Geostatistics with Applications in Earth Sciences

Second Edition

BY
D.D. SARMA

$$U_E(f) = \frac{U_M}{f_N \left| 1 + \sum_{j=1}^{M-1} \gamma_j exp(-i2\pi f_j \Delta_t) \right|^2}$$

$$\gamma(h) = C_0 + C\left[\frac{3}{2}\left(\frac{h}{a}\right) - \frac{1}{2}\left(\frac{h}{a}\right)^3\right] \text{ for } h < a$$

$$= C_0 + C \text{ for } h \geq a.$$

Springer

Geostatistics with Applications in Earth Sciences

Second Edition

Geostatistics with Applications in Earth Sciences

Second Edition

By

D.D. Sarma

Formerly at National Geophysical Research Institute
(Council of Scientific and Industrial Research)
Hyderabad, India

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-1-4020-9379-1 (HB)

Additional material to this book can be downloaded from http://extras.springer.com.

ISBN 978-1-4020-9380-7 (e-book)

Copublished by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands
with Capital Publishing Company, New Delhi, India.

Sold and distributed in North, Central and South America by Springer,
233 Spring Street, New York 10013, USA.

In all other countries, except India, sold and distributed by Springer,
Haberstrasse 7, D-69126 Heidelberg, Germany.

In India, sold and distributed by Capital Publishing Company,
7/28, Mahaveer Street, Ansari Road, Daryaganj, New Delhi, 110 002, India.

www.springer.com

Printed on acid-free paper

Printed in India.

In memory of

My Parents

Preface to the Second Edition

Geostatistics is expanding very fast: concept- and technique-wise. Keeping in view the importance of the subject, it was thought appropriate to bring out the second edition of this book. In this process, Chapter 1 has been expanded incorporating more details on sampling and sampling designs. In Chapter 2, a section on simulation has been introduced with emphasis on Monte-Carlo simulation with worked out examples. In Chapter 5, a procedure to compute variogram in the case of irregular grid has been outlined. Minor modifications have been made in all other chapters. A new chapter on Introduction to Advanced Geostatistics has been introduced with discussions on universal kriging, disjunctive kriging, conditional simulation and median polish kriging. Review Questions are given at the end of each chapter to facilitate a better understanding of the subject by the student/practitioner. The software codes are put in a CD for convenience of the students/practitoner of geostatistics. A few additions have been made in the bibliography making it more exhaustive. This contains references to the concepts and methods presented, in-depth treatment of related topics and possible extensions. My grateful thanks are due to Dr. H.S. Saini, Principal, Guru Nanak Engg. College, Hyderabad for very helpful support. I hope that this edition will be a welcome one.

August 2008 **D.D. Sarma**

Preface to the First Edition

This book has been designed to serve as a text book for post graduate students and research workers in earth sciences who require a background of and a feel for Statistics and the Theory of Regionalised Variables. The book is titled 'Geostatistics with Applications in Earth Sciences'. Although the word geostatistics is used throughout Europe signalling the Theory of Regionalised Variables as propounded by Prof. George Matheron and his colleagues at the Centre de Geostatistique, Fontainebleau, France, still it was considered necessary to include in this book some important classical statistical methods which are essential for modelling the processes concerning earth resource systems for optimum appraisal. Thus, Chapters 1 to 4 deal with the classical statistical methods including a discussion on Box-Jenkins models of Time Series Analysis and Chapters 5 to 8 deal with a discussion on the Theory of Regionalised Variables and restricted upto Kriging (Stationarity Case). Chapter 9 deals exclusively with the software developed for some of the problems. Practical application of these methods in earth sciences is explained at every stage.

In all, it is hoped that this book would serve as a practical guide to geostatistics. The units of measurement used in the examples cited in the text are the real ones. No attempt has been made to convert non-metric units into metric units.

I wish to express my grateful thanks to Dr. Harsh K. Gupta, Secretary, Dept. of Ocean Development, Govt. of India and former Director, National Geophysical Research Institute for the facilities provided to me in the completion of this project and for the Foreword; to Dr. Hari Narain, Former Director, National Geophysical Research Institute, Hyderabad, Former Vice-Chancellor, Banaras Hindu University, Varanasi, Former Surveyor-General, Survey of India and Member, Advisory Council, Directorate General of Hydrocarbons, GOI for the Preface. The Chairman and Managing Director of Bharat Gold Mines Ltd., Kolar gold fields, Karnataka, the Chairman and Managing Director of Chitradurga Copper Corporation, Chitradurga, Karnataka, the Chairman and Managing Director of Hutti Gold Mines Ltd., Hutti, Karnataka, the Chairman and Managing Director of Hindustan Zinc

Ltd., Udaipur, Rajasthan, and the Director-General of the Geological Survey of India have provided with the necessary assay data for stochastic and geostatistical modelling studies carried out by me at the National Geophysical Research Institute. I express my grateful thanks to all these authorities. Acknowledgements are due to my colleagues, Mr. N.H. Prasada Rao and Mr. J.B. Selvaraj for their help in the finalisation of the software programs listed in this volume. Mr. G.R.K. Rao and Mr. C. Shyam Sunder have done an extremely good job in text processing. Mr. M. Jayarama Rao, Mr. O. Prasada Rao of the Maps & Drawings section of NGRI have given their support in tracing the figures listed in the text. Any shortcomings are due to me.

November 2001 **D.D. Sarma**

Contents

Some Important Symbols Used in the Text

$\gamma(h)$	semi-variogram between two points separated by distance h
$\gamma_L(h)$	semi-variogram between two cores each of length L separated by distance h
$\gamma^*(h)$	experimental semi-variogram based on point samples
$\gamma^*_L(h)$	experimental semi-variogram based on core samples
a	range of influence of a semi-variogram
C	sill of a semi-variogram
C_o	nugget effect
m	slope of the linear semi-variogram
$\bar{x}$	sample mean
μ	population mean
s_x, s	standard deviation of x
σ	population std. deviation
$\bar{g}$	mean value of the observations/grades
y	logarithm of the variable
$\bar{y}$	mean value of the logarithms of observations
s_y	standard deviation of the logarithms of observations
Z_N	standard normal deviate in the context of confidence limits for the mean $\bar{z}$
H_L	standard normal deviate in the context of confidence limits for the mean ($\bar{x}$) of logarithms of data.
$\bar{g}_E$	average grade above cutoff-E
S.E	standard error

σ_k^2	kriging variance
Re.V	Regionalised Variable
R.V	Random Variable
RF	Random Function
$E\{Z(x)\}$	expectation of $Z(x)$
$C(h), \sigma(h)$	stationary covariance function
λ_t	weights assigned to various samples in the context of Kriging
$V(x)$	domain V centered at x
$v(x)$	smaller domain v centered at x
$\sigma_{E:}^2 (v/V)$	dispersion variance
μ	Lagrangian parameter
$P\{Z = z_i\}$	probability of Z taking value z_i
μ_k	kth moment about the mean
$r(x, y)$	correlation coefficient between x and y
$M_z(t)$	moment generating function
$\Lambda(z)$	denotes lognormal frequency function
M	in the context of lognormal theory, denotes population mean
θ_1	in the context of moving average process, denotes the parameter
ρ_k	autocorrelation coefficient at lag k
$\phi_1, \phi_2, \ldots \phi_k$	auto-regressive coefficients
ϕ_{kk}	partial autocorrelation coefficient at lag k
$S_F(f)$	spectral density estimates by FFT method
$U_E(f)$	spectral density by maximum entropy method
U_M	updated variance
Z	in the proper context, denotes Regionalised/Random Variable
$Z(x_i), Z_i, x_i$	the value of the regionalised variable at each data point x_i
C_L	sill value of the variogram with core samples of length L as samples
$\sigma^2(o/L)$	within variation in core of length L
s_p	standard deviation of point samples
s_v	standard deviation of samples with volume v

$s^2(0/V)$	sample variance of point samples in volume V
$\bar{\gamma}(x_i,V)$	average variogram between x_i and the volume V
$\bar{\gamma}(v,v)$	average value of the variogram between any two points x and x' sweeping independently throughout the volume v
$\bar{\gamma}(V,V)$	average value of the variogram between any two points y and y' sweeping independently throughout the volume V
$\sigma^2(v/V)$	variance of v in V
$\sigma_E^2(v,V)$	Extension Variance. Error committed when the grade of a sample of volume v is extended to the grade of volume V
σ_{ij}	covariance between neighbourhood samples i and j.
$\forall j$	for all j

1 Statistical Methods in Earth Sciences

1.1 INTRODUCTION

Earth scientists are often involved in taking observations on the earth's surface and its interior. Since earth scientists depend largely on observations, particularly on observations where there is a large portion of uncertainty, Statistics plays a major role in drawing inferences concerning the earth and its interior. Statistical problems, whether perceived or not, still persist when there are elements associated with chance. Earth scientists take advantage of statistical methods in problems involving risk and uncertainty. We need a more variety of statistical techniques in some or all of the following kinds of activities/areas.

1.1.1 Sampling, Data Collection and Sample Design

All items in any field of inquiry constitute a 'universe' or 'population'. From a practical point of view, it is not possible to collect all items in a population for analysis, unless the population itself is a very small one. Therefore, we resort to forming, what is known as 'sample'. Mathematically, if the population size is N and a part of it, say n ($n<N$) is selected according to some rule for studying some characteristic(s) of the population, this set of n units/elements is known as sample. The individual items/elements in a sample, as far as possible, should be representative ones. The selection process is called sampling technique/sampling procedure. Figure 1.1 shows a schematic representation of population, sample and an element.

A researcher must prepare a sample design for his study and its size i.e., the number of sample points/elements.

Data collection and sampling is a very important aspect of any geostatistical study and includes an orderly collection of various types of data. Examples include choosing locations for collecting rock specimens in a given geological region; selecting locations for taking gravity readings

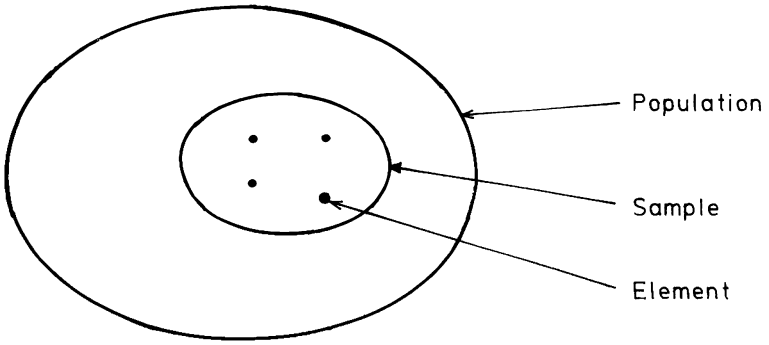


Fig. 1.1 Schematic representation of population, sample and element.

etc., for exploration. There are some aspects which need to be examined before data are collected. Some of these are briefly discussed below. The sampling method could be random sampling, stratified sampling, systematic sampling, cluster sampling etc., depending on the need.

1.1.2 Sample Design and the Various Steps

While developing a sample design, the analyst/researcher must pay attention to the following: (a) *Type of universe:* The first step in developing any sample design is to clearly define the set of objects – typically called the universe or population to be studied. The population can be finite or infinite. While in a finite population, the number of items is limited, the number of items is infinite in an infinite population. Examples of finite population include the number of outcrops in a geological terrain, the number of bore-wells in a region, the number of persons in a city and so on. Examples of infinite population include the number of stars in the sky; and the number of ore samples that can be taken from a gold bearing lode etc. (b) *Sampling unit:* A sampling unit may be a geographical one such as a state, district, village etc. From a geological point of view, it could be a geological district, a rock specimen etc. (c) *Source list:* It is also known as 'sampling frame' from which a sample is to be drawn. Such a list should be comprehensive and reliable. It is important for the source list to be as representative of the population as possible. For example, the source list could be that portion of the lode that exists between two dykes. (d) *Type of sampling:* A researcher must decide on the type of sample to be collected and to be employed. From a geological angle, the question could be whether to collect channel samples, grab samples or chip samples etc. (e) *Sample size:* This refers to the number of items to be selected from the universe/population to constitute a sample. An optimum sample size is one that fulfills the requirements of efficiency, representation, reliability and flexibility. While deciding on the size of the

sample, a researcher must determine the desired precision as also an acceptable confidence level of the estimate. The size of sample variance needs to be considered in relation to population variance. If the variance of the sample is large, then a larger sample size may be needed. The size of population and the parameters of interest in a research study must also be kept in view, while deciding on the size of the sample. (f) *Parameters of interest:* A researcher must address the question of specific population parameters which are of interest. For example, we may be interested in estimating the population mean of mine samples, when the distribution is lognormal, or some other characteristics of the population. Also, a researcher must select a sample design which gives lesser sampling error for a given sample size and cost.

1.1.3 Criteria for Selecting/Drawing a Sample

While selecting a procedure for drawing a sample, a researcher must ensure that it causes relatively small sampling error for a given sample size and cost and also helps in controlling systematic bias in a better way. A systematic bias is the result of one or more of the following factors: (i) inappropriate sampling frame. If the sampling frame is inappropriate, a biased representation of the population and hence a systematic bias occurs, and (ii) defective measuring device. If the measuring device is constantly in error, it will result in a systematic bias in the data collected by using that device.

In mine samples analysis, the analyst plays an important role. An assay value is determined by first crushing a specimen rock sample and then taking a small portion of the same for chemical analysis. If the chosen small portion is not a representative one of the sample, or if the instrument for measuring the assay is biased, then an error or a systematic error can arise.

1.1.4 Characteristics of a Good Sample Design

A good sample design must (i) result in a truly representative sample, (ii) lead to only a small sampling error, (iii) be cost effective, (iv) be one that controls systematic bias, and (v) be one such that the results of the sample study can be applied for the population with a reasonable degree of confidence.

1.1.5 Different Types of Sample Design

There are two different factors on the basis of which different sample designs exist. These factors are: (a) representation basis and (b) element selection technique. In representation basis the samples may be drawn on the basis of (i) probability sampling or (ii) non-probability sampling. While probability sampling is based on the concept of random sampling, non-probability sampling is based on the concept of non-random sampling. A detailed discussion on probability can be seen in Section 1.5.

Representation basis

(i) *Probability sampling*

Probability sampling is also known as 'random sampling' or 'chance sampling'. In this scheme, every item of the universe has an equal chance of inclusion in the sample. Random sampling ensures statistical regularity which means that if on an average the sample chosen is a random one, the sample will have the same composition and characteristics of the universe/population. There are various methods of selecting a random sample.

A random sample can be selected with the aid of a computer software, consulting a table of random numbers, using mid-square method, using a method of coin tossing or a calculator with a random number generator. Most of the books on Statistics contain random number tables. Drawing slips out of hat/box serves as an alternative, if every element in the sampling frame has an equal chance of selection. Mixing the slips thoroughly and returning the slips drawn between every selection ensures unbiasedness, in that, every element is just as likely to be selected as any other element. A table of random numbers is a practical solution when no software programs are readily available. Random number tables contain digits that are not systematically arranged. One can select random numbers row-wise, column-wise or diagonally.

Random numbers can also be generated by coin tossing. The procedure with an unbiased coin is to toss it a number of times. Observe the sequence of heads or tails and compute the number based on this sequence. The number of tosses needed to cover a certain range of numbers and the method of conversion of a sequence of heads and tails to a number on a decimal scale is as follows: suppose it is desired to choose a random number in the range 1-500. First determine the smallest integer k such that $2^k > 500$. In this example, $k = 9$. Then toss an unbiased coin k times. Let the observed sequence of heads (1) and tails (0) be: 001 011 110. A random number is obtained by finding the decimal equivalent of the binary sequence and adding 1 to it. The decimal equivalent of the binary number is $0 \times 2^8 + 0 \times 2^7 + 1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 94$ giving the random number $94 + 1 = 95$. If the number obtained is greater than 501, it is rejected and fresh tosses are made. This way we can generate as many random numbers as are needed.

Mid-square method is one of the methods proposed for use on digital computers to generate random numbers. The method is illustrated as follows: suppose we wish to generate four digit integers and the last number generated was 9837. To obtain the next number in the sequence, we square the one and use the middle four digits of the product. In this case, the product is 96 7665 69 and the next psuedo-number is 7665. In a similar way, the next psuedo numbers in the sequence 7522 and 5804 can be obtained.

(ii) *Non-probabilty sampling*

Non-probability sampling is also known by different names such as, deliberate sampling, purposive sampling and judgement sampling. In this type of sampling, elements for the sample are selected deliberately by the researcher as per his choice (subjective). Suppose a region is surveyed for exploration activity for a gold bearing lode. The area is divided into blocks. Some blocks may not be geologically favourable. Out of the favourable ones, a geologist may like to collect gold bearing rock specimens as per his judgement or choice. In non-probability sampling design, personal element plays a great role. Quota sampling is also an example of non-probability sampling. Under this scheme, an interviewer is simply given quotas to be filled from the different strata with some restrictions on how they are to be filled. In other words, the actual selection of items for the sample is left to the interviewer's discretion. These samples so selected do not possess the characteristic of random samples. However, if the enumerator initially chooses units at random rejecting those that are not needed, this method is equivalent to stratified random sampling.

Element selection technique

In this approach the sample may either be unrestricted or restricted. When each sample element is drawn individually from the population, then the sample so drawn is known as unrestricted sample; whereas all other forms of sampling are covered under the term 'restricted sampling'. The classification is shown below in a tabular form.

Representative Basis		Element selection technique
Probability sampling	Non-probability sampling	
Simple random sampling	Haphazard sampling Convenience sampling	Unrestricted sampling
Stratified random sampling Systematic sampling Cluster sampling etc.	Purposive sampling (such as quota sampling, judgment sampling etc.)	Restricted sampling

1.1.6 Analysis Aspects

The geological data collected for identification of trends, clusters, estimation of a geological variable or for establishing simple or multiple correlations need to be analysed for which geological explanations are also needed. Exploratory data analysis is a simple but a very effective approach in the analysis stage. Extremely useful inferences can be drawn about data and its patterns by this approach.

Exploratory Data Analysis (EDA)

Exploratory Data Analysis (EDA) is both a data analysis perspective and a set of techniques. In EDA, the data guide the choice of analysis rather than the analysis superimposing its structure on the data. Since research is problem-oriented rather than technique-driven, EDA is the first step in the search for evidence, without which confirmatory analysis has nothing to evaluate. As EDA does not follow a rigid structure, it is free to take any path in unraveling the mysteries in the data. A major aspect of exploratory approach lies in the emphasis on visual representations and graphical techniques over summary statistics. Summary statistics may obscure, conceal or even misrepresent the underlying structure of the data leading at times, to erroneous conclusions. For these reasons, data analysis should begin with a visual inspection. After that, it is possible and desirable to cycle between exploratory and confirmatory approaches. Some useful techniques for displaying data are: frequency tables, bar charts, pie charts, histograms, stem-and-leaf displays, transformations etc. Some of these are discussed below.

Frequency tables, Bar charts and Pie charts

Frequency table

Suppose a quartz lode has been sampled at 100 locations and that assaying of the rock specimens indicated 2 gms/tonne of ore at 18 locations, 4 gms at 32 locations, 6 gms at 21 locations, 8 gms at 18 locations and 10 gms at 11 locations. These data can be put in the form of a frequency table as shown in Table 1.1.

Table 1.1 Sample frequency distribution of grade of ore

Grade gms/tonne of ore	*Frequency*	*%age*
2	18	18
4	32	32
6	21	21
8	18	18
10	11	11

Sometimes it is desirable to group the data into convenient intervals; for example in the above case as 2-4 gms, 4-6 gms, and 6-8 gms etc. Also the grade values need not necessarily be integers. The grade could be a real one such as 1.3 gms, 2.7 gms etc. In such a case, a frequency table can be formed choosing appropriate class intervals.

Bar chart

The above frequency distribution data can be displayed in the form of bar chart. In a bar chart each category is depicted by a bar. Bar charts are used

to represent one variable. For the data shown in Table 1.1, the bar chart can be represented as shown in Fig. 1.2.

It may be seen that each bar has equal width but unequal length. The length indicates the magnitude/frequency. Such a chart shows the increase or decrease in the trend. In view of the simplicity, a bar chart is very popular

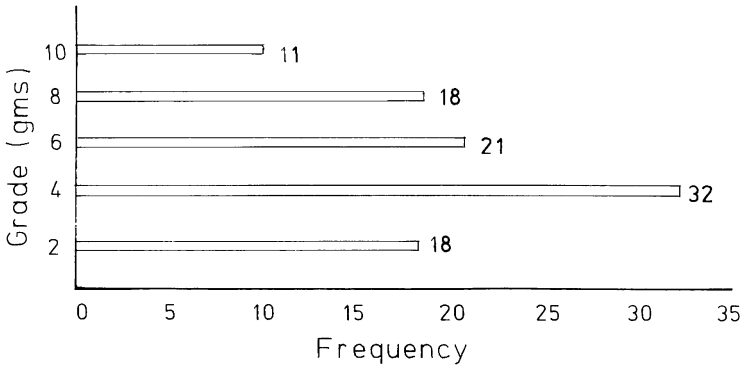


Fig. 1.2 Bar chart corresponding to sample frequency distribution of grade of ore.

in practice. The limitation is that such a classification can display only one category of data. Bar charts can be vertical or horizontal.

Pie chart/diagram

Another type of diagram which is more commonly used is the circular or pie diagram. The pie chart is based on the fact that a circle has 360°. The pie is divided into slices according to the percentage in each category. The pie chart for the data given in Table 1.1 is shown in Fig. 1.1. It clearly shows that the total for all categories adds to 100%.

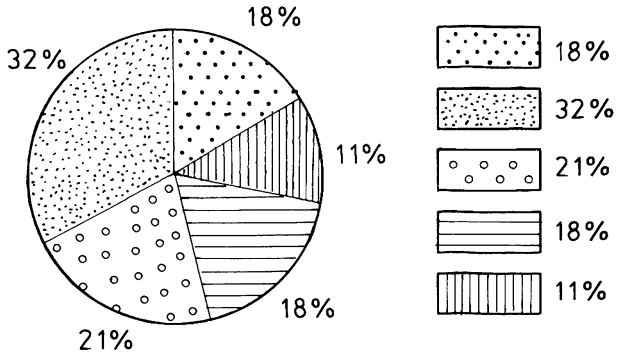


Fig. 1.3 Pie chart corresponding to sample frequency distribution of grade of ore.

Histograms

The histogram is a conventional solution for the display of interval or interval-ratio data. Histograms are used when it is possible to group the variable values into intervals. Histograms are constructed with bars (or asterisks that represent data values). Histograms are useful for (1) displaying all intervals in a distribution, even those without observed values, and (2) examining the shape of the distribution for skewness, kurtosis and other patterns. We can infer from the histogram whether multiple modes exist.

We can also infer whether any subgroups are identifiable and/or any straggling data values are detached from the central concentration. Figure 1.4 shows histogram for a sample set of gold assay values given in Table 1.1. Figure 1.5 shows the histogram for a sample set of copper accumulation values.

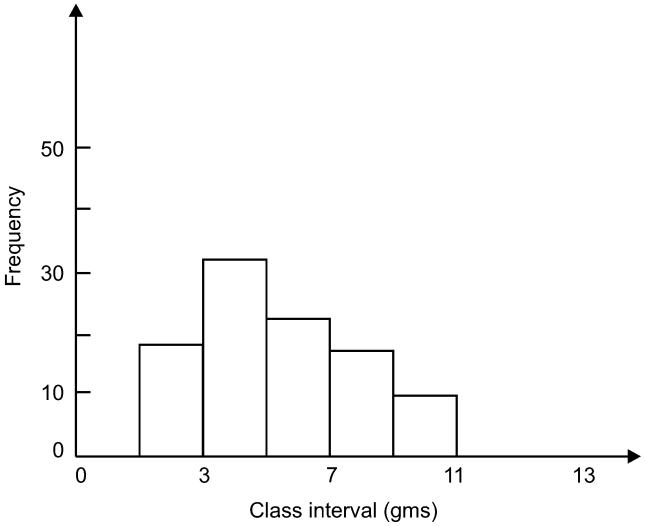


Fig. 1.4 Histogram for a sample frequency distribution of grade.

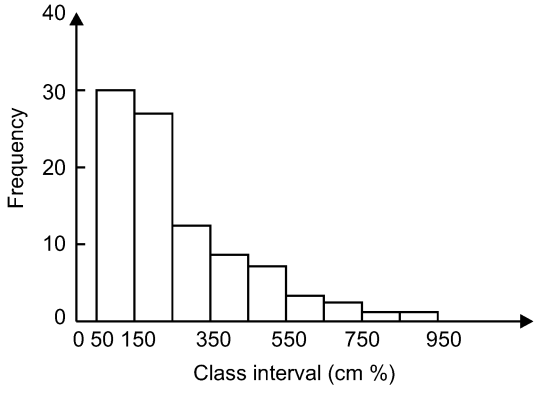


Fig. 1.5 Distribution of a set of copper accumulation values.

Stem-and-Leaf Displays

The stem-and-leaf display technique is closely related to the histogram. Although there are some features which are common between the two, there are several advantages with stem-and-leaf displays. It is easy to construct stem-and-leaf displays by hand for small samples. For large samples, computer programs can be used. In contrast to histograms, where grouping of data into class intervals takes place and thus resulting in loss of information, the stem-and-leaf presents actual data values that can be inspected directly. This feature reveals the distribution of values within the intervals and preserves their rank and order for finding the median, quartiles and other essential statistics. Visualization is the second advantage of stem-and-leaf displays. The range of values is quite clear and both shape and spread attributes are immediate. Patterns within data, clusters of values and outliers are easily observed. In order to develop a stem-and-leaf display, the first digits of each data item are arranged to the left of the vertical line. Next we go back to the data in the order they were recorded and place the last digit for each item to the right of the vertical line. The digit to the right of the decimal point is ignored. The last digit for each item is placed on the horizontal row corresponding to the first digit(s). It is now a simple matter to rank-order the digits in each row, creating the stem-and-leaf display. Consider the accumulation data given in Table 1.3.

Table 1.3 A sample set of gold accumulation values

S.No	*Accumulation (cm-gms)*	*S.No*	*Accumulation (cm-gms)*	*S.No*	*Accumulation (cm-gms)*
1	54.2	18	67.2	35	88.8
2	55.1	19	69.2	36	102.6
3	55.4	20	69.4	37	104.4
4	56.7	21	69.8	38	107.2
5	57.6	22	70.2	39	119.4
6	57.4	23	73.4	40	118.1
7	58.2	24	73.6	41	123.3
8	58.1	25	74.6	42	131.5
9	58.8	26	76.2	43	141.2
10	58.2	27	76.4	44	153.4
11	58.6	28	77.2	45	163.3
12	60.4	29	79.4	46	167.4
13	60.3	30	80.2	47	183.2
14	61.5	31	82.2	48	207.4
15	62.6	32	83.4	49	201.6
16	66.8	33	86.4	50	206.4
17	67.6	34	87.6		

The stem-and-leaf display for the above sample set of data is shown below.

5	4 5 5 6 7 7 8 8 8 8 8
6	0 0 1 2 6 7 7 9 9 9
7	0 3 3 4 6 6 7 9
8	0 2 3 6 7 8
9	- - - - - -
10	2 4 7
11	0 9 8
12	3
13	1
14	1
15	3
16	3 7
17	- - - - - -
18	3
19	- - - - - -
20	7 1 6

Each line or row in the above display is referred to as a stem and each piece of information on the stem as a leaf. The first line or row is: 5 4 5 5 6 7 7 8 8 8 8 8. The meaning attached to this line or row is that there are 11 items in the data whose first digit is 5. The digits are 54, 55, 55, 56, 57, 57, 58, 58, 58, 58, and 58. The second line: 6 0 0 1 2 6 7 7 9 9 9 shows that there are eight items whose first digit is 6. The digits are 60, 60, 61, 62, 66, 67, 67, 69, 69, 69. The digit to the right of the decimal point of any number is ignored (eg., 68.2 is taken as 68). The stem is the digit(s) to the left of the vertical line (6 for this example) and the leaf is the digit(s) to the right of the vertical line: (0, 0, 1, 2, 6, 7, 7, 9, 9, 9). If the stem-and leaf display is turned upright (rotate by 90 degrees to the left), the shape is the same as that of histogram.

1.2 HYPOTHESIS

We define a proposition as a statement about concepts that may be judged as true or false if it refers to observational phenomena. When a proposition is formulated for empirical testing, we call it a hypothesis. As a declarative statement a hypothesis is tentative and conjectural in nature. The purpose of hypothesis testing is to determine the accuracy of the hypotheses set-up (by us) due to the fact that we are basing our decision on a sample data and not on population. The accuracy of hypothesis is evaluated by determining the likelihood that the data reveal true differences and not differences based on random sampling errors.

Hypothesis Testing

There are two approaches for hypothesis testing: (1) The well established classical or sampling theory approach and (2) the Bayesian approach. Bayesian approach is an extension of classical approach. However, it goes beyond classical approach to consider all other available information. The additional information consists of subjective probability estimates stated in terms of one's belief. The subjective estimates are based on general experience. They are expressed as prior probability distributions (apriori) which can be revised after sample information is gathered. The revised estimates which are known as posterior distributions (aposteori) can be further revised by additional information.

Hypothesis testing involves verification of concepts or models of processes believed to explain specific problems. Suppose a sample set of assay values has been drawn from an unlimited number of assay values of a gold bearing lode, using random sampling method. These unlimited number of observations can be termed as population. If we want to know whether the sample mean for grade i.e., the mean computed on a sample set of observations on grade drawn from the population, is significantly different from the population mean, we formulate a null hypothesis H_0*: There is no significant difference between the sample mean and the population mean.* We proceed to test this hypothesis for possible rejection on the basis of available data.

1.3 QUANTIFICATION AND PREDICTION IN EARTH SCIENCES

We need statistical tools and models for solving specific problems such as computing the probability of occurrence of specific types of mineral deposits in a given region; the probability of occurrence of an earthquake in a seismically active zone; the probability of occurrence of an oil reservoir or ground water in a region; the estimation of a gravity field in a region; or the probability of occurrence of a volcanic eruption in an area and so on.

Broadly speaking there are four stages involved in tapping the earth's hidden mineral wealth: (1) Reconnaissance, (2) Exploration, (3) Prospecting and (4) Mining. Out of these, the exploration and prospecting stages give rise to sampling and the samples drawn during these stages form the basic data for analysis, prospect evaluation, forecasting and for drawing inferences regarding the parent population. Towards this end, two types of statistical approaches viz., (i) classical statistics and (ii) geostatistics are suggested. From classical statistics point of view, the samples drawn from a region of interest can be considered as realisations of the random variable—be it grade/accumulation etc. In geostatistics, we use the terminology '*Regionalised Variable*', in short Reg. V. The difference between these two approaches will be discussed in Chapter 5. [Note: Accumulation is the product of the width of the reef multiplied by the grade.]

1.4 THE CONCEPT OF RANDOM VARIABLE

A random variable is a numerically valued variable defined on a sample space (Hoel, p. 15, 1957). As an example, let z' denote the totality of the points obtained in casting an unbiased die. Here these are six in number. Then Z is a random variable (R.V) which assumes 1/6 equally probable values. If a cast results in say, the number 4, by definition, we say that this value is a particular *realisation* of the R.V—result of casting the die. Yet another example is the grade of ore in a mineral deposit. Let Z be random variable (grade of ore) and $z_1, z_2.....z_n$ be the sample values drawn which may be treated as realisations of the random variable Z. We may be interested in finding the probability of Z taking the value $z_j = p(z_j)$. The numbers $p(z_j)$; $j = 1, 2, 3...$ must satisfy the following conditions: $(i)\ p(z_j) \geq 0\ \ \forall j$ and $(ii)\ \sum_{j=1}^{\infty} p(z_j) = 1$. The function p is called the *probability mass function* of the random variable Z and the set $\{p(z_j)\}$ is called the *probability distribution* of the random variable Z.

1.5 PROBABILITY

The word probability is derived from 'probable' which means 'likely'. The intuitive notion of probability is connected with inductive reasoning. Classical probability is the oldest way of defining probabilities. This applies when all possible outcomes of an experiment are equally likely. Suppose there are N equally likely possibilities of which one must occur and there are 'n' favourable ones or successes, then the probability of a success is n/N.

The most widely used concept is the frequency interpretation according to which the probability of an event (the outcome) is the proportion of the time that the events of the same kind will occur in the long run. When the weatherman says that there is a 30 per cent chance of raining (probability 0.30), it means that given the same weather conditions, it will rain 30% of the time. In contrast, the view that is gaining ground is to interpret the probabilities as *personal* or *subjective* evaluations. Such probabilities are governed by one's strength of belief with regard to uncertainties that are involved. In such a case, there is no direct evidence. These are educated guesses or perhaps based on intuition or other subjective factors. In our discussion, we shall follow the axiomatic approach whereby we mean that probabilities are defined as '*mathematical objects*' which behave according to well defined rules.

It is customary to say probability, as an arbitrary number, which ranges from 0 to 1. A classic example of discrete probability used almost universally is related to the experiment of tossing an unbiased coin. We know the probability of obtaining a head or a tail in one throw is 0.5. This means that, in the long run, heads will occur 50% of the time, so also the tails. The possibility of the coin standing on the edge is not considered. If we are

interested in the probability of obtaining only one head in three tosses, then we see from the possible outcomes such a probability as 3/8. The possible outcomes are:

HHH	HTH	TTT	HTT
HHT	THH	THT	TTH

Therefore, we may generalise this and say that the probability of an event A, denoted as $P(A)$, is a number assigned to this event. The number could be interpreted to mean that if the experiment is performed 'N' times and the event occurs 'n' times, with a high degree of certainty, the relative frequency, n/N of the occurrence of A is close to $P(A) = n/N$, provided N is sufficiently large.

The probability distribution that governs experiments such as the tossing of a coin is called the Binomial Distribution whose frequency function, with the usual notation, is given as: ${}^{n}C_{r}\, p^{r} q^{n-r}$ where r represents the number of successes in n trials. Thus the probability of getting two successes from one of the three trials is: ${}^{3}C_{2}(1/2)^{r}(1/2) = 3/8$. It may be seen that this is the same result as the one from the empirical experiment.

1.6 FREQUENCY FUNCTION, JOINT FREQUENCY FUNCTION, CONTINUOUS FREQUENCY FUNCTION AND JOINT CONTINUOUS FREQUENCY FUNCTION

Frequency Function: From the foregoing discussion, we see that the probability of Z assuming a particular value, say z_0, is equal to the number of sample points for which $Z = z_0$ divided by the total number of sample points. The probability is expressed by means of a function called *Frequency Function*. We may define the frequency function $f(z)$ of a random variable Z, as that function which generates probability that the random variable will assume in its range. Taking the example of tossing two coins, if $Z = z_0$ represents the total number of heads obtained, we have the set of values as: $f(0) = 1/4$, $f(1) = 1/2$, $f(2) = 1/4$.

Joint Frequency Function: Usually, many experiments involve several random variables rather than merely one random variable. If we consider two random variables Y and Z, a mathematical model for these two variables will be a function that will give the probability that Y will assume a particular value while at the same time Z will assume another value. A function $f(y, z)$ that gives such probabilities is called the joint frequency function of the two random variables Y and Z. An example of *joint frequency function* of y and z is:

$$f(y, z) = C\, m^{yz}/y!z!.$$

If the variables are unrelated in a probability sense, it means that the probability of one of the variables assuming a particular value is independent

of what the other variable assumes. We call such variables as independent random variables which are distributed independently. Statistically speaking, if the joint frequency function of $(y_1, y_2, \ldots, y_n)$ can be factored in the form $f(y_1)\, f(y_2) \ldots f(y_n)$, where $f_i(y_i)$ is the frequency function of y_i, the random variables $Y_1, Y_2, \ldots, Y_n$ are said to be independently distributed.

As an illustration, suppose that the number of earthquake accidents y occurred in the Latur area (western India) in the month of September, 1993 possessed the frequency function $f(y) = e^{-\lambda}\lambda^y/y!$, where λ is a positive constant. If z denotes the number of accidents due to the occurrence of another subsequent earthquake and it had the same frequency function as y and if y and z were independently distributed, then:

$$f(y, z) = \frac{e^{-\lambda}\lambda^y}{y!} \frac{e^{-\lambda}\lambda^z}{z!} = \frac{e^{-2\lambda}\lambda^{y+z}}{y!\, z!} \tag{1.1}$$

Continuous Frequency Function: A frequency function for a continuous random variable Z is a function $f(z)$ which has the following properties:

(1) $f(z) \geq 0$; (2) $\int_{-\infty}^{+\infty} f(z)dz = 1$ and (3) $\int_{a}^{b} f(z)\, dz = P\{a < z < b\}$, where a and b are any two values of z, with $a < b$. Property (1) is necessary because negative probability has no meaning. Property (2) corresponds to the requirement that the probability of an event that is certain to occur should be equal to one, which is but logical. It is assumed that $f(z)$ is defined to be equal to zero for those values outside the range of the variable.

The graph of the frequency function and the representation of $P\{1 < z < 2\}$ as an area is given in Fig. 1.6. The frequency function for a continuous variable is often called the *Probability Density Function* or simply '*density function*' of the variable. However, it has become common to use '*frequency function*' for both discrete and continuous variables.

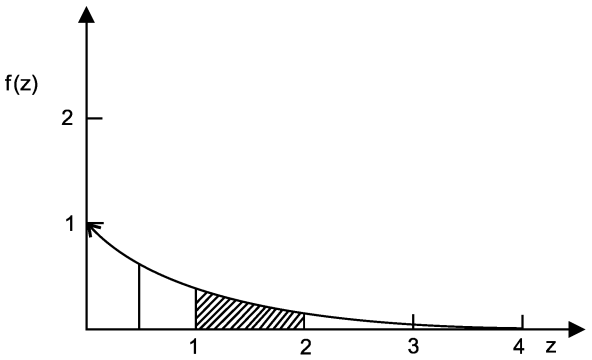


Fig. 1.6 Frequency function for a continuous variable.

In the case of a normal random variable which is continuous (the distribution of which will be discussed later), the frequency function may be written as:

$$f(z) = C \exp\left[-\frac{1}{2}\left(\frac{z_i - a}{b}\right)^2\right]$$

where a and b are parameters and C is a constant that makes $f(z)$ a frequency function.

Joint Continuous Frequency Function: A frequency function for n continuous random variables $z_1, z_2, \ldots, z_n$ is a function $f(z_1, z_2, \ldots, z_n)$ which possesses the following properties:

1. $f(z_1, z_2, \ldots, z_n) \geq 0$; (1.2)

2. $$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(z_1, z_2, \ldots, z_n)\, dz_1, dz_2, \ldots, dz_n = 1 \quad (1.3)$$

3. $$\int_{a_n}^{b_n} \ldots \int_{a_1}^{b_1} f(z_1, z_2, \ldots, z_n)\, dz_1, dz_2, \ldots, dz_n$$
$$= P(a_1 < z_1 < b_1 \ldots a_n < z_n < b_n) \quad (1.4)$$

Review Questions

Q. 1. Discuss the criteria for drawing a sample.
Q. 2. What are the characteristics of a good sample?
Q. 3. List the various steps involved in sample design.
Q. 4. What is the purpose of Exploratory Data Analysis? Explain frequency analysis and bar charts as EDA tools.
Q. 5. What is hypothesis? Explain hypothesis testing.
Q. 6. Explain the concept of Random Variable and Joint Frequency Function.

2 Univariate Statistical Methods, Frequency Analysis and Simulation

2.1 UNIVARIATE STATISTICAL METHODS

We may recall that from the statistical point of view, the totality of possible experimental outcomes may be called 'population' of the outcomes, while a set of data obtained by performing the experiment a number of times is called 'sample' from the population. Since it is time consuming, uneconomical and may not be practical to analyse the whole population, statistical inference consists in drawing conclusions on the basis of its samples drawn. The type of information extracted from a set of data depends upon its input and the model selected.

Geological populations can be sampled and numerical expressions obtained. For example, the tenor of copper in a mineral deposit or the grade of ore in a gold deposit or the elements' concentration in bauxite deposits (Al_2O_3, Fe_2O_3 etc.) at various sampling points or the number of zircon grains in a microscope slide etc., can be obtained for statistical analysis. If we are considering one variable, it becomes univariate as against multivariate corresponding to multivariables. When two variables are considered, the specific term is bivariate.

2.2 FREQUENCY ANALYSIS

Usually, an earth scientist faces the problem of comprehending a huge number of observations. These observations could be realisations of a random variable, say grade of ore. To draw inferences, one must order these observations by grouping and averaging. For this purpose, we resort to frequency analysis. In Table 2.1, a sample set of sixty Fe_2O_3, Al_2O_3 and SiO_2 element concentration values in units of % are given.

Table 2.1 Element concentration values of Fe_2O_3, Al_2O_3 and SiO_2 in a bore hole of a bauxite deposit. (These values are in units of %)

Distance	Fe_2O_3	Al_2O_3	SiO_2	*Distance*	Fe_2O_3	Al_2O_3	SiO_2	*Distance*	Fe_2O_3	Al_2O_3	SiO_2
0.00	24.80	45.01	3.50	10.00	34.40	39.02	1.87	20.00	17.80	49.66	2.72
0.50	22.40	48.69	0.96	10.50	22.80	43.45	2.29	20.50	24.60	45.49	1.86
1.00	19.80	49.93	1.28	11.00	26.60	42.58	3.91	21.00	21.40	48.03	2.32
1.50	27.80	45.17	0.89	11.50	32.40	38.72	3.34	21.50	24.00	45.49	2.30
2.00	25.80	43.77	0.49	12.00	27.40	42.92	4.34	22.00	26.00	44.05	2.34
2.50	23.00	48.69	0.52	12.50	25.80	43.89	3.20	22.50	19.00	49.02	2.95
3.00	34.40	39.38	0.80	13.00	31.40	38.13	3.74	23.00	24.00	32.34	2.75
3.50	29.60	42.26	1.33	13.50	26.80	42.92	3.04	23.50	15.60	41.90	2.04
4.00	30.80	32.40	1.19	14.00	24.40	46.13	1.68	24.00	19.20	47.11	4.63
4.50	35.80	33.53	1.18	14.50	25.60	44.85	2.42	24.50	17.80	48.70	3.07
5.00	40.40	31.74	1.77	15.00	23.80	44.85	3.33	25.00	21.20	46.77	3.07
5.50	40.00	32.68	1.32	15.50	21.60	47.41	2.17	25.50	30.00	49.39	4.10
6.00	15.20	32.56	1.69	16.00	27.60	44.53	3.22	26.00	23.60	45.01	4.17
6.50	17.60	49.98	2.18	16.50	19.80	48.05	2.70	26.50	25.20	42.62	5.21
7.00	24.20	45.83	2.90	17.00	27.80	44.21	2.76	27.00	17.20	50.20	3.25
7.50	19.40	4.69	3.30	17.50	21.20	48.70	2.11	27.50	20.60	48.13	4.73
8.00	35.30	33.95	2.11	18.00	19.00	49.34	2.34	28.00	25.00	43.23	5.34
8.50	31.40	39.02	2.00	18.50	28.60	44.21	1.23	28.50	18.00	56.88	4.55
9.00	28.21	32.52	1.99	19.00	24.80	45.49	2.14	29.00	25.00	44.20	3.54
9.50	34.00	39.02	2.11	19.50	23.40	46.99	1.59	29.50	21.40	46.77	3.99

***Example* 1:** Frequency analysis of Fe_2O_3 element

For the present, we ignore the sampling interval along the borehole, although this will be taken into account later in our discussion. To arrive at the empirical frequency distribution, three steps are implemented:

(i) We choose convenient class intervals,
(ii) group these data into class intervals in terms of tally marks, and
(iii) draw the graph.

In respect of the above mentioned bore hole data giving Fe_2O_3, the class width was chosen as 4.5 units and the first class interval as 9.00-13.50. Following the steps mentioned above, we have the classification of data (Table 2.2) giving the frequency distribution. This approach gives the frequency distribution as shown in Table 2.3.

Table 2.2 Classification of the data

Class Interval (in % units)	*Mid. Pt.*	*Tally Marks*	*Frequency*
9.00–13.50	11.25		0
13.50–18.00	15.75	𝍸 1	6
18.00–22.50	20.25	𝍸 𝍸 1111	14
22.50–27.00	24.75	𝍸 𝍸 𝍸 𝍸 1	21
27.00–31.50	29.25	𝍸 𝍸 11	12
31.50–36.00	33.75	𝍸	5
36.00–40.50	38.25	11	2

Table 2.3 Frequency Distribution

Class Interval (in % units)	*Mid. Pt.*	*Frequency*	*Relative Frequency*	*Cumulative Frequency*	*Relative Cumulative Frequency*
9.00–13.50	11.25	0	0.0	0	0.0
13.50–18.00	15.75	6	10.0	6	10.0
18.00–22.50	20.25	14	23.3	20	33.3
22.50–27.00	24.75	21	35.0	41	68.3
27.00–31.50	29.25	12	20.0	53	88.3
31.50–36.00	33.75	5	8.3	58	96.6
36.00–40.50	38.25	2	3.3	60	100.0

One can change the width of the class interval as per convenience and choice. However, it is to be ensured that the essential information is not lost in this process of grouping. It is customary to agree that a class interval includes measurements upto but not including the upper boundary. Thus an observation having a value 13.49 falls in the first class interval and an observation having a value 13.50 falls in the second class interval. Thus, a measurement that falls on a boundary is placed in the higher of the two intervals. It is important to use the exact class marks; otherwise, a systematic error will be introduced in the computations.

2.3 GRAPHICAL REPRESENTATION OF FREQUENCY DISTRIBUTION

It is often useful to represent a frequency distribution by means of a diagram because it conveys in a nutshell the behaviour patterns of the observations. A diagrammatic representation also facilitates comparison of two or more frequency distributions.

Histograms

In drawing the histograms of a given continuous frequency distribution, we first mark all the class intervals along the X-axis on a suitable scale. In each class interval, erect rectangles with heights proportional to the frequency of the corresponding class interval so that the area of the rectangle is proportional to the frequency of the class. If, however, the classes are of unequal width, the height of the rectangles will be proportional to the ratio of the frequencies to the width of the classes. The diagram of continuous rectangles so obtained is called *Histogram*. The histogram for a sample set of data of Fe_2O_3 element is shown in Fig. 2.1. As may be seen, the histogram is more or less bell shaped and it suggests that a Gaussian (Normal) distribution could be fitted to the above data.

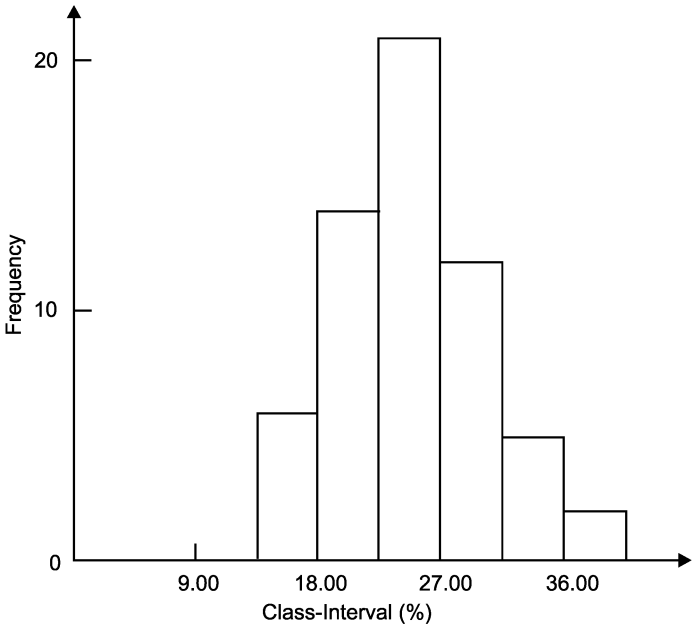


Fig. 2.1 Distribution of a sample set of Fe_2O_3 element values.

Example 2:

Let us consider another example. This one relates to the occurrence of 72 gold assay values in dwts/ton drawn at regular intervals along a drive of *lode* O *of gold field* 1 (1 dwt = 1.55517 gms of gold/ton of ore). At each sampling point, the width of the reef is also noted. The data are given in Table 2.4.

Table 2.4 A sample set of gold assay values along a drive of lode O of gold field 1

Distance (ft.)	*Accumulation (inch-dwt)*	*Grade (dwts)*	*Width (inches)*
403	600	4	150
407	600	4	150
411	480	4	120
415	480	4	120
419	480	4	120
424	342	3	114
428	342	3	114
432	317	5	74
437	317	5	74
440	370	5	74
445	288	4	72
449	240	4	60
453	330	5	66

(*Contd.*)

Distance (ft.)	*Accumulation (inch-dwt)*	*Grade (dwts)*	*Width (inches)*
456	288	4	72
461	504	7	72
466	432	6	72
471	240	4	60
475	504	7	72
479	360	5	72
483	648	9	72
488	360	5	72
492	1020	17	60
501	480	8	60
506	108	2	54
509	120	2	60
513	180	3	60
518	504	7	72
522	504	7	72
524	30	1	30
528	378	6	63
533	240	5	48
537	560	14	40
541	144	4	36
545	264	4	66
550	588	7	84
554	600	10	60
559	630	9	70
564	702	9	78
569	600	10	60
574	792	11	72
577	924	11	84
583	360	3	120
592	472	4	118
596	472	4	118
600	472	4	118
604	1440	12	120
608	1440	12	120
612	1536	12	128
616	1536	12	128
620	1536	12	128
624	792	11	72
632	702	9	78
636	336	7	48
641	1386	21	66
644	1008	14	72
648	672	12	56
652	468	6	78
655	504	7	72
660	288	4	72
663	360	5	72
668	300	5	60
670	648	9	72

673	360	5	72
675	284	4	71
678	288	4	72
683	288	4	72
688	1152	16	72
692	792	11	72
695	340	5	68
697	198	3	66
700	544	8	68
703	2178	18	121

The frequency distributions of the variable accumulation and of the logarithms of the variable are given in Tables 2.5 and 2.6 respectively.

Table 2.5 Empirical frequency distribution of gold assay values for the variable accumulation

Class Interval (inch-dwts)	*Mid. Pt.*	*Freq.*	*Cum. Freq.*
1–175	88.0	4	4
175–350	262.5	16	20
350–525	437.5	24	44
525–700	612.5	11	55
700–875	787.5	6	61
875–1050	962.5	4	65
1050–1225	1137.5	1	66
1225–1400	1312.5	1	67
1400–1575	1487.5	4	71
1575–1750	1662.5	0	71
1750–1925	1837.5	0	71
1925–2100	2012.5	0	71
2100–2275	2187.5	1	72

Table 2.6 Frequency distribution of logarithms of gold assay values for the variable accumulation

Class Interval	*Mid Pt.*	*Freq.*	*Cum. Freq.*
4.00–4.50	4.25	0	0
4.50–5.00	4.75	3	3
5.00–5.50	5.25	6	9
5.50–6.00	5.75	21	30
6.00–6.50	6.25	25	55
6.50–7.00	6.75	10	65
7.00–7.50	7.25	6	71
7.50–8.00	7.75	1	72

As we see, the distribution of gold assay values is far from normal. It is positively skewed—the tail end is to the right. The distribution of the logarithms of the 72 gold assay values is approximately normal. (See Tables 2.5 and 2.6 and Figs 2.2 and 2.3)

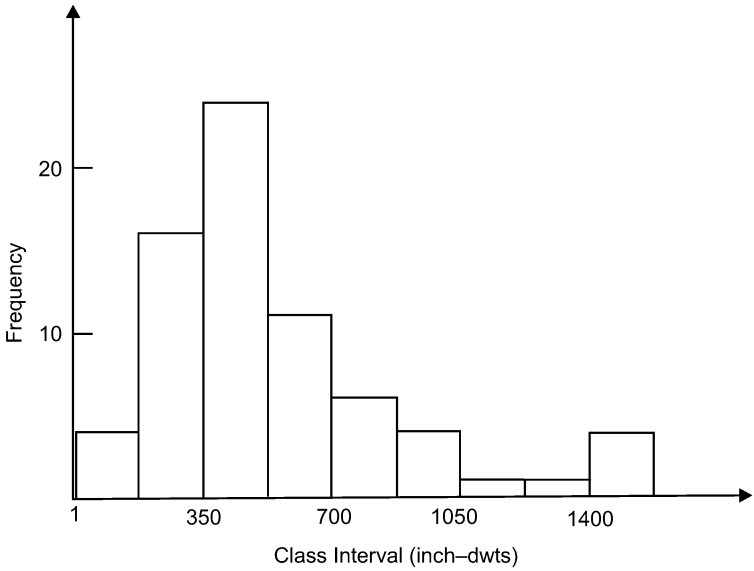


Fig. 2.2 Distribution of a sample set of gold accumulation values in respect of lode O, gold field 1.

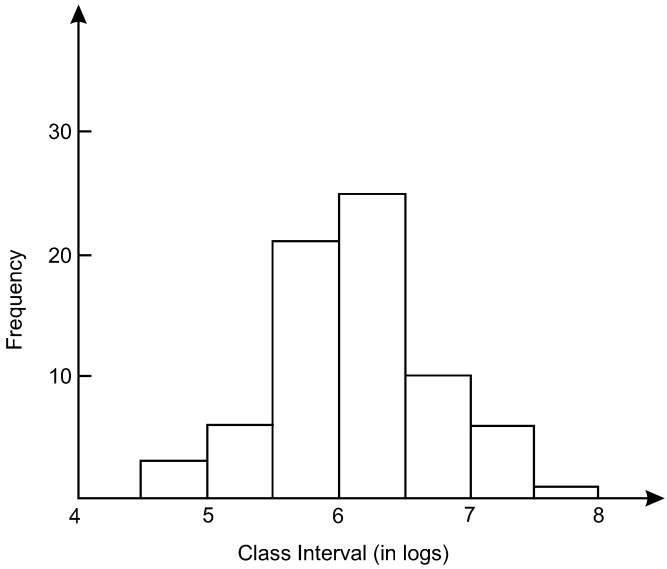


Fig. 2.3 Distribution of logarithms of a sample set of gold accumulation values in respect of lode O, gold field 1.

Let us now study the distribution for the variable grade. The distributions are given in Tables 2.7 and 2.8 for the original variable and the log-transformed variable.

Table 2.7 Frequency distribution of gold assay values for the variable grade

Class Interval (dwts)	*Mid. Pt.*	*Freq.*	*Cum. Freq.*
0.10–2.00	1.05	1	1
2.00–4.00	3.00	8	9
4.00–6.00	5.00	27	36
6.00–8.00	7.00	10	46
8.00–10.00	9.00	8	54
10.00–12.00	11.00	7	61
12.00–14.00	13.00	6	67
14.00–16.00	15.00	2	69
16.00–18.00	17.00	2	71
18.00–20.00	19.00	0	71
20.00–22.00	21.00	1	72

Table 2.8 Frequency distribution of logarithms of gold assay values for the variable grade

Class Interval	*Mid Pt.*	*Freq.*	*Cum. Freq.*
0.00–0.40	0.200	1	1
0.40–0.80	0.600	2	3
0.80–1.20	1.000	6	9
1.20–1.60	1.400	17	26
1.60–2.00	1.800	20	46
2.00–2.40	2.200	15	61
2.40–2.80	2.600	9	70
2.80–3.20	3.000	2	72
3.20–3.60	3.400	0	72

The histograms for the original grade values and the log-transformed grade values are shown in Figs 2.4 and 2.5 respectively.

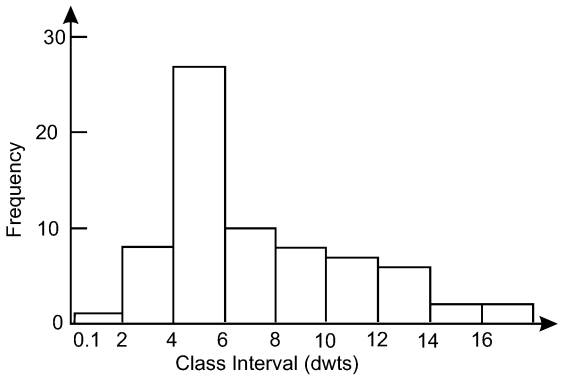


Fig. 2.4 Distribution of a sample set of gold assay values for the variable grade in units of dwts/ton of ore.

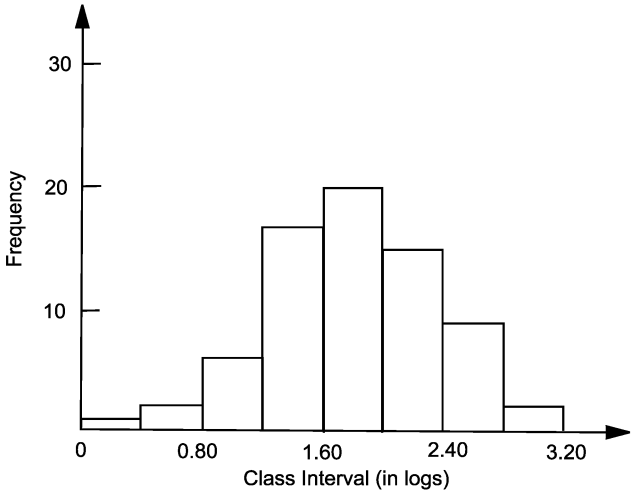


Fig. 2.5 Distribution of logarithms of a sample set of gold assay values for the variable grade.

Example 3:

This example relates to copper assay data drawn from a drive of a copper deposit in southern India. A set of 94 copper assay values in units of % together with the corresponding widths were considered for this type of analysis. This sample set of data is given in Table 2.9.

