

Stochastic Dynamic Macroeconomics: Theory, Numerics and Empirical Evidence

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Preface

This book intends to contribute to the study of alternative paradigms in macroeconomics. As other recent approaches to dynamic macroeconomics, we also build on intertemporal economic behavior of economic agents but stress Keynesian features more than other recent literature in this area. In general, stochastic dynamic macromodels are difficult to solve and to estimate, in particular if intertemporal behavior of economic agents is involved. Thus, beside addressing important macroeconomic issues in a dynamic framework another major focus of this book is to discuss and apply solution and estimation methods to models with intertemporal behavior of economic agents.

The material of this book has been presented by the authors at several universities. Chapters of the book have been presented as lectures at Bielefeld University, Foscari University, Venice, University of Technology, Vienna, University of Aix-en-Provence, Columbia University, New York, New School University, New York, Beijing University, Tsinghua University, Beijing, Chinese University of HongKong, City University of HongKong and European Central Bank. Some chapters of the book have also been presented at the annual conference of the American Economic Association, Society of Computational Economics, and Society of Nonlinear Dynamics and Econometrics. We are grateful for comments by the participants of those conferences. We are also grateful for discussions with Toichiro Asada, Jean-Paul Benassy, Peter Flaschel, Buz Brock, Lars Grüne, Richard Day, Ray Fair, Stefan Mittnik, James Ramsey, Malte Sieveking, Michael Woodford and colleagues of our universities. We thank Uwe Köller for research assistance and Gaby Windhorst for editing and typing the manuscript. Financial support from the Ministry of Education, Science and Technology is gratefully acknowledged.

Introduction and Overview

The dynamic general equilibrium (DGE) model, in particular its more popular version, the Real Business Cycle Model, has become a major paradigm in macroeconomics. It has been applied in numerous fields of economics. Its essential features are the assumptions of intertemporal optimizing behavior of economic agents, competitive markets and price-mediated market clearing through flexible wages and prices. In this type of stochastic dynamic macromodeling only real shocks, such as technology shocks, monetary and government spending shocks variation in tax rates or shifts in preferences generate macro fluctuations.

Recently Keynesian features have been built into the dynamic general equilibrium (DGE) model by preserving its characteristics such as intertemporally optimizing agents and market clearing, but introducing monopolistic competition and sticky prices and wages into the model. In particular, in numerous papers and in a recent book Woodford (2003) has worked out this new paradigm in macroeconomics, which is now commonly called New Keynesian macroeconomics. In contrast to the traditional Keynesian macromodels such variants also presume dynamically optimizing agents and market clearing¹, but sluggish wage and price adjustments.

It is well known that the standard DGE model fails to replicate essential product, labor market and asset market characteristics. In our book, different from the DGE model, its competitive or monopolistic variants, we do not presume clearing of all markets in all periods. As in the monopolistic competition variant of the DGE model we permit nominal rigidities. Yet, by stressing Keynesian features in a model with production and capital accumulation, we demonstrate that even with dynamically optimizing agents not all markets may be cleared.

¹It should be noted that the concept of market clearing in recent New Keynesian literature is not unambiguous. We will discuss this issue in chapter 8.

Solution and Estimation Methods

Whereas models with Keynesian features are worked out and stressed in the chapters of part III of the book, part I and II provide the ground work for those later chapters. In part I and II of the book we build extensively on the basics of stochastic dynamic macroeconomics.

Part I of the book can be regarded as the technical preparation for our theoretical arguments developed in this volume. Here we provide a variety of technical tools to solve and estimate stochastic dynamic optimization models, which is a prerequisite for a proper empirical assessment of the models treated in our book. Solution methods are presented in chapters 1-2 whereas estimation methods along with calibration, the current methods of empirical assessment, are introduced in chapter 3. These methods are subsequently applied in the remaining chapters of the book.

Solving stochastic dynamic optimization models has been an important research topic in the last decade and many different methods have been proposed. Usually, an exact and analytical solution of a dynamic decision problem is not attainable. Therefore one has to rely on an approximate solution, which may also have to be computed by numerical methods. Recently, there have been developed numerous methods to solve stochastic dynamic decision problems. Among the well-known methods are the perturbation and projection methods (Judd (1998)), the parameterized expectations approach (den Haan and Marcet (1990)) and the dynamic programming approach (Santos and Vigo Aguiar (1998) and Grüne and Semmler (2004a)). When an exact and analytical solution to a dynamic optimization problem is not attainable and one has to use numerical methods. A solution method with higher accuracy often requires more complicated procedures and extensive computation time.

In this book, in order to allow for an empirical assessment of stochastic dynamic models we focus on approximate solutions that are computed from two types of first-order conditions: the Euler equation and the equation derived from the Lagrangian. Given these two types of first-order conditions, three types of approximation methods can be found in the literature: the Fair-Taylor method, the log-linear approximation method and the linear-quadratic approximation method. After a discussion on the variety of approximation methods, we introduce a method, which will be repeatedly used in the subsequent chapters. The method, which has been written into a GAUSS procedure, has the advantage of short computation time and easy implementation without sacrificing too much accuracy. We will also compare those methods with the dynamic programming approach.

Often the methods use a smooth approximation of first order conditions,

such as the Euler equation. Sometimes, as, for example, in the model of chapter 7 smooth approximations are not useful if the value function is not differentiable and thus is non-smooth. A method such as employed by Grüne and Semmler (2004a) can then be used.

There has been less progress made regarding the empirical assessment and estimation of stochastic dynamic models. Given the wide application of stochastic dynamic models expected in the future, we believe that the estimation of such type of models will become an important research topic. The discussion in chapters 3-6 can be regarded as an important step toward that purpose. As we will find, our proposed estimation strategy requires to solve the stochastic dynamic optimization model repeatedly, at various possible structural parameters searched by a numerical algorithm within the parameter space. This requires that the solution methods adopted in the estimation strategy should be as little time consuming as possible while not losing too much accuracy. After comparing different approximation methods, we find that the proposed methods of solving stochastic dynamic optimization models, such as used in chapters 3 - 6 most useful. We also will explore the impact of the use of different data sets on the calibration and estimation results.

RBC Model as a Benchmark

In the next part of the book, in part II, we set up a benchmark model, the RBC model, for comparison, in terms of either theory or empirics.

The standard RBC model is a representative agent model, but it is constructed on the basis of neoclassical general equilibrium theory. It therefore assumes that all markets (including product, capital and labor models) are cleared in all periods regardless of whether the model refers to the short- or the long-run. The imposition of market clearing requires that prices are set at an equilibrium level. At the pure theoretical level, the existence of such general equilibrium prices can be proved under certain assumption. Little, however, has been told how the general equilibrium can be achieved. In an economy in which both firms and households are price-takers, implicitly an auctioneer is presumed to exist who adjusts the price towards some equilibrium. Thus, the way of how an equilibrium is brought about is essentially a Walrasian tâtonnement process.

Working with such a framework of competitive general equilibrium is elegant and perhaps a convenient starting point for economic analysis. It nevertheless neglects many restrictions on the behavior of agents, the trading process and the market clearing process, the implementation of technology and the market structure, among many others. In part II of this volume,

we provide a thorough review of the standard RBC model, the representative stochastic dynamic model of competitive general equilibrium type. The review starts with laying out microfoundation, and continues to discuss a variety of empirical issues, such as the estimation of structural parameters, the data construction, the matching with the empirical data, its asset market implications and so on. The issues explored in this part of the book provide the incentives to introduce Keynesian features into a stochastic dynamic model as developed in Part III. Meanwhile, it also provides a reasonable ground to judge new model variants by considering whether they can resolve some puzzles as explored in part II of the book.

Open Ended Dynamics

One of the restrictions in the standard RBC model is that the firm does not face any additional cost (a cost beyond the usual activities at the current market prices) when it makes an adjustment on either price or quantity. For example, changing the price may require the firm to pay a menu cost and also, more importantly, a reputation cost. It is the cost, arising from price and wage adjustments that has become an important focus of New Keynesian research over the last decades.² However, adjustment cost may also come from a change in quantity. In a production economy increasing output requires the firm to hire new workers and add new capacity. In a given period of time, a firm may find more and more difficulties to create new additional capacity. This indicates that there will be an adjustment cost in creating capacity (or capital stock via investment), and further such adjustment cost may also be an increasing function of the size of investment.

In chapter 7, we will introduce adjustment costs into the benchmark RBC model. This may bring about multiple equilibria toward which the economy may move. The dynamics are open ended in the sense that it can move to low level, or high level of economic activity.³ Such an open ended dynamics is certainly one of the important feature of Keynesian economics. In recent times such open ended dynamics have been found in a large number of dynamic models with intertemporal optimization. Those models have been called indeterminacy and multiple equilibria models. Theoretical models of this type are studied in Benhabib and Farmer (1999) and Farmer (2001), and an empirical assessment is given in Schmidt-Grohe (2001). Some of the models

²Important papers in this reserach line are, for example, Calvo (1983) and Rotemberg (1982). For a recent review, see Taylor (1999) and Woodford (2003, ch. 3).

³Keynes (1936) discusses the possibility of such an open ended dynamics in chapter 5 of his book.

are real models, RBC models, with increasing returns to scale and/or more general preferences than power utility that generate indeterminacy. Local indeterminacy and globally multiplicity of equilibria can arise here. Others are monetary macro models, where consumers' welfare is affected positively by consumption and cash balances and negatively by the labor effort and an inflation gap from some target rates. For certain substitution properties between consumption and cash holdings those models admit unstable as well as stable high level and low level steady states. There also can be indeterminacy in the sense that any initial condition in the neighborhood of one of the steady-states is associated with a path toward, or away from, that steady state, see Benhabib et al. (2001).

Overall, the indeterminacy and multiple equilibria models predict an open ended dynamics, arising from sunspots, where the sunspot dynamics are frequently modeled by versions with multiple steady state equilibria, where there are also pure attractors (repellers), permitting any path in the vicinity of the steady state equilibria to move back to (away from) the steady state equilibrium. Although these are important variants of macrodynamic models with optimizing behavior, as, however, recently has been shown⁴ indeterminacy is likely to occur only within a small set of initial conditions. Yet, despite such unsolved problems the literature on open ended dynamics has greatly enriched macrodynamic modeling.

Pursuing this line of research we introduce a simple model where one does not need to refer to model variants with externalities and (increasing returns to scale) and/or to more elaborate preferences to obtain such results. We show that due to the adjustment cost of capital we may obtain non-uniqueness of steady state equilibria in an otherwise standard dynamic optimization version. Multiple steady state equilibria, in turn, lead to thresholds separating different domains of attraction of capital stock, consumption, employment and welfare level. As our solution shows thresholds are important as separation points below or above which it is advantages to move to lower or higher levels of capital stock, consumption, employment and welfare. Our model version thus can explain of how the economy becomes history dependent and moves, after a shock or policy influences, to a low or high level equilibria in employment and output.

Nonclearing Markets

A second important feature of Keynesian macroeconomics concerns the modeling of the labor market. An important characteristic of the DGE model

⁴See Beyn, Pampel and Semmler (2001) and Grüne and Semmler (2004a).

is that it is a market clearing model. For the labor market the DGE model predicts an excessive smoothness of labor effort in contrast to empirical data. The low variation in the employment series is a well-known puzzle in the RBC literature.⁵ It is related to the specification of the labor market as a cleared market. Though in its structural setting, see, for instance, Stockey et al. (1989), the DGE model specifies both sides of a market, demand and supply, the moments of the macro variables of the economy are, however, generated by a one-sided force due to its assumption on wage and price flexibility and thus equilibrium in all markets, including output, labor and capital markets. The labor effort results only from the decision rule of the representative agent to supply labor. In our view there should be no restriction for the other side of the market, the demand, to have effects on the variation of labor effort.

Attempts have been made to introduce imperfect competition features into the DGE model.⁶ In those types of models, producers set the price optimally according to their expected market demand curve. If one follows a Calvo price setting scheme, there will be a gap between the optimal price and the existing price. However, it is presumed that the market is still cleared since the producer is assumed to supply the output according to what the market demands for the existing price. This consideration also holds for the labor market. Here the wage rate is set optimally by the household according to the expected market demand curve for labor. Once the wage has been set, it is assumed to be rigid (or adjusted slowly). Thus, if the expectation is not fulfilled, there will be a gap again between the optimal wage and existing wage. Yet in the New Keynesian models the market is still assumed to be cleared since the household is assumed to supply labor whatever demand is at the given wage rate.⁷

In order to better fit the RBC model's predictions with the labor market data, search and matching theory has been employed⁸ to model the labor market in the context of an RBC model. Informational or institutional search frictions may then explain equilibrium unemployment rates and its rise. Yet, those models still have a hard time to explain the shift of unemployment rates such as, for example, experienced in Europe since the 1980s, as equilibrium unemployment rate.⁹

⁵A recent evaluation of this failure of the RBC model is given in Schmidt-Grohe (2001).

⁶Rotemberg and Woodford (1995, 1999), King and Wollman (1999), Gali (2001) and Woodford (2003) present a variety of models of monopolistic competition with price and wage stickiness.

⁷Yet, as we have mentioned above, this definition of market clearing is not unambiguous.

⁸For further details, see ch. 8.

⁹For an evaluation of the search and matching theory as well as the role of shocks to explain the evolution of unemployment in Europe, see Ljungqvist and Sargent (2003) and

As concerns the labor market along Keynesian lines we pursue an approach that allows for a nonclearing labor market. In our view the decisions with regard to price and quantities can be made separately, both subject to optimal behavior. When the price has been set, and is sticky for a certain period, the price is then given to the supplier when deciding on the quantities. There is no reason why the firm cannot choose the optimal quantity rather than what the market demands, especially when the optimum quantity is less than the quantity demanded by the market. This consideration will allow for nonclearing markets.¹⁰ Our proposed new model helps to study labor market problems by being based on adaptive optimization where households, after a first round of optimization, have to reoptimize when facing constraints in supplying labor in the market. On the other hand, firms may have constraints on the product markets. As we will show in chapters 8 and 9 such a multiple stage optimization model will allow for larger volatility of the employment rates as compared to the standard RBC model, and provides, also a framework to study the secular rise or fall of unemployment.

Technology and Demand Shocks

A further Keynesian feature of macromodels concerns the role of shocks. In the standard DGE model technology shocks are the driving force of the business cycles which is assumed to be measured by the Solow-residual. Since the Solow residual is computed on the basis of observed output, capital and employment, it is presumed that all factors are fully utilized. There are several reasons to distrust the standard Solow residual as a measure of technology shock. First, Mankiw (1989) and Summers (1986) have argued that such a measure often leads to excessive volatility in productivity and even the possibility of technological regress, both of which seem to be empirically implausible. Second, it has been shown that the Solow residual can be expressed by some exogenous variables, for example demand shocks arising from military spending (Hall 1988) and changed monetary aggregates (Evan 1992), which are unlikely to be related to factor productivity. Third, the standard Solow residual can be contaminated if the cyclical variation in factor utilization are significant.

Considering that the Solow-residual cannot be trusted as a measure of

Blanchard (2003)

¹⁰There is indeed a long tradition of macroeconomic modeling with specification of the nonclearing labor markets, see, for instance, Benassy (1995, 2002), Malinvaud (1994), Danthine and Donaldson (1990, 1995) and Uhlig and Xu (1996). Although our approach owes a substantial debt to disequilibrium models, we move beyond this type of literature.

technology shock, researchers have now developed different methods to measure technology shocks correctly. All these methods are focused on the computation of factor utilization. There are basically three strategies. The first strategy is to use an observed indicator to proxy for unobserved utilization. A typical example is to employ electricity use as a proxy for capacity utilization (see Burnside, Eichenbaum and Rebelo 1996). Another strategy is to construct an economic model so that one could compute the factor utilization from the observed variables (see Basu and Kimball 1997 and Basu, Fernald and Kimball 1998). A third strategy uses an appropriate restriction in a VAR estimate to identify a technology shock, see Gali (1999) and Francis and Ramey (2001, 2003).

It is well known that one of the major celebrated arguments of real business cycles theory is that technology shocks are pro-cyclical. A positive technology shock will increase output, consumption and employment. Yet this result is obtained from the empirical evidence, in which the technology shock is measured by the standard Solow-residual. As Gali (1999) and Francis and Ramey (2001, 2003) we also find that if one uses the corrected Solow-residual, the technology shock is negatively correlated with employment and therefore the RBC model loses its major driving force, see chapters 5 and 9.

Puzzles to be Resolved

In order to sum up, we may say that the standard RBC model has left us with major puzzles. The first type of puzzle is related to the asset market and is often discussed under the heading of the equity premium puzzle. Extensive research has attempted to improve on this problem by elaborating on more general preferences and technology shocks. Chapter 6 studies in details the asset price implication of the RBC model. The second puzzle is, as above mentioned, related to the labor market. The RBC model generally predicts an excessive smoothness of labor effort in contrast to empirical data. The model also implies an excessively high correlation between consumption and employment while empirical data only indicates a weak correlation.¹¹ Third, the RBC model predicts a significantly high positive correlation between technology and employment whereas empirical research demonstrates, at least at business cycle frequency, a negative or almost zero correlation. One might name it the technology puzzle. Whereas the first puzzle is studied in chapter 6 of the book, chapters 8-9 of part III of the book are mainly concerned with the latter two puzzles.

¹¹This problem of excessive correlation has, to our knowledge, not sufficiently been studied in the literature. It will be explored in Chapter 5 of this volume.

Finally, we want to note that the research along the line of Keynesian micro-founded macroeconomics has been historically developed by two approaches: one is the tradition of non-clearing market (or disequilibrium analysis), and the other is the New Keynesian analysis of monopolistic competition and sticky (or sluggish) prices. These two approaches will be contrasted in the last two chapters, Chapters 8 and 9. We will find that one can improve on the labor market and technology puzzles once we combine these two approaches. We want to argue that the two traditions can indeed be complementary rather than exclusive, and therefore they can somewhat be consolidated into a more complete system of price and quantity determination within the Keynesian tradition. The main new method we are using here to reconcile the two traditions is a multiple stage optimization behavior, adaptive optimization, where agents, reoptimize once they have perceived and learned about market constraints. Thus, adaptive optimization permits us to properly treat the market adjustment for nonclearing markets which, we hope, allows us to make some progress to match better the model with time series data.

Part I

Solution and Estimation of Stochastic Dynamic Models

Chapter 1

Solution Methods of Stochastic Dynamic Models

1.1 Introduction

The dynamic decision problem of an economic agent whose objective is to maximize his or her utility over an infinite time horizon is often studied in the context of a stochastic dynamic optimization model. To understand the structure of this decision problem we describe it in terms of a recursive decision problem of a dynamic programming approach. Thereafter, we discuss some solution methods frequently employed to solve the dynamic decision problem.

In most cases, an exact and analytical solution of the dynamic programming decision is not attainable. Therefore one has to rely on an approximate solution, which may also have to be computed by numerical methods. Recently, there have been developed numerous methods to solve stochastic dynamic decision problems. Among the well-known methods are the perturbation and projection methods (Judd (1996)), the parameterized expectations approach (den Haan and Marcet (1990)) and the dynamic programming approach (Santos and Vigo Aguiar (1998) and Grüne and Semmler (2004a)). In this book, in order to allow for an empirical assessment of stochastic dynamic models we focus on approximate solutions that are computed from two types of first-order conditions: the Euler equation and the equation derived from the Lagrangian. Given these two types of first-order conditions, three types of approximation methods can be found in the literature: the Fair-Taylor method, the log-linear approximation method and the linear-quadratic approximation method. Still, in most of the cases, an approximate solution cannot be derived analytically, and therefore a numerical algorithm is called

for to facilitate the computation of the solution.

In this chapter, we discuss those various approximation methods and then propose another numerical algorithm that can help us to compute approximate solutions. The algorithm is used to compute the solution path obtained from the method of linear-quadratic approximation with the first-order condition derived from the Lagrangian. While the algorithm takes full advantage of an existing one (Chow 1993), it overcomes the limitations given by the Chow (1993) method.

The remainder of this chapter is organized as follows. We start in Section 2 with the standard recursive method, which uses the value function for iteration. We will show that the standard recursive method may encounter difficulties when being applied to compute a dynamic model. Section 3 establishes the two first-order conditions to which different approximation methods can be applied. Section 4 briefly reviews the different approximation methods in the existing literature. Section 5 presents our new algorithm for dynamic optimization. Appendix I provides the proof of the propositions in the text. Finally, a GAUSS procedure that implements our suggested algorithm is presented in Appendix II.

1.2 The Standard Recursive Method

We consider a representative agent whose objective is to find a control (or decision) sequence $\{u_t\}_{t=0}^{\infty}$ such that

$$\max_{\{u\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t U(x_t, u_t) \right] \quad (1.1)$$

subject to

$$x_{t+1} = F(x_t, u_t, z_t). \quad (1.2)$$

Above, x_t is a vector of m state variables at period t ; u_t is a vector of n control variables; z_t is a vector of s exogenous variables whose dynamics does not depend on x_t and u_t ; E_t is the mathematical expectation conditional on the information available at time t and $\beta \in (0, 1)$ denotes the discount factor.

Let us first make several remarks regarding the formulation of the above problem. First, this formulation assumes that the uncertainty of the model only comes from the exogenous z_t . One popular assumption regarding the dynamics of z_t is to assume that z_t follow an $AR(1)$ process:

$$z_{t+1} = Pz_t + p + \epsilon_{t+1} \quad (1.3)$$

where ϵ_t is independently and identically distributed (*i.i.d.*). Second, this formulation is not restrictive to those structure models with more lags or leads. It is well known that a model with finite lags or leads can be transformed through the use of auxiliary variables into an equivalent model with one lag or lead. Third, the initial condition (x_0, z_0) in this formulation is assumed to be given.

The problem to solve a dynamic decision problem is to seek a time-invariant policy function G mapping from the state and exogenous (x, z) into the control u . With such a policy function (or control equation), the sequences of state $\{x_t\}_{t=1}^{\infty}$ and control $\{u_t\}_{t=0}^{\infty}$ can be generated by iterating the control equation

$$u_t = G(x_t, z_t) \quad (1.4)$$

as well as the state equation (1.2), given the initial condition (x_0, z_0) and the exogenous sequence $\{z_t\}_{t=1}^{\infty}$ generated by (1.3).

To find the policy function G by the recursive method, we first define a value function V :

$$V(x_0, z_0) \equiv \max_{\{u_t\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t U(x_t, u_t) \right] \quad (1.5)$$

Expression (1.5) could be transformed to unveil its recursive structure. For this purpose, we first rewrite (1.5) as follows:

$$\begin{aligned} V(x_0, z_0) &= \max_{\{u_t\}_{t=0}^{\infty}} \left\{ U(x_0, u_0) + E_0 \left[\sum_{t=1}^{\infty} \beta^t U(x_t, u_t) \right] \right\} \\ &= \max_{\{u_t\}_{t=0}^{\infty}} \left\{ U(x_0, u_0) + \beta E_0 \left[\sum_{t=0}^{\infty} \beta^t U(x_{t+1}, u_{t+1}) \right] \right\} \end{aligned} \quad (1.6)$$

It is easy to find that the second term in (1.6) can be expressed as being β times the value V as defined in (1.5) with the initial condition (x_1, z_1) . Therefore, we could rewrite (1.5) as

$$V(x_0, z_0) = \max_{\{u_t\}_{t=0}^{\infty}} \{U(x_0, u_0) + \beta E_0 [V(x_1, z_1)]\} \quad (1.7)$$

The formulation of equ. (1.7) represents a dynamic programming problem which highlights the recursive structure of the decision problem. In every period t , the planner faces the same decision problem: choosing the control variable u_t that maximizes the current return plus the discounted value of the optimum plan from period $t+1$ onwards. Since the problem repeats itself

every period the time subscripts become irrelevant. We thus can write (1.7) as

$$V(x, z) = \max_u \{U(x, u) + \beta E [V(\tilde{x}, \tilde{z})]\} \quad (1.8)$$

where the tilde (\sim) over x and z denotes the corresponding next period values. Obviously, they are subject to (1.2) and (1.3). Equation (1.8) is said to be the Bellman equation, named after Richard Bellman (1957). If we know the function V , we then can solve u via the Bellman equation.

Unfortunately, all these considerations are based on the assumption that we know the function V , which in reality we do not know in advance. The typical method in this case is to construct a sequence of value functions by iterating the following equation:

$$V_{j+1}(x, z) = \max_u \{U(x, u) + \beta E [V_j(\tilde{x}, \tilde{z})]\} \quad (1.9)$$

In terms of an algorithm, the method can be described as follows:

- Step 1. Guess a differentiable and concave candidate value function, V_j .
- Step 2. Use the Bellman equation to find the optimum u and then compute V_{j+1} according (1.9).
- Step 3. If $V_{j+1} = V_j$, stop. Otherwise, update V_j and go to step 1.

Under some regularity conditions regarding the function U and F , the convergence of this algorithm is warranted by the contraction mapping theorem (Sargent 1987, Stockey et al. 1989). However, the difficulty of this algorithm is that in each Step 2, we need to find the optimum u that maximize the right side of equ. (1.9). This task makes it difficult to write a closed form algorithm for iterating the Bellman equation. Researchers are therefore forced to seek different numerical approximation methods.

1.3 The First-Order Conditions

The last two decades have observed various methods of numerical approximation to solve the problem of dynamic optimization.¹ As above stated in

¹Kendrick (1981) can be regarded as a seminal work in this field. For a review up to 1990, see Taylor and Uhlig (1990). For the later development, see Chow (1997), Judd (1999), Ljungqvist and Sargent (2000) and Marimon and Scott (1999). Recent methods of numerically solving the above discrete time Bellman equation (1.9) can be found in Santos and Vigo-Aguiar (1998) and Grüne and Semmler (2004a).

this book we want to focus on the first-order conditions, that are used to derive the decision sequence. One can find two approaches: one is to use the Euler equation and the other the equation derived from the Lagrangian.

1.3.1 The Euler Equation

We start from the Bellman equation (1.8). The first-order condition for maximizing the right side of the equation takes the form:

$$\frac{\partial U(x, u)}{\partial u} + \beta E \left[\frac{\partial F}{\partial u}(x, u, z) \frac{\partial V}{\partial \tilde{x}}(\tilde{x}, \tilde{z}) \right] = 0 \quad (1.10)$$

The objective here is to find $\partial V/\partial x$. Assume V is differentiable and thus from (1.8) it satisfies

$$\frac{\partial V(x, z)}{\partial x} = \frac{\partial U(x, G(x, z))}{\partial x} + \beta E \left[\frac{\partial F}{\partial x}(x, G(x, z), z) \frac{\partial V}{\partial F}(\tilde{x}, \tilde{z}) \right] \quad (1.11)$$

This equation is often called the Benveniste-Scheinkman formula.² Assume $\partial F/\partial x = 0$. The above formula becomes

$$\frac{\partial V(x, z)}{\partial x} = \frac{\partial U(x, G(x, z))}{\partial x} \quad (1.12)$$

Substituting this formula into (1.10) gives rise to the Euler equation:

$$\frac{\partial U(x, u)}{\partial u} + \beta E \left[\frac{\partial F}{\partial u}(x, u, z) \frac{\partial U}{\partial \tilde{x}}(\tilde{x}, \tilde{u}) \right] = 0 \quad (1.13)$$

where the tilde (\sim) over u again denotes the next period value with respect to u .

Note that to use the above Euler equation as the first-order condition for deriving the decision sequence, one must require $\partial F/\partial x = 0$. In economic analysis, one often encounters models, after some transformation, in which x does not appear in the transition law so that $\partial F/\partial x = 0$ is satisfied. We will show this technique in the next chapter using a prototype model as a practical example. However, there are still models in which such transformation is not feasible.

²named after Benveniste and Scheinkman (1979).

1.3.2 Deriving the First-Order Condition from the Lagrangian

Suppose for the dynamic optimization problem as represented by (1.1) and (1.2), we can define the Lagrangian L :

$$L = E_0 \sum_{t=0}^{\infty} \{ \beta^t U(x_t, u_t) - \beta^{t+1} \lambda'_{t+1} [x_{t+1} - F(x_t, u_t, z_t)] \}$$

where λ_t , the Lagrangian multiplier, is a $m \times 1$ vector. Setting the partial derivatives of L to zero with respect to λ_t , x_t and u_t will yield equation (1.2) as well as

$$\frac{\partial U(x_t, u_t)}{\partial x_t} + \beta E_t \left[\frac{\partial F(x_{t+1}, u_{t+1}, z_{t+1})}{\partial x_{t+1}} \lambda_{t+1} \right] = \lambda_t, \quad (1.14)$$

$$\frac{\partial U(x_t, u_t)}{\partial u_t} + E_t \lambda_{t+1} \frac{\partial F(x_t, u_t, z_t)}{\partial u_t} = 0, \quad (1.15)$$

In comparison with the Euler equation, we find that there is an unobservable variable λ_t appearing in the system. Yet, using (1.14) and (1.15), one does not have to transform the model into the setting that $\partial F/\partial x = 0$. This is an important advantage over the Euler equation. Also as we will see in the next chapter, these two types of first-order conditions are equivalent when we appropriately define λ_t in terms of x_t , u_t and z_t .³ This further implies that they can produce the same steady states when being evaluated at their certainty equivalence forms.

1.4 Approximation and Solution Algorithms

1.4.1 The Gauss-Seidel Procedure and the Fair-Taylor Method

The state (1.2), the exogenous (1.3) and the first-order condition derived either as Euler equation (1.13) or from the Lagrangian (1.14) - (1.15) form a dynamic system from which the transition sequences $\{x_{t+1}\}_{t=0}^{\infty}$, $\{z_{t+1}\}_{t=0}^{\infty}$, $\{u_t\}_{t=0}^{\infty}$ and $\{\lambda_t\}_{t=0}^{\infty}$ are implied given the initial condition (x_0, z_0) . Yet, mostly such a system is highly nonlinear, and therefore the solution paths usually are impossible to be computed directly. One popular approach as suggested by Fair and Taylor (1983) is to use a numerical algorithm, called

³See also Chow (1997).

Gauss-Seidel procedure. For the convenience of presentation, the following discussion assumes only the Euler equation to be used.

Suppose the system can be written as the following $m + n$ equations:

$$f_1(y_t, y_{t+1}, z_t, \psi) = 0 \quad (1.16)$$

$$f_2(y_t, y_{t+1}, z_t, \psi) = 0 \quad (1.17)$$

$$\vdots$$

$$f_{m+n}(y_t, y_{t+1}, z_t, \psi) = 0 \quad (1.18)$$

Here y_t is the vector of endogenous variables with $m + n$ dimensions, including both state x_t and control u_t ,⁴ ψ is the vector of structural parameter. Also note that in this formulation we left aside the expectation operator E . This can be done by setting the corresponding disturbance term, if there is any, to their expectation values (usually zero). Therefore the system is essentially not different from the dynamic rational expectation model as considered by Fair and Taylor (1983).⁵ The system, as suggested, can be solved numerically by an iterative technique, called Gauss Seidel procedure, to which we shall now turn.

It is always possible to transform the system (1.16) - (1.18) as follows:

$$y_{1,t+1} = g_1(y_t, y_{t+1}, z_t, \psi) \quad (1.19)$$

$$y_{2,t+1} = g_2(y_t, y_{t+1}, z_t, \psi) \quad (1.20)$$

$$\vdots$$

$$y_{m+n,t+1} = g_{m+n}(y_t, y_{t+1}, z_t, \psi) \quad (1.21)$$

where, $y_{i,t+1}$ is the i th element in the vector y_{t+1} , $i = 1, 2, \dots, m + n$. Given the initial condition $y_0 = y_0^*$, and the sequence of exogenous variable $\{z_t\}_{t=0}^T$, with T to be the prescribed time horizon of our problem, the algorithm starts by setting $t = 0$ and proceeds as follows:

- Step 1. Set an initial guess on y_{t+1} . Call this guess $y_{t+1}^{(0)}$. Compute y_{t+1} according to (1.19) - (1.21) for the given $y_{t+1}^{(0)}$ along with y_t . Denote this new computed value $y_{t+1}^{(1)}$.
- Step 2. If the distance between $y_{t+1}^{(1)}$ and $y_{t+1}^{(0)}$ is less than a prescribed tolerance level, go to Step 3. Otherwise compute $y_{t+1}^{(2)}$ for the given $y_{t+1}^{(1)}$. This procedure will be repeated until the tolerance level is satisfied.

⁴If we use (1.14) and (1.15) as the first-order condition, then there will be $2m + n$ equations and y_t should include x_t, λ_t and u_t .

⁵Our suggested model here is a more simplified version since we only take one lead. See also a similar formulation in Juillard (1996) with one lag and one lead.

- Step 3. Update t by setting $t = t + 1$ and go to Step 1.

The algorithm will continue until t reaches T . This will produce a sequence of endogenous variable $\{y_t\}_{t=0}^T$, which include both decision $\{u_t\}_{t=0}^T$ and state $\{x_t\}_{t=0}^T$.

There is no guarantee that convergence can always be achieved for the iteration in each period. If this is the case, a damping technique can usually be employed to force convergence (see Fair 1984, chapter 7). The second disadvantage of this method is the cost of computation. The procedure requires the iteration and convergence for each period t , $t = 1, 2, 3, \dots, T$. This cost of computation makes it a difficult candidate solving the dynamic optimization problem. The third and the most important problem regards its accuracy of solution. Note that the procedure starts with the given initial condition y_0 , and therefore the solution sequences $\{u_t\}_{t=1}^T$ and $\{x_t\}_{t=1}^T$ depend virtually on the initial condition, which include not only the initial state x_0 but also the initial decisions u_0 . Yet the initial condition for the dynamic decision problem is usually provided only by x_0 (see equation our discussion in the last section). Considering that the weights of u_0 could be important in the value of the objective function (1.1), there might be a problem in accuracy. One possible way to deal with this problem is to start with different initial u_0 . In the next chapter when we turn to a practical problem, we will investigate these issues more thoroughly.

1.4.2 The Log-linear Approximation Method

Solving nonlinear dynamic optimization model with log-linear approximation has been widely used and well documented. This has been proposed in particular by King et al. (1988) and Campbell (1994) in the context of Real Business Cycle models. In principle, this approximation method can be applied to the first-order condition either in terms of the Euler equation or derived from the Lagrangean. Formally, let X_t be the variables, \bar{X} the corresponding steady state. Then,

$$x_t \equiv \ln X_t - \ln \bar{X}$$

is regarded to be the log-deviation of X_t . In particular, $100x_t$ is the percentage of X_t that it deviates from \bar{X} . The general idea of this method is to replace all the necessary equations in the model by approximations, which are linear in the log-deviation form. Given the approximate log-linear system, one then uses the method of undetermined coefficients to solve for the decision rule, which is also in the form of log-linear deviations.

Uhlig (1999) provides a toolkit of such method for solving a general dynamic optimization model. The general procedure involves the following steps:

- Step 1. Find the necessary equations characterizing the equilibrium law of motion of the system. These necessary equations should include the state equation (1.2), the exogenous equation (1.3) and the first-order condition derived either as Euler equation (1.13) or from the Lagrangian (1.14) and (1.15).
- Step 2. Derive the steady state of the model. This requires first parameterizing the model and then evaluating the model at its certainty equivalence form.
- Step 3. Log-linearize the necessary equations characterizing the equilibrium law of motion of the system. Uhlig (1999) suggests the following building block for such log-linearization.

$$X_t \approx \bar{X} e^{x_t} \quad (1.22)$$

$$e^{x_t + ay_t} \approx 1 + x_t + ay_t \quad (1.23)$$

$$x_t y_t \approx 0 \quad (1.24)$$

- Step 4. Solve the log-linearized system for the decision rule (which is also in log-linear form) with the method of undetermined coefficients.

In Chapter 3, we will provide a concrete example to apply the above procedure and to solve a practical problem of dynamic optimization.

Solving a nonlinear dynamic optimization model with log-linear approximation usually does not require a heavy computation in contrast to the Fair-Taylor method. In some cases, the decision rule could be derived even analytically. On the other hand, by assuming a log-linearized decision rule as expressed in (1.4), the solution path does not require the initial condition of u_0 and therefore it should be more accurate in comparison to the Fair-Taylor method. However, the process of log-linearization and solving for the undetermined coefficients are not easy, and usually have to be accomplished by hand. It is certainly desirable to have a numerical algorithm available that can take over at least part of the analytical derivation process.

1.4.3 Linear-quadratic Approximation with Chow's Algorithm

Another important approximation method is the linear-quadratic approximation. Again in principle this method can be applied to the first-order con-

dition either in terms of the Euler equation or derived from the Lagrangian. Chow (1993) was among the first who solves a dynamic optimization model with linear-quadratic approximation applied to the first-order condition derived from the Lagrangian. At the same time, he proposed a numerical algorithm to facilitate the computation of the solution.

Chow's method can be presented in both continuous and discrete time form. Since the models in discrete time are more convenient for empirical and econometric studies, we here only consider the discrete time version. The numerical properties of this approximation method have further been studied in Reiter (1997) and Kwan and Chow (1997).

Suppose the objective of a representative agent can again be written as (1.1), but subject to

$$x_{t+1} = F(x_t, u_t) + \epsilon_{t+1} \quad (1.25)$$

We shall remark that the state equation here is slightly different from the one as expressed by (1.2) in Section 2. Apparently, it is only a special case of (1.2). Consequently, the Lagrangian L should be defined as

$$L = E_0 \sum_{t=0}^{\infty} \{ \beta^t U(x_t, u_t) - \beta^{t+1} \lambda'_{t+1} [x_{t+1} - F(x_t, u_t) - \epsilon_{t+1}] \}$$

Setting the partial derivatives of L to zero with respect to λ_t , x_t and u_t will yield equation (1.25) as well as

$$\frac{\partial U(x_t, u_t)}{\partial x_t} + \beta \frac{\partial F(x_t, u_t)}{\partial x_t} E_t \lambda_{t+1} = \lambda_t, \quad (1.26)$$

$$\frac{\partial U(x_t, u_t)}{\partial u_t} + \beta \frac{\partial F(x_t, u_t)}{\partial u_t} E_t \lambda_{t+1} = 0, \quad (1.27)$$

The linear-quadratic approximation assumes the state equations to be linear and the objective function to be quadratic. In other words,

$$\frac{\partial U(x_t, u_t)}{\partial x_t} = K_1 x_t + K_{12} u_t + k_1 \quad (1.28)$$

$$\frac{\partial U(x_t, u_t)}{\partial u_t} = K_2 u_t + K_{21} x_t + k_2 \quad (1.29)$$

$$F(x_t, u_t) = A x_t + C u_t + b + \epsilon_{t+1} \quad (1.30)$$

Given this linear-quadratic assumption, equation (1.26) and (1.27) can be rewritten as

$$K_1 x_t + K_{12} u_t + k_1 + \beta A' E_t \lambda_{t+1} = \lambda_t \quad (1.31)$$

$$K_2 u_t + K_{21} x_t + k_2 + \beta C' E_t \lambda_{t+1} = 0 \quad (1.32)$$

Assume the transition laws of u_t and λ_t take the linear form:

$$u_t = Gx_t + g \quad (1.33)$$

$$\lambda_{t+1} = Hx_{t+1} + h \quad (1.34)$$

Chow (1993) proves that the coefficient matrices G and H and the vectors g and h satisfy

$$G = -(K_2 + \beta C' HC)^{-1}(K_{21} + \beta C' HA) \quad (1.35)$$

$$g = -(K_2 + \beta C' HC)^{-1}(k_2 + \beta C' Hb + \beta C' h) \quad (1.36)$$

$$H = K_1 + K_{12}G + \beta A' H(A + CG) \quad (1.37)$$

$$h = (K_{12} + \beta A' HC)g + k_1 + \beta A' (Hb + h) \quad (1.38)$$

Generally it is impossible to find, the analytical solution to G , H , g and h is impossible. Thus, an iterative procedure can be designed as follows. First, set the initial H and h . G and g can then be calculated by (1.35) and (1.36). Given G and g as well as the initial H and h , the new H and h can be calculated by (1.37) and (1.38). Using these new H and h , one calculates the new G and g by (1.35) and (1.36) again. The process will continue until convergence is achieved.⁶

In comparison to the log-linear approximation, Chow's method requires less derivations, which must be accomplished by hand. Given the steady state, the only derivation is to obtain the partial derivatives of U and F . Yet even this can be computed with a written procedure in a major software package.⁷ Despite this significant advantage, Chow's method has at least three weaknesses.

First, Chow's method can be a good approximation method only when the state equation is linear or can be transformed into a linear one. Otherwise,

⁶It should be noted that the algorithm suggested by Chow (1993) is much more complicated. For any given x_t , the approximation as represented by (1.30) - (1.32) first takes place around (x_t, u^*) , where u^* is the initial guess on u_t . Therefore, if u_t calculated via the decision rule (1.33), as the result of iterating (1.35) - (1.38), is different from u^* , the approximation will take place again. However, this time it will be around (x_t, u_t) , followed by iterating (1.35) - (1.38). The procedure will continue until convergence of u_t . Since the above algorithm is designed for any given x_t , the resulting decision rule is indeed nonlinear in x_t . In this sense, the method, as pointed by Reiter (1997), is less convenient comparing to other approximation method. In a response to Reiter (1997), Kwan and Chow (1997) propose a one-time linearization around the steady states. Therefore, our above presentation follows the spirit of Kwan and Chow (1997) if we assume the linearization takes place around the steady states.

⁷For instance, in GAUSS, one could use the procedure GRADP for deriving the partial derivatives.

the linearized first-order condition as expressed by (1.31) and (1.32) will not be a good approximation to the non-approximated (1.26) and (1.27), since A' and C' are not good approximations to $\frac{\partial F(x_t, u_t)}{\partial x_t}$ and $\frac{\partial F(x_t, u_t)}{\partial u_t}$.⁸ Reiter (1996) has used a concrete example to show this point. Gong (1997) points out the same problem.

Second, the iteration with (1.35) - (1.38) may exhibit multiple solutions since inserting (1.35) into (1.37) gives a quadratic matrix equation in H . Indeed, one would expect that the number of solutions will increase with the increase of the dimensions of state space.⁹

Third, the assumed state equation (1.25) is only a special case of (1.2). This will create some difficulty when being applied to the model with exogenous variables. One possible way to circumvent this problem is to regard the exogenous variables, if there are any, as a part of state variables. This actually has been done by Chow and Kwan (1998) when the method has been applied to a practical problem. Yet this will increase the dimension of state space and hence intensify the problem of multiple solutions.

1.5 An Algorithm for the Linear-Quadratic Approximation

In this section, we shall present an algorithm for solving a dynamic optimization model with the general formulation as expressed in (1.1) - (1.3). The first-order condition used for this algorithm is derived from the Lagrangian. This indicates that the method does not require the assumption $\partial F/\partial x = 0$. The approximation method we used here is the linear-quadratic approximation, and therefore the method does not require log-linear-approximation, which, in many cases, needs to be accomplished by hand. The established Proposition 1 below allows us to save further derivations when applying the method of undetermined coefficients. Indeed, if we use an existing software procedure to compute the partial derivatives with respect to U and F , the only derivation in applying our algorithm is to derive the steady state. Therefore, our suggested algorithm takes full advantage, yet overcomes all the limitations occurring in Chow's method.

Since our state equation takes the form of (1.2) rather than (1.25), the first-order condition is established by (1.14) and (1.15). Evaluating the first-

⁸Note that the good approximation to $\frac{\partial F(x_t, u_t)}{\partial x_t}$ should be $\frac{\partial^2 F(\bar{x}, \bar{u})}{\partial x^2}(x_t - \bar{x}) + \frac{\partial^2 F(\bar{x}, \bar{u})}{\partial \bar{x} \partial u}(u_t - \bar{u}) + \frac{\partial F(\bar{x}, \bar{u})}{\partial \bar{x}}$. For $\frac{\partial F(x_t, u_t)}{\partial u_t}$ there is a similar problem.

⁹The same problem of multiple solution should also exist for the log-linear approximation method.

order condition along with (1.2) and (1.3) at their certainty equivalence form, we are able to derive the steady states for x_t , u_t , λ_t and z_t , which we shall denote respectively as \bar{x} , \bar{u} , $\bar{\lambda}$ and \bar{z} . Taking the first-order Taylor approximation around the steady state for (1.14), (1.15) and (1.2), we obtain

$$F_{11}x_t + F_{12}u_t + F_{13}E_t\lambda_{t+1} + F_{14}z_t + f_1 = \lambda_t, \quad (1.39)$$

$$F_{21}x_t + F_{22}u_t + F_{23}E_t\lambda_{t+1} + F_{24}z_t + f_2 = 0, \quad (1.40)$$

$$x_{t+1} = Ax_t + Cu_t + Wz_t + b; \quad (1.41)$$

where in particular,

$$\begin{aligned} A &= F_x & C &= F_u \\ b &= F(\bar{x}, \bar{u}, \bar{z}) - F_x\bar{x} - F_z\bar{z} - F_u\bar{u} & W &= F_z \\ F_{11} &= U_{xx} + \beta\bar{\lambda}F_{xx} & F_{12} &= U_{xu} + \beta\bar{\lambda}F_{xu} \\ F_{13} &= \beta A' & F_{14} &= \beta\bar{\lambda}F_{xz} \\ f_1 &= U_x + \beta A'\bar{\lambda} - F_{11}\bar{x} - F_{12}\bar{u} - F_{13}\bar{\lambda} - F_{14}\bar{z} & F_{21} &= U_{ux} + \beta\bar{\lambda}F_{ux} \\ F_{22} &= U_{uu} + \beta\bar{\lambda}F_{uu} & F_{23} &= \beta C' \\ f_2 &= U_u + \beta C'\bar{\lambda} - F_{21}\bar{x} - F_{22}\bar{u} - F_{23}\bar{\lambda} - F_{24}\bar{z} & F_{24} &= \beta\bar{\lambda}F_{uz} \end{aligned}$$

Note that here we define U_x as $\partial U/\partial x$ and U_{xx} as $\partial^2 U/\partial x\partial x$ all to be evaluated at the steady state. The similarity is applied to $U_u, U_{uu}, U_{ux}, U_{xu}, F_{xx}, F_{uu}, F_{xz}, F_{ux}$ and F_{uz} . The objective is to find the linear decision rule and the Lagrangian function:

$$u_t = Gx_t + Dz_t + g \quad (1.42)$$

$$\lambda_{t+1} = Hx_{t+1} + Qz_{t+1} + h \quad (1.43)$$

The following is the proposition regarding the solution for (1.42) and (1.43):

Proposition 1 *Assume u_t and λ_{t+1} follow (1.42) and (1.43) respectively. Then the solution of G, D, g, H, Q and h satisfy*

$$G = M + NH \quad (1.44)$$

$$D = R + NQ \quad (1.45)$$

$$g = Nh + m \quad (1.46)$$

$$HCNH + H(A + CM) + F_{13}^{-1}(F_{12}N - I_m)H + F_{13}^{-1}(F_{11} + F_{12}M) = 0 \quad (1.47)$$

$$[F_{13}^{-1}(I - F_{12}N) - HCN]Q - QP = H(W + CR) + F_{13}^{-1}(F_{14} + F_{12}R) \quad (1.48)$$

$$h = [F_{13}^{-1}(I - F_{12}N) - HCN - I_m]^{-1} [H(Cm + b) + Qp + F_{13}^{-1}(f_1 + F_{12}m)] \quad (1.49)$$

where I_m is the $m \times m$ identity matrix and

$$N = (F_{23}F_{13}^{-1}F_{12} - F_{22})^{-1} F_{23}F_{13}^{-1}, \quad (1.50)$$

$$M = (F_{23}F_{13}^{-1}F_{12} - F_{22})^{-1} (F_{21} - F_{23}F_{13}^{-1}F_{11}), \quad (1.51)$$

$$R = (F_{23}F_{13}^{-1}F_{12} - F_{22})^{-1} (F_{24} - F_{23}F_{13}^{-1}F_{14}), \quad (1.52)$$

$$m = (F_{23}F_{13}^{-1}F_{12} - F_{22})^{-1} (f_2 - F_{23}F_{13}^{-1}f_1). \quad (1.53)$$

Given H , the proposition allows us to solve G , Q and h directly according (1.44), (1.48) and (1.49). Then D and g can be computed from (1.45) and (1.46). The solution to H is implied by (1.47), which is nonlinear (quadratic) in H . Obviously, if the model has more than one state variables, we cannot solve for H analytically. In this case, we shall rewrite (1.47) as

$$H = F(H), \quad (1.54)$$

iterating (1.54) until convergence will give us a solution to H .¹⁰

However, when one encounters the model with one state variable,¹¹ H becomes a scalar, and therefore (1.47) can be written as

$$a_1H^2 + a_2H + a_3 = 0$$

with the two solutions given by

$$H_{1,2} = \frac{1}{2a_1} \left[-a_2 \pm (a_2^2 - 4a_1a_3)^{\frac{1}{2}} \right].$$

In other words, the solutions can be computed without iteration. Further, in most cases, one can also easily identify the proper solution by relying on the economic meaning of λ_t . For example, in all the models that we will present in this book, the state equation is the capital stock. Therefore, λ_t is the shadow price of the capital which should be inversely related to the quantity of capital. This indicates that only the negative solution is a proper solution.

