. We can say that the time-smoothed quantities (4.16) are the true observables of the continuous measurement: the output in the time interval $[s,t]$ is the singular process $\{\dot{W}(r), r \in [s,t]\}$, which we identify with its rigourous formulation $\{X^s_t(k),\ k \in L^2_{\mathrm{loc}}\}$.

Note that a constant k is a possible test function and that, for $k_i(r) = \delta_{ij}$, $i = 1,\dots,m$, one has $X^s_t(k) = W_j(t) - W_j(s)$. So, the increments of W are of type (4.16) and all the other random variables (4.16) are limits of linear combinations of increments. By this, the two sets of random variables $\{X^s_t(k),\ k \in L^2_{\mathrm{loc}}\}$ and $\{W_j(r) - W_j(s), r \in [s,t]\}$ generate the same (augmented) σ-algebra and both can represent the whole output of the continuous measurement in the time interval $[s,t]$.

4.1.2.3 Finite Dimensional Laws and Finite Dimensional Instruments

Under the reference probability $\mathbb{Q}$, the real random variable $X_t^s(k)$ is independent of $\overline{\mathcal{G}}_s^0$ and is normally distributed with zero mean and variance $\sum_{j=1}^m \int_s^t \left|k_j(r)\right|^2 \mathrm{d}r$ (it can be easily obtained by the Itô isometry). More generally, the random vector $\left(X_T^0(k^{(1)}), \ldots, X_T^0(k^{(q)})\right)$ is Gaussian with zero means and covariance matrix

$$\mathrm{Cov}_{\mathbb{Q}}[X_T^0(k^{(i)}),\ X_T^0(k^{(j)})] = \langle k^{(i)} | k^{(j)} \rangle_{L_T^2}\,. \tag{4.17}$$

When the test functions $k^{(1)}, \ldots, k^{(q)}$ are linearly independent, the covariance matrix with elements (4.17) is not singular and the distribution of the vector $\left(X_T^0(k^{(1)}), \ldots, X_T^0(k^{(q)})\right)$ has a density with respect to the Lebesgue measure on $\mathbb{R}^q$.

However, we are interested in the random variables (4.16) under the physical probability $\mathbb{P}_{\rho_0}^T$. Let us consider a single variable $X_t^s(k)$; by the definition of the instruments and Eq. (4.12), its cumulative distribution function is given by

$$\mathbb{P}_{\rho_0}^T\left[X_t^s(k) \le x\right] = \mathrm{Tr}\left\{\mathcal{I}_T^0\left(X_t^s(k) \le x\right)[\rho_0]\right\} = \mathrm{Tr}\left\{\mathcal{I}_t^s\left(X_t^s(k) \le x\right)[\eta(s)]\right\}. \tag{4.18}$$

This distribution is diffuse on the whole real line and a similar statement hold also for $\left(X_T^0(k^{(1)}), \ldots, X_T^0(k^{(q)})\right)$, as the following proposition says.

Proposition 4.6. *For $0 \le s < t \le T$ and $k \ne 0$, the distribution of $X_t^s(k)$ under $\mathbb{P}_{\rho_0}^T$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and its closed support is $\mathbb{R}$, i.e. there exists a density $f_{X_t^s(k)}(x) > 0$, $\forall x \in \mathbb{R}$, such that*

$$\mathbb{P}_{\rho_0}^T\left[X_t^s(k) \le x\right] = \int_{-\infty}^{x} f_{X_t^s(k)}(y)\mathrm{d}y\,, \qquad \forall x \in \mathbb{R}.$$

If $k^{(1)}, \ldots, k^{(q)}$ are linearly independent elements of L_T^2, the distribution of $\left(X_T^0(k^{(1)}), \ldots, X_T^0(k^{(q)})\right)$ under $\mathbb{P}_{\rho_0}^T$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^q$ and its density can be taken strictly positive $\forall x \in \mathbb{R}^q$.

Proof. Let us introduce the laws of $X_t^s(k)$ under $\mathbb{Q}$ and $\mathbb{P}_{\rho_0}^T$:

$$\mathbb{Q}_X(A) := \mathbb{Q}[X_t^s(k) \in A], \qquad \mathbb{P}_X(A) := \mathbb{P}_{\rho_0}^T[X_t^s(k) \in A], \qquad \forall A \in \mathcal{B}(\mathbb{R}).$$

The probability measures $\mathbb{Q}$ and $\mathbb{P}_{\rho_0}^T$ are equivalent, because of Theorem 3.4, and this implies the equivalence of the two laws; indeed, we have

$$0 = \mathbb{Q}_X(N) = \mathbb{Q}[X_t^s(k) \in N] \ \Leftrightarrow\ 0 = \mathbb{P}_{\rho_0}^T[X_t^s(k) \in N] = \mathbb{P}_X(N).$$

The measure $\mathbb{Q}_X$ is a normal distribution on $\mathbb{R}$ and, so, it is equivalent to the Lebesgue measure; then, also $\mathbb{P}_X$ is equivalent to the Lebesgue measure and has a density with respect to it.

By recalling that the closed support of a probability is the smallest closed set with probability one, we have that the support of $\mathbb{P}_X$ is the whole real line as for the normal distribution $\mathbb{Q}_X$. Then, the density of $\mathbb{P}_X$ with respect to the Lebesgue measure can be taken strictly positive everywhere.

The proof of the second part of the proposition follows exactly the same steps as the first one. □

Remark 4.7.

- The distribution of the random vector $\left(X_T^0(k^{(1)}), \ldots, X_T^0(k^{(q)})\right)$ under $\mathbb{P}_{\rho_0}^T$ is said to be a *finite dimensional law* of the generalised stochastic process $\{X_T^0(k),\ k \in L_T^2,\ T \geq 0\}$.
- Analogously, we can define the *finite dimensional instruments*. Let us take the test functions $k^{(1)}, \ldots, k^{(q)}$ in L^2_{loc}. Then, the relation

$$\mathcal{I}_t^s\left(A; k^{(1)}, \ldots, k^{(q)}\right) := \mathcal{I}_t^s\left(\left(X_t^s(k^{(1)}), \ldots, X_t^s(k^{(q)})\right) \in A\right), \quad A \in \mathcal{B}(\mathbb{R}^q),$$

  defines an instrument $\mathcal{I}_t^s\left(\bullet; k^{(1)}, \ldots, k^{(q)}\right)$ with value space $(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q))$.
- Obviously, the distribution of $\left(X_T^0(k^{(1)}), \ldots, X_T^0(k^{(q)})\right)$ can be obtained from the initial state ρ_0 and the finite-dimensional instruments by

$$\mathbb{P}_{\rho_0}^T\left(\left(X_T^0(k^{(1)}), \ldots, X_T^0(k^{(q)})\right) \in A\right) = \text{Tr}\left\{\mathcal{I}_T^0\left(A; k^{(1)}, \ldots, k^{(q)}\right)[\rho_0]\right\}. \quad (4.19)$$

4.2 Characteristic Functional and Characteristic Operator

An essential tool in probability theory is the notion of characteristic function of a distribution on some $\mathbb{R}^q$, say. The analogous notion is useful also in the infinite-dimensional case and for operator-valued measures.

4.2.1 Characteristic Function

4.2.1.1 Characteristic Function of a Distribution

The characteristic function of a joint distribution on $\mathbb{R}^q$ is its Fourier transform; so, if $\mathbb{P}$ is a probability measure on $\left(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q)\right)$, the characteristic function of $\mathbb{P}$ is

$$g(k) = \int_{\mathbb{R}^q} \mathrm{e}^{\mathrm{i}k\cdot x}\, \mathbb{P}(\mathrm{d}x), \qquad k \in \mathbb{R}^q. \quad (4.20)$$

Bochner theorem says that there is a one-to-one correspondence between probability distributions and characteristic functions, which are the complex functions g such that $g(0) = 1$, $k \mapsto g(k)$ is continuous, g is positive definite, which means $\sum_{i,j=1}^m \overline{c_i}\, g(k^{(i)} - k^{(j)}) c_j \geq 0$ for any choice of the integer m, of the complex numbers c_i, and of the vectors $k^{(i)}$.

Another important property is that $g \in L^1\left(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q), \mathrm{d}x\right)$ if and only if the probability is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}x$:

$\mathbb{P}(\mathrm{d}x) = f(x)\mathrm{d}x$. In this case $f \in L^1\big(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q), \mathrm{d}x\big)$ and

$$g(k) = \int_{\mathbb{R}^q} \mathrm{e}^{\mathrm{i}k\cdot x} f(x)\mathrm{d}x\,, \qquad f(x) = \frac{1}{(2\pi)^q}\int_{\mathbb{R}^q} \mathrm{e}^{-\mathrm{i}k\cdot x} g(k)\mathrm{d}k\,. \tag{4.21}$$

4.2.1.2 Characteristic Function of a Random Vector

Also the notion of characteristic function of a random variable is used; it is simply the characteristic function of its distribution. Let X be a q-dimensional random vector: $X : \big(\Omega, \mathcal{F}, \mathbb{P}\big) \to \big(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q)\big)$. The characteristic function ϕ_X of X is by definition the characteristic function of $\mathbb{P}_X$ (Sect. A.1.1.4) or $\phi_X(k) = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}k\cdot X}\big]$. Obviously, from ϕ_X one can reobtain only $\mathbb{P}_X$, by anti-Fourier transform; from the characteristic function of a random variable it is not possible to reconstruct the probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ and the random variable X itself, as a function on Ω.

When the moments of order r of X exist, then its characteristic function is differentiable up to order r and for $1 \le m \le r$

$$\mathbb{E}[X_{j_1}\cdots X_{j_m}] = (-\mathrm{i})^m \frac{\partial^m \phi_X(k)}{\partial k_{j_1}\cdots\partial k_{j_m}}\Big|_{k=0}\,. \tag{4.22}$$

4.2.1.3 Fourier Transform of an Instrument

Similar properties hold for the Fourier transform of an instrument on $\mathbb{R}^q$. Let $\mathcal{I}$ be an instrument with value space $\big(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q)\big)$ and define the map on M_n

$$\mathcal{G}(k) = \int_{\mathbb{R}^q} \mathrm{e}^{\mathrm{i}k\cdot x}\mathcal{I}(\mathrm{d}x), \qquad k \in \mathbb{R}^q; \tag{4.23}$$

$\mathcal{G}(k)$ can be called the characteristic operator of the instrument $\mathcal{I}$.

An analogous of Bochner theorem holds: there is a one-to-one correspondence between instruments and characteristic operators. Characteristic operators are characterised by the properties: $\mathcal{G}(0)$ is trace preserving, $k \mapsto \mathcal{G}(k)$ is continuous, $\mathcal{G}$ is completely positive definite, which means $\sum_{i,j=1}^m \overline{c_i}\,\mathcal{G}(k^{(i)} - k^{(j)})c_j$ is completely positive for any choice of the integer m, of the complex numbers c_i, and of the vectors $k^{(i)}$. These statements are a finite-dimensional version of Theorem 2.2 in [6].

4.2.2 Characteristic Functional and Finite Dimensional Laws

When processes and generalised processes are involved, the notion of characteristic functional enters into play, but the situation is much more complicated [5]. Here we introduce the various notions and results only in the case of our interest.

Let us start by considering the finite-dimensional distributions of Sect. 4.1.2.3. We fix the test functions $k^{(j)} \in L^2_T$, $j = 1, \ldots, q$, with $k^{(1)}, \ldots, k^{(q)}$ linearly

independent, and we denote by $\lambda = (\lambda_1, \ldots, \lambda_q)$ the indeterminate in $\mathbb{R}^q$. By setting $Y_j := X_T^0(k^{(j)})$ we obtain a random vector $Y = \left(Y_1, \ldots, Y_q\right)$ in the probability space $\left(\Omega, \overline{\mathcal{G}}_T^0, \mathbb{P}_{\rho_0}^T\right)$, whose characteristic function is

$$\phi_Y(\lambda) = \mathbb{E}_{\rho_0}^T \left[\mathrm{e}^{\mathrm{i}\lambda \cdot Y}\right]. \tag{4.24}$$

By Proposition 4.6 and Eq. (4.21), the probability distribution of Y has a density, given by

$$f_Y(y) = \frac{1}{(2\pi)^q} \int_{\mathbb{R}^q} \mathrm{e}^{-\mathrm{i}y\cdot\lambda} \Phi_Y(\lambda)\, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_q\,. \tag{4.25}$$

Now, we have that the observables (4.16) are linear in the test function and we have $\lambda \cdot Y = \sum_{j=1}^q \lambda_j X_T^0(k^{(j)}) = X_T^0\left(\sum_{j=1}^q \lambda_j k^{(j)}\right)$. So, if we know the characteristic function of the random variable $X_T^0(k)$ as a functional of k, we know in principle all the finite dimensional distributions; what we obtain in this way is the notion of characteristic functional.

We define the *characteristic functional* of the generalised process $\{\dot{W}_j,\ j = 1, \ldots, m\}$ under the probability $\mathbb{P}_{\rho_0}^T$ by

$$\Phi_t(k|\rho_0) := \mathbb{E}_{\rho_0}^T \left[\exp\left\{ \mathrm{i} \sum_{j=1}^m \int_0^t k_j(s) \mathrm{d}W_j(s) \right\} \right], \qquad T \geq t \geq 0, \tag{4.26}$$

or, by using the observables (4.16),

$$\Phi_t(k|\rho_0) = \mathbb{E}_{\rho_0}^T \left[\mathrm{e}^{\mathrm{i}X_t^0(k)}\right]. \tag{4.27}$$

We are considering $\Phi_t(k|\rho_0)$ as a functional of the test function $k \in L^2_{\mathrm{loc}}$; obviously, it is a function also of $t \geq 0$ and of $\rho_0 \in \mathcal{S}(\mathcal{H})$, but not of T by the consistency of the probabilities.

Then, the characteristic function of the random vector Y given above is

$$\phi_Y(\lambda) = \phi_T\left(\left.\sum_{j=1}^q \lambda_j k^{(j)}\right| \rho_0\right) \tag{4.28}$$

and the characteristic functional gives all the finite-dimensional distributions of $X_T^0(\bullet)$ introduced in Remark 4.7.

As we can take test functions of the form $k_i(t) = \delta_{ij^*} 1_{[0,t^*]}(t)$ for all possible choices of j^* and t^*, we have that the characteristic functional $\Phi_t(\bullet|\rho_0)$ determines all the finite-dimensional distributions of the continuous process W and, by Remark A.13, the whole law of W as a random variable with values in the trajectory space $\left(C_0^m(0,t), \mathcal{C}_t\right)$.

4.2.3 Characteristic Operator

The notion of characteristic operator was introduced and exploited in [1, 2, 7], where the theory of continuous measurements of diffusive type was firstly established; for applications and developments, see also [6, 8–13]. The definition of this new notion is strictly parallel to the definition of characteristic functional.

Definition 4.8. *For $k \in L^2_{\mathrm{loc}}$ we define the characteristic operator $\mathcal{G}(t, r; k)$ of the continuous measurement, defined by the instrument $\mathcal{I}_t^r$ and by the output $\{\dot{W}(s),\ s \in [r, t]\}$, by:* $\forall a,\ \tau \in M_n$

$$\mathrm{Tr}\left\{a\, \mathcal{G}(t, r; k)[\tau]\right\} = \int_\Omega \exp\left\{\mathrm{i} \sum_{j=1}^m \Big(\int_r^t k_j(s) \mathrm{d}W_j(s)\Big)(\omega)\right\} \mathrm{Tr}\left\{a\, \mathcal{I}_t^r(\mathrm{d}\omega)[\tau]\right\}. \tag{4.29}$$

First of all, the characteristic operator is connected to the characteristic functional (4.26) by

$$\Phi_t(k|\rho) = \mathrm{Tr}\left\{\mathcal{G}(t, 0; k)[\rho_0]\right\}. \tag{4.30}$$

From the definition of characteristic operator (Definition 4.8) and Eqs. (4.7), (4.10) we have also

$$\begin{aligned} \mathcal{G}(t, 0; k)[\rho_0] &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\mathrm{i} \sum_{j=1}^m \int_0^t k_j(s) \mathrm{d}W_j(s)\right\} \sigma(t)\right] \\ &= \mathbb{E}^T_{\rho_0}\left[\exp\left\{\mathrm{i} \sum_{j=1}^m \int_0^t k_j(s) \mathrm{d}W_j(s)\right\} \rho(t)\right]. \end{aligned} \tag{4.31}$$

Let us now study the properties of the characteristic operator. By using the representation of the instrument given in Definition 4.1 or the one given in Remark 4.4, we obtain

$$\mathcal{G}(t, r; k)[\tau] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}X_t^r(k)} \mathcal{A}(t, r)[\tau]\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}X_t^r(k)}\, A_t^r\, \tau\, A_t^{r*}\right]. \tag{4.32}$$

For $k \equiv 0$, we get immediately from Eqs. (4.29) and (4.4)

$$\mathcal{G}(t, r; 0) = \mathcal{I}_t^r(\Omega) = \mathcal{T}(t, r). \tag{4.33}$$

Proposition 4.9. *The characteristic operator is* normalised*, in the sense that $\mathcal{G}(t, r; 0)$ preserves the trace, i.e.* $\mathrm{Tr}\left\{\mathcal{G}(t, r; 0)[\tau]\right\} = \mathrm{Tr}\left\{\tau\right\}$, $\forall \tau \in M_n$.

The characteristic operator is completely positive definite*, which means that the maps $\sum_{i,j=1}^q \overline{c_i}\, \mathcal{G}(t, r; k^{(i)} - k^{(j)}) c_j$ are completely positive for any choice of the integer q, of the complex numbers c_i, and of the test functions $k^{(i)}$.*

Proof. The normalisation follows from Eq. (4.33) and the fact that the mean dynamics is trace preserving (Theorem 3.4).

By the definition of characteristic operator we get

$$\sum_{i,j=1}^{q} \overline{c_i}\mathcal{G}(t,r;k^{(i)}-k^{(j)})c_j$$

$$= \int_{\Omega} \left| \sum_{i=1}^{q} \overline{c_i} \exp\left\{ \mathrm{i} \sum_{\ell=1}^{m} \Big(\int_r^t k_\ell^{(i)}(s)\mathrm{d}W_\ell(s) \Big)(\omega) \right\} \right|^2 \mathcal{I}_t^r(\mathrm{d}\omega) \quad (4.34)$$

and the second statement follows from the complete positivity of the instrument. □

Proposition 4.10. *Let us take $t > s > r \geq 0$ and $k \in L^2_{\mathrm{loc}}$. The characteristic operator satisfies the composition law*

$$\mathcal{G}(t,r;k) = \mathcal{G}(t,s;k) \circ \mathcal{G}(s,r;k) \quad (4.35)$$

and the linear equation

$$\mathcal{G}(t,r;k) = \mathrm{Id}_n + \int_r^t \Lambda_s\big(k(s)\big) \circ \mathcal{G}(s,r;k)\,\mathrm{d}s\,, \quad (4.36)$$

$$\Lambda_t(k) := \mathcal{L}(t) + \sum_{j=1}^{m} \left(\mathrm{i}k_j \mathcal{R}_j(t) - \frac{1}{2}k_j^2\,\mathrm{Id}_n \right), \quad (4.37)$$

where Id_n is the identity operator on M_n and $\mathcal{R}_j(t)[\rho] = R_j(t)\rho + \rho R_j(t)^*$, as already defined in Eq. (3.39)*. The solution of* (4.36) *is unique.*

Proof. By the properties of independence and measurability of $\mathcal{A}(t,s)$ given in Theorem 3.4 and by the composition property (3.47), we get

$$\begin{aligned}
\mathcal{G}(t,r;k)[\tau] &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{ \mathrm{i} \sum_{j=1}^{m} \int_r^t k_j(u)\mathrm{d}W_j(u) \right\} \mathcal{A}(t,s) \circ \mathcal{A}(s,r)[\tau] \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{ \mathrm{i} \sum_{j=1}^{m} \int_s^t k_j(u)\mathrm{d}W_j(u) \right\} \right. \\
&\quad \left. \times \mathcal{A}(t,s)\left[\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{ \mathrm{i} \sum_{j=1}^{m} \int_r^s k_j(u)\mathrm{d}W_j(u) \right\} \mathcal{A}(s,r)[\tau] \,\middle|\, \overline{\mathcal{G}}_t^s \right] \right] \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{ \mathrm{i} \sum_{j=1}^{m} \int_s^t k_j(u)\mathrm{d}W_j(u) \right\} \right. \\
&\quad \left. \times \mathcal{A}(t,s)\left[\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{ \mathrm{i} \sum_{j=1}^{m} \int_r^s k_j(u)\mathrm{d}W_j(u) \right\} \mathcal{A}(s,r)[\tau] \right] \right] \right]
\end{aligned}$$

$$= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_s^t k_j(u)\mathrm{d}W_j(u)\right\}\mathcal{A}(t,s)\big[\mathcal{G}(s,r;k)[\tau]\big]\right]$$
$$= \mathcal{G}(t,s;k)\big[\mathcal{G}(s,r;k)[\tau]\big].$$

To prove (4.36), let us introduce the matrix-valued stochastic process

$$x(t) := \exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_r^t k_j(s)\mathrm{d}W_j(s)\right\}\mathcal{A}(t,r)[\tau].$$

By Itô formula and Eq. (3.42), we get

$$\mathrm{d}x(t) = \sum_j \left\{\mathrm{i}k_j(t)x(t)\mathrm{d}W_j(t) - \frac{1}{2}k_j(t)^2x(t)\mathrm{d}t\right\} + \mathcal{L}(t)[x(t)]\mathrm{d}t$$
$$+ \sum_j \left\{\left(R_j(t)x(t) + x(t)R_j(t)^*\right)\mathrm{d}W_j(t) + \mathrm{i}k_j(t)\left(R_j(t)x(t) + x(t)R_j(t)^*\right)\mathrm{d}t\right\}.$$

By the bounds in Assumption 3.3 and the fact that the exponential factor in $x(t)$ has modulus one, the stochastic integrals have zero mean. By the fact that $\mathbb{E}_{\mathbb{Q}}[x(t)] = \mathcal{G}(t,r;k)[\tau]$, we get (4.36), (4.37).

Uniqueness of the solution follows from the fact that (4.36) is a linear equation in a finite-dimensional space and from all the bounds given on the time growth of the coefficients. □

Remark 4.11. In the continuity points t of $\Lambda_t\big(k(t)\big)$, we can also write

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t,s;k) = \Lambda_t\big(k(t)\big)\circ\mathcal{G}(t,s;k)\,,\qquad \mathcal{G}(s,s;k) = \mathrm{Id}_n\,. \tag{4.38}$$

We can say that the Fourier transform of the instruments describing the continuous measurement satisfies a kind of *modified master equation* (recall that $\Lambda_t(0) = \mathcal{L}(t)$).

The fact that the characteristic operator satisfies an evolution equation is the first big advantage with respect to the characteristic function. The second one is that it determines the instruments.

4.2.4 One-to-One Correspondence Between Characteristic Operators and Instruments

Remark 4.12. As the characteristic functional determines the finite-dimensional distributions of $\{X_T^0(k),\ k \in L_T^2,\ T \geq 0\}$, by the discussion in Sect. 4.2.1.3

the characteristic operator (4.29) determines the finite-dimensional instruments $\mathcal{I}_t^s\left(A; k^{(1)}, \ldots, k^{(q)}\right)$ introduced in Remark 4.7.

Equation (4.1) defines the instruments $\mathcal{I}_t^s$ with value space $(\Omega, \overline{\mathcal{G}}_t^s)$. This is an abstract value space. We started with a continuous, standard, m-dimensional Wiener process W living in a stochastic basis with sample space Ω and we constructed the two-times, augmented, natural filtration $\overline{\mathcal{G}}_t^s$ of W. The whole construction implicitly says that no physical consequence depends on the arbitrariness in the choices of Ω, W, etc. We can obtain uniqueness of the correspondence between characteristic operators and instruments only by fixing in some 'canonical' way the value space of the instrument. Let us consider here the canonical realisation of the Wiener process described in Remark A.23.

Remark 4.13. Let $\mathcal{Y} := C_0^m(0, \infty)$ be the space of all $\mathbb{R}^m$-valued continuous functions y on $[0, +\infty)$ starting from the origin at time 0. Let us introduce the canonical projections $\Pi(t) : \mathcal{Y} \to \mathbb{R}^m$, $\Pi(t, y) := y(t)$, $0 \leq t < +\infty$ and the σ-algebras $\mathcal{F}^\Pi(r, t) := \sigma(\Pi(s) - \Pi(r),\ s \in [r, t])$. We can also consider the Wiener measure $\mathbb{P}_W$ on $\left(\mathcal{Y},\ \mathcal{F}^\Pi(0, \infty)\right)$ and its null sets; with these things, we introduce also the augmented σ-algebras $\overline{\mathcal{F}}^\Pi(r, t)$.

Then, we have the following theorem.

Theorem 4.14. *Given a family of operators $\mathcal{G}(t, r; k)$, $0 \leq r \leq t$, $k \in L^2_{\mathrm{loc}}$, defined by Eqs.* (4.36) *and* (4.37)*, there exists a unique instrument $\widehat{\mathcal{I}}_t^r$ on M_n with value space $\left(\mathcal{Y}, \mathcal{F}^\Pi(r, t)\right)$ such that, $\forall a,\ \tau \in M_n$,*

$$\mathrm{Tr}\left\{a\, \mathcal{G}(t, r; k)[\tau]\right\} = \int_{\mathcal{Y}} \exp\left\{\mathrm{i} \sum_{j=1}^{m} \left(\int_r^t k_j(s) \mathrm{d}\Pi_j(s)\right)(y)\right\} \mathrm{Tr}\left\{a\, \widehat{\mathcal{I}}_t^r(\mathrm{d}y)[\tau]\right\}.$$

Moreover, for $t > s > r \geq 0$, the following composition law holds:

$$\widehat{\mathcal{I}}_t^r(F_1 \cap F_2) = \widehat{\mathcal{I}}_t^s(F_2) \circ \widehat{\mathcal{I}}_s^r(F_1), \qquad \forall F_1 \in \mathcal{F}^\Pi(r, s), \quad \forall F_2 \in \mathcal{F}^\Pi(s, t). \quad (4.39)$$

The instrument $\widehat{\mathcal{I}}_t^r$ has a unique extension to $\left(\mathcal{Y}, \overline{\mathcal{F}}^\Pi(r, t)\right)$ and the analog of the composition property (4.39) *holds.*

We do not give the proof here; we only say that the statement can be obtained by putting together (i) Theorem 2.2 in [6] about characteristic operators for instruments on spaces of distributions, (ii) the fact that $C^m(0, T)$ is a 'measurable' subset of the distribution space on $(0, T)$ [5, p. 21], and (iii) the uniqueness of the solution of Eq. (4.36).

Remark 4.15. The instruments introduced in the theorem above are connected to the instruments (4.1) by the following construction: for any $A \in \mathcal{F}^\Pi(r, t)$ we define the event $G_A := \{\omega \in \Omega : (s \mapsto W(s, \omega) - W(r, \omega), s \in [r, t]) \in A\}$; then, $\widehat{\mathcal{I}}_t^r(A) = \mathcal{I}_t^r(G_A)$. Moreover, the instruments $\widehat{\mathcal{I}}_t^r$ and $\mathcal{I}_t^r$ give rise to the same finite-dimensional instruments.

4.3 Moments

When one has to do with random variables or stochastic processes, the moments play often a key role, for theoretical developments and practical applications. The output of a continuous measurement is a stochastic process (in the classical sense) and, so, its moments can be introduced in the usual way for stochastic processes, without any ad hoc "quantum" definition; in particular, the mean and covariance functions can be studied. Our approach through the characteristic operator gives an elegant way of obtaining the expressions of the moments of the output of the continuous measurement [1, 2, 6, 9, 10, 14].

4.3.1 The Mean

By Proposition 3.9 and Eq. (3.63) we get

$$\mathbb{E}^T_{\rho_0}[W_j(t)] = \mathbb{E}^T_{\rho_0}\left[\widehat{W}_j(t) + \int_0^t v_j(s)\mathrm{d}s\right] = \int_0^t \mathbb{E}^T_{\rho_0}[v_j(s)]\mathrm{d}s\,;$$

then, by Eqs. (3.45), (3.46), (3.58), (3.65), we obtain

$$\begin{aligned}\mathbb{E}^T_{\rho_0}[v_j(s)] &= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{Tr}\left\{\left(R_j(s) + R_j(s)^*\right)\sigma(s)\right\}\right] \\ &= \mathrm{Tr}\left\{\left(R_j(s) + R_j(s)^*\right)\mathbb{E}_{\mathbb{Q}}\left[\sigma(s)\right]\right\} = \mathrm{Tr}\left\{\left(R_j(s) + R_j(s)^*\right)\eta(s)\right\}\,.\end{aligned}$$

So, we have

$$\mathbb{E}^T_{\rho_0}[W_j(t)] = \int_0^t \mathrm{Tr}\left\{\left(R_j(s) + R_j(s)^*\right)\eta(s)\right\}\mathrm{d}s\,, \tag{4.40}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}^T_{\rho_0}[W_j(t)] = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\eta(t)\right\}, \tag{4.41}$$

which is the "quantum mean" on the a priori state of the self-adjoint operator $R_j(t)+R_j(t)^*$; we call (4.41) the *a priori mean*.

From these equations we get the mean of an observable defined by the Wiener integral (4.16):

$$\mathbb{E}^T_{\rho_0}\left[X^0_T(k)\right] = \sum_{j=1}^m \int_0^T k_j(s)\,\mathrm{Tr}\left\{\left(R_j(s) + R_j(s)^*\right)\eta(s)\right\}\mathrm{d}s\,. \tag{4.42}$$

Also the conditional means are of interest; by (3.67) we get

$$\mathbb{E}^T_{\rho_0}[W_j(t) - W_j(s)|\mathcal{F}_s] = \int_s^t \mathbb{E}^T_{\rho_0}[v_j(r)|\mathcal{F}_s]\mathrm{d}r$$
$$= \int_s^t \mathrm{Tr}\left\{\left(R_j(r) + R_j(r)^*\right)\mathcal{T}(r,s)[\rho(s)]\right\}\mathrm{d}r. \quad (4.43)$$

Finally, by derivative and limit we get the *a posteriori means*

$$\lim_{s\uparrow t}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}^T_{\rho_0}[W_j(t) - W_j(s)|\mathcal{F}_s] = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\rho(t)\right\}. \quad (4.44)$$

By the fact that $\rho(t)$ is the a posteriori state at time t, we interpret $\dot{W}_j(t)$ as an imprecise measurement of the quantum observable $R_j(r) + R_j(r)^*$ at time t, as already anticipated in Sect. 2.4.

4.3.2 Higher Moments

The best way to have the higher moments is to go through the characteristic functional (4.26) and to obtain them by functional differentiation:

$$\mathbb{E}^T_{\rho_0}\left[\dot{W}_{j_1}(t_1)\dot{W}_{j_2}(t_2)\cdots\dot{W}_{j_q}(t_q)\right] = (-\mathrm{i})^q\,\frac{\delta^q\Phi_t(k|\rho_0)}{\delta k_{j_1}(t_1)\,\delta k_{j_2}(t_2)\cdots\delta k_{j_q}(t_q)}\Big|_{k=0}. \quad (4.45)$$

By choosing for k a step function, different from zero only in suitable "small intervals", and by using the composition law (4.35) and the expression of $\Lambda_t(k)$, we can compute the functional derivatives above. The result for the mean is obviously Eq. (4.40), while for the second moments we get

$$\mathbb{E}^T_{\rho_0}[\dot{W}_j(t)\dot{W}_i(s)] = \delta_{ij}\,\delta(t-s)$$
$$+ 1_{(0,+\infty)}(t-s)\,\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_i(s)\circ\mathcal{T}(s,0)[\rho_0]\right\}$$
$$+ 1_{(0,+\infty)}(s-t)\,\mathrm{Tr}\left\{\mathcal{R}_i(s)\circ\mathcal{T}(s,t)\circ\mathcal{R}_j(t)\circ\mathcal{T}(t,0)[\rho_0]\right\}. \quad (4.46)$$

Let us take $k, h \in L^2_{\mathrm{loc}}$; then, from (4.46) we get

$$\mathbb{E}^T_{\rho_0}\left[X^0_T(k)X^0_T(h)\right] = \sum_{j=1}^m\int_0^T k_j(s)h_j(s)\mathrm{d}s$$
$$+\sum_{i,j=1}^m\int_0^T\mathrm{d}t\int_0^t\mathrm{d}s\,h_j(t)k_i(s)\,\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_i(s)[\eta(s)]\right\}$$
$$+\sum_{i,j=1}^m\int_0^T\mathrm{d}t\int_t^T\mathrm{d}s\,h_j(t)k_i(s)\,\mathrm{Tr}\left\{\mathcal{R}_i(s)\circ\mathcal{T}(s,t)\circ\mathcal{R}_j(t)[\eta(t)]\right\},$$

$$\mathbb{E}^T_{\rho_0}\left[X^0_T(k)X^0_T(h)\right] = \sum_{j=1}^m \int_0^T k_j(s)h_j(s)\mathrm{d}s + \sum_{i,j=1}^m \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\big[h_j(t)k_i(s) + h_i(s)k_j(t)\big]\,\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_i(s)[\eta(s)]\right\}. \quad (4.47)$$

In particular, we have

$$\begin{aligned}\mathbb{E}^T_{\rho_0}[W_j(t)W_i(s)] &= (t\wedge s)\,\delta_{ij} + \int_0^{t\wedge s}\mathrm{d}r_1\int_0^{r_1}\mathrm{d}r_2\,\mathrm{Tr}\big\{\big[\mathcal{R}_j(r_1)\circ\mathcal{T}(r_1,r_2)\circ\mathcal{R}_i(r_2) \\ &\quad + \mathcal{R}_i(r_1)\circ\mathcal{T}(r_1,r_2)\circ\mathcal{R}_j(r_2)\big]\circ\mathcal{T}(r_2,0)[\rho_0]\big\} \\ &+ 1_{(0,+\infty)}(t-s)\int_s^t\mathrm{d}r_1\int_0^s\mathrm{d}r_2\,\mathrm{Tr}\big\{\mathcal{R}_j(r_1)\circ\mathcal{T}(r_1,r_2)\circ\mathcal{R}_i(r_2)\circ\mathcal{T}(r_2,0)[\rho_0]\big\} \\ &+ 1_{(0,+\infty)}(s-t)\int_t^s\mathrm{d}r_1\int_0^t\mathrm{d}r_2\,\mathrm{Tr}\left\{\mathcal{R}_i(r_1)\circ\mathcal{T}(r_1,r_2)\circ\mathcal{R}_j(r_2)\circ\mathcal{T}(r_2,0)[\rho_0]\right\},\end{aligned} \quad (4.48)$$

$$\mathbb{E}^T_{\rho_0}[W_j(t)^2] = t + 2\int_0^t\mathrm{d}r_1\int_0^{r_1}\mathrm{d}r_2\,\mathrm{Tr}\big\{\mathcal{R}_j(r_1)\circ\mathcal{T}(r_1,r_2)\circ\mathcal{R}_j(r_2)\circ\mathcal{T}(r_2,0)[\rho_0]\big\}. \quad (4.49)$$

A rigourous proof of the formulae giving the first two moments, avoiding concepts as functional derivatives of time-ordered products, is based on the following proposition.

Proposition 4.16. *Let us set*

$$\mathcal{J}(t,k) := \sum_j k_j(t)\mathcal{R}_j(t); \quad (4.50)$$

then, the following identities hold:

$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathcal{G}(t,r;\lambda k)\Big|_{\lambda=0} = \int_r^t \mathcal{T}(t,s)\circ\mathcal{J}(s,k)\circ\mathcal{T}(s,r)\,\mathrm{d}s, \quad (4.51)$$

$$\begin{aligned}-\frac{\partial^2\,\mathrm{Tr}\,\{\mathcal{G}(t,r;\lambda_1k+\lambda_2h)[\tau]\}}{\partial\lambda_1\partial\lambda_2}\Big|_{\lambda=0} &= \sum_{j=1}^m\int_r^t k_j(s)h_j(s)\mathrm{d}s\;\mathrm{Tr}\{\tau\} \\ &+ \int_r^t\mathrm{d}s\int_r^s\mathrm{d}u\,\mathrm{Tr}\,\{\mathcal{J}(s,k)\circ\mathcal{T}(s,u)\circ\mathcal{J}(u,h)\circ\mathcal{T}(u,r)[\tau]\} \\ &+ \int_r^t\mathrm{d}s\int_r^s\mathrm{d}u\,\mathrm{Tr}\,\{\mathcal{J}(s,h)\circ\mathcal{T}(s,u)\circ\mathcal{J}(u,k)\circ\mathcal{T}(u,r)[\tau]\}.\end{aligned} \quad (4.52)$$

Proof. Let us set $\mathcal{X}(t,r,k) := \int_r^t \mathcal{T}(t,s)\circ\mathcal{J}(s,k)\circ\mathcal{T}(s,r)\,\mathrm{d}s$ and use the evolution equation (3.48) for $\mathcal{T}(t,s)$; we get

$$\begin{aligned}\mathcal{X}(t,r,k) &= \int_r^t \mathrm{d}s\left(\mathrm{Id}_n + \int_s^t \mathrm{d}u\mathcal{L}(u)\circ\mathcal{T}(u,s)\right)\circ\mathcal{J}(s,k)\circ\mathcal{T}(s,r)\\ &= \int_r^t \mathrm{d}s\,\mathcal{J}(s,k)\circ\mathcal{T}(s,r) + \int_r^t \mathrm{d}u\,\mathcal{L}(u)\circ\int_s^u \mathrm{d}s\,\mathcal{T}(u,s)\circ\mathcal{J}(s,k)\circ\mathcal{T}(s,r)\\ &= \int_r^t \mathrm{d}s\,\mathcal{J}(s,k)\circ\mathcal{T}(s,r) + \int_r^t \mathrm{d}s\,\mathcal{L}(s)\circ\mathcal{X}(s,r,k).\end{aligned}$$

Let us set

$$\mathcal{Y}(t,r,k) := -\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathcal{G}(t,r;\lambda k)\Big|_{\lambda=0}$$

and note that

$$\mathcal{G}(s,r;\lambda k)\Big|_{\lambda=0} = \mathcal{T}(s,r), \qquad \Lambda_t\big(\lambda k(t)\big)\Big|_{\lambda=0} = \mathcal{L}(t),$$
$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\Lambda_t\big(\lambda k(t)\big)\Big|_{\lambda=0} = \mathcal{J}(t,k).$$

Then, from Eq. (4.36) we get

$$\mathcal{Y}(t,r,k) = \int_r^t \mathrm{d}s\,\mathcal{J}(s,k)\circ\mathcal{T}(s,r) + \int_r^t \mathrm{d}s\,\mathcal{L}(s)\circ\mathcal{Y}(s,r,k).$$

Finally we have

$$\mathcal{X}(t,r,k) - \mathcal{Y}(t,r,k) = \int_r^t \mathrm{d}s\,\mathcal{L}(s)\circ\left(\mathcal{X}(s,r,k) - \mathcal{Y}(s,r,k)\right).$$

By the uniqueness of the solution of such a linear equation we get $\mathcal{X}(t,r,k) - \mathcal{Y}(t,r,k) = 0$, which is the identity (4.51).

By using $\mathrm{Tr}\,\{\mathcal{L}(t)[\tau]\} = 0$ and Eq. (4.51) we obtain

$$\begin{aligned}&-\mathrm{i}\,\frac{\partial}{\partial\lambda_1}\,\mathrm{Tr}\,\{\mathcal{G}(t,r;\lambda_1 k + \lambda_2 h)[\tau]\}\Big|_{\lambda_1=0}\\ &\qquad= \int_r^t \mathrm{d}s\;\mathrm{Tr}\left\{\left[\mathcal{J}(s,k) + \mathrm{i}\lambda_2 k(s)\cdot h(s)\right]\circ\mathcal{G}(s,r;\lambda_2 h)[\tau]\right.\\ &\qquad\left. - \mathrm{i}\left[\mathrm{i}\lambda_2\mathcal{J}(s,h) - \frac{1}{2}\,{\lambda_2}^2\,|h(s)|^2\right]\circ\frac{\partial}{\partial\lambda_1}\,\mathcal{G}(s,r;\lambda_1 k + \lambda_2 h)[\tau]\Big|_{\lambda_1=0}\right\}.\end{aligned}$$

By taking the derivative also with respect to λ_2 we get

$$-\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}\,\mathrm{Tr}\left\{\mathcal{G}(t,r;\lambda_1 k+\lambda_2 h)[\tau]\right\}\Big|_{\lambda=0}=\int_r^t \mathrm{d}s\;\mathrm{Tr}\Big\{\mathcal{J}(s,k)\circ\mathcal{X}(s,r;h)[\tau]+k(s)\cdot h(s)\,\mathcal{T}(s,r)[\tau]+\mathcal{J}(s,h)\circ\mathcal{X}(s,r;k)[\tau]\Big\};$$

$\mathcal{T}(s,r)$ being trace preserving, we get the last identity. □

Let us note that

$$\mathbb{E}^T_{\rho_0}\left[X^0_T(k)\right]=-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\Phi_T\left(\lambda k|\rho_0\right)\Big|_{\lambda=0}=\mathrm{Tr}\left\{-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathcal{G}(T,0;\lambda k)\Big|_{\lambda=0}[\rho_0]\right\}.$$

By using Eq. (4.51) and the fact that $\mathcal{T}(T,s)$ is trace preserving, we get again the expression (4.42) for the mean value.

Moreover, we have

$$\begin{aligned}\mathbb{E}^T_{\rho_0}\left[X^0_T(k)X^0_T(h)\right]&=-\frac{\partial^2\Phi_T\left(\lambda_1 k+\lambda_2 h|\rho_0\right)}{\partial\lambda_1\partial\lambda_2}\Big|_{\lambda=0}\\&=-\frac{\partial^2\,\mathrm{Tr}\left\{\mathcal{G}(T,0;\lambda_1 k+\lambda_2 h)[\rho_0]\right\}}{\partial\lambda_1\partial\lambda_2}\Big|_{\lambda=0}.\end{aligned}$$

By using Eq. (4.52), we get the expression (4.47) for the second moment.

4.4 Classical Post-measurement Processing

Up to now we have considered as ideal output of the measurement the generalised process $\dot{W}(t)$, $t\geq 0$, but it is possible that the true signal is some functional of it due to some post-measurement processing or to some imperfections in the measuring apparatus, which modifies in some way the output and eventually loses some information. We can say that there is a classical transformation of the output when the true output $X(t)$ is a functional of $\{\dot{W}(s),\ 0\leq s\leq t\}$, in particular a linear functional, which is the case we shall consider in the following subsections.

4.4.1 Time-Local Transformations

Let us start by transformations which are local in time. Let the output be given by the generalised process $\dot{X}_i(t)=\sum_{j=1}^m C_{ij}(t)\dot{W}_j(t)+a_i(t)$, $i=1,\ldots,m'$, or, in integral form,

$$X_i(t)=\sum_{j=1}^m\int_0^t C_{ij}(s)\mathrm{d}W_j(s)+\int_0^t a_i(s)\mathrm{d}s\,,\qquad i=1,\ldots,m'\,,\tag{4.53}$$

where $C_{ij}(t)$, $a_i(t) \in \mathbb{R}$ and the functions $t \mapsto C(t)$ and $t \mapsto a(t)$ are measurable and bounded in any finite interval.

Obviously, the distribution of the new output up to time T can be obtained from the physical probability $\mathbb{P}^T_{\rho_0}$. To get information on this distribution it is convenient to study the expression of the characteristic operator of the continuous measurement with output $\dot{X}$, which can be defined by analogy with (4.29).

Let us define the integrals with respect to $\mathrm{d}X_i(s)$ as an integral with respect to a Wiener process and an integral with respect to time, either under the physical probability or under the reference probability, i.e. for $k \in L^2_{\mathrm{loc}}$

$$\begin{aligned}\int_r^t k_j(s)\mathrm{d}X_j(s) &= \int_r^t k_j(s)\left[\sum_{i=1}^m C_{ji}(s)\,\mathrm{d}W_i(s) + a_j(s)\,\mathrm{d}s\right] \\ &= \int_r^t k_j(s)\left[\sum_{i=1}^m C_{ji}(s)\mathrm{d}\widehat{W}_j(s) + \left(\sum_{i=1}^m C_{ji}(s)v_i(s) + a_j(s)\right)\mathrm{d}s\right].\end{aligned} \tag{4.54}$$

Then, the characteristic operator for the output $\dot{X}$ in the time interval $[r, t]$ is defined by: $\forall a,\ \tau \in M_n$

$$\mathrm{Tr}\left\{a\,\mathcal{G}_X(t, r; k)[\tau]\right\} := \int_\Omega \exp\left\{\mathrm{i}\sum_{j=1}^{m'}\Big(\int_r^t k_j(s)\mathrm{d}X_j(s)\Big)(\omega)\right\}\mathrm{Tr}\left\{a\,\mathcal{I}_t^r(\mathrm{d}\omega)[\tau]\right\}. \tag{4.55}$$

In particular, we have

$$\begin{aligned}\mathcal{G}_X(t, 0; k)[\rho_0] &= \mathbb{E}^T_{\rho_0}\left[\exp\left\{\mathrm{i}\sum_{j=1}^{m'}\int_0^t k_j(s)\mathrm{d}X_j(s)\right\}\rho(t)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\mathrm{i}\sum_{j=1}^{m'}\int_0^t k_j(s)\mathrm{d}X_j(s)\right\}\sigma(t)\right].\end{aligned} \tag{4.56}$$

By Itô calculus, exactly as in the proof of Proposition 4.10, one gets that $\mathcal{G}_X(t, r; k)$ satisfies an evolution equation analogous to (4.36) with generator

$$\begin{aligned}\Lambda^X_t(k) := \mathcal{L}(t) &+ \mathrm{i}\sum_{j=1}^{m'} k_j\left[\sum_{i=1}^m C_{ji}(t)\mathcal{R}_i(t) + a_j(t)\mathrm{Id}_n\right] \\ &- \frac{1}{2}\sum_{i,j=1}^{m'} k_i\left(C(t)C(t)^{\mathrm{T}}\right)_{ij} k_j\,\mathrm{Id}_n\,.\end{aligned} \tag{4.57}$$

Note that (4.57) is only a slight generalisation of the generator (4.37); this form was found in the original papers [1, 2, 7, 13]. It is possible even to have "infinitely

many" observed channels, with a discrete or continuous label. For instance the label denoting the channel could be the direction of propagation of the photons emitted by the system.

Propositions 4.9, 4.10 and Theorem 4.14 hold true also for the characteristic operator $\mathcal{G}_X(t, r; k)$, with the only change that m is substituted everywhere by m'. In particular, $\mathcal{G}_X(t, r; k)$ satisfies the composition law (4.35) and determines in a unique way a family of instruments on the canonical space of continuous trajectories $\mathcal{Y} := C_0^{m'}(0, \infty)$.

4.4.1.1 Invertible Transformations

Let us now consider the case in which the transformation (4.53) is invertible, i.e. $m' = m$ and

$$|\det C(t)| > c > 0\,, \qquad \forall t \geq 0\,. \tag{4.58}$$

By the invertibility of C, we have

$$W_j(t) = \sum_{i=1}^{m} \int_0^t \left(C(s)^{-1}\right)_{ji} [\mathrm{d}X_i(s) - a_i(s)\mathrm{d}s]\,.$$

Then, observing X or W is the same: the natural σ-algebras of W and X coincide, the measurement is represented by the same family of instruments on Ω and the a posteriori states are the same.

4.4.1.2 Rotation and Translation of the Output

Let us now show that pure rotations of the output and translations can always be reabsorbed in a change of the model.

Let $D(t)$ be a rotation matrix on $\mathbb{R}^m$: $D(t)^{\mathrm{T}} = D(t)^{-1}$; we introduce also $\alpha(t) \in \mathbb{C}^m$. The functions $t \mapsto D(t)$ and $t \mapsto \alpha(t)$ are measurable and such that

$$\sup_{t\in[0,T]} \sum_{j=1}^{m} \left|\alpha_j(t)\right|^2 < +\infty\,, \qquad \forall T > 0\,.$$

Let us define

$$\widetilde{R}_j(t) := \sum_{i=1}^{m} D_{ji}(t) R_i(t) + \alpha_j(t), \tag{4.59}$$

$$\widetilde{H}(t) := H(t) + \frac{\mathrm{i}}{2} \sum_{i,j=1}^{m} D_{ji}(t) \left[\alpha_j(t) R_i(t)^* - \overline{\alpha_j(t)} R_i(t)\right], \tag{4.60}$$

$$\widetilde{W}_j(t) := \sum_{i=1}^{m} \int_0^t D_{ji}(s) \mathrm{d}W_i(s); \tag{4.61}$$

$\widetilde{W}$ is an m-dimensional Wiener process under the original reference probability $\mathbb{Q}$.

Let us construct again all our objects (physical probabilities, a posteriori states, output, etc.), but everywhere we make the substitutions

$$R_j(t) \to \widetilde{R}_j(t)\,, \quad j = 1 \ldots, m, \qquad H(t) \to \widetilde{H}(t)\,, \qquad W \to \widetilde{W}\,.$$

Then, we get that the dynamics does not change,

$$\widetilde{\mathcal{L}}(t) = \mathcal{L}(t)\,, \qquad \widetilde{\mathcal{T}}(t,s) = \mathcal{T}(t,s)\,,$$

and that the characteristic operator of the output $\mathrm{d}\widetilde{W}_j/\mathrm{d}t$, $j = 1, \ldots, m$, turns out to be generated by

$$\begin{aligned}\widetilde{\Lambda}_t(k)[\tau] = \widetilde{\mathcal{L}}(t)[\tau] + \sum_{j=1}^{m} \left(\mathrm{i}k_j \widetilde{\mathcal{R}}_j(t)[\tau] - \frac{1}{2} k_j^2 \tau \right) = \mathcal{L}(t)[\tau] \\ + \mathrm{i} \sum_{j=1}^{m} k_j \left[\sum_{i=1}^{m} D_{ji}(t) \mathcal{R}_i(t)[\tau] + 2\big(\mathrm{Re}\,\alpha_j(t)\big)\tau \right] - \frac{1}{2} \sum_{j=1}^{m} k_j^2 \tau\,. \end{aligned} \tag{4.62}$$

So we have $\widetilde{\Lambda}_t(k) = \Lambda_t^X(k)$ with the choice

$$X_i(t) = \sum_{j=1}^{m} \int_0^t D_{ij}(s) \mathrm{d}W_j(s) + 2\,\mathrm{Re} \int_0^t \alpha_i(s) \mathrm{d}s\,, \qquad i = 1, \ldots, m\,. \tag{4.63}$$

The conclusion is that the new model with the $\widetilde{\ }$-quantities and output $\{\mathrm{d}\widetilde{W}_j/\mathrm{d}t, j = 1, \ldots, m\}$ gives the same physical results (probabilities and a posteriori states) as the original model with output $\dot{X}$ (4.63), which in turn is equivalent to the original model with output $\{\dot{W}_j,\ j = 1, \ldots, m\}$, because of the invertibility of the rotations $D(t)$.

4.4.1.3 Rotation and Loss of Information

In Sect. 3.4.2 we considered the possibility that not all the d channels are observed; the observed channels were the first m ones. A similar situation happens when there is first a rotation of the output and then some component is ignored.

Let us take the situation described above with $\alpha_i(t) = 0$; we have a pure rotation of the output. The generator associated to the model with the rotated output $\sum_{i=1}^{m} D_{ji}(t)\dot{W}_i(t)$ is given by (4.62) without the α's. Then, we assume that $1 \le r < m$ and that the last $m-r$ components of the transformed output are ignored. To ignore a component of the output is to take equal to zero the corresponding

component of the test function in the characteristic operator. So, the generator of the resulting model is

$$\begin{aligned}\widetilde{\Lambda}_t(k)[\tau] =& \widetilde{\mathcal{L}}(t)[\tau] + \sum_{j=1}^{r}\left(\mathrm{i}k_j\widetilde{\mathcal{R}}_j(t)[\tau] - \frac{1}{2}k_j^2\,\tau\right)\\ =& \mathcal{L}(t)[\tau] + \mathrm{i}\sum_{j=1}^{r}k_j\sum_{i=1}^{m}D_{ji}(t)\mathcal{R}_i(t)[\tau] - \frac{1}{2}\sum_{j=1}^{r}k_j^2\,\tau\,,\end{aligned} \tag{4.64}$$

where

$$\widetilde{R}_j(t) = \sum_{i=1}^{m}D_{ji}(t)R_i(t), \qquad \widetilde{H}(t) = H(t). \tag{4.65}$$

4.4.1.4 Decoupled Channels

There are situations in which some of the channels do not carry any information on the quantum system, but are pure noise. A typical situation is when

$$R_j(t) = \beta_j(t)R(t), \tag{4.66}$$

where $R(t)$ is an operator and the $\beta_j(t)$'s are complex functions.

Case $\mathrm{Re}\,\beta(t)$ Parallel to $\mathrm{Im}\,\beta(t)$

If the vector with components $\mathrm{Re}\,\beta_j(t)$ is parallel to the vector with components $\mathrm{Im}\,\beta_j(t)$ it is possible to reduce the model to one effective channel and $m-1$ pure noises. Indeed, it is possible to find a rotation $D(t)$ such that $D(t)\beta(t) = (\zeta(t), 0, \ldots, 0)^{\mathrm{T}}$ and this gives the generator

$$\widetilde{\Lambda}_t(k)[\tau] = \mathcal{L}(t)[\tau] + \mathrm{i}k_1\left(\zeta(t)R(t)\tau + \overline{\zeta(t)}\tau\,R(t)^*\right) - \frac{1}{2}\sum_{j=1}^{m}k_j^2\,\tau\,, \tag{4.67}$$

which says that the components from 2 to m of the rotated output are pure Wiener processes also under the physical probabilities.

Case $\mathrm{Re}\,\beta(t)$ Not Parallel to $\mathrm{Im}\,\beta(t)$

Now $\mathrm{Re}\,\beta(t)$ and $\mathrm{Im}\,\beta(t)$ span a bi-dimensional space and it is possible to find a rotation $D(t)$ such that $D(t)\beta(t) = (\zeta_1(t), \zeta_2(t), 0, \ldots, 0)^{\mathrm{T}}$. In this case we obtain two effective channels and $m-2$ channels decoupled from the quantum system:

$$\widetilde{\Lambda}_t(k)[\tau] = \mathcal{L}(t)[\tau] + \mathrm{i}\sum_{j=1}^{2} k_j \left(\zeta_j(t)R(t)\tau + \overline{\zeta_j(t)}\tau R(t)^*\right) - \frac{1}{2}\sum_{j=1}^{m} k_j^2\,\tau\,. \quad (4.68)$$

4.4.2 Response with Time Delay

In physical applications it is likely that the apparatus has not a perfect time resolution, but that it gives some time-smoothing of the output and some time delay. The typical form of the output process is something like

$$Y_i(t) = \sum_{j=1}^{m}\int_0^t C_{ij}(t,s)\mathrm{d}W_j(s) + \int_0^t a_i(t,s)\mathrm{d}s\,, \qquad i = 1,\ldots,m'\,, \quad (4.69)$$

where $C_{ij}(t,s)$ and $a_i(t,s)$ are deterministic functions. Here we mean that the output at time t is exactly $Y(t)$, not its time derivative. The matrix C can be called the response function of the apparatus.

Apart from the deterministic additive term $\int_0^t a_i(t,s)\mathrm{d}s$, the random variables $Y_i(t)$, $i = 1,\ldots,m'$, $t \geq 0$, defined in (4.69), are a subset of the observables (4.16). We can say that the response of the apparatus eventually introduces some restrictions on the possible test functions. If the σ-algebra generated by the process (4.69) is strictly smaller than the σ-algebra generated by W we are losing information due to the imperfections of the apparatus. If the two σ-algebras are equal (4.69) gives only a transformation of the output, equivalent to W or to the whole set of observables (4.16); the set of a posteriori states does not change.

A good model for the response with time delay is an exponential. Let us take Eq. (4.69) with $m = m'$, $a_i(t,s) = 0$, $C_{ij}(t,s) = \mathrm{e}^{-\varkappa_i(t-s)}D_{ij}$, $\varkappa_i > 0$, D invertible; then, we have that the physical output is

$$Y_i(t) = \sum_{j=1}^{m}\int_0^t \mathrm{e}^{-\varkappa_i(t-s)}D_{ij}\,\mathrm{d}W_j(s)\,, \qquad i = 1,\ldots,m\,. \quad (4.70)$$

But this transformation is invertible; indeed, we have

$$W_j(t) = \sum_{i=1}^{m}(D^{-1})_{ji}\int_0^t [\mathrm{d}Y_i(s) + \varkappa_i Y_i(s)\mathrm{d}s]\,. \quad (4.71)$$

This implies that the natural filtrations of Y and W coincide and there is no loss of information in considering only the physical response with time delay. Then, there is no change in the equations for a posteriori states.

The constant $\varkappa$ controls the time resolution. For $\varkappa_i \downarrow 0$ we get $Y_i(t) \to \sum_{j=1}^{m} D_{ij}W_j(t)$, a very bad time resolution. For $\varkappa_i \uparrow +\infty$ we get the formal limit

$\varkappa_i X_i(t) \to \sum_{j=1}^m D_{ij}\dot{W}_j(t)$, a perfect time resolution (but the resulting process is singular).

4.5 Autocorrelation and Spectrum of the Output Process

4.5.1 The Spectrum of a Stationary Process

4.5.1.1 Mean and Autocorrelation

A real process X with times taking values in the whole real line is said to be *strictly stationary* if all its finite-dimensional distributions depend only on the differences of times and it is said to be *second order stationary* if this is true only for the one- and two-dimensional distributions. For such processes, if the second moments exist, we have that the mean is independent of time

$$\mathbb{E}[X(t)] = \mathbb{E}[X(0)] =: m\,, \qquad \forall t \in \mathbb{R}\,, \tag{4.72}$$

and that the second moment is invariant under time translations

$$\mathbb{E}[X(t+s)X(s)] = \mathbb{E}[X(t)X(0)] =: R^X(t), \qquad \forall t, s \in \mathbb{R}\,. \tag{4.73}$$

The function $R^X(t)$, $t \in \mathbb{R}$, is called the *autocorrelation function* of the process. A process for which Eqs. (4.72) and (4.73) hold true is said to be *wide sense, second order stationary*. The autocorrelation $R^X(t)$ is an even function of t; indeed, by a $-t$ translation and the symmetry of the second moment, we get $R^X(t) = \mathbb{E}[X(t)X(0)] = \mathbb{E}[X(0)X(-t)] = \mathbb{E}[X(-t)X(0)] = R^X(-t)$. Obviously, we have

$$\mathrm{Cov}\,[X(t), X(s)] = R^X(t-s) - m^2\,. \tag{4.74}$$

4.5.1.2 Spectral Density

The *spectrum* (or *spectral density*) of a wide sense, second order stationary stochastic process X is the Fourier transform of its autocorrelation function:

$$S^X(\mu) := \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\mu t} R^X(t)\,\mathrm{d}t\,. \tag{4.75}$$

This formula has to be intended with care, at least in the sense of distributions or in weak sense, see [14, pp. 518–526]. This means that the spectrum is defined by: for any compact support, C^∞-function h

$$\int_{-\infty}^{+\infty} h(\mu) S^X(\mu) \mathrm{d}\mu := \lim_{T\to+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\mu\, h(\mu) \int_{-T}^{+T} \mathrm{e}^{-\mathrm{i}\mu t} R^X(t)\, \mathrm{d}t$$
$$= \int_{-\infty}^{+\infty} \widehat{h}(t) R^X(t) \mathrm{d}t, \tag{4.76}$$

where

$$\widehat{h}(t) := \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\mu t} h(\mu)\, \mathrm{d}\mu.$$

If $\mathrm{Cov}\,[X(t), X(0)] \in L^1(\mathbb{R})$, we can write

$$S^X(\mu) = 2\pi m^2 \delta(\mu) + \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\mu t}\, \mathrm{Cov}\,[X(t), X(0)]\, \mathrm{d}t\,. \tag{4.77}$$

By the properties of the covariance, the function $\mathrm{Cov}\,[X(t), X(0)]$ is positive definite and, by the properties of positive-definite functions, this implies $\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\mu t}\, \mathrm{Cov}\,[X(t), X(0)]\, \mathrm{d}t \geq 0$; then, also $S^X(\mu) \geq 0$.

The expression of the spectrum can be rewritten in a form suitable for generalisations.

Proposition 4.17. *For a wide sense, second order stationary stochastic process X the spectral density* (4.75) *can be written as*

$$S^X(\mu) = \lim_{T\to+\infty} \frac{1}{T}\, \mathbb{E}\left[\left|\int_0^T \mathrm{e}^{-\mathrm{i}\mu t} X(t)\, \mathrm{d}t\right|^2\right]; \tag{4.78}$$

the limit has to be understood in weak sense.

Proof. By the second order stationarity we have

$$\begin{aligned}
\frac{1}{T}\, \mathbb{E}\left[\left|\int_0^T \mathrm{e}^{-\mathrm{i}\mu t} X(t)\, \mathrm{d}t\right|^2\right] &= \frac{1}{T} \int_0^T \mathrm{d}s \int_0^s \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\mu(s-t)}\, \mathbb{E}\,[X(s)X(t)] \\
&\quad + \frac{1}{T} \int_0^T \mathrm{d}s \int_s^T \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\mu(s-t)}\, \mathbb{E}\,[X(s)X(t)] \\
&= \frac{1}{T} \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \left\{\mathrm{e}^{-\mathrm{i}\mu(t-s)}\, \mathbb{E}\,[X(t-s)X(0)] + \mathrm{e}^{\mathrm{i}\mu(t-s)}\, \mathbb{E}\,[X(0)X(t-s)]\right\} \\
&= \frac{1}{T} \int_0^T \mathrm{d}t \int_0^t \mathrm{d}r \left\{\mathrm{e}^{-\mathrm{i}\mu r}\, \mathbb{E}\,[X(r)X(0)] + \mathrm{e}^{\mathrm{i}\mu r}\, \mathbb{E}\,[X(0)X(r)]\right\} \\
&= \int_0^T \mathrm{d}r \left(1 - \frac{r}{T}\right) \left\{\mathrm{e}^{-\mathrm{i}\mu r}\, \mathbb{E}\,[X(r)X(0)] + \mathrm{e}^{\mathrm{i}\mu r}\, \mathbb{E}\,[X(0)X(r)]\right\} \\
&= \int_{-T}^T \mathrm{d}t \left(1 - \frac{|t|}{T}\right) \mathrm{e}^{-\mathrm{i}\mu t}\, R^X(t).
\end{aligned}$$

By considering the limits in weak form, we have

$$\begin{aligned}
&\lim_{T\to+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\mu\, h(\mu)\left\{\int_{-T}^{T}\mathrm{e}^{-\mathrm{i}\mu t}R^X(t)\,\mathrm{d}t-\frac{1}{T}\,\mathbb{E}\left[\left|\int_0^T\mathrm{e}^{-\mathrm{i}\mu t}X(t)\,\mathrm{d}t\right|^2\right]\right\}\\
&\qquad=\lim_{T\to+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\mu\, h(\mu)\int_{-T}^{T}\mathrm{d}t\,\frac{|t|}{T}\,\mathrm{e}^{-\mathrm{i}\mu t}\,R^X(t)\\
&\qquad=\lim_{T\to+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\mu\, h(\mu)\int_0^T\mathrm{d}t\,\frac{\mathrm{i}}{T}\left(\frac{\partial\mathrm{e}^{-\mathrm{i}\mu t}}{\partial\mu}-\frac{\partial\mathrm{e}^{\mathrm{i}\mu t}}{\partial\mu}\right)R^X(t)\\
&\qquad=\lim_{T\to+\infty}\frac{\mathrm{i}}{T}\int_0^T\left[\widehat{h'}(-t)-\widehat{h'}(t)\right]R^X(t)\,\mathrm{d}t=0.
\end{aligned}$$

□

4.5.2 The Spectrum of an Asymptotically Stationary Process

4.5.2.1 Autocorrelation and Spectrum "up to Time T"

Inspired by Eq. (4.78), given a generic real stochastic process $X\equiv\{X(t),\,t\geq 0\}$ (defined only for positive times), we can define the *autocorrelation function* of the process up to time T by

$$R_T^X(t)=\frac{1}{T}\int_0^T\mathbb{E}\left[X(s+|t|)X(s)\right]\mathrm{d}s\,, \tag{4.79}$$

and the *spectrum* by

$$S_T^X(\mu)=\frac{1}{T}\,\mathbb{E}\left[\left|\int_0^T\mathrm{e}^{-\mathrm{i}\mu t}X(t)\,\mathrm{d}t\right|^2\right]\geq 0\,,\qquad \mu\in\mathbb{R}\,. \tag{4.80}$$

Remark 4.18. Properties of $S_T^X(\mu)$.

1. We can write also

$$S_T^X(\mu)=\int_{-T}^{T}\mathrm{e}^{-\mathrm{i}\mu t}\left(1-\frac{|t|}{T}\right)R^X_{T-|t|}(t)\,\mathrm{d}t. \tag{4.81}$$

Indeed, we have

$$\int_{-T}^{T}\mathrm{e}^{-\mathrm{i}\mu t}\left(1-\frac{|t|}{T}\right)R^X_{T-|t|}(t)\,\mathrm{d}t=\int_0^T\mathrm{e}^{-\mathrm{i}\mu t}\left(1-\frac{t}{T}\right)R^X_{T-t}(t)\,\mathrm{d}t+\text{c.c.},$$

$$\begin{aligned}\int_0^T \mathrm{e}^{-\mathrm{i}\mu r}\left(1-\frac{r}{T}\right)R_{T-r}^X(r)\,\mathrm{d}r &= \int_0^T \mathrm{d}r\,\mathrm{e}^{-\mathrm{i}\mu r}\,\frac{1}{T}\int_0^{T-r}\mathrm{d}t\,\mathbb{E}\left[X(t+r)X(t)\right]\\ &= \frac{1}{T}\int_0^T \mathrm{d}r\int_r^T \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\mu r}\;\mathbb{E}\left[X(s)X(s-r)\right]\\ &= \frac{1}{T}\int_0^T \mathrm{d}s\int_0^s \mathrm{d}r\,\mathrm{e}^{-\mathrm{i}\mu r}\;\mathbb{E}\left[X(s)X(s-r)\right]\\ &= \frac{1}{T}\int_0^T \mathrm{d}s\int_0^s \mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\mu(s-t)}\;\mathbb{E}\left[X(s)X(t)\right],\end{aligned}$$

$$\begin{aligned}\int_0^T \mathrm{e}^{-\mathrm{i}\mu r}\left(1-\frac{r}{T}\right)R_{T-r}^X(r)\,\mathrm{d}r &= \frac{1}{T}\int_0^T \mathrm{d}s\int_0^s \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\mu(s-t)}\;\mathbb{E}\left[X(s)X(t)\right]\\ &= \frac{1}{T}\int_0^T \mathrm{d}t\int_t^T \mathrm{d}s\,\mathrm{e}^{\mathrm{i}\mu(s-t)}\;\mathbb{E}\left[X(s)X(t)\right]\\ &= \frac{1}{T}\int_0^T \mathrm{d}s\int_s^T \mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\mu(s-t)}\;\mathbb{E}\left[X(s)X(t)\right].\end{aligned}$$

2. By rewriting Eq. (4.80) as

$$S_T^X(\mu)=\frac{1}{T}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\mu(t-s)}\,\mathbb{E}[X(t)X(s)]$$

and by expressing the second moment as the covariance and the product of the means, we get the decomposition of the spectrum:

$$\begin{aligned}S_T^X(\mu)=\frac{1}{T}\left|\int_0^T \mathrm{e}^{-\mathrm{i}\mu t}\,\mathbb{E}\left[X(t)\right]\,\mathrm{d}t\right|^2\\ +\frac{1}{T}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\mu(t-s)}\,\mathrm{Cov}[X(t),X(s)].\end{aligned}\quad(4.82)$$

Sometimes, the contribution with the product of the means is called the *coherent* part of the *spectrum*, while the contribution of the covariance is called the *incoherent* part.

3. Moreover, we have also

$$\begin{aligned}S_T^X(\mu)-\frac{1}{T}\left|\int_0^T \mathrm{e}^{-\mathrm{i}\mu t}\,\mathbb{E}\left[X(t)\right]\,\mathrm{d}t\right|^2\\ =\frac{1}{T}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\,\mathrm{e}^{\mathrm{i}\mu(t-s)}\,\mathrm{Cov}[X(t),X(s)]\geq 0.\end{aligned}\quad(4.83)$$

Indeed, if we set $Z_T := \frac{1}{\sqrt{T}} \int_0^T \mathrm{e}^{-\mathrm{i}\mu t} X(t)\,\mathrm{d}t$, we get

$$\frac{1}{T}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}\mu(t-s)}\,\mathrm{Cov}[X(t), X(s)] \\ = \mathrm{Cov}\left[Z_T, \overline{Z_T}\right] = \mathrm{Var}\left[\mathrm{Re}\, Z_T\right] + \mathrm{Var}\left[\mathrm{Im}\, Z_T\right] \geq 0.$$

4. An interesting case is when one has asymptotic stationarity (second order, wide sense):

$$\lim_{t\to+\infty} \mathbb{E}[X(t)] = m\,, \tag{4.84a}$$

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathrm{Cov}[X(s+|t|), X(s)]\mathrm{d}s = C(t) \in L^1(\mathbb{R})\,. \tag{4.84b}$$

Then, we have

$$\lim_{T\to+\infty} S_T^X(\mu) = 2\pi m^2 \delta(\mu) + \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\mu t} C(t)\mathrm{d}t\,.$$

When $\lim_{T\to+\infty} S_T^X(\mu)$ exists (at least in a weak sense), as in the case of point 4 above, we can take this limit as a good definition of the spectrum of the process X.

4.5.3 The Spectrum of the Output of the Continuous Measurement

The definition (4.80) can be generalised also to the case of singular processes, as one of the outputs $\dot{W}_j(t)$ of the continuous measurement:

$$S_T^j(\mu) := \frac{1}{T}\,\mathbb{E}_{\rho_0}^T\left[\left|\int_0^T \mathrm{e}^{-\mathrm{i}\mu t}\,\mathrm{d}W_j(t)\right|^2\right]. \tag{4.85}$$

From (4.46), giving the second moments, we get

$$S_T^j(\mu) = 1 + \frac{2}{T}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\, \cos\mu(t-s) \\ \times \mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_j(s)\circ\mathcal{T}(s,0)[\rho_0]\right\}; \tag{4.86}$$

recall that $\mathcal{R}_j(t)[\tau] = R_j(t)\tau + \tau R_j(t)^*$ and $\mathcal{T}(s,0)[\rho_0] = \eta(s)$.

As in the case of generic stochastic processes, it is often useful to isolate the contribution to the spectrum of the "fluctuations" (the incoherent part, due to the covariance) by defining

$$\widetilde{S}_T^j(\mu) = S_T^j(\mu) - \frac{1}{T}\left|\mathbb{E}_{\rho_0}^T\left[\int_0^T \mathrm{e}^{-\mathrm{i}\mu t}\,\mathrm{d}W_j(t)\right]\right|^2 = 1 + \frac{2}{T}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\,\cos\mu(t-s)$$
$$\times\left(\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_j(s)[\eta(s)]\right\} - \mathrm{Tr}\left\{\mathcal{R}_j(t)[\eta(t)]\right\}\mathrm{Tr}\left\{\mathcal{R}_j(s)[\eta(s)]\right\}\right). \tag{4.87}$$

4.5.3.1 Heisenberg Uncertainty Relations

In this subsection, let us consider a family of models in which the operator R_j depends on a phase ϑ_j: $R_j(t)$ is replaced by $R_j(t;\vartheta_j) = \mathrm{e}^{\mathrm{i}\vartheta_j}R_j(t)$. We can say that to change ϑ_j is to change measuring apparatus without changing the mean dynamics. Indeed, it is easy to see that the mean dynamics $\mathcal{T}(t,s)$ does not depends on ϑ_j, while it is the probability law of the output which depends on such phases. In particular, such a dependence is seen in the moments and, so, in the spectrum:

$$S_T^j(\mu;\vartheta_j) = 1 + \frac{2}{T}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\,\cos\mu(t-s)$$
$$\times\,\mathrm{Tr}\left\{\mathcal{R}_j(t;\vartheta_j)\circ\mathcal{T}(t,s)\circ\mathcal{R}_j(s;\vartheta_j)\circ\mathcal{T}(s,0)[\rho_0]\right\}. \tag{4.88}$$

Then, for the spectrum of the fluctuations, the following uncertainty relation holds [16]:

$$\widetilde{S}_T^j(\mu;\vartheta_j)\,\widetilde{S}_T^j(\mu;\vartheta_j \pm \pi/2) \geq 1. \tag{4.89}$$

The proof of this needs the use of the formulation of the theory of continuous measurements based on quantum stochastic calculus. Then, it is a consequence of the Heisenberg–Scrödinger–Robertson uncertainty relations.

Let us stress that the definition of spectrum is the classical one, as given in the theory of stochastic processes. Similarly, the decomposition in coherent and incoherent parts is purely classical, as it is due to the writing of the second moment as product of the means and covariance. On the contrary, the uncertainty relation (4.89) is of pure quantum origin, from Heisenberg principle. What is important to note is that, from a classical probabilistic point of view, Eq. (4.89) connects two quantities referring to two different probabilistic models, as two different values for the phase ϑ_j means two different physical probabilities. It is quantum mechanics which connects the two classical models in a unique quantum model [11, 16] (and the proof of (4.89) is based on the construction of such a quantum model).

4.5.3.2 Cross-Correlations

Sometimes a linear combination of the outputs is of interest, such as the complex process

$$X^c(t) = \sum_{i=1}^m c_i W_i(t), \qquad c_i \in \mathbb{C} \quad i = 1,\ldots,m. \tag{4.90}$$

The spectrum of complex processes is defined similarly to the one of real processes [15] and, in a finite time horizon, it is given by

$$S_T^c(\mu) = \frac{1}{T}\,\mathbb{E}_{\rho_0}^T\left[\left|\int_0^T \mathrm{e}^{-\mathrm{i}\mu t}\,\mathrm{d}X^c(t)\right|^2\right] = \sum_{ij}\overline{c_i}S_T^{ij}(\mu)c_j\,, \tag{4.91}$$

where we have introduced the *spectrum of the cross-correlations*

$$S_T^{ij}(\mu) := \frac{1}{T}\,\mathbb{E}_{\rho_0}^T\left[\int_0^T \mathrm{e}^{\mathrm{i}\mu t}\,\mathrm{d}W_i(t)\int_0^T \mathrm{e}^{-\mathrm{i}\mu s}\,\mathrm{d}W_j(s)\right]. \tag{4.92}$$

Let us stress that, by its definition, the spectrum of a real process is symmetric in μ, while this is not necessarily true for a complex process.

Note that $S_T^{jj}(\mu) \equiv S_T^j(\mu)$ and that

$$\overline{S_T^{ij}(\mu)} = S_T^{ji}(\mu), \qquad \sum_{ij}\overline{c_i}S_T^{ij}(\mu)c_j \geq 0, \quad \forall c_i \in \mathbb{C} \quad i = 1,\ldots,m. \tag{4.93}$$

From Eqs. (4.46), (4.9) we get the expression

$$\begin{aligned} S_T^{ij}(\mu) = \delta_{ij} + \frac{1}{T}\int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\Big(&\mathrm{e}^{\mathrm{i}\mu(t-s)}\,\mathrm{Tr}\left\{\mathcal{R}_i(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_j(s)[\eta(s)]\right\} \\ &+ \mathrm{e}^{-\mathrm{i}\mu(t-s)}\,\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_i(s)[\eta(s)]\right\}\Big). \end{aligned} \tag{4.94}$$

4.5.3.3 Infinite Time Horizon and Ergodic Properties

Also in our case we can rise questions such as the one of point 4 of Remark 4.18. This is the question of the existence of the true spectrum, in an infinite time horizon: does the limit

$$S_{ij}(\mu) = \lim_{T\to+\infty}\frac{1}{T}\,\mathbb{E}_{\rho_0}^T\left[\int_0^T \mathrm{e}^{\mathrm{i}\mu t}\,\mathrm{d}W_i(t)\int_0^T \mathrm{e}^{-\mathrm{i}\mu s}\,\mathrm{d}W_j(s)\right]$$

exist? A further question is about the possibility of estimating the spectrum from a single trajectory: does the ergodic property

$$\lim_{T\to+\infty}\frac{1}{T}\int_{\mu_1}^{\mu_2}\mathrm{d}\mu\int_0^T \mathrm{e}^{\mathrm{i}\mu t}\,\mathrm{d}W_i(t)\int_0^T \mathrm{e}^{-\mathrm{i}\mu s}\,\mathrm{d}W_j(s) = \int_{\mu_1}^{\mu_2} S_{ij}(\mu)\,\mathrm{d}\mu$$

hold, $\mathbb{P}$-a.s. or in some other sense? Here $\mathbb{P}$ is the physical probability and the time horizon is infinite. Obviously the answers to these two questions depend on the Liouvillian and on the operators $R_j(t)$. We touch only the first problem in the case our output is stationary or asymptotically stationary. In Chapters 9 and 10 concrete physical examples will be given.

Stationary Case

Let us assume that there exists a quantum dynamical semigroup $\check{\mathcal{T}}(t)$, having an invariant state $\eta_{\rm eq} : \check{\mathcal{T}}(t)[\eta_{\rm eq}] = \eta_{\rm eq}$, and a time-independent map $\check{\mathcal{R}}_j[\tau] = \check{R}_j\tau + \tau\check{R}_j^*$ such that

$$\mathrm{Tr}\left\{\mathcal{R}_i(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_j(s)\circ\mathcal{T}(s,0)[\eta_{\rm eq}]\right\} = \mathrm{Tr}\left\{\check{\mathcal{R}}_i\circ\check{\mathcal{T}}(t-s)\circ\check{\mathcal{R}}_j[\eta_{\rm eq}]\right\}.$$

With this assumption, when the initial state is $\rho_0 = \eta_{\rm eq}$, the second-order quantum correlation functions appearing inside the integral in (4.94) become time homogeneous.

Then, we define

$$m_i := \mathrm{Tr}\left\{\check{\mathcal{R}}_i[\eta_{\rm eq}]\right\}, \quad C_{ji}(t) := \mathrm{Tr}\left\{\check{\mathcal{R}}_j\circ\check{\mathcal{T}}(t)[\tau_i]\right\}, \quad \tau_i := \check{\mathcal{R}}_i[\eta_{\rm eq}] - m_i\eta_{\rm eq}.$$

and we assume $tC_{ji}(t)$ to be integrable in $(0,+\infty)$. This gives

$$\begin{aligned}
S_T^{ij}(\mu) - \delta_{ij} &= \frac{1}{T}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\Big(\mathrm{e}^{\mathrm{i}\mu s}\,\mathrm{Tr}\left\{\check{\mathcal{R}}_i\circ\check{\mathcal{T}}(s)\circ\check{\mathcal{R}}_j[\eta_{\rm eq}]\right\} \\
&\quad + \mathrm{e}^{-\mathrm{i}\mu s}\,\mathrm{Tr}\left\{\check{\mathcal{R}}_j\circ\check{\mathcal{T}}(s)\circ\check{\mathcal{R}}_i[\eta_{\rm eq}]\right\}\Big) = \int_0^T \mathrm{d}t\left(1-\frac{t}{T}\right) \\
&\quad \times\Big(\mathrm{e}^{\mathrm{i}\mu t}\,\mathrm{Tr}\left\{\check{\mathcal{R}}_i\circ\check{\mathcal{T}}(t)\circ\check{\mathcal{R}}_j[\eta_{\rm eq}]\right\} + \mathrm{e}^{-\mathrm{i}\mu t}\,\mathrm{Tr}\left\{\check{\mathcal{R}}_j\circ\check{\mathcal{T}}(t)\circ\check{\mathcal{R}}_i[\eta_{\rm eq}]\right\}\Big) \\
&= \int_0^T \mathrm{d}t\left(1-\frac{t}{T}\right)\Big(2m_im_j\cos\mu t + \mathrm{e}^{\mathrm{i}\mu t}C_{ij}(t) + \mathrm{e}^{-\mathrm{i}\mu t}C_{ji}(t)\Big) \\
&= 4m_im_j\,\frac{\left(\sin\frac{\mu T}{2}\right)^2}{T\mu^2} + \int_0^T\left(1-\frac{t}{T}\right)\Big(\mathrm{e}^{\mathrm{i}\mu t}C_{ij}(t) + \mathrm{e}^{-\mathrm{i}\mu t}C_{ji}(t)\Big)\mathrm{d}t
\end{aligned}$$

and, by using

$$\lim_{T\to+\infty}\frac{2\,(\sin\mu T/2)^2}{\pi\mu^2 T} = \delta(\mu),$$

we get the existence of the spectrum in the infinite horizon limit:

$$\begin{aligned}
S_{ij}(\mu) := \lim_{T\to+\infty} S_T^{ij}(\mu) &= \delta_{ij} + 2\pi m_im_j\delta(\mu) \\
&\quad + \int_0^{+\infty}\Big(\mathrm{e}^{\mathrm{i}\mu t}C_{ij}(t) + \mathrm{e}^{-\mathrm{i}\mu t}C_{ji}(t)\Big)\mathrm{d}t. \qquad (4.95)
\end{aligned}$$

4.6 Summary

- Instruments: $t > s \geq 0,\ G \in \overline{\mathcal{G}}_t^s, \quad \mathcal{I}_t^s(G) = \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,s)\right]$.
  - Associated POMs: $\mathcal{I}_t^s(G)^*[\mathbb{1}] = E_t^s(G)$.
  - Normalisation: $\mathcal{I}_t^s(\Omega) = \mathcal{T}(t,s)$.
  - Composition property: $G_j \in \overline{\mathcal{G}}_{t_j}^{t_{j-1}},\ S \leq t_0 < t_1 < \cdots < t_n \leq T,$

$$\mathcal{I}_T^S(G_1 \cap G_2 \cap \cdots \cap G_n) = \mathcal{T}(T,t_n) \circ \mathcal{I}_{t_n}^{t_{n-1}}(G_n) \circ \cdots \circ \mathcal{I}_{t_1}^{t_0}(G_1) \circ \mathcal{T}(t_0,S)\,.$$

  - Generalised consistency of the POMs:

$$0 \leq r \leq s < t \leq u, \quad G \in \overline{\mathcal{G}}_t^s \qquad \Rightarrow \qquad E_u^r(G) = \mathcal{T}(s,r)^*[E_t^s(G)].$$

- Instruments and probabilities: $\mathcal{I}_t^0(G)[\rho_0] = \mathbb{E}_{\mathbb{Q}}[1_G\, \sigma(t)]$,

$$\mathbb{P}_{\rho_0}^t(G) = \mathbb{E}_{\mathbb{Q}}\left[1_G \operatorname{Tr}\{\sigma(t)\}\right] = \operatorname{Tr}\left\{E_t^0(G)\rho_0\right\} = \operatorname{Tr}\left\{\mathcal{I}_t^0(G)[\rho_0]\right\}.$$

- A priori states: $\eta(t) = \mathcal{T}(t,0)[\rho_0] = \mathcal{I}_t^0(\Omega)[\rho_0]$.
- A posteriori states:

$$\int_G \rho(t,\omega)\mathbb{P}_{\rho_0}^t(\mathrm{d}\omega) = \mathbb{E}_{\rho_0}^t[1_G\, \rho(t)] = \mathbb{E}_{\mathbb{Q}}[1_G\, \sigma(t)] = \mathcal{I}_t^0(G)[\rho_0].$$

- Test functions: $L^2_{\mathrm{loc}} = L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^m),\ L^2((0,t);\mathbb{R}^m) =: L_t^2$.
- Observables: $k \in L^2_{\mathrm{loc}},\ 0 \leq s < t \leq T, \quad X_t^s(k) = \sum_{j=1}^m \int_s^t k_j(r)\,\mathrm{d}W_j(r)$.
- Diffusive property: if $k^{(1)}, \ldots, k^{(q)}$ are linearly independent elements of L_T^2, the distribution of $\left(X_T^0(k^{(1)}), \ldots, X_T^0(k^{(q)})\right)$ under $\mathbb{P}_{\rho_0}^T$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^q$ and its density can be taken strictly positive $\forall x \in \mathbb{R}^q$.
- Characteristic operator: $\mathcal{G}(t,s;k) = \int_\Omega \mathrm{e}^{\mathrm{i}X_t^s(k)}\,\mathcal{I}_t^s(\mathrm{d}\omega) = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}X_t^s(k)}\mathcal{A}(t,s)\right]$.
  - $\mathcal{G}(t,0;k)[\rho_0] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}X_t^0(k)}\sigma(t)\right] = \mathbb{E}_{\rho_0}^T\left[\mathrm{e}^{\mathrm{i}X_t^0(k)}\rho(t)\right]$.
  - Evolution equation: $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t,s;k) = \Lambda_t\big(k(t)\big) \circ \mathcal{G}(t,s;k)\,, \quad \mathcal{G}(s,s;k) = \mathrm{Id}_n$.
  - Generator of the characteristic operator:

$$\Lambda_t(k) = \mathcal{L}(t) + \sum_{j=1}^m \left(\mathrm{i}k_j\mathcal{R}_j(t) - \frac{1}{2}k_j^2\,\mathrm{Id}_n\right).$$

  - Composition law: $t > s > r \geq 0, \qquad \mathcal{G}(t,r;k) = \mathcal{G}(t,s;k) \circ \mathcal{G}(s,r;k)\,.$

- Characteristic functional: $\Phi_t(k|\rho) = \mathrm{Tr}\{\mathcal{G}(t,0;k)[\rho_0]\}$.
- Moments:

$$\mathbb{E}^T_{\rho_0}\left[\dot{W}_{j_1}(t_1)\dot{W}_{j_2}(t_2)\cdots\dot{W}_{j_q}(t_q)\right] = (-\mathrm{i})^q \left.\frac{\delta^q \Phi_t(k|\rho_0)}{\delta k_{j_1}(t_1)\,\delta k_{j_2}(t_2)\cdots\delta k_{j_q}(t_q)}\right|_{k=0}.$$

  - Mean function: $\mathbb{E}^T_{\rho_0}[\dot{W}_j(t)] = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\eta(t)\right\}$.
  - Second moments:

$$\begin{aligned}\mathbb{E}^T_{\rho_0}[\dot{W}_j(t)\dot{W}_i(s)] = {} & \delta_{ij}\,\delta(t-s) \\ & + 1_{(0,+\infty)}(t-s)\,\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_i(s)\circ\mathcal{T}(s,0)[\rho_0]\right\} \\ & + 1_{(0,+\infty)}(s-t)\,\mathrm{Tr}\left\{\mathcal{R}_i(s)\circ\mathcal{T}(s,t)\circ\mathcal{R}_j(t)\circ\mathcal{T}(t,0)[\rho_0]\right\}.\end{aligned}$$

- Spectrum of the output: $S^j_T(\mu) = \dfrac{1}{T}\,\mathbb{E}^T_{\rho_0}\left[\left|\displaystyle\int_0^T \mathrm{e}^{-\mathrm{i}\mu t}\,\mathrm{d}W_j(t)\right|^2\right]$,

$$\begin{aligned}S^j_T(\mu) = 1 + \frac{2}{T}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\,\cos\mu(t-s) \\ \times\,\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_j(s)\circ\mathcal{T}(s,0)[\rho_0]\right\}.\end{aligned}$$

- Spectrum of the cross-correlations:

$$S^{ij}_T(\mu) = \frac{1}{T}\,\mathbb{E}^T_{\rho_0}\left[\int_0^T \mathrm{e}^{\mathrm{i}\mu t}\,\mathrm{d}W_i(t)\int_0^T \mathrm{e}^{-\mathrm{i}\mu s}\,\mathrm{d}W_j(s)\right],$$

$$\begin{aligned}S^{ij}_T(\mu) = \delta_{ij} + \frac{1}{T}\int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\Big(\mathrm{e}^{\mathrm{i}\mu(t-s)}\,\mathrm{Tr}\left\{\mathcal{R}_i(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_j(s)[\eta(s)]\right\} \\ + \mathrm{e}^{-\mathrm{i}\mu(t-s)}\,\mathrm{Tr}\left\{\mathcal{R}_j(t)\circ\mathcal{T}(t,s)\circ\mathcal{R}_i(s)[\eta(s)]\right\}\Big).\end{aligned}$$

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Chapter 5
The Stochastic Master Equation: Part II

In Chap. 3, starting from the linear formulation of the quantum trajectory theory, it has been shown that the a posteriori states satisfy the nonlinear SDE (3.69). Now we show that this equation can be taken as starting point of the whole theory and we study some of its properties.

