Kalyan Kumar Roy

Potential Theory in Applied Geophysics

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Remembering

Late Prof. P.K. Bhattacharyya
(1920–1967)
Ex Professor of Geophysics
Department of Geology and Geophysics
Indian Institute of Technology
Kharagpur, India

and

Late Prof. Amalendu Roy
(1924–2005)
Ex Deputy Director
National Geophysical Research Institute
Hyderabad, India

Two Great Teachers of Geophysics

Preface

Two professors of Geophysics Late Prof. Prabhat Kumar Bhattacharyya and Late Prof. Amalendu Roy developed the courses on Potential Theory and Electromagnetic Theory in 1950s for postgraduate students of geophysics in the Department of Geology and Geophysics, Indian Institute of Technology, Kharagpur, India. The courses had gone through several stages of additions and alterations from time to time updating during the next 5 decades. Prof. Bhattacharyya died in 1967 and Prof. Amalendu Roy left this department in the year 1961. These subjects still remained as two of the core subjects in the curriculum of M.Sc level students of geophysics in the same department. Inverse theory joined in these core courses much later in late seventies and early eighties. Teaching potential theory and electromagnetic theory for a period of 9 years in M.Sc and predoctoral level geophysics in the same department enthused me to write a monograph on potential theory bringing all the pedagogical materials under one title "Potential Theory in Applied Geophysics". I hope that the book will cater some needs of the postgraduate students and researchers in geophysics. Since many subjects based on physical sciences have some common areas, the students of Physics, Applied Mathematics. Electrical Engineering, Electrical Communication Engineering, Acoustics, Aerospace Engineering etc may find some of the treatments useful for them in preparation of some background in Potential Theory. Every discipline of science has its own need, style of presentation and coverage. This book also has strong bias in geophysics although it is essentially a monograph on mathematical physics. While teaching these subjects, I felt it a necessity to prepare a new book on this topic to cater the needs of the students. Rapid growth of the subject Potential Theory within geophysics prompted me to prepare one more monograph with a strong geophysics bias. The areal coverages are different with at the most 20 to 30% overlap. Every book has a separate identity. Students should go through all the books because every author had his own plans and programmes for projecting his angle of vision.

This book originated mainly from M.Sc level class room teaching of three courses viz. Field Theory – I (Potential Theory), Field Theory -II (Electromagnetic Theory) and Inverse theory in the Department of Geology and Geophysics, I.I.T., Kharagpur, India. The prime motivation behind writing this book was to prepare a text cum reference book on Field Theory (Scalar and Vector Potentials and Inversion of Potential Fields).This book has more detailed treatments on electrical and electromagnetic potentials. It is slightly biased towards electrical methods. The content of this book is structured as follows:

In Chap. 1 a brief introduction on vector analysis and vector algebra is given keeping the undergraduate and postgraduate students in mind. Because important relations in vector analysis are used in many chapters.

In Chap. 2 I have given some introductory remarks on fields and their classifications, potentials, nature of a medium, i.e. isotropic or anisotropic, one, two and three dimensional problems, Dirichlet, Neumann and mixed boundary conditions, tensors, differential and integral homogenous and inhomogenous equations with homogenous and inhomogenous boundary conditions, and an idea about domain of geophysics where treatments are based on potential theory.

In Chap. 3 I briefly discussed about the nature of gravitational field. Newton's law of gravitation, gravitational fields and potentials for bodies of simpler geometric shapes, gravitational field of the earth and isostasy and guiding equations for any treatment on gravitional potentials.

In Chap. 4 Electrostatics is briefly introduced. It includes Coulomb's law, electrical permittivity and dielectrics, electric displacement, Gauss's law of total normal induction and dipole fields. Boundary conditions in electrostatics and electrostatic energy are also discussed.

In Chap. 5 besides some of the basics of magnetostatic field, the similarities and dissimilarities of the magnetostatic field with other inverse square law fields are highlighted. Both rotational and irrotational nature of the field, vector and scalar potentials and solenoidal nature of the field are discussed. All the important laws in magnetostatics, viz Coulomb's law, Faraday's law, Biot and Savart's law, Ampere's force law and circuital law are discussed briefly. Concept of magnetic dipole and magnetostatic energy are introduced here. The nature of geomagnetic field and different types of magnetic field measurements in geophysics are highlighted.

In Chap. 6 most of the elementary ideas and concepts of direct current flow field are discussed. Equation of continuity, boundary conditions, different electrode configurations, depth of penetration of direct current and nature of the DC dipole fields are touched upon.

In Chap. 7 solution of Laplace equation in cartisian, cylindrical polar and spherical polar coordinates using the method of separation of variables are discussed in great details. Bessel's Function, Legendre's Polynomials, Associated Legendre's Polynomial and Spherical Harmonics are introduced. Nature of a few boundary value problems are demonstrated.

In Chap. 8 advanced level boundary value problems in direct current flow field are given in considerable details. After deriving the potentials in different layers for an N-layered earth the nature of surface and subsurface kernel functions in one dimensional DC resistivity field are shown. Solution of Laplace and nonlaplace equations together, solution of these equations using Frobeneous power series, solution of Laplace equations for a dipping contact and anisotropic medium are given.

In Chap. 9 use of complex variables and conformal transformation in potential theory has been demonstrated.A few simple examples of transformation in a complex domain are shown. Use of Schwarz-Christoffel method of conformal transformation in solving two dimensional potential problem of geophysical interest are discussed in considerable detail. A brief introduction is given on elliptic integrals and elliptic functions.

In Chap. 10 Green's theorem, it's first, second and third identities and corollaries of Green's theorem and Green's equivalent layers are discussed. Connecting relation between Green's theorem and Poisson's equation, estimation of mass from gravity field measurement, total normal induction in gravity field, two dimensional nature of the Green's theorem are given.

In Chap. 11 use of electrical images in solving simpler one dimensional potential problems for different electrotrode configurations are shown along with formation of multiple images.

In Chap. 12 after an elaborate introduction on electromagnetic waves and its application in geophysics, I have discussed about a few basic points on Electromagnetic waves, elliptic polarization, mutual inductance, Maxwell's equations, Helmholtz electromagnetic wave equations, propagation constant, skin depth, perturbation centroid frequency, Poynting vector, boundary conditions in electromagnetics, Hertz and Fitzerald vector potentials and their connections with electric and magnetic fields.

In Chap. 13 I have presented the simplest boundary value problems in electromagnetic wave propagations through homogenous half space. Boundary value problems in electromagnetic wave propagations, Plane wave propagation through layered earth (magnetotellurics), propagation of em waves due to vertical oscillating electric dipole, vertical oscillating magnetic dipole, horizontal oscillating magnetic dipole, an infinitely long line source are discussed showing the nature of solution of boundary value problems using the method of separation of variables. Electromagnetic response in the presence of conducting cylindrical and spherical inhomogeneities in an uniform field are discussed. Principle of electrodynamic similitude has been defined.

In Chap. 14 I have discussed the basic definition of Green's function and some of its properties including it's connection with potentials and fields, Fredhom's integral equations and kernel function and it's use for solution of Poisson.s equation.A few simplest examples for solution of potential problems are demonstrated. Basics of dyadics and dyadic Green;s function are given.

In Chap. 15 I have discussed the entry of numerical methods in potential theory. Finite difference, finite element and integral equation methods are mostly discussed. Finite difference formulation for surface and borehole geophysics in DC resistivity domain and for surface geophysics in plane wave electromagnetics (magnetotellurics) domain are discussed. Finite element formulation for surface geophysics in DC resistivity domain using Rayleigh-Ritz energy minimization method, finite element formulation for surface geophysics in magnetotellurics using Galerkin's method, finite element formulation for surface geophysics in magnetotellurics using advanced level elements, Galerkin's method and isoparametric elements are discussed. Integral equation method for surface geophysics in electromagnetics is mentioned briefly.

In Chap. 16 I have discussed on the different approaches of analytical continuation of potential field based on the class lecture notes and a few research papers of Prof. Amalendu Roy. In this chapter I have discussed the use of (a) harmonic analysis for downward continuation, (b) Taylor's series expansion and finite difference grids for downward continuation, (c) Green's theorem and integral equation in upward and downward continuation, (d) Integral equation and areal averages for downward continuation, (e) Integral equation and Lagrange's interpolation formula for analytical continuation.

In Chap. 17 I have discussed a few points on Inversion of Potential field data. In that I covered the following topics briefly, e.g., (a) singular value decomposition(SVD), (b) least squares estimator,(c) ridge regression estimator, (d) weighted ridge regression estimator, (e) minimum norm algorithm for an underdetermined problem, (f) Bachus Gilbert Inversion, (g) stochastic inversion, (h) Occam's inversion, (i) Global optimization under the following heads, (i) Montecarlo Inversion (ii) simulated annealing, (iii) genetic algorithm, (j) artificial neural network, (k) joint inversion.The topics are discussed briefly.Complete discussion on these subjects demands a separate book writing programme. Many more topics do exist besides whatever have been covered.

This book is dedicated to the name of Late Prof.P.K.Bhattacharyya and Late Prof. Amalendu Roy, our teachers, and both of them were great teachers and scholars in geophysics in India. Prof.Amalendu Roy has seen the first draft of the manuscript. I regret that Prof. Amalendu Roy did not survive to see the book in printed form. I requested him for writing the chapter on "Analytical Continuation of Potential Field Data" He however expressed his inability because of his poor health condition. He expired in December 2005 at the age of 81 years. I am grateful to our teacher late Prof. P. K. Bhattacharya whose inspiring teaching formed the basis of this book. He left a group of student to pursue research in future to push forward his ideas.

Towards completion of this monograph most of my students have lot of contributions in one form or the other. My students at doctoral level Dr. O. P. Rathi, Chief Geophysicist, Coal India Limited, Ranchi, India, Dr. D. J. Dutta, Senior Geophysicist, Schlumberger Well Surveying Corporation, Teheran, Iran, Dr. A. K. Singh, Scientist,Indian Institute of Geomagnetism, Mumbai, India, Dr. C. K. Rao, Scientist, Indian Institute of Geomagnetism, Mumbai, India, Dr. N. S. R. Murthy, Infosys, Bangalore,

India, and masters level Dr. P. S. Routh, Assistant Professor of Geophysics, University of Boise at Idaho, USA, Dr. Anupama Venkata Raman, Geophysicist, Exxon, Houston, Texas, USA, Dr. Anubrati Mukherjee, Schlumberger, Mumbai, India, Dr. Mallika Mullick, Institute of Man and Enviroment, Kolkata, India, Mr. Priyank Jaiswal, Graduate Student, Rice University, Houston Texas, USA, Mr. Souvik Mukherjee, Ex Graduate student, University of Utah,Salt Lake City, USA Mrs. Tanima Dutta, graduate student Stanford University, USA have contributions towards development of this volume. My classmate Dr. K. Mallick of National Geophysical Research Institute, Hyderabad have some contribution in this volume. In the References I have included the works of all the scientists whose contributions have helped me in developing this manuscript. Those works are cited in the text.