1.6 A Dynamic Programming Algorithm

In this section we describe a dynamic programming algorithm which enables us to compute optimal value functions as well as optimal trajectories

¹⁰though multiple solutions may exist.

¹¹which is mostly the case in recent economic literature.

of discounted optimal control problems of the type above. An extension to a stochastic decision problem is briefly summarized in appendix III.

The basic discretization procedure goes back to Capuzzo Dolcetta (1983) and Falcone (1987) and is applied with adaptive gridding strategy by Grüne (1997) and Grüne and Semmler (2004a). We consider discounted optimal control problems in discrete time $t \in \mathbb{N}_0$ given by

$$V(x) = \max_{u \in \mathcal{U}} \sum_{t=0}^{\infty} \beta^t g(x(t), u(t)) \quad (1.55)$$

where

$$x_{t+1} = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n$$

For the discretization in space we consider a grid Γ covering the computational domain of interest. Denoting the nodes of the grid Γ by x^i , $i = 1, \dots, P$, we are now looking for an approximation V_h^Γ satisfying

$$V_h^\Gamma(x^i) = T_h(V_h^\Gamma)(x^i) \quad (1.56)$$

for all nodes x^i of the grid, where the value of V_h^Γ for points x which are not grid points (these are needed for the evaluation of T_h) is determined by linear interpolation. For a description of several iterative methods for the solution of (1.56) we refer to Grüne and Semmler (2004a).

For the estimation of the gridding error we estimate the residual of the operator T_h with respect to V_h^Γ , i.e., the difference between $V_h^\Gamma(x)$ and $T_h(V_h^\Gamma)(x)$ for points x which are not nodes of the grid. Thus, for each cell C_l of the grid Γ we compute

$$\eta_l := \max_{x \in C_l} |T_h(V_h^\Gamma)(x) - V_h^\Gamma(x)|$$

Using these estimates we can iteratively construct adaptive grids as follows:

- (0) Pick an initial grid Γ_0 and set $i = 0$. Fix a refinement parameter $\theta \in (0, 1)$ and a tolerance $tol > 0$.
- (1) Compute the solution $V_h^{\Gamma_i}$ on Γ_i
- (2) Evaluate the error estimates η_l . If $\eta_l < tol$ for all l then stop
- (3) Refine all cells C_j with $\eta_j \geq \theta \max_l \eta_l$, set $i = i + 1$ and go to (1).

For more information about this adaptive gridding procedure and a comparison with other adaptive dynamic programming approaches we refer to Grüne and Semmler (2004a) and Grüne (1997).

In order to determine equilibria and approximately optimal trajectories we need an approximately optimal policy, which in our discretization can be obtained in feedback form $u^*(x)$ for the discrete time approximation using the following procedure:

For each x in the gridding domain we choose $u^*(x)$ such that the equality

$$\max_{u \in U} \{hg(x, u) + \beta V_h(x_h(1))\} = \{hg(x, u^*(x)) + \beta V_h(x_h^*(1))\}$$

holds, where $x_h^*(1) = x + hf(x, u^*(x))$. Then the resulting sequence $u_i^* = u^*(x_h(i))$ is an approximately optimal policy and the related piecewise constant control function is approximately optimal.

1.7 Conclusion

This chapter reviews some typical approximation methods to solve a stochastic dynamic optimization model. The approximation methods discussed here use two types of first-order conditions: the Euler equation and the equations derived from the Lagrangian. We find that the Euler equation needs a restriction that the state variable cannot appear as a determinant in the state equation. Although many economic models can satisfy this restriction after some transformation, we still cannot exclude the possibility that sometime the restriction cannot be satisfied.

Given these two types of first-order conditions, we consider the solutions computed by the Fair-Taylor method, the log-linear approximation method and the linear-quadratic approximation method. For all of these different methods, we compare their advantages and disadvantages. We find that the Fair-Taylor method may encounter an accuracy problem due to its additional requirement of the initial condition for the control variable. On the other hand, the methods of log-linear approximation may need an algorithm that can take over some heavy derivation process that otherwise must be analytically accomplished. For the linear-quadratic approximation, we therefore propose an algorithm that could overcome the limitation of existing methods (such as Chow's method). We also have elaborated on dynamic programming as a recently developed method to solve the involved Bellman equation. In the next chapter we will turn to a practical problem and apply the diverse methods.

1.8 Appendix I: Proof of Proposition 1

From (1.39), we obtain

$$E_t \lambda_{t+1} = F_{13}^{-1} (\lambda_t - F_{11} x_t - F_{12} u_t - F_{14} z_t - f_1) \quad (1.57)$$

Substituting the above equation into (1.40) and then solving for u_t , we get

$$u_t = N \lambda_t + M x_t + R z_t + m, \quad (1.58)$$

where N, M, R and m are all defined in the proposition. Substituting (1.58) into (1.41) and (1.55) respectively, we obtain

$$x_{t+1} = S_x x_t + S_\lambda \lambda_t + S_z z_t + s \quad (1.59)$$

$$E_t \lambda_{t+1} = L_x x_t + L_\lambda \lambda_t + L_z z_t + l. \quad (1.60)$$

where

$$S_x = A + CM \quad (1.61)$$

$$S_\lambda = CN \quad (1.62)$$

$$S_z = W + CR \quad (1.63)$$

$$s = Cm + b \quad (1.64)$$

$$L_x = -F_{13}^{-1} (F_{11} + F_{12} M) \quad (1.65)$$

$$L_\lambda = F_{13}^{-1} (I - F_{12} N) \quad (1.66)$$

$$L_z = -F_{13}^{-1} (F_{14} + F_{12} R) \quad (1.67)$$

$$l = -F_{13}^{-1} (f_1 + F_{12} m) \quad (1.68)$$

Now express λ_t in terms of (1.43) and then plug it into (1.59) and (1.60):

$$x_{t+1} = (S_x + S_\lambda H) x_t + (S_z + S_\lambda Q) z_t + s + S_\lambda h, \quad (1.69)$$

$$E_t \lambda_{t+1} = (L_x + L_\lambda H) x_t + (L_z + L_\lambda Q) z_t + l + L_\lambda h. \quad (1.70)$$

Next, taking expectation for the both sides of (1.43) and expressing x_{t+1} in terms of (1.41) while $E_t z_{t+1}$ in terms of (1.3) we obtain

$$E_t \lambda_{t+1} = H A x_t + H C u_t + (H W + Q P) z_t + H b + Q p + h. \quad (1.71)$$

Next, expressing u_t in terms of (1.42) for equations (1.41) and (1.71) respectively, we get

$$x_{t+1} = (A + CG)x_t + (W + CD)z_t + Cg + b \quad (1.72)$$

$$E_t \lambda_{t+1} = H(A + CG)x_t + [H(W + CD) + QP]z_t + H(Cg + b) + Qp + h. \quad (1.73)$$

Comparing (1.69) and (1.70) with (1.72) and (1.73), we obtain

$$S_x + S_\lambda H = A + CG, \quad (1.74)$$

$$L_x + L_\lambda H = H(A + CG), \quad (1.75)$$

$$S_z + S_\lambda Q = W + CD, \quad (1.76)$$

$$L_z + L_\lambda Q = H(W + CD) + QP, \quad (1.77)$$

$$s + S_\lambda h = Cg + b, \quad (1.78)$$

$$l + L_\lambda h = H(Cg + b) + Qp + h. \quad (1.79)$$

These 6 equations determine 6 unknown coefficient matrices and vectors G , H , D , Q , g and h . In particular, H is resolved from (1.75) when $A + CG$ is replaced by (1.74). This gives rise to (1.47) as in Proposition 1 with S_x, S_λ, L_x and L_λ to be expressed by (1.61), (1.62), (1.65) and (1.66). Given H , G is then resolved from (1.74), which allows us to obtain (1.44). Next Q is resolved from (1.77) when $W + CD$ is replaced by (1.76). This gives rise to (1.48) with S_z, S_λ, L_z and L_λ to be expressed by (1.63), (1.62), (1.67) and (1.66). Then D is resolved from (1.76), which allows us to obtain (1.45). Finally, h is resolved from (1.79) when $Cg + b$ is replaced by (1.78). This gives rise to (1.49) with S_λ, L_λ, s and l to be expressed by (1.62), (1.66), (1.64) and (1.66). Finally g is resolved from (1.78), which allows us to obtain (1.46).

1.9 Appendix II: An Algorithm for the LQ-Approximation

The algorithm that we suggest to solve the LQ approximation of chapter 1.5 is written as a GAUSS procedure and available from the authors upon request. We call this procedure DYNPR. The input of this procedure includes

- the steady states of x, u, λ, z and F denoted as $xbar, ubar, lbar, zbar$ and $Fbar$ respectively.
- the first- and the second-order partial derivatives of x, u and z with respect F and U , all evaluated at the steady states. They are denoted

as, for instance, Fx and Fxx for the first- and the second-order partial derivatives of x with respect to F respectively.

- the discount factor β (denoted as $beta$) and the parameters P and p (denoted as BP and sp respectively) appearing in the $AR(1)$ process of z .

The output of this procedure is the decision parameters G, D and g , denoted respectively as BG, BD and sg .

```

PROC(3) = DYNPR (Fx, Fu, Fz, Fxx, Fuu, Fzz, Fxu, Fxz, Fuz, Fux,
                Fzx, Ux, Uu, Uxx, Uuu, Uxu, Uux, Fbar, xbar,
                ubar, zbar, lbar, beta, BP, sp);
LOCAL A, C, W, sb, F11, F12, F13, F14, f1, F21, F22,
       F23, F24, f2, BM, BN, BR, sm, Sx, Slamda, Sz,
       ss, BLx, Bllamda, BLz, sl, sa1, sa2, sa3, BH,
       BG, sh, sg, BQ, BD;

A = Fx;
C = Fu;
W = Fz;
sb = Fbar - A * xbar - C * ubar - W * zbar;
F11 = Uxx + beta * lbar * Fxx;
F12 = Uxu + beta * lbar * Fxu;
F13 = beta * Fx;
F14 = beta * lbar * Fxz;
f1 = Ux + beta * lbar * Fx - F11 * xbar -
     F12 * ubar - F13 * lbar - F14 * zbar;
F21 = Uux + beta * lbar * Fux;
F22 = Uuu + beta * lbar * Fuu;
F23 = beta * (Fu');
F24 = beta * lbar * Fuz;
f2 = Uu + beta * (Fu') * lbar - F21 * xbar -
     F22 * ubar - F23 * lbar - F24 * zbar;
BM = INV(F23*INV(F13) * F12 - F22)*
     (F21 - F23*INV(F13) * F11);
BN = INV(F23*INV(F13) * F12 - F22)*
     F23*INV(F13);
BR = INV(F23*INV(F13) * F12 - F22)*
     (F24 - F23*INV(F13) * F14);
sm = INV(F23*INV(F13) * F12 - F22)*
     (f2 - F23*INV(F13) * f1);
Sx = A + C * BM;

```

```

Slamda = C * BN;
Sz = W + C * BR;
ss = C * sm + sb;
BLx = -INV(F13) * (F11 + F12 * BM);
BLLamda = INV(F13) * (1 - F12 * BN);
BLz = -INV(F13) * (F14 + F12 * BR);
sl = -INV(F13) * (f1 + F12 * sm);
sa1 = Slamda;
sa2 = Sx - BLLamda;
sa3 = -BLx;
BH = (1/(2 * sa1)) * (-sa2 - (sa22 - 4 * sa1 * sa3)0.5);
BG = BM + BN * BH;
BQ = INV(BLLamda - BH * Slamda - BP) *
      (BH * Sz - BLz);
BD = BR + BN * BQ;
sh = INV(BLLamda - BH * Slamda - 1) *
      (BH * ss + BQ * sp - sl);
sg = BN * sh + sm;
RETP(BG, BD, sg);
ENDP;

```

1.9.1 Appendix III: The Stochastic Dynamic Programming Algorithm

Our adaptive approach of chapter 1.6. is easily extended to stochastic discrete time problems of the type

$$V(x) = E \left(\max_{u \in U} \sum_{t=0}^{\infty} \beta^t g(x(t), u(t)) \right) \quad (1.80)$$

where

$$x(t+1) = f(x(t), u(t), z_t), \quad x(0) = x_0 \in R^n \quad (1.81)$$

and the z_t are i.i.d. random variables. This problem can immediately be applied in discrete time with time step $h = 1$.¹²

The corresponding dynamic programming operator becomes

$$T_h(V_h)(x) = \max_{u \in U} E \{ hg(x, u) + \beta V_h(\varphi(x, u, z)) \}, \quad (1.82)$$

¹²For a discretization of a continuous time stochastic optimal control problem with dynamics governed by an Itô stochastic differential equation, see Camilli and Falcone (1995).

where $\varphi(x, u, z)$ is now a random variable.

If the random variable z is discrete then the evaluation of the expectation E is a simple summation, if z is a continuous random variable then we can compute E via a numerical quadrature formula for the approximation of the integral

$$\int_z (hg(x, u) + \beta V_h(\varphi(x, u, z)))p(z)dz,$$

where $p(z)$ is the probability density of z . Grüne and Semmler (2004a) show the application of such a method to such a problem, where z is a truncated Gaussian random variable and the numerical integration is done via the trapezoidal rule.

It should be noted that despite the formal similarity, stochastic optimal control problems have several features different from deterministic ones. First, complicated dynamical behavior like multiple stable steady state equilibria, periodic attractors etc. is less likely because the influence of the stochastic term tends to “smear out” the dynamics in such a way that these phenomena disappear.¹³ Furthermore, in stochastic problems the optimal value function typically has more regularity which allows the use of high order approximation techniques. Finally, stochastic problems can often be formulated in terms of Markov decision problems with continuous transition probability (see Rust (1996) for a details), whose structure gives rise to different approximation techniques, in particular allowing to avoid the discretization of the state space.

In these situations, the above dynamic programming technique, developed by Grüne (1997) and applied in Grüne and Semmler (2004a) may not be the most efficient approach to these problems, and it has to compete with other efficient techniques. Nevertheless, the examples in Grüne and Semmler (2004a). shows that adaptive grids as discussed in chapter 1.6 are by far more efficient than non-adaptive methods if the same discretization technique is used for both approaches. It should also be noted that in the smooth case one can obtain estimates for the error in the approximation of the gradient of V_h from our error estimates, for details we refer to Grüne (2003).

¹³A remark to this extent on an earlier version of our work has been made by Buz Brock and Michael Woodford.

Chapter 2

Solving a Prototype Stochastic Dynamic Model

2.1 Introduction

This chapter turns to a practical problem of dynamic optimization. In particular, we shall solve a prototype model by employing different approximation methods as we have discussed in the last chapter. The model we choose is a Ramsey model (Ramsey 1928) to which the exact solution is computable with the standard recursive method. This will allow us to test the accuracy of approximations by comparing the different approximate solutions to the exact solution.

2.2 The Ramsey Problem

2.2.1 The Model

Ramsey (1928) posed a problem of optimal resource allocation, which is now often used as a prototype model of dynamic optimization.¹ The model presented in this section is essentially that of Ramsey (1928) yet it is augmented by uncertainty. Let C_t denote consumption, Y_t output and K_t the capital stock. Assume that the output is produced by capital stock and it is either consumed or invested, that is, added to the capital stock. Formally,

$$Y_t = A_t K_t^\alpha, \tag{2.1}$$

$$Y_t = C_t + K_{t+1} - K_t, \tag{2.2}$$

¹See, for instance, Stockey et al. (1989, chapter 2), Blanchard and Fisher (1989, chapter 2) and Ljungqvist and Sargent (2000, chapter 2).

where $\alpha \in (0, 1)$ and A_t is the technology which may follow an AR(1) process:

$$A_{t+1} = a_0 + a_1 A_t + \epsilon_{t+1}. \quad (2.3)$$

Here we shall assume ϵ_t to be *i.i.d.* Equation (2.1) and (2.2) indicates that we could write the transition law of capital stock as

$$K_{t+1} = A_t K_t^\alpha - C_t. \quad (2.4)$$

Note that we have assumed here that the depreciation rate of capital stock is equal to 1. This is a simplified assumption by which the exact solution is computable.²

The representative agent is assumed to find the control sequence $\{C_t\}_{t=0}^\infty$ such that

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \ln C_t \quad (2.5)$$

given the initial condition (K_0, A_0) .

2.2.2 The Exact Solution and the Steady States

It is well known that the exact solution for this model – which could be derived from the standard recursive method can be written as

$$K_{t+1} = \alpha\beta A_t K_t^\alpha. \quad (2.6)$$

This further implies from (2.4) that

$$C_t = (1 - \alpha\beta) A_t K_t^\alpha. \quad (2.7)$$

Given the solution paths for C_t and K_{t+1} , we are then able to derive the steady state. It is not difficult to find that one steady state is on the boundary, that is $\bar{K} = 0$ and $\bar{C} = 0$. To obtain a more meaningful interior steady state, we take logarithm for both sides of (2.6) and evaluate the equation at its certainty equivalence form:

$$\log K_{t+1} = \log(\alpha\beta\bar{A}) + \alpha \log K_t. \quad (2.8)$$

At the steady state, $K_{t+1} = K_t = \bar{K}$. Solving (2.8) for $\log \bar{K}$, we obtain $\log \bar{K} = \frac{\log(\alpha\beta\bar{A})}{1-\alpha}$. Therefore,

$$\bar{K} = (\alpha\beta\bar{A})^{1/(1-\alpha)}. \quad (2.9)$$

Given \bar{K} , \bar{C} is resolved from (2.4):

$$\bar{C} = \bar{A}\bar{K}^\alpha - \bar{K} \quad (2.10)$$

²For another similar model where the depreciation rate is also set to 1 and hence the exact solution is computable, see Long and Plosser (1983).

2.3 The First-Order Conditions and Approximate Solutions

To solve the model with different approximation methods, we shall first establish the first-order condition derived from the Ramsey problem. As we have mentioned in the last chapter, there are two types of first-order conditions, the Euler equation and the equation derived from the Lagrangian. Let us first consider the Euler equation.

2.3.1 The Euler Equation

Our first task is to transform the model into a setting so that the state variable K_t does not appear in $F(\cdot)$ as we have discussed in the last chapter. This can be done by assuming K_{t+1} (instead of C_t) as model's decision variable. To achieve a notational consistency in the time subscript, we may denote the decision variable as Z_t . Therefore the model can be rewritten as

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t K_t^\alpha - K_{t+1}). \quad (2.11)$$

subject to

$$K_{t+1} = Z_t$$

Note that here we have used (2.4) to express C_t in the utility function. Also note that in this formulation the state variable in period t is still K_t . Therefore $\partial F/\partial x = 0$ and $\partial F/\partial u = 1$. The Bellman equation in this case can be written as

$$V(K_t, A_t) = \max_{K_{t+1}} \{ \ln(A_t K_t^\alpha - K_{t+1}) + \beta E [V(K_{t+1}, A_{t+1})] \}. \quad (2.12)$$

The necessary condition for maximizing the right hand side of the Bellman equation (2.12) is given by

$$\frac{-1}{A_t K_t^\alpha - K_{t+1}} + \beta E \left[\frac{\partial V}{\partial K_{t+1}}(K_{t+1}, A_{t+1}) \right] = 0. \quad (2.13)$$

Meanwhile applying the Benveniste-Scheinkman formula,

$$\frac{\partial V}{\partial K_t}(K_t, A_t) = \frac{\alpha A_t K_t^{\alpha-1}}{A_t K_t^\alpha - K_{t+1}}. \quad (2.14)$$

Substituting (2.14) into (2.13) allows us to obtain the Euler equation:

$$\frac{-1}{A_t K_t^\alpha - K_{t+1}} + \beta E \left[\frac{\alpha A_{t+1} K_{t+1}^{\alpha-1}}{A_{t+1} K_{t+1}^\alpha - K_{t+2}} \right] = 0,$$

which can further be written as

$$-\frac{1}{C_t} + \beta E \left[\frac{\alpha A_{t+1} K_{t+1}^{\alpha-1}}{C_{t+1}} \right] = 0. \quad (2.15)$$

This Euler equations (2.15) along with (2.4) and (2.3) determine the transition sequences of $\{K_{t+1}\}_{t=1}^{\infty}$, $\{A_{t+1}\}_{t=1}^{\infty}$ and $\{C_t\}_{t=0}^{\infty}$ given the initial condition K_0 and A_0 .

2.3.2 The First-Order Condition Derived from the Lagrangian

Next, we turn to derive the first-order condition from the Lagrangian. Define the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t \ln C_t - \sum_{t=0}^{\infty} E_t \left[\beta^{t+1} \lambda_{t+1} (K_{t+1} - A_t K_t^{\alpha} + C_t) \right].$$

Setting to zero the derivatives of L with respect to λ_t , C_t and K_t , one obtains (2.4) as well as

$$1/C_t - \beta E_t \lambda_{t+1} = 0, \quad (2.16)$$

$$\beta E_t \lambda_{t+1} \alpha A_t K_t^{\alpha-1} = \lambda_t. \quad (2.17)$$

These are the first-order conditions derived from the Lagrangian. Next we try to demonstrate that the two first-order conditions are virtually equivalent. This can be done as follows. Using (2.16) to express $\beta E_t \lambda_{t+1}$ in terms $1/C_t$ and then plug it into (2.17), we obtain $\lambda_t = \alpha A_t K_t^{\alpha-1} / C_t$. This further indicates that

$$E_t \lambda_{t+1} = E_t \left[\alpha A_{t+1} K_{t+1}^{\alpha-1} / C_{t+1} \right] \quad (2.18)$$

Substitute (2.18) back into (2.16), we obtain the Euler equation (2.15).

It is also not difficult to find that the two first-order conditions imply the same steady state as the one derived from the exact solution (see equation (2.9) and (2.10)). Writing either (2.15) or (2.17) in terms of their certainty equivalence form while evaluating them at the steady state, we indeed obtain (2.9).

2.3.3 The Dynamic Programming Formulation

The dynamic programming problem for the Ramsey growth model of chapter 2.2.1 can be written, in its deterministic version, as a basic discrete time growth model

$$V = \max_C \sum_{t=0}^{\infty} \beta^t U(C_t) \quad (2.19)$$

s.t.

$$C_t + K_{t+1} = f(K_t) \quad (2.20)$$

with an one period utility function $U'(C) > 0$, $U''(C) < 0$, $f'(K) > 0$, $f''(K) < 0$.

Let us restate the problem above with K the state variable and K' the control variable, where K' denotes the next period's value of K . Substitute C into the above intertemporal utility function by defining

$$C = f(K) - K' \quad (2.21)$$

We then can express the discrete time Bellman-equation, representing a dynamic programming formulation as

$$V(K) = \max_{K'} \{U[f(K) - K'] + \beta V(K')\} \quad (2.22)$$

By applying the Benveniste-Scheinkman condition³ gives

$$V'(K) = U'(f(K) - K')f'(K) \quad (2.23)$$

Note that K is the state variable and that in equ. (2.22) we have $V(K')$. Notice that from the discrete time form of the envelope condition one again obtains the first order condition of equ. (2.22) as

$$U'[f(K) - K'] + \beta V'(K') = 0$$

which gives by using (2.23) one step forward, i.e. for $V'(K')$.

$$U'[f(K) - K'] = \beta U'[f(K') - K'']f'(K') \quad (2.24)$$

Note that hereby we obtain as a solution a second order difference equation in K , whereby K' denotes the one period and K'' the two period ahead value of K . Yet equ. (2.24) can be written as

$$1 = \beta \frac{U'(C_{t+1})}{U'(C_t)} f'(K_{t+1}) \quad (2.25)$$

³The Benveniste-Scheinkman condition implies that the state variable does not appear in the transition equation, see chapter 1.3 of this book and Ljungquist and Sargent (2000, ch. 2).

which represent the Euler-equation that has extensively been used in economic theory⁴.

If we allow for log-utility as in chapter 2.2.1 the discrete time decision problem is directly analytically solvable. We take the following form of a utility function

$$V = \max_{C_t} \sum_{t=0}^{\infty} \beta^t \ln C_t \quad (2.26)$$

s.t.

$$K_{t+1} = AK_t^\alpha - C_t \quad (2.27)$$

The analytical solution for the value function is

$$V(K) = \tilde{B} + \tilde{C} \ln(K) \quad (2.28)$$

and for the sequence of capital one obtains

$$K_{t+1} = \frac{\beta CA K_t^\alpha}{1 + \beta C} \quad (2.29)$$

with

$$\begin{aligned} \tilde{C} &= \frac{\alpha}{1 - \alpha\beta} \text{ and} \\ \tilde{B} &= \frac{\ln(\tilde{C}(1 - \alpha\beta)A) + \frac{\beta\alpha}{1 - \beta\alpha} \ln(\alpha\beta A)}{1 - \beta} \end{aligned}$$

For the optimal consumption holds

$$C_t = K_{t+1} - AK_t^\alpha \quad (2.30)$$

and for the steady state equilibrium \bar{K} one obtains

$$\frac{1}{\beta} = \alpha A \bar{K}^{\alpha-1} \text{ or} \quad (2.31)$$

$$\bar{K} = \beta\alpha A \bar{K}^\alpha \quad (2.32)$$

⁴The above Euler-equation is also essential not only in stochastic growth but also in finance, to study asset pricing, in fiscal policy to evaluate treasury bonds and testing for sustainability of fiscal policy, see Ljungqvist and Sargent (2000, chs. 2, 7, 10, 17)

2.4 Solving the Ramsey Problem with Different Approximations

2.4.1 The Fair-Taylor Solution

It should be noted that one can apply the Fair-Taylor method either to the Euler equation or to the first-order condition derived from the Lagrangian. Here we shall use the Euler equation. Let us first write equation (2.15) in the form as expressed by (1.19) - (1.21):

$$C_{t+1} = \alpha\beta C_t A_{t+1} K_{t+1}^{\alpha-1}. \quad (2.33)$$

Together with (2.4) and (2.3), they form a recursive dynamic system from which the transition paths of C_t , K_t and A_t can be directly computed. Since the model is simple in its structure, there is no necessity to employ the Gauss-Seidel procedure as suggested in the last chapter.

Before we compute the solution path, we shall first parameterize the model. There are all together 5 structural parameters: α , β , a_0 , a_1 and σ_ϵ . Table 2.1 specifies these parameters and the corresponding interior steady state values:

Table 2.1: Parameterizing the Prototype Model

α	β	a_0	a_1	σ_ϵ	\bar{K}	\bar{C}	\bar{A}
0.3200	0.9800	600.00	0.8000	60.000	23593	51640	3000.0

Given the parameters, as reported in Table 2.1, we provide in Figure 2.1 three solution paths computed by the Fair-Taylor method. These solution paths are compared to the exact solution as expressed in (2.6) and (2.7):

The three solution paths are different due to their initial condition with regard to C_0 . Since we know the exact solution, we thus can choose C_0 close to the exact solution denoted as C_0^* . Note that from (2.7) $C_0^* = (1 - \alpha\beta)A_0K_0^\alpha$. In particular, we allow one C_0 to be equal to C_0^* , and the others to deviate 1% from C_0^* .

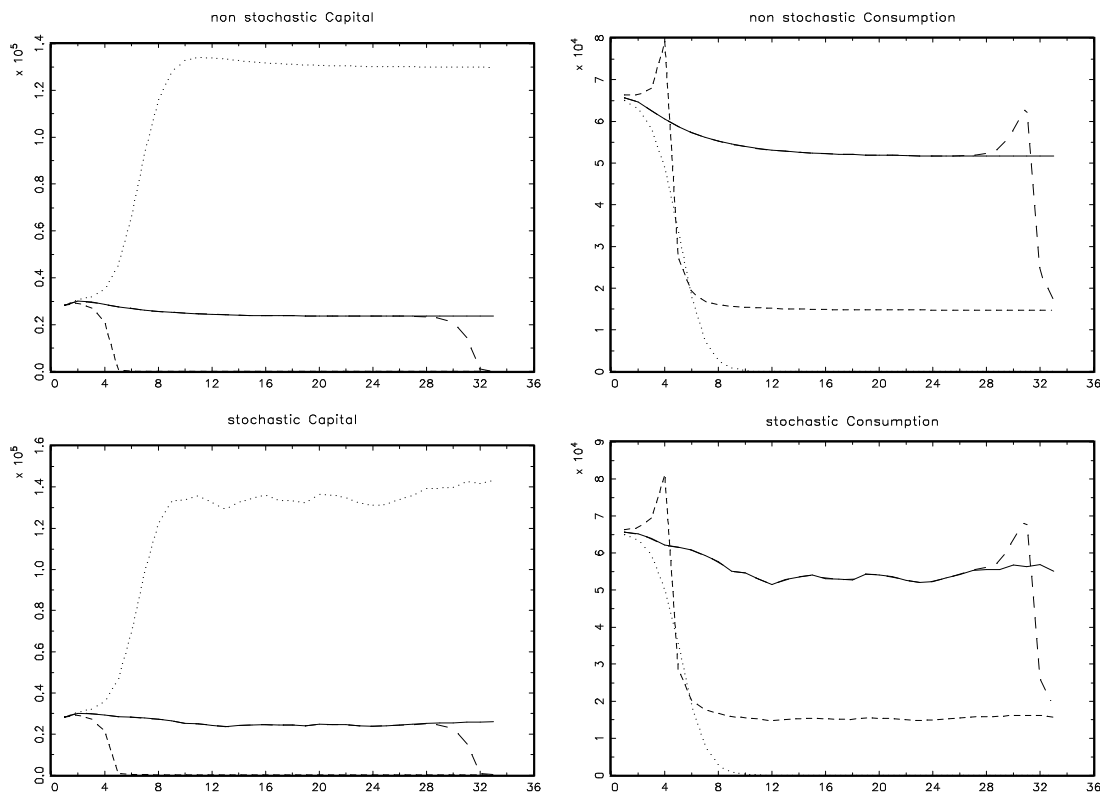


Figure 2.1: The Fair-Taylor Solution in Comparison to the Exact Solution: solid curve the exact solution, dashed and dotted curves the Fair-Taylor solution

The following is a summary of what we have found in this experiment.

- When we choose C_0 above C_0^* (by 1%) the path of K_t quickly reaches to zero and therefore the simulations have to be subject to the constraint $C_t < A_t K_t^\alpha$. In particular, we restrict $C_t \leq 0.99 A_t K_t^\alpha$. This restriction makes K_t never reach zero so that the simulation can be continued. The solution path is shown by one of the dashed curves in the figure.
- When we set C_0 below C_0^* (again by 1%), the path of C_t quickly reaches its lower bound 0. This is shown by the dotted curve in the figure.
- When we set C_0 to C_0^* , the paths of K_t and C_t (shown by the another dashed curve) are close to the exact solution for small t 's. Yet when t goes beyond a certain point, the deviation becomes significant.

What can we learn from this experiment? The exact solution to this problem seems to be the saddle path for the system composed of (2.4), (2.3)

and (2.33). The eventual deviation of the solution starting with C_0^* from the exact solution is likely to be due to the computational errors resulting from our numerical simulation. On the other hand, we have verified our previous concern that the initial condition for the control variable is extremely important for obtaining an appropriate solution path when we employ the Fair-Taylor method.

2.4.2 The Log-linear Solution

As the Fair-Taylor method, the log-linear approximation method can be applied to the first-order condition either from Euler equation or from the Lagrangian. Here we shall again use the Euler equation. Our first task is therefore to log-linearize the state, Euler and the exogenous equations as expressed in (2.4), (2.15) and (2.3). The following is the proposition regarding this log-linearization (the proof is provided in the appendix):

Next we try to find a solution path for c_t , which we shall conjecture as

$$c_t = \eta_{ca}a_t + \eta_{ck}k_t \quad (2.34)$$

The proposition below regards the determination of the two undetermined coefficients η_{ca} and η_{ck} .

Proposition 2 *Let k_t , c_t and a_t denote the log deviations of K_t , C_t and A_t . Then equation (2.4), (2.15) and (2.3) can be log-linearized as*

$$k_{t+1} = \varphi_{ka}a_t + \varphi_{kk}k_t + \varphi_{kc}c_t, \quad (2.35)$$

$$E[c_{t+1}] = \varphi_{cc}c_t + \varphi_{ca}a_t + \varphi_{ck}k_{t+1}, \quad (2.36)$$

$$E[a_{t+1}] = a_1a_t, \quad (2.37)$$

where

$$\begin{aligned} \varphi_{ka} &= \bar{A}\bar{K}^{\alpha-1}, & \varphi_{kk} &= \bar{A}\bar{K}^{\alpha-1}\alpha, \\ \varphi_{kc} &= -(\bar{C}/\bar{K}), & \varphi_{cc} &= \alpha\beta\bar{A}\bar{K}^{\alpha-1}, \\ \varphi_{ca} &= \alpha\beta\bar{A}\bar{K}^{\alpha-1}a_1, & \varphi_{ck} &= \alpha\beta\bar{A}\bar{K}^{\alpha-1}(\alpha-1). \end{aligned}$$

Next we try to find a solution path for c_t which we shall conjecture as

$$c_t = \eta_{ca}a_t + \eta_{ck}k_t \quad (2.38)$$

The proposition below regards the determination of the two undetermined coefficients η_c and η_{ck} .

Proposition 3 Assume c_t follow (2.34). Then η_{ck} and η_{ca} are determined from the following equation

$$\eta_{ck} = \frac{1}{2Q_2} \left(-Q_1 - \sqrt{Q_1^2 - 4Q_0Q_2} \right) \quad (2.39)$$

$$\eta_{ca} = \frac{(\eta_{ck} - \varphi_{ck})\varphi_{ka} - \varphi_{ca}}{\varphi_{cc} - a_1 - \varphi_{kc}(\eta_{ck} - \varphi_{ck})} \quad (2.40)$$

where $Q_2 = \varphi_{kc}$, $Q_1 = (\varphi_{kk} - \varphi_{cc} - \varphi_{kc}\varphi_{ck})$ and $Q_0 = \varphi_{kk}$.

The solution paths of the model can now be computed by relying on (2.34) and (2.35) with a_t to be given by $a_1 a_{t-1} + \epsilon_t / \bar{A}$. All the solution paths are expressed as log deviations. Therefore, to compare the log-linear solution to the exact solution, we shall perform the transformation via $X_t = (1 + x_t) \bar{X}$ for a variable x_t in log deviation form. Using the same parameters as reported in Table 2.1, we show in Figure 2.2 the log-linear solution in comparison to the exact solution.

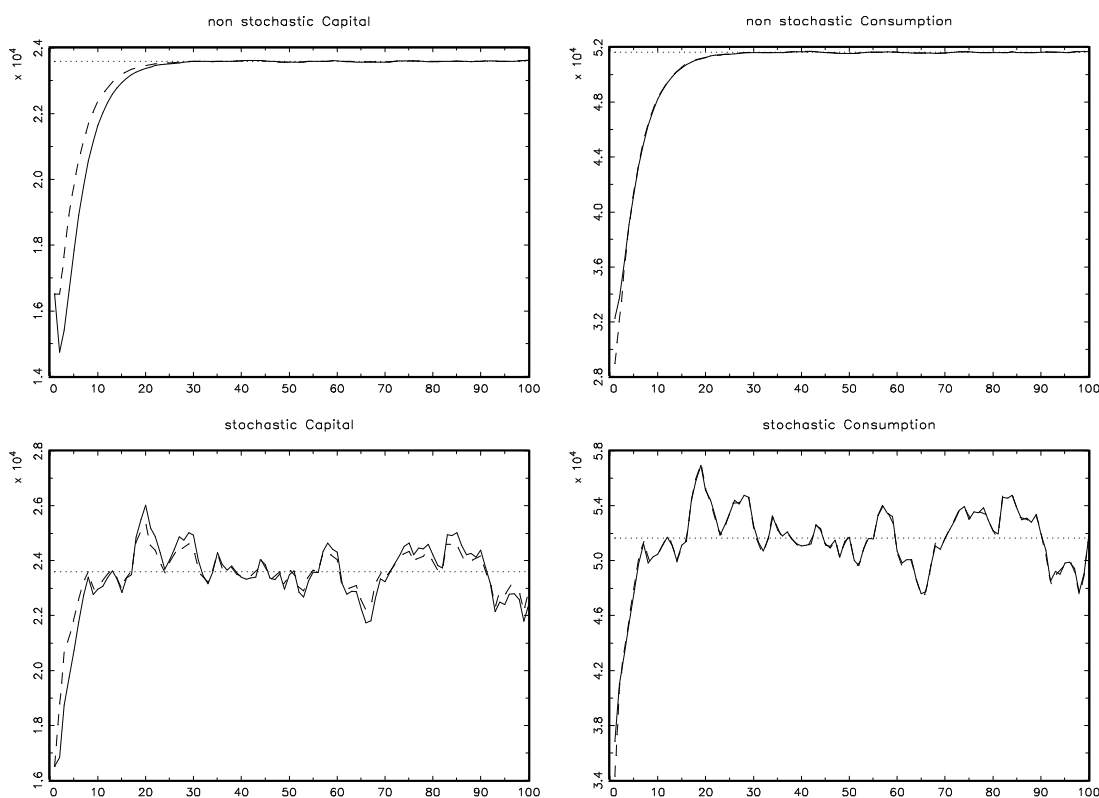


Figure 2.2: The Log-linear Solution in Comparison to the Exact Solution: solid curve the exact solution, dashed curves the log-linear solution

In contrast to the Fair-Taylor solution, one finds that the log-linear solution is quite close to the exact solution except for some initial paths.

2.4.3 The Linear-Quadratic Solution with Chow's Algorithm

To apply Chow's method, we shall first transform the model in which the state equation appears to be linear. This can be done by choosing investment $I_t \equiv A_t K_t^\alpha - C_t$ as the control variable while leaving the capital stock and the technology as the two state variables. When considering this, the model becomes

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \ln(A_t K_t^\alpha - I_t)$$

subject to (2.3) as well as

$$K_{t+1} = I_t \tag{2.41}$$

Due to the insufficiency in the specification of the model with regard to the possible exogenous variable, we have to treat, as suggested in the last chapter, the technology A_t also as a state variable. This indicates that there are two state equations (2.41) and (2.3), both are now in linear form. Next we shall derive the first- and the second-order partial derivatives of the utility function around the steady state. This will allow us to obtain those K_{ij} and k_j ($i, j = 1, 2$) coefficient matrices and vectors as expressed in Chow's first-order conditions (1.31) and (1.32).

Suppose the linear decision rule can be written as

$$I_t = G_{11}K_t + G_{12}A_t + g_1 \tag{2.42}$$

The coefficients G_{11}, G_{12} and g_1 can be computed in principle by iterating (1.35) - (1.38) as discussed in the last chapter. Yet this requires the iteration to be convergent. Unfortunately, this is not attainable for our particular application, even if we start from many different initial conditions.⁵ Therefore, our attempt fails to compute the solution path with Chow's algorithm.

2.4.4 The Linear-Quadratic Solution Using the Suggested Algorithm

When we employ our new algorithm, there is no need to transform the model. Therefore we can define $F = AK^\alpha - C$ and $U = \ln C$. Again our first step is

⁵Reiter (1997) has experienced the same problem.

to compute the first- and second-order partial derivatives with respect to F and U . All these partial derivatives along with the steady states can be used as input in the GAUSS procedure provided in Appendix II of the last chapters. Executing this procedure will allow us to compute the undetermined coefficients in the following decision rule for C_t :

$$C_t = G_{21}K_t + G_{22}A_t + g_2 \quad (2.43)$$

Equation (2.43) along with (2.4) and (2.3) form the dynamic system from which the transition path of C_t, K_t and A_t are computed (see Figure 2.3 for illustration).

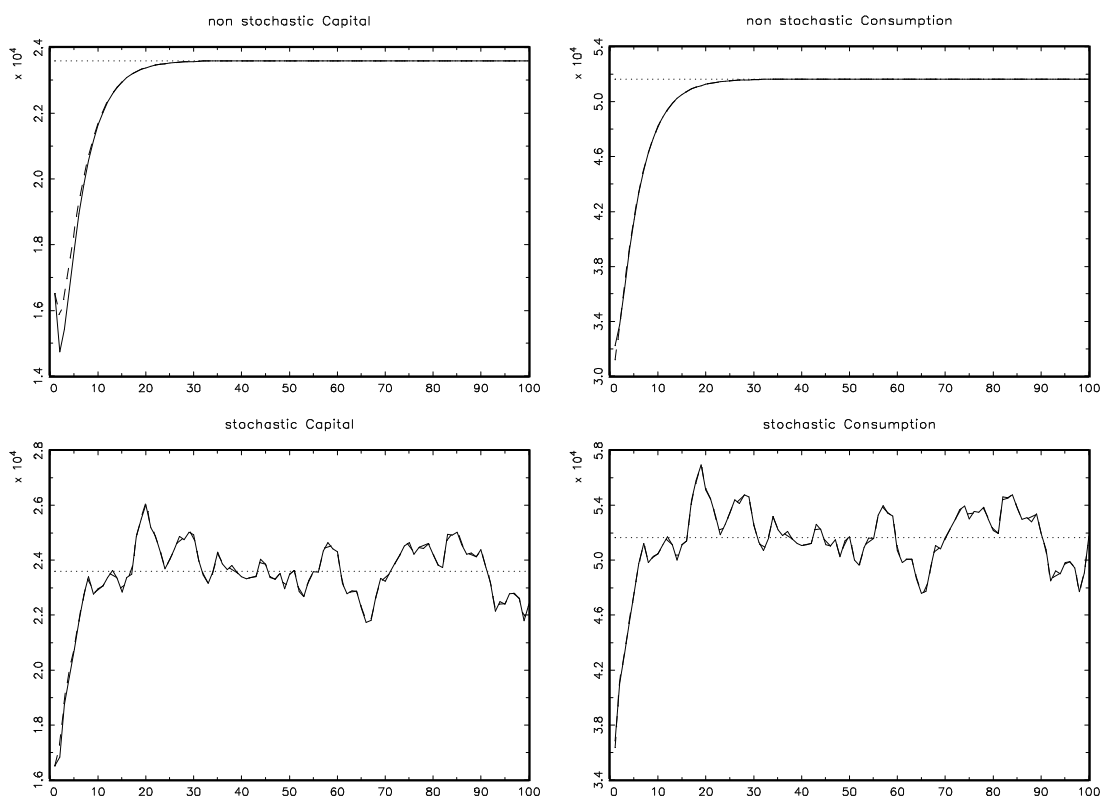


Figure 2.3: The Linear-quadratic Solution in Comparison to the Exact Solution: solid curve for exact solution, dashed curves for linear-quadratic solution

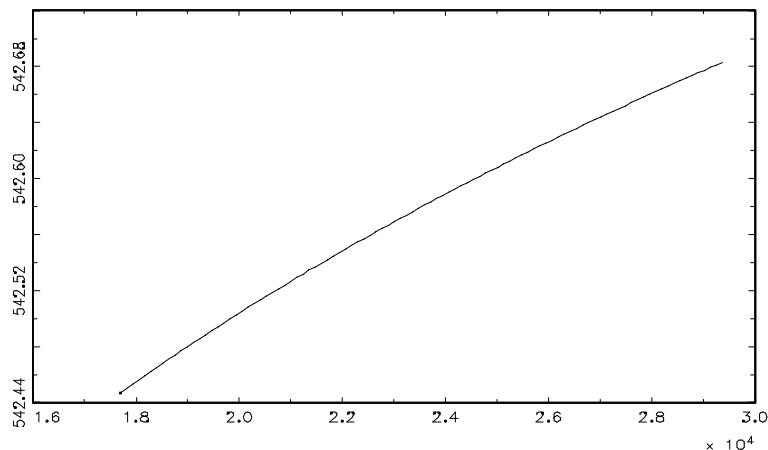


Figure 2.4: Value Function obtained from the Linear-quadratic Solution

In addition, in the figure 2.4, the value function obtained from the linear quadratic solution is shown. As the figure shows the value function is clearly concave in the capital stock, K .

2.4.5 The Dynamic Programming Solution

Next, using as example, we will compare the analytical solution of chapter 2.3.3 with the dynamic programming solution obtained from the dynamic programming algorithm of chapter 1.6. Subsequently we report only results from a deterministic version. Results from a stochastic version are discussed in appendix II.

For the growth model of chapter 2.3.3 we employ the following parameters

$$\alpha = 0.34$$

$$A = 5$$

$$\beta = 0.95$$

we can solve all the above expressions numerically for a grid of the capital stock in the interval $[0.1, 10]$ and the control variable, c , in the interval $[0.1, 5]$. For the parameters chosen we obtain a steady state of the capital stock

$$\bar{K} = 2.07$$

For more details of the solution, see Grüne and Semmler (2004a).⁶

⁶Moreover, as concerns asset pricing log-utility preferences provide us with a very simple

The solution of the growth model with the above parameters, using the dynamic programming algorithm of chapter 1.6 with grid refinement is shown in figures 2.5 and 2.6.

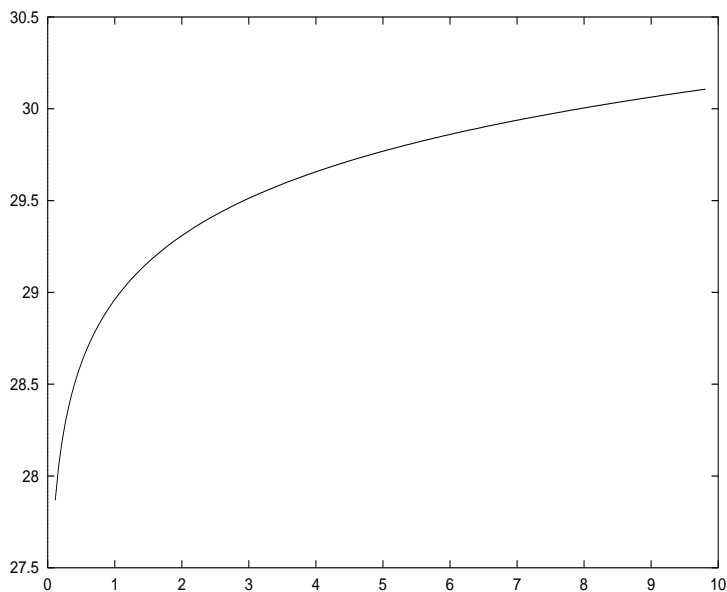


Figure 2.5: Value Function

stochastic discount factor and an analytical expression for the asset price. For $U(C) = \ln(C)$ the asset price is

$$\begin{aligned}
 P_t &= E_t \sum_{t=1}^{\infty} \beta^j \frac{U'(C_{t+j})}{U'(C_t)} \\
 &= E_t \sum_{t=1}^{\infty} \beta^j \frac{C_t \cdot C_{t+j}}{C_{t+j}} = \frac{\beta C_t}{1 - \beta}.
 \end{aligned}$$

For further details, see Cochrane (2001, ch. 9.1) and Grüne and Semmler (2004 b).

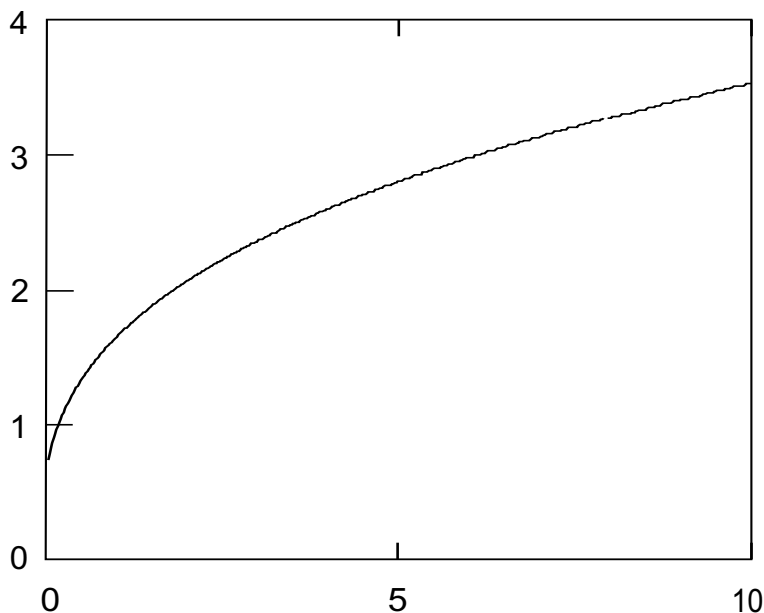


Figure 2.6: Path of Control

As figures 2.5 and 2.6 show the value function and the control, C , are concave in the capital stock, K in figure 2.6 the optimal consumption is shown to depend on the state variable K for a grid of $K, 0 \leq K \leq 10$. Moreover, as observable from figure 2.6 consumption is low when capital stock is low (capital stock can grow) and consumption is high when capital stock is high (capital stock will decrease) where low and high is meant in reference to the optimal steady state capital stock $\bar{K} = 2.07$. As reported in Grüne and Semmler (2004a) the dynamic programming algorithm with adaptive gridding strategy as introduced in chapter 1.6 solves the value function with high accuracy.⁷

2.5 Conclusion

This chapter employs the different approximation methods to solve a prototype dynamic optimization model. Our purpose here is to compare the different approximate solutions to the exact solution, which for this model can be derived analytically by the standard recursive method. As we have found, there have been some difficulties when we apply the Fair-Taylor method and the method of linear-quadratic approximation using Chow's algorithm. Yet

⁷With 100 nodes in capital stock interval the error is $3.2 \cdot 10^{-2}$ and with 2000 nodes the error shrinks to $6.3 \cdot 10^{-4}$, see Grüne and Semmler (2004a).

when we apply the methods of log-linear approximation and linear-quadratic approximation with our suggested algorithm, we find that the approximate solutions are close to the exact solution. At the same time, we also find that the method of log-linear approximation may need an algorithm that can take over some heavy derivations that otherwise must be analytically accomplished. Therefore, our experiment in this chapter verifies our previous concerns (in Chapter 2) with regard to the accuracy and the capability of different approximation methods, including the Fair-Taylor method, the log-linear approximation method and the linear-quadratic approximation method. Although the dynamic programming approach solves the value function with higher accuracy, in the subsequent chapters, when we come to the calibration of the intertemporal decision models, we will work with the linear quadratic approximation of the Chow method since it is better applicable to empirical assessment of the models.