5.1 Quantum Trajectories: The Nonlinear SDE Formulation

In this chapter we show how to found the whole theory on the stochastic master equation (3.69). Let us recall the properties of the operators appearing in (3.69), which we assume to hold through the whole chapter.

Assumption 5.1. The maps $\mathcal{R}_j(t)$, $\mathcal{L}(t)$ are linear operators over the space M_n of $n \times n$ complex matrices τ with the structure

$$\mathcal{R}_j(t)[\tau] = R_j(t)\tau + \tau R_j(t)^*, \qquad \mathcal{L}(t) = \mathcal{L}_0(t) + \mathcal{L}_1(t), \tag{5.1a}$$

where

$$\mathcal{L}_1(t)[\tau] = \sum_{j=1}^{m} \left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{ R_j(t)^* R_j(t), \tau \right\} \right), \tag{5.1b}$$

$$\mathcal{L}_0(t)[\tau] = -\mathrm{i}[H(t), \tau] + \sum_{j=m+1}^{d} \left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{ R_j(t)^* R_j(t), \tau \right\} \right). \tag{5.1c}$$

The coefficients $R_j(t)$, $H(t)$ are non-random linear operators on $\mathcal{H} \equiv \mathbb{C}^n$ and $H(t) = H(t)^*$. The functions $t \mapsto H(t)$ and $t \mapsto R_j(t)$ are measurable and such that $\sup_{t\in[0,T]} \|H(t)\| < +\infty$, $\sup_{t\in[0,T]} \left\| \sum_{j=1}^{d} R_j(t)^* R_j(t) \right\| < +\infty$, $\forall T \in (0, +\infty)$.

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5.1.1 The Nonlinear SDE

In analogy with the Hilbert space case treated in Sect. 2.5, we extend the nonlinear SDE (3.69) from the set of states to the space of all matrices, in a way which allows for application of the standard theorems of existence and uniqueness of the solutions.

For all matrices τ we define

$$n_j(t;\tau) := \begin{cases} \frac{\mathrm{Tr}\{\mathcal{R}_j(t)[\tau]\}}{\|\tau\|_1}, & \text{if } \tau \neq 0, \\ 0, & \text{if } \tau = 0, \end{cases} \tag{5.2a}$$

$$\widehat{n}_j(t;\tau) := \mathcal{R}_j(t)[\tau] - n_j(t;\tau)\tau, \tag{5.2b}$$

where the trace norm $\|\bullet\|_1$ is given in (B.4). Let us note that

$$\mathrm{Tr}\left\{\widehat{n}_j(t;\tau)\right\} = \mathrm{Tr}\left\{\mathcal{R}_j(t)[\tau]\right\}\left(1 - \frac{\mathrm{Tr}\{\tau\}}{\|\tau\|_1}\right), \tag{5.3}$$

which gives $\mathrm{Tr}\left\{\widehat{n}_j(t;\tau)\right\} = 0$ for $\tau \geq 0$, because $\|\tau\|_1 = \mathrm{Tr}\{\tau\}$ by positivity. Moreover,

$$\left|n_j(t;\tau)\right| \leq 2\left\|R_j(t)\right\|, \qquad \forall \tau \in M_n, \tag{5.4}$$

which follows from (B.9), and, for $\tau \in \mathcal{S}(\mathcal{H})$,

$$n_j(t;\tau) = \mathrm{Tr}\left\{\mathcal{R}_j(t)[\tau]\right\}, \qquad \widehat{n}_j(t;\tau) = \mathcal{R}_j(t)[\tau] - \mathrm{Tr}\left\{\mathcal{R}_j(t)[\tau]\right\}\tau, \tag{5.5}$$

because $\|\tau\|_1 = 1$.

Theorem 5.2. *Let $\widehat{B}_j(t)$, $j = 1, \ldots, m$, $t \geq 0$, be a standard Wiener process. Under Assumption 5.1 the nonlinear SDE*

$$\begin{cases} \mathrm{d}\xi(t) = \mathcal{L}(t)[\xi(t)]\mathrm{d}t + \sum_{j=1}^{m} \widehat{n}_j(t;\xi(t))\,\mathrm{d}\widehat{B}_j(t) \\ \xi(0) = \xi_0 \in M_n \end{cases} \tag{5.6}$$

admits strong solutions in the time interval $[0, +\infty)$; pathwise uniqueness and uniqueness in law hold.

Proof. Trivially, measurability conditions A.25 of the coefficients hold. In the proof of Theorem 3.4 we have already proved that the global Lipschitz condition A.32 and the linear growth condition A.34 hold for the coefficient $\mathcal{L}(t)[\tau]$. So, we have to check the same properties on $\widehat{n}_j(t;\tau)$. Recall that, in order to apply Theorem A.36, the relevant norm is the Hilbert–Schmidt one and that we have to fix a finite time interval $[0, T]$.

Let us recall Eqs. (3.52) and (3.53):

$$\left\|R_j(t)\right\|^2 = \left\|R_j(t)^* R_j(t)\right\| \le \ell_T\,, \qquad \sum_{j=1}^{m} \left\|\mathcal{R}_j(t)[\tau]\right\|_2^2 \le 4m\ell_T \left\|\tau\right\|_2^2\,,$$

where ℓ_T is the constant (3.51). Moreover, inequalities (B.8), (B.9) give

$$\left\|\tau\right\|_2 \le \left\|\tau\right\|_1 \le n \left\|\tau\right\|_2\,, \qquad \left|\mathrm{Tr}\{\tau\}\right| \le \left\|\mathbb{1}\right\| \left\|\tau\right\|_1 = \left\|\tau\right\|_1\,.$$

Here and below, we are using also the trivial inequality $(a+b)^2 \le 2a^2 + 2b^2$. From the inequalities above and (5.4), we obtain

$$\begin{aligned}\sum_{j=1}^{m} \left\|\widehat{n}_j(t;\tau)\right\|_2^2 &\le 2\sum_{j=1}^{m}\left(\left\|\mathcal{R}_j(t)[\tau]\right\|_2^2 + \frac{\left|\mathrm{Tr}\left\{\tau\right\}\right|^2}{\left\|\tau\right\|_1^2}\left|\mathrm{Tr}\left\{\mathcal{R}_j(t)[\tau]\right\}\right|^2 \left\|\tau\right\|_2^2\right)\\ &\le 2\sum_{j=1}^{m}\left(\left\|\mathcal{R}_j(t)[\tau]\right\|_2^2 + 4\left\|R_j(t)\right\|^2 \left\|\tau\right\|_2^2\right) \le 16m\ell_T \left\|\tau\right\|_2^2\,.\end{aligned}$$

So, the linear growth condition A.34 holds for all the coefficients of the SDE (5.6).

Finally, we prove the global Lipschitz condition for the nonlinear part of the coefficients; then, the statements on existence and uniqueness follow from Theorem A.36. Let us take two matrices $x \neq 0$, $y \neq 0$ (if one of the two matrices x or y vanishes, the following proof is trivial); we set also $\hat{x} := \frac{x}{\|x\|_1}$, $\hat{y} := \frac{y}{\|y\|_1}$. Then, we have

$$\begin{aligned}\left\|\widehat{n}_j(t;x)\right\| - \widehat{n}_j(t;y)_2 &\le \left\|\mathcal{R}_j(t)\left[x-y\right]\right\|_1\\ &+ \left\|\mathrm{Tr}\left\{\mathcal{R}_j(t)\left[x-y\right]\right\}\hat{x} + \mathrm{Tr}\left\{\mathcal{R}_j(t)\left[\hat{y}\right]\right\}\left(\left\|y\right\|_1\hat{x} - y\right)\right\|_1\\ &\le 2\left\|R_j(t)\right\|\left(2\left\|x-y\right\|_1 + \left\|\left(\left\|y\right\|_1 - \left\|x\right\|_1\right)\hat{x} + x - y\right\|_1\right)\\ &\le 2\left\|R_j(t)\right\|\left(3\left\|x-y\right\|_1 + \left|\left\|y\right\|_1 - \left\|x\right\|_1\right|\right)\\ &\le 8\left\|R_j(t)\right\|\left\|x-y\right\|_1 \le 8\sqrt{\ell_T}\,n\left\|x-y\right\|_2\,.\end{aligned}$$

This ends the proof of existence and uniqueness. □

Assumption 5.3. Let us fix a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\right)$ in usual hypotheses, where $\widehat{B}$ is a continuous Wiener process with increments independent from the past. Let $\xi(t)$ be the continuous solution of (5.6) with initial condition ξ_0.

Moreover, we define the processes

$$z(t) := \exp\left\{\sum_{j=1}^{m}\left[\int_0^t n_j(s;\xi(s))\mathrm{d}\widehat{B}_j(s) + \frac{1}{2}\int_0^t n_j(s;\xi(s))^2\mathrm{d}s\right]\right\}, \tag{5.7}$$

$$\zeta(t) := z(t)\,\xi(t)\,. \tag{5.8}$$

Proposition 5.4. *Under Assumptions 5.1, 5.3, the process* $1/z(t)$ *is a complex* $\mathbb{P}$*-martingale and the stochastic differential of* $\zeta(t)$ *turns out to be*

$$\mathrm{d}\zeta(t) = \mathcal{L}(t)[\zeta(t)]\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t)[\zeta(t)]\left[\mathrm{d}\widehat{B}_j(t) + n_j(t;\xi(t))\mathrm{d}t\right]. \tag{5.9}$$

If $\xi_0^* = \xi_0$*, the process* $\xi(t)$ *is self-adjoint and* $1/z(t)$ *is a positive* $\mathbb{P}$*-martingale with mean 1.*

Proof. By inequality (5.4) we get

$$\sum_{j=1}^{m} \int_0^T \left|n_j(t;\xi(t))\right|^2 \mathrm{d}t \le 4 \sum_{j=1}^{m} \int_0^T \left\|R_j(t)\right\|^2 \mathrm{d}t \le 4m\ell_T T.$$

By Proposition A.42, $1/z(t)$ is a complex martingale. By direct application of Itô calculus, we get the stochastic differential (5.9).

In the case of a self-adjoint initial condition, $\xi^*(t)$ follows the same equation as $\xi(t)$ (B.5) and by pathwise uniqueness the two processes are indistinguishable. In this case $z(t)$ is positive, $1/z(t)$ is a positive martingale and $\mathbb{E}_{\mathbb{P}}[1/z(t)] = \mathbb{E}_{\mathbb{P}}[1/z(0)] = 1$, because the mean of a martingale is constant and $z(0) = 1$. □

Finally, we can prove that the SDE (5.6) is positivity preserving.

Theorem 5.5. *Let the initial condition be positive, i.e.* $\xi_0 \ge 0$, $\xi_0 \ne 0$*. Under Assumptions 5.1, 5.3, the process* $\xi(t)$*, solution of* (5.6)*, is positive and* $\|\xi(t)\|_1 = \|\xi_0\|_1$.

Proof. Let us fix a time interval $[0, T]$, with an arbitrary final time $T > 0$.

Let us define $\mathbb{Q}(\mathrm{d}\omega) := z(T,\omega)^{-1}\mathbb{P}(\mathrm{d}\omega)$. By Proposition 5.4 and the discussion in Sect. A.5.3, $\mathbb{Q}$ is a new probability measure. By Girsanov theorem A.45, under the probability $\mathbb{Q}$, the process $B_j(t) := \widehat{B}_j(t) + \int_0^t n_j(s;\xi(s))\mathrm{d}s$, $t \in [0, T]$, $j = 1, \ldots, m$, is a standard m-dimensional Wiener process.

By Proposition 5.4, the process ζ satisfies, under $\mathbb{Q}$, the linear SDE

$$\mathrm{d}\zeta(t) = \mathcal{L}(t)[\zeta(t)]\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t)[\zeta(t)]\mathrm{d}B_j(t),$$

which coincides with (3.38). By Theorem 3.4 and the positivity of the initial condition, we get $\zeta(t) \ge 0$. Being $z(t) > 0$, we obtain $\xi(t) \ge 0$. By (5.3), Eq. (5.6) is trace preserving when the solution is positive, as one can check by taking the trace. By positivity, the conservation of the trace coincides with the conservation of the trace norm. □

5.1.2 Stochastic Master Equation, A Posteriori States and Output

Now we can state the uniqueness of the solution of the stochastic master equation (3.69). Moreover, we have that the existence is in strong sense, because the stochastic basis is arbitrary.

Theorem 5.6. *Under Assumption 5.1, the stochastic master equation*

$$\begin{cases} \mathrm{d}\rho(t) = \mathcal{L}(t)[\rho(t)]\mathrm{d}t + \sum_{j=1}^{m} \left(\mathcal{R}_j(t)[\rho(t)] - \mathrm{Tr}\left\{\mathcal{R}_j(t)[\rho(t)]\right\}\rho(t)\right)\mathrm{d}\widehat{B}_j(t) \\ \rho(0) = \rho_0 \in \mathcal{S}(\mathcal{H}) \end{cases} \tag{5.10}$$

admits a pathwise unique continuous solution with $\rho(t,\omega) \in \mathcal{S}(\mathcal{H})$. Uniqueness in law holds.

Proof. By Theorems 5.2 and 5.5, Eq. (5.6) has a (pathwise and in law) unique solution $\xi(t)$, which is state-valued when the initial condition is a state. Moreover, if $\xi(t)$ is a state, by (5.5), we have $\widehat{n}_j(t;\xi(t)) = \mathcal{R}_j(t)[\xi(t)] - \mathrm{Tr}\left\{\mathcal{R}_j(t)[\xi(t)]\right\}\xi(t)$. Therefore, (5.10) coincides with (5.6) particularised to the case of a state as initial conditions and, so, the statements of the theorem follow from Theorems 5.2 and 5.5. □

Assumption 5.7. The initial state is $\rho_0 \in \mathcal{S}(\mathcal{H})$ and $\rho(t)$ is the continuous state-valued solution of (5.10) with initial condition ρ_0. The stochastic basis is fixed by Assumption 5.3.

In Sect. 3.5 we obtained the stochastic master equation by a normalisation and a change of probability starting from the linear stochastic master equation. We expect that it is possible to go back by the inverse transformations, starting now from $\rho(t)$ interpreted as a posteriori state. So, we take $\mathbb{P}$ as physical probability, $\{\rho(t),\ t \geq 0\}$ as a posteriori states and the process

$$B_j(t) = \widehat{B}_j(t) + \int_0^t \mathrm{Tr}\left\{\mathcal{R}_j(s)\left[\rho(s)\right]\right\}\mathrm{d}s\,, \qquad j = 1,\dots,m, \tag{5.11}$$

as output. However, we have to show that this interpretation allows to reconstruct the continuous measurement consistently with the linear structure of quantum mechanics. Let us start by showing that it is possible to go back to the linear stochastic master equation.

5.1.3 The Linear SDE

We define

$$p(t) = \exp\left\{\sum_{j=1}^{m}\left[\int_0^t n_j(s;\rho(s))\mathrm{d}\widehat{B}_j(s) + \frac{1}{2}\int_0^t n_j(s;\rho(s))^2\mathrm{d}s\right]\right\}; \tag{5.12}$$

$1/p(t)$ is a positive $\mathbb{P}$-martingale by Proposition 5.4. Then,

$$\mathbb{Q}^T_{\rho_0}(\mathrm{d}\omega) = p(T,\omega)^{-1}\mathbb{P}(\mathrm{d}\omega)\Big|_{\mathcal{F}_T} \tag{5.13}$$

defines a new probability on $(\Omega, \mathcal{F}_T)$. By the martingale property of $1/p(t)$, Girsanov theorem and Itô formula for a product, we immediately have the following results.

Proposition 5.8. *The probabilities $\mathbb{Q}^t_{\rho_0}$, $t \geq 0$, are consistent. Under the law $\mathbb{Q}^T_{\rho_0}$ the process $B(t)$, $t \in [0,T]$, defined by* (5.11)*, is a multidimensional standard Wiener process. The matrix-valued process*

$$\sigma(t) := p(t)\rho(t), \tag{5.14}$$

under the law $\mathbb{Q}^T_{\rho_0}$, satisfies the linear SDE (3.38):

$$\mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\mathrm{d}t + \sum_{j=1}^{m} \mathcal{R}_j(t)\,[\sigma(t)]\,\mathrm{d}B_j(t). \tag{5.15}$$

5.1.4 Characteristic Functional and Instruments

We can use the analog of Eq. (4.31) to introduce the characteristic operator:

$$\mathcal{G}(t,0;k)[\rho_0] = \mathbb{E}_{\mathbb{P}}\left[\exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_0^t k_j(s)\mathrm{d}B_j(s)\right\}\rho(t)\right]. \tag{5.16}$$

Next proposition says that (5.16) defines a linear map on M_n, that it coincides with the one introduced in Sect. 4.2.3 and, so, that it is really the characteristic operator of some instrument.

Proposition 5.9. *Equation* (5.16) *defines a map on $\mathcal{S}(\mathcal{H})$ which respects convex combinations and, so, which can be extended uniquely to a linear map $\mathcal{G}(t,0;k)$ on M_n. Moreover, $\mathcal{G}(t,0;k)$ satisfies the linear evolution equation of Proposition 4.10.*

Proof. Recall that the generator $\Lambda_t(k)$ is defined by Eq. (4.37). Let us set

$$x(t) = \exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_0^t k_j(s)\mathrm{d}B_j(s)\right\}\rho(t).$$

By Itô formula we get

$$\mathrm{d}\exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_0^t k_j(s)\mathrm{d}B_j(s)\right\} = \exp\left\{\mathrm{i}\sum_{j=1}^{m}\int_0^t k_j(s)\mathrm{d}B_j(s)\right\} \times \sum_i\left[\mathrm{i}k_i(t)\mathrm{d}\widehat{B}_i(t) + \mathrm{i}k_i(t)n_i(t;\rho(t))\mathrm{d}t - \frac{1}{2}k_i(t)^2\mathrm{d}t\right],$$

$$\mathrm{d}x(t) = \Lambda_t\big(k(t)\big)[x(t)]\mathrm{d}t + \sum_j\big(\mathcal{R}_j(t)[x(t)] - n_j(t;\rho(t))x(t) + \mathrm{i}k_j(t)x(t)\big)\mathrm{d}\widehat{B}_j(t).$$

By taking the mean value (which exists, because all the coefficients are bounded due to Assumption 5.1) we get

$$\mathcal{G}(t,0;k)[\rho_0] = \rho_0 + \int_0^t \Lambda_s\big(k(s)\big)\big[\mathcal{G}(s,0;k)[\rho_0]\big]\mathrm{d}s,$$

which is a linear equation and has a unique solution. Then, the statements of the proposition follow. □

By this proposition, the characteristic operator constructed here and the one constructed in the previous chapter coincide. By Remark 4.12 they determine uniquely the same 'finite dimensional instruments' and, by Theorem 4.14, also the "canonical" instruments on the trajectory space of Remark 4.13 coincide. This allows to conclude that the two approaches, the one starting from the linear SDE and the one starting from the nonlinear one, are completely equivalent from the point of view of physical predictions.

By putting together Remarks 4.13, 4.15, Theorem 4.14, Proposition 5.9 and the definition of the characteristic operator $\mathcal{G}$ given in (5.16), we get the expression of the instrument on $\big(C_0^m(0,\infty), \overline{\mathcal{F}}^{\Pi}(0,t)\big)$ in terms of the a posteriori states $\rho(t)$ and the physical probability $\mathbb{P}$.

Proposition 5.10. *Given the family of operators $\mathcal{G}(t,0;k)$, $0 \le t$, $k \in L^2_{\mathrm{loc}}$, defined by Eqs.* (5.16)*, there exists a unique instrument $\widehat{\mathcal{I}}_t^r$ on M_n with value space $\big(\mathcal{Y}, \mathcal{F}^{\Pi}(r,t)\big)$ such that, $\forall\tau \in M_n$,*

$$\mathcal{G}(t,r;k)[\tau] = \int_{\mathcal{Y}} \exp\left\{\mathrm{i}\sum_{j=1}^{m}\Big(\int_r^t k_j(s)\mathrm{d}\Pi_j(s)\Big)(y)\right\}\widehat{\mathcal{I}}_t^r(\mathrm{d}y)[\tau]. \tag{5.17}$$

For every $A \in \overline{\mathcal{F}}^{\Pi}(0,t)$ we define $G_A = \{\omega \in \Omega : (s \mapsto B(s;\omega), s \in [0,t]) \in A\}$. Then, we have $G_A \in \mathcal{F}_t$ and

$$\widehat{\mathcal{I}}_t^0(A)[\rho_0] = \int_{G_A} \rho(t,\omega)\mathbb{P}(\mathrm{d}\omega). \tag{5.18}$$

Proof. The first statement is due to Theorem 4.14 and Proposition 5.9, as already discussed.

By arguments similar to those of Sect. 2.5.4, we get that the σ-algebras generated by the processes $\widehat{B}$ and B up to some time t coincide. Moreover, the map which associates to a continuous process its trajectory is measurable by Proposition A.12. These two facts give $G_A \in \mathcal{F}_t$. By comparing Eqs. (5.16) and (5.17), we get (5.18) for a set A of the type $\left(X_t^0(k^{(1)}), \ldots, X_t^0(k^{(q)})\right) \in B$, $B \in \mathcal{B}(\mathbb{R}^q)$, $X_t^0(k) = \sum_j \int_0^t k_j(s)\mathrm{d}b_j(s)$. This kind of sets and the null sets generate the full σ-algebra $\overline{\mathcal{F}}^{\Pi}(0,t)$ and the equality (5.18) between matrix-valued measures can be extended to it. □

Let us note that we know that $\widehat{\mathcal{I}}_t^0$ is an instrument by Theorem 4.14. In particular $\widehat{\mathcal{I}}_t^0(A)$ is a completely positive, linear map on M_n. This fact is not at all evident from the right hand side of Eq. (5.18), where there is a nonlinear dependence on ρ_0 through $\rho(t)$ and also the set G_A depends on ρ_0 because B is defined by Eq. (5.11), which contains $\rho(s)$ for $s \in [0,t]$.

From the construction of this section we see that to obtain the consistency with the axiomatic formulation of quantum mechanics (linear structure, instruments, etc.) is much easier and clearer by starting from the linear stochastic master equation rather than from the nonlinear one. However, in the literature the stochastic master equation is often taken as a starting point for the quantum trajectory theory, for dynamical reduction theories and for unravelling of quantum dynamical semigroups. So, it is useful to know that it is a solid starting point, because of existence and uniqueness of the solutions and because the interpretation in terms of continuous measurement is always possible in agreement with the axiomatic of quantum mechanics. Another advantage of the stochastic master equation is that it allows to study the asymptotic properties for large times of the a posteriori states, as done in the next section.

5.2 Purification of the A Posteriori States

In the case of a complete measurement, which means $m = d$ in Eqs. (5.1) (cf. Sects. 3.1.2 and 3.3), the equation for the a posteriori states preserves pure states [1] and under suitable conditions tends to purify initial mixed states [2, 3].

5.2.1 Preservation of Pure States

By the whole construction of Chaps. 2 and 3 we already know that a stochastic master equation with $m = d$ is equivalent to a stochastic Schrödinger equation and, so, it preserves pure states. Nevertheless, it is instructive to obtain this result directly from the stochastic master equation itself without the long construction through SDEs in Hilbert spaces.

We recall that in the convex set $\mathcal{S}(\mathcal{H})$ the pure states are the one-dimensional projections, the extreme points of $\mathcal{S}(\mathcal{H})$. A measure of "purity" of a state ρ is the so-called *linear entropy* $\mathrm{Tr}\{\rho(\mathbb{1} - \rho)\}$ [4, p. 82]. This quantity is continuous with

respect to ρ, it always belongs to the interval $[0, 1)$, and it is 0 if and only if ρ is a pure state. In the problem we are studying we shall consider the linear entropy $g(t)$ and the mean linear entropy $G(t)$ of the a posteriori state $\rho(t)$:

$$g(t) := \mathrm{Tr}\{\rho(t)(\mathbb{1} - \rho(t))\} \equiv 1 - \mathrm{Tr}\left\{\rho(t)^2\right\}, \qquad G(t) := \mathbb{E}[g(t)].$$

For every random statistical operator $\rho(t)$ we have $0 \leq G(t) < 1$; moreover, $G(t) = 0$ if and only if $\rho(t)$ is a.s. a pure state. So the study of the behaviour of the linear entropy is a way to analyse whether the stochastic master equation preserves pure states or not.

Proposition 5.11. *Under Assumptions 5.1, 5.7, when $m = d$, Eq.* (5.10) *preserves pure states, in the sense that $\rho(t)$ is a.s. a pure state for every pure initial condition.*

Proof. By applying Itô formula to the mean linear entropy we obtain

$$G(t) - G(0) = -\sum_{j=1}^{d} \int_0^t \mathrm{d}s\, \mathbb{E}\Big[\, \mathrm{Tr}\Big\{\big(R_j(s) + R_j(s)^* - n_j(s;\rho(s))\big) \times \rho(s)\big(R_j(s) + R_j(s)^* - n_j(s;\rho(s))\big)\rho(s)\Big\}\Big]. \quad (5.19)$$

But $\rho(s) \geq 0$ because it is a state; therefore, the right hand side of the equation above is not positive. So, we have $0 \leq G(t) \leq G(0) \leq 1$. If the initial state ρ is pure, we have $G(0) = 0$ and, so $G(t) = 0, \forall t \geq 0$. □

5.2.2 From Mixed to Pure States

Let us take the stochastic master equation (5.10) under the hypothesis $m = d$ to guarantee that the equation preserves pure states; we want to study if it is possible to assure also that (5.10) maps asymptotically mixed states into pure ones [3, 5].

Theorem 5.12 ([2, Theorem 2.1]). *Let Assumptions 5.1, 5.7 holds and let us take $m = d$ and $R_j(t) = \mathrm{e}^{\mathrm{i}\omega_j t} R_j(0)$, $\omega_j \in \mathbb{R}$. If for every time t it does not exist a bi-dimensional projection P_t such that, $\forall j$,*

$$P_t \left(R_j(t) + R_j(t)^*\right) P_t = z_j(t) P_t \quad (5.20)$$

for some numbers $z_j(t)$, then Eq. (5.10) *maps asymptotically, for $t \to \infty$, mixed states into pure ones, in the sense that for every initial condition the linear entropy vanishes for long times:*

$$\lim_{t\to\infty} \mathrm{Tr}\{\rho(t)\,(\mathbb{1} - \rho(t))\} = 0, \qquad \text{a.s.} \quad (5.21)$$

Proof. By applying Itô formula to the linear entropy $g(t)$, we obtain

$$g(t) = g(0) - \int_0^t y(s)\mathrm{d}s$$
$$- 2\sum_{j=1}^{d}\int_0^t \mathrm{Tr}\left\{\rho(s)^2\left(R_j(s) + R_j(s)^* - n_j(s;\rho(s))\right)\right\}\mathrm{d}\widehat{B}_j(s) \quad (5.22)$$

with $y(t) = \sum_{j=1}^{d}\mathrm{Tr}\left\{\left(\sqrt{\rho(t)}\left[R_j(t) + R_j^*(t) - n_j(s;\rho(s))\right]\sqrt{\rho(t)}\right)^2\right\}$.

Apparently, $y(t)$ is non-negative; then, from Eq. (5.22) $g(t)$ is a supermartingale and, being bounded between 0 and 1, it is a.s. convergent to some random variable g_∞. Let us study now the process $y(t)$; if $y(t) = 0$, then, by positivity, we have necessarily

$$\sqrt{\rho(t)}\left[R_j(t) + R_j(t)^* - n_j(t;\rho(t))\right]\sqrt{\rho(t)} = 0, \ \forall j. \quad (5.23)$$

But this would only be possible if an orthogonal projection P_t existed such that (5.20) holds for some numbers $z_j(t)$; then for every $\rho(t)$ such that $\rho(t) = P_t\rho(t)P_t$ Eq. (5.23) holds. The state $\rho(t)$ could be mixed only if P_t was at least bi-dimensional. Then, the condition on the non-existence of a bi-dimensional projection P_t given in the statement of the theorem is a sufficient condition to guarantee $y(t) > 0$ unless $\rho(t)$ be pure.

Let us consider now the mean linear entropy $G(t)$; by bounded convergence we have

$$\lim_{t\to\infty} G(t) = \mathbb{E}\left[g_\infty\right] \quad (5.24)$$

and by Eq. (5.22)

$$0 \le G(t) = G(0) - \int_0^t \mathbb{E}\left[y(s)\right]\mathrm{d}s < 1. \quad (5.25)$$

Moreover, by the previous discussion, $\mathbb{E}\left[y(s)\right] = 0$ only if $\rho(s)$ is a.s. a pure state; but if for a certain s, $\rho(s)$ is a.s. pure, then $G(s) = 0$ and by (5.25) it remains zero for all $t > s$ and the a posteriori state remains pure. Also if $\lim_{t\to\infty} G(t) = 0$, the statement of the theorem follows immediately from (5.24). So it remains to show that we can exclude the case in which $\lim_{t\to\infty} G(t) = K > 0$.

Let us assume by contradiction that $\lim_{t\to\infty} G(t) = K > 0$; this implies, because of the monotonicity of the function $G(t)$, that $\lim_{t\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}G(t) = 0$ and so there exists a sequence of times t_k such that

$$\lim_{k\to\infty}\sum_{j=1}^{d}\mathrm{Tr}\left\{\left(\sqrt{\rho(t_k)}\left[R_j(t_k) + R_j(t_k)^* - n_j(t_k;\rho(t_k))\right]\sqrt{\rho(t_k)}\right)^2\right\} = 0 \quad \text{a.s.} \quad (5.26)$$

We prove that this implies $\lim_{k\to\infty}\mathrm{Tr}\left\{\rho(t_k)\left(\mathbb{1} - \rho(t_k)\right)\right\} = 0$ a.s. Let us set

$$A_j(t) = R_j(t) + R_j(t)^* - n_j(t;\rho(t)); \tag{5.27}$$

then from (5.26) we have that $\forall \varepsilon > 0,\ \exists k_0$ such that $\forall k \geq k_0$

$$\sum_{j=1}^{d} \mathrm{Tr}\left\{\left(\sqrt{\rho(t_k)}A_j(t_k)\sqrt{\rho(t_k)}\right)^2\right\} < \varepsilon. \tag{5.28}$$

Let us denote by λ_t the maximum eigenvalue of $\rho(t)$, and by Q_t a monodimensional projection on a subspace of the eigenspace related to λ_t. Let us recall that we have $\lambda_t \geq \frac{1}{n}$. Let us assume $\lambda_t < 1$, otherwise $\rho(t)$ is already a pure state. We can decompose $\rho(t)$ in the following way:

$$\rho(t) = \lambda_t Q_t + (1-\lambda_t)\rho(t)^{\perp}, \tag{5.29}$$

where

$$\rho(t)^{\perp} = \frac{(\mathbb{1} - Q_t)\rho(t)(\mathbb{1} - Q_t)}{\mathrm{Tr}\left\{(\mathbb{1} - Q_t)\rho(t)(\mathbb{1} - Q_t)\right\}}.$$

By using this decomposition, we deduce from (5.28) that $\forall \varepsilon > 0, \exists k_0$ such that $\forall k \geq k_0$

$$\sum_{j=1}^{d}\left(\lambda_{t_k}^2 \mathrm{Tr}\left\{\left(Q_{t_k}A_j(t_k)Q_{t_k}\right)^2\right\} + \left(1-\lambda_{t_k}\right)^2 \mathrm{Tr}\left\{\left(\sqrt{\rho(t_k)^{\perp}}\,A_j(t_k)\sqrt{\rho(t_k)^{\perp}}\right)^2\right\}\right.$$
$$\left. + 2\lambda_{t_k}\left(1-\lambda_{t_k}\right)\mathrm{Tr}\left\{Q_{t_k}A_j(t_k)\rho(t_k)^{\perp}A_j(t_k)Q_{t_k}\right\}\right) < \varepsilon\,. \tag{5.30}$$

Because of the positivity of the elements in Eq. (5.30) we can say that definitively

$$\sum_{j=1}^{d} \mathrm{Tr}\left\{\left(Q_{t_k}A_j(t_k)Q_{t_k}\right)^2\right\} < \frac{\varepsilon}{\lambda_{t_k}^2} \leq \varepsilon n^2. \tag{5.31}$$

Let us recall that our aim is to prove that $\lim_{t\to\infty}(1-\lambda_t) = 0$. We assume by contradiction that $\exists\, h > 0$ such that definitively $(1-\lambda_t) > h$, so that we have also

$$\sum_{j=1}^{d} \mathrm{Tr}\left\{Q_{t_k}A_j(t_k)\rho(t_k)^{\perp}A_j(t_k)Q_{t_k}\right\} < \frac{\varepsilon}{2\lambda_{t_k}(1-\lambda_{t_k})} \leq \frac{\varepsilon n}{2h}, \tag{5.32}$$

$$\sum_{j=1}^{d} \mathrm{Tr}\left\{\left(\sqrt{\rho(t_k)^{\perp}}\,A_j(t_k)\sqrt{\rho(t_k)^{\perp}}\right)^2\right\} < \frac{\varepsilon}{(1-\lambda_{t_k})^2} \leq \frac{\varepsilon}{h^2}. \tag{5.33}$$

Now, let us denote by μ_t the maximum eigenvalue of $\rho(t)^\perp$ and by S_t, a mono-dimensional projection on a subspace of the eigenspace related to μ_t. With the same argument we deduced (5.31), we can prove from (5.33) that definitively

$$\sum_{j=1}^{d} \mathrm{Tr}\left\{ \left(S_{t_k}, A_j(t_k) S_{t_k} \right)^2 \right\} < \frac{\varepsilon}{h^2 \mu_{t_k}^2} \leq \frac{\varepsilon (n-1)^2}{h^2}. \tag{5.34}$$

From Eq. (5.32), by using the decomposition analogous to (5.29), $\rho(t)^\perp = \mu_t S_t + (1-\mu_t)\widetilde{\rho}(t)^\perp$, we have also that definitively

$$\sum_{j=1}^{d} \mathrm{Tr}\left\{ S_{t_k} A_j(t_k) Q_{t_k} A_j(t_k) S_{t_k} \right\} < \frac{\varepsilon n}{2h\mu_{t_k}} \leq \frac{\varepsilon n(n-1)}{2h}. \tag{5.35}$$

So, we have that definitively

$$\begin{aligned} \sum_{j=1}^{d} \left\| (Q_{t_k} + S_{t_k}) A_j(t_k) (Q_{t_k} + S_{t_k}) \right\|^2 &\leq \sum_{j=1}^{d} \mathrm{Tr}\left\{ \left[(Q_{t_k} + S_{t_k}) A_j(t_k) (Q_{t_k} + S_{t_k}) \right]^2 \right\} \\ &= \sum_{j=1}^{d} \mathrm{Tr}\left\{ (Q_{t_k} A_j(t_k) Q_{t_k})^2 + (S_{t_k} A_j(t_k) S_{t_k})^2 + 2 Q_{t_k} A_j(t_k) S_{t_k} A_j(t_k) \right\} \\ &< \varepsilon \left(n + \frac{n-1}{h} \right)^2, \end{aligned} \tag{5.36}$$

where $Q_t + S_t$ is a bi-dimensional projection. So, by (5.27) and (5.36) for every $\varepsilon > 0$ a bi-dimensional projection $P^{\varepsilon}_{t_k(\varepsilon)}$ and some numbers $z^{\varepsilon}_j(t_k)$ would exist such that

$$\left\| P^{\varepsilon}_{t_k(\varepsilon)} \left(R_j(t_k(\varepsilon)) + R_j(t_k(\varepsilon))^* \right) P^{\varepsilon}_{t_k(\varepsilon)} - z^{\varepsilon}_j(t_k) P^{\varepsilon}_{t_k(\varepsilon)} \right\| < \varepsilon\,, \qquad \forall j.$$

Now, because of the compactness of the subset of bi-dimensional projections on a finite dimensional Hilbert space, and because $z^{\varepsilon}_j(t_k)$ is a bounded sequence of numbers, we can take a sequence $\varepsilon_m \downarrow 0$ such that the subsequences $P^{\varepsilon_m}_{t_k(\varepsilon_m)}$, $z^{\varepsilon_m}_j(t_k)$ are convergent. If the sequence of times $\left\{ t_k(\varepsilon_m) \right\}_m$ is convergent to a limit $\hat{t}$, we obtain a bi-dimensional projection $P_{\hat{t}}$ (limit of the convergent subsequence $P^{\varepsilon_m}_{t_k(\varepsilon_m)}$), a time $\hat{t}$ and some numbers $z_j(\hat{t})$ such that $P_{\hat{t}} \left(R_j(\hat{t}) + R_j(\hat{t})^* \right) P_{\hat{t}} = z_j(\hat{t}) P_{\hat{t}}$, $\forall j$, which is in contradiction with the hypothesis (5.20). If the sequence $t_k(\varepsilon_m)$ is divergent, we can extract a subsequence such that $\mathrm{e}^{\mathrm{i}\omega_j t_{k_j}}$ is convergent to a suitable $\mathrm{e}^{\mathrm{i}\omega_j \theta}$ and this gives us again the contradiction.

So if there exists a sequence of times t_k such that (5.26) holds, then we must have also for this sequence that

$$\lim_{k\to\infty} \mathrm{Tr}\,\{\rho(t_k)\,(\mathbb{1} - \rho(t_k))\} = 0 \qquad \text{a.s.} \tag{5.37}$$

and because $\mathrm{Tr}\,\{\rho(t_k)\,(\mathbb{1} - \rho(t_k))\}$ is a bounded sequence, from (5.37) we can deduce that $\lim_{n\to\infty} \mathbb{E}\,[\mathrm{Tr}\,\{\rho(t_k)\,(\mathbb{1} - \rho(t_k))\}] = 0$. This implies that $\lim\limits_{t\to\infty} G(t)$ cannot be a positive quantity. □

Let us note that (5.20) appears to be a technical hypothesis, needed in the proof of the theorem. However, it has also the meaning that the interaction with the measuring apparatus is strong enough and produces always information on the quantum system. Indeed, this hypothesis excludes the extreme case in which the $R_j(t)$ are pure numbers (no interaction between quantum system and apparatus) or the conservative case of Sect. 2.4.4 in which there is dissipation, but no information on the quantum system.

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Chapter 6
Mutual Entropies and Information Gain in Quantum Continuous Measurements

6.1 Introduction

We already saw that there exist peculiar cases (Sects. 2.4.4, 2.5.2.1) in which no information on the quantum system is extracted by the continuous measurement. Obviously, in other cases we get some information on the system, but the question arises of how to quantify the gain in information. The answer coming out from the whole development of classical and quantum information theory is that this can be obtained by means of entropy-like quantities [1–3].

The typical quantity which measures the gain of information is the *mutual entropy*, which is the relative entropy of a state of a composed system (a bipartite system) with respect to the product of its marginals (the reduced states of the subsystems). Such a system can be a classical one, composed of two, or more, classical subsystems, or a quantum one composed of quantum subsystems. But it can be a system of mixed type, composed of classical and quantum subsystems. This is our case: the quantum subsystem is the observed quantum system and the classical subsystem is the output of the measurement [4–6].

The first step will be to formalise the last statement and to show that our $\sigma(t)$, the solution of the linear stochastic master equation, is the state of a classical/quantum system. All these things need the notions of states on algebras (von Neumann algebras in our case) and of *relative entropy* for such general states [2]. We shall try to keep the theory to a minimum, to give only some hints; as a consequence, this chapter is not fully self-contained and some experience of the reader in quantum information is required.

Another key point will be that any quantum measurement (any instrument) is a *channel*, a completely positive map from the quantum states (density operators) to classical/quantum states (probabilities for the output and post-measurement density operator) [4–6].

Various types of entropies and bounds on informational quantities can be introduced and studied in connection with continuous measurements [7–10]. A possible point of view is one of the transmission of classical information, discussed in Sect. 6.6: the quantum system is the carrier of the information, which is encoded in its initial state; the continuous measurement represents the decoding apparatus [9]. Another point of view is to consider the quantum system in itself, not as a

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transmission channel, and we propose and study a couple of mutual entropies giving two indexes of how good the continuous measurement is in extracting information about the quantum system [10] (see Sects. 6.4 and 6.5).

6.2 States and Entropies in the Discrete Case

In this book we have considered only quantum systems associated to finite dimensional Hilbert spaces. However, the output is a whole trajectory and we were compelled to introduce probabilities on general measurable spaces. Anyway, as a first step let us introduce the new notions in the case in which also the output is discrete and finite. Let $\mathcal{H} = \mathbb{C}^n$ be the Hilbert space of the quantum system and $\mathcal{X} = \{x_1, \ldots, x_N\}$ the space of the possible outcomes of the measurement.

6.2.1 Algebras and States

First, we need to introduce the spaces $C(\mathcal{X}; M_n)$ of the functions from $\mathcal{X}$ into M_n and $C(\mathcal{X}) \equiv C(\mathcal{X}; \mathbb{C})$, which are finite C^*-algebras, as M_n; note that $C(\mathcal{X}; M_n) \simeq C(\mathcal{X}) \otimes M_n$. A state on a finite C^*-algebra is a normalised, positive linear functional on the algebra and in our cases we have

- A state ρ on M_n is identified with a statistical operator, i.e. $\rho \in \mathcal{S}(\mathcal{H})$, and ρ applied to an element a of M_n is given by $\langle \rho, a \rangle = \mathrm{Tr}\{\rho a\}$; this is the usual quantum setup.
- A state p on $C(\mathcal{X})$ is a discrete probability density on $\mathcal{X}$ and $\langle p, a \rangle = \sum_{i=1}^{N} p(x_i)\, a(x_i)$, for $a \in C(\mathcal{X})$; this is the classical setup.
- A state Σ on $C(\mathcal{X}; M_n)$ is itself an element of $C(\mathcal{X}; M_n)$ such that $\Sigma(x) \geq 0$ and $\sum_{i=1}^{N} \mathrm{Tr}\{\Sigma(x_i)\} = 1$; the action of the state Σ on an element $a \in C(\mathcal{X}; M_n)$ is given by $\langle \Sigma, a \rangle = \sum_{i=1}^{N} \mathrm{Tr}\{\Sigma(x_i) a(x_i)\}$. Note the quantum/classical hybrid character of this case.

6.2.2 Entropies and Relative Entropies

Entropies and relative entropies can be defined in very general situations [2]. Entropies and relative entropies are non-negative; the relative entropy can be infinite. In the case of our three C^*-algebras the general definitions reduce to

- For $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$, the entropy is

$$S(\rho_i) = -\,\mathrm{Tr}\{\rho_i \ln \rho_i\} =: S_{\mathrm{q}}(\rho_i) \tag{6.1a}$$

  (the von Neumann entropy), and the relative entropy of ρ_1 with respect to ρ_2 is

$$S(\rho_1 \| \rho_2) = \mathrm{Tr}\{\rho_1(\ln \rho_1 - \ln \rho_2)\} =: S_{\mathrm{q}}(\rho_1 \| \rho_2)\,. \tag{6.1b}$$

- In the classical case, for two states p_1, p_2 on $C(\mathfrak{X})$, the entropy is

$$S(p_i) = -\sum_{j=1}^{N} p_i(x_j) \ln p_i(x_j) =: S_{\mathrm{c}}(p_i) \tag{6.2a}$$

(the Shannon information), and the relative entropy is

$$S(p_1 \| p_2) = \sum_{j=1}^{N} p_1(x_j) \ln \frac{p_1(x_j)}{p_2(x_j)} =: S_{\mathrm{c}}(p_1 \| p_2) \tag{6.2b}$$

(the Kullback–Leibler informational divergence).
- For two states Σ_1, Σ_2 on $C(\mathfrak{X}; M_n)$ we have

$$S(\Sigma_i) = -\sum_{j=1}^{N} \mathrm{Tr}\left\{ \Sigma_i(x_j) \ln \Sigma_i(x_j) \right\} = S_{\mathrm{c}}(p_i) + \sum_{j=1}^{N} p_i(x_j) S_{\mathrm{q}}\big(\pi_i(x_j)\big), \tag{6.3a}$$

$$\begin{aligned} S(\Sigma_1 \| \Sigma_2) &= \sum_{j=1}^{N} \mathrm{Tr}\left\{ \Sigma_1(x_j) \big(\ln \Sigma_1(x_j) - \ln \Sigma_2(x_j) \big) \right\} \\ &= S_{\mathrm{c}}(p_1 || p_2) + \sum_{j=1}^{N} p_1(x_j) S_{\mathrm{q}}\big(\pi_1(x_j) \big\| \pi_2(x_j)\big), \end{aligned} \tag{6.3b}$$

$$p_i(x) := \mathrm{Tr}\left\{ \Sigma_i(x) \right\}, \qquad \pi_i(x) := \frac{\Sigma_i(x)}{p_i(x)}\,. \tag{6.4}$$

In both Eqs. (6.3a) and (6.3b) the first step is by definition and the second one by simple computations; in (6.4), when $p_i(x) = 0$, $\pi_i(x)$ is defined arbitrarily.

In the previous formulae we have used the subscripts "c" for "classical" and "q" for "quantum" to emphasise the cases in which the entropy and the relative entropy are of pure classical character or of pure quantum one. Having used the natural logarithm in these definitions, the entropies are in *nats*. To obtain entropies in *bits* one has to divide by $\ln 2$.

6.2.3 Mutual Entropy and χ-Quantities

In classical information theory a key concept is that of mutual information of two random variables X and Y, which is the relative entropy of the joint distribution p_{XY} with respect to the product of its marginals p_X, p_Y:

$$S_{\mathrm{c}}(p_{XY}\|p_X\otimes p_Y) := \sum_{x,y} p_{XY}(x,y)\ln\frac{p_{XY}(x,y)}{p_X(x)p_Y(y)}$$
$$\equiv \sum_x p_X(x)\, S_{\mathrm{c}}(p_{Y|X}(\bullet|x)\|p_Y)$$
$$\equiv \sum_y p_Y(y)\, S_{\mathrm{c}}(p_{X|Y}(\bullet|y)\|p_X)\,, \tag{6.5}$$

$$p_X(x) := \sum_y p_{XY}(x,y), \qquad p_Y(y) := \sum_x p_{XY}(x,y), \tag{6.6a}$$

$$p_{Y|X}(y|x) := \frac{p_{XY}(x,y)}{p_X(x)}\,, \qquad p_{X|Y}(y|x) := \frac{p_{XY}(x,y)}{p_Y(y)}\,. \tag{6.6b}$$

The idea of mutual information can be generalised to all the situations when one has states on a tensor product of algebras. Let C_i, $i = 1,2$ be two finite C^*-algebras; let Π_{12} be a state on $C_1\otimes C_2$. The *marginals* Π_i are the restrictions of Π_{12} to the two factors in the tensor product: $\Pi_i := \Pi_{12}\big|_{C_i}$. More explicitly, Π_1 is the state on C_1 defined by $\langle\Pi_1, A\rangle = \langle\Pi_{12}, A\otimes\mathbb{1}\rangle$, $\forall A\in C_1$, and, similarly, $\langle\Pi_2, A\rangle = \langle\Pi_{12}, \mathbb{1}\otimes B\rangle$, $\forall B\in C_2$. Then, the *mutual information* or the *mutual entropy* of the joint state Π_{12} is its relative entropy with respect to the tensor product of its marginals: $S(\Pi_{12}\|\Pi_1\otimes\Pi_2)$.

For instance, in the case $C_1 = C(\mathcal{X})$, $C_2 = M_n$, a state Σ on $C_1\otimes C_2\simeq C(\mathcal{X};M_n)$ has marginals p and $\overline{\pi} := \sum_{j=1}^N \Sigma(x_j) = \sum_{j=1}^N p(x_j)\pi(x_j)$, where $p(x)$ and $\pi(x)$ are defined as in Eq. (6.4). Then, by Eq. (6.3b) the mutual entropy of Σ with respect to this factorisation is

$$S(\Sigma\|p\otimes\overline{\pi}) = \sum_{j=1}^N p(x_j)S_{\mathrm{q}}\big(\pi(x_j)\big\|\overline{\pi}\big) \equiv S_{\mathrm{q}}(\overline{\pi}) - \sum_{j=1} p(x_j)S_{\mathrm{q}}\big(\pi(x_j)\big). \tag{6.7}$$

In quantum information theory, a couple $\{p,\pi\}$ of a probability p (let us say on the set $\mathcal{X}$) and a family of statistical operators $\pi(x)$ is known as an *ensemble* and

$$\overline{\pi} = \sum_{j=1}^N p(x_j)\pi(x_j) \tag{6.8}$$

is the *average* state of the ensemble. Trivially, the ensemble $\{p,\pi\}$ is equivalent to the state $\Sigma = \{p(x)\pi(x)\}$ on $C(\mathcal{X};M_n)$; the mutual entropy of this state is a key object in quantum information and quantum statistics and it is called the Holevo's χ*-quantity* of the ensemble [3, p. 531]:

$$\chi\{p,\pi\} := \sum_{j=1}^N p(x_j)S_{\mathrm{q}}\big(\pi(x_j)\big\|\overline{\pi}\big) = S(\Sigma\|p\otimes\overline{\pi}). \tag{6.9}$$

Note that a χ-quantity is not negative by the non-negativity of relative entropies.

6.3 The Continuous Measurement as a Channel

Now we go back to the case of the continuous measurements considered in this book.

6.3.1 von Neumann Algebras and Normal States

The Hilbert space is finite dimensional, $\mathcal{H} = \mathbb{C}^n$, and the quantum algebra is again M_n as in the previous section. In the finite dimensional case, M_n has a double role. First, M_n is the linear span of the states, $\mathcal{S}(\mathcal{H}) \subset M_n$; in this role the relevant norm is the trace-norm $\|\bullet\|_1$. But also the usual observables, represented by self-adjoint operators, and the effects are contained in M_n and they span it by linear combinations; now the relevant norm is the operator norm $\|\bullet\|_\infty \equiv \|\bullet\|$.

What we have to change is the "classical algebra". The space of the possible outputs, up to time t, is the measurable space $\left(\Omega, \overline{\mathcal{G}}_t^0\right)$ on which a reference probability measure $\mathbb{Q}$ is given; moreover, all the physical probabilities have a density with respect to $\mathbb{Q}$. The possible outcomes of the measurement are the points of Ω and we can process such outcomes as we want, so that we can think the functions on Ω as the possible observables. In the role of "space of the observables", the algebra $C(\mathcal{X})$ has to be substituted by $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right)$, which is a commutative von Neumann algebra. Then, we have to consider together classical output and quantum system and we have to replace $C(\mathcal{X}; M_n)$ by $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}; M_n\right) \simeq L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right) \otimes M_n$, which is again a von Neumann algebra, but no longer commutative.

Now, notions from the theory of von Neumann algebras become relevant; we recall only a few facts and notions in our concrete cases, without mathematical explanations and proofs, and we suggest the book [2] to the interested reader. Just to recall the terminology, a von Neumann algebra or W^*-algebra $\mathcal{M}$ is a C^*-algebra which is the topological dual of a Banach space, which is called the predual $\mathcal{M}_*$ of $\mathcal{M}$. A state on $\mathcal{M}$ is a continuous, positive, normalised functional on $\mathcal{M}$; this state is called *normal* if it enjoys some more restrictive continuity properties (w^*-continuity) or, equivalently, if it can be represented by an element of $\mathcal{M}_*$.

A normal state on $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right)$ is simply a probability density with respect to $\mathbb{Q}$ and the linear span of such densities is $L^1\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right)$, which substitutes $C(\mathcal{X})$ in the role of the space where the states live. The space $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right)$ is the topological dual of $L^1\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right)$.

Similarly, a normal state Σ on $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}; M_n\right)$ is a matrix-valued density: Σ is a measurable M_n-valued function on Ω (or, better, an equivalence class of functions) such that $\Sigma(\omega) \geq 0$ and $\int_\Omega \mathrm{Tr}\{\Sigma(\omega)\}\,\mathbb{Q}(\mathrm{d}\omega) = 1$. Therefore, the function

$$p(\omega) := \mathrm{Tr}\{\Sigma(\omega)\}$$

is a probability density with respect to $\mathbb{Q}(\mathrm{d}\omega)$, or a state on $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right)$, and

$$\pi(\omega) := p(\omega)^{-1}\Sigma(\omega)$$

is a state on M_n (it belongs to $\mathcal{S}(\mathcal{H})$). We identify Σ with the ensemble $\{p,\ \pi\}$. The quantum state

$$\int_\Omega \mathbb{Q}(\mathrm{d}\omega)\,\overline{\pi} := p(\omega)\pi(\omega)$$

is called the *average state* of the *ensemble* $\Sigma \equiv \{p,\ \pi\}$.

Remark 6.1. Let us consider the solution $\sigma(t)$ of the linear stochastic master equation at a fixed time t. By the discussion above, $\sigma(t)$ is a normal state on the von Neumann algebra $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}; M_n\right) \simeq L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right) \otimes M_n$ (a bipartite classical/quantum system) and it is identified with the ensemble $\{p(t), \rho(t)\}$ (physical probability density and a posteriori states) whose average state is the a priori state $\eta(t)$. The density $p(t)$ and the a priori state $\eta(t)$ are the classical and the quantum *marginals* of the state $\sigma(t)$ on the composed algebra $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right) \otimes M_n$.

6.3.2 Entropies

For what concerns the quantum system, nothing is changed and the definitions of von Neumann entropy and quantum relative entropy are always given by Eqs. (6.1).

However, in the classical continuous case there is no natural analog of the Shannon information (6.2a). For states on general von Neumann algebras the key notion is that of relative entropy. Here we give the definition only in the cases of our interest.

Let p_1 and p_2 be two normal states on $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}\right)$, i.e. two $\mathbb{Q}$-densities. The definition of relative entropy, or Kullback–Leibler divergence, of p_1 with respect to p_2 is similar to (6.2b) and it is given by

$$S_{\mathrm{c}}(p_1 \| p_2) = \int_\Omega p_1(\omega) \ln \frac{p_1(\omega)}{p_2(\omega)}\, \mathbb{Q}(\mathrm{d}\omega). \tag{6.10}$$

As in the discrete case the relative entropy is non-negative and can take the value $+\infty$.

In the case of the composed system with algebra $L^\infty\left(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}; M_n\right)$ let us consider two states $\Sigma_i = \{p_i, \pi_i\}$, $i = 1, 2$. The definition of relative entropy of Σ_1 with respect to Σ_2, analogous to (6.3b), is

$$S(\Sigma_1 \| \Sigma_2) = \int_\Omega \mathrm{Tr}\left\{\Sigma_1(\omega)\big(\ln \Sigma_1(\omega) - \ln \Sigma_2(\omega)\big)\right\} \mathbb{Q}(\mathrm{d}\omega)$$
$$= S_{\mathrm{c}}(p_1 \| p_2) + \int_\Omega p_1(\omega) S_{\mathrm{q}}\big(\pi_1(\omega) \big\| \pi_2(\omega)\big)\, \mathbb{Q}(\mathrm{d}\omega). \qquad (6.11)$$

Let us now consider the ensemble $\Sigma = \{p, \pi\}$ with average state $\overline{\pi}$. Its marginals on the algebras $L^\infty(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q})$ and M_n are p and $\overline{\pi}$, respectively. In analogy with Eq. (6.7), we can consider the relative entropy of Σ with respect to the product of its marginals and we get the mutual entropy

$$S(\Sigma \| p\, \overline{\pi}) = \int_\Omega p(\omega) S_{\mathrm{q}}\big(\pi(\omega) \| \overline{\pi}\big) \mathbb{Q}(\mathrm{d}\omega)$$
$$= S_{\mathrm{q}}(\overline{\pi}) - \int_\Omega p(\omega) S_{\mathrm{q}}\big(\pi(\omega)\big) \mathbb{Q}(\mathrm{d}\omega) = \chi\{p, \pi\}. \qquad (6.12)$$

As already seen in Sect. 6.2.3 in the discrete case, Eq. (6.12) gives the χ-quantity of the ensemble of states $\{p, \pi\}$.

6.3.3 Channels

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two von Neumann algebras. A linear map Λ from the predual $\mathcal{M}_{1*}$ to the predual $\mathcal{M}_{2*}$ is said to be a (quantum) *channel* if it is completely positive and sends states into states (normalisation) [2, p. 137]. The composition of channels gives again a channel. Channels are usually introduced to describe noisy quantum evolutions, but also an instrument can be identified with a channel, as we shall see in the case of continuous measurements.

For instance, a channel from M_n to itself is a completely positive linear map which transforms statistical operators into statistical operators and, so, it is the same as the dynamical map defined in Sect. B.3.3. On the other side, a channel from $L^1(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q})$ into itself is simply a linear map which sends densities into densities (then, in this case, complete positivity is automatic).

A key result which follows from the convexity properties of the relative entropy is *Uhlmann monotonicity theorem* [2, Theorem 1.5 p. 21], which implies that channels decrease the relative entropy.

Theorem 6.2. *If Σ and Π are two normal states on $\mathcal{M}_1$ and Λ is a channel from $\mathcal{M}_{1*} \to \mathcal{M}_{2*}$, then $S(\Sigma \| \Pi) \geq S(\Lambda[\Sigma] \| \Lambda[\Pi])$.*

6.3.3.1 The Channels Associated to the Continuous Measurement

Let us consider the stochastic propagator $\mathcal{A}(t, 0)$ defined by Eq. (3.42); its properties are given in Theorem 3.4. Let us recall that $\mathcal{A}(t, 0; \omega)$ is a completely positive linear map on M_n for every ω. As a function of ω, it can be seen as a linear map from M_n into $L^1(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}; M_n)$, which transforms an initial condition $\rho_0 \in \mathcal{S}(\mathcal{H})$ into the

state $\sigma(t) \in L^1(\Omega, \overline{\mathcal{G}}_t^0, \mathbb{Q}; M_n)$. It can be shown that $\mathcal{A}(t,0)$ is completely positive also as a map with range in the enlarged space; therefore, it is a channel. The same is true, with the obvious changes, for $\mathcal{A}(t,s)$.

Remark 6.3. For $t \geq s \geq 0$, the map $\mathcal{A}(t,s) : M_n \to L^1\big(\Omega, \overline{\mathcal{G}}_t^s, \mathbb{Q}; M_n\big)$ is a channel.

In Chap. 4 (Definition 4.1) we have seen that the channel $\mathcal{A}(t,s)$ defines the instrument $\mathcal{I}_t^s$. It is a general result that any instrument is equivalent to a channel from quantum states to classical/quantum states [4–6].

6.4 A Classical Continuous Information Gain

The aim of this section is to introduce a measure of the effectiveness of the continuous measurement in extracting *classical* information on the quantum system under measurement. Classical information is measured by relative or mutual entropies for probability densities. So, our idea is to introduce a reference probability with density $q(t)$ with respect to $\mathbb{Q}$ such that the Kullback–Leibler divergence

$$S_{\mathrm{c}}\big(p(t)\|q(t)\big) = \mathbb{E}^t_{\rho_0}\left[\ln \frac{p(t)}{q(t)}\right] \tag{6.13}$$

is a good measure of the classical information gain in the time interval $(0,t)$. We want to show that a good choice for the reference density $q(t)$ is

$$q(t) = \exp\left\{\sum_j \left[\int_0^t n_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t n_j(s)^2\,\mathrm{d}s\right]\right\}, \tag{6.14}$$

where the $n_j(t)$ are the a priori quantum means

$$n_j(t) = \mathrm{Tr}\left\{\big(R_j(t) + R_j(t)^*\big)\,\eta(t)\right\} = \mathbb{E}^t_{\rho_0}\big[v_j(t)\big]. \tag{6.15}$$

Let us note that the functions n_j are deterministic. By Girsanov theorem, under $q(T,\omega)\mathbb{Q}(\mathrm{d}\omega)$ the processes W_j have independent increments, as it is under $\mathbb{Q}$ (so, they can be interpreted as noises), but the means have been changed and made equal to the means they have under $\mathbb{P}^T_{\rho_0}$. Precisely, the processes $W_j(t) - \int_0^t n_j(s)\,\mathrm{d}s$ are independent, standard Wiener processes.

We want to show also that, in some sense, $q(t,\omega)\mathbb{Q}(\mathrm{d}\omega)$ is a continuous product of marginals of $\mathbb{P}^t_{\rho_0}(\mathrm{d}\omega) = p(t,\omega)\mathbb{Q}(\mathrm{d}\omega)$ and, so, the classical relative entropy $S_{\mathrm{c}}\big(p(t)\|q(t)\big)$ can be considered as a mutual entropy of $\mathbb{P}^t_{\rho_0}$ with respect to a continuous factorisation.

6.4.1 Product Densities

Let us consider any time s in the time interval $(0,t)$ and let us decompose the von Neumann algebra $L^\infty(\Omega,\overline{\mathcal{G}}_t^0,\mathbb{Q})$ as $L^\infty(\Omega,\overline{\mathcal{G}}_t^0,\mathbb{Q}) = L^\infty(\Omega,\overline{\mathcal{G}}_s^0,\mathbb{Q})\otimes L^\infty(\Omega,\overline{\mathcal{G}}_t^s,\mathbb{Q})$. Now, the density $p(t)$ can be seen as a state on $L^\infty(\Omega,\overline{\mathcal{G}}_t^0,\mathbb{Q})$ and we can consider its marginals $p(0,s)$ and $p(s,t)$ on the two factors $L^\infty(\Omega,\overline{\mathcal{G}}_s^0,\mathbb{Q})$ and $L^\infty(\Omega,\overline{\mathcal{G}}_t^s,\mathbb{Q})$, respectively. These marginals are given by

$$p(0,s)=\mathbb{E}_{\mathbb{Q}}\left[p(t)\middle|\overline{\mathcal{G}}_s^0\right],\qquad p(s,t)=\mathbb{E}_{\mathbb{Q}}\left[p(t)\middle|\overline{\mathcal{G}}_t^s\right]. \tag{6.16}$$

By using the fact that $\{p(t),\ t\geq 0\}$ is a martingale and by taking the trace of Eq. (3.56), we get

$$p(0,s)=p(s)\,,\qquad p(s,t)=\mathrm{Tr}\{\mathcal{A}(t,s)[\eta(s)]\}\,. \tag{6.17}$$

By comparing the last equality with $p(t)=\mathrm{Tr}\{\sigma(t)\}=\mathrm{Tr}\{\mathcal{A}(t,0)[\eta(0)]\}$, we see that $p(s,t)$ is similar to $p(t)$, but with s as initial time, instead of 0, and with $\eta(s)$ as initial state, instead of $\eta(0)$. By this remark and Eq. (3.44), we get

$$p(s,t)=\exp\left\{\sum_j\left[\int_s^t v_j(u,s)\,\mathrm{d}W_j(u)-\frac{1}{2}\int_s^t v_j(u,s)^2\,\mathrm{d}u\right]\right\}, \tag{6.18}$$

where

$$v_j(t,s)=\mathrm{Tr}\left\{\left(R_j(t)+R_j(t)^*\right)\rho(t,s)\right\},\qquad \rho(t,s)=\frac{1}{p(s,t)}\,\mathcal{A}(t,s)[\eta(s)]. \tag{6.19}$$

The random state $\rho(t,s)$ is the a posteriori state for the instrument $\mathcal{I}_t^s$ and the pre-measurement state $\eta(s)$. Note that $v_j(t,0)=v_j(t)$, already defined in (3.45).

Then, we can consider the mutual entropy $S_{\mathrm{c}}\big(p(t)\|p(0,s)p(s,t)\big)$. But the significance of this quantity is dubious, because the time s is completely arbitrary and, moreover, we could divide the time interval into more pieces. For instance, we can take the decomposition $L^\infty(\Omega,\overline{\mathcal{G}}_t^0,\mathbb{Q}) = L^\infty(\Omega,\overline{\mathcal{G}}_r^0,\mathbb{Q})\otimes L^\infty(\Omega,\overline{\mathcal{G}}_s^r,\mathbb{Q})\otimes L^\infty(\Omega,\overline{\mathcal{G}}_t^s,\mathbb{Q})$ and we recognise that $p(0,r)p(r,s)p(s,t)$ is the product of the marginals of $p(t)$ related to this decomposition. Taking a finer generic partition of $(0,t)$ with $t_0=0$ and $t_n=t$, we recognise that $\prod_{j=1}^n p(t_{j-1},t_j)$ is again a product of marginals of $p(t)$. To eliminate arbitrariness, let us consider finer and finer partitions and let us go to a continuous product of marginals.

Let us note that we have $\lim_{s\uparrow t} v_j(t,s)=n_j(t)$, a.s. Then, for an infinitesimal interval we get

$$p(s, s + \mathrm{d}s) = \exp\left\{\sum_j \left[n_j(s)\,\mathrm{d}W_j(s) - \frac{1}{2}\,n_j(s)^2\mathrm{d}s\right]\right\} \tag{6.20}$$

and, so, the density $q(t)$ (6.14) is the continuous product of marginals of $p(t)$.

The fact that it is possible to consider a "continuous product of marginals" is not so unexpected; indeed, the theory of continuous measurements is connected to *infinite divisibility* [11–15].

We have already seen that the marginals of $p(t)$ with respect to the decomposition of the time interval $(0, t)$ into $(0, s)$ and (s, t) are $p(0, s) = p(s)$ and $p(s, t)$ given by Eq. (6.18). The analogous marginals for $q(t)$ are $q(0, s) = q(s)$ and

$$q(s, t) = \exp\left\{\sum_j \left[\int_s^t n_j(u)\,\mathrm{d}W_j(u) - \frac{1}{2}\int_s^t n_j(u)^2\,\mathrm{d}u\right]\right\} = \frac{q(t)}{q(s)}\,. \tag{6.21}$$

6.4.2 The Classical Information Gain

The density $q(t)$ is no more dependent on some arbitrary choice of intermediate times and the measure $q(T, \omega)Q(\mathrm{d}\omega)$ has a distinguished role and can be considered as a reference measure. So, we can introduce the relative entropy (6.13). The density $q(t)$ is a product of marginals of $p(t)$, even the finest product of marginals; then, the quantity (6.13) is a mutual entropy and can be taken as a measure of classical information on the measured system extracted in the time interval $(0, t)$. If we assume that the initial state and the generator of the master equation are completely known, then $q(t)$ is known a priori and we can interpret $S_{\mathrm{c}}\big(p(t)\|q(t)\big)$ as the measure of the mean classical information that we gain when we perform the random experiment described by our continuous measurement and we register the trajectory of the output up to time t. Other reasons can be given to reinforce the interpretation of measure of the information extracted from the observed system.

Proposition 6.4. *The information gain* (6.13) *enjoys the properties: for* $0 \le s \le t$

$$S_{\mathrm{c}}\big(p(t)\|q(t)\big) = S_{\mathrm{c}}\big(p(s)\|q(s)\big) + S_{\mathrm{c}}\big(p(t)\|\,p(s)q(s, t)\big), \tag{6.22}$$

$$0 \le S_{\mathrm{c}}\big(p(s)\|q(s)\big) \le S_{\mathrm{c}}\big(p(t)\|q(t)\big), \tag{6.23}$$

$$S_{\mathrm{c}}\big(p(t)\|\,p(s)q(s, t)\big) = \mathbb{E}_{\mathbb{Q}}\left[p(s)\,\mathbb{E}_{\mathbb{Q}}\left[\frac{p(t)}{p(s)}\ln\frac{p(t)/p(s)}{q(t)/q(s)}\,\middle|\,\overline{\mathcal{G}}_s^0\right]\right]. \tag{6.24}$$

Proof. By Eqs. (3.44), (6.18), (6.14), (6.21) we have $p(0, t) = p(t)$, $q(0, t) = q(t)$, $q(u) = q(t)q(t, u)$, for $0 \le t \le u$. By direct computation of the difference $S_{\mathrm{c}}\big(p(t)\|q(t)\big) - S_{\mathrm{c}}\big(p(s)\|q(s)\big)$ we get (6.22). Then, by the positivity of relative entropies, this equation gives the inequalities (6.23).

By (6.13) and (6.21) we have

$$S_{\mathrm{c}}\big(p(t)\|p(s)q(s,t)\big) = \mathbb{E}_{\mathbb{Q}}\left[p(t)\ln\frac{p(t)}{p(s)q(s,t)}\right] = \mathbb{E}_{\mathbb{Q}}\left[p(t)\ln\frac{p(t)/p(s)}{q(t)/q(s)}\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[p(s)\,\frac{p(t)}{p(s)}\,\ln\frac{p(t)/p(s)}{q(t)/q(s)}\right] = \mathbb{E}_{\mathbb{Q}}\left[p(s)\,\mathbb{E}_{\mathbb{Q}}\left[\frac{p(t)}{p(s)}\ln\frac{p(t)/p(s)}{q(t)/q(s)}\,\middle|\,\overline{\mathcal{G}}_s^0\right]\right].$$

□

Equation (6.23) says that $S_{\mathrm{c}}\big(p(t)\|q(t)\big)$ is non-negative and not decreasing in time, as should be for a measure of an information gain in time. Note that we have also $S_{\mathrm{c}}\big(p(0)\|q(0)\big) = 0$.

The expression (6.24) can be interpreted as a *conditional relative entropy* [1, pp. 22–23]. The quantity $\mathbb{E}_{\mathbb{Q}}\left[\frac{p(t)}{p(s)}\ln\frac{p(t)/p(s)}{q(t)/q(s)}\,\middle|\,\overline{\mathcal{G}}_s^0\right]$ has the same structure as $S_{\mathrm{c}}\big(p(t)\|q(t)\big)$, but it refers to the interval (s,t) and it is constructed with the conditional densities. We can say that Eq. (6.22) expresses in a consistent way a kind of "additivity property" of our measure of information.

Having the explicit exponential forms of the densities $p(t)$ and $q(t)$, we can compute the explicit expression of the information gain.

Proposition 6.5. *The explicit expression of the classical mutual entropy* (6.13) *is*

$$S_{\mathrm{c}}\big(p(t)\|q(t)\big) = \frac{1}{2}\sum_j \int_0^t \mathrm{Var}_{\mathbb{P}^t_{\rho_0}}[v_j(s)]\mathrm{d}s. \tag{6.25}$$

Proof. By Eqs. (3.44) and (6.14) we get

$$\ln\frac{p(t)}{q(t)} = \sum_j\left[\int_0^t \big(v_j(s)-n_j(s)\big)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t \big(v_j(s)^2 - n_j(s)^2\big)\mathrm{d}s\right]$$
$$= \sum_j\left[\int_0^t \big(v_j(s)-n_j(s)\big)\big(\mathrm{d}W_j(s) - v_j(s)\mathrm{d}s\big) + \frac{1}{2}\int_0^t \big(v_j(s)-n_j(s)\big)^2\,\mathrm{d}s\right]$$
$$= \sum_j\left[\int_0^t \big(v_j(s)-n_j(s)\big)\mathrm{d}\widehat{W}_j(s) + \frac{1}{2}\int_0^t \big(v_j(s)-n_j(s)\big)^2\,\mathrm{d}s\right].$$

The first term has zero mean under $\mathbb{P}^T_{\rho_0}$ (or under $\mathbb{P}^t_{\rho_0}$, by consistency). Therefore, Eq. (6.25) follows by taking the $\mathbb{P}^t_{\rho_0}$-mean of $\ln p(t)/q(t)$ and by taking into account Eq. (6.15). □

Remark 6.6.

1. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,S_{\mathrm{c}}\big(p(t)\|q(t)\big) = \frac{1}{2}\sum_j \mathrm{Var}_{\mathbb{P}^t_{\rho_0}}[v_j(t)] \geq 0. \tag{6.26}$$

The positivity of this time derivative follows also from Eq. (6.23).

2. By the properties of relative entropy, $\mathbb{E}^T_{\rho_0}\big[S_{\rm q}\big(\rho(t)\|\eta(t)\big)\big] = 0$ is equivalent to $\rho(t) = \eta(t)$, $\mathbb{P}^T_{\rho_0}$-a.s. By Eqs. (6.15), (6.26), this last relation implies the vanishing of the quantity (6.26). So, we have

$$\mathbb{E}^T_{\rho_0}\Big[S_{\rm q}\big(\rho(t)\|\eta(t)\big)\Big] = 0 \quad \Rightarrow \quad \frac{\rm d}{{\rm d}t}\, S_{\rm c}\big(p(t)\|q(t)\big) = 0\,. \tag{6.27}$$

3. From the definition of $v_j(t)$ (3.45) we see that, if $R_j(t) + R_j(t)^* \propto \mathbb{1}$, then $\mathrm{Var}_{\mathbb{P}^t_{\rho_0}}[v_j(t)] = 0$. This says that in this case no information is extracted from the system by the measurement in channel j, whatever the initial state is.

Other measures of information gain can be proposed, depending on what is considered unknown about the system. For instance in [16] the situation of a dynamics with unknown parameters is considered and some measures are proposed and studied for the information gain on these parameters obtained by various schemes of continuous observation.

6.4.3 An Upper Bound on the Increments of the Classical Information gain

Lemma 6.7. *For $0 \le t \le u$, we have the bound*

$$0 \le S_{\rm c}\big(p(u)\|\,p(t)p(t,u)\big) \le \mathbb{E}^u_{\rho_0}\left[S_{\rm q}\big(\rho(t)\|\eta(t)\big) - S_{\rm q}\big(\rho(u)\|\rho(u,t)\big)\right], \tag{6.28}$$

and the identity

$$S_{\rm c}\big(p(u)\|q(u)\big) - S_{\rm c}\big(p(t)\|q(t)\big) = S_{\rm c}\big(p(u)\|\,p(t)p(t,u)\big) + S_{\rm c}\big(p(t,u)\|q(t,u)\big). \tag{6.29}$$

Proof. Consider the relative entropy $S\big(\sigma(t)\|\,p(t)\eta(t)\big)$, which will be studied in Section 6.5, and apply to both states the channel $\mathcal{A}(u,t)$. By Theorem 6.2 and Definition (6.19) we get the inequality

$$\begin{aligned}
\mathbb{E}^t_{\rho_0}\left[S_{\rm q}\big(\rho(t)\|\eta(t)\big)\right] &= S\big(\sigma(t)\|\,p(t)\eta(t)\big) \ge S\left(\mathcal{A}(u,t)[\sigma(t)]\|\mathcal{A}(u,t)[p(t)\eta(t)]\right)\\
&= S\big(p(u)\rho(u)\|\,p(t)p(t,u)\rho(u,t)\big)\\
&= \mathbb{E}^u_{\rho_0}\left[\mathrm{Tr}\left\{\rho(u)\left(\ln p(u) + \ln\rho(u) - \ln\big(p(t)p(t,u)\big) - \ln\rho(u,t)\right)\right\}\right]\\
&= S_{\rm c}\big(p(u)\|\,p(t)p(t,u)\big) + \mathbb{E}^u_{\rho_0}\left[S_{\rm q}\big(\rho(u)\|\rho(u,t)\big)\right],
\end{aligned}$$

and this gives (6.28).

By Eqs. (3.44), (6.18), (6.14), (6.21) and explicit computation we get (6.29). □

Theorem 6.8 (The bound on the derivative of $S_{\rm c}\big(p(t)\|q(t)\big)$). *The following bound holds, for every t for which the derivatives in the right hand side exist:*

$$0 \le \frac{\mathrm{d}}{\mathrm{d}t}\, S_{\mathrm{c}}\big(p(t)\|q(t)\big) \le -\frac{\mathrm{d}}{\mathrm{d}u}\, \mathbb{E}^T_{\rho_0}\left[S_{\mathrm{q}}\big(\rho(u)\|\rho(u,t)\big)\right]\Big|_{u=t^+}$$
$$\equiv \frac{\mathrm{d}}{\mathrm{d}t}\, \mathbb{E}^T_{\rho_0}\left[S_{\mathrm{q}}\big(\rho(t)\big)\right] - \frac{\mathrm{d}}{\mathrm{d}u}\, \mathbb{E}^T_{\rho_0}\left[S_{\mathrm{q}}\big(\rho(u,t)\big)\right]\Big|_{u=t^+}. \quad (6.30)$$

Proof. From Eqs. (6.20) and (6.21) we get immediately

$$\lim_{u\downarrow t} \frac{S_{\mathrm{c}}\big(p(t,u)\|q(t,u)\big)}{u-t} = 0\,.$$

Then, the second summand in the expression (6.29) of the increment of information becomes negligible with respect to the first when $u \downarrow t$. Therefore, from (6.28) we obtain (6.30). □

What is important in (6.30) is not the precise form of the bound, but the fact that the rate of a classical information $\frac{\mathrm{d}}{\mathrm{d}t}\, S_{\mathrm{c}}\big(p(t)\|q(t)\big)$ is bounded by the quantum quantity in the right hand side. The output is a classical stochastic quantity, the information we gain in observing the output is measured by a classical mutual entropy, but after all the observed system is a quantum one and its "quantumness" poses bounds on the quantity of information that can be extracted.

Remark 6.9. We already saw in Remark 6.6 that $\mathbb{E}^T_{\rho_0}\left[S_{\mathrm{q}}\big(\rho(t)\|\eta(t)\big)\right] = 0$ is equivalent to $\rho(t) = \eta(t)$, $\mathbb{P}^T_{\rho_0}$-a.s.; but this implies $\rho(u) = \rho(u,t), \forall u \ge t$, $\mathbb{P}^T_{\rho_0}$-a.s., because in this case these two stochastic processes, which satisfy the same equation, have the same initial condition at time t. Therefore we have $\mathbb{E}^T_{\rho_0}\left[S_{\mathrm{q}}\big(\rho(u)\|\rho(u,t)\big)\right] = 0$, $\forall u \ge t$, and

$$\mathbb{E}^T_{\rho_0}\left[S_{\mathrm{q}}\big(\rho(t)\|\eta(t)\big)\right] = 0 \;\Rightarrow\; -\frac{\mathrm{d}}{\mathrm{d}u}\, \mathbb{E}^T_{\rho_0}\left[S_{\mathrm{q}}\big(\rho(u)\|\rho(u,t)\big)\right]\Big|_{u=t^+} = 0. \quad (6.31)$$

6.5 The Information Embedded in the A Posteriori States

Another interesting measure of information is the relative entropy of the ensemble $\sigma(t) = \{p(t),\, \rho(t)\}$ with respect to the product of its marginals $p(t)$ and $\eta(t)$; it turns out to be

$$S\big(\sigma(t)\|p(t)\eta(t)\big) = \int_\Omega \mathrm{Tr}\left\{\sigma(t,\omega)\big(\ln\sigma(t,\omega) - \ln p(t,\omega)\eta(t)\big)\right\}\mathbb{Q}(\mathrm{d}\omega)$$
$$= \mathbb{E}^t_{\rho_0}\big[S_{\mathrm{q}}\big(\rho(t)\|\eta(t)\big)\big] = S_{\mathrm{q}}\big(\eta(t)\big) - \mathbb{E}^t_{\rho_0}\big[S_{\mathrm{q}}\big(\rho(t)\big)\big]$$
$$= \chi\{p(t),\rho(t)\}. \quad (6.32)$$

According to the terminology introduced in Sect. 6.2.3, it is a mutual entropy of mixed classical/quantum type and it is also the Holevo's χ-quantity of the ensemble $\sigma(t) \equiv \{p(t),\rho(t)\}$.

This mutual entropy is a sort of quantum information embedded by the measurement in the a posteriori states. When the measurement is not informative, we have $\rho(t,\omega)=\eta(t)$ and $S\big(\sigma(t)\|p(t)\eta(t)\big)=0$. It is zero also if for any reason it happens that $\eta(t)$ is a pure state. For instance, if $\mathcal{T}(t,0)$ has a unique equilibrium state which is pure, then $\lim_{t\to+\infty} S\big(\sigma(t)\|p(t)\eta(t)\big)=0$ even if the measurement is "good".

Let us note that from Eq. (6.32) we have the bound

$$S\big(\sigma(t)\|p(t)\eta(t)\big)\le S_{\mathrm{q}}\big(\eta(t)\big). \tag{6.33}$$

When the von Neumann entropy of the a priori state is not zero, an instantaneous index of "goodness" of the measurement could be $S\big(\sigma(t)\|p(t)\eta(t)\big)\big/S_{\mathrm{q}}\big(\eta(t)\big)$, while a "cumulative" index could be $\displaystyle\int_0^T \frac{S\big(\sigma(t)\|p(t)\eta(t)\big)}{S_{\mathrm{q}}\big(\eta(t)\big)}\,\mathrm{d}t$.

Remark 6.10. The two mutual entropies (6.13) and (6.32) can be obtained from a unique mutual entropy

$$S(\sigma(t)\|q(t)\eta(t))=\int_\Omega Q(\mathrm{d}\omega)\,\mathrm{Tr}\left\{\sigma(t,\omega)\big(\ln\sigma(t,\omega)-\ln q(t,\omega)\eta(t)\big)\right\}. \tag{6.34}$$

Indeed, by direct computation, we get

$$\begin{aligned} S\big(\sigma(t)\|q(t)\eta(t)\big) &= S\big(\sigma(t)\|p(t)\eta(t)\big)+S_{\mathrm{c}}\big(p(t)\|q(t)\big)\\ &= \mathbb{E}^t_{\rho_0}\big[S_{\mathrm{q}}\big(\rho(t)\|\eta(t)\big)\big]+S_{\mathrm{c}}\big(p(t)\|q(t)\big). \end{aligned} \tag{6.35}$$

6.6 Gain of Information on the Initial State: The Input/Output Classical Information

Let us consider a typical setup of quantum communication theory. A message is transmitted by encoding the letters in some quantum states, which are possibly corrupted by a quantum noisy channel; at the end of the channel the receiver attempts to decode the message by performing measurements on the quantum system. So, one has an alphabet A and the letters $\alpha\in A$ are transmitted with some a priori probability $\mathbb{P}_{\mathrm{i}}$. Each letter α is encoded in a quantum state and we denote by $\rho_{\mathrm{i}}(\alpha)$ the state associated to the letter α as it arrives to the receiver, after the passage through the transmission channel. So, $\rho_{\mathrm{i}}(\alpha)$ is the unknown *initial* state for the possible measurements performed by the receiver. While it is usual to consider a finite alphabet, also general continuous parameter spaces are acquiring importance [17–19]. The same setup describes a problem in quantum statistics. The parameter $\alpha\in A$ is unknown and we want to estimate it by performing measurements on the quantum system; the problem is afforded in the Bayesian formulation and the a priori probability $\mathbb{P}_{\mathrm{i}}$ is known. Now, we assume that we have at disposal only the continuous measurement described in the previous chapters and that by means of that measurement we want to decode the transmitted message (to recognise the state

$\rho_{\rm i}(\alpha)$) or to estimate the parameter α. The quantity which measures the effectiveness of the measurement in distinguishing among the possible initial states is the classical input/output information.

Summarising, the possible initial states are the statistical operators $\rho_{\rm i}(\alpha) \in \mathcal{S}(\mathcal{H})$, $\alpha \in A$, with probability distribution $\mathbb{P}_{\rm i}(\mathrm{d}\alpha)$; some σ-algebra $\mathcal{A}$ is understood. Equivalently, $\big(\rho_{\rm i}(\alpha),\ \mathbb{P}_{\rm i}(\mathrm{d}\alpha)\big)$ is the initial ensemble with average initial state $\eta_{\rm i} = \int_A \mathbb{P}_{\rm i}(\mathrm{d}\alpha)\rho_{\rm i}(\alpha)$. Therefore, the possible initial states for the continuous measurement are the statistical operators $\rho_{\rm i}(\alpha)$, $\alpha \in A$, and $\eta_{\rm i}$.

First of all, we can form the physical probability when the initial state is $\rho_{\rm i}(\alpha)$ and we interpret it as the conditional probability on Ω given α:

$$\mathbb{P}_t(\mathrm{d}\omega|\alpha) = p(t,\omega|\alpha)\mathbb{Q}(\mathrm{d}\omega), \qquad p(t,\omega|\alpha) = \mathrm{Tr}\left\{\mathcal{A}(t,0;\omega)[\rho_{\rm i}(\alpha)]\right\}. \tag{6.36}$$

As we know the a priori probability on A, we can form the unconditional probability, which coincides with the physical probability when the initial state is $\eta_{\rm i}$:

$$\mathbb{P}_t(\mathrm{d}\omega) = p(t,\omega)\mathbb{Q}(\mathrm{d}\omega), \qquad p(t,\omega) = \int_A p(t,\omega|\alpha)\mathbb{P}_{\rm i}(\mathrm{d}\alpha) = \mathrm{Tr}\left\{\mathcal{A}(t,0;\omega)[\eta_{\rm i}]\right\}. \tag{6.37}$$

Similarly, we can form the physical joint probability on $A \times \Omega$

$$\mathbb{P}_t(\mathrm{d}\alpha \times \mathrm{d}\omega) = \mathbb{P}_t(\mathrm{d}\omega|\alpha)\mathbb{P}_{\rm i}(\mathrm{d}\alpha) = p(t,\omega|\alpha)\mathbb{P}_{\rm i}(\mathrm{d}\alpha)\mathbb{Q}(\mathrm{d}\omega), \tag{6.38}$$

and the conditional probability on A given ω

$$\mathbb{P}_t(\mathrm{d}\alpha|\omega) = \frac{\mathbb{P}_t(\mathrm{d}\omega|\alpha)\mathbb{P}_{\rm i}(\mathrm{d}\alpha)}{\mathbb{P}_t(\mathrm{d}\omega)} = \frac{p(t,\omega|\alpha)}{p(t,\omega)}\,\mathbb{P}_{\rm i}(\mathrm{d}\alpha). \tag{6.39}$$

The *input/output information* is the relative entropy of the joint probability $\mathbb{P}_t(\mathrm{d}\alpha \times \mathrm{d}\omega)$ with respect to the product $\mathbb{P}_{\rm i}(\mathrm{d}\alpha)\mathbb{Q}(\mathrm{d}\omega)$ of its marginals, that is

$$\begin{aligned} I(t) &:= \int_{A\times\Omega} \mathbb{P}_t(\mathrm{d}\alpha \times \mathrm{d}\omega)\, \ln \frac{\mathbb{P}_t(\mathrm{d}\alpha \times \mathrm{d}\omega)}{\mathbb{P}_{\rm i}(\mathrm{d}\alpha)\mathbb{P}_t(\mathrm{d}\omega)} \\ &= \int_A \mathbb{P}_{\rm i}(\mathrm{d}\alpha) \int_\Omega \mathbb{Q}(\mathrm{d}\omega)\, p(t,\omega|\alpha) \ln \frac{p(t,\omega|\alpha)}{p(t,\omega)}. \end{aligned} \tag{6.40}$$

By using the densities $p(t,\omega|\alpha)$, 1, $p(t,\omega)$ with respect to $\mathbb{P}_{\rm i}(\mathrm{d}\alpha)\mathbb{Q}(\mathrm{d}\omega)$, $\mathbb{P}_{\rm i}(\mathrm{d}\alpha)$, $\mathbb{Q}(\mathrm{d}\omega)$, respectively, we can write

$$I(t) = S_{\rm c}\big(p(t|\bullet)\|\,p(t)\big). \tag{6.41}$$

Now we introduce some useful quantities, namely, the a posteriori state and the a posteriori mean starting from $\rho_{\rm i}(\alpha)$

$$\rho(t,\omega|\alpha) = \frac{\mathcal{A}(t,0;\omega)[\rho_{\rm i}(\alpha)]}{p(t,\omega|\alpha)}\,, \tag{6.42a}$$

$$v_j(t,\omega|\alpha) = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\rho(t,\omega|\alpha)\right\}, \tag{6.42b}$$

and the a posteriori state and a posteriori mean starting from $\eta_{\rm i}$

$$\rho(t,\omega) = \frac{\mathcal{A}(t,0;\omega)[\eta_{\rm i}]}{p(t)}\,, \qquad v_j(t,\omega) = \mathrm{Tr}\left\{\left(R_j(t) + R_j(t)^*\right)\rho(t,\omega)\right\}. \tag{6.43}$$

By using these quantities, as for (3.44), we obtain

$$p(t) = \exp\left\{\sum_{j=1}^{m}\left[\int_0^t v_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t v_j(s)^2\mathrm{d}s\right]\right\}, \tag{6.44a}$$

$$p(t|\alpha) = \exp\left\{\sum_{j=1}^{m}\left[\int_0^t v_j(s|\alpha)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t v_j(s|\alpha)^2\mathrm{d}s\right]\right\}. \tag{6.44b}$$

From (6.44) we get the explicit expression of the input/putput information:

$$I(t) = \frac{1}{2}\int_0^t \mathrm{d}s \int_{A\times\Omega} \mathbb{P}_t(\mathrm{d}\alpha\times\mathrm{d}\omega)\sum_j \left(v_j(s,\omega|\alpha) - v_j(s,\omega)\right)^2. \tag{6.45}$$

Independently from the time-continuous measurements, the input/output information is known to be limited by the famous Holevo bound. Such a bound goes back to a paper by Holevo [20] for the discrete case; the proof for the general case was given by Yuen and Ozawa [18]. Later, Schumacher, Westmoreland and Wootters [21] gave a new formulation of the bound which takes into account also the information left in the a posteriori states. Their work was limited to a very special instrument; the result for a general discrete instrument was obtained in [4] and developed in [5, 22]. The proof for a general instrument, not necessarily discrete, was given in [6] and the result was applied to time-continuous measurements in [9].