The author is grateful to Dr. P. S. Routh, my son in law, for critically going through some of the chapters of this book and making some useful comments. I am grateful to my elder daughter Dr. Baishali Roy, senior geophysicist, Conoco Phillip, Houston, Texas, USA, for computer drafting of many diagrams of this book, collecting some reference materials and purchasing a few books for me, needed to write this monograph. I am grateful to my younger daughter Miss Debanjali Roy, research student, University of Miami, Florida, USA for collecting some literatures for me from the University library. I am grateful to Mr. Priyank Jaiswal, graduate student Rice University, Houston Texas for making arrangement for my visit to Rice University Library. I am grateful to Mr. M. Venkat at Katy, Texas for offering me car ride upto Rice University Library for an extended period. I am grateful to Mr. Subhobroto Sarkar, senior computer engineer, Dell,Salt lake Kolkata for his help in scanning the diagrams. His all round help in providing softwares to computer maintenance is gratefully acknowledged. I am grateful to Ms Lilly Chakraborty and Mr. Sudipta Saha of Printek Point, Technology Market, IIT, Kharagpur for typing the first draft of the manuscript. Second draft of the manuscript was typed jointly by Mr. Dilip Kumar Manna, Technology Cooperative Stores, IIT, Kharagpur and Mr. Rana Roy and his associates at High Tech Point, Jadavpur University Calcutta. I am grateful to Mr. S. P. Hazra, Department of Mining Engineering, Mr. Tapan Sarkar, Department of Geology and Geophysics and Mr. Mukti Ram Bose, Department of Electrical Communication Engineering and Radar Centre, all from IIT, Kharagpur for drafting many diagrams of the book. I am grateful to my wife for her patience and tolerating the troubles she faced for bringing home considerable amount of work and using a part of home as office space.

I am grateful to the Director, IIT, Kharagpur and Dean, Continuing Education Programme, IIT, Kharagpur for financial support regarding preparation of the first draft of the book... The author is grateful to the Vice Chancellor, Jadavpur University for sanctioning an office room in the Department of Geological Sciences such that this type of academic programme can be pursued. I am grateful to Council of Scientific and Industrial Research, New Delhi, India for sanctioning the project titled “Development of a new

magnetotelluric software for detection of lithosphere asthenosphere boundary" (Ref.No.21(0559)/02-EMR-II) to pursue the academic work as an emeritus scientist.

I hope students of physical sciences may find some pages of their interest.

September, 2007

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Contents

1

Elements of Vector Analysis

Since foundation of potential theory in geophysics is based on scalar and vector potentials, a brief introductory note on vector analysis is given. Besides preliminaries of vector algebra, gradient divergence and curl are defined. Gauss's divergence theorem to convert a volume integral to a surface integral and Stoke's theorem to convert a surface integral to a line integral are given. A few well known relations in vector analysis are given as ready references.

1.1 Scalar & Vector

In vector analysis, we deal mostly with scalars and vectors.

Scalars: A quantity that can be identified only by its magnitude and sign is termed as a scalar. As for example distance temperature, mass and displacement are scalars.

Vector: A quantity that has both magnitude, direction and sense is termed as a vector. As for example: Force, field, velocity etc are vectors.

1.2 Properties of Vectors

(i) Sign of a vector. If $\mathrm{A}\vec{\mathrm{B}}$ is vector $\vec{\mathrm{V}}$ then $\mathrm{B}\vec{\mathrm{A}}$ is a vector $-\vec{\mathrm{V}}$

ii) The sum of two vectors (Fig. 1.1)

$$\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{AC}}. \tag{1.1}$$

Here

$$\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{BC}} + \overrightarrow{\mathrm{AB}}. \tag{1.2}$$

iii) The difference of two vectors

$$\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right). \tag{1.3}$$

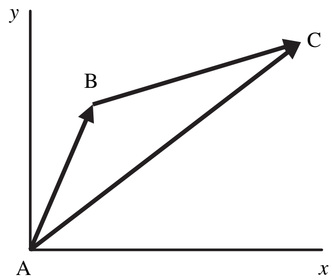


Fig. 1.1. Shows the resultant of two vectors

iv)
$$\vec{A} = b\vec{C} \tag{1.4}$$

i.e., the product of a vector and a scalar is a vector.

v) Unit Vector:
A unit vector is defined as a vector of unit magnitude along the three mutually perpendicular directions $\vec{i}, \vec{j}, \vec{k}$. Components of a vector along the x, y, z directions in a Cartesian coordinate are

$$\vec{A} = \vec{i}A_x + \vec{j}A_y + \vec{k}A_z. \tag{1.5}$$

vi) Vector Components: Three scalars A_x, A_y, and A_z are the three components in a cartisian coordinate system (Fig. 1.2). The magnitude of the vector A is $|A| = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$.
When A makes specific angles α, β and γ with the three mutually perpendicular directions x, y and z, cosines of these angles are respectively given by (Fig. 1.3)

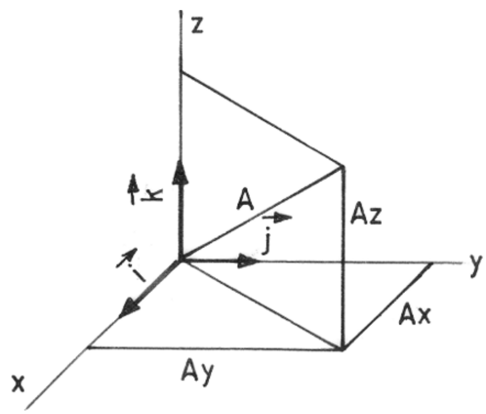


Fig. 1.2. Shows the three components of a vector in a Cartesian coordinate system

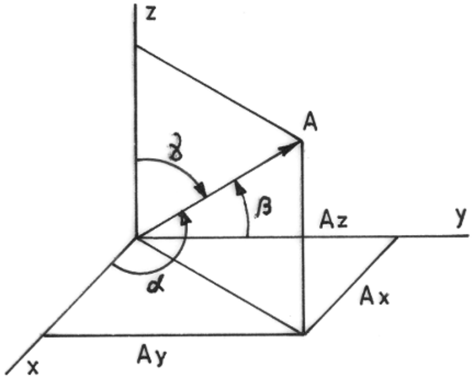


Fig. 1.3. Shows the direction cosines of a vector

$$\cos\alpha = \frac{Ax}{A}, \cos\beta = \frac{A_y}{A} \; and \; \cos\gamma = \frac{A_z}{A}. \tag{1.6}$$

In general cos α, cos β, cos γ are denoted as lx, ly and lz and they are known as direction cosines.

vii) Scalar product or dot product: The scalar product of two vectors is a scalar and is given by (Fig. 1.4)

$$\mathrm{A.B} = \mathrm{AB}\cos\theta \tag{1.7}$$

i.e. the product of two vectors multiplied by cosine of the angles between the two vectors. Some of the properties of dot product are

$$\begin{aligned} &\text{a)} \quad \mathrm{A.B} = \mathrm{B.A}, \\ &\text{b)} \quad \vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0 \text{ and} \\ &\text{c)} \quad \vec{\mathrm{i}}.\vec{\mathrm{i}} = \vec{\mathrm{j}}.\vec{\mathrm{J}} = \vec{\mathrm{k}}.\vec{\mathrm{k}} = 1. \end{aligned} \tag{1.8}$$

Here i, j, k are the unit vectors in the three mutually perpendicular directions.

$$\text{d)} \quad \mathrm{A.B} = \mathrm{A_xB_x} + \mathrm{A_yB_y} + \mathrm{A_zB_z}. \tag{1.9}$$

viii) Vector product or cross product:
The cross product or vector product of two vectors is a vector and its direction is at right angles to the directions of both the vectors (Fig. 1.5).

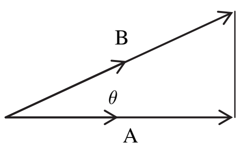


Fig. 1.4. Shows the scalar product of two vectors

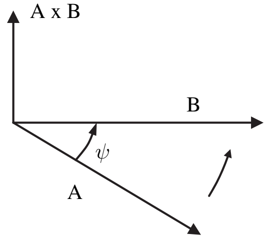


Fig. 1.5. Shows the vector product of two vectors

$$|\mathrm{A} \times \mathrm{B}| = \mathrm{AB} \sin \psi \tag{1.10}$$

where ψ is the angle between the two vectors A and B.

Some of the properties of cross product are

$$\begin{aligned}
&\text{a)}\ \vec{\mathrm{A}} \times \vec{\mathrm{B}} = -\vec{\mathrm{B}} \times \vec{\mathrm{A}},\\
&\text{b)}\ \vec{\mathrm{A}} \times \vec{\mathrm{A}} = 0,\\
&\text{c)}\ \vec{i} \times \vec{j} = \vec{k},\\
&\text{d)}\ \vec{j} \times \vec{k} = \vec{i},\\
&\text{e)}\ \vec{\mathrm{k}} \times \vec{\mathrm{i}} = \vec{\mathrm{j}},
\end{aligned} \tag{1.11}$$

$$\begin{aligned}
&\text{f)}\ \vec{\mathrm{i}} \times \vec{\mathrm{i}} = 0,\\
&\text{g)}\ \vec{j} \times \vec{j} = 0,\\
&\text{h)}\ \vec{\mathrm{k}} \times \vec{\mathrm{k}} = 0 \quad \text{and}
\end{aligned} \tag{1.12}$$

$$\text{i)}\ \vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\,\vec{i} + (A_zB_x - A_xB_z)\,\vec{j} + (A_xB_y - A_yB_x)\,\vec{k}.$$

In the matrix form, it can be written as

$$\vec{\mathrm{A}} \times \vec{\mathrm{B}} = \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ \mathrm{A_x} & \mathrm{A_y} & \mathrm{A_z} \\ \mathrm{B_x} & \mathrm{B_y} & \mathrm{B_z} \end{vmatrix}. \tag{1.13}$$

1.3 Gradient of a Scalar

Gradient of a scalar is defined as the maximum rate of change of any scalar function along a particular direction in a space domain. The gradient is a mathematical operation. It operates on a scalar function and makes it a vector. So the gradient has a direction. This direction coincides with the direction of the maximum slope or the maximum rate of change of any scalar function.