2.6 Appendix I: The Proof of Proposition 2 and 3

2.6.1 The Proof of Proposition 2

For convenience, we shall write (2.4), (2.15) and (2.3) as

$$K_{t+1} - A_t K_t^\alpha - C_t = 0 \quad (2.44)$$

$$E [C_{t+1} - \alpha\beta C_t A_{t+1} K_{t+1}^{\alpha-1}] = 0 \quad (2.45)$$

$$E [A_{t+1} - a_0 - a_1 A_t] = 0 \quad (2.46)$$

Applying (1.22) to the above equations, we obtain

$$\bar{K} e^{k_{t+1}} - \bar{A} \bar{K}^\alpha e^{a_t + \alpha k_t} + \bar{C} e^{c_t} = 0 \quad (2.47)$$

$$E [\bar{C} e^{c_{t+1}} - \alpha\beta \bar{C} \bar{A} \bar{K}^{\alpha-1} e^{a_{t+1} + c_t + (\alpha-1)k_{t+1}}] = 0 \quad (2.48)$$

$$E [\bar{A} e^{a_{t+1}} - a_0 - a_1 \bar{A} e^{a_t}] = 0 \quad (2.49)$$

Applying (1.23), we further obtain from the above:

$$\begin{aligned} \bar{K}(1 + k_{t+1}) - \bar{A} \bar{K}^\alpha (1 + a_t + \alpha k_t) + \bar{C}(1 + c_t) &= 0 \\ E [\bar{C}(1 + c_{t+1}) - \alpha\beta \bar{C} \bar{A} \bar{K}^{\alpha-1} (1 + c_t + a_{t+1} + (\alpha-1)k_{t+1})] &= 0 \\ E [\bar{A}(1 + a_{t+1}) - a_0 - a_1 \bar{A}(1 + a_t)] &= 0 \end{aligned}$$

which can be further written as to be

$$\bar{K}k_{t+1} - \bar{A}\bar{K}^\alpha a_t - \bar{A}\bar{K}^\alpha \alpha k_t + \bar{C}c_t = 0 \quad (2.50)$$

$$E \left[\bar{C}c_{t+1} - \alpha\beta\bar{C}\bar{A}\bar{K}^{\alpha-1} (c_t + a_{t+1} + (\alpha - 1)k_{t+1}) \right] = 0 \quad (2.51)$$

$$E [a_{t+1} - a_1 a_t] = 0 \quad (2.52)$$

Equation (2.52) indicates (2.37). Substituting it into (2.51) to express $E [a_{t+1}]$ and re-arranging (2.50) and (2.51), we obtain (2.35) and (2.36) as indicated in the proposition.

2.6.2 Proof of Proposition 3

Given the conjectured solution (2.34), the transition path of k_{t+1} can be derived from (2.35), which can be written as

$$k_{t+1} = \eta_{ka} a_t + \eta_{kk} k_t \quad (2.53)$$

where

$$\eta_{ka} = \varphi_{ka} + \varphi_{kc} \eta_{ca} \quad (2.54)$$

$$\eta_{kk} = \varphi_{kk} + \varphi_{kc} \eta_{ck} \quad (2.55)$$

Expressing c_{t+1} and c_t in terms of (2.34) while recognizing that $E [a_{t+1}] = a_1 a_t$, we obtain from (2.36):

$$\eta_{ca} a_1 a_t + \eta_{ck} k_{t+1} = \varphi_{cc} (\eta_{ca} a_t + \eta_{ck} k_t) + \varphi_{ca} a_t + \varphi_{ck} k_{t+1}$$

which can further be written as

$$k_{t+1} = \frac{\varphi_{cc} \eta_{ca} + \varphi_{ca} - \eta_{ca} a_1}{\eta_{ck} - \varphi_{ck}} a_t + \frac{\varphi_{cc} \eta_{ck}}{\eta_{ck} - \varphi_{ck}} k_t \quad (2.56)$$

Comparing (2.56) to (2.53) with η_{ka} and η_{kk} to be given by (2.54) and (2.55), we thus obtain

$$\frac{\varphi_{cc} \eta_{ca} + \varphi_{ca} - \eta_{ca} a_1}{\eta_{ck} - \varphi_{ck}} = \varphi_{ka} + \varphi_{kc} \eta_{ca} \quad (2.57)$$

$$\frac{\varphi_{cc} \eta_{ck}}{(\eta_{ck} - \varphi_{ck})} = \varphi_{kk} + \varphi_{kc} \eta_{ck} \quad (2.58)$$

Equation (2.58) gives rise to the following quadratic function in η_{ck} :

$$Q_2 \eta_{ck}^2 + Q_1 \eta_{ck} + Q_0 = 0 \quad (2.59)$$

with Q_2 , Q_1 and Q_0 to be given in the proposition. Solving (2.59) for η_{ck} , we obtain (2.39). Given η_{ck} , η_{ca} is resolved from (2.57), which gives rise to (2.40).

2.7 Appendix II: Dynamic Programming for the Stochastic Version

We here present a stochastic version of a growth model which is based on the Ramsey model of chapter 2.1 but extended to the stochastic case. A model of type goes back to Brock and Mirman (1972). Here the Ramsey 1d model ?? is extended using a second variable modelling a stochastic shock. The model is given by the discrete time equations

$$\begin{aligned} K(t+1) &= A(t)\tilde{A}K(t)^\alpha - C(t) \\ A(t+1) &= \exp(\rho \ln A(t) + z_t) \end{aligned}$$

where α and ρ are real constants and the $z(t)$ are i.i.d. random variables with zero mean. The return function is again $U(C) = \ln C$.

In our numerical computations which follows Grüne and Semmler (2004) we used the parameter values $\tilde{A} = 5$, $\alpha = 0.34$, $\rho = 0.9$ and $\beta = 0.95$. As the case of the Ramsey model, the exact solution is known and given by

$$V(K, A) = B + \tilde{C} \ln K + DA,$$

where

$$B = \frac{\ln((1 - \beta\alpha)\tilde{A}) + \frac{\beta\alpha}{1-\beta\alpha} \ln(\beta\alpha\tilde{A})}{1 - \beta}, \quad \tilde{C} = \frac{\alpha}{1 - \alpha\beta}, \quad D = \frac{1}{(1 - \alpha\beta)(1 - \rho\beta)}$$

We have computed the solution to this problem on the domain $\Omega = [0.1, 10] \times [-0.32, 0.32]$. The integral over the Gaussian variable z was approximated by a trapezoidal rule with 11 discrete values equidistributed in the interval $[-0.032, 0.032]$ which ensures $\varphi(x, u, z) \in \Omega$ for $x \in \Omega$ and suitable $u \in U = [0.5, 10.5]$. For evaluating the maximum in T the set U was discretized with 161 points. Table 2.2 shows the results of the resulting adaptive gridding scheme applied with refinement threshold $\theta = 0.1$ and coarsening tolerance $ctol = 0.001$. Figure 2.7 shows the resulting optimal value function and adapted grid.

# nodes	Error	estimated Error
49	$1.4 \cdot 10^0$	$1.6 \cdot 10^{-1}$
56	$0.5 \cdot 10^{-1}$	$6.9 \cdot 10^0$
65	$2.9 \cdot 10^{-1}$	$3.4 \cdot 10^0$
109	$1.3 \cdot 10^{-1}$	$1.6 \cdot 10^0$
154	$5.5 \cdot 10^{-2}$	$6.8 \cdot 10^{-1}$
327	$2.2 \cdot 10^{-2}$	$2.4 \cdot 10^{-1}$
889	$9.6 \cdot 10^{-3}$	$7.3 \cdot 10^{-2}$
2977	$4.3 \cdot 10^{-3}$	$3.2 \cdot 10^{-2}$

Table 2.2: Number of nodes and errors for our Example

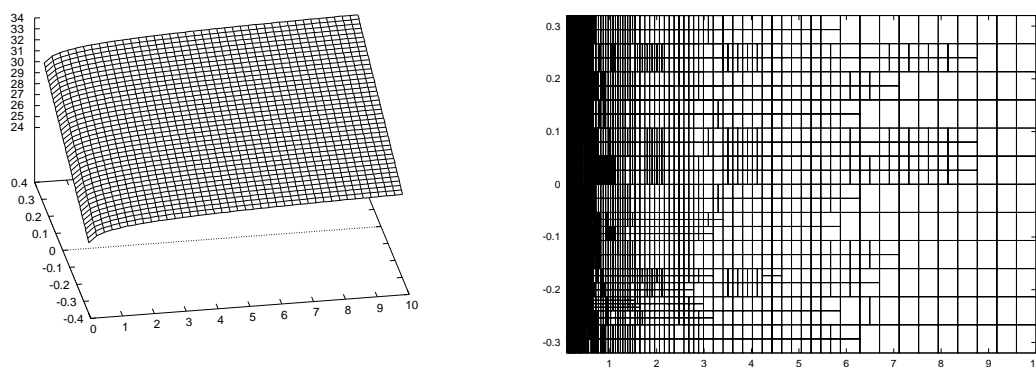


Figure 2.7: Approximated value function and final adaptive grid for our Example

In Santos and Vigo-Aguiar (1995) on equidistant grids with $143 \times 9 = 1287$ and $500 \times 33 = 16500$ nodes, errors of $2.1 \cdot 10^{-1}$ and $1.48 \cdot 10^{-2}$, respectively, were reported. In our adaptive iteration these accuracies could be obtained with 109 and 889 nodes, respectively; thus we obtain a reduction in the number of nodes of more than 90% in the first and almost 95% in the second case, even though the anisotropy of the value function was already taken into account in these equidistant grids. Here again in our stochastic version of the growth model, a steep value function can best be approximated with grid refinement.

Chapter 3

The Estimation and Evaluation of the Stochastic Dynamic Model

3.1 Introduction

Solving a stochastic dynamic optimization model with approximation methods or dynamic programming is only a first step towards the empirical assessment of such a model. Another necessary step is to estimate the model with some econometric techniques. To undertake this step certain approximation methods are more useful than others. Given the estimation, one then can evaluate the model to see how the model's prediction can match the empirical data.

The task of estimation has often been ignored in the current empirical studies of a stochastic dynamic optimization model when a technique, often referred to as calibration, is employed. The calibration approach compares the moment statistics (usually the second moments) of major macroeconomic time series to those obtained from simulating the model.¹ Typically the parameters employed for the model's simulation are selected from independent sources, such as different microeconomic studies. This approach has been criticized because the structural parameters are assumed to be given rather

¹See, e.g., Kydland and Prescott (1982), Long and Plosser (1983), Prescott (1986), Hansen (1985, 1988), King et.al (1988a, 1988b), Plosser (1989) among many others. Recently, other statistics (in addition to the first and second moments) proposed by early business cycle literature, e.g., Burns and Mitchell (1946) and Adelman and Adelman (1959), are also employed for this comparison. See King and Plosser (1994) and Simkins (1994) among others.

than estimated.²

Although this may not create severe difficulties for some currently used stochastic dynamic optimization models, such as the RBC, we believe the problem remains for more elaborated models in the future development of macroeconomic research. Unfortunately, we do not find many econometric studies on how to estimate a stochastic dynamic optimization model except for a few attempts that have been undertaken for some simple cases.³ In this chapter, we shall discuss two estimation methods: the Generalized Method of Moment (GMM) estimation, and the Maximum Likelihood (ML) estimation. Both estimation methods define an objective function to be optimized. Due to the complexity of stochastic dynamic models, it is often unclear how the parameters to be estimated are related to the model's restrictions and hence to the objective function in the estimation. We thus also need to develop an estimation strategy that can be used to recursively search the parameter space in order to obtain the optimum.

Section 2 will first introduce the calibration technique, which has been used in the current empirical study of stochastic dynamic models. As one can find there, a proper application of calibration requires us to define the model's structural parameters correctly. We then in Section 3 consider two possible estimation methods, the GMM and the ML estimations. In Section 4, we propose a strategy to implement these two estimations for estimating a dynamic optimization model. This estimation requires a global optimization algorithm. We, therefore, subsequently in Section 5, introduce a global optimization algorithm, called simulated annealing, which is used for executing the suggested strategy of estimation. Finally, a sketch of the computer program for our estimation strategy will be described in the appendix of this chapter.

3.2 Calibration

The current empirical studies of a stochastic dynamic model often rely on calibration. This approach uses the Monte Carlo method to generate the distribution of some moment statistics implied by the model. For an empirical assessment of the model, solved through some approximation method, we compare these moment statistics to the sample moments computed from the data. Generally, the calibration may include the following steps:

²For an early critique of the parameter selection employed in calibration technique, see Singleton (1988) and Eichenbaum (1991).

³See, e.g., Christiano and Eichenbaum (1992), Burnside et al. (1993), Chow (1993), Chow and Kwan (1998).

- Step 1: Select the model's structural parameters. These parameters may include preference and technology parameters and those that describe the distribution of the random variables in the model, such as σ_ϵ , the standard deviation of ϵ_t as in equation (1.3).
- Step 2: Select the number of times for which the iteration is conducted in the simulation of the model. This number might be the same as the number of observations in the sample. We denote this number by T .
- Step 3: Select the initial condition (x_0, z_0) and use the state equation (1.2), the control equation (1.4) and the exogenous equation (1.3) to compute the solution of the model iteratively for T times. This can be regarded as a one time simulation.
- Step 4: If necessary, detrend the simulated series generated in Step 3 to remove its time trend. Often the HP-filter (see Hodrick and Prescott 1980) is used for this detrending.
- Step 5: Compute the moment statistics of interest using, if necessary, a detrended series generated in Step 4. These moment statistics are mainly those of the second moments such as variances and covariances.
- Step 6: Repeat Step 3 to 5 N times. Here N should be sufficiently large. Then after these N times repeated runs, compute the distributions of these moment statistics. These distributions are mainly represented by their means and their standard deviations.
- Step 7: Compute the same moment statistics from the data sample and check whether it falls within the proper range of the distribution for the moment statistics generated from the Monte Carlo simulation of the model.

Due to the stochastic innovation of ϵ_t , the simulated series should also be cyclically and stochastically fluctuating. The extent and the way of fluctuations, reflected by some second moment statistics, should depend on the model specification and the structural parameters including σ_ϵ , the standard deviation of ϵ_t . Therefore, if we specify the model correctly with the structural parameters, including σ_ϵ , which should also be defined properly, the above comparison of moment statistics of the model and sample should give us a basis to test whether the model can explain the actual business cycles represented by the data.

3.3 The Estimation Methods

The application of the calibration requires techniques to select the structural parameters accurately. This indicates that we need to estimate the model before the calibration. We consider two possible estimation methods: the GMM estimation and the ML estimation.

3.3.1 The Generalized Method of Moments (GMM) Estimation

The GMM estimation starts with a set of orthogonal conditions, representing the population moments established by the theoretical model:

$$E[h(y_t, \psi)] = 0 \quad (3.1)$$

where y_t is a k -dimensional vector of observed random variables at date t ; ψ is a l -dimensional vector of unknown parameters that needs to be estimated; $h(\cdot)$ is a vector-valued function mapping from $R^k \times R^l$ into R^m . Let y_T contain all the observations of the k variables in the sample with size T . The sample average of $h(\cdot)$ can then be written as

$$g_T(\psi; y_T) = \frac{1}{T} \sum_{t=1}^T h(y_t, \psi) \quad (3.2)$$

Notice that $g_T(\cdot)$ is also a vector-valued function with m -dimensions. The idea behind the GMM estimation is to choose an estimator of ψ , denoted $\hat{\psi}$, such that the sample moments $g_T(\hat{\psi}; y_T)$ are as close as possible to the population moments reflected by (3.1). To achieve this, one needs to define a distance function by which that closeness can be judged. Hansen (1982) suggested the following distance function:

$$J(\hat{\psi}; y_T) = \left[g_T(\hat{\psi}; y_T) \right]' W_T \left[g_T(\hat{\psi}; y_T) \right] \quad (3.3)$$

where W_T , called the weighting matrix, is $m \times m$, symmetric, positive definite and depends only on the sample observation y_T . The choice of this weighting matrix defines a metric that makes the distance function a scalar. The GMM estimator of ψ is the value of $\hat{\psi}$ that minimizes (3.3). Hansen (1982) proves that under certain assumption such a GMM estimator is consistent and asymptotically normal. Also from the results established in Hansen (1982), a consistent estimator of the variance-covariance matrix of $\hat{\psi}$ is given by

$$\text{Var}(\widehat{\psi}) = \frac{1}{T}(D_T)^{-1}W_T(D_T')^{-1}, \quad (3.4)$$

where $D_T = \partial g_T(\widehat{\psi})/\partial \psi'$.

There is a great flexibility in the choice of W_T for constructing a consistent and asymptotically normal GMM estimator. In this book, we will adopt the method by Newey and West (1987), where it is suggested that

$$W_T^{-1} = \widehat{\Omega}_0 + \sum_{j=1}^d w(j, d)(\widehat{\Omega}_j + \widehat{\Omega}_j'), \quad (3.5)$$

with $w(j, d) \equiv 1 - j/(1 + d)$, $\widehat{\Omega}_j \equiv (1/T) \sum_{t=j+1}^T g(y_t, \widehat{\psi}^*)g(y_{t-j}, \widehat{\psi}^*)$ and d to be a suitable function of T . Here $\widehat{\psi}^*$ is required to be a consistent estimator of ψ . Therefore, the estimation with the GMM method usually requires two steps as suggested by Hansen and Singleton (1982). First, one chooses a sub-optimal weighting matrix to minimize (3.3) and hence obtains a consistent estimator $\widehat{\psi}^*$. Second, one then uses the consistent estimator obtained in the first step to calculate the optimum W_T through which (3.3) is re-minimized.

3.3.2 The Maximum Likelihood (ML) Estimation

The ML estimation, proposed by Chow (1993) for estimating a dynamic optimization model, starts with an econometric model such as follows:

$$By_t + \Gamma x_t = \epsilon_t \quad (3.6)$$

where B is a $m \times m$ matrix; Γ is a $m \times k$ matrix; y_t in this case is a $m \times 1$ vector of dependent variables; x_t is a $k \times 1$ vector of explanatory variables and ϵ_t is a $m \times 1$ vector of disturbance terms. Note that if we take the expectations on both sides, the model to be estimated here is the same as in the GMM estimation represented by (3.1), except here the functions are linear. Non-linearity may pose a problem to derive the log-likelihood function for (3.6).

Suppose there are T observations. Then the above (3.6) can be re-written as

$$BY' + \Gamma X' = E' \quad (3.7)$$

where Y is $T \times m$; X is $T \times k$ and E is $T \times m$. Assuming normal and serially uncorrelated ϵ_t with the covariance matrix Σ , the concentrated log-likelihood function can be derived (see Chow, 1983, p.170-171) as

$$\log L(\psi) = \text{const.} + n \log |B| - \frac{n}{2} \log |\Sigma| \quad (3.8)$$

with the ML estimator of Σ given by

$$\widehat{\Sigma} = n^{-1}(BY' + \Gamma X')(YB' + X\Gamma') \quad (3.9)$$

The ML estimator of ψ is the one that maximizes $\log L(\psi)$ in (3.8). The asymptotic standard deviation of estimated parameters can be inferred from the following variance-covariance matrix of $\widehat{\psi}$ (see Hamilton 1994 p.143):

$$E(\widehat{\psi} - \psi)(\widehat{\psi} - \psi)' \cong - \left[\frac{\partial^2 L(\psi)}{\partial \psi \partial \psi'} \right]^{-1}. \quad (3.10)$$

3.4 The Estimation Strategy

In practice, using GMM or ML method to estimate a stochastic dynamic model is rather complicated. The first problem that we need to discuss are the restrictions on the estimation.

One proper restriction is the state equation and a first-order condition derived either as Euler equation or from the Lagrangian.⁴ Yet, most first-order conditions are extremely complicated and may include some auxiliary variables, such as the Lagrangian multiplier, which are not observable. This seems to suggest that the restrictions for a stochastic dynamic optimization model should typically be represented by the state equation and the control equation derived from the dynamic optimization problem.

The derivation of the control equations in approximate form from a dynamic optimization problem is a complicated process. As discussed in the previous chapters often a numerical procedure is required. For an approximation method, the linearization of the system at its steady states is needed. The linearization and the derivation of the control equation, possibly through an iterative procedure, make it often unclear how the parameters to be estimated are related to the model's restrictions and hence to the objective function in estimation, such as (3.3) and (3.8). Therefore one is usually incapable of deriving analytically the first-order conditions of minimizing (3.3) or maximizing (3.8) with respect to the parameters. Furthermore, using first-order conditions to minimize (3.3) and maximize (3.8) may only lead to a local optimum, which is quite possible in general, since the system to be estimated is often nonlinear in parameters. Consequently, searching a parameter space becomes the only possible way to find the optimum.

Our search process includes the following recursive steps:

- Step 1. Start with an initial guess on ψ and use an appropriate method of dynamic optimization to derive the decision rules.

⁴The parameters in the state equation could be estimated independently.

- Step 2. Use the state equation and the derived control equation to calculate the value of the objective function.
- Step 3. Apply some optimization algorithm to change the initial guess on ψ and start again with step one.

Using this strategy to estimate a stochastic dynamic model, one needs to employ an optimization algorithm to search the parameter space recursively. The conventional optimization algorithms,⁵ such as Newton-Raphson or related methods, may not serve our purpose well due to the possible existence of multiple local optima. We thus need to employ a global optimization algorithm to execute the estimation process as described above. One possible candidates is the simulated annealing, which shall be discussed in the next section.

3.5 A Global Optimization Algorithm: The Simulated Annealing

The idea of simulated annealing has been initially proposed by Metropolis et al. (1953) and later developed by Vanderbilt and Louie (1984), Bohachevsky et al. (1986) and Corana et al, (1987). The algorithm operates through an iterative random search for the optimal variables of an objective function within an appropriate space. It moves uphill and downhill with a varying step size to escape local optima. The step size is narrowed so that the random search is confined to an ever smaller region when the global optimum is approached.

The simulated annealing algorithm has been tested by Goffe et al. (1992). For this test, Goffe et al. (1992) compute a test function with two optima provided by Judge et al. (1985, p.956-7). By comparing it with conventional algorithms, they find that out of 100 times conventional algorithms are successful 52-60 times to reach the global optimum while simulated annealing is 100 percent efficient. We thus believe that the algorithm may serve our purpose well.

Let $f(x)$, for example, be a function that is to be maximized and $x \in S$, where S is the parameter space with the dimensions equal to the number of structural parameters that need to be estimated. The space S should be defined from the economic viewpoint and by computational convenience. The algorithm starts with an initial parameter vector x^0 . Its value $f^0 = f(x^0)$

⁵For conventional optimization algorithm, see appendix B of Judge et al (1985) and Hamilton (1994, ch. 5).

is calculated and recorded. Subsequently, we set the optimum x and $f(x)$ – denoted by x_{opt} and f_{opt} respectively – to x^0 and $f(x^0)$. Other initial conditions include the initial step-length (a vector with the same dimension as x) denoted by v^0 and an initial temperature (a scalar) denoted by T^0 .

The new variable, x' , is chosen by varying the i th element of x^0 such that

$$x'_i = x_i^0 + r \cdot v_i^0 \quad (3.11)$$

where r is a uniformly distributed random number in $[-1, 1]$. If x' is not in S , repeat (3.11) until x' is in S . The new function value $f' = f(x')$ is then computed. If f' is larger than f^0 , x' is accepted. If not, the Metropolis criteria,⁶ denoted as p , is used to decide on acceptance, where

$$p = e^{(f' - f)/T^0} \quad (3.12)$$

This p is compared to p' , a uniformly distributed random number from $[0, 1]$. If p is greater than p' , x' is accepted. Besides, f' should also be compared to the updated f_{opt} . If it is larger than f_{opt} , both x_{opt} and f_{opt} are replaced by x' and f' .

The above steps (starting with (3.11)) should be undertaken and repeated N_S times⁷ for each i . Subsequently, the step-length is adjusted. The i th element of the new step-length vector (denoted as v'_i) depends on its number of acceptances (denoted as n_i) in its last N_S times of the above repetition and is given by

$$v'_i = \begin{cases} v_i^0 \left[1 + \frac{1}{0.4} c_i (n_i/N_S - 0.6)\right] & \text{if } n_i > 0.6N_S; \\ v_i^0 \left[1 + \frac{1}{0.4} c_i (0.4 - n_i/N_S)\right]^{-1} & \text{if } n_i < 0.4N_S; \\ v_i^0 & \text{if } 0.4N_S \leq n_i \leq 0.6N_S \end{cases} \quad (3.13)$$

where c_i is suggested to be 2 as by Corona et al. (1987) for all i . With the new selected step-length vector, one goes back to (3.11) and hence starts a new round of iteration. Again after another N_S times of such repetitions, the step-length will be re-adjusted. These adjustments as to each v_i should be performed N_T times.⁸ We then come to adjust the temperature. The new temperature (denoted as T') will be

$$T' = R_T T^0 \quad (3.14)$$

with $0 < R_T < 1$.⁹ With this new temperature T' , we should go back again to (3.11). But this time, the initial variable x^0 is replaced by the updated

⁶motivated by thermodynamics.

⁷ N_S is suggested to be 20 as by Corana et al. (1987)

⁸ N_T is suggested to be 100 by Corona et. al. (1987).

⁹ R_T is suggested to be 0.85 by Corana et. al. (1987).

x_{opt} . Of course, the temperature will be reduced further after one additional N_T times of adjusting the step-length of each i .

For convergence, the step-length in (3.11) is required to be very small. In (3.13), whether the new selected step-length is enlarged or not depends on the corresponding number of acceptances. The number of acceptance n_i is not only determined by whether the new selected x_i increases the value of objective function, but also by the Metropolis criteria which itself depends on the temperature. Thus a convergence will ultimately be achieved with the continuous reduction of the temperature. The algorithm will end by comparing the value of f_{opt} for the last N_ϵ times (suggested to be 4) when the temperature is attempted to be re-adjusted.

The simulated annealing algorithm described above has been tested by Goffe et al. (1992). For this test, Goffe et al. (1992) compute a test function with two optima provided by Judge et al. (1985, p. 956-7). By comparing it with conventional algorithms, they find that out of 100 times conventional algorithms are successful 52-60 times to reach the global optimum while simulated annealing is 100 percent efficient. We thus believe that the algorithm may serve our purpose well. In the next chapter, we shall demonstrate the effectiveness of this estimation strategy by estimating a benchmark RBC with simulated data.

3.6 Conclusions

In this chapter, we have first introduced the calibration method, which has often been employed in the assessment of a stochastic dynamic model. We then, based on some approximation methods, have presented an estimation strategy, to estimate stochastic dynamic models employing time series data. We have introduced both the General Method of Moments (GMM) as well as the Maximum Likelihood (ML) estimations as strategies to match the dynamic decision model with time series data. Although both strategies permit to estimate the parameters involved, often a global optimization algorithm, for example the simulated annealing, is needed to be employed to detect the correct parameters.

3.7 Appendix: A Sketch of the Computer Program for Estimation

The algorithm we describe here is written in GAUSS. The entire program consists of three parts. The first part regards some necessary steps in the

data processing after loading the original data. The second part is the procedure that calculates the value of objective function for the estimation. The input of this procedure are the structural parameters, while the activation of this procedure generates the value of objective function. We denote this procedure as $\text{OBJF}(\varphi)$. The third part, which is also the main part of this program, is the simulated annealing. Of these three parts, we shall only describe the simulated annealing.

```

    {Set initial conditions for simulated annealing}
DO UNTIL convergence;
   $t = t + 1$ ;
  DO  $N_T$  times;
     $n = 0$ ;          /*set the vector for recording No. of acceptances*/
    DO  $N_s$  times;
       $i = 0$ ;
      DO UNTIL  $i =$  the dimensions of  $\varphi$ ;
         $i = i + 1$ ;
        HERE:  $\varphi_i = \varphi_i + rv_i$ ;
         $\varphi' =$  {as if current  $\varphi$  except the  $i$ th element to be  $\varphi'_i$ };
        IF  $\varphi'$  is not in  $S$ ;
          GOTO HERE;
          ELSE;
            CONTINUE;
        ENDIF;
         $f = \text{OBJF}(\varphi')$ ; /* $f$  is the value of the objective function to be minimized*/
         $p = \exp[(f' - f)/T]$ ;          /* $p$  is the metropolis criteria*/
        IF  $f' > f$  or  $p > p'$ 
           $\varphi = \varphi'$ ;
           $f = f'$ ;
           $n_i = n_i + 1$ ;
          ELSE;
            CONTINUE;
          ENDIF;
        IF  $f' > f_{opt}$ ;
           $\varphi_{opt} = \varphi'$ ;
           $f_{opt} = f'$ ;
          ELSE;
            CONTINUE;
          ENDIF;
      ENDIF;
    ENDIF;
  ENDIF;

```

```
    ENDO;  
  ENDO;  
   $i = 0$ ;  
  {define the new step-size,  $v'$ , according to  $n_i$ }  
   $v = v'$ ;  
  ENDO;  
  IF change of  $f_{opt} < \varepsilon$  in last  $N_\varepsilon$  times;  
    REPORT  $\varphi_{opt}$  and  $f_{opt}$ ;  
    BREAK  
  ELSE  
     $T = R_T T$ ;  
    CONTINUE;  
  ENDIF;  
  ENDO;
```

Part II

The Standard Stochastic Dynamic Optimization Model

Chapter 4

Real Business Cycles: Theory and the Solutions

4.1 Introduction

The Real Business Cycle model as a prototype of a stochastic dynamic macro-model has influenced quantitative macromodeling enormously in the last two decades. Its concepts and methods have diffused into mainstream macroeconomics. The criticism of the performance of macroeconometric models of Keynesian type in the 1970s and the associated rational expectation revolution pioneered by Lucas (1976) initiated this development. The Real Business Cycle analysis now occupies a major position in the curriculum of many graduate programs. To some extent, the Real Business Cycle approach has become a new orthodoxy of macroeconomics.

The central argument by Real Business Cycle theorists is that economic fluctuations are caused primarily by real factors. Kydland and Prescott (1982) and Long and Plosser (1983) first strikingly illustrate this idea in a simple representative agent optimization model with market clearing, rational expectation and no monetary factors. Stockey, Lucas and Prescott (1989) further illustrate that such type of model could be viewed as an Arrow-Debreu economy so that the model can be established on a solid micro-foundation with many (identical) agents. Therefore, as mentioned above, the RBC analysis can also be regarded as a general equilibrium approach to macrodynamics.

This chapter introduces the RBC model by first describing its microeconomic foundation as set out by Stockey et al. (1989). We then present the standard RBC model as formulated in King et al. (1988). A model of this kind will repeatedly be used in the subsequent chapters in various ways. The

model will then be solved after being parameterized by those standard values of the model's structural parameters.

4.2 The Microfoundation

The standard Real Business Cycle model assumes a representative agent who solves a resource allocation problem over an infinite time horizon via dynamic optimization. It is argued that “the solutions to planning problems of this type can, under appropriate conditions, be interpreted as predictions about the behavior of market economies.” (Stokey et al. 1989, p. 22)

To establish the connection to the competitive equilibrium of the Arrow-Debreu economy,¹ several assumptions should be made for an hypothetical economy. First, the households in the economy are identical, all with the same preference, and firms are also identical, all producing a common output with the same constant returns to scale technology. With this identical assumption, the resource allocation problem can be viewed as an optimization problem of a representative agent.

Second, as in Arrow-Debreu economy, the trading process is assumed to be “once-and-for-all”. The following citation is again from Stokey et al. (1989, p.23).

Finally, assume that all transactions take place in a single once-and-for-all market that meets in period 0. All trading takes place at that time, so all prices and quantities are determined simultaneously. No further trades are negotiated later. After this market has closed, in periods $t = 0, 1, \dots, T$, agents simply deliver the quantities of factors and goods they have contracted to sell and receive those they have contracted to buy.

The third assumption regards the ownership. It is assumed that the household owns all factors of production and all shares of the firm. Therefore, in each period the household sells factor services to the firm. The revenue from selling factors can only be used to buy the goods produced by the firm either for consuming or accumulating as capital. The representative firm owns nothing. In each period it simply hires capital and labor on a rental basis to produce output, sells the output and transfers any profit back to the household.

¹See Arrow and Debreu (1954) and Debreu (1959).

4.2.1 The Decision of the Household

At the beginning of the period 0 when the market is open, the household is given the price sequence $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ at which he (or she) will choose the sequence of output demand and input supply $\{c_t^d, i_t^d, n_t^s, k_t^s\}_{t=0}^{\infty}$ that maximizes the discounted utility:

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_t^d, n_t^s) \right] \quad (4.1)$$

subject to

$$p_t(c_t^d + i_t^d) = p_t(r_t k_t^s + w_t n_t^s) + \pi_t \quad (4.2)$$

$$k_{t+1}^s = (1 - \delta)k_t^s + i_t^d \quad (4.3)$$

Above δ is the depreciation rate; β is the discounted factor; π_t is the expected dividend; c_t^d and i_t^d are the demands for consumption and investment; and n_t^s and k_t^s are the supplies of labor and capital stock. Note that (4.2) can be regarded as a budget constraint. The equality holds due to the assumption $U_c > 0$. Next, we shall consider how the representative household calculates π_t . It is reasonable to assume that

$$\pi_t = p_t(\hat{y}_t - w_t \hat{n}_t - r_t \hat{k}_t) \quad (4.4)$$

where \hat{y}_t , \hat{n}_t and \hat{k}_t are the realized output, labor and capital expected by the household at given price sequence $\{p_t, w_t, r_t\}_{t=0}^{\infty}$. Thus assuming that the household knows the production function while expecting that the market will be cleared at the given price sequence $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, (4.4) can be rewritten as

$$\pi_t = p_t \left[f(k_t^s, n_t^s, \hat{A}_t) - w_t n_t^s - r_t k_t^s \right] \quad (4.5)$$

Above, $f(\cdot)$ is the production function and \hat{A}_t is the expected technology shock. Explaining π_t in (4.2) in terms of (4.5) and then substituting from (4.3) to eliminate i_t^d , we obtain

$$k_{t+1}^s = (1 - \delta)k_t^s + f(k_t^s, n_t^s, \hat{A}_t) - c_t^d \quad (4.6)$$

Note that (4.1) and (4.6) represent the standard RBC model, although it only specifies one side of the markets: output demand and input supply. Given the initial capital stock k_0^s , the solution of this model is the sequence of plans $\{c_t^d, i_t^d, n_t^s, k_{t+1}^s\}_{t=0}^{\infty}$, where k_t^s is implied by (4.6), and

$$c_t^d = G_c(k_t^s, \hat{A}_t) \quad (4.7)$$

$$n_t^s = G_n(k_t^s, \hat{A}_t) \quad (4.8)$$

$$i_t^d = f(k_t^s, n_t^s, \hat{A}_t) - c_t^d \quad (4.9)$$

4.2.2 The Decision of the Firm

Given the same price sequence $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, and also the sequence of expected technology shocks $\{\hat{A}_t\}_{t=0}^{\infty}$, the problem faced by the representative firm is to choose input demands and output supplies $\{y_t^s, n_t^d, k_t^d\}_{t=0}^{\infty}$. However, since the firm simply rents capital and hires labor on a period-by-period basis, its optimization problem is equivalent to a series of one-period maximization (Stokey et al. 1989, p25):

$$\max p_t(y_t^s - r_t k_t^d - w_t n_t^d)$$

subject to

$$y_t^s = f(k_t^d, n_t^d, \hat{A}_t) \quad (4.10)$$

where $t = 0, 1, 2, \dots, \infty$. The solution to this optimization problem satisfies:

$$r_t = f_k(k_t^d, n_t^d, \hat{A}_t)$$

$$w_t = f_n(k_t^d, n_t^d, \hat{A}_t)$$

This first-order condition allow us to derive the following equations of input demands k_t^d and n_t^d :

$$k_t^d = k(r_t, w_t, \hat{A}_t) \quad (4.11)$$

$$n_t^d = n(r_t, w_t, \hat{A}_t) \quad (4.12)$$

4.2.3 The Competitive Equilibrium and the Walrasian Auctioneer

A competitive equilibrium can be described as a sequence of prices $\{p_t^*, w_t^*, r_t^*\}_{t=0}^{\infty}$ at which the two market forces (demand and supply) are equalized in all these three markets, i.e.,

$$k_t^d = k_t^s \quad (4.13)$$

$$n_t^d = n_t^s \quad (4.14)$$

$$c_t^d + i_t^d = y_t^s \quad (4.15)$$

for all t 's, $t = 0, 1, 2, \dots, \infty$. The economy is at the competitive equilibrium if

$$\{p_t, w_t, r_t\}_{t=0}^{\infty} = \{p_t^*, w_t^*, r_t^*\}_{t=0}^{\infty}$$

Using equation (4.6) - (4.12), one can easily prove the existence of $\{p_t^*, w_t^*, r_t^*\}_{t=0}^{\infty}$ that satisfies the equilibrium condition (4.13)-(4.15).

The real business cycles literature usually does not explain how the equilibrium is achieved. Implicitly, it is assumed that there exists an auctioneer in the market, who adjust the price towards the equilibrium. This adjustment process - often named as tâtonnement process as in Walrasian economics - is a common solution to the adjustment problem within the neoclassical general equilibrium framework.

4.2.4 The Contingency Plan

It is not difficult to find that the sequence of equilibrium prices $\{p_t^*, w_t^*, r_t^*\}_{t=0}^\infty$ depends on the expected technology shock $\{\hat{A}_t\}_{t=0}^\infty$. This indeed creates a problem how to express the equilibrium prices and the equilibrium demand and supply which are supposed to be made at the beginning of period 0 when the technology shock from period 1 onward are all unobserved. The Real Business Cycle theorists circumvent this problem skillfully and ingeniously. Their approach is to use the so-called “contingency plan”. As written by Stokey et al.(1989, p17):

In the stochastic case, however, this is not a sequence of numbers but a sequence of contingency plans, one for each period. Specifically, consumption c_t , and end-of-period capital k_{t+1} in each period $t = 1, 2, \dots$ are contingent on the realization of the shocks z_1, z_2, \dots, z_t . This sequence of realization is information that is available when the decision is being carried out but is unknown in period 0 when the decision is being made. Technically, then, the planner chooses among sequence of functions....

Thus the sequence of equilibrium prices and the sequence of equilibrium demand and supply are all contingent on the realization of the shock regardless that the corresponding decisions are all made at the beginning of period 0.

4.2.5 The Dynamics

Assume that the decisions are all contingent on the future shock $\{A_t\}_{t=0}^\infty$, and the prices are all at their equilibrium, the dynamics of our hypothetical economy can be fully described by the following equations regarding the

realized consumption, employment, output, investment and capital stock:

$$c_t = G_c(k_t, A_t) \quad (4.16)$$

$$n_t = G_n(k_t, A_t) \quad (4.17)$$

$$y_t = f(k_t, n_t, A_t) \quad (4.18)$$

$$i_t = y_t - c_t \quad (4.19)$$

$$k_{t+1} = (1 - \delta)k_t + f(k_t, n_t, A_t) - c_t \quad (4.20)$$

given the initial condition k_0 and the sequence of technology shock $\{A_t\}_{t=0}^{\infty}$. This indeed provides another important property of the RBC economy. Although the model specifies the decision behaviors for both household and the firm and therefore the two market forces, demand and supply, for all major three markets: output, capital and labor markets, the dynamics of the economy is reflected by only the household behavior, which concerns only one side of the market forces, the output demand and input supply. The decision of the firm does not have any impact! This is certainly due to the equilibrium feature of the model specification.

4.3 The Standard RBC Model

4.3.1 The Model Structure

The hypothetical economy we have presented in the last section is only for explaining the theory (from microeconomic point of view) behind the standard RBC economy. The model specified in (4.16) - (4.20) is not testable with empirical data, not only because we do not specify the stochastic process of $\{A_t\}_{t=0}^{\infty}$, but also we do not introduce the growth factor. For an empirically testable standard RBC model, we employ here the specifications of a model as formulated by King, et al. (1988). This empirically oriented formulation will be repeatedly used in the subsequent chapters of this volume.

Let K_t denote for the aggregate capital stock, Y_t for aggregate output and C_t for aggregate consumption. The capital stock in the economy follow the transition law:

$$K_{t+1} = (1 - \delta)K_t + Y_t - C_t, \quad (4.21)$$

where δ is the depreciation rate. Assume that the aggregate production function take the form:

$$Y_t = A_t K_t^{1-\alpha} (N_t X_t)^\alpha \quad (4.22)$$

where N_t is per capita working hours; α is the share of labor in the production function; A_t is the temporary shock in technology and X_t the permanent

shock that follows a growth rate γ . Note that here X_t includes not only the growth in labor force, but also the growth in productivity. Apparently, the model is nonstationary due to X_t . To transform the model into a stationary formulation, we divide both sides of equation (4.21) by X_t (when Y_t is expressed by (4.22)):

$$k_{t+1} = \frac{1}{1 + \gamma} \left[(1 - \delta)k_t + A_t k_t^{1-\alpha} (n_t \bar{N} / 0.3)^\alpha - c_t \right], \quad (4.23)$$

where by definition, $k_t \equiv K_t / X_t$, $c_t \equiv C_t / X_t$ and $n_t \equiv 0.3 N_t / \bar{N}$ with \bar{N} to be the sample mean of N_t . Note that n_t is often regarded to be the normalized hours. The sample mean of n_t is equal to 30 %, which, as pointed out by Hansen (1985), is the average percentage of hours attributed to work.

The representative agent in the economy is assumed to make the decision sequence $\{c_t\}_{t=0}^\infty$ and $\{n_t\}_{t=0}^\infty$ so as to

$$\max E_0 \sum_{t=0}^{\infty} \beta^t [\log c_t + \theta \log(1 - n_t)], \quad (4.24)$$

subject to the state equation (4.23). The exogenous variable in this model is the temporary shock A_t , which may follow an AR(1) process:

$$A_{t+1} = a_0 + a_1 A_t + \epsilon_{t+1}, \quad (4.25)$$

with ϵ_t to be an *i.i.d.* innovation.

Note that there is no possibility to derive the exact solution with the standard recursive method. We therefore have to rely on an approximate solution. For this, we shall first derive the first-order conditions.

4.3.2 The First-Order Conditions

As we have discussed in the previous chapters, there are two types of first-order conditions, the Euler equations and the equations from the Lagrangian. The Euler equation is not used in our suggested solution method. We nevertheless still present it here as an exercise and demonstrate that the two first-order conditions are virtually equivalent.

The Euler equation

To derive the Euler equation, our first task is to transform the model into a setting that the state variable k_t does not appear in $F(\cdot)$ as we have discussed in Chapter 1 and 2. This can be done by assuming k_{t+1} (instead of c_t) along

with n_t as model's decision variables. In this case, the objective function takes the form:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(k_{t+1}, n_t, k_t, A_t),$$

where

$$U(k_{t+1}, n_t, k_t, A_t) = \log [(1 - \delta)k_t + y_t - (1 + \gamma)k_{t+1}] + \theta \log(1 - n_t). \quad (4.26)$$

Note that here we have used (4.23) to express c_t in the utility function; y_t is the stationary output via the following equation:

$$y_t = A_t k_t^{1-\alpha} (n_t \bar{N} / 0.3)^\alpha. \quad (4.27)$$

Given such an objective function, the state equation (4.23) can simply be ignored in deriving the first-order condition. The Bellman equation in this case can be written as

$$V(k_t, A_t) = \max_{k_{t+1}, n_t} U(k_{t+1}, n_t, k_t, A_t) + \beta E [V(k_{t+1}, A_{t+1})]. \quad (4.28)$$

The necessary condition for maximizing the right side of Bellman equation (4.28) is given by

$$\frac{\partial U}{\partial k_{t+1}}(k_{t+1}, n_t, k_t, A_t) + \beta E \left[\frac{\partial V}{\partial k_{t+1}}(k_{t+1}, A_{t+1}) \right] = 0; \quad (4.29)$$

$$\frac{\partial U}{\partial n_t}(k_{t+1}, n_t, k_t, A_t) = 0. \quad (4.30)$$

Meanwhile the application of Benveniste-Scheinkman formula gives

$$\frac{\partial V}{\partial k_t}(k_t, A_t) = \frac{\partial U}{\partial k_t}(k_{t+1}, n_t, k_t, A_t) \quad (4.31)$$

Using (4.31) to express $\frac{\partial V}{\partial k_{t+1}}(k_{t+1}, A_{t+1})$ in (4.29), we obtain

$$\frac{\partial U}{\partial k_{t+1}}(k_{t+1}, n_t, k_t, A_t) + \beta E \left[\frac{\partial U}{\partial k_{t+1}}(k_{t+2}, n_{t+1}, k_{t+1}, A_{t+1}) \right] = 0. \quad (4.32)$$

From equation (4.23) and (4.26),

$$\begin{aligned} \frac{\partial U}{\partial k_{t+1}}(k_{t+1}, n_t, k_t, A_t) &= -\frac{1 + \gamma}{c_t}; \\ \frac{\partial U}{\partial k_{t+1}}(k_{t+2}, n_{t+1}, k_{t+1}, A_{t+1}) &= \frac{(1 - \delta)k_{t+1} + (1 - \alpha)y_{t+1}}{k_{t+1}c_{t+1}}; \\ \frac{\partial U}{\partial n_t}(k_{t+1}, n_t, k_t, A_t) &= \frac{\alpha y_t}{n_t c_t} - \frac{\theta}{1 - n_t}. \end{aligned}$$

Substituting the above expressions into (4.32) and (4.30), we establish the following Euler equations:

$$-\frac{1+\gamma}{c_t} + \beta E \left[\frac{(1-\delta)k_{t+1} + (1-\alpha)y_{t+1}}{k_{t+1}c_{t+1}} \right] = 0; \quad (4.33)$$

$$\frac{\alpha y_t}{n_t c_t} - \frac{\theta}{1-n_t} = 0. \quad (4.34)$$

The First-Order Condition Derived from the Lagrangian

Next, we turn to derive the first-order condition from the Lagrangian. Define the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t [\log(c_t) + \theta \log(1-n_t)] - \sum_{t=0}^{\infty} E_t \left\{ \beta^{t+1} \lambda_{t+1} \left[k_{t+1} - \frac{1}{1+\gamma} \left((1-\delta)k_t - A_t k_t^{1-\alpha} (n_t \bar{N}/0.3)^\alpha + c_t \right) \right] \right\}$$

Setting zero the derivatives of L with respect to c_t , n_t , k_t and λ_t , one obtains the following first-order condition:

$$\frac{1}{c_t} - \frac{\beta}{1+\gamma} E_t \lambda_{t+1} = 0; \quad (4.35)$$

$$\frac{-\theta}{1-n_t} + \frac{\alpha \beta y_t}{(1+\gamma)n_t} E_t \lambda_{t+1} = 0; \quad (4.36)$$

$$\frac{\beta}{1+\gamma} E_t \lambda_{t+1} \left[(1-\delta) + \frac{(1-\alpha)y_t}{k_t} \right] = \lambda_t; \quad (4.37)$$

$$k_{t+1} = \frac{1}{1+\gamma} [(1-\delta)k_t + y_t - c_t], \quad (4.38)$$

with y_t again to be given by (4.27).