Theorem 6.11 (The Holevo & SWW bound). *The classical input/output information $I(t)$, introduced in* (6.40)*, is bounded by*

$$0 \le I(t) \le \chi(0) - \chi(t), \tag{6.46}$$

where

$$\chi(t) := \int_\Omega \mathbb{P}_t(\mathrm{d}\omega)\int_A \mathbb{P}_t(\mathrm{d}\alpha|\omega)\, S_{\rm q}\big(\rho(t,\omega|\alpha)\|\rho(t,\omega)\big). \tag{6.47}$$

Proof. The mutual entropy of the initial ensemble is

$$S(\rho_{\rm i}\|\eta_{\rm i}) = \int_A \mathbb{P}_{\rm i}({\rm d}\alpha)\ {\rm Tr}\left\{\rho_{\rm i}(\alpha)\left(\ln\rho_{\rm i}(\alpha) - \ln\eta_{\rm i}\right)\right\}$$

$$= \int_A \mathbb{P}_{\rm i}({\rm d}\alpha)\, S_{\rm q}\big(\rho_{\rm i}(\alpha)\|\eta_{\rm i}\big) = \chi(0).$$

By recalling that $\mathcal{A}(t,0)$ is a channel and by applying the Uhlmann monotonicity theorem 6.2, we get

$$S(\rho_{\rm i}\|\eta_{\rm i}) \geq S\big(\mathcal{A}(t,0)[\rho_{\rm i}]\|\mathcal{A}(t,0)[\eta_{\rm i}]\big).$$

But the we have

$$\begin{aligned} S\big(\mathcal{A}(t,0)[\rho_{\rm i}]\|\mathcal{A}(t,0)[\eta_{\rm i}]\big) &= \int_A \mathbb{P}_{\rm i}({\rm d}\alpha)\int_\Omega \mathbb{Q}({\rm d}\omega)p(t,\omega|\alpha) \\ &\quad\times {\rm Tr}\left\{\rho(t,\omega|\alpha)\left(\ln p(t,\omega|\alpha)\rho(t,\omega|\alpha) - \ln p(t,\omega)\rho(t,\omega)\right)\right\} \\ &= \int_A \mathbb{P}_{\rm i}({\rm d}\alpha)\int_\Omega \mathbb{Q}({\rm d}\omega)p(t,\omega|\alpha)\left\{\ln\frac{p(t,\omega|\alpha)}{p(t,\omega)}\right. \\ &\quad\left. + S_{\rm q}\big(\rho(t,\omega|\alpha)\|\rho(t,\omega)\big)\right\} = I(t) + \chi(t), \end{aligned}$$

which ends the proof. □

The expression $\int_A \mathbb{P}_t({\rm d}\alpha|\omega)\, S_{\rm q}\big(\rho(t,\omega|\alpha)\|\rho(t,\omega)\big)$ is a random chi-quantity; then, $\chi(t)$ is a mean chi-quantity, which represents the information about the initial state left in the a posteriori states.

Moreover, for $t = 0$ we get

$$\chi(0) = \int_A \mathbb{P}_{\rm i}({\rm d}\alpha)\, S_{\rm q}\big(\rho_{\rm i}(\alpha)\|\eta_{\rm i}\big), \tag{6.48}$$

which is the Holevo's chi-quantity of the initial ensemble. The positivity of $\chi(t)$ gives that (6.46) implies

$$I(t) \leq \chi(0), \tag{6.49}$$

which is the Holevo bound. Note that the bound (6.49) is independent of the type of the measurement, while the features of the measurement we are considering appear in the counter-term $-\chi(t)$ in (6.46) through the a posteriori states $\rho(t,\omega|\alpha)$ and $\rho(t,\omega)$ and the density $p(t,\omega|\alpha)$ which appears in the joint probability $\mathbb{P}_t({\rm d}\alpha|\omega)\mathbb{P}_t({\rm d}\omega) = \mathbb{P}_t({\rm d}\alpha\times{\rm d}\omega) = p(t,\omega|\alpha)\mathbb{P}_{\rm i}({\rm d}\alpha)\mathbb{Q}({\rm d}\omega)$.

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Part II
Physical Applications

Chapter 7
Quantum Optical Systems

7.1 How to Construct Physical Models

Up to here we have seen the abstract theory, which involves a certain number of operators on $\mathcal{H}$. The choice of $\mathcal{H}$ and of these operators fixes the physical model. This section provides some hints on how to do this choice.

7.1.1 From the Full Quantum Level to the Reduced Description by SDEs

The construction of physically sensible models of continuously observed quantum systems is not a simple problem. Some useful hints are given by the second approach to continuous measurements described in Chap. 1, the one based on the quantum stochastic differential equation of Hudson and Parthasarathy. In that approach one has to introduce interactions between the quantum system of interest and some quantum fields. Then, the quantum stochastic approach can be translated into the approach with SDEs, as explained in [1, 2]. A more physically oriented presentation can be found in [3], where the authors present both the approaches (with quantum and with classical stochastic differential equations) together with various physical applications.

In the formulation of continuous measurements using classical SDEs, the mathematical model is completely determined when the operators $R_j(t)$ and $H(t)$ are given (cf. Sect. 2.8.1). The translation from the quantum SDEs to the classical SDE formulation gives us a structure for these operators in terms of some fundamental bricks, coming out from the system operators involved in the system/field interaction, the functions characterising the state of the quantum field (say d-dimensional) and some functions involved in the measuring procedure:

- D_k, system operators responsible for the emission of quanta of the fields;
- S_{kj}, system operators such that $\sum_k S_{kj}^* S_{ki} = \sum_k S_{jk} S_{ik}^* = \delta_{ij}$; S is a unitary matrix of system operators and is involved in terms responsible for scattering of quanta;

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- $H_0 = H_0^*$, the free Hamiltonian of the system;
- $f_k(t)$, functions describing some coherent external stimulation (the "state of the fields");
- $h_k(t)$, with $|h_k(t)| = 1$, a time-dependent phase factor appearing when the measuring apparatus uses some interference mechanism, as in the so-called "heterodyne" and "homodyne" photon-detection techniques.

The final result for the operators characterising the reduced description in terms of SDEs is the following structure for $R_k(t)$ and $H(t)$:

$$R_k(t) = \overline{h_k(t)}\Big[D_k + \sum_{j=1}^{d} S_{kj} f_j(t) \Big], \tag{7.1a}$$

$$H(t) = H_0 + \frac{1}{2} \sum_{kj} \Big[\mathrm{i}\overline{f_k(t)} S_{jk}^* D_j - \mathrm{i} D_j^* S_{jk} f_k(t) \Big]. \tag{7.1b}$$

In order to justify a concrete choice of h_k, f_j, D_k, S_{kj}, H_0, one could study the physical consequences of the model; this would eventually provide a phenomenological justification. A better justification would be to construct first the "quantum stochastic" model (the one based on Hudson–Parthasarathy equation). At this higher level, the system–field interaction is physically more transparent, mainly for quantum optical systems. However, this is out of the aims of the present work, as it needs the whole formalism of quantum stochastic calculus.

7.1.2 Observed and Unobserved Channels — The Generators

The operator $H(t)$ represents the effective Hamiltonian of the system, the set of operators $R_k(t)$, $k = 1, \ldots, m$, are involved in the "observed channels" and the set of operators $R_k(t)$, $k = m+1, \ldots, d$, are involved in the "unobserved channels" and in the dissipative interaction with the external world, as a thermal bath. The linear and nonlinear Hilbert stochastic Schrödinger equations (2.28), (2.49), (2.52) depend on all the features of such operators. However, due to the presence of the unobserved channels, the physical consequences of the theory are determined by the stochastic master equations (3.37), (3.69) or, equivalently, by the generator (4.37) of the characteristic operator.

The generator of the characteristic operator can be written in the form: $\forall \tau \in M_n$,

$$\Lambda_t(k)[\tau] := \mathcal{L}(t)[\tau] + \sum_{j=1}^{m} \left(\mathrm{i} k_j \big(R_j(t)\tau + \tau R_j(t)^* \big) - \frac{1}{2} k_j^2 \tau \right), \tag{7.2a}$$

$$\mathcal{L}(t)[\tau] = \mathcal{L}_0(t)[\tau] + \sum_{j=1}^{m} \left(R_j(t) \tau R_j(t)^* - \frac{1}{2} \big\{ R_j(t)^* R_j(t), \tau \big\} \right), \tag{7.2b}$$

$$\mathcal{L}_0(t)[\tau] = -\mathrm{i}[H(t), \tau] + \sum_{k=m+1}^{d} \left(R_k(t)\tau R_k(t)^* - \frac{1}{2}\left\{ R_k(t)^* R_k(t), \tau \right\} \right), \quad (7.2c)$$

where $H(t)$ and $R_k(t)$ are given by Eqs. (7.1). However, the Hamiltonian $H(t)$ and the operators $R_k(t)$, $k = m+1, \ldots, d$, involved in the "unobserved channels", contribute to the physical quantities (probabilities and a posteriori states) only through the reduced Liouville operator (7.2c): to change $H(t)$, $R_k(t)$, $k = m+1, \ldots, d$, without changing $\mathcal{L}_0(t)$ does not modify the physical model.

In particular, an unobserved channel contributes to the physical model only through the expression $R_k(t)\bullet R_k(t)^* - \frac{1}{2}\{R_k(t)^* R_k(t), \bullet\}$, from which the phase factor $h_k(t)$ disappears. This means that the functions $h_k(t)$, $k = m+1, \ldots, d$, can be chosen as one wants, for instance in order to produce some simplification in the form of the stochastic Schrödinger equation.

Moreover, in the finite dimensional case, in which we are working, the best representation of a quantum Liouville operator is often obtained by using only traceless operators in the dissipative part [4]. Therefore, we define

$$g_k(t) := \frac{1}{n}\,\mathrm{Tr}\{D_k\} + \frac{1}{n}\sum_{j=1}^{d} \mathrm{Tr}\{S_{kj}\} f_j(t), \quad (7.3)$$

$$L_k(t) := D_k + \sum_{j=1}^{d} S_{kj} f_j(t) - g_k(t), \quad (7.4)$$

by which we have $\mathrm{Tr}\{L_k(t)\} = 0$ and $R_k(t) = \overline{h_k(t)}\,[L_k(t) + g_k(t)]$. By inserting these expressions and (7.1b) into (7.2b), (7.2c), we obtain ($\forall \tau \in M_n$)

$$\mathcal{L}(t)[\tau] = -\mathrm{i}[H_0 + H_f(t), \tau] + \sum_{k=1}^{d} \left(L_k(t)\tau L_k(t)^* - \frac{1}{2}\left\{ L_k(t)^* L_k(t), \tau \right\} \right), \quad (7.5)$$

$$\begin{aligned} H_f(t) = &-\frac{\mathrm{i}}{2}\sum_{k,j=1}^{d} D_k^* \left(S_{kj} + \frac{1}{n}\,\mathrm{Tr}\left\{S_{kj}\right\} \right) f_j(t) \\ &+ \frac{\mathrm{i}}{2n}\sum_{k=1}^{d} \mathrm{Tr}\left\{D_k^*\right\} \left(D_k + \sum_{j=1}^{d} S_{kj} f_j(t) \right) \\ &+ \frac{\mathrm{i}}{2n}\sum_{i,j,k=1}^{d} \overline{f_i(t)}\,\mathrm{Tr}\left\{S_{ki}^*\right\} S_{kj} f_j(t) + \text{h.c.}; \end{aligned} \quad (7.6)$$

here "+ h.c." means "plus the Hermitian conjugate terms". An interesting special case is when $\mathrm{Tr}\{D_k\} = 0$ and $S_{kj} = \delta_{kj}$; in such a case we have

$$g_k(t) = f_k(t), \qquad L_k(t) = D_k\,, \tag{7.7a}$$

$$H_f(t) = \sum_{k=1}^{d} \left[\mathrm{i}\overline{f_k(t)} D_k - \mathrm{i} D_k^* f_k(t) \right]. \tag{7.7b}$$

7.2 Heterodyne and Homodyne Detection

In order to fix the various quantities entering into play via Eqs. (7.1), we need to be more specific about the physical situation. Typical measuring schemes giving rise to a stochastic description of diffusive type, as the one treated in this book, are the so-called heterodyne and homodyne detection. We start from the description of such a measuring apparatus, which in particular determines the functions $h_k(t)$ in the observed channels.

7.2.1 The Measurement Scheme

The system of interest is some source of light; think of an atom stimulated by a laser, which is the case illustrated in Fig. 7.1. Part of the fluorescence light coming out from the atom is made to interfere with a high-intensity laser signal of frequency ν (the *local oscillator*). The interference between fluorescence light and the local oscillator is obtained by means of a beam splitter (a half-transparent mirror). Then, the light coming out from the splitter is detected by photoelectron counters. The scheme of Fig. 7.1 is called *balanced homodyne detection* [5–8]: each photo-counter receives the light coming out from one of the two output ports of the beam splitter and the two currents are then subtracted. This setup with two photo-counters reduces the noises in the final current.

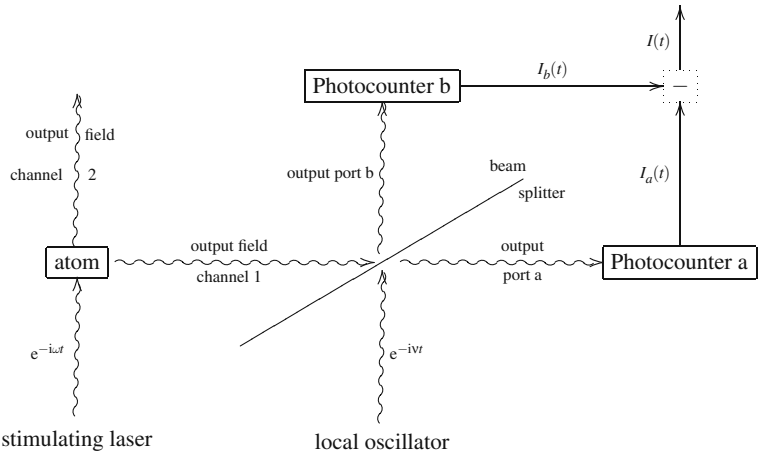

Fig. 7.1 Balanced heterodyne/homodyne detection

Not the whole fluorescence light reaches the beam splitter, it is partly lost in the surrounding free space. Let the light reaching the beam splitter be described by the channel number 1 and the lost light by channel 2. Usually the stimulating laser is directed in such a way that its light does not impinge directly on the beam splitter, so we can say that it acts on channel number 2. Following [9] we can call channel number 2 the *forward channel* and channel number 1 the *side channel*.

The formulation of homodyne/heterodyne detection in terms of quantum stochastic calculus has been given in [10, 11]. Here we give the translation in the formalism of the previous sections.

- The effect of the local oscillator of frequency ν is to fix the function $h_1(t)$ in Eq. (7.1a): $\overline{h_1(t)} = \mathrm{e}^{\mathrm{i}\nu t} h_1(0)$. The constant phase factor $h_1(0)$ can be included in the definitions of the operator D_1 and of the function $f_1(t)$, so that we can take $h_1(0) = 1$.
- The output current is proportional to the rate of variation of the signal $W_1(t)$, but with some time delay depending on the particular features of the apparatus. By representing the response of the apparatus in the simplest natural way, the output current can be written as

$$I(t) = \int_0^t F(t-s)\mathrm{d}W_1(s)\,, \tag{7.8a}$$

  where F is a *detector response function*, say

$$F(t) = k_1\sqrt{\frac{\varkappa}{4\pi}}\exp\left\{-\frac{\varkappa}{2}\,t\right\}, \qquad \varkappa > 0, \quad k_1 \neq 0. \tag{7.8b}$$

  The constants k_1 and $\varkappa$ depend on the measuring apparatus; k_1 has the dimensions of a current and $1/\varkappa$ the dimensions of a time.
  The constant $\varkappa$ controls the time resolution, see Sect. 4.4.2. For $\varkappa \to +\infty$, the current $I(t)$ becomes formally proportional to the singular process $\dot{W}_1(t)$, the past times are not involved: a big value of $\varkappa$ gives a good time resolution.
- The electrical power carried by the current I is proportional to its square, say

$$P(t) = k_2 I(t)^2, \tag{7.9}$$

  where $k_2 > 0$ has the dimensions of a resistance.

Physically, when the local oscillator is not perfectly monochromatic, the phase factor $h_1(t)$ can have some more complicated form and can be stochastic too. Moreover, if the experimenter is able to fine tune the probing laser acting as local oscillator, the functional form of h_1 can be varied; in this way it is the type of measurement which is changed, not the reduced dynamics of the quantum system. This freedom can be used to produce adaptive measurements which give better information on the system and allow to better control it, for instance via a feedback mechanism.

7.2.2 Homodyning Versus Heterodyning

The terms heterodyning and homodyning come from radio technique. The term homodyne detection is reserved for the case in which the local oscillator is in resonance with the carrier frequency of the field reaching the detector. When the local oscillator is out of resonance the term heterodyne detection is used. In the case of an atomic system stimulated by monochromatic light of frequency ω, the carrier frequency in the output is just ω and we can speak of homodyne detection when $\nu = \omega$, where ν is the frequency of the local oscillator.

In a real implementation, homodyning needs also to maintain over the time the coherence between the stimulating laser and the local oscillator. Due to the presence of imperfections and randomness in the real laser light, it is nearly impossible to reach this coherence by using two different sources. Therefore, in order to realise a good homodyne detection in the laboratory, one has to split the light coming out from a single laser into two parts and to use one beam as local oscillator and the other one as stimulating input for the atom [10, 12].

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Chapter 8
A Two-Level Atom: General Setup

8.1 A Two-Level Atom Stimulated by a Laser

A two-level atom is perhaps the simplest quantum system, but in spite of this it has a very rich behaviour. So, it represents an ideal physical system to illustrate the theory presented so far [1].

The fluorescence spectrum of a two-level atom stimulated by an intense monochromatic laser is highly non-trivial; it presents a typical three-peaked structure, which is now known as Mollow spectrum [2]. A measurement scheme which allows to obtain the spectrum is the heterodyne detection, which we discussed in Chap. 7. This simple quantum system presents also another typical quantum phenomenon, the "squeezing" of the fluorescence light, which can be revealed by homodyne detection [3].

The basic assumption is that the frequencies, the polarisations and the energies involved are such that a good description of the dynamics of the atom can be given by using only two non-degenerate levels. This fixes the Hilbert space:

$$\mathcal{H} = \mathbb{C}^2 \,. \tag{8.1}$$

8.1.1 The Pauli Matrices

A very convenient way to treat with operators on $\mathbb{C}^2$ is to use the so-called *Pauli matrices*:

$$\sigma_x \equiv \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y \equiv \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z \equiv \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To face computations involving the Pauli matrices, it is useful to recall the following commutation and anti-commutation rules, which are the characteristic properties of these matrices:

$$[\sigma_i, \sigma_j] = 2\mathrm{i} \sum_k \varepsilon_{ijk} \sigma_k \,, \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \,, \tag{8.2}$$

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where ε_{ijk} is the Ricci symbol

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } ijk \text{ is an even permutation of } 123, \\ -1, & \text{if } ijk \text{ is an odd permutation of } 123, \\ 0, & \text{in the other cases.} \end{cases}$$

Another way of giving the algebraic properties of the Pauli matrices is through the product rule

$$\sigma_i \sigma_j = \mathrm{i} \sum_k \varepsilon_{ijk} \sigma_k + \delta_{ij} \,, \tag{8.3}$$

which is equivalent to (8.2).

Other useful matrices are the "lowering" operator σ_-, the "rising" operator σ_+, the projection on the "up state" P_+ and the projection on the "down state" P_-:

$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us note that

$$\sigma_+ = \sigma_-^* \,, \qquad P_+ = \sigma_+ \sigma_- \,, \qquad P_- = \sigma_- \sigma_+ \,,$$
$$2\sigma_\pm = \sigma_x \pm \mathrm{i}\sigma_y \,, \qquad 2P_\pm = \mathbb{1} \pm \sigma_z \,.$$

8.1.2 The Bloch Representation

The matrices $\mathbb{1}$, σ_x, σ_y, σ_z are linearly independent and form a basis in M_2 called the Pauli basis. Then, any $\tau \in M_2$ can be written as

$$\tau = \frac{1}{2} \left(c_0 \mathbb{1} + \vec{d} \cdot \vec{\sigma} \right), \quad c_0 = \mathrm{Tr}\{\tau\} \in \mathbb{C}, \quad \vec{d} = \mathrm{Tr}\{\vec{\sigma}\, \tau\} \in \mathbb{C}^3. \tag{8.4}$$

As can be easily checked, every positive definite, trace one, 2×2 complex matrix ρ (a statistical operator) can be represented as

$$\rho = \frac{1}{2} \begin{pmatrix} 1+z & x - \mathrm{i}y \\ x + \mathrm{i}y & 1-z \end{pmatrix} = \frac{1}{2} \left(\mathbb{1} + \vec{x} \cdot \vec{\sigma} \right). \tag{8.5}$$

This formula defines a statistical operator if and only if $\vec{x} \in \mathbb{R}^3$ and $|\vec{x}| \leq 1$. The state ρ is pure ($\rho^2 = \rho$) if and only if $|\vec{x}| = 1$.

So, the statistical operators are represented by the points in the unit sphere in the three-dimensional real space, which takes the name of *Bloch sphere* and the pure states are represented by the points in the surface of this sphere. Given ρ, the *Bloch vector* $\vec{x}$ is obtained by

$$x_i = \mathrm{Tr}\{\sigma_i\,\rho\}, \qquad i = 1, 2, 3. \tag{8.6}$$

By using the Bloch representation the distance in trace norm (B.4) between two statistical operators $\rho^{(i)} = \frac{1}{2}\left(\mathbb{1} + \vec{x}^{(i)} \cdot \vec{\sigma}\right)$ becomes very simple and significant; by using (8.3) one obtains easily

$$\left\|\rho^{(1)} - \rho^{(2)}\right\|_1 = \left|\vec{x}^{(1)} - \vec{x}^{(2)}\right|. \tag{8.7}$$

8.1.3 The System Operators

The first problem in constructing a physically sensible model is the choice of all the operators and functions involved in the theory and which have been introduced in abstract terms in Sect. 7.1.

8.1.3.1 The Free Hamiltonian

The free Hamiltonian of the atom can be only a self-adjoint operator with two distinct eigenvalues; the traditional choice is to take it proportional to σ_z.

- The Hamiltonian of the free atom is

$$H_0 = \frac{1}{2}\,\omega_0\,\sigma_z\,, \qquad \omega_0 > 0\,. \tag{8.8}$$

Thus, the canonical basis of $\mathbb{C}^2$ is identified with the eigenvectors of H_0 and this justifies the names given to σ_-, σ_+, P_-, P_+. The parameter ω_0 is the resonance frequency of the atom.

8.1.3.2 The Electromagnetic Channels

The second step in constructing the model is to consider the interaction of the atom with the electromagnetic field. The operator matrix S can be used to describe some kind of scattering of light [4]; but, when the predominant effect responsible for the scattering of light is only emission/absorption, the operator matrix S can be taken to be the identity. Here we consider only this simplified situation; so, we take

- $S_{kj} = \delta_{kj}, \quad \forall k, j.$

When stimulated, the atom emits light in the whole solid angle, but it is enough to distinguish two channels for the outgoing light as schematised in Fig. 7.1: one for the light which reaches the measuring apparatus (the beam splitter and the detectors), say channel 1, and the other one for the light which is lost in the surrounding free space, say channel 2. Usually the laser is well collimated and it involves a small solid angle in the forward direction (channel 2), but as we have already said, in channel 2 we include also all other "forward" or "lateral" directions along which

the light does not reach the detection apparatus. Instead, channel 1 is made up of all the light rays which reach the detector, eventually after some focussing by lenses and mirrors.

In the so-called electric dipole and rotating wave approximations [3, pp. 200, 202], when a photon is emitted into the external electromagnetic field, the atom makes a transition from the upper to the lower level, which means to take the emission operators proportional to σ_-.

- The "emission operators", in the dipole and rotating wave approximation, are given by

$$D_k = \sqrt{\gamma}\,\alpha_k \sigma_- , \qquad \sum_{k=1}^{2} |\alpha_k|^2 = 1 , \quad \gamma > 0, \quad \alpha_k \in \mathbb{C} \quad k = 1, 2 .$$

  The quantity γ has the dimensions of 1/time and represents the *natural linewidth* of the atom. The quantities $|\alpha_1|^2$ and $|\alpha_2|^2$ are the proportions of light in the side and in the forward channels, respectively.

The physical meaning of γ, α_k and of all the other parameters we are introducing can appear not obvious at this stage; only the full analysis of the mathematical model can clarify the role of the various terms and parameters.

As sketched in Fig. 7.1, the stimulating laser acts only in the forward channel; so we take:

- The stimulating laser is perfectly coherent and monochromatic with frequency $\omega > 0$: $\quad f_k(t) = \delta_{2k} \lambda \mathrm{e}^{-\mathrm{i}\omega t}, \quad \lambda \in \mathbb{C}$.

The laser is said to be in resonance with the atom when $\omega = \omega_0$ and out of resonance or "detuned" when $\omega \neq \omega_0$. The intensity of the laser is proportional to $|\lambda|^2$, while the phase of λ gives the phase of the laser at time 0 in the location of the atom.

- The following quantity $\Delta\omega$ is called *detuning*:

$$\Delta\omega := \omega_0 - \omega . \tag{8.9}$$

As already said, the phase factor $h_1(t)$ is produced by the interference with a probing laser, the local oscillator, which is monochromatic of frequency ν and acts in the side channel; moreover, the initial phase factor $h_1(0)$ is included in the definition of D_1. In the unobserved forward channel, the phase factor has no influence on the physical results and can be chosen to simplify some intermediate equation, such as the stochastic Schrödinger equation; we chose $h_2(t)$ equal to the phase factor of channel 1.

- $h_1(t) = h_2(t) = \mathrm{e}^{-\mathrm{i}\nu t}$; $\quad \nu$ is the frequency of the local oscillator.

However, as we shall see in the following computations, what is important is some relative phase between laser and atomic characteristics and the intensity of the

laser as seen by the atom. Concretely, λ will appear in the final results only through suitable combinations with α_1, α_2.

The Contributions of Channels 1 and 2

Let us summarise the contributions of these two first channels.

According to (7.1a) we have

$$R_1(t) = \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma}\,\alpha_1\sigma_-\,, \tag{8.10a}$$

$$R_2(t) = \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma}\,\alpha_2\sigma_- + \lambda\mathrm{e}^{-\mathrm{i}(\omega-\nu)t}\mathbb{1}\,. \tag{8.10b}$$

Only $R_1(t)$ has a direct physical meaning, because the only observed channel is the number 1. The operator $R_2(t)$ contributes to the model only through its role in $\mathcal{L}_0(t)$ (7.2c).

Let us stress that we are in the case $\mathrm{Tr}\{D_k\} = 0$, $S_{kj} = \delta_{kj}$; then, Eqs. (7.5), (7.7) hold and the contribution to $\mathcal{L}(t)$ of channels 1 and 2 reduces to

$$\begin{aligned} &-\mathrm{i}\big[H_f(t), \tau\big] + \sum_{k=1}^{2}\left(D_k\tau D_k^* - \frac{1}{2}\left\{D_k^* D_k, \tau\right\}\right) \\ &\quad = -\mathrm{i}\big[H_f(t), \tau\big] + \gamma\sigma_-\tau\sigma_+ - \frac{\gamma}{2}\left\{P_+, \tau\right\}, \qquad \forall\tau \in M_n\,, \end{aligned} \tag{8.11}$$

$$H_f(t) = \mathrm{i}\overline{f_2(t)}D_2 - \mathrm{i}D_2^* f_2(t) = \sqrt{\gamma}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{i}\overline{\lambda}\alpha_2\,\sigma_- - \sqrt{\gamma}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{i}\lambda\overline{\alpha_2}\,\sigma_+\,.$$

- The following quantity Ω is called *Rabi frequency*:

$$\Omega := 2\sqrt{\gamma}\;|\lambda|\,|\alpha_2|\,. \tag{8.12}$$

Let us stress that of the parameters Ω, $|\lambda|$, $|\alpha_2|$, the physical ones are Ω and $|\alpha_2|$. Indeed, as we shall see, in the mean dynamics only the parameter Ω appears, not $|\lambda|$ and $|\alpha_2|$ by themselves. We already said that $|\alpha_2|^2$ is the percentage of the lost light. If the percentage $|\alpha_2|^2$ of lost light is changed by changing the measuring apparatus, the laser/atom interaction does not change and the mean dynamics cannot change; so, $|\lambda|$ has to be changed to maintain Ω constant.

About the phases of λ and α_2, let us note that in Dirac notation we can write $\sigma_- = |\mathrm{down}\rangle\langle\mathrm{up}|$ and by redefining the relative phase of the two states we can absorb into σ_- the constant phase of the factor multiplying it. This means that, without loss of generality, we can assume

$$\arg\left(\mathrm{i}\overline{\lambda}\,\alpha_2\right) = 0. \tag{8.13}$$

Now we have no more freedom in changing phases. The phase of α_1 will be particularly important in discussing the phenomenon of squeezing in Sect. 9.2, so we put it in evidence.

- An important phase:

$$\vartheta := \arg(\alpha_1)\,. \tag{8.14}$$

Finally we can rewrite the Hamiltonian contribution $H_f(t)$ as

$$H_f(t) = \frac{\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\omega t}\,\sigma_- + \mathrm{e}^{-\mathrm{i}\omega t}\,\sigma_+\right). \tag{8.15}$$

8.1.3.3 Other Dissipative Effects

As already said, unobserved channels can be used also to introduce any kind of dissipative effect. We use the example of the two-level atom to show how to introduce the effect of the interaction with a thermal bath and the "dephasing" effect. In all these terms there is no contribution like f_k and they appear only in the dissipative part of the Liouville operator; so their phases are not important and the coefficients can be taken positive.

- Terms simulating a thermal bath:

$$D_3 = \sqrt{\gamma\,\overline{n}}\,\sigma_-\,, \qquad D_4 = \sqrt{\gamma\,\overline{n}}\,\sigma_+\,, \qquad \overline{n} \geq 0\,.$$

- Dephasing term: $D_5 = \sqrt{\gamma\,k_\mathrm{d}}\,\sigma_z, \qquad k_\mathrm{d} \geq 0.$

The coefficient $\sqrt{\gamma}$ has been introduced in all the terms just by dimensional reasons; in this way k_d and $\overline{n}$ result to be dimensionless. The justification of the form of the terms related to the thermal bath comes from the form of the equilibrium state in the case of no external laser stimulating the atom, cf. Eqs. (8.42), (8.43). Instead, the effect of the dephasing term is to suppress the off-diagonal (with respect to the eigen-basis of the energy H_0) matrix elements of the statistical operator without touching the diagonal ones. This is often called also "decoherence".

As discussed in Sect. 7.1.2, the phase factors $h_k(t)$ in the unobserved channels have no effect and can be chosen arbitrarily. We take

$$h_3(t) = \mathrm{e}^{-\mathrm{i}\nu t}, \qquad h_4(t) = \mathrm{e}^{\mathrm{i}\nu t}, \qquad h_5(t) = 1. \tag{8.16}$$

8.1.3.4 The Final Model

From Eqs. (2.20), (7.1), (7.7b), (8.8), (8.10), (8.11), (8.12), (8.13), (8.14), (8.15), (8.16) we get

$$R_1(t) = \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma}\,\alpha_1\sigma_-\,, \tag{8.17a}$$

$$R_2(t) = \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma}\,\alpha_2\sigma_- + \lambda\mathrm{e}^{-\mathrm{i}(\omega-\nu)t}\mathbb{1}\,, \qquad \gamma > 0, \tag{8.17b}$$

$$R_3(t) = \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma\,\overline{n}}\,\sigma_-\,, \qquad R_4(t) = \mathrm{e}^{-\mathrm{i}\nu t}\sqrt{\gamma\,\overline{n}}\,\sigma_+\,, \qquad \overline{n} \geq 0, \tag{8.17c}$$

$$R_5(t) \equiv R_5 = \sqrt{\gamma\,k_\mathrm{d}}\,\sigma_z\,, \qquad k_\mathrm{d} \geq 0, \tag{8.17d}$$

$$\arg\left(\mathrm{i}\overline{\lambda}\,\alpha_2\right) = 0, \qquad |\alpha_1|^2 + |\alpha_2|^2 = 1, \qquad \alpha_1 = |\alpha_1|\,\mathrm{e}^{\mathrm{i}\vartheta}, \tag{8.17e}$$

$$\Omega = 2\sqrt{\gamma}\;|\lambda|\,|\alpha_2| \geq 0, \tag{8.17f}$$

$$K(t) = -\frac{\mathrm{i}}{2}\,\omega_0\sigma_z - \frac{\gamma}{2}\,P_+ - \frac{\gamma\,(\overline{n}+k_\mathrm{d}) + |\lambda|^2}{2}\,\mathbb{1} - \frac{\mathrm{i}}{2}\,\Omega\mathrm{e}^{-\mathrm{i}\omega t}\sigma_+\,. \tag{8.17g}$$

The Generator of the Reduced Dynamics

Then, the generator of the reduced dynamics (7.5) becomes

$$\begin{aligned}\mathcal{L}(t)[\tau] = &-\frac{\mathrm{i}}{2}\left[\omega_0\sigma_z + \Omega\left(\mathrm{e}^{\mathrm{i}\omega t}\sigma_- + \mathrm{e}^{-\mathrm{i}\omega t}\sigma_+\right),\ \tau\right] + \gamma k_\mathrm{d}\,(\sigma_z\tau\sigma_z - \tau)\\ &+ \gamma\,(\overline{n}+1)\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\,\{P_+,\ \tau\}\right) + \gamma\overline{n}\left(\sigma_+\tau\sigma_- - \frac{1}{2}\,\{P_-,\ \tau\}\right).\end{aligned} \tag{8.18}$$

The Generator of the Characteristic Operator

According to Proposition 4.10 and Theorem 4.14, the characteristic operator determines the instruments and, so, the measurement and the dynamics. Therefore, the whole model is determined by its generator (7.2), which in this case is

$$\Lambda_t(k)[\tau] = \mathcal{L}(t)[\tau] + \mathrm{i}k\big(R_1(t)\tau + \tau R_1(t)^*\big) - \frac{1}{2}\,k^2\,\tau. \tag{8.19}$$

8.2 The Reduced Dynamics of the Two-Level Atom

Let us start by studying the average dynamics of the system (see Sects. 3.2 and 3.2.2). The a priori state $\eta(t)$ satisfies the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta(t) = \mathcal{L}(t)[\eta(t)], \qquad \eta(0) = \rho_0\,, \tag{8.20}$$

where the Liouville operator is given by Eq. (8.18).

8.2.1 The Reduced Dynamics and the Rotating Frame

A peculiar feature of the master equation (8.20) is that any explicit time dependence can be eliminated by a suitable unitary transformation. Let us define the *transformed a priori states* by

$$\check{\eta}(t) := \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\eta(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}. \tag{8.21}$$

The transformation (8.21) can be seen as a time-dependent rotation of a spin around the z-axis; in this sense it is usual to say that a *rotating frame* is used.

Proposition 8.1. *The transformed a priori states* (8.21) *satisfy the master equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\check{\eta}(t) = \check{\mathcal{L}}\,[\check{\eta}(t)]\,, \qquad \check{\eta}(0) = \eta(0) = \rho_0 \in \mathcal{S}(\mathcal{H}), \tag{8.22}$$

where $\check{\mathcal{L}}$ is the generator of a quantum dynamical semigroup and has the explicit expression

$$\begin{aligned}\check{\mathcal{L}}[\tau] = &-\mathrm{i}[\check{H},\tau] + \gamma\,(\overline{n}+1)\,\sigma_-\tau\sigma_+ - \frac{\gamma}{2}\,P_+\tau - \frac{\gamma}{2}\,\tau\,P_+ \\ &+ \gamma\,\overline{n}\sigma_+\tau\sigma_- + \gamma k_{\mathrm{d}}\sigma_z\tau\sigma_z - \gamma\,(\overline{n}+k_{\mathrm{d}})\,\tau,\end{aligned} \tag{8.23}$$

$$\check{H} = \frac{1}{2}\,\Delta\omega\sigma_z + \frac{1}{2}\,\Omega\sigma_x\,. \tag{8.24}$$

Given the transformed a priori states $\check{\eta}(t)$, the true a priori states $\eta(t)$ are

$$\eta(t) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\check{\eta}(t)\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z} = \begin{pmatrix}\check{\eta}_{11}(t) & \mathrm{e}^{-\mathrm{i}\omega t}\check{\eta}_{12}(t)\\ \mathrm{e}^{\mathrm{i}\omega t}\check{\eta}_{21}(t) & \check{\eta}_{22}(t)\end{pmatrix}. \tag{8.25}$$

Proof. By using

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\sigma_-\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z} = \mathrm{e}^{-\mathrm{i}\omega t}\sigma_-\,, \tag{8.26}$$

the adjoint relation and the unitarity of $\mathrm{e}^{\pm\frac{\mathrm{i}}{2}\omega t\sigma_z}$, we obtain Eq. (8.25).

By differentiating (8.21) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\check{\eta}(t) &= \mathrm{i}\,\frac{\omega}{2}\left[\sigma_z,\ \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\eta(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\right] + \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\mathcal{L}(t)[\eta(t)]\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z} \\ &= \mathrm{i}\,\frac{\omega}{2}\,[\sigma_z,\ \check{\eta}(t)] + \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\mathcal{L}(t)\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\check{\eta}(t)\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\right]\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z};\end{aligned}$$

by defining, $\forall \tau \in M_n$,

$$\check{\mathcal{L}}[\tau] := \mathrm{i}\,\frac{\omega}{2}\,[\sigma_z, \tau] + \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\,\mathcal{L}(t)\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\tau\,\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\right]\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}, \tag{8.27}$$

we get (8.22). By this definition of $\check{\mathcal{L}}$ and the fact that $\mathcal{L}(t)$ has a Lindblad structure, one has that also $\check{\mathcal{L}}$ can be written in Lindblad form and, being also time independent, it generates a quantum dynamical semigroup. By the explicit expression of $\mathcal{L}(t)$ and (8.26) again, we obtain (8.23), (8.24). □

Remark 8.2. The operator (8.23) can be written in many ways in the Lindblad form; for instance, it can be written as

$$\check{\mathcal{L}}[\tau] = -\mathrm{i}[\check{H}, \tau] + \sum_{k=1}^{3}\left(V_k\tau V_k^* - \frac{1}{2}\{V_k^* V_k,\ \tau\}\right), \tag{8.28a}$$

$$V_1 := \sqrt{\gamma\,(\overline{n}+1)}\,\sigma_-\,, \qquad V_2 := \sqrt{\gamma\,\overline{n}}\,\sigma_+\,, \qquad V_3 := \sqrt{\gamma k_\mathrm{d}}\,\sigma_z\,. \tag{8.28b}$$

8.2.2 The Bloch Equations

The best way of studying master equations when $\mathcal{H} = \mathbb{C}^2$ is to use the Bloch representation of the statistical operators (8.5). In the following we shall need to solve the master equation (8.22) not only when the initial condition is a statistical operator, but also when it is a generic 2×2 complex matrix. Let us parameterise $\mathrm{e}^{\check{\mathcal{L}}t}[\tau]$ as in (8.4) by writing

$$\tau(t) := \mathrm{e}^{\check{\mathcal{L}}t}[\tau] = \frac{1}{2}\left(c_0\mathbb{1} + \vec{d}(t)\cdot\vec{\sigma}\right). \tag{8.29}$$

We know that c_0 is independent of time because a quantum dynamical semigroup preserves the trace.

By applying $\check{\mathcal{L}}$, given by Eqs. (8.23) and (8.24), to the Pauli basis, we get

$$\check{\mathcal{L}}[\mathbb{1}] = -\gamma\sigma_z\,, \qquad \check{\mathcal{L}}[\sigma_z] = -\Omega\sigma_y - \gamma\,(2\overline{n}+1)\,\sigma_z\,,$$
$$\check{\mathcal{L}}[\sigma_x] = \Delta\omega\sigma_y - \frac{\gamma}{2}\,(2\overline{n}+1+4k_\mathrm{d})\,\sigma_x\,,$$
$$\check{\mathcal{L}}[\sigma_y] = -\Delta\omega\sigma_x + \Omega\sigma_z - \frac{\gamma}{2}\,(2\overline{n}+1+4k_\mathrm{d})\,\sigma_y\,.$$

By these results and the parametrisation (8.29), the master equation (8.22) becomes

$$\begin{cases} \dot{d}_1(t) = -\Delta\omega d_2(t) - \frac{\gamma}{2}\,(2\overline{n}+1+4k_\mathrm{d})\,d_1(t) \\ \dot{d}_2(t) = \Delta\omega d_1(t) - \Omega d_3(t) - \frac{\gamma}{2}\,(2\overline{n}+1+4k_\mathrm{d})\,d_2(t) \\ \dot{d}_3(t) = \Omega d_2(t) - \gamma\,(2\overline{n}+1)\,d_3(t) - \gamma c_0 \end{cases} \tag{8.30}$$

or, in matrix form,

$$\dot{\vec{d}}(t) = -A\vec{d}(t) - \gamma c_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{8.31}$$

$$A = \begin{pmatrix} \frac{\gamma(2\overline{n}+1+4k_\mathrm{d})}{2} & \Delta\omega & 0 \\ -\Delta\omega & \frac{\gamma(2\overline{n}+1+4k_\mathrm{d})}{2} & \Omega \\ 0 & -\Omega & \gamma\,(2\overline{n}+1) \end{pmatrix}. \tag{8.32}$$

It is easy to check that

$$\gamma > 0 \Rightarrow \det A > 0. \tag{8.33}$$

The solution of Eq. (8.31) is

$$\vec{d}(t) = \mathrm{e}^{-At}\vec{d}(0) - \gamma c_0 \int_0^t \mathrm{d}s\, \mathrm{e}^{-A(t-s)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{8.34}$$

which gives, for $\det A \neq 0$,

$$\vec{d}(t) = \mathrm{e}^{-At}\vec{d}(0) - \gamma c_0 \frac{1-\mathrm{e}^{-At}}{A} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{8.35}$$

In the case of a statistical operator as initial condition, which is the case of the transformed a priori states

$$\breve{\eta}(t) = \frac{1}{2}\,(\mathbb{1} + \vec{x}(t)\cdot\vec{\sigma}\,), \tag{8.36}$$

we have the *Bloch equations*[1]:

$$\begin{cases} \dot{x}(t) = -\Delta\omega y(t) - \frac{\gamma}{2}\,(2\overline{n}+1+4k_\mathrm{d})\,x(t), \\ \dot{y}(t) = \Delta\omega x(t) - \Omega z(t) - \frac{\gamma}{2}\,(2\overline{n}+1+4k_\mathrm{d})\,y(t), \\ \dot{z}(t) = \Omega y(t) - \gamma\,(2\overline{n}+1)\,z(t) - \gamma, \end{cases} \tag{8.37}$$

whose solution in matrix form is

$$\vec{x}(t) = \mathrm{e}^{-At}\vec{x}(0) - \gamma \frac{1-\mathrm{e}^{-At}}{A} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{8.38}$$

[1] Felix Bloch and Edward Mills Purcell, Nobel prize in physics 1952 "for their development of new methods for nuclear magnetic precision measurements and discoveries in connection therewith".

Let us stress that the reduced dynamics in the rotating frame depends only on the parameters

$$\gamma > 0, \quad \Delta\omega \in \mathbb{R}, \quad \Omega \geq 0, \quad \overline{n} \geq 0, \quad k_{\mathrm{d}} \geq 0.$$

Finally, the a priori states are given by Eq. (8.25), which gives

$$\eta(t) = \frac{1}{2}\big\{\mathbb{1} + [x(t)\cos\omega t - y(t)\sin\omega t]\,\sigma_x + [x(t)\sin\omega t + y(t)\cos\omega t]\,\sigma_y + z(t)\sigma_z\big\}. \tag{8.39}$$

8.2.2.1 The Rabi Frequency

To understand the meaning of the parameter Ω, let us consider the limit case $\Omega > 0$, but $\gamma = 0$; moreover, we take also $\Delta\omega = 0$, $\overline{n} = 0$, $k_d = 0$. This means that we have no detuning, no dephasing and zero temperature, and the emission of photons is not taken into account. The master equation (8.22) looses all the dissipative terms and only an Hamiltonian contribution survives; the Bloch equations (8.37) reduce to

$$\dot{x}(t) = 0\,, \qquad \dot{y}(t) = -\Omega z(t)\,, \qquad \dot{z}(t) = \Omega y(t)\,,$$

whose solution is $x(t) = x(0)$,

$$y(t) = y(0)\cos\Omega t - z(0)\sin\Omega t\,, \qquad z(t) = z(0)\cos\Omega t + y(0)\sin\Omega t\,.$$

So, the stimulating laser of intensity proportional to Ω^2 induces oscillations of frequency Ω, which takes the name of *Rabi frequency* in honour of I. I. Rabi.[2]

8.2.2.2 No Detuning

Let us consider the case of no detuning, $\Delta\omega = 0$, but now emission is not forgotten, $\gamma > 0$. In this case the eigenvalues of A are easily computed and turn out to be

$$a_0 = \frac{\gamma}{2}\,(2\overline{n} + 1 + 4k_{\mathrm{d}})\,,$$

$$a_{\pm} = \frac{\gamma}{4}\,(6\overline{n} + 3 + 4k_{\mathrm{d}}) \pm \sqrt{\frac{\gamma^2}{16}\,(2\overline{n} + 1 - 4k_{\mathrm{d}})^2 - \Omega^2}\,.$$

The real parts of all the eigenvalues are greater than $\gamma/2$; more precisely, one has

[2] Isidor Isac Rabi, Nobel prize in physics 1944 "for his resonance method for recording the magnetic properties of atomic nuclei".

$$\mathrm{Re}\, a_+ \geq \mathrm{Re}\, a_- \geq \frac{\gamma}{2}\,(2\overline{n}+1) > 0\,, \qquad a_0 \geq \frac{\gamma}{2} > 0\,.$$

Having the matrix A eigenvalues with strictly positive real parts, we get from Eq. (8.38)

$$\vec{x}_{\mathrm{eq}} \equiv \lim_{t\to+\infty} \vec{x}(t) = -\frac{\gamma}{A}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

i.e.

$$\begin{aligned}
&\lim_{t\to+\infty} x(t) = 0\,,\\
&\lim_{t\to+\infty} y(t) = \frac{\Omega\gamma}{\Omega^2 + a_0\gamma\,(2\overline{n}+1)}\,,\\
&\lim_{t\to+\infty} z(t) = \frac{-a_0\gamma}{\Omega^2 + a_0\gamma\,(2\overline{n}+1)}\,.
\end{aligned} \tag{8.40}$$

One can check that, being $\gamma > 0$, the final state is pure, i.e. $\lim_{t\to+\infty} |\vec{x}(t)|^2 = 1$, if and only if $\Omega = 0$ and $\overline{n} = 0$. In this case there is only decay to the ground state, $\lim_{t\to+\infty} \eta(t) = \lim_{t\to+\infty} \check{\eta}(t) = P_-$.

If we need also the transient part of the solution, note that the Bloch equations (8.37) give $x(t) = \mathrm{e}^{-a_0 t}x(0)$ and

$$\dot{y}(t) = -\Omega z(t) - a_0 y(t)\,, \qquad \dot{z}(t) = \Omega y(t) - \gamma\,(2\overline{n}+1)\,z(t) - \gamma\,.$$

The associated homogeneous linear system is

$$\dot{u}(t) = -\Omega v(t) - a_0 u(t)\,, \qquad \dot{v}(t) = \Omega u(t) - \gamma\,(2\overline{n}+1)\,v(t)\,,$$

and, when $4\Omega \neq \gamma\,|2\overline{n}+1-4k_{\mathrm{d}}|$, it has the general solution

$$\begin{aligned}
&u(t) = u_+\mathrm{e}^{-a_+ t} + u_-\mathrm{e}^{-a_- t}\,, \qquad v(t) = v_+\mathrm{e}^{-a_+ t} + v_-\mathrm{e}^{-a_- t}\,,\\
&u_\pm = \pm\frac{\Omega v(0) + (a_0 - a_\mp)\,u(0)}{a_+ - a_-}\,, \qquad v_\pm = \mp\frac{\Omega u(0) + \big[a_\mp - \gamma\,(2\overline{n}+1)\big]\,v(0)}{a_+ - a_-}\,.
\end{aligned}$$

Then, the general solution of the Bloch equations (in the subcase $4\Omega \neq \gamma\,|2\overline{n}+1-4k_{\mathrm{d}}|$) is

$$x(t) = \mathrm{e}^{-a_0 t}x(0), \qquad y(t) = u(t) + y_{\mathrm{eq}}\,, \qquad z(t) = v(t) + z_{\mathrm{eq}}\,,$$

$$u(0) = y(0) - y_{\mathrm{eq}}\,, \qquad v(0) = z(0) - z_{\mathrm{eq}}\,.$$

8.2.2.3 No Laser Stimulation

Let us consider the case of no laser acting on the atom, $\Omega = 0$; again the emission is not forgotten, $\gamma > 0$. In this case the eigenvalues of A are

$$\gamma\,(2\overline{n}+1) \qquad \text{and} \qquad \frac{\gamma}{2}\,(2\overline{n}+1+4k_{\mathrm{d}}) \pm \mathrm{i}\Delta\omega. \tag{8.41}$$

All the eigenvalues have real parts not less than $\gamma/2$. Let us stress that for $\Omega = 0$ the laser frequency ω has no physical meaning and we can take $\Delta\omega = 0$, so that this is a subcase of the previous one.

As in the previous case we get from (8.38) $\lim_{t\to+\infty} \vec{x}(t) = -\frac{\gamma}{A}\begin{pmatrix}0\\0\\1\end{pmatrix}$, i.e.

$$\lim_{t\to+\infty} x(t) = 0\,, \qquad \lim_{t\to+\infty} y(t) = 0\,, \qquad \lim_{t\to+\infty} z(t) = -\frac{1}{2\overline{n}+1}\,.$$

This gives

$$\lim_{t\to+\infty} \breve{\eta}(t) = \frac{\overline{n}}{2\overline{n}+1}\,P_+ + \frac{\overline{n}+1}{2\overline{n}+1}\,P_-\,.$$

From the form of the eigenvalues (8.41), one sees that the main role of the dephasing term is to decrease the decay time of the off-diagonal matrix elements of the a priori states.

In the case $\overline{n} = 0$ the atom decays in the ground state P_-. In the case $\overline{n} > 0$ we can use the black-body like parametrisation

$$\overline{n} = \frac{1}{\mathrm{e}^{\omega_0/(KT)} - 1}\,, \tag{8.42}$$

where K is the Boltzman constant. Then we can write

$$\lim_{t\to+\infty} \breve{\eta}(t) = \lim_{t\to+\infty} \eta(t) = \frac{\mathrm{e}^{-H_0/(KT)}}{\mathrm{Tr}\left\{\mathrm{e}^{-H_0/(KT)}\right\}} \tag{8.43}$$

and T represents the absolute temperature of a thermal bath in contact with the atom. Equation (8.43) says that the equilibrium state of the atom is the *canonical state* at temperature T for a system with Hamiltonian H_0 and this justifies the form of the dissipative terms used to model the interaction with a thermal bath.

8.2.3 The Evolution in the Generic Case

Let us now consider the case of the full matrix A, with possibly all the parameters different from zero: $\gamma > 0$, $\Omega \geq 0$, $\overline{n} \geq 0$, $k_{\mathrm{d}} \geq 0$, $\Delta\omega \in \mathbb{R}$. The eigenvalues

are given by the solutions of a third order algebraic equation for which an explicit formula exists, but it is not easy to use when parameters are present. It is better to look for qualitative results.

Lemma 8.3. *Let us assume* $\gamma > 0$ *and define*

$$B := \gamma\,(2\overline{n}+1)\,b\mathbb{1} - A\,, \qquad b := \min\left\{1, \frac{1}{2} + \frac{2k_{\mathrm{d}}}{2\overline{n}+1}\right\} \geq \frac{1}{2}\,.$$

Then, e^{Bt} *is a semigroup of contractions, i.e.*

$$\left|\mathrm{e}^{Bt}\vec{x}\right| \leq |\vec{x}|\,, \qquad \forall \vec{x} \in \mathbb{R}^3.$$

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\mathrm{e}^{Bt}\vec{x}\right|^2 = \langle \mathrm{e}^{Bt}\vec{x}|\left(B + B^{\mathrm{T}}\right)\mathrm{e}^{Bt}\vec{x}\rangle,$$

$$B + B^{\mathrm{T}} = 2\gamma\,(2\overline{n}+1)\,b\mathbb{1} - A - A^{\mathrm{T}} = \mathrm{diag}\left(\varepsilon', \varepsilon', \varepsilon''\right);$$

from the explicit expressions of ε' and ε'' one can check that $\varepsilon' \leq 0$, $\varepsilon'' \leq 0$. So, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\mathrm{e}^{Bt}\vec{x}\right|^2 \leq 0, \qquad \forall \vec{x} \in \mathbb{R}^3, \quad \forall t \geq 0$$

and this gives the result. What we have proved is a very particular case of the connection between dissipative operators in Hilbert spaces and semigroups of contractions [5, pp. 82–88]. □

In the following we shall need many times the resolvent of the matrix A.

Lemma 8.4. *Let us assume* $\gamma > 0$. *For* $\zeta \in \mathbb{C}$ *with* $\mathrm{Re}\,\zeta \geq 0$, *we have*

$$\det(A+\zeta) = \frac{N(\zeta)}{4}\,, \tag{8.44}$$

$$\begin{aligned} N(\zeta) := {} & 2\Omega^2\,(4\gamma k_{\mathrm{d}} + \zeta) + [\gamma\,(2\overline{n}+1) + \zeta\,] \\ & \times \left\{4\Delta\omega^2 + 2\Omega^2 + [\gamma\,(2\overline{n}+1+4k_{\mathrm{d}}) + 2\zeta\,]^2\right\}, \end{aligned} \tag{8.45}$$

$$\left(\frac{N(\zeta)}{A+\zeta}\right)_{11} = 4\Omega^2 + 2\,[\zeta + \gamma\,(2\overline{n}+1)]\,[2\zeta + \gamma\,(2\overline{n}+1+4k_{\mathrm{d}})]\,, \tag{8.46a}$$

$$\left(\frac{N(\zeta)}{A+\zeta}\right)_{12} = -4\,[\zeta + \gamma\,(2\overline{n}+1)]\,\Delta\omega\,, \qquad \left(\frac{N(\zeta)}{A+\zeta}\right)_{13} = 4\Omega\,\Delta\omega\,, \tag{8.46b}$$

$$\left(\frac{N(\zeta)}{A+\zeta}\right)_{21} = 4\left[\zeta + \gamma\left(2\overline{n}+1\right)\right]\Delta\omega\,, \tag{8.46c}$$

$$\left(\frac{N(\zeta)}{A+\zeta}\right)_{22} = 2\left[\zeta + \gamma\left(2\overline{n}+1\right)\right]\left[2\zeta + \gamma\left(2\overline{n}+1+4k_{\mathrm{d}}\right)\right], \tag{8.46d}$$

$$\left(\frac{N(\zeta)}{A+\zeta}\right)_{23} = -2\Omega\left[2\zeta + \gamma\left(2\overline{n}+1+4k_{\mathrm{d}}\right)\right], \tag{8.46e}$$

$$\left(\frac{N(\zeta)}{A+\zeta}\right)_{31} = 4\Omega\,\Delta\omega\,, \tag{8.46f}$$

$$\left(\frac{N(\zeta)}{A+\zeta}\right)_{32} = 2\Omega\left[2\zeta + \gamma\left(2\overline{n}+1+4k_{\mathrm{d}}\right)\right], \tag{8.46g}$$

$$\left(\frac{N(\zeta)}{A+\zeta}\right)_{33} = 4\Delta\omega^2 + \left[2\zeta + \gamma\left(2\overline{n}+1+4k_{\mathrm{d}}\right)\right]^2. \tag{8.46h}$$

Proof. By direct computations we get

$$\begin{aligned}
4\det\left(A+\zeta\right) &= 4\left[\frac{\gamma}{2}\left(2\overline{n}+1+4k_{\mathrm{d}}\right)+\zeta\right]^2\left[\gamma\left(2\overline{n}+1\right)+\zeta\right] \\
&\quad + 4\Omega^2\left[\frac{\gamma}{2}\left(2\overline{n}+1+4k_{\mathrm{d}}\right)+\zeta\right] + 4\Delta\omega^2\left[\gamma\left(2\overline{n}+1\right)+\zeta\right] \\
&= \left[\gamma\left(2\overline{n}+1\right)+\zeta\right]\left\{4\Delta\omega^2 + 2\Omega^2 + \left[\gamma\left(2\overline{n}+1+4k_{\mathrm{d}}\right)+2\zeta\right]^2\right\} \\
&\quad + 2\Omega^2\left(4\gamma k_{\mathrm{d}}+\zeta\right)
\end{aligned}$$

and Eqs. (8.46). □

8.3 The Equilibrium State η_{eq}

For $\gamma > 0$, the model we have constructed has a unique equilibrium state in the rotating frame. As already said, we take always $\gamma > 0$, which implies $\det A > 0$ and that (8.38) holds.

8.3.1 Convergence to Equilibrium

Let us summarise the situation for the a priori states; we have

$$\eta(t) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\check{\eta}(t)\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\,, \qquad \check{\eta}(t) = \frac{1}{2}\left(\mathbb{1} + \vec{x}(t)\cdot\vec{\sigma}\right), \tag{8.47}$$

$$\eta(0) = \check{\eta}(0) = \rho_0 = \frac{1}{2}\left(\mathbb{1} + \vec{x}(0)\cdot\vec{\sigma}\right) \in \mathcal{S}(\mathcal{H}), \qquad |\vec{x}(0)| \leq 1, \tag{8.48}$$

$$\vec{x}(t) = \mathrm{e}^{-At}\big(\vec{x}(0) - \vec{x}_{\mathrm{eq}}\big) + \vec{x}_{\mathrm{eq}}, \qquad \vec{x}_{\mathrm{eq}} := -\frac{\gamma}{A}\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{8.49}$$

Then, we define

$$\eta_{\mathrm{eq}} := \frac{1}{2}\big(\mathbb{1} + \vec{x}_{\mathrm{eq}}\cdot\vec{\sigma}\big). \tag{8.50}$$

Proposition 8.5. *In the hypotheses above, we have* $\eta_{\mathrm{eq}} \in \mathcal{S}(\mathcal{H})$ *and*

$$\lim_{t\to+\infty}\check{\eta}(t) = \lim_{t\to+\infty}\mathrm{e}^{\check{\mathcal{L}}t}[\rho_0] = \eta_{\mathrm{eq}}\,, \qquad \forall\rho_0 \in \mathcal{S}(\mathcal{H}). \tag{8.51}$$

Moreover, the following identities hold:

$$\left\|\eta(t) - \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\eta_{\mathrm{eq}}\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\right\|_1 = \left\|\check{\eta}(t) - \eta_{\mathrm{eq}}\right\|_1 = \left|\vec{x}(t) - \vec{x}_{\mathrm{eq}}\right|. \tag{8.52}$$

For the convergence to the equilibrium we have the estimates

$$\left|\vec{x}(t) - \vec{x}_{\mathrm{eq}}\right| \leq \mathrm{e}^{-\gamma(2\overline{n}+1)bt}\left|\vec{x}(0) - \vec{x}_{\mathrm{eq}}\right| \leq \mathrm{e}^{-\frac{\gamma}{2}(2\overline{n}+1)t}\left|\vec{x}(0) - \vec{x}_{\mathrm{eq}}\right|, \tag{8.53}$$

where

$$b := \min\left\{1, \frac{1}{2} + \frac{2k_{\mathrm{d}}}{2\overline{n}+1}\right\}.$$

Proof. By using expression (8.7) for the trace norm of the difference and the Bloch representations (8.39), (8.47), (8.50), we get Eq. (8.52).

With the notations of Lemma 8.3 we have $-A = B - \gamma\,(2\overline{n}+1)\,b\mathbb{1}$; then, Eq. (8.49) gives

$$\left|\vec{x}(t) - \vec{x}_{\mathrm{eq}}\right| = \left|\mathrm{e}^{-At}\big(\vec{x}(0) - \vec{x}_{\mathrm{eq}}\big)\right| = \mathrm{e}^{-\gamma(2\overline{n}+1)bt}\left|\mathrm{e}^{Bt}\big(\vec{x}(0) - \vec{x}_{\mathrm{eq}}\big)\right|.$$

By using the fact that e^{Bt} is a contraction and that $b \geq 1/2$, we have the inequalities (8.53).

Equations (8.52) and (8.53) give (8.51). $\check{\eta}(t)$ being a state for all t, then η_{eq} too is a state. □

The proposition above says that there is a unique *equilibrium state* $\eta_{\rm eq}$ (8.50) for the quantum dynamical semigroup $\exp\{\check{\mathcal{L}}t\}$. By equilibrium state we mean that it is a stationary state, i.e.

$$\mathrm{e}^{\check{\mathcal{L}}t}[\eta_{\rm eq}] = \eta_{\rm eq}\,, \tag{8.54}$$

which follows from (8.49) for $\vec{x}(0) = \vec{x}_{\rm eq}$, and that it is reached for large times, whatever the initial state be, as Eq. (8.51) says. We have also some information on the velocity of convergence to the equilibrium. Equations (8.52) and (8.53) say that the states $\check{\eta}(t)$ go to the equilibrium state exponentially with exponent at least $\gamma\,(2\overline{n}+1)\,b$, while the a priori states $\eta(t)$ go to a limit cycle in the same way.

8.3.2 The Explicit Expression of $\eta_{\rm eq}$

To write down the explicit expression of $\eta_{\rm eq}$ it is useful to introduce some shorthand notations:

$$\widetilde{\Gamma}^2 := \Gamma^2 + \frac{8\Omega^2 k_{\rm d}}{2\overline{n}+1}\,, \qquad \Gamma^2 := 2\Omega^2 + \widetilde{\gamma}^2\,, \tag{8.55a}$$

$$\widetilde{\gamma} := \gamma\,(2\overline{n}+1+4k_{\rm d})\,. \tag{8.55b}$$

Then, Lemma 8.4 and Eq. (8.49) give

$$4\det A = N(0) = \gamma\,(2\overline{n}+1)\left(4\Delta\omega^2 + \widetilde{\Gamma}^2\right), \tag{8.56}$$

$$\left(\frac{N(0)}{A}\right)_{13} = 4\Omega\,\Delta\omega\,, \quad \left(\frac{N(0)}{A}\right)_{23} = -2\Omega\widetilde{\gamma}, \quad \left(\frac{N(0)}{A}\right)_{33} = 4\Delta\omega^2 + \widetilde{\gamma}^2\,,$$

$$\vec{x}_{\rm eq} \equiv \lim_{t\to+\infty}\vec{x}(t) = -\frac{\gamma}{A}\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{-1}{(2\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)}\begin{pmatrix}4\Omega\,\Delta\omega\\-2\Omega\widetilde{\gamma}\\4\Delta\omega^2+\widetilde{\gamma}^2\end{pmatrix}. \tag{8.57}$$

The way in which $\widetilde{\Gamma}$ appears in the denominator allows to interpret it as the width of the response in function of the detuning.

By Eqs. (8.50) and (8.57) we get the matrix elements of the equilibrium state:

$$\left(\eta_{\rm eq}\right)_{11} = \frac{\Omega^2\,(1+4k_{\rm d}) + \overline{n}\left(4\Delta\omega^2+\Gamma^2\right)}{(2\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)}, \tag{8.58a}$$

$$\left(\eta_{\rm eq}\right)_{12} = \frac{-\Omega\left(2\Delta\omega + \mathrm{i}\widetilde{\gamma}\right)}{(2\overline{n}+1)\left(4\Delta\omega^2 + \widetilde{\Gamma}^2\right)} = \overline{\left(\eta_{\rm eq}\right)_{21}}, \tag{8.58b}$$

$$\left(\eta_{\rm eq}\right)_{22} = \frac{(\overline{n}+1)\left(4\Delta\omega^2 + \widetilde{\gamma}^2\right) + \Omega^2\left(2\overline{n} + 1 + 4k_{\rm d}\right)}{(2\overline{n}+1)\left(4\Delta\omega^2 + \widetilde{\Gamma}^2\right)}. \tag{8.58c}$$

8.3.3 Some Properties of $\eta_{\rm eq}$

The three components of the vector (8.57) or the three matrix components (8.58) determine completely the equilibrium state $\eta_{\rm eq}$. There exist some functions of these components which are particularly significative and which play a relevant role in the study of the properties of the light emitted by the atom.

Let us start by recalling some traditional terminology [6, p. 148]. Given a state $\rho = \frac{1}{2}\left(\mathbb{1} + \vec{x}\cdot\vec{\sigma}\right)$, the matrix elements $\rho_{11} = \frac{1}{2}(1+z)$ and $\rho_{22} = \frac{1}{2}(1-z)$ are called the *populations of the excited and the ground state levels*, respectively. The off-diagonal elements $\rho_{12} = \frac{1}{2}(x - \mathrm{i}y)$ and $\rho_{21} = \frac{1}{2}(x + \mathrm{i}y)$ are called *coherences*. Then, $z = \rho_{11} - \rho_{22}$ is the *population difference*. In usual situations the ground state is more populated than the excited one, which means $z < 0$; for instance in a thermal state, which corresponds to our equilibrium state with $\Omega = 0$, one has $z = -\frac{1}{2\overline{n}+1} < 0$. In contrast to the usual situations, when $z > 0$ one says that there is a *population inversion*.

By inspection of (8.57) we see that for the equilibrium state of the model of this chapter the coherences vanish only if $\Omega = 0$ and that there is not an inversion of population for the possible choices of the parameters. We shall see a case with population inversion in Chap. 10.

Let us now introduce other interesting parameters.

8.3.3.1 The Vertical and Equatorial Components of the Bloch Vector

Two interesting quantities are the lengths of the vertical and equatorial components of the Bloch vector,

$$\ell_{\parallel} := \left|z_{\rm eq}\right| = \frac{4\Delta\omega^2 + \widetilde{\gamma}^2}{(2\overline{n}+1)\left(4\Delta\omega^2 + \widetilde{\Gamma}^2\right)}, \tag{8.59}$$

$$\ell_{\perp} := \sqrt{x_{\rm eq}^2 + y_{\rm eq}^2} = \frac{2\Omega\sqrt{4\Delta\omega^2 + \widetilde{\gamma}^2}}{(2\overline{n}+1)\left(4\Delta\omega^2 + \widetilde{\Gamma}^2\right)}. \tag{8.60}$$

Let us note that $2\Omega\ell_{\parallel} = \sqrt{4\Delta\omega^2 + \widetilde{\gamma}^2}\,\ell_{\perp}$.

The maximum of $\ell_{\perp}$, as a function of Ω^2, is for

$$2\Omega^2\left(1+\frac{4k_{\rm d}}{2\overline{n}+1}\right)=4\Delta\omega^2+\widetilde{\gamma}^2. \tag{8.61}$$

Under this constraint we have

$$4\Delta\omega^2+\widetilde{\Gamma}^2=2\left(4\Delta\omega^2+\widetilde{\gamma}^2\right),\qquad \left(\eta_{\rm eq}\right)_{11}=\frac{1}{4}+\frac{\overline{n}}{2\left(2\overline{n}+1\right)},$$

$$\ell_\perp{}^2\equiv x_{\rm eq}^2+y_{\rm eq}^2=\frac{1}{2\left(2\overline{n}+1\right)\left(2\overline{n}+1+4k_{\rm d}\right)},$$

$$\ell_\parallel{}^2\equiv z_{\rm eq}^2=\frac{1}{4\left(2\overline{n}+1\right)^2},\qquad z_{\rm eq}=-\frac{1}{2\left(2\overline{n}+1\right)}.$$

The maximum of $\ell_\perp$ with respect to all the parameters is for

$$2\Omega^2=4\Delta\omega^2+\gamma^2,\qquad \overline{n}=0,\qquad k_{\rm d}=0.$$

Under this constraint we have $x_{\rm eq}^2+y_{\rm eq}^2=\frac{1}{2}$ and $z_{\rm eq}^2=\frac{1}{4}$.

8.3.3.2 Linear Entropy

The purity of a state can be measured by the von Neumann entropy (6.1a) which is non-negative and which is zero if and only if the state is pure. But a much easier measure of the purity of a state $\rho=\frac{1}{2}\left(\mathbb{1}+\vec{x}\cdot\vec{\sigma}\right)$ is the *linear entropy* [6, p. 82]

$$\mathrm{LE}_\rho:=\mathrm{Tr}\left\{\rho(\mathbb{1}-\rho)\right\}=\frac{1}{2}\left(1-|\vec{x}|^2\right). \tag{8.62}$$

In the case of a pure state the linear entropy is zero, while the maximum $\frac{1}{2}$ is reached for the completely mixed state $\frac{1}{2}\,\mathbb{1}$: $0\le\mathrm{LE}_\rho\le 1/2$.

The Linear Entropy of the Equilibrium State

In the case of the equilibrium state we have

$$\begin{aligned}\mathrm{LE}_{\eta_{\rm eq}}&=\frac{1-\left|\vec{x}_{\rm eq}\right|^2}{2}\equiv\frac{1-\left(\ell_\perp{}^2+\ell_\parallel{}^2\right)}{2}\\&=\frac{2\overline{n}\left(\overline{n}+1\right)}{\left(2\overline{n}+1\right)^2}+\frac{2\Omega^2\left[\Omega^2+\frac{4k_{\rm d}}{2\overline{n}+1}\left(4\Delta\omega^2+\Gamma^2+\frac{4\Omega^2k_{\rm d}}{2\overline{n}+1}\right)\right]}{\left(2\overline{n}+1\right)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}.\end{aligned} \tag{8.63}$$

Let us note that the equilibrium state is not pure unless $\Omega=0$ and $\overline{n}=0$.

In the case $\overline{n} = 0$, $k_{\mathrm{d}} = 0$, the linear entropy reduces to

$$\mathrm{LE}_{\eta_{\mathrm{eq}}} = \frac{2\Omega^4}{\left(4\Delta\omega^2 + \Gamma^2\right)^2} \equiv \frac{2\Omega^4}{\left(4\Delta\omega^2 + 2\Omega^2 + \gamma^2\right)^2}. \tag{8.64}$$

8.3.3.3 Atomic Squeezing

In [7] Walls and Zoller suggested to define the concept of "atomic squeezing" by starting from the Heisenberg–Robertson uncertainty relations for σ_x and σ_y.

Let $\rho = \frac{1}{2}\left(\mathbb{1} + \vec{x}\cdot\vec{\sigma}\right)$ be a fixed state and, for any self-adjoint operator a, let us denote by $\langle a\rangle := \mathrm{Tr}\{\rho a\}$ the quantum mean of a and by $\Delta a := \sqrt{\mathrm{Tr}\{\rho a^2\} - \langle a\rangle^2}$ its quantum standard deviation. Then, the commutation relations (8.2) give the Heisenberg uncertainty relations $\Delta\sigma_x\,\Delta\sigma_y \geq \frac{1}{2}\left|\langle[\sigma_x, \sigma_y]\rangle\right| = |\langle\sigma_z\rangle|$, which become $\sqrt{(1-x^2)(1-y^2)} \geq |z|$ in terms of the component of the Bloch vector. Let us denote by $\sigma_\perp$ and $\sigma_\top$ two orthogonal equatorial components of $\vec{\sigma}$ such that $\sigma_\perp$ has the minimal quantum variance on ρ. Then, one has

$$\Delta\sigma_\perp\,\Delta\sigma_\top \geq |\langle\sigma_z\rangle| = |z| \tag{8.65}$$

and

$$\begin{aligned}\Delta\sigma_\perp^{\,2} &:= \inf_{n_1^2+n_2^2=1}\left[\mathrm{Tr}\left\{\rho\left(n_1\sigma_x + n_2\sigma_y\right)^2\right\} - \left(\mathrm{Tr}\left\{\rho\left(n_1\sigma_x + n_2\sigma_y\right)\right\}\right)^2\right] \\ &= 1 - \sup_{n_1^2+n_2^2=1}\left[n_1 x + n_2 y\right]^2.\end{aligned}$$

The minimum is reached for $(n_1, n_2) = \pm\frac{(x,y)}{\sqrt{x^2+y^2}}$ and we get

$$\Delta\sigma_\perp^{\,2} = 1 - \left[x^2 + y^2\right].$$

According [7], we say that we have atomic squeezing in ρ when

$$\Delta\sigma_\perp^{\,2} < |\langle\sigma_z\rangle| \equiv |z|\,;$$

this gives $1 - x^2 - y^2 < |z|$. We can call the quantity

$$\mathrm{AS}_\rho := 1 - \left(x^2 + y^2\right) - |z| \tag{8.66}$$

the *atomic squeezing parameter* of the state $\rho = \frac{1}{2}\left(\mathbb{1} + \vec{x}\cdot\vec{\sigma}\right)$ and we say that we have atomic squeezing in this state when $\mathrm{AS}_\rho < 0$. It is easy to check that

$$\min_\rho \mathrm{AS}_\rho = -\frac{1}{4} \tag{8.67}$$

and that this minimum is reached in a state with $x^2+y^2=\frac{3}{4}$ and $z^2=\frac{1}{4}$, which is necessarily a pure state and, so, has zero entropy.

For a pure state $x^2+y^2+z^2=1$ holds and we get $\mathrm{AS}_\rho=|z|\,(|z|-1)$, which is always strictly negative, but in the poles, $z=\pm1$, and on the equator, $z=0$, where it vanishes. Therefore, by representing the states on the Bloch sphere, we see that we have atomic squeezing for states belonging to the region delimited by the surface of the sphere and the surface $\mathrm{AS}_\rho=0$, or $x^2+y^2+|z|=1$. Another way to see this region is: fix $|z|\in(0,1)$; then, $\mathrm{AS}_\rho<0$ for $\sqrt{1-|z|}<\sqrt{x^2+y^2}\leq\sqrt{1-z^2}$. Let us note that

$$\mathrm{AS}_\rho=0 \quad\Leftrightarrow\quad \mathrm{LE}_\rho=\frac{1}{2}\,|z|\,(1-|z|)\,, \tag{8.68a}$$

$$\mathrm{AS}_\rho<0 \quad\Leftrightarrow\quad \mathrm{LE}_\rho<\frac{1}{2}\,|z|\,(1-|z|) \quad\Rightarrow\quad \mathrm{LE}_\rho<\frac{1}{8}\,. \tag{8.68b}$$

The Atomic Squeezing of the Equilibrium State

In the case of $\eta_{\rm eq}$ we have $z_{\rm eq}<0$ and

$$\begin{aligned}\mathrm{AS}_{\eta_{\rm eq}}&=1-\left|\vec{x}_{\rm eq}\right|^2+z_{\rm eq}\left[1+z_{\rm eq}\right]=1-\ell_\perp^2-\ell_\parallel\\&=-\frac{\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)\left[2\overline{n}\left(4\Delta\omega^2+\Gamma^2\right)+2\Omega^2\left(1+4k_{\rm d}\right)\right]}{(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}\\&\quad+\frac{4\overline{n}\,(\overline{n}+1)}{(2\overline{n}+1)^2}+\frac{4\Omega^2\left[\Omega^2+\frac{4k_{\rm d}}{2\overline{n}+1}\left(4\Delta\omega^2+\Gamma^2+\frac{4\Omega^2k_{\rm d}}{2\overline{n}+1}\right)\right]}{(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}.\end{aligned} \tag{8.69}$$

In the case $\overline{n}=0$, $k_{\rm d}=0$, the atomic squeezing parameter reduces to

$$\mathrm{AS}_{\eta_{\rm eq}}=\frac{2\Omega^2}{\left(4\Delta\omega^2+\gamma^2+2\Omega^2\right)^2}\left(2\Omega^2-4\Delta\omega^2-\gamma^2\right). \tag{8.70}$$

In this case the condition $\mathrm{AS}_{\eta_{\rm eq}}<0$ for atomic squeezing becomes

$$0<2\Omega^2<4\Delta\omega^2+\gamma^2, \tag{8.71}$$

while the minimum value of $\mathrm{AS}_{\eta_{\rm eq}}$ is $-1/8$ and it is reached for $6\Omega^2=4\Delta\omega^2+\gamma^2$. Note that we have

$$\mathrm{LE}_{\eta_{\rm eq}}=\begin{cases}0, & \text{for } \Omega^2=0,\\ \frac{1}{32}, & \text{for } 6\Omega^2=4\Delta\omega^2+\gamma^2,\\ \frac{1}{8}, & \text{for } 2\Omega^2=4\Delta\omega^2+\gamma^2.\end{cases} \tag{8.72}$$

While suggestive, the definition of atomic squeezing given in [7] has been criticised and a different definition of spin squeezing or of squeezing of systems with many two-level atoms has been proposed [8]. This alternative definition is very useful and widely used in the literature by its links with "entanglement", but it gives that a single two-level atom cannot be spin squeezed, while the definition of the parameter $\mathrm{AS}_{\eta_{\mathrm{eq}}}$ is at least useful in discussing some global squeezing properties of the emitted light (see Eq. (9.70)), as already found in [7].

8.4 The SDEs for the Two-Level Atom

In order to illustrate the theory, we want to write down explicitly the various SDEs introduced in the previous chapters, just to show them in a concrete case, even if they cannot be solved explicitly.

8.4.1 The SDEs in the Hilbert Space

8.4.1.1 The Linear Stochastic Schrödinger Equation

The starting point, the linear stochastic Schrödinger equation (2.28), is

$$\mathrm{d}\psi(t) = K(t)\psi(t)\,\mathrm{d}t + \sum_{j=1}^{5} R_j(t)\psi(t)\,\mathrm{d}W_j(t)\,. \tag{8.73}$$

By passing to the rotating frame as done in Sect. 8.2.1, it is possible to eliminate part of the time dependence: we set

$$\varphi(t) := \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\,\psi(t)\,. \tag{8.74}$$

By differentiation we get

$$\mathrm{d}\varphi(t) = \check{K}\varphi(t)\,\mathrm{d}t + \sum_{j=1}^{5} \check{R}_j(t)\varphi(t)\,\mathrm{d}W_j(t)\,, \tag{8.75}$$

with

$$\check{K} := \frac{\mathrm{i}\omega}{2}\,\sigma_z + \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z} K(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\,, \tag{8.76a}$$

$$\check{R}_j(t) := \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z} R_j(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\,. \tag{8.76b}$$

By using $\mathrm{e}^{\mathrm{i}\varepsilon\sigma_z} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varepsilon} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varepsilon} \end{pmatrix}$, we can compute the explicit expressions of $\check{K}$ and $\check{R}_j(t)$ and we get

$$\check{K} = -\frac{\mathrm{i}}{2}\,\Delta\omega\sigma_z - \frac{\gamma}{2}\,P_+ - \frac{1}{2}\left[\gamma\,(\overline{n}+k_\mathrm{d}) + |\lambda|^2\right]\mathbb{1} - \frac{\mathrm{i}}{2}\sigma_+\,, \tag{8.77a}$$

$$\check{R}_5(t) = R_5 = \sqrt{\gamma k_\mathrm{d}}\,\sigma_z\,, \qquad \check{R}_j(t) = \sqrt{\gamma}\,\alpha_j(t)\sigma_-\,, \quad j=1,3, \tag{8.77b}$$

$$\check{R}_2(t) = \sqrt{\gamma}\,\alpha_2(t)\sigma_- + \lambda \mathrm{e}^{\mathrm{i}(\nu-\omega)t}\mathbb{1}\,, \qquad \check{R}_4(t) = \sqrt{\gamma}\,\overline{\alpha_3(t)}\,\sigma_+\,, \tag{8.77c}$$

$$\alpha_j(t) := \mathrm{e}^{\mathrm{i}(\nu-\omega)t}\alpha_j\,, \quad j=1,2, \qquad \alpha_3(t) := \mathrm{e}^{\mathrm{i}(\nu-\omega)t}\sqrt{\overline{n}}\,. \tag{8.77d}$$

By using the components, Eq. (8.75) is

$$\begin{aligned}\mathrm{d}\varphi_1(t) = &-\frac{1}{2}\left[\mathrm{i}\Delta\omega + \gamma\,(\overline{n}+1+k_\mathrm{d}) + |\lambda|^2\right]\varphi_1(t)\,\mathrm{d}t - \frac{\mathrm{i}}{2}\,\Omega\varphi_2(t)\,\mathrm{d}t \\ &+ \lambda\mathrm{e}^{\mathrm{i}(\nu-\omega)t}\varphi_1(t)\mathrm{d}W_2(t) + \sqrt{\gamma}\,\overline{\alpha_4(t)}\,\varphi_2(t)\,\mathrm{d}W_4(t) + \sqrt{\gamma k_\mathrm{d}}\,\varphi_1(t)\,\mathrm{d}W_5(t)\,,\end{aligned} \tag{8.78a}$$

$$\begin{aligned}\mathrm{d}\varphi_2(t) = &\frac{1}{2}\left[\mathrm{i}\Delta\omega - \gamma\,(\overline{n}+k_\mathrm{d}) - |\lambda|^2\right]\varphi_2(t)\,\mathrm{d}t + \lambda\mathrm{e}^{\mathrm{i}(\nu-\omega)t}\varphi_2(t)\mathrm{d}W_2(t) \\ &+\sqrt{\gamma}\varphi_1(t)\sum_{j=1}^{3}\alpha_j(t)\,\mathrm{d}W_j(t) - \sqrt{\gamma k_\mathrm{d}}\,\varphi_2(t)\,\mathrm{d}W_5(t)\,.\end{aligned} \tag{8.78b}$$

It seems very difficult to succeed in getting a solution of this equation in closed form, even in the simplest non-trivial cases:
(1) $\Delta\omega = 0$, $\nu = \omega$, $k_\mathrm{d} = 0$, $\overline{n} = 0$

$$\mathrm{d}\varphi_1(t) = -\frac{1}{2}\left(\gamma + |\lambda|^2\right)\varphi_1(t)\,\mathrm{d}t - \frac{\mathrm{i}}{2}\,\Omega\varphi_2(t)\,\mathrm{d}t + \lambda\varphi_1(t)\mathrm{d}W_2(t)\,, \tag{8.79a}$$

$$\mathrm{d}\varphi_2(t) = -\frac{|\lambda|^2}{2}\,\varphi_2(t)\,\mathrm{d}t + \lambda\varphi_2(t)\mathrm{d}W_2(t) + \sqrt{\gamma}\varphi_1(t)\sum_{j=1}^{2}\alpha_j\,\mathrm{d}W_j(t)\,; \tag{8.79b}$$

(2) $\Delta\omega = 0$, $\nu = \omega$, $k_\mathrm{d} = 0$, $\Omega = 0$

$$\mathrm{d}\varphi_1(t) = -\frac{\gamma\,(\overline{n}+1)}{2}\,\varphi_1(t)\,\mathrm{d}t + \sqrt{\gamma}\,\overline{\alpha_3(0)}\,\varphi_2(t)\,\mathrm{d}W_4(t)\,, \tag{8.80a}$$

$$\mathrm{d}\varphi_2(t) = -\frac{\gamma\overline{n}}{2}\,\varphi_2(t)\,\mathrm{d}t + \sqrt{\gamma}\varphi_1(t)\sum_{j=1}^{3}\alpha_j(0)\,\mathrm{d}W_j(t)\,. \tag{8.80b}$$

8.4.1.2 The Probability Density and the New Wiener Process

The density of the physical probability $\widehat{\mathbb{P}}^t_{\psi_0}$ is given by the square norm of the solution of the linear SDE and the new Wiener process is determined by the functions $m_j(t)$, see Eqs. (2.23), (2.24), (2.31). Let us note that both quantities (density and new Wiener process) are the same either if we start from the original SDE (8.73) or if we use the rotated SDE (8.75); indeed, by Eqs. (8.74) and (8.76b) we get

$$\|\psi(t)\|^2 = \|\varphi(t)\|^2\,, \tag{8.81}$$

$$m_j(t) = 2\,\mathrm{Re}\,\frac{\langle\psi(t)|R_j(t)\psi(t)\rangle}{\|\psi(t)\|^2} = 2\,\mathrm{Re}\,\frac{\langle\varphi(t)|\check{R}_j(t)\varphi(t)\rangle}{\|\varphi(t)\|^2}\,. \tag{8.82}$$

8.4.1.3 The Stochastic Schrödinger Equation

Just to have an example of a nonlinear SDE for Hilbert space vectors, let us concretise in our case Eq. (2.52) of type $\ell = 4$, always in the rotating frame:

$$\begin{aligned}
\mathrm{d}X_1(t) = {} & -\frac{1}{2}\left[\mathrm{i}\Delta\omega + \gamma\,(\overline{n}+1)\right] X_1(t)\,\mathrm{d}t - \frac{\mathrm{i}}{2}\,\Omega\,X_2(t)\,\mathrm{d}t \\
& - \gamma\,\frac{|X_2(t)|^2\,|X_1(t)|^2}{2\,\|X(t)\|^4}\,X_1(t)\mathrm{d}t - \frac{\overline{X_2(t)}X_1(t)^2}{\|X(t)\|^2}\,\sqrt{\gamma}\sum_{j=1}^{3}\alpha_j(t)\mathrm{d}\widehat{W}_j(t) \\
& + \gamma\,(\overline{n}-2k_\mathrm{d})\,\frac{|X_2(t)|^4}{\|X(t)\|^4}\,X_1(t)\mathrm{d}t + \mathrm{i}\,\frac{\Omega\,X_1(t)\,\mathrm{Re}\left(\overline{X_1(t)}X_2(t)\right)}{2\,\|X(t)\|^2}\,\mathrm{d}t \\
& + \sqrt{\gamma}\,\overline{\alpha_3(t)}\,\frac{|X_2(t)|^2\,X_2(t)}{\|X(t)\|^2}\,\mathrm{d}\widehat{W}_4(t) + 2\sqrt{\gamma k_\mathrm{d}}\,\frac{|X_2(t)|^2\,X_1(t)}{\|X(t)\|^2}\,\mathrm{d}\widehat{W}_5(t)\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}X_2(t) = {} & \frac{\mathrm{i}}{2}\,\Delta\omega X_2(t)\,\mathrm{d}t - \frac{\mathrm{i}}{2}\,\Omega\,X_1(t)\,\mathrm{d}t - \frac{\gamma}{2}\,\overline{n}\,X_2(t)\mathrm{d}t \\
& + \frac{\gamma\,|X_1(t)|^2\,X_2(t)\left(2\,|X_1(t)|^2 + |X_2(t)|^2\right)}{2\,\|X(t)\|^4}\,\mathrm{d}t + \frac{\mathrm{i}\Omega\,X_2(t)\,\mathrm{Re}\left(\overline{X_1(t)}X_2(t)\right)}{2\,\|X(t)\|^2}\,\mathrm{d}t \\
& + \frac{|X_1(t)|^2\,X_1(t)}{\|X(t)\|^2}\,\sqrt{\gamma}\sum_{j=1}^{3}\alpha_j(t)\,\mathrm{d}\widehat{W}_j(t) + \gamma\,(\overline{n}-2k_\mathrm{d})\,\frac{|X_1(t)|^4}{\|X(t)\|^4}\,X_2(t)\mathrm{d}t \\
& - \sqrt{\gamma}\,\overline{\alpha_3(t)}\,\frac{\overline{X_1(t)}X_2(t)^2}{\|X(t)\|^2}\,\mathrm{d}\widehat{W}_4(t) - 2\sqrt{\gamma k_\mathrm{d}}\,\frac{|X_1(t)|^2\,X_2(t)}{\|X(t)\|^2}\,\mathrm{d}\widehat{W}_5(t)\,.
\end{aligned}$$

If $X(0) = \varphi(0)$ and $\|\varphi(0)\| = 1$ the solution of this nonlinear SDE is linked to $\varphi(t)$ by

$$X(t)=\exp\left\{-\mathrm{i}\sum_{j=1}^{5}\int_0^t \mathrm{Re}\, n_j(s)\,\mathrm{Im}\, n_j(s)\,\mathrm{d}s-\mathrm{i}\sum_{j=1}^{5}\int_0^t \mathrm{Im}\, n_j(s)\,\mathrm{d}\widehat{W}_j(s)\right\}\frac{1}{\|\varphi(t)\|}\,\varphi(t),$$

$$n_j(t)=\sqrt{\gamma}\,\frac{\alpha_j(t)\overline{\varphi_2(t)}\varphi_1(t)}{\|\varphi(t)\|^2}\,,\qquad j=1,3,$$

$$n_2(t)=\sqrt{\gamma}\alpha_2(t)\,\frac{\overline{\varphi_2(t)}\varphi_1(t)}{\|\varphi(t)\|^2}-\lambda\mathrm{e}^{\mathrm{i}(\nu-\omega)t},$$

$$n_4(t)=\sqrt{\gamma}\,\frac{\overline{\alpha_3(t)\varphi_1(t)}\varphi_2(t)}{\|\varphi(t)\|^2}\,,\qquad n_5(t)=\sqrt{\gamma k_\mathrm{d}}\,\frac{|\varphi_1(t)|^2-|\varphi_2(t)|^2}{\|\varphi(t)\|^2}\,,$$

$$\mathrm{d}\widehat{W}_j(t)=\mathrm{d}W_j(t)-2\,\mathrm{Re}\, n_j(t)\,\mathrm{d}t\,,$$

cf. Eqs. (2.22), (2.23), (2.47), (2.50), (2.51).