Let $\phi(\mathrm{x}, \mathrm{y}, \mathrm{z})$ be a scalar function of position in space of coordinate x, y, z. If the coordinates are increased by dx, dy and dz, (Fig. 1.6) then

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial \phi}{\partial \mathrm{y}}\mathrm{dy} + \frac{\partial \phi}{\partial \mathrm{z}}\mathrm{dz}. \tag{1.14}$$

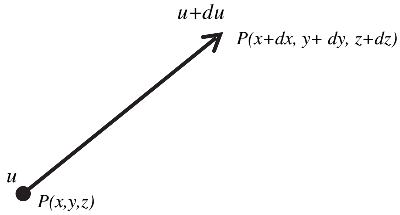


Fig. 1.6. Change of position of a scalar function in a field

If we assume the displacement to be dr, then

$$\overrightarrow{\mathrm{dr}} = \vec{\mathrm{i}}\mathrm{dx} + \vec{\mathrm{j}}\mathrm{dy} + \vec{\mathrm{k}}\mathrm{dz}. \tag{1.15}$$

In vector algebra, the differential operator ∇ is defined as

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \tag{1.16}$$

and the gradient of a scalar function is defined as

$$\mathrm{grad}\ \phi = \vec{\mathrm{i}}\frac{\partial\phi}{\partial \mathrm{x}} + \vec{\mathrm{j}}\frac{\partial\phi}{\partial \mathrm{y}} + \vec{\mathrm{k}}\frac{\partial\phi}{\partial \mathrm{z}}. \tag{1.17}$$

The operator ∇ also when operates on a scalar function $\phi(\mathrm{x},\mathrm{y},\mathrm{z})$, we get

$$\vec{\nabla}\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \tag{1.18}$$

where $\frac{\partial\phi}{\partial \mathrm{x}}, \frac{\partial\phi}{\partial \mathrm{y}}$ and $\frac{\partial\phi}{\partial \mathrm{z}}$ are the rates of change of a scalar function along the three mutually perpendicular directions. We can now write

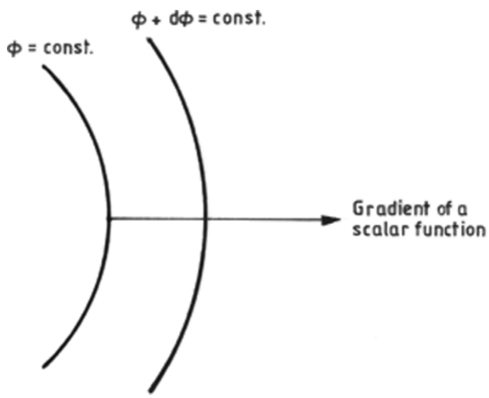


Fig. 1.7. Gradient of a scalar function, the direction of maximum rate of change of a function: Orthogonal to the equipotential lines or surface

$$d\phi = \left(\vec{i}\frac{\partial\phi}{\partial y} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)\left(\vec{i}dx + \vec{j}dy + \vec{k}dz\right) \tag{1.19}$$

$$= (\nabla\phi).\mathrm{dr}$$

where dr is along the normal of the scalar function $\phi(x, y, z) = \text{constant}$. We get the gradient of a scalar function as $d\phi = (\nabla\phi).dr = 0$, when the vector $\nabla\phi$ is normal to the surface $\phi = \text{constant}$. It is also termed as grad ϕ or the gradient of ϕ. (Fig. 1.7).

1.4 Divergence of a Vector

Divergence of a vector is a scalar or dot product of a vector operator ∇ and a vector $\vec{A}$ gives a scalar. That is

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = div\vec{A}. \tag{1.20}$$

This concept of divergence has come from fluid dynamics. Consider a fluid of density $\rho(x, y, z, t)$ is flowing with a velocity V (x, y, z, t). and let $V = v\rho.v$ is the volume. If S is the cross section of a plane surface (Fig. 1.8) then V.S is the mass of the fluid flowing through the surface in an unit time (Pipes, 1958).

Let us assume a small parallelepiped of dimension dx, dy and dz. Mass of the fluid flowing through the face F_1 per unit time is $V_y dx\, dz = (\rho v)_y$ dx dz($S = dxdz$).

Fluids going out of the face F_2 is

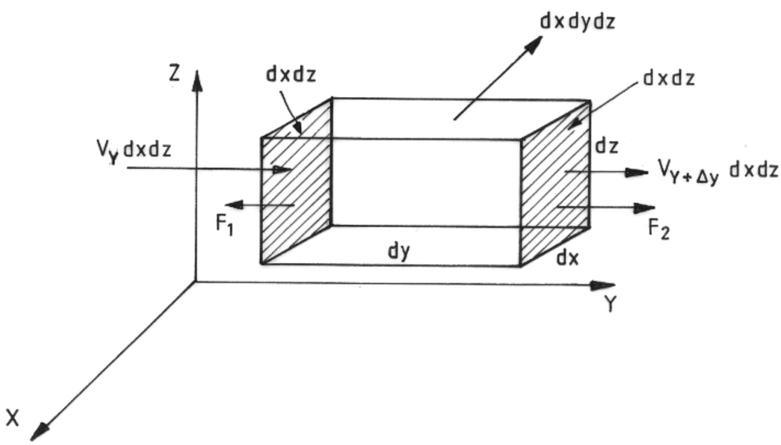


Fig. 1.8. Inflow and out flow of fluid through a parallelepiped to show the divergence of a vector

$$\mathrm{V_{y+dy}dx\ dz} = \left(\mathrm{Vy} + \frac{\partial \mathrm{Vy}}{\partial \mathrm{y}}\mathrm{dy}\right)\mathrm{dx\ dz}. \tag{1.21}$$

Hence the net increase of mass of the fluid per unit time is

$$\mathrm{V_y dx\ dz} - \left(V_y + \frac{\partial V_y}{\partial y}\right) dxdz = \frac{\partial V_y}{\partial y} dxdydz. \tag{1.22}$$

Considering the increase of mass of fluid per unit time entering through the other two pairs of faces, we obtain

$$-\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dxdydz = -(\nabla.\mathrm{V})\mathrm{dx\ dy\ dz} \tag{1.23}$$

as the total increase in mass of fluid per unit time. According to the principle of conservation of matter, this must be equal to the rate of increase of density with time multiplied by the volume of the parallelepiped.

Hence

$$-(\nabla.\mathrm{V})\mathrm{dx\ dy\ dz} = \left(\frac{\partial \rho}{\partial \mathrm{t}}\right)\mathrm{dx\ dy\ dz}. \tag{1.24}$$

Therefore

$$\nabla.\mathrm{V} = -\frac{\partial \rho}{\partial \mathrm{t}}. \tag{1.25}$$

This is known as the equation of continuity in a fluid flow field. This concept is also valid in other fields, viz. direct current flow field, heat flow field etc. Divergence represents the flow outside a volume whether it is a charge or a mass. Divergence of a vector is a dot product between the vector operator ∇ and a vector V and ultimately it generates a scalar.

1.5 Surface Integral

Consider a surface as shown in the (Fig. 1.9). The surface is divided into the representative vectors $\mathrm{ds_1, ds_2, ds_3}$.... etc (Pipes, 1958).

Let $\mathrm{V_1}$ be the value of the vector function of position $\mathrm{V_1(x, y, z)}$ at $\mathrm{ds_i}$.

Then

$$\underset{\substack{\Delta s \to 0 \\ n \to \infty}}{Lim} \sum_{i=1}^{n} V_i dS_i = \iint V.dS. \tag{1.26}$$

The sign of the integral depends on which face of the surface is taken positive. If the surface is closed, the outward normal is taken as positive.

Since

$$d\vec{S} = \vec{i}\ dS_x + \vec{j}\ dS_y + \vec{k}\ dS_z, \tag{1.27}$$

we can write

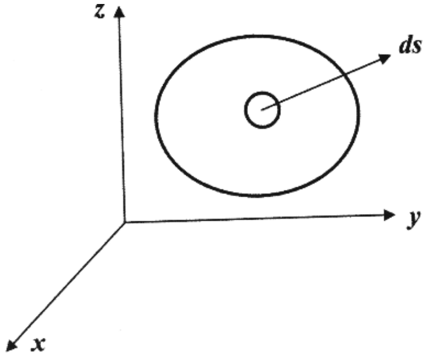


Fig. 1.9. Shows the surface integral as a vector

$$\iint_S \mathrm{V.ds} = \iint_S (\mathrm{V_x\ ds_x + V_y\ ds_y + V_z\ ds_z}). \tag{1.28}$$

Surface integral of the vector V is termed as the flux of V through out the surface.

1.6 Gauss's Divergence Theorem

Gauss's divergence theorem states that volume integral of divergence of a vector A taken over any volume V is equal to the surface integral of A taken over a closed surface surrounding the volume V, i.e.,

$$\iiint_V \left(\nabla.\vec{A}\right) dv = \iint_S \vec{A}.ds. \tag{1.29}$$

Therefore it is an important relation by which one can change a volume integral to a surface integral and vice versa. We shall see the frequent application of this theorem in potential theory.

Gauss's theorem can be proved as follows. Let us expand the left hand side of the (1.29) as

$$\begin{aligned}\iiint_V (\nabla.A)\, dv &= \iiint \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dxdydz \\ &= \iiint_V \frac{\partial A_x}{\partial x}.dxdydz + \iiint_V \frac{\partial A_y}{\partial y} dxdydz \\ &\quad + \iiint_V \frac{\partial A_z}{\partial z} dxdydz. \end{aligned} \tag{1.30}$$

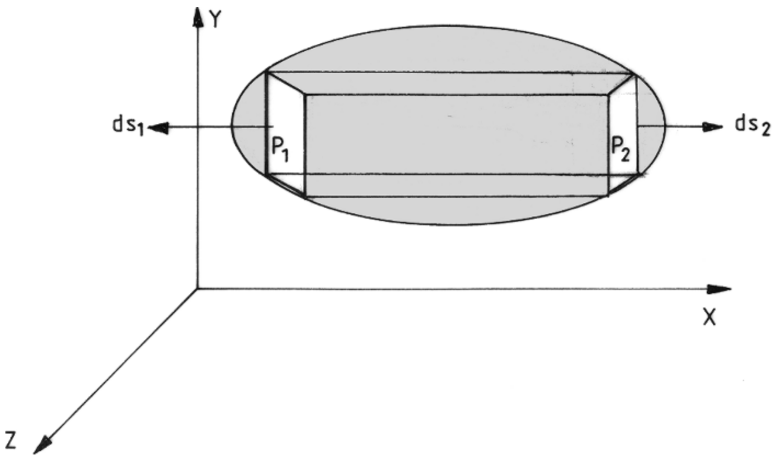


Fig. 1.10. Shows the divergence of a vector

Let us take the first integral on the right hand side. We can now integrate the first integral with respect to x, i.e., along a strip of cross section dy dz extending from P_1 to P_2.(Fig. 1.10).

We thus obtain

$$\iiint\limits_v \frac{\partial A_x}{\partial x} dxdydz = \iint\limits_s \left[A_x\left(x_2, y, z\right) - A_x\left(x_1, y, z\right)\right] dydz. \tag{1.31}$$

Here (x_1, y, z) and (x_2, y, z) are respectively the coordinates of P_1 and P_2. At P_1, we have

$$\mathrm{dy\ dz} = -\mathrm{dS_x},$$

and at P_2

$$\mathrm{dy\ dz} = \mathrm{dS_x}. \tag{1.32}$$

Because the direction of the surface vectors are in the opposite direction.