Next we try to demonstrate that the two first-order conditions: (4.33) - (4.34) and (4.35) - (4.38) are virtually equivalent. This can be done as follows. First, expressing $[\beta/(1+\gamma)]E_t \lambda_{t+1}$ in terms of $1/c_t$ (which is implied by (4.35)), we obtain from (4.37)

$$\lambda_t = \frac{(1-\delta)k_t + (1-\alpha)y_t}{k_t c_t}$$

This further indicates that

$$E_t \lambda_{t+1} = \frac{(1-\delta)k_{t+1} + (1-\alpha)y_{t+1}}{k_{t+1}c_{t+1}} \quad (4.39)$$

Substituting (4.39) into (4.35), we obtain the first Euler equation (4.33). Second, expressing $[\beta/(1+\gamma)]E_t\lambda_{t+1}$ again in terms of $1/c_t$, and substituting it into (4.36), we then verify the second Euler equation (4.34).

4.3.3 The Steady States

Next we try to derive the corresponding steady states. The steady state A_t is simply determined from (4.25). The other steady states are given by the following proposition:

Proposition 4 *Assume A_t has a steady state \bar{A} . Equation (4.35) - (4.38) along with (4.27), when evaluated in terms of their certainty equivalence forms, determines at least two steady states: one is on boundary, denoted as $(\bar{c}_b, \bar{n}_b, \bar{k}_b, \bar{y}_b, \bar{\lambda}_b)$, and the other is interior, denoted as $(\bar{c}_i, \bar{n}_i, \bar{k}_i, \bar{y}_i, \bar{\lambda}_i)$. In particular,*

$$\begin{aligned}\bar{c}_b &= 0, \\ \bar{n}_b &= 1, \\ \bar{\lambda}_b &= \infty, \\ \bar{k}_b &= [\bar{A}/(\delta + \gamma)]^{1/\alpha} (\bar{N}/0.3), \\ \bar{y}_b &= (\delta + \gamma)\bar{k}_b;\end{aligned}$$

and

$$\begin{aligned}\bar{n}_i &= \alpha\phi / [(\alpha + \theta)\phi - (\delta + \gamma)\theta], \\ \bar{k}_i &= \bar{A}^{1/\alpha} \phi^{-1/\alpha} \bar{n}_i (\bar{N}/0.3) \\ \bar{c}_i &= (\phi - \delta - \gamma)\bar{k}_i, \\ \bar{\lambda}_i &= (1 + \gamma)/\beta\bar{c}_i \\ \bar{y}_i &= \phi\bar{k}_i,\end{aligned}$$

where

$$\phi = \frac{(1 + \gamma) - \beta(1 - \delta)}{\beta(1 - \alpha)} \quad (4.40)$$

Note that we have used the first-order condition from the Lagrangian to derive the above two steady states. Since the two first-order conditions are virtually equivalent, we expect that the same steady states can also be derived from the Euler equations.²

²We however leave this exercise to the readers.

4.4 Solving Standard Model with Standard Parameters

To obtain the solution path of standard RBC model, we shall first specify the values of the structural parameters defined in the model. These are reported in Table 4.1. We shall remark that these parameters are close to the standard parameters, which can often be found in the RBC literature.³ More detailed discussion regarding the parameter selection and estimation will be provided in the next chapter.

Table 4.1: Parameterizing the Standard RBC Model

α	γ	β	δ	θ	N	a_0	a_1	σ_ϵ
0.58	0.0045	0.9884	0.025	2	480	0.0333	0.9811	0.0189

The solution method that we shall employ is the method of linear-quadratic approximation with our suggested algorithm as discussed in Chapter 1. Assume that decision rule take the form

$$c_t = G_{11}A_t + G_{12}k_t + g_1 \quad (4.41)$$

$$n_t = G_{21}A_t + G_{22}k_t + g_2 \quad (4.42)$$

Our first step is to compute the first- and the second-order partial derivatives with respect to F and U where

$$F(k, c, n, A) = \frac{1}{1 + \gamma} \left[(1 - \delta)k + Ak^{1-\alpha}(n\bar{N}/0.3)^\alpha - c \right]$$

$$U(c, n) = \log(c) + \theta \log(1 - n)$$

All these partial derivatives along with the steady states can be used as the inputs in the GAUSS procedure provided in Appendix II of Chapter 1. Executing this procedure will allow us to compute the undetermined coefficients G_{ij} and g_i ($i, j = 1, 2$) in the decision rule as expressed in (4.41) and (4.42). In Figure 4.1 and Figure 4.2, we illustrate the solution paths, one for the deterministic and the other for the stochastic case, for the variables k_t , c_t , n_t and A_t .

³Indeed, they are essentially the same as the parameters choosed by King et al. (1988) except the last three parameters, which is related to the stochastic equation (4.25).

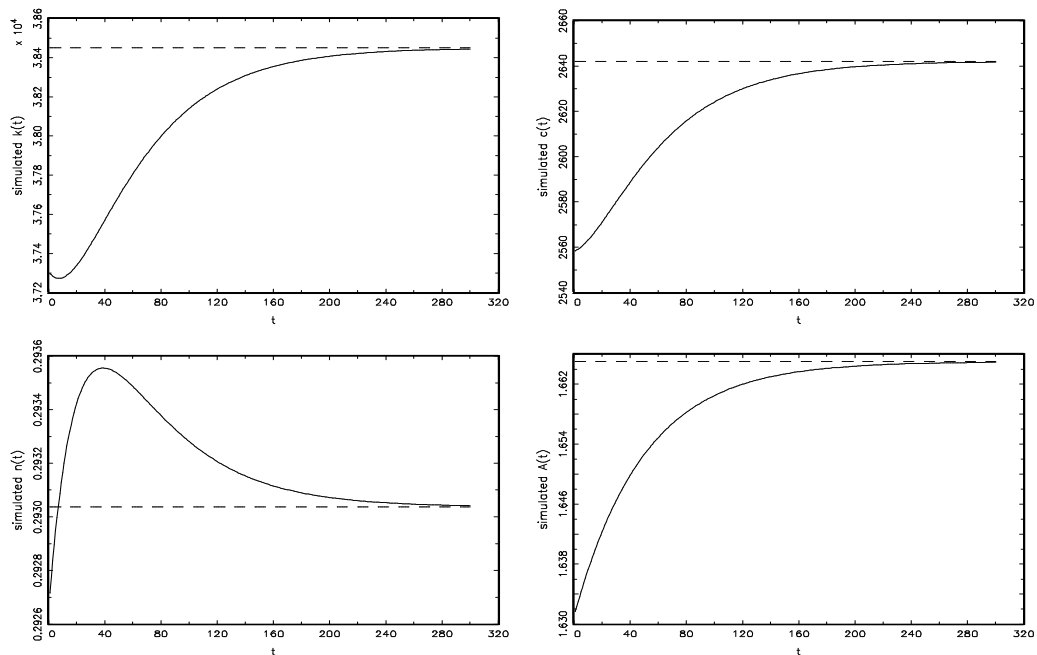


Figure 4.1: The Deterministic Solution to the Benchmark RBC Model for the Standard Parameters

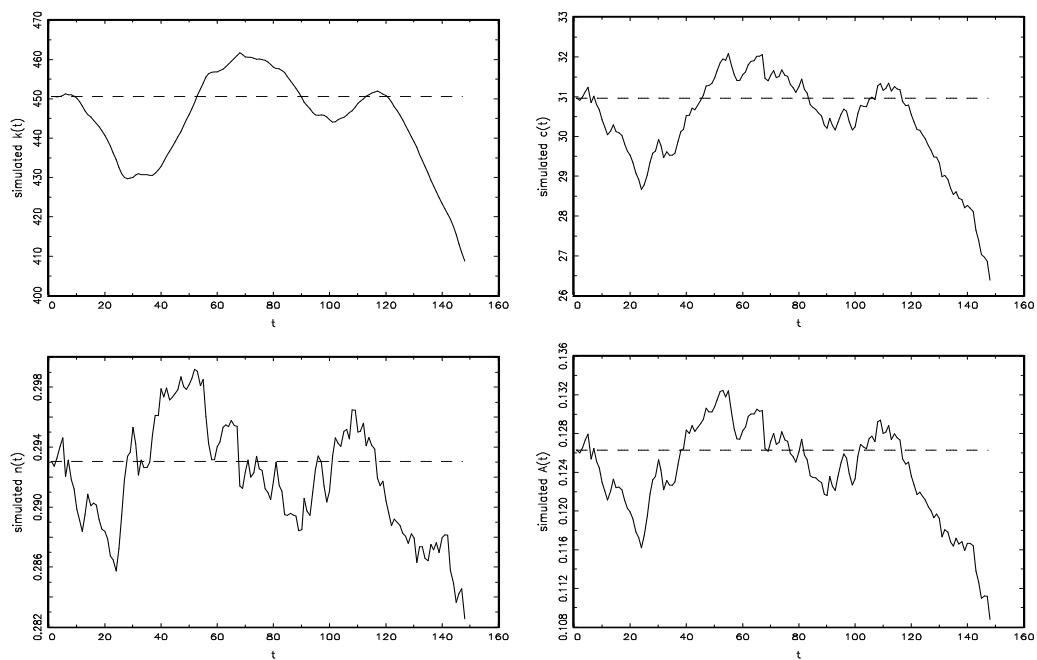


Figure 4.2: The Stochastic Solution to the Benchmark RBC Model for the Standard Parameters

Elsewhere (see Gong and Semmler 2001), we have compared these solu-

tion paths to those computed by Campbell (1994)'s log-linear approximate solution. We find that the two solutions are surprisingly close to the extent that one can hardly observe the differences.

4.5 The Generalized RBC Model

In recent work stochastic dynamic optimization models of general equilibrium type have been presented in the literature that go beyond the standard model as discussed in chapter 4.3. The recent models are more demanding in terms of solution and estimation methods. Although we will not attempt to estimate those more generalized versions it is worth presenting the main structure of the generalized models and to demonstrate how they can be solved by using dynamic programming as introduced in chapter 1.6. Since those generalized versions can easily give rise to multiple equilibria and history dependence, in chapter 7 we will return to these types of models.

4.5.1 The Model Structure

The generalization of the standard RBC model is usually undertaken either with respect to preferences or with respect to the technology. With respect to preferences utility functions such as⁴

$$U(C, N) = \frac{\left[C \exp\left(\frac{-N^{1+\chi}}{1+\chi}\right) \right]^{1-\sigma} - 1}{1-\sigma}. \quad (4.43)$$

are used. The utility function (4.43) with consumption, C and labor effort, N , as arguments is non-separable in consumption and leisure. We can obtain from (4.43) a separable utility function such as⁵

$$U(C, N) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\chi}}{1+\chi} \quad (4.44)$$

which is additively separable in consumption and leisure.

Moreover, by setting $\sigma = 1$ we obtain simplified preferences in log utility and leisure.

$$U(C, N) = \log C - \frac{N^{1+\chi}}{1+\chi} \quad (4.45)$$

⁴See Bennet and Farmer (2000) and Kim (2004).

⁵See Benhabib and Nishimura (1998) and Harrison(2001).

As concerning production technology and markets usually the following generalizations are introduced.⁶ First we can allow for increasing returns to scale. Although the individual-level private technology generates constant returns to scale with

$$Y_i = AK_i^a L_i^b, \quad a + b = 1$$

externalities of the form

$$A = (K^a N^b)^\xi, \quad \xi \geq 0$$

may allow for an aggregate production function with increasing returns to scale, with

$$Y = K^\alpha N^\beta \quad \alpha > 0, \beta > 0, \alpha + \beta \geq 1 \quad (4.46)$$

whereby $\alpha = (1 + \xi)a$, $\beta = (1 + \xi)b$ and Y, K, N represent total output, the aggregate stock of capital and labor hours respectively. The increasing returns to scale technology represented by equ. (4.46) can also be interpreted as a monopolistic competition economy where there are rents arising from inverse demand curves for monopolistic firms.⁷

Another generalization as concerning production technology can be undertaken by introducing adjustment cost of investment.⁸ We may write

$$\frac{\dot{K}}{K} = \varphi\left(\frac{I}{K}\right) \quad (4.47)$$

with the assumption of

$$\varphi(\delta) = 0, \quad \varphi'(\delta) = 1, \quad \varphi''(\delta) \leq 0 \quad (4.48)$$

Hereby δ is the depreciation rate of the capital stock. A functional form that satisfies the three conditions of equ. (4.47) is

$$\delta \left[\left(\frac{I}{\delta K} \right)^{1-\varphi} - 1 \right] / (1 - \varphi) \quad (4.49)$$

For $\varphi = 0$, one has the standard model without adjustment cost, namely

$$\dot{K} = I - \delta K.$$

The above describes a type of a generalized model that we want to solve.

⁶See Kim (2003a).

⁷See Farmer (1999, ch. 7.2.4) and Benhabib and Farmer (1994).

⁸For the following, see Lucas and Prescott (1971) and Kim (2003a) and Boldrin, Christiano and Fisher (2001).

4.5.2 Solving the Generalized RBC Model

We write the model in continuous time and in its deterministic form

$$C_t, N_t = \max \int_{t=0}^{\infty} e^{-\rho t} U(C_t, N_t) dt. \quad (4.50)$$

s.t.

$$\frac{\dot{K}}{K} = \varphi\left(\frac{I_t}{K_t}\right) \quad (4.51)$$

where preferences $U(C_t, N_t)$ are chosen such as represented by equ. (4.45) and the technology such as specified in eqs. (4.46) and (4.47)-(4.49). The latter are used in equ. (4.51).

For solving the model with the dynamic programming algorithm as presented in chapter 1.6 we use the following parameters.

Table 4.2: Parameterizing the General Model

a	b	χ	ρ	δ	ξ	φ
0.3	0.7	0.3	0.05	0.1	0	0.05

Note that in order to stay as close as possible to the standard RBC model we avoid externalities and therefore presume $\xi = 0$. Preferences and technology (by using adjustment cost of capital) take on, however, a more general form. Note that the dynamic decision problem (4.50)-(4.51) are written in continuous time. Its discretization for the use of dynamic programming is undertaken through the Euler procedure. Using then the deterministic variant of our dynamic programming algorithm of ch. 1.6 and a grid for the capital stock, K , in the interval $[0.1, 10]$, we obtain the following value function, representing the total utility along the optimal paths of C and N .

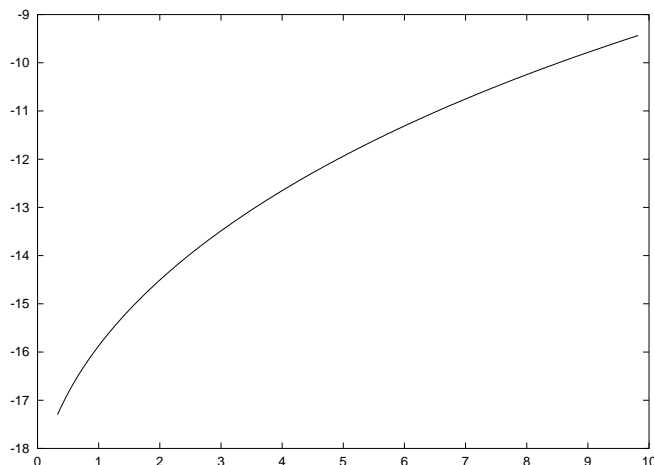


Figure 4.3: Value function for the general model

The out-of-steady state of the two choice variables namely consumption and labor effort, (depending in feedback form on the state variable capital stock, K) are shown in Figure 4.4.

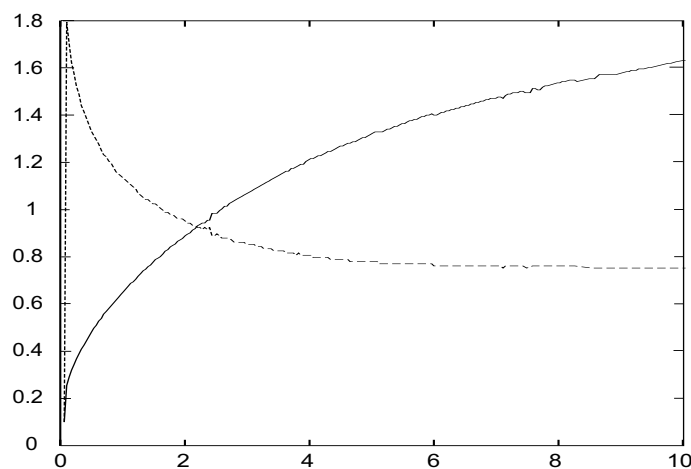


Figure 4.4: Paths of the Choice Variables C and N (depending on K)

As can be clearly observed the value function is concave, see Figure 4.3. Moreover, see Figure 4.4, consumption is low and labor effort high, when capital stock is low (so that capital stock can be built up) and consumption is high and labor effort low (so that capital stock will shrink). The dynamics generated by the response of consumption and labor effort to the state variable capital stock, K , will thus lead to a convergence toward an interior steady state of the capital stock, consumption and labor effort.

4.6 Conclusions

In this chapter we first have introduced the intertemporal general equilibrium model on which the standard RBC model is constructed. Our attempt was to reveal the basis of some intertemporal decision problems behind the RBC model. This will be important for the subsequent part of the book. We then introduce the empirically oriented standard RBC model and, based on our previous chapters, present the solution to the model. This provides us the groundwork for an empirical assessment of the RBC model, the task that will be addressed in the next chapter. We also have presented a generalized RBC model and solved it by using dynamic programming.

4.7 Appendix: The Proof of Proposition 4

Evaluating (4.35) - (4.38) along with (4.27) in their certainty equivalence form, and assuming all the variables to be at their steady states, we obtain

$$\frac{1}{\bar{c}} - \frac{\beta}{1+\gamma}\bar{\lambda} = 0 \quad (4.52)$$

$$\frac{-\theta}{1-\bar{n}} + \beta\bar{\lambda}\frac{\alpha\bar{y}}{(1+\gamma)\bar{n}} = 0 \quad (4.53)$$

$$\frac{\beta}{1+\gamma}\bar{\lambda}\left[(1-\delta) + \frac{(1-\alpha)\bar{y}}{\bar{k}}\right] = \bar{\lambda} \quad (4.54)$$

$$\bar{k} = \frac{1}{1+\gamma}[(1-\delta)\bar{k} + \bar{y} - \bar{c}] \quad (4.55)$$

where from (4.27) \bar{y} is given by

$$\bar{y} = \bar{A}\bar{k}^{1-\alpha}\left(\frac{\bar{n}\bar{N}}{0.3}\right)^\alpha \quad (4.56)$$

The derivation of the boundary steady state is trival. Replace \bar{c} , \bar{n} and $\bar{\lambda}$ with \bar{c}_b , \bar{n}_b and $\bar{\lambda}_b$. We find equation (4.52) - (4.54) are satisfied. Further, \bar{k}_b and \bar{y}_b can be derived from (4.55) and (4.56) given \bar{c}_b , \bar{n}_b and $\bar{\lambda}_b$.

Next we try to derive the interior steady state. For notational convenience, we ignore the subscript i , and therefore all the steady state values are understood as the interior steady states. Let $\frac{\bar{y}}{\bar{k}} = \phi$. By (4.54), we obtain

$\bar{y} = \phi \bar{k}$ where ϕ is defined by (4.40) in the proposition. According to (4.56),

$$\begin{aligned}
\frac{\bar{y}}{\bar{n}} &= \bar{A} \left(\frac{\bar{y}}{\phi} \right)^{1-\alpha} \left(\frac{\bar{n}\bar{N}}{0.3} \right)^\alpha \bar{n}^{-1} \\
&= \bar{A} \left(\frac{\bar{y}}{\bar{n}} \right)^{1-\alpha} \phi^{\alpha-1} \left(\frac{\bar{N}}{0.3} \right)^\alpha \\
&= \bar{A}^{\frac{1}{\alpha}} \phi^{1-\frac{1}{\alpha}} \left(\frac{\bar{N}}{0.3} \right)^\alpha
\end{aligned} \tag{4.57}$$

Therefore,

$$\begin{aligned}
\frac{\bar{k}}{\bar{n}} &= \frac{\bar{y}/\bar{n}}{\bar{y}/\bar{k}} \\
&= \bar{A}^{\frac{1}{\alpha}} \phi^{-\frac{1}{\alpha}} \left(\frac{\bar{N}}{0.3} \right)^\alpha
\end{aligned} \tag{4.58}$$

Expressing \bar{y} in terms of $\phi \bar{k}$ and then expressing \bar{k} in terms of (4.58), we thus obtain from (4.55)

$$\begin{aligned}
\bar{c} &= (\phi - \delta - \gamma) \bar{k} \\
&= (\phi - \delta - \gamma) \bar{A}^{\frac{1}{\alpha}} \phi^{-\frac{1}{\alpha}} \left(\frac{\bar{N}}{0.3} \right)^\alpha \bar{n}
\end{aligned} \tag{4.59}$$

Meanwhile, (4.52) and (4.53) imply that

$$\begin{aligned}
\bar{c} &= \left(\frac{\alpha}{\theta} \right) (1 - \bar{n}) \left(\frac{\bar{y}}{\bar{n}} \right) \\
&= \left(\frac{\alpha}{\theta} \right) (1 - \bar{n}) \bar{A}^{\frac{1}{\alpha}} \phi^{1-\frac{1}{\alpha}} \left(\frac{\bar{N}}{0.3} \right)^\alpha
\end{aligned} \tag{4.60}$$

Equating (4.59) and (4.60), we thus solve the steady state of the labor effort \bar{n} . Once \bar{n} is solved, \bar{k} can be obtained from (4.58), \bar{y} from (4.57) and \bar{c} either from (4.59) or from (4.60). Finally, $\bar{\lambda}$ can be derived from (4.52).

Chapter 5

The Empirics of the Standard Real Business Cycle Model

5.1 Introduction

Many real business cycle theorists believe that the RBC model is empirically powerful in explaining the stylized facts of business cycles. Moreover, some theorists suggest that even a simple RBC model, like the standard model we presented in the last chapter, despite its rather simple structure, can generate the time series to match the macroeconomic moments from empirically observed time series data. As Plosser (1989) pointed out, “the whole idea that such a simple model with no government, no market failures of any kind, rational expectations, no adjustment cost could replicate actual experience this well is very surprising.” (Plosser 1989:...) However, these early assessments have also become the subject to various criticisms. In this chapter, we shall provide a comprehensive empirical assessment of the standard RBC model. Our previous discussion, especially in the first three chapters, has provided a technical preparation for this assessment. We shall first estimate the standard RBC model and then evaluate the calibration results that have been stated by early real business cycle theorists. Yet before we commence with our formal study, we shall first demonstrate the efficiency of our estimation strategy as discussed in Chapter 3.

5.2 Estimation with Simulated Data

In this section, we shall first apply our estimation strategy using simulated data. The simulated data are shown in Figure 4.2 in the last chapter, which is generated from a stochastic simulation of our standard model for the given

standard parameters reported in Table 4.1. The purpose of this estimation is to test whether our suggested estimation strategy works well. If the strategy works well, we expect that the estimated parameters will be close to the standard parameters that we know in advance.

5.2.1 The Estimation Restriction

The standard model implies certain restrictions on the estimation. For the GMM estimation, the model implies the following moment restrictions:¹

$$E[(1 + \gamma)k_{t+1} - (1 - \delta)k_t - y_t + c_t] = 0; \quad (5.1)$$

$$E[c_t - G_{11}k_t - G_{12}A_t - g_{13}] = 0; \quad (5.2)$$

$$E[n_t - G_{21}k_t - G_{22}A_t - g_{23}] = 0; \quad (5.3)$$

$$E\left[y_t - A_t k_t^{1-\alpha} (n_t \bar{N} / 0.3)^\alpha\right] = 0. \quad (5.4)$$

Note that the moment restrictions could be nonlinear as in (5.4). Yet for the ML estimation, the restriction $Bz_t + \Gamma x_t = \varepsilon_t$ (see equation (3.6))² must be linear. Therefore, we shall first linearize (4.27) using Taylor approximation. This gives us

$$y_t = \frac{\bar{y}}{A} A_t + (1 - \alpha) \frac{\bar{y}}{k} k_t + \alpha \frac{\bar{y}}{n} n_t - \bar{y}$$

We thus obtain for the ML estimation:

$$z_t = \begin{bmatrix} k_t \\ c_t \\ n_t \\ y_t \end{bmatrix}, \quad B = \begin{bmatrix} 1 + \gamma & 0 & 0 & 0 \\ -G_{12} & 1 & 0 & 0 \\ -G_{22} & 0 & 1 & 0 \\ -(1 - \alpha) \frac{\bar{y}}{k} & 0 & -\alpha \frac{\bar{y}}{n} & 1 \end{bmatrix}$$

$$x_t = \begin{bmatrix} k_{t-1} \\ c_{t-1} \\ y_{t-1} \\ A_t \\ 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -(1 - \delta) & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -G_{11} & -g_{13} \\ 0 & 0 & 0 & -G_{21} & -g_{23} \\ 0 & 0 & 0 & -\frac{\bar{y}}{A} & \bar{y} \end{bmatrix}$$

¹Note that the parameters in the exogenous equation could be independently estimated since they have no feed back into the other equations. Therefore, there is no necessity to include the exogenous equation into the restriction.

²Note that here we use z_t rather than y_t in order to distinguish it from the output y_t in the model.

5.2.2 Estimation with Simulated Data

Although the model involve many parameters, we shall only estimate the parameters α, β, δ and θ . These are the parameters that are empirically unknown and thus need to be estimated when we later turn to the estimation using empirical data.³ Table 5.1 reports our estimation with the standard deviation included in parenthesis.

Table 5.1: GMM and ML Estimation Using Simulated Data

	α	β	δ	θ
True	0.58	0.9884	0.025	2
ML Estimation	0.5781 (2.4373E-006)	0.9946 (5.9290E-006)	0.0253 (3.4956E-007)	2.1826 (3.7174E-006)
1st Step GMM	0.5796 (6.5779E-005)	0.9821 (0.00112798)	0.02500 (9.6093E-006)	2.2919 (0.006181723)
2nd Step GMM	0.5800 (2.9958E-008)	0.9884 (3.4412E-006)	0.02505 (9.7217E-007)	2.000 (4.6369E-006)

One finds that the estimations from both methods are quite satisfying. All estimated parameters are close to their true parameters, the parameters that we have used to generate the data. This demonstrates the efficiency of our estimation strategy. Yet, the GMM estimation after the second step is more accurate than the ML estimation. This is probably because the GMM estimation does not need to linearize (5.4). However, we should also remark that the difference is minor whereas the time required by the ML estimation is much shorter. The latter holds not only because the GMM needs an additional step, but also each single step of the GMM estimation takes much more time for the algorithm to converge. Approximately 8 hours on a Pentium III computer is required for each step of the GMM estimation whereas approximately only 4 hours is needed for the ML estimation.

In Figure 5.1 and 5.2, we also illustrate the surface of the objective function for our ML estimation. It shows not only the existence of multiple optima, but also that the objective function is not smooth. This verifies the necessity of using simulated annealing in our estimation strategy.

³ \bar{N} can be regarded as the mean of per capital hour. γ does not appear in the model, but create for transforming the model into a stationary version. The parameters with regard to the AR(1) process of A_t have no feedback effect on our estimation restrictions.

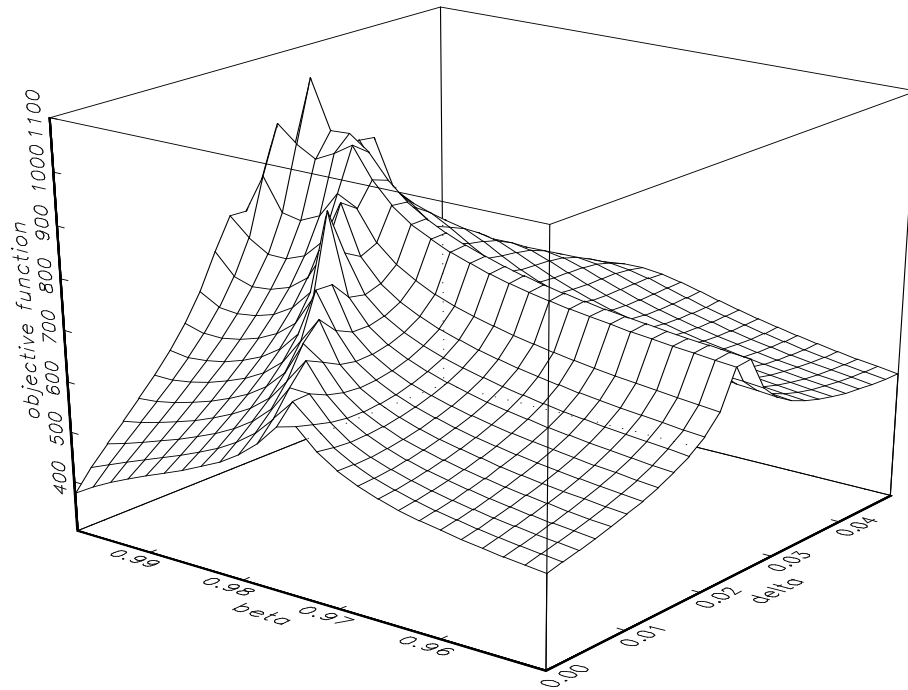


Figure 5.1: The $\beta - \delta$ Surface of the Objective Function for ML Estimation

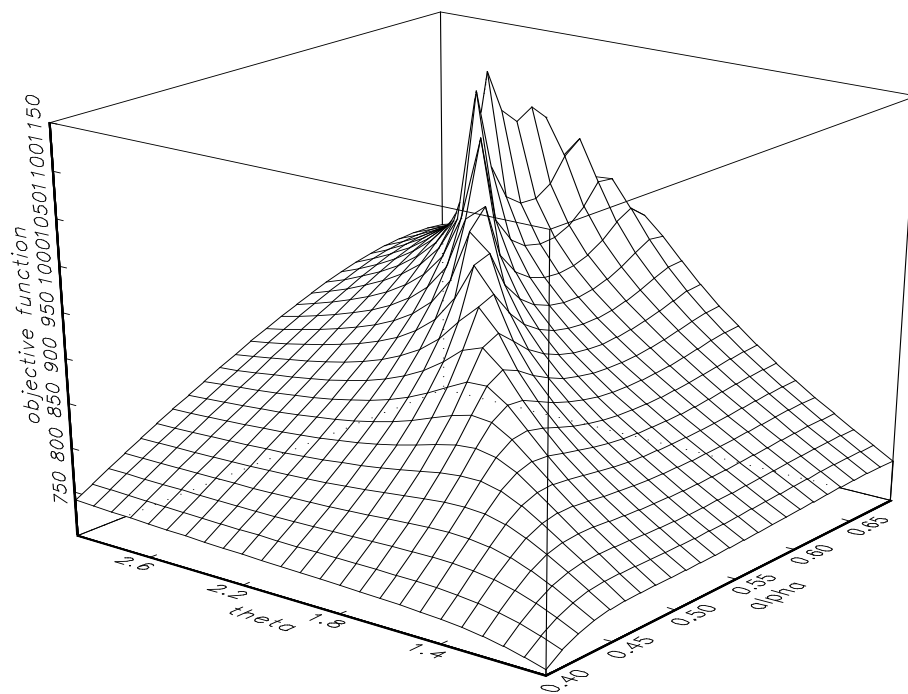


Figure 5.2: The $\theta - \alpha$ Surface of the Objective Function for ML Estimation

Next, we turn to estimating the standard RBC model with the data of U. S. time series data.

5.3 Estimation with Actual Data

5.3.1 The Data Construction

Before estimating the benchmark model with empirical U. S. time series, we shall first discuss the data that shall be used in our estimation. The empirical studies of RBC models often require a considerable re-construction of existing macroeconomic data. The time series employed for our estimation should include A_t , the temporary shock in technology, N_t the labor input (per capita hours), K_t the capital stock, C_t consumption and Y_t output. Here all the data can assume to be obtained from statistical sources except the temporary shock A_t . A common practice is to use the so-called Solow residual for the temporary shock. Assuming that the production function takes the form $Y_t = A_t K_t^{1-\alpha} (N_t X_t)^\alpha$, the Solow residual A_t is computed as follows:

$$A_t = \frac{Y_t}{K_t^{1-\alpha} (N_t X_t)^\alpha} \quad (5.5)$$

$$= \frac{y_t}{k_t^{1-\alpha} N_t^\alpha} \quad (5.6)$$

where X_t follows a constant growth rate:

$$X_t = (1 + \gamma)^t X_0 \quad (5.7)$$

Thus, the Solow residual A_t can be derived if the time series Y_t , K_t , N_t , and the parameter α and γ as well as the initial condition X_0 are given. It should be noted to derive the temporary shock as above deserves some criticism. Indeed, this approach uses macroeconomic data that posits a full-employment assumption, a key assumption in the model, which makes it different from other types of model, such as the business cycle model of Keynesian type. Yet, since this is a common practice we shall here also follow this procedure. Later in this chapter we will, however, deviate from this practice and construct the Solow residual in a different way. This will allow us to explore a puzzle, often called the technology puzzle in the RBC literature.

In addition to the construction of the temporary shock, the existing macroeconomic data (such as those from Citibase) also need to be adjusted to be accommodated to the definition of the variables as defined in the model.

⁴ The national income as defined in the model is simply the sum of consumption C_t and investment, the latter increases the capital stock. Therefore, to make the model's national income account consistent with the actual data, one should also include government consumption in C_t . Further, it is suggested that not only private investment but also government investment and durable consumption goods (as well as inventory and the value of land) should be included in the capital stock. Consequently, the service generated from durable consumption goods and government capital stock should also appear in the definition of Y_t . Since such data are not readily available, one has to compute them based on some assumptions.

Two Different Data Sets

To explore how this treatment of the data construction could affect the empirical assessment of dynamic optimization model, we shall employ two different data sets. The first data set, Data Set I, is constructed by Christiano (1987), which has been used in many empirical studies of RBC model such as Christiano (1988) and Christiano and Eichenbaum (1992). The sample period of this data set is from 1955.1 to 1984.4. All the data are quarterly.

The second data set, Data Set II, is obtained mainly from Citibase except the capital stock which is taken from the Current Survey of Business. This data set is taken without any modification. The sample period for this data set is from 1952.1 to 1988.4.

5.3.2 Estimation with the Christiano Data Set

As we have mentioned before, the shock sequence A_t is computed from the time series of Y_t , X_t , N_t and K_t given the pre-determined α , which we denote as α^* (see equation (5.5)). Here the time series X_t is computed according to equation (5.7) for the given parameter γ and the initial condition X_0 . In this estimation, we set X_0 to 1 and γ to 0.0045.⁵ Meanwhile we shall consider two α^* : one is the standard value 0.58 and the other is 0.66. We shall remark that 0.66 is the estimated α in Christiano and Eichenbaum (1992). Table 5.2 reports the estimations after the second step of GMM estimation.

⁴For a discussion on data definitions in RBC models, see Cooley and Prescott (1995).

⁵We choose 0.0045 for γ to make k_t , c_t and y_t to be stationary (note that we have defined $k_t \equiv \frac{K_t}{X_t}$, $c_t \equiv \frac{C_t}{X_t}$ and $y_t \equiv \frac{Y_t}{X_t}$). Also, γ is the standard parameter as chosen by King et al. (1988).

Table 5.2: Estimation with Christiano's Data Set

	α	β	δ	θ
$\alpha^* = 0.58$	0.5800 (2.2377E-008)	0.9892 (0.0002)	0.0209 (0.0002)	1.9552 (0.0078)
$\alpha^* = 0.66$	0.6600 (9.2393E-006)	0.9935 (0.0002)	0.0209 (0.0002)	2.1111 (0.0088)

As one can observe, all the estimations seem to be quite reasonable. For $\alpha^* = 0.58$, the estimated parameters, though somewhat deviating from the standard parameters used in King et al. (1988), are all within the economically feasible range. For $\alpha^* = 0.66$, the estimated parameters are very close to those in Christiano and Eichenbaum (1992). Even the parameter β estimated here is very close to the β chosen (rather than estimated) by them. In both cases, the estimated α is almost the same as the pre-determined α . This is not surprising due to the way of computing the temporary shocks from the Solow residual. Finally, we should also remark that the standard errors are unusually small.

5.3.3 Estimation with the NIPA Data Set

As in the case of the estimation with Christiano's data set, we again set X_0 to 1 and γ to 0.0045. For the two pre-determined α , we report the estimation results in Table 5.3.

Table 5.3: Estimation with the NIPA Data Set

	α	β	δ	θ
$\alpha^* = 0.58$	0.4656 (71431.409)	0.8553 (54457.907)	0.0716 (89684.204)	1.2963 (454278.06)
$\alpha^* = 0.66$	0.6663 (35789.023)	0.9286 (39958.272)	0.0714 (45174.828)	1.8610 (283689.56)

In contrast to the estimation using the Christiano data set, we find that the estimations here are much less satisfying. They deviate significantly from the standard parameters. Some of the parameters, especially β , are not within the economically feasible range. Furthermore, the estimates are all statistically insignificant due to the huge standard errors. Given such a sharp contrast for the results of the two different data sets, one is forced to think about the data issue involved in the current empirical studies of RBC model. Indeed, this issue is mostly suppressed in the current debate.

5.4 Calibration and Matching to U. S. Time-Series Data

Given the structural parameters, one can then assess the model to see how closely it matches with the empirical data. The current method for assessing a stochastic dynamic optimization model of RBC type is the calibration technique, which has already been introduced in Chapter 3. The basic idea of calibration is to compare the time series moments generated from the model's stochastic simulation to those from a sample economy.

The data generation process for this stochastic simulation is given by the following equations:

$$c_t = G_{11}A_t + G_{12}k_t + g_1 \quad (5.8)$$

$$n_t = G_{21}A_t + G_{22}k_t + g_2 \quad (5.9)$$

$$y_t = A_t k_t^{1-\alpha} (n_t \bar{N} / 0.3)^\alpha \quad (5.10)$$

$$A_{t+1} = a_0 + a_1 A_t + \epsilon_{t+1} \quad (5.11)$$

$$k_{t+1} = \frac{1}{1+\gamma} [(1-\delta)k_t + y_t - c_t] \quad (5.12)$$

where $\epsilon_{t+1} \sim N(0, \sigma_\epsilon^2)$; and G_{ij} and g_i ($i, j = 1, 2$) are all the complicated functions of the structural parameters and can be computed from our GAUSS procedure of solving dynamic optimization problem as presented in Appendix II of Chapter 1.

The structural parameters used for this stochastic simulation are defined as follows. First, we employ those in Table 5.2 at $\alpha^* = 0.58$ for the parameters α, β, δ and θ . The parameter γ is set 0.0045 as usual. The parameters a_0, a_1 and σ_ϵ in the stochastic equation (5.11) are estimated by the OLS method given the time series computed from Solow residue.⁶ The parameter \bar{N} is simply the sample mean of per capita hours N_t . For convenience, we report all these parameters in Table 5.4.

Table 5.4: Parameterizing the Standard RBC Model

α	γ	β	δ	θ	\bar{N}	a_0	a_1	σ_ϵ
0.5800	0.0045	0.9892	0.0209	1.9552	299.03	0.0333	0.9811	0.0189

⁶Note that these are the same as in Table 4.1.

Table 5.5 reports our calibration from 5000 stochastic simulations. In particular, the moment statistics of the sample economy is computed from Christiano's data set, while those for model economy are generated from our stochastic simulation using the data generation process (5.8) - (5.12). Here, the moment statistics include the standard deviations of some major macroeconomic variables and also their correlation coefficients. For the model economy, we can further obtain the distribution of these moment statistics, which can be reflected by their corresponding standard deviations (those in paratheses). Of course, the distributions are derived from our 5000 thousand stochastic simulations. All time series data are detrended by the HP-filter.

Table 5.5: Calibration of Real Business Cycle Model (numbers in parentheses are the corresponding standard deviations)

	Consumption	Capital	Employment	Output
Standard Deviations				
Sample Economy	0.0081	0.0035	0.0165	0.0156
Model Economy	0.0090 (0.0012)	0.0037 (0.0007)	0.0050 (0.0006)	0.0159 (0.0021)
Correlation Coefficients				
Sample Economy				
Consumption	1.0000			
Capital Stock	0.1741	1.0000		
Employment	0.4604	0.2861	1.0000	
Output	0.7550	0.0954	0.7263	1.0000
Model Economy				
Consumption	1.0000 (0.0000)			
Capital Stock	0.2013 (0.1089)	1.0000 (0.0000)		
Employment	0.9381 (0.0210)	-0.1431 (0.0906)	1.0000 (0.0000)	
Output	0.9796 (0.0031)	0.0575 (0.1032)	0.9432 (0.0083)	1.0000 (0.0000)

5.4.1 The Labor Market Puzzle

By observing Table 5.5, we find that among the four key variables the volatilities of consumption, capital stock and output could be regarded as being

somewhat matched. This is indeed one of the major early results of real business cycle theorists. However, the matching does not hold for employment. Indeed, the employment in the model economy is excessively smooth.

These results are further demonstrated by Figure 5.3 and Figure 5.4, where we compare the observed series from the sample economy to the simulated series with innovation given by the observed Solow residual. We shall remark that the excessive smoothness of employment is a typical problem of the standard model that has been addressed many times in literature.

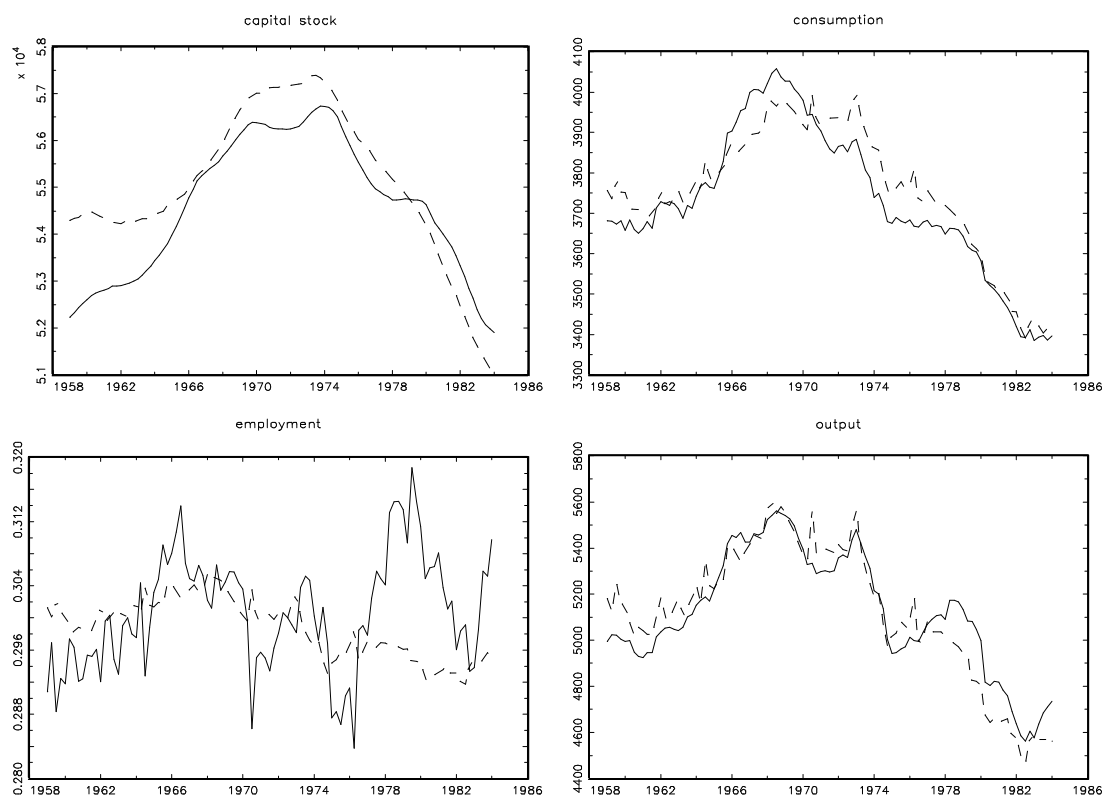


Figure 5.3: Simulated and Observed Series (non detrended): solid line observed and dashed line simulated

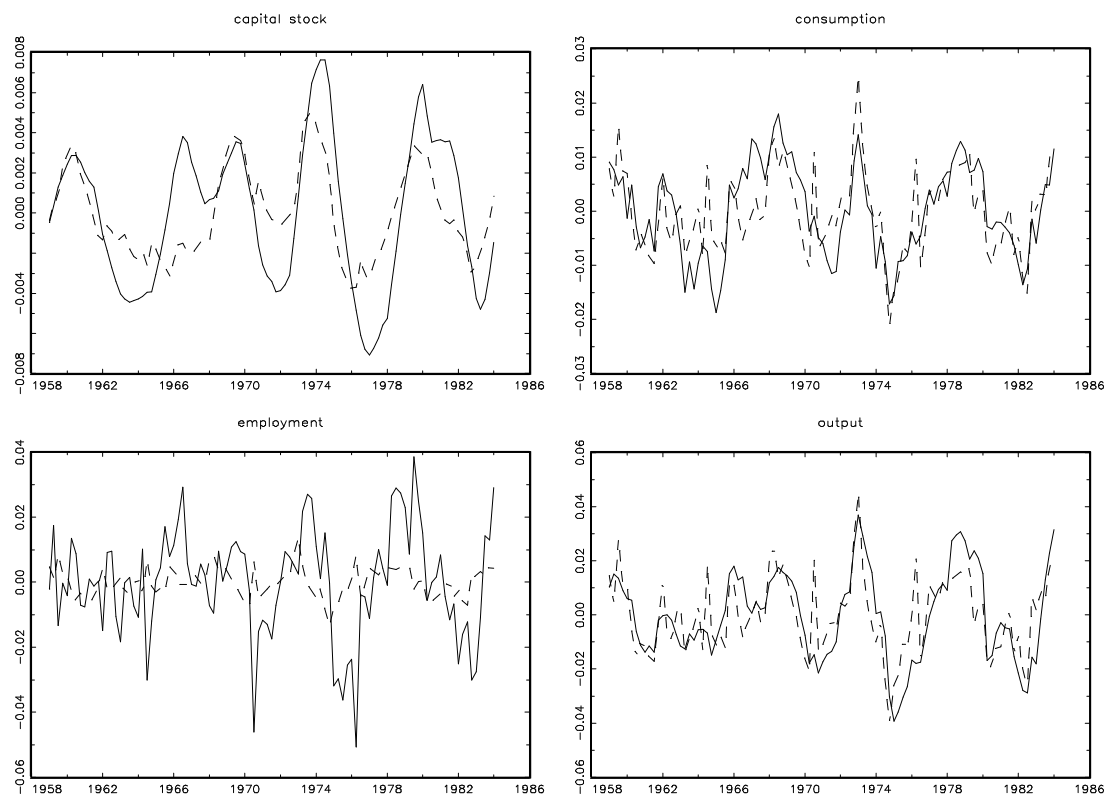


Figure 5.4: Simulated and Observed Series (detrended by HP filter): solid line observed and dashed line simulated

Now let us look at the correlations. In the sample economy, there are basically two significant correlations. One is between consumption and output, and the other is between employment and output. Both of these two correlations have also been found in our model economy. However, in addition to these two correlations, consumption and employment in the model economy are also significantly correlated. We remark that such an excessive correlation has, to our knowledge, not yet been discussed in the literature. The discussions have often been focused on the correlation with output. However, this excessive correlation should not be surprising given that in the RBC model the movements of employment and consumption reflect the movements of the same state variables: capital stock and temporary shock. They, therefore, should be somewhat correlated. The excessive smoothness of labor effort and the excessive correlation between labor and consumption will be taken up in Chapter 8.

5.5 The Issue of the Solow Residual

So far we may argue that one of the major achievements of the standard RBC model is that the model could explain the volatility of some key macroeconomic variables such as output, consumption and capital stock. Meanwhile, these results rely on the hypothesis that the driving force of the business cycles are technology shocks, which are assumed to be measured by the Solow residual. The measurement of technology can impact this result in two ways. One is that the parameters a_0 , a_1 and σ_ϵ in the stochastic equation (5.11) are estimated from the time series computed from Solow residual. These parameters will directly affect the results from our stochastic simulation. The second is that the Solow residual also serves as the sequence of observed innovations that generate the graphs in Figure 5.3 and Figure 5.4. Those innovations are often used in the RBC literature as an additional indicator to support the model and its matching of the empirical data.

Another major presumption of the RBC literature, not yet shown in Table 5.5, but to be shown below, is the technology-driven hypotheses, i.e., the technology is procyclical with output, consumption and employment. Of course, this celebrated result is obtained also from the empirical evidence, in which the technology is measured by the standard Solow residual.

There are several reasons to distrust the standard Solow residual as a measure of technology shock. First, Mankiw (1989) and Summers (1986) have argued that such a measure often leads to excessive volatility in productivity and even the possibility of technological regress, both of which seem to be empirically implausible. Second, It has been shown that the Solow residual can be expressed by some exogenous variables, for example demand shocks arising from military spending (Hall 1988) and changed monetary aggregates (Evan 1992), which are unlikely to be related to factor productivity. Third, the standard Solow residual is not a reliable measure of technology shocks if the cyclical variation in factor utilization are significant.