We recall that, when explicit analytic solutions are not viable, the nonlinear stochastic Schrödinger equation is the best starting point for numerical simulations.

8.4.2 The Stochastic Master Equation

From the theoretical point of view the linear and nonlinear stochastic master equations are very important because they contain the whole information on probabilities and a priori and a posteriori states of the continuously observed quantum system; moreover, they involve only physically relevant parameters.

8.4.2.1 The Linear Stochastic Master Equation

Let us start from the linear stochastic master equation for trace-class operators; again the transformation to the rotating frame eliminates some explicit time dependence in the coefficients. By defining

$$\check{\sigma}(t):=\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}\sigma(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z},\tag{8.83}$$

we get the linear SDE in operator form

$$\mathrm{d}\check{\sigma}(t)=\check{\mathcal{L}}\,[\check{\sigma}(t)]\,\mathrm{d}t+\left[\check{R}_1(t)\check{\sigma}(t)+\check{\sigma}(t)\check{R}_1(t)^*\right]\mathrm{d}W_1(t).\tag{8.84}$$

Because of the unitarity of the transformation (8.83), to use $\sigma(t)$ or $\check{\sigma}(t)$ to construct the probability density is the same; the probability density, giving the physical probabilities $\mathbb{P}^t_{\rho_0}$, is

$$p(t) := \mathrm{Tr}\{\sigma(t)\} = \mathrm{Tr}\,\{\check{\sigma}(t)\}\,. \tag{8.85}$$

By using the representation on the Pauli basis

$$\check{\sigma}(t) = \left[p(t) + \vec{\mathsf{s}}(t)\cdot\vec{\sigma}\,\right], \tag{8.86}$$

the linear SDE (8.84) becomes

$$\mathrm{d}p(t) = \sqrt{\gamma}\,[\mathsf{s}_1(t)\,\mathrm{Re}\,\alpha_1(t) + \mathsf{s}_2(t)\,\mathrm{Im}\,\alpha_1(t)]\,\mathrm{d}W_1(t), \tag{8.87a}$$

$$\begin{aligned}\mathrm{d}\vec{\mathsf{s}}(t) = -A\vec{\mathsf{s}}(t)\,\mathrm{d}t - \begin{pmatrix}0\\0\\\gamma\,p(t)\end{pmatrix}\mathrm{d}t \\ + \sqrt{\gamma}\begin{pmatrix}[p(t)+\mathsf{s}_3(t)]\,\mathrm{Re}\,\alpha_1(t)\\ [p(t)+\mathsf{s}_3(t)]\,\mathrm{Im}\,\alpha_1(t)\\ -[\mathsf{s}_1(t)\,\mathrm{Re}\,\alpha_1(t)+\mathsf{s}_2(t)\,\mathrm{Im}\,\alpha_1(t)]\end{pmatrix}\mathrm{d}W_1(t),\end{aligned} \tag{8.87b}$$

where

$$\alpha_1(t) = \mathrm{e}^{\mathrm{i}(\nu-\omega)t+\mathrm{i}\vartheta}\,|\alpha_1|\,. \tag{8.88}$$

8.4.2.2 The Nonlinear Stochastic Master Equation for Density Matrices

Let us now consider the a posteriori states

$$\rho(t) = \frac{1}{p(t)}\,\sigma(t) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\check{\rho}(t)\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}, \tag{8.89}$$

where

$$\check{\rho}(t) = \frac{1}{p(t)}\,\check{\sigma}(t)\,. \tag{8.90}$$

As in the general case, we get the nonlinear SDE in operator form

$$\mathrm{d}\check{\rho}(t) = \check{\mathcal{L}}\,[\check{\rho}(t)]\,\mathrm{d}t + \left(\check{R}_1(t)\check{\rho}(t) + \check{\rho}(t)\check{R}_1(t)^* - v(t)\check{\rho}(t)\right)\mathrm{d}\widehat{W}_1(t), \tag{8.91}$$

where

$$v(t) = 2\,\mathrm{Re}\,\mathrm{Tr}\,\{R_1(t)\rho(t)\} = 2\,\mathrm{Re}\,\mathrm{Tr}\left\{\check{R}_1(t)\check{\rho}(t)\right\}, \tag{8.92}$$

and

$$\widehat{W}_1(t) = W_1(t) - \int_0^t v(s)\,\mathrm{d}s \tag{8.93}$$

is a Wiener process under the physical probability.

By using the Bloch representation

$$\check{\rho}(t) = \frac{1}{2}\left[1 + \vec{\mathrm{r}}(t) \cdot \vec{\sigma}\right], \tag{8.94}$$

we get

$$v(t) = \sqrt{\gamma}\left[\mathrm{r}_1(t)\,\mathrm{Re}\,\alpha_1(t) + \mathrm{r}_2(t)\,\mathrm{Im}\,\alpha_1(t)\right] \tag{8.95}$$

and

$$\begin{aligned}\mathrm{d}\vec{\mathrm{r}}(t) = &-A\vec{\mathrm{r}}(t)\,\mathrm{d}t - \begin{pmatrix}0\\0\\\gamma\end{pmatrix}\mathrm{d}t \\ &+\sqrt{\gamma}\begin{pmatrix}[1+\mathrm{r}_3(t)-\mathrm{r}_1(t)^2]\,\mathrm{Re}\,\alpha_1(t) - \mathrm{r}_1(t)\mathrm{r}_2(t)\,\mathrm{Im}\,\alpha_1(t)\\ [1+\mathrm{r}_3(t)-\mathrm{r}_2(t)^2]\,\mathrm{Im}\,\alpha_1(t) - \mathrm{r}_1(t)\mathrm{r}_2(t)\,\mathrm{Re}\,\alpha_1(t)\\ -[1+\mathrm{r}_3(t)]\left[\mathrm{r}_1(t)\,\mathrm{Re}\,\alpha_1(t) + \mathrm{r}_2(t)\,\mathrm{Im}\,\alpha_1(t)\right]\end{pmatrix}\mathrm{d}\widehat{W}_1(t).\end{aligned} \tag{8.96}$$

8.4.3 Linear Entropy and Atomic Squeezing

We have two typical parameters for states, the linear entropy (8.62) and the atomic squeezing (8.66), and four types of states, a priori and a posteriori states both before and after the transformation to the rotating frame, Eqs. (8.47), (8.89), (8.94). But the linear entropy is invariant under unitary transformations, which gives

$$\mathrm{LE}_{\eta(t)} = \mathrm{LE}_{\check{\eta}(t)} = \frac{1}{2}\left(1 - |\vec{x}(t)|^2\right), \tag{8.97a}$$

$$\mathrm{LE}_{\rho(t)} = \mathrm{LE}_{\check{\rho}(t)} = \frac{1}{2}\left(1 - |\vec{\mathrm{r}}(t)|^2\right), \tag{8.97b}$$

and the atomic squeezing parameter is invariant under rotations around the z-axis, which gives

$$\mathrm{AS}_{\eta(t)} = \mathrm{AS}_{\check{\eta}(t)} = 1 - \left(x(t)^2 + y(t)^2\right) - |z(t)|\,, \tag{8.97c}$$

$$\mathrm{AS}_{\rho(t)} = \mathrm{AS}_{\check{\rho}(t)} = 1 - \left(\mathrm{r}_1(t)^2 + \mathrm{r}_2(t)^2\right) - |\mathrm{r}_3(t)|\,. \tag{8.97d}$$

Let us introduce the means of the a posteriori linear entropy and atomic squeezing parameter:

$$\langle \mathrm{LE}\rangle(t) := \mathbb{E}^t_{\rho_0}\left[\mathrm{LE}_{\check{\rho}(t)}\right], \qquad \langle \mathrm{AS}\rangle(t) := \mathbb{E}^t_{\rho_0}\left[\mathrm{AS}_{\check{\rho}(t)}\right]. \tag{8.98}$$

Then, we get easily

$$\langle \mathrm{LE} \rangle(t) = \mathrm{LE}_{\check{\eta}(t)} - \sum_{j=1}^{3} \mathrm{Var}^t_{\rho_0} \left[\mathsf{r}_j(t) \right] \leq \mathrm{LE}_{\check{\eta}(t)} \,, \tag{8.99a}$$

$$\langle \mathrm{AS} \rangle(t) = \mathrm{AS}_{\check{\eta}(t)} - \sum_{j=1}^{2} \mathrm{Var}^t_{\rho_0} \left[\mathsf{r}_j(t) \right] - \left(\mathbb{E}^t_{\rho_0} \left[|\mathsf{r}_3(t)| \right] - \left| \mathbb{E}^t_{\rho_0} \left[\mathsf{r}_3(t) \right] \right| \right) \leq \mathrm{AS}_{\check{\eta}(t)} \,. \tag{8.99b}$$

Linear Entropy and Purification

The stochastic differential of the a posteriori linear entropy is

$$\begin{aligned} \mathrm{dLE}_{\check{\rho}(t)} = \frac{1}{2} \Big\{ -\widetilde{\gamma} \left[1 - |\vec{\mathsf{r}}(t)|^2 \right] + \gamma \, |\alpha_2|^2 \left[1 + \mathsf{r}_3(t) \right]^2 + 2\overline{n}\gamma \left(1 + \mathsf{r}_3(t)^2 \right) \\ + 4k_{\mathrm{d}}\gamma \left(1 - \mathsf{r}_3(t)^2 \right) + \left[1 - |\vec{\mathsf{r}}(t)|^2 \right] v(t)^2 \Big\} \mathrm{d}t - \left[1 - |\vec{\mathsf{r}}(t)|^2 \right] v(t) \, \mathrm{d}\widehat{W}_1(t) \,. \end{aligned}$$

Recall that $|\vec{\mathsf{r}}(t)|^2 \leq 1$ and $|\alpha_1(t)|^2 = |\alpha_1|^2 \leq 1$, which implies

$$\left[\mathsf{r}_1(t) \, \mathrm{Re}\, \alpha_1(t) + \mathsf{r}_2(t) \, \mathrm{Im}\, \alpha_1(t) \right]^2 \leq |\alpha_1|^2 \left[\mathsf{r}_1(t)^2 + \mathsf{r}_2(t)^2 \right] \leq |\vec{\mathsf{r}}(t)|^2 \leq 1 \,.$$

Then, the time derivative of the mean a posteriori linear entropy is given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathrm{LE} \rangle(t) \leq & -\frac{\gamma}{2} \, |\alpha_1|^2 \, \mathbb{E}^t_{\rho_0} \left[\left(1 - |\vec{\mathsf{r}}(t)|^2 \right) \left(1 - \mathsf{r}_1(t)^2 - \mathsf{r}_2(t)^2 \right) \right] \\ & + \frac{\gamma}{2} \, |\alpha_2|^2 \, \mathbb{E}^t_{\rho_0} \left[|\vec{\mathsf{r}}(t)|^2 + \mathsf{r}_3(t)^2 + 2\mathsf{r}_3(t) \right] \\ & + \overline{n}\gamma \mathbb{E}^t_{\rho_0} \left[|\vec{\mathsf{r}}(t)|^2 + \mathsf{r}_3(t)^2 \right] + 2k_{\mathrm{d}}\gamma \mathbb{E}^t_{\rho_0} \left[\mathsf{r}_1(t)^2 + \mathsf{r}_2(t)^2 \right] . \end{aligned}$$

In the case $\alpha_2 = 0, k_{\mathrm{d}} = 0, \overline{n} = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathrm{LE} \rangle(t) \leq 0 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathrm{LE} \rangle(t) = 0 \quad \Leftrightarrow \quad |\vec{\mathsf{r}}(t)| = 1 \text{ a.s.}$$

Moreover, the hypotheses of Theorem 5.12 hold and

$$\lim_{t \to +\infty} \langle \mathrm{LE} \rangle(t) = 0,$$

which means that the a posteriori states become pure a.s.

8.4.4 The Instruments in the Rotating Frame

Equation (8.84) shows that, in the rotating frame, the only possible non-homogeneity in time is in $\check{R}_1(t)$, defined by Eqs. (8.77b) and (8.88):

$$\check{R}_1(t) = \sqrt{\gamma}\,|\alpha_1|\,\mathrm{e}^{\mathrm{i}(\nu-\omega)t+\mathrm{i}\vartheta}\sigma_-. \tag{8.100}$$

This is true also for instruments and characteristic operators.

In analogy with Eq. (4.10)

$$\check{\mathcal{I}}_t^0(G)[\rho_0] = \mathbb{E}_{\mathbb{Q}}[1_G\,\check{\sigma}(t)], \qquad G \in \overline{\mathcal{G}}_t^0\,, \tag{8.101}$$

where $\overline{\mathcal{G}}_t^0$ is the augmented natural filtration of W_1. Then, by Eqs. (8.84), (8.83), (8.101), the original instruments, whose characteristic operator is generated by the operator (8.19), are given by

$$\mathcal{I}_t^0(G)[\rho_0] = \mathbb{E}_{\mathbb{Q}}[1_G\,\sigma(t)] = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z}\check{\mathcal{I}}_t^0(G)[\rho_0]\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}. \tag{8.102}$$

Finally, by (4.29) one can introduce the characteristic operator $\check{\mathcal{G}}(t,0;k)$ of the instruments $\check{\mathcal{I}}_t^0(\bullet)$ and its generator turns out to be

$$\check{\Lambda}_t(k)[\tau] = \check{\mathcal{L}}[\tau] + \mathrm{i}k\left(\check{R}_1(t)\tau + \tau\check{R}_1(t)^*\right) - \frac{1}{2}\,k^2\,\tau. \tag{8.103}$$

8.5 Summary

8.5.1 Bloch Representation of States and Terminology

- A statistical operator in the Bloch representation:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-\mathrm{i}y \\ x+\mathrm{i}y & 1-z \end{pmatrix} = \frac{1}{2}\,(\mathbb{1} + \vec{x}\cdot\vec{\sigma})\,, \qquad \vec{x}\in\mathbb{R}^3, \quad |\vec{x}| \le 1.$$

- Population of the excited state level: $\rho_{11} = \frac{1}{2}\,(1+z)$.
- Population of the ground state level: $\rho_{22} = \frac{1}{2}\,(1-z)$.
- Population difference: $z = \rho_{11} - \rho_{22}$.
- Coherences: $\rho_{12} = \frac{1}{2}\,(x-\mathrm{i}y)$ and $\rho_{21} = \frac{1}{2}\,(x+\mathrm{i}y)$.
- Atomic squeezing parameter: $\mathrm{AS}_\rho := 1 - \left(x^2+y^2\right) - |z|$.
- Linear entropy: $\mathrm{LE}_\rho := \mathrm{Tr}\,\{\rho(\mathbb{1}-\rho)\} = \frac{1}{2}\left(1-|\vec{x}|^2\right)$.

8.5.2 The Model and the Parameters

In the following all the formulae are given in the rotating frame.

8.5.2.1 The A Priori Dynamics

- Liouville operator:

$$\check{\mathcal{L}}[\tau] = -\mathrm{i}[\check{H}, \tau] + \gamma\,(\overline{n}+1)\,\sigma_-\tau\sigma_+ - \frac{\gamma}{2}\,P_+\tau - \frac{\gamma}{2}\,\tau P_+ + \gamma\,\overline{n}\sigma_+\tau\sigma_- + \gamma k_\mathrm{d}\sigma_z\tau\sigma_z - \gamma\,(\overline{n}+k_\mathrm{d})\,\tau.$$

- Effective Hamiltonian: $\check{H} = \frac{1}{2}\,\Delta\omega\sigma_z + \frac{1}{2}\,\Omega\sigma_x$.
- Coefficient of the thermal disturbance: $\overline{n} \geq 0$.
- Coefficient of the dephasing disturbance: $k_\mathrm{d} \geq 0$.
- Resonance frequency of the atom: $\omega_0 > 0$.
- Frequency of the stimulating laser: $\omega > 0$.
- Detuning: $\Delta\omega = \omega_0 - \omega$.
- Rabi frequency: $\Omega \geq 0$.
- Natural linewidth: $\gamma > 0$.
- A priori states: $\check{\eta}(t) = \frac{1}{2}\,(\mathbb{1} + \vec{x}(t)\cdot\vec{\sigma})$.
- Master equation: $\dfrac{\mathrm{d}}{\mathrm{d}t}\,\check{\eta}(t) = \check{\mathcal{L}}[\check{\eta}(t)]$.
- Bloch equations: $\dfrac{\mathrm{d}}{\mathrm{d}t}\,\vec{x}(t) = A\vec{x}(t) - \gamma\begin{pmatrix}0\\0\\1\end{pmatrix}$.
- Evolution matrix: $(\gamma > 0 \Rightarrow \det A > 0)$

$$A = \begin{pmatrix} \frac{\gamma(2\overline{n}+1+4k_\mathrm{d})}{2} & \Delta\omega & 0 \\ -\Delta\omega & \frac{\gamma(2\overline{n}+1+4k_\mathrm{d})}{2} & \Omega \\ 0 & -\Omega & \gamma\,(2\overline{n}+1) \end{pmatrix}.$$

8.5.2.2 The Equilibrium State

- Equilibrium state: $\eta_\mathrm{eq} = \frac{1}{2}\big(\mathbb{1} + \vec{x}_\mathrm{eq}\cdot\vec{\sigma}\big)$,

$$\vec{x}_\mathrm{eq} \equiv \lim_{t\to+\infty}\vec{x}(t) = -\frac{\gamma}{A}\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{-1}{(2\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)}\begin{pmatrix}4\Omega\,\Delta\omega\\-2\Omega\,\widetilde{\gamma}\\4\Delta\omega^2+\widetilde{\gamma}^2\end{pmatrix}.$$

- Some short hand notations: $\widetilde{\gamma} = \gamma\,(2\overline{n}+1+4k_\mathrm{d})$,

$$\Gamma^2 = 2\Omega^2 + \widetilde{\gamma}^2\,, \qquad \widetilde{\Gamma}^2 = \Gamma^2 + \frac{8\Omega^2 k_\mathrm{d}}{2\overline{n}+1}\,.$$

- Matrix elements of the equilibrium state:

$$\left(\eta_{\mathrm{eq}}\right)_{11} = \frac{\Omega^2\left(1+4k_{\mathrm{d}}\right)+\overline{n}\left(4\Delta\omega^2+\Gamma^2\right)}{(2\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)},$$

$$\left(\eta_{\mathrm{eq}}\right)_{12} = \frac{-\Omega\left(2\Delta\omega+\mathrm{i}\widetilde{\gamma}\right)}{(2\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)} = \overline{\left(\eta_{\mathrm{eq}}\right)_{21}},$$

$$\left(\eta_{\mathrm{eq}}\right)_{22} = \frac{(\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)+\Omega^2\left(2\overline{n}+1+4k_{\mathrm{d}}\right)}{(2\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)}.$$

- Lengths of the vertical and equatorial components of the Bloch vector:

$$\ell_{\parallel} := \left|z_{\mathrm{eq}}\right| = \frac{4\Delta\omega^2+\widetilde{\gamma}^2}{(2\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)},$$

$$\ell_{\perp} := \sqrt{x_{\mathrm{eq}}^2+y_{\mathrm{eq}}^2} = \frac{2\Omega\sqrt{4\Delta\omega^2+\widetilde{\gamma}^2}}{(2\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)}.$$

- Linear entropy:

$$\begin{aligned}\mathrm{LE}_{\eta_{\mathrm{eq}}} &= \frac{1-\left|\vec{x}_{\mathrm{eq}}\right|^2}{2} \equiv \frac{1-\left(\ell_{\perp}{}^2+\ell_{\parallel}{}^2\right)}{2} \\ &= \frac{2\overline{n}\,(\overline{n}+1)}{(2\overline{n}+1)^2} + \frac{2\Omega^2\left[\Omega^2+\frac{4k_{\mathrm{d}}}{2\overline{n}+1}\left(4\Delta\omega^2+\Gamma^2+\frac{4\Omega^2k_{\mathrm{d}}}{2\overline{n}+1}\right)\right]}{(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}.\end{aligned}$$

- Atomic squeezing parameter:

$$\begin{aligned}\mathrm{AS}_{\eta_{\mathrm{eq}}} &= 1-\left|\vec{x}_{\mathrm{eq}}\right|^2+z_{\mathrm{eq}}\left[1+z_{\mathrm{eq}}\right] = 1-\ell_{\perp}^2-\ell_{\parallel} \\ &= -\frac{\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)\left[2\overline{n}\left(4\Delta\omega^2+\Gamma^2\right)+2\Omega^2\left(1+4k_{\mathrm{d}}\right)\right]}{(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2} \\ &\quad + \frac{4\overline{n}\,(\overline{n}+1)}{(2\overline{n}+1)^2} + \frac{4\Omega^2\left[\Omega^2+\frac{4k_{\mathrm{d}}}{2\overline{n}+1}\left(4\Delta\omega^2+\Gamma^2+\frac{4\Omega^2k_{\mathrm{d}}}{2\overline{n}+1}\right)\right]}{(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}.\end{aligned}$$

8.5.2.3 The Measurement

- Generator of the characteristic operator:

$$\check{\Lambda}_t(k)[\tau] = \check{\mathcal{L}}[\tau] + \mathrm{i}k\left(\check{R}_1(t)\tau + \tau\,\check{R}_1(t)^*\right) - \frac{1}{2}\,k^2\,\tau.$$

- Side channel, the channel of the detected light: channel 1.
- Forward channel, the channel of the lost light and of the stimulating laser: channel 2.
- Detection operator: $R_1(t) = \mathrm{e}^{\mathrm{i}(\nu-\omega)t}\sqrt{\gamma}\,\alpha_1\sigma_-$.
- Measurement phase: $\vartheta = \arg\alpha_1$.
- Proportions of light in the side and in the forward channels: $|\alpha_1|^2$ and $|\alpha_2|^2$ with $|\alpha_1|^2 + |\alpha_2|^2 = 1$.
- Frequency of the local oscillator: ν.
- Output current: $I(t) = \int_0^t F(t-s)\mathrm{d}W_1(s)$.
- Detector response function: $F(t) = k_1\sqrt{\frac{\varkappa}{4\pi}}\exp\left\{-\frac{\varkappa}{2}t\right\}, \quad \varkappa > 0, \quad k_1 \neq 0.$
- Electrical power carried by the current $I(t)$: $P(t) = k_2 I(t)^2, \quad k_2 > 0.$

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Chapter 9
A Two-Level Atom: Heterodyne and Homodyne Spectra

In this chapter we study the atomic spectra, a concept which depends on the detection type, heterodyning or homodyning. These spectra give information on the atom, the atom/field interaction and the properties of the emitted light. In particular we shall see the important phenomena of dynamical Stark effect and of squeezing of the fluorescence light. Heterodyne and homodyne spectra are essentially connected to the second moments of the output current $I(t)$ (7.8), whose expression we write again here:

$$I(t) = \int_0^t F(t-s)\mathrm{d}W_1(s)\,, \tag{9.1a}$$

$$F(t) = k_1\sqrt{\frac{\varkappa}{4\pi}}\exp\left\{-\frac{\varkappa}{2}\,t\right\}, \qquad \varkappa > 0, \quad k_1 \neq 0. \tag{9.1b}$$

9.1 Heterodyne Detection and Mollow Spectrum

We consider first the case of heterodyne detection: the stimulating laser has frequency ω and the local oscillator has frequency ν and it is produced by a different source, see Fig. 7.1. The frequency ν can be changed by tuning the source producing the light for the local oscillator or by changing such a source, but this amounts to a change of measuring apparatus.

9.1.1 The Output Current and the Electrical Power

In Eq. (9.1a) the output current is written in terms of the output signal $\mathrm{d}W$ and of the detector response function F (9.1b). The electrical power carried by the current I is proportional to its square and it is given by (7.9). In the formulation of the theory based on the linear stochastic master equation there is no explicit dependence on ν in the stochastic processes $I(t)$ and $P(t)$, while it is the physical probability, determined by the linear SDE of Sect. 8.4.2, which depends on ν. Let us denote the

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expectation with respect to the physical probability up to a large time T and with initial state ρ_0 by $\mathbb{E}_{\rho_0}^{T,\nu}$. By the consistency of the probabilities, we have for the mean power

$$\mathbb{E}_{\rho_0}^{T,\nu}[P(t)] \equiv \mathbb{E}_{\rho_0}^{t,\nu}[P(t)], \qquad 0 \le t \le T.$$

Now we want to study the large time behaviour of the mean power as a function of ν. In the heterodyne detection scheme the local oscillator and the stimulating light come out from two different lasers; it is impossible to maintain a stable relative phase between the two. Neither the stimulating laser of frequency ω nor the local oscillator of frequency ν is perfectly monochromatic and coherent and some smoothing effect is always present. To take into account such a smoothing effect also in our idealisation of the lasers, the limit for large times has to be taken in the sense of distributions, and this kills the rapidly oscillating terms. Then, the mean power at large times for heterodyne detection is

$$P_{\mathrm{het}}(\nu) = \lim_{t\to+\infty} \mathbb{E}_{\rho_0}^{t,\nu}[P(t)] = \lim_{t\to+\infty} k_2 \mathbb{E}_{\rho_0}^{t,\nu}\left[I(t)^2\right] \qquad \text{limit in the sense of distributions in } \nu. \quad (9.2)$$

As a function of ν, $P_{\mathrm{het}}(\nu)$ is the mean observed *power spectrum.*

In the case of the two-level atom we can get an analytic expression for $P_{\mathrm{het}}(\nu)$.

9.1.1.1 The Moments of the Output Current and the Mean Power

The first two moments of the current are of particular importance and suffice to get the mean observed power. From Eq. (4.42) we get the mean of the current

$$\begin{aligned}\mathbb{E}_{\rho_0}^{t,\nu}[I(t)] &= \int_0^t \mathrm{d}s\, F(t-s)\,\mathrm{Tr}\left\{\mathcal{R}_1(s)\circ\mathcal{T}(s,0)[\rho_0]\right\} \\ &= k_1\sqrt{\frac{\varkappa}{4\pi}}\int_0^t \mathrm{d}s\, \exp\left\{-\frac{\varkappa}{2}(t-s)\right\}\mathrm{Tr}\left\{\left(R_1(s)+R_1(s)^*\right)\eta(s)\right\}. \quad (9.3)\end{aligned}$$

The autocorrelation function of the current is obtained from Eq. (4.47):

$$\begin{aligned}\frac{4\pi}{k_1^2}\,\mathbb{E}_{\rho_0}^{t,\nu}[I(t)I(s)] &= \mathrm{e}^{-\frac{\varkappa}{2}|t-s|} - \mathrm{e}^{-\frac{\varkappa}{2}(t+s)} \\ &+ \left(\int_{s\wedge t}^{s\vee t}\mathrm{d}t'\int_0^{s\wedge t}\mathrm{d}s' + 2\int_0^{s\wedge t}\mathrm{d}t'\int_0^{t'}\mathrm{d}s'\right)\varkappa\mathrm{e}^{-\frac{\varkappa}{2}\left[(t+s)-\left(t'+s'\right)\right]} \\ &\times \mathrm{Tr}\left\{\left(R_1(t')+R_1(t')^*\right)\mathcal{T}(t',s')\left[R_1(s')\eta(s')+\eta(s')R_1(s')^*\right]\right\}. \quad (9.4)\end{aligned}$$

Then, the mean output power is given by

$$\mathbb{E}^{t,\nu}_{\rho_0}[P(t)] = k_2\mathbb{E}^{t,\nu}_{\rho_0}\left[I(t)^2\right]$$

$$= \frac{k_1^2k_2}{4\pi}\left(1-\mathrm{e}^{-\varkappa t}\right) + \frac{k_1^2k_2\varkappa}{2\pi}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\,\mathrm{e}^{-\varkappa[t-(s+r)/2]}$$

$$\times \mathrm{Tr}\left\{\left(R_1(s)+R_1(s)^*\right)\mathcal{T}(s,r)\left[R_1(r)\eta(r)+\eta(r)R_1(r)^*\right]\right\}. \quad (9.5)$$

By introducing the variance of $I(t)$ under the physical probability we have also

$$0 \le k_2\,\mathrm{Var}^{t,\nu}_{\rho_0}[I(t)] = \mathbb{E}^{t,\nu}_{\rho_0}[P(t)] - k_2\left(\mathbb{E}^{t,\nu}_{\rho_0}[I(t)]\right)^2 = \frac{k_1^2k_2}{4\pi}\left(1-\mathrm{e}^{-\varkappa t}\right)$$

$$+\frac{k_1^2k_2\varkappa}{2\pi}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\,\mathrm{e}^{-\varkappa[t-(s+r)/2]}\,\mathrm{Tr}\Big\{\left(R_1(s)+R_1(s)^*-2\,\mathrm{Re}\,\mathrm{Tr}\{R_1(s)\eta(s)\}\right)$$

$$\times\,\mathcal{T}(s,r)\big[R_1(r)\eta(r)+R_1(r)^*\eta(r)-2\,\mathrm{Re}\,\mathrm{Tr}\{R_1(r)\eta(r)\}\eta(r)\big]\Big\}. \quad (9.6)$$

9.1.1.2 The Power Spectrum

Proposition 9.1. *The mean observed power spectrum* (9.2) *can be written as*

$$P_{\mathrm{het}}(\nu) = \frac{k_1^2k_2}{4\pi} + k_1^2k_2\,|\alpha_1|^2\;\Sigma(\nu-\omega)\,, \quad (9.7)$$

where

$$\Sigma(\mu) = \frac{\gamma}{\pi}\,\mathrm{Re}\int_0^{+\infty}\mathrm{e}^{-\left(\frac{\varkappa}{2}+\mathrm{i}\mu\right)t}\,\mathrm{Tr}\left\{\mathrm{e}^{\check{\mathcal{L}}t}\left[\sigma_-\eta_{\mathrm{eq}}\right]\sigma_+\right\}\mathrm{d}t\,. \quad (9.8)$$

Proof. Let us recall that we have $\rho_0 = \eta(0) = \check{\eta}(0)$, $\alpha_1(t) = \mathrm{e}^{\mathrm{i}(\nu-\omega)t}\alpha_1$, and

$$R_1(t) = \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma}\alpha_1\sigma_-\,, \qquad \check{R}_1(t) = \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t\sigma_z}R_1(t)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t\sigma_z} = \alpha_1(t)\sigma_-\,;$$

moreover, we set $\check{\mathcal{R}}(t)[\tau] = \check{R}_1(t)\tau + \tau\check{R}_1(t)^*$.

Then, from Eq. (9.5) we get

$$\mathbb{E}^{t,\nu}_{\rho_0}[P(t)] = p_1(t) + p_2(\nu,t) + 2\,\mathrm{Re}\,p_3(\nu,t) + 2\,\mathrm{Re}\,p_4(\nu,t),$$

where

$$p_1(t) = \frac{k_1^2k_2}{4\pi}\left(1-\mathrm{e}^{-\varkappa t}\right), \quad (9.9)$$

$$p_2(\nu,t) = \frac{k_1^2 k_2 \varkappa}{2\pi} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\, \mathrm{e}^{-\varkappa\left(t-\frac{s+r}{2}\right)} \times \mathrm{Tr}\left\{\check{\mathcal{R}}(s)\circ \mathrm{e}^{\check{\mathcal{L}}(s-r)}\circ \check{\mathcal{R}}(r)\left[\mathrm{e}^{\check{\mathcal{L}}r}[\rho_0]-\eta_{\mathrm{eq}}\right]\right\}, \tag{9.10}$$

$$p_3(\nu,t) = \frac{k_1^2 k_2 \varkappa\gamma\,|\alpha_1|^2}{2\pi} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\, \mathrm{e}^{-\varkappa\left(t-\frac{s+r}{2}\right)+\mathrm{i}(\nu-\omega)(s-r)} \times \mathrm{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}(s-r)}\left[\eta_{\mathrm{eq}}\sigma_+\right]\right\} = \frac{k_1^2 k_2 \varkappa\gamma\,|\alpha_1|^2}{2\pi} \int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2 \times \mathrm{e}^{-\varkappa(s_2+s_1/2)+\mathrm{i}(\nu-\omega)s_1}\,\mathrm{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}s_1}\left[\eta_{\mathrm{eq}}\sigma_+\right]\right\} = p_3^1(\nu,t) - p_3^2(\nu,t),$$

$$p_3^1(\nu,t) = \frac{k_1^2 k_2 \gamma\,|\alpha_1|^2}{2\pi} \int_0^t \mathrm{d}s\, \mathrm{e}^{\left(-\frac{\varkappa}{2}+\mathrm{i}(\nu-\omega)\right)s}\,\mathrm{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}s}\left[\eta_{\mathrm{eq}}\sigma_+\right]\right\}, \tag{9.11}$$

$$p_3^2(\nu,t) = \frac{k_1^2 k_2 \gamma\,|\alpha_1|^2}{2\pi} \int_0^t \mathrm{d}s_1\, \mathrm{e}^{-\varkappa(t-s_1/2)+\mathrm{i}(\nu-\omega)s_1}\,\mathrm{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}s_1}\left[\eta_{\mathrm{eq}}\sigma_+\right]\right\}$$
$$= \frac{k_1^2 k_2 \gamma\,|\alpha_1|^2}{2\pi}\, \mathrm{e}^{\left(-\frac{\varkappa}{2}+\mathrm{i}(\nu-\omega)\right)t} \int_0^t \mathrm{d}s\, \mathrm{e}^{-\left(\frac{\varkappa}{2}+\mathrm{i}(\nu-\omega)\right)s}\,\mathrm{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}(t-s)}\left[\eta_{\mathrm{eq}}\sigma_+\right]\right\}, \tag{9.12}$$

$$p_4(\nu,t) = \frac{k_1^2 k_2 \varkappa\gamma\alpha_1^2}{2\pi} \times \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\, \mathrm{e}^{-\varkappa\left(t-\frac{s+r}{2}\right)+\mathrm{i}(\nu-\omega)(s+r)}\,\mathrm{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}(s-r)}\left[\sigma_-\eta_{\mathrm{eq}}\right]\right\} = \frac{k_1^2 k_2 \varkappa\gamma\alpha_1^2}{2\pi} \int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2\, \mathrm{e}^{-\varkappa(s_2+s_1/2)+\mathrm{i}(\nu-\omega)(2t-2s_2-s_1)} \times \mathrm{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}s_1}\left[\sigma_-\eta_{\mathrm{eq}}\right]\right\} = p_4^1(\nu,t) - p_4^2(\nu,t),$$

$$p_4^1(\nu,t) = \frac{k_1^2 k_2 \gamma\alpha_1^2 \varkappa}{2\pi[\varkappa+2\mathrm{i}(\nu-\omega)]} \times \int_0^t \mathrm{d}s_1\, \mathrm{e}^{-\frac{\varkappa}{2}s_1+\mathrm{i}(\nu-\omega)(2t-s_1)}\,\mathrm{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}s_1}\left[\sigma_-\eta_{\mathrm{eq}}\right]\right\} = \mathrm{e}^{2\mathrm{i}(\nu-\omega)t}\, p_4^3(\nu,t),$$

$$p_4^3(\nu,t)=\frac{k_1^2k_2\gamma\alpha_1^2\varkappa}{2\pi[\varkappa+2\mathrm{i}(\nu-\omega)]}\int_0^t \mathrm{d}s\,\mathrm{e}^{-\frac{\varkappa}{2}s-\mathrm{i}(\nu-\omega)s}\,\mathrm{Tr}\left\{\sigma_-\mathrm{e}^{\check{\mathcal{L}}s}\left[\sigma_-\eta_{\mathrm{eq}}\right]\right\}, \quad (9.13)$$

$$\begin{aligned}p_4^2(\nu,t)&=\frac{k_1^2k_2\alpha_1^2\varkappa\gamma}{2\pi[\varkappa+2\mathrm{i}(\nu-\omega)]}\int_0^t \mathrm{d}s_1\,\mathrm{e}^{-\varkappa(t-s_1/2)+\mathrm{i}(\nu-\omega)s_1}\\&\quad\times\mathrm{Tr}\left\{\sigma_-\mathrm{e}^{\check{\mathcal{L}}s_1}\left[\sigma_-\eta_{\mathrm{eq}}\right]\right\}=\frac{k_1^2k_2\alpha_1^2\varkappa\gamma}{2\pi[\varkappa+2\mathrm{i}(\nu-\omega)]}\,\mathrm{e}^{\left(-\frac{\varkappa}{2}+\mathrm{i}(\nu-\omega)\right)t}\\&\quad\times\int_0^t \mathrm{d}s\,\mathrm{e}^{-\left(\frac{\varkappa}{2}+\mathrm{i}(\nu-\omega)\right)s}\,\mathrm{Tr}\left\{\sigma_-\mathrm{e}^{\check{\mathcal{L}}(t-s)}\left[\sigma_-\eta_{\mathrm{eq}}\right]\right\}. \quad (9.14)\end{aligned}$$

Summarising, we have

$$\begin{aligned}\mathbb{E}^{t,\nu}_{\rho_0}[P(t)]&=p_1(t)+p_2(\nu,t)\\&\quad+2\,\mathrm{Re}\left[p_3^1(\nu,t)-p_3^2(\nu,t)+\mathrm{e}^{2\mathrm{i}(\nu-\omega)t}\,p_4^3(\nu,t)-p_4^2(\nu,t)\right],\end{aligned}$$

where the various contributions are given by Eqs. (9.9), (9.10), (9.11), (9.12), (9.13), (9.14).

Because $\varkappa$ is strictly positive and the quantity $\mathrm{Tr}\left\{\sigma_-\mathrm{e}^{\check{\mathcal{L}}t}\left[\eta_{\mathrm{eq}}\sigma_\pm\right]\right\}$ is bounded in t, we get

$$p_1(t)\xrightarrow{t\to+\infty}\frac{k_1^2k_2}{4\pi}, \qquad p_3^2(\nu,t)\xrightarrow{t\to+\infty}0, \qquad p_4^2(\nu,t)\xrightarrow{t\to+\infty}0,$$

$$p_3^1(\nu,t)\xrightarrow{t\to+\infty}\quad\frac{k_1^2k_2\gamma\,|\alpha_1|^2}{2\pi}\int_0^{+\infty}\mathrm{d}s\,\mathrm{e}^{\left(-\frac{\varkappa}{2}+\mathrm{i}(\nu-\omega)\right)s}\,\mathrm{Tr}\left\{\sigma_-\mathrm{e}^{\check{\mathcal{L}}s}\left[\eta_{\mathrm{eq}}\sigma_+\right]\right\}.$$

Moreover, we have

$$\begin{aligned}|p_2(\nu,t)|&\le\frac{2k_1^2k_2\varkappa\gamma\,|\alpha_1|^2}{\pi}\int_0^t\mathrm{d}s\int_0^s\mathrm{d}r\,\mathrm{e}^{-\varkappa\left(t-\frac{s+r}{2}\right)}\left\|\mathrm{e}^{\check{\mathcal{L}}r}[\rho_0]-\eta_{\mathrm{eq}}\right\|_1\\&=\frac{4k_1^2k_2\gamma\,|\alpha_1|^2}{\pi}\int_0^t\mathrm{d}r\left(\mathrm{e}^{-\varkappa(t-r)/2}-\mathrm{e}^{-\varkappa t}\right)\left\|\mathrm{e}^{\check{\mathcal{L}}r}[\rho_0]-\eta_{\mathrm{eq}}\right\|_1\\&\le\frac{4k_1^2k_2\gamma\,|\alpha_1|^2}{\pi}\int_0^t\mathrm{d}s\,\mathrm{e}^{-\varkappa s/2}\left\|\mathrm{e}^{\check{\mathcal{L}}(t-s)}[\rho_0]-\eta_{\mathrm{eq}}\right\|_1\xrightarrow{t\to+\infty}0.\end{aligned}$$

Finally, by Lebesgue lemma, for every Schwartz function h, we have

$$\lim_{t\to+\infty}\int_{-\infty}^{+\infty}h(\nu)p_4^1(\nu,t)\,\mathrm{d}\nu=\lim_{t\to+\infty}\int_{-\infty}^{+\infty}h(\nu)\mathrm{e}^{2\mathrm{i}(\nu-\omega)t}\,p_4^3(\nu,+\infty)\,\mathrm{d}\nu=0. \quad (9.15)$$

Therefore, only the contributions from $p_1(t)$ and $p_3^1(t)$ survive in the limit and we get Eq. (9.1) with

$$\Sigma(\mu) = \frac{\gamma}{\pi} \operatorname{Re} \int_0^{+\infty} \mathrm{e}^{(-\frac{\varkappa}{2}+\mathrm{i}\mu)s} \operatorname{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}t}\left[\eta_{\mathrm{eq}}\sigma_+\right]\right\} \mathrm{d}t \,,$$

which can be rewritten in the form (9.8) by taking the complex conjugate under the operation of taking the real part. □

Equation (9.15) is the only point where the limit in the sense of distributions is needed. Such a limit washes out the rapidly oscillating terms with $\nu \neq \omega$ and considers irrelevant the term with exactly $\nu = \omega$; this is physically justified for heterodyne detection. On the contrary, homodyne detection would mean to consider important just the term with $\nu = \omega$, as we shall see in Sect. 9.2.

It is important to note that the limit is independent of the initial state ρ_0 and it depends only on the mean dynamics through $\check{\mathcal{L}}$ and η_{eq}.

9.1.1.3 The Shot Noise

The constant term $\frac{k_1^2 k_2}{4\pi}$ in $P_{\mathrm{het}}(\nu)$ is a white noise contribution and it is of quantum origin. When the atom is not stimulated, $\Omega^2 = 0$ and $\overline{n} = 0$, we get $\eta_{\mathrm{eq}} = P_-$ and $\eta_{\mathrm{eq}}\sigma_+ = \sigma_-\eta_{\mathrm{eq}} = 0$; then, Eq. (9.8) gives $\Sigma(\mu) \equiv 0$. In this case only the noise term $\frac{k_1^2 k_2}{4\pi}$ survives in $P_{\mathrm{het}}(\nu)$. Only the photons coming out from the local oscillator can reach the counters and, so, it is interpreted as *shot noise*.

9.1.2 The Fluorescence Spectrum

The function $\Sigma(\mu)$ is interpreted as the *fluorescence spectrum*; recall that $\mu = \nu - \omega$. Note that to change ν one needs to change the probing laser, so we can say that a change of ν is a change of measuring apparatus.

Equation (9.8) can be easily transformed to

$$\begin{aligned}\Sigma(\mu) = \frac{\gamma}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\mu t}\, \mathrm{e}^{-\varkappa|t|/2} \Big(& 1_{[0,+\infty)}(t) \operatorname{Tr}\left\{\mathrm{e}^{\check{\mathcal{L}}|t|}\left[\sigma_-\eta_{\mathrm{eq}}\right]\sigma_+\right\} \\ & + 1_{(-\infty,0)}(t) \operatorname{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}|t|}\left[\eta_{\mathrm{eq}}\sigma_+\right]\right\}\Big) \mathrm{d}t. \end{aligned} \tag{9.16}$$

This new expression says that the spectrum of the fluorescence light is, apart from the smoothing factor $\mathrm{e}^{-\varkappa|t|/2}$, the Fourier transform of

$$1_{(-\infty,0)}(t) \operatorname{Tr}\left\{\sigma_- \mathrm{e}^{\check{\mathcal{L}}|t|}\left[\eta_{\mathrm{eq}}\sigma_+\right]\right\} + 1_{[0,+\infty)}(t) \operatorname{Tr}\left\{\mathrm{e}^{\check{\mathcal{L}}|t|}\left[\sigma_-\eta_{\mathrm{eq}}\right]\sigma_+\right\},$$

which is called "the two-times dipole–dipole atomic quantum correlation function at equilibrium". This is the traditional way to present the spectrum, without relating

it to a POM or to an instrument representing the measuring scheme. The above expression is in turn derived from suitable quantum correlations for the outgoing electromagnetic field; see, for instance, the presentation given in Sect. 11.2 in [1].

By integrating Eq. (9.16) over μ (note that the integrand in (9.16) is continuous in t) we obtain

$$\int_{-\infty}^{+\infty} \Sigma(\mu)\,\mathrm{d}\mu = \gamma\,\mathrm{Tr}\left\{P_+\eta_{\mathrm{eq}}\right\} \equiv \frac{\gamma}{2}\big(1+z_{\mathrm{eq}}\big). \tag{9.17}$$

The quantity $\mathrm{Tr}\left\{P_+\eta_{\mathrm{eq}}\right\} = \big(\eta_{\mathrm{eq}}\big)_{11}$ is the population of the excited state level (Sect. 8.3.3) and, by multiplying it by γ, we get the rate of emission of photons in the equilibrium state. The choice of the normalisation of $\Sigma(\mu)$ is to have the integral exactly equal to the rate of emission and not only proportional to it.

As already said in Sects. 4.4.2 and 7.2.1, a big value of $\varkappa$ gives a good time resolution. However, from (9.16) we see that a big value of $\varkappa$ tends to mask the dependence of the frequency μ and gives a flat spectrum $\Sigma(\mu)$. To have a good frequency resolution we need a small $\varkappa$. So, we can say that in heterodyne detection $\varkappa$ represents an *instrumental width*.

9.1.2.1 A Decomposition of the Spectrum

In order to obtain the explicit expression of the spectrum $\Sigma(\mu)$ from Eq. (9.8) we have to compute $\mathrm{e}^{\check{\mathcal{L}}t}\left[\sigma_-\eta_{\mathrm{eq}}\right]$. Moreover, it is useful to decompose $\sigma_-\eta_{\mathrm{eq}}$ in a part proportional to η_{eq}, which is left invariant by $\mathrm{e}^{\check{\mathcal{L}}t}$, and in the remaining part, which is sent to zero by the dynamical semigroup for $t \to +\infty$. To do this, let us define some useful shorthand notations:

$$\tau := \sigma_-\eta_{\mathrm{eq}} - \mathrm{Tr}\left\{\sigma_-\eta_{\mathrm{eq}}\right\}\eta_{\mathrm{eq}}, \tag{9.18}$$

$$\vec{d}(0) := \mathrm{Tr}\left\{\vec{\sigma}\,\tau\right\} = \frac{1}{2}\,\vec{a}^{(1)} + \frac{\mathrm{i}}{2}\,\vec{a}^{(2)}, \tag{9.19}$$

$$\vec{a}^{(1)} := \begin{pmatrix} 1+z_{\mathrm{eq}} - x_{\mathrm{eq}}^{\,2} \\ -x_{\mathrm{eq}}y_{\mathrm{eq}} \\ -x_{\mathrm{eq}}\big(1+z_{\mathrm{eq}}\big) \end{pmatrix}, \qquad \vec{a}^{(2)} := \begin{pmatrix} x_{\mathrm{eq}}y_{\mathrm{eq}} \\ -\big(1+z_{\mathrm{eq}}\big) + y_{\mathrm{eq}}^{\,2} \\ y_{\mathrm{eq}}\big(1+z_{\mathrm{eq}}\big) \end{pmatrix}. \tag{9.20}$$

By construction we have $\mathrm{Tr}\{\tau\} = 0$ and we get $c_0 = 0$ in Eq. (8.29); then, Eqs. (8.29), (8.34) give

$$\vec{d}(t) := \mathrm{Tr}\left\{\vec{\sigma}\,\mathrm{e}^{\check{\mathcal{L}}t}[\tau]\right\} = \mathrm{e}^{-At}\vec{d}(0), \qquad \mathrm{e}^{\check{\mathcal{L}}t}[\tau] = \frac{1}{2}\,\vec{d}(t)\cdot\vec{\sigma}\,. \tag{9.21}$$

Proposition 9.2. *The fluorescence spectrum* (9.8) *can be decomposed as*

$$\Sigma(\mu) = \Sigma_{\mathrm{el}}(\mu) + \Sigma_{\mathrm{inel}}(\mu), \tag{9.22}$$

where

$$\Sigma_{\rm el}(\mu) = \frac{\varkappa\gamma\left|{\rm Tr}\left\{\sigma_-\eta_{\rm eq}\right\}\right|^2}{2\pi\left|q\right|^2} = \Sigma_{\rm el}(-\mu)\,, \tag{9.23}$$

$$\Sigma_{\rm inel}(\mu) = \frac{\gamma}{\pi}\,{\rm Re}\int_0^{+\infty} {\rm e}^{-qt}\,{\rm Tr}\left\{{\rm e}^{\check{\mathcal{L}}t}\left[\tau\right]\sigma_+\right\}{\rm d}t\,, \tag{9.24}$$

$$q := \frac{\varkappa}{2} + {\rm i}\mu\,. \tag{9.25}$$

We have also

$$\begin{aligned}\Sigma_{\rm inel}(\mu) &= \frac{1}{2\pi}\,{\rm Re}\left[\left(\frac{\gamma}{A+q}\,\vec{d}(0)\right)_1 + {\rm i}\left(\frac{\gamma}{A+q}\,\vec{d}(0)\right)_2\right]\\ &= \frac{\gamma}{4\pi}\sum_{i=1}^{2}\left(\frac{1}{\left(A+\frac{\varkappa}{2}\right)^2+\mu^2}\,\vec{c}^{\,(i)}(\mu)\right)_i,\end{aligned} \tag{9.26}$$

$$\vec{c}^{\,(1)}(\mu) = \left(A+\frac{\varkappa}{2}\right)\vec{a}^{\,(1)} + \mu\vec{a}^{\,(2)}, \qquad \vec{c}^{\,(2)}(\mu) = -\left(A+\frac{\varkappa}{2}\right)\vec{a}^{\,(2)} + \mu\vec{a}^{\,(1)}. \tag{9.27}$$

Proof. By writing $\sigma_-\eta_{\rm eq} = \tau + {\rm Tr}\left\{\sigma_-\eta_{\rm eq}\right\}\eta_{\rm eq}$, we get

$${\rm e}^{\check{\mathcal{L}}t}[\sigma_-\eta_{\rm eq}] = {\rm e}^{\check{\mathcal{L}}t}[\tau] + {\rm Tr}\left\{\sigma_-\eta_{\rm eq}\right\}\eta_{\rm eq}.$$

By inserting this expression into (9.8) we obtain Eqs. (9.22), (9.23), (9.24), (9.25).

By using (9.21) we get the first equality in (9.26). Finally, by using

$$\frac{1}{A+q} = \frac{A+\frac{\varkappa}{2}-{\rm i}\mu}{\left(A+\frac{\varkappa}{2}\right)^2+\mu^2}$$

and the fact that A has real matrix elements, we get the second equality in (9.26). □

The Elastic and Inelastic Parts of the Spectrum

The term $\Sigma_{\rm el}(\mu)$ is interpreted as the *elastic* part of the spectrum. From Eq. (9.23), we get that for $\varkappa \downarrow 0$ the quantity $\Sigma_{\rm el}(\mu)$ becomes proportional to a Dirac delta $\delta(\mu) \equiv \delta(\nu-\omega)$. This means that the energy of the photons is conserved in the processes responsible for this part of the spectrum and this fact justifies the attribute of "elastic" given to this part of the spectrum. Instead, the other part of the spectrum does not develop any singularity even in the limit $\varkappa \downarrow 0$ and, so, it is termed *inelastic*.

Proposition 9.3. *The explicit expressions of the elastic and inelastic parts of the spectrum are*

$$\Sigma_{\rm el}(\mu) = \frac{\gamma \ell_\perp{}^2 \varkappa}{8\pi\,|q|^2} = \frac{\gamma\Omega^2\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)}{\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)^2(2\overline{n}+1)^2}\,\frac{\varkappa}{2\pi\left(\mu^2+\frac{\varkappa^2}{4}\right)}, \tag{9.28}$$

$$\Sigma_{\rm inel}(\mu) = \frac{\gamma}{\pi\,(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}\ {\rm Re}\ \frac{4\Omega^2 N_1(q)+N_2(q)}{N(q)}\,, \tag{9.29}$$

where

$$\begin{aligned} N_1(q) := {} & \left[\Omega^2(2\overline{n}+1+4k_{\rm d})+\overline{n}\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)\right] \\ & \times\left[\Omega^2(2\overline{n}+1+4k_{\rm d})+4\overline{n}\Delta\omega^2+(\overline{n}+1)\,\widetilde{\gamma}^2+q\,(\widetilde{\gamma}-2{\rm i}\Delta\omega)\right] \\ & +4\Delta\omega^2\left[\Omega^2(2\overline{n}+4k_{\rm d})+\overline{n}\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)\right]-2{\rm i}\Omega^2\Delta\omega\widetilde{\gamma}\,, \end{aligned} \tag{9.30a}$$

$$\begin{aligned} N_2(q) := {} & (2q+\widetilde{\gamma}+2{\rm i}\Delta\omega)\,[q+\gamma\,(2\overline{n}+1)]\left\{2\overline{n}\,(2\overline{n}+1)\left(4\Delta\omega^2+\Gamma^2\right)^2\right. \\ & \left.+\,4\Omega^2\Big[\Omega^2(1+4k_{\rm d})^2+\overline{n}\left(4\Delta\omega^2+\Gamma^2\right)(1+8k_{\rm d})+2k_{\rm d}\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)\Big]\right\}. \end{aligned} \tag{9.30b}$$

Moreover, we have

$$\Sigma_{\rm el}(\nu-\omega) = \frac{1}{k_1^2\,|\alpha_1|^2}\lim_{t\to+\infty}\left(\mathbb{E}^{t,\nu}_{\rho_0}\left[I(t)\right]\right)^2, \tag{9.31a}$$

$$\Sigma_{\rm inel}(\nu-\omega) = \frac{1}{k_1^2\,|\alpha_1|^2}\lim_{t\to+\infty}{\rm Var}^{t,\nu}_{\rho_0}\left[I(t)\right]-\frac{1}{4\pi\,|\alpha_1|^2}\,; \tag{9.31b}$$

the limits are again in the sense of the distributions in ν.

When $\Delta\omega=0$, the inelastic spectrum is an even function of μ and can be written as

$$\begin{aligned} \Sigma_{\rm inel}(\mu) = {} & \frac{\Omega^2+\overline{n}\gamma\widetilde{\gamma}}{2\Omega^2+(2\overline{n}+1)\,\gamma\widetilde{\gamma}}\cdot\frac{\gamma\,(\widetilde{\gamma}+\varkappa)}{\pi\left[(\widetilde{\gamma}+\varkappa)^2+4\mu^2\right]} \\ & +\frac{\gamma}{4\pi}\left(\frac{B}{B^2+\mu^2}\begin{pmatrix}1+z_{\rm eq}-y_{\rm eq}^2\\ -y_{\rm eq}\left(1+z_{\rm eq}\right)\end{pmatrix}\right)_1, \end{aligned} \tag{9.32a}$$

$$B = \begin{pmatrix}\frac{\widetilde{\gamma}+\varkappa}{2} & \Omega\\ -\Omega & (2\overline{n}+1)\,\gamma+\frac{\varkappa}{2}\end{pmatrix}, \tag{9.32b}$$

$$y_{\rm eq} = \frac{2\gamma\Omega}{2\Omega^2 + (2\overline{n}+1)\,\gamma\widetilde{\gamma}}, \qquad 1 + z_{\rm eq} = \frac{2\left(\Omega^2 + \overline{n}\gamma\widetilde{\gamma}\right)}{2\Omega^2 + (2\overline{n}+1)\,\gamma\widetilde{\gamma}}. \tag{9.32c}$$

The quantities $N(\bullet)$, Γ^2, $\widetilde{\Gamma}^2$, $\widetilde{\gamma}$ were already defined by Eqs. (8.45), (8.55), but we write them again here:

$$N(q) = 2\Omega^2\,(2q + \widetilde{\gamma}) + [\gamma\,(2\overline{n}+1) + q]\left[4\Delta\omega^2 + (2q+\widetilde{\gamma})^2\right], \tag{9.33}$$

$$\widetilde{\Gamma}^2 = \gamma^2 + \frac{8\Omega^2 k_{\rm d}}{2\overline{n}+1}, \qquad \gamma^2 = 2\Omega^2 + \widetilde{\gamma}^2, \qquad \widetilde{\gamma} = \gamma\,(2\overline{n}+1+4k_{\rm d}). \tag{9.34}$$

Proof. The explicit expressions in (9.28) are easily obtained from (9.23) by using (8.50), (8.60).

By the results in Lemma 8.4 and the first equality in (9.26), we have

$$\pi\,|N(q)|^2\;\Sigma_{\rm inel}(\mu) = \gamma\,{\rm Re}\,\overline{N(q)}\Big\{2\Omega^2 d_1(0) + (2q+\widetilde{\gamma}+2{\rm i}\Delta\omega)\,[(q+\gamma\,(2\overline{n}+1))\,(d_1(0)+{\rm i}d_2(0)) - {\rm i}\Omega d_3(0)]\Big\},$$

Then, we set

$$N_1(q) := \frac{(2\overline{n}+1)^2}{4\Omega}\,\left(4\Delta\omega^2 + \widetilde{\Gamma}^2\right)^2\,[2\Omega d_1(0) - {\rm i}\,(2q+\widetilde{\gamma}+2{\rm i}\Delta\omega)\,d_3(0)],$$

$$N_2(q) := (2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2 \times (2q+\widetilde{\gamma}+2{\rm i}\Delta\omega)\,(q+\gamma\,(2\overline{n}+1))\,(d_1(0)+{\rm i}d_2(0));$$

by Eqs. (8.57), (9.19) and by direct computations we find the expressions (9.29), (9.30).

By using $\lim_{t\to+\infty} {\rm e}^{\check{\mathcal{L}}s}\,[\rho_0] = \eta_{\rm eq}$ and $\mu = \nu - \omega$, we get the mean value of the current:

$$\begin{aligned}\mathbb{E}^{t,\nu}_{\rho_0}\,[I(t)] &= \int_0^t {\rm d}s\; F(t-s)\,\frac{\rm d}{{\rm d}s}\,\mathbb{E}^{t,\nu}_{\rho_0}[W_1(s)] \\ &= 2k_1\sqrt{\frac{\varkappa\gamma}{4\pi}}\,|\alpha_1|\,{\rm Re}\int_0^t {\rm d}s\,{\rm e}^{-\frac{\varkappa}{2}(t-s)+{\rm i}(\nu-\omega)s}\;{\rm Tr}\left\{\sigma_-{\rm e}^{\check{\mathcal{L}}s}\,[\rho_0]\right\} \\ &\overset{t\to+\infty}{\asymp} 2k_1\sqrt{\frac{\varkappa\gamma}{4\pi}}\,|\alpha_1|\,{\rm Re}\,{\rm Tr}\left\{\sigma_-\eta_{\rm eq}\right\}\,\frac{1}{q}\left({\rm e}^{-\varkappa t/2} - {\rm e}^{{\rm i}(\nu-\omega)t}\right).\end{aligned}$$

From this equation we have, in the sense of distributions,

$$\frac{1}{k_1^2\,|\alpha_1|^2}\lim_{t\to+\infty}\left(\mathbb{E}_{\rho_0}^{t,\nu}\left[I(t)\right]\right)^2=\frac{\varkappa\gamma}{4\pi}\,\frac{2\left|\mathrm{Tr}\left\{\sigma_-\eta_{\mathrm{eq}}\right\}\right|^2}{|q|^2}\,;$$

by comparing with (9.23) we get (9.31a).

By Eqs. (9.2), (9.7), (9.22), (9.31a) we get easily (9.31b).

Let us now consider the case $\Delta\omega = 0$. From Eq. (8.40) we get $x_{\mathrm{eq}} = 0$ and (9.32c). We insert these expressions in Eqs. (9.19), (9.27) and use the second expression of the spectrum in Eq. (9.26). The fact that now the matrix A (8.32) is block-diagonal gives the splitting of the spectrum in two terms; explicit computations give the result. □

The Coherent and Incoherent Parts of the Spectrum

Equations (9.31) give rise to a second interpretation of the decomposition of the spectrum. Equation (9.31b) says that the inelastic part of the spectrum is purely due to fluctuations and by this it is also called the *incoherent* part. Similarly, the elastic part of the spectrum is due to the square of the mean value of the current and it is called the *coherent* part.

The Lorentz Shape

In the following sections we shall study the heterodyne spectrum, then the homodyne one. Sometimes we shall have to compare the shapes of the various spectral densities for the two-level atom with the Lorentzians; so, here we recall what is this curve. The Lorentzian is the most common shape for atomic spectral lines and it is given by

$$\nu\mapsto a\,\frac{c/2\pi}{(\nu-\nu_0)^2+c^2/4}\,,\qquad a>0,\quad c>0.$$

The area below the curve is a, the height is $\frac{2a}{\pi c}$ and the *linewidth* (the width at half height) is c.

When used as a probability density, the function $\nu\mapsto\frac{c/2\pi}{(\nu-\nu_0)^2+c^2/4}$ is called Cauchy density. Note that this density is symmetric around ν_0, but no moment exists.

9.1.2.2 Examples

Remark 9.4. From Proposition 9.3 we get easily the expression of the spectrum in some extreme cases.

- Case $\overline{n}=0$, $k_{\mathrm{d}}=0$, $\Omega>0$. From Eqs. (9.28), (9.29), (9.30) we obtain

$$N(q)=2\Omega^2\,(2q+\gamma)+(\gamma+q)\left[4\Delta\omega^2+(2q+\gamma)^2\right],$$

$$\Sigma_{\mathrm{el}}(\mu) = \frac{\varkappa\gamma\Omega^2\left(4\Delta\omega^2+\gamma^2\right)}{2\pi\,|q|^2\left(4\Delta\omega^2+\Gamma^2\right)^2}\,, \tag{9.35a}$$

$$\Sigma_{\mathrm{inel}}(\mu) = \frac{4\gamma\Omega^4}{\pi\left(4\Delta\omega^2+\Gamma^2\right)^2}\,\mathrm{Re}\,\frac{\Omega^2+2(q+\gamma)^2}{N(q)}\,. \tag{9.35b}$$

Note that in this case, under the transformation $\mu \to -\mu$, one has $q \to \overline{q}$, $N(q) \to \overline{N(q)}$; this implies

$$\Sigma_{\mathrm{inel}}(\mu) = \Sigma_{\mathrm{inel}}(-\mu).$$

– In the limit $\varkappa \downarrow 0$ we get the original result by Mollow [2]:

$$\Sigma_{\mathrm{el}}(\mu) \overset{\varkappa=0}{=} \frac{\gamma\Omega^2\left(4\Delta\omega^2+\gamma^2\right)}{\left(4\Delta\omega^2+\Gamma^2\right)^2}\,\delta(\mu), \tag{9.36a}$$

$$\Sigma_{\mathrm{inel}}(\mu) \overset{\varkappa=0}{=} \frac{4\gamma^2\Omega^4\left(\Omega^2+2\gamma^2+2\mu^2\right)}{\pi\left(4\Delta\omega^2+\gamma^2\right)|N(\mathrm{i}\mu)|^2}\,, \tag{9.36b}$$

$$|N(\mathrm{i}\mu)|^2 = \gamma^2\left(\Gamma^2+4\Delta\omega^2-8\mu^2\right)^2+\mu^2\left(2\Gamma^2+3\gamma^2+4\Delta\omega^2-4\mu^2\right)^2. \tag{9.36c}$$

– In resonance ($\Delta\omega = 0$) the incoherent Mollow spectrum becomes

$$\Sigma_{\mathrm{inel}}(\mu) \overset{\varkappa=0}{=} \frac{4\gamma^2\Omega^4\left(\Omega^2+2\gamma^2+2\mu^2\right)}{\pi\left(2\Omega^2+\Gamma^2\right)|N(\mathrm{i}\mu)|^2}\,, \tag{9.37a}$$

$$|N(\mathrm{i}\mu)|^2 = \gamma^2\left(\Gamma^2-8\mu^2\right)^2+\mu^2\left(2\Gamma^2+3\gamma^2-4\mu^2\right)^2. \tag{9.37b}$$

Moreover, for small Ω, we get the very simple expression

$$\lim_{\Omega\downarrow 0}\frac{\Sigma_{\mathrm{inel}}(\mu)}{\Omega^4} \overset{\varkappa=0}{=} \frac{1}{2\pi\left(\mu^2+\gamma^2/4\right)^2}\,, \tag{9.38}$$

which is a squared Lorentzian.

• Case $\Omega = 0, \overline{n} > 0$: $\quad \widetilde{\gamma} = \gamma\,(2\overline{n}+1+4k_{\mathrm{d}}), \quad \Sigma_{\mathrm{el}}(\mu) = 0,$

$$\Sigma_{\mathrm{inel}}(\nu-\omega) = \frac{2\gamma\overline{n}\,(\varkappa+\widetilde{\gamma})}{\pi\,(2\overline{n}+1)\left[(\varkappa+\widetilde{\gamma})^2+4\,(\nu-\omega_0)^2\right]}\,, \tag{9.39a}$$

$$\Sigma_{\text{inel}}(\nu-\omega) \overset{\varkappa=0}{=} \frac{2\gamma\overline{n}\,\widetilde{\gamma}}{\pi\,(2\overline{n}+1)\left[\widetilde{\gamma}^2+4\,(\nu-\omega_0)^2\right]}\,. \tag{9.39b}$$

Note the shape which is purely Lorentzian with thermal and dephasing broadening.

As we shall see in the plots, the inelastic fluorescence spectrum $\Sigma_{\text{inel}}(\mu)$ develops a typical three-peaked structure for a large intensity of the stimulating laser. In honour of Mollow, who predicted this behaviour before the experimental verification [2], the heterodyne spectrum of a two-level atom is often called *Mollow spectrum*. The insurgence of the three-peaked structure is also called *dynamical Stark effect*.

To take into account also the shot noise and to have a better way to compare the results of heterodyne and homodyne detection schemes, let us consider the mean observed power spectrum $P_{\text{het}}(\nu)$ for $k_1^2 k_2 = 4\pi$ (in order to set the noise level to 1). So, we write

$$P_{\text{het}}(\mu-\omega)\big|_{k_1^2k_2=4\pi} = 1+4\pi\,|\alpha_1|^2\,\Sigma(\mu) =: |\alpha_2|^2+|\alpha_1|^2\,S_{\text{het}}(\mu), \tag{9.40a}$$

$$S_{\text{het}}(\mu) = S^{\text{el}}_{\text{het}}(\mu)+S^{\text{inel}}_{\text{het}}(\mu), \tag{9.40b}$$

$$S^{\text{el}}_{\text{het}}(\mu) := 4\pi\,\Sigma_{\text{el}}(\mu), \qquad S^{\text{inel}}_{\text{het}}(\mu) := 1+4\pi\,\Sigma_{\text{inel}}(\mu). \tag{9.40c}$$

These definitions are given to have $S_{\text{het}}(\mu)$, $S^{\text{el}}_{\text{het}}(\mu)$, $S^{\text{inel}}_{\text{het}}(\mu)$ all independent from $|\alpha_1|^2$, as in the extreme case when all the light reaches the beam splitter.

Remark 9.5. From Eqs. (9.40), the positivity of P_{het} and (9.31b), we get

$$S^{\text{el}}_{\text{het}}(\mu) \geq 0, \qquad S^{\text{inel}}_{\text{het}}(\mu) \geq 0, \tag{9.41}$$

but not $\Sigma_{\text{inel}}(\mu) \geq 0$, which however holds in all the examples. Only going through homodyne detection we get Eq. (9.61), which gives

$$\Sigma_{\text{inel}}(\mu)+\Sigma_{\text{inel}}(-\mu) \geq 0. \tag{9.42}$$

So, $\Sigma_{\text{inel}}(\mu)$ is surely positive at least when it is an even function of μ.

In Fig. 9.1 the total spectrum $S_{\text{het}}(\mu)$ is given for some choices of the parameters in the case of no temperature, no dephasing and response parameter $\varkappa = 0.4$. In all the figures the natural linewidth γ is taken equal to one in order to fix the scale. Experimental results, confirming the three-peaked structure, are given in [3–7].

When only the inelastic part of the spectrum $S^{\text{inel}}_{\text{het}}$ is considered, the limit of vanishing $\varkappa$ can be taken; some examples are plotted in Fig. 9.2.

In the case $\Omega = 0$, the spectrum is given by a pure Lorentzian curve with a temperature-dependent width $\widetilde{\gamma}$, see Eq. (9.39b). When Ω^2 is relatively big, the effect of temperature and dephasing is of "smoothing" and distortion of the spectrum, as can be seen from the examples of Fig. 9.3. The case with $\overline{n} > 0$, but $\Delta\omega = 0, k_{\text{d}} = 0$, was studied (inside the traditional approach) also in [8].

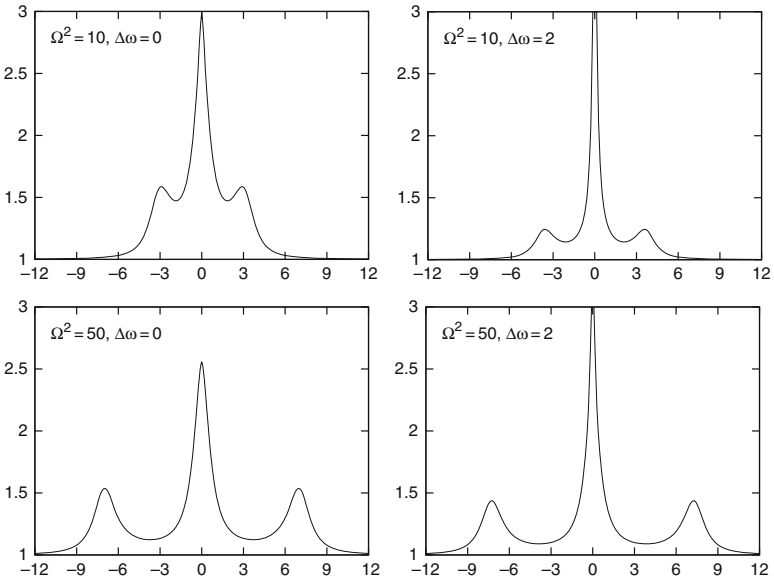

Fig. 9.1 The spectrum $S_{\text{het}}(x)$ for $\gamma = 1$, $\varkappa = 0.4$, $\Delta\omega = 0$ or 2, $\Omega^2 = 10$ or 50, $\overline{n} = 0$, $k_{\text{d}} = 0$

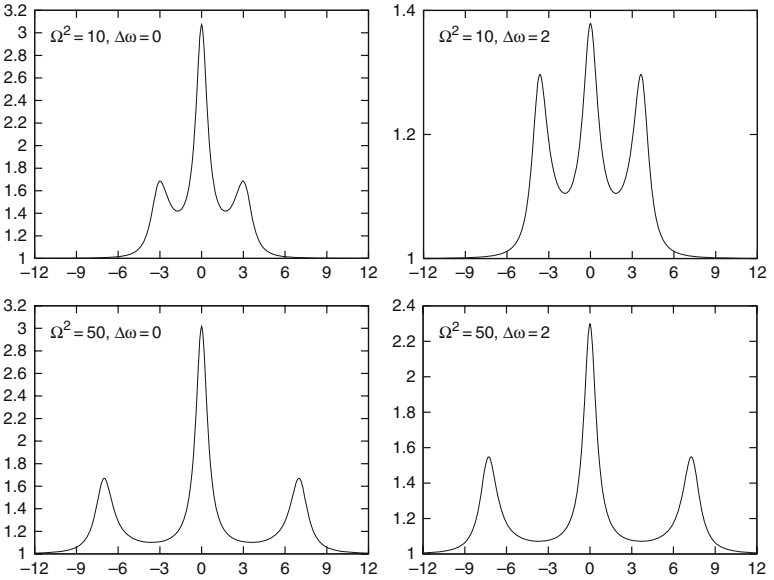

Fig. 9.2 The inelastic spectrum $S_{\text{het}}^{\text{inel}}(x)$ for $\gamma = 1$, $\varkappa = 0$, $\Delta\omega = 0$ or 2, $\Omega^2 = 10$ or 50, $\overline{n} = 0$, $k_{\text{d}} = 0$

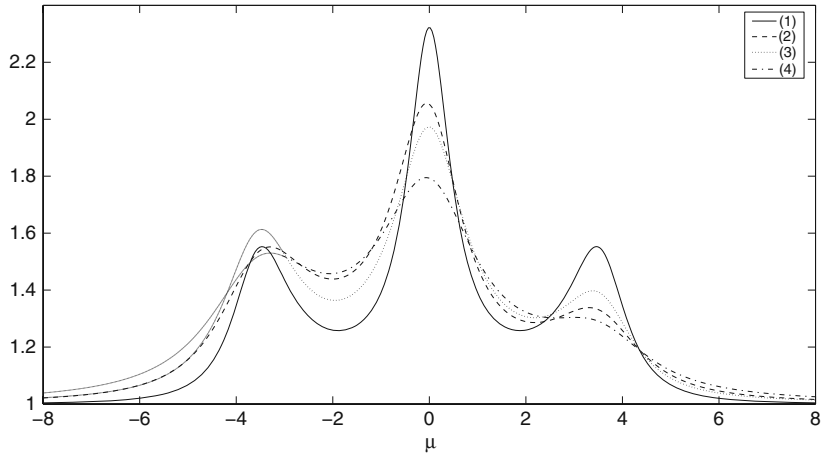

Fig. 9.3 The inelastic spectrum $S_{\rm het}^{\rm inel}(x)$ for $\gamma = 1$, $\varkappa = 0$, $\Delta\omega = -1$, $\Omega^2 = 12$, and (1) $\overline{n} = 0$ $k_{\rm d} = 0$; (2) $\overline{n} = 0.3$ $k_{\rm d} = 0$; (3) $\overline{n} = 0$ $k_{\rm d} = 0.2$; (4) $\overline{n} = 0.3$ $k_{\rm d} = 0.2$

Other features of the heterodyne spectrum are discussed in Sect. 9.2.2.2; see the comments to Fig. 9.8.

The power spectrum can be obtained also directly from the quantum stochastic formulation of the continuous measurements, without going through the classical SDEs [9–11]. This treatment is presented in [12], where the spectrum is studied also in the case of a non-perfectly monochromatic stimulating laser and of $S \neq \mathbb{1}$, but with $\overline{n} = 0$, $k_{\rm d} = 0$. While theoretical results can be obtained in different ways, the approach with classical SDEs allows for numerical simulations. As soon as one needs a more elaborated model, say when more than two atomic levels are involved in an essential way, immediately analytical computations become unfeasible. In this case the approach with SDEs gives a way to simulate the processes $\widehat{\psi}(t)$, $W(t)$ and to obtain numerically the moments needed for quantities like the power (9.2).

9.1.2.3 Intensity

Let us consider now the total intensity of the spectrum (the integral over all the frequencies of its difference from the noise level) and the intensities of its elastic and inelastic components. In order to have dimensionless quantities we define

$$\Pi_{\rm TOT}(\Omega, \Delta\omega) := \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty} [S_{\rm het}(\mu) - 1]\,{\rm d}\mu = \frac{2}{\gamma}\int_{-\infty}^{+\infty} \Sigma(\mu)\,{\rm d}\mu\,, \qquad (9.43a)$$

$$\Pi_{\rm el}(\Omega, \Delta\omega) := \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty} S_{\rm het}^{\rm el}(\mu)\,{\rm d}\mu = \frac{2}{\gamma}\int_{-\infty}^{+\infty} \Sigma_{\rm el}(\mu)\,{\rm d}\mu\,, \qquad (9.43b)$$

$$\Pi_{\rm inel}(\Omega,\Delta\omega) := \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty}\left[S_{\rm het}^{\rm inel}(\mu)-1\right]{\rm d}\mu = \frac{2}{\gamma}\int_{-\infty}^{+\infty}\Sigma_{\rm inel}(\mu)\,{\rm d}\mu$$
$$= \Pi_{\rm TOT}(\Omega,\Delta\omega) - \Pi_{\rm el}(\Omega,\Delta\omega)\,. \quad (9.43c)$$

By Eqs. (8.59), (8.60), (8.69), (9.17), (9.28) we get

$$\Pi_{\rm TOT}(\Omega,\Delta\omega) = 1 + z_{\rm eq} = 1 - \ell_{\parallel} = 2\,\frac{\Omega^2\,(1+4k_{\rm d}) + \overline{n}\,\left(4\Delta\omega^2+\Gamma^2\right)}{(2\overline{n}+1)\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)}, \quad (9.44a)$$

$$\Pi_{\rm el}(\Omega,\Delta\omega) = \frac{x_{\rm eq}^2+y_{\rm eq}^2}{2} = \frac{\ell_{\perp}^2}{2} = \frac{2\Omega^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)}{(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}\,, \quad (9.44b)$$

$$\Pi_{\rm inel}(\Omega,\Delta\omega) = \frac{1-\ell_{\parallel}+{\rm AS}_{\eta_{\rm eq}}}{2} = \frac{1-\left|\vec{x}_{\rm eq}\right|^2+\left(1+z_{\rm eq}\right)^2}{2}$$
$$= \frac{2}{(2\overline{n}+1)^2}\left\{\overline{n}\,(\overline{n}+1) + \frac{\Omega^2\left[\Omega^2+\frac{4k_{\rm d}}{2\overline{n}+1}\left(4\Delta\omega^2+\Gamma^2+\frac{4\Omega^2k_{\rm d}}{2\overline{n}+1}\right)\right]}{\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}\right.$$
$$\left. + \frac{\left[\Omega^2\,(1+4k_{\rm d})+\overline{n}\,\left(4\Delta\omega^2+\Gamma^2\right)\right]^2}{\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}\right\}. \quad (9.44c)$$

Once again we see that $\widetilde{\Gamma}$ plays the role of width with respect to the detuning.

In the case of $\overline{n}=0$ and $k_{\rm d}=0$, these formulae become simpler:

$$\Pi_{\rm TOT}(\Omega,\Delta\omega) = \frac{2\Omega^2}{4\Delta\omega^2+\gamma^2+2\Omega^2}, \quad (9.45a)$$

$$\Pi_{\rm el}(\Omega,\Delta\omega) = \frac{2\Omega^2\left(4\Delta\omega^2+\gamma^2\right)}{\left(4\Delta\omega^2+\gamma^2+2\Omega^2\right)^2}\,, \quad (9.45b)$$

$$\Pi_{\rm inel}(\Omega,\Delta\omega) = \frac{4\Omega^4}{\left(4\Delta\omega^2+\gamma^2+2\Omega^2\right)^2}\,. \quad (9.45c)$$

Note that the total intensity, as a function of the frequency ω of the stimulating laser, is a Lorentzian γ whose width is $\Gamma=\sqrt{\gamma^2+2\Omega^2}$.

9.2 Homodyne Detection

We continue to consider the ideal case of perfectly coherent and monochromatic stimulating laser and local oscillator, but now in the homodyne case. This means that we have exactly $\nu = \omega$ and that the light stimulating the atom and the light feeding the local oscillator come out from the same source to maintain phase coherence during the time.

In the rotating frame, for $\nu = \omega$, any explicit time dependence disappears in all the coefficients, see Eqs. (8.87), (8.88), (8.96). In particular, Eq. (8.88) gives $\alpha_1(t) = \alpha_1(0) = |\alpha_1|\, \mathrm{e}^{\mathrm{i}\vartheta}$, where the phase ϑ is defined by Eq. (8.14).

The results on squeezing and homodyne spectrum were obtained in [13–17].

9.2.1 The Spectrum of the Homodyne Current

In Section 9.1 we have studied the mean power $P_{\mathrm{het}}(\nu)$ in heterodyne detection, mainly for small values of $\varkappa$ (bad time resolution). Let us study now the spectrum of the homodyne current $I(t)$ according to the general approach of the theory of stochastic processes of Sect. 4.5. As we shall see in the following, to have significant spectra we shall need a good time resolution in the expression of $I(t)$ (big values of $\varkappa$). Let us stress also that in the case of the heterodyne detection what has been studied is the power of the output signal $I(t)$, which means a quantity proportional to $I(t)^2$ for large times. Instead, in the case of homodyne detection the signal is processed by a spectrum analyser and the Fourier transform of $I(t)$ is obtained.

According to the definition of spectrum of an asymptotically stationary stochastic process given in Sect. 4.5.2, the spectrum of the output current $I(t)$ is given by

$$S_{\mathrm{I}}(\mu;\vartheta) := \lim_{T\to+\infty} \frac{1}{T}\, \mathbb{E}^{T,\omega}_{\rho_0}\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu t} I(t)\, \mathrm{d}t\right|^2\right]; \tag{9.46}$$

the limit has to be taken in the sense of distributions, as discussed in Section 4.5.

Proposition 9.6. *The spectrum of the homodyne current can be expressed as*

$$S_{\mathrm{I}}(\mu;\vartheta) = \frac{k_1^2 \varkappa}{\pi\left(\varkappa^2 + 4\mu^2\right)}\, S_{\mathrm{hom}}(\mu;\vartheta), \tag{9.47}$$

where

$$S_{\mathrm{hom}}(\mu;\vartheta) = \lim_{T\to+\infty} \frac{1}{T}\, \mathbb{E}^{T,\omega}_{\eta_{eq}}\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\, \mathrm{d}W_1(s)\right|^2\right]; \tag{9.48}$$

$S_{\mathrm{hom}}(\mu;\vartheta)$ *is independent of* $\varkappa$ *and of the initial state* ρ_0.