Table 2.9 A sample set of copper assay values

Distance (m)	*Accumulation (cm%) (Tenor × Thickness)*	*Tenor (%)*	*Thickness (cm)*
1.00	158.90	4.54	35.00
2.00	360.60	6.01	60.00
3.00	96.32	2.24	43.00
4.00	117.20	2.93	40.00
5.00	135.66	3.23	42.00
6.00	140.50	2.81	50.00
7.00	241.40	7.10	34.00
8.00	82.60	2.36	35.00
9.00	22.25	0.89	25.00
10.00	318.00	6.36	50.00
11.00	105.50	4.22	25.00
12.00	32.55	2.17	15.00
13.00	254.10	8.47	30.00
14.00	490.05	8.91	55.00
15.00	123.20	6.16	20.00
16.00	115.50	3.85	30.00
17.00	231.00	7.70	30.00
18.00	58.00	0.40	145.00

19.00	237.90	1.30	183.00
20.00	84.00	0.40	210.00
21.00	52.00	2.60	20.00
22.00	41.00	0.20	205.00
23.00	43.00	0.20	215.00
24.00	118.00	5.90	20.00
25.00	186.00	6.20	30.00
26.00	480.00	8.00	60.00
27.00	131.00	13.10	10.00
28.00	51.00	0.60	85.00
29.00	58.00	5.80	10.00
30.00	25.50	1.70	15.00
31.00	232.00	11.60	20.00
32.00	332.00	16.60	20.00
33.00	260.00	13.00	20.00
34.00	171.50	4.90	35.00
35.00	123.00	4.10	30.00
36.00	99.00	2.20	45.00
37.00	63.00	0.30	210.00
38.00	42.00	0.20	210.00
39.00	222.00	7.40	30.00
40.00	150.00	5.00	30.00
41.00	192.50	3.50	55.00
42.00	44.00	0.20	220.00
43.00	30.00	0.20	150.00
44.00	405.00	4.50	90.00
45.00	100.00	0.50	200.00
46.00	80.00	0.40	200.00
47.00	160.00	1.00	160.00
48.00	150.00	1.00	150.00
49.00	75.00	0.30	250.00
50.00	78.00	0.30	260.00
51.00	262.50	1.05	230.00
52.00	300.00	2.00	150.00
53.00	171.60	2.86	60.00
54.00	396.00	6.60	60.00
55.00	184.80	1.54	120.00
56.00	1044.00	5.80	180.00
57.00	156.00	0.80	195.00
58.00	199.80	3.33	60.00
59.00	165.00	3.30	50.00
60.00	56.00	1.40	40.00
61.00	468.00	3.90	120.00
62.00	498.00	4.15	120.00
63.00	237.00	2.37	100.00
64.00	1298.70	9.99	130.00
65.00	686.70	7.63	90.00
66.00	199.80	2.22	90.00
67.00	445.20	3.71	120.00

(*Contd.*)

Distance (m)	*Accumulation (cm%) (Tenor × Thickness)*	*Tenor (%)*	*Thickness (cm)*
68.00	645.60	8.07	80.00
69.00	264.45	1.23	215.00
70.00	81.00	0.36	225.00
71.00	95.40	1.06	90.00
72.00	156.00	2.60	60.00
73.00	43.20	2.16	20.00
74.00	149.60	1.87	80.00
75.00	79.35	0.69	115.00
76.00	214.80	1.79	120.00
77.00	135.80	0.97	140.00
78.00	453.60	3.78	120.00
79.00	219.00	1.46	150.00
80.00	272.00	1.60	170.00
81.00	236.60	1.82	130.00
82.00	313.60	1.96	160.00
83.00	162.00	0.81	200.00
84.00	304.00	1.27	240.00
85.00	271.80	1.51	180.00
86.00	266.40	1.48	180.00
87.00	110.40	0.92	120.00
88.00	61.80	1.03	60.00
89.00	258.00	2.58	100.00
90.00	393.40	2.81	140.00
91.00	227.91	2.13	107.00
92.00	216.00	1.44	150.00
93.00	322.00	1.79	180.00
94.00	222.30	1.17	190.00

The frequency distributions for the variable accumulation and logarithms of accumulation values are given in Tables 2.10 and 2.11 respectively.

Table 2.10 Frequency distribution of copper assay values for the variable accumulation

Class Interval (cm%)	*Mid. Pt.*	*Freq.*	*Cum. Freq.*
1.0–50.0	25.5	9	9
50.0–150.0	100	30	39
150.0–250.0	200	27	66
250.0–350.0	300	14	80
350.0–450.0	400	6	86
450.0–550.0	500	5	91
550.0–650.0	600	1	92
650.0–750.0	700	1	93
750.0–850.0	800	0	93
850.0–950.0	900	0	93
950.0–1050.0	1000	1	94

Table 2.11 Frequency distribution of logarithms of copper assay values for the variable accumulation

Class Interval	*Mid Pt.*	*Freq.*	*Cum. Freq.*
2.25–3.00	2.625	0	0
3.00–3.75	3.375	6	6
3.75–4.50	4.125	17	23
4.50–5.25	4.875	28	51
5.25–6.00	5.625	32	83
6.00–6.75	6.375	9	92
6.75–7.50	7.125	2	94
7.50–8.25	7.875	0	94

The histograms for the original variable accumulation and the log-transformed variable are shown in Figs 2.6 and 2.7 respectively.

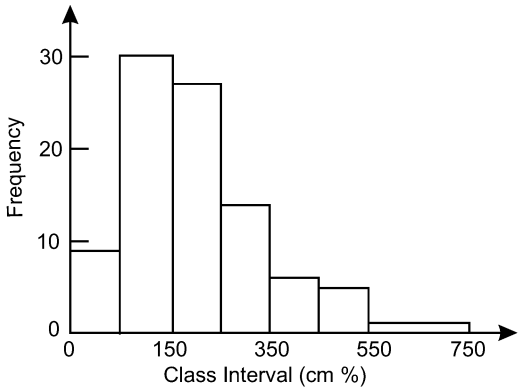


Fig. 2.6 Distribution of a set of copper accumulation values in units of cm %.

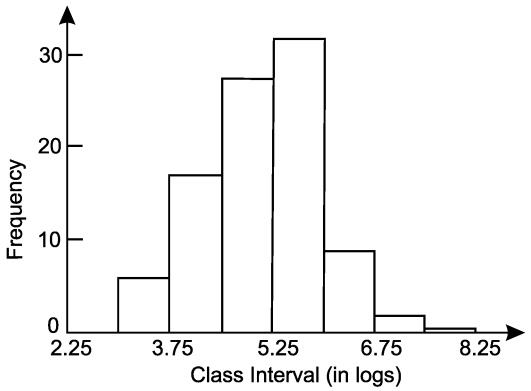


Fig. 2.7 Distribution of logarithms of a set of copper accumulation values.

The frequency distributions for variable tenor and the logarithms of tenor are given in Tables 2.12 and 2.13 respectively.

Table 2.12 Frequency distribution of copper assay values for the variable tenor

Class Interval (in % units)	*Mid. Pt.*	*Freq.*	*Cum. Freq.*
0.1–1.0	0.55	20	20
1.0–2.0	1.50	22	42
2.0–3.0	2.50	16	58
3.0–4.0	3.50	8	66
4.0–5.0	4.50	6	72
5.0–6.0	5.50	4	76
6.0–7.0	6.50	5	81
7.0–8.0	7.50	4	85
8.0–9.0	8.50	4	89
9.0–10.0	9.50	1	90
10.0–11.0	10.50	0	90
11.0–12.0	11.50	1	91
12.0–13.0	12.50	0	91
13.0–14.0	13.50	2	93
14.0–15.0	14.50	0	93
15.0–16.0	15.50	0	93
16.0–17.0	16.50	1	94

Table 2.13 Frequency distribution of logarithms of copper assay values for the variable tenor

Class Interval	*Mid. Pt.*	*Freq.*	*Cum. Freq.*
–2.00 to –1.30	–1.65	5	5
–1.30 to –0.60	–0.95	8	13
–0.60 to 0.10	–0.25	12	25
0.10 to 0.80	0.45	23	48
0.80 to 1.50	1.15	21	69
1.50 to 2.20	1.85	20	89
2.20 to 2.90	2.55	5	94
2.90 to 3.60	3.25	0	94

The respective histograms for the above distributions are shown in Figs 2.8 and 2.9.

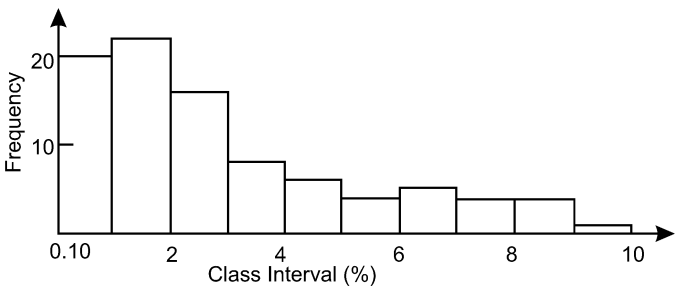


Fig. 2.8 Distribution of a set of copper tenor values in units of %.

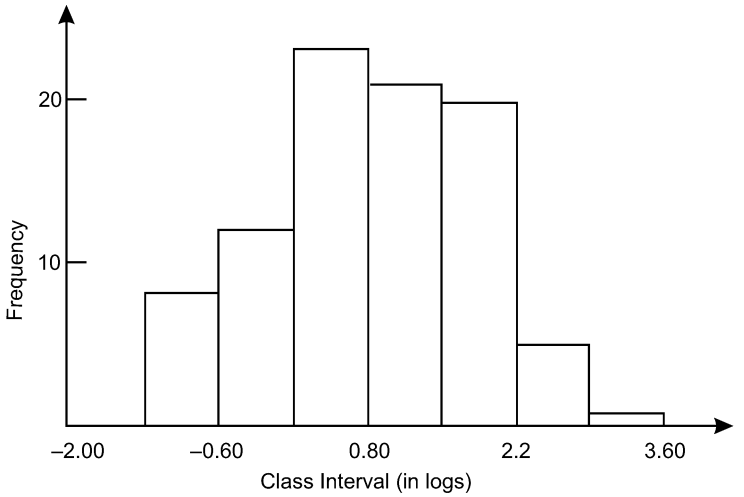


Fig. 2.9 Distribution of logarithms of a set of copper tenor values.

2.4 ARITHMETIC REPRESENTATION OF EMPIRICAL DISTRIBUTIONS

2.4.1 Measure of Central Location

To arrive at an accurate quantitative information about the underlying distribution, lower-order moments of the distribution, as defined below, are usually computed. In many problems, one is concerned with the first four moments only and more often with the first two moments. For data grouped into various class intervals, let x_i be the class mark for the ith class interval, f_i the observed frequency for the ith interval, k the number of intervals and N the sum of the absolute frequencies. The rth moment about origin A of an empirical frequency distribution is given by: $\mu'_r = \frac{1}{N} \sum_{i=1}^{k} (x_i - A)^r f_i$, where $\Sigma f_i = N$. For ungrouped data, x_i represents the value of the ith observation, f_i is equal to unity and the summation is over the total number of observations (n).

The first moment as a measure of location: The first moment (μ'_1) about the origin A is defined as

$$\mu'_1 = \frac{1}{N} \sum_{i=1}^{k} (x_i - A) f_i \tag{2.1}$$

and therefore $\bar{x}$ (mean) $= A + \mu'_1$. When $A = 0$, we have $\bar{x} = \frac{1}{N} \sum_{i=1}^{n} x_i f_i$.

For unclassified data: $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$.

Note: The *r*th moment about the mean $\bar{x}$ is denoted as $\mu_r = \frac{1}{N}\sum_i (x_i - \bar{x})^r f_i$.

In particular, $\mu_0 = \frac{1}{N}\sum(x_i - \bar{x})^0 f_i = 1$; and μ_1 – the first moment about the mean $\bar{x} = \frac{1}{N}\sum_i (x_i - \bar{x})^1 f_i = 0$.

The following are the mean values in respect of Fe_2O_3 element concentration values:

(i) Mean of ungrouped data = 25.17% and
(ii) Mean of grouped data = 24.9%.

2.4.2 Measures of Dispersion

Out of various measures of dispersion, the second moment is more often used to quantify the dispersion in a series of data. It is customary to assume variation as variation of data about a measure of location. Usually, mean is used as the measure of location. As mentioned above the *r*th moment about the mean ($\bar{x}$) of an empirical frequency distribution is expressed as:

$$\mu_r = \frac{1}{N}\sum_{i=1}^{k}(x_i - \bar{x})^r f_i \tag{2.2}$$

It follows that the second moment μ_2 is: $\frac{1}{N}\Sigma(x_i - \bar{x})^2 f_i$. It is appropriate to have the measure of variation in the same units as of the data. Therefore $\sqrt{\mu_2}$ is usually chosen. This is known as **standard deviation** and may be denoted as *s*. Thus:

$$s = \sqrt{\frac{1}{N}\sum_{i=1}^{k}(x_i - \bar{x})^2 f_i} \tag{2.3}$$

The variance (μ_2) for grouped data of Fe_2O_3 = 29.63% and the standard deviation = 5.44%.

2.4.3 Skewness and Kurtosis

Skewness: Skewness means lack of symmetry. A distribution is said to be symmetrical if the frequencies are symmetrically distributed about the mean. For such a distribution, the mean, mode and median coincide. A distribution is said to be positively skewed, if the tail end is more to the right. If the tail end is towards the left, the distribution is said to be negatively skewed. This is to say that frequencies are not symmetrically distributed about the mean.

Kurtosis: A frequency curve may be symmetrical but it may fail to be equally flat topped with the normal curve. The relative flatness of top of a distribution vis-à-vis the normal curve is called kurtosis. It is represented as β_2.

β_1, β_2 *and* γ *Coefficients*

These coefficients are defined as:

$$\beta_1 = \mu_3^2/\mu_2^3;\ \gamma_1 = +\sqrt{\beta_1} \qquad (2.4)$$

$$\beta_2 = \mu_4/\mu_2^2$$

Measure of skewness based on moments

Skewness: Based on β_1 and β_2 coefficients, the coefficient of skewness is defined as:

$$C_{sk} = [\sqrt{\beta_1}\,(\beta_2+3)]/[2(5\beta_2 - 6\beta_1 - 9)]$$

Since β_2 cannot be negative being a squared expression, C_{sk} is zero, if and only if $\beta_1 = 0$. Therefore, for practical purposes β_1 is taken as a measure of skewness.

Kurtosis: A measure of relative flatness is given as $\beta_2 = \mu_4/\mu_2^2$. For a normal distribution $\beta_2 = 3$. Hence for any distribution, the quantity $\gamma_2 = \beta_2 - 3$ is called the excess of Kurtosis. Curves with values of $\beta_2 < 3$ are called *Platykurtic* while those with values of $\beta_2 > 3$ are called *Leptokurtic*.

2.5 CORRELATION AND REGRESSION

In a bivariate distribution where two variables are involved, we may be interested to find out if any correlation exists between two variates (variables) under study. The existence of a change in one variable, say X, in sympathy with a change in another variable say Y, is called correlation. In this case we say that the variables X and Y are correlated. If the change is in the same direction, the correlation is said to be positive. If the variables deviate in the opposite directions, the correlation is said to be negative. Correlation is said to be perfect if the deviation in one variable is followed by a corresponding deviation in the other.

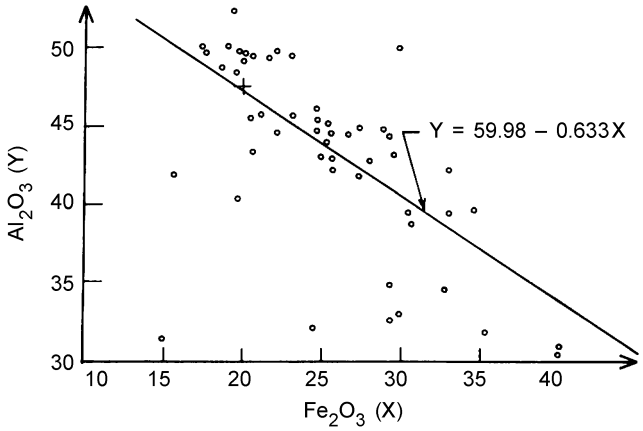


Fig. 2.10 Scatter diagram of Fe_2O_3 (X) and Al_2O_3 (Y) values.

Scatter Diagram: The approximate form of the relationship between the two variables X and Y can be guessed by studying the graph of the data called Scatter Diagram. By means of the graph one can easily discern whether there is any pronounced relationship between the two variables and, if so, whether the relationship can be treated as approximately linear. Figure 2.10 shows the scatter diagram of the two variables Fe_2O_3 and Al_2O_3 of the bauxite sample given in Table 1.

If we change the scale to the form $u_i = (x_i - \bar{x})/s_x$ and $v_i = (y_i - \bar{y})/s_y$, where $\bar{x}$ and $\bar{y}$ are the mean values of Fe_2O_3 and Al_2O_3 respectively and s_x and s_y are the respective standard deviations, the scatter diagram turns out to be Fig. 2.11. This procedure may be termed as standardisation.

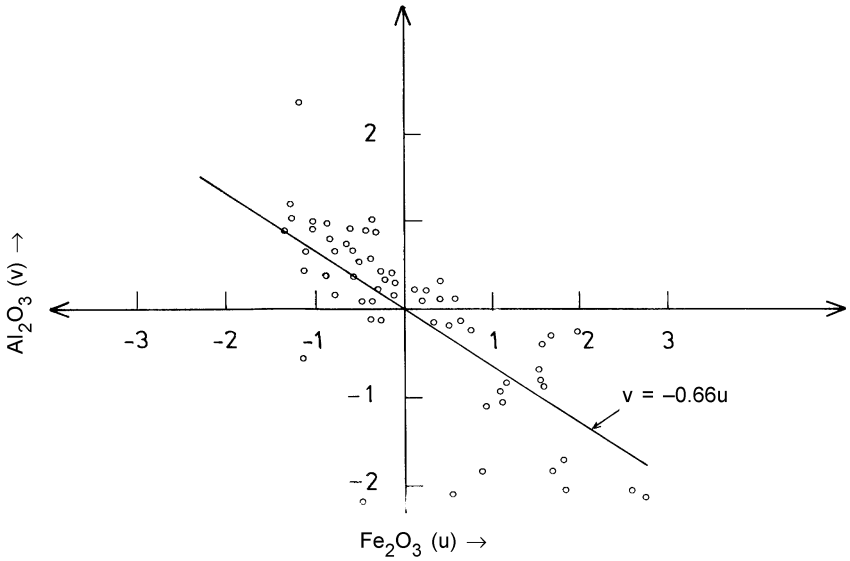


Fig. 2.11 Scatter diagram of standardised Fe_2O_3 (u) and Al_2O_3 (v) values.

2.5.1 Correlation Coefficient

A measure of the intensity or degree of linear relationship between the variables X and Y may be given as:

$$r(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} \tag{2.5}$$

$$= \frac{E[\{X - E(X)\}\{Y - E(Y)\}]}{\sqrt{E[\{X - E(X)\}]^2 E[\{Y - E(Y)\}]^2}} \tag{2.6}$$

estimated as:
$$= \frac{\frac{1}{n}\Sigma\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)}{\sqrt{\frac{1}{n}\Sigma\left(x_i - \bar{x}\right)^2 \frac{1}{n}\Sigma\left(y_i - \bar{y}\right)^2}} \tag{2.7}$$

In the above discussion on r, the following points are noteworthy: (1) r lies between –1 and +1 and (2) r is independent of the origin and scale.

Referring to the bauxite example, the correlation coefficient between Fe_2O_3 and Al_2O_3 works out to –0.66. The sample size is 60. Thus there is a strong negative correlation between Fe_2O_3 and Al_2O_3 which indicates that as Fe_2O_3 increases Al_2O_3 decreases and vice versa.

2.5.2 Regression

(a) Linear

The regression technique is used to study the relationship between two variables in the hope that any relationship that we find can be used to assist in making estimates/predictions of one variable. Thus, if there is a relationship between say geomagnetic intensity (X) and the number of road accidents (Y), we may like to predict Y for a given value of X. Here X is the independent variable and Y the dependent variable. We may have a relation as: $Y = a + bX$, where 'a' is the intercept and 'b' is the slope. In terms of r, σ_x and σ_y, this relation works out to:

$$Y - \bar{Y} = r\frac{\sigma_Y}{\sigma_X}(X - \bar{X}) \tag{2.8}$$

$$Y = \left(-r\frac{\sigma_Y}{\sigma_X}\bar{X} + \bar{Y}\right) + r\frac{\sigma_Y}{\sigma_X}X \tag{2.9}$$
$$= a + bX$$

The regression equation between Fe_2O_3 and Al_2O_3 works out to $Y = 59.98 - 0.633X$ where X represents Fe_2O_3 and Y represents Al_2O_3.

When we take the standardised values of $Fe_2O_3(u)$ and Al_2O_3 (v), the regression of v on u works out to $v = -0.66u$.

(b) Multiple Linear Regression

Sometimes it happens that there is no single variable sufficiently closely related to the variable being estimated to yield good results. It is possible that when several variables are taken jointly, the estimate of the derived variable may be satisfactory. For example, Al_2O_3 may be estimated taking other variables Fe_2O_3, SiO_2, TiO_2, etc. For this purpose, let Y be the dependent variable and $X_1, X_2, ..., X_k$ represent the independent variables and consider the problem of estimating the variable Y as a function of the remaining variables. If Y stands for the variable to be estimated, the function may be written as:

$$\hat{Y} = k_0 + k_1X_1 + k_2X_2 + \ldots + k_nX_n + E \tag{2.10}$$

where k's are regression coefficients and E is random error. The k's are determined on the basis of available data. These unknown coefficients can be estimated by the method of least-squares assuming that a set of values of the $n + 1$ variables are available. We are now required to find the values k_is such that $\Sigma(Y_i - \hat{Y})^2$ is minimised. For ease in computations, it is often convenient to work with the variable measured from their sample means than with the variables themselves. Writing $y = Y - \overline{Y}$ and $x_j = X_j - \overline{X}_j$ (j = 1, 2, …, k) and denoting $\hat{y} = \hat{Y} - \overline{Y}$, we have:

$$Y - \hat{Y} = y + \overline{Y} - (\hat{y} + \overline{Y}) = y - \hat{y}. \tag{2.11}$$

It may be noted that minimising $\Sigma(Y - \hat{Y})^2$ is equivalent to minimising $\Sigma(y - \hat{y})^2$. If we represent X's and Y in terms of x's and y, eqn. (2.10) above can be written in the form:

$$\hat{y} = A_o + A_1x_1 + A_2x_2 + \ldots + A_kx_k + \in \tag{2.12}$$

where $\in$ is random error based on transformed units.

For obtaining A_0, A_1, …, A_k, we form a set of normal equations:

$$\left.\begin{aligned} A_0n + A_1\Sigma x_1 + \ldots + A_kx_k &= \Sigma y \\ A_0\Sigma x_1 + A_1\Sigma x_1^2 + \ldots + A_k\Sigma x_1x_k &= \Sigma x_1y \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & \\ A_0\Sigma x_k + A_1\Sigma x_kx_1 + \ldots + A_k\Sigma x_k^2 &= \Sigma x_ky \end{aligned}\right\} \tag{2.13}$$

which when written in matrix notation takes the form:

$$\begin{bmatrix} n & \Sigma x_1 & \Sigma x_2 & \ldots & \Sigma x_k \\ \Sigma x_1 & \Sigma x_1^2 & \Sigma x_1x_2 & \ldots & \Sigma x_1x_k \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \Sigma x_k & \Sigma x_kx_1 & \Sigma x_kx_2 & & \Sigma x_k^2 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \ldots \\ A_k \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma x_1y \\ \\ \Sigma x_ky \end{bmatrix} \tag{2.14}$$

As we see, the k's in the regression equation model (2.10) are estimated by A's in (2.13)—the sample partial regression coefficients. They are called partial regression coefficients because each coefficient reflects the rate of change in the dependent variable for a unit change in that particular independent variable, provided all other independent variables remain constant. Some authors prefer to use the following notation (example of three variables):

$$\hat{Y} = a_0 + a_{1.23}X_1 + a_{2.13}X_2 + \ldots + a_{3.12}X_3 + \in \tag{2.15}$$

to emphasise the above point.

(c) Polynomial Regression

Let us assume a situation where the independent variable is X and the dependent variable is Y. If there are no compelling reasons to fit a curve of

a certain type to describe the relationship, polynomials are usually selected because of their simplicity and flexibility. Usually, the degree of the curve is determined by a look at the scatter diagram. If we are interested in fitting a curve of degree—say k, we write:

$$\hat{Y} = A_0 + A_1X + A_2X^2 + \ldots + A_kX^k + \in \tag{2.16}$$

Following the approach detailed in multiple regression analysis, we may write the normal equations for polynomial regression in terms of original variables Y and X as:

$$\begin{aligned} A_0\, n + A_1\Sigma X + \ldots + \Sigma A_k X^k &= \Sigma Y \\ A_0\Sigma X + A_1\Sigma X^2 + \ldots + \Sigma A_k X^{k+1} &= \Sigma XY \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & \\ A_0\Sigma X^k + A_1\Sigma X^{k+1} + \ldots \Sigma A_k X^{2k} &= \Sigma X^k Y \end{aligned} \tag{2.17}$$

In matrix notation, it may be written as:

$$\begin{bmatrix} n & \Sigma X & \ldots & \Sigma X^k \\ \Sigma X & \Sigma X^2 & \ldots & \Sigma X^{k+1} \\ \ldots & \ldots & \ldots & \ldots \\ \Sigma X^k & \Sigma X^{k+1} & & \Sigma X^{2k} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \ldots \\ A_k \end{bmatrix} = \begin{bmatrix} \Sigma Y \\ \Sigma XY \\ \ldots \\ \Sigma X^k Y \end{bmatrix} \tag{2.18}$$

It will suffice to have $k + 1$ distinct X values, since a polynomial of degree k is uniquely determined by $k + 1$ points.

Again referring to the bauxite elements example, suppose, we may be interested in fitting a fourth order polynomial of the following type between Al_2O_3 (Y) and Fe_2O_3 (X). Thus:

$$\hat{Y} = A_0 + A_1X + A_2X^2 + A_3X^3 + A_4X^4 + \in \tag{2.19}$$

For the data mentioned above, and solving the normal equations, we have:

$$\hat{Y} = -85.67 + 4.41\ X + 0.10\ X^2 - 0.01\ X^3 + 0.0001\ X^4 + \text{error term} \tag{2.20}$$

(d) Other Regression Functions

There are other regression functions of the type:

$Y = Ce^{pX}$, where C and p are parameters. These can be fitted as per the procedures described above.

2.6 SIMULATION

2.6.1 Introduction

Simulation is the representative model for real situations. In laboratories, we often perform a number of experiments on simulated models to predict the behaviour of real system under true environments. The environment in a

museum of natural history is a good example of simulation. Thus we may say that simulation is a representation of reality through the use of a model or other device which will react in the same manner as reality under a given set of conditions. Simulation is mainly of two types: (a) Analog simulation (or environmental simulation) and (b) Computer simulation (system simulation).

For complex and intricate problems of managerial decision making, the analog or actual experimentation with the system may be uneconomical also. Under these situations, the complex system is formulated into a mathematical model for which computer modelling is done. Such a type of simulation is called computer simulation or system simulation. Simulation models can be classified into four categories: (a) Deterministic models, (b) Stochastic models (c) Static models and (d) Dynamic models.

2.6.2 Advantages of Simulation

1. Simulation techniques allow experimentation with a model of the real-life system instead of the actual operating system. Sometimes experimenting with the actual system could prove to be too expensive and in several cases too disruptive. For example, if we compare two different ways of providing food service in a hospital, the confusion that may arise from operation of two different systems long enough to get valid observations might be too great. The operation of large computer centre under a number of different operating alternatives might be too costly to be feasible. Similarly, the experimentation on the earth's electro-magnetic field may be too expensive or infeasible at times.

2. Sometimes there may not be enough time to allow the actual system to operate extensively. For example, if we want to study long-term trends in a geological population, it is not possible to wait for desired number of years to see the results. The interesting feature is that simulation allows time to be incorporated into an analysis. In a computer situation of geological analysis, the geologist can compress the results of several years or periods into a few minutes of running time.

3. The non-geologist can comprehend simulation more easily than a complex mathematical model. Simulation does not require simplifications and assumptions to the extent needed in analytical solutions. A simulation model is easier to explain to management personnel since it is a description of behaviour of some system or a geological process.

4. Simulation enables a geologist to provide insights into certain managerial problems where analytical solutions of a model are not possible or where the actual environment is difficult to observe. For example, simulation is used in space flights or in the launching of a satellite or in studying a geological population.

2.6.3 Limitations of Simulation Techniques

(i) Optimum results cannot be produced by simulation. Since the model mostly deals with uncertainties, the results of simulation are only reliable

approximations involving statistical errors. (ii) Another drawback lies in the quantification of the variables. In many situations, it is not possible to quantify all the variables which affect the behaviour of the system. (iii) In very large and complex geological situations involving many variables, it becomes difficult to develop the computer program on account of large number of variables and the inter-relationships that are involved amongst them. The number of variables may be too large and may exceed beyond the capacity of the available computer. (iv) Simulation should be limited to complex situations and not applied to some simple problems which can otherwise be solved by more appropriate techniques of mathematical programing.

2.6.4 Generation of Random Numbers

Random numbers play a very important role in simulation as could be seen in the following examples. Generation of random numbers was discussed in Chapter I.

2.6.5 Monte-Carlo Simulation

Among the methods of simulation, Monte-Carlo technique has become so important that the term and simulation are often assumed to be synonymous. However, it is only a special technique of simulation. The technique of Monte-Carlo involves the selection of random observations within the simulation model.

This technique is restricted to application involving random numbers to solve deterministic and stochastic problems. The principle behind this technique is replacement of actual universe represented by a statistical universe by another universe described by some assumed probability distribution and then sampling from this theoretical population by means of random numbers. In fact, this process is the generation of simulated statistics that can be explained in simple terms as choosing a random number and substituting this value in standard probability density function to obtain a random variable or simulated statistics. Let us now look at a few examples of the application of simulation.

2.7 SOME APPLICATIONS OF SIMULATION

2.7.1 Applications to Inventory Control

Let us illustrate this technique with the help of the following simple inventory problem.

Example 1:

A book store wishes to carry 'Geostatistics' in stock. Demand is probabilistic and replenishment of stock takes two days (i.e., if an order is placed on March 1, it will be delivered at the end of the day on March 3). The probabilities of demand are given as:

Demand (daily):	0	1	2	3	4
Probability:	0.05	0.10	0.30	0.45	0.10

Each time an order is placed, the store incurs an ordering cost of Rs. 10 per order. The store also incurs a carrying cost of Rs. 1.0 per book per day. The inventory carrying cost is calculated on the basis of stock at the time of each day. The manager of the book store wishes to compare two options for his inventory decision.

A: Order five books when the inventory at the beginning of the day plus orders outstanding is less than eight books.
B: Order eight books when the inventory at the beginning of the day plus orders outstanding is less than eight. Currently (beginning of the first day) the store has stock of eight books plus six books ordered two days ago and expected to arrive next day.

Using Monte-Carlo simulation for 10 cycles, recommend which option the manager should choose. The two digit random numbers are given below: 89, 34, 78, 63, 61, 81, 39, 16, 13, 73.

Solution:

Table 2.14 Demand, probabilities and random numbers for geostatistics book

Demand	*Prob.*	*Cum.Prob.*	*Random Nos.*
0	0.05	0.05	00-04
1	0.10	0.15	05-14
2	0.30	0.45	15-44
3	0.45	0.90	45-89
4	0.10	1.00	90-99

Stock in hand = 8, and stock on order = 6 (expected next day).

Option A

Table 2.15 Demand generation corresponding to given random numbers and stock position

Random No.	*Demand sales*	*Opt. Stock in hand*	*Receipt*	*Cl. stock in hand*	*Opt. stock on order*	*Order Qty.*	*Cl. Stock on order*
89	3	8	-	5	6	-	6
34	2	5	6	9	-	-	-
78	3	9	-	6	-	5	5
63	3	6	-	3	5	-	5
61	3	3	-	0	5	5	10
81	3	0	5	2	5	5	10
39	2	2	-	0	10	-	10
16	2	0	5	3	5	-	5
13	1	3	5	7	0	5	5
73	3	7	-	4	5	-	5

No. of orders made = 4, Ordering cost = Rs. 4 × 10 = Rs. 40
Closing stock of 10 days (in hand) = 39, carrying cost = Rs. 39 × 1.00 = 39.00.
Cost for 10 days = Rs. 79.00.

Option B

Table 2.16 Open and closing stock positions for option B

Sales	*Opt. stock in hand*	*Receipt*	*Closing stock in hand*	*Opt. stock on order*	*Order Qty.*	*Closing stock on order*
3	8	-	5	6	-	6
2	5	6	9	-	-	-
3	9	-	6	-	8	8
3	6	-	3	8	-	8
3	3	-	0	8	-	8
3	3	8	5	-	8	8
2	5	-	3	8	-	8
2	3	-	1	8	-	8
1	1	8	8	-	-	-
3	8	-	5	-	8	8

No. of orders made = 3, ordering cost = Rs. 30. Closing stock of 10 days (in hand) = 45, carrying cost = Rs. 45 × 1.00 = Rs. 45.00. Since option B has lower cost, viz., Rs. 30+45 = Rs. 75/-, manager should choose option B.

Example 2

Consider an inventory situation in a manufacturing concern. If the number of sales per day follows a Poisson distribution with mean 5, then generate 30 days of sales by Monte-Carlo method.

Solution

Here the sales follow Poisson distribution with mean equal to 5. So we calculate the probabilities for demand from 0 to 12. The probability for sales s is given by $P(X=s) = (e^{-m} m^{s}) / s!$, where $m = 5$. The cumulative probabilities for s = 0, 1, 2,..12 are computed and shown in Table 2.17.

Table 2.17 Generation of probabilities based on Poisson law

Value of s	*Cumulative probability*	*Random no. range*	*Value of s*	*Cumulative probability*	*Random no. range*
0	.0067	00	7	.8666	76–86
1	.0404	01-03	8	.9319	87–92
2	.1247	04-11	9	.9682	93–96
3	.2650	12-26	10	.9763	97
4	.4405	27-43	11	.9845	98
5	.6160	44-61	12	1.0000	99
6.	.7622	62-75			

Now we take 30 two-digit random numbers from random number tables and read the corresponding values of sales s from random number class-intervals listed in Table 2.17. These values will give us the sales for 30 days. The values are tabulated here (Table 2.18).

Table 2.18 Generated sales corresponding to generated random numbers

Random number	*Sales (s)*	*Random number*	*Sales (s)*	*Random number*	*Sales (s)*
10	02	81	07	46	05
48	05	64	06	57	05
01	00	79	07	32	04
50	05	16	03	55	05
11	02	46	05	95	09
01	00	69	06	85	07
53	05	17	03	39	04
60	05	92	08	33	04
20	03	23	03	09	02
11	02	68	06	93	09

2.7.2 Applications to Gold Mineralisation

Simulation for the Variable Accumulation (cm-gms)

Accumulation is available for 19 blocks of ore each of 30 m × 30 m. These blocks are drawn from various locations in a mine. The problem is to employ the simulation technique and generate the accumulation figures.

Table 2.19 Application of simulation technique to gold accumulation

Block No. (1)	*Accumulation (cm-gms) (2)*	*Probability (3)*	*Cumulative probability (4)*	*Random no. Range (5)*	*Random numbers (6)*	*Expected mean accumulation (7)*
1	3.12	0.0122	0.0122	0.00-0121	0013	3.12
2	16.54	0.0646	0.0767	0122-0766	5636	20.42
3	20.81	0.0812	0.1579	0767-1578	1933	15.77
4	15.77	0.0615	0.2195	1579-2194	8087	5.52
5	10.94	0.0427	0.2622	2195-2621	5850	20.42
6	35.52	0.1386	0.4008	2622-4007	4799	30.49
7	30.49	0.1190	0.5198	4008-5197	3503	35.52
8	20.42	0.0797	0.5995	5198-5994	8960	29.76
9	9.03	0.0352	0.6347	5995-6346	8228	1.23
10	14.65	0.0572	0.6919	6347-6918	7466	12.42
11	4.24	0.0165	0.7085	6919-7084	1741	15.77
12	6.76	0.0264	0.7348	7085-7347	8589	29.76
13	2.76	0.0108	0.7456	7348-7455	7105	6.76

14	12.42	0.0485	0.7941	7456-7940	5135	30.49
15	5.52	0.0215	0.8158	7941-8155	3040	35.52
16	1.23	0.0048	0.8204	8156-8203	0150	16.54
17	1.23	0.0048	0.8252	8204-8251	0914	20.81
18	29.76	0.1161	0.9414	8252-9413	3645	35.52
19	15.02	0.0586	1.0000	9414-9999	1473	20.81

Review Questions

Q. 1. (a) Using the data in Table 2.1, draw scatter diagram between Fe_2O_3 and SiO_2 values.

(b) Fit a linear model of the type $Y = A + BX$ using least squares approach with SiO_2 as independent variable (X) and Fe_2O_3 as dependent variable (Y).

(c) Compute correlation coefficient and fit a regression equation of Y on X. Bring it to the form $Y = A + BX$. ($Y = Fe_2O_3$ and $X = SiO_2$)

Q. 2. Construct frequency distributions and draw histograms for the data on Fe_2O_3 and SiO_2 values choosing appropriate class-intervals.

Q. 3. What is simulation? List out the advantages and limitations of simulation.

3 Some Statistical Distributions

This chapter contains discussion on Normal and Lognormal distributions which have wide applications in Earth Sciences.

3.1 THE NORMAL DISTRIBUTION

In a number of ways Normal Distribution is the most widely used distribution and is the cornerstone of modern statistical theory and analysis. It was investigated incidently in the 18th century when scientists observed an astonishing degree of regularity in errors of measurement. It was found that the patterns (which may also be called as distributions) could be closely approximated by continuous curves which were referred to as 'Normal curve of errors' and attributed to the laws of chance.

The mathematical properties of such normal curves were first studied by Abratiam de Moivre (1667–1745), Pierre Laplace (1749–1827) and Karl Gauss (1777–1865). However, it appears that through historical error, it was credited to Gauss only.

3.1.1 Salient Features of Normal Distribution and Normal Probability Law

The salient features of the Normal Distribution and Normal Probability law are as follows:

1. A random variable Z is said to have a normal distribution with parameters μ (called mean) and σ^2 (called variance) if its density function has the following probability law:

$$f(z;\ \mu,\ \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left\{\frac{z-\mu}{\sigma}\right\}^2\right]; \qquad (3.1)$$

for $-\infty < z < +\infty$, $\sigma > 0$, $-\infty < \mu < +\infty$.

2. The curve is bell shaped and symmetrical about the line $z = \mu$.
3. The mean, median and mode of the distribution coincide about the line $z = \mu$. As z increases numerically, $f(z)$ decreases rapidly; the maximum probability occurring at the point $z = \mu$ is given by: $\frac{1}{\sigma\sqrt{2\pi}}$.
4. All odd moments about the mean vanish. Thus : $\mu_{2r+1} = 0$ ($r = 0, 1, 2, \ldots$) and $\mu_{2r} = 1, 3, 5 \ldots (2r - 1)\sigma^{2r}$; ($r = 0, 1, 2, \ldots$).
5. $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ is zero

   $\beta_2 = \frac{\mu_4}{\mu_2^2}$ is 3.
6. Since $f(z)$ being the probability, it can never be negative.
7. Linear combinations of independent normal variates also follow a normal variate.
8. The points of inflexion of the curve are given by

$$\left[z = \mu \pm \sigma, f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2} \right].$$

   i.e., they are equi-distant from the mean at a distance σ.
9. Mean distribution about mean is $\frac{4}{5}\sigma$ approximately.
10. The total area under the curve is equal to 1.
11. The moment-generating function of the normal distribution is given by:

$$M_z(t) = e^{\mu t + \sigma^2 t^2/2}$$

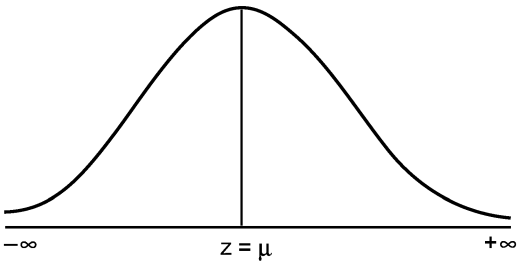


Fig. 3.1 Normal probability curve.

12. The normal distribution with mean $\mu = 0$ and variance = 1 [which is possible when we standardise; i.e., we make the transformation $z' = (z - \mu)/\sigma$] is referred to as the Standard Normal Distribution.

3.1.2 Confidence Limits for the Mean ($\bar{z}$)

Let z_i ($i = 1, 2, \ldots n$) be a random sample of n observations from a population involving a single unknown parameter, say θ. Let $p(z, \theta)$ be the probability function of the parent distribution from which the sample is drawn and let us suppose that this distribution is continuous. Let $L = f(z_1, z_2, \ldots z_n)$, a function of the sample values, be an estimate of the population parameter θ, with a sampling distribution given by $g(L/\theta)$, and obtain the value of the statistic from a given sample. We now wish to make a reasonable statement with certain amount of confidence about the unknown parameter θ in the population, from which the sample has been drawn. We can address this problem by the technique of *confidence interval* due to Neyman.

We choose, once for all, some small value of α (5% or 1%) and then determine two constants say, k_1 and k_2 such that $P(k_1 < \theta < k_2/L) = 1 - \alpha$. The quantities k_1 and k_2, so determined, are known as the *confidence limits* or *fiducial limits*. The interval $[k_1, k_2]$ within which the unknown value of the population parameter is expected to lie, is called the *confidence interval*. $(1 - \alpha)$ is called the *confidence coefficient*. For example, if we take $\alpha = 0.05$ (or 0.01), we shall get 95% (or 99%) confidence limits. In order to find k_1 and k_2, let S_1 and S_2 be two statistics such that:

$$P(S_1 > \theta) = \alpha_1$$

and

$$P(S_2 < \theta) = \alpha_2 \tag{3.2}$$

where α_1 and α_2 are constants independent of θ. The expressions in (3.2) can be combined to give $P(S_1 < \theta < S_2) = 1 - \alpha$, where $\alpha = \alpha_1 + \alpha_2$. Statistics S_1 and S_2 defined in (3.2), may be taken as k_1 and k_2. As an example, if we take a large sample from a normal population with mean μ and standard deviation σ:

$$Z_N = \frac{\bar{z} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

and $$P(-1.96 < Z_N < 1.96) = 0.95 \text{ (from normal tables)}$$

$$\Rightarrow \quad P\left(-1.96 < \frac{\bar{z} - \mu}{\sigma/\sqrt{n}} < 1.96\right) = 0.95$$

$$\Rightarrow \quad P\left[\bar{z} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{z} + 1.96\frac{\sigma}{\sqrt{n}}\right] = 0.95$$

Thus, $\bar{z} \pm 1.96 \dfrac{\sigma}{\sqrt{n}}$ are 95% confidence limits for the unknown parameter μ, the population mean; and the interval $\left[\bar{z} - 1.96\dfrac{\sigma}{\sqrt{n}}, \bar{z} + 1.96\dfrac{\sigma}{\sqrt{n}}\right]$ is called the 95% confidence interval. Similarly, the 99% confidence limits for μ, are $z \pm 2.58 \dfrac{\sigma}{\sqrt{n}}$ and the 99% confidence interval for μ is:

$$\left[\bar{z}-2.58\frac{\sigma}{\sqrt{n}},\bar{z}+2.58\frac{\sigma}{\sqrt{n}}\right].$$

3.2 THE LOGNORMAL DISTRIBUTION AND PROPERTIES

Let z_i (i = 1, 2, … n) be a random sample of n observations from a population involving a single unknown parameter, say M. Consider an essentially positive variate Z ($0 < z < \infty$) such that $X = \log_e Z$ is normally distributed with mean ξ and variance σ^2. We say that Z is lognormally distributed or Z is a Λ variate and write Z is $\Lambda(\xi, \sigma^2)$ and the corresponding X-variate is $N(\xi, \sigma^2)$. The two-parameter lognormal frequency function is given by:

$$\Lambda(z) = \frac{1}{z\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\log_e z-\xi}{\sigma}\right)^2\right.$$

or equivalently as (3.3)

$$\Lambda(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{\sigma^2}{2}-\xi-\frac{1}{2\sigma^2}\left(\log_e z-\xi+\sigma^2\right)^2\right]$$

with parameters ξ and σ^2 [See Krige (1951)]. The population mean

$$M = \exp\left(\xi+\frac{\sigma^2}{2}\right)$$

and the population variance = M^2 [exp(σ^2) – 1].

The normalised form of the distribution is obtained by substituting $X = \log_e Z$ such that $-\infty < x < +\infty$ to yield the frequency function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x-\xi)^2\right]$$

with mean ξ and variance σ^2.

3.2.1 Estimates for the Mean

Finney (1941) derived the maximum likelihood estimator G for the mean M of a lognormal distribution.

This estimator, based on a sample of size n, can be expressed as:

$$G^* = \exp(\bar{x})\psi_n\left(\frac{s_x^2}{2}\right) \tag{3.4}$$

$$\psi_n\left(\frac{s_x^2}{2}\right) = \left[1+\left(\frac{n-1}{n}\right)\frac{s_x^2}{2}+\frac{(n-1)^2}{n^2(n-1)}\left(\frac{s_x^2}{2}\right)^2\frac{1}{2!}+\frac{(n-1)^5}{n^3(n+1)(n+3)}\left(\frac{s_x^2}{2}\right)^3\frac{1}{3!}+\ldots\right] \tag{3.5}$$

where $$s_x^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_1 - \bar{x})^2$$

and $$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \, .$$

s_x^2 is an unbiased estimator of the population variance. x_is are the natural logarithms of the observations.

Sichel (1952) claims that he had also arrived at the same expression in 1949 for the mean of lognormal distribution independently of Finney (1941). According to Sichel (1966), the estimator for the mean of lognormal population can be expressed as:

$$G = \gamma_n(V) \exp(\bar{x}),$$

where $\gamma_n(V)$ is a mathematical expression containing Gamma and Bessel functions. Expanding $\gamma_n(V)$ and writing it as a comparatively simpler series, G can be written as:

$$G = \exp(\bar{x}) \left[1 + \frac{V}{2} + \frac{V^2(n-1)}{2!2^2(n+1)} + \frac{(n-1)^2 V^2}{2^2 3!(n+1)(n+3)} + \ldots \right] \tag{3.6}$$

where $$V = \frac{1}{n} \Sigma (x_i - \bar{x})^2$$

is unadjusted estimator for the population variance σ^2.

One can see that by expressing $s_x^2 = \left(\frac{n}{n-1} V \right)$ and substituting in (3.5) we arrive at equation (3.6). Therefore, G and G^* are same; only the expressions are different. It is known that the estimator G^* or G is an unbiased estimator for the mean M, and is also a maximum likelihood estimator of M. The estimator has minimum variance. The variance of G is

$$G^2 \left\{ e^{V/(n-1)} \left[1 + \frac{V^2}{2(n-1)} + \frac{V^4}{2!2^2(n-1)(n+1)} + \ldots \right] - 1 \right. \tag{3.7}$$

As n, the sample size, becomes large, G can be expressed as G_1 (asymptotic approximation) where

$$G_1 = \exp\left(\bar{x} + \frac{V}{2} \right). \tag{3.8}$$

The variance of G_1 is approximately

$$G_1^2 \left\{ \exp\left[\frac{V}{n-1} + \frac{V^2 n}{2(n-1)^2} \right] - 1 \right\}, \tag{3.9}$$

the actual variance being:

$$M^2[\exp(\sigma^2/n + \sigma^4/2n) - 1] \tag{3.10}$$

If the population variance is known a priori, the estimate G_1 for M (the mean grade of ore) can be modified by replacing V suitably. If we represent this by G_2, we have:

$$G_2 = \exp\left[\bar{x} + \frac{(n-1)\sigma^2}{2n}\right] \tag{3.11}$$

G_2 is lognormally distributed with M as mean and variance:

$$M^2[\exp(\sigma^2/n) - 1] \cong (G_2)^2[\exp(\sigma^2/n) - 1] \tag{3.12}$$

and $\log_e G_2$ is normally distributed with ξ as mean and variance σ^2/n.

3.2.2 Confidence Limits for the Mean

Sichel (1966) derived confidence limits for the mean of a lognormal distribution upto samples of size 1000 and variance upto 6.0. Expressing

$$G = e^{\bar{x} + \log_e \gamma_n(V)} \tag{3.13}$$

it can be shown that $U = \log_e G = \bar{x} + \log_e \gamma_n(V)$ is approximately normally distributed, or G is approximately lognormally distributed. The approximate sampling distribution of G is

$$\frac{1}{\sigma_G\sqrt{2\pi}}\frac{1}{G}\exp\left[-\frac{1}{2}\left(\frac{\log_e G - \xi_G}{\sigma_G}\right)^2\right] \tag{3.14}$$

where $$\xi_G = \xi + \frac{1}{2}(\sigma^2 - \sigma_G^2) = \log_e G - \sigma_G^2/2$$

and $$\sigma_G^2 = \frac{\sigma^2}{n} + \log_e \gamma_n\left(\frac{n-1}{n^2}\sigma^4\right).$$

Hence from the transformation in (3.13), we have the probability density $f(u)$ as:

$$\frac{1}{\sigma_G\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{u - \xi_G}{\sigma_G}\right)^2\right] \tag{3.15}$$

The lower (L_1) and upper (L_2) confidence limits for the mean M are given by

$$L_1 = G\exp\left[\frac{\hat{\sigma}_G^2(V)}{2} - T_P\,\hat{\sigma}_G(V)\right] \tag{3.16}$$

and
$$L_2 = G \exp\left[\frac{\hat{\sigma}_G^2(V)}{2} + T_{1-p}\,\hat{\sigma}_G(V)\right] \tag{3.17}$$

where
$$\hat{\sigma}_G^2(V) = \frac{V}{n-1} + \log_e \gamma_n\left(\frac{V^2}{n-1}\right)$$

and T_p is the value of a standard deviate with proportion p in the tail of the variate

$$T = \frac{\log_e G - \log_e M}{\hat{\sigma}_G(V)} + \frac{\hat{\sigma}_G(V)}{2} \tag{3.18}$$

The expression for the exact distribution of T may be seen in Sichel (1966).

In this connection, the following observations may be made:

(i) Central confidence limits with $p = 0.05$ in each tail or lower limits with $p = 0.10$ in the left tail, calculated and given in Sichel (1966) may be used for finding the confidence limits for G.

(ii) When n is large and σ^2 is unknown, the estimate G_1 is adopted instead of G.

Another approach suggested by Dr. Finney in a discussion on the paper by Krige (1951) may also be considered for finding the confidence limits. Using the above method, the confidence limits for G_1, an estimate of the mean M, are computed corresponding to a chosen level of significance from the formula:

$$\exp\left[\bar{x} \pm \frac{H_L s}{\sqrt{n}} + \frac{\sigma^2}{2}\right] \tag{3.19}$$

where $\bar{x}$ is the mean of the natural logarithms of the data, s is the sample standard deviation based on natural logarithms, σ^2 is the population (true) variance of the natural logarithms of the data, n is the size of (large) sample and H_L is the standardised Normal variate. When σ^2 is not known, the problem is complicated. However, from a practical point of view, s^2 may be substituted for σ^2 in the above expression.

(iii) When σ^2 is known, following Krige (1961), the estimate G_2 is adopted instead of G. Confidence limits for the estimate G_2 of the mean are determined on the basis of normal law by utilising the formulae, viz.,

$$w = (\log_e z - \xi)/\sigma$$
$$M = \exp(\xi + \sigma^2/2)$$

and reducing to:

$$\log_e\left(\frac{z_i}{M}\right) = w_i\sigma - \frac{\sigma^2}{2}$$

and substituting G_2 for z_i and σ^2/n for σ^2 with L_1 and L_2 as the lower and upper confidence limits corresponding to the normal fractile w_i. 90% confidence limits based on this method are tabulated by Krige (1961, p. 14). These may be used for finding the 90% confidence limits for G_2. Some more details on lognormal theory are reported by Aitchison and Brown (1957) in their monograph *'The Log Normal Distribution'*.

3.3 THE CHI-SQUARE (χ^2) TEST

Quite often, results obtained in samples do not always agree exactly with the theoretical results expected according to probability laws. In such situations, we wish to know whether the observed results match more or less with the expected (or theoretical) ones. When we discuss in terms of 'frequencies' in a statistical distribution, we have observed frequencies (O_i), and expected or theoretical frequencies (E_i) based on the assumed/governing probability distribution.

Chi-square (χ^2)

A measure of the discrepancy existing between observed and expected frequencies is given by (Chi-square) statistic and is given by:

$$\chi^2 = \sum_{i=1}^{r} \left(\frac{O_i - E_i}{E_i} \right)^2 \tag{3.20}$$

where the summation is over the number (r) of class-intervals considered. For $\chi^2 = 0$, there is perfect agreement between observed and expected frequencies while for $\chi^2 > 0$, there is no perfect agreement between observed and expected frequencies. The sampling distribution of the Chi-square statistic is approximated very closely by the Chi-square distribution, the probability density function of which is given as:

$$f(\chi^2) = \frac{1}{2^{(\nu/2)}\Gamma(\nu/2)} (\chi^2)^{\frac{\nu-2}{2}} e^{\frac{-\chi^2}{2}} \text{ for } \chi^2 > 0 \tag{3.21}$$
$$= 0 \text{ elsewhere.}$$

Expected and theoretical investigations show that the approximation is satisfactory for $e_i \geq 5$ and $r \geq 5$. If $r < 5$, it is better to have e_i larger than 5. In the above expression, ν stands for the number of degrees of freedom $= r - 1$ where r = number of class intervals. However, $\nu = r - m - 1$, if the expected frequencies can be computed only by estimating 'm' population parameters from sample statistics. For each class interval, there corresponds a frequency.

Significance Test

In practice, expected frequencies are computed on the basis of the null hypothesis H_0: *There is no significant difference between observed and*

expected frequencies. If under this hypothesis, the computed value of χ^2 based on (3.20) is greater than some critical value (such as $\chi^2_{0.95}$ which is the critical value at the 0.05 significance level), we would conclude that the observed frequencies differ significantly from expected frequencies and therefore we reject H_0 at the corresponding level of significance. Otherwise, we would have no ground to reject H_0. Since the Chi-square distribution arises in many important applications, integrals of its density have been extensively tabulated. These may be seen in any standard books on Statistics. For example for 5 degrees freedom and at 5% level of significance χ^2 is 11.07.

It may be noted that $\chi^2_{(\alpha, \nu)}$ is such that the area to its right under the Chi-square curve with ν degrees of freedom is equal to α. That is to say, $\chi^2_{(\alpha, \nu)}$ is such that if Z is a random variable having a Chi-square distribution with ν degrees of freedom, then $P\ (Z \geq \chi^2_{\alpha, \nu}) = \alpha$ (see Fig. 3.2).

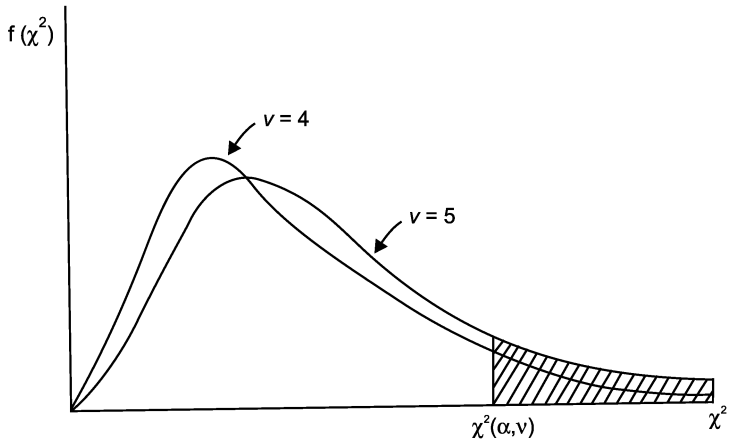


Fig. 3.2 Typical distributions for 4 and 5 degrees of freedom (ν). The distribution for $\nu = 5$ also shows the critical region containing 5% of area under the curve. Critical value of $\chi^2 = 11.07$.

3.4 APPLICATIONS

Let us now apply this test to the distributions of Fe_2O_3, and the log-transformed distribution of gold and copper assay values given in Tables 2.3, 2.6, 2.8, 2.11 and 2.13 respectively. We may recall the criterion that the expected frequencies in each class interval should be ≥ 5. Since this condition was not satisfied in some cases, such cells were combined with one or more other cells. The expected frequencies were also adjusted accordingly.

The Fitting of Normal Distribution and Testing for Goodness of Fit

3.4.1 Bauxite Example: Distribution of Fe_2O_3 Element

Let us now fit normal distribution to the empirical distribution of Fe_2O_3 discussed in Chapter 2. The frequency distribution and the expected frequencies are as tabulated below.

Table 3.1 Fitting normal distribution to the distribution of Fe_2O_3 values

Class interval (in %)	*Mid pt.*	*Observed frequency (O_i)*	*Expected frequency (E_i)*	$\frac{(O_i-E_i)^2}{E_i}$
9.00–13.50	11.25	0 } 6	0.98 } 6.05	0.00
13.50–18.00	15.75	6	5.07	
18.00–22.50	20.25	14	13.63	0.01
22.50–27.00	24.75	21	19.22	0.16
27.00–31.50	29.25	12	14.23	0.35
31.50–36.00	33.75	5 }	5.52 }	
36.00–40.50	38.25	2 } 7	1.12 } 6.76	0.00
40.50–45.00	42.75	0	0.12	
	Total	60	59.89	0.52

The computed χ^2 given by $\sum_{i=1}^{r} \frac{(O_i - E_i)^2}{E_i}$ gives a value of 0.52 while the theoretical χ^2 value at 5% level of significance and for $(r - m - 1) = (5 - 2 - 1) = 2$ degrees of freedom (d.f) is 5.991. Since the computed Chi-square is less than the theoretical value, the fitting of normal distribution to the observed distribution of Fe_2O_3 appears to be justified (Fig. 3.3).