Therefore

$$\iiint\limits_v \frac{\partial A_x}{\partial x} dxdydz = \iint\limits_s A_x\ dS_x. \tag{1.33}$$

where the surface integral on the right hand side is evaluated on the whole surface. This way we can get

$$\iiint\limits_v \frac{\partial A_y}{\partial y} dxdydz = \iint\limits_s A_y\ dS_y \text{ and} \tag{1.34}$$

$$\iiint\limits_v \frac{\partial A_z}{\partial z} dxdydz = \iint\limits_s A_z\ dS_z. \tag{1.35}$$

If we add these three components, we get Gauss's theorem

$$\iiint_{\mathrm{v}} (\nabla.\mathrm{V})\,\mathrm{dv} = \iint_{\mathrm{s}} (\mathrm{A_x\ dS_x + A_y\ dS_y + A_z\ dS_z}) = \iint_{\mathrm{s}} \mathrm{A.dS}. \quad (1.36)$$

1.7 Line Integral

Let A be a vector field in a space and MN is a curve described in the sense M to N. Let the continuous curve MN be subdivided into infinitesimal vector elements (Fig. 1.11)

$$\mathrm{dl_1, dl_2, dl_3} - - - - - - - - \mathrm{dl_{n.}}.$$

The sum of these scalar products, is

$$\sum_M^N \vec{A}_r\ \overrightarrow{dl}_r = \int_M^N A.dl. \quad (1.37)$$

This sum along the entire length of the curve is known as the line integral of A along the curve MN. In terms of Cartesian coordinate system, we can write

$$\int_M^N A.dl = \int_M^N (A_x dx + A_y dy + A_z dz). \quad (1.38)$$

Let A be the gradient of ϕ, a scalar function of position, then

$$\vec{A} = \vec{\nabla}\phi \quad (1.39)$$

and

$$\int_M^N A.dl = \int_M^N (\nabla\phi) dl = \int_M^N \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right). \quad (1.40)$$

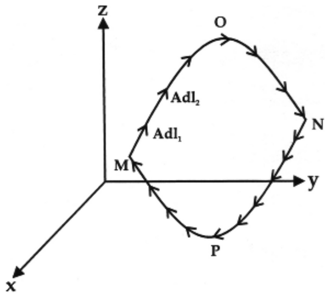


Fig. 1.11. Shows the paths of movement for line integral of a vector

Since

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi, \tag{1.41}$$

we get

$$\int_M^N A.dl = \int_M^N d\phi = \phi_N - \phi_M \tag{1.42}$$

where the value of ϕ_M and ϕ_N are the values of ϕ at the points M and N. Therefore, the line integral of the gradient of any scalar function of position ϕ around a closed curve vanishes. Because if the curve is closed, the point M and N are coincident and the line integral is equal to $\phi_M - \phi_N$ and is equal to zero. In other words

$$\int_{MON} A.dl + \int_{NPM} A.dl = 0.$$

Hence

$$\int_{MON} \vec{A}.dl = -\int_{NPM} \vec{A}.dl. \tag{1.43}$$

This concept of line integral with a vector function, which is a gradient of another scalar function, is used later to define potential in a scalar potential field.

1.8 Curl of a Vector

Curl or circulation of a vector operates on a vector and generates another vector (Fig. 1.12). It is written as $\nabla \times \vec{A}$, i.e., it is a cross product of the vector operator ∇ and a vector A. Curl of a vector can be explained using the concept of line integral. If A is a vector, the curl or rot of A (circulation or rotation) is defined as the vector function of space obtained by taking the vector product of the operator ∇ *and* $\vec{A}$ and its direction is at right angles to the original vector. It is written as $\nabla \times \vec{A}$. So we can write

$$\begin{aligned}\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = \vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\ + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).\end{aligned} \tag{1.44}$$

It can be written in a matrix form as

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{J} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{1.45}$$

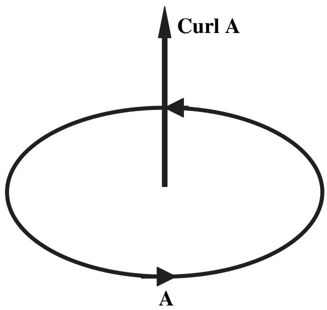


Fig. 1.12. Shows the circulation of a vector

If $A = \nabla\phi$, then

$$\nabla \times A = \nabla \times (\nabla\phi) = \vec{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) + \vec{j}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \vec{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right) = 0. \tag{1.46}$$

Curl of a vector is zero if that vector can be defined as a gradient of another scalar function.

1.9 Line Integral in a Plane (Stoke's Theorem)

To show the connection between line integral and curl of a vector, let us compute the line integral of a vector field $\vec{A}$ around an infinitesimal rectangle of sides Δx and Δy lying in the xy plane as shown in the (Fig. 1.13).

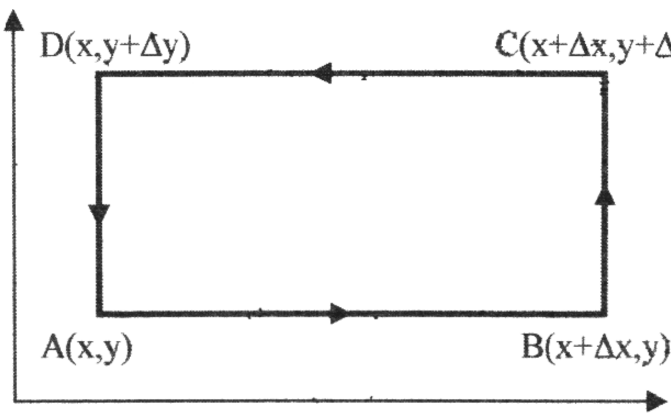


Fig. 1.13. Shows the line integral in a plane to explain stokes theorem

We can compute $\oint$ A.dl around the rectangle writing down the contributions towards this integral from different sides as follows:

$$\begin{aligned}
&\text{Along AB} \rightarrow \mathrm{A_x \Delta x}\\
&\text{Along BC} \rightarrow \left(A_y + \frac{\partial A_y}{\partial x}\Delta x\right)\Delta y\\
&\text{Along CD} \rightarrow -\left(A_x + \frac{\partial A_x}{\partial y}\Delta y\right)\Delta x \qquad (1.47)\\
&\text{Along DA} \rightarrow -A_y\Delta y.
\end{aligned}$$

Here Δx and Δy are infinitesimally small. Adding the various contributions, we obtain

$$\oint_{ABCD} A.dl = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta x\Delta y \qquad (1.48)$$

It can be written as

$$\oint_{\mathrm{ABCD}} \mathrm{A.dl} = \left(\vec{\nabla} \times \vec{\mathrm{A}}\right)_{\mathrm{z}} \mathrm{ds_{xy}} \qquad (1.49)$$

where $\nabla \times \vec{A}_z$ is the z-component of the curl of $\vec{\mathrm{A}}$ and $\mathrm{ds_{xy}}$ is the area of the rectangle ABCD.

If we take a closed surface s in the xy plane (Fig. 1.14) and the space is divided into several rectangular elements of infinitesimally small areas, the sum of the line integrals of the various meshes is given by

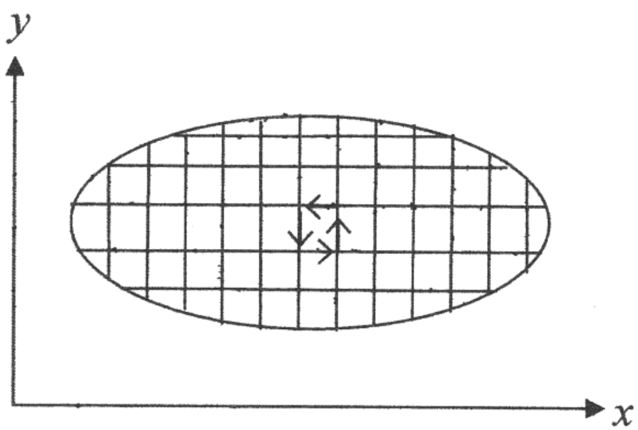


Fig. 1.14. Shows that the vectors only on the boundary remains and the vectors inside get cancelled

$$\sum_{r=1}^{\infty} \oint_{r} A.dl = \sum_{r=1}^{\infty} (\nabla \times A)_z \, ds_{xy}. \tag{1.50}$$

Contributions to the line integrals of the adjoining meshes cancel each other. Only the line integrals in the periphery remain.

Hence

$$\sum_{r=1}^{\infty} \oint_{r} A.dl = \oint_{c} A.dl \tag{1.51}$$

and

$$\oint \mathrm{A.dl} = \iint_{\mathrm{s}} (\nabla \times \mathrm{A})_{\mathrm{z}} \, \mathrm{ds_{xy}}. \tag{1.52}$$

This relation is valid for all the components. Therefore we can write

$$\oint \mathrm{A.dl} = \iint (\nabla \times \mathrm{A})\mathrm{ds}. \tag{1.53}$$

This is Stoke's theorem. It states that surface integral of a curl of a vector is equal to the line integral of the vector itself. It is a mathematical tool to convert surface integrals to line integrals and vice versa.

1.10 Successive Application of the Operator ∇

In vector analysis, for successive application of operator ∇, we can take the vector $\nabla \times \vec{B}$ for $\nabla \times \nabla \times \vec{A}$. where $\vec{\mathrm{B}}$ is $\nabla \times \vec{A}$. If we expand this equation in cartisian coordinate, we get

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{A}_x & \vec{A}_y & \vec{A}_z \end{vmatrix}. \tag{1.54}$$

Here $\mathrm{A_x}$, $\mathrm{A_y}$ and $\mathrm{A_z}$ are respectively the x, y and z components of A in a cartisian coordinate. Equation (1.54) is

$$\text{Curl } \vec{A} = \vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{1.55}$$

and curl $\vec{\mathrm{B}}$ is

$$\nabla \times \nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{vmatrix} \tag{1.56}$$

$$\Rightarrow \vec{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\right\}$$
$$+\vec{\mathrm{j}}\left\{\frac{\partial}{\partial \mathrm{z}}\left(\frac{\partial \mathrm{A_z}}{\partial \mathrm{y}}-\frac{\partial \mathrm{A_y}}{\partial \mathrm{z}}\right)-\frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial \mathrm{A_y}}{\partial \mathrm{x}}-\frac{\partial \mathrm{A_x}}{\partial \mathrm{y}}\right)\right\}$$
$$+\vec{\mathrm{k}}\left\{\frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial \mathrm{A_x}}{\partial \mathrm{z}}-\frac{\partial \mathrm{A_z}}{\partial \mathrm{x}}\right)-\frac{\partial}{\partial \mathrm{y}}\left(\frac{\partial \mathrm{A_z}}{\partial \mathrm{y}}-\frac{\partial \mathrm{A_y}}{\partial \mathrm{z}}\right)\right\} \quad (1.57)$$
$$\Rightarrow \vec{i}\left\{\frac{\partial A_y}{\partial y\partial x}-\frac{\partial^2 A_x}{\partial y^2}-\frac{\partial^2 A_x}{\partial z^2}+\frac{\partial^2 A_z}{\partial z\partial x}\right\}$$
$$+\vec{\mathrm{j}}\left\{\frac{\partial^2 \mathrm{A_z}}{\partial \mathrm{z}\partial \mathrm{y}}-\frac{\partial^2 \mathrm{A_y}}{\partial \mathrm{z}^2}-\frac{\partial \mathrm{A_y}}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{A_x}}{\partial \mathrm{x}\partial \mathrm{y}}\right\}$$
$$+\vec{\mathrm{k}}\left\{\frac{\partial^2 \mathrm{A_y}}{\partial \mathrm{y}\partial \mathrm{x}}-\frac{\partial^2 \mathrm{A_z}}{\partial \mathrm{x}^2}-\frac{\partial^2 \mathrm{A_z}}{\partial \mathrm{y}^2}-\frac{\partial^2 \mathrm{A_y}}{\partial \mathrm{y}\partial \mathrm{z}}\right\}. \quad (1.58)$$

The ith component can be written as

$$\Rightarrow \vec{\mathrm{i}}\left\{\frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial \mathrm{A_x}}{\partial \mathrm{x}}+\frac{\partial \mathrm{A_y}}{\partial \mathrm{y}}+\frac{\partial \mathrm{A_z}}{\partial \mathrm{z}}\right)-\left(\frac{\partial^2 \mathrm{A_x}}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{A_x}}{\partial \mathrm{y}^2}+\frac{\partial^2 \mathrm{A_x}}{\partial \mathrm{z}^2}\right)\right\}. \quad (1.59)$$

Writing the $\vec{j}$ *th* and $\vec{k}$ *th* component as in (1.59), we can write

$$\nabla\times\nabla\times\vec{A}=\nabla\left(\nabla.\vec{A}\right)-\nabla^2\vec{A}$$
$$\text{or curl curl }\vec{\mathrm{A}}=\text{grad div }\vec{\mathrm{A}}-\nabla^2\vec{\mathrm{A}}. \quad (1.60)$$

Equation (1.60) is an important relation and is used quite often in electromagnetic theory.