Proof. By using the expression of the current (9.1) we get

$$\int_0^T \mathrm{e}^{\mathrm{i}\mu t} I(t)\,\mathrm{d}t = k_1\sqrt{\frac{\varkappa}{4\pi}}\int_0^T \mathrm{d}W_1(s)\int_s^T \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\mu t-\frac{\varkappa}{2}(t-s)}$$

$$= \frac{k_1\sqrt{\varkappa}}{\sqrt{\pi}\,(\varkappa-2\mathrm{i}\mu)}\int_0^T \left(\mathrm{e}^{\mathrm{i}\mu s}-\mathrm{e}^{\left(\mathrm{i}\mu-\frac{\varkappa}{2}\right)T}\mathrm{e}^{\frac{\varkappa}{2}s}\right)\mathrm{d}W_1(s).$$

By inserting this expression into (9.46) and by using the second moment formula (4.46), expressed in the rotating frame, we have

$$\frac{\pi\left(\varkappa^2+4\mu^2\right)}{k_1^2\varkappa}\;\; S_{\mathrm{I}}(\mu;\vartheta)-S_{\mathrm{hom}}(\mu;\vartheta)$$

$$= \lim_{T\to+\infty}\frac{1}{T}\,\mathbb{E}_{\rho_0}^{T,\omega}\left[\left|\int_0^T\left(\mathrm{e}^{\mathrm{i}\mu s}-\mathrm{e}^{\mathrm{i}\mu T-\frac{\varkappa}{2}(T-s)}\right)\mathrm{d}W_1(s)\right|^2\right]$$

$$-\lim_{T\to+\infty}\frac{1}{T}\,\mathbb{E}_{\eta_{\mathrm{eq}}}^{T,\omega}\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\,\mathrm{d}W_1(s)\right|^2\right]$$

$$= \lim_{T\to+\infty}\frac{1}{T}\left\{\int_0^T\left[\left|\mathrm{e}^{\mathrm{i}\mu s}-\mathrm{e}^{\mathrm{i}\mu T-\frac{\varkappa}{2}(T-s)}\right|^2-1\right]\mathrm{d}s\right.$$

$$+2\,\mathrm{Re}\int_0^T\mathrm{d}t\int_0^t\mathrm{d}s\left[\left(\mathrm{e}^{-\mathrm{i}\mu t}-\mathrm{e}^{-\mathrm{i}\mu T-\frac{\varkappa}{2}(T-t)}\right)\left(\mathrm{e}^{\mathrm{i}\mu s}-\mathrm{e}^{\mathrm{i}\mu T-\frac{\varkappa}{2}(T-s)}\right)\right.$$

$$\times\,\mathrm{Tr}\left\{\check{\mathcal{R}}_1\circ\mathrm{e}^{\check{\mathcal{L}}(t-s)}\circ\check{\mathcal{R}}_1[\eta(s)]\right\}$$

$$\left.\left.-\,\mathrm{e}^{-\mathrm{i}\mu(t-s)}\,\mathrm{Tr}\left\{\check{\mathcal{R}}_1\circ\mathrm{e}^{\check{\mathcal{L}}(t-s)}\circ\check{\mathcal{R}}_1[\eta_{\mathrm{eq}}]\right\}\right]\right\}$$

$$= \lim_{T\to+\infty}\left[\Delta_1(T,\mu)+\Delta_2(T,\mu)+\Delta_3(T,\mu)\right],$$

where

$$\check{\mathcal{R}}_1[\tau] = \sqrt{\gamma}\,|\alpha_1|\left(\mathrm{e}^{\mathrm{i}\vartheta}\sigma_-\tau+\mathrm{e}^{-\mathrm{i}\vartheta}\tau\sigma_+\right),$$

$$\Delta_1(T,\mu) = \frac{1}{T}\int_0^T\left(\mathrm{e}^{-\varkappa(T-s)}+2\,\mathrm{Re}\,\mathrm{e}^{\left(\mathrm{i}\mu-\frac{\varkappa}{2}\right)(T-s)}\right)\mathrm{d}s$$

$$= \frac{1}{T}\left(\frac{1-\mathrm{e}^{-\varkappa T}}{\varkappa}+2\,\mathrm{Re}\,\frac{1-\mathrm{e}^{\left(\mathrm{i}\mu-\frac{\varkappa}{2}\right)T}}{\frac{\varkappa}{2}-\mathrm{i}\mu}\right)\xrightarrow{t\to+\infty}0,$$

$$\Delta_2(T,\mu) = \frac{2}{T}\,\mathrm{Re}\int_0^T\mathrm{d}t\int_0^t\mathrm{d}s\left(\mathrm{e}^{-\varkappa\left(T-\frac{t+s}{2}\right)}-\mathrm{e}^{\mathrm{i}\mu(T-t)-\frac{\varkappa}{2}(T-s)}\right.$$

$$\left.-\mathrm{e}^{-\mathrm{i}\mu(T-s)-\frac{\varkappa}{2}(T-t)}\right)\mathrm{Tr}\left\{\check{\mathcal{R}}_1\circ\mathrm{e}^{\check{\mathcal{L}}(t-s)}\circ\check{\mathcal{R}}_1[\eta(s)]\right\},$$

$$\Delta_3(T,\mu) = \frac{2}{T}\,\mathrm{Re}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}\mu(t-s)}\,\mathrm{Tr}\left\{\check{\mathcal{R}}_1 \circ \mathrm{e}^{\check{\mathcal{L}}(t-s)} \circ \check{\mathcal{R}}_1[\eta(s)-\eta_{\mathrm{eq}}]\right\}.$$

By using

$$\left\|\check{\mathcal{R}}_1\right\| \le 2\,|\alpha_1|\,\sqrt{\gamma} \le 2\sqrt{\gamma}, \qquad \left\|\mathrm{e}^{\check{\mathcal{L}}(t-s)}\right\| = 1, \qquad \|\eta(s)\|_1 = 1, \tag{9.49}$$

we get $\left|\mathrm{Tr}\left\{\check{\mathcal{R}}_1 \circ \mathrm{e}^{\check{\mathcal{L}}(t-s)} \circ \check{\mathcal{R}}_1[\eta(s)]\right\}\right| \le 4\gamma$. Then, we have

$$\begin{aligned}|\Delta_2(T,\mu)| &\le \frac{8\gamma}{T}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\left(\mathrm{e}^{-\varkappa\left(T-\frac{t+s}{2}\right)} + \mathrm{e}^{-\frac{\varkappa}{2}(T-s)} + \mathrm{e}^{-\frac{\varkappa}{2}(T-t)}\right)\\ &= \frac{16\gamma}{T\varkappa^2}\left(1-\mathrm{e}^{-\frac{\varkappa}{2}T}\right) \xrightarrow{t\to+\infty} 0.\end{aligned}$$

The limit on $\Delta_3(T,\mu)$ needs to be done explicitly in the sense of distributions. Let h be any Schwartz function and consider its Fourier transform $\widehat{h}(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\mu t} h(\mu)\mathrm{d}\mu$. Then, also $\widehat{h}$ is a Schwartz function and $\int_0^{+\infty} \left|\widehat{h}(t)\right| \mathrm{d}t < +\infty$. Then, we have

$$\begin{aligned}&\int_{-\infty}^{+\infty} h(\mu)\Delta_3(T,\mu)\\ &= \frac{1}{T}\int_0^T \mathrm{d}s \int_s^T \mathrm{d}t\, \widehat{h}(t-s)\,\mathrm{Tr}\left\{\check{\mathcal{R}}_1 \circ \mathrm{e}^{\check{\mathcal{L}}(t-s)} \circ \check{\mathcal{R}}_1[\eta(s)-\eta_{\mathrm{eq}}]\right\} + \text{c.c.}\\ &= \frac{1}{T}\int_0^T \mathrm{d}s \int_0^{T-s} \mathrm{d}r\, \widehat{h}(r)\,\mathrm{Tr}\left\{\check{\mathcal{R}}_1 \circ \mathrm{e}^{\check{\mathcal{L}}r} \circ \check{\mathcal{R}}_1[\eta(s)-\eta_{\mathrm{eq}}]\right\} + \text{c.c.}\end{aligned}$$

By (8.52), (8.53), (9.49) we get

$$\left|\mathrm{Tr}\left\{\check{\mathcal{R}}_1 \circ \mathrm{e}^{\check{\mathcal{L}}r} \circ \check{\mathcal{R}}_1[\eta(s)-\eta_{\mathrm{eq}}]\right\}\right| \le 4\gamma\,\mathrm{e}^{-\frac{\gamma}{2}t}\left|\vec{x}(0)-\vec{x}_{\mathrm{eq}}\right|$$

and then we have

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} h(\mu)\Delta_3(T,\mu)\right| &\le \frac{2}{T}\int_0^T \mathrm{d}s \int_0^{T-s} \mathrm{d}r\left|\widehat{h}(r)\right| 4\gamma\,\mathrm{e}^{-\frac{\gamma}{2}s}\left|\vec{x}(0)-\vec{x}_{\mathrm{eq}}\right|\\ &\le \frac{8\gamma}{T}\left|\vec{x}(0)-\vec{x}_{\mathrm{eq}}\right| \int_0^{+\infty}\left|\widehat{h}(r)\right|\mathrm{d}r \int_0^T \mathrm{e}^{-\frac{\gamma}{2}s}\,\mathrm{d}s\\ &= \frac{4}{T}\left|\vec{x}(0)-\vec{x}_{\mathrm{eq}}\right| \int_0^{+\infty}\left|\widehat{h}(r)\right|\mathrm{d}r\left(1-\mathrm{e}^{-\frac{\gamma}{2}T}\right) \xrightarrow{t\to+\infty} 0.\end{aligned}$$

Finally, the existence of the limit in the definition (9.48) of $S_{\mathrm{hom}}(\mu;\vartheta)$ follows from the explicit computations of Propositions 9.7 and 9.9. □

We have defined the homodyne spectrum $S_{\rm hom}(\mu;\vartheta)$ by eliminating the inessential modulating pre-factor from the expression of $S_{\rm I}(\mu;\vartheta)$. Note that we have also

$$S_{\rm hom}(\mu;\vartheta) = \lim_{\varkappa\to+\infty} \frac{\pi\varkappa}{k_1^2}\, S_{\rm I}(\mu;\vartheta). \tag{9.50}$$

While we started from the definition of $S_{\rm I}$, which is the spectrum of the regular process $I(t)$, by eliminating the pre-factor we ended with $S_{\rm hom}$ given by (9.48), in agreement with the definition (4.85) in Section 4.5.3 of the spectrum of a singular process.

By expressing the second moments as square of the mean plus variance, we get the usual decomposition into the coherent (elastic) part and the incoherent (inelastic) one:

$$S_{\rm hom}(\mu;\vartheta) = S^{\rm el}_{\rm hom}(\mu;\vartheta) + S^{\rm inel}_{\rm hom}(\mu;\vartheta), \tag{9.51a}$$

$$S^{\rm el}_{\rm hom}(\mu;\vartheta) := \lim_{T\to+\infty} \frac{1}{T} \left| \mathbb{E}^{T,\omega}_{\rho_{\rm eq}} \left[\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\, \mathrm{d}W_1(s) \right] \right|^2, \tag{9.51b}$$

$$\begin{aligned} S^{\rm inel}_{\rm hom}(\mu;\vartheta) := \lim_{T\to+\infty} \frac{1}{T} \Bigg(& \mathbb{E}^{T,\omega}_{\rho_{\rm eq}} \left[\left| \int_0^T \mathrm{e}^{\mathrm{i}\mu s}\, \mathrm{d}W_1(s) \right|^2 \right] \\ & - \left| \mathbb{E}^{T,\omega}_{\rho_{\rm eq}} \left[\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\, \mathrm{d}W_1(s) \right] \right|^2 \Bigg). \end{aligned} \tag{9.51c}$$

9.2.1.1 The Spectrum of the Mean Current

Proposition 9.7. *The elastic part of the homodyne spectrum is given by*

$$\begin{aligned} S^{\rm el}_{\rm hom}(\mu;\vartheta) &= 2\pi\gamma\, |\alpha_1|^2 \left| \mathrm{Tr}\left\{ \left(\mathrm{e}^{\mathrm{i}\vartheta}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta}\sigma_+ \right) \eta_{\rm eq} \right\} \right|^2 \delta(\mu) \\ &= 2\pi\gamma\, |\alpha_1|^2 \left(x_{\rm eq}\cos\vartheta + y_{\rm eq}\sin\vartheta \right)^2 \delta(\mu) \\ &= |\alpha_1|^2\, \frac{8\pi\,\Omega^2\gamma\,(-2\Delta\omega\cos\vartheta + \widetilde{\gamma}\sin\vartheta)^2}{(2\overline{n}+1)^2\left(4\Delta\omega^2 + \widetilde{\Gamma}^2\right)^2}\, \delta(\mu). \end{aligned} \tag{9.52}$$

Moreover, the following relation with the elastic part of the heterodyne power spectrum holds:

$$\frac{1}{2\pi}\int_0^{2\pi} S^{\rm el}_{\rm hom}(\mu;\vartheta)\,\mathrm{d}\vartheta = 4\pi\, |\alpha_1|^2\, \Sigma_{\rm el}(\mu)\Big|_{\varkappa=0} = |\alpha_1|^2\, S^{\rm el}_{\rm het}(\mu)\Big|_{\varkappa=0}. \tag{9.53}$$

Proof. By recalling that

$$\lim_{T\to+\infty} \frac{2\,(\sin \mu T/2)^2}{\pi\,\mu^2 T} = \delta(\mu)$$

and by using the expression of the mean function of W_1 and the fact that the initial state is the equilibrium one, we get

$$\begin{aligned}
\frac{1}{T\,|\alpha_1|^2}\left|\mathbb{E}^{T,\omega}_{\rho_{\rm eq}}\left[\int_0^T \mathrm{e}^{-\mathrm{i}\mu s}\,\mathrm{d}W_1(s)\right]\right|^2 &= \frac{\gamma}{T}\left|\int_0^T \mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\mu t}\,2\,\mathrm{Re}\,\mathrm{Tr}\left\{\check{\eta}(t)\sigma_-\mathrm{e}^{\mathrm{i}\vartheta}\right\}\right|^2 \\
&= \frac{4\gamma\,(\sin \mu T/2)^2}{\mu^2 T}\left|\mathrm{Tr}\left\{\left(\mathrm{e}^{\mathrm{i}\vartheta}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta}\sigma_+\right)\eta_{\rm eq}\right\}\right|^2 \\
&\xrightarrow{T\to+\infty} 2\pi\gamma\left|\mathrm{Tr}\left\{\left(\mathrm{e}^{\mathrm{i}\vartheta}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta}\sigma_+\right)\eta_{\rm eq}\right\}\right|^2\delta(\mu) \\
= 8\pi\gamma\left(\mathrm{Re}\,\mathrm{e}^{\mathrm{i}\vartheta}\left(\eta_{\rm eq}\right)_{12}\right)^2\delta(\mu) &= 2\pi\gamma\,|\alpha_1|^2\left(x_{\rm eq}\cos\vartheta + y_{\rm eq}\sin\vartheta\right)^2\delta(\mu) \\
&= \frac{8\pi\,\Omega^2\gamma\,(-2\Delta\omega\cos\vartheta + \widetilde{\gamma}\sin\vartheta)^2}{(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}\,\delta(\mu).
\end{aligned}$$

The last statement follows by comparing this result with Eqs. (9.28), (9.40) in the limit $\varkappa \downarrow 0$. □

Remark 9.8. The elastic component of the homodyne spectrum is a δ-spike around $\mu = 0$ whose peculiar feature is the dependence of its intensity on ϑ.

Let us note that the intensity of *the elastic component vanishes for* $\Omega = 0$ *or for* $\widetilde{\gamma}\sin\vartheta = 2\Delta\omega\cos\vartheta$.

As discussed in Section 8.1.3, the angle ϑ is a relative phase among the laser light, the local oscillator and the fluorescence light. Therefore, it can be varied by changing the optical paths from the laser source to the atom, from the laser source to the beam splitter and from the atom to the beam splitter.

9.2.1.2 The Spectrum of the Fluctuations of the Current

Proposition 9.9. *The inelastic part of the homodyne spectrum can be written as*

$$S^{\rm inel}_{\rm hom}(\mu;\vartheta) = |\alpha_2|^2 + |\alpha_1|^2\,S^{\rm red}_{\rm hom}(\mu;\vartheta), \tag{9.54}$$

where the "reduced" inelastic homodyne spectrum $S^{\rm red}_{\rm hom}(\mu;\vartheta)$ *is not negative and does not depend on* α_1 *and* α_2.

The explicit expression of the reduced component is given by

$$S^{\rm red}_{\rm hom}(\mu;\vartheta) = 1 + \vec{u}(\vartheta)\cdot\left(\frac{2\gamma A}{A^2+\mu^2}\,\vec{t}(\vartheta)\right), \tag{9.55}$$

$$\vec{u}(\vartheta) = (\cos\vartheta,\ \sin\vartheta,\ 0)\,, \tag{9.56}$$

$$\vec{t}(\vartheta) = \mathrm{Tr}\left\{\vec{\sigma}\left(\mathrm{e}^{\mathrm{i}\vartheta}\tau + \mathrm{e}^{-\mathrm{i}\vartheta}\tau^*\right)\right\} = \begin{pmatrix} \left(1 + z_{\mathrm{eq}} - x_{\mathrm{eq}}^2\right)\cos\vartheta - x_{\mathrm{eq}}y_{\mathrm{eq}}\sin\vartheta \\ \left(1 + z_{\mathrm{eq}} - y_{\mathrm{eq}}^2\right)\sin\vartheta - x_{\mathrm{eq}}y_{\mathrm{eq}}\cos\vartheta \\ -\left(1 + z_{\mathrm{eq}}\right)\left(x_{\mathrm{eq}}\cos\vartheta + y_{\mathrm{eq}}\sin\vartheta\right) \end{pmatrix}, \tag{9.57}$$

with $\tau = \sigma_-\eta_{\mathrm{eq}} - \mathrm{Tr}\left\{\sigma_-\eta_{\mathrm{eq}}\right\}\eta_{\mathrm{eq}}$.

An alternative expression of the reduced inelastic homodyne spectrum is

$$S^{\mathrm{red}}_{\mathrm{hom}}(\mu;\vartheta) = \frac{1}{2}\left[S^{\mathrm{inel}}_{\mathrm{het}}(\mu) + S^{\mathrm{inel}}_{\mathrm{het}}(-\mu)\right]_{\varkappa=0} + \mathrm{Re}\left[\mathrm{e}^{2\mathrm{i}\vartheta}b(\mu)\right], \tag{9.58}$$

where

$$b(\mu) := (1,\ -\mathrm{i},\ 0)\cdot\left(\frac{2\gamma A}{A^2+\mu^2}\,\vec{d}(0)\right). \tag{9.59}$$

Proof. By using the definition (9.51c) and the expression of the second moments (4.46), we get

$$\begin{aligned} S^{\mathrm{inel}}_{\mathrm{hom}}(\mu;\vartheta) &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\mu(t-s)}\,\frac{\partial^2}{\partial t\,\partial s}\,\mathrm{Cov}^{T,\omega}_{\rho_{\mathrm{eq}}}\left[W_1(t),\,W_1(s)\right] \\ &= 1 + 2\gamma\,|\alpha_1|^2\lim_{T\to+\infty}\int_0^T\left(1-\frac{t}{T}\right)g(t)\cos\mu t\,\mathrm{d}t, \end{aligned}$$

$$g(t) := \mathrm{Tr}\left\{\sigma_\vartheta\,\mathrm{e}^{\check{\mathcal{L}}t}[\mathrm{e}^{\mathrm{i}\vartheta}\tau + \mathrm{e}^{-\mathrm{i}\vartheta}\tau^*]\right\}, \qquad \sigma_\vartheta := \mathrm{e}^{\mathrm{i}\vartheta}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta}\sigma_+.$$

By the definition (9.54) of $S^{\mathrm{red}}_{\mathrm{hom}}(\mu;\vartheta)$, we get from the result above

$$S^{\mathrm{red}}_{\mathrm{hom}}(\mu;\vartheta) = 1 + 2\gamma\lim_{T\to+\infty}\int_0^T\left(1-\frac{t}{T}\right)g(t)\cos\mu t\,\mathrm{d}t.$$

From here we see that the reduced component depends on γ and on all the parameters appearing in $\check{\mathcal{L}}$ and η_{eq}; therefore, it does not depend on α_1 and α_2. By its definition, the quantity $S^{\mathrm{inel}}_{\mathrm{hom}}(\mu;\vartheta)$ is not negative for any choice of the parameters, in particular for α_1 and α_2. By taking $\alpha_2 = 0$ and, therefore, $|\alpha_1| = 1$ in Eq. (9.54), we get that also $S^{\mathrm{red}}_{\mathrm{hom}}(\mu;\vartheta)$ is not negative for any choice of the parameters.

By using the Bloch representation, we get

$$g(t) = \vec{u}(\vartheta) \cdot \left(\mathrm{e}^{-At}\vec{t}(\vartheta)\right).$$

By Lemma 8.3, $g(t)$ decays exponentially and the term with t/T vanishes for $T \to +\infty$; so, we obtain

$$\begin{aligned} S^{\mathrm{red}}_{\mathrm{hom}}(\mu;\vartheta) - 1 &= \gamma \int_0^{+\infty} \mathrm{d}t \left(\mathrm{e}^{\mathrm{i}\mu t} + \mathrm{e}^{-\mathrm{i}\mu t}\right) \vec{u}(\vartheta) \cdot \left(\mathrm{e}^{-At}\vec{t}(\vartheta)\right) \\ &= \gamma \vec{u}(\vartheta) \cdot \left[\left(\frac{1}{A - \mathrm{i}\mu} + \frac{1}{A + \mathrm{i}\mu}\right)\vec{t}(\vartheta)\right]. \end{aligned}$$

By using

$$\frac{1}{A + \mathrm{i}\mu} + \frac{1}{A - \mathrm{i}\mu} = \frac{2A}{A^2 + \mu^2},$$

we get the expression (9.55).

By comparing (9.19) and (9.57) we get

$$\vec{t}(\vartheta) = 2\cos\vartheta\, \mathrm{Re}\,\vec{d}(0) - 2\sin\vartheta\, \mathrm{Im}\,\vec{d}(0).$$

By inserting this and the explicit expression of $\vec{u}(\vartheta)$ in (9.55), we obtain

$$S^{\mathrm{red}}_{\mathrm{hom}}(\mu;\vartheta) - 1 = \mathrm{Re}\left[\left(1 + \mathrm{e}^{2\mathrm{i}\vartheta},\ \mathrm{i}\left(1 - \mathrm{e}^{2\mathrm{i}\vartheta}\right),\ 0\right) \cdot \left(\frac{2\gamma A}{A^2 + \mu^2}\,\vec{d}(0)\right)\right]. \quad (9.60)$$

By comparing this expression with Eqs. (9.26), (9.27) we obtain Eqs. (9.58), (9.59). □

Other Approaches

The spectrum of the fluorescence light was already obtained in [13], for the case $\Delta\omega = 0$, $\overline{n} = 0$, $k_{\mathrm{d}} = 0$ (see also [1, Sect. 11.4]), and in [14], for the case $\overline{n} = 0$. The quantities which are studied in those papers are, in our notations, $S^{\mathrm{red}}_{\mathrm{hom}}(x;0) - 1$ and $S^{\mathrm{red}}_{\mathrm{hom}}(x;\pi/2) - 1$. The approach of [1, 13, 14] does not use quantum measuring theory and, so, there is no proof of positivity of $S^{\mathrm{red}}_{\mathrm{hom}}(x;\vartheta)$ (the fluctuation spectrum with shot noise included). However, the final formulae are the same, apart from a different normalisation and the fact that shot noise is not included. The approach used in [1, 13, 14] is similar to the traditional one for the case of heterodyne spectrum: a suitable quantum correlation function for the outgoing electromagnetic field is defined and the spectrum is by definition its Fourier transform; then, the fields are eliminated in favour of the atomic variables and the "quantum regression theorem" [1, p. 118] is used to get the final result.

9.2.1.3 Uncertainty Relations

From Eq. (9.58) one obtains immediately that $S^{\rm red}_{\rm hom}(\mu;\vartheta)$ is periodic in ϑ of period π and that

$$\frac{1}{2}\left[S^{\rm red}_{\rm hom}(\mu;\vartheta)+S^{\rm red}_{\rm hom}(\mu;\vartheta+\pi/2)\right]=\frac{1}{2}\left[S^{\rm inel}_{\rm het}(\mu)+S^{\rm inel}_{\rm het}(-\mu)\right]_{\varkappa=0}\geq 1. \quad (9.61)$$

In Remark 9.4 we have seen that for $\overline{n}=0,\ k_{\rm d}=0$ the inelastic heterodyne spectrum $S^{\rm inel}_{\rm het}$ is an even function of μ. In this case we get

$$S^{\rm inel}_{\rm het}(\mu)\big|_{\varkappa=0}=\frac{1}{2}\left[S^{\rm red}_{\rm hom}(\mu;\vartheta)+S^{\rm red}_{\rm hom}(\mu;\vartheta+\pi/2)\right]. \quad (9.62)$$

As discussed in Sect. 4.5.3.1, by using the approach to continuous measurements of [12], (based on quantum stochastic calculus) and the Heisenberg uncertainty relations, one can obtain also the uncertainty relation for spectra (4.89) [16], which for the inelastic homodyne spectrum becomes

$$S^{\rm red}_{\rm hom}(\mu;\vartheta)\,S^{\rm red}_{\rm hom}(\mu;\vartheta+\pi/2)\geq 1. \quad (9.63)$$

9.2.2 Examples of Spectra and Squeezing

Let us comment the main features of the inelastic homodyne spectrum. Recall the notation

$$\Gamma^2=2\Omega^2+\widetilde{\gamma}^2\,,\qquad \widetilde{\gamma}=\gamma\,(2\overline{n}+1+4k_{\rm d})\,.$$

9.2.2.1 Noise and Thermal Light

No Signal

For $\Omega=0$ and $\overline{n}=0$ we get $S^{\rm inel}_{\rm hom}(\mu;\vartheta)=S^{\rm red}_{\rm hom}(\mu;\vartheta)=1$. In this case there is no fluorescence light in the long run and we see the spectrum of a pure white noise (shot noise due to the local oscillator).

Pure Thermal Light

For $\Omega=0$ and $\overline{n}>0$ we get $b(\mu)=0$ and, so, there is no dependence on ϑ. Then, from Eq. (9.39b), we obtain

$$S^{\rm red}_{\rm hom}(\mu;\vartheta)=1+4\gamma\,\frac{\overline{n}}{2\overline{n}+1}\left[\frac{\widetilde{\gamma}}{\widetilde{\gamma}^2+4\,(\mu-\Delta\omega)^2}+\frac{\widetilde{\gamma}}{\widetilde{\gamma}^2+4\,(\mu+\Delta\omega)^2}\right]>1. \quad (9.64)$$

In this case there is only thermal light with carrier frequency ω_0, while the local oscillator is at frequency ω. The result are two temperature-dependent Lorentzian peaks at $\mu = \pm\Delta\omega$. The white noise contribution is always present.

Squeezing of the Fluorescence Light

When $\Omega > 0$, $S^{\rm inel}_{\rm hom}(\mu;\vartheta)$ and $S^{\rm red}_{\rm hom}(\mu;\vartheta)$ become ϑ-dependent. This phase dependence is the first peculiar feature which differentiates homodyne and heterodyne detection. Moreover, it is possible that the spectrum goes below the shot noise level (which is 1 by our choice of normalisation). This is a typical quantum phenomenon known as *squeezing*. From the probabilistic point of view, this fact means that in the decomposition $\mathrm{d}W_1(t) = \mathrm{d}\widehat{W}_1(t) + v_1(t)\mathrm{d}t$ some negative correlation between the white noise component $\mathrm{d}\widehat{W}_1(t)$ and the smooth contribution $v_1(t)\mathrm{d}t$ has been developed.

The minimal value of $S^{\rm red}_{\rm hom}(\mu;\vartheta)$ with respect to ϑ is given by

$$S^{\rm min}_{\rm hom}(\mu) := \min_{\vartheta} S^{\rm red}_{\rm hom}(\mu;\vartheta) = 1 + 2\pi\left[\Sigma_{\rm inel}(\mu) + \Sigma_{\rm inel}(-\mu)\right]_{\varkappa=0} - |b(\mu)|\,. \quad (9.65)$$

To have $S^{\rm min}_{\rm hom}(\mu) < 1$ implies obviously that $S^{\rm red}_{\rm hom}(\mu;\vartheta) < 1$ for some ϑ and it says that the light in the modes around μ is squeezed [16]. So, the condition for squeezing is

$$|b(\mu)| > 2\pi\left[\Sigma_{\rm inel}(\mu) + \Sigma_{\rm inel}(-\mu)\right]_{\varkappa=0}\,.$$

Let us stress that, due to the positivity of $S^{\rm red}_{\rm hom}(\mu;\vartheta)$, we have always

$$|b(\mu)| \leq 1 + 2\pi\left[\Sigma_{\rm inel}(\mu) + \Sigma_{\rm inel}(-\mu)\right]_{\varkappa=0}\,. \quad (9.66)$$

Global Squeezing

Let us define

$$\Pi_{\rm hom}(\vartheta) := \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty}\left[S^{\rm red}_{\rm hom}(\mu,\vartheta) - 1\right]\mathrm{d}\mu; \quad (9.67)$$

we get

$$\begin{aligned}\Pi_{\rm hom}(\vartheta) &= 2\,\mathrm{Re}\left(\mathrm{Tr}\{\sigma_+\tau\} + \mathrm{e}^{2\mathrm{i}\vartheta}\,\mathrm{Tr}\{\sigma_-\tau\}\right)\\ &= 2\,\mathrm{Tr}\{P_+\eta_{\rm eq}\} - 2\left|\mathrm{Tr}\{\sigma_-\eta_{\rm eq}\}\right|^2 - 2\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\vartheta}\,\mathrm{Tr}\{\sigma_-\eta_{\rm eq}\}\right)^2,\end{aligned}$$

or

$$\Pi_{\rm hom}(\vartheta) = 1 - \left(x_{\rm eq}\cos\vartheta + y_{\rm eq}\sin\vartheta\right)^2 + z_{\rm eq}\,. \quad (9.68)$$

From Eqs. (9.43c), (9.61), (9.67), or from Eqs. (9.44), (9.68), we relate $\Pi_{\rm hom}$ with the heterodyne inelastic intensity $\Pi_{\rm inel}(\Omega, \Delta\omega)$:

$$\frac{1}{2}\left[\Pi_{\rm hom}(\vartheta) + \Pi_{\rm hom}(\vartheta + \pi/2)\right] = \Pi_{\rm inel}(\Omega, \Delta\omega) = 1 + z_{\rm eq} - \frac{x_{\rm eq}^{\,2} + y_{\rm eq}^{\,2}}{2}\,. \quad (9.69)$$

As a synthetic index of squeezing, let us introduce the minimum value of $\Pi_{\rm hom}(\vartheta)$ with respect to ϑ:

$$\Pi_{\rm hom} := \inf_{\vartheta} \Pi_{\rm hom}(\vartheta) = 2\,{\rm Tr}\{P_+\eta_{\rm eq}\} - 4\left|{\rm Tr}\{\sigma_-\eta_{\rm eq}\}\right|^2$$
$$= 1 - \left|\vec{x}_{\rm eq}\right|^2 + z_{\rm eq}\left(1 + z_{\rm eq}\right) = {\rm AS}_{\eta_{\rm eq}}. \quad (9.70)$$

The quantity $\Pi_{\rm hom}$ can be positive or negative. We can say that we have "global squeezing" in the fluorescence light when $\Pi_{\rm hom} < 0$; note that this condition is equivalent to the condition of atomic squeezing (8.69). Indeed, the first theoretical predictions of squeezing of the fluorescence light were based on equations of the type of (9.68), (9.70), which were obtained in the approximation of a two-level atom interacting with a single-mode quantum field [18–20].

Let us study the case $\overline{n} = 0$ and $k_{\rm d} = 0$, in which we have

$$\Pi_{\rm hom} = \frac{2\Omega^2\left[2\Omega^2 - \left(4\Delta\omega^2 + \Gamma^2\right)\right]}{\left(4\Delta\omega^2 + \Gamma^2\right)^2}\,.$$

It is easy to check that $\Pi_{\rm hom} < 1$ and to find its minimum with respect to Ω. This minimum is reached for $6\Omega^2 = 4\Delta\omega^2 + \gamma^2$, where $\Pi_{\rm hom}$ takes the value $-1/8$. In the same point we have $z_{\rm eq} = -3/4$, $x_{\rm eq}^2 + y_{\rm eq}^2 = 3/8$, $1 - \left|\vec{x}_{\rm eq}\right|^2 = 1/16$.

So, for $\overline{n} = 0$ and $k_{\rm d} = 0$, we have $-\frac{1}{8} \le \Pi_{\rm hom} < 1$. The maximum of $x_{\rm eq}^2 + y_{\rm eq}^2$ is for $2\Omega^2 = 4\Delta\omega^2 + \gamma^2$, the minimum of $\Pi_{\rm hom}$ is for $6\Omega^2 = 4\Delta\omega^2 + \gamma^2$; an intermediate choice is $\Omega^2 \simeq \Delta\omega^2$ and as a matter of fact this is a working choice for obtaining a good squeezing in the spectra, when $\Delta\omega$ is not too small (see Figs. 9.4, 9.5).

Let us stress that one can have local squeezing, in the sense of the previous paragraph, also when the condition for global squeezing is violated, see Fig. 9.6.

9.2.2.2 Some Homodyne Spectra

Pure Fluorescence

For $k_{\rm d} = 0$, $\overline{n} = 0$, $\Omega > 0$, $\Sigma_{\rm inel}(\mu)\big|_{\varkappa=0}$ is given by Eq. (9.36b) and it is an even function of μ. Moreover, by some long computations, one gets

$$b(\mu) = c(\mu) + c(-\mu), \quad (9.71a)$$

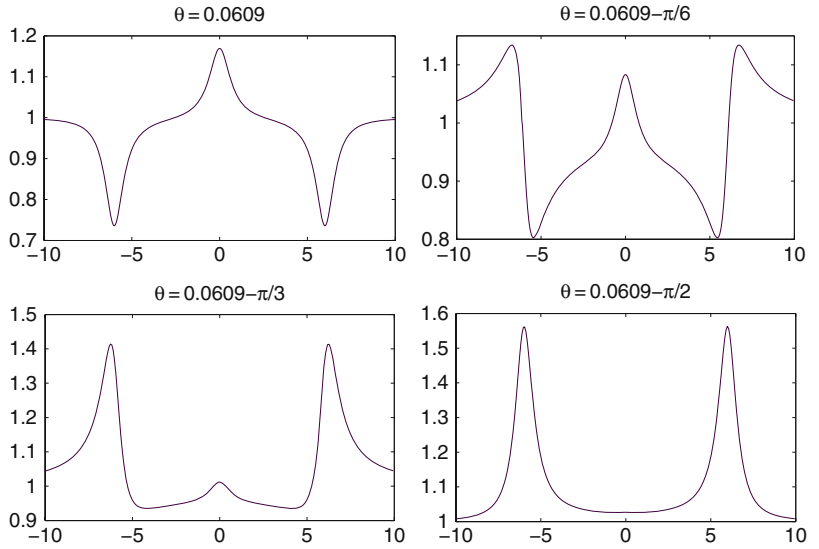

Fig. 9.4 $S^{\mathrm{red}}_{\mathrm{hom}}(x;\vartheta)$ with $\gamma = 1$, $\Omega = 4.3709$, $\Delta\omega = -4.1455$, $\overline{n} = 0$, $k_{\mathrm{d}} = 0$. The minimum is 0.7358 at $\mu = \pm 6$ for $\vartheta = 0.0609$

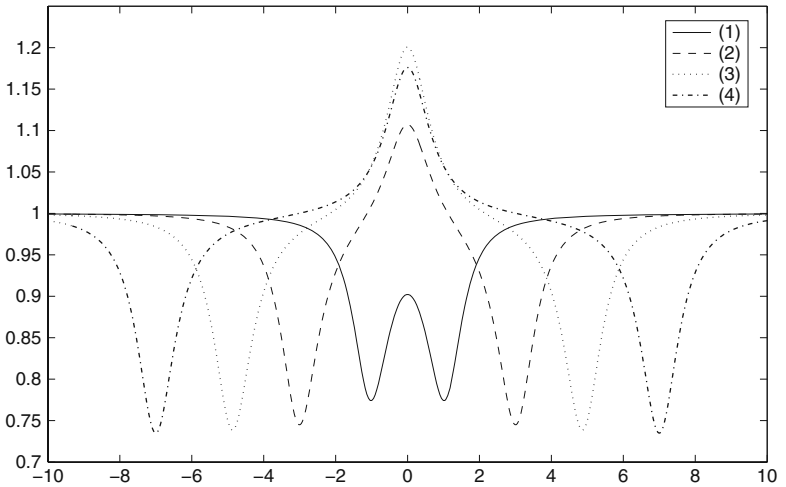

Fig. 9.5 $S^{\mathrm{red}}_{\mathrm{hom}}(x;\vartheta)$ with $\gamma = 1$, $\overline{n} = 0$, $k_{\mathrm{d}} = 0$ and (1) $\vartheta = -0.3480$, $\Delta\omega = 0.8279$, $\Omega = 0.7445$, value of the minimum 0.7742; (2) $\vartheta = -0.1204$, $\Delta\omega = 2.1469$, $\Omega = 2.1649$, value of the minimum 0.7449; (3) $\vartheta = -0.0729$, $\Delta\omega = 3.2746$, $\Omega = 3.6357$, value of the minimum 0.7373; (4) $\vartheta = -0.0522$, $\Delta\omega = 4.8191$, $\Omega = 5.1055$, value of the minimum 0.7349

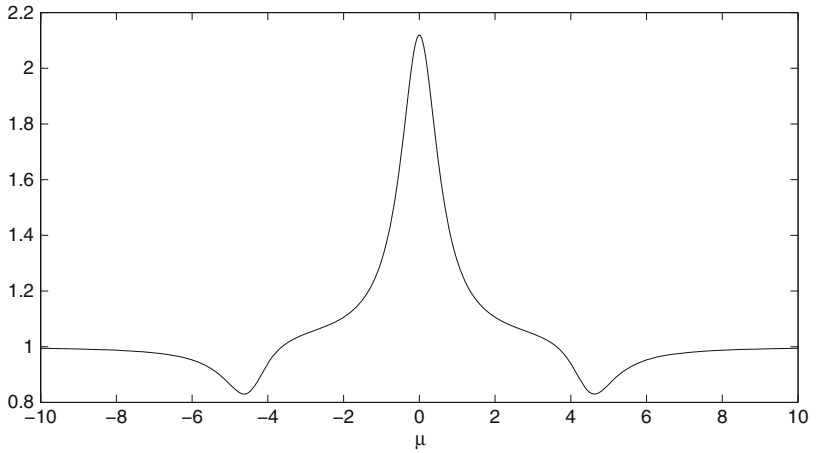

Fig. 9.6 $S^{\mathrm{red}}_{\mathrm{hom}}(\mu; -0.1786)$ with $\gamma = 1, \overline{n} = 0, k_{\mathrm{d}} = 0, \Delta\omega = 2, \Omega = 4$; the value of the minimum is 0.8299 at $\mu = \pm 4.6239$

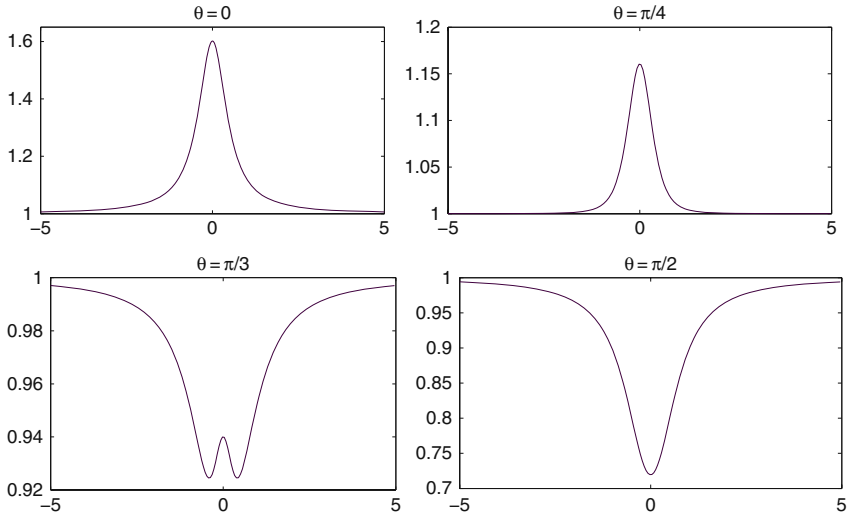

Fig. 9.7 $S^{\mathrm{red}}_{\mathrm{hom}}(x; \vartheta)$ with $\gamma = 1, \Omega = 0.2976, \Delta\omega = 0, \overline{n} = 0, k_{\mathrm{d}} = 0$. The minimum is 0.7195 for $\vartheta = \pi/2$

$$c(\mu) := \frac{4\gamma\Omega^2}{\left(4\Delta\omega^2+\Gamma^2\right)^2 N(\mathrm{i}\mu)}\Big[2\Omega^4 - 2\mu\Omega^2(2\Delta\omega+\mathrm{i}\gamma) + (2\Delta\omega+\mathrm{i}\gamma)^2(\mathrm{i}\gamma+2\Delta\omega-2\mu)(\mathrm{i}\gamma-\mu)\Big], \tag{9.71b}$$

$$N(\mathrm{i}\mu) = 2\mathrm{i}\mu\Omega^2 + (\gamma+\mathrm{i}\mu)\left[4\Delta\omega^2+2\Omega^2+(\gamma+2\mathrm{i}\mu)^2\right]. \tag{9.71c}$$

Resonance fluorescence

For $k_\mathrm{d}=0$, $\overline{n}=0$, $\Delta\omega=0$, $\Omega>0$, we get

$$4\pi\,\Sigma_\mathrm{inel}(\mu) \overset{\varkappa=0}{=} \frac{8\gamma^2\Omega^4\left(\Gamma^2+3\gamma^2+4\mu^2\right)}{\Gamma^2\,|N(\mathrm{i}\mu)|^2}, \tag{9.72a}$$

$$b(\mu) = \frac{8\gamma^2\Omega^2}{\Gamma^4\,|N(\mathrm{i}\mu)|^2}\Big[\Gamma^2\left(\gamma^4+2\Omega^4\right) + \mu^2\left(\gamma^4+\mu^2\Gamma^2+4\mu^2\Omega^2-6\Omega^2\Gamma^2-4\Omega^4\right)\Big], \tag{9.72b}$$

$$|N(\mathrm{i}\mu)|^2 = \gamma^2\left(\Gamma^2-8\mu^2\right)^2 + \mu^2\left(2\Gamma^2+3\gamma^2-4\mu^2\right)^2, \tag{9.72c}$$

$$S^\mathrm{red}_\mathrm{hom}(\mu;\vartheta) = 1 + 4\pi\,\Sigma_\mathrm{inel}(\mu)\big|_{\varkappa=0} + b(\mu)\cos 2\vartheta. \tag{9.72d}$$

An example of the resulting spectrum is given in Fig. 9.7, where the parameters are chosen to have a good squeezing in $\mu=0$, which happens for $\vartheta=\pi/2$.

By (9.72d) or by (9.62) and (9.40c) we get

$$0 \le S^\mathrm{inel}_\mathrm{het}(\mu) - 1 = \frac{S^\mathrm{red}_\mathrm{hom}(\mu;\vartheta)-1}{2} + \frac{S^\mathrm{red}_\mathrm{hom}(\mu;\vartheta+\pi/2)-1}{2}. \tag{9.73}$$

With the choice of parameters done in Fig. 9.7 one has $\frac{S^\mathrm{red}_\mathrm{hom}(\mu;0)-1}{2} > 0$ and $\frac{S^\mathrm{red}_\mathrm{hom}(\mu;\pi/2)-1}{2} < 0$; so, $S^\mathrm{inel}_\mathrm{het}(\mu)-1$ is the difference of two positive quantities and this combination must give a particularly narrow line. As noticed in [21], the presence of squeezing in the homodyne spectrum must be associated to the narrowing of the lines in the heterodyne spectrum; this happens in resonance for small Ω. To see this effect in Fig. 9.8 we plot the heterodyne inelastic spectrum with the parameters of Fig. 9.7 and the heterodyne inelastic spectrum with a very small Ω and we compare them with the same spectrum with $\Omega=1$ and with a Lorentzian with natural linewidth ($\gamma=1$); all the curves are normalised to unit area. So, in Fig. 9.8 line (1) is $\frac{1/2\pi}{\mu^2+1/4}$, while line (2), (3) and (4) are $\frac{\left(2\Omega^2+\gamma^2\right)^2}{2\gamma\Omega^4}\,\Sigma_\mathrm{inel}(\mu)$ for $\varkappa=0$, $\gamma=1$, $\Delta\omega=0$, $\overline{n}=0$, $k_\mathrm{d}=0$ and line (2) with $\Omega=1$, line (3) with $\Omega=0.2976$, line (4)

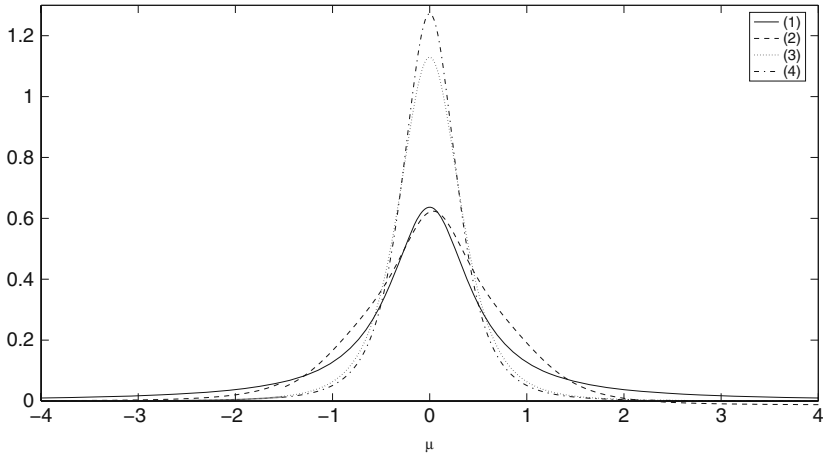

Fig. 9.8 Line (1) is the Lorentzian $\mu \mapsto \frac{1/2\pi}{\mu^2+1/4}$. The other *lines* are the normalised heterodyne inelastic spectrum (with $\gamma = 1$, $\Delta\omega = 0$, $\overline{n} = 0$, $k_{\mathrm{d}} = 0$, $\varkappa = 0$) for: (2) $\Omega = 1$; (3) $\Omega = 0.2976$; (4) $\Omega = 0.01$

with $\Omega = 0.01$. Note the heavy tails of the Lorentzian, with respect to the spectral lines, and the sharpness of lines (3) and (4).

$\Delta\omega \neq 0$: tunable squeezing

An example of the inelastic homodyne spectrum is plotted in Fig. 9.4 for four values of ϑ. The choice of the parameters ϑ, Ω and $\Delta\omega$ is to have a region with a pronounced squeezing around $\mu = \pm 6$. Note that $\Omega = 4.3709$ is near $|\Delta\omega| = 4.1455$, while the maximum global squeezing is at $\sqrt{2\Delta\omega^2/3 + \gamma^2/6} \simeq 3.409$ and the limit for vanishing global squeezing is at $\sqrt{2\Delta\omega^2 + \gamma^2/2} \simeq 5.9051$.

Another example is given in Fig. 9.5, where the parameters are chosen to have the minima in $\mu = \pm 1, \pm 3, \pm 5, \pm 7$. The possibility of choosing the position of the minima by controlling the laser parameters is very important; following [14], we say that the two-level atom can produce *tunable two-mode squeezed light*.

Local Squeezing Versus Global Squeezing

As already stressed, we can have local squeezing even when there is no global squeezing, see Fig. 9.6 where $2\Omega^2 = 32 > 17 = 4\Delta\omega^2 + \gamma^2$.

The best global squeezing, which is for $6\Omega^2 = 4\Delta\omega^2 + \gamma^2$, does not correspond to the best local squeezing, see Fig. 9.9.

The Effect of the Thermal Bath and of the Dephasing Term

An example of the homodyne spectrum with and without $\overline{n}$ and k_{d} is given in Fig. 9.10. The parameters $\overline{n}$ and k_{d} are fixed a priori (their true values depend on

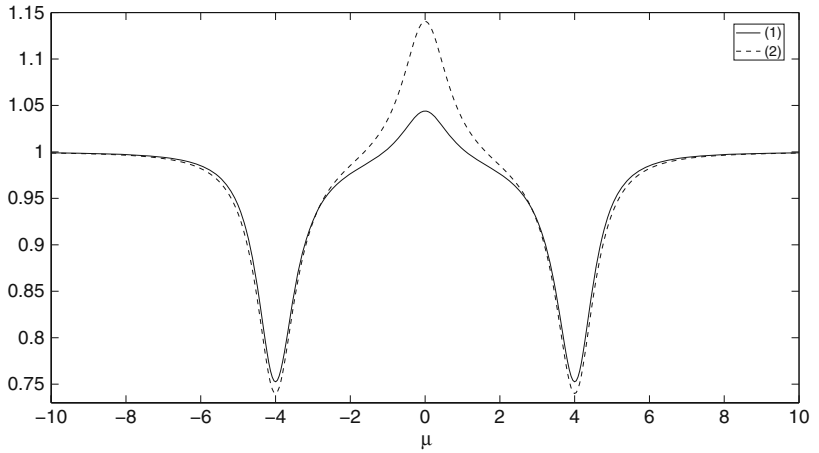

Fig. 9.9 The homodyne spectrum $S^{\mathrm{red}}_{\mathrm{hom}}(\mu;\vartheta)$ for $\gamma = 1, \overline{n} = 0, k_{\mathrm{d}} = 0$, and (1) with $\Delta\omega = 3.1151$, $\Omega = 2.5760 = \sqrt{2\Delta\omega^2/3 + 1/6}$, $\vartheta = -0.0816$, the value of the minimum is 0.7525 at $\mu = \pm 4$; (2) with $\Delta\omega = 2.8077$, $\Omega = 2.9001 \neq 2.3285 = \sqrt{2\Delta\omega^2/3 + 1/6}$, $\vartheta = -0.0909$, the value of the minimum is 0.7399 at $\mu = \pm 4$

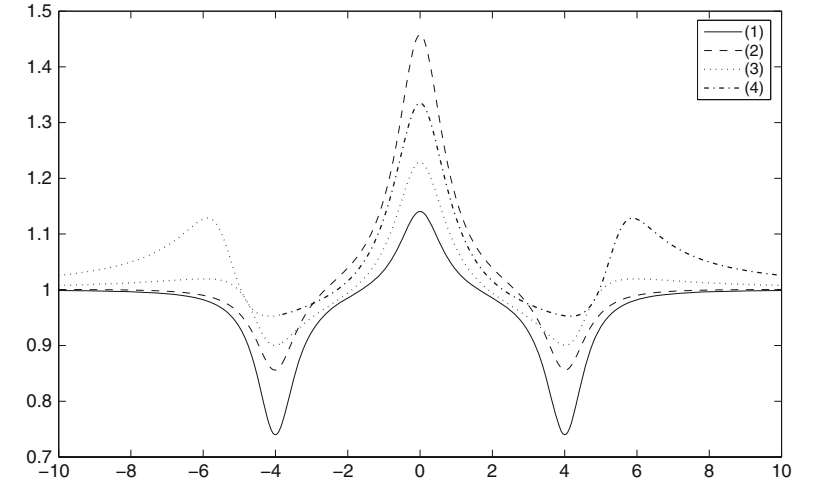

Fig. 9.10 $S^{\mathrm{red}}_{\mathrm{hom}}(x;\vartheta)$ with $\gamma = 1$ and (1) $\overline{n} = 0$, $k_{\mathrm{d}} = 0$, $\vartheta = -0.0909$, $\Delta\omega = 2.8077$, $\Omega = 2.9001$; (2) $\overline{n} = 0.05$, $k_{\mathrm{d}} = 0$, $\vartheta = -0.0918$, $\Delta\omega = 2.5762$, $\Omega = 3.0980$; (3) $\overline{n} = 0$, $k_{\mathrm{d}} = 0.1$, $\vartheta = 0.0742$, $\Delta\omega = 3.2347$, $\Omega = 2.9225$; (4) $\overline{n} = 0.05$, $k_{\mathrm{d}} = 0.1$, $\vartheta = 0.3710$, $\Delta\omega = 3.8009$, $\Omega = 3.7841$

the experimental conditions) and the other are chosen to have the best minimum in $\mu = \pm 4$. From this figure one sees that the squeezing is very sensitive to any small perturbation.

9.3 Summary

9.3.1 Atomic and Measurement Quantities

Here we recall some results and notations of Chaps. 7 and 8.

- Equilibrium state: $\eta_{\rm eq} := \frac{1}{2}\big(\mathbb{1} + \vec{x}_{\rm eq} \cdot \vec{\sigma}\big)$.
- Side channel, the channel of the detected light: channel 1.
- Forward channel, the channel of the lost light and of the stimulating laser: channel 2.
- Detection operator: $R_1(t) = \mathrm{e}^{\mathrm{i}(\nu-\omega)t}\sqrt{\gamma}\,\alpha_1\sigma_-$.
- Output current: $I(t) = \int_0^t F(t-s)\mathrm{d}W_1(s)$.
- Detector response function: $F(t) = k_1\sqrt{\frac{\varkappa}{4\pi}}\exp\left\{-\frac{\varkappa}{2}\,t\right\}$.
- Electrical power carried by the current $I(t)$: $P(t) = k_2 I(t)^2$.
- Parameters and notations:
  - Coefficient of the thermal disturbance: $\overline{n} \geq 0$;
  - Coefficient of the dephasing disturbance: $k_{\rm d} \geq 0$;
  - Resonance frequency of the atom: $\omega_0 > 0$;
  - Frequency of the stimulating laser: $\omega > 0$;
  - Detuning: $\Delta\omega = \omega_0 - \omega$;
  - Rabi frequency: $\Omega \geq 0$;
  - Natural linewidth: $\gamma > 0$;
  - Measurement phase: $\vartheta = \arg\alpha_1$;
  - Proportions of light in the side and in the forward channels: $|\alpha_1|^2$ and $|\alpha_2|^2$ with $|\alpha_1|^2 + |\alpha_2|^2 = 1$;
  - Frequency of the local oscillator: ν;
  - Instrumental width: $\varkappa > 0$;
  - Response constants: $k_1 \neq 0$, $k_2 > 0$;
  - Atomic squeezing parameter at equilibrium: $\mathrm{AS}_{\eta_{\rm eq}} = 1 - \left(x_{\rm eq}^2 + y_{\rm eq}^2\right) - \left|z_{\rm eq}\right|$;
  - Linewidth with thermal and dephasing broadening: $\widetilde{\gamma} = \gamma\,(2\overline{n} + 1 + 4k_{\rm d})$;
  - Width of the response to the detuning: $\widetilde{\Gamma}^2 = \Gamma^2 + \frac{8\Omega^2 k_{\rm d}}{2\overline{n}+1}$, $\qquad \Gamma^2 = 2\Omega^2 + \widetilde{\gamma}^2$.

9.3.2 Heterodyne Spectral Density

The atom and the local oscillator are driven by two distinct lasers; the phase coherence cannot be maintained. The observed quantity is the electrical power at large times, without any spectral analysis. No spectral quantity depend on the angle ϑ. To have a good spectrum $\varkappa$ must be small.

- Mean power spectrum (limit in the sense of distributions in ν):

$$P_{\rm het}(\nu) = \lim_{t\to+\infty} \mathbb{E}^{t,\nu}_{\rho_0}\left[P(t)\right] = \lim_{t\to+\infty} k_2\mathbb{E}^{t,\nu}_{\rho_0}\left[I(t)^2\right] = \frac{k_1^2 k_2}{4\pi} + k_1^2 k_2\,|\alpha_1|^2\;\Sigma(\nu-\omega).$$

- Shot noise: $\frac{k_1^2 k_2}{4\pi}$.
- Fluorescence spectrum (suitably normalised to be independent of k_1, k_2, $|\alpha_1|^2$):

$$\Sigma(\mu) = \frac{\gamma}{\pi}\,\mathrm{Re}\int_0^{+\infty} \mathrm{e}^{-\left(\frac{\varkappa}{2}+\mathrm{i}\mu\right)t}\,\mathrm{Tr}\left\{\mathrm{e}^{\check{\mathcal{L}}t}\left[\sigma_-\eta_{\mathrm{eq}}\right]\sigma_+\right\}\mathrm{d}t = \Sigma_{\mathrm{el}}(\mu) + \Sigma_{\mathrm{inel}}(\mu)\,.$$

- Elastic or coherent part of the fluorescence spectrum:

$$\begin{aligned}\Sigma_{\mathrm{el}}(\mu) &= \frac{1}{k_1^2\,|\alpha_1|^2}\lim_{t\to+\infty}\left(\mathbb{E}_{\rho_0}^{t,\mu+\omega}\left[I(t)\right]\right)^2 \\ &= \frac{\gamma\,\Omega^2\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)}{\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2(2\overline{n}+1)^2}\,\frac{\varkappa}{2\pi\left(\mu^2+\frac{\varkappa^2}{4}\right)}.\end{aligned}$$

- Inelastic or incoherent part of the fluorescence spectrum: $(q = \mathrm{i}\mu + \varkappa/2)$

$$\begin{aligned}\Sigma_{\mathrm{inel}}(\mu) &= \frac{1}{k_1^2\,|\alpha_1|^2}\lim_{t\to+\infty}\mathrm{Var}_{\rho_0}^{t,\mu+\omega}\left[I(t)\right] - \frac{1}{4\pi\,|\alpha_1|^2} \\ &= \frac{\gamma}{\pi\,(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)^2}\,\mathrm{Re}\,\frac{4\Omega^2 N_1(q)+N_2(q)}{N(q)}\,,\end{aligned}$$

$$\begin{aligned}N_1(q) = {}&\left[\Omega^2(2\overline{n}+1+4k_{\mathrm{d}})+\overline{n}\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)\right] \\ &\times\left[\Omega^2(2\overline{n}+1+4k_{\mathrm{d}})+4\overline{n}\Delta\omega^2+(\overline{n}+1)\,\widetilde{\gamma}^2+q\,(\widetilde{\gamma}-2\mathrm{i}\Delta\omega)\right] \\ &+4\Delta\omega^2\left[\Omega^2(2\overline{n}+4k_{\mathrm{d}})+\overline{n}\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)\right]-2\mathrm{i}\Omega^2\Delta\omega\widetilde{\Gamma}\,,\end{aligned}$$

$$\begin{aligned}N_2(q) = {}&(2q+\widetilde{\gamma}+2\mathrm{i}\Delta\omega)\left[q+\gamma\,(2\overline{n}+1)\right]\Big\{2\overline{n}\,(2\overline{n}+1)\left(4\Delta\omega^2+\Gamma^2\right)^2 \\ &+4\Omega^2\Big[\Omega^2(1+4k_{\mathrm{d}})^2+\overline{n}\left(4\Delta\omega^2+\Gamma^2\right)(1+8k_{\mathrm{d}})+2k_{\mathrm{d}}\left(4\Delta\omega^2+\widetilde{\gamma}^2\right)\Big]\Big\},\end{aligned}$$

$$N(q) = 2\Omega^2(2q+\widetilde{\gamma})+\left[\gamma\,(2\overline{n}+1)+q\right]\left[4\Delta\omega^2+(2q+\widetilde{\gamma})^2\right].$$

- Parity: $\Sigma_{\mathrm{el}}(\mu)$ is an even function of μ; the condition $\Delta\omega = 0$ or the condition $\overline{n} = 0$ and $k_{\mathrm{d}} = 0$ imply that also $\Sigma_{\mathrm{inel}}(\mu)$ is an even function of μ.
- Normalised heterodyne spectrum:

$$P_{\rm het}(\mu-\omega)\big|_{k_1^2k_2=4\pi} = 1 + 4\pi\,|\alpha_1|^2\,\Sigma(\mu) = |\alpha_2|^2 + |\alpha_1|^2\,S_{\rm het}(\mu),$$

$$S_{\rm het}(\mu) = S^{\rm el}_{\rm het}(\mu) + S^{\rm inel}_{\rm het}(\mu),$$

$$S^{\rm el}_{\rm het}(\mu) = 4\pi\,\Sigma_{\rm el}(\mu), \qquad S^{\rm inel}_{\rm het}(\mu) = 1 + 4\pi\,\Sigma_{\rm inel}(\mu).$$

- Positivity: $S^{\rm el}_{\rm het}(\mu) \geq 0$, $S^{\rm inel}_{\rm het}(\mu) \geq 0$, $\Sigma_{\rm inel}(\mu) + \Sigma_{\rm inel}(-\mu) \geq 0$.
- Intensities:

$$\Pi_{\rm TOT}(\Omega,\,\Delta\omega) = \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty}[S_{\rm het}(\mu)-1]\,{\rm d}\mu = \frac{2}{\gamma}\int_{-\infty}^{+\infty}\Sigma(\mu)\,{\rm d}\mu = 1 + z_{\rm eq}\,,$$

$$\Pi_{\rm el}(\Omega,\,\Delta\omega) = \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty}S^{\rm el}_{\rm het}(\mu)\,{\rm d}\mu = \frac{2}{\gamma}\int_{-\infty}^{+\infty}\Sigma_{\rm el}(\mu)\,{\rm d}\mu = \frac{x_{\rm eq}^2 + y_{\rm eq}^2}{2}\,,$$

$$\begin{aligned}\Pi_{\rm inel}(\Omega,\,\Delta\omega) &= \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty}\left[S^{\rm inel}_{\rm het}(\mu)-1\right]{\rm d}\mu \\ &= \frac{2}{\gamma}\int_{-\infty}^{+\infty}\Sigma_{\rm inel}(\mu)\,{\rm d}\mu = \frac{1-\left|\vec{x}_{\rm eq}\right|^2 + \left(1+z_{\rm eq}\right)^2}{2}.\end{aligned}$$

- The power spectrum $P_{\rm het}(\nu)$, the fluorescence spectrum $\Sigma(\mu)$ and the normalised heterodyne spectrum $S_{\rm het}(\mu)$ are independent of the initial state ρ_0.

9.3.2.1 Particular Cases of the Heterodyne Spectral Density

- For $\Omega = 0$, $\overline{n} > 0$, $\Sigma_{\rm el}$ is zero and $\Sigma_{\rm inel}$ reduces to a single Lorentzian (9.39) centred on ω_0, of width $\widetilde{\gamma} + \varkappa$, and intensity (area below the curve) $\frac{\gamma\overline{n}}{2\overline{n}+1}$.
- For $\overline{n} = 0$, $k_{\rm d} = 0$, the spectrum goes from a single line for small Ω to a triplet structure for large Ω, see Eqs. (9.35) and Fig. 9.1.
- For $\overline{n} = 0$, $k_{\rm d} = 0$, $\varkappa = 0$, the spectrum Σ is known as Mollow spectrum, see Eqs. (9.36) and Fig. 9.2.
- For $\overline{n} = 0$, $k_{\rm d} = 0$, $\varkappa = 0$, and Ω small, the inelastic spectrum presents a single line with sub-natural narrowing, see Fig. 9.8.
- If $\overline{n}$ and $k_{\rm d}$ are not too big with respect to Ω/γ, their effect is of smoothing and distorting the spectrum, see Fig. 9.3.

9.3.3 Homodyne Spectral Density

The atom and local oscillator are driven by a single laser; the phase coherence is maintained. The observed quantity is the output electrical current at large times and

it is processed by a spectrum analyser. To have a good spectrum $\varkappa$ must be big. No quantity depends on k_2.

- Homodyne detection: $\nu = \omega$.
- Spectral density of the homodyne current:

$$S_{\mathrm{I}}(\mu;\vartheta) = \lim_{T\to+\infty} \frac{1}{T}\, \mathbb{E}_{\rho_0}^{T,\omega}\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu t} I(t)\,\mathrm{d}t\right|^2\right] = \frac{k_1^2\varkappa}{\pi\left(\varkappa^2+4\mu^2\right)}\, S_{\mathrm{hom}}(\mu;\vartheta).$$

- Normalised homodyne spectrum (independent of k_1 and $\varkappa$):

$$\begin{aligned} S_{\mathrm{hom}}(\mu;\vartheta) = \lim_{\varkappa\to+\infty} \frac{\pi\varkappa}{k_1^2}\, S_{\mathrm{I}}(\mu;\vartheta) &= \lim_{T\to+\infty} \frac{1}{T}\, \mathbb{E}_{\rho_0}^{T,\omega}\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\,\mathrm{d}W_1(s)\right|^2\right] \\ &= S_{\mathrm{hom}}^{\mathrm{el}}(\mu;\vartheta) + S_{\mathrm{hom}}^{\mathrm{inel}}(\mu;\vartheta). \end{aligned}$$

- Elastic or coherent part of the homodyne spectrum:

$$\begin{aligned} S_{\mathrm{hom}}^{\mathrm{el}}(\mu;\vartheta) &= \lim_{T\to+\infty} \frac{1}{T}\left|\mathbb{E}_{\rho_0}^{T,\omega}\left[\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\,\mathrm{d}W_1(s)\right]\right|^2 \\ &= 2\pi\gamma\,|\alpha_1|^2\left|\mathrm{Tr}\left\{\left(\mathrm{e}^{\mathrm{i}\vartheta}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta}\sigma_+\right)\eta_{\mathrm{eq}}\right\}\right|^2\delta(\mu) \\ &= 2\pi\gamma\,|\alpha_1|^2\left(x_{\mathrm{eq}}\cos\vartheta + y_{\mathrm{eq}}\sin\vartheta\right)^2\delta(\mu) \\ &= |\alpha_1|^2\,\frac{8\pi\Omega^2\gamma\,(-2\Delta\omega\cos\vartheta + \widetilde{\gamma}\sin\vartheta)^2}{(2\overline{n}+1)^2\left(4\Delta\omega^2+\widetilde{\Gamma}^2\right)^2}\,\delta(\mu). \end{aligned}$$

- Connection with the elastic part of the heterodyne power spectrum:

$$\frac{1}{2\pi}\int_0^{2\pi} S_{\mathrm{hom}}^{\mathrm{el}}(\mu;\vartheta)\,\mathrm{d}\vartheta = 4\pi\,|\alpha_1|^2\,\Sigma_{\mathrm{el}}(\mu)\Big|_{\varkappa=0} = |\alpha_1|^2\,S_{\mathrm{het}}^{\mathrm{el}}(\mu)\Big|_{\varkappa=0}.$$

- Inelastic or incoherent part of the homodyne spectrum:

$$\begin{aligned} S_{\mathrm{hom}}^{\mathrm{inel}}(\mu;\vartheta) = \lim_{T\to+\infty} \frac{1}{T}\Bigg(&\mathbb{E}_{\rho_0}^{T,\omega}\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\,\mathrm{d}W_1(s)\right|^2\right] \\ &- \left|\mathbb{E}_{\rho_0}^{T,\omega}\left[\int_0^T \mathrm{e}^{\mathrm{i}\mu s}\,\mathrm{d}W_1(s)\right]\right|^2\Bigg) = |\alpha_2|^2 + |\alpha_1|^2\, S_{\mathrm{hom}}^{\mathrm{red}}(\mu;\vartheta). \end{aligned}$$

- Reduced inelastic homodyne spectrum (independent of $|\alpha_1|^2$):

$$S_{\text{hom}}^{\text{red}}(\mu;\vartheta) = 1 + \vec{u}(\vartheta)\cdot\left(\frac{2\gamma A}{A^2+\mu^2}\,\vec{t}(\vartheta)\right), \qquad \vec{u}(\vartheta) = (\cos\vartheta,\ \sin\vartheta,\ 0)\,,$$

$$\vec{t}(\vartheta) = \begin{pmatrix} \left(1+z_{\text{eq}}-x_{\text{eq}}^2\right)\cos\vartheta - x_{\text{eq}}y_{\text{eq}}\sin\vartheta \\ \left(1+z_{\text{eq}}-y_{\text{eq}}^2\right)\sin\vartheta - x_{\text{eq}}y_{\text{eq}}\cos\vartheta \\ -\left(1+z_{\text{eq}}\right)\left(x_{\text{eq}}\cos\vartheta + y_{\text{eq}}\sin\vartheta\right) \end{pmatrix}.$$

- Connection with the inelastic part of the heterodyne power spectrum:

$$\frac{1}{2}\left[S_{\text{hom}}^{\text{red}}(\mu;\vartheta) + S_{\text{hom}}^{\text{red}}(\mu;\vartheta+\pi/2)\right] = \frac{1}{2}\left[S_{\text{het}}^{\text{inel}}(\mu) + S_{\text{het}}^{\text{inel}}(-\mu)\right]_{\varkappa=0} \geq 1.$$

- Normalised shot noise level: 1.
- Parity: both the elastic and inelastic parts of the homodyne spectrum are even functions of μ.
- Positivity: $S_{\text{hom}}^{\text{el}}(\mu;\vartheta) \geq 0$, $S_{\text{hom}}^{\text{red}}(\mu;\vartheta) \geq 0$.
- Uncertainty relations: $S_{\text{hom}}^{\text{red}}(\mu;\vartheta)\, S_{\text{hom}}^{\text{red}}(\mu;\vartheta+\pi/2) \geq 1$.
- Squeezing: $S_{\text{hom}}^{\text{red}}(\mu;\vartheta) < 1$ for some μ and ϑ.
- Intensity:

$$\Pi_{\text{hom}}(\vartheta) = \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty}\left[S_{\text{hom}}^{\text{red}}(\mu,\vartheta)-1\right]\mathrm{d}\mu = 1-\left(x_{\text{eq}}\cos\vartheta + y_{\text{eq}}\sin\vartheta\right)^2 + z_{\text{eq}}\,;$$

$$\frac{1}{2}\left[\Pi_{\text{hom}}(\vartheta)+\Pi_{\text{hom}}(\vartheta+\pi/2)\right] = \Pi_{\text{inel}}(\Omega,\Delta\omega) = 1 + z_{\text{eq}} - \frac{x_{\text{eq}}^2+y_{\text{eq}}^2}{2}\,;$$

$$\Pi_{\text{hom}} = \inf_{\vartheta}\Pi_{\text{hom}}(\vartheta) = 1 - \left|\vec{x}_{\text{eq}}\right|^2 + z_{\text{eq}}\left(1+z_{\text{eq}}\right) = \text{AS}_{\eta_{\text{eq}}}.$$

- $\text{AS}_{\eta_{\text{eq}}} < 0 \Leftrightarrow \Pi_{\text{hom}} < 0 \Rightarrow S_{\text{hom}}^{\text{red}}(\mu,\vartheta) < 1$ for some μ and ϑ, i.e. the fluorescence light is squeezed.
- All the spectral densities $S_{\text{I}}(\mu;\vartheta)$, $S_{\text{hom}}(\mu;\vartheta)$, $S_{\text{hom}}^{\text{red}}(\mu;\vartheta)$ are independent of the initial state ρ_0.

9.3.3.1 Particular Cases of the Homodyne Spectral Density

- For $\Omega = 0$, $\overline{n} > 0$, $S_{\text{hom}}^{\text{el}}$ is zero and $S_{\text{hom}}^{\text{red}}$ reduces to the sum of shot noise and two Lorentzians (9.64).
- For $\Delta\omega = 0$, $\overline{n} = 0$, $k_{\text{d}} = 0$, and Ω not too large, there is some squeezing around $\mu = 0$, see Fig. 9.7.

- Tunable squeezing: the parameters can be chosen to have squeezing around a preassigned value of μ, see Figs. 9.4 and 9.5. The minima are symmetric with respect to $\mu = 0$.
- Squeezing is possible also if $\Pi_{\text{hom}} > 0$, see Fig. 9.6.
- The best local squeezing does not correspond to the minimum of Π_{hom}, see Fig. 9.9.
- If $\overline{n}$ and k_{d} are not too big with respect to Ω/γ, their effect is of smoothing the spectrum and of cancelling the squeezing, see Fig. 9.10.

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Chapter 10
Feedback

10.1 Introduction

The larger part of this chapter is devoted to applications (again a two-level atom is used as a prototype), but first some general theoretical issues are developed, namely feedback and control of quantum systems.

10.1.1 Feedback and Control of Quantum Systems

When in a dynamical system, classical or quantum, deterministic or stochastic, the experimenter can change some part of the dynamics, or at least some parameter, this freedom can be used to obtain some behaviour of the system, to keep its state in some region... This is *control*.

When the experimenter (the controller) decides a priori how to use his freedom in modifying the dynamics, one speaks of *open loop control*. In the previous chapter we assumed to have the freedom of changing the intensity and frequency of the laser and of changing the optical paths; we can say that we were assuming to have control on Ω, $\Delta\omega$ and ϑ. Then, we used this freedom to enforce the best possible squeezing around a frequency chosen a priori, as done in Figs. 9.4, 9.5, 9.9; so, that was a case of open loop control.

When the system produces an output, we can use the output, directly or after some processing, to change the dynamics of the system: this is called *feedback*. When one uses some feedback mechanism to control the system, one speaks of *closed loop control* or *feedback control*. A typical situation of this type is when the system is observed and the control strategy is decided on the basis of the observed output. If the output is stochastic, likely the control strategy will be stochastic: this is *stochastic control through feedback*.

Open loop control has been developed also for closed and open quantum systems: one assumes that there is some freedom in choosing the Hamiltonian or the Liouville operator of the system and studies the problem of using this freedom to enforce a prescribed behaviour to the system. The theory of quantum continuous measurements, which gives the stochastic output of the observation and the

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evolution equation for the conditional state, opens the possibility of closed loop control in continuous time for quantum systems [1].

10.1.2 Feedback Control of a Quantum Optical System

The first step of a control problem is to establish what can be controlled by the experimenter and what is unalterable. Of course, a good mathematical model has to reflect what can be really done in the laboratory. In this chapter we study one of the various feedback schemes on the atom evolution, based on the outgoing photocurrent, which have been proposed [2–5].

Next section introduces an important model of closed loop control, the Markovian feedback scheme à la Wiseman and Milburn [2–4]. The evolution of a continuously observed open quantum system is modified on the basis of the output by an additional Hamiltonian action with intensity proportional to the instantaneous signal. Then, Sect. 10.3 shows that such a feedback control can be reasonably realised for a quantum optical system.

More precisely, we consider a continuously observed open quantum system described by the formalism of Sect. 7.1. We suppose that the experimenter wants to control the system to a certain extent, but that he is not entitled to change the intrinsic dynamics of the system; so, he cannot change the emission operators D_k, the scattering operators S_{kj} and the free Hamiltonian H_0. In order to control the quantum system, he can change the input signal f (or at least some of its components), as done for instance in [6, 7]. But also the phase factors $h_k(t)$ can be adjusted to some extent by the experimenter, as discussed in Sect. 7.2.1.

When controls depend on the past measured signal, the functions f, h become stochastic processes and the involved SDEs eventually need to be generalised by including intrinsically stochastic coefficients [8]. A physical overview of quantum control based on the stochastic master equation is given in [9].

In this framework, suitable choices produce just a stochastic effective Hamiltonian and realise a Wiseman–Milburn feedback control of the quantum optical system. We show this fact in the case of a two-level atom monitored by detecting its fluorescence light and, then, we present three applications: control of the atom dynamics, driving the atom to a preassigned asymptotic state or to a preassigned asymptotic unitary dynamics [5, 10]; control of the squeezing properties of the fluorescence light; [11, 12] and control of the atomic decay rates that is of the linewidth of the homodyne spectrum [12].

10.2 The Feedback Scheme of Wiseman and Milburn

Let us present the very ideal scheme of fast feedback due to Wiseman and Milburn [2–4]. The original derivation [4] was partly based on quantum Langevin equations (quantum stochastic differential equations); to be consistent with the rest of the book, here we base the presentation only on the classical SDEs of quantum trajectory theory. We use an heuristic approach based on an analysis of the dynamics

in infinitesimal time intervals; the advantage of this approach is that it guarantees that the feedback acts on the system after the events that produced it and that, so, causality is respected.

10.2.1 Observation Without Feedback

Let us analyse what happens in an infinitesimal interval in the situation of Chapter 3. We use a tilde for the quantities related to the case without feedback.

The fundamental equation is the linear stochastic master equation (3.38) with Liouville operator given by (3.40a), (3.40b), (3.40c). The stochastic evolution map (or propagator) $\widetilde{\mathcal{A}}(t,s)$ of such an SDE satisfies Eq. (3.42) and the composition property (3.47). Let us consider now an infinitesimal time interval $(t, t+\mathrm{d}t)$. The evolution of the non-normalised a posteriori states $\widetilde{\sigma}(t)$ is given by

$$\widetilde{\sigma}(t+\mathrm{d}t) = \widetilde{\mathcal{A}}(t+\mathrm{d}t, t)[\widetilde{\sigma}(t)], \tag{10.1}$$

where the infinitesimal propagator satisfies

$$\widetilde{\mathcal{A}}(t+\mathrm{d}t, t) - \mathrm{Id}_n = \widetilde{\mathcal{L}}(t)\mathrm{d}t + \sum_{j=1}^{m} \widetilde{\mathcal{R}}_j(t)\mathrm{d}W_j(t), \tag{10.2}$$

which is the infinitesimal version of (3.42). By the Itô rules, we have that (10.2) is equivalent to

$$\widetilde{\mathcal{A}}(t+\mathrm{d}t, t) = \exp\left\{\widetilde{\mathcal{L}}(t)\mathrm{d}t - \frac{1}{2}\sum_{j=1}^{m}\widetilde{\mathcal{R}}_j(t)^2\mathrm{d}t + \sum_{j=1}^{n}\widetilde{\mathcal{R}}_j(t)\mathrm{d}W_j(t)\right\}. \tag{10.3}$$

The Liouville operator appearing here is given by $\widetilde{\mathcal{L}}(t) = \widetilde{\mathcal{L}}_0(t) + \widetilde{\mathcal{L}}_1(t)$,

$$\widetilde{\mathcal{L}}_1(t)[\tau] = \sum_{j=1}^{m}\left(\widetilde{R}_j(t)\tau\widetilde{R}_j(t)^* - \frac{1}{2}\left\{\widetilde{R}_j(t)^*\widetilde{R}_j(t), \tau\right\}\right),$$

$$\widetilde{\mathcal{L}}_0(t)[\tau] = -\mathrm{i}[\widetilde{H}(t), \tau] + \sum_{j=m+1}^{d}\left(\widetilde{R}_j(t)\tau\widetilde{R}_j(t)^* - \frac{1}{2}\left\{\widetilde{R}_j(t)^*\widetilde{R}_j(t), \tau\right\}\right),$$

while $\widetilde{\mathcal{R}}_j(t)[\tau] = \widetilde{R}_j(t)\tau + \tau\widetilde{R}_j(t)$.

The a posteriori states (3.46) are obtained by normalisation,

$$\widetilde{\rho}(t+\mathrm{d}t) = \mathrm{Tr}\left\{\widetilde{\mathcal{A}}(t+\mathrm{d}t, t)[\widetilde{\rho}(t)]\right\}^{-1}\ \widetilde{\mathcal{A}}(t+\mathrm{d}t, t)[\widetilde{\rho}(t)],$$

and the probability density (3.62) can be written as

$$\mathrm{Tr}\left\{\widetilde{\sigma}(t+\mathrm{d}t)\right\} = \mathrm{Tr}\left\{\widetilde{\mathcal{A}}(t+\mathrm{d}t, t)[\widetilde{\sigma}(t)]\right\} = \mathrm{Tr}\left\{\widetilde{\mathcal{A}}(t+\mathrm{d}t, t)[\widetilde{\rho}(t)]\right\}\mathrm{Tr}\left\{\widetilde{\sigma}(t)\right\}.$$

So, we can say that the contribution of the interval $(t,\ t+\mathrm{d}t)$ to the probability density is $\mathrm{Tr}\left\{\widetilde{\mathcal{A}}(t+\mathrm{d}t,t)[\widetilde{\rho}(t)]\right\}$.

10.2.2 Introduction of the Feedback

The idea of Wiseman and Milburn is to introduce a feedback proportional to the instantaneous observed signal. Considering only the output of channel 1, the instantaneous signal is formally $\dot{W}_1(t)$, which can be positive or negative; this allows only for a feedback of Hamiltonian type. So, the contribution of the feedback to the propagator in an infinitesimal interval is assumed to be $\mathrm{e}^{\mathcal{M}(t)\mathrm{d}W_1(t)}$ with

$$\mathcal{M}(t)[\tau] = -\mathrm{i}[M(t),\tau], \qquad M(t)^* = M(t). \tag{10.4}$$

Typically, the Hamiltonian $M(t)$ depends on some control parameters which allow to tune the effects of the feedback.

By taking into account that the feedback must act after the signal was produced, the new infinitesimal propagator will be

$$\mathcal{A}(t+\mathrm{d}t,t) = \mathrm{e}^{\mathcal{M}(t)\mathrm{d}W_1(t)} \circ \widetilde{\mathcal{A}}(t+\mathrm{d}t,t). \tag{10.5}$$

As before, the a posteriori states are given by

$$\rho(t+\mathrm{d}t) = \left(\mathrm{Tr}\left\{\mathcal{A}(t+\mathrm{d}t,t)[\rho(t)]\right\}\right)^{-1} \mathcal{A}(t+\mathrm{d}t,t)[\rho(t)]$$

and the probability density satisfies

$$\mathrm{Tr}\left\{\sigma(t+\mathrm{d}t)\right\} = \mathrm{Tr}\left\{\mathcal{A}(t+\mathrm{d}t,t)[\sigma(t)]\right\} = \mathrm{Tr}\left\{\mathcal{A}(t+\mathrm{d}t,t)[\rho(t)]\right\}\mathrm{Tr}\left\{\sigma(t)\right\}.$$

The contribution of the interval $(t,\ t+\mathrm{d}t)$ to the probability density is

$$\mathrm{Tr}\left\{\mathcal{A}(t+\mathrm{d}t,t)[\rho(t)]\right\} = \mathrm{Tr}\left\{\widetilde{\mathcal{A}}(t+\mathrm{d}t,t)[\rho(t)]\right\}.$$

To compute the new evolution linear SDE we take into account (10.2) and

$$\mathrm{e}^{\mathcal{M}(t)\mathrm{d}W_1(t)} - \mathrm{Id}_n = \mathcal{M}(t)\mathrm{d}W_1(t) + \frac{1}{2}\,\mathcal{M}(t)^2\mathrm{d}t.$$

We obtain

$$\begin{aligned}\mathcal{A}(t+\mathrm{d}t,t) - \mathrm{Id}_n &= \widetilde{\mathcal{L}}(t)\mathrm{d}t + \sum_{j=1}^{m}\widetilde{\mathcal{R}}_j(t)\mathrm{d}W_j(t) + \mathcal{M}(t)\mathrm{d}W_1(t)\\ &\quad + \frac{1}{2}\,\mathcal{M}(t)^2\mathrm{d}t + \mathcal{M}(t)\circ\widetilde{\mathcal{R}}_1(t)\mathrm{d}t = \mathcal{L}(t)\mathrm{d}t + \sum_{j=1}^{m}\mathcal{R}_j(t)\mathrm{d}W_j(t),\end{aligned}$$

with

$$\mathcal{R}_j(t) = \widetilde{\mathcal{R}}_j(t) + \delta_{j1}\mathcal{M}(t), \qquad R_j(t) = \widetilde{R}_j(t) - \mathrm{i}\delta_{j1}M(t), \tag{10.6}$$

$$\mathcal{L}(t) = \widetilde{\mathcal{L}}(t) + \frac{1}{2}\,\mathcal{M}(t)^2 + \mathcal{M}(t) \circ \widetilde{\mathcal{R}}_1(t) = \mathcal{L}_0(t) + \mathcal{L}_1(t), \tag{10.7}$$

$$\mathcal{L}_0(t) = \widetilde{\mathcal{L}}_0(t), \tag{10.8}$$

$$\mathcal{L}_1(t)[\tau] = -\frac{\mathrm{i}}{2}\left[M(t)\widetilde{R}_1(t) + \widetilde{R}_1(t)^* M(t),\ \tau\right] + \sum_{j=1}^{m}\left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{R_j(t)^* R_j(t),\ \tau\right\}\right). \tag{10.9}$$

Thus, both the a posteriori and the a priori evolutions are still Markovian and the key equation is still the linear stochastic master equation (3.38), but with the new operators given above.

It is just a peculiar feature of the Wiseman–Milburn feedback scheme to modify only the operators in the linear stochastic master equation, without altering the structure of the dynamics. This allows to study the effectiveness of feedback and the role of control parameters using just the theory already developed.

Next section goes back to consider our two-level atom and shows how a Wiseman–Milburn feedback can be realised.

10.3 The Two-Level Atom with Feedback

As an example of a quantum system with control and feedback, we consider a two-level atom stimulated by a laser. The feedback acts on the laser through an electromodulator [10, 13]. The ideal experimental configuration is given in Fig. 10.1.

The measuring apparatus is made by two homodyne detectors. Part of the emitted light reaches the detectors and part is lost in the free space. The fraction of light detected by one of the detectors depends on its efficiency, on the spanned solid

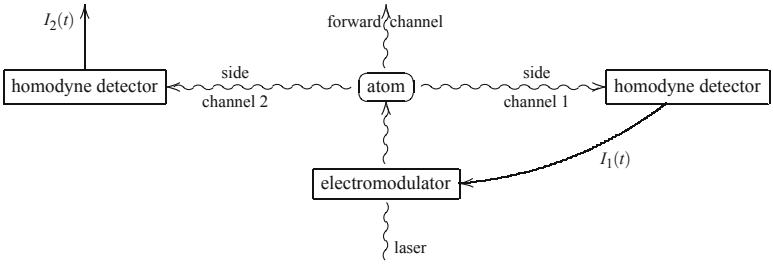

Fig. 10.1 Channel 0: forward channel with laser; channel 1: side channel with feedback; channel 2: side channel without feedback

angle and can eventually be enhanced by using a focussing mirror, as suggested in [10], where the ideal experimental setup is described in the case of a single detector. So, the fluorescence light is divided into three parts according to the direction of propagation; following the terminology introduced in Chap. 9, we call *side channel* k ($k = 1, 2$) the directions reaching the detector k, and *forward channel*, or channel 0, the directions of the lost light. The stimulating laser is well collimated in such a way that it does not hit the detectors; so, we can say that it acts in the forward channel. We denote the effective fractions of light emitted in the forward and in the two side channels by $|\alpha_0|^2$, $|\alpha_1|^2$, $|\alpha_2|^2$, respectively; obviously,

$$|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1.$$

For $k = 1, 2$, we can also interpret $|\alpha_k|^2$ as the total efficiency of the detector k. Moreover, the initial phase of the local oscillator in each detector is denoted by ϑ_k and it is included in the parameter $\alpha_k \in \mathbb{C}$ by setting

$$\vartheta_k = \arg \alpha_k \, .$$

To change ϑ_k means to change the measuring apparatus. The output current $I_1(t)$ from channel 1 is used to drive the electromodulator. The light in channel 2 can be analysed by homodyne detection or can be used for any other scope. With respect to the situation of the previous chapters, we can say that the old channel 2 splits into channel 0 and the new channel 2.

10.3.1 The Model

The free Hamiltonian and the contributions of the unobserved channels 3, 4, 5 are as in Chap. 8, Eqs. (8.8), (8.17). Therefore, we have

$$H_0 = \frac{\omega_0}{2} \sigma_z \, , \tag{10.10}$$

while the channels 3, 4, 5 contribute only to the dissipative part of the Liouvillian with the term (cf. Eqs. (8.18))

$$\gamma k_{\mathrm{d}} (\sigma_z \tau \sigma_z - \tau) + \gamma \overline{n} (\sigma_- \tau \sigma_+ + \sigma_+ \tau \sigma_- - \tau).$$

10.3.1.1 No Feedback

In the case of no feedback, the contribution of channel 1 too is as in Chap. 8, with $\nu = \omega$,

$$\widetilde{R}_1(t) = \mathrm{e}^{\mathrm{i}\omega t} \sqrt{\gamma} \, \alpha_1 \sigma_- \, . \tag{10.11}$$

The old contribution of channel 2, $R_2(t) = \mathrm{e}^{\mathrm{i}\omega t}\sqrt{\gamma}\,\alpha_2\sigma_- + \lambda\mathbb{1}$, splits into the contributions of channel 0 and channel 2

$$\widetilde{R}_0(t) = \mathrm{e}^{\mathrm{i}\omega t}\sqrt{\gamma}\,\alpha_0\sigma_- + \lambda\mathbb{1}, \qquad \widetilde{R}_2(t) = \mathrm{e}^{\mathrm{i}\omega t}\sqrt{\gamma}\,\alpha_2\sigma_- \,. \tag{10.12}$$

Now, we have

$$\alpha_i = \mathrm{e}^{\mathrm{i}\vartheta_i}\,|\alpha_i|\ , \qquad |\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1. \tag{10.13}$$

Channel 0 contributes to the Hamiltonian part of the Liouvillian with

$$H_f(t) = \sqrt{\gamma}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{i}\overline{\lambda}\alpha_0\,\sigma_- - \sqrt{\gamma}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{i}\lambda\overline{\alpha_0}\,\sigma_+ \tag{10.14}$$

and to the dissipative part with $|\alpha_0|^2\,\gamma\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\{P_+,\tau\}\right)$.