Considering that the Solow residual cannot be trusted as a measure of the technology shock, researchers have now developed different methods to measure technology correctly. All these methods are focused on the computation of factor utilization. There are basically three strategies. The first strategy is to use an observed indicator to proxy for unobserved utilization. A typical example is to employ electricity use as a proxy for capacity utilization (see Burnside, Eichenbaum and Rebelo 1996). Another strategy is to construct an economic model so that one could compute the factor utilization from the observed variables (see Basu and Kimball 1997 and Basu, Fernald and Kimball 1998). A third strategy identifies the technology shock through an VAR estimate, see Gali (1999) and Francis and Ramey (2001, 2003).

Recently, Gali (1999) and Francis and Ramey (2001) have found that if one uses the corrected Solow residual – if one identifies the technology shock correctly – the technology shock is negatively correlated with employment and therefore the celebrated discovery of the RBC literature must be rejected. Also, if the corrected Solow residual is significantly different from the standard Solow residual, one may find that the standard RBC model, using the Solow residual, can match well the variations in output, consumption and capital stock not because the model has been constructed correctly, but because it uses a problematic measure of technology.

All these are important problematic issues that are related to the Solow residual. Indeed, if they are confirmed, the real business cycles model, as driven by technology shocks, may not be a realistic paradigm for macroeconomic analysis any more. In this section, we will refer to all of this recent research employing our available data set. We will first follow Hall (1988) and Evan (1992) to test the exogeneity of Solow residual. Yet in this test, we simply use government spending, which is available in our Christiano's data set. We will construct a measurement of the technology shock that represents a corrected Solow residual. This construction needs data on factor utilization. Unlike other current research, we use empirically observed data series, the capacity utilization of manufacturing, IPXMCAQ, obtained from Citibase. Given our new measurement, we then explore whether the RBC model is still able to explain the business cycles, in particular the variation in consumption, output and capital stock. We shall also look at whether the technology still moves procyclically with output, consumption and employment.

5.5.1 Testing the Exogeneity of the Solow Residual

Apparently, a critical assumption of the Solow residual to be a correct measurement of the technology shock is that A_t should be purely exogenous. In other words, the distribution of A_t cannot be altered by the change in other exogenous variables such as the variables of monetary and fiscal policy. Therefore, testing the exogeneity of the Solow residual becomes our first investigation to explore whether the Solow residual is a correct measure of the technology shock.

One possible way to test the exogeneity is to employ the Granger causality test. This is also the approach taken by Evan (1992). For this purpose, we shall investigate the following specification:

$$A_t = c + \alpha_1 A_{t-1} + \cdots + \alpha_p A_{t-p} + \beta_1 g_{t-1} + \cdots + \beta_p g_{t-p} + \varepsilon_t \quad (5.13)$$

where g_t in this test is government spending, as an aggregate demand variable, over X_t . If the Solow residual is exogenous, g_t should not have any

explanatory power for A_t . Therefore our null hypothesis is

$$H_0 : \beta_1 = \dots = \beta_p = 0 \quad (5.14)$$

The rejection of the null hypothesis is sufficient for us to refute the assumption that A_t is strictly exogenous. It is well known that the result of any empirical test for Granger causality can be surprisingly sensitive to the choice of lag length p . The test therefore will be conducted for different lag lengths p 's. Table 5.6 provides the corresponding F -statistics computed for the different p 's.

Table 5.6: F -Statistics for Testing Exogeneity of Solow Residual

	F - statistics	degrees of freedom
$p = 1$	9.5769969	(1, 92)
$p = 2$	4.3041035	(2, 90)
$p = 3$	3.2775435	(4, 86)
$p = 4$	2.3825632	(6, 82)

From Table 5.6, one finds that at 5% significance level we can reject the null hypothesis for all the lag lengths p 's.⁷

5.5.2 Corrected Technology Shocks

The analysis in our previous section indicates that the hypothesis can be rejected that the standard Solow residual be strictly exogenous. On the other hand, those policy variables, such as government spending, which certainly represents a demand shock, may have explanatory power for the variation of the Solow residual. This finding is consistent with the results in Hall (1988) and Evan (1992). Therefore we may have sufficient reason to distrust the Solow residual to be a good measure of the technology shock.

Next, we present a simple way of how to extract a technology shock from macroeconomic data. If we look at the computation of the Solow residual, e.g., equation (5.5), we find two strong assumptions inherent in the formulation of the Solow residual. First, it is assumed that the capital stock is fully utilized. Second, it is further assumed that the population follows a constant growth rate which is a part of γ . In other words, there is no variation in population growth. Next we shall consider the derivation of the corrected Solow residual by relaxing those strong assumptions.

⁷Although we are not able to obtain the same at 1% significance level.

Let u_t denote the utilization of capital stock, which can be measured by IPXMQ from Citibase. The observed output is thus produced by the utilized capital and labor service (expressed in terms of total observed working hours) via the production function:

$$Y_t = \tilde{A}_t (u_t K_t)^{1-\alpha} (Z_t E_t H_t)^\alpha \quad (5.15)$$

Above, \tilde{A}_t is the corrected Solow residual (which is our new measure of temporary shock in technology); E_t is the number of workers employed; H_t denotes the hours per employed worker,⁸ and Z_t is the permanent shock in technology. Note that in this formulation, we interpret the utilization of labor service only in terms of their working hours and therefore ignore their actual effort, which is more difficult to be observed.

Let \bar{L}_t denote the permanent shock to population so that $X_t = Z_t \bar{L}_t$ while L_t denotes the observed population so that $\frac{E_t H_t}{L_t} = N_t$. Dividing both sides of (5.15) by X_t , we then obtain

$$y_t = \tilde{A}_t (u_t k_t)^{1-\alpha} (l_t N_t)^\alpha \quad (5.16)$$

Above $l_t \equiv \frac{L_t}{\bar{L}_t}$. Given equation (5.16), the corrected Solow residual \tilde{A}_t can be computed as

$$\tilde{A}_t = \frac{y_t}{(u_t k_t)^{1-\alpha} (l_t N_t)^\alpha} \quad (5.17)$$

Comparing this with equation (5.5), one finds that our corrected Solow residual \tilde{A}_t will match the standard Solow residual A_t if and only if both u_t and l_t equal 1. Figure 5.5 compares these two time series: one for non-detrended and the for detrended series.

⁸Note that this is different from our notation N_t before, which is the hours per capita.

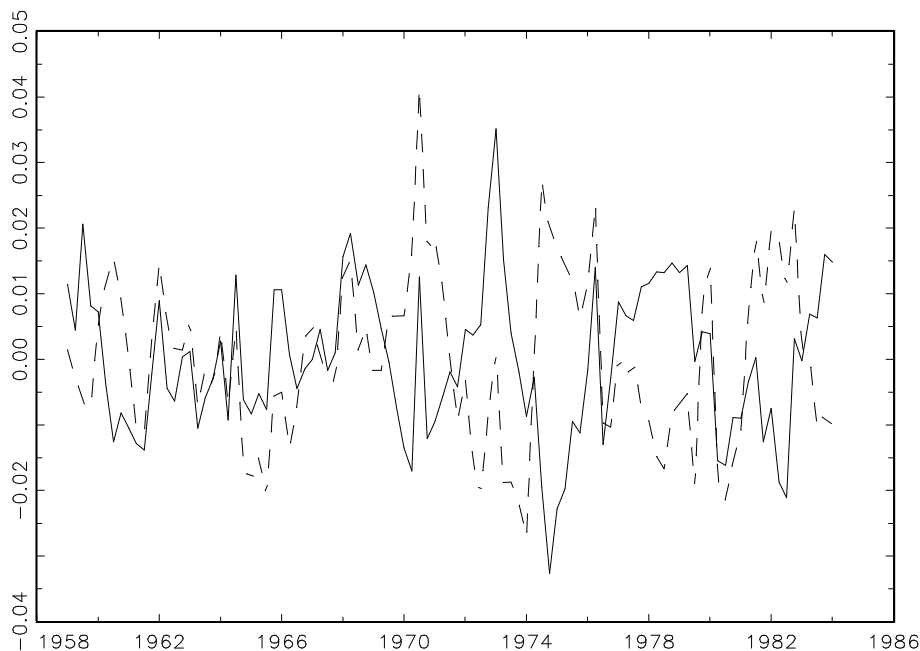


Figure 5.5: The Solow Residual: standard (solid curve) and corrected (dashed curve)

As one can observe in the figure 5.5, the two series follow basically the same trend while their volatilities are almost the same.⁹ However, in the short run, they rather move in different directions if we compare the detrended series.

5.5.3 Business Cycles with Corrected Solow Residual

Next we shall use the corrected Solow residual to test the technology-driven hypothesis. In Table 5.7, we report the cross-correlations of the technology shock to our four key economic variables: output, consumption, employment and capital stock. These correlations are compared for three economies: the RBC economy (whose statistics is computed from 5000 simulations), the Sample Economy I (in which the technology shock is represented by the standard Solow residual) and the Sample Economy II (to be represented by the corrected Solow residual). The data series are again detrended by the HP-filter.

⁹A similar volatility is also found in Burnside et al. (1996).

Table 5.7: The Cross-Correlation of Technology

	output	consumption	employment	capital stock
RBC Economy	0.9903 (0.0031)	0.9722 (0.0084)	0.9966 (0.0013)	-0.0255 (0.1077)
Sample Economy I	0.7844	0.7008	0.1736	-0.2142
Sample Economy II	-0.3422	-0.1108	-0.5854	0.0762

If we look at the Sample Economy I, where the standard Solow residual is employed, we find that the technology shock is procyclical to output, consumption and employment. This result is exactly predicted by the RBC Economy and represents what has been called the technology-driven hypothesis. However, if we use the corrected Solow residual, as in Sample Economy II, we find a somewhat opposite result, especially for employment. We, therefore, can confirm the findings of the recent research by Basu, et al. (1998), Gali (1999) and Francis and Ramey (2001, 2003).

To test whether the model can still match the observed business cycles, we provide in Figure 5.6 a one time simulation with the observed innovation given by the corrected Solow residual.¹⁰ Comparing Figure 5.6 to Figure 5.4, we find that the results are in sharp contrast to the prediction as referred in the standard RBC model.

¹⁰Here the structural parameters are still the standard ones as given in Table 5.4.

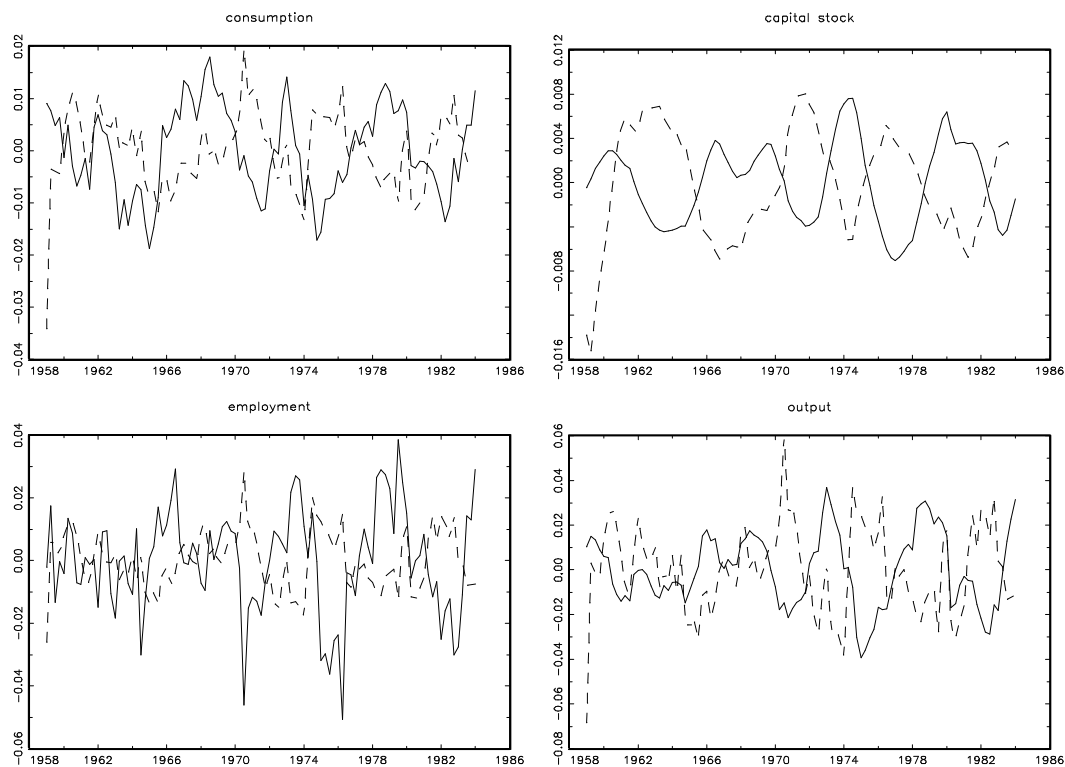


Figure 5.6: Sample and Predicted Moments with Innovation Given by Corrected Solow Residual

5.6 Conclusions

The standard RBC model has been regarded as a model that replicates the basic moment properties of U.S. macroeconomic time series data despite its rather simple structure. Prescott (1986) summarizes the moment implications as indicating “the match between theory and observation is excellent, but far from perfect”. Indeed, many have felt that the RBC research has at least passed the first test. Yet this early assessment should be subject to certain qualification.

In the first place, this early assessment builds on the reconstruction of U.S. macroeconomic data. Through its necessity to accommodate the data to the model’s implication, such data reconstruction seems to force the first moments of certain macroeconomic variables of the U.S. economy to be matched by the model’s steady state at the given economically feasible standard pa-

rameters. The unusual small standard errors of the estimates seem to confirm this suspicion.

Second, although one may celebrate the fit of the variation of consumption, output and capital stock when the reconstructed data series are employed, we still cannot ignore the problems of excessive smoothness of labor effort and excessive correlation between labor and consumption. Both of these two problems are related to the labor market specification of the RBC model. For the model to be able to replicate employment variation, it seems necessary to make improvement upon the labor market specification. One possible approach for such improvement is to allow for wage stickiness and nonclearing of the labor market, a task that we will turn to in Chapter 8.

Third, the celebrated fit of the variation in consumption, output and capital stock may rely on the uncorrected measure of technology. As we have shown in Figure 5.6, the match does not exist any more when we use the corrected Solow residual as the observed innovations. This incorrect measure of technology takes us to the technology puzzle: the procyclical technology, driving the business cycle, may not be a very plausible hypothesis. As King et. al (1999) pointed out, “it is the final criticism that the Solow residual is a problematic measure of technology shock that has remained the Achilles heel of the RBC literature.” In Chapter 9, we shall address the technology puzzle again by introducing monopolistic competition into an stochastic dynamic macro model.

Chapter 6

Asset Market Implications of Real Business Cycles

6.1 Introduction

In this chapter, we shall study asset price implications of the standard RBC model. The idea of employing a basic stochastic growth model to study asset prices goes back to Brock and Mirman (1972) and Brock (1978, 1982). Asset prices contain valuable information about intertemporal decision making and dynamic models explaining asset pricing are of great importance in current research. We here want to study a production economy with asset market and spell out its implications for asset prices and returns. In particular we will explore to what extent it can replicate the empirically found risk-free interest rate, equity premium and Sharpe-ratio.

Modelling asset price and risk premia in models with production is much more challenging than in exchange economies. Most of the asset pricing literature has followed Lucas (1978) and Mehra and Prescott (1985) in computing asset prices from the consumption based asset pricing models with an exogenous dividend streams. Production economies offer a much richer, and realistic environment. First, in economies with an exogenous dividend stream and no savings consumers are forced to consume their endowment. In economies with production where asset returns and consumption are endogenous consumers can save and hence transfer consumption between periods. Second, in economies with an exogenous dividend stream the aggregate consumption is usually used as a proxy for equity dividends. Empirically, this is not a very sensible modelling choice. Since there is a capital stock in production economies, there is a more realistic modelling of equity dividends is possible.

Although recently further extension of the baseline stochastic growth model of RBC type were developed to match better actual asset market characteristics ¹ we will in the current paper by and large restrict ourselves to the baseline model. The theoretical framework in this chapter is taken from Lettau (1999), Lettau and Uhlig (1999) and Lettau, Gong and Semmler (2001) where the closed-form solutions for risk premia of equity, long-term real bonds, the Sharpe-ratio and the risk-free interest rates are presented in a log-linearized RBC model as developed by Campbell (1994). Those equations can be used as additional moment restrictions in the estimation process. We introduce the asset pricing restrictions step-by-step to clearly demonstrate the effect of each new restriction.

First, we estimate the model using only the restrictions of real variables as in Chapter 5. The data employed for this estimation are taken again from Christiano (1987).² We then add our first asset pricing restriction, the risk-free interest rate. We use the observed 30-day T-bill rate to match the one-period risk-free interest rate implied by the model.³ The second asset pricing restriction concerns the risk-return trade-off as measured by the Sharpe-ratio, or the price of risk. This variable determines how much expected return agents require per unit of financial risk. Hansen and Jagannathan (1991) and Lettau and Uhlig (1999) show how important the Sharpe-ratio⁴ is in evaluating asset prices generated by different models. Introducing the Sharpe-ratio as moment restriction in the estimation procedure requires an iterative procedure to estimate the risk aversion parameter. We find that the Sharpe-ratio restriction affects the estimation of the model drastically. For each estimation, we compute the implied premia of equity and long-term real bond. Those values are then compared to the stylized facts of asset markets.

The estimation technique in this chapter follows the Maximum Likelihood (ML) method as discussed in Chapter 4. All the estimations are again conducted through the numerical algorithm, the simulated annealing. In addition, we introduce a diagnostic procedure developed by Watson (1993) and Diebold, Ohanian and Berkowitz (1995) to test whether the moments predicted by the model, for the estimated parameters, can match the moments of the actual macroeconomic time series. In particular, we use the variance-covariance matrix of the estimated parameters to infer the intervals of the

¹See, for example Jerman (1998), Boldrin, Christiano and Fisher (2001) and Grüne and Semmler (2004b).

²Using Christiano's data set, we implicitly assume that the standard model can, to some extent, replicate the moments of the real variables. Of course, as the previous chapter has shown the standard model fails also along some real dimensions.

³Using 30-day rate allows us to keep inflation uncertainty at a minimum.

⁴See also Sharpe (1964)

moment statistics and to study whether the actual moments derived from the sample data fall within this interval.

The rest of the chapter is organized as follows. In section 2, we use the standard RBC model and log-linearization as proposed by Campbell (1994) and derive the closed-form solutions for the financial variables. Section 3 presents the estimation for the model specified by different moments restrictions. In Section 4, we interpret our results and contrast the asset market implications of our estimates to the stylized facts of the asset market. Section 5 compares the second moments of the time series generated from the model to the moments of actual time series data. Section 6 concludes.

6.2 The Standard Model and Its Asset Pricing Implications

6.2.1 The Standard Model

We follow Campbell (1994) and use the notation Y_t for output, K_t for capital stock, A_t for technology, N_t for normalized labor input and C_t for consumption. The maximization problem of a representative agent is assumed to take the form⁵

$$\text{Max} \quad E_t \sum_{i=0}^{\infty} \beta^i \left[\frac{C_{t+i}^{1-\gamma}}{1-\gamma} + \theta \log(1 - N_{t+i}) \right]$$

subject to

$$K_{t+1} = (1 - \delta)K_t + Y_t - C_t$$

with Y_t given by $(A_t N_t)^\alpha K_t^{1-\alpha}$. The first order condition is given by the following Euler equation:

$$C_t^{-\gamma} = \beta E_t \{ C_{t+1}^{-\gamma} R_{t+1} \} \quad (6.1)$$

$$\frac{1}{\theta(1 - N_t)} = \alpha \frac{A_t^\alpha}{C_t} \left(\frac{K_t}{N_t} \right)^{(1-\alpha)} \quad (6.2)$$

⁵Note that, as in our previous modelling, we apply here the power utility as describing the preferences of the representative household. For the model of asset market implications other preferences, for example, habit formation are often employed, see Jerman (1998), Boldri, Christiano and Fisher (2001) and Cochrane (2001, ch. 21) and Grüne and Semmler (2004b).

where R_{t+1} is the gross rate of return on investment in capital, which is equal to the marginal product of capital in production plus undepreciated capital:

$$R_{t+1} \equiv (1 - \alpha) \left(\frac{A_{t+1} N_{t+1}}{K_{t+1}} \right)^\alpha + 1 - \delta.$$

We allow firms to issue bonds as well as equity. Since markets are competitive, real allocations will not be affected by this choice, i.e. the Modigliani-Miller theorem is presumed to hold. We denote the leverage factor (the ratio of bonds outstanding and total firm value) as ζ .

At the steady state, the technology, consumption, output and capital stock all grow at a common rate $G = A_{t+1}/A_t$. Hence, (6.1) becomes

$$G = \beta R$$

where R is the steady state of R_{t+1} . Taking log for both sides, we can further write the above equation as

$$\gamma g = \log(\beta) + r. \quad (6.3)$$

where $g \equiv \log G$ and $r \equiv \log R$. This defines the relation among g, r, β and γ . In the rest of the chapter, we use g, r , and γ as parameters to be determined, the implied value for the discount factor β can then be deduced from (6.3).

6.2.2 The Log-linear Approximate Solution

Outside the steady state, the model characterizes a system of nonlinear equations in the logs of technology a_t , consumption c_t , labor n_t and capital stock k_t . Note that here we use the lower case letter as the corresponding log variables of the capital letter. In the case of incomplete capital depreciation $\delta < 1$, the exact analytical solution to the model is not feasible. We therefore seek instead an approximate analytical solution. Assume that the technology shock follows an AR(1) process:

$$a_t = \phi a_{t-1} + \varepsilon_t \quad (6.4)$$

with ε_t to be the *i.i.d.* innovation: $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. Campbell (1994) shows that the solution, using the log-linear approximation method, can be written as

$$c_t = \eta_{ck} k_t + \eta_{ca} a_t \quad (6.5)$$

$$n_t = \eta_{nk} k_t + \eta_{na} a_t \quad (6.6)$$

and the law of motion of capital is

$$k_t = \eta_{kk}k_{t-1} + \eta_{ka}a_{t-1} \quad (6.7)$$

where $\eta_{ck}, \eta_{ca}, \eta_{nk}, \eta_{na}, \eta_{kk}$, and η_{ka} are all the complicated functions of the parameters $\alpha, \delta, r, g, \gamma, \phi$ and \bar{N} (the steady state value of N_t).

6.2.3 The Asset Price Implications

The standard RBC model as presented above has strong implications for asset pricing. First, the Euler equation (6.1) implies the following expression regarding the risk-free rate R_t^f :⁶

$$R_t^f = \{ \beta E_t [(C_{t+1}/C_t)^{-\gamma}] \}^{-1}.$$

Writing the equation in the log form, we obtain the risk-free rate in logs as⁷

$$r_t^f = \gamma E_t \Delta c_{t+1} - \frac{1}{2} \gamma^2 \text{Var} \Delta c_{t+1} - \log \beta. \quad (6.8)$$

Using the process of consumption, capital stock and technology as expressed in (6.5), (6.7) and (6.4) while ignoring the constant term involving the discount factor and the variance of consumption growth, we derive from (6.8) (see Lettau et al. (2001) for the details.):

$$r_t^f = \gamma \frac{\eta_{ck} \eta_{ka}}{1 - \eta_{kk} L} \varepsilon_{t-1}, \quad (6.9)$$

where L is the lag operator. Matching this process implied by the model to the data will give us the first asset market restriction. The second asset market restriction will be the Sharpe-ratio which summarizes the risk-return trade-off:

$$SR_t = \max_{\text{all assets}} \frac{E_t [R_{t+1} - R_{t+1}^f]}{\sigma_t [R_{t+1}]}. \quad (6.10)$$

Since the model is log-linear and has normal shocks, the Sharpe-ratio can be computed in closed form as:⁸

$$SR = \gamma \eta_{ca} \sigma_\varepsilon. \quad (6.11)$$

⁶For further details, see Cochrane (2001, chs. 1.2 and 2.1). We also want to note that in RBC models the risk-free rate is generally too high, and its standard deviation is much too low compared to the data, see Hornstein and Uhlig (2001).

⁷Note that here we use the formula $E e^x = e^{E x + \sigma_x^2 / 2}$.

⁸See Lettau and Uhlig (1999) for the details.

Lastly, we consider the risk premia of equity (EP) and long-term real bonds (LTBP). These can be computed on the basis of the log-linear solutions (6.5)-(6.7) as:

$$LTBP = -\gamma^2 \beta \frac{\eta_{ck} \eta_{ka}}{1 - \beta \eta_{kk}} \eta_{ca}^2 \sigma_\varepsilon^2 \quad (6.12)$$

$$EP = \left(\frac{\eta_{dk} \eta_{nk} - \eta_{da} \eta_{kk}}{1 - \beta \eta_{kk}} - \gamma \beta \frac{\eta_{ck} \eta_{kk}}{1 - \beta \eta_{kk}} \right) \gamma \eta_{ca}^2 \sigma_\varepsilon^2. \quad (6.13)$$

Again we refer to Lettau (1999) and Lettau, Gong and Semmler (2001) for details of those computations.

6.2.4 Some Stylized Facts

Table 6.1 summarizes some key facts on asset markets and real economic activity for the US economy. A successful model should be consistent with these basic moments of real and financial variables. In addition to the well-known stylized facts on macroeconomic variables, we will consider the performance of the model concerning the following facts of asset markets.

Table 6.1: Asset Market Facts and Real Variables

	Standard Deviation	Mean
GDP	1.72	
Consumption	1.27	
Investment	8.24	
Labor Input	1.59	
T-Bill	0.86	0.19
SP 500	7.53	2.17
Equity Premium	7.42	1.99
Long Bond Premium	0.21	4.80
Sharpe Ratio		0.27

Note: Standard Deviations for the real variables are taken from Cooley and Prescott (1995). The series are H-P filtered. Asset market data are from Lettau (1999). All data are from the U.S. economy at quarterly frequency. Units are per cent per quarters. The Sharpe-ratio is the mean of equity premium divided by its standard deviation.

The table shows that the equity premium is roughly 2% per quarter. The Sharpe-ratio, which measures the risk-return trade-off, equals 0.27 in

post-war data of the U.S.. The standard deviation of the real variables reveal the usual hierarchy in volatility with investment being most volatile and consumption the smoothest variable. Among the financial variables the equity price and equity premium exhibit the highest volatility, roughly six times higher than consumption.

6.3 The Estimation

6.3.1 The Structural Parameters to be Estimated

The RBC model presented in section 2 contains seven parameters, α , δ , r , g , γ , ϕ , and \bar{N} . Recall that the discount factor is determined in (6.3) for given values of g , r and γ . The parameter θ is simply dropped due to our log-linear approximation. Of course, we would like to estimate as many parameters as possible. However, some of the parameters have to be pre-specified. The computation of technology shocks requires the values for α and g . In this paper we use the standard values of $\alpha = 0.667$ and $g = 0.005$. \bar{N} is specified as 0.3. The parameter ϕ is estimated independently from (6.4) by OLS regression. This leaves the risk aversion parameter γ , the average interest rate r and the depreciation rate δ to be estimated. The estimation strategy is similar to Christiano and Eichenbaum (1992). However, they fix the discount factor and the risk aversion parameter without estimating them. In contrast, the estimation of these parameters is central to our strategy, as we will see shortly.

6.3.2 The Data

For the real variables of the economy, we use the data set as constructed by Christiano (1987). The data set covers the period from the third quarter of 1955 through the fourth quarter of 1983 (1955.1-1983.4). As we have demonstrated in the last chapter, the Christiano data set can match the real side of the economy better than the commonly used NIPA data set. For the time series of the risk-free interest rate, we use the 30-day T-bill rate to minimize unmodeled inflation risk.

To make the data suitable for estimation, we are required to detrend the data into their log-deviation form. For a data observation X_t , the detrended value x_t is assumed to take the form $\log(X_t/\bar{X}_t)$, where \bar{X}_t is the value of X_t on its steady state path, i.e., $\bar{X}_t = (1 + g)^{t-1}\bar{X}_1$. Therefore, for the given g , which could be calculated from the sample, the computation of x_t depends on \bar{X}_1 , the initial \bar{X}_t . We compute this initial condition based on

the consideration that the mean of x_t is equal to 0. In other words,

$$\begin{aligned} \frac{1}{T} \sum_{i=1}^T \log(X_t/\bar{X}_t) &= \frac{1}{T} \sum_{i=1}^T \log(X_t) - \frac{1}{T} \sum_{i=1}^T \log(\bar{X}_t) \\ &= \frac{1}{T} \sum_{i=1}^T \log(X_t) - \frac{1}{T} \sum_{i=1}^T \log(\bar{X}_1) - \frac{1}{T} \sum_{i=1}^T \log[(1+g)^{t-1}] \\ &= 0 \end{aligned}$$

Solving the above equation for \bar{X}_1 , we obtain

$$\bar{X}_1 = \frac{1}{T} \exp \left\{ \sum_{i=1}^T \log(X_t) - \sum_{i=1}^T \log[(1+g)^{t-1}] \right\}.$$

6.3.3 The Moment Restrictions of Estimation

For the estimation in this chapter, we use the maximum likelihood (ML) method as discussed in Chapter 3. In order to analyze the role of each restriction, we introduce the restrictions step-by-step. First, we constrain the risk aversion parameter r to unity and use only moment restrictions of the real variables, i.e. (6.5) - (6.7) so we can compare our results to those in Christiano and Eichenbaum (1992). The remaining parameters thus to be estimated are δ and r . We call this Model 1 (M1). The matrices for the ML estimation are given by

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -\eta_{ck} & 1 & 0 \\ -\eta_{ck} & 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -\eta_{kk} & -\eta_{ka} & 0 \\ 0 & 0 & -\eta_{ca} \\ 0 & 0 & -\eta_{ma} \end{bmatrix},$$

$$y_t = \begin{bmatrix} k_t \\ c_t \\ n_t \end{bmatrix}, \quad x_t = \begin{bmatrix} k_{t-1} \\ a_{t-1} \\ a_t \end{bmatrix}.$$

After considering the estimation with the moment restrictions only for real variables, we add restrictions from asset markets one by one. We start by including the following moment restriction of the risk-free interest rate in estimation while still keeping risk aversion fixed at unity:

$$E \left[b_t - r_t^f \right] = 0$$

where b_t denotes the return on the 30-day T-bill and the risk-free rate r_t^f is computed as in (6.9). We refer to this version as Model 2 (M2). In this case the matrices B and Γ and the vectors x_t and y_t can be written as

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\eta_{nk} & 1 & 0 & 0 \\ -\eta_{nk} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -\eta_{kk} & -\eta_{ka} & 0 & 0 \\ 0 & 0 & -\eta_{ca} & 0 \\ 0 & 0 & -\eta_{na} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$y_t = \begin{bmatrix} k_t \\ c_t \\ n_t \\ b_t \end{bmatrix}, \quad x_t = \begin{bmatrix} k_{t-1} \\ a_{t-1} \\ a_t \\ r_t^f \end{bmatrix}.$$

Model 3 (M3) uses the same moment restrictions as Model 2 but leaves the risk aversion parameter r to be estimated rather than fixed to unity.

Finally, we impose that the dynamic model should generate a Sharpe-ratio of 0.27 as measured in the data (see Table 1). We take this restriction into account in two different ways. First, as a shortcut, we fix the risk aversion at 50, a value suggested in Lettau and Uhlig (1999) for generating a Sharpe-ratio of 0.27 using actual consumption data. Given this value, we estimate the remaining parameter δ and r . This will be called Model 4 (M4). In the next version, Model 5 (M5), we are simultaneously estimating γ while imposing a Sharpe-ratio restriction of 0.27. Recall from (6.11) that the Sharpe-ratio is a function of risk aversion, the standard deviation of the technology shock and the elasticity of consumption with respect to the shock η_{ca} . Of course, η_{ca} is itself a complicated function of γ . Hence, the Sharpe-ratio restriction becomes

$$\gamma = \frac{0.27}{\eta_{ca}(\gamma)\sigma_\varepsilon}. \quad (6.14)$$

This equation provides the solution of γ , given the other parameters δ and r . Since it is nonlinear in γ , we, therefore, have to use an iterative procedure to obtain the solution. For each given δ and r , searched by the simulated annealing, we first set an initial γ , denoted by γ_0 . Then the new γ , denoted by γ_1 , is calculated from (6.14), which is equal to $0.27/[\eta_{ca}(\gamma_0)\sigma_\varepsilon]$. This procedure is continued until convergence.

We summarize the different cases in Table 6.2, where we start by using only restrictions on real variables and fix risk aversion to unity (M1). We add the risk-free rate restriction keeping risk aversion at one (M2), then estimate

it (M3). Finally we add the Sharpe-ratio restriction, fixing risk aversion at 50 (M4) and estimate it using an iterative procedure (M5). For each model we also compute the implied values of the long-term bond and equity premium using (6.12) and (6.13).

Table 6.2: Summary of Models

Models	Estimated Parameters	Fixed Parameters	Asset Restrictions
M1	r, δ	$\gamma = 1$	none
M2	r, δ	$\gamma = 1$	risk-free rate
M3	r, δ, γ		risk-free rate
M4	r, δ	$\gamma = 50$	risk-free rate, Sharpe-ratio
M5	r, δ, γ		risk-free rate, Sharpe-ratio

6.4 The Estimation Results

Table 6.3 summarizes the estimations for the first three models. Standard errors are in parentheses. Entries without standard errors are preset and hence are not estimated.

Table 6.3: Summary of Estimation Results⁹

Models	δ	r	γ
M1	0.0189 (0.0144)	0.0077 (0.0160)	prefixed to 1
M2	0.0220 (0.0132)	0.0041 (0.0144)	prefixed to 1
M3	0.0344 (0.0156)	0.0088 (0.0185)	2.0633 (0.4719)

Consider first Model 1, which only uses restrictions on real variables. The depreciation rate is estimated to be just below 2% which close to Christiano and Eichenbaum's (1992) results. The average interest rate is 0.77% per quarter or 3.08% on an annual basis. The implied discount factor computed from (6.3) is 0.9972. These results confirm the estimates in Christiano and Eichenbaum (1992). Adding the risk-free rate restriction in Model 2 does not significantly change the estimates. The discount factor is slightly higher while the average risk-free rate decreases.

However the implied discount factor now exceeds unity, a problem also encountered in Eichenbaum et al. (1988). Christiano and Eichenbaum (1992)

⁹The standard errors are in parenthesis.

avoid this by fixing the discount factor below unity rather than estimating it. Model 3 is more general since the risk aversion parameter is estimated instead of fixed at unity. The ML procedure estimates the risk aversion parameter to be roughly 2 and significantly different from 1, the value implied from log-utility function. Adding the risk-free rate restriction increases the estimates of δ and r somewhat. Overall, the model is able to produce sensible parameter estimates when the moment restriction for the risk-free rate is introduced.

While the implications of the dynamic optimization model concerning the real macroeconomic variables could be considered as fairly successful, the implications for asset prices are dismal. Table 6.4 computes the Sharpe-ratio as well as risk premia for equity and long term real bond using (6.11) - (6.13). Note that these variables are not used in the estimation of the model parameters. The leverage factor ζ is set to 2/3 for the computation of the equity premium.¹⁰

Table 6.4: Asset Pricing Implications

Models	SR	LT BPrem	EqPrem
M1	0.0065	0.000%	-0.082%
M2	0.0065	-0.042%	-0.085%
M3	0.0180	-0.053%	-0.091%

Table 6.4 shows that the RBC model is not able to produce sensible asset market prices when the model parameters are estimated from the restrictions derived only from the real side of the model (or, as in M3, adding the risk-free rate). The Sharpe-ratio is too small by a factor of 50 and both risk premia are too small as well, even negative for certain cases. Introducing the risk-free rate restriction improves the performance only a little bit. Next, we will try to estimate the model by adding the Sharpe-ratio moment restrictions. The estimation is reported in Table 6.5.

Table 6.5: Matching the Sharpe-Ratio

Models	δ	r	γ
M4	1	0	prefixed to 50
M5	1	1	60

¹⁰This value is advocated in Benninga and Protopapadakis (1990).

Model 4 fixes the risk aversion at 50. As explained in Lettau and Uhlig (1999), such a high level of risk aversion has the potential to generate reasonable Sharpe-ratios in consumption CAPM models. The question now is how the moment restrictions of the real variables are affected by such a high level of risk aversion. The first row of Table 6.5 shows that the resulting estimates are not sensible. The estimates for the depreciation factors and the steady-state interest rate converge to the pre-specified constraints,¹¹ or the estimation does not settle down to an interior optimum. This implies that the real side of the model does not yield reasonable results when risk aversion is 50. High risk aversion implies a low elasticity of intertemporal substitution so that agents are very reluctant to change their consumption over time.

Trying to estimate risk aversion while matching the Sharpe-ratio gives similar results. It is not possible to estimate the RBC model with simultaneously satisfying the moment restrictions from both the real side and the financial side of the model, as shown in the last row in Table 7.5. Again the parameter estimates do converge to pre-specified constraints. The depreciation rate converges again to unity as does the steady-state interest rate r . The point estimate of risk aversion parameter is high (60). The reason is of course that a high Sharpe-ratio requires high risk aversion. The tension between the Sharpe-ratio restriction and the real side of the model causes the estimation to fail. It demonstrates again that the asset pricing characteristics that one find in the data are fundamentally incompatible with the standard RBC model.

6.5 The Evaluation of Predicted and Sample Moments

Next we provide a diagnostic procedure to compare the second moments predicted by the model with the moments implied by the sample data. Our objective here is to ask whether our RBC model can predict the actual moments of the time series for both the real and asset market. The moments are revealed by the spectra at various frequencies. We remark that a similar diagnostic procedure can be found in Watson (1993) and Diebold et. al (1995).

Given the observations on k_t, a_t and the estimated parameters of our log-linear model, the predicted c_t and n_t can be constructed from the right hand side of (6.5) - (6.6) with k_t and a_t to be their actual observations. We now

¹¹We constraint the estimates to lie between 0 and 1.

consider the possible deviations of our predicted series from the sample series. We hereby employ our most reasonable estimated Model 3. We can use the variance-covariance matrix of our estimated parameters to infer the intervals of our forecasted series hence also the intervals of the moment statistics that we are interested in.

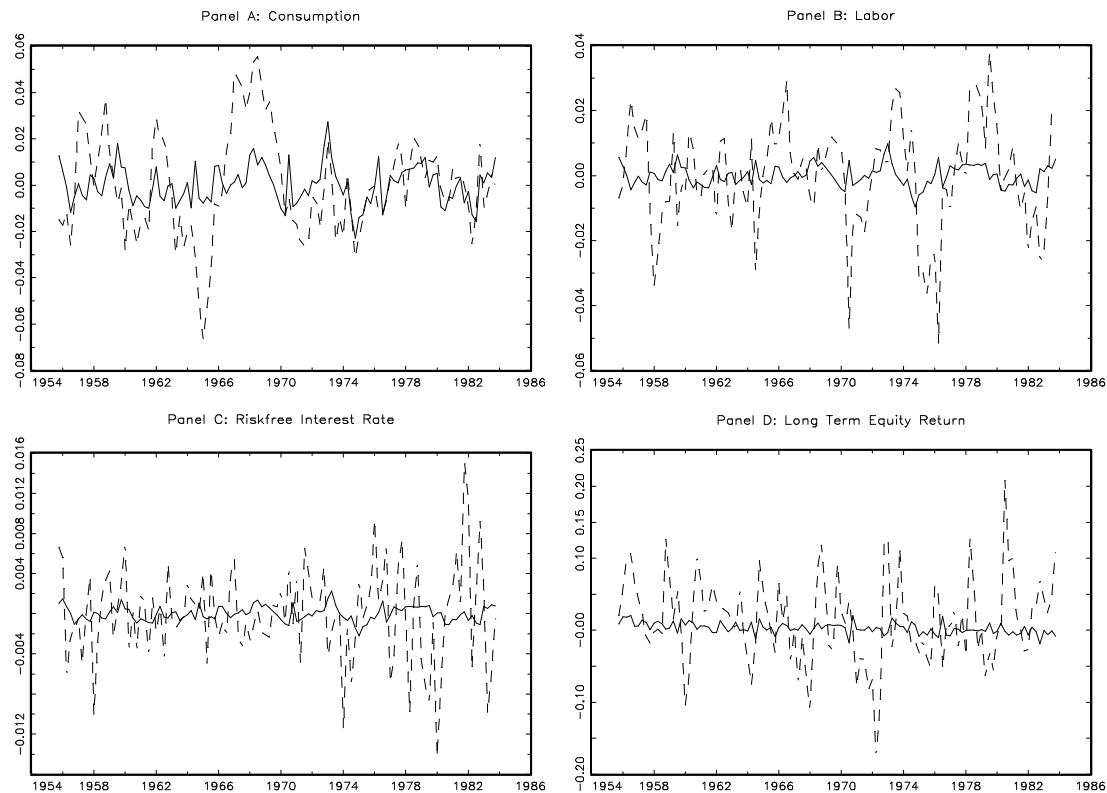


Figure 6.1: Predicted and Actual Series: solid lines (predicted series), dotted lines (actual series) for A) consumption, B) labor, C) risk-free interest rate and D) long term equity excess return; all variables HP detrended (except for excess equity return)

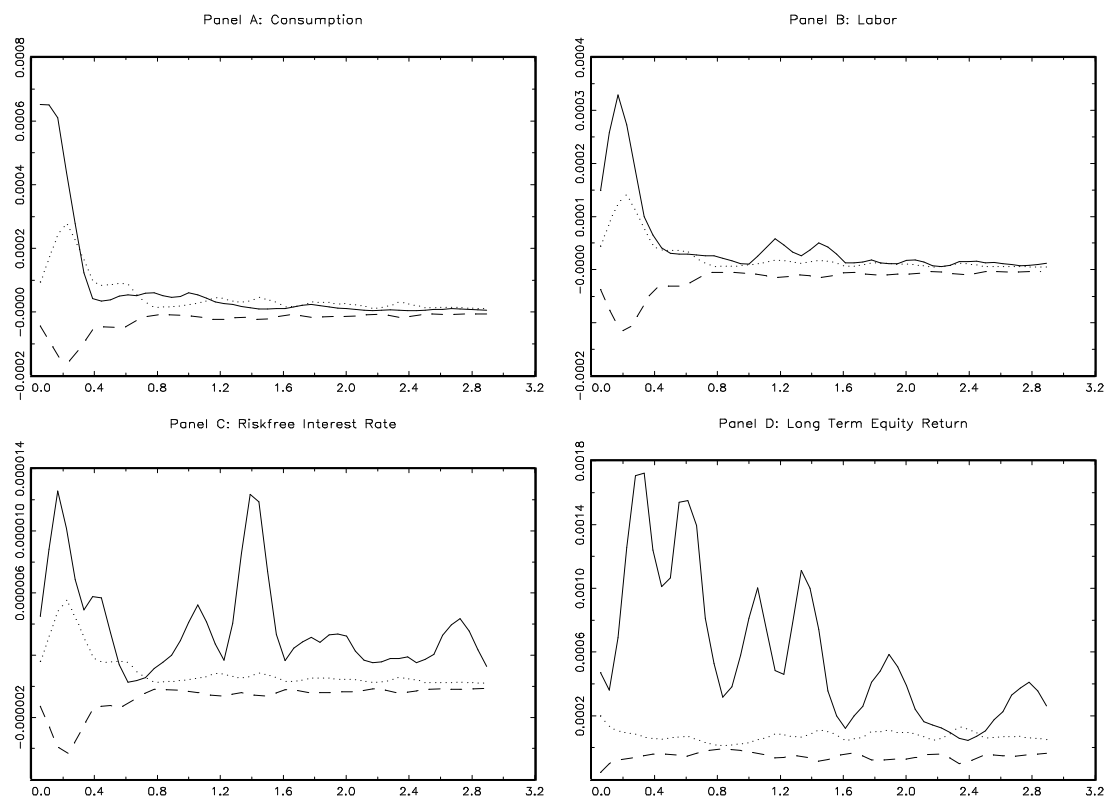


Figure 6.2: The Second Moment Comparison: solid line (actual moments), dashed and dotted lines (the intervals of predicted moments) for A) consumption, B) labor, C) risk-free interest rate and D) long-term equity excess return; all variables detrended (except excess equity return)

Figure 6.1 presents the Hodrick-Prescott (HP) filtered actual and predicted time series data on consumption, labor effort, risk-free rate and equity return. As shown in Chapter 5, the consumption series can somewhat be matched whereas the volatility in the labor effort as well as in the risk-free rate and equity excess return cannot be matched. The insufficient match of the latter three series are further confirmed by Figure 6.2 where we compare the spectra calculated from the data samples to the intervals of the spectra predicted, at 5% significance level, by the models.

A good match of the actual and predicted second moments of the time series would be represented by the fact that the solid line falls within the interval of the dashed and dotted lines. In particular the time series for

labor effort, risk-free interest rate and equity return fail to do so.

6.6 Conclusions

Asset prices contain valuable information about intertemporal decision making of economic agents. This chapter has estimated the parameters of a standard RBC model taking the asset pricing implications into account. We introduce model restrictions based on asset pricing implications in addition to the standard restrictions of the real variables and estimate the model by using ML method. We use the risk-free interest rate and the Sharpe-ratio in matching actual and predicted asset market moments and compute the implicit risk premia for long real bonds and equity. We find that though the inclusion of the risk-free interest rate as a moment restriction can produce sensible estimates, the computed Sharpe-ratio and the risk premia of long-term real bonds and equity are in general counterfactual. The computed Sharpe-ratio is too low while both risk premia are small and even negative. Moreover, the attempt to match the Sharpe-ratio in the estimation process can hardly generate sensible estimates. Finally, given the sensible parameter estimates, the second moments of labor effort, risk-free interest rate and long-term equity return predicted by the model do not match well the corresponding moments of the sample economy.

We conclude that the standard RBC model cannot match the asset market restrictions, at least with the standard technology shock, constant relative risk aversion (CRRA) utility function and no adjustment costs. Other researchers have looked at some extensions of the standard model such as technology shocks with a greater variance, other utility functions, for example, utility functions with habit formation, and adjustment costs of investment. The latter line of research has been pursued by Jerman (1998) and Boldrin, Christiano and Fisher (1996, 2001).¹² Those extensions of the standard model are, to least to a certain extent, more successful in replicating stylized asset market characteristics, yet these extensions frequently use extreme parameter values to be able to match the asset price characteristics of the model with the data. Moreover, the approximation methods for solving the models might not be very reliable since accuracy tests for the used approximation methods are still missing.¹³

¹²see also Wöhrmann, Semmler and Lettau (2001) where time varying characteristics of asset prices are explored.

¹³See Grüne and Semmler (2004b).

Part III

Beyond the Standard Model — Model Variants with Keynesian Features

Chapter 7

Multiple Equilibria and History Dependence

7.1 Introduction

One of the important features of Keynesian economics is that there is no unique equilibrium toward which the economy moves. The dynamics are open ended in the sense that it can move to low level, or high level of economic activity and expectations and policy may become important to tild the dynamics to one or the other outcomes.¹

In recent times such type of dynamics have been found in a large number of dynamic models with intertemporal optimization. Those models have been called indeterminacy models. Theoretical models of this type are reviewed in Benhabib and Farmer (1999) and Farmer (12001) and an empirical assessment is given in Schmidt-Grohe (2002). Some of the models are real models, RBC models, with increasind returns to scale and or more generate preferences, as introduced in chapter 4.5, that can exhibit locally stable steady state equilibria giving rise to sun spot phenomena.² Multiplicity of equilibria can also arise here as a consequence of increasing returns to scale and/or more general preferences. Others are monetary models, where consumers' welfare is affected positively by consumption and cash balances and negatively by the labor effort and an inflation gap from some target rates. For certain substitution properties between consumption and cash holdings those models admit unstable as well as stable high level and low level steady

¹In Keynes (1936) such an open ended dynamic is described in Chapter 5 of his book; Keynes describes here how higher or lower "long term positions" associated with higher or lower output and employment might be generated by expectational forces.

²See, for example Kim (2004).

states. Here can be indeterminacy in the sense that any initial condition in the neighborhood of one of the steady-states is associated with a path forward or away from that steady state, see Benhabib et al. (2001).

When indeterminacy models exhibit multiple steady state equilibria, where a middle one is an attractor (repellor), than this permits any path in the vicinity of the steady state equilibria to move back to (away from) the steady state equilibrium. Despite some unresolved issues in the literature on multiple equilibria and indeterminacy³ it has greatly enriched macrodynamic modelling.