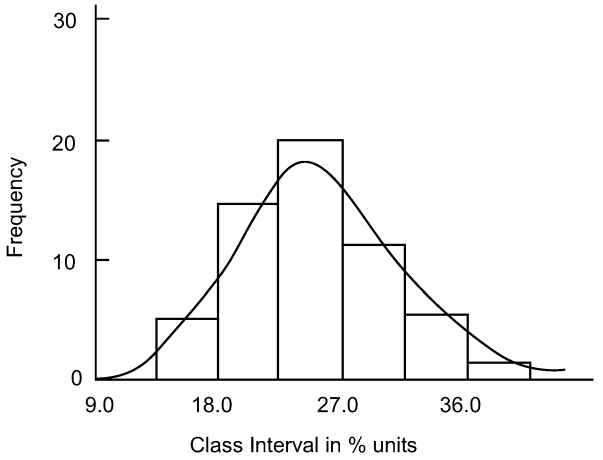


Fig. 3.3 Observed and fitted distributions for the Fe_2O_3 values.

Estimates for parameters

Sample mean ($\bar{x}$) = 25.17%

Standard Deviation (s) = 5.52%

3.4.2 Gold Ore Distribution

We have seen in Chapter 2 that the distribution of 72 gold assay values for grade (in units of dwts/ton of ore) and accumulation (in units of inch-dwts) followed a positively skewed distribution and can be approximated to a two-parameter lognormal distribution. The distributions of grade values, accumulation values and logarithms of grade and accumulation values (i.e. when $X = \log_e Z$) are shown in Chapter 2. The details of relevant fittings of the distribution are given below:

Accumulation

Table 3.2 Fitting of lognormal distribution to the gold accumulation values

Class interval (inch-dwt)	*Mid pt.*	*logs of class limits*	*Observed frequency (O_i)*		*Expected frequency (E_i)*		$\frac{(O_i - E_i)^2}{E_i}$
1–175	88.0	0.0–5.165	4	20.0	4.32	25.83	1.32
175–350	262.5	5.165–5.858	16		21.51		
350–525	437.5	5.858–6.263	24		18.72		1.49
525–700	612.5	6.263–6.551	11		7.71		1.40
700–875	787.5	6.551–6.774	6	10.0	5.22	8.87	0.14
875–1050	962.5	6.774–6.957	4		3.65		
1050–1225	1137.5	6.957–7.111	1	7.0	3.62	10.20	1.00
1225–1400	1312.5	7.111–7.244	1		1.93		
1400–1575	1487.5	7.244–7.362	4		1.45		
1575–1750	1662.5	7.362–7.467	0		1.11		
1750–1925	1837.5	7.467–7.563	0		0.87		
1925–2100	2012.5	7.563–7.650	0		0.68		
2100–2275	2187.5	7.650–7.730	1		0.54		
		Total	72		71.33		5.35

As we see, the computed value of $\chi^2 = 5.35$. This is less than the theoretical value of 5.991 for 2 degrees of freedom at 5% level of significance. Therefore, we cannot reject the hypothesis of the distribution being a lognormal one.

We will now fit a normal distribution to the logarithms of the accumulation values. This is a slight variation to the above approach. Here, we take logarithmic class intervals and distribute the logarithms of accumulation values and fit a normal distribution. Table 3.3 shows the fit.

Table 3.3 Fitting of normal distribution to the logarithms of gold accumulation values

Class interval (log. units)	*Mid pt.*	*Observed frequency (O_i)*		*Expected frequency (E_i)*		$\frac{(O_i - E_i)^2}{E_i}$
4.00–4.50	4.25	0		0.29		
4.50–5.00	4.75	3	9	2.15	11.01	0.367
5.00–5.50	5.25	6		8.66		0.817
5.50–6.00	5.75	21		18.83		0.250
6.00–6.50	6.25	25		22.14		0.369
6.50–7.00	6.75	10		14.09		1.137
7.00–7.50	7.25	6	17	4.84	19.83	0.404
7.50–8.00	7.75	1		0.90		
	Total	72		71.81		3.344

The observed Chi-square value is 3.344, while the theoretical Chi-square value at 5% level of significance and (4 – 2 – 1) = 1 degree of freedom is 3.841. Since the observed Chi-square is less than the theoretical Chi-square, we infer that the fit of normal distribution to the logarithms of the accumulation values is justified.

Grade

We shall now discuss the fitting of lognormal distribution to the gold assay values which are in units of dwts/ton of ore. Table 3.4 shows the details of this fitting.

Table 3.4 Fitting a lognormal distribution to grade values

Class interval (dwts)	*Mid pt.*	*logs of class limits*	*Observed frequency (O_i)*		*Expected frequency (E_i)*		$\frac{(O_i - E_i)^2}{E_i}$
0.1–2.0	1.05	–2.30 to 0.69	1.0	9.0	4.03	13.61	1.56
2.0–4.0	3.0	–0.69 to 1.39	8.0		9.58		
4.0–6.0	5.0	1.39 to 1.79	27.0		21.47		1.42
6.0–8.0	7.0	1.79 to 2.08	10.0		11.69		0.24
8.0–10.0	9.0	2.08 to 2.30	8.0		9.19		0.15
10.0–12.0	11.0	2.30 to 2.48	7.0		5.86		0.22
12.0–14.0	13.0	2.48 to 2.64	6.0		3.69		
14.0–16.0	15.0	2.64 to 2.77	2.0		2.33		
16.0–18.0	17.0	2.77 to 2.89	2.0		1.48		
18.0–20.0	19.0	2.89 to 2.99	0.0	11.0	0.96	9.79	0.15
20.0–22.0	21.0	3.00 to 3.09	1.0		0.63		
22.0–24.0	23.0	3.09 to 3.18	0.0		0.41		
24.0–26.0	25.0	3.18 to 3.26	0.0		0.29		
		Total	72.0		71.61		3.74

The observed Chi-square value is 3.74, while the theoretical Chi-square value at 5% level of significance for (6 – 2 – 1) = 3 d.f is 7.815. Since the observed Chi-square value is less than the theoretical value, we infer that the fit of lognormal distribution to the gold assay values is justified.

As before we shall discuss the fitting of normal distribution to the logarithms of the gold assay values. Here the class intervals are in logarithmic units. This is a slight variation to the above approach. Table 3.5 shows the details of this fitting.

Table 3.5 Fitting normal distribution to a sample set of logarithms of gold assay values

Class interval (log. units)	*Mid pt.*	*Observed frequency (O_i)*		*Expected frequency (E_i)*		$\frac{(O_i - E_i)^2}{E_i}$
0.00–0.40	0.20	1		0.45		
0.40–0.80	0.60	2	9	2.32	10.36	0.18
0.80–1.20	1.00	6		7.59		
1.20–1.60	1.40	17		15.47		0.15
1.60–2.00	1.80	20		19.70		0.00
2.00–2.40	2.20	15		15.67		0.02
2.40–2.80	2.60	9		7.79		
2.80–3.20	3.00	2	11	2.42	10.68	0.00
3.20–3.60	3.40	0		0.47		
	Total	72		71.88		0.35

Since the observed Chi-square value is less than 5.991 which is the theoretical Chi-square value at 5% level of significance and for (5 – 2 – 1) = 2 degrees of freedom, we infer that the fitting of normal distribution to the logarithms of grade values is justified. The other statistical parameters are as follows.

Estimates for Parameters

Type of estimate	*Accumulation (inch-dwts)*	*Grade (dwts/ton of ore)*
Sample mean	566.35	7.56
Standard deviation	388.59	4.00
Lognormal estimate	660.22	7.63
$\sqrt{\text{Variance}}$	76.92	4.90

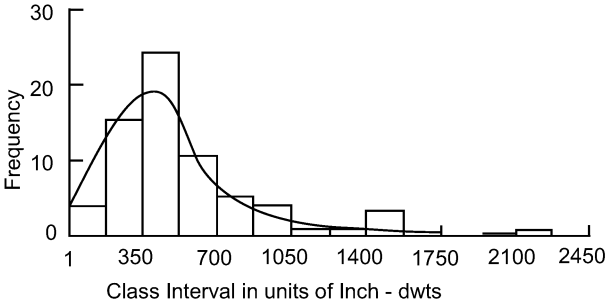


Fig. 3.4 Observed and fitted distributions to the gold accumulation values.

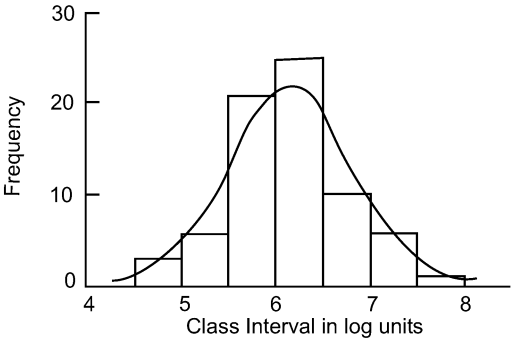


Fig. 3.5 Observed and fitted distributions to the logarithms of gold accumulation values.

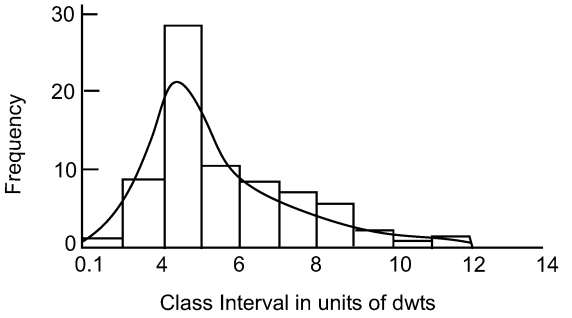


Fig. 3.6 Observed and fitted distributions to gold grade values.

3.4.3 Copper Example

We now discuss another example related to copper mineralisation. The sample set consists of 94 assay values. In this case also we have seen that the distribution can be approximated by a two-parameter lognormal distribution. We shall now give the details of the fitting of lognormal distribution to the accumulation and grade (tenor) values. The same steps were followed as in the gold ore example. Table 3.6 shows the details of fitting lognormal distribution to copper accumulation values.

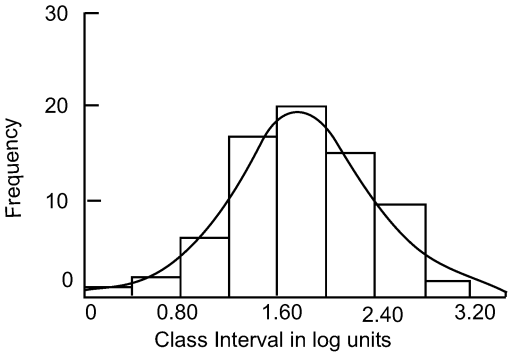


Fig. 3.7 Observed and fitted distributions to the logarithms of gold grade values.

Accumulation

Table 3.6 Fitting of lognormal distribution to copper accumulation values

Class interval (cm%)	*Mid Pt.*	*Observed frequency* (O_i)		*Expected frequency* (E_i)		$\frac{(O_i - E_i)^2}{E_i}$
1–50	25.50	9		11.45		0.524
50–150	100.00	30		31.99		0.123
150–250	200.00	27		24.50		0.255
250–350	300.00	14		9.01		2.760
350–450	400.00	5		4.94		
450–550	500.00	5		3.27		
550–650	600.00	1		2.28		
650–750	700.00	1	13	1.65	14.05	0.078
750–850	800.00	0		1.23		
850–950	900.00	0		0.95		
950–1050	1000.00	1		0.74		
	Total	94		93.60		3.740

Since the computed Chi-square value (3.740) is less than the theoretical Chi-square value (5.991) at 5% level of significance and (5 – 2 – 1) = 2 degrees of freedom, we infer that the fitting of log normal distribution to the copper accumulation values is justified.

As before we shall show the details of fitting a normal distribution to the logarithms of copper accumulation values. The details are shown in Table 3.7.

Since the computed Chi-square value (1.79) is less than the theoretical Chi-square value (5.991) at 5% level of significance and (5 – 2 – 1) = 2 degrees of freedom, we infer that the fitting of normal distribution to the logarithms of copper accumulation values is justified.

Table 3.7 Fitting of normal distribution to the logarithms of copper accumulation values

Class interval (log. units)	*Mid Pt.*	*Observed frequency* (O_i)		*Expected frequency* (E_i)		$\frac{(O_i - E_i)^2}{E_i}$
2.25 - 3.00	2.625	0	6	0.59	5.25	0.11
3.00 - 3.75	3.375	6		4.66		
3.75 - 4.50	4.125	17		17.49		0.01
4.50 - 5.25	4.875	28		31.27		0.34
5.25 - 6.00	5.625	32		26.72		1.04
6.00 - 6.75	6.375	9	11	10.51	12.93	0.29
6.75 - 7.50	7.125	2		2.12		
7.50 - 8.25	7.875	0		0.20		
8.25 - 9.00	8.625	0		0.06		
9.00 - 9.75	9.375	0		0.04		
	Total	94		93.46		1.79

Tenor

Table 3.8 shows the details of fitting a lognormal distribution to the copper tenor values in units of %.

Table 3.8 Fitting of lognormal distribution to copper tenor values

Class interval (log. units)	*Mid pt.*	*Observed frequency* (O_i)		*Expected frequency* (E_i)		$\frac{(O_i - E_i)^2}{E_i}$
0.10–1.00	0.55	20		27.75		2.16
1.00–2.00	1.50	22		23.51		0.09
2.00–3.00	2.50	16		13.52		0.45
3.00–4.00	3.50	8		8.25		0.00
4.00–5.00	4.50	6	10	5.34	9.97	0.00
5.00–6.00	5.50	4		4.63		
6.00–7.00	6.50	5	14	3.56	10.69	1.02
7.00–8.00	7.50	4		2.86		
8.00–9.00	8.50	4		2.39		
9.00–10.00	9.50	1		1.06		
10.00–11.00	10.50	0		0.82		
	Total	94		93.69		3.72

Since the computed Chi-square value (3.72) is less than the theoretical value (7.815) at 5% level of significance and for 3 degrees of freedom, the fitting of lognormal distribution to copper tenor values is justified.

As before, we shall fit a normal distribution to the logarithms of copper assay values. The details are shown in Table 3.9.

Table 3.9 Fitting of normal distribution to a sample set of logarithms of copper assay values

Class interval (log. units)	*Mid pt.*	*Observed frequency* (O_i)		*Expected frequency* (E_i)		$\frac{(O_i - E_i)^2}{E_i}$
–2.00 to –1.30	–1.65	5	13	2.44	10.22	0.76
–1.30 to –0.60	–0.95	8		7.78		
–0.60 to 0.10	–0.25	12		13.50		0.16
0.10 to 0.80	0.45	23		23.31		0.00
0.80 to 1.50	1.15	21		21.95		0.04
1.50 to 2.20	1.85	20		13.76		2.83
2.20 to 2.90	2.55	5	5	3.75	7.69	0.94
2.90 to 3.60	3.25	0		1.40		
3.60 to 4.30	3.95	0		0.30		
4.30 to 5.00	4.65	0		0.04		
	Total	94		93.98		4.73

Since the computed Chi-square value (4.73) is less than the theoretical Chi-square value (7.815) at 5% level of significance and for 3 degrees of freedom, we infer that the fitting of normal distribution to the logarithms of copper tenor values is justified.

Estimates for Parameters

Type of estimate	*Accumulation (cm%)*	*Tenor (%)*
Sample mean	222.6	3.43
Std. deviation	195.0	3.22
Lognormal estimates	261.3	3.72
$\sqrt{\text{Variance}}$	432.8	6.07

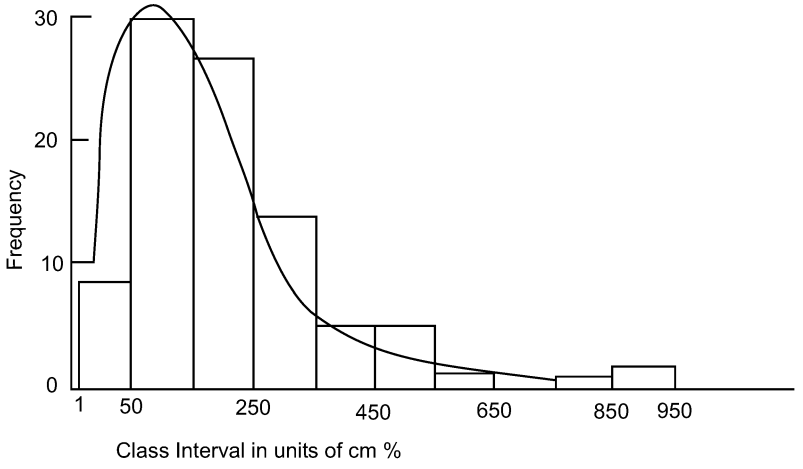


Fig. 3.8 Observed and fitted distributions to copper accumulation values.

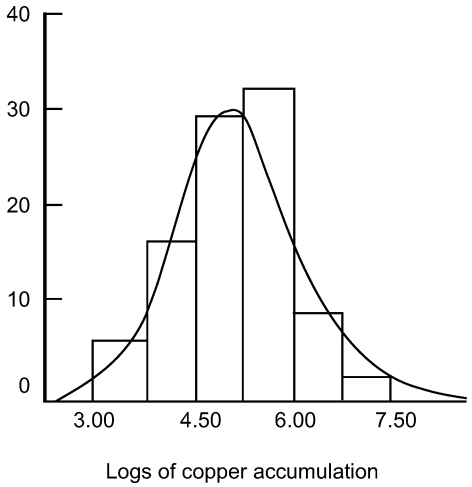


Fig. 3.9 Observed and fitted distributions to the logarithms of copper accumulation values.

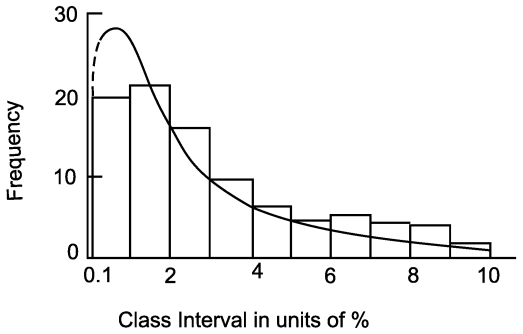


Fig. 3.10 Observed and fitted distributions to copper tenor values.

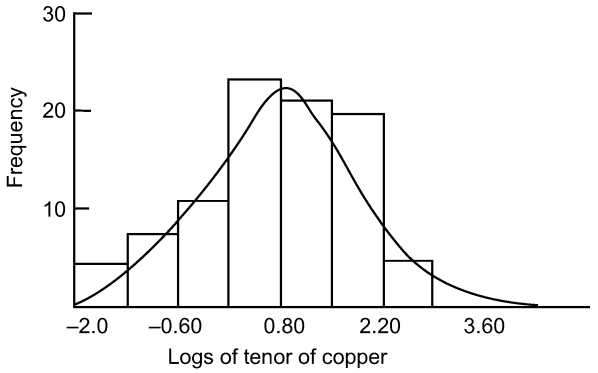


Fig. 3.11 Observed and fitted distributions to the logarithms of tenor of copper values.

3.5 CASE OF REJECTION OF NORMAL DISTRIBUTION

So far we have discussed the cases of fitting normal distributions to Fe_2O_3 values, and also fitting normal and lognormal distributions in the following cases.

S.No.	*Deposit*	*Variable*	*Fitted distribution*
1.	Gold ore	Accumulation (cm-gms)	Lognormal
2.	Gold ore	Accumulation (cm-gms)	Normal to logarithms accumulation
3.	Gold ore	Grade (gms/tonne)	Lognormal
4.	Gold ore	Grade (gms/tonne)	Normal to logarithms of grade
5.	Copper	Accumulation (cm-%)	Lognormal
6.	Copper	Accumulation (cm-%)	Normal to logarithms of accumulation
7.	Copper	Tenor (%)	Lognormal
8.	Copper	Tenor (%)	Normal to logarithms of tenor

For a change, let us first fit *normal distribution to copper tenor values* and see what would result from the application of Chi-square test. The copper assay values which are 94 in number were listed in Chapter 2.

The observed distribution and the fitted normal distribution to these data together with the application of Chi-square test are shown in Table 3.10.

Table 3.10 Fitting of normal distribution to a set of copper assay values

Class interval	*Mid point*	*Observed frequency*		*Expected frequency*		$\frac{(O_i - E_i)^2}{E_i}$
0.00 – 1.00	0.5	20		06.12		31.48
1.00 - 2.00	1.5	22		11.50		9.58
2.00 - 3.00	2.5	16		16.72		0.00
3.00 - 4.00	3.5	08		17.10		5.08
4.00 – 5.00	4.5	06		16.31		6.52
5.00 – 6.00	5.5	05		10.02		2.52
6.00 – 7.00	6.5	05		07.50		0.83
7.00 – 8.00	7.5	04		03.50		
8.00 - 9.00	8.5	03		02.39		
9.00 – 10.0	9.5	01		00.99		
10.0 – 11.0	10.5	00	12	00.81	8.57	1.37
11.00-12.0	11.5	01		00.52		
12.00-13.0	12.5	00		00.23		
13.00-14.0	13.5	02		00.09		
13.00-15.0	14.5	00		00.03		
15.00-16.0	15.5	00		00.01		
16.00-17.0	16.5	01		00.00		
	Total	94		93.64		57.38

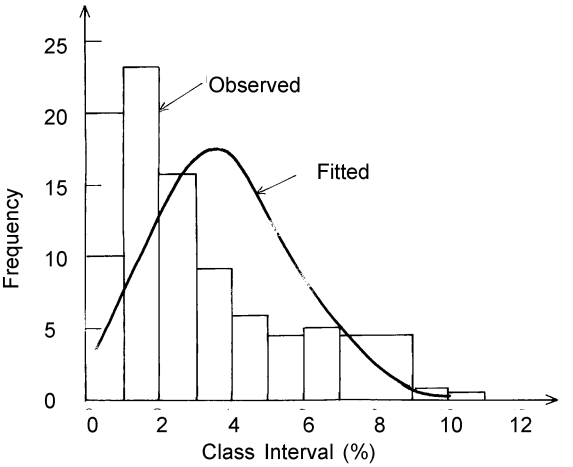


Fig. 3.12 Observed distribution and fitted normal distribution to the copper tenor values.

Chi-square (χ^2) at 5% level of significance for $(r - m - 1) = (8 - 2 - 1) = 5$ d.f is 11.07 while the computed one is 57.38. Since the computed value is greater than the theoretical (table), we infer that the fitting of *normal* distribution to the observed frequencies of copper tenor values is not justified. This example is given to illustrate the rejection of the fit of normal distribution to the copper tenor values, whereas, as we have seen, the fit of a lognormal distribution to the same set of copper tenor values and the fit of a normal distribution to the logarithms of the same tenor values have been justified.

Review Questions

Q. 1. For the frequency distributions generated in respect of elements Fe_2O_3 and SiO_2 given under Review Questions in Chapter 2, fit normal law and obtain expected frequencies.

Q. 2. Apply Chi-square test using the criterion that the frequencies in any cell class should not be < 5. What inference can be drawn?

4 Stochastic Modelling (Time Series Analysis) and Forecasting

4.1 INTRODUCTION

One of the objectives of statistical analysis of sequences of data is to draw inferences about the properties of the population from which these sequences of samples are drawn. Prediction of future observations is done by constructing relevant models based on stochastic process concepts. Stochastic processes can be classified as stationary and non-stationary. Special classes of linear models of stationary stochastic processes are:

1. Auto-regressive processes (AR),
2. Moving-average (MA) and
3. Auto-regressive and moving average processes (ARMA).

4.1.1 Stochastic Processes

Stochastic process may be described as a phenomenon unfolding in 'time' according to certain probability laws. Here, the word 'time' is used as a real variable which may not always stand for time. When such inference is subjected to certain laws of probability, it can be described in terms of random variables $Z_1, Z_2, Z_3, \ldots Z_n$, each Z corresponding to one instant of time. As the number of time units or instants increase abundantly, we are obliged to consider the situation of a multivariate. The assemblage of these random variables together with their probability distributions is called as Stochastic Process. A geological process may be viewed as a stochastic process because it is associated with different geochemical elements—each of which can be treated as *Random Variable* having a probability distribution. Here the observations are not in time, but in space. Even then, we can apply the time-domain models of stochastic processes (time series analysis) to geological processes. The applicability of a stochastic process and in particular, the subclasses, viz. AR, MA and ARMA, depends on the behaviour of the relevant autocorrelation function (acf) and the partial autocorrelation function (pacf). An excellent treatment of these models may be seen in Box and Jenkins (1976).

4.1.2 The Autocorrelation Function (acf)

For a sequence of observations $z_1, z_2, \ldots, z_n$, the autocorrelation coefficient at lag 'k' is defined as: $\rho_k = E[(z_i - \mu)(z_{i+k} - \mu)] / \sqrt{E(z_i - \mu)^2 E(z_{i+k} - \mu)^2}$, where E stands for the expectation or expected value. For a stationary process, the variance is the same at time $t + k$ as at t. For $k = 0$, $\rho_0 = 1$. The plot of autocorrelation coefficients $\rho_1, \rho_2, \ldots \rho_k$ as a function of lag (k) is called the autocorrelation function of the process. Generally speaking, if the 'acf' is of (i) infinite damped exponentials and/damped sine waves form, the process is autoregressive; (ii) if it cuts the X-axis (finite), it is moving average (MA), and (iii) if it is infinite damped exponentials and/or damped sine waves after q-p first lags, then the process is autoregressive and moving average model (ARMA).

Standard Error of Autocorrelation Estimates

In the process of identification of the appropriate model, it is necessary to verify in the first instance, whether the population autocorrelation coefficient ρ_k is zero beyond a certain lag k. Bartlett (1946) has given an approximate expression for the variance of the estimated autocorrelation coefficient (r_k) of a stationary Normal process and this can be used for the said purpose.

$$Var\ (r_k) \cong \frac{1}{N} \sum_{v=-\infty}^{+\infty} \{\rho_v^2 + \rho_{v+k}\ \rho_{v-k} - 4\rho_k \rho_v \rho_{v-k} + 2\rho_v^2 \rho_k^2\} \qquad (4.1)$$

The variance of the estimated autocorrelations r_k at lags $k >$ some value q beyond which the theoretical autocorrelation function may be treated as petered out. Bartlett's approximation gives:

$$Var\ (r_k) \cong \frac{1}{N} \{1 + 2 \sum_{v=1}^{q} \rho_v^2\},\ k > q \qquad (4.2)$$

$$\text{Standard Error (S.E)} = \sqrt{Var(r_k)}\ .$$

If the assumption is that the series is completely random, we have q = 0. Therefore, for all lags, r_k is zero and hence $Var\ (r_k) \simeq \frac{1}{N}$. S.E = $\sqrt{Var}$. Employing these statistics, 95% confidence limits (±1.96 S.E) can be worked out for the autocorrelations. Any points exceeding these limits can be considered as significant.

Partial Autocorrelation Function (pacf)

The quantity ϕ_{kk} regarded as a function of the lag k is called the partial autocorrelation function (see eqn. 4.5 below).

The partial autocorrelation coefficients may be estimated by fitting successively autoregressive processes of order 1, 2, 3 … by least squares and picking up the estimates $\hat{\phi}_{11}, \hat{\phi}_{22}, \hat{\phi}_{33}, \ldots$ of the last coefficient fitted at each

stage. There are, of course, other methods of estimation; (i) if the 'pacf' is finite or cuts-off, the process is autoregressive, (ii) if it is infinite i.e. dominated by damped exponentials and/or sine waves, the process is moving average and (iii) if it is infinite i.e., dominated by damped exponentials and/or sine waves after first p-q lags or tails-off, it is auto-regressive and moving average (integrated) model. For more details, please see Box and Jenkins (1976). Thus, depending on the information provided by acf and pacf, we decide on the class of models to be chosen, viz., AR, MA, or ARMA.

Standard Error of Partial Autocorrelation Estimates

Quenouille (1949) has shown that on the hypothesis that the process is autoregressive of order p, the estimated partial autocorrelations of order $p + 1$ and higher are approximately independently distributed with variance $\cong \frac{1}{N}$ for $k \geq p + 1$. The standard error (S.E) of the estimated partial autocorrelation ϕ_{kk} is

$$\text{S.E } (\phi_{kk}) = \sqrt{\frac{1}{N}} \quad \text{for } k \geq p + 1$$

Employing this, 95% confidence limits can be worked out for the partial autocorrelation coefficients. Any points exceeding these limits can be considered as significant.

4.2 STOCHASTIC MODELLING (TIME SERIES ANALYSIS)

A linear random process may be described by a linear filter model of the type:

$$\tilde{z}_t = a_t + \Psi_1 a_{t-1} + \Psi_2 a_{t-2} + \ldots \tag{4.3}$$

where $\tilde{z}_t = z_t - \mu$, Ψ_1, Ψ_2, … are filter coefficients and μ = mean of the process. The sequence a_t (usually referred to by communication engineers as white noise series) has zero mean and variance σ_a^2. The μ is a parameter that describes the level of the process z_t. If the process is stationary, μ is the mean of z_t process. We may also assume that μ has been estimated ($\bar{z}$) and removed from the time-series. Thus $\tilde{z}_t = z_t - \bar{z}$. The process $\tilde{z}_t$ will, therefore, describe a zero mean process. The model defined by equation (4.3) implies that $\tilde{z}_t$ can be written alternatively as a weighted set of past values of a_t, plus an added shock noise/error i.e.,

$$\tilde{z}_t = \pi_1 \tilde{z}_{t-1} + \pi_2 \tilde{z}_{t-2} + \ldots + a_t \tag{4.4}$$

and a finite version may be written as:

$$\tilde{z}_t = \phi_{p1} \tilde{z}_{t-1} + \phi_{p2} \tilde{z}_{t-2} + \ldots + \phi_{pp} \tilde{z}_{t-p} + a_t \tag{4.5}$$

or simply,

$$\tilde{z}_t = \phi_1 \tilde{z}_{t-1} + \phi_2 \tilde{z}_{t-2} + \ldots + \phi_p \tilde{z}_{t-p} + a_t$$

The model represented by (4.5) is known as autoregressive model of order p i.e., AR(p). It may be seen that this is a regression model and z is said to be regressed on the previous values itself; hence the name 'autoregressive'. The autoregressive model is a special case of the linear filter model represented by (4.3). For example we estimate $\tilde{z}_{t-1}$ from the RHS of (4.5) by substituting

$$\tilde{z}_{t-1} = \phi_1 \tilde{z}_{t-2} + \phi_2 \tilde{z}_{t-2} + \phi_3 \tilde{z}_{t-3} + \ldots + a_{t-1} \tag{4.6}$$

Similarly, we substitute for $\tilde{z}_{t-2}$ and so on. Ultimately we arrive at infinite series in the a's.

From (4.5), we write, AR(1) as:

$$\tilde{z}_t = \phi_1 \tilde{z}_{t-1} + a_t \tag{4.7}$$

4.2.1 Physical Significance in Relation to Estimation of Blocks of Ore

Consider the set-up given in Fig. 4.1.

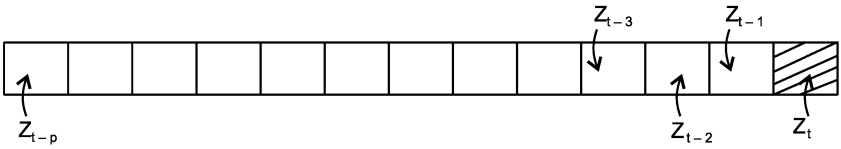


Fig. 4.1 An example of auto-regressive set-up of a sequence of blocks of ore.

We are interested in estimating the average grade of the block, z_t (shaded area), given the previous grades of the blocks z_{t-1}, z_{t-2}, ..., z_{t-k}. Let us suppose that the block values are detrended by subtracting the mean value $\bar{z}$ from each. If AR(1) model is chosen, it means that the block z_t is estimated by giving proper weightage to the immediately preceding block z_{t-1}. Suppose, the average grade $\bar{z}$ is 4.0 gms and the block $\tilde{z}_{t-1}$ has a grade of 6 gms/tonne of ore and the weight coefficient ϕ_1 (AR coefficient) is 0.8. Then, the estimated value for z_t is $0.8 \times 6.0 = 4.8 + 4.0 = 8.8$ gms/tonne of ore. Let us extend the logic to AR(2) model. We write $\tilde{z}_t = \phi_1 \tilde{z}_{t-1} + \phi_2 \tilde{z}_{t-2} + a_t$. In order to estimate z_t, we need the grades of the immediately preceding two blocks. If the grades of $\tilde{z}_{t-1}$, $\tilde{z}_{t-2}$ blocks are say 8 gms and 6 gms respectively and the weight coefficients are 0.6 and 0.5 respectively, the estimated grade for z_t is $8.0 \times 0.6 + 6.0 \times 0.5 + 4.0 = 11.8$ gms/tonne of ore. If the order is p, in order to estimate z_t, we consider the immediately preceding p blocks of z_t. Every estimate has an error associated with it. Following Box and Jenkins (1976), for AR(1) model, the variance function at lead time 'L' for AR(1) model is:

$$V(L) = \frac{\sigma_a^2 \left(1 - \phi_1^{2L}\right)}{\left(1 - \phi_1^2\right)} = \sigma_z^2 (1 - \phi_1^{2L}) \tag{4.8}$$

4.2.2 Estimation of Parameters of A.R Process of Order p [AR(p)]

Let us consider equation (4.5) which is known as autoregressive model of order p. In this model, the current value of the process is expressed as a finite linear aggregate of previous values of the process and a shock (error) a_t. Equation (4.5) can also be written as:

$$(1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p \mathrm{B}^p)\, \tilde{z}_t = a_t \tag{4.9}$$

or $\phi(B)\, \tilde{z}_t = a_t$ following the notation of backward shift operator viz., $Bz_t = z_{t-1}$ and $B^j z_t = z_{t-j}$. For stationarity, the roots of the polynomial $\phi(B)$ must lie outside the unit circle. These roots are sometimes known as zeroes.

The AR coefficients $\phi_1, \phi_2, \ldots. \phi_p$ are also known as the prediction error coefficients. In the case of AR(1), $\phi_1 = \rho_1$, the autocorrelation coefficient at lag 1 itself. One way of evaluating the AR coefficients of order ≥ 2 is by solving the Yule-Walker equations (Yule, 1927 and Walker, 1931).

Yule-Walker Scheme

For an Mth order AR process, the Y–W scheme may be expressed as:

$$\begin{bmatrix} \hat{\rho}_0 & \hat{\rho}_1 & \cdots & \hat{\rho}_{M-1} \\ \hat{\rho}_1 & \hat{\rho}_0 & \cdots & \hat{\rho}_{M-2} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{\rho}_{M-1} & \hat{\rho}_{M-2} & \cdots & \hat{\rho}_0 \end{bmatrix} \begin{bmatrix} \hat{\phi}_{M1} \\ \hat{\phi}_{M2} \\ \cdots \\ \hat{\phi}_{MM} \end{bmatrix} = \begin{bmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \\ \cdots \\ \hat{\rho}_M \end{bmatrix} \tag{4.10}$$

A simple way of obtaining the prediction error coefficients is to compute the estimates $\hat{\phi}_{11}, \hat{\phi}_{22}, \ldots \hat{\phi}_{MM}$ and invoke the Levinson-Durbin algorithm (Levinson, 1947 and Durbin, 1960). For example, in the case of third order AR process, we have the relationship:

$$\begin{bmatrix} \hat{\phi}_{31} \\ \hat{\phi}_{32} \end{bmatrix} = \begin{bmatrix} \hat{\phi}_{21} \\ \hat{\phi}_{22} \end{bmatrix} - \hat{\phi}_{33} \begin{bmatrix} \hat{\phi}_{22} \\ \hat{\phi}_{21} \end{bmatrix} \tag{4.11}$$

The AR coefficients obtained by the above method have some shortcomings viz., (*i*) the *Y–W* estimates of the AR coefficients are sensitive to rounding errors, and (*ii*) the AR coefficients should be estimated in a manner that is maximally noncommittal with respect to unavailable information. It is argued that the estimation of the autocorrelation coefficients assumes that $z_t = 0$ for $|t| > N$, an assumption that contradicts the principle of maximum entropy.

Burg Scheme

To avoid the shortcomings of Yule-Walker scheme, Burg (1967, 1968) suggested a method which does not involve the prior estimation of the autocovariances. The residual sum of squares Σa_t^2 is minimised with respect to

$\hat{\phi}_{MM}$. Let us again consider a third order process. The residual sum of squares is:

$$S(\hat{\phi}_{33}) = \sum_{i=4}^{N} [z_t - \phi_{31}\, z_{t-1} - \phi_{32}\, z_{t-2} - \phi_{33} z_{t-3}]^2 \qquad (4.12)$$

Actually, Burg suggested that the prediction error power (residual sum of squares) can be calculated, by running the predictor error filter over the data in a forward and backward fashion. Thus, for a third order process:

$$\begin{aligned} P_4 &= \text{Prediction error power} \\ &= \frac{1}{2(N-3)} \Sigma (z_t - \hat{\phi}_{33} z_{t-3} - \hat{\phi}_{32} z_{t-2} - \hat{\phi}_{31} z_{t-1})^2 \\ &\quad + (z_{t-3} - \hat{\phi}_{33} z_t - \hat{\phi}_{32} z_{t-1} - \hat{\phi}_{31} z_{t-2})^2 \qquad (4.13) \end{aligned}$$

and $\hat{\phi}_{33}$ is determined by solving: $\delta P_4 / \delta(\hat{\phi}_{33}) = 0$. This essentially means that no assumptions are made concerning the extensions of data outside the parameter space. After obtaining $\hat{\phi}_{33}$, a recursive scheme as detailed by Andersen (1974) may be used for obtaining the other coefficients. The above logic is applicable to $\bar{z}_t$ values as well.

4.2.3 Moving Average Process [MA(q)]

Eqn. (4.3) can be rewritten for a finite length filter, such that the first 'q' of the weights are non-zero. Thus:

$$\tilde{z}_t = a_t - \theta_1\, a_{t-1} - \theta_2\, a_{t-2} \ldots - \theta_q\, a_{t-q} \qquad (4.14)$$

where $\theta_1, \theta_2, \ldots \theta_q$ are the respective weights. This is the moving average (MA) process of order q. A finite order MA process is always stationary.

MA process of order 1—MA(1)

We write:

$$\tilde{z}_t = a_t - \theta_1\, a_{t-1} \qquad (4.15)$$

The process is stationary for all values of θ_1. θ_1 must lie in the range $-1 < \theta_1 < 1$ for the process to be invertible. θ_1 can be obtained from the relation:

$$\begin{aligned} \rho_k &= -\theta_1/[1 + \theta_1^2] \text{ for } k = 1; \qquad (4.16) \\ \text{and} &= 0 \qquad\quad \text{for } k \geq 2 \end{aligned}$$

where ρ_1 is the autocorrelation coefficient for lag 1.

4.2.4 Auto-regressive and Moving Average Process of Order p, q [ARMA(p, q)]

The general form of ARMA (p, q) which is an integrated form of AR and MA models may be written as:

$$\tilde{z}_t = \phi_1 \tilde{z}_{t-1} + \phi_2 \tilde{z}_{t-2} + \ldots + \phi_p \tilde{z}_{t-1} + a_t$$
$$-\theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q} \qquad (4.17)$$

Equivalently $\phi(B)\ \tilde{z}_t = \theta(B)$. This represents a mixed auto-regressive moving average (ARMA) process of order p and q. For the process to be stationary, the characteristic equation $\phi(B) = 0$ has all the roots lying outside the unit circle. Similarly, the roots of $\theta(B) = 0$ must lie outside the unit circle for the process to be invertible. The method of estimation of AR parameters was elaborately discussed by Box and Jenkins (1976) and Ulrych and Bishop (1975). The method of estimation of the parameters of an ARMA process was discussed by quite a number of authors. Notable among them are Hannan (1969), Box and Jenkins (1976), Robinson (1983). The method of estimation of the parameters for MA (1st and 2nd order) and ARMA (1st and 2nd order) processes is relatively simple. Following Box and Jenkins (1976), we write the first order ARMA process as:

Auto-regressive and Moving Average Process of order 1 [ARMA(1,1)]

$$\tilde{z}_t = \phi_1' \tilde{z}_{t-1} + a_t - \theta_1' a_{t-1} \qquad (4.18)$$

The parameters ϕ_1' and θ_1' are obtained by solving:

$$\rho_1 = [(1 - \phi_1' \theta_1')(\phi_1' - \theta_1')/(1 + \theta_1'^2 - 2\phi_1' \theta_1')];$$
$$\rho_2 = \phi_1' \rho_1 \qquad (4.19)$$

If the process is non-stationary, we differentiate the series and the differenced series may then become stationary. The stage of differencing may either be 1, 2, 3, … depending upon where the minimum variance lies. In this case, we use the terminology, ARI (p, d, 0), IMA (0, d, q), and ARIMA (p, d, q).

4.3 APPLICATIONS

4.3.1 Bauxite Example (Fe_2O_3)

Autocorrelation Coefficients (acf)

The autocorrelation coefficients for a sample set of Fe_2O_3 element values in units of % are given in Table 4.1.

Table 4.1 Autocorrelation coefficients for Fe_2O_3 element values

Lag	1	2	3	4	5	6	7	8
$r(k)$	0.64	0.46	0.36	0.31	0.28	0.27	0.02	0.01

Partial Autocorrelation Coefficients (pacf)

The estimated partial autocorrelation coefficients are given in Table 4.2.

Table 4.2 Partial autocorrelation coefficients for a sample set of Fe_2O_3 element values

Lag	1	2	3	4	5	6	7	8
φ_{kk}	0.300	0.134	0.082	–0.180	0.126	0.004	0.260	0.291

The acf can be approximated as having an exponential pattern with an exponential decay and the pacf has a cut off indicating that the candidate model could be an AR. The graphs of acf and pacf together with 2σ limits are shown in Figs 4.2(A) and 4.3(A). In Table 4.3, the standard errors of estimates for the variable are given for various models.

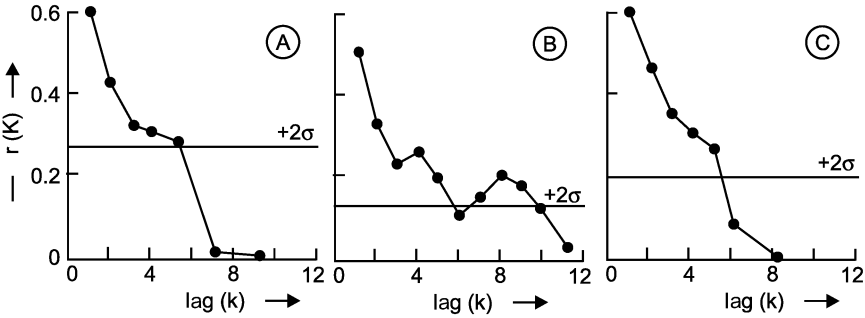


Fig. 4.2 Autocorrelation functions for (A) Fe_2O_3 element values, (B) gold accumulation values of lode *O* of gold field 1 and (C) copper accumulation values.

Table 4.3 Standard errors of estimates for Fe_2O_3 in units of % by AR (1) to AR (8), MA(1) and ARMA(1, 1) models

AR(1)	AR(2)	AR(3)	AR(4)	AR(5)	AR(6)	AR(7)	AR(8)	MA(1)	ARMA(1,1)
5.50	5.47	5.46	5.38	5.38	5.42	5.85	5.91	6.89	5.96

On a comparison of the standard errors of estimates based on these statistics, AR(4) model may be preferred. Here, again, from parsimony point of view, AR(1) may be selected. By solving the relevant Y-W scheme for the AR coefficients, we have:

$$\text{AR(1):}\quad 0.64\,\tilde{z}_{t-1} + a_t \tag{4.20}$$

$$\text{AR(4):}\quad 0.26\,\tilde{z}_{t-1} + 0.13\,\tilde{z}_{t-2} + 0.13\,\tilde{z}_{t-3} - 0.18\,\tilde{z}_{t-4} + a_t \tag{4.21}$$

4.3.2 Gold Mineralisation

Let us first compute the autocorrelation function (acf) and the partial autocorrelation function (pacf) for the sample accumulation values of lode *O* of gold field 1. These values are in units of inch-dwts. Table 4.4 gives these values upto 10 lags.

Table 4.4 Autocorrelation coefficients for gold accumulation values

Lag	1	2	3	4	5	6	7	8	9	10
$r(k)$	0.50	0.32	0.23	0.27	0.20	0.10	0.17	0.20	0.18	0.15

The autocorrelation function is approximately a decaying exponential and suggests that the candidate model can be AR. The values for the partial autocorrelation function upto lag 8 are computed and are shown in Table 4.5.

Table 4.5 Partial autocorrelation coefficients for a sample set of gold accumulation values

Lag	1	2	3	4	5	6	7	8
$\varphi(kk)$	0.50	0.09	0.06	0.14	0.005	0.7	0.15	0.12

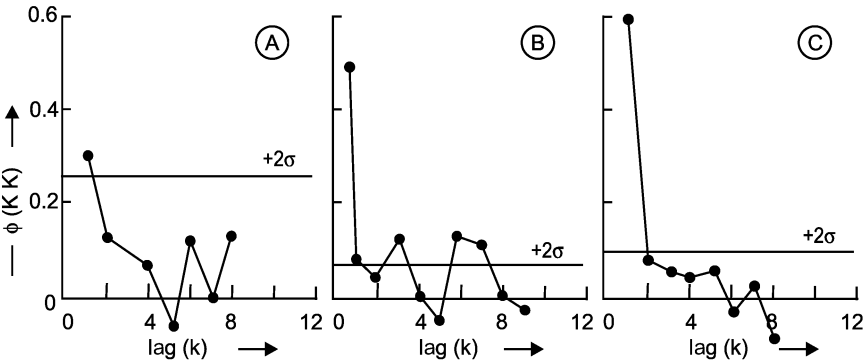


Fig. 4.3 Partial autocorrelation functions for (A) Fe_2O_3 element values, (B) gold accumulation values of lode *O* gold field 1 and (C) copper accumulation values.

The graphs of acf and pacf with the corresponding 2σ limits are shown in Figs 4.2B and 4.3B respectively. The partial autocorrelation function can be viewed as having a cut-off. These two characteristics of 'acf'' and 'pacf' indicate that the candidate model could be AR. Table 4.6 shows the standard errors of estimates by AR(1) to AR(8), MA(1) and ARMA(1,1) models.

Table 4.6 Standard errors of estimates for gold accumulation in units of inch-dwts by AR, MA and ARMA models

AR(1)	AR(2)	AR(3)	AR(4)	AR(5)	AR(6)	AR(7)	AR(8)	MA(1)	ARMA(1,1)
334	332	334	337	340	342	343	344	14868	344

On the basis of this criterion, AR(2) model may be preferred. From parsimony point of view, AR(1) may also be preferred. The fitted AR(1) and AR(2) models are

$$\text{AR(1): } 0.500\ \tilde{z}_{t-1} + a_t \tag{4.22}$$

$$\text{AR(2): } 0.454\ \tilde{z}_{t-1} + 0.088\ \tilde{z}_{t-2} + a_t \tag{4.23}$$

4.3.3 Copper Example

The autocorrelation coefficients for the sample set of copper accumulation values are as follows. These values are in units of cm%.

Table 4.7 Autocorrelation coefficients for a sample set of copper accumulation values

Lag	1	2	3	4	5	6	7	8
$r(k)$	0.65	0.47	0.37	0.31	0.28	0.10	0.02	0.00

The partial autocorrelation coefficients upto lag 8 are as follows.

Table 4.8 Partial autocorrelation coefficients for copper accumulation values

Lag	1	2	3	4	5	6	7	8
ϕ_{kk}	0.607	0.081	0.063	0.059	0.058	–0.226	0.024	–0.082

The graphs of acf and the pacf with the corresponding 2σ limits are shown in Figs 4.2C and 4.3C respectively. It appears that the acf is having an exponential decay and the pacf is having a cut-off indicating that the candidate model could be AR.

Table 4.9 gives the standard errors of estimates in units of cm% for models AR(1) to AR(8), MA(1) and ARMA(1,1). The exact model can be decided on the basis of minimum standard error of the estimate.

Table 4.9 Standard errors of estimates by AR, MA and ARMA models

AR(1)	AR(2)	AR(3)	AR(4)	AR(5)	AR(6)	AR(7)	AR(8)	MA(1)	ARMA(1,1)
163	166	167	169	166	164	165	164	—	164

On a comparison of the standard errors, AR(1) model appears to be an appropriate model. The model is:

$$\text{AR(1): } 0.65\,\bar{z}_{t-1} + a_t \tag{4.24}$$

4.4 SPECTRAL ANALYSIS (FREQUENCY DOMAIN)

4.4.1 Spectrum, Discrete Fourier Transform (DFT) and Fast Fourier Transform (FFT)

The great interest in spectrum analysis of time series lies in locating significant peaks in the spectrum. The peaks bear important relationship to the physics of the phenomenon being studied. Haykin (1983) opined that in the characterisation of second order weakly stationary stochastic processes, use of spectral density is often preferred to the correlation function because a spectral representation may reveal such useful information as hidden frequencies or close spectral estimates. Significant spikes indicate high

concentration of energy and when interpreted imply possible periodicities in mineralisation/phenomena under study.

The spectrum of a stochastic process $Z(T)$ is defined as the Fourier transform of autocorrelation function. Here, T stands for the data length sampled at intervals of Δ_T seconds leading to N data points. If T stands for distance, Δ_T is the corresponding sampling interval. Thus,

$$S_z(f) = \int_{-\infty}^{\infty} C_z(h) \exp(ifh)dh \tag{4.25}$$

where $$C_z(h) = \frac{1}{2T} \int_{-T}^{T} Z(T)\, Z(T + h)dT.$$

The spectrum can also be defined in terms of the generalised Fourier transform of a stochastic process. Broadly speaking, the various techniques that are used for spectrum estimation can be classified as: (i) Fourier transformation of autocorrelation function, (ii) averaging square of the magnitude of the Fourier transform of the time series (direct method) and (iii) bank of sharply tuned filters. The second method is more often used and the same has been followed in this study. We shall briefly discuss this method. There are two variations of this method viz., (A) Frequency averaging and (B) Sectioning of time series.

A. Frequency Average

In this approach, the entire time series is subjected to Fourier transform. The energy in small non-overlapping frequency intervals is averaged. This averaged energy is the estimate of the spectrum at the centre of the frequency band.

B. Sectioning of Time Series

In this approach, a long time series is divided into 's' segments and each segment is Fourier transformed and magnitude squared. The magnitude squared Fourier transform is averaged over all segments. This gives us an estimate of the spectrum.

Both approaches of the direct method are equivalent. However, from a computational point of view, the first approach is not economical because it requires storing of the entire data in the core memory of the computer and the time taken for Fourier transformation of a long segment is more. It is, therefore, advisable to segment the data into 's' segments and proceed. The complex Fourier coefficients are obtained as:

$$A(\eta) = \sum_{T=0}^{N-1} z(T) \exp(-2\pi i \eta T/N) \tag{4.26}$$

Eqn. (4.26) is often called the discrete Fourier transform (DFT). The function of time may be recovered exactly by the following inverse Fourier transform.

$$z(T) = \frac{1}{N} \sum_{\eta=0}^{N-1} A(\eta) \exp(+2\pi i \eta T/N) \tag{4.27}$$

Here, the frequency in radians is: $f = \dfrac{\eta\pi}{N\Delta_T}$, $\eta = 0, 1, 2, \ldots N-1$. The spectrum which is obtained from a finite length of data is distorted in some way and an attempt is made to minimise this effect. Usually, a window function $D(N)$ is included in the expression (4.27). Thus, we may rewrite (4.27) as:

$$z(T) = \frac{1}{N} \sum_{\eta=0}^{N-1} A(\eta)D(N) \exp(+2\pi i \eta T/N) \tag{4.28}$$

Numerical evaluation of DFT requires arithmetic operations which include complex addition and multiplications. When N is very large, the computation required to carry out Fourier transformation becomes very large. Much of the computational work can be minimised by taking advantage of the symmetry properties of the kernel functions leading DFT to Fast Fourier Transform (FFT) method. Towards this end, the Cooley-Tukey and Sande-Tukey algorithms are very helpful. After making the Fourier transform, the raw spectral densities are obtained as:

$$S_F(f) = \frac{\Delta_T}{N} |A(\eta)|^2 \tag{4.29}$$

where Δ_T is the sampling interval. The spectral densities given by equation (4.29) are not consistent as the sample size increases and to circumvent this drawback, Bartlett's principle is applied (Kaneswich, 1975, p. 100). Therefore, the data of N points are segmented into 's' sub-series of r data points each and necessary zeroes appended to the sub-series to facilitate the power of 2 algorithm workable. The size of the appended series is r'. The spectral densities are obtained by averaging the individual spectral densities $S_{F,k}(f)$ for $k = 1, 2, \ldots s$ at frequency f, and multiplying the estimate by the quotient (r'/r) – being the compensation factor; r = number of points of the sub-series without appending zeroes. Thus:

$$S_F(f) = \left[\frac{1}{S} \sum_{k=1}^{S} S_{F,k}(f) \right] (r'/r) \tag{4.30}$$

The importance of these techniques lies in the identification of significant spikes in the spectra. Table 4.10 gives the significant spectral density estimates for various applications.

4.4.2 Maximum Entropy Method

We will now briefly discuss another widely used method viz., the MEM. This needs a little introduction to bring out the relation between information and entropy.

Information and Entropy

The relationship between information and entropy may be written as:

$$I = k \ln (1/p_i) \tag{4.31}$$

where I = information; p_i = probability of occurrence of the 'i'th event and k is a constant which is 1 when the base of logarithm is 2. Assuming that we observe the above system for a long time T, we may expect p_1T of m_1 things, p_2T of m_2 things, etc., to have happened in the time interval T. The total information about the system will then be:

$$I_{\text{Total}} = k[p_1T \ln (1/p_1) + p_2T \ln (1/p_2) + \ldots]$$

The average information per time interval is represented by H and is referred to as 'entropy'. Following Shannan (1948):

$$H = I_{\text{Total}}/T = -k\Sigma p_i \ln p_i \tag{4.32}$$

It is clear that entropy is described by a set of probabilities and is a measure of uncertainty. The entropy ranges from zero to unity. Thus entropy may be viewed as a measure of disorder in the system or a measure of our ignorance about the actual structure of the system (Brillouin, 1956). The method of determining the maximum entropy probability distribution was outlined by Ulrych and Bishop (1975). Following Jaynes (1963, 1968), we may say that the process $Z(T)$ takes values $z_1, z_2, \ldots z_n$. The probability distribution $p_i = p(z_i)$ that is considered with the information but is maximally free of other constraints is the one that maximises the entropy.

$$H = -\sum_i p_i \log_2 p_i \tag{4.33}$$

subject to $\sum_i p_i = 1$ and $\Sigma p_i f_k(z_i) = \langle f_k z_i \rangle$ for $k = 1, 2, \ldots m$

$$m < n$$

4.4.3 Spectral Density and Entropy

The relationship between the entropy and the spectral density $S(f)$ of stationary Gaussian process allows us to write:

$$H = \frac{1}{4f_N} \int_{-f_N}^{f_N} \log\, S(f)\, df \tag{4.34}$$

Rewriting the above in terms of autocorrelations of the process, we have:

$$H = \frac{1}{4f_N} \int_{-f_N}^{f_N} \log \left[\sum_{-\infty}^{+\infty} \rho(k) \exp(-i2\pi f_k \Delta_t) \right] d_f \tag{4.35}$$

where ρ_k is the autocorrelation at lag k, f_N is the Nyquist frequency and Δ_t is the uniform sampling interval rate. Maximising equation (4.35) with respect to the unknown ρ_k, with the constraint that $S(f)$ must also be consistent with the known autocorrelations $\rho(0)$, $\rho(1)$, … $\rho(M–1)$, results in the MEM spectrum estimate. This estimate expresses maximum uncertainty with respect to the unknown information. This variational procedure leads to the well known expression for the MEM spectral density (Smylie et al., 1973, Edward and Fitelson, 1973) which for a real linear process $Z(T)$ is:

$$U_E(f) = \frac{U_M}{f_N \left| 1 + \sum_{j=1}^{M-1} \gamma_j \exp(-i2\pi f_j \Delta_t) \right|^2} \tag{4.36}$$

In eqn. (4.36), U_M is constant (updated variance) and the γ_j are the prediction error coefficients that are to be determined from data. The shortcoming of the MEM spectral estimate has been the lack of a quantitative method of determining the length of the prediction error filter γ_j. However, the work of Akaike (1969, 1970) on the determination of the order of the AR process is a step forward in overcoming this problem. This was briefly discussed in Ulrych and Bishop (1975). The following conclusions of their article are noteworthy:

(*i*) MEM spectrum analysis is equivalent to fitting an AR model to the random process (van den Bos, 1971), and

(*ii*) that the representation of a stochastic process by an AR model is that representation that exhibits the maximum entropy. Eqn. (4.36) identifies $N + 1$ point prediction error filter (or prediction error coefficients) as $1, \gamma_1, \gamma_2, \ldots \gamma_j$. The coefficients are usually written as: $1, -\alpha_1, -\alpha_2, \ldots -\alpha_j$. The prediction error coefficients can be estimated by either of the two schemes, viz., (1) Yule-Walker equations and (2) Burg Methods which were discussed in the previous section.

Spectra obtained by applying FFT and MEM (Burg scheme) are shown in Figs 4.4 and 4.5 for Fe_2O_3 element values, gold and copper sample accumulation values.