By setting $\Omega = 2\sqrt{\gamma}\ |\lambda|\,|\alpha_0|$ and by taking $\arg\left(\mathrm{i}\overline{\lambda}\,\alpha_0\right) = 0$, we get

$$H_f(t) = \frac{\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\omega t}\,\sigma_- + \mathrm{e}^{-\mathrm{i}\omega t}\,\sigma_+\right), \qquad \widetilde{\mathcal{L}}(t) = \widetilde{\mathcal{L}}_0(t) + \widetilde{\mathcal{L}}_1(t), \tag{10.15a}$$

$$\begin{aligned}\widetilde{\mathcal{L}}_1(t)[\tau] &= \sum_{j=1}^{2}\left(\widetilde{R}_j(t)\tau\widetilde{R}_j(t)^* - \frac{1}{2}\left\{\widetilde{R}_j(t)^*\widetilde{R}_j(t),\tau\right\}\right)\\ &= \left(|\alpha_1|^2 + |\alpha_2|^2\right)\gamma\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\{P_+,\tau\}\right),\end{aligned} \tag{10.15b}$$

$$\begin{aligned}\widetilde{\mathcal{L}}_0(t)[\tau] = -\mathrm{i}[H_0 + H_f(t),\tau] &+ |\alpha_0|^2\,\gamma\left(\sigma_-\tau\sigma_+ - \frac{1}{2}\{P_+,\tau\}\right)\\ &+ \gamma k_\mathrm{d}\,(\sigma_z\tau\sigma_z - \tau) + \gamma\overline{n}\,(\sigma_-\tau\sigma_+ + \sigma_+\tau\sigma_- - \tau)\,.\end{aligned} \tag{10.15c}$$

10.3.1.2 The Feedback

The feedback changes the laser through the electromodulator. We consider a very ideal situation. As in Sect. 10.2.2, we suppose that the output current is just $I_1(t) = \dot{W}_1(t)$ (infinite time resolution) and that the feedback can be instantaneously based on $\dot{W}_1(t)$. We assume that the feedback modifies the function $\widetilde{f}_k(t) = \delta_{0k}\lambda\mathrm{e}^{-\mathrm{i}\omega t}$, describing the stimulating laser, to $f_k(t) = \delta_{0k}\left(\lambda + g\dot{W}_1(t)\right)\mathrm{e}^{-\mathrm{i}\omega t}$. Then, the effect of the feedback is summarised by the replacement $\lambda \to \lambda + g\dot{W}_1(t)$ inside $H_f(t)$. So, we get an Hamiltonian feedback à la Wiseman and Milburn:

$$H_f(t) \longrightarrow H_f(t) + M(t)\dot{W}_1(t), \tag{10.16}$$

$$M(t) = c\sqrt{\gamma}\,\mathrm{e}^{\mathrm{i}(\omega t+\varphi)}\,\sigma_- + \mathrm{h.c.}, \qquad c \in \mathbb{R}, \qquad \varphi \in [0,\pi). \tag{10.17}$$

We have introduced the feedback control parameter c and the feedback phase φ by the following rule. Take $\arg(\mathrm{i}\overline{g}\,\alpha_0) \in [0, 2\pi)$; then, if $\arg(\mathrm{i}\overline{g}\,\alpha_0) \in [0, \pi)$ define $c = |g|\,|\alpha_0|$ and $\varphi = \arg(\mathrm{i}\overline{g}\,\alpha_0)$, while, if $\arg(\mathrm{i}\overline{g}\,\alpha_0) \in [\pi, 2\pi)$ define $c = -|g|\,|\alpha_0|$ and $\varphi = \arg(\mathrm{i}\overline{g}\,\alpha_0) - \pi$.

According to Eqs. (10.6), (10.7), (10.8), (10.9), we have

$$R_1(t) = \widetilde{R}_1(t) - \mathrm{i}M(t), \qquad R_2(t) = \widetilde{R}_2(t), \tag{10.18a}$$

$$R_0(t) = \widetilde{R}_0(t), \qquad \mathcal{L}_0(t) = \widetilde{\mathcal{L}}_0(t), \tag{10.18b}$$

$$\mathcal{L}(t) = \widetilde{\mathcal{L}}(t) + \frac{1}{2}\,\mathcal{M}(t)^2 + \mathcal{M}(t) \circ \widetilde{\mathcal{R}}_1(t) = \mathcal{L}_0(t) + \mathcal{L}_1(t), \tag{10.18c}$$

$$\begin{aligned}\mathcal{L}_1(t)[\tau] = &-\frac{\mathrm{i}}{2}\left[M(t)\widetilde{R}_1(t) + \widetilde{R}_1(t)^* M(t),\ \tau\right] \\ &+ \sum_{j=1}^{m}\left(R_j(t)\tau R_j(t)^* - \frac{1}{2}\left\{R_j(t)^* R_j(t),\, \tau\right\}\right).\end{aligned} \tag{10.18d}$$

Of course, the same feedback based on heterodyne detection in channel 1 ($\nu \neq \omega$) also leads to a Wiseman–Milburn feedback scheme, but with a different time dependence of the modified operators. The choice of homodyne detection in channel 1 gives a particularly simple time dependence which allows for a successful introduction of the rotating frame.

Let us remark also that, even if the feedback is based on the singular stochastic process $\dot{W}_1$, the mathematical formulation of the model is not affected by this singularity, essentially because we do not observe the light directly in the forward channel.

Rotating frame

As in the previous chapters it is possible to eliminate any explicit time dependence by passing to the rotating frame. By inserting the explicit expressions of the various operators in Eqs. (10.18) and by passing to the rotating frame we get the coefficients of the observed channels

$$\check{R}_1 = \sqrt{\gamma}\,(\alpha_1\,\sigma_- - \mathrm{i}c\,\sigma_\varphi), \qquad \check{R}_2 = \sqrt{\gamma}\,\alpha_2\,\sigma_-, \tag{10.19}$$

and the Liouville operator

$$\begin{aligned}\check{\mathcal{L}}[\rho] = &-\mathrm{i}\left[\frac{\Delta\omega_c}{2}\,\sigma_z + \frac{\Omega}{2}\,\sigma_x\ ,\ \rho\right] + \gamma k_\mathrm{d}\,(\sigma_z\,\rho\,\sigma_z - \rho) \\ &+ \gamma\overline{n}\left(\sigma_+\,\rho\,\sigma_- - \frac{1}{2}\,\{P_-\ ,\ \rho\}\right) + \gamma(\overline{n} + 1 - |\alpha_1|^2)\end{aligned}$$

$$\times \left(\sigma_- \rho \sigma_+ - \frac{1}{2}\{P_+ , \rho\}\right) + \gamma(\alpha_1 \sigma_- - \mathrm{i}c\,\sigma_\varphi)\,\rho\,(\overline{\alpha}_1 \sigma_+ + \mathrm{i}c\,\sigma_\varphi)$$
$$- \frac{\gamma}{2}\left\{\Big(|\alpha_1|^2 - 2c|\alpha_1|\sin(\vartheta_1 - \varphi)\Big)P_+ + c^2 , \rho\right\}, \tag{10.20}$$

$$\Delta\omega_c := \Delta\omega + \gamma c\,|\alpha_1|\cos(\vartheta_1 - \varphi). \tag{10.21}$$

Here and in the following we are using the notation

$$\vec{u}(\theta) := \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \tag{10.22}$$

$$\sigma_\theta := \mathrm{e}^{\mathrm{i}\theta}\sigma_- + \mathrm{e}^{-\mathrm{i}\theta}\sigma_+ = \sigma_x \cos\theta + \sigma_y \sin\theta = \vec{u}(\theta)\cdot\vec{\sigma}\,. \tag{10.23}$$

10.3.2 A Priori States

Let us start by studying the mean dynamics; we use the Bloch representation of the a priori states, as in Sect. 8.2.2:

$$\check{\eta}(t) = \frac{1}{2}\,(1 + \vec{x}(t)\cdot\vec{\sigma})\,. \tag{10.24a}$$

Then, the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\check{\eta}(t) = \check{\mathcal{L}}[\check{\eta}(t)]$$

gives

$$\vec{x}(t) = \mathrm{e}^{-At}\vec{x}(0) + \gamma\,[2c\,|\alpha_1|\sin(\vartheta_1 - \varphi) - 1]\int_0^t \mathrm{d}s\,\mathrm{e}^{-A(t-s)}\begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{10.24b}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & \Omega \\ 0 & -\Omega & a_{33} \end{pmatrix}, \qquad k = 4k_\mathrm{d} + 2\overline{n} + 1, \tag{10.25a}$$

$$\begin{aligned} a_{11} &= \frac{\gamma}{2}\,k + 2\gamma c^2 \sin^2\varphi + 2c\gamma|\alpha_1|\sin\varphi\cos\vartheta_1 \\ &= 2\gamma k_\mathrm{d} + \gamma\overline{n} + \frac{\gamma}{2}\left[1 - |\alpha_1|^2\cos^2\vartheta_1 + (2c\sin\varphi + |\alpha_1|\cos\vartheta_1)^2\right] \geq 0, \end{aligned} \tag{10.25b}$$

$$\begin{aligned} a_{22} &= \frac{\gamma}{2}\, k + 2\gamma c^2 \cos^2\varphi - 2c\gamma|\alpha_1|\cos\varphi \sin\vartheta_1 \\ &= 2\gamma k_\mathrm{d} + \gamma\overline{n} + \frac{\gamma}{2}\left[1 - |\alpha_1|^2 \sin^2\vartheta_1 + (2c\cos\varphi - |\alpha_1|\sin\vartheta_1)^2\right] \geq 0, \end{aligned} \tag{10.25c}$$

$$\begin{aligned} a_{33} &= \gamma\,(1 + 2\overline{n}) + 2\gamma c^2 - 2c\gamma|\alpha_1|\sin(\vartheta_1 - \varphi) \\ &= 2\gamma\overline{n} + \frac{\gamma}{2}\left[2 - |\alpha_1|^2 \sin^2(\vartheta_1 - \varphi) + (2c - |\alpha_1|\sin(\vartheta_1 - \varphi))^2\right] \\ &\geq 2\gamma\overline{n} + \frac{\gamma}{2} > 0, \end{aligned} \tag{10.25d}$$

$$a_{12} = \Delta\omega + 2c\gamma|\alpha_1|\sin\varphi\sin\vartheta_1 - \gamma c^2 \sin 2\varphi, \tag{10.25e}$$

$$a_{21} = -\Delta\omega - 2c\gamma|\alpha_1|\cos\varphi\cos\vartheta_1 - \gamma c^2 \sin 2\varphi. \tag{10.25f}$$

Let us note the relation

$$a_{11} + a_{22} = a_{33} + 4\gamma k_\mathrm{d}\,. \tag{10.26}$$

As for the case without control, it is useful to study the symmetric part of the matrix A. Now, we have

$$\frac{1}{2}\,(a_{12} + a_{21}) = -\gamma c\,|\alpha_1|\cos(\vartheta_1 + \varphi) - \gamma c^2 \sin 2\varphi\,, \tag{10.27a}$$

$$\frac{1}{2}\left(A + A^\mathrm{T}\right) = \begin{pmatrix} B & 0 \\ 0 & a_{33} \end{pmatrix}, \tag{10.27b}$$

$$B = \begin{pmatrix} a_{11} & \frac{1}{2}\,(a_{12} + a_{21}) \\ \frac{1}{2}\,(a_{12} + a_{21}) & a_{22} \end{pmatrix}, \tag{10.27c}$$

$$\det B = \gamma^2 c^2 \left(k - |\alpha_1|^2\right) + \gamma^2 \left(\frac{k}{2} + c\,|\alpha_1|\sin\left(\varphi - \vartheta_1\right)\right)^2 \geq 0, \tag{10.27d}$$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \det B + \Delta{\omega_c}^2, \tag{10.27e}$$

$$\det A = a_{33}\left(\det B + \Delta{\omega_c}^2\right) + \Omega^2 a_{11} \geq 0\,, \tag{10.27f}$$

$$\det \frac{1}{2}\left(A + A^\mathrm{T}\right) \equiv a_{33} \det B \geq 0\,. \tag{10.27g}$$

If $\det A > 0$, then

$$\vec{x}(t) = \mathrm{e}^{-At}\vec{x}(0) + \gamma\,[2c\,|\alpha_1|\sin(\vartheta_1 - \varphi) - 1]\;\frac{1 - \mathrm{e}^{-At}}{A} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{10.28}$$

The asymptotic behaviour of the a priori state $\check{\eta}(t)$ depends on the determinant of A, which is related to the determinant of B. Equations (10.25b) and (10.25d) give $a_{11} \geq 0$, $a_{33} > 0$; then, from (10.27f), one has

$$\det A = 0 \quad \Rightarrow \quad \det B = 0. \tag{10.29}$$

The case $\det B > 0$

$\det B > 0$ implies that

- $a_{ii} > 0, \quad B \geq 0,$
- $\det\left(A + A^{\mathrm{T}}\right) > 0, \quad A + A^{\mathrm{T}} \geq 0,$
- $\det A > 0,$
- $-A$ is a dissipative operator.

In this case, for every initial preparation ρ_0 of the atom, the a priori state $\check{\eta}(t)$ asymptotically reaches the stationary state η_{eq}, which will be computed in Eq. (10.33).

The case $\det B = 0$

$\det B = 0$ is equivalent to

$$\overline{n} = 0, \quad k_{\mathrm{d}} = 0, \quad |\alpha_1| = 1, \quad 2c\sin(\vartheta_1 - \varphi) = 1. \tag{10.30}$$

In this case we have

$$a_{33} > 0, \qquad |c| \geq \frac{1}{2}, \qquad \vec{x}(t) = \mathrm{e}^{-At}\vec{x}(0), \tag{10.31a}$$

$$\cos 2(\vartheta_1 - \varphi) = 1 - \frac{1}{2c^2}, \qquad \cos^2(\vartheta_1 - \varphi) = 1 - \frac{1}{4c^2}, \tag{10.31b}$$

$$a_{11} = \frac{\gamma}{2} + \gamma c^2\left[\cos 2(\vartheta_1 - \varphi) - \cos 2\vartheta_1\right] = 2\gamma c^2 \sin^2\vartheta_1, \tag{10.31c}$$

$$a_{22} = \frac{\gamma}{2} + \gamma c^2\left[\cos 2(\vartheta_1 - \varphi) + \cos 2\vartheta_1\right] = 2\gamma c^2 \cos^2\vartheta_1, \tag{10.31d}$$

$$a_{33} = 2\gamma c^2, \tag{10.31e}$$

$$a_{12} = \Delta\omega + \gamma c\cos(\vartheta_1 - \varphi) - \gamma c^2 \sin 2\vartheta_1, \tag{10.31f}$$

$$a_{21} = -\Delta\omega - \gamma c\cos(\vartheta_1 - \varphi) - \gamma c^2 \sin 2\vartheta_1, \tag{10.31g}$$

$$\frac{1}{2}(a_{12} + a_{21}) = -\gamma c^2 \sin 2\vartheta_1, \tag{10.31h}$$

$$\det A = 2\gamma c^2\left[\Omega^2 \sin^2\vartheta_1 + (\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi))^2\right]. \tag{10.31i}$$

The case $\det B = 0$, $\det A > 0$

Lemma 10.1. *When* $\overline{n} = 0$, $k_{\mathrm{d}} = 0$, $|\alpha_1| = 1$, $2c\sin(\vartheta_1 - \varphi) = 1$, $\det A \neq 0$, *the real parts of the eigenvalues of the matrix A are strictly positive.*

Proof. The characteristic polynomial $p(x)$ of the matrix A is

$$-p(x) := x^3 - 4\gamma c^2 x^2 + \left\{\Omega^2 + 4\gamma^2 c^4 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2\right\} x - 2\gamma c^2\left\{\Omega^2\sin^2\vartheta_1 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2\right\}.$$

The coefficients all being real, the three roots x_i, $i = 1, 2, 3$, can be all real or one real and two complex conjugate;[1] moreover, one has

$$\begin{aligned} x_1 + x_2 + x_3 &= 4\gamma c^2, \\ x_1x_2 + x_2x_3 + x_3x_1 &= \Omega^2 + 4\gamma^2 c^4 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2, \\ x_1x_2x_3 &= 2\gamma c^2\left\{\Omega^2\sin^2\vartheta_1 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2\right\}. \end{aligned}$$

By the third equation all the roots are different from zero.

Let us consider the case of real roots. By the third equation the roots are all positive or one positive and two negative. Let us assume $x_3 \leq x_2 < 0$, $x_1 > 0$. By the first equation $x_1 > 4\gamma c^2$, by the second one $x_2x_3 > \Omega^2 + 4\gamma^2 c^4 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2$ and by inserting these two results into the third equation we get

$$\Omega^2\left(1 + \cos^2\vartheta_1\right) + 8\gamma c^2 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2 < 0,$$

which is false. So, in the case of three real roots, all the roots are positive.

Let us consider the case of x_1 real and $\overline{x_3} = x_2$. Then, we have

$$\begin{aligned} x_1 + 2\,\mathrm{Re}\,x_2 &= 4\gamma c^2, \\ 2x_1\,\mathrm{Re}\,x_2 + |x_2|^2 &= \Omega^2 + 4\gamma^2 c^4 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2, \\ x_1\,|x_2|^2 &= 2\gamma c^2\left\{\Omega^2\sin^2\vartheta_1 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2\right\}. \end{aligned}$$

By the third equation $x_1 > 0$. Let us assume $\mathrm{Re}\,x_2 \leq 0$. By the first equation $x_1 \geq 4\gamma c^2$, by the second one $|x_2|^2 \geq \Omega^2 + 4\gamma^2 c^4 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2$ and by inserting these two results into the third equation we get

$$\Omega^2\left(1 + \cos^2\vartheta_1\right) + 8\gamma c^2 + [\Delta\omega + \gamma c\cos(\vartheta_1 - \varphi)]^2 \leq 0,$$

which is false too. So, the real parts of all the roots are positive. □

[1] For third order algebraic equations and many other things visit the web site: *EqWorld: The World of Mathematical Equations*, http://eqworld.ipmnet.ru

Also in this case, for every initial preparation ρ_0 of the atom, the a priori state $\check{\eta}(t)$ asymptotically reaches the stationary state (10.33), but now $\vec{x}_{\rm eq} = \vec{0}$ and, so,

$$\eta_{\rm eq} = \frac{1}{2}\,\mathbb{1}.$$

10.3.3 The Case of Many Stationary States

Let us consider the case $\det A = 0$, which, by (10.29), (10.30), (10.31i), is equivalent to

$$\overline{n} = 0, \qquad k_{\rm d} = 0, \qquad |\alpha_1| = 1, \qquad \Omega \sin\vartheta_1 = 0, \tag{10.32a}$$

$$2c\sin(\vartheta_1 - \varphi) = 1, \qquad \Delta\omega = -\gamma c\cos(\vartheta_1 - \varphi). \tag{10.32b}$$

In this case we also have

$$4\Delta\omega^2 + \gamma^2 = 4\gamma^2 c^2,$$

$$a_{11} = 2\gamma c^2 \sin^2\vartheta_1, \qquad a_{22} = 2\gamma c^2 \cos^2\vartheta_1,$$

$$a_{33} = 2\gamma c^2, \qquad a_{12} = a_{21} = -\gamma c^2 \sin 2\vartheta_1.$$

Subcase with $\Omega = 0$

In this case we get

$$A = 2\gamma c^2 \begin{pmatrix} \sin^2\vartheta_1 & -\sin\vartheta_1\cos\vartheta_1 & 0 \\ -\sin\vartheta_1\cos\vartheta_1 & \cos^2\vartheta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, any state with Bloch vector $\varepsilon \begin{pmatrix} \cos\vartheta_1 \\ \sin\vartheta_1 \\ 0 \end{pmatrix}$, $\varepsilon \in [-1, 1]$, is invariant.

Moreover, the Liouville operator (10.20) reduces to

$$\check{\mathcal{L}}[\rho] = \gamma c^2 \left(\left(\mathrm{e}^{\mathrm{i}\vartheta_1}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta_1}\sigma_+ \right) \rho \left(\mathrm{e}^{\mathrm{i}\vartheta_1}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta_1}\sigma_+ \right) - \rho \right).$$

Subcase with $\sin\vartheta_1 = 0$

In this case we get

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\gamma c^2 & \Omega \\ 0 & -\Omega & 2\gamma c^2 \end{pmatrix}.$$

Then, any state with Bloch vector $\begin{pmatrix} \varepsilon \\ 0 \\ 0 \end{pmatrix}$, $\varepsilon \in [-1, 1]$, is invariant.

Moreover, the Liouville operator (10.20) reduces to

$$\check{\mathcal{L}}[\rho] = -\frac{\mathrm{i}\Omega}{2}\,[\sigma_x\,,\ \rho] + \gamma c^2\,(\sigma_x\,\rho\,\sigma_x - \rho)\,.$$

10.3.4 Equilibrium

By the results of Sect. 10.3.2, we have that when $\det A > 0$ there is a unique stable stationary state. Such an equilibrium state is given by

$$\eta_{\mathrm{eq}} = \frac{1}{2}\left(\mathbb{1} + \vec{x}_{\mathrm{eq}}\cdot\vec{\sigma}\right), \qquad \vec{x}_{\mathrm{eq}} := \frac{\gamma}{A}\begin{pmatrix} 0 \\ 0 \\ 2c|\alpha_1|\sin(\vartheta_1-\varphi)-1 \end{pmatrix}. \tag{10.33}$$

Parameters

The choice of γ fixes the scale; we take $\gamma = 1$.

The dynamical matrix A (10.25) and the equilibrium Bloch vector $\vec{x}_{\mathrm{eq}}$ depend on

$$\begin{gathered} \Delta\omega \in \mathbb{R}, \qquad \Omega \geq 0, \qquad k_{\mathrm{d}} \geq 0, \qquad \overline{n} \geq 0, \\ c \in \mathbb{R}, \qquad |\alpha_1| \in [0,1], \qquad \vartheta_1 \in (-\pi,\pi], \qquad \varphi \in [0,\pi). \end{gathered}$$

The sign of z_{eq}

Obviously, we have $z_{\mathrm{eq}} = 0$ iff $2c|\alpha_1|\sin(\vartheta_1-\varphi) = 1$. Let u be a real vector; then,

$$\langle u|A^{-1}u\rangle = \frac{1}{2}\langle A^{-1}u|(A^{\mathrm{T}}+A)A^{-1}u\rangle \geq 0.$$

But we have $z_{\mathrm{eq}} = \gamma\,[2c|\alpha_1|\sin(\vartheta_1-\varphi)-1]\,\langle u|A^{-1}u\rangle$ with $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Therefore,

$$z_{\mathrm{eq}} > 0 \qquad \text{iff} \qquad 2c|\alpha_1|\sin(\vartheta_1-\varphi) > 1. \tag{10.34}$$

10.3.5 The Nonlinear Stochastic Master Equation

The a posteriori state $\check{\rho}(t)$ is governed by the nonlinear stochastic master equation

$$\begin{aligned} \mathrm{d}\check{\rho}(t) = {} & \check{\mathcal{L}}[\check{\rho}(t)]\mathrm{d}t + \sqrt{\gamma}\,\mathcal{D}(\alpha_1\,\sigma_- - \mathrm{i}c\,\sigma_\varphi)[\check{\rho}(t)]\mathrm{d}\widehat{W}_1(t) \\ & + \sqrt{\gamma}\,\mathcal{D}(\alpha_2\,\sigma_-)[\check{\rho}(t)]\mathrm{d}\widehat{W}_2(t), \end{aligned} \tag{10.35}$$

where the map $\mathcal{D}(a)$ is

$$\mathcal{D}(a)[\rho] = a\,\rho + \rho\,a^* - \mathrm{Tr}\left\{(a + a^*)\rho\right\}\rho. \tag{10.36}$$

By using the Bloch representation

$$\check{\rho}(t) = \frac{1}{2}\left[1 + \vec{\mathrm{r}}(t)\cdot\vec{\sigma}\right], \tag{10.37}$$

we get the nonlinear SDE

$$\begin{aligned}
\mathrm{d}\vec{\mathrm{r}}(t) = -A\vec{\mathrm{r}}(t)\,\mathrm{d}t + \gamma \begin{pmatrix} 0 \\ 0 \\ 2c\,|\alpha_1|\sin(\vartheta_1 - \phi) - 1 \end{pmatrix}\mathrm{d}t \\
+ \sqrt{\gamma}\sum_{k=1}^{2}|\alpha_k| \begin{pmatrix} [1 + \mathrm{r}_3(t) - \mathrm{r}_1(t)^2]\cos\vartheta_k - \mathrm{r}_1(t)\mathrm{r}_2(t)\sin\vartheta_k \\ [1 + \mathrm{r}_3(t) - \mathrm{r}_2(t)^2]\sin\vartheta_k - \mathrm{r}_1(t)\mathrm{r}_2(t)\cos\vartheta_k \\ -[1 + \mathrm{r}_3(t)]\,[\mathrm{r}_1(t)\cos\vartheta_k + \mathrm{r}_2(t)\sin\vartheta_k] \end{pmatrix}\mathrm{d}\widehat{W}_k(t) \\
+ 2c\sqrt{\gamma}\begin{pmatrix} \mathrm{r}_3(t)\sin\varphi \\ -\mathrm{r}_3(t)\cos\varphi \\ \mathrm{r}_2(t)\cos\varphi - \mathrm{r}_1(t)\sin\varphi \end{pmatrix}\mathrm{d}\widehat{W}_1(t).
\end{aligned} \tag{10.38}$$

Moreover, the two output signals can be written as

$$I_k(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\widehat{W}_k(t) + \sqrt{\gamma}\,|\alpha_k|\,[\mathrm{r}_1(t)\cos\vartheta_k + \mathrm{r}_2(t)\sin\vartheta_k]\,. \tag{10.39}$$

10.4 Control of the Atomic State

In the field of quantum communication and quantum computing it is important to be able to manipulate q-bits (two-state systems). In the case of our atom, we can ask if it is possible to prepare it in any pure state, chosen a priori, that is to choose the control parameters in such a way that, in the rotating frame, the atom is frozen in a preassigned pure state $h_0 \in \mathbb{C}^2$. More precisely, the problem is to find a set of values for the free parameters such that, in the rotating frame, both the a priori state $\check{\eta}(t)$ and the a posteriori state $\check{\rho}(t)$ asymptotically reach $\eta_{\mathrm{eq}} = |h_0\rangle\langle h_0|$, $\|h_0\| = 1$. This problem was raised and studied in [5, 10, 13]; with respect to that papers we have more freedom in the choice of parameters, but we allow only for fast feedback of the homodyne signal and we do not consider a feedback based on the full knowledge of the a posteriori state. Next proposition gives the possible choices of the parameters which accomplish this task.

Proposition 10.2. *Suppose that* $\det A > 0$. *Then the atomic state can be driven to a preassigned asymptotic pure state* $\eta_{\mathrm{eq}} = \frac{1}{2}\left(\mathbb{1} + \vec{x}_{\mathrm{eq}}\cdot\vec{\sigma}\right)$, $\left|\vec{x}_{\mathrm{eq}}\right| = 1$, *provided that the control parameters are chosen as follows:*

- $\eta_{\rm eq} = P_-$, *i.e.* $\vec{x}_{\rm eq} = (0,\ 0,\ -1)^{\rm T}$, *if and only if* $\overline{n} = 0$, $\Omega = 0$, $c = 0$, *which corresponds to the Liouville operator*

$$\check{\mathcal{L}}[\rho] = -\frac{\mathrm{i}\Delta\omega}{2}\,[\sigma_z\,,\ \rho] + \gamma k_{\rm d}\,(\sigma_z\,\rho\,\sigma_z - \rho) + \gamma\left(\sigma_-\,\rho\,\sigma_+ - \frac{1}{2}\,\{P_+\,,\ \rho\}\right);$$

- $\eta_{\rm eq} = P_+$, *i.e.* $\vec{x}_{\rm eq} = (0,\ 0,\ 1)^{\rm T}$, *if and only if* $|\alpha_1|^2 = 1$, $\overline{n} = 0$, $\Omega = 0$, $|c| = 1$, $\sin(\vartheta_1 - \varphi) = c$, *which corresponds to the Liouville operator*

$$\check{\mathcal{L}}[\rho] = -\frac{\mathrm{i}\Delta\omega}{2}\,[\sigma_z\,,\ \rho] + \gamma k_{\rm d}\,(\sigma_z\,\rho\,\sigma_z - \rho) + \gamma\left(\sigma_+\,\rho\,\sigma_- - \frac{1}{2}\,\{P_-\,,\ \rho\}\right);$$

- $y_{\rm eq} \neq 0$, *if and only if* $|\alpha_1|^2 = 1$, $k_{\rm d} = 0$, $\overline{n} = 0$,

$$\cos(\vartheta_1 - \varphi) = \frac{\cos(\vartheta_1 + \varphi)\left(x_{\rm eq}^2 - y_{\rm eq}^2\right) + 2\sin(\vartheta_1 + \varphi)x_{\rm eq}y_{\rm eq}}{(1 + z_{\rm eq})^2}\,, \tag{10.40a}$$

$$\frac{1}{c} = \sin(\vartheta_1 - \varphi) + \frac{2x_{\rm eq}y_{\rm eq}\cos(\vartheta_1 + \varphi) - \sin(\vartheta_1 + \varphi)\left(x_{\rm eq}^2 - y_{\rm eq}^2\right)}{(1 + z_{\rm eq})^2}\,, \tag{10.40b}$$

$$\Omega = \frac{\gamma\left(1 - z_{\rm eq}^2\right)}{2y_{\rm eq}}\,(1 - 2c\sin(\vartheta_1 - \varphi))\,, \tag{10.40c}$$

$$\Delta\omega = \gamma c\cos(\vartheta_1 - \varphi) - \Omega\,\frac{x_{\rm eq}z_{\rm eq}}{1 - z_{\rm eq}^2}. \tag{10.40d}$$

A pure equilibrium state with $y_{\rm eq} = 0$, $z_{\rm eq} \neq \pm 1$ *cannot be reached by any choice of the parameters.*

Proof. As $\det A > 0$, the necessary and sufficient condition to have a pure state $\eta_{\rm eq} = P_0 = |h_0\rangle\langle h_0|$ $\big(\|h_0\| = 1\big)$ as equilibrium state is

$$\check{\mathcal{L}}[P_0] = 0. \tag{10.41}$$

Let h_1 be a normalised vector, orthogonal to h_0 $\big(\langle h_i|h_j\rangle = \delta_{ij}\big)$. By the fact that $\check{\mathcal{L}}$ preserves self-adjointness and annihilates the trace, the condition (10.41) is equivalent to

$$\langle h_1|\check{\mathcal{L}}[P_0]h_1\rangle = 0, \qquad \langle h_1|\check{\mathcal{L}}[P_0]h_0\rangle = 0. \tag{10.42}$$

By explicit computations we get from (10.20)

$$\langle h_1|\check{\mathcal{L}}[P_0]h_1\rangle = \gamma k_{\mathrm{d}}\,|\langle h_1|\sigma_z h_0\rangle|^2 + \gamma\overline{n}\,|\langle h_1|\sigma_+ h_0\rangle|^2 + \gamma\left(\overline{n}+1-|\alpha_1|^2\right)|\langle h_1|\sigma_- h_0\rangle|^2 + \gamma\left|\langle h_1|\left(\alpha_1\sigma_- - \mathrm{i}c\sigma_\varphi\right)h_0\rangle\right|^2, \quad (10.43)$$

$$\begin{aligned}\langle h_1|\check{\mathcal{L}}[P_0]h_0\rangle = &-\frac{\mathrm{i}\Delta\omega_c}{2}\langle h_1|\sigma_z h_0\rangle - \frac{\mathrm{i}\Omega}{2}\langle h_1|\sigma_x h_0\rangle + \gamma k_{\mathrm{d}}\langle h_1|\sigma_z h_0\rangle\langle h_0|\sigma_z h_0\rangle \\ &+ \gamma\overline{n}\left(\langle h_1|\sigma_+ h_0\rangle\langle h_0|\sigma_- h_0\rangle - \frac{1}{2}\langle h_1|P_- h_0\rangle\right) \\ &+ \gamma\left(\overline{n}+1-|\alpha_1|^2\right)\left(\langle h_1|\sigma_- h_0\rangle\langle h_0|\sigma_+ h_0\rangle - \frac{1}{2}\langle h_1|P_+ h_0\rangle\right) \\ &+ \gamma\langle h_1|\left(\alpha_1\sigma_- - \mathrm{i}c\sigma_\varphi\right)h_0\rangle\langle h_0|\left(\overline{\alpha_1}\sigma_+ + \mathrm{i}c\sigma_\varphi\right)h_0\rangle \\ &- \frac{\gamma}{2}\left(|\alpha_1|^2 - 2c\,|\alpha_1|\sin(\vartheta_1-\varphi)\right)\langle h_1|P_+ h_0\rangle.\end{aligned}$$

By the positivity of the terms in (10.43) and the fact that $\gamma > 0$, we get that conditions (10.42) are equivalent to

$$k_{\mathrm{d}}\langle h_1|\sigma_z h_0\rangle = 0, \qquad \overline{n}\langle h_1|\sigma_+ h_0\rangle = 0, \quad (10.44a)$$

$$\left(\overline{n}+1-|\alpha_1|^2\right)\langle h_1|\sigma_- h_0\rangle = 0, \qquad \langle h_1|\left(\alpha_1\sigma_- - \mathrm{i}c\sigma_\varphi\right)h_0\rangle = 0, \quad (10.44b)$$

$$\mathrm{i}\Delta\omega_c\langle h_1|\sigma_z h_0\rangle + \mathrm{i}\Omega\langle h_1|\sigma_x h_0\rangle + \gamma\left(1-2c\,|\alpha_1|\sin(\vartheta_1-\varphi)\right)\langle h_1|P_+ h_0\rangle = 0. \quad (10.44c)$$

When $P_0 = P_-$, we get immediately $\overline{n} = 0$, $c = 0$, $\Omega = 0$ and the expression of the Liouvillian follows from (10.20).

When $P_0 = P_+$, we get easily $\overline{n} = 0$, $|\alpha_1|^2 = 1$, $\alpha_1 = \mathrm{i}c\mathrm{e}^{\mathrm{i}\varphi}$, $\Omega = 0$, from which we have the statement of the proposition for this case.

When $P_0 \neq P_\pm$, we take $h_0 = \begin{pmatrix}\sqrt{1-|\beta|^2}\\ \beta\end{pmatrix}$, $h_1 = \begin{pmatrix}-\overline{\beta}\\ \sqrt{1-|\beta|^2}\end{pmatrix}$, with $0 < |\beta|^2 < 1$. By $\vec{x}_{\mathrm{eq}} = \langle h_0|\vec{\sigma}h_0\rangle$ we obtain the correspondence

$$x_{\mathrm{eq}} = 2\sqrt{1-|\beta|^2}\,\mathrm{Re}\,\beta, \qquad y_{\mathrm{eq}} = 2\sqrt{1-|\beta|^2}\,\mathrm{Im}\,\beta,$$
$$z_{\mathrm{eq}} = 1-2\,|\beta|^2, \qquad \beta = \frac{x_{\mathrm{eq}}+\mathrm{i}y_{\mathrm{eq}}}{\sqrt{2(1+z)}}.$$

This gives

$$2\langle h_1|P_+ h_0\rangle = \langle h_1|\sigma_z h_0\rangle = -2\beta\sqrt{1-|\beta|^2} \neq 0, \qquad \langle h_1|\sigma_+ h_0\rangle = -\beta^2 \neq 0,$$

$$\langle h_1|\sigma_- h_0\rangle = 1 - |\beta|^2 \neq 0, \qquad \langle h_1|\sigma_x h_0\rangle = 1 - |\beta|^2 - \beta^2.$$

Then, conditions (10.44) are equivalent to

$$k_\mathrm{d} = 0, \qquad \overline{n} = 0, \qquad |\alpha_1|^2 = 1$$

and

$$\left(1 - \mathrm{i}c\mathrm{e}^{\mathrm{i}(\varphi-\vartheta_1)}\right)\left(1 - |\beta|^2\right) + \mathrm{i}c\beta^2\mathrm{e}^{-\mathrm{i}(\varphi+\vartheta_1)} = 0,$$

$$\mathrm{i}\Omega\left(1 - \beta^2 - |\beta|^2\right) - 2\mathrm{i}\Delta\omega\beta\sqrt{1-|\beta|^2} - \gamma\beta\sqrt{1-|\beta|^2}\left(1 + 2\mathrm{i}c\mathrm{e}^{\mathrm{i}(\vartheta_1-\varphi)}\right) = 0.$$

The last two equations can be written as

$$(1 + z_\mathrm{eq})^2\cos(\vartheta_1 - \varphi) = \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}(\vartheta_1+\varphi)}\left(x_\mathrm{eq}^2 - y_\mathrm{eq}^2 + 2\mathrm{i}x_\mathrm{eq}y_\mathrm{eq}\right)\right], \tag{10.45a}$$

$$\frac{1}{c} = \sin(\vartheta_1 - \varphi) + \frac{\mathrm{Im}\left[\mathrm{e}^{-\mathrm{i}(\vartheta_1+\varphi)}\left(x_\mathrm{eq}^2 - y_\mathrm{eq}^2 + 2\mathrm{i}x_\mathrm{eq}y_\mathrm{eq}\right)\right]}{(1 + z_\mathrm{eq})^2}, \tag{10.45b}$$

$$\gamma\left(1 + 2\mathrm{i}c\mathrm{e}^{\mathrm{i}(\vartheta_1-\varphi)}\right) = 2\mathrm{i}\Omega\,\frac{x_\mathrm{eq}z_\mathrm{eq} - \mathrm{i}y_\mathrm{eq}}{1 - z_\mathrm{eq}^2} - 2\mathrm{i}\Delta\omega. \tag{10.45c}$$

In the case $y_\mathrm{eq} = 0$, $z_\mathrm{eq} \neq \pm 1$ we get

$$1 = 2c\sin(\vartheta_1 - \varphi), \qquad \gamma c\cos(\vartheta_1 - \varphi) = \Omega\,\frac{x_\mathrm{eq}z_\mathrm{eq}}{1 - z_\mathrm{eq}^2} - \Delta\omega,$$

$$\sin(\vartheta_1 - \varphi) = -\frac{\sin(\vartheta_1 + \varphi)x_\mathrm{eq}^2}{(1 + z_\mathrm{eq})^2}, \qquad \cos(\vartheta_1 - \varphi) = \frac{\cos(\vartheta_1 + \varphi)x_\mathrm{eq}^2}{(1 + z_\mathrm{eq})^2}.$$

But the last equations imply $z_\mathrm{eq} = 0$, $x_\mathrm{eq} = \pm 1$ and, then, we get $\vartheta_1 = 0$, $\Delta\omega = -\gamma c\cos\phi$, $1 + 2c\sin\phi = 0$, which corresponds to $\det A = 0$ and it is excluded by $\det A > 0$.

In the case $y_\mathrm{eq} \neq 0$, which implies $z_\mathrm{eq} \neq \pm 1$, Eqs. (10.45) give (10.40). □

Apart from the trivial case $P_0 = P_-$, it is possible to freeze the atom in a pure state in an exact way only in the very ideal case $|\alpha_1| = 1$. This is an ideal case not only because it is nearly impossible to collect all the fluorescence light in channel 1, as needed to have $|\alpha_1| = 1$, but also because this implies $|\alpha_0| = 0$, so that there is no room in the forward channel for the stimulating laser. To reach $|\alpha_1| = 1$ without $\Omega = 0$ and $c = 0$, one has to think that the intensity of the laser goes to infinity to

compensate for $|\alpha_0| \to 0$. A less ambitious task, which we do not develop here, is to take the value of $|\alpha_1|^2$ as given by the experimental situation and to use control to get the maximal purity obtainable in any fixed direction in the Bloch sphere [5, 10, 13].

Another subject in quantum information is the one of the "decoherence free subspaces". Any quantum system which one could use in quantum computations or quantum communication is inevitably open and suffers decoherence (loss of purity, loss of "entanglement") due to dissipation. One can try to reduce dissipation by better isolating the system, or by finding some subspaces of the Hilbert space of the system free from dissipation. A single two-level atom is a very poor model from the point of view of the structure of the Hilbert space; however, we can use it to give a toy model with decoherence free subspaces.

Let us consider the case $\det A = 0$, presented in Sect. 10.3.4. Again it is a very ideal case because we need $|\alpha_1| = 1, \overline{n} = 0, k_\mathrm{d} = 0$.

In the case $\sin\vartheta_1 = 0, 2c\sin(\vartheta_1 - \varphi) = 1, \Delta\omega = -\gamma c\cos(\vartheta_1 - \varphi)$, the Liouville operator reduces to

$$\check{\mathcal{L}}[\rho] = -\frac{\mathrm{i}\Omega}{2}\,[\sigma_x\,,\ \rho] + \gamma c^2\,(\sigma_x\,\rho\,\sigma_x - \rho)\,.$$

The master equation with such a generator leaves invariant any state of the form $\frac{1}{2}\,(\mathbb{1} + x\sigma_x)$. At the Hilbert space level, the two subspaces of the eigenvectors of σ_x are decoherence free.

In the case $\Omega = 0, 2c\sin(\vartheta_1 - \varphi) = 1, \Delta\omega = -\gamma c\cos(\vartheta_1 - \varphi)$, the Liouville operator reduces to

$$\check{\mathcal{L}}[\rho] = \gamma c^2\left(\left(\mathrm{e}^{\mathrm{i}\vartheta_1}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta_1}\sigma_+\right)\rho\left(\mathrm{e}^{\mathrm{i}\vartheta_1}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta_1}\sigma_+\right) - \rho\right)$$

and the decoherence free subspaces are the eigenspaces of $\mathrm{e}^{\mathrm{i}\vartheta_1}\sigma_- + \mathrm{e}^{-\mathrm{i}\vartheta_1}\sigma_+$.

10.5 Control of the Squeezing of Fluorescence Light

Feedback, besides controlling the atomic state, can be used to control the light emitted by the atom; in particular we are interested in the squeezing properties of the light in the side channels 1 and 2. With the help of the incoherent spectrum of the homodyne photocurrents we can analyse the squeezing properties of the light detected in the two side channels, and thus we can investigate the effect of the control parameters [11, 12].

When $|\alpha_2|^2 = 0$, the fluorescence light which is not lost in the forward channel is gathered in a unique side channel, so that the squeezing is analysed just for that light which is also detected for the feedback loop. This means that the eventually squeezed light would not be available for other purposes. Thus in this case a unique homodyne detector is employed and $|\alpha_1|^2$ is its efficiency. When $|\alpha_2|^2 > 0$, the fluorescence light which is not lost in the forward channel is split in the two side channels. The homodyne detection of the light in channel 1 allows either to do

the spectral analysis and to detect squeezing, or to make the feedback control. The light in channel 2 is detected for squeezing analysis, and it could be employed for different uses.

To have the spectrum we need a stationary output process, at least for large times. So, we want to have an equilibrium state and we need $\det A > 0$. We assume also that there is always some light in the forward channel, where the laser acts, and some light in the side channel 1, to be used for feedback.

Assumption 10.3. We assume $|\alpha_0|^2 > 0$ and $|\alpha_1|^2 > 0$.

This implies that $|\alpha_1|^2$ cannot reach the value 1 and, due to the conditions (10.32), that $\det A > 0$.

10.5.1 The Spectral Densities

Now we have two channels and, so, the spectra of the two outputs and the spectrum of the cross-correlations. By defining

$$m_i := \mathrm{Tr}\left\{\check{\mathcal{R}}_i[\eta_{\mathrm{eq}}]\right\}, \quad C_{ji}(t) := \mathrm{Tr}\left\{\check{\mathcal{R}}_j \circ \check{\mathcal{T}}(t)[\check{\tau}_i]\right\}, \quad \check{\tau}_i := \check{\mathcal{R}}_i[\eta_{\mathrm{eq}}] - m_i\eta_{\mathrm{eq}},$$

by Eq. (4.94) we get the spectra

$$S_{ij}(\mu) = S_{ij}^{\mathrm{el}}(\mu) + S_{ij}^{\mathrm{inel}}(\mu), \qquad S_{ij}^{\mathrm{el}}(\mu) = 2\pi m_i m_j \delta(\mu), \tag{10.46a}$$

$$S_{ij}^{\mathrm{inel}}(\mu) = \delta_{ij} + \int_0^{+\infty} \left(\mathrm{e}^{\mathrm{i}\mu t} C_{ij}(t) + \mathrm{e}^{-\mathrm{i}\mu t} C_{ji}(t)\right) \mathrm{d}t. \tag{10.46b}$$

In the present case we have

$$\check{R}_1 = \sqrt{\gamma}\left(\alpha_1\sigma_- - \mathrm{i}c\sigma_\varphi\right), \qquad \check{R}_2 = \sqrt{\gamma}\,\alpha_2\sigma_-, \tag{10.47}$$

$$m_i = \sqrt{\gamma}\,|\alpha_i|\left(x_{\mathrm{eq}}\cos\vartheta_i + y_{\mathrm{eq}}\sin\vartheta_i\right). \tag{10.48}$$

Here below, in the computations of the spectra, the following notation will be used:

$$\tau := \sigma_-\eta_{\mathrm{eq}} - \mathrm{Tr}\left\{\sigma_-\eta_{\mathrm{eq}}\right\}\eta_{\mathrm{eq}} = \frac{1}{2}\,\vec{d}(0)\cdot\vec{\sigma}, \tag{10.49}$$

$$\vec{d}(0) := \mathrm{Tr}\left\{\vec{\sigma}\,\tau\right\} = \frac{1}{2}\begin{pmatrix} 1 + z_{\mathrm{eq}} - x_{\mathrm{eq}}\left(x_{\mathrm{eq}} - \mathrm{i}y_{\mathrm{eq}}\right) \\ -\mathrm{i}\left[1 + z_{\mathrm{eq}} - \mathrm{i}y_{\mathrm{eq}}\left(x_{\mathrm{eq}} - \mathrm{i}y_{\mathrm{eq}}\right)\right] \\ -\left(1 + z_{\mathrm{eq}}\right)\left(x_{\mathrm{eq}} - \mathrm{i}y_{\mathrm{eq}}\right) \end{pmatrix}. \tag{10.50}$$

10.5.2 Channel 1: The In-Loop Light

The spectrum of the light in the channel used for feedback has a peculiar status; for this light the Heisenberg like relation (9.63) does not hold, because the involved phase ϑ_1 now appears also in the dynamics, due to the feedback loop. When the spectrum is below 1 we keep on speaking of "squeezed" in-loop light, but, perhaps, it would be better to speak of sub-shot noise spectrum of the in-loop light.

By making the previous expressions more explicit, the elastic and inelastic components of the homodyne spectrum of output 1 are

$$\begin{aligned} S_{11}^{\mathrm{el}}(\mu) \equiv S_1^{\mathrm{el}}(\mu;\vartheta_1) &= 2\pi\gamma\,|\alpha_1|^2\left|\mathrm{Tr}\left\{\sigma_{\vartheta_1}\eta_{\mathrm{eq}}\right\}\right|^2\delta(\mu) \\ &= 2\pi\gamma\,|\alpha_1|^2\left(x_{\mathrm{eq}}\cos\vartheta_1 + y_{\mathrm{eq}}\sin\vartheta_1\right)^2\delta(\mu) \\ &= 2\pi\gamma\,|\alpha_1|^2\left(\vec{x}_{\mathrm{eq}}\cdot\vec{u}(\vartheta_1)\right)^2\delta(\mu), \end{aligned} \tag{10.51}$$

$$\begin{aligned} S_{11}^{\mathrm{inel}}(\mu) \equiv S_1^{\mathrm{inel}}(\mu;\vartheta_1) &= 1 + 2\gamma\,|\alpha_1|^2\int_0^{+\infty}\cos\mu t\,\mathrm{Tr}\left\{\sigma_{\vartheta_1}\mathrm{e}^{\check{\mathcal{L}}t}[\tau_1]\right\}\mathrm{d}t \\ &= 1 + 2\,|\alpha_1|^2\left(\frac{\gamma A}{A^2+\mu^2}\,\vec{t}_1\right)\cdot\vec{u}(\vartheta_1), \end{aligned} \tag{10.52}$$

where

$$\tau_1 = \mathrm{e}^{\mathrm{i}\vartheta_1}\tau + \mathrm{e}^{-\mathrm{i}\vartheta_1}\tau^* + \mathrm{i}\widetilde{c}[\eta_{\mathrm{eq}},\,\sigma_\varphi], \qquad \widetilde{c} = \begin{cases} \frac{c}{|\alpha_1|}, & \text{if } \alpha_1 \neq 0, \\ 0, & \text{if } \alpha_1 = 0, \end{cases} \tag{10.53}$$

$$\vec{t}_1 = \mathrm{Tr}\{\tau_1\vec{\sigma}\} = \begin{pmatrix} \left(1 + z_{\mathrm{eq}} - x_{\mathrm{eq}}^2\right)\cos\vartheta_1 - x_{\mathrm{eq}}y_{\mathrm{eq}}\sin\vartheta_1 \\ \left(1 + z_{\mathrm{eq}} - y_{\mathrm{eq}}^2\right)\sin\vartheta_1 - x_{\mathrm{eq}}y_{\mathrm{eq}}\cos\vartheta_1 \\ -\left(1 + z_{\mathrm{eq}}\right)\left(x_{\mathrm{eq}}\cos\vartheta_1 + y_{\mathrm{eq}}\sin\vartheta_1\right) \end{pmatrix} + 2\widetilde{c}\,\vec{u}(\varphi)\times\vec{x}_{\mathrm{eq}}\,. \tag{10.54}$$

Examples of the inelastic spectrum of output 1 are given in Figs. 10.2 and 10.3. The choice of the linewidth is $\gamma = 1$ and the percentage of fluorescence light in channel 1 is taken to be $|\alpha_1|^2 = 0.45$ which is a reasonable value.

In Fig. 10.2 the parameters are chosen to have big squeezing in predetermined positions; the "smoothing" effect of $\overline{n}$ and k_{d} is also shown. The strategy has been to consider k_{d} and $\overline{n}$ assigned by external conditions and to fix the position where we want the minimum; then, the other parameters are taken by numerically minimising the spectrum in the fixed position. The best squeezing for the in-loop light turns out to be always for $\Omega = 0$. Moreover, lines (1)–(4) are for the case $\overline{n} = 0$ and $k_{\mathrm{d}} = 0$ and lines (5) and (6) for the case in which these parameters are small, but not vanishing: $\overline{n} = 0.01$ and $k_{\mathrm{d}} = 0.05$. The values of the parameters are given in

the caption of Fig. 10.2; for what concerns the minima, the situation is summarised by the following table, where the numbers in brackets are the numbers of the lines in the figure:

Position of the minimum	Value of the minimum	Value of the minimum
	$\overline{n} = 0, k_\mathrm{d} = 0$	$\overline{n} = 0.01, k_\mathrm{d} = 0.05$
$\mu = 0$	(1) 0.4354	(5) 0.5019
$\mu = \pm 1$	(2) 0.5617	(6) 0.6078
$\mu = \pm 2$	(3) 0.5887	(7) 0.6367
$\mu = \pm 4$	(4) 0.5998	(8) 0.6494

Other examples of spectra below and above shot noise are given in Fig. 10.3. Here the parameters are optimal for squeezing in channel 2 and are taken from Fig. 10.4. This figure shows that a situation of optimal squeezing in channel 2 does not necessarily correspond to sub-shot noise in channel 1 or, at least, not to optimal sub-shot noise.

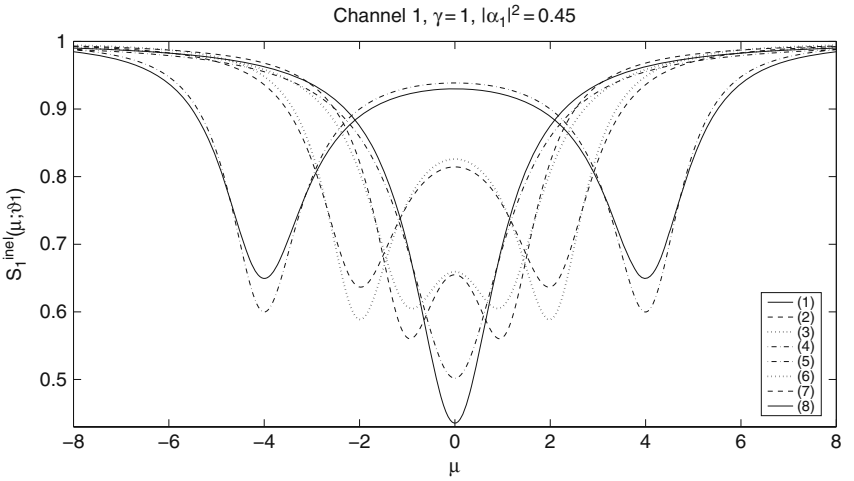

Fig. 10.2 $\Omega = 0$. (1) $k_\mathrm{d} = 0$, $\overline{n} = 0$, $\Delta\omega = 0$, $c = 0.2936$, $\vartheta_1 = -\pi/2$, $\varphi = 0$, value of the minimum 0.4354 at $\mu = 0$; (2) $k_\mathrm{d} = 0$, $\overline{n} = 0$, $\Delta\omega = 1.0386$, $c = 0.3700$, $\vartheta_1 = -1.2763$, $\varphi = -0.1236$, value of the minimum 0.5617 at $\mu = \pm 1$; (3) $k_\mathrm{d} = 0$, $\overline{n} = 0$, $\Delta\omega = 2.0540$, $c = 0.3765$, $\vartheta_1 = -1.3145$, $\varphi = -0.0846$, value of the minimum 0.5887 at $\mu = \pm 2$; (4) $k_\mathrm{d} = 0$, $\overline{n} = 0$, $\Delta\omega = 4.0372$, $c = 0.3776$, $\vartheta_1 = -1.3790$, $\varphi = -0.0211$, value of the minimum 0.5998 at $\mu = \pm 4$; (5) $k_\mathrm{d} = 0.05$, $\overline{n} = 0.01$, $\Delta\omega = 0$, $c = 0.3255$, $\vartheta_1 = -\pi/2$, $\varphi = 0$, value of the minimum 0.5019 at $\mu = 0$; (6) $k_\mathrm{d} = 0.05$, $\overline{n} = 0.01$, $\Delta\omega = 1.0335$, $c = 0.4020$, $\vartheta_1 = -1.2864$, $\varphi = -0.1136$, value of the minimum 0.6078 at $\mu = \pm 1$; (7) $k_\mathrm{d} = 0.05$, $\overline{n} = 0.01$, $\Delta\omega = 2.0715$, $c = 0.4115$, $\vartheta_1 = -1.3001$, $\varphi = -0.0799$, value of the minimum 0.6367 at $\mu = \pm 2$; (8) $k_\mathrm{d} = 0.05$, $\overline{n} = 0.01$, $\Delta\omega = 4.0534$, $c = 0.4138$, $\vartheta_1 = -1.3611$, $\varphi = -0.0189$, value of the minimum 0.6494 at $\mu = \pm 4$

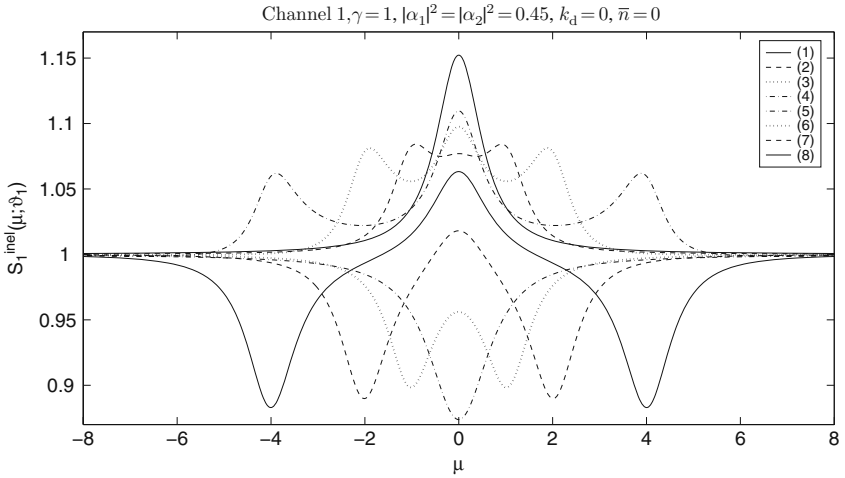

Fig. 10.3 (1) $\Delta\omega = 0$, $\Omega = 0.2698$, $c = -0.0896$, $\vartheta_1 = -\pi/2$, $\varphi = 0$; (2) $\Delta\omega = 0.7781$, $\Omega = 0.7967$ $c = -0.1123$, $\vartheta_1 = -0.5315$, $\varphi = 1.0393$; (3) $\Delta\omega = 1.3852$, $\Omega = 1.5464$, $c = -0.1301$, $\vartheta_1 = -0.3722$, $\varphi = 1.1986$; (4) $\Delta\omega = 2.6310$, $\Omega = 3.0721$, $c = -0.1348$, $\vartheta_1 = -0.2305$, $\varphi = 1.3403$; (5) $\Delta\omega = 0$, $\Omega = 0.2976$, $c = 0$, $\vartheta_1 = -\pi/2$; (6) $\Delta\omega = 0.8279$, $\Omega = 0.7445$, $c = 0$, $\vartheta_1 = -0.3480$; (7) $\Delta\omega = 1.4937$, $\Omega = 1.4360$, $c = 0$, $\vartheta_1 = -0.1784$; (8) $\Delta\omega = 2.8077$, $\Omega = 2.9001$, $c = 0$, $\vartheta_1 = -0.0909$

10.5.3 Channel 2

With channel 2, two new parameters enter into play: $\vartheta_2 \in (-\pi, \pi]$ and $|\alpha_2|$ with $0 < |\alpha_1|^2 + |\alpha_2|^2 < 1$.

We get again the elastic and inelastic components of the homodyne spectrum

$$\begin{aligned} S_{22}^{\mathrm{el}}(\mu) \equiv S_2^{\mathrm{el}}(\mu;\vartheta_2) &= 2\pi\gamma\,|\alpha_2|^2\left|\mathrm{Tr}\left\{\sigma_{\vartheta_2}\eta_{\mathrm{eq}}\right\}\right|^2\delta(\mu) \\ &= 2\pi\gamma\,|\alpha_2|^2\left(x_{\mathrm{eq}}\cos\vartheta_2 + y_{\mathrm{eq}}\sin\vartheta_2\right)^2\delta(\mu) \\ &= 2\pi\gamma\,|\alpha_2|^2\left(\vec{x}_{\mathrm{eq}}\cdot\vec{u}(\vartheta_2)\right)^2\delta(\mu), \end{aligned} \tag{10.55}$$

$$\begin{aligned} S_{22}^{\mathrm{inel}}(\mu) \equiv S_2^{\mathrm{inel}}(\mu;\vartheta_2) &= 1 + 2\gamma\,|\alpha_2|^2\int_0^{+\infty}\cos\mu t\,\mathrm{Tr}\left\{\sigma_{\vartheta_2}\mathrm{e}^{\check{L}t}[\tau_2]\right\}\mathrm{d}t \\ &= 1 + 2\,|\alpha_2|^2\left(\frac{\gamma A}{A^2+\mu^2}\,\vec{t}_2\right)\cdot\vec{u}(\vartheta_2), \end{aligned} \tag{10.56}$$

where

$$\tau_2 = \mathrm{e}^{\mathrm{i}\vartheta_2}\tau + \mathrm{e}^{-\mathrm{i}\vartheta_2}\tau^*, \tag{10.57}$$

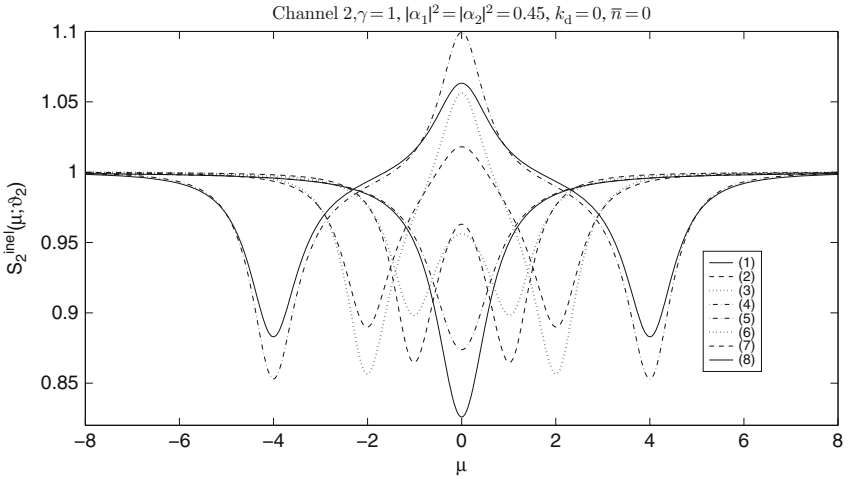

Fig. 10.4 (1) $\Delta\omega = 0$, $\Omega = 0.2698$, $c = -0.0896$, $\vartheta_1 = -\pi/2$, $\vartheta_2 = -\pi/2$, $\varphi = 0$, value of the minimum 0.8259 at $\mu = 0$; (2) $\Delta\omega = 0.7781$, $\Omega = 0.7967$ $c = -0.1123$, $\vartheta_1 = -0.5315$, $\vartheta_2 = -0.2498$, $\varphi = 1.0393$, value of the minimum 0.8647 at $\mu = \pm 1$; (3) $\Delta\omega = 1.3852$, $\Omega = 1.5464$, $c = -0.1301$, $\vartheta_1 = -0.3722$, $\vartheta_2 = -0.1105$, $\varphi = 1.1986$, value of the minimum 0.8565 at $\mu = \pm 2$; (4) $\Delta\omega = 2.6310$, $\Omega = 3.0721$, $c = -0.1348$, $\vartheta_1 = -0.2305$, $\vartheta_2 = -0.0497$, $\varphi = 1.3403$, value of the minimum 0.8530 at $\mu = \pm 4$; (5) $\Delta\omega = 0$, $\Omega = 0.2976$, $c = 0$, $\vartheta_2 = -\pi/2$, value of the minimum 0.8738 at $\mu = 0$; (6) $\Delta\omega = 0.8279$, $\Omega = 0.7445$, $c = 0$, $\vartheta_2 = -0.3480$, value of the minimum 0.8984 at $\mu = \pm 1$; (7) $\Delta\omega = 1.4937$, $\Omega = 1.4360$, $c = 0$, $\vartheta_2 = -0.1784$, value of the minimum 0.8898 at $\mu = \pm 2$; (8) $\Delta\omega = 2.8077$, $\Omega = 2.9001$, $c = 0$, $\vartheta_2 = -0.0909$, value of the minimum 0.8830 at $\mu = \pm 4$

$$\vec{t}_2 = \mathrm{Tr}\{\tau_2 \vec{\sigma}\} = \begin{pmatrix} \left(1 + z_{\mathrm{eq}} - x_{\mathrm{eq}}^2\right)\cos\vartheta_2 - x_{\mathrm{eq}} y_{\mathrm{eq}} \sin\vartheta_2 \\ \left(1 + z_{\mathrm{eq}} - y_{\mathrm{eq}}^2\right)\sin\vartheta_2 - x_{\mathrm{eq}} y_{\mathrm{eq}} \cos\vartheta_2 \\ -\left(1 + z_{\mathrm{eq}}\right)\left(x_{\mathrm{eq}} \cos\vartheta_2 + y_{\mathrm{eq}} \sin\vartheta_2\right) \end{pmatrix}. \tag{10.58}$$

Examples of the inelastic spectrum of output 2 are given in Figs.10.4 and 10.5 for $\gamma = 1$, $|\alpha_1|^2 = |\alpha_1|^2 = 0.45$, $k_d = 0$, $\overline{n} = 0$. In Fig. 10.4 the parameters are chosen to have optimal squeezing in predetermined positions; the case $c = 0$ is also shown to see the effect of feedback. The situation of the minima is the following one:

Position of the minimum	Value of the minimum $c > 0$	Value of the minimum $c = 0$
$\mu = 0$	(1) 0.8259	(5) 0.8738
$\mu = \pm 1$	(2) 0.8647	(6) 0.8984
$\mu = \pm 2$	(3) 0.8565	(7) 0.8898
$\mu = \pm 4$	(4) 0.8530	(8) 0.8830

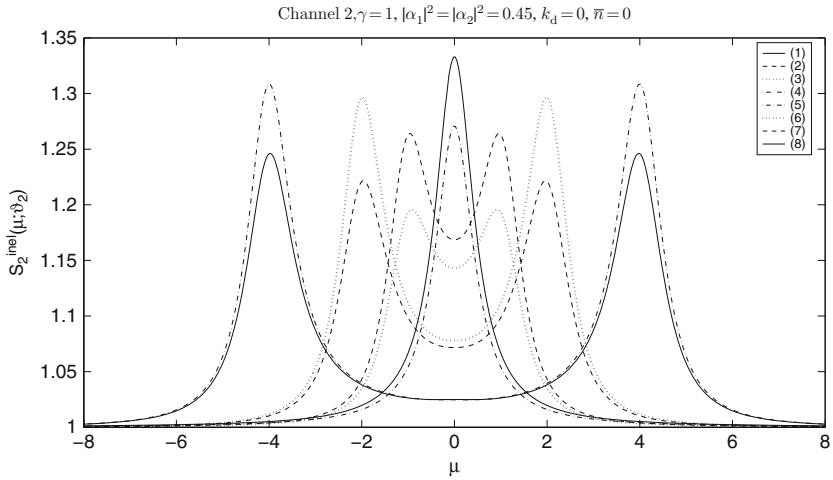

Fig. 10.5 As Fig. 10.4 with ϑ_2 shifted by $\pi/2$

In Fig. 10.5 the same parameters are used, but ϑ_2, which is shifted by $\pi/2$ to see the effect of the uncertainty relation (9.63). Figures 10.4 and 10.5 give the spectra of two complementary quadrature of the electromagnetic field.

10.5.4 The Spectrum of the Cross-Correlations

As discussed in Sect. 4.5.3, when linear combinations of the outputs are studied, also the spectrum of the cross-correlations (4.92) enters into play. In the present case from Eq. (4.94) we get

$$S_{12}^{\mathrm{el}}(\mu) = S_{21}^{\mathrm{el}}(\mu) = 2\pi\gamma\,|\alpha_1|\,|\alpha_2|\,\left(x_{\mathrm{eq}}\cos\vartheta_1 + y_{\mathrm{eq}}\sin\vartheta_1\right) \times \left(x_{\mathrm{eq}}\cos\vartheta_2 + y_{\mathrm{eq}}\sin\vartheta_2\right)\delta(\mu), \quad (10.59)$$

$$S_{12}^{\mathrm{inel}}(\mu) = \overline{S_{21}^{\mathrm{inel}}(\mu)} = \gamma\,|\alpha_1|\,|\alpha_2|\left[\vec{u}(\vartheta_1)\cdot\left(\frac{1}{A-\mathrm{i}\mu}\,\vec{t}_2\right) + \vec{u}(\vartheta_2)\cdot\left(\frac{1}{A+\mathrm{i}\mu}\,\vec{t}_1\right)\right], \quad (10.60a)$$

$$\mathrm{Re}\,S_{12}^{\mathrm{inel}}(\mu) = \vec{u}(\vartheta_1)\cdot\left(\frac{\gamma\,|\alpha_1|\,|\alpha_2|\,A}{A^2+\mu^2}\,\vec{t}_2\right) + \vec{u}(\vartheta_2)\cdot\left(\frac{\gamma\,|\alpha_1|\,|\alpha_2|\,A}{A^2+\mu^2}\,\vec{t}_1\right), \quad (10.60b)$$

$$\mathrm{Im}\,S_{12}^{\mathrm{inel}}(\mu) = \vec{u}(\vartheta_1)\cdot\left(\frac{\gamma\,|\alpha_1|\,|\alpha_2|\,\mu}{A^2+\mu^2}\,\vec{t}_2\right) - \vec{u}(\vartheta_2)\cdot\left(\frac{\gamma\,|\alpha_1|\,|\alpha_2|\,\mu}{A^2+\mu^2}\,\vec{t}_1\right). \quad (10.60c)$$

Let us note that the positivity condition (4.93) implies that the two eigenvalues of the matrix $\begin{pmatrix} S_{11}^{\text{inel}}(\mu) & S_{12}^{\text{inel}}(\mu) \\ S_{21}^{\text{inel}}(\mu) & S_{22}^{\text{inel}}(\mu) \end{pmatrix}$ are real and non-negative:

$$\frac{S_{11}^{\text{inel}}(\mu) + S_{22}^{\text{inel}}(\mu)}{2} \pm \sqrt{\left(\frac{S_{11}^{\text{inel}}(\mu) - S_{22}^{\text{inel}}(\mu)}{2}\right)^2 + \left|S_{12}^{\text{inel}}(\mu)\right|^2} \geq 0. \tag{10.61}$$

10.5.4.1 Alternative Expressions

The expressions of all the spectral densities introduced in this chapter can be written in a unified manner also as

$$S_{ij}^{\text{el}}(\mu) = 2\pi\gamma\,|\alpha_i|\,\left|\alpha_j\right|\left(\vec{x}_{\text{eq}}\cdot\vec{u}(\vartheta_i)\right)\left(\vec{x}_{\text{eq}}\cdot\vec{u}(\vartheta_j)\right)\delta(\mu), \tag{10.62}$$

$$S_{ij}^{\text{inel}}(\mu) = \delta_{ij} + \gamma\,|\alpha_i|\,\left|\alpha_j\right|\left[\vec{u}(\vartheta_i)\cdot\left(\frac{1}{A-\mathrm{i}\mu}\,\vec{t}_j\right) + \vec{u}(\vartheta_j)\cdot\left(\frac{1}{A+\mathrm{i}\mu}\,\vec{t}_i\right)\right]. \tag{10.63}$$

10.5.5 Global Squeezing of Light and Atomic Squeezing

10.5.5.1 Global Squeezing Parameter of the Light

As in the previous chapter, we can introduce the global squeezing parameter for the two channels:

$$\Pi_k(\vartheta_k) = \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty}\left[S_k^{\text{inel}}(\mu;\vartheta_k) - 1\right]\mathrm{d}\mu = |\alpha_k|^2\,\vec{t}_k\cdot\vec{u}(\vartheta_k). \tag{10.64}$$

Channel 2

The explicit expression of the global squeezing parameter is

$$\Pi_2(\vartheta_2) = |\alpha_2|^2\left[1 + z_{\text{eq}} - \left(\vec{x}_{\text{eq}}\cdot\vec{u}(\vartheta_2)\right)^2\right]. \tag{10.65}$$

This channel is as the observed channel of the previous chapter: the phase ϑ_2 appears only in the spectrum, not in the dynamics. So, one can minimise with respect to ϑ_2 and an expression for the best global squeezing can be obtained:

$$\begin{aligned}\Sigma_2 = \min_{\vartheta_2}\Pi_2(\vartheta_2) &= |\alpha_2|^2\left[1 + z_{\text{eq}} - \left(x_{\text{eq}}^2 + y_{\text{eq}}^2\right)\right] \\ &= |\alpha_2|^2\left[\text{AS}_{\eta_{\text{eq}}} + z_{\text{eq}} + \left|z_{\text{eq}}\right|\right].\end{aligned} \tag{10.66}$$

The difference with respect to the situation of the previous chapter is that feedback allows also for cases in which $z_{\rm eq}$ is positive and there is atomic squeezing. As we have always $1-\left(x_{\rm eq}^2+y_{\rm eq}^2\right)\geq 0$, we have

$$z_{\rm eq}\geq 0 \quad \Rightarrow \quad \Sigma_2\geq 0. \tag{10.67}$$

So, when the equilibrium state is in the upper hemisphere of the Bloch sphere, there is no global squeezing in channel 2, even if the atomic state is squeezed.

Channel 1

The explicit expression of the global squeezing parameter is now more complicate:

$$\begin{aligned}\Pi_1(\vartheta_1) &= |\alpha_1|^2\left[1+z_{\rm eq}-\left(\vec{x}_{\rm eq}\cdot\vec{u}(\vartheta_1)\right)^2\right]+2c\,|\alpha_1|\,\vec{x}_{\rm eq}\cdot\left(\vec{u}(\vartheta_1)\times\vec{u}(\varphi)\right)\\ &= |\alpha_1|^2\left[1+z_{\rm eq}-\left(\vec{x}_{\rm eq}\cdot\vec{u}(\vartheta_1)\right)^2\right]-2c\,|\alpha_1|\,z_{\rm eq}\sin(\vartheta_1-\varphi).\end{aligned} \tag{10.68}$$

10.5.5.2 Fluorescence Light Squeezing Versus Atomic Squeezing

The feedback loop can be really efficient also to enhance the atomic squeezing. For example, in the ideal situation $|\alpha_1|=1$, $k_{\rm d}=\overline{n}=0$, with $\gamma=1$, $\Delta\omega=3$, $\Omega=4$, $\vartheta_1=\pi/2$, $c=1.3372$, $\varphi=-\pi/40$, we get $\mathrm{AS}_{\eta_{\rm eq}}=-0.2414$, which is very close to the bound $-1/4$. In this case $\eta_{\rm eq}$ is almost pure so that also the a posteriori state $\rho(t)$ is frozen in a neighbourhood of $\eta_{\rm eq}$ and $\mathrm{AS}_{\rho(t)}$ is minimised, too.

There are no simple relations among the squeezing properties of fluorescence light in channel 1, of fluorescence light in channel 2, of atomic a priori equilibrium state and of atomic a posteriori state. Indeed, by changing the parameters of our model, we can observe a wide variety of behaviours.

The only clear link is the one mentioned above: if $\mathrm{AS}_{\eta_{\rm eq}}\simeq -1/4$, then $\eta_{\rm eq}$ is almost pure and $\rho(t)$ is frozen in a neighbourhood of $\eta_{\rm eq}$, so that $\mathrm{AS}_{\rho(t)}$ is minimised, too, and the fluorescence light squeezing disappears as there is no incoherent scattering of light. One can check that actually only the coherent scattering survives, giving a δ-contribution in $\mu=0$ to the complete spectrum. If the freezing of the atom is only approximate, one can check that all the spectra tend to become flatter and the fluorescence light squeezing tends to disappear.

There is also the link (10.66) between $\mathrm{AS}_{\eta_{\rm eq}}$ and Σ_2. This gives a direct relation between atomic and fluorescence light squeezing in the absence of feedback. Indeed, in this case we have $z_{\rm eq}\leq 0$, so that $\Sigma_2=|\alpha_2|^2\,\mathrm{AS}_{\eta_{\rm eq}}$, and we can consider the case $|\alpha_1|=0$, so that $|\alpha_2|^2$ is the fraction of the whole detected light. This relation is essentially the same as found by Walls and Zoller considering a single mode for the emitted light [14]. However, the relation is not fundamental, as the feedback loop can give $z_{\rm eq}>0$ and in this case we have always $\Sigma_2\geq 0$ even if $\mathrm{AS}_{\eta_{\rm eq}}<0$.

There is no relation between fluorescence light squeezing revealed in channel 1 and in channel 2, even if we fix the constraint $|\alpha_1|=|\alpha_2|$. For example, the lowest

minima of S_1^{inel} are found for $c > 0$ and $\Omega = 0$, but, every time $\Omega = 0$, the light in channel 2 is not squeezed as it can be proved that $S_2^{\text{inel}} \geq 1$ for every μ and every ϑ_2. Another example is the one of Fig. 10.3, which we already discussed. This figure gives the plots of the incoherent spectrum with the parameters which are optimal for squeezing in channel 2 and it shows that with these parameters the squeezing in channel 1 is far from optimality.

It is worth mentioning also the ideal case $|\alpha_1| = 1$, $\gamma = 1$, $k_{\text{d}} = \overline{n} = 0$, with $\Delta\omega = 0$, $\Omega = 0$, $\vartheta_1 = \pi/2$, $c = 1.2818$, $\varphi = 0$. Then we have an extremely visible squeezing in channel 1 (S_1^{inel} reaches 0.3183), there is no squeezing of the atomic a priori equilibrium state ($\text{AS}_{\eta_{\text{eq}}} = 0.0922$), while numerical simulations show that the a posteriori state $\rho(t)$ tends to become pure (as $|\alpha_1| = 1$) with $\text{AS}_{\rho(t)}$ stochastically moving between $-1/4$ and 0.

10.5.5.3 No Feedback

When $c = 0$, the two channels 1 and 2 are completely equivalent, apart from the possibility of different detection parameters α_1 and α_2. Being $c = 0$, each channel, taken singularly, is equivalent to the observed channel of Sect. 9.2; what is new here is the presence of the cross-correlations with their spectrum. In this case, the matrix A is given by Eq. (8.32), the equilibrium state by Eqs. (8.49), (8.50) and

$$\vec{t}_j = \left(1 + z_{\text{eq}}\right) \vec{u}(\vartheta_j) - \vec{x}_{\text{eq}} \cdot \vec{u}(\vartheta_j) \begin{pmatrix} x_{\text{eq}} \\ y_{\text{eq}} \\ 1 + z_{\text{eq}} \end{pmatrix}.$$

Note that we are in the case of mutually proportional detection operators, the situation of Sect. 4.4.1.4, Eq. (4.66): $\check{R}_j = \sqrt{\gamma}\alpha_j\sigma_-$, $j = 1, 2$. If $\vartheta_1 - \vartheta_2 = 0,\ \pi$, with a rotation, the output $(W_1,\ W_2)$ can be transformed in a new output with a component which reduces to pure noise. Otherwise, we have two effective channels, also after any rotation.

10.6 Control and Line Narrowing

After the first observation of squeezing, Gardiner predicted that stimulating a two-level atom with squeezed light would inhibit the phase decay of the atom. The squeezed light would break the equality between the transverse decay rates for the two quadratures of the atom and one decay rate could be made arbitrarily small, producing an observable narrow line in the spectrum of the atom's fluorescence light. This was seen as a "direct effect of squeezing" and thus as a measure of the squeezing of the incident light [15]. With our notation, the "transverse decay rates" are the decay rates of the x and y components of the Bloch vector of the a priori states and, so, they are a_{11} and a_{22}, at least when $a_{12} = a_{21} = 0$.

Later, Wiseman showed that this atomic line narrowing is not only characteristic of squeezed light, but also can be produced by immersing a two-level atom in

"in-loop squeezed" light [16]. The difference is that in the Gardiner case the other decay rate becomes larger, while in the Wiseman case it is left unchanged.

Actually, the same atomic line narrowing can be obtained stimulating a two-level atom even with non-squeezed light, that is with a coherent monochromatic laser in presence of a (Wiseman–Milburn) feedback scheme based on the (homodyne) detection of the fluorescence light, which is the case studied in this chapter. The presentation of this phenomenon is particularly clear in two special cases in which the matrix A turns out to be block-diagonal; the next two subsections are dedicated to such two cases.

It is interesting to note that the atomic master equation in [16] and our equation for a priori states, in the case $\Delta\omega = 0$, $\Omega = 0$, $\varphi = \pi/2$, $\vartheta_1 = \pi$, are equal, in spite that Wiseman's results refer to a different physical situation.

10.6.1 A First Case of Line Narrowing

The first way to have A in block-diagonal form is to take $\Omega = 0$; we ask also $a_{12} = a_{21}$, $\forall c$. As we have $0 < |\alpha_1| < 1$ and $\gamma > 0$, this condition is equivalent to $\Delta\omega = 0$ and $\vartheta_1 - \varphi = \pm\pi/2$, as one sees by using the expressions (10.25e), (10.25f) of a_{12} and a_{21}. By changing the signs to c and to the free angles, the two cases can be seen to be physically equivalent and we fix the upper sign. So, we have

$$\gamma > 0, \qquad 0 < |\alpha_1| < 1, \qquad \Omega = 0, \qquad \Delta\omega = 0, \qquad \varphi = \vartheta_1 - \frac{\pi}{2}\,. \tag{10.69}$$

Under these conditions we get that the matrix A is positive definite and block-diagonal:

$$A = \begin{pmatrix} B & 0 \\ 0 & a_{33} \end{pmatrix} \geq 0, \qquad B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \geq 0, \tag{10.70a}$$

$$a_{11} = \frac{\gamma}{2}\,k + 2\gamma c\,(c - |\alpha_1|)\cos^2\vartheta_1\,, \tag{10.70b}$$

$$a_{22} = \frac{\gamma}{2}\,k + 2\gamma c\,(c - |\alpha_1|)\sin^2\vartheta_1\,, \tag{10.70c}$$

$$a_{12} = a_{21} = 2\gamma c\,(c - |\alpha_1|)\sin\vartheta_1\cos\vartheta_1\,, \tag{10.70d}$$

$$a_{33} = \gamma\,(1 + 2\overline{n}) + 2\gamma c^2 - 2\gamma c|\alpha_1|. \tag{10.70e}$$

Recall that $k = 4k_{\mathrm{d}} + 2\overline{n} + 1$.

By direct computation we get the eigenvalues and the eigenvectors of B:

$$b_1 = \frac{\gamma}{2}\left[k - |\alpha_1|^2 + (2c - |\alpha_1|)^2\right] > 0, \qquad \vec{e}_1 = \begin{pmatrix} \cos\vartheta_1 \\ \sin\vartheta_1 \end{pmatrix}; \tag{10.71a}$$

$$b_2 = \frac{\gamma}{2}\,k > 0, \qquad \vec{e}_2 = \begin{pmatrix} \sin\vartheta_1 \\ -\cos\vartheta_1 \end{pmatrix}. \tag{10.71b}$$

10.6.1.1 Transverse Decay Rates and Eigenvalues of B

The matrix B gives the evolution of the x, y components of the a priori state, which in this case is decoupled from the evolution of the z component; moreover, B being symmetric and positive, these two eigenvalues represent the two transverse decay rates. Note that for $c = 0$ these two rates are equal and their value is $\gamma k/2$ ($\gamma/2$ for $\overline{n} = k_\mathrm{d} = 0$). When c is different from 0 and from $|\alpha_1|$, this equality is broken, b_2 maintains the original value and b_1 can become smaller or bigger. The smallest transverse decay rate is obtained for $c = |\alpha_1|/2$ and its value is $b_1 = \frac{\gamma}{2}\left(k - |\alpha_1|^2\right)$, which reduces to $b_1 = \frac{\gamma}{2}\left(1 - |\alpha_1|^2\right)$ for $\overline{n} = k_\mathrm{d} = 0$.

10.6.1.2 Equilibrium State and Global Squeezing

Again by direct computation we obtain the equilibrium state and the global squeezing parameters of the fluorescence light, which turn out to be all independent of ϑ_1 and ϑ_2:

$$x_\mathrm{eq} = 0, \qquad y_\mathrm{eq} = 0, \qquad z_\mathrm{eq} = \frac{-1 + 2c|\alpha_1|}{2\overline{n} + 2c^2 + 1 - 2c|\alpha_1|}, \tag{10.72}$$

$$\Pi_1 \equiv \Pi_1(\vartheta_1) = |\alpha_1|^2\left(1 + z_\mathrm{eq}\right) - 2c\,|\alpha_1|\,z_\mathrm{eq} = \frac{c\,|\alpha_1| + |\alpha_1|^2\left(\overline{n} - c^2\right)}{\overline{n} + c^2 + \frac{1}{2} - c|\alpha_1|}, \tag{10.73}$$

$$\Pi_2 \equiv \Pi_2(\vartheta_2) = |\alpha_2|^2\left(1 + z_\mathrm{eq}\right) = \frac{|\alpha_2|^2\left(\overline{n} + c^2\right)}{\overline{n} + c^2 + \frac{1}{2} - c|\alpha_1|} \geq 0. \tag{10.74}$$

10.6.1.3 Spectral Densities

The quantities appearing in the expressions of the spectral densities can be computed; we have in particular

$$\vec{t}_1 = \left(1 + z_\mathrm{eq} - 2\widetilde{c}z_\mathrm{eq}\right)\vec{u}(\vartheta_1), \qquad \vec{t}_2 = \left(1 + z_\mathrm{eq}\right)\vec{u}(\vartheta_2).$$

Then, we get the expressions of the spectra.

First of all, with these choices of the parameters all the elastic spectral densities vanish:

$$S_1^{\rm el}(\mu;\vartheta_1)=0,\qquad S_2^{\rm el}(\mu;\vartheta_2)=0,\qquad S_{12}^{\rm el}(\mu)=0.$$

Then, the inelastic spectral densities become:

$$S_1^{\rm inel}(\mu;\vartheta_1)=1+\Pi_1\,\frac{2\gamma b_1}{b_1^2+\mu^2}\,, \tag{10.75a}$$

$$S_2^{\rm inel}(\mu;\vartheta_2)=1+\Pi_2\left[\frac{2\gamma b_1}{b_1^2+\mu^2}\,\cos^2(\vartheta_1-\vartheta_2)+\frac{2\gamma b_2}{b_2^2+\mu^2}\,\sin^2(\vartheta_1-\vartheta_2)\right], \tag{10.75b}$$

$$\mathrm{Re}\,S_{12}^{\rm inel}(\mu)=\left(\frac{|\alpha_1|\,\Pi_2}{|\alpha_2|}+\frac{|\alpha_2|\,\Pi_1}{|\alpha_1|}\right)\frac{\gamma b_1}{b_1^2+\mu^2}\,\cos(\vartheta_1-\vartheta_2)\,, \tag{10.75c}$$

$$\mathrm{Im}\,S_{12}^{\rm inel}(\mu)=\left(\frac{|\alpha_1|\,\Pi_2}{|\alpha_2|}-\frac{|\alpha_2|\,\Pi_1}{|\alpha_1|}\right)\frac{\gamma\mu}{b_1^2+\mu^2}\,\cos(\vartheta_1-\vartheta_2)\,. \tag{10.75d}$$

All the spectra are linear combinations of white noise and Lorentzians of widths b_1 and b_2; the possible presence of squeezing is controlled by the signs of the coefficients.

10.6.1.4 Best Line Narrowing

The width b_2 is independent of the control parameters and we get the best line narrowing by minimising b_1. The smallest value of b_1 is reached when $c=|\alpha_1|/2$; so, we have

$$b_1=\frac{\gamma}{2}\left(k-|\alpha_1|^2\right),\qquad b_2=\frac{\gamma}{2}\,k. \tag{10.76}$$

The other relevant quantities are given by

$$z_{\rm eq}=-\frac{1-|\alpha_1|^2}{1+2\overline{n}-|\alpha_1|^2/2}\,,\qquad \frac{2\overline{n}}{1+2\overline{n}}<1+z_{\rm eq}<1\,, \tag{10.77}$$

$$\Pi_1=|\alpha_1|^2\,,\qquad \Pi_2=\frac{|\alpha_2|^2\left(4\overline{n}+|\alpha_1|^2\right)}{2+4\overline{n}-|\alpha_1|^2}=|\alpha_2|^2\left(1+z_{\rm eq}\right). \tag{10.78}$$

The case $\vartheta_1 = \vartheta_2$

When we have also $\vartheta_1 = \vartheta_2$, only the sharpest Lorentzian survives in the spectrum of the light in channel 2 and we have

$$S_1^{\text{inel}}(\mu;\vartheta_1) = 1 + |\alpha_1|^2 \frac{2\gamma b_1}{b_1^2 + \mu^2}\,, \tag{10.79a}$$

$$S_2^{\text{inel}}(\mu;\vartheta_1) = 1 + |\alpha_2|^2 \left(1 + z_{\text{eq}}\right) \frac{2\gamma b_1}{b_1^2 + \mu^2}\,, \tag{10.79b}$$

$$S_{12}^{\text{inel}}(\mu) = \overline{S_{21}^{\text{inel}}(\mu)} = \frac{|\alpha_1|\,|\alpha_2|\,\gamma}{b_1^2 + \mu^2}\left[2b_1 + z_{\text{eq}}\,(b_1 + \mathrm{i}\mu)\right]. \tag{10.79c}$$

Figures 10.6 (lines 2 and 3) and 10.7 give the spectra of the fluctuations of the light in the channels 1 and 2 and the real and imaginary parts of the spectrum of the cross-correlations. Both channels present positive Lorentzian peaks, but the cross correlations are strong, a fact which suggest to study the spectrum of some linear combination of the two outputs.