Pursuing this line of research we show that one does not need to refer to increasing returns to scale or specific preferences to obtain such results. We show that due to the adjustment cost of capital we may obtain non-uniqueness of steady state equilibria in an otherwise standard dynamic optimization version. Multiple steady state equilibria, in turn, may lead to thresholds separating different domains of attraction of capital stock, consumption, employment and welfare level. As our solution shows thresholds are important as separation points below or above which it is advantages to move to lower or higher levels of capital stock, consumption, employment and welfare. Our model version thus can explain of how the economy becomes history dependent and moves, after a shock or policy influences, to a low or high level equilibria in employment and output.

Recently, numerous stochastic growth model have employed adjustment cost of capital. In non-stochastic dynamic models adjustment cost has already been used in Eisner and Stroz (1963), Lucas (1967) and Hayashi (1982). Authors in this tradition have also distinguished the absolute adjustment cost depending on the level of investment from the adjustment cost depending on investment relative to capital stock (Uzawa 1968, Asada and Semmler 1995).⁴ In stochastic growth models adjustment cost has been used in Boldrin, Christiano and Fisher (2001) and adjustment cost associated with the rate of change of investment can be found in Christiano, Eichenbaum and Evans (2001). In this chapter we want to show that adjustment cost in a standard RBC model can give rise to multiple steady state equilibria. The existence of multiple steady state equilibria entails thresholds that separate different domains of attraction for welfare and employment and allow

³Although these are important variants of macrodynamic models with optimizing behavior, as, however, recently has been shown, see Beyn, Pampel and Semmler (2001) and Grüne and Semmler (2004a), indeterminacy is likely to occur solely at a point in these models, at a threshold, and not within a set as the indeterminacy literature often claims.

⁴In Feichtinger et al. (2000) it is shown that relative adjustment cost, where investment as well as capital stock enters the adjustment costs, likely to generate multiplicity of steady state equilibria.

for an open ended dynamics depending on the initial conditions and policy influences impacting the initial conditions.

The remainder of this chapter is organized as follows. Section 2 presents the model. Section 3 studies the adjustment cost function which gives rise to multiple equilibria and section 4 demonstrates the existence of a threshold⁵ that separates different domains of attraction. Section 5 concludes the chapter. The proof of the propositions in the text is provided in the appendix.

7.2 The Model

The model we present here is the standard stochastic growth model of RBC type, as in King et al. (1988), augmented by adjustment cost. The state equation for the capital stock takes the form:

$$K_{t+1} = (1 - \delta)K_t + I_t - Q_t \quad (7.1)$$

where

$$I_t = Y_t - C_t \quad (7.2)$$

and

$$Y_t = A_t K_t^{1-\alpha} (N_t X_t)^\alpha \quad (7.3)$$

Above, K_t , Y_t , I_t , C_t and Q_t are the level of capital stock, output, investment, consumption and adjustment cost, all in real terms. N_t is per capita working hours; A_t is the temporary shock in technology; and X_t is the permanent (including both population and productivity growth) shock that follows a growth rate γ . The model is non-stationary due to X_t . To transform the model into a stationary version we need to detrend the variables. For this purpose, we divide both sides of equation (7.1) - (7.3) by X_t :

$$k_{t+1} = \frac{1}{1 + \gamma} [(1 - \delta)k_t + i_t - q_t]$$

$$i_t = y_t - c_t \quad (7.4)$$

$$y_t = A_t k_t^{1-\alpha} (n_t \bar{N} / 0.3)^\alpha \quad (7.5)$$

Above, k_t , c_t , i_t , y_t and q_t are the detrended variables for K_t , C_t , I_t , Y_t and

Q_t .⁶; n_t is defined to be $\frac{0.3N_t}{\bar{N}}$ with \bar{N} denoting the sample mean of N_t . Note that here n_t is often regarded to be the normalized hours with its sample

⁵In the literature those thresholds have been called Skiba-points (see Skiba, 1978).

⁶In particular, $k_t \equiv \frac{K_t}{X_t}$, $c_t \equiv \frac{C_t}{X_t}$, $i_t \equiv \frac{I_t}{X_t}$, $y_t \equiv \frac{Y_t}{X_t}$ and $q_t \equiv \frac{Q_t}{X_t}$.

mean equal to 30 %. We shall assume that the detrended adjustment cost q_t depends on detrended investment i_t :

$$q_t = q(i_t)$$

The objective function takes the form

$$\max E_0 \sum_{t=0}^{\infty} \beta^t [\log c_t + \theta \log(1 - n_t)]$$

To solve the model, we first form the Lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t [\log(c_t) + \theta \log(1 - n_t)] - \sum_{t=0}^{\infty} E_t \left\{ \beta^{t+1} \lambda_{t+1} \left[k_{t+1} - \frac{1}{1 + \gamma} [(1 - \delta)k_t + i_t - q(i_t)] \right] \right\}$$

Setting to zero the derivatives of L with respect to c_t , n_t , k_t and λ_t , we obtain the following first-order conditions:

$$\frac{1}{c_t} - \frac{\beta}{1 + \gamma} E_t \lambda_{t+1} [1 - q'(i_t)] = 0 \quad (7.6)$$

$$\frac{-\theta}{1 - n_t} + \frac{\beta}{1 + \gamma} E_t \lambda_{t+1} \frac{\alpha y_t}{n_t} [1 - q'(i_t)] = 0 \quad (7.7)$$

$$\frac{\beta}{1 + \gamma} E_t \lambda_{t+1} \left[(1 - \delta) + \frac{(1 - \alpha)y_t}{k_t} [1 - q'(i_t)] \right] = \lambda_t \quad (7.8)$$

$$k_{t+1} = \frac{1}{1 + \gamma} [(1 - \delta)k_t + i_t - q(i_t)] \quad (7.9)$$

with i_t and y_t to be given by (7.4) and (7.5) respectively.

The following proposition concerns the steady states

Proposition 5 *Assume A_t has a steady state \bar{A} . Equation (7.4) - (7.9), when evaluated at their certainty equivalence form, determine the following steady states:*

$$[b\phi(\bar{i}) - 1][\bar{i} - q(\bar{i})] - a\phi(\bar{i})^{1-\frac{1}{\alpha}} - q(\bar{i}) = 0 \quad (7.10)$$

$$\bar{k} = \frac{1}{\gamma + \delta} [\bar{i} - q(\bar{i})] \quad (7.11)$$

$$\bar{n} = \left[\frac{\phi(\bar{i})}{A} \right]^{\frac{1}{\alpha}} \bar{k} \frac{0.3}{\bar{N}} \quad (7.12)$$

$$\bar{y} = \phi(\bar{i}) \bar{k} \quad (7.13)$$

$$\bar{c} = \bar{y} - \bar{i} \quad (7.14)$$

$$\bar{\lambda} = \frac{(1 + \gamma)}{\beta \bar{c} [1 - q'(\bar{i})]} \quad (7.15)$$

where

$$a = \frac{\alpha}{\theta} A^{\frac{1}{\alpha}} (\bar{N}/0.3) \quad (7.16)$$

$$b = \frac{(1 + \frac{\alpha}{\theta})}{\gamma + \delta} \quad (7.17)$$

$$\phi(\bar{i}) = \frac{m}{1 - q'(\bar{i})} \quad (7.18)$$

and

$$m = \frac{(1 + \gamma) - (1 - \delta)\beta}{\beta(1 - \alpha)}. \quad (7.19)$$

Note that equation (7.10) determines the solution of \bar{i} , depending on the assumption that all other steady states can be uniquely solved via (7.11) - (7.15). Also if $q(i)$ is linear, equ. (7.18) indicates that $\phi(\cdot)$ is constant, and then from (7.10) \bar{i} is uniquely determined. Therefore, if $q(i)$ is linear, no multiple steady state equilibria will occur.

7.3 The Existence of Multiple Steady States

Many non-linear forms of $q(i)$ may lead to a multiplicity of equilibria. Here we shall only consider that $q(i)$ takes the logistic form:

$$q(i) = \frac{q_0 \exp(q_1 i)}{\exp(q_1 i) + q_2} - \frac{q_0}{1 + q_2} \quad (7.20)$$

Figure (7.1) shows a typical shape of $q(i)$ while Figure (7.2) shows the corresponding derivative $q'(i)$ with the parameters given in Table 7.1:

Table 7.1: The Parameters in the Logistic Function

q_0	q_1	q_2
2500	0.0034	500

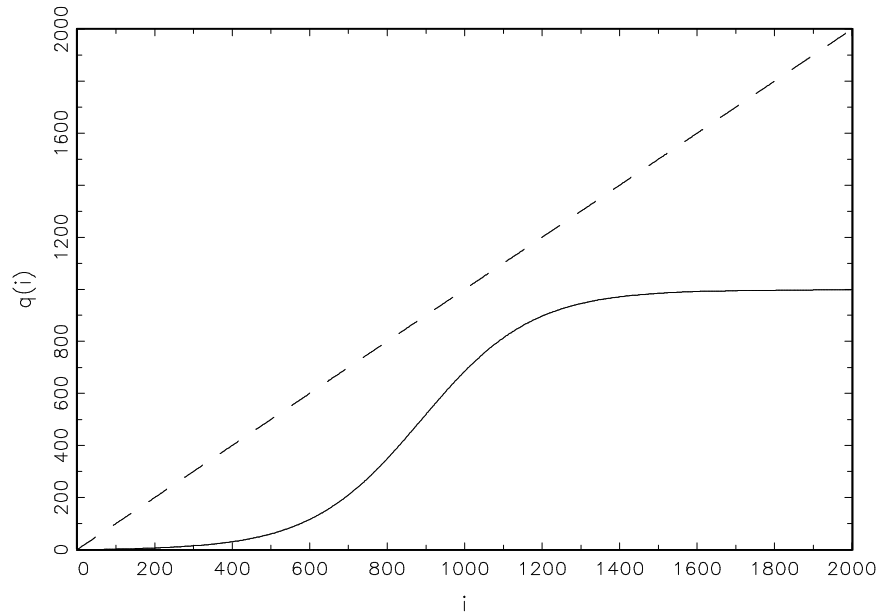


Figure 7.1: The Adjustment Cost Function

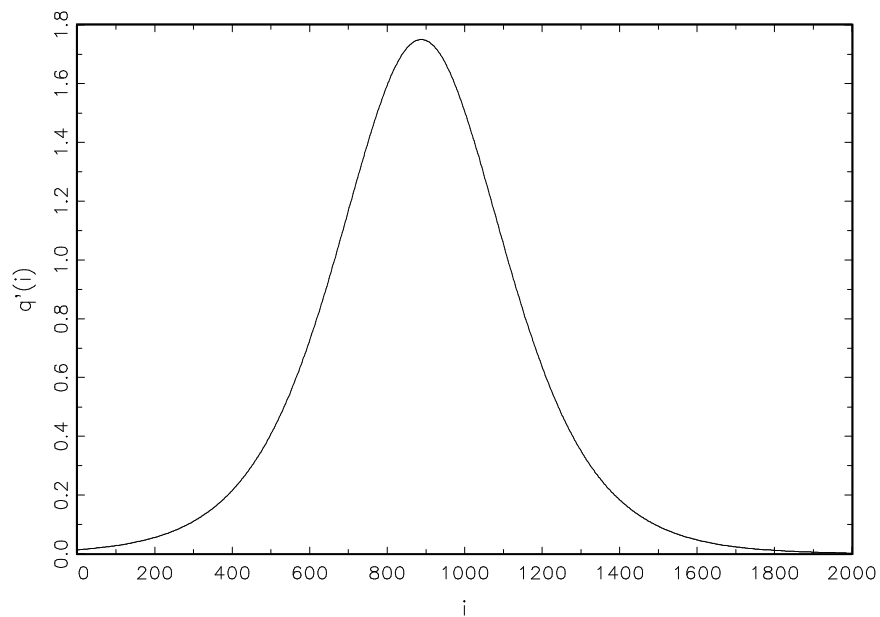


Figure 7.2: The Derivatives of the Adjustment Cost

Note that in equation (7.20) we posit a restriction such that $q(0) = 0$. Another restriction, which is reflected in Figure (7.1), is that

$$q(i) < i \quad (7.21)$$

indicating that the adjustment cost should never be larger than the investment itself. Both restrictions seem reasonable.

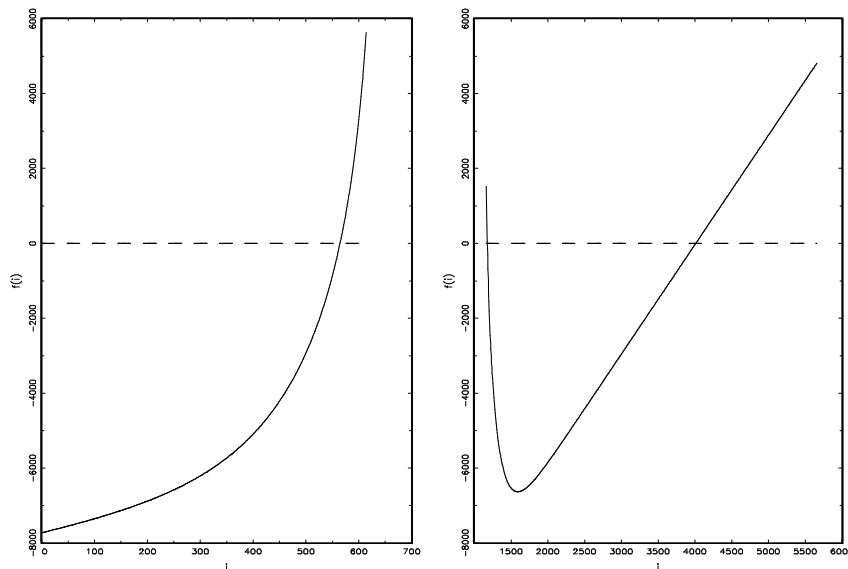
The two critical points, i_1 and i_2 , in Figure (7.1) and (7.2) need to be discussed. These are the two points at which $q'(i) = 1$. Meanwhile between i_1 and i_2 , $q'(i) > 1$. When $q'(i) > 1$, equation (7.18) indicates that $\phi(i)$ is negative since from (7.19) $m > 0$. A negative $\phi(i)$ will lead to a complex $\phi(i)^{1-\frac{1}{\alpha}}$ in (7.10). We therefore obtain two feasible ranges for the existence of steady states of i : one is $(0, i_1)$ and the other is $(i_2, +\infty)$.

The following proposition concerns the existence of multiplicity of equilibria.

Proposition 6 *Let $f(i) \equiv [b\phi(i) - 1][i - q(i)] - a\phi(i)^{1-\frac{1}{\alpha}} - q(i)$, where $q(i)$ takes the form as in (7.20) subject to (7.21). Assume $bm - 1 > 0$.*

- *There exists one and only one i , denoted \bar{i}_1 in the range $(0, i_1)$ such that $f(i) = 0$.*
- *In the range $(i_2, +\infty)$, if there are some i 's at which $f(i) < 0$, then there must exist two i 's, denoted as \bar{i}_2 and \bar{i}_3 such that $f(i) = 0$.*

We shall first remark that the assumption $bm - 1 > 0$ is plausible given the standard parameters for b and m also $f(i) = 0$ is indeed the equation (2.10). Therefore, this proposition indicates a condition under which multiple steady states will occur. In particular, if there exist some i 's in $(i_2, +\infty)$ at which $f(i) < 0$, three equilibria will occur. A formal mathematical proof of the existence of this condition is intractable. In Figure 7.3, we show the curve of $f(\cdot)$ given the empirically plausible parameters as reported in Table 7.1 and other standard parameters as given in Table 7.2. The curves cut the zero line three times, indicating three steady states of i .

Figure 7.3: Multiplicity of Equilibria: $f(i)$ functionTable 7.2: The Standard Parameters of RBC Model⁷

α	γ	β	δ	θ	\bar{N}	\bar{A}
0.5800	0.0045	0.9930	0.2080	2.0189	480.00	1.7619

We use a numerical method to compute the three steady states of i : \bar{i}_1, \bar{i}_2 and \bar{i}_3 . Given these steady states, the other steady states are computed by (7.11) - (7.15). Table 7.3 uses essentially the same parameters as reported in Table 5.1. The result of the computations of the three steady states are:

⁷The \bar{N} below is calculated on the assumption of 12 weeks per quarter and 40 working hours per week. \bar{A} is derived from $A_t = a_0 + a_1 A_{t-1} + \varepsilon_t$, with estimated a_0 and a_1 are given respectively by 0.0333 and 0.9811.

Table 7.3: The Multiple Steady States

	Corresponding to \bar{i}_1	Corresponding to \bar{i}_2	Corresponding to \bar{i}_3
\bar{i}	564.53140	1175.7553	4010.4778
\bar{k}	18667.347	11672.642	119070.11
\bar{n}	0.25436594	0.30904713	0.33781724
\bar{c}	3011.4307	2111.2273	5169.5634
\bar{y}	3575.9621	3286.9826	9180.0412
$\bar{\lambda}$	0.00083463435	0.0017500565	0.00019568017
\bar{V}	1058.7118	986.07481	1101.6369

Note that above, \bar{V} is the value of the objective function at the corresponding steady states. Therefore it reflects one corresponding welfare level. The steady state corresponding to \bar{i}_2 deserves some discussion. The welfare of \bar{i}_1 is larger than \bar{i}_2 , its corresponding steady states in capital, output, and consumption are all greater than those corresponding to \bar{i}_2 , yet its corresponding steady state in labor effort and thus employment is larger for \bar{i}_2 . This already indicates that \bar{i}_2 may be inferior in terms of the welfare, at least compared to \bar{i}_1 . On the other hand \bar{i}_1 and \bar{i}_3 exhibit also differences in welfare and employment.

7.4 The Solution

An analytical solution to the dynamics of the model with adjustment cost is not feasible. We, therefore, have to rely on an approximate solution. For this, we shall first linearize the first-order conditions around the three sets of steady states as reported in Table 7.3. Then by applying an approximation method as discussed in chapter 2, we obtain three sets of linear decision rules for c_t and n_t corresponding to our three sets of steady states. For notational convenience, we shall denote them as decision rule Set 1, Set 2 and Set 3 corresponding to \bar{i}_1 , \bar{i}_2 and \bar{i}_3 . Assume that A_t stays at its steady state \bar{A} so that we only consider the deterministic case. The i th set of decision rule can then be written as

$$c_t = G_c^i k_t + g_c^i \quad (7.22)$$

$$n_t = G_n^i k_t + g_n^i \quad (7.23)$$

where $i = 1, 2, 3$. We therefore can simulate the solution paths by using the above two equations together with (7.4), (7.5) and (7.9). The question then

arises as to which set of decision rule, as expressed by (7.22) and (7.23), should be used. The likely conjecture is that this will depend on the initial condition k_0 . For example, if k_0 is close to the \bar{k}_1 , the steady state of k corresponding to \bar{i}_1 , we would expect that the decision rule 1 is appropriate. This consideration further indicates that there must exist some thresholds for k_0 at which intervals are divided regarding which set of decision rule should be applied. To detect such thresholds, we shall compute the value of the objective functions starting at different k_0 for our three decision rules. Specifically, we compute V , where

$$V \equiv \sum_{t=0}^{\infty} \beta^t [\log(c_t) + \theta \log(1 - n_t)]$$

We should choose the range of k_0 's that covers the three steady states of \bar{k} 's as reported in Table 7.3. In this exercise, we choose the range $[8000, 138000]$ for k_0 . Figure 7.4 compares the welfare performance of our three sets of linear decision rules.

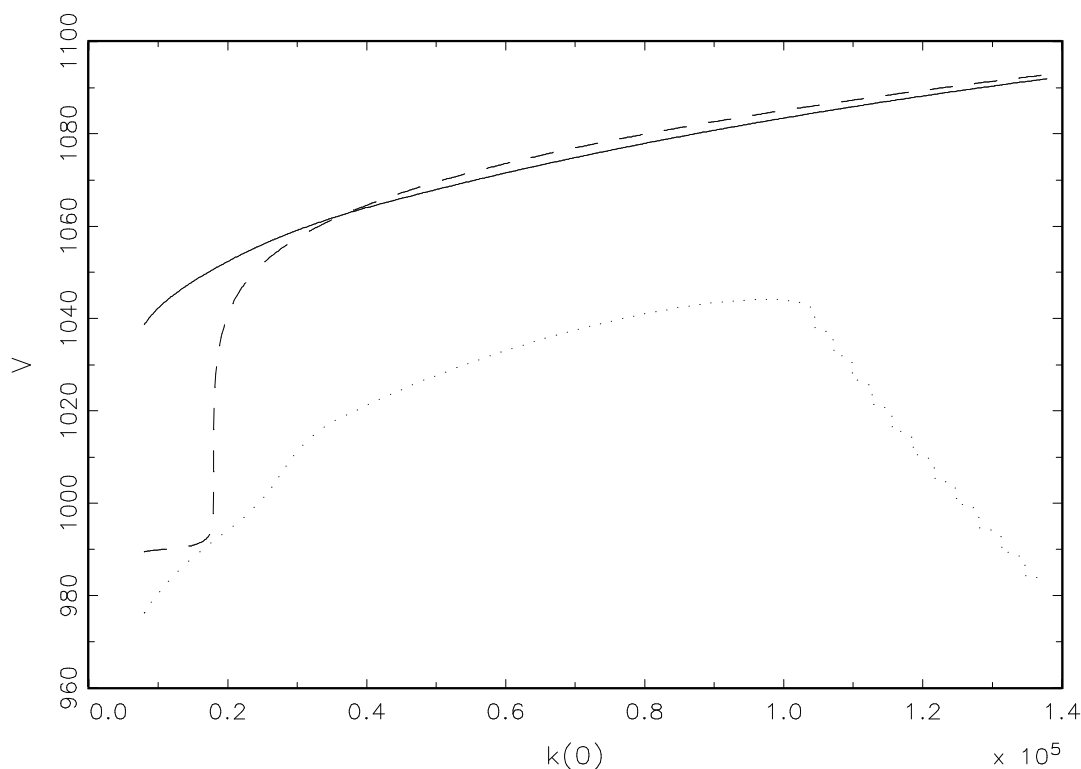


Figure 7.4: The Welfare Performance of three Linear Decision Rules (solid, dotted and dashed lines for decision rule Set 1, Set 2 and Set 3 respectively)

From Figure 7.4, we first realize that the value of the objective function is always lower for decision rule Set 2. This is likely to be caused by its inferior welfare performance at the steady states for which we compute the decision rule. However, there is an intersection of the two welfare curves corresponding to the decision rules Set 1 and Set 3. This intersection, occurring around $k_0 = 36900$, can be regarded as a threshold. If $k_0 < 36900$, the household should choose decision rule Set 1 since it will allow the household to obtain a higher welfare. On the other hand, if $k_0 > 36900$, the household may choose decision rule Set 3, since this leads to a higher welfare.

7.5 Conclusion

This chapter shows that the introduction of adjustment cost of capital may lead to non-uniqueness of steady state equilibria in an otherwise standard RBC model. Multiple steady state equilibria, in turn, lead to thresholds separating different domains of attraction of capital stock, consumption, employment and welfare level. As our simulation shows thresholds are important as separation points below or above which it is optimal to move to lower or higher levels of capital stock, consumption, employment and welfare. Our model thus can easily explain of how an economy become history dependent and moves, after a shock, to a low or high level equilibrium in employment and output. A variety of further economic models giving rise to multiple equilibria and thresholds are presented in Grüne and Semmler (2004a).

The above model stays as close as possible to the standard RBC model except, for illustrative purpose, a specific form of adjustment cost of capital was introduced. On the other hand, dynamic models giving rise to indeterminacy usually have to presume some weak externalities and increasing returns and/or more general preferences. Kim (2004) discusses to what extent weak externalities in combination with more complex preferences will produce indeterminacy. He in fact shows that if for the generalized RBC model, as studied in chapter 4.5., there is a weak externality, $\xi > 0$, then the model generates local indeterminacy. Overall, as shown in many recent contributions, the issue of multiple equilibria and indeterminacy is an important macroeconomic issue and should be pursued in further research in the future.

7.6 Appendix: The Proof of Propositions 5 and 6

7.6.1 The Proof of Proposition 5

The certainty equivalence of equation (7.4) - (7.9) takes the form

$$\frac{1}{\bar{c}} - \frac{\beta}{1+\gamma} \bar{\lambda} [1 - q'(\bar{i})] = 0 \quad (7.24)$$

$$\frac{-\theta}{1-\bar{n}} + \frac{\beta}{1+\gamma} \bar{\lambda} \frac{\alpha \bar{y}}{\bar{n}} [1 - q'(\bar{i})] = 0 \quad (7.25)$$

$$\frac{\beta}{1+\gamma} \left[(1-\delta) + \frac{(1-\alpha)\bar{y}}{\bar{k}} [1 - q'(\bar{i})] \right] = 1 \quad (7.26)$$

$$\bar{k} = \frac{1}{1+\gamma} [(1-\delta)\bar{k} + \bar{i} - q(\bar{i})] \quad (7.27)$$

$$\bar{i} = \bar{y} - \bar{c} \quad (7.28)$$

$$\bar{y} = \overline{Ak}^{1-\alpha} (\bar{n}\bar{N}/0.3)^\alpha \quad (7.29)$$

From (7.27),

$$\bar{k} = \frac{1}{\gamma + \delta} [\bar{i} - q(\bar{i})] \quad (7.30)$$

which is equation (7.11). From (7.26),

$$\begin{aligned} \frac{\bar{y}}{\bar{k}} &= \frac{(1+\gamma) - (1-\delta)\beta}{\beta(1-\alpha) [1 - q'(\bar{i})]} \\ &= \phi(\bar{i}) \end{aligned} \quad (7.31)$$

which is equation (7.13) with $\phi(\bar{i})$ given by (refequ7.18) and (7.19). Further from (7.29),

$$\frac{\bar{y}}{\bar{k}} = \overline{Ak}^{-\alpha} (\bar{n}\bar{N}/0.3)^\alpha \quad (7.32)$$

Using (7.31) to express $\frac{\bar{y}}{\bar{k}}$, we derive from (7.32)

$$\bar{n} = \left[\frac{\phi(\bar{i})}{A} \right]^{\frac{1}{\alpha}} \bar{k} \frac{0.3}{\bar{N}} \quad (7.33)$$

which is (7.12). Next, using (7.28) to express \bar{i} while using $\phi(\bar{i})\bar{k}$ for \bar{y} . We derive from (7.30) $(\gamma + \delta)\bar{k} = \phi(\bar{i})\bar{k} - \bar{c} - q(\bar{i})$, which is equivalent to

$$\bar{c} = (\phi(\bar{i}) - \gamma - \delta)\bar{k} - q(\bar{i})$$

$$= c_1(\phi(\bar{i}), \bar{k}, q(\bar{i})) \quad (7.34)$$

Meanwhile from (7.24) and (7.25):

$$\frac{\beta}{1+\gamma} \bar{\lambda} [1 - q'(\bar{i})] = \frac{1}{\bar{c}} = \frac{\theta}{(1-\bar{n})\alpha} \left(\frac{\bar{y}}{\bar{n}}\right)^{-1}$$

The first equation is equivalent to (7.15) while the second equation indicates

$$\bar{c} = \frac{(1-\bar{n})\alpha}{\theta} \left(\frac{\bar{y}}{\bar{n}}\right) \quad (7.35)$$

where from (7.29),

$$\begin{aligned} \frac{\bar{y}}{\bar{n}} &= \bar{A} \left(\frac{\bar{k}}{\bar{n}}\right)^{1-\alpha} (\bar{N}/0.3)^\alpha \\ &= \bar{A} \left(\frac{\bar{y}}{\bar{n}}\right)^{1-\alpha} \left(\frac{\bar{y}}{\bar{k}}\right)^{\alpha-1} (\bar{N}/0.3)^\alpha \\ &= \bar{A}^{\frac{1}{\alpha}} \phi(\bar{i})^{1-\frac{1}{\alpha}} (\bar{N}/0.3) \end{aligned} \quad (7.36)$$

Substitute (7.36) into (7.35) and then express \bar{n} in terms of (7.33):

$$\begin{aligned} \bar{c} &= \frac{(1-\bar{n})\alpha}{\theta} \bar{A}^{\frac{1}{\alpha}} \phi(\bar{i})^{1-\frac{1}{\alpha}} (\bar{N}/0.3) \\ &= \frac{\alpha}{\theta} \bar{A}^{\frac{1}{\alpha}} (\bar{N}/0.3) \phi(\bar{i})^{1-\frac{1}{\alpha}} - \frac{\alpha}{\theta} \bar{A}^{\frac{1}{\alpha}} (\bar{N}/0.3) \phi(\bar{i})^{1-\frac{1}{\alpha}} \left[\frac{\phi(\bar{i})}{\bar{A}}\right]^{\frac{1}{\alpha}} \bar{k} \frac{0.3}{\bar{N}} \\ &= a \phi(\bar{i})^{1-\frac{1}{\alpha}} - \frac{\alpha}{\theta} \phi(\bar{i}) \bar{k} \\ &= c_2(\phi(\bar{i}), \bar{k}) \\ &= c_2(\phi(\bar{i}), \bar{k}) \end{aligned} \quad (7.37)$$

with a given by (7.16).

Let $c_1(\cdot) = c_2(\cdot)$. We thus obtain

$$(\phi(\bar{i}) - \gamma - \delta) \bar{k} - q(\bar{i}) = a \phi(\bar{i})^{1-\frac{1}{\alpha}} - \frac{\alpha}{\theta} \phi(\bar{i}) \bar{k}$$

which is equivalent to

$$\left[\left(1 + \frac{\alpha}{\theta}\right) \phi(\bar{i}) - \gamma - \delta \right] \bar{k} = a \phi(\bar{i})^{1-\frac{1}{\alpha}} + q(\bar{i})$$

Using (7.30) for \bar{k} , we obtain

$$\left[\left(1 + \frac{\alpha}{\theta}\right) \phi(\bar{i}) - \gamma - \delta \right] \frac{1}{\gamma + \delta} (\bar{i} - q(\bar{i})) = a \phi(\bar{i})^{1-\frac{1}{\alpha}} + q(\bar{i}) \quad (7.38)$$

Equation (7.37) is equivalent to (7.10) with b given by (7.17).

7.6.2 The Proof of Proposition 6

Note that within our two ranges $(0, i_1)$ and $(i_2, +\infty)$ $\phi(i)$ is positive and hence $f(i)$ is continuous and differentiable. In particular,

$$f'(i) = \phi'(i) \left\{ b[i - q(i)] + a\left(\frac{1}{\alpha} - 1\right)\phi(i)^{-\frac{1}{\alpha}} \right\} + (bm - 1) \quad (7.39)$$

where

$$\phi'(i) = \frac{mq''(i)}{[1 - q'(i)]^2} \quad (7.40)$$

We shall first realized that a , b and m are all positive as indicated by (7.16) and (7.17) and (7.19), and therefore the term $b[i - q(i)] + a(\frac{1}{\alpha} - 1)\phi(i)^{-\frac{1}{\alpha}}$ is positive. Meanwhile in the range $(0, i_1)$, $q''(i) > 0$ and hence $f'(i) > 0$. However, in the range $(i_2, +\infty)$, $q''(i) < 0$ and hence $f'(i)$ can either be positive or negative due to the sign of $(bm - 1)$.

Let us first consider the range $(0, i_1)$. Assume $i \rightarrow 0$. In this case, $q(i) \rightarrow 0$ and therefore $f(i) \rightarrow -a\phi(0)^{1-\frac{1}{\alpha}} < 0$. Next assume $i \rightarrow i_1$, In this case, $q'(i) \rightarrow 1$ and therefore $\phi(i) \rightarrow +\infty$. Since $1 - \frac{1}{\alpha}$ is negative, this further indicates $\phi(i)^{1-\frac{1}{\alpha}} \rightarrow 0$. Therefore $f(i) \rightarrow +\infty$. Since $f'(i) > 0$, by the intermediate value theorem, there exists one and only one i such that $f(i) = 0$. We thus have proved the first part of the proposition.

Next we turn to the range $(i_2, +\infty)$. To verify the second part of the proposition, we only need to prove that $f(i) \rightarrow +\infty$ and $f(i) < 0$ when $i \rightarrow i_2$ and $f(i) \rightarrow +\infty$ and $f(i) > 0$ when $i \rightarrow +\infty$. Consider first $i \rightarrow i_2$, Again in this case, $q'(i) \rightarrow 1$, $\phi(i) \rightarrow +\infty$, and $\phi(i)^{1-\frac{1}{\alpha}} \rightarrow 0$. Therefore $f(i) \rightarrow +\infty$. Meanwhile from (7.39), $\phi'(i) \rightarrow -\infty$ (since $q''(i) < 0$) and therefore $f'(i) < 0$. Consider now $i \rightarrow +\infty$, In this case, $q'(i) \rightarrow 0$ and $q(i) \rightarrow q^m$ where q^m is the upper limit of $q(i)$. This indicates that $[i - q(i)] \rightarrow +\infty$. Therefore $f(i) \rightarrow +\infty$. Meanwhile, since $q''(i) \rightarrow 0$ and hence $\phi'(i) \rightarrow 0$. Therefore $f'(i) \rightarrow (bm - 1)$, which is positive. We thus have proved the second part of the proposition.

Chapter 8

Business Cycles with Nonclearing Labor Market

8.1 Introduction

As discussed in the previous chapters, especially in Chapter 5, the standard real business cycle (RBC) model, despite its rather simple structure, can explain the volatilities of some macroeconomic variables such as output, consumption and capital stock. However, to explain the actual variation in employment the model generally predicts an excessive smoothness of labor effort in contrast to empirical data. This problem of excessive smoothness in labor effort is well-known in the RBC literature. A recent evaluation of this failure of the RBC model is given in Schmidt-Grohe (2001). There the RBC model is compared to indeterminacy models, as developed by Benhabib and his co-authors. Whereas in RBC models the standard deviation of the labor effort is too low, in indeterminacy models it turns out to be excessively high. Another problem in RBC literature related to this, is that the model implies a excessively high correlation between consumption and employment while empirical data only indicates a weak correlation. This problem of excessive correlation has, to our knowledge, not sufficiently been studied in the literature. It has preliminarily been explored in Chapter 5 of this volume. Lastly, the RBC model predicts a significantly high positive correlation between technology and employment whereas empirical research demonstrates, at least at business cycle frequency, a negative or almost zero correlation.

These are the major issues that we shall take up from now on. We want to note that the labor market problems, the lack of variation in the employment and the high correlation between consumption and employment in the standard RBC model, may be related to the specification of the labor

market, and therefore we could name it the labor market puzzle. In this chapter we are mainly concerned with this puzzle. The technology puzzle, that is, the excessively high correlation between technology and employment, preliminarily discussed in Chapter 5, will be taken up in Chapter 9.

Although in the specification of its model structure (see Chapter 4), the real business cycle model specifies both sides, the demand and supply, of a market, the moments of the economy are however reflected by the variation on one side of markets due to its general equilibrium nature for all markets (including output, labor and capital markets). For the labor market, the moments of labor effort result from the decision rule of the representative household to supply labor. The variations in labor and consumption both reflect the moments of the two state variables, capital and technology. It is therefore not surprising why employment is highly correlated with consumption and why the variation of consumption is a smooth as labor effort. This further suggests that to resolve the labor market puzzle in a real business cycle model, one has to make improvement upon labor market specifications. One possible approach for such improvement is to introduce the Keynesian feature into the model and to allow for wage stickiness and a nonclearing labor market.

The research along the line of micro-founded Keynesian economics has been historically developed by the two approaches: one is the disequilibrium analysis, which had been popular before 1980's and the other is the New Keynesian analysis based on monopolistic competition. Attempts have now been made recently that introduce the Keynesian features into a dynamic optimization model. Rotemberg and Woodford (1995, 1999), King and Wollman (1999), Gali (1999) and Woodford (2003) present a variety of models with monopolistic competition and sticky price. On the other hand, there are models of efficiency wages where nonclearing labor market could occur.¹ We shall remark that in those studies with nonclearing labor market, an explicit labor demand function is introduced from the decision problem of the firm side. However, the decision rule with regard to labor supply in these models is often dropped because the labor supply no longer appears in the utility function of the household. Consequently, the moments of labor effort become purely demand-determined.²

In this chapter, we will present a stochastic dynamic optimization model of RBC type but argumented by Keynesian features along the line of above

¹See Danthine and Donaldson (1990, 1995), Benassy (1995) and Uhlig and Xu (1996) among others.

²The labor supply in the these models is implicitly assumed to be given exogenously, and normalized to 1. Hence nonclearing of the labor market occurs if the demand is not equal to 1.

consideration. In particular, we shall allow for wage stickiness³ and non-clearing labor market. However, unlike other recent models that drop the decision rule of labor supply, we view the decision rule of the labor effort as being derived from a dynamic optimization problem as a quite natural way to determine desired labor supply.⁴ With the determination of labor demand, derived from the marginal product of labor and other factors,⁵ the two basic forces in the labor market can be formalized. One of the advantages of this formulation, as will become clear, is that a variety of employment rules could be adopted to specify the realization of actual employment when a nonclearing market emerges.⁶ We will assess this model by employing U.S. and German macroeconomic time series data.

Yet before we formally present the model and its calibration we want to note that there is a similarity of our approach chosen here and the New Keynesian analysis. New Keynesian literature presents models with imperfect competition and sluggish price and wage adjustments where labor effort is endogenized. Important work of this type can be found in Rotemberg and Woodford (1995, 1999), King and Wollman (1999), Gali (1999), Erceg, Henderson and Levin (2000) and Woodford (2003). However, the market in those models are still assumed to be cleared since the producer supplies the output according to what the market demands at the existing price. A similar consideration is also assumed to hold for the labor market. Here the wage rate is set optimally by a representative of the household according to

³Already Keynes (1936) had not only observed a wide-spread phenomenon of downward rigidity of wages but has also attributed strong stabilizing properties of wage stickiness.

⁴One could perceive a change in secular forces concerning labor supply from the side of households, for example, changes in preferences, demographic changes, productivity and real wage, union bargaining, evolution of wealth, taxes and subsidies which all affect labor supply. Some of those secular forces are often mentioned in the work by Phelps, see Phelps (1997) and Phelps and Zoega (1998). Recently, concerning Europe, generous unemployment compensation and related welfare state benefits have been added to the list of factors affecting the supply of labor, intensity of job search and unemployment. For an extensive reference to those factors, see Blanchard and Wolfers (2000) and Ljungqvist and Sargent (1998, 2003).

⁵On the demand side one could add beside the pure technology shocks and the real wage, the role of aggregate demand, high interest rates (Phelps 1997, Phelps and Zoega 1998), hiring and firing cost, capital shortages and slow down of growth, for example, in Europe. See Malinvaud (1994) for a more extensive list of those factors .

⁶Another line of recent research on modeling unemployment in a dynamic optimization framework can be found in the work by Merz (1999) who employs search and matching theory to model the labor market, see also Ljungqvist and Sargent (1998, 2003). Yet, unemployment resulting from search and matching problems can rather be viewed as frictional unemployment (see Malinvaud (1994) for his classification of unemployment). As will become clear, this will be different from the unemployment that we will discuss in this chapter.

the expected market a demand curve for labor. Once the wage has been set, it is assumed to be sticky for some time period and only a fraction of wages are set optimally in each period. In those models there will be a gap again between the optimal wage and existing wage, yet the labor market is still cleared since the household is assumed to supply labor whatever the market demand is at the given wage rate.⁷

In this chapter, we shall present a dynamic model that allows for a non-cleared labor market, which could be seen to be caused by staggered wage as described by Taylor (1980), Calvo (1983) or other theories of sluggish wage adjustment. The objective to construct a model such as ours is to approach the two aforementioned labor market problems coherently within a single model of dynamic optimization. Yet, we wish to argue that the New Keynesian and approach are complementary rather than exclusive, and therefore they can somewhat be consolidated as a more complete system for price and quantity determination within the Keynesian tradition. For further details of this consolidation, see Chapter 9. In the current chapter we are only concerned with a nonclearing of the labor market as brought into the academic discussion by the disequilibrium school. We will derive the nonclearing of the labor market from optimizing behavior of economic agents but it will be a multiple stage decision process that will generate the nonclearing of the labor market.⁸

The remainder of this chapter is organized as follows. Section 2 presents the model structure. Section 3 estimates and calibrates our different model variants for the U.S. economy. Section 4 undertakes the same exercise for the German economy. Section 5 concludes. Appendices I and II in this chapter contain some technical derivation of the adaptive optimization procedure whereas Appendix III undertakes a welfare comparison of the different model variants.

⁷See, for example, Woodford (2003, ch. 3). There are also traditional Keynesian models that allow for disequilibria, see Benassy (1984) among others. Yet, the well-known problem of these earlier disequilibrium models was that they disregard intertemporal optimizing behavior and never specify who sets the price. This has now been resolved by the modern literature of monopolistic competition as can be found in Woodford (2003). However, while resolving the price setting problem, the decision with regard to quantities seems to be unresolved. The supplier may no longer behave optimally concerning their supply decision, but simply supplies whatever the quantity the market demands for at the current price.

⁸For models with multiple steps of optimization in the context of learning models, see Dawid and Day (2003), Sargent (1998) and Zhang and Semmler (2003).

8.2 An Economy with Nonclearing Labor Market

We shall still follow the usual assumptions of identical households and identical firms. Therefore we are considering an economy that has two representative agents: the representative household and the representative firm. There are three markets in which the agents exchange their products, labor and capital. The household owns all the factors of production and therefore sells factor services to the firm. The revenue from selling factor services can only be used to buy the goods produced by the firm either for consuming or for accumulating capital. The representative firm owns nothing. It simply hires capital and labor to produce output, sells the output and transfers the profit back to the household.

Unlike the typical RBC model, in which one could assume an once-for-all market, we, however, in this model shall assume that the market to be reopened at the beginning of each period t . This is necessary for a model with nonclearing markets in which adjustments should take place which leads us to a multiple stage adaptive optimization behavior. Yet, let us first describe how prices and wages are set.

8.2.1 The Wage Determination

As usual we presume that both the household and the firm express their desired demand and supply on the basis of given prices, including the output price p_t , the wage rate w_t and the rental rate of the capital stock r_t , we shall first discuss how the period t prices are determined at the beginning of period t . Note that there are three commodities in our model. One of them should serve as a numeraire, which we assume to be the output. Therefore, the output price p_t always equals 1. This indicates that the wage w_t and the rental rate of capital stock r_t are all measured in terms of the physical units of output.⁹ As to the rental rate of capital r_t , it is assumed to be adjustable so as to clear the capital market. We can then ignore its setting. Indeed, as will become clear, one can imagine any initial value of the rental rate of capital when the firm and the household make the quantity decisions and express their desired demand and supply. This leaves us to focus the discussion only on the wage setting. Let us first discuss how the wage rate

⁹For our simple representative agent model without money, this simplification does not affect our major result derived from our model. Meanwhile, it will allow us to save some effort to explain the nominal price determination, a focus in the recent New Keynesian literature.

might be set.

Most recent literature, in discussing wage setting,¹⁰ assumes that it is the supplier of labor, the household or its representative, that sets the wage rate whereas the firm is simply a wage taker. On the other hand, there are also models that discuss how firms set the wage rate.¹¹ In actual bargaining it is likely, as Taylor (1999) has pointed out, that wage setting is an interacting process between firms and households. Despite this variety of wage setting models, we, however, follow the recent approach. We may assume that the wage rate is set by a representative of the household which acts as a monopolistic agent for the supply of labor effort as Woodford (2003, ch. 3) has suggested. Woodford (2003, p.221) introduces different wage setting agents and monopolistic competition since he assumes heterogenous households as different suppliers of differentiated types of labor. In appendix I, in close relationship to Woodford (2003, ch.3), Erceg et al (2000) and Christiano et al. (2001) we present a wage setting model, where wages are set optimally, but a fraction of wages may be sticky. We neglect, however, differentiated types of labor and refer only to aggregate wages.

We want to note, however, that recently many theories have been developed to explain wage and price stickiness. There is the so-called menu cost for changing prices (though this seems more appropriate for the output price). There is also a reputation cost for changing prices and wages.¹² In addition, changing the price, or wage, needs information, computation and communication, which may be costly.¹³ All these efforts cause costs which may be summarized as adjustment costs of changing the price or wage. The adjustment cost for changing the wage may provide some reason for the representative of the household to stick to the wage rate even if it is known that current wage may not be optimal. One may also derive this stickiness of wages from wage contracts as in Taylor (1980) with the contract period to be longer than one period.

Since workers, or their respective representative, enter usually into long term employment contracts involving labor supply for several periods with a variety of job security arrangements and termination options, a wage contract may also be understood from an asset price perspective, namely as derivative security based on a fundamental underlying asset such as the asset price of the firm. In principle a wage contract could be treated as a debt contract with

¹⁰See, for instance, Erceg, Henderson and Levin (2000), Christiano, Eichenbaum and Evans (2001) and Woodford (2003) among others.

¹¹These are basically the efficiency wage models that are mentioned in the introduction.

¹²This is emphasized by Rotemberg (1982)

¹³See the discussion in Christiano, Eichenbaum and Evans (2001) and Zbaracki, Ritson, Levy, Dutta and Bergen (2000).

similar long term commitment as exists for other liabilities of the firm.¹⁴ As in the case of the pricing of corporate liabilities the wage contract, the value of the derivative security, would depend on some specifications in contractual agreements. Yet, in general it can be assumed to be arranged for several periods.

As noted above we do not have to posit that the wage rate, w_t , to be completely fixed in contracts and never responds to the disequilibrium in the labor market. One may imagine that the dynamics of the wage rate, for example, follows the updating scheme as suggested in Calvo's staggered price model (1983) or in Taylor's wage contract model (1980). In Calvo's model, for example, there is always a fraction of individual prices to be adjusted in each period t .¹⁵ This can be expressed in our model as the expiration of some wage contracts, to be reviewed in each time period and therefore new wage contracts will be signed in each t . The new signed wage contracts should respond to the expected market conditions not only in period t but also through t to $t + j$, where j can be regarded as the contract period.¹⁶ Through such a pattern of wage dynamics, wages are only partially adjusted.

Explicit formulation of wage dynamics of a Calvo type of updating scheme, particularly with differentiated types of labor, is studied in Erceg et al (2000), Christiano et al. (2001) and Woodford (2003, ch. 3) and briefly sketched, as underlying our model, for an aggregate wage in appendix I of this chapter. A more explicit treatment is not needed here. Indeed, as will become clear in section 3, the empirical study of our model does not rely on how we formulate the wage dynamics. All we need to presume is that, wage contracts are only partially adjusted, giving rise to a sticky aggregate wage.

8.2.2 The Household's Desired Transactions

The next step in our multiple stage decision process is to model the quantity decisions of the households. When the price, including the wage, has been set, the household is then going to express its desire of demand for goods and supply of factors. We define the household's desired demand and supply as those that can allow the household to obtain the maximum utility on the condition that these demand and supply can be realized at the given set of prices. We can express the household's desired demand and supply as a sequence of output demand and factor supply $\{c_{t+i}^d, i_{t+i}^d, n_{t+i}^s, k_{t+i+1}^s\}_{i=0}^{\infty}$,

¹⁴For such a treatment of the wages as derivative security, see Uhlig (2003). For further details of the pricing of such liabilities, see Grüne and Semmler (2004c).

¹⁵These are basically those prices that have not been adjusted for some periods and there the adjustment costs (such as the reputation cost) may not be high.

¹⁶This type of wage setting is used in Woodford (2003, ch. 4) and Erceg et al. (2000).

where i_{t+i} is referred to investment. Note that here we have used the superscripts d and s to refer to the agent's desired demand and supply. The decision problem for the household to derive its demand and supply can be formulated as

$$\max_{\{c_{t+i}, n_{t+i}\}_{i=0}^{\infty}} E_t \left[\sum_{i=0}^{\infty} \beta^i U(c_{t+i}^d, n_{t+i}^s) \right] \quad (8.1)$$

subject to

$$c_{t+i}^d + i_{t+i}^d = r_{t+i} k_{t+i}^s + w_{t+i} n_{t+i}^s + \pi_{t+i} \quad (8.2)$$

$$k_{t+i+1}^s = (1 - \delta) k_{t+i}^s + i_{t+i}^d \quad (8.3)$$

Above π_{t+i} is the expected dividend. Note that (8.2) can be regarded as a budget constraint. The equality holds due to the assumption $U_c > 0$. Next, we shall consider how the representative household calculates π_{t+i} .