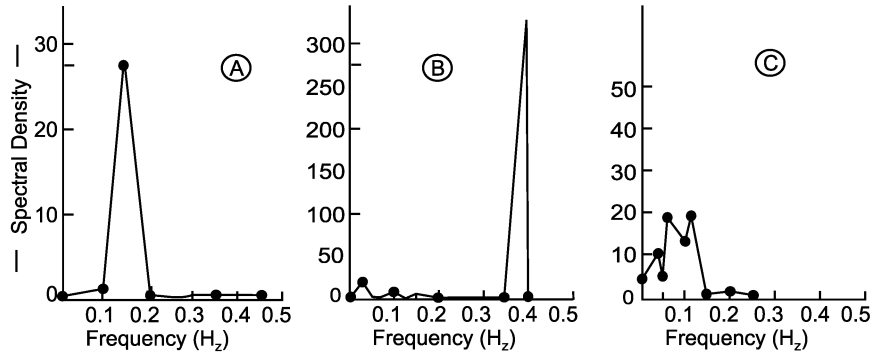


Fig. 4. 4 MEM spectra for (A) Fe_2O_3 element values, (B) gold accumulation values of lode *O* of gold field 1; and (C) copper accumulation values.

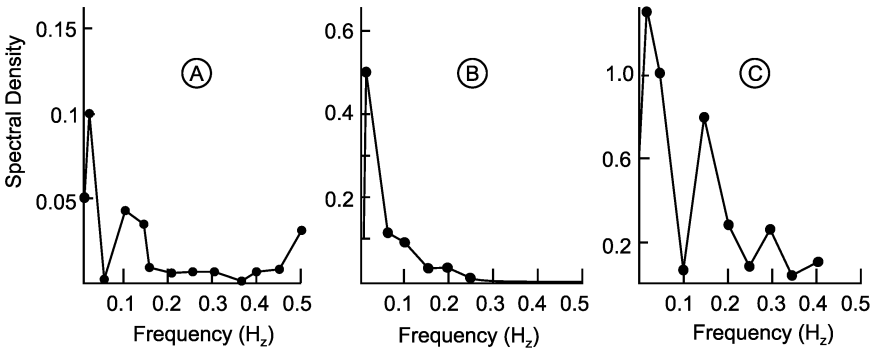


Fig. 4.5 FFT spectra for (A) Fe_2O_3 element values, (B) gold accumulation values of lode *O* of gold field 1 and (C) copper accumulation values.

Table 4.10 Significant spectral density estimates by MEM and FFT

	MEM		*FFT*	
	Freq.	*Spec. Density*	*Freq.*	*Spec. Density*
(*i*) Data: Fe_2O_3 values	0.15	31.51	0.0156	0.10
			0.13	0.045
(*ii*) Data: Gold accumulation	0.38	408.00	0.05	3.90
values			0.29	0.11
			0.40	0.21
(*iii*) Data: Copper accumulation	0.05	12.00	0.015	0.80
values	0.07	18.20	0.300	0.28
	0.12	18.20		

Note: The spectral analysis techniques are applicable both in time and spatial domains. In the context of spatial domain, samples are taken or observations made in space at regular intervals.

Review Questions

Q. 1. Compute auto-correlation function for the average grades of ore given below up to lag 5 and plot the same.

Year	1960	'61	'62	'63	'64	'65	'66	'67
Avg. grade	9.30	8.67	9.09	9.19	7.94	8.01	8.13	7.42
Year	1968	'69	'70	'71	'72	'73	'74	'75
Avg. grade	6.54	4.98	6.49	6.41	5.14	5.07	5.66	5.33
Year	1976	'77	'77	'78	'79	'80	'81	'82
Avg. grade	5.32	4.87	4.65	4.06	3.74	4.90	5.03	4.60

Q. 2. Obtain the AR coefficients using Y-W equations.

Q. 3. Fit AR(1) model and using this model, obtain the estimates for the years 1982 and 1983.

Q. 4. Bring out the advantage of spectrum analysis over correlation approach.

Q. 5. Discuss FFT and MEM spectral analysis methods. What are the advantages and disadvantages of MEM methods?

5 Concepts of Regionalised Variables and Variogram Modelling

5.1 INTRODUCTION

Usually, information about a geological phenomenon under study is sketchy or very limited. Therefore, we need a model to be able to draw valid inferences about mineralised zones that have not been sampled. In Chapter 4, we discussed models based on time series analysis (stochastic modelling) and applied the same to realistic situations. There are also methods based on trend surface analysis which is a regression method. Here a dependent variable, Z, is predicted, given the locational co-ordinates (x, y). The implicit assumption underlying these types of regression methods is that the surface under study can be represented, locally, by a fairly simple deterministic function such as a polynomial, plus a random error component. However, most geological variables display a considerable amount of short scale variation in addition to the large scale trends that can reasonably be described by deterministic functions. Therefore, a dichotomy into deterministic and random may not always be correct.

To tackle problems of this type associated with geological and other related variables, the term *Regionalised Variable* was coined by Prof. G. Matheron (Matheron, 1963) to emphasise on the two aspects of the variables: (i) a random aspect which accounts for local variations, and (ii) a structured aspect which reflects large scale tendencies of the phenomenon. It may be mentioned here that much of the literature on the theory of Regionalised Variables is available in French and mostly available either as Technical Reports or Course Manuals at the Centre de Geostatistique, Fountainebleau, France. Some manuals and books in English language dealing with this type of approach, called *Geostatistics*, are by Matheron (1971), Journel and Huijbregts (1978), M. David (1977), Armstrong (1986), Galli et al. (1987), I. Clark (1979), A.G. Royle (1971), and Rendu (1978). There are a number of research papers published on this subject. Some of these are listed under Bibliography.

As mentioned above, statistical models such as trend surfaces have two parts viz., *the deterministic part* and *the error part.* A better way of looking at randomness is to think of it as fluctuations around a fixed surface. This may be called as 'drift'. These fluctuations are not errors, but full-fledged features of the phenomenon having structures of their own needing a *Structural Analysis.*

Random Functions

The observed value at each data point x_i is considered as the outcome $z(x_i)$ of a random variable $Z(x)$, whose mean at point x_i is called the drift $m(x_i)$. The observed values can also be thought of as being the outcomes (or realisations) of the corresponding random variables $Z(x)$. In mathematical terms, as mentioned in Chapter 4, the assemblage of all these random variables together with their respective probability distributions is called a *Random Function*—the synonyms being *Stochastic Processes and Random Fields.* A random function has the same type of relationship with one of its realisations as a random variable has with the numerical outcome in a single trial. A random function is characterised by its finite dimensional distributions. Also, we have to make some assumptions about the characteristics of these distributions such as stationarity.

5.2 STATIONARITY AND INTRINSIC HYPOTHESIS

5.2.1 Stationarity

In Statistics, it is common to assume that the process under study is stationary, i.e. *its distribution is invariant under translation.* In the same way, a stationary random function is homogenous and self-repeating in space. The *strict sense* stationarity requires all the moments to be invariant under translation. Let us further elaborate this. A stochastic process is said to be strictly stationary if its properties are unaffected by change of time origin; that is, the joint probability distribution associated with n observations $z_{t1}, z_{t2}, \ldots z_{tn}$ made at any set of times $t_1, t_2, t_3, \ldots t_n$ is the same as that associated with n observations $z_{t1+k}, z_{t2+k}, z_{t3+k}, \ldots z_{tn+k}$ made at times $t_1 + k, t_2 + k, t_3 + k, \ldots t_n + k$.* Thus for a discrete stochastic process to be strictly stationary, the joint distribution of any set of observations must remain unaffected (invariant under translation) by shifting all the times of observations either in forward or backward direction by an integer amount τ (time difference). But since this cannot be verified from the limited experimental data, we usually require *the first two moments* (the mean and variance) to be invariant under

* The above concepts discussed in the context of time domain are applicable in the context of spatial domain as well, where samples are drawn or observations made at regular intervals. The observations $z_{t1+k}, z_{t2+k}, z_{t3+k}, \ldots z_{tn+k}$ could now be termed as $z_{s1+h}, z_{s2+h}, z_{s3+h}, \ldots z_{sn+h}$ made at spatial points $s_{1+h}, s_{2+h}, \ldots s_{n+h}$ respectively, where h is the sampling interval.

translation. This is called *weak* (*weak sense*) *stationarity*. Some authors refer to this as *second order stationarity* (*stationarity of second order*). In spatial context, we require: (i) The expected value (or mean) of the function $Z(x)$ to be constant for all points x. That is, $E[Z(x)] = m(x) = m$ which is independent of x and h and (ii) the covariance between any two points x and $x + h$ is independent of x. It depends only on the vector h. Thus: $E[Z(x)\ Z(x + h)] - m^2 = C(h)$. In particular, when $h = 0$, the covariance comes back to the ordinary variance of $Z(x)$ which must also be constant. Stationarity of the covariance implies stationarity of variance, and the variogram which will be introduced below.

Weaker Second Order Stationarity: In reality, the above assumptions are more often not satisfied. When there is a marked trend, the mean value cannot be assumed to be constant. For the present, we shall only consider cases where the mean is constant. However, even when this is true, the covariance need not exist. An example of this kind was found by Krige (1978) for the gold assay values in South Africa. Therefore this hypothesis needs to be further relaxed.

5.2.2 Intrinsic Hypothesis

A random function is said to be intrinsic if (i) the mathematical expectation exists and does not depend on the support point x, i.e., $E\{Z(x)\} = m$ and (ii) for any vector h, *the increment* $[Z(x + h) - Z(x)]$ has a finite variance which is independent of the point x. In other words, $E[Z(x + h) - Z(x)] = 0$ and Var $[Z(x + h)–Z(x)] = 2\gamma(h)$, a finite value which does not depend on x. The function $\gamma(h)$ is called the *semi-variogram*. For short we call this simply as *variogram*. It is the basic tool for structural interpretation of phenomena as well as for estimation. Before we discuss more about the semi-variogram, it is important to see how to decide whether a particular variable is stationary or not.

5.2.3 Stationarity in Actual Practice

In actual practice, variogram is used upto a certain distance. This limit could be the extent of a homogenous zone within a deposit which can be considered as stationary up to this distance. The problem can be resolved by considering a series of *sliding neighbourhoods* within which the expected value, and the variogram can be considered to be stationary.

5.3 VARIOGRAM

The spatial correlations of an intrinsic random function are characterised by semi-variogram function defined as [Matheron (1967)]:

$$\gamma(h) = 0.5 \text{ Var } [Z(x + h) - Z(x)]. \tag{5.1}$$

Since it has been assumed that the mean of $Z(x + h) - Z(x)$ is zero, $\gamma(h)$ is just half the mean square value of the difference. That is:

$$\gamma(h) = 0.5E[Z(x + h) - Z(x)]^2 \tag{5.2}$$

$$\hat{\gamma}(h) = \frac{1}{2N}\sum_{i=1}^{N}[Z(x_i + h) - Z(x_i)]^2 \text{ in the discrete case.}$$

N = Number of pairs

Here x and $x + h$ refer to points in 3-dimensional space. For a fixed angle, the variogram gives different values as the distance increases. When the angle is changed, the variogram discloses the directional features, if any, of the phenomenon such as its anisotropy. Arithmetically, the variogram is simple to understand. The differences between, say, assay values derived from rock samples and separated by distance 'h' (lag k) are squared and divided by twice the number of differences (pairs) found. It may be described as follows:

1. It starts at 0 [for $h = 0$, $Z(x + h) = Z(x)$]
2. It generally increases with h.
3. It rises upto a certain level called the *sill* and then flattens out in some cases. In other cases, it could even just go on rising. Figure 5.1A shows a typical variogram.

5.3.1 Properties of Variogram

1. *Range*

The rate of increase of $\gamma(h)$ is an indicator of the rate at which the 'influence' of a sample decreases with increasing distances from the sample site. This critical distance is called the *range* of the variogram. It gives a more precise definition to the notion of *'zone of influence'*. Theoretically speaking, this *limiting value* of the variogram called 'sill' is exactly the variance of the population. When there is no correlation between $Z(x + h)$ and $Z(x)$ we have:

$$\gamma(h) = 0.5 \text{ Var } [Z(x + h) - Z(x)]$$

$$= 0.5 \{\text{Var } [Z(x + h)] + \text{Var}[Z(x)]\} = \frac{2\sigma^2}{2} = \sigma^2. \qquad (5.3)$$

It is not necessary for all variograms to reach a sill. Some variograms keep on increasing with h [see Fig. 5.1B]. The range need not be the same in all directions which reflects the anisotropy of the phenomenon. Also, for a given direction there can be more than one range. This occurs when there are several nested structures acting at different scales of distance. Figure 5.1 shows examples of bounded and unbounded variograms.

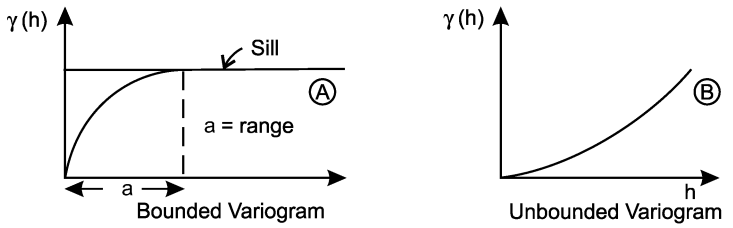


Fig. 5.1 (A) Bounded and (B) Unbounded variograms.

2. *Behaviour near the Origin*

We have examined the behaviour of the variogram for large distances. Four types of behaviour near the origin can be identified. These are shown in Fig. 5.2.

(a) *A parabolic shape:* This indicates that the regionalized variable (Re. V.) is highly continuous and even differentiable. At times a parabolic shape can be associated with the presence of a drift.

(b) *A linear shape:* In this case, the Re. V is continuous but not differentiable, and thus less regular than in (a). For example, seam thickness for coal or gravity field in a given area exhibit this type of variogram.

(c) *A discontinuity at the origin:* When h tends to zero, $\gamma(h)$ does not tend to zero. This means that the variable is not even continuous in the mean square. It is, therefore, highly irregular at short distances. Most geological variables including metal grades such as gold, copper, lead and zinc show this type of behaviour. This jump at the origin is called nugget effect because it was first noticed in gold deposits in South Africa where it was associated with the presence of nuggets in the ore. Here, the grade changes abruptly from zero outside to a high value (nugget) inside it. The term nugget effect is used to describe short range variability even though it may be due to some other factors such as measurement errors, errors in location, etc.

(d) *A flat curve:* A curve which is more or less parallel to X-axis. This represents pure randomness or white noise. The regionalized variables $Z(x + h)$ and $Z(x)$ are uncorrelated for all values of h, no matter how close they are. This is the limiting case of a total lack of structure.

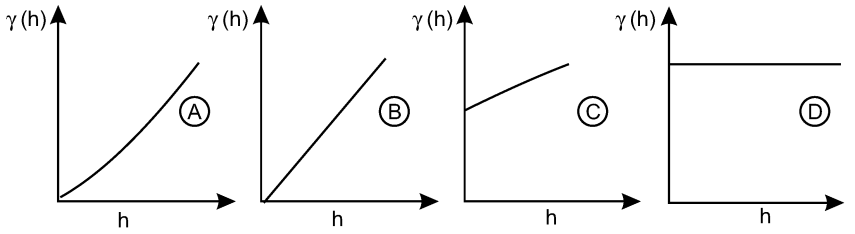


Fig. 5.2 Behaviour of variogram at the origin: (A) Highly continuous, (B) Continuous, (C) Discontinuous and (D) Purely random.

5.3.2 Anisotropies

When the variogram is calculated for all pairs of points in certain directions such as North-South, East-West etc., it sometimes shows different types of behaviour signifying the presence of anisotropy. If this does not occur, the variogram depends only on the magnitude of the distance between points h

and is said to be isotropic. Two different types of anisotropy can be distinguished: geometric anisotropy and zonal anisotropy.

1. Geometrical anisotropy (also called 'elliptic' anisotropy)

This occurs when $\gamma(h) \neq \gamma(r)$ and a simple linear transformation of the co-ordinates is sufficient to restore isotropy. Typical situations are shown in Fig. 5.3A. In this figure, the variograms have the same sill in all directions even though their ranges are different. In Fig. 5.3B, the variograms are both linear but have different slopes. If the curve is an ellipse (2–D), the anisotropy is said to be geometrical (or elliptic). In this case, by a proper change of coordinates, we can convert the ellipse into a circle to eliminate the anisotropy. This transformation is particularly simple when the major axes of the ellipse coincide with the co-ordinate axes (Fig. 5.4A). Then, if the equation of the variogram in direction 1 is $\gamma_1(h)$, the overall variogram after correcting for the anisotropy is of the form:

$$\gamma(h) = \gamma_1 \sqrt{(x_1 - x_2)^2 - b^2 (y_1 - y_2)^2}$$

where b is the anisotropy ratio, viz., $b = \dfrac{\text{range 1}}{\text{range 2}}$ or $b = \dfrac{\text{slope 1}}{\text{slope 2}}$.

When calculating the variogram, it is important to use at least four directions so that there are no chances of missing the anisotropy completely (Fig. 5.4B).

2. Zonal (or stratified) anisotropy

This is a complex type of anisotropy. For example, we may come across strongly marked changes or variation in the values between footwall and hanging wall or various strata in a mine. This in between variation in the zones/strata leads to what may be called zonal/stratified anisotropy. In such cases, the usual practice is to split the variogram into two components — an isotropic one plus another which depends only on the vertical component.

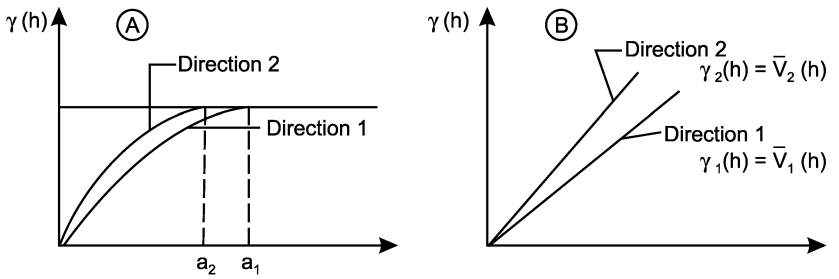


Fig. 5.3 Anisotropic situations: (A) Variogram with the same sill and (B) Linear variogram with different slopes.

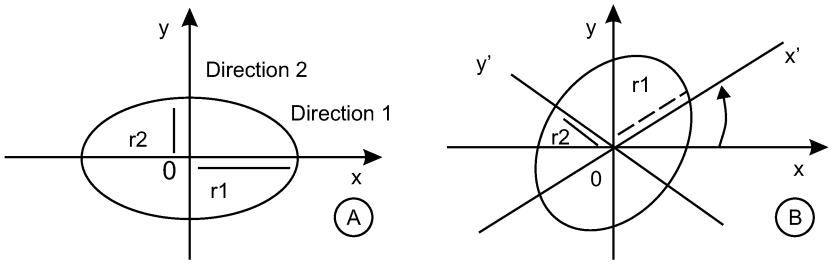


Fig. 5.4 Examples of elliptical anisotropy: (A) Main axes coincide with the co-ordinate axes and (B) Main axes do not coincide with the co-ordinate axes.

5.3.3 Some Practical Points on Variograms

1. It is desirable that the data set, on the basis of which a variogram is constructed consists of at least 50 samples/assay values of the same volume, shape and orientation and these are drawn from a strike length/segment at a more or less regular sampling interval. It is better if the samples are drawn intersecting the whole thickness of the deposit.
2. If a break occurs in the strike length/segment of a deposit from where the samples are drawn (due to a fault etc.), the variogram computations should recognise this.
3. When samples are available from different levels of a deposit, the variogram for each level is produced, and then their average variogram is determined. This averaging is important, because in practice individual variograms may differ widely.
4. Only the values of $\gamma(h)$ near the origin and in respect of the first few lags (of distance) can be regarded as very inportant. At increasing distances, the variations between the local semi-variogram and the intrinsic function $\gamma(h)$ can fluctuate widely.

5.3.4 Presence of a Drift

From a theoretical point of view, for large distances (lags), the variogram must increase very slowly. To be more specific $\{\gamma(h)/h^2\} \to 0$ as $h \to \infty$. However, in practice, we often find variograms which increase more rapidly than h^2. This indicates the presence of a drift (Fig. 5.5).

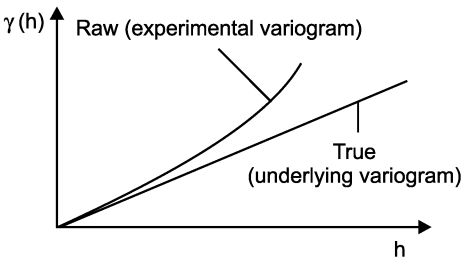


Fig. 5.5 Effect of linear drift on variogram.

The experimental variogram shown in Fig. 5.5 gives us an estimate of $0.5\,E[Z(x+h)-Z(x)]^2$, which is called the raw variogram, instead of the true (or underlying) variogram. These two coincide only if the increments have a zero mean. Otherwise:

$$E[Z(x+h)-Z(x)]^2 = \text{Var}[Z(x+h)-Z(x)] + \{E[Z(x+h)-Z(x)]\}^2$$

$$\text{raw variogram} = \text{underlying variogram} + (\text{bias term})^2$$

Consequently, when there is a drift, the empirical variogram overestimates the underlying variogram.

5.3.5 Proportional Effect

Proportional effect usually occurs with data which are lognormally distributed. The variograms for different zones have the same shape but the sill of the variograms in rich zones is much higher than in poor ones. Figure 5.6 shows the variogram $\gamma_1(h)$ for poor zones and $\gamma_2(h)$ for rich zones. Often the sill turns out to be proportional to the square of the local mean. So the underlying variogram model can be found by dividing each of the local variograms by the square of the local mean and then averaging them before fitting a variogram model.

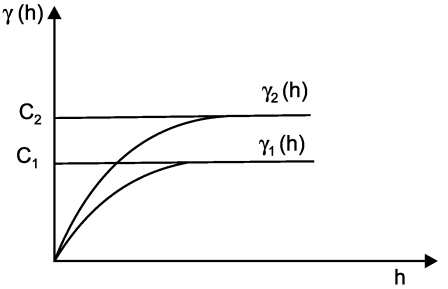


Fig. 5.6 An example of proportional effect. Variogram in rich and poor zones.

5.3.6 Other Features

At times, the behaviour of the variogram shows various other features such as nested structures, periodicities or a hole effect.

Nested Structures: These indicate the presence of variations at different scales such as sample collection, petrographic analysis etc. (Fig. 5.7A).

Periodicity: Variograms, like covariances, can exhibit periodic behaviour. Sedimentary gold deposits sometimes exhibit this type of variogram (Fig. 5.7B).

Hole Effect: In some cases, there could be a bump in the variogram (which would correspond to a hole in the covariance (Fig. 5.7C). Generally, this is caused by less number of points used in computing the experimental variogram value for a specific distance or due to *natural fluctuations* in the variogram.

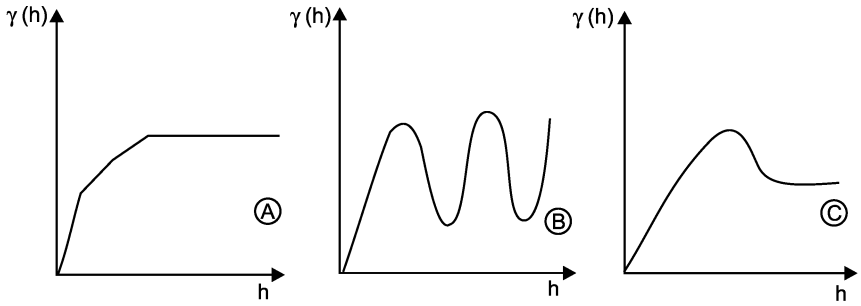


Fig. 5.7 Variogram features: (A) Nested structures, (B) Periodicity and (C) Hole effect.

5.4 COMMONLY USED VARIOGRAM MODELS

The following are some commonly used variogram models:

1. *Power functions:* $\alpha|h|^p$ with $0 < p < 2$. A particular case of this model is when we have the linear model $\gamma(h) = \alpha|h|$.

2. *Spherical model:* $$\gamma(h) = \left\{C\left[1.5\left(\frac{h}{a}\right) - \frac{1}{2}\left(\frac{h}{a}\right)^3\right]\right\}; \text{ for } |h| < a \qquad (5.4)$$
$$= C \qquad ; \text{ for } |h| \geq a$$
where C = sill and a = range. If there is nugget effect C_0, the same is added to the model. The spherical model is the most commonly used model. The tangent at the origin intersects the sill at a point with an abscissa $2a/3$.

3. *Exponential model:* $\gamma(h) = C[1 - \exp(h/a)]$ where C and a stand for sill and range respectively. For practical purposes, the range can be taken as $3a$. The tangent at the origin intersects the sill at a point with an abscissa a.

4. *Gaussian model:* $\gamma(h) = C\left(1 - \exp\left[-\left(\frac{h}{a}\right)^2\right]\right)$ where C and a, as usual, stand for sill and range. The practical range is $1.73a$. The Gaussian model represents an extremely continuous phenomenon. Experience shows that numerical instabilities often occur when this is used without a nugget effect. Figure 5.8 shows these models.

5. *Cubic model:*
$$\gamma(h) = C\left(7\frac{h^2}{a^2} - \frac{35h^3}{4a^3} + \frac{7h^5}{2a^5} - \frac{3h^7}{4a^7}\right) \text{ for } |h| \leq a$$
$$= C \qquad \text{for } |h| > a$$

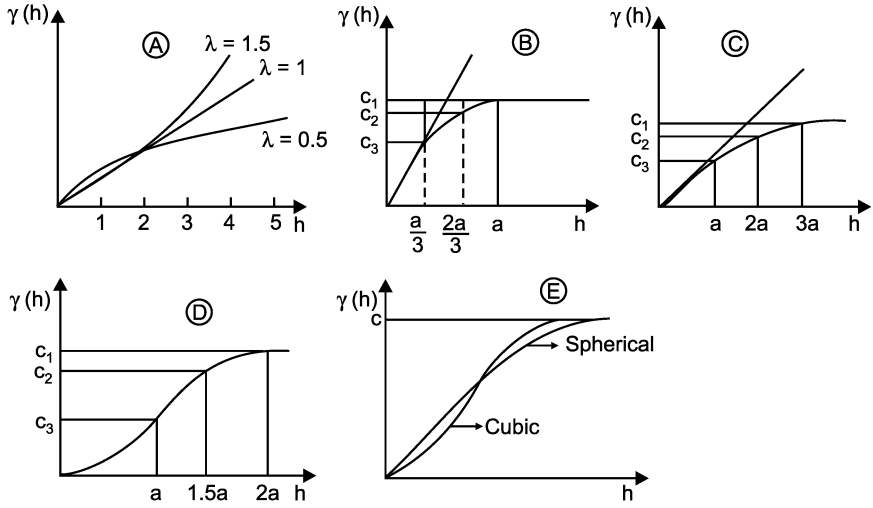


Fig. 5.8 Examples of the forms of some commonly used variogram models: (A) Power functions, (B) Spherical model, (C) Exponential model, (D) Gaussian model and (E) Cubic model.

5.5 CHANGE OF SUPPORT, REGULARISATION AND ESTIMATION VARIANCE

Sometimes point samples $z(x)$ may not be available. We may have core samples having a definite volume v – [support $v(x)$], centred on a point x. The grade of $z_v(x)$ is the mean value of the point grade $z(x)$ in $v(x)$ i.e.,

$$z_v(x) = \frac{1}{v} \int_{v(x)} z(x)dx \tag{5.5}$$

The mean value $z_v(x)$ is said to be the regularisation of the point variable $z(x)$ over the volume $v(x)$. Let us extend this logic to RF $Z(x)$. The regularisation of the point RF $Z(x)$ over the volume $v(x)$ is a RF which may be denoted by $Z_v(x)$. Thus:

$$Z_v(x) = \frac{1}{v} \int_{v(x)} Z(x)dx \tag{5.6}$$

If we are interested to derive the regularised variogram $2\gamma_v(h)$ from the point variogram $2\gamma(h)$, one way out is to consider the expression of the regularized variogram as the variance of the estimation of the mean grade $Z_v(x)$ by the mean grade $Z_v(x + h)$ separated by the vector h. This estimation variance is then given by the general formula (Journel and Huijbregts, p. 77, 1978):

$$2\gamma_v(h) = 2\,\overline{\gamma}[v(x),\ v(x + h)] - \overline{\gamma}[v(x),\ v(x)] - \overline{\gamma}[v(x + h),\ v(x + h)] \tag{5.7}$$

Since the point variogram $\gamma(h)$ is stationary, $\bar{\gamma}[v(x), v(x)]$ and $\bar{\gamma}[v(x+h), v(x+h)]$ are equal and, therefore, we have:

$$2\gamma_v(h) = 2\bar{\gamma}[v(x), v(x+h)] - 2\bar{\gamma}[v(x), v(x)]$$
$$\gamma_v(h) = \bar{\gamma}(v, v_h) - \bar{\gamma}(v, v) \qquad (5.8)$$

where v_h denotes the support v derived from v by the vector h, $\bar{\gamma}(v, v_h)$ represents the mean value of the point semi-variogram $\gamma(u)$ when one of the extremities of the vector u describes the support v and the other extremity independently describes the derived support v_h. Figure 5.9 shows a comparison of hypothetical variograms with point samples and a support v_h.

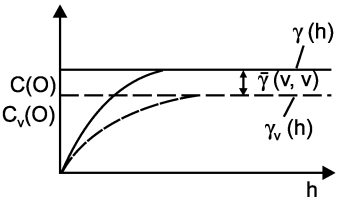


Fig. 5.9 Effect of support on the variograms.

For distances h which are very large in comparison with the dimension of support v, the mean value $\bar{\gamma}(v, v_h)$ is approximately equal to the value $\gamma(h)$ of the point semi-variogram and we have a very useful approximation:

$$\gamma_v(h) \cong \gamma(h) - \bar{\gamma}(v, v) \text{ for } h > v. \qquad (5.9)$$

Regularisation by Cores along a Bore Hole

We may extend the foregoing logic to cores along a bore hole. This type of regularisation corresponds to the construction of the variogram of the mean grade of core samples along the length of a bore hole. We assume that all the core samples have the same length 'L' and the same cross sectional area 'a' i.e., the RF regularised on the support $v = (a \times L)$ of the core samples may be expressed as in expression (5.6). When the diameter of the core is small compared to length 'L', the regularisation effect of the cross-sectional area 's' of the core samples can be neglected. The mean value over the length L of the core samples can then be written as:

$$Z_v(x) = Z_L(x) = \frac{1}{L}\int_{L(x)} Z(x)dx \qquad (5.10)$$

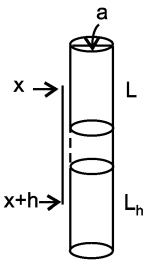


Fig. 5.10 Cores of length L with support v.

Figure. 5.10 shows a core of length 'L' with supports $v = a \times L$.

When the cross-sectional area 's' of the core samples is negligible, two core samples can be considered as two aligned segments L and L_h of the

same length L and separated by a distance 'h'. The regularised semi-variogram can be expressed as (Journel and Huijbregts, p. 80, 1978):

$$\gamma_{L(h)} = \overline{\gamma}(L, L_h) - \overline{\gamma}(L, L) \tag{5.11}$$

We will discuss more about regularisation with real examples in the next chapter.

5.6 EXAMPLES OF VARIOGRAM COMPUTATION

Example 1

Let us consider the following set of gold assay values (grade in units of dwts/ton of ore) of samples, each separated by a distance of 3 ft taken from a segment of a gold bearing lode '*O*' of gold field 1, southern India.

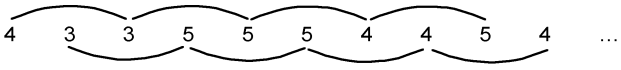
Data in dwts/ton of ore: (one dwt = 1.55517 gms ton of ore)

4	3	3	5	5	5	4	4	5	4
5	17	8	2	2	3	7	7	1	6
10	9	9	10	11	12	11	3	3	4

Employing formula (5.2) mentioned in the previous section for the computation of variogram, we have:

$$\begin{aligned}\overline{\gamma}(3) = [&(4–3)^2 + (3–3)^2 + (3–5)^2 + (5–5)^2 + (5–5)^2 + (5–4)^2 + (4–4)^2 + (4–5)^2 \\ &+ (5–4)^2 + (4–5)^2 + (5–17)^2 + (17–8)^2 + (8–2)^2 + (2–2)^2 + (2–3)^2 \\ &+ (3–7)^2 + (7–7)^2 + (7–1)^2 + (1–6)^2 + (6–10)^2 + (10–9)^2 + (9–9)^2 \\ &+ (9–10)^2 + (10–11)^2 + (11–12)^2 + (12–11)^2 + (11–3)^2 + (3–3)^2 \\ &+ (3–4)^2]/(29\times2) \\ = 7.48\end{aligned}$$

The set-up for $\overline{\gamma}(6)$ is:



The eligible pairs are:

(4, 3), (3, 5), (3, 5), (5, 5), (5, 4), (5, 4), (4, 5), (4, 4) …

The number of pairs is 28. The $\overline{\gamma}(6)$ = 13.86. In a similar way, $\overline{\gamma}(9)$, $\overline{\gamma}(12)$, etc. can be computed. Table below shows values for various distances.

h (ft)	*No. of pairs*	*Gamma*
3	29	7.48
6	28	13.86
9	27	14.37
12	26	13.77
15	25	12.58
18	24	14.73
21	23	18.98
24	22	15.86
27	21	13.52
30	20	15.95

Now let us consider a set-up where some values say 4th, 19th and 22nd are missing.

4	3	3	–	5	5	4	4
5	4	5	17	8	2	2	3
7	7	–	6	10	–	9	10
11	12	11	3	3	4		

It can be seen that the number of pairs for $\bar{\gamma}(3)$ is 23. Table below shows the statistics for the set-up.

h (ft)	*No. of pairs*	*Gamma*
3	23	8.00
6	22	14.66
9	22	15.95
12	21	15.17
15	20	13.48
18	19	12.76
21	18	11.42
24	17	10.71
27	17	16.03
30	15	17.03

Figure 5.11 shows the variograms and fitted spherical models for the above mentioned data set-ups.

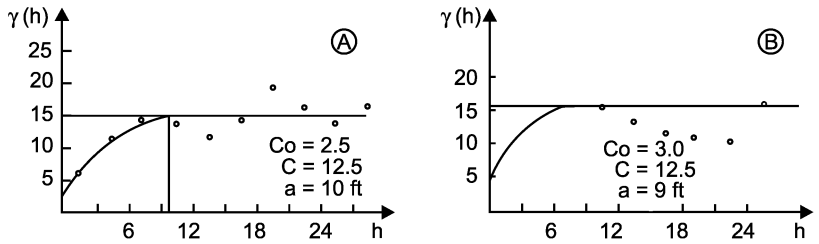


Fig. 5.11 Variogram and fitted spherical models for (A) complete set of data and (B) for the same with a few missing values.

Examples 2 to 4

Let us now come back to the original data sets of gold, copper and Fe_2O_3 element of the Bauxite deposit mentioned in Chapter 2. Figures 5.12, 5.13 and 5.14 show the experimental variograms, which are spherical in shape in all the three cases. Hence, spherical models of the following form were fitted to each of these.

$$\gamma(h) = C_0 + C\left[\frac{3}{2}\left(\frac{h}{a}\right) - \frac{1}{2}\left(\frac{h}{a}\right)^3\right] \text{ for } h < a \qquad (5.12)$$

and

$$= C_0 + C \text{ for } h \geq a.$$

Example	*Variogram parameters for Grade (dwts/ton)*			*Variogram parameters for Accumulation (inch-dwts)*		
	C_0	C	a	C_0	C	a
Gold	6.0	9.0	14 ft	40,000	1,55,000	18 ft
Copper	4.0	5.5	2 m	2,30,000	1,20,000	2.5 m
Fe_2O_3	15.0	13.0	2.4 m	–	–	–

The variograms were constructed in $N \rightarrow S$ direction (strikewise direction with 0° angle and with a deviation of 10°). Figures 5.13 to 5.15 show these variograms.

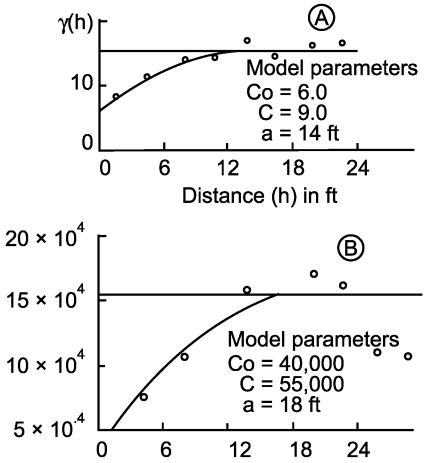


Fig. 5.12 Variograms for (A) grade and (B) accumulation values of a sample set of gold assays from lode *O*, Gold field 1.

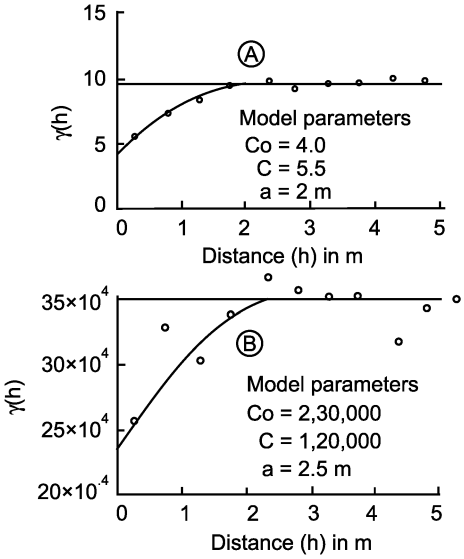


Fig. 5.13 Variogram for (A) tenor – % and (B) accumulation – cm %, values of a sample set of copper assays from a copper deposit.

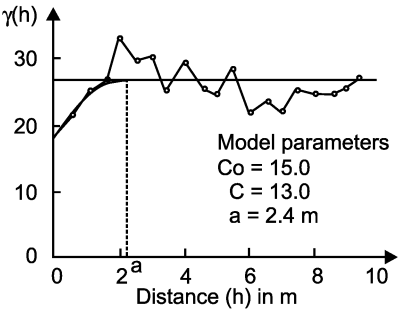


Fig. 5.14 Variogram and fitted model for Fe_2O_3 values of a bauxite deposit.

5.7 EXAMPLES OF VARIOGRAMS IN OTHER FIELDS OF EARTH SCIENCES

Gravity

Figure 5.15 shows the variogram computed for a set of observations relating to gravity field. The units are milli-gals and h is in units of kms. The variogram is linear and can be expressed as:

$$\gamma(h) = 0.954|h|$$

Groundwater Hydrology

Figures 5.16 and 5.17 show the variogram for the variables (a) Transmissivity and (b) Specific capacity (SC). The variograms are spherical and hence spherical models have been fitted. The model parameters are shown in the respective figures.

Lead-Zinc Mineralisation

Figure 5.18 shows the variograms for Lead and Zinc in respect of Rajpura-Dariba mineralisation in Rajasthan of western India. Here again, the variograms are spherical and hence spherical models have been fitted. It may be observed that the nugget effect for zinc is less compared to the one for lead. The variogram for zinc is well defined than that for lead. The model parameters are shown in the relevant figure.

One interesting feature that may be observed is that the variograms of all *metallic deposits* exhibit the same type of behaviour and are spherical in nature.

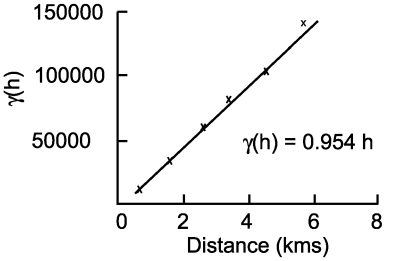


Fig. 5.15 Variogram for a set of gravity observations.

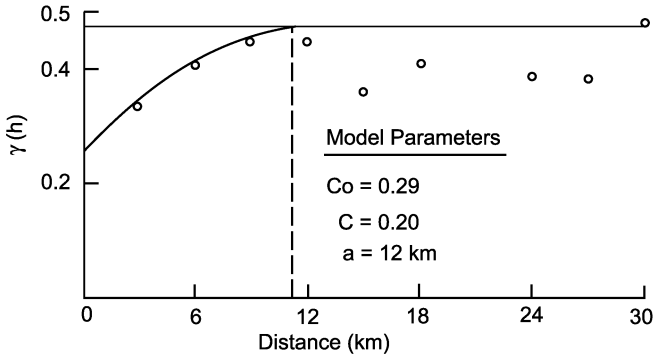


Fig. 5.16 Variograms for a sample set of logarithmic values of Transmissivity.

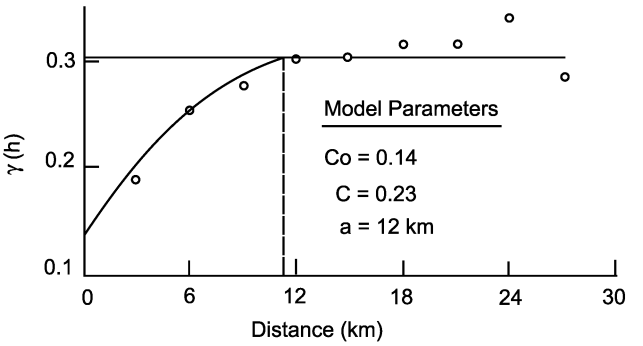


Fig. 5.17 Variograms for sample set of logarithmic values of Specific capacity.

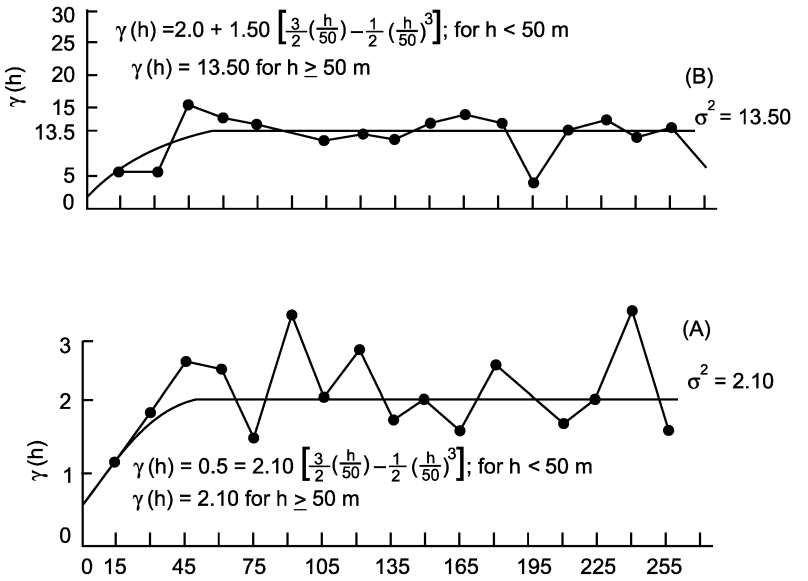


Fig. 5.18 Variograms for sample sets of (a) Lead and (b) Zinc assays in units of %.

5.8 COMPUTATION OF VARIOGRAM IN THE CASE OF IRREGULAR GRID

In the case of irregular grid, i.e., when the data points are irregularly spaced, the variogram is computed by grouping the classes of angles and distances. In practice, the computations are as follows: Each point, say, x_i (i = 1, 2, ... N) is taken as origin. Then for each other point, say x_j ($j \neq 1$), the squared difference viz., $[Z(x_i) - Z(x_j)]^2$ is computed. We observe in what direction the vector (x_i, x_j) is, and in what class of distance the absolute value of the distance falls. We now add the squared differences to the relevant sub-total, and the index to the count of elements in that subtotal is increased by 1. We delete the point already considered as the origin and start with another point. The process is repeated till all the points are exhausted. At the end, we divide all totals by twice the number of terms that are associated. That gives us the semi-variogram (variogram in short).

Review Questions

Q. 1. (i) Explain Stationarity and (ii) Intrinsic hypothesis.

Q. 2. (a) What are the properties of variogram?
(b) Explain geometric and zonal anisotrophies.

Q. 3. Explain proportional effect. How can it be tackled?

Q. 4. (a) What is regularisation?
(b) Explain regularisation by cores along a bore hole.

Q. 5. Explain the procedure to compute variogram when the data are irregularly spaced.

Q. 6. (a) Construct variograms for the following grid data in the four directions N, S, E, W. • indicates missing value.

4	____	8	____	7	____	8	____	6	____	5	____	8	____	7
8	____	7	____	7	____	•	____	•	____	•	____	•	____	6
10	____	4	____	6	____	4	____	7	____	•	____	•	____	•
8	____	10	____	6	____	8	____	6	____	5	____	3	____	11
9	____	7	____	8	____	6	____	4	____	6	____	7	____	4
5	____	9	____	8	____	7	____	5	____	6	____	5	____	4

(b) Compute the mean variogram.

6 Regularised Models, Volume-Variance Relationships and Economics

6.1 INTRODUCTION

So far we have discussed various types of (semi) variograms and model fitting with examples from earth sciences. We notice that the variograms for gold and copper were based on point/punctual samples. Point samples do not possess any support/volume. However, in the case of variogram for Fe_2O_3 element of bauxite mineral, the variogram was based on core samples of length 0.5 m. The assay values represent the averages for core lengths of 0.5 m. Table 6.1 gives these values as also the average values for 1 m and 1.5 m lengths.

The variograms constructed on 0.5 m, 1.0 m and 1.5 m core lengths were different, although the basic structure remained the same. The variograms for 0.5 m, 1.0 m and 1.5 m lengths were one below the other. The sill value was the highest for 0.5 m and lowest for 1.5 m. Figure 6.1 shows these variograms.

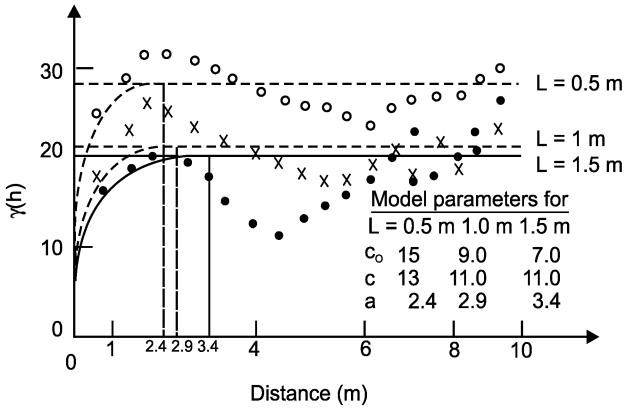


Fig. 6.1 Variograms based on cores of length 0.5 m, 1.0 m and 1.5 m in respect of Fe_2O_3 element values in a bauxite deposit.

Table 6.1 Assay values for core lengths of 0.5 m, 1.0 m, and 1.5 m in respect of Fe_2O_3 element in percentage

Avg. for 0.5 m	*Avg. for 1.0 m*	*Avg. for 1.5 m*	*Avg. for 0.5 m*	*Avg. for 1.0 m*	*Avg. for 1.5 m*
24.80			23.80		
22.40	23.60		21.60	22.70	
19.80		22.33	27.60		24.33
27.80	23.60		19.80	23.70	
25.80			27.80		
23.00	24.40	25.53	21.20	24.50	22.93
34.40			19.00		
29.60	32.00		28.60	23.80	
30.80		31.60	24.80		24.13
35.80	33.30		23.40	24.10	
40.40			17.80		
40.00	40.20	38.73	24.60	21.20	21.93
15.20			21.40		
17.60	16.40		24.00	22.70	
24.20		19.00	26.00		23.80
19.40	21.80		19.00	22.50	
35.30			24.00		
31.40	33.35	28.70	15.60	19.80	19.53
28.21			19.20		
31.00	29.60		17.80	18.50	
34.40		24.53	21.20		19.40
22.80	28.60		30.00	25.60	
26.60			23.60		
32.40	29.50	27.26	25.20	24.40	26.20
27.40			17.20		
25.80	26.60		20.60	18.90	
31.40		28.20	25.00		20.93
26.80	29.10		18.00	21.50	
24.40			26.20		
25.60	25.00	25.60	21.40	23.80	21.86

This variation in sill and other parameters was due to averaging the assay values for these lengths. It should be noted that we are comparing average grades, but not individual grades. Under conditions of stationarity, the sill of the variogram, if it exists, is equal to the sample variance, s^2. If we are dealing with point/punctual samples, we can estimate the sill of the variogram (when there is no nugget effect) and compare it with the variance, s^2. The sill of the variogram computed with $L = 0.5$ m, will be less than the sill of the variogram for point samples. Similarly the sill $C_{0.5}$ will be less than $C_{1.0}$ and so on. There exists a mathematical relationship between the point model γ and the model for samples of lengths γ_L. These relationships are discussed below with reference to a few models and different situations.

6.2 DIFFERENT SITUATIONS

Case 1

Given the semi-variogram for point samples, we could produce the model for any other sample of length L.

Examples

(*i*) *Linear Model:* A linear model for point samples may be written as: $\gamma(h) = mh$, where m is the slope of the semi-variogram. The semi-variogram for samples of length 'L' is given as:

$$\gamma_L(h) = \frac{mh^2}{3L^2}\ (3L - h) \quad \text{for } h < L. \tag{6.1}$$

$$= m\left(h - \frac{L}{3}\right) \qquad \text{for } h \geq L. \tag{6.2}$$

If we have an experimental semi-variogram, γ_L^*, for samples of length L, we can derive the model for point samples γ. Since the slopes of core model and point model are one and the same, the slope 'm' of the experimental model will be the same as for the point model. Assuming that there is no nugget effect and employing the above formula for variograms of core lengths L, and extending the line of the core model until it intersects the semi-variogram axis, an intercept of $-mL/3$ is produced. If a nugget effect C_0 exists, this needs to be added to the model.

Figure 6.2 shows a hypothetical point model and the regularised model without nugget effect. In the examples discussed below, for simplicity, nugget effect is assumed to be absent.

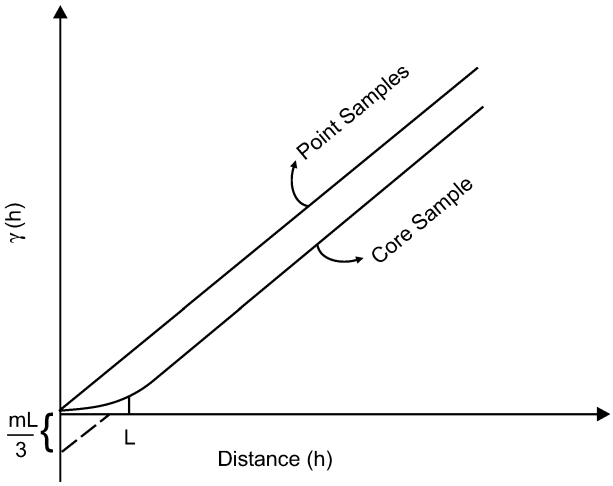


Fig. 6.2 Regularisation of a linear semi-variogram by core length (L).

(*ii*) *Exponential Model:* If the point variogram followed an exponential model with sill C, then:

$$\gamma(h) = C\left[1-\exp\left(\frac{-h}{a}\right)\right] \text{ for } h \geq 0. \tag{6.3}$$

For cores of length L, the theoretical model becomes:

$$\gamma_L(h) = C\left\{2\frac{a}{L}+\frac{a^2}{L^2}\left[1-\exp\left(\frac{L}{a}\right)\right]\right\}\left\{\exp\left(\frac{-h}{a}\right)\left[1-\exp\left(\frac{L}{a}\right)-2\right]\right\}$$

for $h \geq L$ (6.4)

Figure 6.3 shows a hypothetical point exponential model and the regularised model for a sample length, L. We notice that C_L is lower than C. The relationship between C and C_L may be written as:

$$C_L = 2C\left\{\frac{a}{L}-\frac{a^2}{L^2}\left[1-\exp\left(\frac{-L}{a}\right)\right]\right\} \tag{6.5}$$

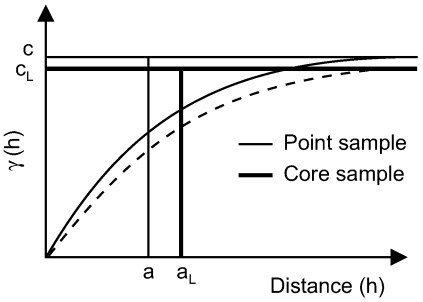


Fig. 6.3 Regularisation of an exponential semi-variogram by core length.

Table 6.2 gives C_L as a function of C for various values of L and a.

Table 6.2 C_L as a function of C

L	a	C_L
3	9	0.906C
4	16	0.922C
5	25	0.936C

Thus, the new sill (C_L) in the above case is lower than the corresponding point sill (C). It may also be observed that the range of influence a_L is longer than the range for points. In fact $a_L = a + L$.

(*iii*) *Spherical Model:* If the point variogram followed a spherical model with sill C, then:

$$\gamma(h) = C\left[\frac{3}{2}\left(\frac{h}{a}\right) - \frac{1}{2}\left(\frac{h}{a}\right)^3\right] \quad \text{for } h < a$$
$$= C \qquad \text{for } h \geq a$$

For cores of length L, the theoretical semi-variogram model is complex since there is a discontinuity in the model (see also Clark, 1979). However, using the relation:

$$C_L = C\left[1 - \frac{L}{2a} + \frac{L^3}{20a^3}\right] \quad \text{for } L < a$$

$$C_L = \frac{Ca}{L}\left[\frac{15}{20} - \frac{4}{20}\frac{a}{L}\right] \quad \text{for } L \geq a$$

we can obtain the estimates for C_L. Again, we recall $a_L = a + L$. We can, thus, generate values for C_L for various values of L and a and construct the variogram for core lengths.

The above discussion is applicable to a situation where we know the point model and we wish to obtain the 'regularised' model for cores of length 'L'.

Case 2

Given the experimental variogram for cores of a given length L, we wish to find the point model for use in estimation.

(*i*) *Linear Model:* In this case, the slope of both the point model and the core length model are one and the same. Keeping this slope and producing a line passing through the origin gives the point model.

(*ii*) *Exponential Model:* Let us suppose that an experimental variogram $\gamma_L(h)$ for core samples of length L and which follows an exponential model is available. The sill value C_L will be greater than most of the experimental points on the graph. We can get an estimate of the range a, since $a = a_L - L$. We now have the first order estimate for a. Substituting this in equation (6.5) for C_L and reversing it, we can get a value for C – the point sill. *These can be treated as first order estimates*. Having obtained these estimates of the values for a and C, we can *cross-validate,* by producing the model values $\gamma_L^*(h)$ for $\gamma_L(h)$ using equation (6.4). If our estimated values for a and C are reasonably good, the theoretical values of $\gamma_L(h)$ should match with the experimental semi-variogram $\gamma_L^*(h)$. Joining the $\gamma_L^*(h)$ values, we get a smooth curve for cores. If there is any variation between this curve and the variogram γ_L, the values for a and C can be modified till the model parameters give a good fit.

(*iii*) *Spherical Model:* This will be treated more or less in the same way as the above mentioned exponential model. The sill for the cores will be

lower than that for the point samples. The regularisation aspect will be discussed for the Fe_2O_3 example in the following pages.

6.3 STEPS TO BE FOLLOWED FOR THE DECONVOLUTION PROBLEM

1. A point model $\gamma(h)$ is derived from an inspection of the variogram $\gamma_L(h)$ experimentally available.
2. The regularised theoretical variogram $\gamma_L^*(h)$, computed based on the derived point model, is then obtained and compared with the experimental variogram γ_L. The values for a and C of the point model are then adjusted in such a way so as to bring $\gamma_L^*(h)$ in line with $\gamma_L(h)$.
3. When once the point model $\gamma(h)$ is decided, we may repeat the exercise for obtaining the theoretical expression $\gamma_{L'}^*$, over a possible second length L' (or support). This can be checked with the corresponding experimental curve $\gamma_{L'}^*$.

6.4 EXAMPLE: Fe_2O_3 ELEMENT VALUES FOR CORE LENGTH OF L = 0.5 m

Experimental variogram for cores of length L is available and we wish to derive the point model.

Case 1

(*i*) Since the ***Linear Model*** is a simple one, we now discuss the exponential and spherical models.

(*ii*) ***Exponential Model:*** Let us assume that there is no nugget effect and C_L = 28 and a_L = 2.4 m. Now the point model parameters may be derived as: a = 2.4 – 0.5 = 1.9 m. We have the formula for C_L as:

$$C_L = 2C\left\{\frac{a}{L} - \frac{a^2}{L^2}\left[1 - \exp\left(\frac{-L}{a}\right)\right]\right\}$$

$$= 2C\left\{\frac{1.9}{0.5} - \frac{3.61}{0.25}\left[1 - \exp\left(\frac{-0.5}{1.9}\right)\right]\right\}$$

$$= 2C\ \{3.80 - 14.44\ [1 - \exp\ (-0.263)]\}$$

$$= 2C\ \{3.80 - 14.44\ [1 - 0.768]\}$$

$$= 2C\ \{3.80 - 14.44\ (0.232)\}$$

$$= 2C\ \{3.80 - 3.35\} = 2C(0.45)$$

$$= 0.90C;\ \text{when } C_L = 28,\ \hat{C} \cong 31.$$

Given the derived values for C and a, we can estimate C_L and a_L, and if need be, some adjustments can be made.

(*iii*) ***Spherical Model:*** The sill for the cores variogram will be lower than that for the 'points'.

We have

$$C_L = C\left(1 - \frac{L}{2a} + \frac{L^3}{20a^3}\right) \quad \text{for } L < a \tag{6.6}$$

and

$$C_L = \frac{Ca}{L}\left(\frac{15}{20} - \frac{4}{20}\frac{a}{L}\right) \text{ for } L \geq a$$

As the formula for the semi-variogram for cores is complex, we use the tabulated values of Table 6.4 for obtaining the variogram for point samples. This procedure serves as an approximation. Making use of this table, the regularised semi-variogram for core length L can be obtained assuming that the original point semi-variogram had a range a and sill unity. Isobel Clark (1977) published an elegant computer subroutine in FORTRAN for this regularisation aspect.