We choose to diagonalise the matrix of the spectra at $\mu = 0$. Let us denote by $\vec{w}^{\pm}$ the eigenvectors of the matrix $\begin{pmatrix} S_{11}^{\text{inel}}(0) & S_{12}^{\text{inel}}(0) \\ S_{21}^{\text{inel}}(0) & S_{22}^{\text{inel}}(0) \end{pmatrix}$ and by $S_{\pm}^{\text{inel}}(0)$ the eigenvalues (10.61); we have

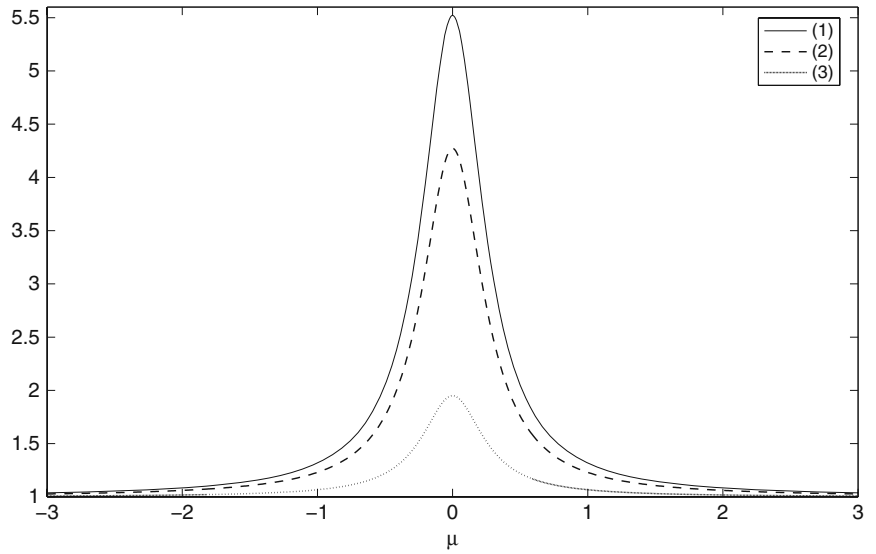

Fig. 10.6 $\Omega = 0$, $\Delta\omega = 0$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$, $\gamma = 1$, $\varphi = \vartheta_1 - \frac{\pi}{2}$, $c = |\alpha_1|/2$, $\vartheta_1 = \vartheta_2$, $\overline{n} = 0$, $k_{\text{d}} = 0$: (1) $S_{+}^{\text{inel}}(\mu)$; (2) $S_{11}^{\text{inel}}(\mu)$; (3) $S_{22}^{\text{inel}}(\mu)$

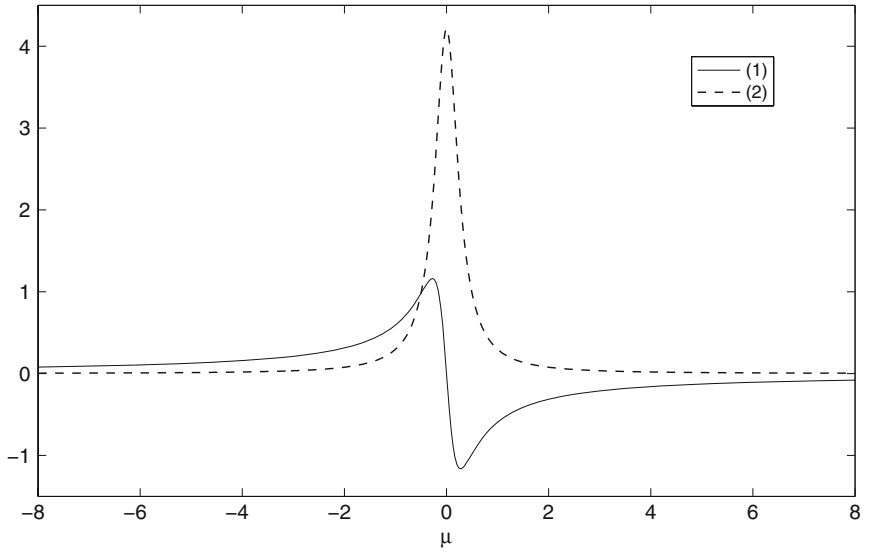

Fig. 10.7 The spectrum of cross-correlations for $\Omega = 0$, $\Delta\omega = 0$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$, $\gamma = 1$, $\varphi = \vartheta_1 - \frac{\pi}{2}$, $c = |\alpha_1|/2$, $\vartheta_1 = \vartheta_2$, $\overline{n} = 0$, $k_\mathrm{d} = 0$: (1) $\mathrm{Im}\, S_{12}^{\mathrm{inel}}(\mu)$; (2) $\mathrm{Re}\, S_{12}^{\mathrm{inel}}(\mu)$

$$S_{\pm}^{\mathrm{inel}}(0) = 1 + \frac{\gamma}{b_1}\Big\{|\alpha_1|^2 + |\alpha_2|^2\left(1 + z_{\mathrm{eq}}\right) \pm \sqrt{\left[|\alpha_1|^2 + |\alpha_2|^2\left(1 + z_{\mathrm{eq}}\right)\right]^2 + |\alpha_1|^2\,|\alpha_2|^2\, z_{\mathrm{eq}}^2}\Big\}, \quad (10.80a)$$

$$w_1^{\pm} = \frac{h}{\sqrt{h^2 + \ell_{\pm}^2}}, \qquad w_2^{\pm} = \frac{\ell_{\pm}}{\sqrt{h^2 + \ell_{\pm}^2}}, \qquad h = |\alpha_1|\,|\alpha_2|\left(2 + z_{\mathrm{eq}}\right), \quad (10.80b)$$

$$\ell_{\pm} = |\alpha_2|^2\left(1 + z_{\mathrm{eq}}\right) - |\alpha_1|^2 \pm \sqrt{\left[|\alpha_1|^2 + |\alpha_2|^2\left(1 + z_{\mathrm{eq}}\right)\right]^2 + |\alpha_1|^2\,|\alpha_2|^2\, z_{\mathrm{eq}}^2}. \quad (10.80c)$$

Then, we introduce the spectra related to these two eigenvectors:

$$S_{\pm}^{\mathrm{inel}}(\mu) := \sum_{i,j=1}^{2} w_i^{\pm} S_{ij}^{\mathrm{inel}}(\mu) w_j^{\pm}. \quad (10.81)$$

Figure 10.6 (line 1) gives the plot $S_{+}^{\mathrm{inel}}(\mu)$, which presents a positive narrow pick, as $S_{11}^{\mathrm{inel}}(\mu)$ and $S_{22}^{\mathrm{inel}}(\mu)$. On the contrary $S_{-}^{\mathrm{inel}}(\mu)$, plotted in Fig. 10.8, presents a sub-shot noise narrow pick.

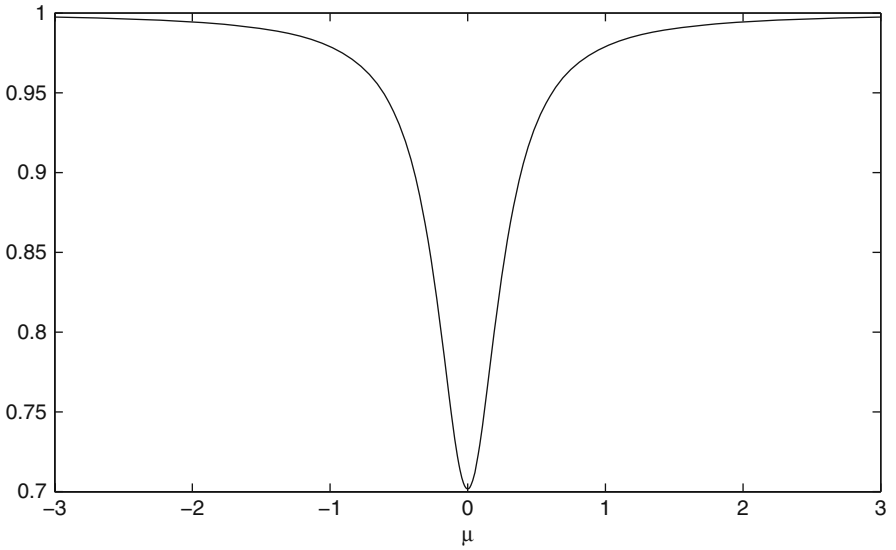

Fig. 10.8 $\Omega = 0$, $\Delta\omega = 0$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$, $\gamma = 1$, $\varphi = \vartheta_1 - \frac{\pi}{2}$, $c = |\alpha_1|/2$, $\vartheta_1 = \vartheta_2$, $\overline{n} = 0$, $k_\mathrm{d} = 0$: $S_-^{\mathrm{inel}}(\mu)$

We have shown that it is possible to have the line narrowing by stimulating the atom with coherent, non-squeezed light (but with feedback). However, Fig. 10.8 shows that, as a matter of fact, sub-shot noise fluctuations are present, at least in the emitted light, together with line narrowing.

Diagonalising the matrix of the spectra at a different value of μ can give rise to very different forms for the spectra. Let us choose the position of the half-height of the Lorentzian, $\mu = b_1$. We denote by $\vec{v}^{\pm}$ the eigenvectors of the matrix $\begin{pmatrix} S_{11}^{\mathrm{inel}}(b_1) & S_{12}^{\mathrm{inel}}(b_1) \\ S_{21}^{\mathrm{inel}}(b_1) & S_{22}^{\mathrm{inel}}(b_1) \end{pmatrix}$ and by $\widetilde{S}_{\pm}^{\mathrm{inel}}(b_1)$ the eigenvalues; we have

$$\widetilde{S}_{\pm}^{\mathrm{inel}}(b_1) = 1 + \frac{\gamma}{2b_1}\Bigg\{|\alpha_1|^2 + |\alpha_2|^2\left(1 + z_{\mathrm{eq}}\right) \pm \sqrt{\left[|\alpha_1|^2 + |\alpha_2|^2\left(1 + z_{\mathrm{eq}}\right)\right]^2 + 2\,|\alpha_1|^2\,|\alpha_2|^2\,z_{\mathrm{eq}}^2}\Bigg\}, \quad (10.82a)$$

$$v_1^{\pm} = \frac{g}{\sqrt{|g|^2 + l_{\pm}^2}}, \qquad v_2^{\pm} = \frac{l_{\pm}}{\sqrt{|g|^2 + l_{\pm}^2}}, \qquad g = |\alpha_1|\,|\alpha_2|\left(1 + z_{\mathrm{eq}}\,\frac{1+\mathrm{i}}{2}\right), \quad (10.82b)$$

$$l_{\pm} = \frac{1}{2}\Bigg\{|\alpha_2|^2\left(1 + z_{\mathrm{eq}}\right) - |\alpha_1|^2 \pm \sqrt{\left[|\alpha_1|^2 + |\alpha_2|^2\left(1 + z_{\mathrm{eq}}\right)\right]^2 + 2\,|\alpha_1|^2\,|\alpha_2|^2\,z_{\mathrm{eq}}^2}\Bigg\}. \quad (10.82c)$$

Then, we introduce the spectra related to these two eigenvectors:

$$\widetilde{S}_{\pm}^{\text{inel}}(\mu) := \sum_{i,j=1}^{2} v_i^{\pm} S_{ij}^{\text{inel}}(\mu) v_j^{\pm} . \tag{10.83}$$

Figure 10.9 gives the two spectra $\widetilde{S}_{\pm}^{\text{inel}}(\mu)$ which are clearly asymmetric and present a narrow pick and a sub-shot noise well.

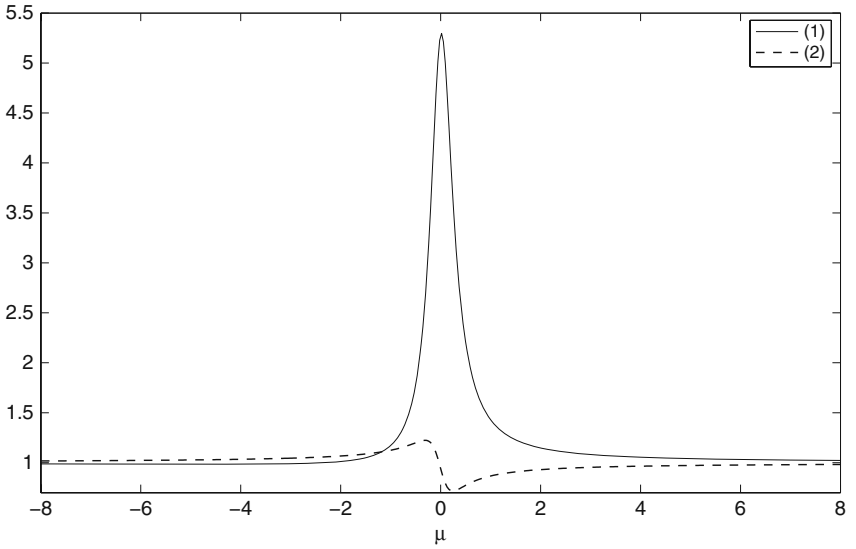

Fig. 10.9 $\Omega = 0$, $\Delta\omega = 0$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$, $\gamma = 1$, $\varphi = \vartheta_1 - \frac{\pi}{2}$, $c = |\alpha_1|/2$, $\vartheta_1 = \vartheta_2$, $\overline{n} = 0$, $k_{\text{d}} = 0$: (1) $\widetilde{S}_{+}^{\text{inel}}(\mu)$; (2) $\widetilde{S}_{-}^{\text{inel}}(\mu)$

10.6.1.5 Good Squeezing

Being $\Pi_2 \geq 0$, there is no squeezing in channel 2, while the in-loop light is squeezed when $\Pi_1 < 0$, which happens for

$$c \notin \left[\frac{1}{2} - \sqrt{\frac{1}{4} + |\alpha_1|^2 \overline{n}},\ \frac{1}{2} + \sqrt{\frac{1}{4} + |\alpha_1|^2 \overline{n}} \right],$$

which becomes $c \notin [0, 1]$ for $\overline{n} = 0$. Good in-loop squeezing is obtained for c near $-\frac{|\alpha_1|}{2}$, but if we take $\varphi = \vartheta_1 + \pi/2$ the good squeezing is for positive c. Indeed, Figure 10.2 shows good squeezing for $\Omega = 0$, $\Delta\omega = 0$, $\varphi = 0$, $\vartheta_1 = -\pi/2$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$: line (1) with $\overline{n} = 0$, $k_{\text{d}} = 0$, $c = 0.2936$ and line (5) with $\overline{n} = 0.01$, $k_{\text{d}} = 0.05$, $c = 0.3255$.

If we take exactly

$$c = -\frac{|\alpha_1|}{2} \tag{10.84}$$

we get

$$b_1 = \frac{\gamma}{2}\left(k + 3\,|\alpha_1|^2\right), \qquad b_2 = \frac{\gamma}{2}\,k, \tag{10.85a}$$

$$x_{\rm eq} = 0, \qquad y_{\rm eq} = 0, \qquad z_{\rm eq} = -\frac{1 + |\alpha_1|^2}{1 + 2\overline{n} + 3|\alpha_1|^2/2}, \tag{10.85b}$$

$$\Pi_1 \equiv \Pi_1(\vartheta_1) = |\alpha_1|^2\left(1 + 2z_{\rm eq}\right) = \frac{|\alpha_1|^2\left(2\overline{n} - 1 - |\alpha_1|^2/2\right)}{1 + 2\overline{n} + 3|\alpha_1|^2/2}\,, \tag{10.85c}$$

$$\Pi_2 \equiv \Pi_2(\vartheta_2) = |\alpha_2|^2\left(1 + z_{\rm eq}\right) = \frac{|\alpha_2|^2\left(2\overline{n} + |\alpha_1|^2/2\right)}{1 + 2\overline{n} + 3|\alpha_1|^2/2}\,, \tag{10.85d}$$

$$\frac{|\alpha_1|\,\Pi_2}{|\alpha_2|} + \frac{|\alpha_2|\,\Pi_1}{|\alpha_1|} = \frac{|\alpha_1|\,|\alpha_2|\,(4\overline{n} - 1)}{1 + 2\overline{n} + 3|\alpha_1|^2/2}\,, \tag{10.85e}$$

$$\frac{|\alpha_1|\,\Pi_2}{|\alpha_2|} - \frac{|\alpha_2|\,\Pi_1}{|\alpha_1|} = \frac{|\alpha_1|\,|\alpha_2|\left(1 + |\alpha_1|^2\right)}{1 + 2\overline{n} + 3|\alpha_1|^2/2}\,. \tag{10.85f}$$

The inelastic spectrum is given by Eqs. (10.75). For $\gamma = 1$, $|\alpha_1|^2 = |\alpha_2|^2 = 0.45$ (which gives $c \simeq -0.3354$), $\overline{n} = 0$, $k_{\rm d} = 0$, we obtain $S_{11}^{\rm inel}(0) \simeq 0.4398$, which is comparable with the minimum of 0.4354 for line (1) in Fig. 10.2.

Note that for $\Omega = 0$ the presence of sub-shot noise in channel 1 is connected to line broadening, as shown by the expression of b_1 in (10.85a).

10.6.2 A Second Case of Line Narrowing

In order to have the dynamical matrix A of block-diagonal form for all possible values of c and Ω, we have to ask $a_{12} = a_{21} = 0$, $\forall c$. Under $0 < |\alpha_1| < 1$ and $\gamma > 0$, this condition is equivalent to $\Delta\omega = 0$, $\varphi = 0$, $\vartheta_1 = \pm\pi/2$ or to $\Delta\omega = 0$, $\varphi = \pi/2$, $\vartheta_1 = 0,\ \pi$, as one see immediately from Eqs. (10.25e), (10.25f).

Under this condition the matrix A is block-diagonal with a_{11} isolated from the block (2,3); this isolated element is responsible for a Lorentzian peak of width a_{11}

in the inelastic spectrum. With the first choice of the angles we get $a_{11} = \frac{\gamma}{2}\, k$, while the second choice gives $a_{11} = \frac{\gamma}{2}\, k + 2\gamma c^2 \pm 2\gamma c\, |\alpha_1|$. So, only with the second choice we can control the linewidth of the peak. Moreover, by changing the sign to c one sees that the two possibilities for ϑ_1 are physically equivalent and we fix the case $\vartheta_1 = \pi$. So, we take

$$\gamma > 0, \qquad 0 < |\alpha_1| < 1, \qquad \Delta\omega = 0, \qquad \varphi = \frac{\pi}{2}\,, \qquad \vartheta_1 = \pi\,. \tag{10.86}$$

Under these conditions we get (recall that $k = 4k_{\rm d} + 2\overline{n} + 1$)

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & C \end{pmatrix}, \qquad C = \begin{pmatrix} a_{22} & \Omega \\ -\Omega & a_{33} \end{pmatrix}, \tag{10.87a}$$

$$a_{11} = \frac{\gamma}{2}\, k + 2\gamma c^2 - 2\gamma c\, |\alpha_1|\,, \qquad a_{22} = \frac{\gamma}{2}\, k, \tag{10.87b}$$

$$a_{33} = \gamma\,(1 + 2\overline{n}) + 2\gamma c^2 - 2\gamma c|\alpha_1|, \tag{10.87c}$$

$$x_{\rm eq} = 0, \qquad y_{\rm eq} = \frac{\gamma\Omega\,(1 - 2c\,|\alpha_1|)}{a_{22}a_{33} + \Omega^2}\,, \qquad z_{\rm eq} = -\frac{\gamma a_{22}\,(1 - 2c\,|\alpha_1|)}{a_{22}a_{33} + \Omega^2}\,, \tag{10.87d}$$

$$\Pi_2(\vartheta_2) = |\alpha_2|^2 \left(1 + z_{\rm eq} - y_{\rm eq}^2 \sin^2\vartheta_2\right), \tag{10.87e}$$

$$\Pi_1(\pi) = |\alpha_1|^2 \left(1 + z_{\rm eq}\right) - 2c\,|\alpha_1|\, z_{\rm eq}. \tag{10.87f}$$

Then, the spectral densities become

$$S_1^{\rm el}(\mu;\pi) = 0, \qquad S_2^{\rm el}(\mu;\vartheta_2) = 2\gamma\,|\alpha_2|^2\, y_{\rm eq}^2 \sin^2\vartheta_2\, \delta(\mu), \tag{10.88a}$$

$$S_1^{\rm inel}(\mu;\pi) = 1 + |\alpha_1| \left[|\alpha_1| \left(1 + z_{\rm eq}\right) - 2cz_{\rm eq}\right] \frac{2\gamma a_{11}}{a_{11}^2 + \mu^2}\,, \tag{10.88b}$$

$$\begin{aligned} S_2^{\rm inel}(\mu;\vartheta_2) = 1 + |\alpha_2|^2 \Bigg\{ & \left(1 + z_{\rm eq}\right) \frac{2\gamma a_{11}}{a_{11}^2 + \mu^2}\, \cos^2\vartheta_2 + \sin^2\vartheta_2 \\ & \times \left[\left(1 + z_{\rm eq} - y_{\rm eq}^2\right) \left(\frac{2\gamma A}{A^2 + \mu^2}\right)_{22} - \left(1 + z_{\rm eq}\right) \left(\frac{2\gamma A}{A^2 + \mu^2}\right)_{23} \right] \Bigg\}, \end{aligned} \tag{10.88c}$$

$$S_{12}^{\text{inel}}(\mu) = -\frac{2\gamma\,|\alpha_2|\cos\vartheta_2}{a_{11}^2+\mu^2}\left\{\left[|\alpha_1|\left(1+z_{\text{eq}}\right)-cz_{\text{eq}}\right]a_{11}+\mathrm{i}\mu c z_{\text{eq}}\right\}. \tag{10.88d}$$

10.6.2.1 The Subcase with $c = |\alpha_1|/2$, $\vartheta_2 = \pi$

By taking $c = |\alpha_1|/2$ we obtain the minimal value for the transverse decay rate a_{11}, while for $\vartheta_2 = \pi$ only the sharpest Lorentzian survives in channel 2. Now, we get

$$a_{11} = \frac{\gamma}{2}\left(k-|\alpha_1|^2\right), \qquad a_{22} = \frac{\gamma}{2}k, \qquad a_{33} = \gamma\left(1+2\overline{n}-\frac{|\alpha_1|^2}{2}\right), \tag{10.89a}$$

$$x_{\text{eq}} = 0, \qquad y_{\text{eq}} = \frac{\gamma\Omega\left(1-|\alpha_1|^2\right)}{a_{22}a_{33}+\Omega^2}, \qquad z_{\text{eq}} = -\frac{\gamma a_{22}\left(1-|\alpha_1|^2\right)}{a_{22}a_{33}+\Omega^2}, \tag{10.89b}$$

$$\Pi_1(\pi) = |\alpha_1|^2, \qquad \Pi_2(\pi) = |\alpha_2|^2\left(1+z_{\text{eq}}\right), \tag{10.89c}$$

$$S_1^{\text{el}}(\mu;\pi) = 0, \qquad S_2^{\text{el}}(\mu;\pi) = 0, \tag{10.90a}$$

$$S_1^{\text{inel}}(\mu;\pi) = 1+|\alpha_1|^2\,\frac{2\gamma a_{11}}{a_{11}^2+\mu^2}, \tag{10.90b}$$

$$S_2^{\text{inel}}(\mu;\pi) = 1+|\alpha_2|^2\left(1+z_{\text{eq}}\right)\frac{2\gamma a_{11}}{a_{11}^2+\mu^2}, \tag{10.90c}$$

$$S_{12}^{\text{inel}}(\mu) = \frac{\gamma\,|\alpha_1|\,|\alpha_2|}{a_{11}^2+\mu^2}\left[\left(2+z_{\text{eq}}\right)a_{11}+\mathrm{i}\mu z_{\text{eq}}\right]. \tag{10.90d}$$

Note that $\frac{|\alpha_1|^2}{2-|\alpha_1|^2} \leq 1+z_{\text{eq}} < 1$ and that the supremum of this quantity is 1, which is reached for $\Omega^2 \to +\infty$.

One of the eigenvalues of the 2×2 matrix $\left(S_{ij}^{\text{inel}}(0)\right)$ is slightly below 1; the case of $\left(S_{ij}^{\text{inel}}(a_{11})\right)$ is similar. This means that there exist linear combinations of the two outputs whose spectra go slightly below the shot noise, as in the case of Sect. 10.6.1.

10.7 Summary

We summarise only what concerns the two level atom, not the general introduction of the feedback. We give only quantities in the rotating frame.

10.7.1 Atomic and Measurement Quantities

- Forward channel, the channel of the lost light and of the stimulating laser: channel 0.
- Side channel 1, the channel of the detected, in-loop light: channel 1.
- Side channel 2, the channel of the free, detectable light: channel 2.
- Detection operators: $\check{R}_1 = \sqrt{\gamma}\,(\alpha_1\,\sigma_- - \mathrm{i}c\,\sigma_\varphi)$, $\check{R}_2 = \sqrt{\gamma}\,\alpha_2\,\sigma_-$.
- Coefficient of the thermal disturbance: $\overline{n} \geq 0$.
- Coefficient of the dephasing disturbance: $k_\mathrm{d} \geq 0$.
- Resonance frequency of the atom: $\omega_0 > 0$.
- Frequency of the stimulating laser and of the local oscillator: $\omega > 0$.
- Detuning: $\Delta\omega = \omega_0 - \omega$.
- Rabi frequency: $\Omega \geq 0$.
- Natural linewidth: $\gamma > 0$.
- Measurement phases: $\vartheta_i = \arg\alpha_i \in (-\pi, \pi]$, $i = 1, 2$.
- Proportions of light in the two side channels and in the forward channel: $|\alpha_0|^2$, $|\alpha_1|^2$ and $|\alpha_2|^2$ with $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1$, $0 < |\alpha_1|^2 < 1$.
- Feedback control parameter: $c \in \mathbb{R}$.
- Feedback phase: $\varphi \in [0, \pi)$.
- $\sigma_\theta = \mathrm{e}^{\mathrm{i}\theta}\sigma_- + \mathrm{e}^{-\mathrm{i}\theta}\sigma_+ = \vec{u}(\theta)\cdot\vec{\sigma}$; $\vec{u}(\theta) = \big(\cos\theta\ \sin\theta\ 0\big)^\mathrm{T}$.
- Dynamical matrix: $A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & \Omega \\ 0 & -\Omega & a_{33} \end{pmatrix}$, $\det A > 0$, $k = 2\overline{n} + 1 + 4k_\mathrm{d}$,
  - $a_{11} = \frac{\gamma}{2}\,k + 2\gamma c^2 \sin^2\varphi + 2c\gamma|\alpha_1|\sin\varphi\cos\vartheta_1 \geq 0$,
  - $a_{22} = \frac{\gamma}{2}\,k + 2\gamma c^2 \cos^2\varphi - 2c\gamma|\alpha_1|\cos\varphi\sin\vartheta_1 \geq 0$,
  - $a_{33} = \gamma\,(1 + 2\overline{n}) + 2\gamma c^2 - 2c\gamma|\alpha_1|\sin(\vartheta_1 - \varphi) \geq 2\gamma\overline{n} + \frac{\gamma}{2} > 0$,
  - $a_{12} = \Delta\omega + 2c\gamma|\alpha_1|\sin\varphi\sin\vartheta_1 - \gamma c^2\sin 2\varphi$,
  - $a_{21} = -\Delta\omega - 2c\gamma|\alpha_1|\cos\varphi\cos\vartheta_1 - \gamma c^2\sin 2\varphi$,
  - $a_{11} + a_{22} = a_{33} + 4\gamma k_\mathrm{d}$.
- Equilibrium state: $\eta_\mathrm{eq} := \frac{1}{2}\big(\mathbb{1} + \vec{x}_\mathrm{eq}\cdot\vec{\sigma}\big)$, $\vec{x}_\mathrm{eq} := \frac{\gamma}{A}\begin{pmatrix} 0 \\ 0 \\ 2c|\alpha_1|\sin(\vartheta_1 - \varphi) - 1 \end{pmatrix}$.
- Sign of z_eq: $z_\mathrm{eq} > 0$ iff $2c|\alpha_1|\sin(\vartheta_1 - \varphi) > 1$.

10.7.2 Homodyne Spectral Densities

We have three homodyne spectra: the spectrum of the in-loop light in channel 1, the spectrum of the free light in channel 2 and the spectrum of the cross-correlations.

- $S_{ij}(\mu) = S_{ij}^\mathrm{el}(\mu) + S_{ij}^\mathrm{inel}(\mu)$, $i, j = 1, 2$. When $c = 0$, S_{ii}, S_{ii}^el and S_{ii}^inel become identical to the spectral densities S_hom, $S_\mathrm{hom}^\mathrm{el}$ and $S_\mathrm{hom}^\mathrm{inel}$ of the previous chapter.
- Normalised shot noise level: 1.
- Elastic or coherent densities:

$$S_{ij}^{\mathrm{el}}(\mu) = 2\pi m_i m_j \delta(\mu), \qquad m_i = \sqrt{\gamma}\,|\alpha_i|\left(x_{\mathrm{eq}}\cos\vartheta_i + y_{\mathrm{eq}}\sin\vartheta_i\right).$$

- Inelastic or incoherent spectrum for channel 1, the in-loop light:

$$\begin{aligned} S_{11}^{\mathrm{inel}}(\mu) \equiv S_1^{\mathrm{inel}}(\mu;\vartheta_1) &= 1 + 2\gamma\,|\alpha_1|^2 \int_0^{+\infty} \cos\mu t\,\mathrm{Tr}\left\{\sigma_{\vartheta_1} \mathrm{e}^{\check{\mathcal{L}}t}[\tau_1]\right\}\mathrm{d}t \\ &= 1 + 2\,|\alpha_1|^2 \left(\frac{\gamma A}{A^2+\mu^2}\,\vec{t}_1\right)\cdot\vec{u}(\vartheta_1), \end{aligned}$$

$$\tau_1 = \mathrm{e}^{\mathrm{i}\vartheta_1}\tau + \mathrm{e}^{-\mathrm{i}\vartheta_1}\tau^* + \mathrm{i}\widetilde{c}[\eta_{\mathrm{eq}},\,\sigma_\varphi], \qquad \widetilde{c} = \begin{cases} \frac{c}{|\alpha_1|}, & \text{if } \alpha_1 \neq 0, \\ 0, & \text{if } \alpha_1 = 0, \end{cases}$$

$$\vec{t}_1 = \mathrm{Tr}\{\tau_1\vec{\sigma}\} = \begin{pmatrix} \left(1 + z_{\mathrm{eq}} - x_{\mathrm{eq}}^2\right)\cos\vartheta_1 - x_{\mathrm{eq}}y_{\mathrm{eq}}\sin\vartheta_1 \\ \left(1 + z_{\mathrm{eq}} - y_{\mathrm{eq}}^2\right)\sin\vartheta_1 - x_{\mathrm{eq}}y_{\mathrm{eq}}\cos\vartheta_1 \\ -\left(1+z_{\mathrm{eq}}\right)\left(x_{\mathrm{eq}}\cos\vartheta_1 + y_{\mathrm{eq}}\sin\vartheta_1\right) \end{pmatrix} + 2\widetilde{c}\,\vec{u}(\varphi)\times\vec{x}_{\mathrm{eq}}\,.$$

- Inelastic or incoherent spectrum for channel 2, the free-space light:

$$\begin{aligned} S_{22}^{\mathrm{inel}}(\mu) \equiv S_2^{\mathrm{inel}}(\mu;\vartheta_2) &= 1 + 2\gamma\,|\alpha_2|^2 \int_0^{+\infty} \cos\mu t\,\mathrm{Tr}\left\{\sigma_{\vartheta_2} \mathrm{e}^{\check{\mathcal{L}}t}[\tau_2]\right\}\mathrm{d}t \\ &= 1 + 2\,|\alpha_2|^2 \left(\frac{\gamma A}{A^2+\mu^2}\,\vec{t}_2\right)\cdot\vec{u}(\vartheta_2), \end{aligned}$$

$$\tau_2 = \mathrm{e}^{\mathrm{i}\vartheta_2}\tau + \mathrm{e}^{-\mathrm{i}\vartheta_2}\tau^*,$$

$$\vec{t}_2 = \mathrm{Tr}\{\tau_2\vec{\sigma}\} = \begin{pmatrix} \left(1 + z_{\mathrm{eq}} - x_{\mathrm{eq}}^2\right)\cos\vartheta_2 - x_{\mathrm{eq}}y_{\mathrm{eq}}\sin\vartheta_2 \\ \left(1 + z_{\mathrm{eq}} - y_{\mathrm{eq}}^2\right)\sin\vartheta_2 - x_{\mathrm{eq}}y_{\mathrm{eq}}\cos\vartheta_2 \\ -\left(1+z_{\mathrm{eq}}\right)\left(x_{\mathrm{eq}}\cos\vartheta_2 + y_{\mathrm{eq}}\sin\vartheta_2\right) \end{pmatrix}.$$

- Spectrum of the inelastic cross-correlations:

$$\mathrm{Re}\,S_{12}^{\mathrm{inel}}(\mu) = \vec{u}(\vartheta_1)\cdot\left(\frac{\gamma\,|\alpha_1|\,|\alpha_2|\,A}{A^2+\mu^2}\,\vec{t}_2\right) + \vec{u}(\vartheta_2)\cdot\left(\frac{\gamma\,|\alpha_1|\,|\alpha_2|\,A}{A^2+\mu^2}\,\vec{t}_1\right),$$

$$\mathrm{Im}\,S_{12}^{\mathrm{inel}}(\mu) = \vec{u}(\vartheta_1)\cdot\left(\frac{\gamma\,|\alpha_1|\,|\alpha_2|\,\mu}{A^2+\mu^2}\,\vec{t}_2\right) - \vec{u}(\vartheta_2)\cdot\left(\frac{\gamma\,|\alpha_1|\,|\alpha_2|\,\mu}{A^2+\mu^2}\,\vec{t}_1\right).$$

- Positivity: $\frac{S_{11}^{\text{inel}}(\mu)+S_{22}^{\text{inel}}(\mu)}{2} \pm \sqrt{\left(\frac{S_{11}^{\text{inel}}(\mu)-S_{22}^{\text{inel}}(\mu)}{2}\right)^2 + \left|S_{12}^{\text{inel}}(\mu)\right|^2} \geq 0.$
- Intensity:

$$\Pi_k(\vartheta_k) = \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty}\left[S_k^{\text{inel}}(\mu;\vartheta_k) - 1\right]\mathrm{d}\mu = |\alpha_k|^2\,\vec{t}_k\cdot\vec{u}(\vartheta_k). \tag{10.91}$$

- Global squeezing parameter in channel 2:

$$\Sigma_2 = \min_{\vartheta_2}\Pi_2(\vartheta_2) = |\alpha_2|^2\left[\mathrm{AS}_{\eta_{\text{eq}}} + z_{\text{eq}} + \left|z_{\text{eq}}\right|\right]; \tag{10.92}$$

  in the case with feedback the direct proportionality between the global squeezing parameter of the light and the atomic squeezing parameter is lost, because z_{eq} can be positive.

10.7.3 Important Features of the Homodyne Spectral Densities

- Tunable squeezing: the parameters can be chosen to have squeezing around a preassigned value of μ, see Figs. 10.2 and 10.4. The minima are symmetric with respect to $\mu = 0$.
- In channel 1 it is possible to have spectra with big shot-noise wells, see Fig. 10.2.
- Feedback can enhance the squeezing of the light in channel 2, see Fig. 10.4.
- There is no evidence of particular connections between squeezing in the two channels, see Fig. 10.3
- The Heisenberg-type relations hold for the light in channel 2, see Fig. 10.5.
- Cross-correlations can be strong, see Fig. 10.7.
- Feedback can produce line narrowing, see Fig. 10.6. We say that there is line narrowing when the spectrum presents a peak of width less than the standard width γk.
- Cross-correlation can connect the line narrowing and the presence of sub-shot noise, see Figs. 10.8 and 10.9.

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Appendix A
Ordinary SDEs

We collect in this appendix some notions and theorems in Itô stochastic calculus and SDEs needed in our presentation of the theory of quantum open systems and continuous measurements. We use as sources the books [1–3], but one can find these notions in any good book on SDEs.

In probability theory it is usual to consider only real processes. However, we want to apply stochastic calculus to quantum mechanics, where complex Hilbert spaces enter into play. Therefore, we shall present directly SDEs for complex processes; with respect to our sources [1–3], this nearly amounts to a mere change of notations.

A.1 Probability Spaces and Random Variables

Let us start by recalling the terminology and a few notions of probability theory.

A.1.1 Probability Spaces

A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F})$ is a measurable space and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. The set Ω consists of all the possible results ω of a classical random experiment and it is called the *sample space*. The set $\mathcal{F}$ is called the *space of the events* and it consists of subsets A of Ω, each A describing an event which occurs every time the experiment produces an outcome $\omega \in A$. Before the experiment, $\mathbb{P}(A)$ evaluates the probability that an event A occurs. After the experiment, a single ω has been produced, all the events A with $\omega \in A$ have occurred and the others have not.

A.1.1.1 Measurable Spaces

A *measurable space* is a couple $(E, \mathcal{E})$, where E is a non-empty set and $\mathcal{E}$ is a σ-algebra of subsets of E. A *σ-algebra*, called also *σ-field*, is a family of subsets of E such that

(a) $E \in \mathcal{E}$,
(b) $A \in \mathcal{E} \Rightarrow E \setminus A \in \mathcal{E}$,
(c) $\{A_j, j = 1, \ldots\} \subset \mathcal{E} \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathcal{E}$.

If E is the total set, $E \setminus A$ is the complement of A and it is also denoted by A^{c}.

Let us recall that the intersection of σ-algebras is a σ-algebra. If $\mathcal{A}$ is a collection of subsets of A, we denote by $\sigma(\mathcal{A})$ the σ-algebra generated by $\mathcal{A}$, i.e. the smallest σ-algebra containing all the sets in $\mathcal{A}$. When more classes of sets are involved, say $\mathcal{A}$, $\mathcal{B}$, ..., we use the notations $\mathcal{A} \vee \mathcal{B} \vee \cdots \equiv \sigma(\mathcal{A}, \mathcal{B}, \ldots)$ for the σ-algebra generated by the union $\mathcal{A} \cup \mathcal{B} \cup \ldots$.

Borel σ-Algebra

If E is a topological space, the Borel σ-algebra $\mathcal{B}(E)$ is the σ-algebra generated by the open sets. We always understand that the spaces $\mathbb{C}^d$ or $\mathbb{R}^d$ are equipped with the Borel σ-algebras $\mathcal{B}(\mathbb{C}^d)$ or $\mathcal{B}(\mathbb{R}^d)$. If A is a subset of $\mathbb{C}^d$, we denote by $\mathcal{B}(A)$ the σ-algebra $A \cap \mathcal{B}(\mathbb{C}^d)$, i.e. all the subsets of A of the form $A \cap C$ for some $C \in \mathcal{B}(\mathbb{C}^d)$. We call $\mathcal{B}(A)$ again the Borel σ-algebra of A.

Product σ-Algebra

Let $(E, \mathcal{E})$ and $(E', \mathcal{E}')$ be two measurable spaces. By $\mathcal{E} \otimes \mathcal{E}'$ we mean the σ-algebra on $E \times E'$ generated by the rectangles: $\mathcal{E} \otimes \mathcal{E}' := \sigma\big(A \times B : \ A \in \mathcal{E}, \ B \in \mathcal{E}'\big)$.

Measurable Function

Let $(E, \mathcal{E})$ and $(E', \mathcal{E}')$ be two measurable spaces. A function $f : E \to E'$ is said to be *measurable*, or $\mathcal{E}/\mathcal{E}'$-measurable, if the inverse image of any measurable set is measurable, i.e. $A \in \mathcal{E}' \Rightarrow f^{-1}(A) \in \mathcal{E}$. To declare the measurability of a function, we also write $f : (E, \mathcal{E}) \to (E', \mathcal{E}')$.

Theorem A.1. *Let $(E, \mathcal{E})$ and $(E^{'}, \mathcal{E}^{'})$ be measurable spaces.*

1. *If $f : (E, \mathcal{E}) \to (E^{'}, \mathcal{E}^{'})$ and $g : (E^{'}, \mathcal{E}^{'}) \to (E^{''}, \mathcal{E}^{''})$, then $g \circ f : (E, \mathcal{E}) \to (E^{''}, \mathcal{E}^{''})$.*
2. *If $f : E \to E'$, $\mathcal{A}' \subset \mathcal{E}'$ with $\sigma(\mathcal{A}') = \mathcal{E}'$, and $f^{-1}(A') \in \mathcal{E}$, $\forall A' \in \mathcal{A}'$, then f is $\mathcal{E}/\mathcal{E}'$-measurable.*

Borel Isomorphic Spaces

Let $(E, \mathcal{E})$ and $(E^{'}, \mathcal{E}^{'})$ be measurable spaces. An invertible function f from E to E' is called *bimeasurable* if both f and its inverse are measurable. Two measurable spaces are said to be *Borel isomorphic* (B-isomorphic) if there exists a bimeasurable map from one of the spaces to the other. The Borel isomorphism is an equivalence relation.

A.1.1.2 Standard Measurable Spaces

The notion of standard measurable space (or standard Borel space) is needed for the composition of instruments (Sect. B.4.3), for the dilation of instruments (Theorem B.10), for the separability of L^p spaces (Remark A.4), for Kolmogorov's extension theorem [4, p. 18] and the related problems of extension of probabilities discussed in Sect. A.5.5 [4].

Definition A.2. *A measurable space is called* standard *(also* standard Borel space*) if it is B-isomorphic to a Borel subset of* $\mathbb{R}$ *with its Borel* σ*-algebra.*

Theorem A.3.

1. *Every standard measurable space is B-isomorphic to a compact subset of* $\mathbb{R}$.
2. *Every standard measurable space is B-isomorphic to one of the following standard measurable spaces:* $[0, 1]$, $\mathbb{N}$, $\{1, 2, \ldots, n\}$ $(n = 1, 2, \ldots)$.
3. *Let E be a Polish space (complete, separable, metric space). Then,* $\big(E, \mathcal{B}(E)\big)$ *is standard.*
4. *Let* $(E, \mathcal{E})$ *and* $(E', \mathcal{E}')$ *be two standard measurable spaces. Then, the measurable space* $(E \times E',\ \mathcal{E} \otimes \mathcal{E}')$ *is standard. This statement holds also for the product of countably many standard measurable spaces.*

A.1.1.3 Probability Measures

A *probability measure* $\mathbb{P}$ on $\mathcal{F}$, or on $(\Omega, \mathcal{F})$ if we want to underline also the sample space, is a non-negative, σ-additive measure with mass 1, i.e.

(a) $\mathbb{P} : \mathcal{F} \to [0, 1]$,
(b) $\mathbb{P}(\Omega) = 1$,
(c) $\{A_j,\ j = 1, \ldots\} \subset \mathcal{F}$ with $A_i \cap A_j = \emptyset$ for $i \neq j \Rightarrow \mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}\left(A_j\right)$.

A property p is said to hold *almost surely* (a.s.) or *with probability one* in Ω if $\exists A \in \mathcal{F}$ such that $A \supset \{\omega : p \text{ is false at } \omega\}$ and $\mathbb{P}(A) = 0$.

The expression "almost surely" in probability theory corresponds to the expression "almost everywhere" in measure theory.

A.1.1.4 Random Variables

- Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a *random variable* X with values in a measurable space $(E, \mathcal{E})$ is a $\mathcal{F}/\mathcal{E}$-measurable function from Ω to E, i.e. $X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$.
- The *distribution* or *law* of the random variable X is the probability measure $\mathbb{P}_X$ on $(E, \mathcal{E})$ defined by $\mathbb{P}_X(A) := \mathbb{P}[X \in A] \equiv \mathbb{P}[\{\omega \in \Omega : X(\omega) \in A\}]$, $A \in \mathcal{E}$.

The random variable X describes a quantity, observed during the random experiment, taking a value $x \in E$ which depends on the general outcome $\omega \in \Omega$. At the

end of the experiment, if ω is observed, then we know that X has taken the value $x = X(\omega)$, while a priori evaluations on X are given by its distribution.

- Given a function $X : \Omega \to (E, \mathcal{E})$, the smallest σ-algebra in Ω making X measurable is $\sigma(X) = \{X^{-1}(A) | A \in \mathcal{E}\}$; $\sigma(X)$ is called the *σ-algebra generated by the random variable X*. Similarly, the σ-algebra generated by a family of random variables is $\sigma(X_i,\ i \in I) := \bigvee_{i\in I} \sigma(X_i)$.

The symbol $X^{-1}(A)$ denotes the inverse image of $A \in \mathcal{E}$, i.e. $X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\}$. The image of a set F is $X(F)$, i.e. $X(F) = \{e \in E : \exists \omega \in F \text{with} X(\omega) = e\}$. Let us note that these definitions give

$$\forall A \in \mathcal{E} : \mathbb{P}_X(A) = \mathbb{P}[X^{-1}(A)], \tag{A.1a}$$

$$\forall F \in \sigma(X) : \mathbb{P}_X\big(X(F)\big) = \mathbb{P}[F]. \tag{A.1b}$$

Independence

Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the events F_i, $i \in I$, are said to be *independent* if

$$\mathbb{P}[F_{i_1} \cap \cdots \cap F_{i_k}] = \mathbb{P}[F_{i_1}] \cdots \mathbb{P}[F_{i_k}] \tag{A.2}$$

for all choices of the integer $k \geq 2$ and of the indices $i_1, \ldots, i_k \in I$. The sub-σ-algebras $\mathcal{F}_i$, $i \in I$, are said to be *independent* if equality (A.2) holds for all choices of the integer $k \geq 2$, of the indices $i_1, \ldots, i_k \in I$ and of the events $F_{i_1} \in \mathcal{F}_{i_1}, \ldots, F_{i_k} \in \mathcal{F}_{i_k}$. The random variables X_i, $i \in I$, are said to be *independent* if the σ-algebras $\sigma(X_i)$, $i \in I$, generated by them are independent.

A complex or real random variable X is said to be *integrable* if $\int_\Omega |X(\omega)|\, \mathbb{P}(\mathrm{d}\omega) < +\infty$. Then its *mean value* (expectation) exists, we denote it by $\mathbb{E}[X]$ and it is given by

$$\mathbb{E}[X] = \int_\Omega X(\omega)\mathbb{P}(\mathrm{d}\omega) = \int_{\mathbb{C}} z\, \mathbb{P}_X(\mathrm{d}z) = \int_{\mathbb{R}} x\, \mathbb{P}_{\mathrm{Re}\, X}(\mathrm{d}x) + \mathrm{i} \int_{\mathbb{R}} y\, \mathbb{P}_{\mathrm{Im}\, X}(\mathrm{d}y).$$

The mean provides a very synthetic a priori evaluation on X. If X is a $\mathbb{C}^d$-valued random variable, by $\mathbb{E}[X]$ we denote the vector of the mean values of the components of X.

L^p-spaces

As usual, two random variables are considered to be *equivalent* if they are equal a.s. and we denote by $L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \in [1, \infty)$, the Banach spaces of the (equivalence classes of) p-integrable complex random variables. Recall that the norm in L^p is $\mathbb{E}[|X|^p]^{1/p}$ and that L^2 is a Hilbert space with inner product $\langle X | Y \rangle = \mathbb{E}[\overline{X}\, Y]$.

Moreover, $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the Banach space of the (equivalence classes of) essentially bounded complex random variables with the supremum norm. When $\mathcal{F} = \mathcal{G} \vee \mathcal{N}$, where $\mathcal{G}$ is a σ-algebra and $\mathcal{N}$ is a collection of events of probability zero, we have $L^p(\Omega, \mathcal{F}, \mathbb{P}) = L^p(\Omega, \mathcal{G}, \mathbb{P})$ by a natural identification.

Remark A.4. If $\mathbb{P}$ is a probability on a standard measurable space $(\Omega, \mathcal{F})$, then the spaces $L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \in [1, \infty)$, are separable. The same statement holds if $\mathcal{F} = \mathcal{G} \vee \mathcal{N}$, where $(\Omega, \mathcal{G})$ is a standard measurable space and $\mathcal{N}$ is a collection of events of probability zero [4, p. 20].

A.1.2 Densities, Absolute Continuity, Conditional Expectations

A.1.2.1 The Radon-Nikodym Theorem

Given a measurable space $(\Omega, \mathcal{F})$ and two measures $\mathbb{P}$ and $\mathbb{Q}$ on it, the measure $\mathbb{Q}$ is said to be *absolutely continuous* with respect to $\mathbb{P}$, and we write $\mathbb{Q} \ll \mathbb{P}$, if $\mathbb{P}(F) = 0$ implies $\mathbb{Q}(F) = 0$. When $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$, the two measures are said to be *equivalent*, and we write $\mathbb{Q} \sim \mathbb{P}$.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a non-negative random variable Z with $\mathbb{E}[Z] = 1$ can be used as a density to define a new probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$:

$$\forall F \in \mathcal{F} \qquad \mathbb{Q}(F) := \int_F Z(\omega)\, \mathbb{P}(\mathrm{d}\omega) \equiv \mathbb{E}[Z 1_F]. \tag{A.3}$$

As $\mathbb{P}(F) = 0$ implies $\mathbb{Q}(F) = 0$, then $\mathbb{Q} \ll \mathbb{P}$. The Radon–Nikodym theorem for probabilities gives the reverse statement: if $\mathbb{P}$ and $\mathbb{Q}$ are two probability measures on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} \ll \mathbb{P}$, then there exists a random variable $Z \geq 0$ such that Eq. (A.3) holds; Z is unique $\mathbb{P}$-a.s. The random variable Z is called *Radon–Nikodym derivative* and the following notation is used:

$$Z(\omega) = \frac{\mathbb{Q}(\mathrm{d}\omega)}{\mathbb{P}(\mathrm{d}\omega)}. \tag{A.4}$$

Let us denote by $\mathbb{E}_\mathbb{Q}$ the expectation with respect to $\mathbb{Q}$. Then, we have that $X \in L^1(\mathbb{Q})$ if and only if $XZ \in L^1(\mathbb{P})$ and that $\mathbb{E}_\mathbb{Q}[X] = \mathbb{E}[XZ]$, for $X \in L^1(\mathbb{Q})$. If $Z \neq 0$ $\mathbb{P}$-a.s., let us define Z^{-1} to be the reciprocal of Z over the set $\{Z \neq 0\}$ and zero elsewhere. Then, Z^{-1} is a non-negative random variable with

$$\begin{aligned} \mathbb{E}_\mathbb{Q}[Z^{-1}] &= \int_{\{Z \neq 0\}} Z^{-1}(\omega) \mathbb{Q}(\mathrm{d}\omega) \\ &= \int_{\{Z \neq 0\}} Z^{-1}(\omega)\, Z(\omega)\, \mathbb{P}(\mathrm{d}\omega) = \mathbb{P}(Z \neq 0) = 1, \end{aligned}$$

and, for every $F \in \mathcal{F}$, we have

$$\mathbb{P}(F) = \mathbb{E}[1_{F\cap\{Z\neq 0\}}] = \mathbb{E}[ZZ^{-1}1_F] = \mathbb{E}_{\mathbb{Q}}[Z^{-1}1_F] = \int_F Z(\omega)^{-1}\,\mathbb{Q}(\mathrm{d}\omega)\,.$$

Therefore, one has also $\mathbb{P} \ll \mathbb{Q}$ and the two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent: $\mathbb{P} \sim \mathbb{Q}$. In this case, given two random variables, the notions of equivalence with respect to $\mathbb{P}$ and $\mathbb{Q}$ coincide.

A.1.2.2 Conditional Expectations

A key tool in the theory of stochastic processes is the notion of conditional expectation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X a (complex or real) integrable random variable and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. If $\mathcal{F}$ consists of all the events regarding a random experiment, then $\mathcal{G}$ could be a subset of events whose occurrence is already known at an intermediate experimental step, before the experiment comes to its end. In this case the a priori expectation $\mathbb{E}[X]$ is updated at the intermediate step on the basis of the acquired information described by $\mathcal{G}$. Mathematically, such an updated expectation is found by looking for a random variable Y (because it is not known before of the experiment), $\mathcal{G}$-measurable (because it can depend only on the information described by $\mathcal{G}$), such that $\int_G Y(\omega)\mathbb{P}(\mathrm{d}\omega) = \int_G X(\omega)\mathbb{P}(\mathrm{d}\omega)$ for all $G \in \mathcal{G}$ (because it has to describe the expectation for X on the basis of the a priori evaluations on the experiment $\mathbb{P}$ and of the new information $\mathcal{G}$). By the Radon–Nikodym theorem there exists a $\mathbb{P}$-a.s. unique, integrable, $\mathcal{G}$-measurable random variable Y satisfying these properties. This random variable Y is denoted by $\mathbb{E}[X|\mathcal{G}]$ and it is called the *conditional expectation* of X given the σ-algebra $\mathcal{G}$. In a more symbolic way:

if $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, the *conditional expectation* of X given $\mathcal{G}$ is defined by
$\mathbb{E}[X|\mathcal{G}] \in L^1(\Omega, \mathcal{G}, \mathbb{P})$ and $\mathbb{E}\left[1_G\,\mathbb{E}[X|\mathcal{G}]\right] = \mathbb{E}[1_G X]$ for all $G \in \mathcal{G}$.

When only square-integrable random variables are considered, conditional expectations can be seen as orthogonal projections. The space $L^2(\Omega, \mathcal{G}, \mathbb{P})$ is a closed subspace of the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}[\bullet|\mathcal{G}]$ is the orthogonal projection from $L^2(\Omega, \mathcal{F}, \mathbb{P})$ to $L^2(\Omega, \mathcal{G}, \mathbb{P})$. For a presentation of the main properties of conditional expectations see [2, p. 9] or [3, pp. 51–56].

Analogously, we can update the probability of an event $F \in \mathcal{F}$. If the probability of F is evaluated with $\mathbb{P}(F)$ before of the experiment, then, once all the information described by $\mathcal{G} \subset \mathcal{F}$ is acquired, the probability of F is updated with

$$\mathbb{P}(F|\mathcal{G}) := \mathbb{E}[1_F|\mathcal{G}]. \tag{A.5}$$

This is a random variable and it is $\mathcal{G}$-measurable as it depends on the events in $\mathcal{G}$ which occur.

A.2 Filtrations and Processes

A.2.1 Stochastic Processes

A stochastic process is a family $X = \{X(t),\ t \in I\}$ of random variables defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in some measurable space (the *state space*). As for any family of random variables, $\sigma\big(X(t),\ t \in I\big)$ denotes the σ-algebra generated by the process. Two or more processes are said to be independent if the σ-algebras generated by them are independent. A stochastic process provides the mathematical model of a quantity observed continuously in time during a random experiment.

We consider only (real or complex) finite dimensional stochastic process, with the time t in $[0, +\infty)$ or sometimes in $[t_0, T]$. If X is a stochastic process and $\omega \in \Omega$ a sample point, the function $t \mapsto X(t, \omega)$ is a *sample path* realisation or *trajectory* of the process X. A process X is *continuous* if all its trajectories are continuous functions, *right continuous* if all its trajectories are right continuous functions, etc. A process X is *integrable* if $X(t) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for all t.

Definition A.5. *Two d-dimensional processes X and Y, defined in two possibly different probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$, are said to have* the same law *if they have the same finite dimensional distributions, i.e. for any choice of the integer n and of the times $0 \le t_1 < t_2 < \cdots < t_n$ the random vectors $(X(t_1), \ldots, X(t_n))$ and $(Y(t_1), \ldots, Y(t_n))$ have the same distribution on $\mathbb{C}^{nd}$ (or $\mathbb{R}^{nd}$).*

When $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega', \mathcal{F}', \mathbb{P}')$, the process Y is said to be a modification *or* version *of X if $Y(t)$ and $X(t)$ are equivalent random variables for every t:*

$$\mathbb{P}[X(t) = Y(t)] = 1\,, \qquad \forall t \ge 0\,.$$

The two processes are called indistinguishable *if almost all their sample paths agree, i.e. the complement of the set $\big\{\omega \in \Omega : X(t, \omega) = Y(t, \omega),\ \forall t \ge 0\big\}$ is contained in a measurable set of probability zero.*

Here above we give the definition of equality in law of processes, based on finite dimensional distributions. The following proposition gives some other justification of such a notion.

Proposition A.6 (see [3, Prop. 1.12, p. 28]). *Let us have a d-dimensional process $X(t)$, $t \in [0, T]$ (eventually $T = +\infty$), defined on $(\Omega, \mathcal{F})$, but considered under two probabilities $\mathbb{P}_1$ and $\mathbb{P}_2$. Assume that X has the same finite dimensional distributions under both probabilities. Then, $\mathbb{P}_1$ and $\mathbb{P}_2$ coincide on the σ-algebra $\sigma\big(X(t),\ t \in [0, T]\big)$.*

When $\{\omega \in \Omega : X(t, \omega) = Y(t, \omega),\ \forall t \ge 0\}$ is measurable, which is usually true when some regularity properties hold for the trajectories of the two processes, we can say that X and Y are indistinguishable if

$$\mathbb{P}[X(t) = Y(t),\ \forall t \ge 0] = 1\,.$$

If X and Y are indistinguishable, then one is a modification of the other. When Y is a modification of X, then the two processes have the same law.

If X and Y are one the modification of the other and both are a.s. continuous, then they are indistinguishable.

A.2.2 Filtrations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *filtration* is a family $(\mathcal{F}_t)_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{F}$, i.e. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s < t < +\infty$. Sometimes, $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ is said to be a *stochastic basis*. Typically a filtration describes the accumulation of information during time: each $\mathcal{F}_t$ is the collection of all the events which we can decide whether they have been verified or not up to time t.

Let us denote by $\mathcal{N}$ the class of all $\mathbb{P}$-null sets in $\mathcal{F}$, i.e.

$$\mathcal{N} := \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}\,.$$

- The filtration is said to be *right continuous* if $\mathcal{F}_t = \mathcal{F}_{t_+}$ for all $t \geq 0$, where $\mathcal{F}_{t_+}$ is the σ-algebra of events decidable immediately after t, i.e.

$$\mathcal{F}_{t_+} := \bigcap_{s:s>t} \mathcal{F}_s\,. \tag{A.6}$$

- The stochastic basis (or the filtration) is said to satisfy the *usual conditions* if the filtration is right continuous and $\mathcal{F}_0$ contains $\mathcal{N}$. Obviously $\mathcal{N} \subset \mathcal{F}_0$ implies $\mathcal{N} \subset \mathcal{F}_t$, $\forall t \geq 0$.

Given a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ we can construct a stochastic basis $\big(\Omega, \mathcal{F}, (\widetilde{\mathcal{F}}_t), \mathbb{P}\big)$, satisfying the usual conditions, by setting $\widetilde{\mathcal{F}}_t := \mathcal{F}_{t_+} \vee \mathcal{N}$. In the following we shall try to emphasise where usual conditions are useful.

A.2.3 Adapted Processes

Definition A.7. *A stochastic process X is said to be* adapted *to a filtration $(\mathcal{F}_t)$ if, for all $t \geq 0$, the random variable $X(t)$ is $\mathcal{F}_t$-measurable. In this case also the expression "non-anticipating process" is used.*

The notion of adapted process has both a technical and a physical meaning. It catches the idea of not anticipating the future.

The request that $\mathcal{F}_0$ contains $\mathcal{N}$ is useful when we want to modify a process to gain some good property, but without loosing the property of being adapted with respect to $(\mathcal{F}_t)$: if Y is a modification of an $(\mathcal{F}_t)$-adapted process X, then Y is guaranteed to be $(\mathcal{F}_t)$-adapted provided $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. For instance, let Y be a continuous modification of an adapted process X. This means that Y is a process with continuous trajectories such that $\mathbb{P}[X_t = Y_t] = 1$ for all t. To know that X is

adapted does not allow to conclude that Y is adapted, because X_t and Y_t could differ over a set not belonging to $\mathcal{F}_t$. But if $\mathcal{N} \subset \mathcal{F}_t$ this is not possible and Y turns out to be adapted.

A.2.3.1 Natural Filtration and Augmented Natural Filtration

Given a process X let us define $\mathcal{F}_t^X := \sigma\,(X(s) : 0 \le s \le t)$, i.e. $\mathcal{F}_t^X$ is the smallest σ-algebra with respect to which $X(s)$ is measurable for every $s \in [0, t]$: $\left(\mathcal{F}_t^X\right)$ is called the *natural filtration* generated by X. A process is always adapted to its natural filtration $(\mathcal{F}_t^X)$ and it is adapted to another filtration $(\mathcal{F}_t)$ if and only if $\mathcal{F}_t \supset \mathcal{F}_t^X$. It is often useful to add the $\mathbb{P}$-null sets $\mathcal{N}$ to the natural filtration $\left(\mathcal{F}_t^X\right)$ of a process X: the filtration $\left(\overline{\mathcal{F}}_t^X\right)$, defined by $\overline{\mathcal{F}}_t^X := \mathcal{F}_t^X \vee \mathcal{N}$, is called the *augmented natural filtration* of X. There is an important difference between the natural filtration of a process and its augmentation. To define $\mathcal{F}_t^X$ we need the sample space Ω, the state space E with its σ-algebra $\mathcal{E}$ and the functions $X_s : \Omega \to E$ with $s \in [0, t]$. To define $\overline{\mathcal{F}}_t^X$ we need also $\mathcal{N}$, the events with zero probability, and, so, we need also the σ-algebra $\mathcal{F}$ and the probability $\mathbb{P}$.

Definition A.8. *A process X is called* measurable *if the function $[0, +\infty) \times \Omega \ni (t, \omega) \mapsto X(t, \omega)$ is $\mathcal{B}([0 + \infty)) \otimes \mathcal{F}$-measurable.*

We need the joint measurability in t and ω, for instance, when we want to exchange an integral over time and an expectation by invoking Fubini theorem (see Sect. A.3.2).

Definition A.9. *A process X is called* progressively measurable *or* progressive *if for every $T \ge 0$ the function $[0, T] \times \Omega \ni (t, \omega) \mapsto X(t, \omega)$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$-measurable.*

Trivially, a progressive process is adapted and measurable. The converse is true up to a modification; moreover, an adapted process having trajectories with certain regularity properties is progressively measurable [1, Propositions 1.12 and 1.13 p. 5].

Proposition A.10. *If the stochastic process X is measurable and adapted to the filtration $(\mathcal{F}_t)$, then it has a progressively measurable modification.*

If the stochastic process X is adapted to the filtration $(\mathcal{F}_t)$ and every sample path is right continuous or else every sample path is left-continuous, then X is also progressively measurable with respect to $(\mathcal{F}_t)$.

With respect to the idea of adapted process, the notion of progressive process is more technical, but it is not so much restrictive due to the proposition above. It is useful to have a progressive process when one has to consider the stopped process (see Remark A.17) or the integral over time of a process and the result has to be adapted (see Sect. A.3.2).

A.2.4 The Law of a Continuous Process

Any continuous process can be represented in the space of continuous functions (the space of its possible trajectories) and this fact can be used to interpret such a process as a single random variable [3, Exercise E1.4, pp. 29–30, 275]. By enlarging the trajectory space, also processes with jumps could be treated [4, pp. 20–22], but we do not need them in this book. As we have always to do with continuous processes starting from zero, we impose this condition, while it is trivial to extend the whole construction to the generic case.

Remark A.11 (The canonical trajectory space for a continuous process starting from zero). Let I be the finite interval $[0, T]$ for some $T > 0$, or $I = [0, +\infty)$. Let $C_0^d(I)$ be the space of all $\mathbb{R}^d$-valued continuous functions on I starting from zero. Then, let us define the coordinate mapping process $\Pi(t, c) := c(t)$, $t \in I$, $c \in C_0^d(I)$.

The σ-algebra $\mathcal{F}^\Pi$ generated by the process $\Pi(t)$, $t \in I$, is the σ-algebra $\mathcal{C}_I$ generated by the cylinder sets, i.e. the sets of the form $\{c \in C_0^d(I) : (c(t_1), \dots, c(t_n)) \in A\}$, $n \geq 1$, $A \in \mathcal{B}(\mathbb{R}^{d\times n})$. Moreover, it is possible to define a metric on $C_0^d(I)$ which makes it a complete separable metric space with Borel σ-algebra $\mathcal{B}\big(C_0^d(I)\big) = \mathcal{C}_I = \mathcal{F}^\Pi$ (the topology is the one of uniform convergence on the compact sets). Then, by Theorem A.3, the measurable space $\Big(C_0^d(I),\ \mathcal{C}_I\Big)$ is standard.

Proposition A.12 (A continuous process as a single random variable). *Let $X(t)$, $t \in I$, be a continuous, adapted, d-dimensional real process in the stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ with $X(0) = 0$. Let us introduce the application $\Xi_X : \Omega \to C_0^d(I)$ by $\Xi_X(\omega) = \{t \mapsto X(t, \omega)\}$, i.e. Ξ_X associates to every sample point ω the relative trajectory of X. Then, Ξ_X is a random variable with values in $\Big(C_0^d(I),\ \mathcal{C}_I\Big)$.*

We can say that the map Ξ_X identifies the process X with the coordinate mapping process Π.

Remark A.13. We can call *law of the continuous process X* the distribution of the random variable Ξ_X; the distribution of a random variable is defined in Sect. A.1.1.4. It turns out that the distribution of Ξ_X is uniquely determined by finite dimensional distributions of the process X and, so, this definition of law is equivalent to the definition of law of a process as the collection of the finite dimensional distributions (Definition A.5).

A.2.5 Martingales and Stopping Times

Definition A.14. *A one-dimensional complex process X is said to be an* $(\mathcal{F}_t)$-martingale *if it is $(\mathcal{F}_t)$-adapted, integrable and*

$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s) \quad \mathbb{P}\text{-a.s.}, \qquad \text{for every } 0 \leq s < t\,.$$

When the process X is real, adapted and integrable, it is called a supermartingale *if* $\mathbb{E}[X(t)|\mathcal{F}_s] \leq X(s)$*, and it is called a* submartingale *if* $\mathbb{E}[X(t)|\mathcal{F}_s] \geq X(s)$.

We can interpret $\mathbb{E}[X(t)|\mathcal{F}_s]$ by saying that past and present (the present is s) are frozen and we take the mean of $X(t)$ only with respect to all the stochasticity entering into play in the future.

Remark A.15. The mean value of a martingale is constant in time. Moreover (see [2] Theorem 3.13 p. 16), if the filtration satisfies the usual conditions, every martingale has a modification, which is again a martingale, having right continuous trajectories with left limits.

Definition A.16. *A random variable* $\tau : \Omega \to [0, +\infty]$ *is called a* stopping time*, or, better, an* $(\mathcal{F}_t)$*-stopping time, if* $\{\tau \leq t\} \in \mathcal{F}_t$ *for all* $t \geq 0$.

A stopping time describes the occurrence instant of a random phenomenon observed during the random experiment related to $(\mathcal{F}_t)$.

Remark A.17. If τ is a finite stopping time and X is a measurable process, then $\omega \mapsto X(\tau(\omega), \omega)$ is a random variable. In this statement the joint measurability in (t, ω) of X is crucial in order that $X(\tau)$ be $\mathcal{F}$-measurable. Moreover, if τ is a stopping time, then $\tau \wedge t \equiv \min\{\tau, t\}$ is a finite stopping time and, if X is a progressive process, $X(t \wedge \tau)$ is an $\mathcal{F}_t$-measurable random variable and the *stopped process* $\{X(t \wedge \tau), t \geq 0\}$ is a progressive process. Again the progressive character of X is crucial in order that the stopped process be adapted.

Definition A.18. *A process X is a* local martingale*, with respect to a filtration* $(\mathcal{F}_t)$*, if there exists an increasing sequence of stopping times* τ_n *such that* $\tau_n \underset{n\to+\infty}{\longrightarrow} +\infty$ *a.s. and* $\{X(t \wedge \tau_n), t \geq 0\}$ *is an* $(\mathcal{F}_t)$*-martingale for all* n.

Remark A.19. Every martingale is a local martingale; it is enough to take $\tau_n = +\infty$ for all n. If X is a local martingale, $X(0)$ is integrable; indeed, $X(0) = X(0 \wedge \tau_n)$ and $X(\bullet \wedge \tau_n)$ is a martingale. However, we do not know if $X(t)$ is integrable. There exist either non-integrable local martingales or integrable local martingales which are not martingales. Under usual conditions, by modification, every local martingale can be taken right continuous, and so progressive.

Proposition A.20. *A non-negative local martingale is a supermartingale. It is a martingale if and only if its mean is constant in time.*

A.2.6 The Wiener Process

A Wiener process W is a Gaussian process, with independent and stationary increments, with mean zero and variance proportional to t, or covariance matrix proportional to $t\mathbb{1}$ in the multidimensional case. It is usual to take it exactly equal to $t\mathbb{1}$ (standard Wiener process). At the price of a modification, it is always possible to obtain continuous trajectories. Moreover, for the developments of stochastic calculus, where adapted processes are integrated with respect to Wiener processes, it is

convenient to include the filtration in the definition of Wiener process, in order to allow the filtration $(\mathcal{F}_t)$ to be bigger than $(\mathcal{F}_t^W)$. Without loss of generality, we have the following definition.

Definition A.21. *Let* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\right)$ *be a stochastic basis. A* d*-dimensional* Wiener process $W \equiv \{W_j(t), t \geq 0, j = 1, \ldots, d\}$ *is a continuous,* $\mathbb{R}^d$*-valued, adapted process with the following properties:*

(i) $W(0) = 0$ a.s.;
(ii) for $0 \leq s < t < +\infty$ the increment $W(t) - W(s)$ is normal with vector of means 0 and covariance matrix $(t - s)\mathbb{1}$;
(iii) for $0 \leq s < t < +\infty$ the increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$.

It would be equivalent to define a one-dimensional Wiener process and to say that a d-dimensional Wiener process is a collection of d independent one-dimensional Wiener processes.

Remark A.22. Let W be a Wiener process in $\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\right)$. Then W is a Wiener process also in $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^W), \mathbb{P}\right)$ and in $\left(\Omega, \mathcal{F}, (\overline{\mathcal{F}}_t^W), \mathbb{P}\right)$. The stochastic basis $\left(\Omega, \mathcal{F}, (\overline{\mathcal{F}}_t^W), \mathbb{P}\right)$ automatically satisfies the usual conditions [3, pp. 58–60].

The following remark summarises the canonical construction of the Wiener process; see for instance [1, pp. 59-60, 71].

Remark A.23. Let $\left(C_0^d(0, \infty), \mathcal{C}_\infty\right)$ be the trajectory space of the d-dimensional continuous functions starting from zero and let Π be the coordinate mapping process, both introduced in Remark A.11. There exists a unique probability measure $\mathbb{P}_W$ (the *Wiener measure*) on $\left(C_0^d(0, \infty), \mathcal{C}_\infty\right)$ such that Π is a d-dimensional Wiener process with respect to the filtration $(\mathcal{F}_t^\Pi)$ and thus also with respect to the filtration $(\overline{\mathcal{F}}_t^\Pi)$, where we mean the augmentation with respect to $\left(C_0^d(0, \infty), \mathcal{C}_\infty, \mathbb{P}_W\right)$.

In the mathematical literature, what we are calling Wiener process is termed also Brownian motion. However in the physical literature the Brownian motion is a physical phenomenon which can be described by various mathematical models (even quantum models); the simplest and most famous mathematical model of the Brownian motion is a process proportional to Wiener process, with a temperature dependent variance.

A.3 Stochastic Calculus

Let us take now W to be a d-dimensional, continuous, standard Wiener process defined in a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\right)$ satisfying the usual conditions.

A.3.1 Classes of Integrands

For complex-valued measurable adapted processes X, we want to introduce the integrals $\int_a^b X(t)\mathrm{d}t$ and $\int_a^b X(t)\mathrm{d}W_j(t)$ for $b \geq a \geq 0$, $j = 1, \ldots, d$. These two integrals will define $\mathcal{F}_b$-measurable random variables, up to equivalence. The integrals will not change if the integrand process X is replaced with a process X' such that $X'(t,\omega) = X(t,\omega)$ almost everywhere with respect to "Lebesgue measure" $\otimes\, \mathbb{P}$. Therefore two complex measurable adapted processes X and X' are called *equivalent* if $X'(t,\omega) = X(t,\omega)$ almost everywhere with respect to "Lebesgue measure" $\otimes\, \mathbb{P}$, that is if

$$\mathbb{P}\left[\int_0^{+\infty} \left|X(t) - X'(t)\right| \mathrm{d}t = 0\right] = 1\,.$$

Notice that if X' is a measurable modification of X, then X' is equivalent to X. Then, thanks to Proposition A.10, without loss of generality we can always consider progressively measurable integrands and we introduce the following spaces for $p \geq 1$.

- $\mathcal{M}^p$ is the linear space of the (equivalence classes of) progressively measurable complex processes X such that

$$\int_0^t \mathbb{E}\left[|X(s)|^p\right] \mathrm{d}s < +\infty\,, \qquad \forall t \geq 0\,. \tag{A.7}$$

- $\mathcal{L}^p$ is the linear space of the (equivalence classes of) progressively measurable complex processes X such that

$$\mathbb{P}\left[\int_0^t |X(s)|^p \,\mathrm{d}s < +\infty\right] = 1\,, \qquad \forall t \geq 0\,. \tag{A.8}$$

Of course $\mathcal{M}^p \subset \mathcal{L}^p$, and for $p < p'$ we have $\mathcal{M}^{p'} \subset \mathcal{M}^p$ and $\mathcal{L}^{p'} \subset \mathcal{L}^p$. As usual, we do not distinguish between an equivalence class and a single representative of the class.

A.3.2 Integrals on Time

When $X \in \mathcal{L}^1$, by the definition of $\mathcal{L}^1$, the trajectories of X are Lebesgue measurable and there exists a set Ω_T such that $\mathbb{P}(\Omega_T) = 1$ and $\int_{t_0}^T |X(s,\omega)|\,\mathrm{d}s < +\infty$ for $\omega \in \Omega_T$. Moreover, by the continuity of the usual integrals over time, the function $[t_0, T] \ni t \mapsto \int_{t_0}^t X(s,\omega)\mathrm{d}s$ is continuous for $\omega \in \Omega_T$. By Fubini theorem, $\omega \mapsto \int_{t_0}^t X(s,\omega)\mathrm{d}s$ is an a.s. finite measurable function ($t \geq t_0$). By usual hypotheses, we have that $\Omega_T \in \mathcal{F}_0 \subset \mathcal{F}_t$; then, we can modify the definition of this integral over time by taking it to be zero on ${\Omega_T}^{\mathrm{c}}$ and in this way we obtain a

process which is $(\mathcal{F}_t)$-adapted, finite everywhere and continuous as a function of t. Let us stress that the $\mathcal{F}_t$-measurability condition needs the progressive character of X; adaptedness and measurability would imply only that the integral is a.s. equal to a random variable having this property.

When $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfies the usual conditions, as we assume, and $X \in \mathcal{L}^1$, by $t \mapsto \int_{t_0}^t X(s)\mathrm{d}s$ we always mean a (everywhere defined) continuous, adapted (and, so, progressive) version of the process.

When $X \in \mathcal{M}^1$, the mean value of the integral exists and, thanks to the joint (t, ω)-measurability and Fubini theorem,

$$\mathbb{E}\left[\int_{t_0}^t X(s)\mathrm{d}s\right] = \int_{t_0}^t \mathbb{E}[X(s)]\mathrm{d}s\,.$$

A.3.3 Stochastic Integrals

A process G is said to be *simple* if there exist a sequence of times $0 = t_1 < t_2 < \cdots$ (without finite limit points) and a sequence of random variables $G_1, G_2, \ldots$ such that G_α is $\mathcal{F}_{t_\alpha}$-measurable and $G(t) = \sum_{\alpha=1}^{+\infty} G_\alpha 1_{[t_\alpha, t_{\alpha+1})}(t)$. For a simple process the stochastic integral with respect to a component W_j of the Wiener process is defined by

$$\int_0^t G(s)\mathrm{d}W_j(s) = \sum_{\alpha=1}^{n(t)-1} G_\alpha \left[W_j(t_{\alpha+1}) - W_j(t_\alpha)\right] + G_{n(t)}\left[W_j(t) - W_j(t_{n(t)})\right], \tag{A.9}$$

where $n(t)$ is the integer such that $t_{n(t)} \le t < t_{n(t)+1}$; if the initial time is not zero we define

$$\int_{t_0}^t G(s)\mathrm{d}W_j(s) = \int_0^t 1_{[t_0,t]}(s)G(s)\mathrm{d}W_j(s)\,, \qquad t \ge t_0 \ge 0\,.$$

The crucial point in Eq. (A.9) is that G_α and $\left[W_j(t_{\alpha+1}) - W_j(t_\alpha)\right]$ are independent random variables. Then, by suitable limits, the integral is defined as an $\mathcal{F}_t$-measurable random variable for any $G \in \mathcal{L}^2$ and every t. As a function of t the integral process is a local martingale. When $G \in \mathcal{M}^2$, the integral process turns out to be a square-integrable martingale. Moreover the integral process has a continuous modification and, as discussed in Sect. A.2.3, when $\mathcal{F}_0$ contains all the null events (as in the case of the usual conditions which we assume), this continuous modification can be taken to be adapted.

Let us collect the main results on the Itô integral, under usual conditions.

1. When $G \in \mathcal{L}^2$ and $t \ge t_0 \ge 0$, the stochastic integral $\int_{t_0}^t G(s)\mathrm{d}W_j(s)$ is a well-defined complex-valued random variable (up to equivalence), and

   a. the stochastic integral is linear in G;

b. the process $t \mapsto \int_{t_0}^t G(s)\mathrm{d}W_j(s)$ is a continuous, adapted (and so progressive) local martingale;
c. $\int_{t_0}^t ZG(s)\mathrm{d}W_j(s) = Z\int_{t_0}^t G(s)\mathrm{d}W_j(s)$ for any $\mathcal{F}_{t_0}$-measurable random variable Z.

2. When $G, G' \in \mathcal{M}^2$

a. the process $t \mapsto \int_{t_0}^t G(s)\mathrm{d}W_j(s)$ is a continuous square-integrable martingale,
b. $\mathbb{E}\left[\int_{t_0}^t G(s)\mathrm{d}W_j(s)\right] = 0$,
c. $\mathbb{E}\left[\int_{t_0}^t G(s)\mathrm{d}W_j(s)\int_{t_0}^t G'(r)\mathrm{d}W_i(r)\right] = \delta_{ij}\int_{t_0}^t \mathbb{E}\left[G(s)G'(s)\right]\mathrm{d}s$ and, in particular, $\mathbb{E}\left[\left|\int_{t_0}^t G(s)\mathrm{d}W_j(s)\right|^2\right] = \int_{t_0}^t \mathbb{E}\left[|G(s)|^2\right]\mathrm{d}s$.

Property 2(c) is known as Itô isometry.

A.3.4 Itô Formula

A process $\{X(t),\ t \geq t_0 \geq 0\}$ is called *Itô process* if it is a continuous, adapted process such that, for every $t \geq t_0$,

$$X_i(t) = X_i(t_0) + \int_{t_0}^t F_i(s)\,\mathrm{d}s + \sum_{j=1}^d \int_{t_0}^t G_{ij}(s)\,\mathrm{d}W_j(s)\,, \qquad i = 1, \ldots, n,$$

with $X_i(t_0)$ being $\mathcal{F}_{t_0}$-measurable and $F_i \in \mathcal{L}^1$, $G_{ij} \in \mathcal{L}^2$. It is usual to say that X has initial value $X_i(t_0)$ and it admits the *stochastic differential*

$$\mathrm{d}X_i(t) = F_i(t)\,\mathrm{d}t + \sum_{j=1}^d G_{ij}(t)\,\mathrm{d}W_j(t)\,.$$

The stochastic differential is unique, up to an equivalence. The product of Itô processes is an Itô process and we have

$$\begin{aligned} X_i(t)X_k(t) =& X_i(t_0)X_k(t_0) + \int_{t_0}^t \Bigg[F_i(s)X_k(s) \\ &+ X_i(s)F_k(s) + \sum_{j=1}^d G_{ij}(s)G_{kj}(s)\Bigg]\mathrm{d}s \\ &+ \sum_{j=1}^d \int_{t_0}^t \left[G_{ij}(s)X_k(s) + X_i(s)G_{kj}(s)\right]\mathrm{d}W_j(s)\,. \end{aligned} \tag{A.10}$$

We can remember this formula by saying that the stochastic differential of the product is

$$\mathrm{d}\big(X_i(t)X_k(t)\big) = X_i(t)\big(\mathrm{d}X_k(t)\big) + \big(\mathrm{d}X_i(t)\big)X_k(t) + \big(\mathrm{d}X_i(t)\big)\big(\mathrm{d}X_k(t)\big), \qquad \text{(A.11)}$$

where the last term, called the *Itô correction*, has to be computed by using the expressions of the two differentials $\mathrm{d}X_i(t)$ and $\mathrm{d}X_k(t)$ and the *Itô product table*:

$$(\mathrm{d}t)^2 = 0\,, \qquad \mathrm{d}t\,\mathrm{d}W_j(t) = 0\,, \qquad \mathrm{d}W_j(t)\,\mathrm{d}W_\ell(t) = \delta_{i\ell}\,\mathrm{d}t\,. \qquad \text{(A.12)}$$

Equation (A.10) is Itô formula for a product. By generalising it to polynomials and by using Taylor expansions and suitable limits one gets the general Itô formula.

Theorem A.24. *Let $X_i(t_0)$ be a real $\mathcal{F}_{t_0}$-measurable random variable, let $F_i \in \mathcal{L}^1$ and $G_{ij} \in \mathcal{L}^2$, $i = 1, \dots, n$, $j = 1, \dots, d$, be real processes for $t \geq t_0 \geq 0$ and let X be the Itô process with components*

$$X_i(t) = X_i(t_0) + \int_{t_0}^{t} F_i(s)\,\mathrm{d}s + \sum_{j=1}^{d} \int_{t_0}^{t} G_{ij}(s)\,\mathrm{d}W_j(s)\,.$$

Let $f(x,t)$, $x \in \mathbb{R}^n$, $t \in [t_0, +\infty)$ be a complex function twice continuously differentiable in x and once in t. Let us set

$$f_t(x,t) := \frac{\partial f(x,t)}{\partial t}\,, \qquad f_i(x,t) := \frac{\partial f(x,t)}{\partial x_i}\,, \qquad f_{ik}(x,t) := \frac{\partial^2 f(x,t)}{\partial x_i \partial x_k}\,.$$

Then $f\big(X(t),t\big)$ is an Itô process and for $t \geq t_0 \geq 0$

$$\begin{aligned} f\big(X(t),t\big) = {} & f\big(X(t_0),t_0\big) + \int_{t_0}^{t} \Big[f_t\big(X(s),s\big) + \sum_i f_i\big(X(s),s\big)F_i(s) \\ & + \frac{1}{2}\sum_{ikj} f_{ik}\big(X(s),s\big)\,G_{ij}(s)G_{kj}(s) \Big]\mathrm{d}s \\ & + \sum_{ij} \int_{t_0}^{t} f_i\big(X(s),s\big)G_{ij}(s)\,\mathrm{d}W_j(s)\,. \end{aligned} \qquad \text{(A.13)}$$

Note that in this theorem the "spatial" variables are real. If we have a function of an n-dimensional complex process X and we want to apply Itô formula, it is enough to consider X as a $2n$-dimensional real process.

A.4 Ordinary Stochastic Differential Equations

In this appendix we consider two types of stochastic differential equations (SDEs) for processes X taking values in a finite dimensional complex Hilbert space. Here we introduce the first type, an SDE with coefficients which are non-random functions of the unknown process taken only at the last time. The second type of SDEs we consider is the Doléans equation (A.26).

A.4.1 The Main Class of SDEs

Hypothesis A.25. Let b and σ_j, $j = 1, \ldots, d$, be (Borel) measurable deterministic functions from $\mathbb{C}^n \times [t_0, T]$ to $\mathbb{C}^n$.

We consider the SDE

$$\mathrm{d}X(t) = b\big(X(t), t\big)\mathrm{d}t + \sum_{j=1}^{d} \sigma_j\big(X(t), t\big)\mathrm{d}W_j(t) \tag{A.14}$$

for processes X with values in $\mathbb{C}^n \equiv \mathcal{H}$. The term in $\mathrm{d}t$ is called the drift and b is the *drift coefficient*, and the term in $\mathrm{d}W(t)$ is called the diffusion term and σ is the *diffusion coefficient*.

We can assume that the Wiener process and the stochastic basis are given and that they are the ones introduced at the beginning of Sect. A.3. Then, an initial condition at time t_0 for (A.14) is an $\mathcal{F}_{t_0}$-measurable, $\mathcal{H}$-valued random variable η.

A *solution* of (A.14) with initial condition $X(t_0) = \eta$ is an Itô process satisfying (a.s., $\forall t \geq t_0$)

$$X(t) = \eta + \int_{t_0}^{t} b\big(X(s), s\big)\mathrm{d}s + \sum_{j=1}^{d} \int_{t_0}^{t} \sigma_j\big(X(s), s\big)\mathrm{d}W_j(s). \tag{A.15}$$

This solution is said to be *unique* if any other Itô process X' satisfying (A.15) is indistinguishable from X.

A.4.2 Types of Solutions

There is another way to see at SDE (A.14). Usually, by the physical problem, by the situation to be modelled, only b, σ and the law of the initial condition are given; the whole probabilistic structure has to be constructed. Now, we can distinguish among different concepts of existence and of uniqueness.

Definition A.26. *A* weak solution *of the SDE* (A.14) *with an initial condition with law μ is a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfying the usual conditions, with a*

Wiener process W, an $\mathcal{H}$-valued $\mathcal{F}_{t_0}$-measurable random variable $\eta \sim \mu$ and an adapted, continuous process X such that for every $t \geq t_0$ Eq. (A.15) *holds.*

Definition A.27. *The SDE* (A.14) *admits* strong solutions *if, for any choice of a stochastic basis satisfying usual conditions with a Wiener process W and for every $x_0 \in \mathcal{H}$, there exists a continuous adapted process X such that Eq.* (A.15) *holds with $\eta = x_0$.*

Definition A.28. *The solution of the SDE* (A.14) *is* unique in law *if, taken any two solutions $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), P\big)$, W, η, X and $\big(\Omega', \mathcal{F}', (\mathcal{F}'_t), P'\big)$, W', η', X', with $\eta \sim \eta'$, then the processes X and X' have the same law.*

Definition A.29. *The solution of the SDE* (A.14) *is* pathwise unique *if, taken any two solutions $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), P\big)$, W, η, X and $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), P\big)$, W, η, X', then the processes X and X' are indistinguishable.*

Remark A.30.

- Pathwise uniqueness implies uniqueness in law.
- The inclusion of the usual conditions in the definition of solution is only to have a sufficient condition to guarantee that the solution can be taken both continuous and adapted.
- In the case of strong solutions with deterministic initial condition x_0, it is always possible to take $\big(\overline{\mathcal{F}}_t^W\big)$ as filtration. In this case there exists a modification X' of X which is $\big(\mathcal{F}_t^W\big)$-adapted. This fact can be interpreted by saying that $X'(t)$ is a functional of x_0 and of the increments $W(s) - W(t_0)$ for $s \in [t_0, t]$.
- When a weak solution X exists, but the equation does not admit strong solutions, X is adapted to a filtration which is strictly greater than $\big(\overline{\mathcal{F}}_t^W\big)$.

Proposition A.31 ([1, Prop. 3.20 and Cor. 3.23, pp. 309–311]). *The existence of weak solutions and pathwise uniqueness imply strong existence.*

A.4.3 Sufficient Conditions for Existence and Uniqueness

There are various sufficient sets of hypotheses which imply existence and uniqueness of the solution of our SDE. Let us collect the hypotheses and the main results we need.

Hypothesis A.32 (Global Lipschitz condition). There exists a constant $L(T) > 0$ such that

$$\|b(x,t) - b(y,t)\|^2 + \sum_j \big\|\sigma_j(x,t) - \sigma_j(y,t)\big\|^2 \leq L(T)\,\|x - y\|^2 \qquad \text{(A.16)}$$

for all $x, y \in \mathbb{C}^n$ and $t \in [t_0, T]$.

Hypothesis A.33 (Local Lipschitz condition). For every $N > 0$ there exists a constant $L(N, T) > 0$ such that

$$\|b(x,t)-b(y,t)\|^2+\sum_j\left\|\sigma_j(x,t)-\sigma_j(y,t)\right\|^2\le L(N,T)\,\|x-y\|^2 \quad \text{(A.17)}$$

for all $t\in[t_0,T]$ and for all $x,y\in\mathbb{C}^n$ with $\|x\|\le N$, $\|y\|\le N$.