Assuming that the household knows the production function $f(\cdot)$ while it expects that all its optimal plans can be fulfilled at the given price sequence $\{p_{t+i}, w_{t+i}, r_{t+i}\}_{i=0}^{\infty}$, we thus obtain

$$\pi_{t+i} = f(k_{t+i}^s, n_{t+i}^s, A_{t+i}) - w_{t+i} n_{t+i}^s - r_{t+i} k_{t+i}^s \quad (8.4)$$

Explaining π_{t+i} in (8.2) in terms of (8.4) and then substituting from (8.3) to eliminate i_t^d , we obtain

$$k_{t+i+1}^s = (1 - \delta) k_{t+i}^s + f(k_{t+i}^s, n_{t+i}^s, A_{t+i}) - c_{t+i}^d \quad (8.5)$$

For the given technology sequence $\{A_{t+i}\}_{i=0}^{\infty}$, equations (8.1) and (8.5) form a standard intertemporal decision problem. The solution to this problem can be written as:

$$c_{t+i}^d = G_c(k_{t+i}^s, A_{t+i}) \quad (8.6)$$

$$n_{t+i}^s = G_n(k_{t+i}^s, A_{t+i}) \quad (8.7)$$

We shall remark that although the solution appears to be a sequence $\{c_{t+i}^d, n_{t+i}^s\}_{i=0}^{\infty}$ only (c_t^d, n_t^s) along with (i_t^d, k_t^s) , where $i_t^d = f(k_t^s, n_t^s, A_t) - c_t^d$ and $k_t^s = k_t$, are actually carried into the market by the household for exchange due to our assumption of re-opening of the market.

8.2.3 The Firm's Desired Transactions

As in the case of the household, the firm's desired demand for factors and supply of goods are those that maximize the firm's profit under the condition that all its intentions can be carried out at the given set of prices. The

optimization problem for the firm can thus be expressed as being to choose the input demands and output supply (n_t^d, k_t^d, y_t^s) that maximizes the current profit:

$$\max y_t^s - r_t k_t^d - w_t n_t^d$$

subject to

$$y_t^s = f(A_t, k_t^d, n_t^d) \quad (8.8)$$

For regular conditions on the production function $f(\cdot)$, the solution to the above optimization problem should satisfy

$$r_t = f_k(k_t^d, n_t^d, A_t) \quad (8.9)$$

$$w_t = f_n(k_t^d, n_t^d, A_t) \quad (8.10)$$

where $f_k(\cdot)$ and $f_n(\cdot)$ are respectively the marginal products of capital and labor. Next we shall consider the transactions in our three markets. Let us first consider the two factor markets.

8.2.4 Transaction in the Factor Market and Actual Employment

We have assumed the rental rate of capital r_t to be adjustable in each period and thus the capital market is cleared. This indicates that

$$k_t = k_t^s = k_t^d$$

As concerning the labor market, there is no reason to believe that firm's demand for labor, as implicitly expressed in (8.10) should be equal to the willingness of the household to supply labor as determined in (8.7) given the way the wage determination is explained in section 8.2.1. Therefore, we cannot regard the labor market to be cleared. An illustration of this statement, though in a simpler version, is given in Appendix I.¹⁷

Given a nonclearing labor market, we shall have to specify what rule should apply regarding the realization of actual employment.

¹⁷Strictly speaking, the so-called labor market clearing should be defined as the condition that the firm's willingness to demand factors is equal to the household's willingness to supply factors. Such concept has somehow disappeared in the new Keynesian literature in which the household supplies the labor effort according to the market demand and therefore it does not seem to face excess demand or supply. Yet, even in this case, the household's willingness to supply labor effort is not necessarily equal to its actual supply or the market demand. At some point the marginal disutility of work may be higher than the pre-set wage. This indicates that even if there are no adjustment costs so that the household can adjust the wage rate at every time period t , the disequilibrium in the labor market may still exist. In Appendix I these points are illustrated in a static version of the working of the labor market.

Disequilibrium Rule: When disequilibrium occurs in the labor market either of the following two rules will be applied:

$$n_t = \min(n_t^d, n_t^s) \quad (8.11)$$

$$n_t = \omega n_t^d + (1 - \omega)n_t^s \quad (8.12)$$

where $\omega \in (0, 1)$.

Above, the first is the famous short-side rule when nonclearing of the market occurs. It has been widely used in the literature on disequilibrium analysis (see, for instance, Benassy 1975, 1984, among others). The second might be called the compromising rule. This rule indicates that when nonclearing of the labor market occurs both firms and workers have to compromise. If there is excess supply, firms will employ more labor than what they wish to employ.¹⁸ On the other hand, when there is excess demand, workers will have to offer more effort than they wish to offer.¹⁹ Such mutual compromises may be due to institutional structures and moral standards of the society.²⁰ Given the rather corporate relationship of labor and firms in Germany, for example, this compromising rule might be considered a reasonable approximation. Such a rule that seems to hold for many other countries was already discussed early in the economic literature, see Meyers (1968) and also Solow (1979).

We want to note that the unemployment we discuss here is certainly different from the frictional unemployment as often discussed in search and matching models. In our representative agent model, the unemployment is mainly due to adaptive optimization of the household given the institutional

¹⁸This could also be realized by firms by demanding the same (or less) hours per worker but employing more workers than being optimal. This case corresponds to what is discussed in the literature as labor hoarding where firms hesitate to fire workers during a recession because it may be hard to find new workers in the next upswing, see Burnside et al. (1993). Note that in this case firms may be off their marginal product curve and thus this might require wage subsidies for firms as has been suggested by Phelps (1997).

¹⁹This could be achieved by employing the same number of workers but each worker supplying more hours (varying shift length and overtime work); for a more formal treatment of this point, see Burnside et al. (1993).

²⁰Note that if firms are off their supply schedule and workers off their demand schedule, a proper study would have to compute the firms' cost increase and profit loss and the workers' welfare loss. If, however, the marginal cost for firms is rather flat (as empirical literature has argued, see Blanchard and Fischer, 1989) and the marginal disutility is also rather flat the overall loss may not be so high. The departure of the value function – as measuring the welfare of the representative household from the standard case – is studied in Gong and Semmler (2001). Results of this study are reported in Appendix III of this chapter.

arrangements of the wage setting (see Chapter 8.5.). The cause for frictional unemployment can arise from informational and institutional search and matching frictions where welfare state and labor market institutions may play a role.²¹ Yet the frictions in the institutions of the matching process are likely to explain only a certain fraction of observed unemployment.²²

8.2.5 Actual Employment and Transaction in the Product Market

After the transactions in these two factor markets have been carried out, the firm will engage in its production activity. The result is the output supply, which, instead of (8.8), is now given by

$$y_t^s = f(k_t, n_t, A_t). \quad (8.13)$$

Then the transaction needs to be carried out with respect to y_t^s . It is important to note that when the labor market is not cleared, the previous consumption plan as expressed by (8.6) becomes invalid due to the improper budget constraint (8.2), which further bring the improper transition law of capital (8.5), for deriving the plan. Therefore, the household will be required to construct a new consumption plan, which should be derived from the following optimization program:

$$\max_{(c_t^d)} U(c_t^d, n_t) + E_t \left[\sum_{i=1}^{\infty} \beta^i U(c_{t+i}^d, n_{t+i}^s) \right] \quad (8.14)$$

subject to

$$k_{t+1}^s = (1 - \delta)k_t + f(k_t, n_t, A_t) - c_t^d \quad (8.15)$$

$$k_{t+i+1}^s = (1 - \delta)k_{t+i}^s + f(k_{t+i}^s, n_{t+i}^s, A_{t+i}) - c_{t+i}^d \quad (8.16)$$

$$i = 1, 2, \dots$$

Note that in this optimization program the only decision variable is about c_t^d and the data includes not only A_t and k_t but also n_t , which is given by

²¹For a recent position representing this view, see Ljungqvist and Sargent (1998, 2003). For comments on this view, see Blanchard (2003), see also Walsh (2002) who employs search and matching theory to derive the persistence of real effects resulting from monetary policy shocks.

²²Already Hicks (1963) has called this frictional unemployment. Recently, one important form of a mismatch in the labor market seems to be the mismatch of skills, see Greiner, Rubart and Semmler (2003).

either (8.11) or (8.12). We can write the solution in terms of the following equation (see Appendix II of this chapter for the detail):

$$c_t^d = G_{c2}(k_t, A_t, n_t) \quad (8.17)$$

Given this adjusted consumption plan, the product market should be cleared if the household demand $f(k_t, n_t, A_t) - c_t^d$ for investment. Therefore, c_t^d in (8.17) should also be the realized consumption.

8.3 Estimation and Calibration for U. S. Economy

This section provides an empirical study, for the U. S. economy, of our model as presented in the last section. However, the model in the last section is only for illustrative purpose. It is not the model that can be tested with empirical data, not only because we do not specify the forms of production function, utility function and the stochastic process of A_t , but also we do not introduce the growth factor into the model. For an empirically testable model, we here still employ the model as formulated by King, Plosser and Rebelo (1988).

8.3.1 The Empirically Testable Model

Let K_t denote for capital stock, N_t for per capita working hours, Y_t for output and C_t for consumption. Assume that the capital stock in the economy follow the transition law:

$$K_{t+1} = (1 - \delta)K_t + A_t K_t^{1-\alpha} (N_t X_t)^\alpha - C_t, \quad (8.18)$$

where δ is the depreciation rate; α is the share of labor in the production function $F(\cdot) = A_t K_t^{1-\alpha} (N_t X_t)^\alpha$; A_t is the temporary shock in technology and X_t the permanent shock that follows a growth rate γ .²³ The model is nonstationary due to X_t . To transform the model into a stationary setting, we divide both sides of equation (8.18) by X_t :

$$k_{t+1} = \frac{1}{1 + \gamma} \left[(1 - \delta)k_t + A_t k_t^{1-\alpha} (n_t \bar{N}/0.3)^\alpha - c_t \right], \quad (8.19)$$

where $k_t \equiv K_t/X_t$, $c_t \equiv C_t/X_t$ and $n_t \equiv 0.3N_t/\bar{N}$ with \bar{N} to be the sample mean of N_t . Note that n_t is often regarded to be the normalized hours. The

²³Note that X_t includes both population and productivity growth.

sample mean of n_t is equal to 30 %, which, as pointed out by Hansen (1985), is the average percentage of hours attributed to work. Note that the above formulation also indicates that the form of $f(\cdot)$ in the previous section may follow

$$f(\cdot) = A_t k_t^{1-\alpha} (n_t \bar{N} / 0.3)^\alpha \quad (8.20)$$

while $y_t \equiv Y_t / X_t$ with Y_t to be the empirical output.

With regard to the household preference, we shall assume that the utility function takes the form

$$U(c_t, n_t) = \log c_t + \theta \log(1 - n_t) \quad (8.21)$$

The temporary shock A_t may follow an AR(1) process:

$$A_{t+1} = a_0 + a_1 A_t + \epsilon_{t+1}, \quad (8.22)$$

where ϵ_t is an independently and identically distributed (*i.i.d.*) innovation: $\epsilon_t \sim N(0, \sigma_\epsilon^2)$.

8.3.2 The Data Generating Process

For our empirical test, we consider three model variants: the standard RBC model, as a benchmark for comparison, and the two labor market disequilibrium models with the disequilibrium rules as expressed in (8.11) and (8.12) respectively. Specifically, we shall call the standard model the Model I; the disequilibrium model with short side rule (8.11) the Model II; and the disequilibrium model with the compromising rule (8.12) the Model III.

For the standard RBC model, the data generating process include (8.19), (8.22) as well as

$$c_t = G_{11} A_t + G_{12} k_t + g_1 \quad (8.23)$$

$$n_t = G_{21} A_t + G_{22} k_t + g_2 \quad (8.24)$$

Note that here (8.23) and (8.24) are the linear approximations to (8.6) and (8.7) when we ignore the superscripts s and d . The coefficients G_{ij} and g_i ($i = 1, 2$ and $j = 1, 2$) are the complicated functions of the model's structural parameters, α , β , among others. They are computed as in Chapter 5 by the numerical algorithm using the linear-quadratic approximation method presented in Chapter 1 and 2. Given these coefficients and the parameters in equation (8.22), including σ_ϵ , we can simulate the model to generate stochastically simulated data. These data can then be compared to the sample moments of the observed economy.

Obviously, the standard model does not allow for nonclearing of the labor market. The moments of the labor effort are solely reflected by the decision rule (8.24) which is quite similar in its structure to the other decision rule given by (8.23), i.e., they are both determined by k_t and A_t . This structural similarity are expected to produce two labor market puzzles as aforementioned:

- First, the volatility of the labor effort can not be much different from the volatility of consumption, which generally appears to be smooth.
- Second, the moments of labor effort and consumption are likely to be strongly correlated.

To define the data generating process for our disequilibrium models, we shall first modify (8.24) as

$$n_t^s = G_{21}A_t + G_{22}k_t + g_2 \quad (8.25)$$

On the other hand, the equilibrium in the product market indicates that c_t^d in (8.17) should be equal to c_t . Therefore, this equation can also be approximated as

$$c_t = G_{31}A_t + G_{32}k_t + G_{33}n_t + g_3 \quad (8.26)$$

In the appendix, we provide the details how to compute the coefficients G_{3j} , $j = 1, 2, 3$, and g_3 .

Next we consider the labor demand derived from the production function $F(\cdot) = A_t K_t^{1-\alpha} (N_t X_t)^\alpha$. Let $X_t = Z_t L_t$, with Z_t to be the permanent shock resulting purely from productivity growth, and L_t from population growth. We shall assume that L_t has a constant growth rate μ and hence Z_t follows the growth rate $(\gamma - \mu)$. The production function can be written as $Y_t = A_t Z_t^\alpha K_t^{1-\alpha} H_t^\alpha$, where H_t equals $N_t L_t$, which can be regarded as total labor hours. Taking the partial derivative with respect to H_t and recognizing that the marginal product of labor is equal to the real wage, we thus obtain

$$w_t = \alpha A_t Z_t k_t^{1-\alpha} (n_t^d \bar{N} / 0.3)^{\alpha-1}$$

This equation is equivalent to (8.10). It generates the demand for labor as

$$n_t^d = (\alpha A_t Z_t / w_t)^{1/(1-\alpha)} k_t (0.3 / \bar{N}). \quad (8.27)$$

Note that the per capita hours demanded n_t^d should be stationary if the real wage w_t and productivity Z_t grow at the same rate. This seems to be roughly consistent with the U.S. experience that we shall now calibrate.

Thus, for the nonclearing market model with short side rule, Model II, the data generating process includes (8.19), (8.22), (8.11), (8.25), (8.26) and (8.27) with w_t given by the observed wage rate. We thereby do not attempt to give the actually observed sequence of wages a further theoretical foundation.²⁴ For our purpose it suffices to take the empirically observed series of wages. For Model III, we use (8.12) instead of (8.11).

8.3.3 The Data and the Parameters

Before we calibrate the models we shall first specify the parameters. There are altogether 10 parameters in our three variants: $a_0, a_1, \sigma_\varepsilon, \gamma, \mu, \alpha, \beta, \delta, \theta$, and ω . We first specify α and γ respectively at 0.58 and 0.0045, which are standard. This allows us to compute the data series of the temporary shock A_t . With this data series, we estimate the parameters a_0, a_1 and σ_ε . The next three parameters β, δ and θ are estimated with the GMM method by matching the moments of the model generated by (8.19), (8.23) and (8.25). The estimation is conducted by a global optimization algorithm called simulated annealing. These parameters have already been estimated in Chapter 5, and therefore we shall employ them here. For the new parameters, we specify μ at 0.001, which is close to the average growth rate of the labor force in U.S.; the parameter ω in Model III is set to 0.1203. It is estimated by minimizing the residual sum of square between actual employment and the model generated employment. The estimation is executed by a conventional algorithm, the grid search. Table 8.1 illustrates these parameters:

Table 8.1: Parameters Used for Calibration

a_0	0.0333	σ_ε	0.0185	μ	0.0010	β	0.9930	θ	2.0189
a_1	0.9811	γ	0.0045	α	0.5800	δ	0.2080	ω	0.1203

The data set used in this section is taken from Christiano (1987). The wage series are obtained from Citibase. It is re-scaled to match the model's implication.²⁵

²⁴One however might apply here the efficiency wage theory or other theories such as the staggered contract theory that justify the wage stickiness.

²⁵Note that this re-scaling is necessary because we do not exactly know the initial condition of Z_t , which we set equal to 1. We re-scaled the wage series in such a way that the first observation of employment is equal to the demand for labor as specified by equation (8.27).

8.3.4 Calibration

Table 8.2 reports our calibration from 5000 stochastic simulations. The results in this table are confirmed by Figure 8.1, where a one time simulation with the observed innovation A_t are presented.²⁶ All time series are detrended by the HP-filter.

²⁶Due to the discussion on Solow residual in Chapter 5, we shall now understand that A_t computed as the Solow residual may reflect also the demand shock in addition to the technology shock.

Table 8.2: Calibration of the Model Variants: U.S. Economy (numbers in parentheses are the corresponding standard errors)

	Consumption	Capital	Employment	Output
Standard Deviations				
Sample Economy	0.0081	0.0035	0.0165	0.0156
Model I Economy	0.0091 (0.0012)	0.0036 (0.0007)	0.0051 (0.0006)	0.0158 (0.0021)
Model II Economy	0.0137 (0.0098)	0.0095 (0.0031)	0.0545 (0.0198)	0.0393 (0.0115)
Model III Economy	0.0066 (0.0010)	0.0052 (0.0010)	0.0135 (0.0020)	0.0197 (0.0026)
Correlation Coefficients				
Sample Economy				
Consumption	1.0000			
Capital Stock	0.1741	1.0000		
Employment	0.4604	0.2861	1.0000	
Output	0.7550	0.0954	0.7263	1.0000
Model I Economy				
Consumption	1.0000 (0.0000)			
Capital Stock	0.2043 (0.1190)	1.0000 (0.0000)		
Employment	0.9288 (0.0203)	-0.1593 (0.0906)	1.0000 (0.0000)	
Output	0.9866 (0.00332)	0.0566 (0.1044)	0.9754 (0.0076)	1.0000 (0.0000)
Model II Economy				
Consumption	1.0000 (0.0000)			
Capital Stock	0.4944 (0.1662)	1.0000 (0.0000)		
Employment	0.4874 (0.1362)	-0.0577 (0.0825)	1.0000 (0.0000)	
Output	0.6869 (0.1069)	0.0336 (0.0717)	0.9392 (0.0407)	1.0000 (0.0000)
Model III Economy				
Consumption	1.0000 (0.0000)			
Capital Stock	0.4525 (0.1175)	1.0000 (0.0000)		
Employment	0.6807 (0.0824)	-0.0863 (0.1045)	1.0000 (0.0000)	
Output	0.8924 (0.0268)	0.0576 (0.0971)	0.9056 (0.0327)	1.0000 (0.0000)

First we want to remark that the structural parameters that we used here for calibration are estimated by matching the Model I Economy to the Sample Economy. The result, reflected in Table 8.2, is therefore somewhat biased in favor of the Model I Economy. It is not surprising that for most variables the moments generated from the Model I Economy are closer to the moments of the Sample Economy. Yet even in this case, there is an excessive smoothness of the labor effort and the employment series of the data cannot be matched. For our time period, 1955.1 to 1983.4, we find 0.32 in the Model I Economy as the ratio of the standard deviation of labor effort to the standard deviation of output. This ratio is roughly 1 in the Sample Economy. The problem is, however, resolved in our Model II and Model III Economies representing sticky wages and labor market nonclearing. There the ratio is 1.38 and 0.69 for the Model II and Model III Economies respectively.

Further evidence on the better fit of the nonclearing labor market models – as concerns the volatility of the macroeconomic variables – is also demonstrated in the Figure 8.1 where the horizontal figures show, from top to bottom, actual (solid line) and simulated data (dotted line) for consumption, capital stock, employment and output, the three columns representing the figures for Model I, Model II and Model III Economies. As observable, in particular the Model III Economy fits, along most dimensions, best the actual data. As can be seen from the separate figures, the volatility of employment has been greatly increased for both Model II and Model III. In particular, the volatility in the Model III Economy is close to the one in the Sample Economy, although too high a volatility is observable in the Model II Economy which may reflect our assumption that there are no search and matching frictions (which, of course, in the actual economy will not hold). We therefore may conclude that Model III is the best in matching the labor market volatility.

We want to note that the failure of the standard model to match the volatility of employment of the data is also described in the recent paper by Schmidt-Grohe (2001). For her employed time series data 1948.3 - 1997.4, Schmidt-Grohe (2001) finds that the ratio of the standard deviation of employment to the standard deviation of output is roughly 0.95, close to our Sample Economy. Yet for the standard RBC model, the ratio is found to be 0.49, which is too low compared to the empirical data. For the indeterminacy model, originating in the work by Benhabib and co-authors, she finds the ratio to be 1.45, which seems too high. As noted above, a similarly high ratio of standard deviations can also be observed in our Model II Economy where the short side rule leads to excessive fluctuations of the labor effort.

Next, let us look at the cross-correlations of the macroeconomic variables. In the Sample Economy, there are two significant correlations we can observe:

the correlation between consumption and output, roughly 0.75, and between employment and output, about 0.72. These two strong correlations can also be found in all of our simulated economies. However, in our Model I Economy and this only holds for the Model I Economy (the standard RBC model) in addition to these two correlations, consumption and employment are, with 0.93, also strongly correlated. Yet, empirically, this correlation is weak, about 0.46.

The latter result of the standard model is not surprising given that movements of employment as well as consumption reflect the movements in the state variables capital stock and the temporary shock. They, therefore, should be somewhat correlated. We remark here that such an excessive correlation has, to our knowledge, not explicitly been discussed in the RBC literature, including the recent study by Schmidt-Grohe (2001). Discussions have often focused on the correlation with output.

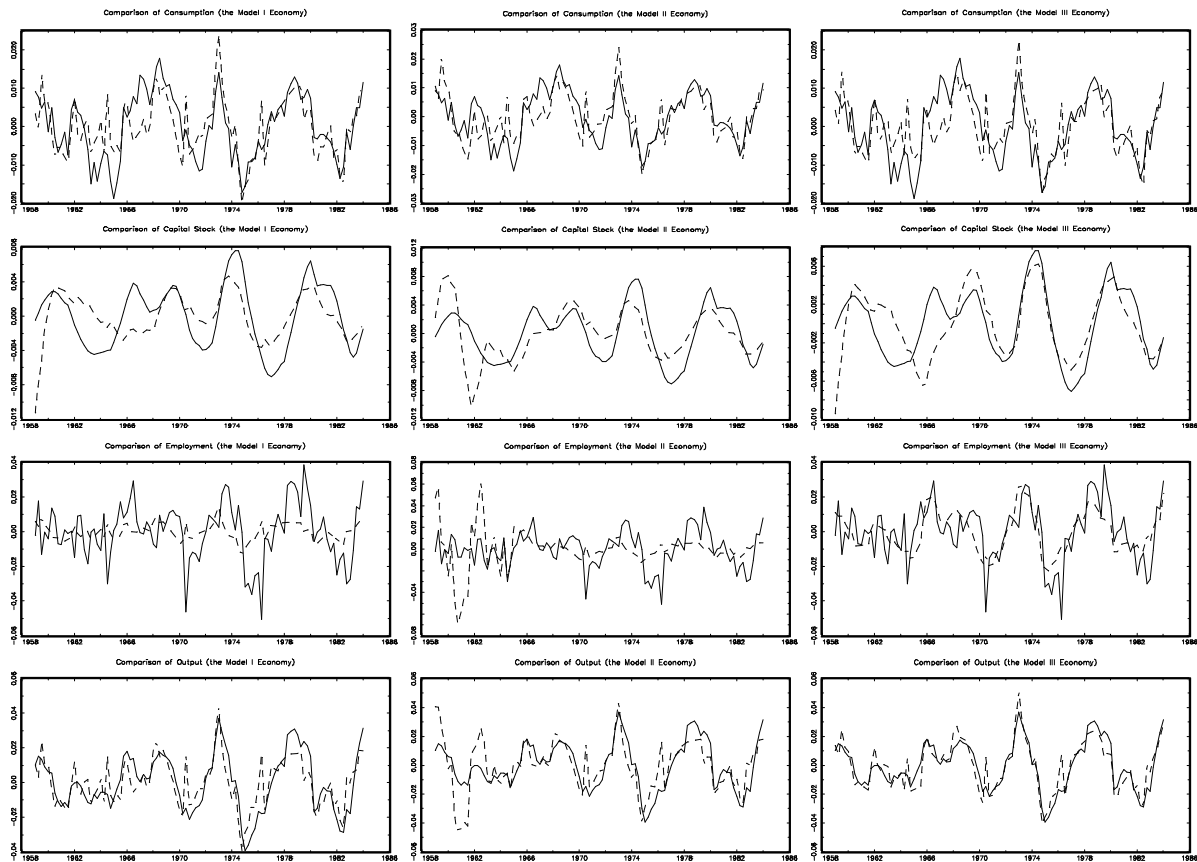


Figure 8.1: Simulated Economy versus Sample Economy: U.S. Case (solid line for sample economy, dotted line for simulated economy)

A success of our nonclearing labor market models, see the Model II and III Economies, is that employment is no longer significantly correlated with consumption. This is because we have made a distinction between the demand and supply of labor, whereas only the latter, labor supply, reflects the moments of capital and technology as consumption does. Since the realized employment is not necessarily the same as the labor supply, the correlation with consumption is therefore weakened.

8.4 Estimation and Calibration for the German Economy

Above we have employed a model with nonclearing labor market for the U. S. economy. We have seen that one of the major reasons that the standard model can not appropriately replicate the variation in employment is its lack of introducing the demand for labor. Next, we pursue a similar study of German economy. For this purpose we shall first summarize some stylized facts on the German economy compared to the U.S. economy.

8.4.1 The Data

Our subsequent study of the German economy employs the time series data from 1960.1 to 1992.1. We thus have included a short period after the unification of Germany (1990 - 1991). We use again quarterly data. The time series data on GDP, consumption, investment and capital stock are OECD data, see OECD (1998a), the data on total labor force is also from the OECD (1998b). The time series data on total working hours is taken from Statistisches Bundesamt (1998). The time series on the hourly real wage index is from OECD (1998a).

8.4.2 The Stylized Facts

Next, we want to compare some stylized facts. Figures 8.2 and 8.3 compare 6 key variables relevant for the models for both the German and U.S. economies. In particular, the data in Figure 8.3 are detrended by the HP-filter. The standard deviations of the detrended series are summarized in Table 8.3.

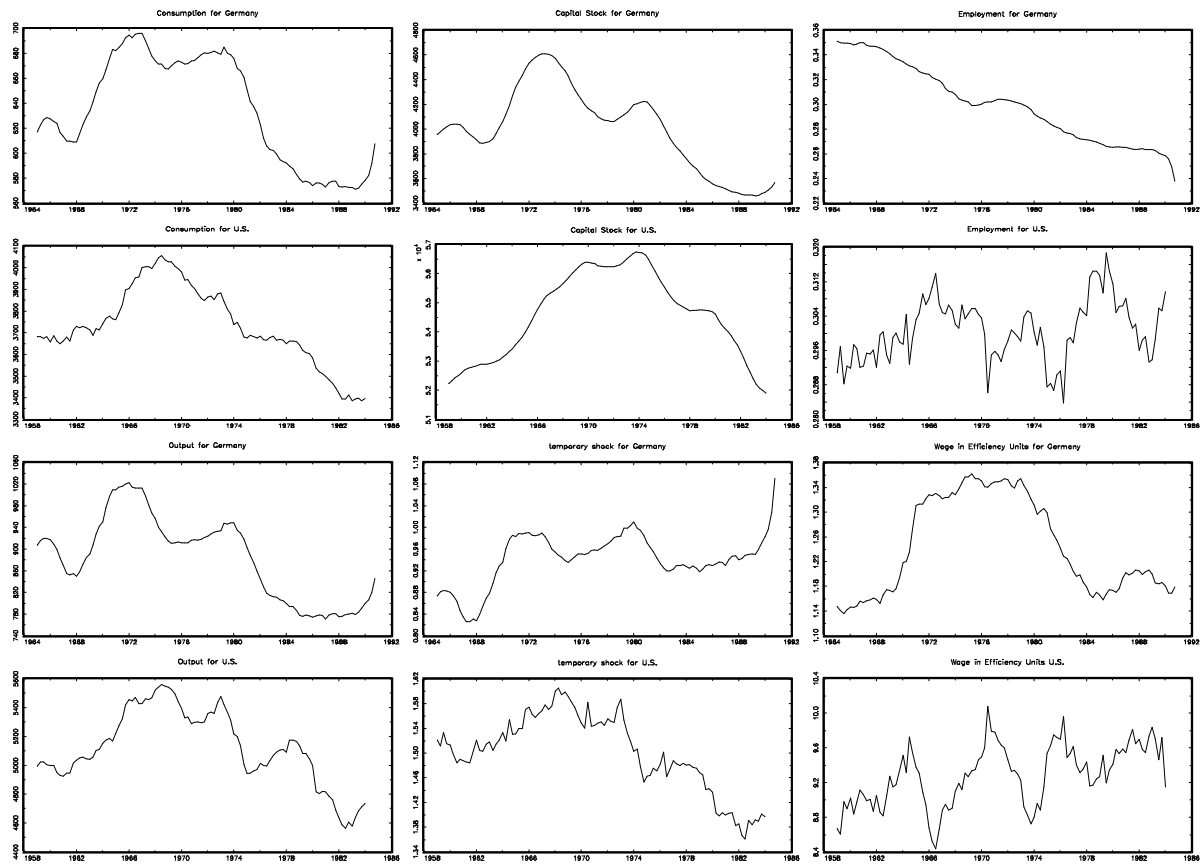


Figure 8.2: Comparison of Macroeconomic Variables U. S. versus Germany

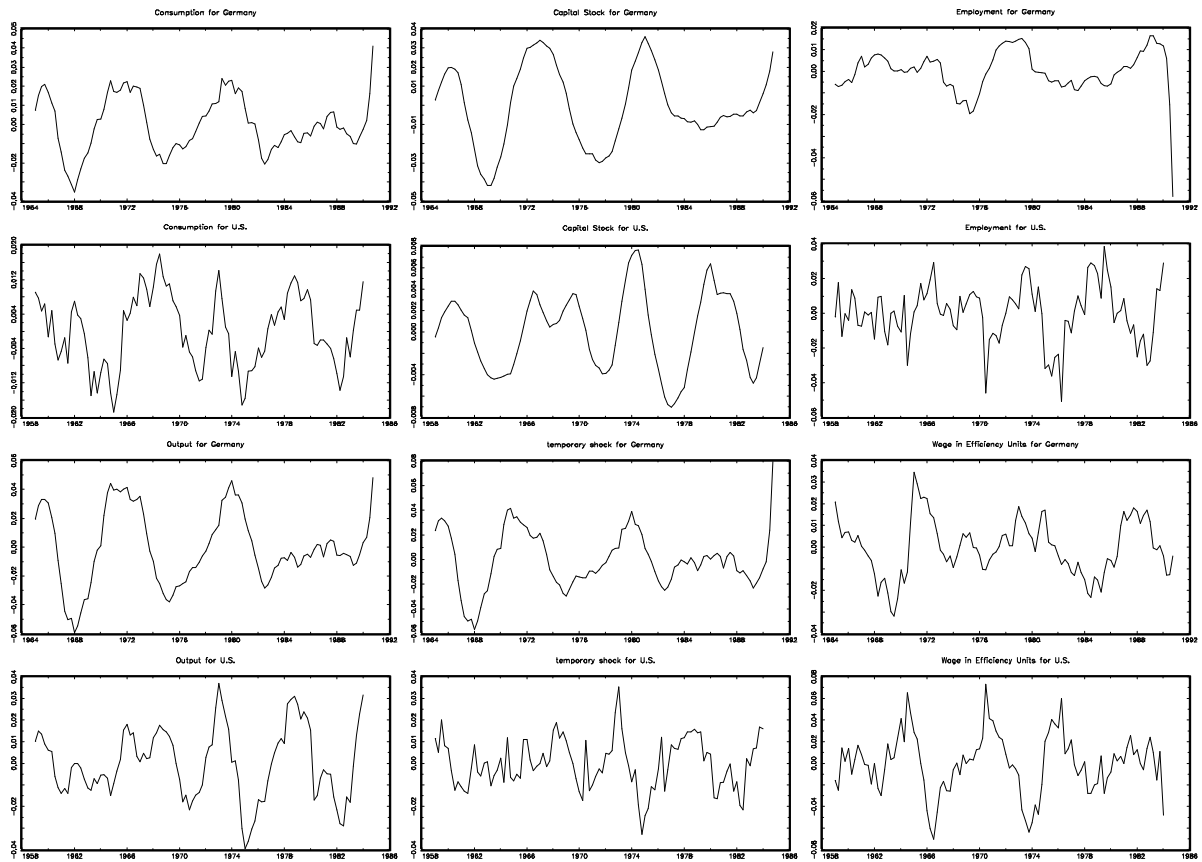


Figure 8.3: Comparison of Macroeconomic Variables: U. S. versus Germany (data series are detrended by the HP-filter)

Table 8.3: The Standard Deviations (U.S. versus Germany)

	Germany (detrended)	U.S. (detrended)
consumption	0.0146	0.0084
capital stock	0.0203	0.0036
employment	0.0100	0.0166
output	0.0258	0.0164
temporary shock	0.0230	0.0115
efficiency wage	0.0129	0.0273

Several remarks are at place here. First, employment and the efficiency wage are among the variables with the highest volatility in the U. S. economy. However, in the German economy they are the smoothest variables. Second, the employment (measured in terms of per capita hours) is declining over time in Germany (see Figure 8.2 for the non-detrended series), while in the U.S. economy, the series is approximately stationary. Third, in the U. S. economy, the capital stock and temporary shock to technology are both relatively smooth. In contrast, they are both more volatile in Germany. These results might be due to our first remark regarding the difference in employment volatility. The volatility of output must be absorbed by some factors in the production function. If employment is smooth, the other two factors have to be volatile.

Should we expect that such differences will lead to different calibration of our model variants? This will be explored next.

8.4.3 The Parameters

For the German economy, our investigation showed that an AR(1) process does not match well the observed process of A_t . Instead, we shall use an AR(2) process:

$$A_{t+1} = a_0 + a_1 A_t + a_2 A_{t-1} + \varepsilon_{t+1}$$

The parameters used for calibration are given in Table 8.4. All of these parameters are estimated in the same way as those for the U.S. economy.

Table 8.4: Parameters used for Calibration (German Economy)

a_0	0.0044	γ	0.0083	δ	0.0538
a_1	1.8880	μ	0.0019	θ	2.1507
a_2	-0.8920	α	0.6600	ω	0
σ_ε	0.0071	β	0.9876		

It is important to note that the estimated ω in this case is on the boundary 0, indicating the weight of the demand is zero in the compromising rule (8.12). In other words, the Model III Economy is almost identical to the Model I Economy. This seems to provide us with the conjecture that the Model I Economy, the standard model, will be the best in matching German labor market.

8.4.4 Calibration

As for the U.S. economy we provide in Table 8.5 for the German economy the calibration result from 5000 time stochastic simulations. In Figure 8.4 we again compare the one-time simulation with the observed A_t for our model variants. Again all time series here are detrended by the HP-filter. ²⁷

²⁷Note that we do not include the Model III Economy for calibration. Due to the zero value of the weighting parameter ω , the Model III Economy is equivalent to the Model I Economy.

Table 8.5: Calibration of the Model Variants: German Economy (number in parentheses are the corresponding standard errors)

	Consumption	Capital	Employment	Output
Standard Deviations				
Sample Economy	0.0146	0.0203	0.0100	0.0258
Model I Economy	0.0292 (0.0106)	0.0241 (0.0066)	0.0107 (0.0023)	0.0397 (0.0112)
Model II Economy	0.1276 (0.1533)	0.0425 (0.0238)	0.0865 (0.1519)	0.4648 (0.9002)
Correlation Coefficients				
Sample Economy				
Consumption	1.0000			
Capital Stock	0.4360	1.0000		
Employment	0.0039	-0.3002	1.0000	
Output	0.9692	0.5423	0.0202	1.0000
Model I Economy				
Consumption	1.0000 (0.0000)			
Capital Stock	0.7208 (0.0920)	1.0000 (0.0000)		
Employment	0.5138 (0.1640)	-0.1842 (0.1309)	1.0000 (0.0000)	
Output	0.9473 (0.0200)	0.4855 (0.1099)	0.7496 (0.1028)	1.0000 (0.0000)
Model II Economy				
Consumption	1.0000 (0.0000)			
Capital Stock	0.6907 (0.1461)	1.0000 (0.0000)		
Employment	0.7147 (0.2319)	0.3486 (0.4561)	1.0000 (0.0000)	
Output	0.8935 (0.1047)	0.5420 (0.2362)	0.9130 (0.1312)	1.0000 (0.00000)

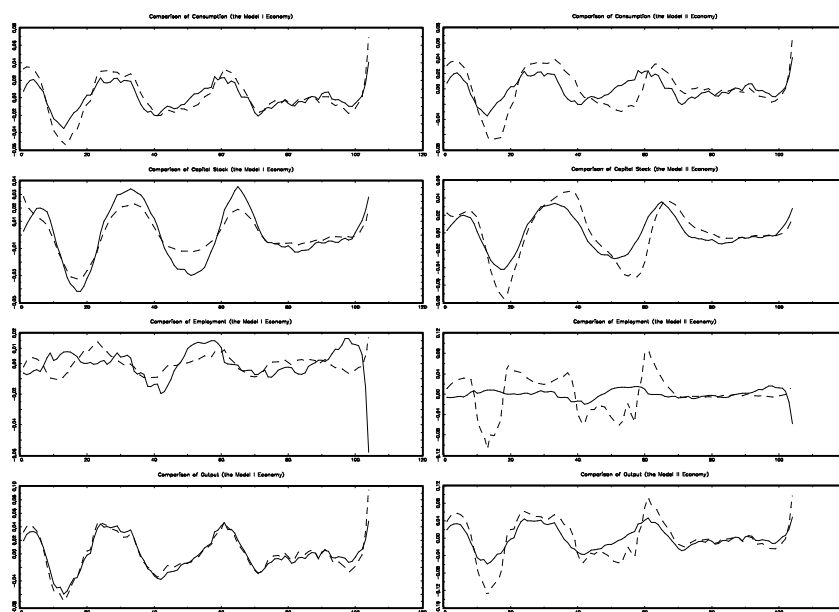


Figure 8.4: Simulated Economy versus Sample Economy: German Case (solid line for sample economy, dotted line for simulated economy)

In contrast to U.S. economy we find some major differences. First, there is a difference concerning the variation of employment. The standard problem of excessive smoothness with respect to employment in the benchmark model no longer holds for the German economy. This is likely to be due to the fact that employment itself is smooth in the German economy (see Table 8.3 and Figure 8.3). We shall also note that the simulated labor supply in Germany is smoother than in the U. S. (see Figure 8.5). In most labor market studies the German labor market is often considered less flexible than the U. S. labor market. In particular, there are stronger influences of labor unions and various legal restrictions on firms' hiring and firing decisions.²⁸ Such influences and legal restriction will give rise to the smoother employment series in contrast to the U. S. Such influences and legal restriction, or what Solow (1979) has termed the moral factor in the labor market, may also be viewed as a readiness to compromise as our Model III suggests. Those factors

²⁸See, for example, Nickell (1997) and Nickell (2003), and see already Meyers (1964).

will indeed give rise to a smooth employment series.

Further, if we look at the labor demand and supply in Figure 8.5, the supply of labor is mostly the short side in the Germany economy whereas in U.S. economy demand is dominating in most periods. Note that here we must distinguish the supply that is actually provided in the labor market and the “supply” that is specified by the decision rule in the standard model. It might reasonably be argued that due to the intertemporal optimization subject to the budget constraints the supply specified by the decision rule may only approximate the decisions from those households for which unemployment is not expected to pose a problem on their budgets. Such households are more likely to be currently employed and protected by labor unions and legal restrictions. In other words, currently employed labor decides, through the optimal decision rule, about labor supply and not those who are currently unemployed. Such a shortcoming of single representative intertemporal decision model could presumably be overcome by a intertemporal model with heterogenous households.²⁹

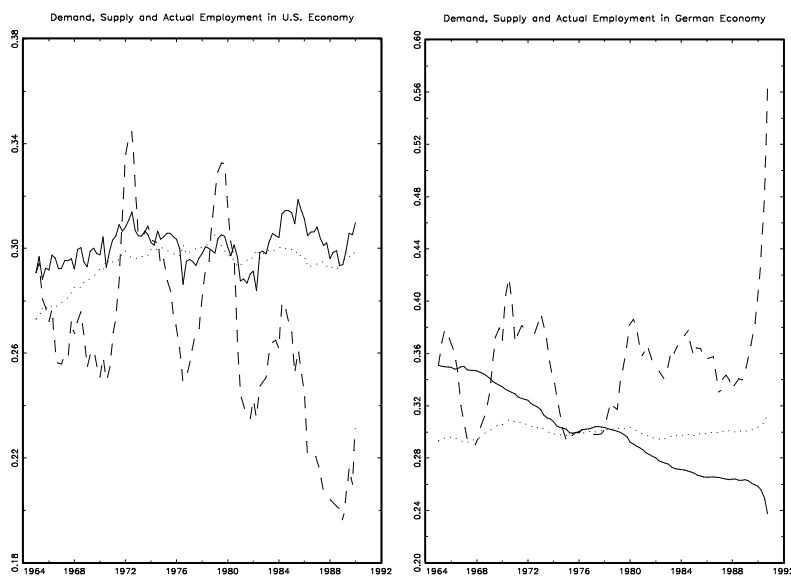


Figure 8.5: Comparison of demand and supply in the labor market (solid line for actual, dashed line for demand and dotted line for supply)

²⁹See, for example, Uhlig and Xu (1996).

The second difference concerns the trend in employment growth and unemployment of the U.S. and Germany. So far we only have shown that our model of nonclearing labor market seems to match better than the standard RBC model the variation in employment. This in particular seems to be true for the U.S. economy. We did not attempt to explain the trend of the unemployment rate neither for the U.S. nor for Germany. We want to note that the time series data (U. S. 1955.1 - 1983.1, Germany 1960.1 - 1992.1) are from a period where the U.S. had higher – but falling – unemployment rates, whereas Germany had still lower but rising unemployment rates. Yet, since the end of the 1980s the level of the unemployment rate in Germany has considerably moved up, partly due to the unification of Germany after 1989.

8.5 Differences in Labor Market Institutions

In Chapter 8.2 we have introduced rules that might be thought to be operative when there is a nonclearing labor market. In this respect, as our calibration in section 3 has shown, the most promising route to model, and to match, stylized facts of the labor market, through a microbased labor market behavior, is the compromising model. One hereby may pay attention to some institutional characteristics of the labor market presumed in our model.

The first is the way how the agency representing the household sets the wage rate. If the household sets the wage rate, as if it were monopolistic competitor, then at this wage rate the household's willingness to supply labor is likely to be less than the market demand for labor unless the household sufficiently under-estimates the market demand when it conducts its optimization for wage setting. Such a way of wage setting may imply unemployment and it is likely to be the institutional structure that gives the representative household (or the representative of the household, such as unions), the power to bargain with the firm in wage setting.³⁰ Yet, there could be, of course, other reasons why wages do not move to a labor market clearing level – such as efficiency wage, insider – outsider relationship, or wages determined by standards of fairness as Solow (1979) has noted and so on.

On the other hand, there can be labor market institutions, for example corporatist structures, also measured by our ω , which affect the actual employment. Our ω expresses how much weight is given to the desired labor supply or desired labor demand. A small ω means that the agency, repre-

³⁰This is similar to Woodford's (2003, ch. 3) idea of a deviation of the efficient and natural level of output where the efficient level is achieved only in a competitive economy with no frictions.

senting the household, has a high weight in determining the outcome of the employment compromise. A high ω means that the firm's side is stronger in employment negotiations. As our empirical estimations in Gong, Ernst and Semmler (2004) have shown the former case, a low ω , is very characteristic of Germany, France and Italy whereas a larger ω is found for U.S. and the U.K.³¹

Given the rather corporatist relationship of labor and the firm in some European countries, with some considerable labor market regulations through legislature and union bargaining (rules of employment protection, hiring and firing restrictions, extension of employment even if there is a shortfall of sales etc.)³², our ω may thus measure differences concerning labor market institutions between the U.S. and European countries. This has already been stated in the 1960s by Meyers. He states: "One of the differences between the United States and Europe lies in our attitude toward layoffs... When business falls off, he [the typical American employer] soon begins to think of reduction in work force... In many other industrial countries, specific laws, collective agreements, or vigorous public opinion protect the workers against layoffs except under the most critical circumstances. Despite falling demand, the employer counts on retraining his permanent employees. He is obliged to find work for them to do... These arrangements are certainly effective in holding down unemployment". (Meyers, 1964:)

Thus, we wish to argue that the major international difference causing employment variation does arise less from real wage stickiness (due to the presence of unions and the extend and duration of contractual agreements between labor and the firm)³³ but rather it seems to be the degree to which compromising rules exist and which side dominates the compromising rule. A lower ω , defining, for example, the compromising rule in Euro-area countries, can show up as difference in the variation of macroeconomic variables. This is demonstrated in Chapter 8.4 for the German economy.

We there could observe that first, employment and the efficiency wage (defined as real wage divided productions) are among the variables with the

³¹In the paper by Gong, Ernst and Semmler (2004) it is also shown that the ω is strongly negatively correlated with labor market institutions.

³²This could also be realized by firms by demanding the same (or less) hours per worker but employing more workers than being optimal. The case would then correspond to what is discussed in the literature as labor hoarding where firms hesitate to fire workers during a recession because it may be hard to find new workers in the next upswing, see Burnside et al. (1993). Note that in this case firms may be off their marginal product curve and thus this might require wage subsidies for firms as has been suggested by Phelps (1997).

³³In fact real wage rigidities in the U.S. are almost the same as in European countries, see Flaschel, Gong and Semmler (2001).

highest volatility in the U. S. economy. However, in the German economy they are the smoothest variables. Second, in the U. S. economy, the capital stock and temporary shock to technology are both relatively smooth. In contrast, they are both more volatile in Germany. These results are likely to be due to our first remark regarding the difference in employment volatility. The volatility of output must be absorbed by some factors in the production function. If employment is smooth, the other two factors have to be volatile.

Indeed, recent Phillips curve studies do not seem to reveal much difference in real wage stickiness between Germany and the U.S., although the German labor market is often considered less flexible.³⁴ Yet, there are differences in another sense. In Germany, there are stronger influences of labor unions and various legal restrictions on firms' hiring and firing decisions shorter work week even for the same pay etc. ³⁵ Such influences and legal restriction will give rise to the smoother employment series in contrast to the U.S.. Such influences and legal restriction, or what Solow (1979) has termed the moral factor in the labor market, may also be viewed as a readiness to compromise as our Model III suggests. Those factors will indeed give rise to a lower ω and a smoother employment series.³⁶

So far we only have shown that our model of nonclearing labor market seems to match better the variation in employment than the standard RBC model. Yet, we did not attempt to explain the secular trend of the unemployment rate neither for the U.S. nor for Germany. We want to express a conjecture of how our model can be used to study the trend shift in employment. We want to note that the time series data for the table 8.3 (U.S. 1955.1-1983.1, Germany 1960.1-1992.1) are from a period where the U.S. had higher – but falling – unemployment rates, whereas Germany had still lower but rising unemployment rates. Yet, since the end of the 1980s the level of the unemployment rate in Germany has considerably moved up, partly, of course due to the unification of Germany after 1989.

One recent attempt to better fit the RBC model's predictions with labor

³⁴See Flaschel, Gong and Semmler (2001).

³⁵See, for example, Nickell (1997) and Nickell et al. (2003), and see already Meyers (1964).

³⁶It might reasonably be argued that, due to intertemporal optimization subject to the budget constraints, the supply specified by the decision rule may only approximate the decisions of those households for which unemployment is not expected to pose a problem on their budgets. Such households are more likely to be currently employed represented by labor unions and covered by legal restrictions. In other words, currently employed labor decides, through the optimal decision rule, about labor supply and not those who are currently unemployed. Such a feature could presumably be better studied by an intertemporal model with heterogeneous households, see, for example, Uhlig and Xu (1996).

market data has employed search and matching theory.³⁷ Informational or institutional search frictions may then explain the equilibrium unemployment rate and its rise. Yet, those models usually observe that there has been a shift in matching functions due to evolution of unemployment rates such as, for example, experienced in Europe since the 1980s, and that the model itself fails to explain such a shift.³⁸

In contrast to the literature on institutional frictions in the search and matching process we think that the essential impact on the trend in the rate of unemployment seems to stem from both changes of preferences of households as well as a changing trend in the technology shock.³⁹ Concerning the latter, as shown in Chapters 5 and 9, the Solow residual, as it used in RBC models as the technology shock, greatly depends on endogenous variables (such as capacity utilization). Thus exogenous technology shocks constitute only a small fraction of the Solow residual. We thus might conclude that cyclical fluctuations in output and employment are not likely to sufficiently be explained by productivity shocks alone. Gali (1999) and Francis and Ramey (2001, 2003) have argued that other shocks, for example demand shocks, are important as well.