1.11 Important Relations in Vector Algebra

Some important relations in vector algebra, needed in potential theory, are presented in this section. A couple of relations are derived in the text. The other relations can be derived. They are

i)
$$\vec{\mathrm{a}}.\vec{\mathrm{b}}\times\vec{\mathrm{c}}=\vec{\mathrm{b}}.\vec{\mathrm{c}}\times\vec{\mathrm{a}}=\vec{\mathrm{c}}.\vec{\mathrm{a}}\times\vec{\mathrm{b}} \quad (1.61)$$

ii)
$$\vec{\mathrm{a}}\times\left(\vec{\mathrm{b}}\times\vec{\mathrm{c}}\right)=(\vec{\mathrm{a}}.\vec{\mathrm{c}})\,\vec{\mathrm{b}}-\left(\vec{\mathrm{a}}.\vec{\mathrm{b}}\right)\vec{\mathrm{c}} \quad (1.62)$$

iii)
$$\left(\vec{\mathrm{a}}\times\vec{\mathrm{b}}\right).\left(\vec{\mathrm{c}}\times\vec{\mathrm{d}}\right)=\vec{\mathrm{a}}.\vec{\mathrm{b}}\times\left(\vec{\mathrm{c}}\times\vec{\mathrm{d}}\right) \quad (1.63)$$
$$=\vec{\mathrm{a}}.\left(\vec{\mathrm{b}}.\vec{\mathrm{d}}\vec{\mathrm{c}}-\vec{\mathrm{b}}.\vec{\mathrm{c}}\vec{\mathrm{d}}\right)$$
$$=(\vec{\mathrm{a}}.\vec{\mathrm{c}})\left(\vec{\mathrm{b}}.\vec{\mathrm{d}}\right)-\left(\vec{\mathrm{a}}.\vec{\mathrm{d}}\right)\left(\vec{\mathrm{b}}\vec{\mathrm{c}}\right)$$

iv) $$\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left(\vec{a} \times \vec{b}.\vec{d}\right) \vec{c} - \left(\vec{a} \times \vec{b}.\vec{c}\right) \vec{d} \tag{1.64}$$

v) $$\vec{\nabla} (\phi + \psi) = \nabla\phi + \nabla\psi \tag{1.65}$$

vi) $$\vec{\nabla} (\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \tag{1.66}$$

vii) $$\vec{\nabla}. \left(\vec{a} + \vec{b}\right) = \vec{\nabla}.\vec{a} + \vec{\nabla}.\vec{b} \tag{1.67}$$

viii) $$\vec{\nabla}. (\phi\vec{a}) = \phi\nabla.\vec{a} + \vec{a}.\nabla\phi \tag{1.68}$$

ix) $$\vec{\nabla}. \left(\vec{a} \times \vec{b}\right) = \vec{b}.\vec{\nabla}.\vec{a} - \vec{a}.\vec{\nabla} \times \vec{b} \tag{1.69}$$

x) $$\vec{\nabla} \times \left(\vec{a} + \vec{b}\right) = \vec{\nabla} \times \vec{a} + \vec{\nabla} \times \vec{b} \tag{1.70}$$

xi) $$\vec{\nabla} \times (\phi\vec{a}) = \phi\vec{\nabla}\vec{a} + \nabla\phi \times \vec{a} \tag{1.71}$$

xii) $$\vec{\nabla} \left(\vec{a}.\vec{b}\right) = (\vec{a}.\nabla) \vec{b} + \left(\vec{b}.\nabla\right) \vec{a} + \left(\vec{a} \times \left(\nabla \times \vec{b}\right) + \vec{b} \times (\nabla \times \vec{a})\right) \tag{1.72}$$

xiii) $$\vec{\nabla} \times \left(\vec{a} \times \vec{b}\right) = \vec{a}.\nabla.\vec{b} - \vec{b}\nabla.\vec{a} + \left(\vec{b}.\nabla\right) \vec{a} - (\vec{a}.\nabla) \vec{b} \tag{1.73}$$

xiv) $$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \nabla\nabla.\vec{A} - \nabla^2 \vec{A} \tag{1.74}$$

xv) $$\vec{\nabla} \times (\nabla\phi) = 0 \tag{1.75}$$

xvi) $$\vec{\nabla}.\vec{\nabla} \times \vec{a} = 0. \tag{1.76}$$

2

Introductory Remarks

In this chapter the basic ideas of (a) fields and their classifications (ii) potentials (iii) boundary value problems and boundary conditions (iv) dimensionality of a geophysical problem (v) nature of a medium in earth science (isotropy and anisotropy, homogeneity and inhomogeneity) (vi) Tensors (vi)Nature of equations encountered in solving geophysical problems (vii) Areas of geophysics controlled by potential theory are introduced.

2.1 Field of Force

At any point in a medium, an unit mass or an unit charge or an unit magnetic pole experiences a certain force. This force will be a force of attraction in the case of a gravitational field. It will be a force of attraction or repulsion when two unit charges or two magnetic poles of opposite or same polarity are brought close to each other. These forces are fields of forces (Figs. 2.1, 2.2, 2.3).

A body at a point external to a single body or a group of bodies will experience force(s) of attraction in a gravitational field. These forces will be exerted by a body or a group of bodies on a mass placed at a point. Every mass in the space is associated with a gravitational force of attraction. This force has both a direction and a magnitude. For gravitational field, the force of attraction will be between two masses along the line joining the bodies (Fig. 2.1). For electrostatic, magnetostatic and direct current flow fields, the direction of a field will be tangential to any point of observation. These forces are the fields of forces. These fields are either global fields or are man made artificial local fields. Thus we arrive at a conception of a field of force.

These fields are used to quantitatively estimate some physical properties at every point in a medium. Important physical fields used by geophysicists are (i)Gravitational field, (ii) Magnetostatic field, (iii) Electromagnetic field, (iv) Direct current flow field, (v) Electrostatic field, (vi) Heat flow field, (vii) Fluid flow field, (viii) Earth's natural electromagnetic field etc. Each of these

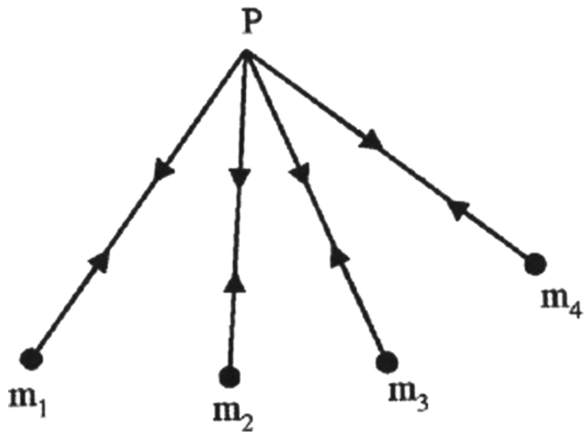


Fig. 2.1. Shows the gravitational attraction at a point P due to the masses m_1, m_2, m_3, and m_4

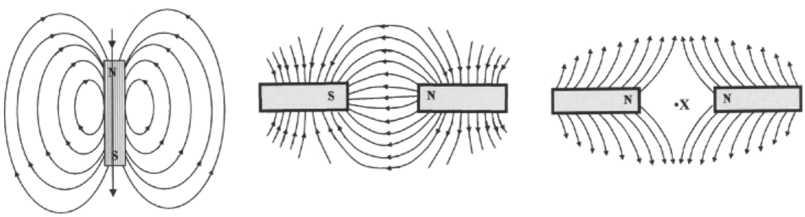


Fig. 2.2. Shows the magnetic field due to a bar magnet and the forces of attraction and repulsion in the vicinity of two unlike and like magnetic poles

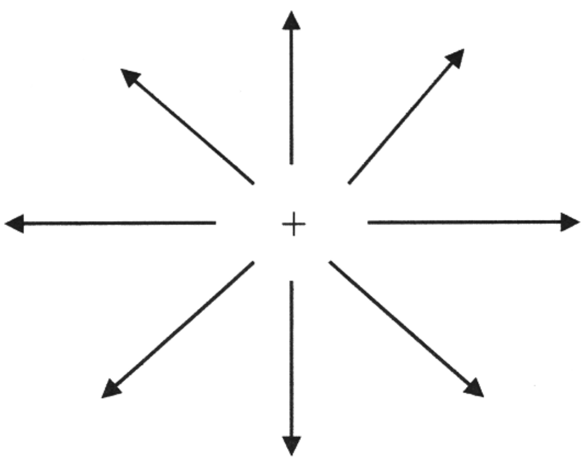


Fig. 2.3. Shows the electrostatic field due to a point charge

fields is associated with one or two physical properties e.g., gravity field is associated with density or density contrasts of the earth's materials, direct current flow field is associated with electrical resistivity or electrical conductivity, electrostatic field is associated with dielectric constant and electrical permittivity, magnetostatic field is associated with magnetic permeability and electromagnetic field is associated with electrical conductivity, electrical permittivity and magnetic permeability of a medium. Physicists and geophysicists use these fields to determine certain physical properties or their variations in a medium. Mathematically they are expressed as a function of space.

2.2 Classification of Fields

These fields can be classified in many ways. Some of these classifications are as follows:

2.2.1 Type A Classification

(i) Naturally occurring fields
(ii) Artificially created man made fields

Gravitational field, earth's magnetic field due to dynamo current in the core of the earth, earth's extraterrestrial electromagnetic field originated due to interaction of the solar flare with the earth's magnetosphere, electric fields generated due to electrochemical and electrokinetic activities within the earth at shallow depths are naturally occurring fields. These fields are present always. No man made sources are needed to generate these fields. They are receiving energy from one form of the natural source or the other.

Direct current flow fields, electromagnetic field, are mostly artificial man made fields. Artificial source of energy is required to generate these fields. Normally we use batteries(cells) or gasoline generators to generate these powers for sending current through the ground.