Hypothesis A.34 (Linear growth condition). There exists a constant $M(T)>0$ such that

$$\|b(x,t)\|+\left(\sum_j\left\|\sigma_j(x,t)\right\|^2\right)^{1/2}\le M(T)\,(1+\|x\|) \quad \text{(A.18)}$$

for all $x\in\mathbb{C}^n$ and $t\in[t_0,T]$.

Theorem A.35 (Monotone condition [2, pp. 58–59]). *There exists a constant* $C(T)\ge 0$ *such that*

$$\mathrm{Re}\langle x|b(x,t)\rangle+\frac{1}{2}\sum_j\left\|\sigma_j(x,t)\right\|^2\le C(T)\left(1+\|x\|^2\right) \quad \text{(A.19)}$$

for all $x\in\mathbb{C}^n$ *and* $t\in[t_0,T]$.

- The linear growth condition A.34 implies the monotone condition A.35.
- The global Lipschitz condition A.32 implies the local Lipschitz condition A.33.
- For time independent coefficients, i.e. $b(x,t)=b(x)$ and $\sigma_j(x,t)=\sigma_j(x)$, the global Lipschitz condition A.32 implies the linear growth condition A.34.

Theorem A.36 ([2, Theor. 3.5, p. 58];[1, Prop. 3.20, p. 309]). *Under Hypotheses A.25, A.33, A.35 the SDE* (A.14) *admits strong solutions in* $[t_0,T]$. *Pathwise uniqueness and uniqueness in law hold. If Hypotheses A.25, A.33, A.35 hold for every* $T>0$, *then the SDE* (A.14) *admits a unique strong solution in* $[t_0,\infty)$.

When the assumptions of the existence-and-uniqueness theorem hold for every $T>0$, the unique strong solution in $[t_0,\infty)$ is called a global solution.

A.4.4 L^p-Estimates on the Solution

Let us consider again a fixed stochastic basis $\left(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P}\right)$ satisfying the usual conditions and a standard continuous Wiener process W defined in this stochastic basis.

Theorem A.37 ([2, Theorem. 3.5, p. 58]; [3, Theorem. 8.10, p. 169]). *Under Hypotheses A.25, A.33, A.35 the SDE* (A.15) *with initial condition* $\eta\in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathcal{H})$ *has a pathwise unique solution* X *in* $[t_0,T]$. *The solution satisfies*

$$\int_{t_0}^{T}\mathbb{E}\left[\|X(t)\|^2\right]\mathrm{d}t<+\infty\,. \quad \text{(A.20)}$$

Theorem A.38 ([3, Propositions 8.13, 8.15, p. 172]; [2, Theor. 4.3, 4.4, p. 61]). *Let $p \geq 2$ and $\eta \in L^p(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathcal{H})$. If Hypotheses A.25, A.34 hold and it exists a pathwise unique solution X of* (A.15)*, then there exists a constant $c\big(p, T, M(T)\big)$ such that, for all $t_0 \leq s < t \leq T$,*

$$\mathbb{E}\left[\sup_{t_0 \leq t \leq T} \|X(t)\|^p\right] \leq c\big(p, T, M(T)\big)\left(1 + \mathbb{E}\left[\|\eta\|^p\right]\right). \tag{A.21}$$

$$\mathbb{E}\left[\|X(t) - X(s)\|^p\right] \leq c\big(p, T, M(T)\big)\left(1 + \mathbb{E}\left[\|\eta\|^p\right]\right)(t-s)^{p/2}. \tag{A.22}$$

A.5 Change of Measure and Girsanov Transformation

As for the whole appendix, also the content of this section is standard in the theory of stochastic processes and in stochastic calculus. See for instance [1, pp. 156–158, 191–196, 198–201]; [2, 270–272]; [3, pp. 78, 144–149, 180–181]. However, due to the relevance of this material in the theory of continuous measurements and to the fact that it is not well known among peoples working in quantum measurement theory, we think that it is useful and instructive to recall at least some of the proofs.

A.5.1 A Characterisation of the Wiener Process

The following theorem is essentially the same as the Lévy characterisation of the Wiener process; it only avoids to introduce explicitly the notion of "quadratic variation of a local martingale". We take it from [3, Theorem 4.26, p. 78].

Theorem A.39. *Let $B(t)$, $t \geq 0$, be a continuous d-dimensional adapted real process defined in a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let us assume that $B(0) = 0$ and that the complex process $\exp\left\{\mathrm{i}\sum_j \lambda_j B_j(t) + \frac{1}{2}|\lambda|^2 t\right\}$ is a martingale for all $\lambda \in \mathbb{R}^d$. Then, B is a d-dimensional Wiener process with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. If B is not continuous, but the stochastic basis satisfies usual conditions, then there is a continuous modification of B which is a Wiener process.*

Proof. The martingale property

$$\mathbb{E}\left[\exp\left\{\mathrm{i}\sum_j \lambda_j B_j(t) + \frac{1}{2}|\lambda|^2 t\right\}\middle|\mathcal{F}_s\right] = \exp\left\{\mathrm{i}\sum_j \lambda_j B_j(s) + \frac{1}{2}|\lambda|^2 s\right\}$$

can be rewritten as

$$\mathbb{E}\left[\exp\left\{\mathrm{i}\sum_j \lambda_j \big(B_j(t) - B_j(s)\big)\right\}\middle|\mathcal{F}_s\right] = \exp\left\{-\frac{1}{2}|\lambda|^2 (t-s)\right\},$$

which implies the independence of the increments from the past and their "Gaussianity" with covariance matrix t times the identity. Therefore, if B is continuous, all the properties which characterise a Wiener process hold. Otherwise, under usual conditions, there exists a continuous, adapted modification of B which is a Wiener process. □

A.5.2 Exponential of Itô Processes

Let W be a d-dimensional Wiener process defined in a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfying the usual conditions.

A.5.2.1 A Doléans SDE

Let us take some stochastic processes $F \in \mathcal{L}^1$, $G_j \in \mathcal{L}^2$, $j = 1, \dots, d$, and let us introduce the complex Itô process

$$X(t) := \sum_{j=1}^{d} \int_0^t G_j(s)\, \mathrm{d}W_j(s) + \int_0^t F(s)\, \mathrm{d}s\,. \tag{A.23}$$

Then, we consider the exponential of X times a generic constant:

$$Z(t) := z_0 \exp\{X(t)\}\,, \qquad z_0 \in \mathbb{C}. \tag{A.24}$$

The process Z is an Itô process by Theorem A.24 and, by Itô formula, we get

$$Z(t) = z_0 + \sum_{j=1}^{d} \int_0^t Z(s) G_j(s)\, \mathrm{d}W_j(s) + \int_0^t Z(s) \left[F(s) + \frac{1}{2} \sum_{j=1}^{d} G_j(s)^2 \right] \mathrm{d}s\,, \tag{A.25}$$

that is

$$\begin{cases} \mathrm{d}Z(t) = \displaystyle\sum_{j=1}^{d} Z(t) G_j(t)\, \mathrm{d}W_j(t) + Z(t) \left[F(t) + \frac{1}{2} \sum_{j=1}^{d} G_j(t)^2 \right] \mathrm{d}t\,, \\ Z(0) = z_0\,. \end{cases} \tag{A.26}$$

Remark A.40. The integrals in Eq. (A.23) are well defined and, so, $\mathbb{P}[|X(t)| < +\infty] = 1$. For $z_0 \neq 0$, this implies $\mathbb{P}[Z(t) = 0] = 0$ and, so, $Z(t)^{-1}$ is a bona fide random variable. Obviously, $Z(t)^{-1} = \exp\{-X(t)\}/z_0$ and by Itô formula we get

$$Z(t)^{-1} = \frac{1}{z_0} - \sum_{j=1}^{d} \int_0^t Z(s)^{-1} G_j(s)\, \mathrm{d}W_j(s) + \int_0^t Z(s)^{-1} \left[-F(s) + \frac{1}{2} \sum_{j=1}^{d} G_j(s)^2 \right] \mathrm{d}s\,. \tag{A.27}$$

Formula (A.26) can be seen as a stochastic equation for Z. However, it is not of the type (A.14) because the coefficients are intrinsically random and not only through the solution. In particular, the stochastic basis is implicitly a priori assigned together with F and the G_j's. Therefore, given a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfying the usual condition, with a Wiener process W and the processes $F \in \mathcal{L}^1$, $G_j \in \mathcal{L}^2$, a *solution* of the SDE (A.26) is a an adapted, continuous process Z satisfying (A.25) for every $t \geq 0$. One-dimensional linear equations like (A.26) are known as *Doléans equations* (the general Doléans equation involves also contributions with jumps). The solution of the SDE (A.26) is *unique in law* if any two solutions Z and Z' have the same law. The solution of the SDE (A.26) is *pathwise unique* if any two solutions Z and Z' are indistinguishable. By construction we already have the existence of a solution of (A.26). The pathwise uniqueness can be obtained from a simple direct proof, without invoking a general theory of stochastic differential equations with random coefficients.

Proposition A.41. *For $F \in \mathcal{L}^1$, $G_j \in \mathcal{L}^2$, $j = 1, \ldots, d$, the process Z, defined by Eqs.* (A.23)*,* (A.24)*, is the pathwise unique solution of the Doléans equation* (A.26)*.*

Proof. Let Y be another solution of (A.26). This means that Y is a continuous, adapted process, that $YG_j \in \mathcal{L}^2$, $Y\Big[F + \frac{1}{2}\sum_{j=1}^{d} {G_j}^2\Big] \in \mathcal{L}^1$ and that (A.25) holds with Z replaced by Y. Then $Y(0)\exp\{-X(0)\} = z_0$, and Itô formula and Eq. (A.27) imply $\mathrm{d}\big(Y(t)\exp\{-X(t)\}\big) = 0$. Being continuous processes we obtain that $Y(t)\exp\{-X(t)\} = z_0$ for every $t \geq 0$ a.s., so that Y and Z are indistinguishable. □

A.5.2.2 Exponential Martingales

Let us take now $z_0 = 1$ and $F = -\frac{1}{2}\sum_{j=1}^{d} {G_j}^2$. From Eqs. (A.23), (A.24), (A.26), (A.27) we get

$$Z(t) = \exp \sum_{j=1}^{d} \left\{ \int_0^t G_j(s)\, \mathrm{d}W_j(s) - \frac{1}{2} \int_0^t G_j(s)^2\, \mathrm{d}s \right\}, \tag{A.28}$$

$$Z(t)^{-1} = \exp \sum_{j=1}^{d} \left\{ -\int_0^t G_j(s)\, \mathrm{d}W_j(s) + \frac{1}{2} \int_0^t G_j(s)^2\, \mathrm{d}s \right\}, \tag{A.29}$$

$$Z(t) = 1 + \sum_{j=1}^{d} \int_0^t Z(s) G_j(s)\, \mathrm{d}W_j(s)\,, \tag{A.30}$$

$$Z(t)^{-1} = 1 - \sum_{j=1}^{d} \int_0^t Z(s)^{-1} G_j(s)\, \mathrm{d}W_j(s) + \sum_{j=1}^{d} \int_0^t Z(s)^{-1} G_j(s)^2 \mathrm{d}s\,. \tag{A.31}$$

By (A.30), Z reduces to a stochastic integral plus a constant and, so, it is a continuous local martingale.

Proposition A.42 ([3, Propositions 7.19 and 7.20, p. 145]). *If $G_j \in \mathcal{L}^2$, $j = 1, \ldots, d$, and for some positive constants T and K one has $\sum_{j=1}^{d} \int_0^T \left|G_j(s)\right|^2 \mathrm{d}s \leq K$, then $\{Z(t),\ t \in [0, T]\}$, defined by* (A.28)*, is a square-integrable complex martingale and*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Z(t)|^p\right] < +\infty\,, \qquad \forall p \geq 1\,. \tag{A.32}$$

If the processes G_j are real, Z is a non-negative local martingale and, by Proposition A.20, it is a supermartingale. It is possible to give sufficient conditions to have a martingale, less restrictive than those of Proposition A.42. One of the best known is the following Novikov condition.

Theorem A.43. *If $G_j \in \mathcal{L}^2$, $j = 1, \ldots, d$, are real processes such that*

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\sum_j \int_0^T G_j(t)^2\, \mathrm{d}t\right\}\right] < +\infty\,, \qquad \forall T \in [0, +\infty), \tag{A.33}$$

then Z, defined by (A.28)*, is a positive real martingale.*

The proofs of Proposition A.42 and Theorem A.43 are technical and require martingale inequalities and estimates on stochastic integrals.

A.5.3 Positive Martingales and Change of Measure

Consider now a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\right)$ and a non-negative martingale $Z = \{Z(t),\ t \geq 0\}$ with $\mathbb{E}[Z(t)] = 1$. For every fixed $t \geq 0$, the random variable $Z(t)$ can be used as a density to define a new probability measure $\mathbb{Q}_t$ on $(\Omega, \mathcal{F}_t)$ (cf. Eq. (A.3)):

$$\forall F \in \mathcal{F}_t \qquad \mathbb{Q}_t(F) := \int_F Z(t, \omega)\mathbb{P}(\mathrm{d}\omega) \equiv \mathbb{E}[Z(t)1_F]. \tag{A.34}$$

Let us denote by $\mathbb{P}_t$ the restriction of $\mathbb{P}$ to $\mathcal{F}_t$. Because in the definition of $\mathbb{Q}_t$ the set F belongs to $\mathcal{F}_t$, we can write $\mathbb{Q}_t(F) = \int_F Z(t,\omega)\mathbb{P}_t(\mathrm{d}\omega)$. Therefore, $\mathbb{Q}_t$ is absolutely continuous with respect to $\mathbb{P}_t$: $\mathbb{Q}_t \ll \mathbb{P}_t$. Let us denote by $\mathbb{E}_{\mathbb{Q}_t}$ the expectation with respect to $\mathbb{Q}_t$, while $\mathbb{E}$ is the mean with respect to $\mathbb{P}$ or $\mathbb{P}_t$.

Being Z a martingale, all the probabilities $\mathbb{Q}_t$, $t \geq 0$, are consistent, in the sense that

$$\mathbb{Q}_t(F) = \mathbb{Q}_s(F)\,, \qquad \forall t, s : t \geq s \geq 0\,, \quad \forall F \in \mathcal{F}_s\,. \tag{A.35}$$

Indeed, 1_F is $\mathcal{F}_s$-measurable and one has

$$\begin{aligned}\mathbb{Q}_t(F) = \mathbb{E}[Z(t)1_F] = \mathbb{E}\left[\mathbb{E}[Z(t)1_F|\mathcal{F}_s]\right] \\ = \mathbb{E}\left[\mathbb{E}[Z(t)|\mathcal{F}_s]1_F\right] = \mathbb{E}[Z(s)1_F] = \mathbb{Q}_s(F)\,.\end{aligned}$$

Proposition A.44. *Fixed $T > 0$, let Z and $\mathbb{Q}_T$ be as above. Let Y be an adapted, complex process. The process $\{Y(t),\ t \in [0,T]\}$ is a $\mathbb{Q}_T$-martingale if and only if $\{Z(t)Y(t),\ t \in [0,T]\}$ is a $\mathbb{P}_T$-martingale.*

Proof. Let ZY be a $\mathbb{P}_T$-martingale. Then, $\forall t, s : T \geq t \geq s \geq 0, \forall F \in \mathcal{F}_s$, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_T}\left[1_F\,\mathbb{E}_{\mathbb{Q}_T}[Y(t)|\mathcal{F}_s]\right] &= \mathbb{E}_{\mathbb{Q}_T}[1_F Y(t)] = \mathbb{E}[Z(t)1_F Y(t)] \\ &= \mathbb{E}\left[1_F\,\mathbb{E}[Z(t)Y(t)|\mathcal{F}_s]\right] = \mathbb{E}[1_F Z(s)Y(s)] \\ &= \mathbb{E}_{\mathbb{Q}_T}[1_F Y(s)],\end{aligned}$$

which implies $\mathbb{E}_{\mathbb{Q}_T}[Y(t)|\mathcal{F}_s] = Y(s)$.

Let Y be a $\mathbb{Q}_T$-martingale. Then, $\forall t, s : T \geq t \geq s \geq 0, \forall F \in \mathcal{F}_s$, we have

$$\begin{aligned}\mathbb{E}\left[1_F\,\mathbb{E}[Z(t)Y(t)|\mathcal{F}_s]\right] &= \mathbb{E}[1_F Z(t)Y(t)] = \mathbb{E}_{\mathbb{Q}_T}[1_F Y(t)] \\ &= \mathbb{E}_{\mathbb{Q}_T}\left[1_F\,\mathbb{E}_{\mathbb{Q}_T}[Y(t)|\mathcal{F}_s]\right] = \mathbb{E}_{\mathbb{Q}_T}[1_F Y(s)] \\ &= \mathbb{E}[Z(s)1_F Y(s)],\end{aligned}$$

which implies $\mathbb{E}[Z(t)Y(t)|\mathcal{F}_s] = Z(s)Y(s)$. □

A.5.4 Girsanov Theorem

Let W be a d-dimensional Wiener process defined in a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfying the usual conditions.

Theorem A.45. *Assume that the processes $G_j \in \mathcal{L}^2$, $j = 1,\ldots,d$, are real and such that $Z(t)$ defined by* Eq. (A.28) *is a martingale and introduce the probabilities $\mathbb{Q}_t$ by Eq.* (A.3). *Define a continuous processes $\widehat{W}$ by*

$$\widehat{W}_j(t) := W_j(t) - \int_0^t G_j(s)\mathrm{d}s\,, \qquad j = 1,\ldots,d, \quad 0 \leq t < +\infty\,. \tag{A.36}$$

Then, for each fixed $T \geq 0$, $\{\widehat{W}(t),\ t \in [0,T]\}$ is a d-dimensional Wiener process in $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), \mathbb{Q}_T)$.

Proof. We take the proof from [3, pp. 146–147].

By Theorem A.39 it is enough to prove that $Y_\lambda(t) := \exp \sum_j \left[\mathrm{i}\lambda_j \widehat{W}_j(t) + \frac{1}{2}\lambda_j{}^2 t\right]$ is a $\mathbb{Q}_T$-martingale, $\forall \lambda \in \mathbb{R}^d$. By Proposition A.44, we have to check whether $X_\lambda(t) := Z(t)Y_\lambda(t)$ is a $\mathbb{P}$-martingale.

By Eqs. (A.28) and (A.36) we have

$$\begin{aligned} X_\lambda(t) &= \exp \sum_j \left\{ \int_0^t G_j(s)\,\mathrm{d}W_j(s) \right. \\ &\qquad \left. - \frac{1}{2}\int_0^t G_j(s)^2\,\mathrm{d}s + \mathrm{i}\lambda_j W_j(t) - \mathrm{i}\lambda_j \int_0^t G_j(s)\,\mathrm{d}s + \frac{1}{2}\lambda_j{}^2 t \right\} \\ &= \exp \sum_j \left\{ \int_0^t \left[G_j(s) + \mathrm{i}\lambda_j\right]\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t \left[G_j(s) + \mathrm{i}\lambda_j\right]^2 \mathrm{d}s \right\}. \end{aligned}$$

By Eq. (A.30), $X_\lambda(t)$ is a local martingale and we already know that it is a martingale for $\lambda = 0$.

If the random variable $\sum_j \int_0^T \left|G_j(s)\right|^2 \mathrm{d}s$ is bounded, $X_\lambda(t)$ is a $\mathbb{P}$-martingale by Proposition A.42. Otherwise, we can prove that $X_\lambda(t)$, $0 \leq t \leq T$, is a martingale by "localisation" techniques.

We introduce the stopping times

$$\tau_n := \inf\left\{ t \leq T : \left| \sum_j \int_0^t G_j(s)\,\mathrm{d}W_j(s) \right| + \sum_j \int_0^t \left|G_j(s)\right|^2 \mathrm{d}s > n \right\}; \quad \text{(A.37)}$$

when the set of times in the right hand side is empty, we take $\tau_n = T$. Then, we have

$$\lim_{n\to+\infty} \tau_n = T\,, \qquad \mathbb{P}\text{-a.s.} \quad \text{(A.38)}$$

Let us define

$$G_j^n(s,\omega) := 1_{[0,\tau_n(\omega))}(s)\, G_j(s,\omega)\,, \quad \text{(A.39)}$$

$$X_\lambda^n(t) := \exp \sum_j \left\{ \int_0^t \left[G_j^n(s) + \mathrm{i}\lambda_j\right]\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t \left[G_j^n(s) + \mathrm{i}\lambda_j\right]^2 \mathrm{d}s \right\}. \quad \text{(A.40)}$$

Due to the structure (A.40) and the definitions (A.37), (A.39), $X_\lambda^n(t)$ is a martingale and by (A.38) it converges a.s.:

$$\mathbb{E}\left[X_\lambda^n(t)\middle|\mathcal{F}_s\right] = X_\lambda^n(s)\,, \qquad 0 \le s \le t \le T\,, \tag{A.41}$$

$$\lim_{n\to+\infty} X_\lambda^n(t) = X_\lambda(t) \qquad \mathbb{P}\text{-a.s.} \tag{A.42}$$

If we prove that $X_\lambda^n(t)$ converges in L^1 too, we can take the limit in the martingale relation (A.41) and we conclude that X_λ is a $\mathbb{P}$-martingale.

To prove this L^1-convergence, let us consider $Z(t \wedge \tau_n)$ and define

$$Y_\lambda^n(t) := \exp \sum_j \left\{ \mathrm{i}\lambda_j W_j(t) - \mathrm{i}\lambda_j \int_0^t G_j^n(s)\,\mathrm{d}s + \frac{1}{2}{\lambda_j}^2 t \right\}.$$

Then, we have

$$X_\lambda^n(t) = Z(t \wedge \tau_n) Y_\lambda^n(t)\,, \tag{A.43}$$

$$\lim_{n\to+\infty} Y_\lambda^n(t) = Y_\lambda(t) \quad \mathbb{P}\text{-a.s.}, \qquad |Y_\lambda(t)| = \left|Y_\lambda^n(t)\right| = \exp\left\{\frac{1}{2}\,\|\lambda\|^2\, t\right\}, \tag{A.44}$$

$$\lim_{n\to+\infty} Z(t \wedge \tau_n) = Z(t) \quad \mathbb{P}\text{-a.s.}, \tag{A.45}$$

$$\mathbb{E}[Z(t \wedge \tau_n)|\mathcal{F}_s] = Z(s \wedge \tau_n)\,, \qquad \mathbb{E}[Z(t)] = \mathbb{E}[Z(t \wedge \tau_n)] = 1\,. \tag{A.46}$$

By setting $H_n(t) := \min\{Z(t \wedge \tau_n), Z(t)\}$, we have $\lim_{n\to+\infty} H_n(t) = Z(t)$ a.s. But $0 \le H_n(t) \le Z(t)$ and, by Lebesgue theorem,

$$\lim_{n\to+\infty} \mathbb{E}[H_n(t)] = \mathbb{E}[Z(t)]. \tag{A.47}$$

By (A.46) and (A.47) we get

$$\begin{aligned}\mathbb{E}\left[|Z(t) - Z(t \wedge \tau_n)|\right] = \mathbb{E}\left[\big(Z(t) - H_n(t)\big) + \big(Z(t \wedge \tau_n) - H_n(t)\big)\right] \\ = 2\,\mathbb{E}\left[Z(t) - H_n(t)\right] \underset{n\to+\infty}{\longrightarrow} 0\,.\end{aligned}$$

Therefore,

$$L^1\text{-}\lim_{n\to+\infty} Z(t \wedge \tau_n) = Z(t)\,. \tag{A.48}$$

By the two conditions (A.44) and the L^1-convergence (A.48), we get the L^1-convergence $L^1\text{-}\lim_{n\to+\infty} Z(t \wedge \tau_n) Y_\lambda^n(t) = Z(t) Y_\lambda(t)$ or $L^1\text{-}\lim_{n\to+\infty} X_\lambda^n(t) = X_\lambda(t)$, which is what we needed. □

Remark A.46. In the hypotheses of Theorem A.45, according to the discussion of Sect. A.5.3, the probability measures $\{\mathbb{Q}_t,\ t \in [0, T]\}$ are consistent and $\mathbb{Q}_t \sim \mathbb{P}_t$. In particular, given $T > 0$, let us consider the two equivalent probability measures $\mathbb{P} = \mathbb{P}_T$ and $\mathbb{Q}_T$ on the σ-algebra $\mathcal{F}_T$. As said in Sect. A.1.2, given two random variables, the notions of equivalence with respect to $\mathbb{P}_T$ and $\mathbb{Q}_T$ coincide. Similarly,

given two stochastic processes, the notions of modification, indistinguishableness and equivalence with respect to $\mathbb{P}_T$ and $\mathbb{Q}_T$ coincide, as well as the spaces of integrand processes $\mathcal{L}^p$ do not change if $\mathbb{P}_T$ is replaced by $\mathbb{Q}_T \sim \mathbb{P}_T$.

At this point we can introduce integrals with respect to $\mathrm{d}\widehat{W}_j(t)$ with two possible interpretations: we can consider this integral as an Itô integral in the probability space $(\Omega, \mathcal{F}_T, \mathbb{Q}_T)$ or we can write $\mathrm{d}\widehat{W}_j(t) = \mathrm{d}W_j(t) - G_j(t)\mathrm{d}t$ and we can interpret the integral as the sum of an Itô integral with respect to the probability $\mathbb{P}$ plus a Lebesgue integral. The two interpretations coincide, [5, pp. 195–196], [2, pp. 270–272].

Proposition A.47. *Assume the hypotheses of Theorem A.45 and suppose that $X \in \mathcal{L}^2$. We define under $\mathbb{P}$ the Itô process $\int_0^t X(s)\mathrm{d}W_j(s)$ and under $\mathbb{Q}_T$ the Itô process $\int_0^t X(s)\mathrm{d}\widehat{W}_j(s)$, $0 \le t \le T$. For $j = 1, \dots, d$ we have*

$$\int_0^t X(s)\mathrm{d}\widehat{W}_j(s) = \int_0^t X(s)\mathrm{d}W_j(s) - \int_0^t X(s)G_j(s)\mathrm{d}s, \qquad \forall t \in [0, T],\ \mathbb{P}\text{-a.s. and } \mathbb{Q}_T\text{-a.s.} \tag{A.49}$$

By this proposition, if $X(t)$ is the solution of the SDE

$$\mathrm{d}X(t) = b\big(X(t), t\big)\mathrm{d}t + \sum_{j=1}^{d} \sigma_j\big(X(t), t\big)\mathrm{d}W_j(t)\,, \qquad 0 \le t \le T\,,$$

in $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$, then it solves also the new SDE, for $0 \le t \le T$,

$$\mathrm{d}X(t) = \left[b\big(X(t), t\big) + \sum_{j=1}^{d} G_j(t)\sigma_j\big(X(t), t\big) \right] \mathrm{d}t + \sum_{j=1}^{d} \sigma_j\big(X(t), t\big)\mathrm{d}\widehat{W}_j(t)\,,$$

in $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{Q}_T)$.

A.5.5 Extension of the Local Probability Measures

Let us introduce the final σ-algebra

$$\mathcal{F}_\infty := \bigvee_{t>0} \mathcal{F}_t \subset \mathcal{F}\,.$$

An interesting problem is to see if it is possible to extend the consistent probabilities $\mathbb{Q}_t$, $t > 0$, to a unique probability $\mathbb{Q}$ on $\mathcal{F}_\infty$.

A case in which we have a positive answer is when there exists a random variable $Z_\infty \in L^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$ such that $Z(t) = \mathbb{E}[Z_\infty|\mathcal{F}_t]$. By the Doob's martingale convergence theorem this is equivalent to the uniform integrability of the martingale

Z [2, p. 13] or to the fact that $Z(t)$ converges a.s. and in L^1 as well, for $t \to \infty$. Indeed, by setting $\mathbb{Q}(\mathrm{d}\omega) = Z_\infty(\omega)\mathbb{P}(\mathrm{d}\omega)$ over $(\Omega, \mathcal{F}_\infty)$ we get immediately $\mathbb{Q}\big|_{\mathcal{F}_t} = \mathbb{Q}_t$ and $\mathbb{Q}$ is the wanted extension. But our martingale Z is given by Eq. (A.28) and the existence of Z_∞ implies the existence, in some sense, of the integrals $\sum_j \int_0^{+\infty} G_j(s)^2 \mathrm{d}s$ and this can be a too restrictive requirement.

A typical example for which the extension of the probabilities $\mathbb{Q}_t$ to $\mathcal{F}_\infty$ does not exist is the following one. Let us take $d = 1$ and $G(s) = \mu \in \mathbb{R}$, $\mu \neq 0$. This gives $Z(t) = \exp\left\{\mu W(t) - \frac{1}{2}\mu^2 t\right\}$, which is a mean-one martingale, converging to zero a.s. (but not in L^1). We get also $\widehat{W}(t) = W(t) - \mu t$, which is a Wiener process under the new probabilities.

Let us consider the sets

$$A = \left\{\lim_{t\to\infty} \frac{1}{t} W(t) = 0\right\},$$

$$B = \left\{\lim_{t\to\infty} \frac{1}{t} \widehat{W}(t) = 0\right\} = \left\{\lim_{t\to\infty} \frac{1}{t} W(t) = \mu\right\} \subset \Omega \setminus A\,.$$

By the law of large numbers, we have

$$\mathbb{P}(A) = 1\,, \qquad \mathbb{P}(B) \leq \mathbb{P}(\Omega \setminus A) = 0\,.$$

By usual conditions we have $A,\ \Omega \setminus A,\ B,\ \Omega \setminus B \in \mathcal{F}_0 \subset \mathcal{F}_t \subset \mathcal{F}_\infty$. Since $\mathbb{Q}_t$ is equivalent to $\mathbb{P}\big|_{\mathcal{F}_t}$, we have also

$$\mathbb{Q}_t(A) = 1\,, \qquad \mathbb{Q}_t(B) = 0\,. \tag{A.50}$$

On the other end, if a probability $\mathbb{Q}$ on $\mathcal{F}_\infty$ extending $\mathbb{Q}_t$ would exist, we should have, again by the law of large numbers,

$$\mathbb{Q}(A) = 0\,, \qquad \mathbb{Q}(B) = 1\,.$$

But this is in contradiction with Eq. (A.50) and $\mathbb{Q}$ cannot exist.

A less ambitious program is to extend the law of $\{W(t),\ 0 \leq t \leq T\}$ under $\mathbb{Q}_T$ to a unique law for $\{W(t),\ 0 \leq t < +\infty\}$. To obtain this result we use the representation of a continuous process as a single random variable with values in the trajectory of the continuous functions with the σ-algebra generated by the canonical projections, see Sect. A.2.4 and Remark A.23.

By hypothesis $\{W(t),\ 0 \leq t < +\infty\}$ has continuous trajectories starting from zero. In Remark A.11 the trajectory space $C_0^d(0, \infty) =: \boldsymbol{C}$ was introduced; with the σ-algebra $\mathcal{B}(\boldsymbol{C}) =: \mathcal{C}$, the measurable space $(\boldsymbol{C}, \mathcal{C})$ is standard. By Proposition A.12 the map $\Xi_W : \Omega \to \boldsymbol{C}$, defined by $\Xi_W(\omega) = \{t \mapsto W(t, \omega)\}$ is $\mathcal{F}_\infty^W/\mathcal{C}$-measurable, where $(\mathcal{F}_t^W)$ is the natural (**not** augmented) filtration of the process W and $\mathcal{F}_\infty^W = \bigvee_{t>0} \mathcal{F}_t^W$. Under $\mathbb{P}$ the law of the random variable Ξ_W is the Wiener measure (Remark A.23).

In $\boldsymbol{C}$ the canonical projections $\Pi(t) : \boldsymbol{C} \to \mathbb{R}^d$, $\Pi(t; c) = c(t)$ were introduced; we set $\mathcal{C}_t = \sigma\{\Pi(s), s \in [0, t]\}$. By the arguments given in [1, p. 192], one shows that $F \in \mathcal{F}_t^W \Leftrightarrow \exists A \in \mathcal{C}_t : \{\Xi_W \in A\} = F$. By defining $\mathbb{Q}_T^W(A) := \mathbb{Q}_T(\Xi_W \in A)$, $A \in \mathcal{C}_T$, one constructs a consistent set of probabilities on $\boldsymbol{C}$. By a version of Kolmogorov's extension theorem, which holds for standard Borel spaces [4, p. 18], one gets that there exists a unique probability $\mathbb{Q}^W$ on $(\boldsymbol{C}, \mathcal{C})$ such that $\mathbb{Q}^W\big|_{\mathcal{C}_T} = \mathbb{Q}_T^W$. By the correspondence $F \in \mathcal{F}_\infty^W \Leftrightarrow \exists A \in \mathcal{C} : \{\Xi_W \in A\} = F$, one induces a probability $\mathbb{Q}$ on $(\Omega, \mathcal{F}_\infty^W)$ and it turns out $\mathbb{Q}\big|_{\mathcal{F}_T^W} = \mathbb{Q}_T\big|_{\mathcal{F}_T^W}$. The conclusion is that there exists a unique probability $\mathbb{Q}$ on $(\Omega, \mathcal{F}_\infty^W)$ such that, for any fixed T, $\mathbb{Q}$ and $\mathbb{Q}_T$ coincide when restricted both to $\mathcal{F}_T^W$.

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Appendix B
Some Notions of Quantum Mechanics

B.1 Notations

The axiomatic formulation of quantum mechanics involves complex separable Hilbert spaces and operators over them. We work only in finite dimensional Hilbert spaces, in order to avoid the analytical complications of operator theory.

B.1.1 The Hilbert Space $\mathcal{H} = \mathbb{C}^n$

Let $\mathcal{H}$ be a complex Hilbert space of finite dimension n. Choosing a basis $\{e_i\}_{i=1}^n$, $\mathcal{H}$ is identified with $\mathbb{C}^n$ and every vector $\psi \in \mathcal{H}$ is identified with the n-tuple of its coordinates (ψ_i), where $\psi = \sum_{i=1}^n \psi_i \, e_i$. For $\psi, \varphi \in \mathcal{H}$, the inner product, conjugate linear in the first variable and linear in the second one, is

$$\langle \varphi | \psi \rangle = \sum_{i=1}^{n} \overline{\varphi_i} \, \psi_i \, ,$$

where the overbar denotes the complex conjugation. Then, the square norm is

$$\| \psi \|^2 = \langle \psi | \psi \rangle = \sum_{i=1}^{n} |\psi_i|^2 \, .$$

B.1.2 Operators

With the same choice as above of a basis in $\mathcal{H}$, a linear operator A on $\mathcal{H}$ is identified with an $n \times n$ complex matrix: $(A\psi)_i = \sum_j A_{ij} \, \psi_j$. We denote by M_n the set of the $n \times n$ complex matrices and by $\mathbb{1}$ the identity operator: $\mathbb{1}_{ij} = \delta_{ij}$.

The *adjoint operator* A^* of A, defined by $\langle \varphi | A\psi \rangle = \langle A^* \varphi | \psi \rangle$, $\forall \psi, \varphi \in \mathcal{H}$, is given by the transposed, complex-conjugated matrix $(A^*)_{ij} = \overline{A_{ji}}$.

An *orthogonal projection* P is a self-adjoint idempotent operator, i.e. $P^* = P$, $P^2 = P$. It has only 0 and 1 as eigenvalues.

A self-adjoint operator $A^* = A$ has real eigenvalues, say a_k, and can be written as $A = \sum_k a_k P_k$, with $a_k \neq a_l$ for $k \neq l$, and where the operators P_k are its eigen-projections: ${P_k}^2 = P_k = {P_k}^*$, $P_k P_l = \delta_{kl} P_l$, $\sum_k P_k = \mathbb{1}$.

A *positive operator* $A \geq 0$ is an operator such that $\langle \psi | A\psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$, or, equivalently, a self-adjoint operator with non-negative eigenvalues.

B.1.2.1 Trace, Commutator and Anti-Commutator

The *trace* of an operator A is $\mathrm{Tr}\{A\} = \sum_i A_{ii}$, which does not depend on the chosen basis. Recall that, for every $a, b \in \mathbb{C}$,

$$\mathrm{Tr}\{aA + bB\} = a\,\mathrm{Tr}\{A\} + b\,\mathrm{Tr}\{B\}, \qquad \mathrm{Tr}\{AB\} = \mathrm{Tr}\{BA\}.$$

Given two operators, a very important quantity is the *commutator* between them:

$$[A, B] := AB - BA\,. \tag{B.1}$$

We shall use the commutator only as a useful notation, but its importance comes from its analogies with the Poisson brackets in classical mechanics. Another useful shorthand notation is the *anti-commutator*:

$$\{A, B\} := AB + BA\,.$$

B.1.2.2 Norms and Duality

Three useful norms can be introduced on M_n. Of course, as M_n is finite dimensional, they are all metrically equivalent and induce the Euclidean topology. The first one is the *operator norm*, which is denoted in two different ways, according to the context:

$$\|A\| \equiv \|A\|_\infty := \sup_{\psi \in \mathcal{H}:\|\psi\|=1} \|A\psi\|\,; \tag{B.2}$$

it is equal to the square root of the greatest eigenvalue of A^*A. The second one is the *Hilbert-Schmidt norm*:

$$\|A\|_2 := \sqrt{\mathrm{Tr}\{A^*A\}} = \sqrt{\sum_{ij} |A_{ij}|^2}\,. \tag{B.3}$$

The third one is the *trace norm*:

$$\|A\|_1 := \mathrm{Tr}\left\{\sqrt{A^*A}\right\}. \tag{B.4}$$

Note that

$$B \geq 0 \quad \Rightarrow \quad \|B\|_1 = \mathrm{Tr}\{B\}.$$

Moreover, one has

$$\left\|A^*\right\| = \|A\|\,, \qquad \left\|A^*\right\|_2 = \|A\|_2\,, \qquad \left\|A^*\right\|_1 = \|A\|_1\,. \tag{B.5}$$

Every operator A defines a linear functional on M_n under the correspondence $A \mapsto \mathrm{Tr}\{A\bullet\}$. This correspondence allows to identify $(M_n, \|\bullet\|_\infty)$ with the dual space of $(M_n, \|\bullet\|_1)$. Then, we have

$$\|A\|_\infty = \sup_{B\in M_n:\|B\|_1=1} |\mathrm{Tr}\{AB\}|\,, \qquad \|B\|_1 = \sup_{A\in M_n:\|A\|_\infty=1} |\mathrm{Tr}\{AB\}|\,. \tag{B.6}$$

Coherently, the *adjoint of a linear map* $\mathcal{O} : (M_n, \|\bullet\|_1) \to (M_n, \|\bullet\|_1)$ is the linear map $\mathcal{O}^* : (M_n, \|\bullet\|_\infty) \to (M_n, \|\bullet\|_\infty)$,

$$\mathrm{Tr}\{\mathcal{O}^*[A]\,B\} = \mathrm{Tr}\{A\,\mathcal{O}[B]\}, \qquad \forall A, B \in M_n\,. \tag{B.7}$$

B.1.2.3 Inequalities

The following inequalities hold:

$$\|A\| = \|A\|_\infty \leq \|A\|_2 \leq \|A\|_1 \leq n\ \|A\|_\infty\,, \tag{B.8}$$

$$|\langle\varphi|A\psi\rangle| \leq \|A\|\,\|\psi\|\,\|\varphi\|\,, \qquad |\mathrm{Tr}\{AB\}| \leq \begin{cases} \|A\|\,\|B\|_1\,, \\ \|A\|_2\,\|B\|_2\,. \end{cases} \tag{B.9}$$

B.1.3 Dirac Notations

If ψ is a vector in $\mathcal{H}$, the "ket" $|\psi\rangle$ denotes ψ itself thought as a column vector and the "bra" $\langle\psi|$ denotes the transposed conjugated vector:

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}, \qquad \langle\psi| = \left(\overline{\psi_1}\ \overline{\psi_2}\ \ldots\ \overline{\psi_n}\right).$$

Therefore, $|\psi\rangle\langle\psi'|$ is the rank-one operator $\varphi \mapsto \langle\psi'|\varphi\rangle\psi$ with matrix elements

$$\left(|\psi\rangle\langle\psi'|\right)_{ij} = \psi_i\,\overline{\psi'_j}\,.$$

Note that, for every operator A, $\mathrm{Tr}\{A\,|\psi\rangle\langle\varphi|\} = \langle\varphi|A\psi\rangle$.

If $\psi \in \mathcal{H}$ and $\psi \neq 0$, $\frac{|\psi\rangle\langle\psi|}{\|\psi\|^2}$ is the orthogonal projection on the Hilbert ray containing ψ.

B.2 The Hilbert Space Formulation of Quantum Mechanics

Let us start with the Hilbert space formulation of quantum mechanics: a state of the quantum system is represented by a normalised vector ψ in $\mathcal{H}$. The initial state depends on the way the system has been experimentally prepared and it determines the probability distributions of every measurement performed on the system.

B.2.1 Observables

A measurement on a quantum system can produce different results with some probability distribution depending on the state ψ. An event regarding such a measurement, which can occur or cannot, is represented by an *effect* E: a self-adjoint operator E such that $0 \leq E \leq \mathbb{1}$ [2]. We denote by $[0, \mathbb{1}]$ the set of all effects. Obviously, $E \leq \mathbb{1}$ means $\mathbb{1} - E \geq 0$. Denoting by the same symbol E both the event and the associated effect, the probability that E occurs, with the system in the state ψ, is

$$\mathbb{P}[E] = \langle\psi|E\psi\rangle.$$

Note that, for every effect E, the operator $\mathbb{1} - E$ is the effect associated to the complementary event $\neg E$.

The two effects E and $\neg E$ represent the simplest experiment, often called a yes–no experiment: the measurement has two possible outcomes, the outcome 'yes' associated with the effect E and the outcome 'no' associated with the effect $\neg E$. $\mathbb{P}[E] = \langle\psi|E\psi\rangle$ gives the probability of the outcome 'yes' when the pre-measurement state is ψ and $\mathbb{P}[\neg E] = 1 - \mathbb{P}[E]$ is the probability of the outcome 'no'.

A first generalisation is to have a discrete set of possible outcomes x_k; this experiment is represented by a collection of effects E_k, such that $\sum_k E_k = \mathbb{1}$. Then, the probability of observing the outcome x_k, with the system in the state ψ, is $\mathbb{P}[x_k] = \langle\psi|E_k\,\psi\rangle$.

More generally, the possible outcomes of an experiment form a set Ω and the possible events are the elements of a σ-algebra $\mathcal{F}$ of subsets of Ω. Then, the experiment is represented by a normalised effect-valued measure or positive operator-valued measure on the measurable space $(\Omega, \mathcal{F})$; the acronyms POV measure, or POVM, or POM are used.

Definition B.1. *Let $(\Omega, \mathcal{F})$ be a measurable space; a* positive operator-valued measure *is a map E from $\mathcal{F}$ into the set of the effects such that it is normalised and σ-additive, i.e. $E(\Omega) = \mathbb{1}$ and for any sequence $F_1, F_2, \ldots$ of incompatible events (disjoint sets) one has $E\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} E\,(F_k)$.*

The interpretation is: the probability of an outcome in F, for a set $F \in \mathcal{F}$, with the system in the state ψ, is

$$\mathbb{P}_\psi(F) = \langle \psi | E(F)\psi \rangle. \tag{B.10}$$

Note that $\mathbb{P}_\psi(\bullet)$ is indeed a probability distribution on $(\Omega, \mathcal{F})$.

The random outcomes of the experiment can be also thought as the values of some observable X. In this case, we often write $\mathbb{P}[X \in F] = \langle \psi | E(F)\psi \rangle$, we interpret it as the probability that the observable X takes a value in the set F and we call X the *output* of the measurement. With this interpretation in mind the POM itself is called *observable* or *generalised observable*. It is always possible to think the possible outcomes of an experiment as the values of an observable X; so, we indifferently speak of an observable X or of a POM E, of the outcomes of an experiment or of the values of an observable.

The measurable space $(\Omega, \mathcal{F})$ is called the *value space* of the observable X, or of the POM E. For the notion of measurable space and for its use in probability theory see Section A.1.1. Of course, every effect E determines the Bernoulli observable X which is 1, if E occurs, and 0, if $\neg E$ occurs.

Remark B.2.

- For every real θ, the states ψ and $e^{i\theta}\psi$ assign the same distribution to any observable X and thus there is no experimental way to distinguish between the two states.
- In infinite dimensional Hilbert spaces $\mathcal{H}$ the series in Definition B.1 has to be strongly convergent.
- In finite or infinite dimensional Hilbert spaces $\mathcal{H}$, if one adds the requirement that all the elements of a POM are orthogonal projections, it turns out that all the elements of the POM commute. The resulting measure is called a *projection-valued measure* (pvm). The Spectral Theorem gives a bijective correspondence between real observables X associated to pvms E on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ and self-adjoint operators A in $\mathcal{H}$:

$$A = \int_{\mathbb{R}} x \, E(\mathrm{d}x).$$

By restricting the theory to pvms only, we go back to the old formulation of quantum mechanics: observables are associated to self-adjoint operators and probabilities are obtained through spectral measures. In this case, the Spectral Theorem gives also a direct formula for the moments of a real observable X: denoted by $\mathbb{E}_\psi$ the expectation with respect to $\mathbb{P}_\psi$ and taken a measurable real function g, we have

$$\mathbb{E}_\psi[g(X)] = \langle \psi | g(A)\psi \rangle.$$

- In a finite dimensional Hilbert space a pvm is necessarily discrete, but this does not hold for a generic POM in any kind of Hilbert space nor for a pvm in the infinite dimensional case.
- Let us consider a pvm E on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$ and define

$$A_k := \int_{\mathbb{R}^d} x_k \, E(\mathrm{d}x_1 \cdots \mathrm{d}x_d).$$

  It turns out that the operators A_k are self-adjoint and commuting. Moreover, E is the joint spectral measure of such operators. By restricting the theory to pvms only and going back to the old formulation of quantum mechanics, only observables associated to commuting self-adjoint operators can be jointly measured.

B.2.2 The Abstract Schrödinger Equation

The Schrödinger equation gives the evolution of the state vector in the Hilbert space of the system. For a closed system the abstract Schrödinger equation is $\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar} H\psi(t)$, where H is a self-adjoint operator called Hamiltonian, which depends on the system itself and on the experimental conditions, and $\hbar = h/(2\pi)$ is the reduced Planck constant. It is usual in theoretical physics to choose the units in such a way that $\hbar = 1$. When the evolution is not autonomous, but without dissipation, one simply introduces a time-dependent Hamiltonian and the Schrödinger equation becomes

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = -\mathrm{i}H(t)\psi(t)\,, \qquad H(t) = H(t)^*\,, \quad \psi(0) = \psi \in \mathcal{H}\,, \quad \|\psi\| = 1\,. \tag{B.11}$$

The map $\psi \mapsto \psi(t)$ defines a unitary operator for every t.

A possible way to introduce evolutions with dissipation is the use of stochastic Hamiltonians or of SDEs of the type discussed in Section 2.4.4. However, a more traditional approach is the use of evolution equations for "statistical operators", as discussed in the remaining of this appendix.

B.3 The Statistical Formulation of Quantum Mechanics

Any physical statement is a probabilistic one, and probabilities are obtained from effects and states. Anyway, there are situations which are not well described by a unit vector and require a generalisation of the notion of state. For example, this happens when a preparation procedure introduces some classical uncertainty on the initial state, when the system undergoes some noisy evolution due to the interaction with the external world, when a measuring apparatus acts on the system, or when (see, for example, [2 Sect. 2.2]) the system under consideration is a subsystem of a composite quantum system whose state is pure but not factorised. The need to

include such situations in the mathematical description leads to the statistical formulation of quantum mechanics [1, 2, 4, 5].

B.3.1 Statistical Operators

Let us consider now the case in which there is uncertainty on the initial state of the system. Let us say that the state is one of the normalised vectors ψ_α with probability $p(\alpha)$; obviously, $p(\alpha) \geq 0$, $\sum_\alpha p(\alpha) = 1$.

Let E be any effect. According to the rules of Sect. B.2.1, the probability that E occurs, conditioned on knowing that the state is ψ_α, is $\mathbb{P}[E|\alpha] = \langle\psi_\alpha|E\psi_\alpha\rangle$. Then, the unconditional probability will be

$$\begin{aligned}\mathbb{P}[E] &= \sum_\alpha \mathbb{P}[E|\alpha]p(\alpha) = \sum_\alpha p(\alpha)\langle\psi_\alpha|E\psi_\alpha\rangle \\ &= \sum_\alpha p(\alpha)\,\mathrm{Tr}\left\{|\psi_\alpha\rangle\langle\psi_\alpha|\,E\right\} = \mathrm{Tr}\left\{\sum_\alpha p(\alpha)|\psi_\alpha\rangle\langle\psi_\alpha|\,E\right\}.\end{aligned}$$

By introducing the projections $P_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$ and by setting

$$\rho = \sum_\alpha p(\alpha)|\psi_\alpha\rangle\langle\psi_\alpha| = \sum_\alpha p(\alpha)P_\alpha\,, \tag{B.12}$$

for every event E we can write

$$\mathbb{P}[E] = \mathrm{Tr}\{\rho E\}. \tag{B.13}$$

Thus the operator ρ defined by Eq. (B.12) determines the probability $\mathbb{P}[E]$ of any event E regarding any measurement on the quantum system and hence it can represent the preparation of the system itself. The operator ρ is self-adjoint, positive of trace one. Vice versa, any operator ρ with these three properties can be written in the form (B.12), even if this decomposition is not unique. One of the choices is to take the spectral decomposition of ρ. Being self-adjoint, ρ has eigenvalues λ_k and eigen-projections P_k; if some eigenvalue is degenerate, repeat it according to its multiplicity and chose mutually orthogonal one-dimensional eigen-projections. Then, we can write $\rho = \sum_k \lambda_k P_k$; this decomposition is of the form (B.12) because $\rho \geq 0 \Rightarrow \lambda_k \geq 0$ and $\mathrm{Tr}\{\rho\} = 1 \Rightarrow \sum_k \lambda_k = 1$. We call an operator with these three properties a *statistical operator* or *density matrix*.

Let us denote by $\mathcal{S}(\mathcal{H})$ the set of all statistical operators on $\mathcal{H}$:

$$\mathcal{S}(\mathcal{H}) := \left\{\text{all operators } \rho \text{ on } \mathcal{H} \text{ such that: } \rho^* = \rho,\ \rho \geq 0,\ \mathrm{Tr}\{\rho\} = 1\right\}. \tag{B.14}$$

$\mathcal{S}(\mathcal{H})$ is a closed convex set. This means that, if $\mathbb{Q}$ is a probability measure on a measurable space $(A, \mathcal{A})$ and $\rho(\alpha)$ a measurable family of statistical operators, then

$$\rho = \int_A \rho(\alpha)\, \mathbb{Q}(\mathrm{d}\alpha) \in \mathcal{S}(\mathcal{H}). \tag{B.15}$$

The pair $\{\mathbb{Q}, \rho(\bullet)\}$ is called a *demixture* of ρ and it describes a possible uncertainty mechanism which produces the state ρ.

Remark B.3 (Pure and mixed states). In the language of statistical operators, each vector state ψ clearly corresponds to the one-dimensional projection $|\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H})$, with no more ambiguity about the state phase. These are called *pure states* because they do not admit any non-trivial demixture. All the other states are called *mixtures* or *mixed states*. Since a projection belongs to $\mathcal{S}(\mathcal{H})$ if and only if it is one-dimensional, a state ρ is pure if and only if $\rho^2 = \rho$.

Summarising, a state of a quantum system is represented by a statistical operator $\rho \in \mathcal{S}(\mathcal{H})$, an event related to a measurement on the system is represented by an effect $E \in [0, \mathbb{1}]$, and the probability that it occurs is $\mathrm{Tr}\{\rho E\}$. Then, if we have a generic measurement represented by a POM E, Eq. (B.10) becomes

$$\mathbb{P}_\rho(F) = \mathrm{Tr}\{\rho E(F)\}. \tag{B.16}$$

B.3.2 The von Neumann Equation

Let us consider now a quantum system which evolves without dissipation according to the Schrödinger equation (B.11), but let us take the mixture (B.12) as initial state. Every vector ψ_α evolves into $\psi_\alpha(t)$, with $\mathrm{d}\psi_\alpha(t)\big/\mathrm{d}t = -\mathrm{i}H(t)\psi_\alpha(t)$. Then, ρ evolves into

$$\rho(t) = \sum_\alpha p(\alpha)|\psi_\alpha(t)\rangle\langle\psi_\alpha(t)|. \tag{B.17}$$

Note in particular that the map $\rho \mapsto \rho(t)$ is well defined for every t, because it is independent of the decomposition (B.12); thus the map turns out to respect the convex structure of $\mathcal{S}(\mathcal{H})$, that is it admits a linear extension to M_n.

Thanks to the self-adjointness of $H(t)$, by differentiation we get the von Neumann equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho(t) = -\mathrm{i}[H(t), \rho(t)]. \tag{B.18}$$

Because of the analogy of the commutator in quantum mechanics with the Poisson brackets in classical mechanics, Eq. (B.18) is the quantum analog of the Liouville equation in classical mechanics and, by this, it is called also the Liouville–von Neumann equation and the operator $-\mathrm{i}[H(t), \bullet]$ is called the Liouvillian.

B.3.3 Master Equation and Quantum Dynamical Semigroups

Let us consider now an open quantum system which evolves interacting with the external world. If it is possible to predict the final system state on the basis of its initial state, this general dynamics has to be represented by a linear map $\mathcal{T}$ on M_n which sends statistical operators into statistical operators. In particular, such a map has to be positive ($\mathcal{T}[\tau] \geq 0$, $\forall \tau \geq 0$), but, due to some physical and mathematical considerations [2], one asks something more, the so-called complete positivity [6, 7].

Definition B.4. *A linear map $\mathcal{A}$ from M_n into itself is* completely positive *if for all integers m and for all choices of vectors ϕ_i, ψ_i, $i = 1, \ldots, m$, one has*

$$\sum_{i,j=1}^{m} \langle \phi_i | \mathcal{A}[|\psi_i\rangle\langle\psi_j|]\phi_j\rangle \geq 0\,. \tag{B.19}$$

Thus, a general state transformation for a quantum open system is represented by a *dynamical map* $\mathcal{T}$, which is a trace preserving, completely positive, linear map on M_n. In this order of ideas, a memoryless evolution in continuous time for an autonomous quantum open system is represented by a semigroup of dynamical maps $\mathcal{T}(t) = \mathrm{e}^{\mathcal{L}t}$, $t \geq 0$, which is called *quantum dynamical semigroup* or *quantum Markov semigroup*. The most general infinitesimal generator $\mathcal{L}$ of a quantum dynamical semigroup in the case of a finite dimensional Hilbert space (which is our case) was found in [8]. In the case of a separable Hilbert space, the most general infinitesimal generator of a uniformly continuous quantum dynamical semigroup was found in [9], while important results on the generator of an arbitrary quantum dynamical semigroup can be found in [1, 10].

Theorem B.5. *A map $\mathcal{L}$ is the infinitesimal generator of a quantum dynamical semigroup on M_n if and only if it has the structure: $\forall \tau \in M_n$,*

$$\begin{aligned}\mathcal{L}[\tau] &= -\mathrm{i}H\tau + \mathrm{i}\tau H + \frac{1}{2}\sum_{j=1}^{d}\left(2R_j\tau R_j^* - R_j^*R_j\tau - \tau R_j^*R_j\right)\\ &= -\mathrm{i}[H,\tau] + \frac{1}{2}\sum_{j=1}^{d}\left([R_j\tau, R_j^*] + [R_j, \tau R_j^*]\right),\end{aligned} \tag{B.20}$$

where d is some integer, R_j, H are operators on $\mathcal{H}$, H is self-adjoint.

When the evolution is memoryless, but the system is not autonomous, the dynamical generator becomes time dependent and the time change of the state is given by the so-called (quantum) *master equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \mathcal{L}(t)[\rho(t)], \qquad \rho(0) = \rho \in \mathcal{S}(\mathcal{H})\,, \tag{B.21}$$

where $\mathcal{L}(t)$ has the structure (B.20) for any t. Let us stress that a quantum master equation is a direct generalisation of the Liouville–von Neumann equation to the case in which dissipation is involved. Again, $\mathcal{L}(t)$ is called *Liouville operator* or *Liouvillian*.

B.4 Instruments

If we want to perform more than one measurement on a quantum system, its initial state ρ and the POM associated to the first measurement alone do not give a description exhaustive enough of the first measurement: they give the probability distribution of the first outcome, but not the system state after the first measurement, conditioned on the response, which is needed to evaluate the conditional probability distribution of the outcome of an arbitrary second measurement. We have to push the mathematical description of the measurement process at a higher level, introducing the action of the measurement on the system itself, i.e. giving the transformation from the pre-measurement state ρ to the post-measurement state conditioned on an arbitrary event E, which could occur in the experiment. This state describes the preparation of the system consisting of a preliminary preparation ρ, followed by the measurement and by the discarding of the system if E does not occur. Or else, in the frequency interpretation of probability, this state describes the sub-ensemble of independent systems prepared according to ρ, subjected to the measurement process and filtered on the occurrence of E.

B.4.1 Operations and Events

Let us consider again an event E related to a measurement on the system. If we want not only the probability of E, but also the state after the measurement conditioned on the response, an effect is no more enough. Now the fundamental object is an operation.

Definition B.6. *An* operation *is a completely positive, linear map* $\mathcal{O} : M_n \to M_n$ *such that* $\mathrm{Tr}\{\mathcal{O}[\tau]\} \leq \mathrm{Tr}\{\tau\}$, $\forall \tau \geq 0$.

To ask that the completely positive map $\mathcal{O}$ is trace decreasing is equivalent to the request that $\mathcal{O}^*[\mathbb{1}]$ is an effect. If the map $\mathcal{O}$ is completely positive, also $\mathcal{O}^*$ is such. The structure of the operations is given by the following theorem and it is known as Kraus decomposition.

Theorem B.7. *Any operation* $\mathcal{O}$ *on* M_n *can be represented as*

$$\mathcal{O}[\tau] = \sum_k A_k \tau A_k{}^*, \qquad \forall \tau \in M_n\,, \tag{B.22}$$

with $A_k \in M_n$ *and*

$$\sum_k A_k{}^* A_k \leq \mathbb{1}\,. \tag{B.23}$$

On the contrary, if Eq. (B.23) *holds, then Eq.* (B.22) *defines an operation. Of course,* $\mathcal{O}^*[\mathbb{1}] = \sum_k A_k{}^* A_k$.

The symbol $\sum_k$ denotes a converging series or a finite sum. Given an operation, the Kraus decomposition is not unique. Being M_n an n^2-dimensional space, at most n^2 elements A_k are needed to represent an operation. The Kraus decomposition holds also in the case of a separable Hilbert space; in this case the series in Eq. (B.23) has to be strongly convergent.

The operator $\mathcal{O}^*[\mathbb{1}]$ is the effect associated to the event E, while the whole map $\mathcal{O}$ describes the state change. Indeed, the physical interpretation is the following one. Let $\rho \in \mathcal{S}(\mathcal{H})$ be the pre-measurement state; then, the probability of the event E is

$$\mathbb{P}[E] = \mathrm{Tr}\{\mathcal{O}[\rho]\} = \mathrm{Tr}\{\mathcal{O}^*[\mathbb{1}]\rho\}, \tag{B.24}$$

and the post-measurement state, conditioned on the occurrence of E, is

$$\rho(E) := \mathcal{O}[\rho] \big/ \mathrm{Tr}\{\mathcal{O}[\rho]\} \in \mathcal{S}(\mathcal{H}). \tag{B.25}$$

The meaning of the post-measurement state is apparent when a sequence of two measurements is considered, because it allows to construct the joint probabilities. Let E' be a generic effect, which is measured after the measurement represented by the operation $\mathcal{O}$. For systems with pre-measurement state ρ we have a probability $\mathbb{P}[E] = \mathrm{Tr}\{\mathcal{O}[\rho]\}$ of observing E in the first measurement and, after a first measurement with E occurred, the state is $\rho(E)$. According to the usual rule (B.13), now the probability that E' occurs in the second measurement is $\mathrm{Tr}\left\{\rho(E)\, E'\right\}$. This is the conditional probability of E' in the second measurement given E in the first one, so we have

$$\mathbb{P}[E'|E] = \mathrm{Tr}\left\{\rho(E)\, E'\right\} = \frac{\mathrm{Tr}\left\{\mathcal{O}[\rho]\, E'\right\}}{\mathbb{P}[E]}\,. \tag{B.26}$$

Having the conditional probability $\mathbb{P}[E'|E]$ and the probability $\mathbb{P}[E]$ of the conditioning event, we construct the joint probability of the event E in the first measurement and of the event E' in the second one by

$$\mathbb{P}[E, E'] = \mathbb{P}[E'|E]\,\mathbb{P}[E] = \mathrm{Tr}\left\{\mathcal{O}[\rho]\, E'\right\} = \mathrm{Tr}\left\{\mathcal{O}^*[E']\,\rho\right\}. \tag{B.27}$$

Obviously, the probability of the complementary event $\neg E$ in the first measurement is $\mathbb{P}[\neg E] = 1 - \mathbb{P}[E] = 1 - \mathrm{Tr}\{\mathcal{O}[\rho]\}$, but the post-measurement state, conditional on the event $\neg E$, is not uniquely determined by $\mathcal{O}$: one needs a second operation $\tilde{\mathcal{O}}$ such that $\tilde{\mathcal{O}}^*[\mathbb{1}] + \mathcal{O}^*[\mathbb{1}] = \mathbb{1}$ and many choices with this property

are possible; just as many different operations $\mathcal{O}$ are associated to a same effect $E = \mathcal{O}^*[\mathbb{1}]$.

Note that an operation $\mathcal{O}$ is linear, but the map $\rho \mapsto \rho(E)$ is not. Given a mixed pre-measurement state

$$\rho = p(1)\,\rho_1 + p(2)\,\rho_2, \tag{B.28}$$

the post-measurement state, conditioned on the occurrence of E, is

$$\begin{aligned}\rho(E) =& \frac{p(1)\mathbb{P}_{\rho_1}[E]}{p(1)\mathbb{P}_{\rho_1}[E] + p(2)\mathbb{P}_{\rho_2}[E]}\,\rho_1(E) + \frac{p(2)\mathbb{P}_{\rho_2}[E]}{p(1)\mathbb{P}_{\rho_1}[E] + p(2)\mathbb{P}_{\rho_2}[E]}\,\rho_2(E)\\ =& p(1|E)\,\rho_1(E) + p(2|E)\,\rho_2(E),\end{aligned} \tag{B.29}$$

that is, if the quantum system is prepared in the state ρ_α with probability $p(\alpha)$, then, after the occurrence of E, the state of the system $\rho(E)$ is the convex combination of the post-measurement states $\rho_\alpha(E)$ with the probabilities $p(\alpha|E)$ of the cases α conditioned by the occurrence of E.

B.4.2 Instruments and Observables

If we want the system state after the measurement of an observable X, then the notion of *instrument* enters into play. An instrument gives both the probability distribution for X and the state change due to the measurement [1, 11, 12].

Definition B.8. *Let $(\Omega, \mathcal{F})$ be a measurable space. An* instrument $\mathcal{I}$ *is a normalised operation-valued measure, i.e.*

1. *$\mathcal{I}(F)$ is an operation, $\forall F \in \mathcal{F}$,*
2. *(normalisation)* $\mathrm{Tr}\,\{\mathcal{I}(\Omega)[\tau]\} = \mathrm{Tr}\,\{\tau\},\ \forall \tau \in M_n$,
3. *(σ-additivity) for every countable family $\{F_i\}$ of disjoint sets in $\mathcal{F}$*

$$\mathcal{I}\Big(\bigcup_i F_i\Big) = \sum_i \mathcal{I}(F_i).$$

From the previous properties and Definition B.1, the map

$$E_{\mathcal{I}}(F) := \mathcal{I}(F)^*[\mathbb{1}] \tag{B.30}$$

turns out to be a POM, which is interpreted as *the observable X associated with the instrument $\mathcal{I}$*. Thus, for every set $F \in \mathcal{F}$, given the pre-measurement state ρ, the probability of a result $X \in F$ is denoted by $\mathbb{P}_\rho(F)$ and it is given by

$$\mathbb{P}_\rho(F) = \mathrm{Tr}\,\{\rho\, E_{\mathcal{I}}(F)\} = \mathrm{Tr}\left\{\rho\, \mathcal{I}(F)^*[\mathbb{1}]\right\} = \mathrm{Tr}\{\mathcal{I}(F)[\rho]\}\,. \tag{B.31}$$

Moreover, the instrument gives the state after the measurement, conditioned upon the result $X \in F$. It is the conditional state [compare with (B.25)]

$$\rho(F) := \frac{\mathcal{I}(F)[\rho]}{\mathbb{P}_\rho(F)}\,, \tag{B.32}$$

which is a statistical operator if ρ is a state.

Many different instruments $\mathcal{I}$ are associated to a same observable X, as well as many different operations $\mathcal{O}$ are associated to a same effect E, reflecting the fact that many different experimental procedures can measure the same observable, but they may perturb the system in different ways.

B.4.2.1 A Priori and A Posteriori States

When we take the whole Ω in the role of F, we get $\mathbb{P}_\rho(\Omega) = 1$ and the conditional state becomes

$$\rho(\Omega) = \mathcal{I}(\Omega)[\rho]. \tag{B.33}$$

We can call $\rho(\Omega)$ the *a priori state*: if we know the pre-measurement state ρ and the instrument $\mathcal{I}$, then $\rho(\Omega)$ is the state we can "a priori" attribute to our system soon after the measurement, if we do not know the result of the measurement, that is, if we do not filter on the basis of the observed results. Note that $\mathcal{I}(\Omega)$ is a dynamical map.

Let us now consider the case when in (B.32) the set F shrinks to an "infinitesimally small" set $\mathrm{d}\omega$ around the value $\omega \in \Omega$: the quantity $\rho(\omega) = \mathcal{I}(\mathrm{d}\omega)[\rho]/\mathbb{P}_\rho(\mathrm{d}\omega)$ represents the state conditioned upon the result $X \in \mathrm{d}\omega$. The quantity $\rho(\omega)$ is the state one can attribute to those systems for which the result ω has been actually found in the measurement and for this reason we call it the *a posteriori* state [13]. Note that, while the conditional state $\rho(F)$ is a function defined on $\mathcal{F}$, the a posteriori state $\rho(\omega)$ is a function defined just on Ω. The mathematical definition, due to Ozawa [13], is the following one.

Definition B.9. *A family of statistical operators* $\{\rho(\omega),\, \omega \in \Omega\}$ *is said to be a family of* a posteriori states*, for a pre-measurement state* ρ *and an instrument* $\mathcal{I}$ *with value space* $(\Omega, \mathcal{F})$*, if the function* $\omega \mapsto \rho(\omega)$ *is measurable and,* $\forall F \in \mathcal{F}$*,*

$$\int_F \rho(\omega)\, \mathbb{P}_\rho(\mathrm{d}\omega) = \mathcal{I}(F)[\rho]. \tag{B.34}$$

For any instrument $\mathcal{I}$ and any pre-measurement state ρ, a family of a posteriori states $\rho(\omega)$ always exists and it is unique $\mathbb{P}_\rho$-a.s. [14] (the statement holds also in an infinite dimensional Hilbert space).

From a probabilistic point of view, the $\mathcal{F}$-measurable function $\rho : \Omega \to \mathcal{S}(\mathcal{H})$ is a random variable, the random a posteriori state, which is not known in advance,

but which will depend on the possible outcomes ω, occurring with probability distribution $\mathbb{P}_\rho$. Let us note that the a priori state is the expectation of the a posteriori state with respect to $\mathbb{P}_\rho$:

$$\mathcal{I}(\Omega)[\rho] = \int_\Omega \rho(\omega)\, \mathbb{P}_\rho(\mathrm{d}\omega)\,.$$

With the terminology introduced after Eq. (B.15), we can say that $\left\{\mathbb{P}_\rho,\ \rho(\bullet)\right\}$ is a demixture of the a priori state $\mathcal{I}(\Omega)[\rho]$.

Let us stress that Eq. (B.34) defines the a posteriori states once the instrument $\mathcal{I}$ and the pre-measurement state ρ are given. On the contrary, if $\rho(\omega)$ and $\mathbb{P}_\rho(\mathrm{d}\omega)$ are given for any pre-measurement state ρ and are known to come out from an instrument, then, Eq. (B.34) allows to reconstruct the instrument $\mathcal{I}$.

Note that, just as for operations (Sect. B.4.1), the map $\rho \mapsto \rho(\omega)$ is not linear. It can easily be checked that, given a mixed pre-measurement state

$$\rho = p(1)\,\rho_1 + p(2)\,\rho_2,$$

the a posteriori states are

$$\rho(\omega) = p(1)\,\frac{\mathbb{P}_{\rho_1}(\mathrm{d}\omega)}{\mathbb{P}_\rho(\mathrm{d}\omega)}\,\rho_1(\omega) + p(2)\,\frac{\mathbb{P}_{\rho_2}(\mathrm{d}\omega)}{\mathbb{P}_\rho(\mathrm{d}\omega)}\,\rho_2(\omega), \tag{B.35}$$

where $\mathbb{P}_{\rho_\alpha}(\mathrm{d}\omega)/\mathbb{P}_\rho(\mathrm{d}\omega)$ is the Radon–Nikodym derivative (Sect. A.1.2.1) of the probability $\mathbb{P}_{\rho_\alpha}$ with respect to the probability $\mathbb{P}_\rho = p(1)\mathbb{P}_{\rho_1} + p(2)\mathbb{P}_{\rho_2}$. Of course, $\mathbb{P}_{\rho_\alpha} \ll \mathbb{P}_\rho$. The interpretation is similar to Sect. B.4.1: if the quantum system is prepared in the state ρ_α with probability $p(\alpha)$, then, after the occurrence of the result ω, the state of the system $\rho(\omega)$ is the convex combination of the post-measurement states $\rho_\alpha(\omega)$ with the probabilities of the cases α conditioned by the occurrence of $X = \omega$. In this book we are often interested in the case $\mathbb{P}_\rho \sim \mathbb{Q}$ for every ρ in $\mathcal{S}(\mathcal{H})$, where $\mathbb{Q}$ is some mathematical reference probability. Than we also have

$$\rho(\omega) = p(1)\,\frac{\mathbb{P}_{\rho_1}(\mathrm{d}\omega)}{\mathbb{Q}(\mathrm{d}\omega)}\,\frac{\mathbb{Q}(\mathrm{d}\omega)}{\mathbb{P}_\rho(\mathrm{d}\omega)}\,\rho_1(\omega) + p(2)\,\frac{\mathbb{P}_{\rho_2}(\mathrm{d}\omega)}{\mathbb{Q}(\mathrm{d}\omega)}\,\frac{\mathbb{Q}(\mathrm{d}\omega)}{\mathbb{P}_\rho(\mathrm{d}\omega)}\,\rho_2(\omega). \tag{B.36}$$

B.4.2.1.1 Von Neumann Instruments

An example of instrument is given by the von Neumann measurement postulate. In the case of an observable X with values x_k and pvm $E_k = |k\rangle\langle k|$, $k = 1, \ldots, n$, that is $\Omega = \{x_1 \ldots, x_n\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$ (the set of the parts of Ω), the von Neumann instrument is

$$\mathcal{I}(F)[\tau] := \sum_{x_k \in F} E_k\, \tau\, E_k \equiv |k\rangle\langle k|\tau|k\rangle\langle k|, \qquad F \subseteq \Omega,$$

and, for every initial state ρ, the a posteriori states are always the pure states

$$\rho(x_k) = |k\rangle\langle k|.$$

In the case of degenerate spectrum (when some eigenspace has dimension greater than 1), the E_k become multi-dimensional projections. By restating the von Neumann–Lüders postulate in modern language, again the instrument is $\mathcal{I}(F)[\tau] := \sum_{x_k \in F} E_k\, \tau\, E_k$, but now the a posteriori state given the result x_k is $\rho(x_k) = \frac{E_k \rho E_k}{\mathrm{Tr}\{E_k \rho\}}$, which is not necessarily pure, unless ρ is pure.

B.4.2.2 Joint Distributions

When the measurement of an observable X is described by an instrument, and not simply by a POM, the conditional states after the measurement are available to construct the probability distribution of any other measurement X' following the measurement of X, together with the joint distribution of X and X'. The construction is similar to the one given in Eqs. (B.24), (B.25), (B.26), (B.27) in the case of operations.

Let X' be given by the POM E' on $(\Omega', \mathcal{F}')$. Then, for systems with pre-measurement state ρ and first outcome $X \in F$, the conditional probability $\mathbb{P}_\rho(F'|F)$ of a second result $X' \in F'$ is

$$\mathbb{P}_\rho(F'|F) = \mathbb{P}_{\rho(F)}(F') = \mathrm{Tr}\left\{\rho(F)\, E'(F')\right\} = \frac{\mathrm{Tr}\left\{\mathcal{I}(F)[\rho]\, E'(F')\right\}}{\mathbb{P}_\rho(F)}\,. \tag{B.37}$$

Thus, for every $F \in \mathcal{F}$, we get the conditional distribution $\mathbb{P}_\rho(\bullet|F)$ on $(\Omega', \mathcal{F}')$ for X', with the following interpretation. Before the measurement process starts, the a priori evaluation on the result of X' is given by $\mathbb{P}_\rho(\bullet|\Omega)$. Nevertheless, if after the first measurement we know that $X \in F$, then we attribute the state (B.32) to the system between the first and the second measurement, and we update the evaluation on the result of X' with the conditional distribution $\mathbb{P}_\rho(\bullet|F)$.

Obviously, having the conditional probability $\mathbb{P}_\rho(F'|F)$ and the probability $\mathbb{P}_\rho(F)$ of the conditioning event, we construct the joint probability $\mathbb{P}_\rho(F, F')$ of the result $X \in F$ in the first measurement and of the result $X' \in F'$ in the second one by

$$\mathbb{P}_\rho(F, F') = \mathbb{P}_\rho(F'|F)\,\mathbb{P}_\rho(F) = \mathrm{Tr}\left\{\mathcal{I}(F)[\rho]\, E'(F')\right\} = \mathrm{Tr}\left\{\mathcal{I}(F)^*[E'(F')]\,\rho\right\}. \tag{B.38}$$

This formula defines the joint probability distribution of X and X' on the rectangles $F \times F' \subset \Omega \times \Omega'$ for every $F \in \mathcal{F}$ and $F' \in \mathcal{F}'$. By standard arguments it can be proved that this probability has a unique extension to the whole σ-algebra $\mathcal{F} \otimes \mathcal{F}'$ generated by the rectangles. In particular, with the aid of the a posteriori states, the existence of the extension can be obtained with the same techniques which allow to

generate the multi-time probabilities of a Markov process from its initial law and its transition probabilities.

B.4.2.3 Dilations of Instruments

Ozawa [12] proved the following dilation theorem for instruments (it holds also when $\mathcal{H}$ is infinite dimensional and separable). The formulation of the theorem and its interpretation need the notions of tensor product of Hilbert spaces, product states, partial trace and composite quantum systems [2, Sect. 2.2]. But we do not use this theorem in the main text; it is given here only for interpretation and as an hint for further developments (it is essential to understand the construction of continuous measurements via quantum stochastic differential equations). For these reasons we do not give all the implied definitions here.

Theorem B.10. *Let $\mathcal{I}$ be an instrument with a standard Borel space $(\Omega, \mathcal{F})$ as value space. Then, there exist a separable Hilbert space $\mathcal{K}$, a projection-valued measure E on $\mathcal{K}$ with the same value space, a statistical operator σ on $\mathcal{K}$, a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ such that*

$$\mathcal{I}(F)[\rho] = \mathrm{Tr}_{\mathcal{K}} \left\{ (\mathbb{1} \otimes E(F))\, U\, (\rho \otimes \sigma)\, U^* \right\}, \qquad \forall F \in \mathcal{F}, \quad \forall \rho \in \mathcal{S}(\mathcal{H}). \tag{B.39}$$

Conversely, the right hand side of Eq. (B.39) *defines an instrument for any choice of $\mathcal{K}$, E, σ, U; E can be also a generic POM on $\mathcal{K}$.*

The value space $(\Omega, \mathcal{F})$ has been taken standard Borel (see p. 265) to prove that the Hilbert space $\mathcal{K}$ can be taken separable.

The interpretation of such a dilation is the following. The measuring apparatus is a quantum system represented in the Hilbert space $\mathcal{K}$ and prepared in the state σ. The system of interest is prepared in a generic state ρ, completely uncorrelated with the measuring apparatus, so that the initial state of the composed system is $\rho \otimes \sigma$. Measured system and measuring apparatus interact and evolve during some time interval as a closed system; the unitary evolution is represented by U. Then, some observable $E(\bullet)$ of the apparatus is measured. Such an observable plays the role of a pointer and gives information on the system of interest. Such a scheme is sometimes referred to as *indirect measurement*. Note that E can always be taken to be a projection-valued measure. Therefore, we can say that, by introducing indirect measurements, the old formulation of quantum mechanics can be recovered (cf. the last point of Remark B.2).

The second approach to continuous measurements quoted in Sect. 1.2, but not treated here, is indeed based on a dilation of the type (B.39): quantum stochastic differential equations are used to construct U and Bose fields are used to construct $E(\bullet)$.

B.4.3 A Sequence of Measurements

To understand measurements in continuous time (the main topic of this book), it is perhaps useful to start from the case of measurements at discrete times and to consider sequences of instruments.

B.4.3.1 Two Instruments

Let us consider now the measurement of an observable X_1 followed by the measurement of a second observable X_2, under the hypothesis that both of them are represented by an instrument, say $\mathcal{I}_j$ on $(\Omega_j, \mathcal{G}_j)$ for X_j. We assume any time specification to be included in the definition of the instruments. The joint probability (B.38) is equivalently given by

$$\mathbb{P}_\rho(F_1, F_2) = \mathrm{Tr}\left\{\mathcal{I}_2(F_2) \circ \mathcal{I}_1(F_1)[\rho]\right\}. \tag{B.40}$$

In this situation it is natural to look not only for the joint distribution of X_1 and X_2, but also for an instrument $\mathcal{I}_{12}$ on $(\Omega_1 \times \Omega_2, \mathcal{G}_1 \otimes \mathcal{G}_2)$, such that

$$\mathcal{I}_{12}(F_1 \times F_2) = \mathcal{I}_2(F_2) \circ \mathcal{I}_1(F_1), \qquad \forall F_j \in \mathcal{G}_j.$$

In this way the entire measurement process is described by a single instrument. Nevertheless, the existence of $\mathcal{I}_{12}$ is usually get under some extra condition. A sufficient and not very restrictive one is the requirement that $(\Omega_1, \mathcal{G}_1)$ and $(\Omega_2, \mathcal{G}_2)$ are both standard Borel spaces. Standard Borel spaces are presented in Sect. A.1.1.2; the composition of instruments is discussed in [1, 4].

B.4.3.2 Many Instruments

Let us consider a sequence of measurements of n observables X_j represented by the instruments $\mathcal{I}_j$ with value spaces $(\Omega_j, \mathcal{G}_j)$. We assume that the measurements are performed in the natural order ($\mathcal{I}_2$ after $\mathcal{I}_1$ and so on) and that any time specification is included in the definition of the instruments. Then, the space of the results is $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$, with the σ-algebra $\mathcal{F} = \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \cdots \otimes \mathcal{G}_n$. According to Eq. (B.40), the joint probability $\mathbb{P}_\rho(F_1, F_2, \ldots, F_n)$ of the sequence of results $X_1 \in F_1$, $X_2 \in F_2, \ldots, X_n \in F_n$, with $F_j \in \mathcal{G}_j$, when the pre-measurement state is ρ, is

$$\mathbb{P}_\rho(F_1, F_2, \ldots, F_n) = \mathrm{Tr}\left\{\mathcal{I}_n(F_n) \circ \cdots \circ \mathcal{I}_2(F_2) \circ \mathcal{I}_1(F_1)[\rho]\right\}. \tag{B.41}$$

By assuming that the measurable spaces $(\Omega_j, \mathcal{G}_j)$ are standard, we get the existence of an instrument $\mathcal{I}_{\mathrm{Tot}} = \mathcal{I}_n \circ \cdots \circ \mathcal{I}_1$ on $(\Omega, \mathcal{F})$ such that

$$\mathcal{I}_{\mathrm{Tot}}(F_1 \times F_2 \times \cdots \times F_n) = \mathcal{I}_n(F_n) \circ \cdots \circ \mathcal{I}_2(F_2) \circ \mathcal{I}_1(F_1), \qquad \forall F_j \in \mathcal{G}_j.$$

Similarly, the composition $\mathcal{I}_t \circ \cdots \circ \mathcal{I}_1$ defines a single instrument also for $t \leq n$.

Having a sequence of n measurements, we can consider any step between the first and the last measurement. Therefore, for every $t = 1, \ldots, n$, we consider the conditional probability of the results $X_{t+1} \in F_{t+1}, \ldots, X_n \in F_n$, given the previous results $X_1 \in F_1, \ldots, X_t \in F_t$,

$$\begin{aligned}\mathbb{P}_\rho(F_{t+1}, \ldots, F_n | F_1, \ldots, F_t) =& \frac{\mathbb{P}_\rho(F_1, F_2, \ldots, F_n)}{\mathbb{P}_\rho(F_1, \ldots, F_t)} \\ =& \mathbb{P}_{\rho(F_1,\ldots,F_t)}(F_{t+1}, \ldots, F_n) \\ =& \operatorname{Tr}\{\mathcal{I}_n(F_n) \circ \cdots \circ \mathcal{I}_{t+1}(F_{t+1})[\rho(F_1, \ldots, F_t)]\} ,\end{aligned}$$

where the conditional state

$$\rho(F_1, \ldots, F_t) = \frac{\mathcal{I}_t(F_t) \circ \cdots \circ \mathcal{I}_1(F_1)[\rho]}{\mathbb{P}_\rho(F_1, \ldots, F_t)} \tag{B.42}$$

represents the system state after the first t measurements conditioned upon the results $X_1 \in F_1, \ldots, X_t \in F_t$.

Shrinking again the sets F_j to sets $\mathrm{d}\omega_j$ around the values $\omega_j \in \Omega_j$, we find the state

$$\rho(\omega_1, \ldots, \omega_t) = \frac{\mathcal{I}_t(\mathrm{d}\omega_t) \circ \cdots \circ \mathcal{I}_1(\mathrm{d}\omega_1)[\rho]}{\mathbb{P}_\rho(\mathrm{d}\omega_1, \ldots, \mathrm{d}\omega_t)} ,$$

that is the a posteriori state at step t, which can be attributed to those systems for which the instrument $\mathcal{I}_t \circ \cdots \circ \mathcal{I}_1$ has measured the value $(\omega_1, \ldots, \omega_t)$ for the observable $(X_1, \ldots, X_t)$.

When t goes from 1 to n and the length of the outcome $(\omega_1, \ldots, \omega_t)$ increases, instead of considering an increasing sequence of spaces $\Omega_1, \Omega_1 \times \Omega_2, \ldots$, it is useful to take always the same space $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ and to introduce here the increasing family of σ-algebras $\mathcal{F}_t = (\mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \cdots \otimes \mathcal{G}_t) \times (\Omega_{t+1} \times \cdots \times \Omega_n)$, $1 \leq t \leq n$. This is an example of filtration of σ-algebras in discrete time; the analogous notion in continuous time is presented in Sect. A.2.2.

Then, all the a posteriori states can be defined on the same space Ω by

$$\rho(t, \omega) := \rho(\omega_1, \ldots, \omega_t) ,$$

where, as a function of the complete result $\omega = (\omega_1, \ldots, \omega_n)$, each $\rho(t)$ depends only on the first t coordinates of ω. Thus each $\rho(t)$ is measurable with respect to the corresponding σ-algebra $\mathcal{F}_t$, and, from a probabilistic point of view, the family of random states $\rho(t) : (\Omega, \mathcal{F}, \mathbb{P}_\rho) \to \mathcal{S}(\mathcal{H})$, is a discrete time stochastic process adapted to the filtration $\mathcal{F}_t$ (cf. Definition A.7). The filtration describes the increasing acquirement of information during the measurement steps, while the stochastic process describes the evolution of the a posteriori state, due to the action of the instruments. The evolution is therefore related to the experimental results observed

so far. The index t is discrete because this measurement consists of n independent instruments applied one after the other. A generalisation of this probabilistic structure to continuous time allows to describe measurements with an output continuous in time. Such a generalisation leads to consider a space of results Ω consisting of trajectories $\omega(t)$, $t \geq 0$, with its natural (cylindric) σ-algebra $\mathcal{F}$ and its natural filtration $\mathcal{F}_t$, $t \geq 0$, where each $\mathcal{F}_t$ is generated by $\omega(s)$, $0 \leq s \leq t$ (cf. Remark A.23). Then, for every pre-measurement state $\rho \in \mathcal{S}(\mathcal{H})$, a probability distribution $\mathbb{P}_\rho$ is introduced on $(\Omega, \mathcal{F})$ together with a stochastic process $\rho(t) : (\Omega, \mathcal{F}, \mathbb{P}_\rho) \to \mathcal{S}(\mathcal{H})$ of a posteriori states which, for every $t \geq 0$, will allow to reconstruct the instrument acting on the system during the time interval $[0, t]$. Roughly speaking, as the process is adapted, every $\rho(t, \omega)$ actually depends only on $\omega(s)$, $0 \leq s \leq t$, and it is the state which can be attributed at time t to those systems, with pre-measurement state ρ, for which the trajectory $\omega(s)$, $0 \leq s \leq t$, has been observed so far.

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Acronyms & Symbols

a.s.	almost surely, see Sect. A.1.1.3
c.c.	complex conjugate term
h.c.	Hermitian conjugate term
POM	positive operator valued measure, see Sect. B.2.1
pvm	projection valued measure, see Sect. B.2.1
SDE	stochastic differential equation
*	adjoint
$\overline{\quad}$	complex conjugate
$(\ )^{\mathrm{T}}$	transpose of a matrix
$\mathbb{1}$	identity operator on a Hilbert space, identity $n \times n$ matrix
A_t^s	propagator of the linear stochastic Schrödinger equation
$\mathcal{A}(t, s)$	propagator of the linear stochastic master equation
$\mathbb{C}$	complex numbers
$(\mathcal{D}_t^s)$	natural filtration of the increments of the d-dimensional Wiener process
$(\overline{\mathcal{D}}_t^s)$	augmented natural filtration of the increments of the d-dimensional Wiener process
$(\mathcal{G}_t^s)$	natural filtration of the increments of the m-dimensional output W
$(\overline{\mathcal{G}}_t^s)$	augmented natural filtration of the increments of the m-dimensional output W
$\mathbb{E}_{\mathbb{Q}}$	the expectation with respect to the probability measure $\mathbb{Q}$
$\widehat{\mathbb{E}}_{\psi_0}^T$	the expectation with respect to the probability measure $\widehat{\mathbb{P}}_{\psi_0}^T$
$\mathbb{E}_{\rho_0}^T$	the expectation with respect to the probability measure $\mathbb{P}_{\rho_0}^T$
$\mathcal{H}$	the Hilbert space of the quantum system of interest
Id_n	identity map on M_n
$\mathcal{L}^p$	a class of processes, see Sect. A.3.1
$\mathcal{M}^p$	a class of processes, see Sect. A.3.1
M_n	the space of the $n \times n$ complex matrices
$\mathcal{N}$	the class of null events

$\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$	stochastic basis of reference
$\widehat{\mathbb{P}}^T_{\psi_0}$	physical probability up to time T, Hilbert space formulation
$\mathbb{P}^T_{\rho_0}$	physical probability up to time T, statistical formulation
$\mathbb{R}$	real numbers
$\sigma\{\cdots\}$	σ-algebra generated by $\cdots$
$\mathcal{S}(\mathcal{H})$	the convex set of the statistical operators on $\mathcal{H}$
$\mathcal{T}(t, s)$	propagator of the quantum master equation

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