Illustration

Let us consider the variogram for Fe_2O_3 element in the bauxite example. The value of the sill (C_L) is 28(%)^2, and a_L as 2.4 m. Since $a_L = a + L$, $a = 2.4 - 0.5 = 1.9$. Employing the formula:

$$C_L = C - \frac{CL}{2a} + \frac{CL^3}{20a^3}, \text{ we have:}$$

$$28 = C\left[1 - \frac{L}{2a} + \frac{L^3}{20a^3}\right]$$

$$= C\left[1 - \frac{0.5}{3.8} + \text{negligible term}\right]$$

$$= C[1 - 0.132] = 0.868C$$

$$\hat{C} = 28/0.868 = 32.3(\%)^2$$

and

$$\hat{a} = 1.9 \text{ m.}$$

With these derived values $\hat{C}$ and $\hat{a}$ for C and a, let us get the model values for the semi-variogram of core length 0.5 m *to cross-validate* the experimental values. For this, we consult Table 6.4. The tabulated γ_L values are for various normalised values of a/L and h/L. In our case $a/L = (1.9/0.5) = 3.8$. The entries in this table correspond to various sample lengths L. That is, $h/L = 1$ means $h = 0.5$ m; $h/L = 2$ means $h = 1.0$ m and so on. For $h/L = 1$ and $a/L = 3.8$, we have the table value as 0.254. This is for a semi-variogram with a sill unity. For a sill of 32.3 we have $\gamma_{0.5} = 0.254 \times 32.3 = 8.20$. Similarly, $\gamma_{1.0} = 0.582 \times 32.3 = 18.8$; $\gamma_{1.5} = 0.795 \times 32.3 = 25.67$;

$\gamma_{2.0} = 0.853 \times 32.3 = 27.87$ and so on. Table 6.3 gives the derived γ_L^* values and the experimental variogram values for a core length of 0.5 m interval.

Table 6.3 Experimental and derived variogram values for a core length (L) of 0.5 m

L	γ_L	γ_L^*
0.5	21.14	8.20
1.0	25.93	18.79
1.5	26.53	25.67
2.0	34.38	27.87
2.5	30.61	28.00

Let us see if we can improve upon this. Suppose $C_L = 26\%$ (instead of 28%) and $a_L = 2.20$ (instead of 2.40). We have:

$$a_L = a + L$$

$$a = 2.20 - 0.50 = 1.70 \text{ m}$$

$$C_L = C\left[1 - \frac{0.5}{3.4} + \frac{(0.5)^3}{20a^3}\right] = C[1 - 0.147 + \text{negligible term}]$$

$$26 = C[1 - 0.147] = C[0.853]$$

Therefore, $C = (26/0.853) = 30.48(\%)^2$. Now, for $a/L = 1.70/0.5 = 3.4$ m and various h/L values, the $\gamma_L^*(h)$ values are shown in Table 6.5.

Figure 6.4 shows the experimental variogram and the fitted regularised model for 0.5 m cores. There seems to be a good fit. Therefore, we may accept the improved value for C and a as $30.48(\%)^2 \cong 30.5(\%)^2$ and 1.70 m respectively instead of $32.3(\%)^2$ and 1.9 m.

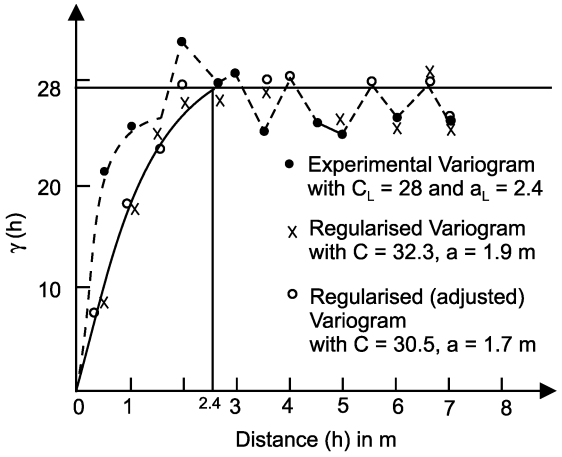


Fig. 6.4 Experimental and regularised variogram models for Fe_2O_3 element.

Table 6.4 Regularised variogram for ore lengths L in respect of Spherical Model with range and sill = 1.0 for various distances

	$h/L \rightarrow$																		
$a/L\downarrow$	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
0.50	.300	.313	.325	.325	.325	.325	.325	.325	.325	.325	.325	.325	.325	.325	.325	.325	.325	.325	.325
0.75	.375	.406	.438	.438	.438	.438	.438	.438	.438	.438	.438	.438	.438	.438	.438	.438	.438	.438	.438
1.00	.450	.500	.550	.550	.550	.550	.550	.550	.550	.550	.550	.550	.550	.550	.550	.550	.550	.550	.550
1.25	.456	.535	.614	.615	.615	.615	.615	.615	.615	.615	.615	.615	.615	.615	.615	.615	.615	.615	.615
1.50	.463	.571	.678	.679	.681	.681	.681	.681	.681	.681	.681	.681	.681	.681	.681	.681	.681	.681	.681
1.75	.438	.570	.703	.711	.719	.719	.719	.719	.719	.719	.719	.719	.719	.719	.719	.719	.719	.719	.719
2.00	.412	.570	.728	.742	.756	.756	.756	.756	.756	.756	.756	.756	.756	.756	.756	.756	.756	.756	.756
2.25	.383	.553	.722	.751	.779	.779	.780	.780	.780	.780	.780	.780	.780	.780	.780	.780	.780	.780	.780
2.50	.355	.536	.717	.760	.802	.803	.803	.803	.803	.803	.803	.803	.803	.803	.803	.803	.803	.803	.803
2.75	.331	.512	.693	.753	.812	.816	.819	.819	.819	.819	.819	.819	.819	.819	.819	.819	.819	.811	.803
3.00	.307	.488	.669	.746	.822	.829	.835	.835	.835	.835	.835	.835	.835	.835	.835	.835	.835	.819	.803
3.25	.288	.464	.640	.728	.817	.832	.846	.846	.846	.846	.846	.846	.846	.846	.846	.846	.846	.839	.831
3.50	.269	.440	.610	.711	.812	.835	.858	.858	.858	.858	.858	.858	.858	.858	.858	.858	.858	.858	.858
3.75	.254	.418	.582	.689	.795	.829	.863	.865	.867	.867	.867	.867	.867	.867	.867	.867	.867	.867	.867
4.00	.239	.397	.555	.666	.778	.823	.868	.872	.876	.876	.876	.876	.876	.876	.876	.876	.876	.876	.876
4.25	.227	.379	.531	.643	.755	.810	.864	.873	.882	.882	.882	.882	.882	.882	.882	.882	.882	.882	.882
4.50	.215	.361	.507	.620	.733	.797	.861	.875	.889	.889	.889	.889	.889	.889	.889	.889	.889	.889	.889
4.75	.205	.330	.456	.582	.709	.779	.849	.871	.893	.893	.895	.895	.895	.895	.895	.895	.895	.895	.895
5.00	.194	.299	.404	.545	.686	.761	.836	.866	.896	.898	.900	.900	.900	.900	.900	.900	.900	.900	.900
5.25	.186	.301	.416	.540	.664	.742	.819	.856	.893	.899	.905	.904	.904	.904	.904	.904	.904	.904	.904
5.50	.178	.303	.428	.535	.642	.722	.802	.846	.890	.900	.909	.908	.908	.908	.908	.908	.908	.908	.908
5.75	.171	.291	.412	.517	.622	.702	.783	.832	.881	.896	.911	.912	.913	.913	.913	.913	.913	.913	.913
6.00	.163	.280	.396	.498	.601	.683	.764	.818	.872	.893	.914	.915	.917	.917	.917	.917	.917	.917	.917

(*Contd.*)

(*Table 6.4 contd.*)

	$h/L \rightarrow$																		
$a/L\downarrow$	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
6.25	.157	.270	.382	.482	.582	.664	.745	.802	.859	.885	.911	.916	.920	.920	.920	.920	.920	.920	.920
6.50	.151	.259	.368	.466	.564	.645	.726	.786	.845	.877	.909	.916	.923	.923	.923	.923	.923	.923	.923
6.75	.146	.251	.356	.451	.547	.627	.708	.769	.830	.866	.902	.913	.924	.925	.926	.926	.926	.926	.926
7.00	.141	.243	.344	.437	.530	.610	.690	.752	.814	.854	.895	.910	.926	.928	.929	.929	.929	.929	.929
7.25	.137	.235	.334	.424	.515	.594	.673	.735	.798	.841	.885	.905	.924	.928	.931	.931	.931	.931	.931
7.50	.132	.228	.323	.412	.500	.577	.655	.719	.782	.828	.874	.898	.923	.928	.933	.933	.933	.933	.933
7.75	.128	.221	.313	.400	.486	.563	.639	.703	.766	.814	.862	.889	.918	.926	.934	.935	.936	.936	.936
8.00	.124	.214	.304	.388	.472	.548	.623	.687	.751	.800	.849	.880	.912	.924	.936	.937	.938	.938	.938
8.25	.120	.208	.295	.377	.460	.534	.608	.672	.735	.785	.836	.869	.903	.919	.934	.937	.939	.939	.939
8.50	.117	.202	.287	.367	.447	.520	.593	.656	.720	.771	.822	.858	.894	.914	.933	.937	.941	.941	.941
8.75	.113	.197	.280	.358	.436	.508	.579	.642	.705	.757	.808	.846	.884	.906	.929	.935	.942	.942	.943
9.00	.110	.191	.272	.349	.425	.495	.566	.628	.690	.742	.794	.834	.874	.899	.924	.934	.943	.944	.945
9.25	.107	.186	.265	.340	.414	.484	.553	.615	.676	.729	.781	.822	.863	.890	.917	.930	.942	.944	.946
9.50	.104	.181	.258	.331	.404	.473	.541	.602	.663	.715	.767	.809	.851	.881	.910	.926	.941	.944	.947
9.75	.102	.177	.252	.324	.395	.462	.529	.589	.650	.702	.754	.797	.839	.870	.901	.919	.937	.942	.948
10.00	.099	.172	.246	.316	.386	.451	.517	.576	.636	.688	.741	.784	.827	.860	.892	.913	.933	.941	.949

Table 6.5 Experimental variogram and fitted regularised model values for core lengths of 0.5 m

L	γ_L	γ_L^*
0.5	21.14	8.20
1.0	25.93	18.59
1.5	26.53	24.75
2.0	34.38	26.15
2.5	30.61	26.15

6.5 DISPERSION VS BLOCK SIZE

Consider a set of assay values determined from point samples and grouped into blocks of chosen sizes. While the mean values of these individual assay values from point samples and those of the blocks remain the same, the dispersions decrease with the increase in the size of the averaging blocks. The same is true with point samples and cores of length L.

6.5.1 Example: Lode O: gold field 1

In this lode, the units of measurements are dwts/ton* of ore for grade, inches for width of the reef and inch-dwts for accumulation. A set of sample assay values (72) pertaining to this lode were given in Chapter 2 under Table 2.4. This deposit was developed by drives approximately 100 ft in depth. Chip (point) samples are collected for every 3 ft (approximately) during development and along the strike and assayed. A total of 2,154 sample values were utilised for this discussion. Figure 6.5A, B and C show the histograms based on 10 ft × 10 ft averaged values (2154 in number), 30 ft × 10 ft averaged values (718 in number), and 100 ft × 10 ft averaged values (215 in number). Blocks of 10 ft × 10 ft values may also be viewed as point

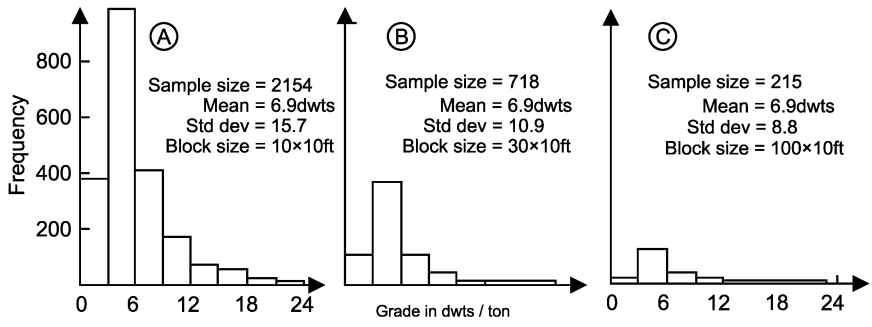


Fig. 6.5 Distribution of grade values in lode O of gold field 1 when the block size is (A) 10 × 10 ft, (B) 30 × 10 ft and (C) 100 × 10 ft.

*1 dwt = 1.55517 gms/ton and 1 inch-dwt = 3.95013 cm-gms

samples as the area is small. It can be seen from Fig. 6.5, based on 10 ft averaged values, that while 17.9% of the total values (385 out of 2154) remained below 3 dwts/ton, the corresponding figures for 30 ft averaged block and 100 ft averaged block values were 16.6% (119 out of 718) and 10.2% (22 out of 215) respectively. Further, while the mean value remained the same at 6.9 dwts, the standard deviation was 15.7% for 10 ft values, 10.9% for 30 ft values and 8.8% for 100 ft average values.

6.5.2 Example: Lode Z: gold field 2

In this lode, the units of measurement are: gms/tonne of ore for grade, cms for width of the reef and cm-gms for accumulation. A total of 4,179 grade values distributed over four levels—each separated in depth by approximately 30 m have been considered. These samples were drawn at 1 m interval of distance. These can also be viewed as point samples drawn from blocks of size 1 m × 3 m. Figures 6.6 *a* and *b* shows the distribution of assay values and averaged over blocks of size 30 m × 30 m. These averaged block values were 65 in number. It is clear from the distribution of 1 m × 3 m assay values that nearly 62.6% of the total values lie below 3 gms, while 52.3% of the total values lie below 3 gms as per the distribution of 30 m × 30 m blocks. Further, while the mean value for grade remained the same viz., 3.30 gms, for both 1 m × 3 m block values as well as block averages, the standard deviations were 3.19 and 1.69 for 1 m × 3 m and 30 m × 30 m block values respectively. The standard deviation for block values has come down, as is to be expected, because of the average effect.

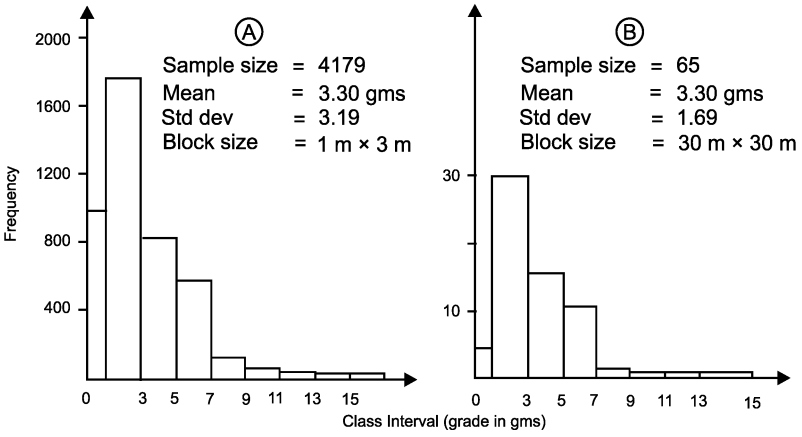


Fig. 6.6 Distribution of grade values in lode Z of gold field 2 with blocks of size 1 × 3 m and 30 × 30 m.

The above discussion indicates that if we used the histogram of 1 m × 3 m block values, we would have over-estimated the percentage of blocks over the cut-off value of 3 gms, since the histogram of 30 m × 30 m block average values is less spread out. This is reflected in the respective standard deviations.

Therefore, defining the ore depends on the unit of selection in terms of size and shape. If our deposit is divided in terms of blocks of size 30 m × 30 m which is the usual practice, then it is proper to draw inferences on the percentage of blocks below cut-off based on this unit. Let us discuss this point a little further.

Let us suppose that in a deposit, we have core samples of length L and blocks with volume v. We also assume that we have the variograms $\gamma_L(h)$ and $\gamma_v(h)$ with sill values C_L and C_v respectively. Since the average values $\bar{x}_L$ and $\bar{x}_v$ (for cores and the averaged block values respectively) are one and the same, we represent this as $\bar{x}$. Also, given the variogram model for the point samples, we can arrive at the variogram model for cores of length L and vice-versa. If we consider a core of length L and two points p and p', separated by a distance h, we can compute the difference in grades for *various possible pairs* of p and p', which exist within this core length. Here the diameter of the core is negligible compared to core length. Hence the sample can be considered as having zero volume and the core length as a straight line. This gives us a measure of the variability of the grades and leads us to the computation of variogram function, which in turn gives us the variance of the grades within the core length L. This *within variance* in the core length L is the one which is removed if we considered only the average grade over L. Mathematically, this within variation $[C - C_L]$ may be expressed as:

$$\sigma^2(o/L) = \frac{1}{L^2}\int_0^L\int_0^L \gamma(p - p')dpdp' \tag{6.7}$$

The same logic is applicable to point samples *vs* panels each of size $L \times B$, i.e. $2D$ panels**.** In this case, we have to evaluate quadruple integrals since the points p and p' can move throughout the whole panel. This is discussed in more detail in Chapter 7.

6.6 THE WITHIN VARIATION IN CORE LENGTH *L*: DIFFERENT CASES

In the case of linear variogram, i.e., when $\gamma(h) = mh$, this within variation reduces to $mL/3$. In the case of exponential variogram of the type: $\gamma(h) = C[1 - \exp(-h/a)]$ for $h \geq 0$, the within variation reduces to: $C\left[1-2\frac{a}{L}+\frac{a^2}{L^2}\{1-\exp(-L)a\}\right]$, and corresponds exactly with the difference in the point and regularised semi-variograms. For spherical models, this within variation is: $\frac{C}{20}\ \frac{L}{a}\left(10-\frac{L^2}{a^2}\right)$ for $L < a$ and

$\frac{C}{20}\left(20-15\frac{l}{a}+4\frac{a^2}{L^2}\right)$ for $L \geq a$ (see Clark, 1979).

6.7 DISTRIBUTIONS BASED ON CORE SAMPLE STATISTICS AND DERIVED ONES FOR POINT SAMPLES

Let us now study the economic implications posed in tonnage and grade above the cut-off grade due to a change in the support. For this, we should know the distribution of samples. Usually the element concentrations in massive types of ore bodies such as iron and bauxite (elements Fe_2O_3, Al_2O_3 etc.) follow approximately *normal distribution*. The element concentrations in precious mineral deposits such as gold, uranium etc., follow a positively skewed distribution. We have seen that in the case of Fe_2O_3 element of the bauxite example, the distribution is approximately normal. We recall the following statistics with respect to these examples.

Element Fe_2O_3: Core length 0.5 m; Distribution: Normal; Sample size = 60

Case	*Mean*	*Std. dev.*
1. Distribution of cores of 0.5 m	25.17%	$s = \sqrt{28}\% = 5.29\%$
2. Distribution of point samples (derived)	25.17%	$s = \sqrt{30.48}\% = 5.52\%$

Given the mean and the variance, the distributions can be worked by standardising the variable. The distributions of Fe_2O_3 with mean 25.17% and standard deviations 5.52% (derived) for point samples and 5.29% for core samples are shown in Fig. 6.7.

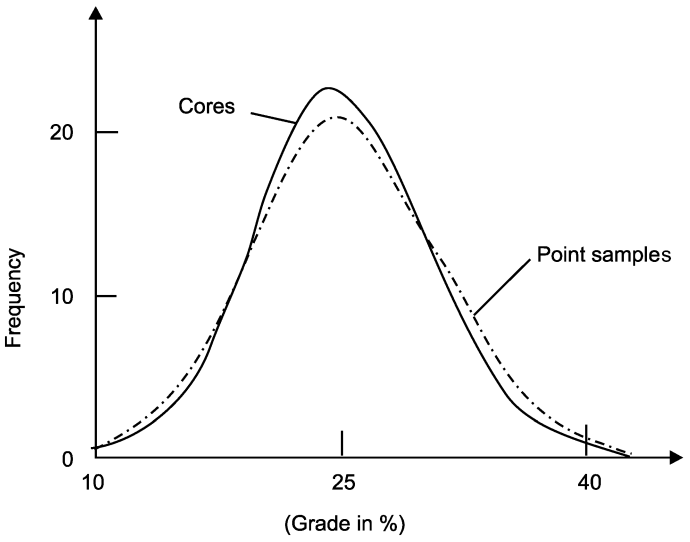


Fig. 6.7 Distribution of Fe_2O_3 element values with mean 25.17% and variances (A) $28.0(\%)^2$ (cores) and (B) $30.48(\%)^2$ (derived sill value for point samples).

Table 6.6 Fitting normal distribution to Fe_2O_3 element values with $\bar{x}$ = 25.17% and standard deviation (s_v) = 5.29% (core samples) in case 1

x	$\frac{x-\bar{x}}{s_v}$	*Probability*	y
10	–15.2/5.29 = –2.86	0.0067 × 60 =	0.40
15	–10.2/5.29 = –1.92	0.063 × 60 =	3.78
20	–5.2/5.29 = –1.03	0.235 × 60 =	14.10
25	0.2/5.29 = 0.03	0.3414 × 60 =	20.48
30	4.8/5.29 = 0.85	0.2689 × 60 =	16.13
35	9.8/5.29 = 1.85	0.0721 × 60 =	4.32
40	14.8/5.29 = 2.69	0.0107 × 60 =	0.31
			59.52

Table 6.7 Fitting normal distribution to Fe_2O_3 element values with $\bar{x}$ = 25.17% and standard deviation (s_p) = 5.52% (derived for point samples) in case 2

x	$\frac{x-\bar{x}}{s_p}$	*Probability*	y
10	–15.2/5.52 = –2.75	0.009 × 60	0.5
15	–10.2/5.52 = – 1.85	0.072 × 60	4.3
20	–5.2/5.52 = – 0.94	0.256 × 60	14.4
25	0.2/5.52 = 0.04	0.368 × 60	21.1
30	4.8/5.52 = 0.87	0.213 × 60	12.8
35	9.8/5.52 = 1.77	0.083 × 60	5.6
40	14.8/5.52 = 2.68	0.012 × 60	0.7
			59.4

From the graph it may be seen that the spread is more in the case of point samples. We would now see how the difference in the point sample and core sample distributions of Fe_2O_3 element affects the grade-tonnage computations.

Grade Tonnage Computations and Economics

i) *Based on derived values for points:*

Let E be the cut-off = 20%

Mean = 25.17%

Std. Deviation (s_p) = 5.52%

$$t = \frac{E-\bar{x}}{s_p} = \frac{20-25.17}{5.52} = -0.936$$

Consulting the standard normal tables (which are available in any standard text book on Statistics), we have $\phi(t) = 0.256$ so that p (the proportion of ore above cut-off) = 1 – 0.178 = 0.824. The average grade of this ore which is above cut off is: $\bar{x}_E = \bar{x} + \frac{s_p}{p}\phi(t)$, where $\phi(t)$ is the height of the standard normal curve at value t, i.e., $\phi(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2)$. Here $\phi(t) = \phi(-0.936) = 0.256$. Therefore, $\bar{x}_E = 25.17 + \frac{5.52}{0.824} \times 0.256 = 25.17 + 1.71 = 26.88\% \cong 26.9\%$. Thus, 82.4% of the ore lies above the cut-off of 20% and this has an average grade of approximately 26.9%.

ii) *Based on Core Samples of Length 0.5 m:*

E (cut off) = 20%

Mean $\bar{x}$ = 25.17%

Std. Deviation (s_v) = 5.29%

The proportion p of the distribution above cut-off is given by: Pr ($\bar{x} > E$). The standardised variable $t = (E - \bar{x})/s_v$ works out to : (20 – 25.17)/ 5.29 = (– 0.977). Consulting the standard normal tables, $\phi(t)$ – the proportion of ore lying below cut-off i.e., the area in the standard normal curve lying below $t = 0.162$ so that the proportion of ore lying above the cut-off = 1 – $\phi(t)$ = 0.838. We now would like to know the average grade of this 83.8%. The average grade above cut-off is given by: $\bar{x}_E = \bar{x} + \frac{s_v}{p}\phi(t)$, where $\phi(t)$ is the height of standard normal curve at $t = -0.977$.

With respect to our example, we have: $\phi(t) = \phi(-0.977) = 0.247$ so that $\bar{x}_E = \bar{x} + \frac{s_v}{p}\phi(t) = 25.17 + \frac{5.29}{0.838} \times 0.247 = 25.17 + 1.56 = 26.79\%$. Therefore, 83.8% of the ore lying above the cut-off has an average grade of 26.79%.

iii) *Based on actual 1 m core length variogram values:*

E (cut off) = 20%, Mean = 25.17%, Std. Dev. = 4.58

$t = -5.17/4.58 = -1.163$; $\phi(t) = 0.23$,

and $p = 0.877$. $\phi(t) = \phi(-1.13)$.

$$\bar{x}_E = \bar{x} + \frac{s_p}{p}\phi(t) = 25.17 + \frac{4.58}{0.87} \times 0.211 = 25.17 + 1.02 = 26.27\%.$$

Therefore, 87.7% of the ore will be above the cut-off and this average grade is approximately 26.27%. Table 6.8 gives a comparison of these observations.

Table 6.8 A comparison of the estimates

Base	*Avg. grade (%)*	*Std. Dev.*	*Cut-off (%)*	*Proportion of ore above cut-off*	*Avg. grade (%)*
1. Point samples (derived)	25.17	5.52	20	82.4	26.90
2. Core samples (0.5 m length)	25.17	5.29	20	83.8	26.79
3. Core samples (1 m samples)	25.17	4.58	20	87.7	26.27

The difference between point sample and core sample estimates is of course less as the core sample length is only 0.5 m. However, the above example points out to the fact that (i) tonnage should be calculated taking into account the volume (support) of the sample considered, i.e., whether it is a point sample with no volume or a core sample of length L or blocks of size (L, B, W), as the case may be, and *(ii) tonnage calculated with a volume v is more with lesser grade compared to the one by point samples viz., with no volume, where the tonnage is less with higher grade.*

6.8 CASE OF LOGNORMAL DISTRIBUTION AND BLOCKS OF SIZE *V*

In section 6.5.2, we have considered the distribution of 4,179 grade values in units of gms from lode Z of gold field 2, distributed over four levels—each separated in depth by approximately 30 m. The sampling interval is 1 m. These samples can be viewed as point samples. Following are the statistics:

	Point samples		*Panels/Blocks of size 30 × 30 m*	
I	*Original samples*	*Log transformed samples*	*Original samples*	*Log transformed samples*
Mean	3.30	1.20	3.30	1.200
Variance	10.20	1.30	5.20 (derived)	0.510 (derived)
S.D. (s_p)	3.19	1.14	2.28 (s_v)	0.714

It is clear that the variance is more in the case of point samples as is to be expected. Let us now look at the variogram. The sill value based on point samples is 10.2. The standard deviation is $\sqrt{10.2} = 3.19$ which is comparable to the values by direct computations. Given the variogram for point samples, we can now derive the sill value and the range for blocks of size 30 × 30 m. The variogram of point samples is shown in Fig. 6.8.

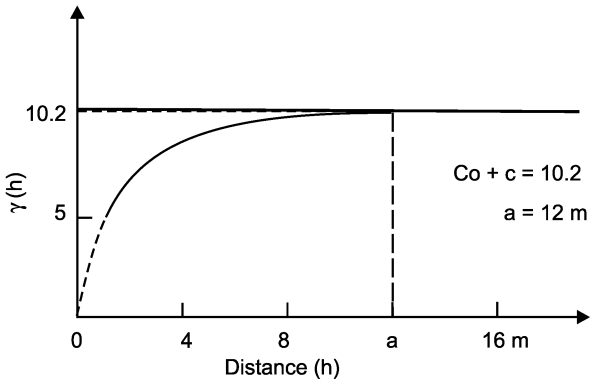


Fig. 6.8 Variograms based on point samples drawn from four levels of lode *Z*, gold field 2.

We now know how to derive the sill value for blocks of size 30 × 30 m and also the range. The range for point samples is 12 m. Therefore $a_L = a + L = 30 + 12 = 42$ m. We use Table 6.9 to get the necessary factor to work out the sill value C_L for blocks of size 30 m × 30 m. This table gives the factors for standardised spherical model with range = 1 and sill = 1 applicable to panels (2*D*). From *D* (*L*, *B*) tables we have:

$D\left(\frac{30}{42}, \frac{30}{42}\right) = D(0.71, 0.71) = 0.51$. The sill value for blocks of size 30 m × 30 m = 10.20 × 0.51 = 5.20 . The standard deviation is $\sqrt{5.20} = 2.28$. The details of the distribution when $\bar{x} = 3.30$ and $s_v = 2.28$ and $N = 4{,}179$ are given below. Here N is taken as 4179, just for a comparison of the distribution of the blocks with the point samples distribution. The actual number of blocks each of size 30 m × 30 m is 65 only.

6.8.1 Distributions Based on Point Samples and Derived Statistics for Blocks

Case 1: Based on original point samples: $\bar{x} = 3.30$ gms and $s_p = 3.19$, $N = 4179$.

x	$\frac{x-\bar{x}}{s_p}$	Ordinate (Y)
1	– 2.30/3.19 = –0.72	1249
3	– 0.30/3.19 = –0.01	1650
6	+ 2.70/3.19 = 0.85	944
9	+ 5.70/3.19 = 1.78	232
12	+ 8.70/3.19 = 2.72	100
15	+11.70/3.50 = 3.66	4
	Total	4179

Case 2: Based on derived statistics for blocks of size 30 m × 30 m: $\bar{x}$ = 3.30, derived standard deviation value (s_v) = 2.28 and N = 4179.

x	$\frac{x-\bar{x}}{s_p}$	Ordinate (Y)
1	–2.30/2.28 = –1.01	1036
3	–0.30/2.28 = –0.13	1737
6	2.70/2.28 = 1.18	1048
9	5.70/2.28 = 2.50	323
12	8.70/2.28 = 3.81	35
	Total	4179

The distributions based on the above statistics for point samples and blocks of size 30 × 30 m are shown in Fig. 6.9. It may be mentioned that at this stage we are not interested in looking at the economic aspects with these statistics *as the distribution is lognormal.* We will look at this with statistics based on logarithms.

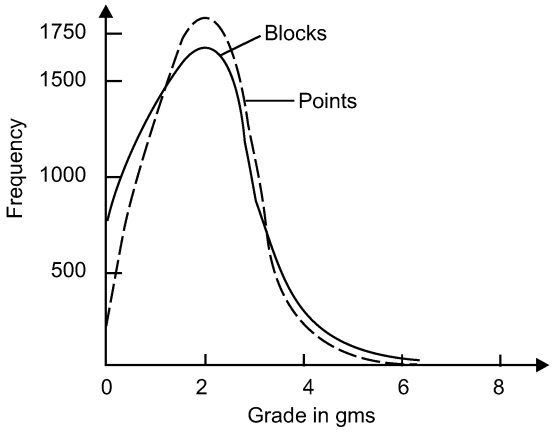


Fig. 6.9 Distribution based on point samples and blocks each of size 30 × 30 m.

6.8.2 Grade Tonnage Computations Based on Log-transformed Statistics and Economic Implications

Case 1: Based on point samples : Mean of logarithms $\bar{y}$ = 1.20, Standard deviation of logarithms of sample values s_y = 1.14 and N = 4179.

Let E (cut-off) = 2.5 gms is the cut-off. $l_n(E)$ = 0.916. We now calculate the proportion above cut-off (0.916) and the average grade of the ore above this cut-off. Since the distribution is lognormal we need to work with the mean grade of ore and the standard deviation based on logarithms of the assay values. Now $t = (\log_e E - \bar{y})/s_y$ where $y = \log_e x$ and P the proportion of the ore above cut-off is $1 - \phi(t)$.

The average grade above cut-off is given by $\bar{g}_E = \frac{Q}{P}\bar{g}$ where $Q = 1 - \phi(t - s_y)$, and $\bar{g} = (\bar{x})$ = mean = 3.30 gms. Since the cut-off is 2.5 gms/ton, we have $t = \frac{\log_e E - \bar{y}}{s_y} = \frac{0.916 - 1.20}{1.14} = -0.249$. Therefore, the proportion of ore below the cut-off = 0.4013 and P the proportion of the ore above the cut-off = $1 - \phi(t) = 0.5987$. This means the point samples tell us that 59.87% of the deposit will be above cut-off. The average value of the ore above this cut-off is given by (Q/P) where $Q = 1 - \phi(t - s_y)$.

$$Q = 1 - \phi(-0.249 - 1.14) = 1 - \phi(-1.389) = 1 - 0.0823 = 0.9177.$$

$$\bar{g}_E = \frac{0.9177}{0.5987} \times 3.30 = 5.05 \text{ gms.}$$

That is 59.87% of the deposit will be above cut-off and this has an average grade of 5.05 gms.

Case 2: With derived values for blocks of size 30 m × 30 m: The sill value based on logarithms of *individual samples* is 1.00 with range $a = 12$. The derived sill value (s_y^2) based on logarithms for blocks of size 30 m × 30 m = 1.00 × 0.51 = 0.510; $s_y = 0.714$. Range $a_L = a + L = 42$ m.

The variogram based on logarithm individual samples is shown in Fig. 6.10.

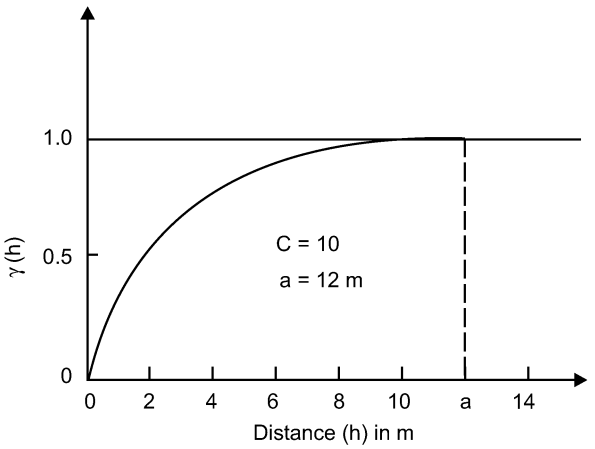


Fig. 6.10 Variogram based on logarithms of grade values.

We now have : $\bar{g} = 3.30$, $\bar{y}$(logs of blocks) = 1.20; $s_y^2 = 0.51$ and $s_y = 0.714$.

With cut-off E = 2.5 gms, we have: $t = \dfrac{\log_e E - \bar{y}}{s_y} = \dfrac{0.916 - 1.20}{0.714} = -0.397.$

$\phi(t)$ – proportion of ore below cut off = 0.3446 and P – proportion of ore above cut off = 0.6554. $Q = 1 - \phi(-0.397 - 0.714) = 1 - \phi(-1.111) = 1 - 0.1335 = 0.8665$. Therefore :

$$\bar{g}_E = \left(\frac{Q}{P}\bar{g}\right) = \frac{0.8665}{0.6554} \times 3.30 = 4.36 \text{ gms.}$$

That is 65.54% of the ore will have a grade above the cut-off of 2.5 gms and the average of this percentage is 4.36 gms. *Thus, we see that the selection made on a block unit produces a larger tonnage with lower grade than estimated from point/punctual samples where we see smaller tonnage with higher grade.*

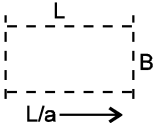
Table 6.9 gives values of a function D to be used in $2D$ panels of length L and breadth B for a standardised spherical model with range = 1.0 and sill = 1.0.

Review Questions

Q. 1. Given a point model, discuss the procedure for obtaining the regularised model for cores of length L.

Q. 2. Discuss volume-variance relationship.

Table 6.9 Values of function $D(L, B)$ to be used in $2D$ panels of length L and breadth B for a standardised spherical model with range = 1.0 and sill = 1.0



↓ B/a	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.10	.078	.099	.120	.142	.165	.188	.211	.234	.256	.278	.300	.321	.342	.363	.383	.402	.422	.439	.457
0.15	.099	.118	.138	.159	.181	.202	.224	.246	.268	.289	.311	.331	.352	.372	.392	.411	.430	.447	.465
0.20	.120	.138	.155	.176	.196	.216	.237	.259	.280	.301	.321	.341	.362	.382	.401	.419	.438	.456	.473
0.25	.142	.159	.176	.194	.213	.234	.254	.274	.294	.315	.335	.355	.374	.394	.412	.431	.449	.466	.483
0.30	.165	.181	.196	.213	.231	.251	.270	.289	.309	.329	.349	.368	.387	.405	.424	.442	.460	.477	.493
0.35	.188	.202	.216	.234	.251	.269	.288	.308	.328	.346	.364	.382	.401	.419	.438	.455	.472	.488	.504
0.40	.211	.224	.237	.254	.270	.288	.305	.326	.348	.363	.379	.397	.415	.433	.451	.468	.484	.500	.516
0.45	.234	.246	.259	.274	.289	.307	.324	.343	.362	.378	.395	.413	.430	.447	.465	.481	.498	.513	.528
0.50	.256	.268	.280	.294	.309	.326	.342	.359	.376	.394	.411	.428	.445	.462	.479	.495	.511	.526	.541
0.55	.278	.289	.301	.315	.329	.345	.361	.377	.394	.410	.427	.444	.461	.477	.493	.509	.525	.539	.553
0.60	.300	.311	.321	.335	.349	.364	.379	.395	.411	.427	.443	.460	.476	.492	.507	.523	.538	.552	.566
0.65	.321	.331	.341	.355	.368	.382	.397	.413	.428	.444	.460	.475	.491	.506	.521	.537	.551	.565	.579
0.70	.342	.352	.362	.374	.387	.401	.415	.430	.445	.461	.476	.491	.506	.521	.536	.551	.565	.578	.591
0.75	.363	.372	.382	.393	.405	.419	.433	.447	.462	.477	.492	.506	.521	.536	.550	.564	.578	.591	.604
0.80	.383	.392	.401	.412	.424	.438	.451	.465	.479	.493	.507	.521	.536	.550	.564	.577	.591	.604	.616
0.85	.402	.411	.419	.431	.442	.455	.468	.481	.495	.509	.523	.536	.551	.564	.577	.590	.604	.616	.628

(*Contd*)

B/a ↓ \ L/a →	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.90	.422	.430	.438	.449	.460	.472	.484	.498	.511	.525	.538	.551	.565	.578	.591	.604	.616	.628	.640
0.95	.439	.447	.456	.466	.477	.488	.500	.513	.526	.539	.552	.565	.578	.591	.604	.616	.628	.640	.651
1.00	.457	.465	.473	.483	.493	.504	.516	.528	.541	.553	.566	.579	.591	.604	.616	.628	.640	.651	.662
1.10	.488	.496	.503	.513	.522	.533	.544	.555	.567	.579	.591	.603	.615	.626	.638	.650	.661	.671	.681
1.20	.520	.527	.534	.543	.551	.562	.572	.582	.593	.604	.616	.627	.638	.649	.660	.671	.682	.691	.701
1.30	.546	.553	.559	.567	.576	.585	.595	.605	.615	.626	.637	.647	.658	.668	.679	.689	.699	.708	.717
1.40	.572	.578	.584	.592	.600	.609	.618	.627	.637	.647	.657	.667	.677	.687	.697	.707	.716	.725	.733
1.50	.593	.599	.604	.612	.620	.628	.637	.646	.655	.664	.674	.683	.693	.702	.712	.721	.730	.738	.747
1.60	.614	.620	.625	.632	.639	.647	.655	.664	.673	.682	.691	.700	.709	.718	.727	.735	.744	.752	.760
1.70	.632	.637	.642	.649	.655	.663	.671	.679	.688	.696	.705	.714	.722	.731	.739	.748	.756	.763	.771
1.80	.650	.655	.659	.665	.672	.679	.687	.695	.703	.711	.719	.727	.736	.744	.752	.760	.767	.775	.782
1.90	.664	.669	.674	.680	.686	.693	.700	.708	.715	.723	.731	.739	.747	.755	.762	.770	.777	.784	.791
2.00	.679	.684	.688	.694	.700	.706	.713	.720	.728	.735	.743	.750	.758	.766	.773	.780	.787	.794	.800
2.25	.707	.711	.715	.721	.726	.732	.738	.745	.752	.758	.765	.772	.779	.786	.793	.799	.806	.811	.817
2.50	.735	.739	.743	.747	.752	.757	.763	.769	.775	.781	.788	.794	.800	.807	.813	.819	.824	.829	.835
2.75	.755	.758	.762	.766	.770	.776	.781	.786	.792	.798	.804	.809	.815	.821	.827	.832	.837	.842	.847
3.00	.775	.778	.781	.785	.789	.794	.799	.804	.809	.814	.820	.825	.830	.836	.841	.846	.851	.855	.860
3.25	.789	.793	.796	.799	.803	.808	.812	.817	.822	.826	.831	.836	.841	.846	.851	.856	.860	.865	.869
3.50	.804	.807	.810	.813	.817	.821	.825	.829	.834	.839	.843	.847	.852	.857	.861	.865	.870	.874	.878
3.75	.816	.818	.821	.824	.827	.831	.835	.839	.844	.848	.852	.856	.861	.865	.870	.873	.877	.881	.885
4.00	.827	.830	.832	.835	.838	.842	.845	.849	.853	.857	.861	.865	.870	.874	.878	.882	.885	.888	.892
4.50	.844	.846	.848	.851	.854	.857	.860	.863	.867	.870	.874	.878	.882	.886	.890	.893	.896	.899	.903
5.00	.860	.862	.864	.867	.869	.872	.874	.877	.881	.884	.887	.891	.894	.898	.901	.904	.907	.910	.913

7 The Concepts of Dispersion, Extension and Estimation Variances

Before we discuss the estimation procedure known as 'Kriging', we should get familiarised with concepts of Variances of Dispersion, Extension and Estimation:

7.1 VARIANCE OF DISPERSION

Here we discuss:

1. The variance of point samples within any volume V, and
2. The dispersion variance v within a volume V.

7.1.1 Variance of Point Samples within Volume *V*

Let $Z'(x)$ be a random function and $Z(x)$, the variable under consideration, be a realisation of the random function. Assuming that all the values of $Z(x)$ were available in V, the mean and variance of $Z(x)$ may be written as:

$$m_V = \frac{1}{V}\int_V Z(x)dx \tag{7.1}$$

$$s^2(o/V) = \frac{1}{V}\int_V [Z(x) - m_V]^2 dx \tag{7.2}$$

where 'o' stands for point sample.

Since we can have many realisations of $Z(x)$, the expected value $s^2(o/V)$ over all these possible realisations may be written as:

$$\sigma^2(o/V) = E[s^2(o/V)] \tag{7.3}$$

This variance is related to the variogram of $Z'(x)$ as:

$$\sigma^2(o/V) = \frac{1}{V^2}\int_V dx \int_V \gamma(x - y)dy \tag{7.4}$$

If V is replaced by L, the core length, we have:

$$\sigma^2(o/L) = \frac{1}{L^2} \int dx \int \gamma(x - y)dy$$

The integral represents the average value of the variogram when x and y move independently within V.This can be expressed as: $\overline{\gamma}(V, V)$. Therefore:

$$\sigma^2(o/V) = \overline{\gamma}(V, V) \tag{7.5}$$

If V represents the deposit D itself, the variance of the point samples in the deposit can be written as:

$$\sigma^2(o/V) = \sigma^2(o/D) = \frac{1}{D^2} \int_D dx \int_D \gamma(x - y)dy \tag{7.6}$$

= Variance in the deposit

= Sill value ($C_o + C$) in a spherical variogram.

The point samples do not possess any volume. In equation (7.6), x and y are two dummy variables used for integration of the variogram function over the volume of interest. In fact, if v is a volume in 3-D, the above equation involves sextuple integrals. If v is a panel (2-D), then it reduces to quadruple integrals. How do we evaluate this?

Practical Approach to Evaluate Variance of Point Samples within a Panel

Suppose we are interested in evaluating the variance of point samples within a panel of area A of 10 m by 10 m, i.e., σ^2 (o/A). Also, suppose that the initial point is at (0,0) (see Fig. 7.1). Therefore, given the variogram function, the variance of point samples within this area is obtained by evaluating:

$$\frac{1}{(10 \times 10)^2} \int_0^{10} \int_0^{10} dxdy \int_0^{10} \int_0^{10} \gamma\left(\sqrt{x^2 + y^2}\right) dxdy \tag{7.7}$$

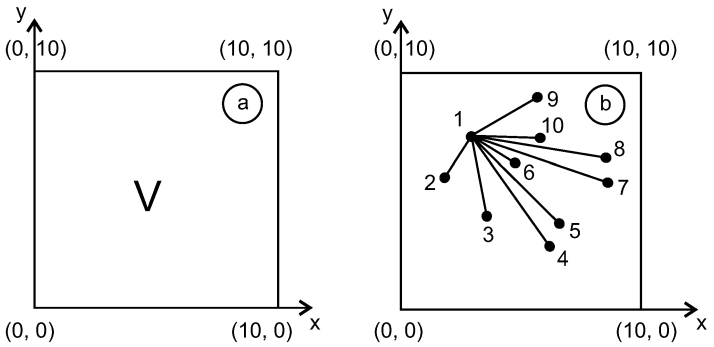


Fig. 7.1 (a) Hypothetical sample area,
(b) 10-point numerical approximation for integration.

Usually, a numerical approximation is used to get the result. In this direction, let us consider a randomly chosen discrete number of points in this area. We now compute the distances between the first point and the remaining points and substituting these distances in the variogram function, the variogram values for various values of h (distance) are obtained. The above steps are repeated for each and every point in the area. Figure 7.1 shows an illustration of this procedure. The variogram values are added and the sum is divided by the total number of points. The resulting value represents the average value of the variogram in the block.

7.1.2 Variance of *v* within *V*

Let v be a smaller volume in V. We have $Z_v(x) = \frac{1}{v}\int_v Z(x+u)du$. We will now express the dispersion of $Z_v(x)$ when it is moved within a larger volume V. For example, v can be a section of drill-hole core within a block V in a deposit; or v can be a block in a deposit D (see Figs 7.2 and 7.3). The variance of v within V is denoted as $\sigma^2(v/V)$.

$$\sigma^2(v/V) = \mathrm{E}\left[\frac{1}{V}\int_v \{Z_v(x) - m_v\}^2 dx\right] \tag{7.8}$$

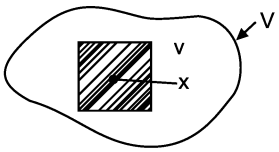


Fig. 7.2 An example of the presence of a smaller volume v in a bigger volume V.

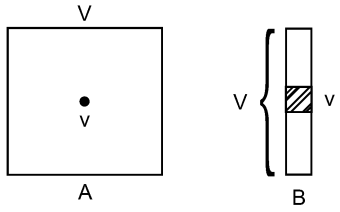


Fig. 7.3 (A) An example of the presence of a drill hole in a typical block of volume V, and (B) a core v in a bore hole.

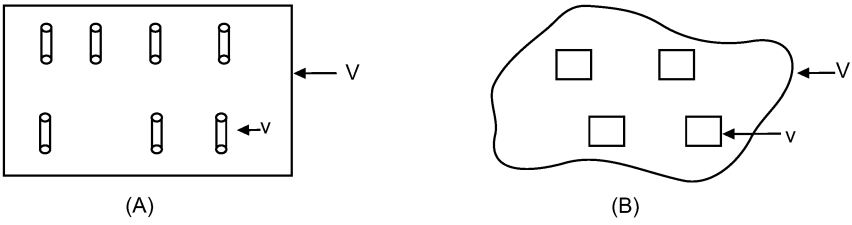


Fig. 7.4 General situation of the presence of (A) boreholes v_i in a bigger volume V and (B) blocks each of volume v in a deposit of volume V.

In terms of variogram, this can be written as

$$\sigma^2(v/V) = \frac{1}{V^2}\int_V\int_V \gamma(x-y)dx\ dy - \frac{1}{v^2}\int_v\int_v \gamma(x-y)dx\ dy \tag{7.9}$$

$$= \bar{\gamma}(V, V) - \bar{\gamma}(v, v) \tag{7.10}$$

$$= \sigma^2(o/V) - \sigma^2(o/v)$$

$$\text{Alternatively, } \sigma^2(o/V) = \sigma^2(o/v) + \sigma^2(v/V) \tag{7.11}$$

Remark: Relation (7.11) is also known as Krige's relation, a term coined in honour of Dr. D.G. Krige who found it operating in the gold deposits of Witwatersrand, South Africa. If v is a core section, V a block and D a deposit, the relation says that the variance of a core section v within a deposit D is equal to the variance of the core section within a block V + the variance of the block within the deposit D.

$$\sigma^2_{v/D} = \sigma^2_{v/V} + \sigma^2_{V/D} \text{ if } v \subset V \subset D$$

$$\text{It may be noted that } \sigma^2(o/D) > \sigma^2_{v/D} > \sigma^2_{V/D} \tag{7.12}$$

7.2 EXTENSION VARIANCE

Suppose we want to estimate the average value of $Z(x)$ over a given domain V of the field. We write: $Z(V) = \frac{1}{V}\int_V Z(x)dx$. Let us also suppose that the information is available only on domain v. *This domain v may be a drill hole/core of length L*, and V a block; or v may be a block and V a deposit and so on. The actual value of $Z(v) = \frac{1}{v}\int_v Z(x)dx$ may be known. In many situations, we simply take $Z(v)$ as an estimate of $Z(V)$. Obviously, we are committing an error, *an error committed by extending the grade of a known volume to an unknown volume V*. How do we quantify this error? Figures 7.5 and 7.6 show examples of the concepts of $\bar{\gamma}(v, V)$, $\bar{\gamma}(V, V)$, $\bar{\gamma}(v, v)$ and $\bar{\gamma}(x_i, V)$.

We know that under the assumption that $Z(x)$ is intrinsic, $Z(v)$ is an unbiased estimator of $Z(V)$.

$$E[Z(v)] = \frac{1}{v}\int_v E[Z(x)dx] = \frac{1}{v}\int_v m\ dx = m = E[Z(V)] \tag{7.13}$$

$$\text{Then } E[Z(v) - Z(V)]^2 = \text{Var}\ [Z(v) - Z(V)]$$

$$\sigma^2_E(v, V) = E[Z(v)]^2 + E[Z(V)]^2 - 2E[Z(v)Z(V)] \tag{7.14}$$

$$= \sigma^2(v, v) - 2\sigma(v, V) + \sigma^2(V, V) \tag{7.15}$$

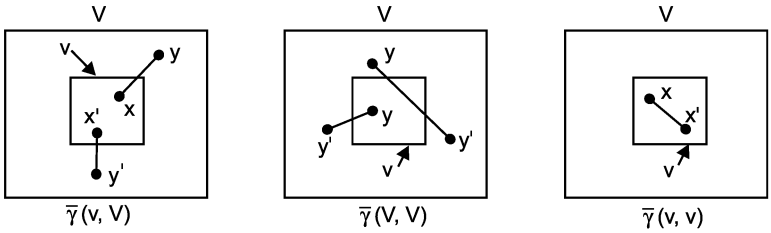


Fig. 7.5 Concepts of $\bar{\gamma}(v, V)$, $\bar{\gamma}(V, V)$ and $\bar{\gamma}(v, v)$.

$\sigma^2_E(v, V)$ or simply σ^2_E is the error committed when the grade of a known volume v is extended to infer the grade of a bigger volume V. *This is known as extension variance.* In some texts, the following notation is used:

$$\bar{C}(v, v) - 2\bar{C}(v, V) + \bar{C}(V, V) \qquad (7.16)$$

where $\bar{C}$ represents the average covariance.

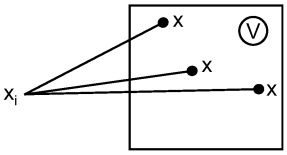


Fig. 7.6 Concept of $\bar{\gamma}(x_i, V)$.

[If $v < V < D$, we may slightly change the above notation and write:

$$\sigma^2_E = \sigma^2(v/D) - 2\sigma(v, V/D) + \sigma^2(V/D)$$

If v is a point sample,

$$\sigma^2_E = \sigma^2(0/D) - 2\sigma(0, V/D) + \sigma^2(V/D) \qquad (7.17)$$

When the covariance $C(h)$ exists, the semi-variogram $\gamma(h)$ also exists and these two are related as follows:

$$C(h) = C(0) - \gamma(h)$$

Hence (7.16) can also be written as:

$$[C(0) - \bar{\gamma}(v, v)] - 2[C(0) - \bar{\gamma}(v, V)] + [C(0) - \bar{\gamma}(V, V)]$$

$$= -\bar{\gamma}(v, v) + 2\bar{\gamma}(v, V) - \bar{\gamma}(V, V)$$

$$= 2\bar{\gamma}(v, V) - \bar{\gamma}(V, V) - \bar{\gamma}(v, v) \qquad (7.18)$$

$$= \frac{2}{NV}\sum_{i=1}^{N}\int_V \gamma(x - y_i)dy - \frac{1}{V^2}\int_V dy\int_V \gamma(y - y')dy' - \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma(x_i - x_j) \qquad (7.19)$$

Here, v is replaced by N discrete samples; $\bar{\gamma}(V, V)$ is the average variogram of volume V with respect to the same volume V; $\bar{\gamma}(v, V)$ is the average variogram of volume v with respect to volume V; $\bar{\gamma}(v, v)$ is the average variogram of volume v with respect to the same volume. Further, rewriting the relation at (7.18) above, we have:

$$\sigma_E^2(v, V) = [\bar{\gamma}(v, V) - \bar{\gamma}(V, V)] + [\bar{\gamma}(v, V) - \bar{\gamma}(v, v)]$$

It is clear that the variance decreases as (i) the sampling is more representative of the domain V to be estimated. In the limit when $v \to V$, $\sigma_E^2(v, V) \to 0$; and (ii) when the variogram is more regular. Another important factor to be noted is that the extension variance involves only the variogram and the geometry of the problem and not the actual values taken by the variable under study. The above formula holds good for any shape of v and V.

7.3 ESTIMATION VARIANCE

We have seen that extension variance $\sigma_E^2(v, V)$ is different from the dispersion variance $\sigma_E^2(v/V)$. The dispersion variance measures the dispersion of samples of size v within the domain V. The extension variance signifies the error attached to a given sampling pattern. It refers to the variance (error) one incurs when the grade of a smaller value v is extended to infer the grade of a bigger value say V.

We may recall that if v is a point sample and V is a block in a deposit D, the extension variance may be written as equation (7.17):

$$\sigma^2(o/D) - 2\sigma(o, V/D) + \sigma^2(V/D).$$

We compute the above variances/covariances using the numerical approximation techniques. When these are computed, we have the desired extension error for a given block and given variogram function. Usually, more than one diamond-drill hole or point sample is used for estimation. If a *number of sample grades* are extended to a block, the error one incurs is called *Estimation Variance* instead of *Extension Variance.* When a number of sample grades are used to estimate the block grade, we usually make use of the average grade of all N samples. In that case, the estimation variance $\sigma_E^2(v, V)$ or simply σ_E^2 may be represented as in eqn. (7.19).

We notice that a sample grade v is replaced by an average grade of N samples. Therefore, the estimation variance is now equal to *twice* the *average value* of the variogram between samples and the block V, *minus* the variance of points within a block V, *minus* the average value of the variogram between the samples. Usage of average value of N samples implies that the samples are assigned equal weight. For example, a sample which is farther away from the point/block under estimation is given the same weightage as the one at a lesser distance. In reality, different weights need to be given to different samples located at varying distances. The holes/samples closer to the block have a greater influence and hence need to be given greater weightage than those that are farther. Kriging procedure takes this into consideration. If different weights $\lambda_i (i = 1, N)$ are assigned to different holes/samples instead of equal weight $\frac{1}{N}$, equation (7.19) above can be reframed. In terms of covariance, it can be written as:

$$\sigma_K^2 = \sigma_E^2 = \sigma_V^2 - 2\sum_{i=1}^{N} \lambda_i \sigma_{Vx_i} + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \lambda_j \sigma_{x_i x_j} \tag{7.20}$$

In terms of variogram (γ) notation, we have:

$$\sigma_K^2 = \sigma_E^2 = 2\sum_{i=1}^{N} \lambda_i \overline{\gamma}(x_i, V) - \overline{\gamma}(V, V) - \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \lambda_j \gamma(x_i, x_j) \tag{7.21}$$

σ_K^2 is called the *Estimation Variance.*

Review Questions

Q. 1. Explain the terms: (a) Dispersion variance (b) Extension variance and (c) Estimation variance.

Q. 2. Discuss the relationship of extension variance to the dispersion variance.

8 Kriging Variance and Kriging Procedure

8.1 TOWARDS KRIGING VARIANCE

In order to derive Kriging Variance, we proceed as follows: we assume that $Z'(x)$ — the random function is defined on a point support and is second order stationary. It follows that $E[Z(x)] = m$, and the covariance, defined as $E[Z(x + h)Z(x)] - m^2 = C(h)$ exists. We know that $E[\{Z(x + h) - Z(x)\}^2] = 2\gamma(h)$. We are interested in the mean $Z_V(x_0) = \frac{1}{V} \int Z(x)\, dx$. The data comprises a set of grade values $Z(x_i)$, in short x_i, $i = 1$ to N. The grades are defined either on point supports, core supports, etc. They could also be mean grades $Z_{Vi}(x_i)$ defined on the supports V_i centered on the points x_i. It is possible that the N supports could be different from each other. Under the assumption of stationarity, the expectation of these data is m. That is, $E(Z_i) = m$.

Let Z_V^* be a linear estimator of Z_V with a combination of N data points. Thus $Z_V^* = \Sigma\lambda_i x_i$. We want this estimator to be (i) unbiased and (ii) optimal. For this, under conditions of stationarity, we should have (i) $E(Z_V^*) = m = E(Z_V)$, *i.e.*, $E(Z_V - Z_V^*) = 0$. This is possible if $\Sigma\lambda_i = 1$; and (ii) $Var(Z_V^* - Z_V)$ is minimum.