2.2.2 Type B Classification

(i) Scalar potential field
(ii) Vector potential field

Scalar potential fields are those where the potentials are scalars but a field, being a gradient of potential is a vector. Gravitational field, direct current flow field, electrostatic field, heat flow field, stream lined fluid flow fields are scalar potential fields. In a vector potential field, both the potential and field are vectors. Magnetostatic field and, electromagnetic fields have both scalar and vector potentials (see Chap. 5 and Chaps. 12, and 13)

2.2.3 Type C Classification

(i) Static field
(ii) Stationary field
(iii) Variable field

Electrostatic field is a static field (see Chap. 4) where the charges do not move and stay at a particular point. Direct current flow field and magnetostatic fields are stationary fields (see Chaps. 5 and 6) where the charges are moving at a constant rate so that both the electric and magnetic fields of constant magnitude are generated. Electromagnetic field is a time varying field where both magnitude and direction of electric and magnetic vectors are changing constantly depending upon the frequency of the variable field (see Chap. 12).

2.2.4 Type D Classification

(i) Rotational field (Fig. 2.2a)
(ii) Irrotational field (Fig. 2.1)

If curl of a vector is zero, then the field is an irrotational field. Gravitational field, direct current flow fields are irrotational fields, e.g,

$$\text{curl } \vec{\mathrm{E}} = \nabla \times \vec{\mathrm{E}} = 0$$
$$\text{curl } \vec{\mathrm{g}} = \nabla \times \vec{\mathrm{g}} = 0.$$

Here $\vec{\mathrm{E}}$ is the electric field (see Chaps. 4 and 6) $\overrightarrow{\mathrm{g}}$ is the acceleration due to gravity (see Chap. 3). If curl of a field vector is non zero, then it is a rotational field. Magnetostatic field and electromagnetic field are rotational fields because the curl of a field vector is not zero. Here $\text{curl}\vec{\mathrm{H}} = \vec{\mathrm{J}}$ in magnetostatics (see Chap. 5) and $\text{curl}\vec{\mathrm{H}} = \vec{\mathrm{J}} + \frac{\partial \vec{\mathrm{D}}}{\partial \mathrm{t}}$ in electromagnetics (see Chap. 12). $\vec{\mathrm{H}}$, $\vec{\mathrm{J}}$, $\vec{\mathrm{D}}$ and $\frac{\partial \vec{D}}{\partial t}$ are respectively the intensity of the magnetic field, the current density, (see Chap. 6), the displacement vector (see Chap. 4) and $\frac{\mathrm{d}\vec{\mathrm{D}}}{\partial \mathrm{t}}$ displacement current vector (see Chap. 12).

2.2.5 Type E Classification

(i) Conservative field (Fig. 2.4)
(ii) Non-conservative field

If potential difference between two points in a field is independent of the path through which an unit charge or an unit mass moves from a point A to a point B, (Fig. 2.4) it is called a conservative field (see Sect. 2.3). In a conservative field when an unit charge or an unit mass move around a close loop, the net work done by the mass or charge will be zero.Otherwise the field will be a non-conservative field.

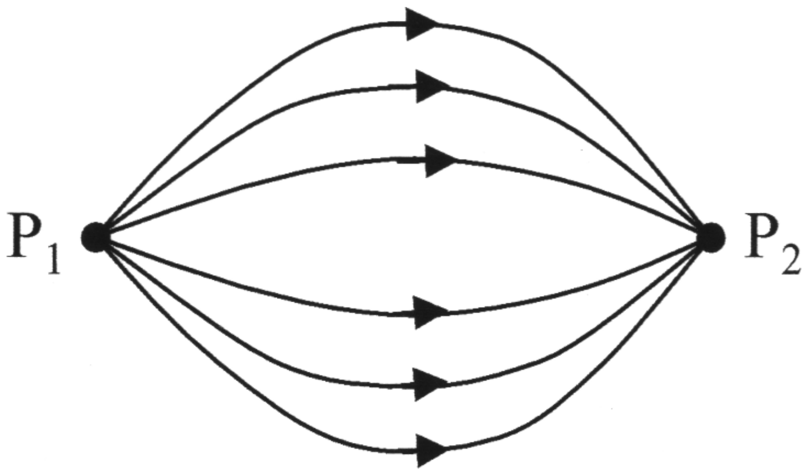


Fig. 2.4. Work done to move from P1 to P2 is independent of paths followed

2.2.6 Type F Classification

(i) Solenoidal
(ii) Nonsolenoidal field

A force field, which has zero divergence through out the entire region of investigation, is called a solenoidal field. Gravity field, direct current flow field, steady state heat flow and fluid flow fields in a source free region are solenoidal field. Magnetostatic field is always a divergence free or solenoidal field (Figs. 2.5 and 2.6). For a magnetostatic field, div $\vec{\mathrm{B}} = 0$ or div $\vec{\mathrm{H}} = 0$ is always true and is therefore a solenoidal field. For a gravitational and an electrostatic fields div $\vec{g}$ and div $\vec{\mathrm{E}}$ will be zero if they do not contain any mass or charge as the case may be in the space domain. (see Chaps. 3, 4, 5).

Potential (discussed later in this chapter) problem, that include the source function, satisfy Poisson's equation. These fields are not divergence free. They are called non-solenoidal field. To solve boundary value problems in potential theory using finite element or finite difference method in a direct current domain we generally use Poisson's equations (see Chap. 15).

2.2.7 Type G Classification

(i) Newtonian potential field
(ii) Non Newtonian potential field

Newtonian potentials are those, which satisfy $\frac{1}{\mathrm{r}}$ relation. The potential at a point is inversely proportional to the first power of the distance (Sect. 2.3). Gravitational potential $\phi = \mathrm{G}\frac{\mathrm{M}}{\mathrm{R}}$ is a Newtonian potential where ϕ is the

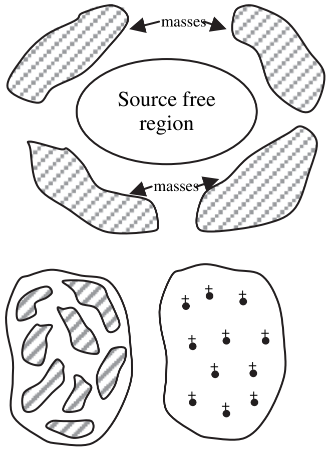


Fig. 2.5. (**A**). Shows the region devoid of any masses; (**B**). Shows the region containing masses; (**C**). Shows the region containing charges

potential at a point due to a mass M at a distance R from the point of observation and G is the universal gravitational constant.(see Chap. 3).

Non-Newtonian potentials are those which do not satisfy $\frac{1}{r}$ variation of potential with distance. The potentials at a point due to a line source and a dipole source are non-Newtonian potentials that way. Potentials are logarithmic for a line source and follow inverse square law for a dipole source (see Chaps. 3, 4, 6). But all electro chemical and electrokinetic potentials are non-Newtonian potentials.

2.2.8 Type H Classification

(i) Dipole field
(ii) Non dipole field

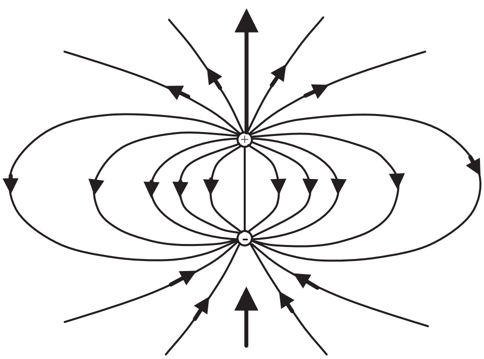


Fig. 2.6. Shows the field due to an electric dipole

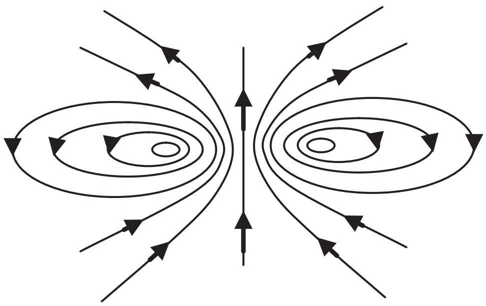


Fig. 2.7. Shows the field due to a magnetic dipole

Dipole fields are those where potentials are inversely proportional to the square of the distance. Fields generated by a loop carrying current and two very closely placed current electrodes generate dipole fields (Figs. 2.6 and 2.7). Fields generated by two widely separated sources are non-dipole fields. Gravitational fields, direct current flow fields with widely separated current electrodes are non-dipole fields.

2.2.9 Type I Classification

(i) Laplacian fields
(ii) Non Laplacian fields

Gravity, electrostatic, direct current flow fields in a source free regions satisfy Laplace equation $\nabla^2\phi = 0$; where ϕ is the potential at a point (Chaps. 3, 4, 6, 7). Non Laplacian fields include extra nonlaplacian terms in a differential equation.

(1) Scalar potential fields, where the sources are included, satisfy the Poisson's equation. For example, the gravitational field, electrostatic and direct current flow field satisfy the following Poisson's equations:
   (i) $\nabla^2\phi_g = 4\pi Gm$ (Gravity Field)
   (ii) div grad$\phi = \nabla^2\phi = -\frac{\rho}{\in}$ (Electrostatic Field)
   (iii) div$\vec{E}$ = div grad$\phi = \nabla^2\phi = -\rho$ (Direct Current Flow Field)

   For mathematical modelling, Poisson's equations are used for solution of boundary value problems in geophysics. In a source free region they satisfy Laplace equation.
(2) Potential problems for a transitional earth where a physical property changes continuously along a particular direction generates a non-Laplacian terms in the governing equation (see Chap. 8) Laplacian and non-Laplacian equations are solved together with the introduction of proper boundary conditions.
(3) For electromagnetic field, the guiding equations are Helmholtz wave equations. Electric field, magnetic field, scalar and vector potentials satisfy

the equation $\nabla^2 F = \gamma^2 F$ where F stands for magnetic field, electric field, vector potentials and or scalar potential. γ is the propagation constant (see Chap. 12). For $\gamma = 0$, i.e., for zero frequency, the Helmholtz equation changes to a Laplace equation. So electromagnetic field is a non-Laplacian field.

(4) A large number of non-Laplacian potentials exist and are being used by the geophysicists. These potentials are known as self-potentials and they are of electrochemical and electrokinetic origin. Potentiometers used for measuring Laplacian potentials can also measure these self-potentials. Important members of this family are (i) liquid junction potential, (ii) membrane potential, (iii) oxidation-reduction potential, and (iv) electrode potential of electrochemical origin and (i) streaming potential, (ii) eletrofiltration potential and (iii) thermal potentials of electrokinetic origin. A source of energy is required to generate these potentials. In the case of direct current field a battery or a generator is used to create the field. For self-potentials, electrochemical cells are generated within the earth to sustain the flow of current for a long time. These electrochemical cells originate at the contact of the two different electrolytes of different chemical activities or they may be at different oxidation-reduction environment. These redox cells of oxidation-reduction origin sustain flow of current across an ore body for many years In any thermodynamic system there is always some free energy which can be easily converted into work. Some potentials are generated when one form of energy is converted into another form of energy. For example, mechanical energy is converted into electrical energy for generation of streaming potential in a fractured rock zone inside a borehole. Due to maintenance of high pressure in a borehole, the mud filtrate enters into the formations through fractures and generates streaming potentials. For electrofiltration potentials, gravitational energy is converted into electrical energy. When fluid moves through a porous medium under a gravity gradient, the potentials are developed and can be measured. Thermal gradients in a geothermal area generate potentials where thermal energy is converted to an electrical energy. Geothermal gradient cells are created. These potentials satisfy Nerst equation, Handerson's equation and Oxidation-Reduction equations. They form a big group of non-Laplacian potentials. Self-potentials as such is a major topic in geophysics and fairly lengthy discussion is necessary to do any justice to this topic.