Yet, in the long run, the change in the trend of the unemployment rate is likely to be related to the long-run trend in the true technology shock. Empirical evidence on the role of lagging implementation and diffusion of new technology for low employment growth in Germany can be found in Heckman (2003) and Greiner, Semmler and Gong (2004). In the context of our model this would have the effect that labor demand, given by equation (8.27) may fall short of labor supply given by equation (8.24). This is likely to occur in the long-run if the productivity Z_t in equation (8.27) starts tending to grow at a lower rate which many researchers recently have maintained to have happened in Germany, and other European countries, since the 1980s.⁴⁰ Yet, as recent research has stressed, for example, the work by Phelps, see Phelps (1997) and Phelps and Zoega (1998), there have also been secular changes on the supply side of labor due to changes in preferences of households.⁴¹ Some of those factors affecting the households' supply of labor have been discussed

³⁷See Merz (1999) and Ljungqvist and Sargent (1998, 2003).

³⁸For an evaluation of the search and matching theory as well as the role of shocks to explain the evolution of unemployment in Europe, see Blanchard and Wolfers (2000) and Blanchard (2003).

³⁹See Campbell (1994) for a modelling of a trend in technology shocks.

⁴⁰Of course, the trend in the wage rate is also important in the equation for labor demand (in equation 25). For an account of the technology trend, see Flaschel, Gong and Semmler (2001), and for an additional account of the wage rate, see Heckman (2003).

⁴¹Phelps and his co-authors have pointed out that an important change in the households' preferences in Europe is that households now rely more on assets instead of labor income.

above.

8.6 Conclusions

Market clearing is a prominent feature in the standard RBC model which commonly presumes wage and price flexibility. In this chapter, we have introduced an adaptive optimization behavior and a multiple stage decision process that, given wage stickiness, results in a nonclearing labor market in an otherwise standard stochastic dynamic model. Nonclearing labor market is then a result of different employment rules derived on the basis of a multiple stage decision process. Calibrations have shown that such model variants will produce a higher volatility in employment, and thus fit the data significantly better than the standard model.⁴²

As concerning international aspects of our study we presume that different labor market institutions result in different weights defining the compromising rule. The results for Euro-area economies, for example, for Germany in contrast to the U.S., are consistent with what has been found in many other empirical studies with regard to the institutions of the labor market.

Finally, with respect to the trend of lower employment growth in some European countries as compared to the U.S. since the 1980s, our model suggests that one has to study more carefully the secular forces affecting the supply and the demand of labor as modeled in our multiple stage decision process of section 2. In particular, on the demand side for labor, the slow down of technology seems to have been a major factor for the low employment growth in Germany and other countries in Europe.⁴³ On the other hand there has also been changes in the preferences of households. Our study has provided a framework that allows to also follow up such issues.⁴⁴

⁴²Appendix III computes the welfare loss of our different model variants of nonclearing labor market. There we find that similarly to Sargent and Ljungqvist (1998), that the welfare losses are very small.

⁴³See Blanchard and Wolfers (2000), Greiner, Semmler and Gong (2004) and Heckman (2003)

⁴⁴For further discussion, see also Chapter 9.

8.7 Appendix I: Wage Setting

Suppose now that at the beginning of t the household (of course with certain probability denoted as $1 - \xi$) decides to set up a new wage rate w_1^* given the data (A_t, k_t) , and the sequence of expectations on $\{A_{t+i}\}_{i=1}^{\infty}$ where A_t and k_t are referred to as the technology and capital stock respectively. If the household knows the production $f(A_t, k_t, n_t)$, where n_t the labor effort so that it may also know the firm's demand for labor, the decision problem of the household with regard to wage setting may be expressed as follows:

$$\max_{w_t^*, \{c_{t+i}\}_{i=0}^{\infty}} E_t \left[\sum_{i=0}^{\infty} (\xi\beta)^i U(c_{t+i}, n(w_t^*, k_{t+i}, A_{t+i})) \right] \quad (8.28)$$

subject to

$$k_{t+i+1} = (1 - \delta)k_{t+i} + f(A_{t+i}, k_{t+i}, n(w_t^*, k_{t+i}, A_{t+i})) - c_{t+i} \quad (8.29)$$

Above ξ^i is the probability that the new wage rate w_t^* will still be effective in period $t + i$. Obviously, this probability will be reduced when i become larger. $U(\cdot)$ is the household's utility function, which depends on consumption c_{t+i} and the labor effort $n(w_t^*, k_{t+i}, A_{t+i})$. Note that here $n(w_t^*, k_{t+i}, A_{t+i})$ is the function of firm's demand for labor, which is derived from the condition of marginal product equal to the wage rate:

$$w_t^* = f_n(A_{t+i}, k_{t+i}, n_{t+i})$$

We shall remark that although the decision is mainly about the choice of w_t^* , the sequence of $\{c_{t+i}\}_{i=0}^{\infty}$ should also be considered for the dynamic optimization. Of course there is no guarantee that the household will actually implement this sequence $\{c_{t+i}\}_{i=0}^{\infty}$. However, as argued by recent New Keynesian literature, there is only a certain probability (due to the adjustment cost in changing the wage) that the household will set a new wage rate in period t . Therefore, the observed wage dynamics w_t may follow Calvo's updating scheme:

$$w_t = (1 - \xi)w_t^* + \xi w_{t-1}$$

Such a wage indicates that there exists a gap between optimum wage w_t^* and the observed wage w_t .

It should be noted that in recent New Keynesian literature where the wage is set in a similar way as we have discussed here, the concept of non-clearing labor market somehow disappeared. In this literature, the household

is assumed to supply the labor effort according to the market demand at the existing wage rate and therefore does not seem to face the problem of excess demand or supply. Instead, what New Keynesian economists are concerned with is the gap between the optimum price and actual price, whose existence is caused by the adjustment cost in changing prices. In correspondence to the gap between optimum and actual price, there also exists a gap between optimum output and actual output.

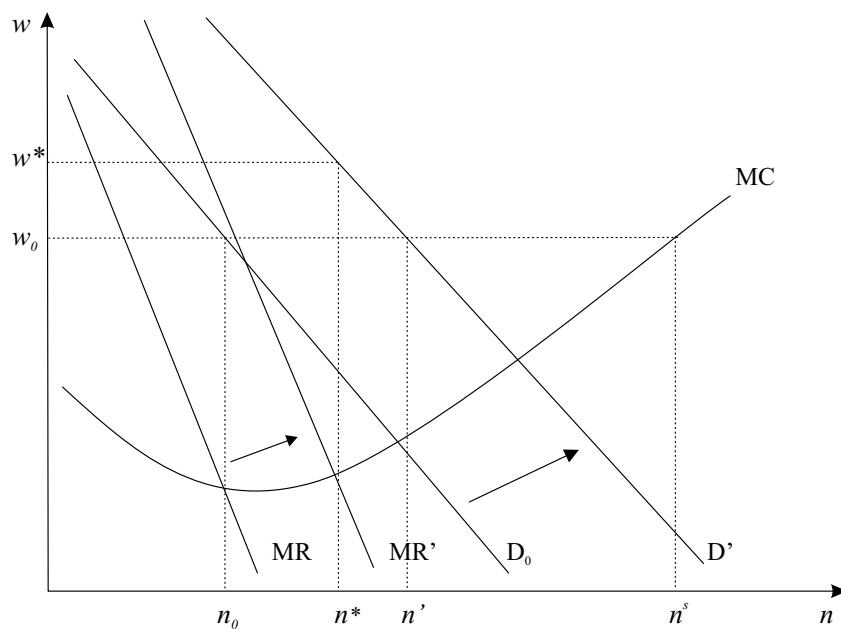


Figure 8.6: A Static Version of the Working of the Labor Market

Some clarifications may be obtained by referring to a static version of our view on the working of the labor market. In figure 8.6, the supplier (or the household, in the labor market case) first (say, at the beginning of period 0) sets its price optimally according to the expected demand curve D_0 . Let us denote this price as w_0 . Consider now the situation that the supplier's expectation on demand is not fulfilled. Instead of n_0 , the market demand at w_0 is n' . In this case, the household may reasonably believe that the demand curve should be D' and therefore the optimum price should be w^* while the optimum supply should be n^* . Yet, due to the adjustment cost in changing prices, the supplier may stick to w_0 . This produces the gaps between optimum price w^* and actual price w_0 and between optimum supply n^* and actual supply n' .

However, the existence of price and output gaps does not exclude the

existence of a disequilibrium or nonclearing market. New Keynesian literature presumes that at the existing wage rate, the household supplies labor effort whatever the market demand for labor is. Note that in figure 8.6 the household's willingness to supply labor is n^s . In this context the marginal cost curve, MC, can be interpreted as marginal disutility of labor which has also an upward slope since we use the standard log utility function as in the RBC literature. This then means that the household's supply of labor will be restricted by a wage rate below, or equal, to the marginal disutility of work. If we define the labor market demand and supply in a standard way, that is, at the given wage rate there is a firm's willingness to demand labor and the household's willingness to supply labor, and a nonclearing labor market can be very general phenomena. This indicates that even if there are no adjustment costs so that the household can adjust the wage rate in every t (so that there is no price and quantity gaps as we have mentioned earlier), the disequilibrium in the labor market may still exist.

Appendix II: Adaptive Optimization and Consumption Decision

For the problem (8.14) - (8.16), we define the Lagrangian:

$$\begin{aligned}
 L = & E_t \left\{ [\log c_t^d + \theta \log(1 - n_t)] + \right. \\
 & \left. \lambda_t \left[k_{t+1}^s - \frac{1}{1 + \gamma} [(1 - \delta)k_t^s + f(k_t^s, n_t, A_t) - c_t^d] \right] \right\} + \\
 & E_t \left\{ \sum_{i=1}^{\infty} \beta^i [\log(c_{t+i}^d) + \theta \log(1 - n_{t+i}^s)] + \right. \\
 & \left. \beta^i \lambda_{t+i} \left[k_{t+1+i}^s - \frac{1}{1 + \gamma} [(1 - \delta)k_{t+i}^s + f(k_{t+i}^s, n_{t+i}^s, A_{t+i}) - c_{t+i}^d] \right] \right\}
 \end{aligned}$$

Since the decision is only about c_t^d , we thus take the partial derivatives of L with respect to c_t^d , k_{t+1}^s and λ_t . This gives us the following first-order

condition:

$$\frac{1}{c_t^d} - \frac{\lambda_t}{1 + \gamma} = 0, \quad (8.30)$$

$$\frac{\beta}{1 + \gamma} E_t \left\{ \lambda_{t+1} \left[(1 - \delta) + (1 - \alpha) A_{t+1} (k_{t+1}^s)^{-\alpha} (n_{t+1}^s \bar{N} / 0.3)^\alpha \right] \right\} = \lambda_t, \quad (8.31)$$

$$k_{t+1}^s = \frac{1}{1 + \gamma} \left[(1 - \delta) k_t^s + A_t (k_t^s)^{1-\alpha} (n_t \bar{N} / 0.3)^\alpha - c_t^d \right]. \quad (8.32)$$

Recall that in deriving the decision rules as expressed in (8.23) and (8.24) we have postulated

$$\lambda_{t+1} = H k_{t+1}^s + Q A_{t+1} + h, \quad (8.33)$$

$$n_{t+1}^s = G_{21} k_{t+1}^s + G_{22} A_{t+1} + g_2, \quad (8.34)$$

where H, Q, h, G_{21}, G_{22} and g_2 have all been resolved previously in the household optimization program. We therefore obtain from (8.33) and (8.34)

$$E_t \lambda_{t+1} = H k_{t+1}^s + Q(a_0 + a_1 A_t) + h, \quad (8.35)$$

$$E_t n_{t+1}^s = G_2 k_{t+1}^s + D_2(a_0 + a_1 A_t) + g_2. \quad (8.36)$$

Our next step is to linearize (8.30) - (8.32) around the steady states. Suppose they can be written as

$$F_{c1} c_t + F_{c2} \lambda_t + f_c = 0, \quad (8.37)$$

$$F_{k1} E_t \lambda_{t+1} + F_{k2} E_t A_{t+1} + F_{k3} k_{t+1}^s + F_{k4} E_t n_{t+1}^s + f_k = \lambda_t, \quad (8.38)$$

$$k_{t+1}^s = A k_t + W A_t + C_1 c_t^d + C_2 n_t + b. \quad (8.39)$$

Expressing $E_t \lambda_{t+1}$, $E_t n_{t+1}^s$ and $E_t A_{t+1}$ in terms of (8.35), (8.36) and $a_0 + a_1 A_t$ respectively, we obtain from (8.38)

$$\kappa_1 k_{t+1}^s + \kappa_2 A_t + \kappa_0 = \lambda_t, \quad (8.40)$$

where, in particular,

$$\kappa_0 = F_{k1}(Q a_0 + h) + F_{k2} a_0 + F_{k4}(G_{22} a_0 + g_2) + f_k, \quad (8.41)$$

$$\kappa_1 = F_{k1} H + F_{k3} + F_{k4} G_{21}, \quad (8.42)$$

$$\kappa_2 = F_{k1} Q a_1 + F_{k2} a_1 + F_{k4} G_{22} a_1. \quad (8.43)$$

Using (8.37) to express λ_t in (8.40), we further obtain

$$\kappa_1 k_{t+1}^s + \kappa_2 A_t + \kappa_0 = -\frac{F_{c1}}{F_{c2}} c_t^d - \frac{f_c}{F_{c2}}, \quad (8.44)$$

which is equivalent to

$$k_{t+1}^s = -\frac{\kappa_2}{\kappa_1}A_t - \frac{F_{c1}}{F_{c2}\kappa_1}c_t^d - \frac{\kappa_0}{\kappa_1} - \frac{f_c}{F_{c2}\kappa_1}. \quad (8.45)$$

Comparing the right side of (8.39) and (8.45) will allow us to solve c_t^d as

$$c_t^d = -\left(\frac{F_{c1}}{F_{c2}\kappa_1} + C_1\right)^{-1} \left[Ak_t + \left(\frac{\kappa_2}{\kappa_1} + W\right) A_t + C_2 n_t + \left(b + \frac{\kappa_0}{\kappa_1} + \frac{f_c}{F_{c2}\kappa_1}\right) \right].$$

Appendix III: Welfare Comparison of the Model Variants

In this appendix we want to undertake a welfare comparison of our different model variants. We follow here Ljungqvist and Sargent (1998) and compute the welfare implication of the different model variants. Yet, whereas they concentrate on the steady state, we compute the welfare also outside the steady state. We here restrict our welfare analysis to the U. S. model variants. It is sufficient to consider only the equilibrium (benchmark) model, and the two models with nonclearing labor market. They are given by Simulated Economy I, II, and III. A likely conjecture is that the benchmark model should always be superior to the other two variants because the decisions on labor supply - which are optimal for the representative agent - are realized in all periods.

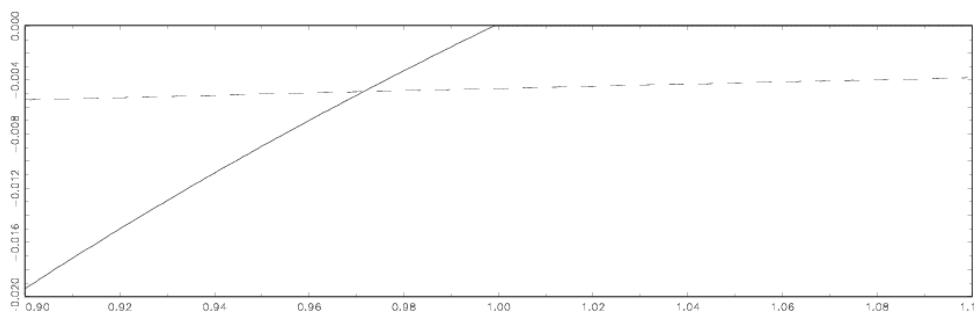
However, we believe that this may not generically be the case. The point here is that the model specification in variants II and III, is somewhat different from the the benchmark model due to the distinction between expected and actual moments with respect to our state variable, the capital stock. In the models of nonclearing market the representative agent may not rationally expect those moments of the capital stock. The expected moments are represented by equation (8.5) while the actual moments are expressed by equation (8.5). They are not necessary equal unless the labor efforts of those two equations are equal. Also, in addition to A_t , there is another external variable w_t , entering into the models, which will affect the labor employed (via demand for labor) and hence eventually the welfare performance. The welfare result due to these changes in the specification may therefore deviate from what one would expect.

Our exercise here is to compute the values of the objective function for all our three models, given the sequence of our two decision variables, consumption and employment. Note that for our models variants with nonclearing

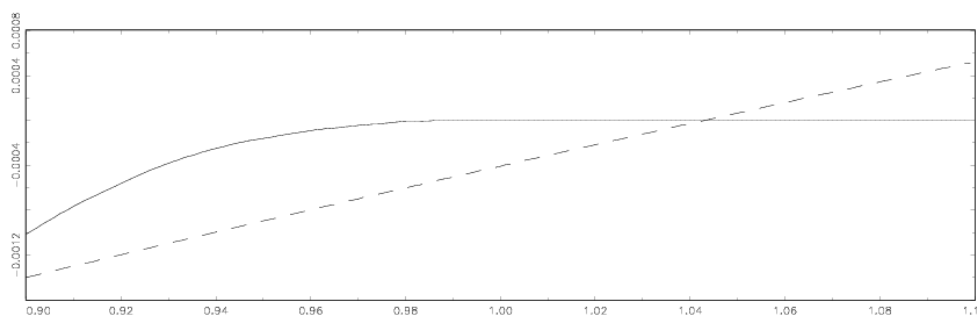
labor market, we use realized employment, rather than the decisions on labor supply, to compute the utility functional. More specifically, we calculate V , where

$$V \equiv \sum_{t=0}^{\infty} \beta^t U(c_t, n_t)$$

where $U(c_t, n_t)$ is given by $\log(c_t) + \theta \log(1 - n_t)$. This exercise here is conducted for different initial conditions of k_t denoted by k_0 . We choose the different k_0 based on the grid search around the steady state of k_t . Obviously, the value of V for any given k_0 will also depend on the external variable A_t and w_t (though in the benchmark model, only A_t appears). We consider two different ways to treat these external variables. One is to set both external variables at their steady state levels for all t . The other is to employ their observed series entering into the computation. Figure 8.7 provides the welfare comparison of the two versions.



(a) Welfare Comparison with External Variable set at their Steady State (Solid Line for Model II; Dashed Line for Model III)



(b) Welfare Comparison with External Variable set at their Observed Series (Solid Line for Model II; Dashed Line for Model III)

Figure 8.7: Welfare Comparison of Model II and III

In Figure 8.7(a), 8.7(b), the percentage deviations of V from the corresponding values of benchmark model is plotted for both Model II and Model III given for various k_0 around the steady states. The various k_0 's are expressed in terms of the percentages deviation from the steady state of k_t .

It is not surprising to find that in most cases the benchmark model is the best in its welfare performance, since most of the values are negative. However, it is important to note that the deviations from the benchmark model are very small. Similar results have been obtained by Ljungqvist and Sargent (1998), they, however, compare only the steady states. Meanwhile, not always is the benchmark model the best one. When k_0 is sufficiently high, close to or higher than the steady state of k_t , the deviations become 0 for the Model II. Furthermore, in the case of using observed external variables, the Model III will be superior in its welfare performance when k_0 is larger than its steady state, see lower part of the figure.

Chapter 9

Monopolistic Competition, Nonclearing Markets and Technology Shocks

In the last chapter we have found that if we introduce some non-Walrasian features into an intertemporal decision model with the household's wage setting, sluggish wage and price adjustments and adaptive optimization the labor market may not be cleared. This model then naturally generates higher volatility of employment and a low correlation between employment and consumption. Next we relate our approach of nonclearing labor market to the theory of monopolistic competition in the product market as developed in New Keynesian economics.

In many respects, the specifications in this chapter are the same as for the model of the last chapter. We shall still follow the assumptions with respect to ownership, adaptive optimization and nonclearing labor markets. The assumption of re-opening of the market shall also be adopted here. This is necessary for a model with nonclearing markets where adjustments should take place in real time.

9.1 The Model

As mentioned in chapter 8, price and wage stickiness is an important feature in New Keynesian literature. As concerning wage stickiness Keynes (1936) has attributed strong stabilizing effects to wage stickiness. Recent literature uses monopolistic competition theory to give a foundation to nominal stickiness.

Since both household and firm make their quantity decisions on the basis

of the given prices, including the output price p_t , the wage rate w_t and the rental rate of capital stock r_t , we shall first discuss how in our model the period t prices are determined at the beginning of period t .

Here again, as in the model of the last chapter there are three commodities. One of them should serve as a numeraire, which we assume to be the output. Therefore, the output price p_t always equals 1. This indicates that the wage w_t and the rental rate of capital stock r_t are all measured in terms of the physical units of output.¹ As to the rental rate of capital r_t , it is assumed to be adjustable and to clear the capital market. We can then ignore its setting. Here, we shall follow all the specifications on price and wage setting as in presented the chapter 8.2.1.

9.1.1 The Household's Desired Transactions

When the prices, including wages, have been set the household is going to express its desired demand and supply. We define the household's willingness as those demand and supply that can allow the household to obtain the maximum utility on the condition that these demand and supply can be realized at the given set of prices. We can express this as a sequence of output demand and factor supply $\{c_{t+i}^d, i_{t+i}^d, n_{t+i}^s, k_{t+i+1}^s\}_{i=0}^{\infty}$, where i_{t+i} is referred to investment. Note that here we have used the superscripts d and s to refer to the agent's desired demand and supply. The decision problem for the household to derive its desired demand and supply is very similar as in the last chapter and can be formulated as

$$\max_{\{c_{t+i}^d, n_{t+i}^s\}_{i=0}^{\infty}} E_t \left[\sum_{i=0}^{\infty} \beta^i U(c_{t+i}^d, n_{t+i}^s) \right] \quad (9.1)$$

subject to

$$k_{t+i+1}^s = (1 - \delta)k_{t+i}^s + f(k_{t+i}^s, n_{t+i}^s, A_{t+i}) - c_{t+i}^d \quad (9.2)$$

All the notations have been defined in the last chapter. For the given technology sequence $\{A_{t+i}\}_{i=0}^{\infty}$, the solution of optimization problem can be written as:

$$c_{t+i}^d = G_c(k_{t+i}^s, A_{t+i}) \quad (9.3)$$

$$n_{t+i}^s = G_n(k_{t+i}^s, A_{t+i}) \quad (9.4)$$

¹For our simple representative agent model without money, this simplification does not effect our major result derived from our model. Meanwhile, it will allow us to save the effort to work on the nominal price determination, a main focus in the recent new Keynesian literature.

We shall remark that although the solution appears to be a sequence $\{c_{t+i}^d, n_{t+i}^s\}_{i=0}^{\infty}$ only (c_t^d, n_t^s) along with (i_t^d, k_t^s) , where $i_t^d = f(k_t^s, n_t^s, A_t) - c_t^d$ and $k_t^s = k_t$, are actually carried by the household into the market for exchange due to our assumption of re-opening market.

9.1.2 The Quantity Decisions of the Firm

The problem of our representative firm in period t is to choose the current input demand and output supply (n_t^d, k_t^d, y_t^s) to maximize the current profit. However in this chapter, we no longer assume that the product market is in perfect competition. Instead, we shall assume that our representative firm behaves as a monopolistic competitor, and therefore it should face a perceived demand curve for its product, see the discussion above. Thus given the output price, which shall always be 1 (since it serves as a numeraire), the firm has a perceived constraint on the market demand for its product. We shall denote this perceived demand as \hat{y}_t .

On the other hand, given the prices of output, labor and capital stock $(1, w_t, r_t)$, the firm should also have its own desired supply y_t^* . This desired supply is the amount that allows the firm to obtain a maximum profit on the assumption that all its output can be sold. Obviously, if the expected demand \hat{y}_t is less than the firm's desired supply y_t^* , the firm will choose \hat{y}_t . Otherwise, it will simply follow the short side rule to choose y_t^* as in the general New Keynesian model.

Thus, for our representative firm, the optimization problem can be expressed as

$$\max \min(\hat{y}_t, y_t^*) - r_t k_t^d - w_t n_t^d \quad (9.5)$$

subject to

$$\min(\hat{y}_t, y_t^*) = f(A_t, k_t, n_t) \quad (9.6)$$

For the regular condition on the production function, the solutions should satisfy

$$k_t^d = f_k(r_t, w_t, A_t, \hat{y}_t) \quad (9.7)$$

$$n_t^d = f_n(r_t, w_t, A_t, \hat{y}_t) \quad (9.8)$$

where r_t and w_t are respectively the prices (in real term) of capital and labor.²

We are now considering the transactions in our three markets. Let us first consider the two factor markets.

²The detail will be provided in the appendix of this chapter.

9.1.3 Transaction in the Factor Market

Since the rental rate of capital stock r_t is adjusted to clear the capital market when the market is re-opened in period t , we have

$$k_t = k_t^s = k_t^d \quad (9.9)$$

Due to the monopolistic wage setting and the sluggish wage adjustment, there is no reason to believe that the labor market will be cleared, see the discussion in the last chapter. Therefore, we shall again define a realization rule with regard to actual employment. As we have discussed in the last chapter, the most frequent rule that has been used is the short side rule, that is,

$$n_t = \min(n_t^d, n_t^s)$$

Thus, when a disequilibrium occurs, only the short side of demand and supply will be realized. Another important rule that we have discussed in the last chapter is the compromising rule. The latter rule means that when disequilibrium occurs in the labor market both firms and workers have to compromise. In particular, we again formulate this rule as

$$n_t = \omega n_t^d + (1 - \omega)n_t^s \quad (9.10)$$

where $\omega \in (0, 1)$. Our study in the last chapter indicates that the short side rule seems to be empirically less satisfying than the compromising rule. Therefore, in this chapter we shall only consider the compromising rule.

9.1.4 The Transaction in the Product Market

After the transactions in those two factor markets have been carried out, the firm will engage in its production activity. The result is the output supply, which is now given by

$$y_t^s = f(k_t, n_t, A_t) \quad (9.11)$$

One remark should be added here. Equation (9.11) indicates that the firm's actual produced output is not necessarily constrained by equation (9.6), and therefore one may argue that the output determination does follow eventually the Keynesian way, that is, the output is constrained by demand. However, the Keynesian way of output determination is still reflected in the firm's demand for inputs, capital and labor (see equation (9.7) and (9.8)). On the other hand, if the produced output is still constrained by (9.6), one may

encounter the difficulty either in terms of feasibility when y_t^s in (9.11) is less than $\min(\hat{y}_t, y_t^*)$ or in terms of inefficiency when y_t^s is larger than $\min(\hat{y}_t, y_t^*)$.³

Given that the output is determined by (9.11), the transaction then needs to be carried out with respect to y_t^s . It is important to note here that when disequilibrium occurs in the labor market the previous consumption plan as expressed by (9.3) becomes invalid due to the improper rule of capital accumulation (9.2) for deriving the plan. Therefore, the household will construct a new plan as expressed below:

$$\max_{(c_t^d)} E_t \left[\sum_{i=0}^{\infty} \beta^i U(c_{t+i}^d, n_{t+i}^s) \right] \quad (9.12)$$

$$\text{s.t. } k_{t+1}^s = \frac{1}{1+\gamma} [(1-\delta)k_t^s + f(k_t, n_t, A_t) - c_t^d] \quad (9.13)$$

$$k_{t+i+1}^s = \frac{1}{1+\gamma} [(1-\delta)k_{t+i}^s + f(k_{t+i}^s, n_{t+i}^s, A_{t+i}) - c_{t+i}^d], \quad (9.14)$$

$$i = 1, 2, \dots$$

Above, k_t equals k_t^s as expressed by (9.9) and n_t is given by (9.10) with n_t^s and n_t^d are implied by (9.4) and (9.8) respectively. As we have demonstrated in the last chapter, the solution to this further step in the optimization problem can be written in terms of the following equation:

$$c_t^d = G_{c2}(k_t, A_t, n_t) \quad (9.15)$$

Given this consumption plan, the product market should be cleared if the household demands the amount $f(k_t, n_t, A_t) - c_t^d$ for investment. Therefore, c_t^d in (9.15) should also be the realized consumption.

9.2 Estimation and Calibration for U.S. Economy

9.2.1 The Empirically Testable Model

This section provides an empirical study of our theoretical model above presented which again, in order to make it empirically more realistic, has to include economic growth.

³Note that here when $y_t^s < \min(\hat{y}_t, y_t^*)$ there will be no sufficient inputs to produce $\min(\hat{y}_t, y_t^*)$. On the other hand, when $y_t^s > \min(\hat{y}_t, y_t^*)$, not all inputs will be used in production, and therefore resources are somewhat wasted.

Let K_t denote for capital stock, N_t for per capita working hours, Y_t for output and C_t for consumption. Assume the capital stock in the economy follows the transition law:

$$K_{t+1} = (1 - \delta)K_t + A_t K_t^{1-\alpha} (N_t X_t)^\alpha - C_t, \quad (9.16)$$

where δ is the depreciation rate; α is the share of labor in the production function $F(\cdot) = A_t K_t^{1-\alpha} (N_t X_t)^\alpha$; A_t is the temporary shock in technology and X_t the permanent shock that follows a growth rate γ . Dividing both sides of equation (9.16) by X_t , we obtain

$$k_{t+1} = \frac{1}{1 + \gamma} \left[(1 - \delta)k_t + A_t k_t^{1-\alpha} (n_t \bar{N}/0.3)^\alpha - c_t \right], \quad (9.17)$$

where $k_t \equiv K_t/X_t$, $c_t \equiv C_t/X_t$ and $n_t \equiv 0.3N_t/\bar{N}$ with \bar{N} to be the sample mean of N_t . Note that the above formulation also indicates that the form of $f(\cdot)$ in the last section may take the form

$$f(\cdot) = A_t k_t^{1-\alpha} (n_t \bar{N}/0.3)^\alpha \quad (9.18)$$

With regard to the household preference, we shall assume that the utility function takes the form

$$U(c_t, n_t) \log c_t + \theta \log(1 - n_t) \quad (9.19)$$

The temporary shock A_t may follow an AR(1) process:

$$A_{t+1} = a_0 + a_1 A_t + \epsilon_t, \quad (9.20)$$

where ϵ_t is an independently and identically distributed (*i.i.d.*) innovation: $\epsilon_t \sim N(0, \sigma_\epsilon^2)$.

Finally, we shall assume that the output expectation \hat{y}_t be simply equal to y_{t-1} , that is,

$$\hat{y}_t = y_{t-1} \quad (9.21)$$

where $y_t = Y_t/X_t$, so that the expectation is fully adaptive to the actual output in the last period.⁴

⁴Of course, one can also consider other forms of expectation. One possibility is to assume the expectation to be rational so that it is equal to the steady state of y_t . Indeed, we also have done the same empirical study, yet the result is less satisfying.

9.2.2 The Data Generating Process

For our empirical assessment, we consider two model variants: the standard model, as a benchmark for comparison, and our model with monopolistic competition and nonclearing labor market. Specifically, we shall call the benchmark model the Model I and the model with monopolistic competition the Model IV (in distinction from the Model II and Model III in Chapter 8).

For the benchmark dynamic optimization model, the Model I, the data generating process include (9.17), (9.20) as well as

$$c_t = G_{11}A_t + G_{12}k_t + g_1 \quad (9.22)$$

$$n_t = G_{21}A_t + G_{22}k_t + g_2 \quad (9.23)$$

Note that here (9.22) and (9.23) are the linear approximations to (9.3) and (9.4). The coefficients G_{ij} and g_i ($i = 1, 2$ and $j = 1, 2$) are the complicated functions of the model's structural parameters, α , β , δ , among others. They are computed by the numerical algorithm using the linear-quadratic approximation method.⁵

To define the data generating process for our model with monopolistic competition and nonclearing labor market, the Model IV, we shall first modify (9.23) as

$$n_t^s = G_{21}A_t + G_{22}k_t + g_2 \quad (9.24)$$

On the other hand, the equilibrium in product market indicates that c_t^d in (9.15) should be equal to c_t , and therefore this equation can also be approximated by

$$c_t = G_{31}A_t + G_{32}k_t + G_{33}n_t + g_3 \quad (9.25)$$

The computation of coefficients g_3 and G_{3j} , $j = 1, 2, 3$, are the same as in Chapter 8.

Next we consider the demand for labor n_t^d derived from the firm's optimization problem (9.5) - (9.8), which shall now be augmented by the growth factor for our empirical test. The following proposition concerns the derivation of n_t^d .

Proposition: *When the capital market is cleared, the firm's demand for labor can be expressed as*

$$n_t^d = \begin{cases} \left(\frac{0.3}{N}\right) \left(\frac{\hat{y}_t}{A_t}\right)^{1/\alpha} \left(\frac{1}{k_t}\right)^{(1-\alpha)/\alpha} & \text{if } \hat{y}_t < y_t^* \\ \frac{0.3}{N} \left(\frac{\alpha A_t Z_t}{w_t}\right)^{\frac{1}{1-\alpha}} k_t & \text{if } \hat{y}_t \geq y_t^* \end{cases} \quad (9.26)$$

⁵The algorithm used here is again from Chapter 1 of this volume.

where

$$y_t^* = (\alpha A_t Z_t / w_t)^{\alpha / (1 - \alpha)} k_t A_t \quad (9.27)$$

Note that the first n_t^d in the above equation responds to the condition that the expected demand is less than the firm's desired supply, and the second to the condition otherwise. The proof of this proposition is provided in the appendix to this chapter. Thus, for Model IV, the data generating process includes (9.17), (9.20), (9.10), (9.24), (9.25), (9.26) and (9.21) with w_t given by the observed wage rate. Here again we do not need to attempt to give the actually observed sequence of wages a further theoretical foundation. For our purpose it suffices to take the empirically observed series of wages.

9.2.3 The Data and the Parameters

We here only employ time series data of the U.S. economy. To calibrate the models, we shall first specify the structural parameters. There are altogether 10 structural parameters in Model IV: a_0 , a_1 , σ_ε , γ , μ , α , β , δ , θ and ω . All these parameters are essentially the same as we have employed in Chapter 8 (see Table 8.1) except for the ω . We choose ω to be 0.5203. This is estimated according to our new model by minimizing the residual sum of square between actual employment and the model generated employment. The estimation is again executed by a conventional algorithm, the grid search. Note that here again we need a rescaling of the wage series in the estimation of ω .⁶

9.2.4 Calibration

Table 9.1 provides the result of our calibrations from 5000 stochastic simulations. This result is further confirmed by Figure 9.1, where a one time simulation with the observed innovation A_t are presented.⁷ All time series are detrended by the HP-filter.

⁶Note that there is a need of rescaling the wage series in the estimation of ω . This rescaling is necessary because we do not exactly know the initial condition of Z_t , which we set equal to 1. We have followed the same rescaling procedure as we did in Chapter 8.

⁷Of course, for this exercise one should still consider A_t , the observed Solow residual, to include not only the technology shock, but also the demand shock among others.

Table 9.1: of the Model Variants (numbers in parentheses are the corresponding standard deviations)

	Consumption	Capital	Employment	Output
Standard Deviations				
Sample Economy	0.0081	0.0035	0.0165	0.0156
Model I Economy	0.0091	0.0036	0.0051	0.0158
	(0.0012)	(0.0007)	(0.0006)	(0.0021)
Model IV Economy	0.0071	0.0058	0.0237	0.0230
	(0.0015)	(0.0018)	(0.0084)	(0.0060)
Correlation Coefficients				
Sample Economy				
Consumption	1.0000			
Capital Stock	0.1741	1.0000		
Employment	0.4604	0.2861	1.0000	
Output	0.7550	0.0954	0.7263	1.0000
Model I Economy				
Consumption	1.0000			
	(0.0000)			
Capital Stock	0.2043	1.0000		
	(0.1190)	(0.0000)		
Employment	0.9288	-0.1593	1.0000	
	(0.0203)	(0.0906)	(0.0000)	
Output	0.9866	0.0566	0.9754	1.0000
	(0.0033)	(0.1044)	(0.0076)	(0.0000)
Model IV Economy				
Consumption	1.0000			
	(0.0000)			
Capital Stock	0.3878	1.0000		
	(0.1515)	(0.0000)		
Employment	0.4659	0.0278	1.0000	
	(0.1424)	(0.1332)	(0.0000)	
Output	0.8374	0.0369	0.8164	1.0000
	(0.0591)	(0.0888)	(0.1230)	(0.0000)

9.2.5 The Labor Market Puzzle

Despite the bias towards Model I Economy, due to the selection of the structural parameters, we find that the labor effort is much more volatile than

in the Model I Economy the benchmark model. Indeed, comparing to the benchmark model, the Model I economy, the volatility of labor effort in our Model IV economy has much been increased if anything the volatility of the labor effort is too high. This result is, however, not surprising since the agents face two constraints – one in the labor market and one in the product market. Also the excessive correlation between labor and consumption has been weakened.

Further evidence on the better fit of our Model IV economy — as concerns the volatility of the macroeconomic variables — is also demonstrated in the Figure 9.1 where the horizontal figures show, from top to bottom, actual (solid line) and simulated data (dotted line) for consumption, capital stock, employment and output. The two columns of figures, from the left to the right, represent the figures for Model I and Model IV economies respectively. As observable, the employment series in Model IV economy can fit the data better than the Model I economy.

This resolution to the labor market puzzle should not be surprising because we specify the structure of labor market essentially the same way as in the last chapter. However, in addition to the labor market disequilibrium as specified in the last chapter, we also allow in this chapter for monopolistic competition in the product market. In addition to impacting the volatility of labor effort, this may provide the possibility to resolve the another puzzle, that is, the technology puzzle also arising in the market clearing RBC model.

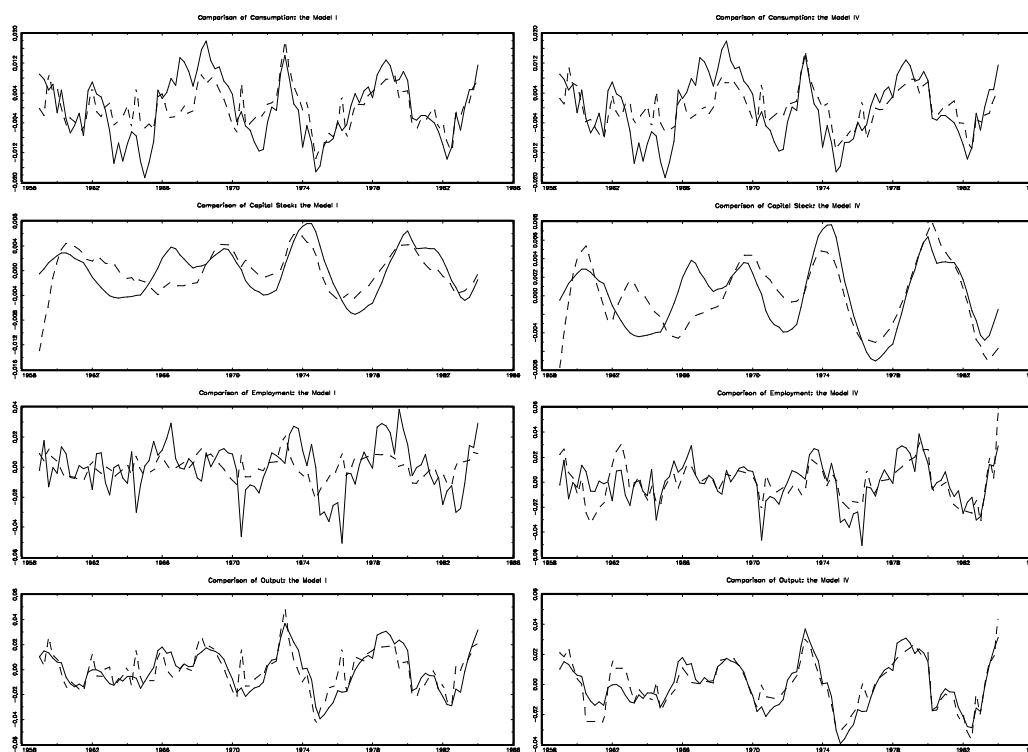


Figure 9.1: Simulated Economy versus Sample Economy: U.S. Case (solid line for sample economy, dotted line for simulated economy)

9.2.6 The Technology Puzzle

In economic literature, one often discusses the technology in terms of its persistent and temporary effects on the economy. One possibility to investigate the persistent effect in our models here is to look at the steady states. Given that at the steady state all the markets will be cleared, our Model IV economy should have the same steady state as in the benchmark model. For the convenience of our discussion, we rewrite these steady states in the following

equations (see the proof of Proposition 4 in Chapter 4):

$$\begin{aligned}\bar{n} &= \alpha\phi / [(\alpha + \theta)\phi - (\delta + \gamma)\theta] \\ \bar{k} &= \bar{A}^{1/\alpha} \phi^{-1/\alpha} \bar{n} (\bar{N}/0.3) \\ \bar{c} &= (\phi - \delta - \gamma)\bar{k} \\ \bar{y} &= \phi\bar{k}\end{aligned}$$

where

$$\phi = [(1 + \gamma) - \beta(1 - \delta)] / \beta(1 - \alpha)$$

From the above equation, one finds that technology has the positive persistent effect on output, consumption and capital stock,⁸ yet zero effect on employment.

Next, we shall look at the temporary effect of the technology shock. Table 9.2 records the cross correlation of the temporary shock A_t from our 5000 thousand stochastic simulation. As one can find there, the two models predicts rather different correlations. In the Model I (RBC) economy, technology A_t has a temporary effect not only on consumption and output, but also on employment, which are all strongly positive. Yet in our Model IV Economy with monopolistic competition and nonclearing labor market, we find that the correlation is much weaker with respect to employment. This is consistent with the widely discussed recent finding that technology has near-zero (and even negative) effect on employment.

Table 9.2: The Correlation Coefficients of Temporary Shock in Technology.

	output	consumption	employment	capital stock
Model I Economy	0.9903 (0.0031)	0.9722 (0.0084)	0.9966 (0.0013)	-0.0255 (0.1077)
Model IV Economy	0.8397 (0.0512)	0.8510 (0.0507)	0.4137 (0.1862)	-0.1264 (0.1390)

At the given expected market demand, an improvement in technology (reflected as an increase in labor productivity) will reduce the demand for labor, if the firm follows the Keynesian way of output determination, that is, the output is determined by demand. In this case, less labor is required to produce the given amount of output. Technical progress, therefore, may

⁸This long run effect of technology is also revealed by recent time series studies in the context of a variety of endogenous growth models, see Greiner, Semmler and Gong (2004).

have an adverse effect on employment at least in the short run. This stylized fact cannot be explained in the RBC framework since at the given wage rate, the demand for labor is simply determined by the marginal product, which should be increased with the improvement in technology. This chapter thus demonstrates that if we follow the Keynesian way of quantity determination in a monopolistic competition model, the technology puzzle explored in standard market clearing models would disappear.

9.3 Conclusions

In the last chapter, we have shown how households may be constrained in the product market in buying consumption goods by the firms' actual demand for labor. Then a noncleared labor market was derived from a multiple stage decision process of households where we have neglected that firms may also be demand constrained on the product market. The proposition in this chapter which shows the firms' constraint in the product market, explains this additional complication that can arise due to the interaction of the labor market and the product market constraints.

We have then shown in this chapter how the firms' constraints on the product market may explain the technology puzzle, namely that positive technology shocks may have, only a weak effect on employment in the short run - a phenomenon inconsistent with equilibrium business cycle models, where technology shocks and employment are predicted to be positively correlated. This result was obtained in an economy with monopolistic competition, as in New Keynesian economics, where prices and wages are set by a monopolistic supplier and are sticky, resulting in an updating scheme of prices and wages where only a fraction of prices and wages are optimally set each time period. Yet we have also introduced a nonclearing labor market, resulting from a multiple stage decision problem, where then the households' constraint on the labor market spills over to the product market and the firms' constraint on the product market generates employment constraints. We could show that such a model matches better time series data of the U.S. economy.

9.4 Appendix: Proof of the Proposition

Let $X_t = Z_t L_t$, with Z_t to be the permanent shock resulting purely from productivity growth, and L_t from population growth. We shall assume that L_t has a constant growth rate μ and hence Z_t follows the growth rate $(\gamma - \mu)$. The production function can be written as $Y_t = A_t Z_t^\alpha K_t^{1-\alpha} H_t^\alpha$, where H_t equals $N_t L_t$ and can be regarded as total labor hours.

Let us first consider the firm's willingness to supply Y_t^* , $Y_t^* = X_t y_t^*$, under the condition that the rental rate of capital r_t clears the capital market while the wage rate w_t is given. In this case, the firm's optimization problem can be expressed as

$$\max Y_t^* - r_t K_t^d - w_t H_t^d$$

subject to

$$Y_t^* = A_t (Z_t)^\alpha (K_t^d)^{1-\alpha} (H_t^d)^\alpha$$

The first order condition tell us that

$$(1 - \alpha) A_t (Z_t)^\alpha (K_t^d)^{-\alpha} (H_t^d)^\alpha = r_t \quad (9.28)$$

$$\alpha A_t (Z_t)^\alpha (K_t^d)^{1-\alpha} (H_t^d)^{\alpha-1} = w_t \quad (9.29)$$

from which we can further obtain

$$\frac{r_t}{w_t} = \left(\frac{1 - \alpha}{\alpha} \right) \frac{H_t^d}{K_t^d} \quad (9.30)$$

Since the rental rate of capital r_t is assumed to clear the capital market, we can thus replace K_t^d in the above equations by K_t . Since w_t is given, and therefore the demand for labor can be derived from (9.29):

$$H_t^d = \left(\frac{\alpha A_t}{w_t} \right)^{\frac{1}{1-\alpha}} (Z_t)^{\frac{\alpha}{1-\alpha}} K_t$$

Dividing both sides of the above equation by X_t , and then reorganizing, we obtain

$$n_t^d = \frac{0.3}{\bar{N}} \left(\frac{\alpha A_t Z_t}{w_t} \right)^{\frac{1}{1-\alpha}} k_t$$

We shall regard this labor demand as the demand when the firm desired activities are carried out, which is indeed the first equation in (9.26). Given this n_t^d , the firm's desire to supply y_t^* can be expressed as

$$\begin{aligned} y_t^* &= A_t k_t^{1-\alpha} (n_t^d \bar{N} / 0.3)^\alpha \\ &= A_t k_t \left(\frac{\alpha A_t Z_t}{w_t} \right)^{\frac{\alpha}{1-\alpha}} \end{aligned} \quad (9.31)$$

This is the equation (9.27) as expressed in the proposition.

Next, we consider the case that the firm's supply is constrained by the expanded demand \hat{Y}_t , $\hat{Y}_t = X_t y_t$. In other words, $\hat{y}_t < y_t^*$ where y_t^* is given by (9.31). In this case, the firm's profit maximization problem is equivalent to the following minimization problem:

$$\min r_t K_t^d + w_t H_t^d$$

subject to

$$\hat{Y}_t = A_t (Z_t)^\alpha (K_t^d)^{1-\alpha} (H_t^d)^\alpha \quad (9.32)$$

The first-order condition will still allows us to obtain (9.30). Using equation (9.32) and (9.30), we obtain the demand for capital K_t^d and labor H_t^d as

$$K_t^d = \left(\frac{\hat{Y}_t}{A_t Z_t^\alpha} \right) \left[\left(\frac{w_t}{r_t} \right) \left(\frac{1-\alpha}{\alpha} \right) \right]^\alpha$$

$$H_t^d = \left(\frac{\hat{Y}_t}{A_t Z_t^\alpha} \right) \left[\left(\frac{w_t}{r_t} \right) \left(\frac{\alpha}{1-\alpha} \right) \right]^{1-\alpha}$$

Dividing both sides of the above two equations by X_t , we obtain

$$k_t^d = \left(\frac{\hat{y}_t}{A_t} \right) \left[\left(\frac{w_t}{r_t Z_t} \right) \left(\frac{1-\alpha}{\alpha} \right) \right]^\alpha \quad (9.33)$$

$$n_t^d = \left(\frac{0.3 \hat{y}_t}{A_t \bar{N}} \right) \left[\left(\frac{r_t Z_t}{w_t} \right) \left(\frac{\alpha}{1-\alpha} \right) \right]^{1-\alpha} \quad (9.34)$$

Since the real rental of capital r_t will clear the capital market, we can replace k_t^d in (9.33) by k_t . Substituting it into (9.34) for explaining r_t , we obtain

$$n_t^d = \left(\frac{0.3}{\bar{N}} \right) \left(\frac{\hat{y}_t}{A_t} \right)^{1/\alpha} \left(\frac{1}{k_t} \right)^{(1-\alpha)/\alpha}$$

This is the second equation in (9.26).

Chapter 10

Conclusions

In this book, we try to contribute to the current research in stochastic dynamic macroeconomics. We recognize that the stochastic dynamic optimization model is important in macroeconomics, we consider the current standard model of model, the real business cycle model, only to be a simple and starting point for macrodynamic analysis. For the model to explain the real world more effectively, some Keynesian features should be introduced. We have shown that with such an introduction the model can be enriched while it becomes possible to resolve the most important puzzles, the labor market puzzle and the technology puzzle, in RBC economy.

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