Condition 1 implies:

$$E(Z_V^* - Z_V) = E(\Sigma\lambda_i x_i - Z_V) = \Sigma\lambda_i E[(x_i) - E(Z_V)] \tag{8.1}$$

$$\Sigma\lambda_1 m - m = m\ \Sigma\lambda_i - m$$

$$= 0 \text{ since } \Sigma\lambda_i = 1$$

Condition 2 implies:

$$\text{Var}\ (Z_V^* - Z_V) = E[Z_V^* - Z_V]^2 - [E(Z_V^* - Z_V)]^2$$

$$= E[Z_V^* - Z_V]^2 - 0 = E[Z_V^* - Z_V]^2 \tag{8.2}$$

$$= E[\Sigma\lambda_i z_i - Z_V]^2 \text{ is minimum.}$$

Note: x_i is actually $Z(x_i)$; also denoted as Z_i.

Now consider, as an example: $\left(\sum_{i=1}^{3} \lambda_i x_i\right)^2$ which is square of a linear combination of three variables.

$$\begin{aligned} &= (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)^2 \\ &= \lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 + \lambda_3^3 x_3^2 \\ &\quad + 2\lambda_1\lambda_2 x_1 x_2 + 2\lambda_2\lambda_3 x_2 x_3 + 2\lambda_3\lambda_1 x_3 x_1 \\ &= \sum_{i=1}^{3} \lambda_i^2 x_i^2 + 2\sum_{i=1}^{3} \sum_{i \neq j=1}^{3} \lambda_i \lambda_j x_i x_j \quad i > j \end{aligned} \tag{8.3}$$

Therefore,

$$\begin{aligned} E(\Sigma\lambda_i x_i - Z_V)^2 &= E(\Sigma\lambda_i x_i)^2 - 2E(\Sigma\lambda_i x_i Z_V) + E(Z_V)^2 \\ &= [E(\Sigma\lambda_i^2 x_i^2 + 2\Sigma\Sigma\lambda_i\lambda_j x_i x_j)] - 2E(\Sigma\lambda_i x_i Z_V) + E[(Z_V)^2] \\ &= E\left(\sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j x_i x_j\right) - 2E(\Sigma\lambda_i x_i Z_V) + E(Z_V)^2 \\ &= \Sigma\Sigma\lambda_i\lambda_j \sigma_{x_i x_j} - 2\Sigma\lambda_i \sigma_{x_i Z_V} + \sigma_V^2 \end{aligned}$$

For convenience, we replace Z_V by V and write:

$$\sigma_E^2 = \sigma_V^2 - 2\sum_{i=1}^{N} \lambda_i \sigma_{Vx_i} + \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j \sigma_{x_i x_j} \tag{8.4}$$

σ_E^2 is called the estimation variance. In terms of 'γ' notation, this variance may be written as:

$$\sigma_E^2 = 2\sum_{i=1}^{N} \lambda_i \bar{\gamma}(x_i, V) - \bar{\gamma}(V, V) - \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j \gamma(x_i - x_j) \tag{8.5}$$

Eqn. (8.4) says that the estimation variance is equal to the block variance, *minus* twice the weighted covariance function values between the samples $Z(x_i)$ and the block V plus the weighted covariance function values between samples.

There are different methods of estimating the in situ resources of a deposit. In the context of estimating a mineral value property, let us distinguish between *local estimation* and *global estimation.* In case of local estimation, we try to estimate the mean value of a regionalised variable (e.g. grade/ tenor/accumulation) over a limited domain—the dimensions of which are small compared to the dimensions of the quasi-stationary (homogeneous)

zones of the deposit. (Journel and Huijbregts, 1978). In global estimation, distances larger than the limits of quasi-stationarity are considered. When we consider such larger distances, it is possible that we come across non-homogeneous zones/mineralisation. *Kriging,* which is one of the methods of estimation, may be defined as a local estimation technique which gives the *Best Linear Unbiased Estimate (BLUE)* of the unknown characteristics studied.

8.2 KRIGING PROCEDURE

Our interest is to minimise the estimation variance σ_E^2. This is possible by choosing the appropriate weights (λ_i), with the constraint that the sum total of the weights ($\Sigma\lambda_i$) must be equal to unity. *This is a constrained optimisation problem.*

$$\text{Thus:} \qquad \text{Min } \sigma_E^2 = \sigma_V^2 - 2\sum_{i=1}^{N} \sigma_{Vx_i} + \Sigma\Sigma\lambda_t\lambda_j\sigma_{x_tx_j} \tag{8.6}$$

$$\text{or} \qquad \text{Min } \sigma_E^2 = L(\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n) \text{ subject to: } \sum_{i=1}^{N} \lambda_i = 1.$$

This constrained optimisation problem can be solved by the method of lagrangian multipliers. Let us now introduce a lagrangian multiplier in equation (8.6) above:

$$L(\lambda_1,\lambda_2, \ldots, \lambda_N, \mu) = \sigma_V^2 - 2\Sigma\sigma_{vx_t} + \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda_j\sigma_{x_tx_j} - 2\mu\left(\sum_{i=1}^{N}\lambda_i - 1\right) \tag{8.7}$$

In equation (8.7), we have used 2μ instead of μ. Introducing μ or 2μ does not alter the expression on RHS, as any way the third term in the RHS expression becomes zero. One may observe that with the constraint equation viz., $\Sigma\lambda_i = 1$, we have $N + 1$ equations but there are only N original variables $\lambda_1, \lambda_2, \ldots, \lambda_N$. To balance the system of equations we needed another variable μ – *the Lagrangian multiplier.*

To determine the stationary points, we differentiate with respect to $\lambda_t's$, and μ to obtain a set of simultaneous equations having $N + 1$ equations which would yield values for $\lambda_i's$ and μ. For the sake of convenience, we may write σ_V^2 as σ_o^2, $\sigma_{x_tx_j}$ as σ_{ij} and $\sigma_{vx_i} = \sigma_{oi}$. Thus:

$$L(\lambda_1, \lambda_2, \ldots, \lambda_N, \mu) = \sigma_o^2 - 2\sum_{i=1}^{N}\lambda_i\sigma_{oi} + \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda_j\sigma_{ij} - 2\mu\left(\sum_{i=1}^{N}\lambda_i - 1\right) \tag{8.8}$$

On expansion:

$$\begin{aligned} L(\lambda_s', \mu) = {} & \sigma_o^2 \\ & - 2\lambda_1\sigma_{o1} - 2\lambda_2\sigma_{o2} - \ldots - 2\lambda_N\sigma_{oN} \\ & + \lambda_1^2\sigma_{11} + \lambda_2^2\sigma_{22} + \ldots + \lambda^2{}_N\sigma_{NN} \\ & + 2\lambda_1\lambda_2\sigma_{12} + 2\lambda_1\lambda_3\sigma_{13} + \ldots + 2\lambda_1\lambda_N\sigma_{1N} \\ & + 2\lambda_2\lambda_3\sigma_{23} + 2\lambda_2\lambda_4\sigma_{24} + \ldots + 2\lambda_2\lambda_N\sigma_{2N} \\ & + 2\lambda_3\lambda_4\sigma_{34} + 2\lambda_3\lambda_5\sigma_{35} + \ldots + 2\lambda_1\lambda_N\sigma_{3N} \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \ldots\ldots\ldots\ldots + 2\lambda_{N-1}\lambda_N\,\sigma_{N-1,\,N} \\ & - 2\mu(\lambda_1 + \lambda_2 + \ldots + \lambda_N - 1) \end{aligned} \tag{8.9}$$

To obtain the stationary point, partial differentiation is carried out and the resulting expression is set equal to zero. Thus:

$$\frac{\partial L}{\partial \lambda_1} = -2\sigma_{o1} + 2\lambda_1\sigma_{11} + 2\lambda_2\sigma_{12} + \ldots + 2\lambda_N\sigma_{1N} - 2\mu = 0$$

or $$= -\sigma_{o1} + \sum_{j=1}^{N}\lambda_1\sigma_{11} + \sum_{j=2}^{N}\lambda_i\sigma_{1j} - \mu = 0$$

or $$= \sum_{j=1}^{N}\lambda_j\sigma_{1j} - \mu = \sigma_{o1} \tag{8.10}$$

Similarly, $\dfrac{\partial L}{\partial \lambda_2} = \sum_{j=1}^{N}\lambda_j\sigma_{2j} - \mu = \sigma_{o2}$; and $\dfrac{\partial L}{\partial \lambda_3} = \sum_{j=1}^{N}\lambda_j\sigma_{3j} - \mu = \sigma_{o3}$

Generalising, we have:

$$\frac{\partial L}{\partial \lambda_i} = \sum_{j=1}^{N}\lambda_j\sigma_{ij} - \mu = \sigma_{oi}$$

Finally, $\dfrac{\partial L}{\partial_\mu}$ gives $\sum_{j=1}^{N}\lambda_j = 1$ (8.11)

The resulting set of $N + 1$ equations are

$$\sum_{j=1}^{N}\lambda_j\sigma_{ij} - \mu = \sigma_{oi} \text{ for } i = 1, 2, \ldots, N$$

$$\sum_{j=1}^{N}\lambda_j = 1 \tag{8.12}$$

In matrix notation the system of equations represented by (8.12) may be written as:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & .... & \sigma_{1N} & -1 \\ \sigma_{21} & \sigma_{22} & .... & \sigma_{2N} & -1 \\ & & .... & & \\ \sigma_{N1} & \sigma_{N2} & .... & \sigma_{NN} & -1 \\ 1 & 1 & .... & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ .... \\ \sigma_N \\ \mu \end{pmatrix} = \begin{pmatrix} \sigma_{01} \\ \sigma_{02} \\ .... \\ \sigma_{0N} \\ 1 \end{pmatrix} \tag{8.13}$$

$$A \qquad X \qquad B$$

Therefore, $X = A^{-1}B$ (8.14)

If V is used instead of 0, the elements of the column on RHS change as: σ_{V1}, σ_{V2}, $\ldots\sigma_{VN}$; alternatively, as σ_{1V}, σ_{2V}, . . . σ_{NV}.

When the optimal weights λ_i's and the lagrangian multiplier μ are known, the *kriging variance* for the block V can be computed from equation (8.6) viz.,

$$\sigma_K^2 = \sigma_V^2 - 2\sum_{i=1}^{N} \lambda_i \sigma_{Vx_i} + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \lambda_j \sigma_{x_i x_j}$$

$$= \sigma_o^2 - 2\sum_{i=1}^{N} \lambda_i \sigma_{oi} + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \lambda_j \sigma_{ij} \tag{8.15}$$

(if V is replaced by o and $\sigma_{x_i x_j}$ by σ_{ij}). The above equation (8.15) can be simplified by utilising the computed value for μ. Consider again the system of equations represented at (8.12). We have:

$$\sum_{j=1}^{N} \lambda_i \sigma_{ij} - \mu = \sigma_{oi} \text{ for } i = 1, 2, \ldots, N$$

Multiplying both sides of the above equation by λ_i and summing up, we have:

$$\sum_{j=1}^{N}\sum_{i=1}^{N} \lambda_i \lambda_j \sigma_{ij} - \mu \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \sigma_{oi} \lambda_i \tag{8.16}$$

$$= \sum_{j=1}^{N}\sum_{i=1}^{N} \lambda_i \lambda_j \sigma_{ij} = \sum_{i=1}^{N} \sigma_{oi} \lambda_i + \mu, \text{ since } \Sigma\lambda_i = 1 \tag{8.17}$$

Substituting this value in (8.15), we have

$$\sigma_K^2 = \sigma_o^2 - 2\sum_{i=1}^{N} \lambda_i \sigma_{oi} + \sum_{i=1}^{N} \lambda_i \sigma_{oi} + \mu$$

(after replacing σ_V^2 by σ_o^2)

$$= \sigma_o^2 - \sum_{i=1}^{N} \lambda_i \sigma_{oi} + \mu \tag{8.18}$$

This minimum estimation variance is called *kriging variance.*

8.3 KRIGING SYSTEM AND KRIGING VARIANCE IN TERMS OF γ NOTATION

In terms of variogram notation, we write the kriging system of equations as:

$$\left.\begin{aligned} \sum_{j=1}^{N} \lambda_j \gamma(x_i - x_j) + \mu &= \bar{\gamma}(x_i, V) \text{ for } i = 1, 2, \ldots, N \\ \sum_{j=1}^{N} \lambda_j &= 1 \end{aligned}\right\} \tag{8.19}$$

Kriging variance is given by:

$$\sigma_k^2 = \Sigma \lambda_i \bar{\gamma}(x_i, V) - \bar{\gamma}(V, V) + \mu \tag{8.20}$$

When V is a point sample, we have $\bar{\gamma}(x_i, V) = \gamma(x_i - x_o)$; and $\bar{\gamma}(V, V) = \gamma(0) = 0$. In terms of matrix notation the kriging system of equations may be written as:

$$\underset{\alpha}{\begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N} & 1 \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N} & 1 \\ & \cdots\cdots & & & \cdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{NN} & 1 \\ 1 & 1 & \cdots\cdots & 1 & 0 \end{pmatrix}} \underset{X}{\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \cdots \\ \lambda_N \\ \mu \end{pmatrix}} = \underset{\beta}{\begin{pmatrix} \bar{\gamma}(x_1, V) \\ \bar{\gamma}(x_2, V) \\ \cdots\cdots \\ \bar{\gamma}(x_N, V) \\ 1 \end{pmatrix}} \tag{8.21}$$

It has to be ensured that γ is a proper variogram model. Then α is always non-singular and the solution is simply $X = \alpha^{-1}\beta$ and the variance is $\sigma_k^2 = X^T\beta - \bar{\gamma}(V, V)$. In (8.21), all the diagonal elements $\gamma_{11}, \gamma_{22}, \ldots \gamma_{NN}$ are equal to zero. Hence α is not positive definite.

Note: Usually, the kriging system of equations with covariance notation is used.

8.4 SIMPLE KRIGING (ALSO KNOWN AS LINEAR KRIGING WITH KNOWN EXPECTATION)

The discussion in Sections 8.2 and 8.3 goes under the name of ordinary kriging (O.K.) also known as linear kriging with unknown expectation. We may recall that in ordinary kriging, we have imposed the constraint that the sum of the weights i.e., $\Sigma \lambda_i = 1$, estimated the mean as $\Sigma \lambda_i Z(x_i)$, and derived the kriging variance. In cases where the mean is known from past experience and need not be estimated, we proceed as follows.

Consider a regionalized variable $G(x)$ with zero mean related to $Z(x)$, the original regionalized variable, as $Z(x) = G(x) + m$, where m is the mean of $Z(x)$. We have the estimator of $G(x)$ viz., G_V^* as: $\Sigma \lambda'_i G(x_i)$.

For this estimator G_V^* to be unbaised, we write:

$$E[G_V^* - G_V] = E[\Sigma\lambda_i' \, G(x_i) - G_V]$$

As the mean of $G(x)$ is zero, we have $E[\Sigma\lambda_i' \, G(x_i)] = 0$. This is possible only when there is no condition imposed on the sum of the weights.

Let us look at the variance of the estimation error:

$$\text{Var}[G_V^* - G_V] = E[\Sigma\lambda_i' G(x_i) - G_V]^2$$

$$= \sum_j \sum_i \Sigma\lambda_i'\lambda_j'\sigma_{ij} + \sigma_{00} - 2\Sigma\, \lambda_i'\sigma_{iV}$$

(as per σ notation)

Since there is no condition that the sum of the weights should add up to unity, the system is not unbalanced and therefore there is no need to introduce the lagrangian multilier. Therefore the kriging system of equations in this case of simple kriging comes to:

$$\sum_{i=1}^{N} \lambda_i'\sigma_{ij} = \sigma_{iV}, \quad \text{for } i = 1, 2, \ldots\ldots N$$

The corresponding krigning variance is given by:

$$\sigma_{sk}^2 = \sigma_0^2 - \Sigma\lambda_i'\sigma_{oi}$$

The above kriging system gives us the kriging weights to estimate G_V. However, our interest is to estimate Z_V. This can be done by replacing $G(x)$ by $Z(x) - m$

$$Z_V^* = G_V^* + m$$
$$= \Sigma\lambda_i'[Z(x_i) - m] + m$$
$$= \Sigma\lambda_i' Z(x_i) + m[1 - \Sigma\lambda_i']$$

The term $[1 - \Sigma\lambda_i']$ is called the 'weight of the mean' in simple kriging.

It may be noted that the utility of simple kriging is limited, as in paractice, the mean is not known and needs to be estimated. However, as in the case of south African gold mines or in the case of Kolar gold fields in India, where mining activity has been carried out for a number of years, we can expect the mean to be known for each mine or for each of the regions in the same mine.

8.5 EXAMPLES

8.5.1 Punctual Kriging

Consider the set-up as given in Fig. 8.1. We want to estimate the grade at point z_0 surrounded by four data points z_1, z_2, z_3 and z_4.

Let us for the sake of simplicity assume the following linear model for the variogram:

$$\gamma(h) = 0.01h \text{ for } h < 30 \text{ m}$$
$$= 4.20 \text{ for } h \geq 30 \text{ m} \tag{8.42}$$

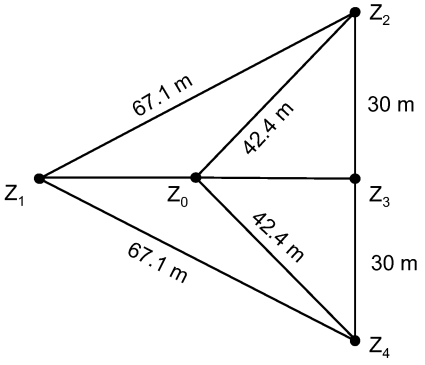


Fig. 8.1 A four-point set up for punctual kriging.

Step 1

Let us compute the covariance between each of the sample points using the relation $\sigma(h) = \sigma(0) - \gamma(h)$. We know that:

$$\sigma(1, 1) = \sigma(2, 2) = \sigma(3, 3) = \sigma(4, 4) = 4.20$$
$$\sigma(1, 2) = \sigma(2, 1) = \sigma(1, 4) = \sigma(4, 1) = 4.20 - 0.67 = 3.53$$
$$\sigma(2, 4) = \sigma(4, 2) = 4.2 - 0.01(60) = 3.60$$
$$\sigma(2, 3) = \sigma(3, 2) = 4.2 - 0.01(30) = 3.90$$
$$\sigma(3, 4) = \sigma(4, 3) = 4.20 - 0.01(30) = 3.90$$
$$\sigma(1, 3) = \sigma(3, 1) = 3.60$$
$$\sigma(0, 1) = 3.9; \sigma(0, 2) = 4.20 - 0.42 = 3.78; \sigma(0, 3) = 3.90$$
$$\sigma(0, 4) = 4.20$$

The system of equations is:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & -1 \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & -1 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & -1 \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & -1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \mu \end{pmatrix} = \begin{pmatrix} \sigma_{01} \\ \sigma_{02} \\ \sigma_{03} \\ \sigma_{04} \\ 1 \end{pmatrix} = \begin{pmatrix} \sigma_{V1} \\ \sigma_{V2} \\ \sigma_{V3} \\ \sigma_{V4} \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4.20 & 3.53 & 3.60 & 3.53 & -1 \\ 3.53 & 4.20 & 3.90 & 3.60 & -1 \\ 3.60 & 3.90 & 4.20 & 3.90 & -1 \\ 3.53 & 3.60 & 3.90 & 4.20 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \mu \end{pmatrix} \begin{pmatrix} 3.900 \\ 3.776 \\ 3.900 \\ 3.776 \\ 1 \end{pmatrix}$$

Inverting the matrix we have:

$$\lambda_1 = 0.38,\ \lambda_2 = 0.12,\ \lambda_3 = 0.38,\ \lambda_4 = 0.12 \text{ and } \mu = -0.04.$$

We have the kriging variance as (eqn. 8.18)

$$\sigma_k^2 = \sigma_o^2 + \mu - \Sigma\lambda_i\sigma_{oi} = 4.2+(-0.04) - (0.38 \times 3.90 + 0.12 \times 3.78 + 0.38 \times 3.90 + 0.12 \times 3.76)$$

$$= 4.2 - 0.04 - (1.75 + 0.45 + 1.21 + 0.45)$$

$$= 4.2 - 0.04 - 3.87 = 0.29 \text{ (gms)}^2$$

$$\sigma_k = 0.54 \text{ gms}$$

If the grades at points z_1, z_2, z_3, z_4 are say, 4, 6, 2, 1 gms/tonne, the kriged estimate at point z_0 is $0.38 \times 4 + 0.12 \times 6 + 0.38 \times 2 + 0.12 \times 1 = 3.02$ gms/tonne.

8.5.2 Block Kriging

In the case of mineralisations, the problem is one of estimating the grade/ accumulation of a block of ore of defined size,

(i) in the neighbourhood of a set of samples (S_i) whose grades are available. We may also identify a sample (S_1) in the block V_1 with grade for this sample; (see Fig. 8.2) or

(ii) in the neighbourhood of blocks of ore of actual dimensions and the average or estimated grades of each of these neighbourhood blocks are available.

Let us consider the first possibility, viz., the neighbourhood consists of a set of samples and the grades in gms/tonne of these samples are available. Figure 8.2 shows a typical block with a sample point S_1 and this block is surrounded by five other blocks represented by their sample points (S_2, S_3, . . ., S_6). Figure 8.3 shows an enlarged block of 30 m × 30 m. The variogram model is defined as:

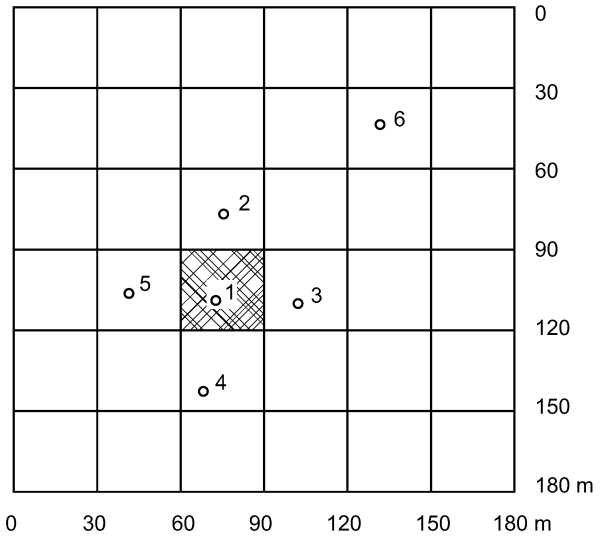


Fig. 8.2 A block of 30 m × 30 m surrounded by five blocks each with the same dimension.

$$\gamma(h) = 35.0 + 55\left[\frac{3}{2}\left(\frac{h}{a}\right) - \frac{1}{2}\left(\frac{h}{a}\right)^3\right] \quad \text{for } h < a$$

and $\gamma(h) = Co + C$ for $h \geq a$.

$= 90$ m

a, the range of influence is 150 m.

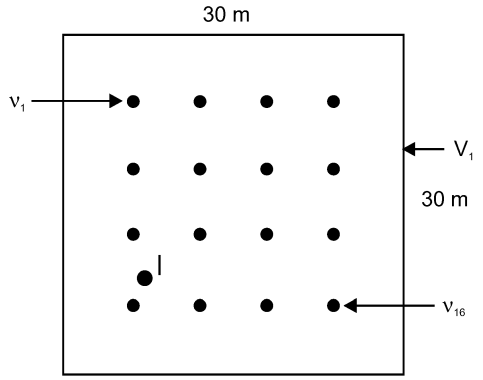


Fig. 8.3 Enlarged studied block of 30 m × 30 m.

The following steps are followed to compute the kriging variance and the kriged estimate.

Step 1

$\gamma(h)$ and covariance values are computed between each of these sample points (S_i). We recall: $\sigma(h) = \sigma(0) - \gamma(h)$. These values are as given in Table 8.1.

Step 2

The covariance between the block under consideration (V_1) and the sample point 1 in it is now computed. To achieve this, the block under consideration is discretised into 16 equi-spaced points v_1, v_2, … v_{16} spread over the block. We first compute the $\gamma(h)$ values between each of these discretised points and the sample point in the block (V_1), sum them up and averaged. The covariance is obtained from the relation $\sigma(h) = \sigma(0) - \bar{\gamma}(h)$. Figure 8.4 shows the sample points 1 to 6 and the 16 discretised points (v_1,v_2, . . ., v_{16}) within each of the blocks.

Table 8.1 Computed $\gamma(h)$ and $\sigma(h)$ values between the samples

Sample point i-j	*Distance (m)*	$\gamma(h)$	$\sigma(h)$
1–2, 2–1	32	50.4	39.6
1–3, 3–1	30	48.9	41.1
1–4, 4–1	60	69.7	20.3
1–5, 5–1	30	48.9	41.1
1–6, 6–1	96	86.4	3.6
2–3, 3–2	64	72.1	17.9
2–4, 4–2	95	86.1	3.9
2–5, 5–2	50	63.3	26.7
2–6, 6–2	75	78.0	12.0
3–4, 4–3	70	75.4	14.6
3–5, 5–3	62	70.9	19.1
3–6, 6–3	65	72.6	17.4
4–5, 5–4	75	78.0	12.0
4–6, 6–4	140	90.0	0.0
5–6, 6–5	120	90.0	0.0

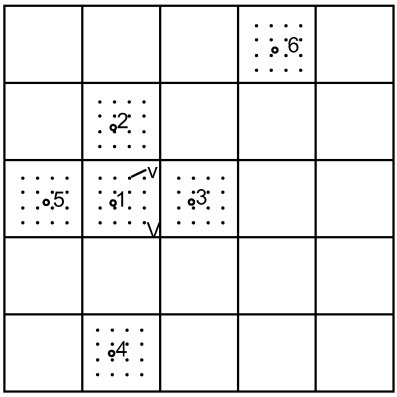


Fig. 8.4 Studied block V_1 and five neighbourhood blocks with discrete points $v_1, v_2, \ldots, v_{16}$ in each of them.

Covariance between sample point 1 and the block V_1

$$\sigma_{1V1} = \sigma(0) - \frac{1}{16}[\gamma(pt_1 - v_1) + \gamma(pt_1 - v_2) + \gamma(pt_1 - v_3) + \gamma(pt_1 - v_4) + \gamma(pt_1 - v_5) + \gamma(pt_1 - v_6) + \ldots + \gamma(pt_1 - v_{16})]$$

$$= 90 - \frac{1}{16}[\gamma(18) + \gamma(16.8) + \gamma(21.6) + \gamma(22.0) + \gamma(10.8) + \gamma(10.8) + \gamma(14.5) + \gamma(18.0) + \gamma(6.0) + \gamma(12) + \gamma(18.0) + \gamma(6.0) + \gamma(6.0) + \gamma(12.0) + \gamma(18.0)].$$

The same procedure is followed for samples 2 to 6 vis-a-vis the block under consideration (V).

Covariance between sample point 2 and the block V_1

$$\sigma_{2V1} = \sigma(0) - \frac{1}{16}[\gamma(pt_2 - v_1) + \gamma(pt_2 - v_2) + \ldots + \gamma(pt_2 - v_{16})]$$

$$= \sigma(0) - \frac{1}{16}[\gamma(12) + \gamma(9) + \gamma(15) + \gamma(16.5) + \gamma(16.5) + \gamma(18) + \gamma(19.5) + \gamma(24) + \gamma(24) + \gamma(24.5) + \gamma(24) + \gamma(27) + \gamma(36) + \gamma(28.5) + \gamma(30) + \gamma(33)]$$

Covariance between sample point 3 and the block V_1

$$\sigma_{3V1} = \sigma(0) - \frac{1}{16}[\gamma(42) + \gamma(39) + \gamma(33) + \gamma(27) + \gamma(39) + \gamma(36) + \gamma(30) + \gamma(24) + \gamma(36) + \gamma(33) + \gamma(27) + \gamma(21) + \gamma(39) + \gamma(33) + \gamma(27) + \gamma(21)]$$

Covariance between sample point 4 and the block V_1

$$\sigma_{4V1} = \sigma(0) - \frac{1}{16}[\gamma(78) + \gamma(75) + \gamma(78) + \gamma(81) + \gamma(72) + \gamma(69) + \gamma(75) + \gamma(78) + \gamma(66) + \gamma(63) + \gamma(69) + \gamma(72) + \gamma(57) + \gamma(54) + \gamma(60) + \gamma(63)]$$

Covariance between sample point 5 and the block V_1

$$\sigma_{5V1} = \sigma(0) - \frac{1}{16}[\gamma(24) + \gamma(36) + \gamma(36) + \gamma(42) + \gamma(21) + \gamma(27) + \gamma(33) + \gamma(39) + \gamma(24) + \gamma(30) + \gamma(36) + \gamma(42) + \gamma(18) + \gamma(24) + \gamma(30) + \gamma(36)]$$

Covariance between sample point 6 and the block V_1

$$\sigma_{6V1} = \sigma(0) - \frac{1}{16}[\gamma(71) + \gamma(68) + \gamma(62) + \gamma(59) + \gamma(84) + \gamma(81) + \gamma(75) + \gamma(69) + \gamma(90) + \gamma(87) + \gamma(81) + \gamma(76) + \gamma(93) + \gamma(90) + \gamma(84) + \gamma(81)]$$

The relevant matrix is formulated as:

$$\begin{pmatrix} 90.0 & 39.6 & 41.1 & 20.3 & 41.1 & 03.6 & -1 \\ 39.6 & 90.0 & 17.9 & 03.9 & 26.7 & 12.0 & -1 \\ 41.1 & 17.9 & 90.0 & 14.6 & 19.8 & 17.4 & -1 \\ 20.3 & 03.9 & 14.6 & 90.0 & 19.1 & 00.0 & -1 \\ 41.1 & 26.9 & 19.1 & 12.0 & 90.0 & 00.0 & -1 \\ 03.6 & 12.0 & 17.4 & 00.0 & 00.0 & 90.0 & -1 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \mu \end{pmatrix} = \begin{pmatrix} 53.7 \\ 47.5 \\ 39.9 \\ 15.2 \\ 40.7 \\ 10.9 \\ 1 \end{pmatrix}$$

Solving the above, we have $\lambda_1 = 0.28$, $\lambda_2 = 0.30$, $\lambda_3 = 0.20$, $\lambda_4 = 0.03$, $\lambda_5 = 0.18$, $\lambda_6 = 0.02$ and $\mu = -0.70$.

The kriging variance $\sigma_K^2 = \sigma_o^2 - \Sigma\lambda_i\sigma_{oi} + \mu$

$$= 90 - 0.28\ (53.7) + 0.30\ (47.5) + 0.20\ (39.9) + 0.01\ (15.2) + 0.18\ (40.7) + 0.04(10.9) - 0.70$$

$$= 90 - [15.04 + 14.25 + 7.98 + 0.15 + 7.32 + 0.22] - 0.70$$

$$= 90 - 44.96 - 0.70 = 44.34 \text{ (gms)}^2$$

$$\sigma_k = 6.65 \text{ gms.}$$

Kriged Estimate

Let us now assign mean grades to these samples/neighbouring blocks. We also know the weights attached to each of these blocks. Table below gives these details:

Sample pt./Block No	*Weight* λ_i	*Known grade (gms/tonne)*
2	0.28	9
3	0.30	10
4	0.20	7
5	0.18	8
6	0.02	12

The kriged estimate for grade for the block V (with sample point 1 in it):

$0.28 \times 9 + 0.30 \times 10 + 0.20 \times 12 + 0.18 \times 8 + 0.02 \times 12 = 9.6$ gms/tonne.

Note:

(i) The sample points referred to above can also be the central points of each of the blocks.

(ii) It is not necessary to include a sample point with a grade value in the block (V_1) as is done in the above example. In that case, the neighbourhood, the matrix and the weights to be assigned to the neighbouring holes/blocks change.

(iii) For estimating another block, the same procedure is followed. However, for another block, say (V_2) , the neighbourhood of the samples and the weights for these samples change.

(iv) Instead of considering the sample points as neighbourhood, *the blocks with actual dimensions* can be the neighbourhood. In this case, each of these blocks may be discretised and the covariance between each of these discretised points and the block under consideration, say (V_1), is computed and averaged. Obviously, the number of computations are more.

(v) The neighbourhood can also be cores instead of blocks or samples.

Review Questions

Q. 1. Explain the significance of kriging.

Q. 2. Explain kriging procedure.

Q. 3. Distinguish between simple kriging (SK) and ordinary kriging (OK).

9 Introduction to Advanced Geostatistics

9.1 INTRODUCTION

So far we have discussed, on the classical statistics front, various types of estimators starting with arithmetic mean, lognormal estimator, estimators based on auto-regressive processes, moving average process and ARMA processes. From the point of view of the theory of regionalized variables, we have also discussed kriging estimator in the stationary case. It was Dr. D.G. Krige, a mining engineer from South Africa, who first introduced the concept of moving averages to overcome the problem of systematic over-estimation of ore reserves. Professor Matheron improved on this concept and gave a mathematical orientation to a method bringing the concept of regionalized variables and variogram. In honour of that pioneering mining engineer, Prof. Matheron coined the word "kriging' for the method he has developed.

As we have seen, kriging is an approach/method to find the best linear estimator under the assumption of second order stationarity. In reality, a geological process may not be stationary. In those cases where the process is non-stationary, we have to use non-linear functions of the sample data. Non-linear estimators may be more accurate than the linear kriging estimators as they try to address the phenomena in more realistic terms. The derivation of non-linear estimators needs the estimation of unknown functions which are non-linear combinations of unknown values. These non-linear estimators can be categorized as those based on disjunctive kriging or conditional simulation. Broadly, we may classify these geostatistical techniques as follows:

	Stationary	*Non-stationary*
Linear	Ordinary/simple kriging	Universal kriging; Kriging using IRF-K
Non-linear	Disjunctive kriging	
	Simulation	Simulation of IRF-K

Subsequent to the introduction of kriging methodology, various developments have taken place keeping in view the non-stationarity of the phenomena. In one approach the prerequisites were weakened by admitting non-stationarity and the existence of a drift, $E[Z(x)] = m(x) = \Sigma A_p G_p$, for $p = 1, 2, \ldots k$). This can be expressed in a known form. This approach is known as 'universal kriging' or 'unbiased kriging' of order, k. In the other approach, the prerequisites were strengthened by requiring knowledge of not only the covariance but also of the k-variate distribution of Random Function (RF) – $Z'(X)$. Non-linear estimators based on conditional expectation or disjunctive kriging still carry the tag 'kriging', since all these estimators may be viewed as extensions of kriging aimed at estimating the unknown value of $Z(x_0)$. Thus the larger the set on to which the projection is done, the nearer will be the corresponding kriging estimator to the unknown value (Journel and Huijbregts, 1977). Some of these approaches are discussed below.

9.2 NON-STATIONARY GEOSTATISTICS

9.2.1 Universal Kriging

Universal kriging is a method of estimating a regionalized variable in the non-stationary case i.e, when trends or more precisely drifts, are present. This approach is also known as 'unbiased kriging' of order k. Let the regionalized variable $Z(x)$ be a realization of a non-stationary R.F $\boldsymbol{Z}'(x)$. Let $m(x) = E[Z(x)]$. This function $m(x)$ is called the drift. This drift can be represented by a polynomial of the form $\Sigma A_p G^p(x)$, for $p = 1, 2, \ldots k$, where G^ps are monomials. The linear drift in terms of the coordinates x_1 and x_2 may be written as $m(x_1, x_2) = A_0 + A_1 x_1 + A_2 x_2$. In the case of a quadratic drift we write: $m(x_1, x_2) = A_0 + A_1 x_1 + A_2 x_2 + A_3 x_1 x_2 + A_4 x_1^2 + A_5 x_2^2$. The coefficients $A_0, A_1, \ldots A_k$ are unknown and there is no need to estimate them. Suffice it to introduce them into the kriging system with some conditions. Let us recall point estimation where we write $Y_0 = Z(x_0)$. The unbiased estimate Y_0^* of Y_0 implies: $E(Y_0^* - Y_0) = \sum_{i=1}^{N} \lambda_i m(x_i) - \sum m(x_0) = 0$ (9.1)

Since $m(x_i) = \sum_{p=1}^{k} A_p G^p(x_0)$ and substituting this in (9.1) above, we have:

$$\sum_{p=1}^{k} A_p [\sum_{i=1}^{N} \lambda_i G^p(x_i) - G^p(x_0)] = 0 \quad (9.2)$$

This needs identity in A_p and the unbiased conditions are:

$$\sum_{i=1}^{N} \lambda_i G^p(x_i) = G^p(x_0) \quad \text{for } p = 1, 2, \ldots k \quad (9.3)$$

Further derivations are the same as for ordinary kriging in the presence of stationarity.

We see that the weights λ_i are solutions of a linear system which has now k lagrange parameters $\mu_1, \mu_2, \ldots \mu_k$. This linear system has a unique solution, if only the k vectors $G^p(x_i)$, $i = 1, 2, \ldots N$, $p = 1, 2, \ldots k$ are linearly independent. Thus, $\sum_{p=1}^{k} H_p \, G^p(x_i) = 0$ for all $i \Rightarrow H_p = 0 \; \forall \; k$. The kriging system in the case of non-stationarity can now be formulated as follows (see also C-78, 1979):

$$\sum_{j=1}^{N} \lambda_j \, \lambda_j \, \gamma(x_i - x_j) + \sum_{p=1}^{k} \mu_p \, G^p(x_i) = \gamma(x_i - x_0) \text{ for } i = 1, 2, \ldots N \qquad (9.4)$$

$$\sum_{j=1}^{N} \lambda_j \; G^p(x_j) = G^p(x_0) \text{ for } p = 1, 2, \ldots k \qquad (9.5)$$

and the kriging variance is given as:

$$\sigma_K^2 = \text{Var}\,[Z^*(x_0) - Z(x_0)] = \sum_{j=1}^{N} \lambda_j \, \gamma(x_j - x_0) + \sum_{p=1}^{k} \mu_p \, G^p(x_0) \qquad (9.6)$$

9.2.2 Disjunctive Kriging

Disjunctive kriging (DK) refers to a procedure for obtaining certain non-linear estimators which are needed to tackle the usually observed problems of the type of 'disapperaring tonnage' during mining and over-estimation of total ore tonnages. Essentially, the DK procedure deals with deriving a probability distribution of an estimate of grade/accumulation or any other attribute of interest within any size volume rather than relying on a single estimate itself. The DK estimator which is a non-linear estimator is certainly a better one than any of the linear estimators. In reality, most probability distributions are non-linear in their shape.

The estimator based on conditional expectation of several variables is also a non-linear estimator. This estimator also answers in a better way the 'disappearing tonnage' problem during mining, and the over-estimation in total tonnages, when the block model is used for ore reserve estimation (see Chapter 6). However, the derivation of this estimator requires a knowledge of the joint probability distributions in the $(n+1)$ variables which is an impossible task. Therefore, Matheron (1976) and Marechal (1976) suggested disjunctive kriging as an alternative estimator. The procedure for DK estimator is relatively simpler to obtain and it is based on available data only. The DK estimator procedure still requires the knowledge of all the bivariate probability distributions for the $(n+1)$ variables i.e., two variables at a time. This is

naturally much easier to tackle than to arrive at the knowledge of the probability distributions of the (n+1) variables. The first step in this procedure of DK is therefore to estimate these bivariate probability distributions using the available information. The next step is to derive the conditional probability distribution (transfer function) using the estimated bivariate distributions. Disjunctive kriging has wide applications; it was used to estimate the recovery and ash content of washed coal (Armstrong, 1980).

9.2.3 Disjunctive Kriging Estimator

With the usual notation, we state the problem of estimating point grades or block grades from neighbourhood data (say DDH assays) as follows: Let $Z(x_0)$ correspond to the point (block) grade to be estimated and let $Z(x_1)$, $Z(x_2)$, … $Z(x_n)$ correspond to the neighbouring DDH assays/other relevant data. For short, we represent $Z(x_0)$, $Z(x_1)$, $Z(x_2)$ ... $Z(x_n)$ as Z_0, Z_1, Z_2, ..Z_n respectively. We now write:

$$g(Z_1, Z_2, \ldots Z_n) = E[Z_0/Z_1, Z_2, \ldots Z_n) \tag{9.7}$$

The problem is to find a function $g(Z_1, Z_2, \ldots Z_n)$ which is an unbiased minimum variance estimator of Z_0. As we see, this function g in n variables can be obtained as the conditional expectation of Z_0 given the variables Z_1, Z_2, … Z_n. Since we are estimating Z_0, we write: $Z_0^* = g(Z_1, Z_2, \ldots Z_n) = E[Z_0/Z_1, Z_2, \ldots Z_n]$.

This conditional expectation depends on knowing the joint probability function in n + 1 variables, Z_0, Z_1, Z_2 … Z_n. However it is not always possible to know or construct this joint probability density function.We recall that in kriging we make use of the variogram and DDH data/neighbouring assay data. In DK we try to obtain more information using variogram and DDH/the neighbouring assay data, without making any assumptions on the probability distributions. However, transformation of data to standard normal distribution form is followed just for computational ease. While the kriging estimator is obtained as a linear combination of Z_i's, the DK estimator is obtained *as a linear combination of a function*. Thus in kriging, we write: $Z_0^* = \Sigma\, P_i\, Z_i$ and in DK, we write $Z_0^* = \Sigma\, g_i\, (Z_i)$ which is no longer a constant but a function or a sequence of functions. Thus the DK estimator is a non-linear estimator that is more general than a kriging estimator, but retaining two important properties:

1. Each term in the sum depends on only one of the variables Z_1, Z_2, … Z_n. Each variable can be kriged separately. Hence the name '*disjunctive*'.
2. If we insist that the DK estimator should be an unbiased one, then the functions g_i are selected such that the mean square error can be expressed in terms of covariances.

The g_is in the DK estimator Z_0* are chosen in such a way that the mean square error is minimum. Towards this, we need to solve the following system of **n** equations.

$$E[Z_0/Z_1] = \sum_{i=1}^{n} E[g_i (Z_i)/Z_1] \qquad (9.8)$$

$$\ldots\ldots \ldots\ldots \ldots\ldots \ldots\ldots \ldots\ldots$$

$$\ldots\ldots \ldots\ldots \ldots\ldots \ldots\ldots \ldots\ldots$$

$$E[Z_0/Z_n] = \sum_{i=1}^{n} E[g_i (Z_i)/Z_n]$$

The unknown quantities in (9.8) above are g_is which need to be computed. This is possible only when we know the bivariate distributions:

$$G_{0i} (Z_0, Z_1) \qquad \text{for } i = 1, 2, \ldots n$$

$$G_{0i} (Z_i, Z_j) \qquad \text{for } i \neq j,\ i = 1, 2, \ldots n$$

As we see this is a much more weaker assumption than assuming that we know the joint probability distributions for the n+1 variables. Suffice it to know that a knowledge of the bivariate distributions is needed to compute g_is in (9.8) above. This is a very important development in the DK procedure. A detailed discussion on DK precedure may be seen in Matheron (1976), Journel and Huijbregts (1977) and Kim et al. (1977).

9.3 ESTIMATION BASED ON CONDITIONAL SIMULATION

As mentioned earlier, addressing the problem of estimating recoverable reserves is by the method of conditional simulation. The idea is to simulate the grades within a deposit so that the simulated grades have the same values as the observations at sample points and have the same statistical distributions as spatial correlation.

We have discussed some aspects of simulation with examples in Chapter 2. We now discuss conditional simulation as applicable to mine grades. Here simulation is carried out in two stages: (1) Non-conditional simulation of grades so that these have the same histogram and the same variograms. (2) Conditionalisation: At each point two estimates of the grade are obtained by kriging utilising the actual data and the simulated data.

The conditional simulated values $Z_{CS}(x)$ are obtained as the sum of $Z_S(x)$ obtained by using ordinary simulation plus $Z_K(x)$ obtained by kriging the actual values minus Z_{SK} obtained by using simulated values. Conditional simulation can also be used to show the relationship between recovered reserves and those obtained by the chosen method. [See also Armstrong (1981)].

9.4 KRIGING NONSTATIONARY DATA—THE MEDIAN POLISH METHOD

We recall that the problem is one of kriging in the presence of non-stationarity having a trend. Yet another approach used to deal with estimation in such situation is median polish kriging (Cresie, 1986). In the case of intrinsic hypothesis, we have:

$$E(Z_{x+h} - Z_x) = 0 \text{ and } \text{Var}(Z_{x+h} - Z_x) = 2\,\gamma(h);\ x,\ x + h \in D \tag{9.9}$$

We may modify this intrinsic hypothesis so that there is non-stationarity in the mean. Let $E(Z) = d(x)$; $d(x)$ is called the drift. How do we krig in the presence of non-constant drift? There are two ways. One way is that $d(x)$ may be represented as a polynomial of finite order. The second one stipulates that a certain finite-order differences of Z's is weakly stationary. We model taking these differences and later reconvert the output into original units as in time series. The first one called Universal Kriging, has already been discussed in this chapter. Here the order k of the polynomial and the variogram γ of the error need to be known. The second method of kriging known as intrinsic random functions of order k (Matheron, 1973) has a more general model assumption than the first one. However, in practice it reduces to guessing an order k and estimating a generalized covariance function from kth order differences.

We now discuss median polish kriging introduced by Cressie (1996) who adopted the approach of Tukey (1977) and Emerson and Hoaglin (1983) for median polish. Median polish is a quick, easy and resistant alternative to a two-way analysis by means so that the decomposition is preserved.

9.4.1 Median Polish Kriging

We know from time series that an observed surface can be decomposed as:

Observed surface = Large scale variation + Small scale variation

The drift $d(x)$ is thought of as due to large scale variation and the stationary error is due to small scale variation. Most geological problems have a small scale variation which need to be modeled. However, it is very difficult to assign contributions to the different sources. As we know, the aim of kriging is to predict a value Z_0 or Z_B from data Z_{xi}, $i = 1, 2, \ldots n$, exploiting the association between neighbouring observations. We know that small scale variability can be modeled as: data = fit + residual. The residual is analysed as a fresh data set to give: residual = new fit + new residual and so on. Cressie (1984) suggested that the fit f_{ij} at location $x(i, j)$ be obtained by median polish. The fit f_{ij} is expressed as:

$$f_{ij} = a + r_i + c_j \tag{9.10 a}$$

and the residual sum from median polish as:

$$R_{ij} = Z_{ij} - f_{ij} \qquad (9.10\ b)$$

R_{ij} has the property that $\text{med}_i\ \{R_{ij}\} = 0 = \text{med}_j\ (R_{ij})$. The row effects $\{r_i\}$ and column effects $\{c_j\}$ fitted by median polish are such that $\text{med}_i\ \{r_i\} = 0 = \text{med}_j\ \{c_j\}$. To this end, one may have to go in for a number of iterations.

This spatial analysis by rows and columns allows us to estimate the large scale variation. The advantage of the median polish algorithm is that the median component takes care of outliers and relatively bias-free residuals. The removal of trend by median poish is grid-oriented. The above model can be further improved by the addition of one extra quadratic term in the fit. Now the usual geostatistical analysis is carried out on the residuals. First we obtain the appropriate variogram model for the residuals and utilizing the model, the residuals are kriged. The estimated value at point x_i is obtained by adding the estimated large-scale variation (the fit) to an estimated value of the small-scale variation (residuals) obtained by ordinary kriging. The median-based analysis of spatial data reduces the bias. The regularity of the grid means that most of the data configurations used for kriging the unknown Z_0 or Z_B are balanced and remain the same throughout the domain. The final result is a robust and an accurate kriging estimator in the presence of non-stationarity.

In short, median polish kriging (MPK) proceeds as follows: An observation at spatial location x_i is given by: $Z(x_i) = f_x + R_x$, where f_x is the drift directly estimated by the median polish fit given by (9.10a, b) above and R_x as a regionalized variable (estimating the error). If Z_0 is to be predicted then MPK says krige to obtain the predicted value of R_{x0} and add this back to the estimated drift f_{x0} giving the predictor: $\hat{Z}_{x0} = f_{x0} + R_{x0.}$ Cressie (1996) has given an algorithm for median polish.

9.4.2 Example

The average grade values in respect of various blocks of ore in different strata in a gold mine are given hereafter. The unit of measurement is gms/ tonne of ore. The distance between one block of ore and another is 30 m. The results, after each successive step, are detailed for this example.

Note: This example is for grade, although the variograms for residuals are given for both grade and accumulation.

Sample grade values

Sample grade values (Z_{ij}) of $p \times q$ matrix with additional row and additional column.

Block →	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	q + 1col	Row Med
	3.26	4.22	3.86	2.47	7.73	8.17	4.47	1.29	2.38	1.06	1.69	0.86	4.09	1.1	-	0	2.865
	-	5.71	8	9.11	4.67	7.66	3.88	2.13	1.96	1.13	5.47	3.78	2.7	7.18	2.85	0	4.275
	0.47	5.69	10.8	20.36	24.28	8.27	2.74	1.76	0.77	1.07	5.12	4.22	2.89	7.78	-	0	4.67
	-	5.25	9.23	15.88	37.61	23.45	12.2	1.18	1.09	0.59	1.37	7.2	7.53	1.85	0.84	0	6.225
	1.28	2.8	5.79	12.04	27.81	49.55	22.4	6.48	1.4	1.05	1.55	15.82	13	1.29	1.03	0	5.790
p+1 row	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

1st Iteration: Subtract row medians from respective rows and add row median to q+1 column. The resultant matrix is:

Block →	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	q+1 col
	0.395	1.355	0.995	–0.4	4.865	5.305	1.61	–1.58	–0.49	–1.81	–1.18	–2.005	1.23	–1.8	-	2.865
	-	1.435	3.725	4.835	0.395	3.385	–0.4	–2.15	–2.32	–3.15	1.195	–0.495	–1.58	2.91	–1.43	4.275
	–4.20	1.02	6.13	15.69	19.61	3.6	–1.93	–2.91	–3.9	–3.6	0.45	–0.45	–1.78	3.11	-	4.67
	-	–0.975	3.005	9.655	31.39	17.23	5.94	–5.05	–5.14	–5.64	–4.86	0.975	1.31	–4.4	–5.39	6.225
	–4.51	–2.99	0	6.25	22.02	43.76	16.6	0.69	–4.39	–4.74	–4.24	10.03	7.2	–4.5	–4.76	5.790
p + 1 row	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Col. Med	–4.20	1.02	3.005	6.25	19.61	5.305	1.61	–2.15	–3.9	–3.6	–1.18	–0.45	1.23	–1.8	–4.76	4.67

Subtract column medians from respective columns, add column median at p + 1th row and compute row median. The resultant matrix is:

Block →	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	q + 1th col	Row Med
	4.595	0.335	–2.01	–6.65	–14.7	0	0	0.57	3.415	1.795	0	–1.555	0	0	-	2.865	0
	-	0.415	0.72	–1.42	–19.2	–1.92	–2	0	1.585	0.455	2.37	–0.045	–2.8	4.67	3.335	4.275	0.2075
	0	0	3.125	9.44	0	–1.705	–3.54	–0.77	0	0	1.625	0	–3.01	4.88	-	4.67	0
	-	–1.995	0	3.405	11.78	11.92	4.33	–2.9	–1.24	–2.04	–3.68	1.425	0.08	–2.6	–0.63	6.225	–0.3125
	–0.31	–4.01	–3.005	0	2.41	38.46	15	2.835	–0.49	–1.14	–3.07	10.48	5.98	–2.7	0	5.790	0
p + 1 row	–4.20	1.02	3.005	6.25	19.61	5.305	1.61	–2.15	–3.9	–3.6	–1.18	–0.45	1.23	–1.8	–4.76	4.67	–0.45

2nd Iteration: Subtract row medians from respective rows and add row median to q + 1 column. The resultant matrix is:

Block →	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	q + 1 col
	4.60	0.335	–2.01	–6.65	–14.7	0	0	0.57	3.415	1.795	0	–1.555	0	0	-	2.865
	-	0.208	0.512	–1.62	–19.4	–2.128	–2.21	–0.21	1.378	0.248	2.163	–0.253	–3.01	4.46	3.128	4.4825
	0	0	3.125	9.44	0	–1.705	–3.54	–0.77	0	0	1.625	0	–3.01	4.88	-	4.67
	-	–1.683	0.313	3.718	12.09	12.23	4.64	–2.59	–0.92	–1.72	–3.37	1.738	0.39	–2.3	–0.31	5.9125
	–0.31	–4.01	–3.005	0	2.41	38.46	15	2.835	–0.49	–1.14	–3.07	10.48	5.98	–2.7	0	5.79
p+1 row	–4.20	1.02	3.005	6.25	19.61	5.305	1.61	–2.15	–3.9	–3.6	–1.18	–0.45	1.23	–1.8	–4.76	4.67
Col. Med	0	0	0.313	0	0	0	0	–0.21	0	0	0	0	0	0	0	4.67

Subtract column medians from the respective columns and add column medians to p + 1th row. The resultant matrix is:

Block →	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	q + 1 col	Row Med
	4.60	0.33	–2.32	–6.65	–14.75	0.00	0.00	0.78	3.42	1.80	0.00	–1.56	0.00	0.00	-	–1.81	0.00
	-	0.21	0.20	–1.62	–19.42	–2.13	–2.21	0.00	1.38	0.25	2.16	–0.25	–3.01	4.46	3.13	–0.19	0.10
	0.00	0.00	2.81	9.44	0.00	–1.71	–3.54	–0.56	0.00	0.00	1.63	0.00	–3.01	4.88	-	0.00	0.00
	-	–1.68	0.00	3.72	12.09	12.23	4.64	–2.38	–0.92	–1.72	–3.37	1.74	0.39	–2.30	–0.31	1.24	–0.16
	–0.31	–4.01	–3.32	0.00	2.41	38.46	14.99	3.04	–0.49	–1.14	–3.07	10.48	5.98	–2.74	0.00	1.12	0.00
p+1 row	–4.20	1.02	3.32	6.25	19.61	5.31	1.61	–2.35	–3.90	–3.60	–1.18	–0.45	1.23	–1.8	–4.76	9.34	–0.45
Col. Med	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	

Combined effect due to rows and columns. } 9.34

Since the row and column median values are zero or nearly zero, it is taken that median polishing has converged. We denote the residuals as R_{ij}. Now the fitted values are given by $f_{ij} = a + r_i + c_j$ eg: 1st cell value = 9.34–1.81–4.20 = 3.33

Block →	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
	3.33	8.55	10.85	13.78	27.14	12.84	9.14	5.18	3.63	3.93	6.36	7.08	8.76	5.77	-
	-	10.17	12.47	15.40	28.76	14.46	10.76	6.80	5.25	5.55	7.98	8.70	10.38	7.39	4.39
	5.14	10.36	12.66	15.59	28.95	14.65	10.95	6.99	5.44	5.74	8.17	8.89	10.57	7.58	-
	-	11.60	13.90	16.83	30.19	15.89	12.19	8.23	6.68	6.98	9.41	10.13	11.81	8.82	5.82
	6.26	11.48	13.78	16.71	30.07	15.77	12.07	8.11	6.56	6.86	9.29	10.01	11.69	8.70	5.70

The residuals $R_{ij} = Z_{ij} - f_{ij}$ are:

Block →	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
	–0.07	–4.33	–6.99	–11.31	–19.41	–4.67	–4.67	–3.89	–1.25	–2.87	–4.67	–6.22	–4.67	–4.67	-
	-	–4.46	–4.47	–6.29	–24.09	–6.80	–6.88	–4.67	–3.29	–4.42	–2.51	–4.92	–7.68	–0.21	–1.54
	–4.67	–4.67	–1.86	4.77	–4.67	–6.38	–8.21	–5.23	–4.67	–4.67	–3.05	–4.67	–7.68	0.21	-
	-	–6.35	–4.67	–0.95	7.42	7.57	–0.02	–7.05	–5.59	–6.39	–8.04	–2.93	–4.28	–6.97	–4.98
	–4.98	–8.68	–7.99	–4.67	–2.26	33.79	10.32	–1.63	–5.16	–5.81	–7.74	5.81	1.31	–7.41	–4.67

We now construct the variogram for the residuals R_{ij}. The experimental variogam parameters are: $C_o = 8$, $C = 34$, $C_o + C = 42$ and $a = 90$m. The variogram is shown in Fig. 9.1

The kriged residual values are:

Block →	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
	–3.98	–3.95	–5.34	–7.32	–20.96	–10.1	–5.91	–4.11	–3.77	–3.62	–3.68	–4.76	–6.27	–1.9	-
	-	–4.89	–9.39	–9.89	–17.2	–5.68	–5.09	–4.08	–3.12	–3.44	–5.6	–5.16	–1.7	–4.6	–4.06
	–3.00	–2.14	–2.36	–4.98	–9.91	–3.48	–4.47	–5.15	–4.2	–4.99	–4.4	–4.33	–3.32	–2.1	-
	-	–5.66	–4.36	0.43	–2.53	7.53	1.52	–2.73	–5.64	–6.44	–1.69	–3.98	–5.82	–3.2	–2.93
	–6.19	–3.87	–4.63	–0.54	0.78	0.39	6.75	–0.29	–4.9	–4.83	–4.61	–6.39	–2.5	–1	–2.93

The median polish kriged estimates are now obtained as = fitted values $(a + r_i + c_j)$ + kriged residuals R_{ij}

Block →	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
	–0.65	4.60	5.51	6.46	6.18	2.74	3.23	1.07	–0.14	0.31	2.68	2.32	2.49	3.91	-
	-	5.28	3.08	5.51	11.61	8.78	5.67	2.72	2.13	2.11	2.38	3.54	8.68	2.78	0.33
	2.14	8.22	10.30	10.61	19.04	11.17	6.48	1.84	1.24	0.75	3.77	4.56	7.25	5.52	-
	-	5.94	9.54	17.26	27.66	23.42	13.71	5.50	1.04	0.54	7.72	6.15	5.99	5.63	2.89
	0.07	7.61	9.15	16.17	30.85	16.16	18.82	7.82	1.66	2.03	4.68	3.62	9.19	7.67	2.77

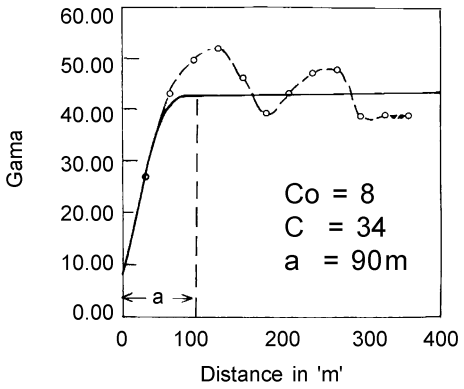


Fig. 9.1 Variogram based on residuals. Variable : Grade.