2.2.10 Type J Classification

(i) Global field
(ii) Local field

Gravity field is a global field, which is present in the entire universe. It includes different stars, planets and satellites. Magnetic field of the earth and earth's natural electromagnetic field of extra terrestrial origin, heat flow field etc. are global fields.

Fields generated by artificial man made sources for geophysical exploration are local fields. Direct current flow field and electromagnetic fields generated by man made sources are local fields.

2.2.11 Type K Classification

(i) Microscopic field
(ii) Macroscopic field

Inter atomic and inter molecular fields are microscopic fields. Understanding of these fields is in the domain of physics. Geophysicists are interested in studying these macroscopic fields.

2.3 Concept of Potential

If we have a vector $\vec{F}$ and a small element of length $\overrightarrow{dl}$ along which one wants to move in a field then the amount work done is given by force multiplied by distance i.e., $\vec{F}.\overrightarrow{dl}$. If we move from point P_1 to P_2 (Fig. 2.8) then the work down is $\int_{P_1}^{P_2} \vec{F}.\overrightarrow{dl}$. If this work done is path dependent, then the field is non-conservative. Otherwise it is a conservative field.

Here

$$dw = \int_{P_1}^{P_2} \vec{F}.\overrightarrow{dl} \tag{2.1}$$

where dw is the element of work done and it is the change in potential energy. Potential at a point in a field is defined as the amount of work done to bring an unit mass or charge from infinity to that point. Potential energy at P_2 – Potential energy at P_1 will be the amount of work done to move from P_1 to P_2 (Fig. 2.9). The potential difference

$$\phi_2 - \phi_1 = \int_{P_1}^{P_2} \vec{F}.\overrightarrow{dl} \tag{2.2}$$

$$= \int_{P_1}^{P} \frac{m}{r^2}.dr = \left| -\frac{m}{r} \right|_{r_1}^{r_2} = m\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{2.3}$$

The potential difference depends upon the end points and not on the path. If $\phi = 0$, when the reference point is at infinity, $\phi_2 = \frac{m}{r}$. Therefore the potential at a point at a distance r is $\frac{m}{r}$ multiplied by a constant. These constants vary from one type of field to the other. Next three chapters deal

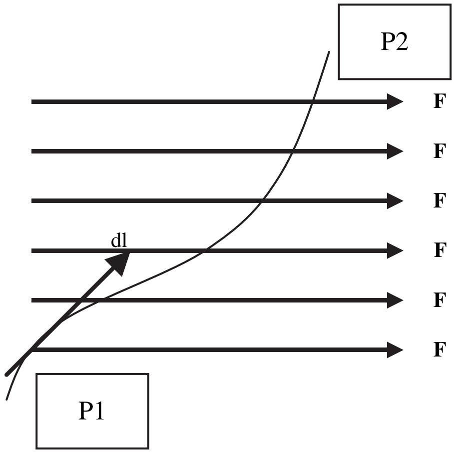


Fig. 2.8. Shows the path of movement of an unit mass or an unit charge in an uniform gravitational or electrostatic field

with these constants. Potential at any point in a gravitational field due to a given distribution of masses is the work done by attraction of masses on a particle of unit mass as it moves along any path from infinite distance up to the point considered.

For a scalar potential field, the principle of superposition is valid. Principle of superposition states that the potential at a point due to a number of mass or charge distributions can be added algebraically. Potential at a point due to the combined effect of volume, surface, linear distributions and several discrete point masses is

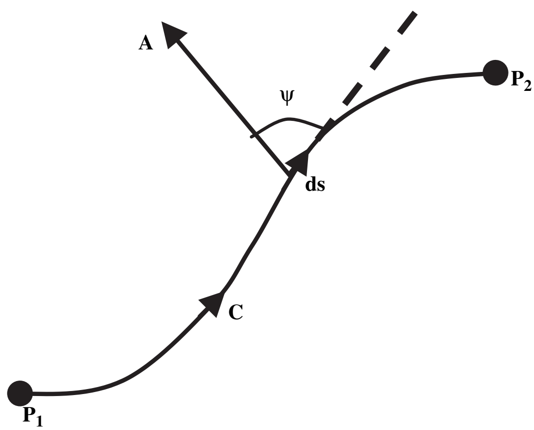


Fig. 2.9. Shows the work done to move from point P_1 to P_2 in the field $\vec{F}$; ds is the elementary movement making an angle Ψ at that point

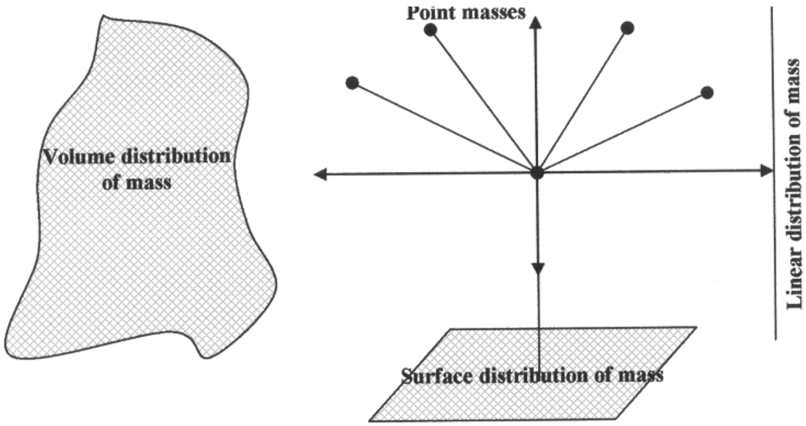


Fig. 2.10. Shows the gravitational attraction at a point due to volume, surface, line and point distribution of masses

$$\phi = \iiint_{\nu} \frac{\rho d\nu}{r} + \iint_{s} \frac{\sigma ds}{r} + \oint_{l} \frac{\lambda dl}{r} + \sum_{i=1}^{n} \frac{mi}{r} \tag{2.4}$$

where ρ, σ, λ are respectively the volume, surface and linear density of mass. m_i is the ith particle in a family of n number of particles.(Fig. 2.10).

2.4 Field Mapping

The direction of the field lines due to gravitational field, magnetostatic field, electrostatic field, direct current flow field are shown in the Figs. 2.1, 2.2, 2.3, 2.7 and 2.8. Figure 2.11 shows a section of a field line where a small element dl is chosen. Here

$$\frac{f_y}{f_x} = \tan\psi = \frac{dy}{dx} \tag{2.5}$$

where fy and fx are the components of the field lines along the y and x directions.

Ψ is the angle made by the field lines at the point dl with the x axis.

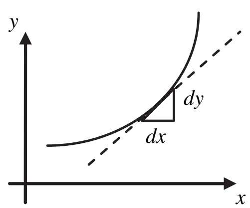


Fig. 2.11. Shows that the field vector at any point is tangential to the field direction

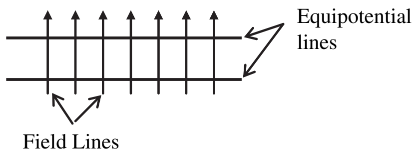


Fig. 2.12. Shows the orthogonal nature of field lines and equipotential lines

Therefore, equation for a field line is

$$\frac{dx}{f_x} = \frac{dy}{f_y} \tag{2.6}$$

For a movement of an unit charge or an unit mass making an angle α with the direction of the field, the element of the potential will be $-\mathrm{d}\phi = \vec{\mathrm{f}}.\overrightarrow{\mathrm{dl}} = \vec{\mathrm{f}}\,\overrightarrow{\mathrm{dl}}\cos\alpha$ (Fig. 2.11) where α is the angle between $\vec{\mathrm{f}}$ and $\overrightarrow{\mathrm{dl}}$. When $\alpha = 0$, $\vec{\mathrm{f}} = \left(\frac{\partial\phi}{\partial \mathrm{l}}\right)_{\max}$ and ϕ = constant when $\alpha = \pi/2$. This shows that the field lines and equipotential lines are at right angles in a conservative field and the gradient of a potential is

$$-grad\ \phi = \vec{a}\max\nabla\phi = \underset{\Delta l \to 0}{Lim}\left(\frac{\Delta\phi}{\Delta\ell}\right)_{\max} \tag{2.7}$$

where $\vec{\mathrm{a}}_{\max}$ is the unit vector along the direction of field line. Figure 2.12 shows the nature of field lines and equipotential lines for an uniform field where theoretically the source and sink are at infinite distance away. Figure 2.13 shows the nature of the field lines and equipotential lines for a point source and sink at finite distance away.

and equation for the equipotential surface is

$$\mathrm{d}\phi(\mathrm{x},\mathrm{y},\mathrm{z}) = 0 \tag{2.8}$$

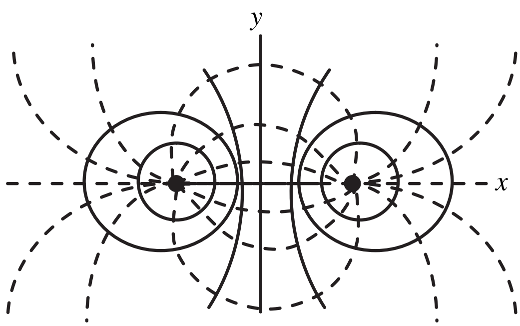


Fig. 2.13. Field lines and equipotential lines for a point source and sink at a finite distance away

In a spherical polar co-ordinate system (Fig. 2.14) (see Chap. 7), the vectors are denoted in the direction of R, θ and ψ. The unit vectors are $\vec{a}_R, \vec{a}_\theta, \vec{a}_\psi$.