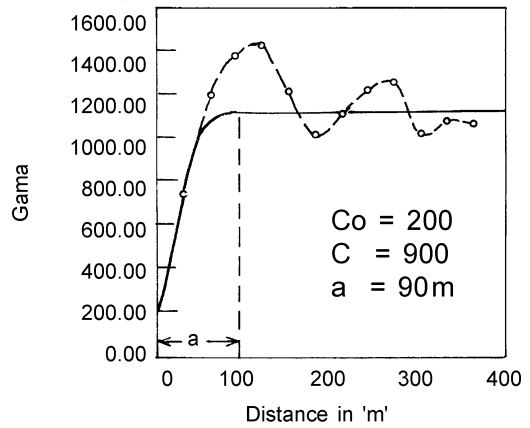


Fig. 9.2 Variogram based on residuals. Variable : Accumulation.

Review Questions

Q. 1. Discuss Universal Kriging and Disjunctive Kriging.

Q. 2. For the grade values given in the example, compute the variogram, obtain the kriged estimates and compare the same with median polish kriged estimates.

10 Computer Software

The following statistical and geostatistical software programs written in FORTRAN can be implemented with ease on a PC. These can be also implemented on other systems with slight modifications.

1. NORMAL.FOR
   This program computes the frequency distribution for specified class-intervals. The program prints the histogram by invoking the subroutine HISTO.FOR.

2. LN.FOR
   This program is a modification of NORMAL.FOR. It exhibits the frequency distribution for the original class intervals, and computes the expected frequencies for each clsss interval taking the logarithms of the specified and generated class intervals, on the basis of normal probability law. Further, it prints the histogram by invoking the sub-routine HISTO.FOR.

3. AR.FOR
   This program takes care of AR modelling upto order 8. It can be modified to take care of higher orders. The AR parameters are computed by invoking Yule-Walker scheme detailed in Chapter 4. The standard error is computed for each order by comparing the original and the estimated values. The appropriate order is chosen based on the criterion of minimum standard error.

4. MA1.FOR
   This program is for Moving Average model of order 1 – MA(1). It first computes the parameter θ_1 and then estimates grade/accumulation, as the case may be, based on this model. The standard error is computed by comparing the original and estimated values.

5. VGRAM.FOR
   This program, besides computing the basic statistical parameters and $\gamma(h)$ values for a set of data, prints the variogram by invoking the subroutine VPRINT.FOR. The angle between one data point and the other and the distance between them are taken into consideration for inclusion or otherwise of the data point in the analysis The details of variogram computations are given in Chapter 5. Provision is made in the program to compute the variogram and other parameters for the entire data or for a segment of the same.

6. ORDKRIG.FOR
   This program is for ordinary kriging using spherical variogram model. It computes the kriged estimates based on the variogram parameters provided. The neighbourhood (number of sample points/blocks) need to be specified. The program can be modified to include kriging based on other variogram models. The block variance is computed using sixteen point approximation.

   The source codes of all these listed software programs are appended and a CD containing these source codes is inserted at the inside back cover of the book.

PROGRAM: NORMAL.FOR

```
C      ************************************************************
C      FITTING A NORMAL DISTRIBUTION TO LOGS OF OBSERVATIONS AND
C      TESTING:NORMAL.FOR
C      FOR PROCESSING ACCUMULATION/THICKNESS/GRADE AS THE CASE MAY BE
C      THIS PROGRAM READS ONE VALUE PER RECORD AND PROCESSES.
C      ************************************************************
       CHARACTER*1  PNEW
       DIMENSION  A(31),AN(104),FR(100),AV(100),FL(100),EL(100)
       DIMENSION  PCR(100),ALEVEL(6)
       DIMENSION  R(2),T(2),P(32),PP(2),Q(52),E(52)
C      ************************************************************
C      DISCRETE VALUES FOR COMPUTING THE NORMAL PROBABILITY
C      ************************************************************
       DATA A/1.,6.0,6.6666,8.4,10.2857,12.2222,14.1818,16.1538,
     1 18.1333,20.1176,22.1052,24.0952,26.0869,28.08,30.0740,32.0689,
     2 34.0645,36.0606,38.0571,40.0540,42.0512,44.0487,46.0465,48.0444,
     3 50.0425,52.0408,54.0392,56.0377,58.0363,60.0350,62.03/
C      ************************************************************
       PNEW=CHAR(12)
       OPEN(UNIT=6,FILE='NORMAL.DAT',STATUS='OLD')
       OPEN(UNIT=2,FILE='NORMAL.RES',STATUS='UNKNOWN')
1111   FORMAT(A)
C      ************************************************************
C      READS CLASS LIMITS AN(1) AND AN(2) FOR ACCUMULATION
C      ************************************************************
5      READ(6,51)(ALEVEL(I),I=1,6)
51     FORMAT(6A4)
       WRITE(2,52)  (ALEVEL(I),I=1,6)
52     FORMAT(5X,6A4/)
       READ(6,111)N1,KEY1,KEY2
111    FORMAT(3I2)
       READ(6,*)AN(1),AN(2),W
       N2=N1*2
C      ************************************************************
C      USUALLY AN(1)=0.0, AN(2)=0.5 MAY SUFFICE  FOR ALL  CASES
C      OF ACCUMULATION AND AN(1)=0.0 AND AN(2)=0.2 FOR ALL CASES
C      OF GRADE/THICKNESS.
C      ************************************************************
200    DO 100 K=3,N2,2
       AN(K)=AN(K-1)
       AN(K+1)=AN(K)+W
100    CONTINUE
C      ************************************************************
C      N2 STANDS FOR THE NUMBER OF CLASS INTERVALS
C      C IS A CONSTANT TO BE ADDED IN CASE THE DISTRIBUTION
C      IS A 3-PARAMETER LOGNORMAL DISTRIBUTION. C CAN BE 0 ALSO.
```

```
C     ************************************************************
      READ(6,696)C
696   FORMAT(F10.0)
      SM1=0.
      SM2=0.
      SM3=0.0
      SM4=0.0
      ESUM=0.0
      CHI=0.0
C     ************************************************************
C     FR(I)  STANDS  FOR  THE  FREQUENCIES
C     ************************************************************
      DO  8  I=1,N1
8     FR(I)=0.
      N=0
1000  READ(6,1001)XC,YC,Z1,Z2,Z3,Z4
1001  FORMAT(6F10.2)
      IF(XC.EQ.99.99)GO  TO  1005
      IF(KEY1.EQ.1)GO  TO  444
      IF(KEY1.EQ.2)GO  TO  445
      IF(KEY1.EQ.3)GO  TO  446
444   Z=Z1
      GO  TO  555
445   Z=Z2
      GO  TO  555
446   Z=Z3
555   K=1
      Z=Z+C
C     ************************************************************
C     IF  THE  FIT  IS  FOR  UNTRANSFORMED  OBSERVATIONS,  THEN  KEY2=0;
C     IF THE FIT IS FOR LOGTRANSFORMED OBSERVATION THEN KEY2=1,2,..
C     ************************************************************
      IF(KEY2.EQ.0.0)  GO  TO  666
      PROD=ALOG(Z)
      GO  TO  667
666   PROD=Z
667   DO  9  I=1,N1
      IF(PROD.GE.AN(K).AND.PROD.LT.AN(K+1))FR(I)=FR(I)+1
9     K=K+2
      N=N+1
      GO  TO  1000
1005  TOTAL=N
C     ************************************************************
C     COMPUTES  THE  MEAN  OF  THE  LOGS  OF  (Z+C)  OBSERVATIONS  AND
C     VARIANCE  THERE  ON.
C     ************************************************************
      WRITE  (2,780)
780   FORMAT(15X,'FITTING  NORMAL  DISTRIBUTION'/)
```

```
      WRITE (2,779)
779   FORMAT(13X,'CLASS INTERVAL',2X, 'MID POINT',2X, 'OBS.FREQ')
      K=1
      DO 14 IP=1,N2,2
      FLIMT=AN(IP)
      ELIMT=AN(IP+1)
      FL(K)=FLIMT
      EL(K)=ELIMT
      AV(K)=(FLIMT+ELIMT)/2.
      WRITE(2,778) AN(IP),AN(IP+1),AV(K),FR(K)
778   FORMAT(10X,F7.2,2X,'TO',F7.2,F10.1,F10.1)
14    K=K+1
      DO 140 I=1,N1
      SM1=SM1+FR(I)*AV(I)
140   SM2=SM2+FR(I)*(AV(I)**2)
      ZBAR=SM1/TOTAL
      V2=SM2/TOTAL-ZBAR**2
      SD2=SQRT(V2)
      DO 141 I=1,N1
      SM3=SM3+FR(I)*(AV(I)-ZBAR)**3
141   SM4=SM4+FR(I)*(AV(I)-ZBAR)**4
      U3=SM3/TOTAL
      U4=SM4/TOTAL
      WRITE(2,1111)PNEW
      IF(KEY1.EQ.1)GO TO 3000
      IF(KEY1.EQ.2)GO TO 3001
      WRITE(2,2002)
2002  FORMAT(///30X,'HISTOGRAM FOR GRADE'///)
      GO TO 4000
3000  WRITE(2,2000)
2000  FORMAT(///30X,'HISTOGRAM FOR ACCUMULATION'///)
      GO TO 4000
3001  WRITE(2,2001)
2001  FORMAT(///30X,'HISTOGRAM FOR THICKNESS'///)
4000  CALL HISTO(N1,FR)
      WRITE(2,1111)PNEW
      WRITE(2,1009)
1009  FORMAT(/15X,'OBSERVED AND EXPECTED FREQUENCIES'/)
      WRITE(2,301)
      DO 107 I=1,N1
      PCR(I)=(FR(I)/TOTAL)*100.
      T(1)=(FL(I)-ZBAR)/SD2
      T(2)=(EL(I)-ZBAR)/SD2
      IF(T(1))280,282,282
280   IF(-T(1)-5.)282,281,281
281   T(1)=-5.
      GO TO 284
282   IF(T(1)-5.)284,283,283
```

```
283    T(1)=5.
284    IF(T(2))285,287,287
285    IF(-T(2)-5.)287,286,286
286    T(2)=-5.
       GO TO 289
287    IF(T(2)-5.)289,288,288
288    T(2)=5.
289    DO 15 L=1,2
       R(L)=T(L)
       IF(R(L))802,803,803
802    R(L)=-R(L)
803    X=R(L)
       P(1)=1.
       DO 804 II=2,31
       III=II-1
804    P(II)=(P(III)/A(II))*X*X
       P(32)=X*.39894228
       SUM=0.0
       D=1.
       DO 805 JJ=1,31
       SUM=SUM+P(JJ)*D
805    D=-D
15     PP(L)=.5+SUM*P(32)
       IF(T(1))20,21,21
20     IF(T(2))22,22,23
22     Q(I)=PP(1)-PP(2)
       GO TO 50
23     Q(I)=PP(1)+PP(2)-1.
       GO TO 50
21     IF(T(2))23,24,24
24     Q(I)=PP(2)-PP(1)
50     E(I)=Q(I)*TOTAL
       ESUM=ESUM+E(I)
       IF(E(I).EQ.0.0)GO TO 1007
1008   CHI=CHI+(FR(I)-E(I))*(FR(I)-E(I))/E(I)
301    FORMAT(3X,'FREQ.',2X,'%AGE',3X,'NORMAL PROB.LIMITS',
     1 6X,'Q(I)',2X,'EXP.FREQ'/)
1007   WRITE(2,998)FR(I),PCR(I),T(1),T(2),Q(I),E(I)
998    FORMAT(F6.0,1X,F8.1,F8.2,'TO',F8.2,3X,F8.3,F8.1)
107    CONTINUE
       TN=TOTAL
       WRITE(2,1111)PNEW
       WRITE(2,300)TN
300    FORMAT(//20X,'TOTAL NO. OF POINTS = 'F5.0/)
       ZAR=ZBAR+V2/2.
       T1=EXP(ZAR)-C
       FACT1=1.+V2/2.+(V2*V2*(TN-1.))/(8.0*(TN+1.))+
     1 ((V2**3)*((TN-1.)**2))/(48.0*(TN+1.)*(TN+3.))
```

```
      FACT2=((V2**4)*((TN-1.)**3))/(384.*(TN+1.)*(TN+3.)*(TN+5.))+
     1 (V2**5)*((TN-1.)**4))/(3840.*(TN+1.)*(TN+3.)*(TN+5.)*(TN+7.))
      T2=EXP(ZBAR)*(FACT1+FACT2)-C
      VE=((T2**2)*(EXP(V2/TN)-1.))
      GM=EXP(ZBAR)
      CS=U3**2/V2**3
      CK=U4/V2**2
      WRITE(2,82)ZBAR
82    FORMAT(5X,'A.M. BASED ON FREQUENCY DISTN.,(ZBAR)..='F8.2/)
      WRITE(2,83)SD2
83    FORMAT(5X,'STD. DEVIATION FROM ZBAR.,.....='F8.2/)
      WRITE(2,92)ESUM
92    FORMAT(5X,'SUM OF THE EXPECTED FREQUENCIES.......='F8.2/)
      WRITE(2,84)CHI
84    FORMAT(5X,'OBSERVED CHI-SQUARE VALUE.,.....='F8.2/)
      WRITE(2,85)T1
85    FORMAT(5X,'KRIGE'S ESTIMATOR. .....='F8.2/)
      WRITE(2,89)T2
89    FORMAT(5X,'SIEHEL'S ESTIMATOR.. .....='F8.2/)
      WRITE(2,86)GM
86    FORMAT(5X,'GEOMETRIC MEAN .....='F8.2/)
      WRITE(2,88)VE
88    FORMAT(5X,'VARIANCE OF T-ESTIMATOR .....='F8.2/)
      WRITE(2,90)CS
90    FORMAT(5X,'COEFFICIENT OF SKEWNESS .....='F8.2/)
      WRITE(2,91)CK
91    FORMAT(5X,'COEFFICIENT OF KURTOSIS .....='F8.2/)
      GO TO 5
      END
C     ************************************************************
C     SUBROUTINE HISTO.FOR FOR HISTOGRAM
C     ************************************************************
      SUBROUTINE HISTO(N1,FR)
      CHARACTER*1 ICHAR1,ICHAR2
      DIMENSION B(104,100),FR(104)
      DATA ICHAR1,ICHAR2/'H',' '/
      DO 4 I=1,N1
      IFR=FR(I)
      DO 3 J=1,100
      IF(J-IFR)1,1,2
1     B(I,J)=ICHAR1
      GO TO 3
2     B(I,J)=ICHAR2
3     CONTINUE
4     CONTINUE
      WRITE(2,5)
5     FORMAT(7X,'0',9X,'10',8X,'20',8X,'30',8X,'40',8X,'50',
     1 8X,'60',8X,'70',8X,'80')
```

```
      WRITE(2,6)
6     FORMAT(1X,80(1H-))
      DO 9 I=1,N1
      WRITE(2,8)(B(I,J),J=1,80)
      WRITE(2,7)FR(I),(B(I,J),J=1,80)
7     FORMAT(1X,F6.1,2X,100A1)
      WRITE(2,8)(B(I,J),J=1,80)
8     FORMAT(9X,100A1)
9     CONTINUE
      RETURN
      END
```

PROGRAM: LN.FOR

```
C     ************************************************************
C     FITTING A LOG-NORMAL DISTRIBUTION AND TESTING:LN.FOR
C     SUBROUTINE REQUIRED AND APPENDED IS HISTO.FOR
C     ************************************************************
      CHARACTER*1 PNEW
      DIMENSION A(31),AN(104),FR(100),AV(100),FL(100),EL(100)
      DIMENSION AAV(100),S1(100),S2(100),S3(100),S4(100),PCR(100)
      DIMENSION R(2),T(2),P(32),PP(2),Q(52),E(52),ALEVEL(6)
C     ************************************************************
C     THE NORMAL PROBABILITIES ARE CALCULATED ON THE BASIS OF
C     DISCRETE APPROXIMATION
C     ************************************************************
      DATA A/1.,6.0,6.6666,8.4,10.2857,12.2222,14.1818,16.1538,
     1 18.1333,20.1176,22.1052,24.0952,26.0869,28.08,30.0740,32.0689,
     2 34.0645,36.0606,38.0571,40.0540,42.0512,44.0487,46.0465,48.0444,
     3 50.0425,52.0408,54.0392,56.0377,58.0363,60.0350,62.03/
C     ************************************************************
      PNEW=CHAR(12)
      OPEN(UNIT=6,FILE='LN.DAT',STATUS='OLD')
      OPEN(UNIT=2,FILE='LN.RES',STATUS='UNKNOWN')
C     ************************************************************
C     GENERATING CLASS LIMITS AN(I) FOR ACCUMULATION
C     ************************************************************
      READ(6,51)(ALEVEL(I),I=1,6)
51    FORMAT(6A4)
      READ(6,*)KEY,N1
      READ(6,697)AN(1),AN(2),W
697   FORMAT(3F10.0)
      N2=N1*2
      DO 100 K=3,N2,2
      AN(K)=AN(K-1)
      AN(K+1)=AN(K)+W
100   CONTINUE
C     ************************************************************
C     N2 STANDS FOR THE NUMBER OF CLASS INTERVALS
```

```
C      C IS A CONSTANT TO BE ADDED IN CASE THE DISTRIBUTION
C      IS A 3-PARAMETER LOGNORMAL DISTRIBUTION. C CAN BE 0 ALSO)
C      ************************************************************
5      READ(6,696)C
696    FORMAT(F10.0)
       SM1=0.0
       SM2=0.0
       SM3=0.0
       SM4=0.0
       ESUM=0.0
       CHI=0.0
C      ************************************************************
C      FR(I) STANDS FOR THE FREQUENCIES
C      ************************************************************
       DO 8 I=1,N1
8      FR(I)=0.
       N=0
       JK=1
1000   READ(6,1001)XC,YC,Z1,Z2,Z3
1001   FORMAT(5F10.0)
       IF(XC.EQ.99.99)GO TO 1005
       IF(KEY.EQ.1)GO TO 444
       IF(KEY.EQ.2)GO TO 445
       IF(KEY.EQ.3)GO TO 446
444    Z=Z1
       GO TO 555
445    Z=Z2
       GO TO 555
446    Z=Z3
555    K=1
       Z=Z+C
666    PROD=Z
C      ************************************************************
C      DISTRIBUTES THE OBSERVATIONS INTO VARIOUS CLASS INTERVALS
C      ************************************************************
       DO 9 I=1,N1
       IF(PROD.GE.AN(K).AND.PROD.LT.AN(K+1))FR(I)=FR(I)+1
9      K=K+2
       N=N+1
       GO TO 1000
1005   TOTAL=N
C      ************************************************************
C      COMPUTES THE MEAN OF THE LOGS OF Z+C OBSERVATIONS AND
C      VARIANCE THERE ON. THIS IS BASED ON FREQUENCY DISTRIBUTION.
C      ************************************************************
       WRITE(2,791)
791    FORMAT(4X,'FITTING LOGNORMAL DISTRIBUTION'/)
       WRITE(2,792)(ALEVEL(I),I=1,6)
```

```
792    FORMAT(4X,6A4/)
       WRITE(2,178)
178    FORMAT(4X,'CLASS LIMITS',4X,'AVG.OF CL',3X,'LOGARITHMIC LMTS',
     1 7X,'AVG',4X,'OBS.FREQ')
       K=1
       DO 14 IP=1,N2,2
       AV(K)=(AN(IP)+AN(IP+1))/2.
       FLOG=ALOG(AN(IP))
       ELOG=ALOG(AN(IP+1))
       FL(K)=FLOG
       EL(K)=ELOG
       AAV(K)=(FLOG+ELOG)/2.
C      ************************************************************
C      FREQUENCY DISTRIBUTION AND OTHER RELEVANT STATISTICS
C      ************************************************************
       WRITE(2,778)AN(IP),AN(IP+1),AV(K),FLOG,ELOG,AAV(K),FR(K)
778    FORMAT(F7.0,'TO',F7.0,F10.1,F10.1,'TO',F10.1,2F10.1)
14     K=K+1
       DO 140 I=1,N1
       S1(I)=FR(I)*AV(I)
       S2(I)=FR(I)*AAV(I)
       S3(I)=FR(I)*AV(I)**2
       S4(I)=FR(I)*AAV(I)**2
       SM1=SM1+S1(I)
       SM2=SM2+S2(I)
       SM3=SM3+S3(I)
140    SM4=SM4+S4(I)
C      ************************************************************
C      XBAR AND VAR1 ARE MEAN AND STANDARD DEVIATION BASED ON
C      UNTRANSFORMED OBSERVATIONS BUT COMPUTED ON FREQUENCY
C      DISTRIBUTION APPROACH.
C      ZBAR AND VAR2 ARE MEAN AND STANDARD DEVIATION BASED
C      TRANSFORMED OBSERVATIONS BUT COMPUTED ON FREQUENCY
C      DISTRIBUTION APPROACH.
C      ************************************************************
       XBAR=SM1/TOTAL-C
       ZBAR=SM2/TOTAL
       VAR1=SM3/TOTAL-XBAR**2
       VAR2=SM4/TOTAL-ZBAR**2
       SD1=SQRT(VAR1)
       SD2=SQRT(VAR2)
C      ************************************************************
C      CALLING SUBROUTINE HISTO.FOR FOR HISTOGRAM
C      ************************************************************
       WRITE(2,1111)PNEW
       IF(KEY.EQ.1)GO TO 3000
       IF(KEY.EQ.3)GO TO 3001
       WRITE(2,2002)
```

```
2002   FORMAT(///25X,'HISTOGRAM FOR GRADE'///)
       GO TO 4000
3000   WRITE(2,2000)
2000   FORMAT(///25X,'HISTOGRAM FOR ACCUMULATION'///)
       GO TO 4000
3001   WRITE(2,2001)
2001   FORMAT(///25X,'HISTOGRAM FOR THICKNESS'///)
4000   CALL HISTO(N1,FR)
C      ************************************************************
C      OBSERVED AND EXPECTED FREQUENCIES AND OTHER STATISTICS
C      ************************************************************
       WRITE (2,1111)PNEW
       WRITE(2,1009)
1009   FORMAT(/15X,'OBSERVED AND EXPECTED FREQUENCIES'/)
       WRITE(2,978)
978    FORMAT(5X,'OBS.FREQ',3X,'%AGE',4X,'PROBABILITY LMTS',5X,'PROB',
     1 3X,'EXP.FREQ')
       DO 107 I=1,N1
       PCR(I)=(FR(I)/TOTAL)*100.
       T(1)=(FL(I)-ZBAR)/SD2
       T(2)=(EL(I)-ZBAR)/SD2
       IF(T(1))280,282,282
280    IF(-T(1)-5.)282,281,281
281    T(1)=-5.
       GO TO 284
282    IF(T(1)-5.)284,283,283
283    T(1)=5.
284    IF(T(2))285,287,287
285    IF(-T(2)-5.)287,286,286
286    T2=-5.
       GO TO 289
287    IF(T(2)-5.)289,288,288
288    T(2)=5.
289    DO 15 L=1,2
       R(L)=T(L)
       IF(R(L))802,803,803
802    R(L)=-R(L)
803    X=R(L)
       P(1)=1.
       DO 804 II=2,31
       III=II-1
804    P(II)=(P(III)/A(II))*X*X
       P(32)=X*.39894228
       SUM=0.0
       D=1.
       DO 805 JJ=1,31
       SUM=SUM+P(JJ)*D
805    D=-D
```

```
15      PP(L)=.5+SUM*P(32)
        IF(T(1))20,21,21
20      IF(T(2))22,22,23
22      Q(I)=PP(1)-PP(2)
        GO TO 50
23      Q(I)=PP(1)+PP(2)-1.
        GO TO 50
21      IF(T(2))23,24,24
24      Q(I)=PP(2)-PP(1)
50      E(I)=Q(I)*TOTAL
        ESUM=ESUM+E(I)
        IF(E(I).EQ.0.0)GO TO 1007
1008    CHI=CHI+(FR(I)-E(I))*(FR(I)-E(I))/E(I)
1007    WRITE(2,998)FR(I),PCR(I),T(1),T(2),Q(I),E(I)
998     FORMAT(5X,F6.0,1X,F8.1,F8.1,' TO ',F8.1,F8.3,2X,F8.1)
107     CONTINUE
C       ************************************************************
C       ESTIMATION OF PARAMETERS
C       ************************************************************
        ZAR=ZBAR+VAR2/2.
        EST=EXP(ZAR)-C
        POP=((EST**2)*(EXP(VAR2)-1.))/TOTAL
        DEM1=EXP(ZBAR-VAR2)
        WRITE(2,79)TOTAL
        WRITE(2,80)XBAR
        WRITE(2,81)SD1
        WRITE(2,82)ZBAR
        WRITE(2,83)SD2
        WRITE(2,84)ESUM
        WRITE(2,85)CHI
        WRITE(2,86)EST
        WRITE(2,88)POP
79      FORMAT(//4X,'TOTAL NUMBER OF OBSERVATIONS.....='F8.2/)
80      FORMAT(4X,'MEAN BASED ON FREQUENCY DISTRIBUTION.....='F8.2/)
81      FORMAT(4X,'STANDARD DEVIATION BASED ON FREQ.DISTRN..='F8.2/)
82      FORMAT(4X,'MEAN BASED ON AVERAGE OF LOGS OF CLS....='F8.2/)
83      FORMAT(4X,'STANDARD DEVIATION BASED ON ABOVE MEAN...='F8.2/)
84      FORMAT(4X,'SUM OF EXPECTED FREQUENCIES.............='F8.2/)
85      FORMAT(4X,'COMPUTED CHI-SQUARE.....................='F8.2/)
86      FORMAT(4X,'KRIGES ESTIMATE.........................='F8.2/)
88      FORMAT(4X,'POPULATION VARIANCE.....................='F8.2/)
1111    FORMAT(A)
        GO TO 5
        END
C       ************************************************************
C       SUBROUTINE HISTO.FOR FOR HISTOGRAM
C       ************************************************************
        SUBROUTINE HISTO(N1,FR)
```

```
      CHARACTER*1  ICHAR1,ICHAR2,  B(104,100)
      DIMENSION  FR(104)
      DATA  ICHAR1,ICHAR2  /'H',' '/
      DO  4  I=1,N1
      IFR=FR(I)
      DO  3  J=1,100
      IF(J-IFR)1,1,2
1     B(I,J)=ICHAR1
      GO  TO  3
2     B(I,J)=ICHAR2
3     CONTINUE
4     CONTINUE
      WRITE(2,5)
5     FORMAT(7X,'0',9X,'10',8X,'20',8X,'30',8X,'40',8X,'50',
     1 8X,'60',8X,'70',8X,'80')
      WRITE(2,6)
6     FORMAT(1X,80(1H-))
      DO  9  I=1,N1
      WRITE(2,8)(B(I,J),J=1,80)
      WRITE(2,7)FR(I),(B(I,J),J=1,80)
7     FORMAT(1X,F6.1,2X,100A1)
      WRITE(2,8)(B(I,J),J=1,80)
8     FORMAT(9X,100A1)
9     CONTINUE
      RETURN
      END
```

PROGRAM: AR.FOR

```
C     ************************************************************
C     ESTIMATION  BY  AUTO-REGRESSIVE  METHOD.
C     THE  AR  COEFFICIENTS  ARE  OBTAINED  BY  SOLVING  THE  YULE-
C     WALKER  SCHEME.  THIS  PROGRAM  WORKS  FOR  ANY  ORDER.
C     NP=LLL-1  GIVES  THE  DESIRED  ORDER.
C     ************************************************************
C     THIS PROGRAM COMPUTES THE AUTO-CORRELATION COEFFICIENTS ALSO
C     ************************************************************
C     SUBROUTINES  REQUIRED  ARE  YWS.FOR  (WHICH  INCLUDES  CRP.FOR
C     CDOT.FOR)  AND  SRL.FOR)
C     ************************************************************
C     XX(I)  ARE  THE  LOG  TRANSFORMED  INPUT  VALUES  WHICH  ARE
C     DETRENDED  SUBSEQUENTLY  BY  REMOVING  THE  MEAN
C     ACV(N+1)  ARE  THE  AUTOCOVARIANCE  COEFFICIENTS
C     FFPE(N+1)  ARE  THE  FINAL  PREDICTION  ERROR
C     COEFFICIENTS  USED  TO  DETERMINE  THE  MAXIMUM
C     LENGTH  OF  THE  OPERATOR  ACCORDING  TO  AKAIKE'S  CRITERION
C     ALPHA(I)=-GG(I)  ARE  THE  PREDICTION  ERROR  COEFFICIENTS
C     ************************************************************
C     READS  ONE  RECORD  PER  CARD  I.E.,  XE,YN,Z1(ACCUMULATION)
```

```
C       Z2(GRADE),Z3(THICKNESS)
C       KEY = 1 MEANS LOG.TRANSFORMATION AND DEVIATIONS
C       TAKEN FROM THE MEAN
C       KEY=0 MEANS NO. LOG. TRANSFORMATION BUT DEVIATIONS
C       TAKEN FROM THE MEAN
C       ------------------------------------------------------------
        DIMENSION WP(50),R(50)
        DIMENSION XX(1200),ACV(1200),FFPE(1200),GG(1200),XXC(1200)
        DIMENSION XM(1200),XXT(1200),ALPHA(10)
        DIMENSION ALEVEL(10)
        COMMON NX,M1
C       ************************************************************
        OPEN(UNIT=6,FILE='AR.DAT',STATUS='OLD')
        OPEN(UNIT=2,FILE='AR.RES',STATUS='UNKNOWN')
C       ************************************************************
        PNEW=CHAR(12)
3       WRITE(2,9999)PNEW
9999    FORMAT(A)
C       ************************************************************
        READ(6,22)(ALEVEL(I),I=1,10)
22      FORMAT(10A4)
        WRITE(2,33)  (ALEVEL(I),I=1,10)
33      FORMAT(1X,10A4)
C       ************************************************************
C       KEY=0 MEANS COMPUTATIONS ON UNTRANSFORMED OBSERVATIONS
C       OTHERWISE ON LOG-TRANSFORMED OBSERVATIONS
C       KKK IS THE UPPER LIMIT OF LLL. EG:IF LLL=5 THEN ORDER OF
C       THE PROCESS IS 4
C       ************************************************************
        READ(6,*)KEY,KKK
        NX=0
        SUM=0.0
C       ************************************************************
C       READS DATA, PERFORMS LOGARITHMIC TRANSFORMATION IF NEEDED
C       AND COMPUTES MEAN
C       ************************************************************
5       READ(6,40)XE,YN,Z1,Z2,Z3
40      FORMAT(5F10.2)
        IF(XE.EQ.99.99)GO TO 66
        NX=NX+1
        XX(NX)=Z1
        XM(NX)=XX(NX)
        IF(KEY.EQ.0)GO TO 223
        XX(NX)=ALOG(XM(NX))
223     SUM=SUM+XX(NX)
        GO TO 5
66      AN=NX
        AM=SUM/AN
```

```
C     ******************************************************
C     DETRENDS DATA BY SUBTRACTING MEAN FROM EACH DATA POINT
C     ******************************************************
      DO 19 I=1,NX
19    XX(I)=XX(I)-AM
C     ******************************************************
C     XX(I) ARE THE DETRENDED VALUES: IF CONTL=1 THEN XX(I)
C     ARE LOG. TRANSFORMED AND DETRENDED VALUES WHILE
C     XM(I) ARE THE ORIGINAL VALUES.  OTHERWISE XM(I) ARE THE
C     ORIGINAL VALUES AND XX(I) ARE THE DETRENDED VALUES.
C     ******************************************************
      IF (KEY.EQ.1) GO TO 776
C     WRITE(2,498)
C498  FORMAT(1X,'PRINTS ACCUMULATION DATA'/)
      WRITE(2,555) (XM(I),I=1,10)
      WRITE(2,772)AM
772   FORMAT(/14X,'MEAN OF DATA='F8.2)
      GO TO 778
776   WRITE (2,497)
497   FORMAT(/1X,'PRINTS  DATA & LOG.TRANS.DEV.TAKEN DATA'/)
      WRITE(2,555)(XM(I),XX(I),I=1,10)
555   FORMAT(4X,10F8.3)
      WRITE (2,777)AM
777   FORMAT(/2X,'MEAN OF TRANSFORMED DATA='F8.2)
C     ******************************************************
C     COMPUTES AUTO-CORRELATION THROUGH SUB-ROUTINE 'SRL.FOR'
C     ******************************************************
778   WRITE(2,41)
41    FORMAT(/2X,'AUTO.CORRELATION COEFFICIENT FOR ZERO LAG'/)
C     ******************************************************
      IF(KEY.EQ.0.0)GO TO 444
C     ******************************************************
      CALL SRL(XX,CPO,RZERO,WP,R)
C     ******************************************************
      GO TO 445
C     ******************************************************
444   CALL SRL (XM,CPO,RZERO,WP,R)
C     ******************************************************
445   WRITE(2,42)CPO,RZERO
42    FORMAT(3X,F10.2,4X,F8.2/)
      WRITE(2,43)
43    FORMAT(2X,'AUTOCORRELATION COEFFICIENTS FOR LAGS 1 TO M')
      IF(KEY.EQ.0.0)GO TO 446
C     ******************************************************
      CALL SRL(XX,CPO,RZERO,WP,R)
C     ******************************************************
      GO TO 447
C     ******************************************************
446   CALL SRL(XM,CPO,RZERO,WP,R)
```

```
C      ******************************************************
447    WRITE(2,44)(L,R(L),L=1,M1)
44     FORMAT(7(1X,I3,1X,F6.2))
       LLL=2
667    LEXT=LLL-1+LLL
       IF(LEXT.GT.NX)LEXT=NX
       NP=LLL-1
       WRITE(2,442)NP
442    FORMAT(/2X,'FOR ORDER OF THE PROCESS='I2)
       WRITE(2,443)
443    FORMAT(2X,27('-')/)
C      ************************************************************
100    CALL YWS(NX,XX,LLL,GG,ACV,LEXT,FFPE,PPM)
C      ************************************************************
       WRITE(2,75)
75     FORMAT(/2X,'PREDICTION COEFFICIENTS: ALPHA(I)=-GG(I)'/
    1  2X,40('-')/)
C      PRINTS PREDICTION ERROR COEFFICIENTS
       ALPHA(1)=1.0
       DO 888 I=2,LLL
888    ALPHA(I)=-GG(I)
       WRITE(2,15)(ALPHA(I),I=1,LLL)
15     FORMAT(8F10.4/)
C      ************************************************************
C      COMPUTES EXPECTED VALUES ON THE BASIS OF AUTO-REG. EQN.,
C      AND BACK TRANSFORMS TO ORIGINAL VALUES.
C      ORDER CAN BE UPTO LLL=N/2+1 OR UPTO ANY OTHER VALUE
C      ************************************************************
       WRITE(2,707)
707    FORMAT(/2X,'OBSERVED AND EXPECTED VALUES:'/2X,29('-')/)
       DO 600 I=1,NX
600    XXC(I)=0.0
C      ************************************************************
       IF(KEY.EQ.0.0)GO TO 999
       DO 86 K=LLL,NX
       DO 85 I=2,LLL
       KM=K-I+1
85     XXC(K)=XXC(K)+ALPHA(I)*XX(KM)
       XXC(K)=XXC(K)+AM
86     XXT(K)=EXP(XXC(K))
C      ************************************************************
C      IF THE ORDER IS P THEN THE ESTIMATION IS FROM THE
C      P+1TH POINT/SAMPLE.   IN THIS PROGRAM IN THE PREDICTED
C      VALUES LIST, THE FIRST P POINTS/SAMPLES ARE THE ORIGINAL
C      VALUES.
C      ************************************************************
       DO 26 IK=1,NP
26     XXT(IK)=XM(IK)
       WRITE(2,500)(XM(I),XXT(I),I=1,10)
```

```
500   FORMAT(3(F10.3,2X,F10.3))
      GO TO 38
C     ************************************************************
999   DO 87 K=LLL,NX
      DO 88 I=2,LLL
      KM=K-I+1
88    XXC(K)=XXC(K)+ALPHA(I)*XM(KM)
87    XXT(K)=XXC(K)+AM
C     ************************************************************
      DO 28 K=1,NP
28    XXT(K)=XM(K)
      WRITE(2,501)(XM(I),XXT(I),I=1,10)
501   FORMAT(3(F10.2,2X,F10.2))
C     ************************************************************
38    S3=0.0
      VE=0.0
      NNX=NX-(LLL-1)
      DO 81 I=LLL,NX
      S3=S3+XXT(I)
81    VE=VE+(XM(I)-XXT(I))**2
      AM3=S3/NNX
      VE=VE/(NNX-1)
      SE=SQRT(VE)
333   WRITE(2,82)AM3,SE
82    FORMAT(/2X,'MEAN OF ESTS.=',F8.1,3X,'STANDARD ERROR='F10.2)
666   LLL=LLL+1
      IF(LLL.LE.KKK)GO TO 667
      GO TO 3
      END
C     ************************************************************
      SUBROUTINE YWS(NPTS,A,LAG,GG,FY,IEXT,FPE,UPVAR)
C     ************************************************************
C     THIS SUBROUTINE WAS ORIGINALLY WRITTEN BY T.J.ULRICH (1975)
C     MODIFIED HERE TO SUIT PC ENVIRONMENT.
C     ------------------------------------------------------------
      DIMENSION GG(NPTS),HH(500),DFY(500)
      DIMENSION A(NPTS),FPE(NPTS),FY(IEXT)
      CALL CRP (NPTS,A,NPTS,A,LAG,FY)
      ANPTS=NPTS
      DO 1 I=1,LAG
      FY(I)=FY(I)/ANPTS
1     DFY(I)=FY(I)
      GG(1)=1.0
      GG(2)=-DFY(2)/DFY(1)
      FPE(1)=((ANPTS+1)/(ANPTS-1))*FY(1)
      TEMP=FPE(1)
      FPE(1)=1.0
      DO 6 KK=2,LAG
```

```
      V=0.0
      D=0.0
      DO  2  L=1,KK
      V=V+GG(L)*DFY(L)
      LZ=KK+1-L
2     D=D+GG(LZ)*DFY(L+1)
      UPVAR=V
      IF(KK.EQ.NPTS)GO  TO  3
      AKK=KK
      FPE(KK)=((ANPTS+(AKK+1))/(ANPTS-(AKK+1)))*V
      FPE(KK)=FPE(KK)/TEMP
3     IF(KK.EQ.LAG)GO  TO  7
      GG(KK+1)=-D/V
      DO  4  I=1,KK
      IZ=KK+2-I
4     HH(I)=GG(I)+GG(KK+1)*GG(IZ)
      DO  5  I=2,KK
5     GG(I)=HH(I)
6     CONTINUE
7     LAG1=LAG+1
      DO  9  J=LAG1,IEXT
      SUM=0
      DO  8  I=2,LAG
      JZ=J+1-I
8     SUM=SUM-DFY(JZ)*GG(I)
      DFY(J)=SUM
9     FY(J)=SUM
      RETURN
      END
C     ************************************************************
      SUBROUTINE  CRP(LX,X,LY,Y,LC,C)
C     ************************************************************
C     CRP  COMPUTES  THE  CROSS  PRODUCTS  C(LC)
C     X(LX)  AND  Y(LY)  ARE  THE  INPUTS
C     ************************************************************
      DIMENSION  X(2),Y(2),C(2)
      DO  10  I=1,LC
      IZ=LY+I-1
      IF(LX.LT.IZ)GO  TO  5
      IY=IZ-I+1
      GO  TO  10
5     IY=LX-I+1
10    CALL  CDOT(IY,X(I),Y,C(I))
      RETURN
      END
C     ************************************************************
      SUBROUTINE  CDOT(L,X,Y,ANS)
C     ************************************************************
C     CDOT  COMPUTES  THE  DOT  PRODUCT
```

```
      DIMENSION X(2),Y(2)
      ANS=0.0
      IF(L)30,30,10
10    DO 20 I=1,L
20    ANS=ANS+X(I)*Y(I)
30    RETURN
      END
C     ************************************************************
C     SUBROUTINE SRL.FOR FOR COMPUTING AUTO CORR. COEFFICIENTS
      (SRL.FOR)
C     ************************************************************
      SUBROUTINE SRL(X,WPO,RZO,W,RR)
      DIMENSION X(500),RR(100),F1(100),T1(100),S1(100)
      DIMENSION C1(100),G1(100),W(100)
      COMMON NN,M
      AN=NN
15    JP=0.0
      F=0.0
      T=0.0
      G=0.0
      S=0.0
      C=0.0
      NJP=NN-JP
C     ************************************************************
C     COMPUTES MEAN LAGGED PRODUCTS AND AUTO.CORRL.COEFF.FOR P=0
C     ************************************************************
      DO 80 I=1,NJP
      IJP=I+JP
      F=F+X(I)
      T=T+X(IJP)
      S=S+X(IJP)**2
      G=G+X(I)**2
80    C=C+X(I)*X(IJP)
      AJP=NJP
      WPO=C/AJP
      FACT1=(AJP*C)-F*T
      A2=(AJP*G)-F*F
      FACT2=SQRT(A2)
      A3=(AJP*S)-T*T
      FACT3=SQRT(A3)
      RZO=FACT1/(FACT2*FACT3)
C     WRITE(2,16)WPO,RZO
C16   FORMAT(4X,F10.2,4X,F10.2)
C     ************************************************************
C     COMPUTES MEAN LAGGED PRODUCTS AND AUTO CORRELATION
C     COEFFICIENTS FOR P = 1 TO M
C     ************************************************************
      KM=NN/2
      IF(KM.GT.25)KM=25
      M=KM
```

```
      DO  90  JP=1,M
      F1(JP)=0.0
      T1(JP)=0.0
      S1(JP)=0.0
      G1(JP)=0.0
      C1(JP)=0.0
      NJP=NN-JP
      DO  111  I=1,NJP
      IJP=I+JP
      F1(JP)=F1(JP)+X(I)
      T1(JP)=T1(JP)+X(IJP)
      S1(JP)=S1(JP)+X(IJP)**2
      C1(JP)=C1(JP)+X(I)*X(IJP)
111   G1(JP)=G1(JP)+X(I)**2
C     WRITE  (2,19)F1(JP),T1(JP),S1(JP),C1(JP),G1(JP)
19    FORMAT(5E15.8)
      AJP=NJP
      W(JP)=C1(JP)/AJP
      FACT4=(AJP*C1(JP)-F1(JP)*T1(JP))
      A5=(AJP*G1(JP)-F1(JP)*F1(JP))
      FACT5=SQRT(A5)
      A6=(AJP*S1(JP)-T1(JP)*T1(JP))
      FACT6=SQRT(A6)
      RR(JP)=FACT4/(FACT5*FACT6)
C     WRITE(2,20)W(JP),RR(JP)
20    FORMAT(E15.8,F12.2)
90    CONTINUE
      RETURN
      END
```

PROGRAM: MA1.FOR

```
C     ************************************************************
C     THIS  MODEL  PROCESSES  DETRENDED(I.E,  DEVIATIONS  FROM  MEAN
C     TAKEN  RAW  DATA  (WHEN  CONTL=0.0)
C     OR  LOG  TRANSFORMED  DEVIATIONS  TAKEN  DATA  (CONTL=1.0).
C     MOVING  AVERAGE  MODEL  (FIRST  ORDER)
C     ************************************************************
C     SUBROUTINE  REQUIRED  AND  APPENDED  HERE  IS  SRL.FOR
C     ************************************************************
      DIMENSION  XX(2500),E(2500),A(2500),R(100),WP(100)
      DIMENSION  XM(2500),ALPHA(10)
      COMMON  N,M1
C     ************************************************************
      OPEN(UNIT=6,FILE='MA1.DAT',STATUS='OLD')
      OPEN(UNIT=2,FILE='MA1.RES',STATUS='UNKNOWN')
      PNEW=CHAR(12)
3     WRITE(2,1111)PNEW
1111  FORMAT(A)
```

```
994     READ(6,77)ALPHA
77      FORMAT(10A4)
        WRITE(2,76)ALPHA
76      FORMAT(/4X,10A4/)
C       ************************************************************
        READ(6,*)CONTL
C       ************************************************************
        N=0
        SUM=0.0
        SUM1=0.0
C       ************************************************************
C       READS DATA,PERFORMS LOG TRANSFORMATION, IF NEEDED
C       AND COMPUTES THE APPROPRIATE MEAN.
C       ************************************************************
5       READ(6,5001)XE,YN,Z1,Z2,Z3
5001    FORMAT(5F10.2)
        IF(XE.EQ.99.99)GO TO 30
        N=N+1
        XX(N)=Z1
        XM(N)=XX(N)
        IF(CONTL.EQ.0.0)GO TO 223
        XX(N)=ALOG(XX(N))
223     SUM=SUM+XX(N)
        GO TO 5
C       ************************************************************
30      AN=N
        AM=SUM/AN
        DO 764 I=1,N
764     SUM1=SUM1+(XX(I)-AM)**2
        VAR=SUM1/(AN-1.0)
C       ************************************************************
C       FROM NOW ON XX(I) ARE THE DEVIATIONS TAKEN DATA
C       ************************************************************
        DO 301 I=1,N
301     XX(I)=XX(I)-AM
C       ************************************************************
C       XX(I) ARE THE DETRENDED VALUES: IF CONTL=1 THEN XX(I)
C       ARE LOG.TRANSFORMED AND DETRENDED VALUES.
C       IF CONTL=0.0 THEN XX(I) ARE THE DETRENDED VALUES.
C       XM(I) ARE THE ORIGINAL VALUES.
C       ************************************************************
        WRITE(2,497)
497     FORMAT(/1X,'FIRST 10 DATA & LOG.TRANS.DEV. TAKEN DATA'/
     1 1X,'OR SIMPLY THE FIRST 10 DATA & DEVIATIONS TAKEN DATA'/)
        WRITE(2,555)(XM(I),XX(I),I=1,10)
555     FORMAT(5(2F7.2))
```

```
      WRITE(2,797)N
797   FORMAT(/14X,'NO. OF DATA POINTS=',I4)
C     ************************************************************
      IF(CONTL.EQ.1.0)GO TO 776
      WRITE(2,772)AM,VAR
772   FORMAT(/5X,'MEAN OF DATA=',F8.2,2X,'VARIANCE=',F10.2)
      GO TO 778
776   WRITE(2,777)AM,VAR
777   FORMAT(/5X,'MEAN OF TRANS.DATA=',F8.2,2X,'VARIANCE=',F6.2)
C     ************************************************************
C     COMPUTES AUTO-CORRELATION THROUGH SUB-ROUTINE 'SRL.FOR'
C     ************************************************************
778   WRITE(2,41)
41    FORMAT(/2X,'MEAN LAGGED PRODUCT  AND R(0)'/)
C     ************************************************************
      CALL SRL(XX,CPO,RZERO,WP,R)
C     ************************************************************
42    FORMAT(3X,F10.2,4X,F8.2/)
      WRITE(2,43)
43    FORMAT(2X,'MEAN LAGGED PRODUCTS AND R(K) FOR LAGS'/)
C     ************************************************************
      CALL SRL(XX,CPO,RZERO,WP,R)
C     ************************************************************
447   WRITE(2,44)(L,R(L),L=1,M1)
44    FORMAT(7(1X,I3,1X,F6.2))
781   D=AM
      B=1.0/R(1)
      TT=(1.0/R(1)**2-4.0)
C     ************************************************************
C     WHEN DISCRIMINANT IS NEGATIVE OR ZERO
C     ************************************************************
      IF(TT)22,22,23
22    WRITE(2,111)
111   FORMAT(1X,'DISCRIMINANT IS <= ZERO - CAN NOT PROCEED')
      GO TO 994
C     ************************************************************
C     WHEN THE DISCRIMINANT IS POSITIVE
C     ************************************************************
23    T11=(-B+SQRT(TT))/2.
      T12=(-B-SQRT(TT))/2.
400   WRITE(2,12)
12    FORMAT(/24X,'DISCRIMINANT')
      WRITE(2,13)TT
13    FORMAT(24X,F8.3)
      WRITE(2,14)
14    FORMAT(/23X,'THETA(1)',7X,'THETA(2)',6X,'MEAN')
      WRITE(2,15)T11,T12,AM
```

```
15    FORMAT(21X,F8.2,7X,F8.2,5X,F8.2/)
C     ************************************************************
C     CHOOSING THE APPROPRIATE VALUE FOR THETA.
C     THETA SHOULD BE < 1.0
C     ************************************************************
      N1=N-1
      IF(ABS(T11).GE.1.0.AND.ABS(T12).GE.1.0) GO TO 333
      IF(ABS(T11).LE.1.0.AND.ABS(T12).LE.1.0) GO TO 455
      GO TO 200
455   CONTINUE
      IF(ABS(T11).LT.ABS(T12))T=ABS(T11)
      IF(ABS(T11).GT.ABS(T12))T=ABS(T12)
      GO TO 456
200   CONTINUE
      IF(ABS(T11).LT.ABS(T12))T=ABS(T11)
      IF(ABS(T11).GT.ABS(T12))T=ABS(T12)
C     ************************************************************
C     COMPUTES EXPECTED VALUES
C     ************************************************************
456   E(1)=XX(1)
      A(1)=XX(1)-E(1)
C     ************************************************************
      DO 56 I=1,N1
      II=I+1
      E(II)=-T*A(I)
      A(II)=XX(II)-E(II)
56    CONTINUE
C     ************************************************************
      IF(CONTL.EQ.0.0)GO TO 79
C     ************************************************************
      E(1)=XM(1)
      DO 58 I=2,N
      F=E(I)+AM
      E(I)=EXP(F)
58    CONTINUE
      GO TO 59
C     ************************************************************
79    E(1)=XM(1)
      DO 80 I=2,N
      F=E(I)+AM
      E(I)=F
80    CONTINUE
C     ************************************************************
59    WRITE(2,28)
28    FORMAT(8X,'XM(I)',14X,'T',9X,'E(I)',11X,'A(I)'/)
C     ************************************************************
C     DO 9 I=1,N
C9    WRITE(2,11)XM(I),T,E(I),A(I)
```

```
11     FORMAT(4X,F10.2,4X,F10.2,4X,F10.2,4X,F10.2)
C      ************************************************************
C      ERROR ANALYSIS:COMPUTES STANDARD DEVIATION OF ERRORS
C      THE FIRST DATA PT. IS NOT TAKEN INTO ACCOUNT AS IT IS
C      EQUATED TO THE OBSERVED ONE.
C      ************************************************************
       S3=0.0
       S4=0.0
       DO 50 I=2,N
       S3=S3+E(I)
       S4=S4+(XM(I)-E(I))**2
50     CONTINUE
       NN=N-1
       AM3=S3/(NN)
       AS4=S4/(NN-1)
       SE=SQRT(AS4)
       CV=(SE/AM3)*100.0
       WRITE(2,84)AM3
84     FORMAT(/2X,'MEAN OF ESTIMATES=',F10.2)
       WRITE(2,51)SE,CV
51     FORMAT(/2X,'STANDARD ERROR=',F8.2,2X,'COEF.OF VARIATION=',F8.2)
       GO TO 994
333    WRITE(2,999)
999    FORMAT(1X,'BOTH THE VALUES OF THETA ARE >1.0')
       GO TO 994
       END
C      ************************************************************
C      SUBROUTINE SRL.FOR FOR COMPUTING AUTO CORRELATION COEFFICIENTS
C      ************************************************************
       SUBROUTINE SRL(X,WPO,RZO,W,RR)
       DIMENSION X(1500),RR(100),F1(100),T1(100),S1(100)
       DIMENSION C1(100),G1(100),W(100)
       COMMON NN,M
       AN=NN
15     JP=0.0
       F=0.0
       T=0.0
       G=0.0
       S=0.0
       C=0.0
       NJP=NN-JP
C      ************************************************************
C      COMPUTES MEAN LAGGED PRODUCTS AND AUTO CORRL. COEFF.
C      ************************************************************
C      WHEN P=0
C      ************************************************************
       DO 80 I=1,NJP
```

```
      IJP=I+JP
      F=F+X(I)
      T=T+X(IJP)
      S=S+X(IJP)**2
      G=G+X(I)**2
80    C=C+X(I)*X(IJP)
      AJP=NJP
      WPO=C/AJP
      FACT1=(AJP*C)-F*T
      A2=(AJP*G)-F*F
      FACT2=SQRT(A2)
      A3=(AJP*S)-T*T
      FACT3=SQRT(A3)
      RZO=FACT1/(FACT2*FACT3)
C     WRITE(2,16)WPO,RZO
C16   FORMAT(4X,F10.2,4X,F10.2)
C     ************************************************************
C     COMPUTES MEAN LAGGED PRODUCTS AND AUTO CORRELATION
C     COEFFICIENTS WHEN P = 1 TO M
C     ************************************************************
      KM=NN/2
      IF(KM.GT.25)KM=25
      M=KM
      DO 90 JP=1,M
      F1(JP)=0.0
      T1(JP)=0.0
      S1(JP)=0.0
      G1(JP)=0.0
      C1(JP)=0.0
      NJP=NN-JP
      DO 111 I=1,NJP
      IJP=I+JP
      F1(JP)=F1(JP)+X(I)
      T1(JP)=T1(JP)+X(IJP)
      S1(JP)=S1(JP)+X(IJP)**2
      C1(JP)=C1(JP)+X(I)*X(IJP)
111   G1(JP)=G1(JP)+X(I)**2
C     ************************************************************
C     PRINTS THE RESULTS OF COMPUTATIONS
C     ************************************************************
C     WRITE (2,19)F1(JP),T1(JP),S1(JP),C1(JP),G1(JP)
19    FORMAT(5E15.8)
      AJP=NJP
      W(JP)=C1(JP)/AJP
      FACT4=(AJP*C1(JP)-F1(JP)*T1(JP))
      A5=(AJP*G1(JP)-F1(JP)*F1(JP))
      FACT5=SQRT(A5)
      A6=(AJP*S1(JP)-T1(JP)*T1(JP))
```

```
      FACT6=SQRT(A6)
      RR(JP)=FACT4/(FACT5*FACT6)
C     ************************************************************
C     PRINTS  COVARIANCES  AND  AUTOCORRELATIONS
C     ************************************************************
C     WRITE(2,20)W(JP),RR(JP)
20    FORMAT(E15.8,F12.2)
90    CONTINUE
      RETURN
      END
```

PROGRAM: VGRAM.FOR

```
C     PROGRAM  FOR  VARIOGRAM  ANALYSIS
C     ************************************************************
      CHARACTER*1  PNEW
      DIMENSION  TOT(20,5),TVAR(20,5),DP(10),Z(3),DATA(1000,4)
      COMMON  VAR(20,5),HEAD(10),LOG,CLASS,DLIM,ANG,SMEAN,VARIANCE
      COMMON  L,SPR,SCALE,TMAX,IRVAG,STD,N,MSET,INDEX
      COMMON  /IO/IOUT,TITLE(10),MAXN
      COMMON  /DAT2/SELN,YMIN,YMAX,XMIN,XMAX
      DATA  MAXN,IOUT/1000,2/
      DATA  TOT,TSUM,TSUM2/102*0.0/
C     ************************************************************
C     SUBROUTINE  INCLUDED  IN  THE  SAME  FILE:VPRINT
C     DP:  DATA  PARTICULARS  FOR  REFERENCE  PURPOSE
C     ************************************************************
      PNEW=CHAR(12)
      OPEN(UNIT=6,FILE='VGRAM.DAT',STATUS='OLD')
      OPEN(UNIT=2,FILE='VGRAM.RES',STATUS='UNKNOWN')
11111 FORMAT(A)
C     ************************************************************
C     DATA  REFERENCE
C     ************************************************************
      READ(6,2221)DP
2221  FORMAT(10A4)
      WRITE(2,2222)DP
2222  FORMAT(23X,10A4)
C     ************************************************************
C     NUMBER  OF  SETS  TO  BE  PROCESSED
C     ************************************************************
      READ(6,222)MSET
222   FORMAT(I3)
C     ************************************************************
C     TITLE  REFERENCE
C     ************************************************************
      READ(6,20)HEAD
```

```
20     FORMAT(10A4)
C      ************************************************************
C      IF LOG=1 MEANS, THE PROGRAM WORKS ON LOGS.OF DATA; OTHERWISE
C      ON  UNTRANSFORMED  DATA
C      AVR=1  MEANS  AVERAGE  VARIOGRAM  IS  ALSO  COMPUTED
C      LL=0  MEANS  GAMMA  VALUES;  1  MEANS  MOMENT  CENTRE
C      SCALE:1.O MEANS AUTOMATIC SCALING; OTHERWISE NO AUTOMATIC SCALING
C      IRVAG:1  RELATIVE  VARIOGRAM  NEEDED;  OTHERWISE  NOT  NEEDED
C      CLASS:  CLASS  INTERVAL  TO  BE  USED  TO  GROUP  DISTANCES
C      ANGLE:ANGLE  IN  WHICH  THE  VARIOGRAM  IS  TO  BE  COMPUTED
C      0=E-W;45=NE-SW;90=N-S;-45=NW-SE
C      SPR:ALLOWABLE  ANGLE  DEVIATION  IN  DEGREES
C      MAX:MAXIMUM  VARIOGRAM  VALUE  ON  Y-AXIS  OF  THE  GRAPH  WHEN
C      AUTOMATIC  SCALING  IS  NOT  USED.  IF  AUTOMATIC  SCALING  IS  OPTED,
C      IGNORE  THIS  BY  GIVING  0  VALUE  TO  MAX.
C      ************************************************************
       READ(6,*)LOG,LL,IRVAG,AVR,SCALE,CLASS,ANG,SPR,TMAX
       INDEX=1
       L=LL+3
       IF(LL.GT.0)L=4
       DLIM=20.0*CLASS
       DANG=ANG*0.017453292519943
       CST=COS(DANG)
       SIT=SIN(DANG)
       DSPR=SPR/2.*0.017453292519943
       CSPR=COS(DSPR)
C      ************************************************************
C      SPECIFY  PARAMETERS  FOR  CHOOSING  THE  DESIRED  AREA  FROM
C      THE  OVERALL  ONE  FOR  VARIOGRAM  COMPUTATIONS
C      YMIN:MINIMUM VALUE OF Y(DESIRED); YMAX:MAXIMUM VALUE OF Y(DESIRED)
C      XMIN:MINIMUM VALUE OF X (DESIRED); XMAX:MAXIMUM VALUE OF X(DESIRED)
C      IF SELN=1.0 (SELECTION OPTION), THE BOUNDARIES OF THE DESIRED AREA
C      ARE TO BE SPECIFIED. OTHERWISE TAKE WHATEVER HAS BEEN SPECIFIED.
C      ************************************************************
       READ(6,*)SELN,YMIN,YMAX,XMIN,XMAX
C      ************************************************************
       ISET=1
C      ************************************************************
C      DATA:XE=X-COORD[DATA(N,1)],YN=Y-COORD[DATA(N,2)]
C      ACCUMULATION/GRADE/THICKNESS REPRESENTED AS Z(1),Z(2), OR Z(3).
C      STORED  UNDER  [DATA(N,3)].
C      ************************************************************
C      IF  KEY=1  MEANS  DATA  IN  PROPER  FORMAT-PROCESS
C      IF  KEY=2  MEANS  NORMAL  END  OF  JOB.  NO  MORE  DATA  SETS.
```

```
C      IF KEY=3 ERRORS IN DATA; ABRUPT END.
C      ************************************************************
75     N=0
C      KEY=1
       IF(INDEX.GT.MSET) GO TO 988
C      ************************************************************
C      READS THE TITLE OF THE SET
C      ************************************************************
       READ(6,10)TITLE
10     FORMAT(10A4)
C      ************************************************************
C      READS X-COORD,Y-COORD AND THE RELEVANT ASSAY VALUE
C      ************************************************************
80     READ(6,787)XE,YN,Z1
787    FORMAT(3F10.0)
C      ************************************************************
C      RELOCATES THE DATA
C      ************************************************************
       Z(1)=Z1
       Z(2)=YN
       Z(3)=XE
C      ************************************************************
       IF(Z1.EQ.99.99)GO TO 885
       IF(SELN.EQ.0.0) GO TO 81
       IF(Z(2).LT.YMIN.OR.Z(2).GT.YMAX) GO TO 80
       IF(Z(3).LT.XMIN.OR.Z(3).GT.XMAX) GO TO 80
81     N=N+1
       IF(N.GT.MAXN) GO TO 83
       DATA(N,1)=Z(1)
       DATA(N,2)=Z(2)
       DATA(N,3)=Z(3)
C      ************************************************************
C      SPARE COLUMN
C      ************************************************************
       DATA(N,4)=0.0
       GO TO 80
885    WRITE(2,333)
333    FORMAT(1X,'DATA IN PROPER ORDER - PROCESS')
       GO TO 988
83     WRITE(2,210)N
210    FORMAT('INPUT DATA ERROR',3X,I5,'DATA POINTS PRESENT')
       KEY=3
988    IF(INDEX.GT.MSET)GO TO 888
       IF(ISET.GT.1)WRITE(2,11111)PNEW
       IF(ISET.GT.1)GO TO 5556
       WRITE(2,223)MSET
223    FORMAT(/30X,'NO.OF DATA SETS = ',I4/)
5556   WRITE(2,777)ISET
C      ******************************************************
```

```
C       PRINTS  A  SAMPLE  INPUT
C       ************************************************************
        IF(ISET.GT.MSET)  GO  TO  888
777     FORMAT(28X,'INPUT  DATA  FOR  SET=',I3/)
        WRITE(2,188)(DATA(I,1),DATA(I,2),DATA(I,3),I=1,6)
188     FORMAT(17X,3F15.2)
888     CONTINUE
C       ************************************************************
C       TO  WORK  ON  LOGARITHMS  OF  DATA  (LOG=1);  OTHERWISE  ON
C       ORIGINAL  DATA.
C       ************************************************************
C       TO  WORK  ON  LOGARITHMS  OF  DATA  (LOG=1);  OTERWISE  ON
C       ORIGINAL  DATA
C       ************************************************************
C       GO  TO  (85,185,285)KEY
85      IF(LOG.NE.1)GO  TO  91
        DO  90  I=1,N
        DATA(I,1)=ALOG(DATA(I,1))
90      CONTINUE
91      NI=N-1
        DH=0.0002
        DH=DH/2.0
        DO  130  I=1,NI
        Q1=DATA(I,1)
        Y1=DATA(I,2)
        X1=DATA(I,3)
        II=I+1
        DO  130  J=II,N
        Q2=DATA(J,1)
        Y2=DATA(J,2)
        X2=DATA(J,3)
        X=X1-X2
        Y=Y1-Y2
        Q=Q1-Q2
        DIST=SQRT(Y*Y+X*X)
        IF(DIST.GT.DLIM.OR.DIST.LT.DH)GO  TO  130
C       ************************************************************
C       COMPARES DIRECTION WITH ACCEPTABLE DIRECTIONS. PROVISIONS OF
C       DISTANCE  TOLERANCE  OF  50  UNITS  IS  MADE.
C       ************************************************************
        DOT=(X*CST+Y*SIT)/DIST
        ADOT=ABS(DOT)
        IF(ADOT.LT.CSPR)GO  TO  130
C       ************************************************************
C       THE PROGRAM TAKES CARE OF THE DISTANCE BETWEEN TWO POINTS.
C       CLASSIFIES  THIS   DISTANCE  CONSIDERING  DIS+/-  A.
C       SPECIFICALLY FOR A SAMPLING INTERVAL OF 100FT, ALL POINTS FALLING
C       BETWEEN  50  -  149  WILL  BE  GROUPED  TOGETHER  AND  SO  ON.
```

```
C       THE  PTS.,  BELOW  49FT  DISTANCE  ARE  NOT  CONSIDERED.
C       ************************************************************
444     K=(DIST/CLASS+0.5)
        IF(K.GT.20)GO  TO  130
C       ************************************************************
C       KEEP TRACK OF DATA POINTS USED IN THE CALCULATION OF THE VARIOGRAM
C       ************************************************************
        DATA(I,4)=1
        DATA(J,4)=1
        Q=SIGN(Q,DOT)
        QSQR=Q*Q
        DISQ=DIST*QSQR
        TOT(K,1)=DIST+TOT(K,1)
        TOT(K,2)=Q+TOT(K,2)
        TOT(K,3)=QSQR+TOT(K,3)
        TOT(K,4)=DISQ+TOT(K,4)
        TOT(K,5)=TOT(K,5)+1.0
130     CONTINUE
C       ************************************************************
C       CALCULATE  MEAN  AND  SD  OF  SAMPLES
C       ************************************************************
        SUM=0.0
        SUM2=0.0
        DO  100  I=1,N
        IF(DATA(I,4).NE.1)GO  TO  100
        SUM=SUM+DATA(I,1)
        SUM2=SUM2+DATA(I,1)**2
100     CONTINUE
        VARIANCE=(FLOAT(N)*SUM2-SUM*SUM)/(FLOAT(N)*FLOAT(N-1))
        SMEAN=SUM/FLOAT(N)
        SMU2=SMEAN*SMEAN
        STD=SQRT(VARIANCE)
        DO  150  I=1,20
        AN=TOT(I,5)
        IF(AN.EQ.0.0)GO  TO  150
        VAR(I,1)=TOT(I,1)/AN
        VAR(I,2)=TOT(I,2)/AN
        IF(IRVAG.EQ.0)GO  TO  148
        TOT(I,3)=TOT(I,3)/SMU2
        TOT(I,4)=TOT(I,4)/SMU2
148     VAR(I,3)=TOT(I,3)/(2.0*AN)
        IF(TOT(I,4).EQ.0.)GO  TO  149
        VAR(I,4)=TOT(I,4)/(TOT(I,1)*2.0)
149     VAR(I,5)=TOT(I,5)
150     CONTINUE
C       ************************************************************
C       PRINTS  RESULTS  OF  VARIOGRAM
C       ************************************************************
        CALL  VPRINT
```

```
C     ************************************************************
C     COMBINE DATA IF AVERAGE VARIOGRAM IS TO BE CALCULATED.
C     ************************************************************
      IF(INDEX.GT.1)GO TO 555
      IF(AVR.EQ.0.0)GO TO 170
      DO 343 I=1,20
      DO 343 J=1,5
      TVAR(I,J)=0.0
343   CONTINUE
555   DO 160 I=1,20
      DO 160 J=1,5
      TVAR(I,J)=TVAR(I,J)+TOT(I,J)
160   CONTINUE
      TSUM=SUM+TSUM
      TSUM2=SUM2+TSUM2
      NN=NN+N
C     ************************************************************
C     INITIALISATION FOR TOTALS AND VARIOGRAM VALUES
C     ************************************************************
170   DO 180 I=1,20
      DO 180 J=1,5
      TOT(I,J)=0.0
      VAR(I,J)=0.0
180   CONTINUE
      INDEX=INDEX+1
      ISET=ISET+1
      GO TO 75
185   IF(AVR.EQ.0.0)GO TO 194
C     ************************************************************
C     CALCULATE AVERAGE VARIOGRAM
C     ************************************************************
      VARIANCE=(FLOAT(NN)*TSUM2-TSUM*TSUM)/(FLOAT(NN)*FLOAT(NN-1))
      SMEAN=TSUM/FLOAT(NN)
      STD=SQRT(VARIANCE)
      N=NN
      DO 190 I=1,20
      AN=TVAR(I,5)
      IF(AN.EQ.0.0)GO TO 190
      VAR(I,1)=TVAR(I,1)/AN
      VAR(I,2)=TVAR(I,2)/AN
      VAR(I,3)=TVAR(I,3)/(AN*2.0)
      IF(TVAR(I,4).EQ.0)GO TO 189
      VAR(I,4)=TVAR(I,4)/(TVAR(I,1)*2.0)
189   VAR(I,5)=AN
190   CONTINUE
C     ************************************************************
C     PRINT OUT RESULTS OF AVERAGE VARIOGRAM
C     ************************************************************
```

```
      CALL VPRINT
194   WRITE(2,195)
195   FORMAT(//20X,'NORMAL END OF JOB')
      GO TO 285
200   WRITE(2,191)
191   FORMAT('END OF FILE ENCOUNTERED.CHECK FOR ERRORS, IF ANY')
285   STOP
      END
C     ************************************************************
      SUBROUTINE VPRINT
C     ************************************************************
C     SUBROUTINE VPRINT PRINTS OUT THE RESULTS OF
C     THE VARIOGRAM CALCULATION
C     ************************************************************
      DIMENSION C(11),D(50),E(13),ILOG(2)
      COMMON VAR(20,5),HEAD(10),LOG,CLASS,DLIM,ANG,SMEAN,VARIANCE
      COMMON L,SPR,SCALE,TMAX,IRVAG,STD,NN,MSET,INDEX
      COMMON/IO/IOUT,TITLE(10),MAXN
      COMMON/DAT2/SELN,YMIN,YMAX,XMIN,XMAX
      CHARACTER A(102)
      DATA ILOG/'NO','YES'/
      DATA D/20*1H ,1HG,1HA,1HM,1HM,1HA,1H ,1H ,1H*,1HH,1H*,20*1H/
      DATA E/1HM,1HO,1HM,1HE,1HN,1HT,1H ,1HC,1HE,1HN,1HT,1HE,1HR/
      IF(L.EQ.3) GO TO 8
      DO 7 I=1,13
      D(I+18)=E(I)
7     CONTINUE
8     C(1)=0
      DO 85 I=2,11
      C(I)=CLASS*2.0+C(I-1)
85    CONTINUE
C     ************************************************************
C     RESULTS OF VARIOGRAM COMPUTATIONS
C     ************************************************************
      IF(INDEX.GT.MSET)GO TO 300
      WRITE(2,656)
656   FORMAT(/60X,'VARIOGRAM'/60X,11(1H-))
      WRITE(2,20)TITLE
20    FORMAT(21X,10A4)
      NL=LOG+1
      NR=IRVAG+1
      WRITE(2,20)HEAD
      GO TO 304
      WRITE(2,333)
333   FORMAT(50X,'AVERAGE VARIOGRAM'/50X,17(1H-))
304   WRITE(2,25)
25    FORMAT(76X,'DATA USED IN COMPUTATIONS')
```

```
      WRITE(2,30)ANG,SPR,SMEAN,CLASS,VARIANCE,DLIM,STD,
     1  ILOG(NL),ILOG(NR),NN
30    FORMAT(' DIRECTION =',F6.0,18X,'WINDOW=',F5.0,25X,'MEAN',
     1 7X,'=',F9.1,/' CLASS SIZE =',1X,F5.0,55X,'VARIANCE =',F9.1
     2 /' MAX DISTANCE =',F7.0,54X,'STD DEVIN  =',F9.1,
     3 /' LOGARITHMS  - ',A5,18X,'RELATIVE VARIOGRAM  -',A5,12X,
     4 'SAMPLE SIZE=',I8)
      IF(SELN.NE.0.0) WRITE(2,110) YMIN,YMAX,XMIN,XMAX
110   FORMAT(' CO-ORDINATE SELECTION- NORTH(',F5.0,2X,'TO',F6.0,')'
     1 ,3X,'EAST(',F5.0,2X,'TO',F6.0,')')
39    WRITE(2,40)
40    FORMAT(13X,'LAG',15X,'NO. OF PAIRS',15X,'DRIFT',
      115X,'GAMMA (H)',10X,'MOMENT CENTER',4X,'AVERAGE DISTANCE')
      DO 50 I=1,20
      LOW=(I-1)*CLASS
      LUP=I*CLASS
      WRITE(2,45)LOW,LUP,VAR(I,5),VAR(I,2),VAR(I,3),VAR(I,4),VAR(I,1)
45    FORMAT(' ',4X,I6,'——',I6,10X,F8.0,5X,3(11X,E10.3),10X, 1F10.1)
50    CONTINUE
      WRITE(2,11111)PNEW
C     ************************************************************
C     PRINTS VARIOGRAM
C     ************************************************************
      IF(INDEX.GT.MSET) GO TO 600
      WRITE(2,320) HEAD
320   FORMAT(/21X,10A4)
      GO TO 700
600   WRITE(2,333)
700   CONTINUE
C     ************************************************************
C     IF AUTOMATIC SCALING IS NOT DESIRED SKIP TO 56
C     ************************************************************
      IF(SCALE.EQ.0.0.AND.TMAX.NE.0) GO TO 56
C     ************************************************************
C     FINDS MAXIMUM VALUE OF VARIOGRAM
C     ************************************************************
      DO 95 I=1,102
      A(I)=' '
95    CONTINUE
      TMAX=VAR(1,L)
      DO 55 I=2,20
      IF(VAR(I,L).GT.TMAX) TMAX=VAR(I,L)
55    CONTINUE
56    UNIT=CLASS/5.
      DIV=TMAX/50.
      TEMP=TMAX
```

```
      DO 75 K=1,50
      TOP=TEMP
      BOT=TEMP-DIV
      DO 65 I=1,20
      IF(VAR(I,5).EQ.0.0) GO TO 65
      IF((VAR(I,L).GE.TOP).OR.VAR(I,L).LE.BOT) GO TO 65
      J=VAR(I,1)/UNIT+1
      A(J)='X'
      IF(VAR(I,5).LT.5) A(J)='*'
65    CONTINUE
C     ARRAY 'A' CONTAINS BLANKS AS  WELL AS DATA POINT MARKERS
      WRITE(2,60) D(K),TEMP,A
60    FORMAT(' ',2X,A1,2X,E10.3,' *',102A1)
      TEMP=BOT
      DO 70 I=1,102
      A(I)=' '
70    CONTINUE
75    CONTINUE
      WRITE(2,80)
80    FORMAT(' ',17X,10('*-----*-----'),'*')
      WRITE(2,90)C
90    FORMAT(' ',13X,11(F6.0,4X))
      RETURN
      END
```

PROGRAM: ORDKRIG.FOR

```
C     PROGRAM FOR PERFORMING ORDINARY (BLOCK) KRIGING AND TO PROCESS
C     TWO VARIABLES BASED ON SPHERICAL VARIOGRAM MODEL.
C     THIS PROGRAM IS EXECUTABLE ON PCS.
C     ************************************************************
C     SUBROUTINES REQUIRED AND INCLUDED HERE ARE KRIG,BLOCK,COVAR,
C     ARRANGE,WEIGHTS,VGAM
C     ************************************************************
C     THIS PROGRAM PROCESSES DATA TAKEN FROM A GRID WHOSE DIMENSIONS
C     ARE IN ONE OF THE FOUR TYPES DEPENDING ON THE CONTROL INFORMATION
C     GIVEN BELOW:
C     ************************************************************
C     SLACHES CAN BE USED TO SKIP DATA. EG. 1 SLASH MEANS ONLY
C     ALTERNATIVE POINTS DATA ARE STORED.
C     THE VALUES READ FOR YMAX AND XMIN ARE THE COORDINATES OF
C     THE TOP LEFT POSITION OF THE AREA UNDER CONSIDERATION.
C     ************************************************************
C     THE IDENTIFICATION OF BLOCKS IN A GRID FORM FOR KRIGING IS
C     HANDLED BY 'CONTROL' TAKING 1,2,3 OR 4 AS THE SITUATION NEEDED.
C     ************************************************************
C     NOTE: THE Y COORDINATE OF THE NORTH-EAST CORNER IS Y-MAX
C     THE X COORDINATE OF THE NORTH-EAST CORNER IS X-MIN
```

```
C     ************************************************************
      COMMON /SYS/IPT,IOUT,IPR,KEY1
      COMMON /BLK/WX2,WY2,WX4,WY4,WX8,WY8
      COMMON /GAM/CO(2),C(2),RANGE(2),CST(2),SIT(2),HAF(2),VAF(2)
      COMMON /KRG/NDATA(51),GA(51),R(51)
      COMMON /DRILL/DPT(3000),YN(3000),XE(3000),ZL(3000),GD(3000,3)
C     ************************************************************
      DIMENSION  B(3),DIST(51),BLKVAR(2),VARKG(2),ANG(2)
      DIMENSION  GR(2),COMMENT(20)
      INTEGER  CONTROL
      DATA  IPT,IOUT,IPR/1,5,3/
C     ************************************************************
      OPEN(UNIT=6,FILE='KRIG.DAT',STATUS='OLD')
      OPEN(UNIT=2,FILE='KRIG.RES',STATUS='UNKNOWN')
C     ************************************************************
C     READS  HEADER  CARD  INFORMATION
C     ************************************************************
      READ(6,111)COMMENT
111   FORMAT(20A4)
      READ(6,6666)CONTROL
6666  FORMAT(I2)
      READ(6,*)NBC,NEC,NBR,NER
      READ(6,*)YMAX,XMIN,WIDY,WIDX,RMAX
      READ(6,*)NK,KEY1,KEY2
C     ************************************************************
C     NK  STANDS  FOR  THE  SPECIFIED  NUMBER  OF  NEIGHBOURHOOD  DATA
C     POINTS/HOLES  USED  FOR  KRIGING.
C     IF  KEY1= 1,  THEN  THE  AUGMENTED  MATRIX  AND  THE  HOLES  USED  FOR
C     KRIGING  EACH  BLOCK  ARE  PRINTED.  IF  KEY2=1  THEN  THE  ENTIRE
C     DRILL  HOLE/  ASSAY  DATA  ARE  PRINTED  PRIOR  TO  KRIGING.
C     ************************************************************
      WRITE(2,15)YMAX,XMIN,NBC,NEC,NBR,NER
15    FORMAT(1X,'DESCRIPTION  OF  AREA  TO  BE  KRIGED'/,5X,
     1 'NORTHING  OF  ROW  1',  T25,F10.0,/5X,'EASTING  OF  COL  1'
     2 ,T25,F10.0,/5X,'KRIGED  AREA  IS  BOUNDED  BY'/
     3 5X,'COLUMNS',T25,I5,'  TO',I5,/,5X,'ROWS',T25,I5,
     4 '  TO',I5)
      WRITE(2,16)WIDY,WIDX
16    FORMAT(1X,'BLOCK   DIMENSIONS  ARE  (',F5.0,'X',F5.0,')')
      WRITE(2,17)RMAX,NK
17    FORMAT(1X,'MAX  RANGE  OF  INFLUENCE  OF  HOLE  ',T45,F6.0,/,
    1 1X,'  NUMBER  OF  HOLES  USED  TO  KRIG  A  BLOCK',T44,I6)
C     ************************************************************
C     READ  IN  THE  5TH  AND  6TH  CARDS  -  FOR  VARIOGRAM  PARAMETERS
C     HAF(I)  AND  VAF(I)  STAND  FOR  HORIZONTAL  AND  VERTICAL  ANISOTROPY
C     FACTORS.HAF:SET  EQUAL  TO  1,IF  OTHERWISE  NOT  KNOWN;  VAF:SET
C     EQUAL  TO  0,  IN  2D  CASES.
C     ANG(I)  STANDS  FOR  DEVIATION  ALLOWED  IN  THE  ANGLE.
```

```
C     ************************************************************
C     READ CO=NUGGET EFFECT; C=SILL, RANGE, HORIZONTAL(HAF) AND
C     VERTICAL(VAF) ANISOTROPIC FACTORS.
C     ************************************************************
      DO 26 I=1,2
      READ(6,*)CO(I),C(I),RANGE(I),ANG(I),HAF(I),VAF(I)
C     ************************************************************
C     PRINTS THE PARAMETERS READ
C     ************************************************************
      WRITE(2,25)CO(I),C(I),RANGE(I),ANG(I),HAF(I),VAF(I)
25    FORMAT(1X,'VARIOGRAM PARAMETERS',/5X,'CO',T25,F10.2,/,
     1 5X,'C',T22,F13.2,/,5X,'RANGEE',T25,F10.2,/,5X,'ANGLE OF ROTATION',
     2 T25,F10.2,/5X,'HOR. ANISOTROPY', T25,F10.2,/,
     3 5X,'VERT. ANISOTROPY', T25,F10.2)
      RA=ANG(I)*0.017453292
      CST(I)=COS(RA)
      SIT(I)=SIN(RA)
26    CONTINUE
C     ************************************************************
      WX2=WIDX/2.0
      WY2=WIDY/2.0
      WX4=WIDX/4.0
      WY4=WIDY/4.0
      WX8=WIDX/8.0
      WY8=WIDY/8.0
C     ************************************************************
      I=0
40    I=I+1
C     ************************************************************
C     READS DATA WITH X,Y COORDINATES. HERE Z1=ACCUMULATION,
C     Z2=GRADE, Z3=THICKNESS. THE ORDER CAN BE CHANGED IF NEEDED.
C     ************************************************************
      READ(6,30)XE(I),YN(I),Z1,Z2,Z3
30    FORMAT(5F10.0)
      EL=0
      GD(I,1)=Z2
      GD(I,2)=Z1
      GD(I,3)=Z3
      ZL(I)=EL
      DPT(I)=I
      IF(XE(I).EQ.0.0) GO TO 50
      GO TO 40
50    NN=I-1
      IF(KEY2.EQ.0)GO TO 57
C     ************************************************************
C     WRITES DATA WITH COORDINATES. REMOVE C BEFORE WRITE STATEMENT
C     ************************************************************
```

```
C       DO  54  I=1,NN
C       WRITE(2,52)DPT(I),XE(I),YN(I),ZL(I),GD(I,1),GD(I,3),GD(I,2)
52      FORMAT(' ',F10.0,5X,3F10.0,3F10.2)
54      CONTINUE
C       ************************************************************
C       CALCULATE  THE  VARIANCE  OF  A   BLOCK  IN  THE  DEPOSIT
C       ************************************************************
57      CALL  BLOCK(BLKVAR(1),1)
        CALL  BLOCK(BLKVAR(2),2)
        WRITE(2,56)BLKVAR
56      FORMAT(/5X,'BLOCK  VARIANCES:',T25,F12.5,T40,F12.5/)
C       ************************************************************
C       IDENTIFICATION  OF  BLOCKS  IN  GRID  ENVIRONMENT.  FOUR  SITUATIONS
C       ARISE.  1)  Y  IS  DECREASING  DOWNWARDS  AND  X  IS  INCREASING
C       FROM  LEFT  TO  RIGHT,  2)  Y  IS  INCREASING  DOWNWARDS  AND  X  IS
C       DECREASING  FROM  LEFT  TO  RIGHT,  3)  Y  IS  DECREASING  DOWNWARDS
C       AND  X  IS  DECREASING  FROM  LEFT  TO  RIGHT  4)  Y  IS  INCREASING
C       DOWNWARDS  AND  X  IS  INCREASING  FROM  LEFT  TO  RIGHT.
C       'CONTROL'  TAKES  CARE  OF  THESE  VARIOUS  SITUATIONS.
C       ************************************************************
        GO  TO  (11,22,33,44)CONTROL
11      YM=YMAX+0.5*WIDY
        XM=XMIN-0.5*WIDX
        GO  TO  55
22      YMIN=YMAX
        XMAX=XMIN
        YM=YMIN-0.5*WIDY
        XM=XMAX+0.5*WIDX
        GO  TO  55
33      XMAX=XMIN
        YM=YMAX+0.5*WIDY
        XM=XMAX+0.5*WIDX
        GO  TO  55
44      YMIN=YMAX
        YM=YMIN-0.5*WIDY
        XM=XMIN-0.5*WIDX
        GO  TO  55
C       ************************************************************
C       NOTE:IF  B(3)  VALUE  IS  NOT  AVAILABLE  THEN  SET  TO  ZERO.  IT  IS  NOW  2D.
C       THE  VARIABLES  USED  HERE  ARE  GRADE  AND  ACCUMULATION  (ACCU).THESE
C       CAN  BE  CHANGED.
C       ************************************************************
55      B(3)=0.0
        WRITE(2,62)
62      FORMAT(1X,20X,'  KRIGED  RESULTS  ')
        WRITE(2,667)
667     FORMAT(4X,'X-CORD',4X,'Y-CORD',6X,'GRADE',5X,'S.E',6X,'ACCU.' 1,6X,'S.E')
```

```
C     ************************************************************
C     ROW IS FIXED COLUMN VARIES
C     ************************************************************
      DO 80 J=NBR,NER
      GO TO(66,77,88,99)CONTROL
66    B(2)=YM-J*WIDY
      GO TO 222
77    B(2)=YM+J*WIDY
      GO TO 222
88    B(2)=YM-J*WIDY
      GO TO 222
99    B(2)=YM+J*WIDY
C     ************************************************************
C     AND DOES FOR VARIOUS COLUMNS
C     ************************************************************
222   DO 80 K= NBC,NEC
      GO TO (666,777,888,999)CONTROL
666   B(1)=XM+K*WIDX
      GO TO 5555
777   B(1)=XM-K*WIDX
      GO TO 5555
888   B(1)=XM-K*WIDX
      GO TO 5555
999   B(1)=XM+K*WIDX
C     ************************************************************
C     FIND NEAREST NK HOLES FOR KRIGING. EXCLUDES THE BLOCK
C     IF THE COORDINATES OF THE BLOCK TO BE ESTIMATED MATCH WITH
C     THE COORDINATES OF THE 'NEAREST' NEIGHBOURHOOD BLOCK/HOLE.
C     ************************************************************
5555  R2=RMAX2
      II=0
      DO 60 L=1,NN
      IF(B(2).EQ.YN(L).AND.B(1).EQ.XE(L))GO TO 60
      DY=B(2)-YN(L)
      DX=B(1)-XE(L)
      DZ=B(3)-ZL(L)
C     ************************************************************
C     COMPUTES THE DISTANCE BETWEEN TWO HOLES
C     ************************************************************
      DIS=(DX*CST(2)+DY*SIT(2))**2+(HAF(2)*(DY*CST(2)-DX*SIT(2)))**2
     1 +(VAF(2)*DZ)**2
      IF(DIS.GT.R2)GO TO 60
      IF(II.EQ.0)GO TO 100
      IF(II.GE.NK.AND.DIS.GT.DIST(II))GO TO 60
100   II=II+1
      DIST(II)=DIS
      NDATA(II)=L
      IF(II.EQ.1)GO TO 60
```

```
C     ************************************************************
C     NDATA STANDS FOR THE NEIGHBOURHOOD DATA.HERE THE PROVISION IS
C     FOR 51 POINTS WHICH IS REASONABLY GOOD. ARRANGE NEIGHBOURHOOD
C     DATA INCREASING ORDER OF DISTANCE BY CALLING SUBROUTINE ARRANGE
C     ************************************************************
      CALL ARRANGE(DIST,NDATA,II)
      IF(II.GT.NK)II=NK
60    CONTINUE
      IF(II.LE.2)GO TO 80
C     ************************************************************
C     PRINTS THE NEIGHBOURHOOD ACTUALLY TAKEN FOR KRIGING DEPENDING
C     THE SPECIFIED LIMIT, NK.
C     ************************************************************
C     WRITE(2,1000)
1000  FORMAT(3X,'NEIGHBOURHOOD DATA ACTUALLY USED FOR KRIGING')
      DO 9999 I=1,II
C     WRITE(2,9998)XE(NDATA(I)),YN(NDATA(I)),GD(NDATA(I),1),GD(NDATA(I),2)
9998  FORMAT(4F10.2)
9999  CONTINUE
C     ************************************************************
C     KRIG THE BLOCK IF THERE ARE MORE THAN TWO DATA POINTS
C     BY INVOKING KRIG.FOR
C     ************************************************************
      DO 76 M=1,2
      CALL KRIG(B,II,M,ERROR)
C     ************************************************************
C     IF AN ERROR OCCURS IN THE MATRIX, THEN:
C     ************************************************************
      IF(ERROR.EQ.0)GO TO 61
      WRITE(2,73)J,K
73    FORMAT('ERROR IN MATRIX SOLUTION BLOCK',2I5)
      DO 900 KN=1,II
      LL=NDATA(KN)
      WRITE(2,85)XE(LL),YN(LL)
85    FORMAT(2F10.0)
900   CONTINUE
      GO TO 80
C     ************************************************************
C     CALCULATE THE KRIGED ESTIMATE AND THE KRIGING VARIANCE
C     ************************************************************
61    AG=0
      GR(M)=0
      IF(KEY1.GE.1) GO TO 707
65    FORMAT(1X,'HOLE',T11,'NORTH',T22,'EAST',T32,'ELEV',T42,'GRADE',
     1 T52,'WEIGHT')
      DO 70 L=1,II
      LL=NDATA(L)
```

```
      AG=AG+GA(L)*R(L)
C     ************************************************************
C     KRIGES FOR THE FIRST AND SECOND VARIABLES
C     ************************************************************
      GR(M)=GR(M)+GD(LL,M)*R(L)
70    CONTINUE
      VARKG(M)=BLKVAR(M)-AG-R(II+1)
      VARKG(M)=SQRT(VARKG(M))
76    CONTINUE
C     ************************************************************
C     NOTE: FOR 2D KRIGING, THE ELEVATION OF ORE IN THE NEAREST
C     HOLE IS ASSIGNED TO THE BLOCK.
C     ************************************************************
      KK=NDATA(1)
      ELEV=ZL(KK)
      WRITE(2,81)B(1),B(2),(GR(I),VARKG(I),I=1,2)
81    FORMAT(2F10.0,4F10.2)
80    CONTINUE
707   STOP
      END
C     ************************************************************
C     SUBROUTINE BLOCK.FOR FOR COMPUTING BLOCK VARIANCE
C     ************************************************************
      SUBROUTINE BLOCK(BLKVAR,M)
C     ************************************************************
C     THIS PROGRAM COMPUTES THE BLOCK VARIANCE IN THE DEPOSIT
C     FOR A GIVEN SIZE BLOCK USING 16 EQUALLY SPACED DATA POINT
C     APPROXIMATION.
C     ************************************************************
      COMMON/BLK/WX2,WY2,WX4,WY4,WX8,WY8
      COMMON/GAM/CO(2),C(2),RANGE(2),CST(2),SIT(2),HAF(2),VAF(2)
      DIMENSION P1(3),P2(3)
      GMOY=0.
      P1(1)=WX2
      P1(2)=WY2
      P1(3)=0.0
      P2(3)=0.0
      DO 10 I=1,4
      P2(1)=WX2+(I-3)*WX4+WX8
      DO 10 J=1,4
      P2(2)=WY2+(J-3)*WY4+WY8
      CALL COVAR(P1,P2,GH,M)
      GMOY=GMOY+GH
10    CONTINUE
      BLKVAR=CO(M)+C(M)-GMOY/16.
      RETURN
      END
```

```
C     ************************************************************
C     SUBROUTINE KRIG.FOR FOR COMPUTING KRIGED ESTIMATES FOR GRIDDED
      BLOCKS
C     ************************************************************
      SUBROUTINE KRIG(B,NS,M,ERROR)
      COMMON /SYS/IPT,IOUT,IPR,KEY1
      COMMON /KRG/NDATA(51),GA(51),R(51)
      COMMON /DRILL/DPT(3000),YN(3000),XE(3000),ZL(3000),GD(3000,3)
      COMMON /GAM/CO(2),C(2),RANGE(2),CST(2),SIT(2),HAF(2),VAF(2)
      DIMENSION P1(3),P2(3),B(3),A(51,52)
      DO 15 K=1,NS
      KK=NDATA(K)
      P1(1)=XE(KK)
      P1(2)=YN(KK)
      P1(3)=ZL(KK)
      K1=K+1
      IF(K1.GT.NS)GO TO 11
      DO 10 L=K1,NS
      LL=NDATA(L)
      P2(1)=XE(LL)
      P2(2)=YN(LL)
      P2(3)=ZL(LL)
      CALL VGAM(P1,P2,GH,M)
      A(K,L)=CO(M)+C(M)-GH
      A(L,K)=CO(M)+C(M)-GH
10    CONTINUE
11    A(K,K)=CO(M)+C(M)
      CALL COVAR(B,P1,GMOY,M)
      NS1=NS+1
      NS2=NS+2
      A(K,NS2)=CO(M)+C(M)-GMOY
      GA(K)=A(K,NS2)
      A(K,NS1)=1.0
      A(NS1,K)=1.0
15    CONTINUE
      A(NS1,NS2)=1.0
      A(NS1,NS1)=0.0
      IF(KEY1.NE.1)GO TO 20
C     WRITE(2,90)(GA(IX),IX=1,NS2)
      WRITE(2,18)
18    FORMAT(1H0,'AUGMENTED MATRIX')
      DO 19 IX=1,NS1
C     WRITE(2,90)(A(IX,IY),IY=1,NS2)
19    CONTINUE
C90   FORMAT(15F7.2)
20    CALL WEIGHTS(A,R,NS1,ERROR)
C     ************************************************************
C     R CONTAINS THE SOLUTION
```

```
C     ************************************************************
      RETURN
      END
C     ************************************************************
C     SUBROUTINE ARRANGE.FOR FOR ARRANGING THE NEIGHBOURHOOD DATA
C     IN ASCENDING ORDER
C     ************************************************************
      SUBROUTINE ARRANGE(DS,NDATA,N)
C     THIS IS A SORTING ROUTING IN INCREASING ORDER
      DIMENSION DS(1),NDATA(1)
      DNEW=DS(N)
      NEW=NDATA(N)
      N1=N-1
      DO 20 I=1,N1
      K=I
      IF(DNEW.LT.DS(I))GO TO 30
20    CONTINUE
      RETURN
30    JK=0
      DO 40 I=K,N1
      J=N1-JK
      JK=JK+1
      DS(J+1)=DS(J)
      NDATA(J+1)=NDATA(J)
40    CONTINUE
      DS(K)=DNEW
      NDATA(K)=NEW
C     WRITE(2,51)(DS(K),K=1,N)
51    FORMAT(10F8.2)
      RETURN
      END
C     ************************************************************
C     SUBROUTINE WEIGHTS.FOR FOR COMPUTING THE LAMDAS AND
C     THE LAGRANGIN PARAMETER 'MU'
C     ************************************************************
      SUBROUTINE WEIGHTS(A,X,N,ERROR)
C     THIS SUBROUTINE COMPUTES LAMDAS AND 'MU'
      DIMENSION A(51,52),X(51)
      ERROR=0
      MP=N+1
      DO 666 I=1,N
C     WRITE(2,555)A(I,I)
555   FORMAT(1X,F10.2)
666   CONTINUE
      DO 10 I=1,N
      IP=I+1
      DO 10 J=1,N
      IF(I.EQ.J)GO TO 10
```

```
      IF(A(I,I).EQ.0)GO TO 30
      F=(-A(J,I))/A(I,I)
      DO 9 K=IP,MP
9     A(J,K)=A(J,K)+F*A(I,K)
10    CONTINUE
      DO 20 I=1,N
      X(I)=A(I,N+1)/A(I,I)
C     WRITE *, X(I)
20    CONTINUE
      ALAM=0.0
      DO 40 I=1,N-1
      ALAM=ALAM+X(I)
40    CONTINUE
      NNN=N-1
      WRITE(2,888)NNN,ALAM
888   FORMAT (1X, 'NEIGHBOURHOOD=',I4,'   SUM OF LAMDAS=',F5.2)
      RETURN
30    ERROR=1
      END
C     ************************************************************
C     SUBROUTINE COVARIANCE.FOR COMPUTES THE COVARIANCE BETWEEN A
C     NEIGHBOURHOOD DATA POINT/ HOLE AND A BLOCK USING 16 POINT
C     APPROXIMATION.
C     ************************************************************
      SUBROUTINE COVAR(P1,P2,GH,M)
C     THE REFERENCE POINT FOR A BLOCK IS ITS CENTRE.
      COMMON /GAM/CO(2),C(2),RANGE(2),CST(2),SIT(2),HAF(2),VAF(2)
      COMMON /BLK/WX2,WY2,WX4,WY4,WX8,WY8
      DIMENSION D(3),P1(3),P2(3)
      GMOY=0
      D(3)=P1(3)
      D(1)=P1(1)-2*WX4-WX8
      DO 10 I=1,4
      D(1)=D(1)+WX4
      D(2)=P1(2)-2*WY4-WY8
      DO 10 J=1,4
      D(2)=D(2)+WY4
      CALL VGAM(D,P2,GH,M)
      GMOY=GMOY+GH
10    CONTINUE
      GH=GMOY/16
      RETURN
      END
C     ************************************************************
C     SUBROUTINE VGAM.FOR FOR COMPUTING THE VARIOGRAM VALUES FOR
C     VARIOUS DISTANCES. M=1 MEANS G-T PRODUCT;M=2 MEANS GRADE;
C     ************************************************************
      SUBROUTINE VGAM(P1,P2,GH,M)
C     THIS ROUTINE COMPUTES THE VARIOGRAM VALUE FOR A GIVEN DISTANCE
```

```
C       USING  THE  SPHERICAL  MODEL.
        COMMON/GAM/CO(2),C(2),RANGE(2),CST(2),SIT(2),HAF(2),VAF(2)
        DIMENSION  P1(3),P2(3)
        XD=P1(1)-P2(1)
        YD=P1(2)-P2(2)
        ZD=P1(3)-P2(3)
        D=SQRT((XD*CST(M)+YD*SIT(M))**2+HAF(M)*HAF(M)*(YD*CST(M)-
     1  XD*SIT(M))**2+VAF(M)*VAF(M)*ZD*ZD)
        IF(D.GT.RANGE(M))GO  TO  20
        GH=CO(M)+C(M)*(1.5*D/RANGE(M)-0.5*(D/RANGE(M))**3)
        IF(D.EQ.0.0)GH=0
        RETURN
20      GH=CO(M)+C(M)
        RETURN
        END
```

Bibliography

Bibliography is divided into two sections:

I. Deals with bibliographical list in the field of mathematical statistics and related fields—specially those relevant to geological sciences (this includes the references cited in the text also); and
II. Deals with a bibliographical list relevant to geological data analysis, and geostatistical methodology as propounded by the French School.

SECTION I

Mathematical Statistical Methods

Aitchison, J. and Brown, J.A.C., 1957. The Lognormal Distribution. Cambridge University Press, London, 176 p.

Bartlett, M.S., 1946. On the Theoretical Specification of Sampling Properties of Autocorrelated Time Series. *Jrl. Royal Stat. Soc.*, **B8**, 27.

Box, G.E.P. and Jenkins, G.M., 1976. Time Series Analysis, Forecasting and Control, Holden-Day, San Francisco, California, 575 p., 2nd Edn.

Cooley, W. and Lohnes, P.R., 1971. Multivariate Data Analysis. Wiley and Sons, New York, 564 p.

Cooley, J. and Tukey, S.W., 1965. An Algorithm for the Machine Calculation of Complex Fourier Series. *Math. Comp.*, **19**, 297-301.

Cramer, H., 1945. Mathematical Methods of Statistics (First published in Sweden, Uppasala by Almqvist & Wiksells). Princeton University Press. Princeton, 579 p.

Cressie, N.A.C., 1996. Statistics for Spatial Data. John Wiley and Sons, New York, 900 p.

Doob, J.L., 1953. Stochastic Processes. John Wiley and Sons, New York, 654 p.

Feller, W., 1957. An Introduction to Probability Theory and its Applications, Vol. I, 2nd Edn. Asia Publishing House, New Delhi, 461 p.

Feller, W., 1966. An Introduction to Probability Theory and its Applications, Vol. II. Wiley Eastern Pvt. Ltd., New Delhi, 626 p.

Hoel, P.G., 1957. Introduction to Mathematical Statistics, 2nd Edn. Asia Publishing House, 331 p.

Jenkins, G.M. and Watts, D.G., 1968. Spectral Analysis and its Applications. Holden-Day, San Francisco, 525 p.

Naidu, P.S., 1995. Modern Spectrum Analysis of Time Series. CRC Press, New York, 399 p.

Papoulis, A., 1984. Probability, Random Variables and Stochastic Processes, 2nd Edition, McGraw Hill, Ind. Editions. 576 p.

Parzen, E., 1962. Stochastic Process. Holden-Day, 324 p.

Quenouille, M.H., 1949. Approximate Tests of Correlation in Time Series. *Jrl. Royal Stat. Soc., B11*, 68.

Rao, C.R., 1973. Linear Statistical Inference and its Applications. Wiley Eastern Private Ltd., New Delhi, 625 p.

Tukey, J.W., 1977. Exploratory Data Analysis, Reading, MA. Addison Wesley.

Yaglom, A.M., 1962. An Introduction to the Theory of Stationary Random Functions. Translated from Russian by R.A. Silverman. Prentice Hall, Englewood Cliffs, NJ (as of 1973, published by Dover, New York), 235 p.

SECTION II

Statistics and Mathematics Applied to Geological Data Analysis, and Geostatistical Methodology

Agterberg, F., 1970. Autocorrelation Functions in Geology. *In:* Geostatistics Colloquim (D.Merriam, ed.). Plenum Press, New York, pp 113-141.

Agterberg, F.P., 1974. Geomathematics - Mathematical Background and Geoscience Applications. Elsevier, Amsterdam, 596 p.

Akaike, H., 1969. Fitting Auto-regressive Models for Prediction. *Ann. Inst., Stat. Math*, **21**, 243-247.

Akaike, H., 1970. Statistical Predictor Identification. *Ann. Inst., Stat. Math.*, **22**, 203-217.

Armstrong, M., 1980. A Geostatistical Approach to Predicting the Washability Characteristics of Coal in Situ. Unpublished note N.687 Centre de Geostatistique, Fontainebleau, Ecole des Mines, Paris.

Armstrong, M., 1981. Basic Geostatistics Applied to Coal. Unpublished report: C-87, Centre De Geostatistique, Fontainebleau.

Armstrong, M., 1984. Common Problems Seen in Variograms. *Math. Geology*, **16(3)**, 305-313.

Armstrong, M. (Ed), 1988. Basic Linear Geostatistics. Springer-Verlag (Berlin), 256 p.

Andersen, N., 1974. On the Calculation of Filter Coefficients for Maximum Entropy Spectral Analysis. *Geophysics*, **39(1)**, 6.

Brooker, P.I., 1975a. Avoiding Unnecessary Drilling. Australasian IMM Proceedings, No.253, pp 21-23.

Brooker, P.I., 1975b. Optimal Block Estimation by Kriging. Australasian IMM Proceedings, No.253, pp 15-19.

Brillouin, L., 1956. Science and Information Theory, Academic New York, pp 159-161.

Burg, J.P., 1967. Maximum Entropy Spectral Analysis. Paper presented at the 37th Annual International Meeting; Soc. of Explo. Geophysics, Oklahoma City, Okla.

Burg, J.P., 1968. A New Analysis Technique for Time Series Data. Paper presented at Advanced Study Institute on Signal Processing; Nato, Ensched, Netherlands.

Clark, Isobel, 1977. Regularization of a Semi Variogram. *Computers and Geostatistics*, **3(2)**, 341-346.

Clark, Isobel, 1979. Practical Geostatistics. Applied Science Publishers Ltd., London, 129 p.

CPT-78, 1979. Unpublished Technical Report: Basic Introduce to Geostatistics. Fontainebleau, France.

Cressie, N.A.C., (1996). Statistics for Spatial Data. Iowa State University, John Wiley & Sons, New York, 900p.

David, M., 1970a. Geostatistical Ore Estimation – A Step by Step Case Study. *In:* Decision Making in the Mineral Industry. Spl. Volume no.12, CIMM. Montreal, pp 185-191.

David, M., 1977. Geostatistical Ore Reserve Estimation. Elsevier Sci. Pub. Co., Amsterdam, 364 p.

Davis, J.C., 1973. Statistics and Data Analysis in Geology. John Wiley and Sons, New York, 550 p.

Dowd, P., 1971. Application of Geostatistics. Internal Report. Zinc Corporation, NBHC Broken Hill, Australia.

Dowd, P., 1976. Application of Dynamic and Stochastic Programming to Optimise Cut-off Grades and Production Rates. *Transactions of the Institution for Mining and Metallurgy*, **85**, A21-A31.

Durbin, J., 1960. The Fitting of Time Series Models. *Rev. Int. Inst. Statist.*, **28**, 233-244.

Edward, J.A. and Fitelson, M.M., 1973. Notes on Maximum Entropy Processing. IEEE. Trans. Inform. Theory, IT-19, 232 p.

Emerson, J.D. and Hoaglin, D.C. 1983. Analysis of Two-way Tables by Medians. *In:* Understanding Robust and Exploratory Data Analysis. John Wiley & Sons., New York, pp. 166-210.

Finney, D.J., 1941. On the Distribution of a Variate whose Logarithm is Normally Distributed. *Suppl. J. Royal Stat. Soc.*, **7(2)**, 151.

Galli, A., Beucher, H., De Fouquet, C., Dong, A., Langlais, V., Leioch, G. and Renord, D., 1987. Applied Geostatistics for the Petroleum Industry. Course Manual, Centre de Geostatistique 404 p.

Hass, A. and Jousselin, C., 1975. Geostatistics in the Petroleum Industry. *In:* 'Geostat 75', pp 333-347.

Hass, A. and Viallix, J.R., 1976. Krigeage Applied to Geophysics. *Geophysical Prospecting*, **24**, 49-69.

Hannan, E.J., 1969. The Estimation of Mixed Moving Average Autoregressive Systems. *Biometrica*, **56(3)**, pp 579-593.

Haykin, S., 1983. Non-linear Methods of Spectral Analysis, Vol. 34. Springer-Verlag, Berlin, 263 p.

Huijbregts, Ch. and Journel, A., 1972. Estimation of Lateritic Type Ore Bodies. *In:* Procs. of the 10th Int. APCOM Symp., Johannesburg, pp 207-212.

Huijbregts, Ch., 1973. Regionalised Variables and Application to Quantitative Analysis of spatial data. *In:* Proceedings of NATO A.S.I. 'Display and Analysis of Spatial Data'. John Wiley and Sons, London and New York, pp 38-53.

Jaynes, E.T., 1963. New Engineering Applications of Information Theory. Intl. Proceedings of the First Symposium on Engineering Applications of Random Function Theory and Probability (eds J.L. Bogdanoff and F. Kozin). John Wiley, New York, pp 163-203.

Jaynes, E.T., 1963. Prior Probabilities. IEEE Trans. Systems Sci. Cybern., SEC-4, pp 227-241.

Journel, A., 1973. Geostatistics in Sequential Exploration. *Mining Engineering*, Oct., 1973 pp 44-48.

Journel, A., 1975. From Geological Reconnaissance to Exploitation—A Decade of Applied Geostatistics. *CIMM Bulletin*, June, 1975.

Journel, A., 1977. Kriging in Terms of Projections, Mathematical Geology. *Journal of the IAMG*, **9**, 563-586.

Journel, A. and Huijbregts, Ch. J., 1978. Mining Geostatistics. Economic Press, 600 p.

Jowett, G.H., 1957. Statistical Analysis Using Local Properties of Smoothly Heteromorphic Stochastic Series. *Biometrika*, **44**, 454-463.

Kanasewich, E.R., 1975. Time Sequence Analysis in Geophysics. The Univ. of Alberta Press, 364 p.

Kim, Y.C., Myers, D.E and Knudsen, H.P., 1977. Advanced geostatistics in ore reserve estimation and mine planning (Practitioner's guide). Unpublished Report—QJBX-65(77) prepared for U.S Dept of Energy, Colorado.

Koch, G.S. and Link, R.F. 1970. Statistical Analysis of Geological Data, Vol. 1. John Wiley and Sons, New York, 375 p.

Koch, G.S. and Link, R.F., 1971. Statistical Analysis of Geological Data, Vol. 2. John Wiley and Sons, New York, 438 p.

Krige, D.G., 1951. A Statistical Approach to Some Basic Mine Valuation Problems on the Witwatersrand. *Jrl. Chem., Met., and Min., Soc. of S.Afn.*, **52,** 119-139.

Krige, D.G., 1961. Developments in the valuation of Gold Mining Properties from Borehole Results. 7th Commonwealth Mining and Met. Congress, preprint.

Krige, D.G., 1966. Two Dimensional Weighted Moving Average Trend Surfaces for Ore Valuation. *In:* Procs. of the Symp., on Mathematics, Statistics and Computer Applications in Ore Valuation. South Afr., Inst. of Min., and Met., Johannesburg.

Krige, D.G., 1970. The Role of Mathematical Statistics in Improved Ore Valuation Techniques in South African Gold Mines. *In:* Topics in Mathematical Geology. Consultants Bureau, New York and London.

Krige, D.G., 1975. A Review of the Development of Geostatistics in South Africa. *In:* 'Geostat 75', pp 279-293.

Krige, D.G., 1976. Some Basic Considerations in the Application of Geostatistics to Gold Ore Valuation. Jrl. of the South Afr. Inst., of Min., and Met., 76, pp 383-391.

Levinson, H., 1947. The Wiener RMS (root mean square) Error Criterion in Filter Design and Prediction, *J.M.Math.Phys*., **25**, 261-278.

Marechal, A., 1967. The practice of transfer functions: Numerical methods and their applications. Procs. of NATO ASI., Rome 1975.

Matheron, G., 1963. Principles of Geostatistics. *Economic Geology*, **58**, 1245-1266.

Matheron, G., 1967. Kriging or Polynomial Interpolation Procedures? CIMM Transactions, 70, pp 240-244.

Matheron G., 1970. Random Functions and Their Applications in Geology. *In:* Geostatistics—A Colloquim (D.Merriam, ed.). Plenum Press, New York, pp 79-87.

Matheron G., 1971. The Theory of Regionalized Variables and Applications. Les Cahiers du Centre de Morphologie Mathematique. Fasc. 5, CG Fontainebleau.

Matheron, G., 1973. The Intrinsic Random Functions and Their Applications, Advances in Applied Probability. 5, pp 439-468.

Matheron, G., 1975. A Simple Substitute for Conditional Expectation. Disjunctive Kriging, in 'Geostat 75', pp 221-236.

Myers, D.E., 1982. Matrix Formulation of Cokriging. *Math Geology*, **14(3)**, 249-257.

Newton, M.J., 1973. The Application of Geostatistics to Mine Sampling Patterns. *In:* Procs. of the 11th Int. APCOM Symp. Univ. of Arizona, Tucson, April 1973. D44-D58.

Rendu, J.M., 1970. Bayesian Decision Theory Applied to Mineral Exploration and Mine Valuation, *In:* 'Geostat 75', pp 435-445.

Rendu, J.M., 1978. An Introduction to Geostatistical Methods of Ore Valuation. South African Int., of Min., & Met., Johannesburg, 82 p.

Royle, A.G., 1975. A Practical Introduction to Geostatistics, University of Leeds, Course Material, Leeds.

Royle, A.G., 1977. How to Use Geostatistics for Ore Reserve Classification. World Mining, Feb., pp 52-56.

Robinson, E., 1983. Iterative Least-Squares Procedure for ARMA Spectral Estimation. *In:* Topics in Applied Physics (Ed. Haykin, S.), Springer Verlag, N.Y.

Sahu, B.K., 2005. Statistical Models in Earth Sciences. B.S. Publications, Hyderabad, 211 p.

Sarma, D.D., 1970. A Statistical Study in Mining Explorations as Related to Kolar Gold Fields (India). *Geoexploration*, **8**, 19-35.

Sarma, D.D., 1979. Statistical Appraisal of Ore Valuation (with applications to Kolar Gold Fields). Andhra University Press & Publications. Waltair, Series No. 157, 167 p.

Sarma, D.D., 1990. Stochastic Modelling of Gold Mineralisation in the Champion Lode System of Kolar Gold Fields (India). *Math. Geology*, **22(3)**, 261-279.

Sarma, D.D., 1995. Stochastic and Geostatistical Modelling of Gold Mineralisation in the Zone-I reef of Hutti Gold Mines, Karnataka. NGRI Tech. Rept. No. NGRI-95-Exp-185.

Sarma, D.D., 1997. Stochastic Characterisation of Copper Mineralisation. A Case Study. *Jrl. of Geophysics*, **XIX(1)**, 53-60.

Sarma, D.D., 1997. Geostatistical Modelling of Gold Mineralisation in the Oakley's reef of Hutti Gold Mines, Karnataka. NGRI Tech. Rept. No. NGRI-97- Exp-206.

Sarma, D.D., 1998a. A Geostatistical Analysis of Gold Mineralisation in Oakley's Reef, Hutti Gold Mines, Karnataka, India. Procs. of the Symp. Applin. of Geophysics to Environmental and Engg. Problems (SAGEEP), held at Chicago, III, pp 257-266.

Sarma, D.D., 1998b. Stochastic and Geostatistical Modelling of Gold Mineralisation in the Strike Reef (HW) of Hutti Gold Mines, Karnataka.

Sarma, D.D., 1999. Stochastic and Geostatistical Modelling of Gold Mineralisation in the Strike Reef (FW) of Hutti Gold Mines, Karnataka. NGRI Tech. Rept. No. NGRI-98-Exp-230(B).

Shannan, C.E., 1948. A Mathematical Theory of Communicaton. *Bell Syst. Tech. J.*, **27**, 379-423.

Sichel, H.S., 1947. An Experimental and Theoretical Investigation of Bias Error in Mine Sampling with Special Reference to Narrow Gold Reefs. Trans. Inst. Mining Met., **56**, 403-474.

Sichel, H.S., 1952. New Methods in the Statistical Evaluation of Mine Sampling Data. Trans. Inst. Min. and Met., March, 261-288.

Smylie, D.E., Clarke, G.K.C. and Ulrych, T.J., 1973. Analysis of Irregularities in the Earth's Rotation. *In:* Methods in Computational Physics, Vol. 13, Academic, New York, pp 391-430.

Sichel, H.S., 1952. New Methods in the Statistical Evaluation of Mine Sampling Data. *In:* Transactions of the Institution for Mining and Metallurgy, pp 261-288.

Sichel, H.S., 1966. The Estimation of Means and Associated Confidence Limits for Small Samples from Lognormal Populations. *In:* Proceedings of the Symposium of the South African Institute of Mining and Metallurgy, Johannesburg, preprint.

Sichel, H.S., 1972. Statistical Valuation of Diamondiferous Deposits. *In:* Procs. of the 10^{th} APCOM Symp, Johannesburg, pp 17-25.

Ulrych, T.J. and Bishop, T.N., 1975. Maximum Entropy Spectral Analysis and Auto-regressive Decomposition. Reviews of Geophysics and Space Physics, pp 183-200.

Ulrych, T.J. and Ooe, M., 1970. Auto-regressive and Mixed Auto-regressive Moving Average Models and Spectra. *In:* Topics in Applied Physics (Ed., Haykin, S.), Springer-Verlag, New York, pp 73-125.

Van den Bos, A., 1971. Alternative Interpretation of Maximum Entropy Spectral Analysis. IEEE Trans. Inform. Theory, IT, 17, pp 493-494.

Walker, G., 1931. On Periodicity in Series of Related Terms, Procs. Roy. Soc., London, Series-A, 131, pp 518-532.

Yule, G.U., 1927. On a Method of Investigating Periodicities in Disturbed Series in Special Reference to Wolfer's Son spot numbers, Phil. Trans. Roy. Soc., London, Series-A, 226, pp 267-298.

Tukey, J.W., 1977. Exploratory Data Analysis, Reading, MA: Addison Wesley.

Index