Here

$$\overrightarrow{dl} = \vec{a}_R.dR + \vec{a}_\theta.Rd\theta + \vec{a}_\psi R.Sin\theta.d\psi \tag{2.9}$$

and the field components are

$$f_R = -\frac{\partial\phi}{\partial R}, f_\theta = -\frac{1}{R}\frac{\partial\phi}{\partial\theta}$$
$$f_\psi = -\frac{1}{R\ Sin\theta}\frac{\partial\phi}{\partial\psi}. \tag{2.10}$$

In a cylindrical polar co-ordinate system (Fig. 2.15) $r^2 = \rho^2 + z^2$ in (ρ, ψ, z) system. In this system the unit vectors are

$$\overrightarrow{dl} = \vec{a_\rho}.\vec{dr} + \vec{a_\psi}.rd\psi + \vec{a_z}.dz \tag{2.11}$$

So the field components are

$$f_\rho = -\frac{\partial\phi}{\partial\rho}, f_\psi = -\frac{1}{\rho}\frac{\partial\phi}{\partial\psi} \text{ and } f_z = -\frac{\partial\phi}{\partial z} \tag{2.12}$$

Equations for field lines in spherical polar and cylindrical polar co-ordinates are respectively given by

$$\frac{dR}{f_R} = \frac{Rd\theta}{f_\theta} = \frac{R\,Sin\,\theta d\varphi}{f_\psi} \tag{2.13}$$

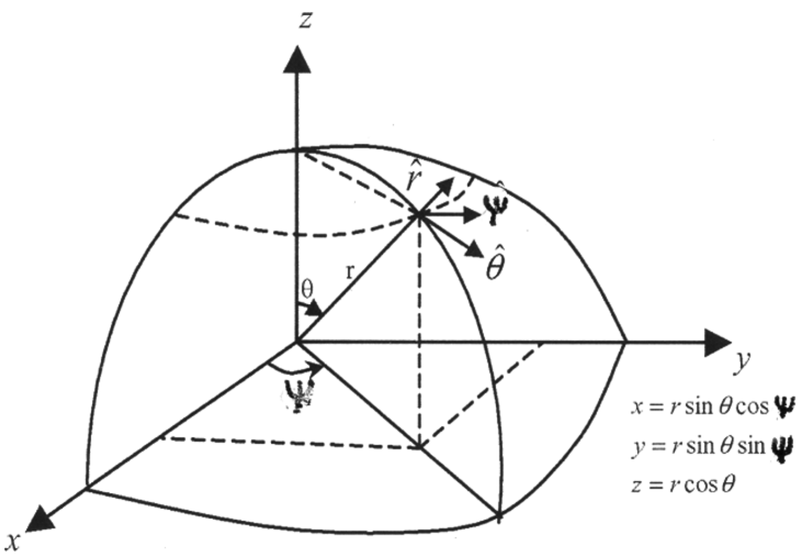


Fig. 2.14. Shows the layout of the spherical polar coordinates

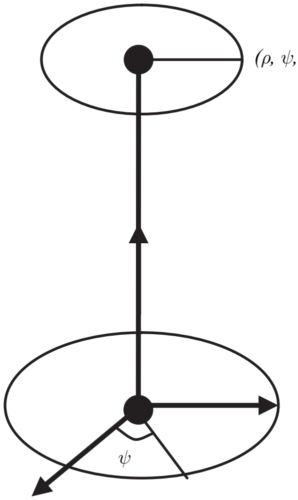


Fig. 2.15. Shows the layout of the cylindrical polar coordinates

and

$$\frac{d\rho}{f_\rho} = \frac{\rho d\psi}{f_\psi} = \frac{dz}{f_z}. \tag{2.14}$$

In the case of a point mass

$$\phi = -\mathrm{G}\frac{\mathrm{m}}{\mathrm{R}} \text{ and } \mathrm{f} = -\mathrm{G}\frac{\mathrm{m}}{\mathrm{R}^2} \tag{2.15}$$

Equipotential surface for a point source is given by

$$\mathrm{G}\frac{\mathrm{m}}{\mathrm{R}} = \mathrm{Cons\,tan\,t}.\ \text{Therefore}, \mathrm{R} = \text{Constant} \tag{2.16}$$

For a point mass or a point source of charge, the equipotential surface will be spherical. In a plane paper the equipotential surface is a circle with the centre at m and radius R. In the xy plane

$$\mathrm{f} = -\mathrm{G}\frac{\mathrm{m}}{\mathrm{x}^2 + \mathrm{y}^2} \tag{2.17}$$

$$\mathrm{f_x} = -\mathrm{G}\frac{\mathrm{mx}}{(\mathrm{x}^2 + \mathrm{y}^2)^{3/2}} \tag{2.18}$$

$$\mathrm{f_y} = -\mathrm{G}\frac{\mathrm{my}}{(\mathrm{x}^2 + \mathrm{y}^2)^{3/2}} \tag{2.19}$$

Therefore,

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{y}}{\mathrm{x}} \text{ or } \mathrm{y} = \mathrm{mx} \tag{2.20}$$

i.e., the field lines are radial lines passing through the centre.

2.5 Nature of a Solid Medium

A solid medium is said to be an homogeneous and isotropic medium when any physical property is same at every point as well as in every direction x, y, z of the medium. If a physical property (say electrical conductivity or electrical permittivity or magnetic permeability or moduli of elasticity) are different at different points and they are different in different directions, the medium is an inhomogeneous and anisotropic medium. Let ρ_1 and ρ_2 be the resistivities along the three mutually perpendicular directions x,y and z at two point A and B in a medium. (Fig. 2.16). Then for

(a) an homogeneous and isotropic medium

$$\text{(i) } \rho_1 = \rho_2 \quad \text{(ii) } \rho_{1x} = \rho_{1y} = \rho_{1z} = \rho_{2x} = \rho_{2y} = \rho_{2z} \tag{2.21}$$

(b) an inhomogeneous and isotropic medium

$$\text{(i) } \rho_1 \neq \rho_2 \quad \text{(ii)} \rho_{1x} = \rho_{1y} = \rho_{1z} \text{ and } \rho_{2x} = \rho_{2y} = \rho_{2z} \tag{2.22}$$

(c) an homogeneous and anisotropic medium

$$\begin{aligned} &\text{(i) } \rho_1 = \rho_2 \quad \text{(ii) } \rho_{1x} \neq \rho_{1y} \neq \rho_{1Z} \quad \text{(iii) } \rho_{2x} \neq \rho_{2y} \neq \rho_{2z} \\ &\text{(iv) } \rho_{1x} = \rho_{2x}, \quad \rho_{1y} = \rho_{2y} \quad \text{ and } \quad \rho_{1z} = \rho_{2z} \end{aligned} \tag{2.23}$$

(d) an inhomogeneous and anisotropic earth

$$\text{(i) } \rho_1 \neq \rho_2 \quad \text{(ii) } \rho_{1x} \neq \rho_{1y} \neq \rho_{1z} \neq \rho_{2x} \neq \rho_{2y} \neq \rho_{1z}. \tag{2.24}$$

For a homogeneous and isotropic medium (Fig. 2.16a), electrical conductivity or electrical permittivity are scalar quantities. For an inhomogeneous and anisotropic earth, these properties become 3×3 tensors. For a homogeneous and isotropic earth $\vec{J}\left(=\sigma\vec{E}\right)$ is in the same directions as $\vec{E}$ (see Chap. 6). For

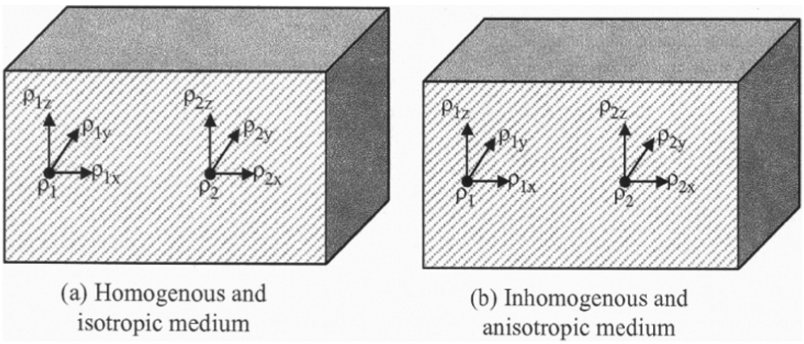


Fig. 2.16(a,b). Show models of homogenous and isotropic and inhomogenous and anisotopic medium

an anisotropic medium (Fig. 2.16b), $\vec{J}$ has the directive property and generally the direction of $\vec{J}$ and $\vec{E}$ are different. Therefore for a rectangular coordinate system, the modified version of Ohm's law can be written as:

$$
\begin{aligned}
J_x &= \sigma_{xx}\, E_x + \sigma_{xy}\, E_y + \sigma_{xz}\, E_z \\
J_y &= \sigma_{yx}\, E_x + \sigma_{yy}\, E_y + \sigma_{yz}\, E_z \\
J_z &= \sigma_{zx}\, E_x + \sigma_{zy}\, E_y + \sigma_{zz}\, E_z
\end{aligned} \tag{2.25}
$$

where σ_{ik} may be defined as the electrical conductivity in the direction κ when the current flow is in the direction i. It is a 3×3 tensor.When $\sigma_{ik} = \sigma_{ki,}$. conductivity of an anisotropic medium is a symmetric tensor having six components.

2.6 Tensors

Any physical entity which are expressed using n subscripts or superscripts is a tensor of order n and is expressed as $T_{123456.....n}$. Any physical property of the earth, say electrical conductivity σ or electrical permittivity $\in$ or magnetic permeability μ is a scalar in a perfectly homogeneous and isotropic medium. In an inhomogeneous and anisotropic medium scalars become tensors. As for example in a direct current flow field (see Chap. 6), $\vec{J} = \sigma\vec{E}$ in a homogeneous and isotropic medium where $\vec{J}$ is the current density in amp/meter^2, σ is the electrical conductivity in mho/meter and $\vec{E}$ is the electric field in volt/meter. In an inhomogeneous and anisotropic medium $\vec{J} = \sigma_{ij}\vec{E}$ where $\vec{J}$ is no more in the same direction as $\vec{E}$ and σ is replaced by σ_{ij} to accommodate the effect of change in direction and magnitude. σ_{ij} is a tensor of rank 2. The effect of direction dependence is given in equation (2.25). This tensor σ_{ij} has 9 components with $i = x, y, z$ and $j = x, y, z$ for cartisian coordinate in an Euclidean geometry. Here σ_{xy} is the electrical conductivity for current density along the x direction and contribution of electrical field in the y direction. The directional dependence changes a scalar to a tensor of rank 2 which can be expressed in the form of a matrix. So a tensor of rank or order n can be written as $T_{i_1 i_2 i_3 i_4 \ldots i_n}$. With this brief introduction about the nature of a tensor, we discuss very briefly some of the basic properties of a tensor.

The fundamental definition involved in tensor analysis is connected with the subject of coordinate transformation. Let us consider a set of variables (x^1, x^2, x^3) which are related to another set of variables (z^1, z^2, z^3) where 1, 2, 3 are superscripts. The relation between the two variables is of the form

$$
\begin{aligned}
z^1 &= a_1^1 x^1 + a_2^1 x^2 + a_3^1 x^3 \\
z^2 &= a_1^2 x^1 + a_2^2 x^2 + a_3^2 x^3 \\
z^3 &= a_1^3 x^1 + a_2^3 x^2 + a_3^3 x^3
\end{aligned} \tag{2.26}
$$

where the coefficients are constants. In this case two sets of variables (x^1, x^2, x^3) and (z^1, z^2, z^3) are related by a linear transformation. This transformation can be written as

$$z^r = \sum_{n=1}^{n=3} a_n^r x^n \text{ where } r = 1, 2, 3. \tag{2.27}$$

Equation (2.27) can be written as

$$z^r = a_n^r u^n \quad \text{for } r = 1, 2, 3. \tag{2.28}$$

So we can define a nine component second order tensor as t_{ij} for $i = 1, 2, 3$ and $j = 1, 2, 3$ in the unprimed frame and nine components in the primed frame if the components are related by the coordinate transformation law.

$$T_{ij} = \mu_{ik}.\mu_{ji} T_{iJ} \tag{2.29}$$

In the present context we redefine scalars and vectors as follows. A physical entity is called a scalar if it has only a single component say α in the coordinate system x_i and measured along u_i and this component does not change when it is expressed in $x_i^/$ and it is measured along $u_i^/$, i.e.

$$\alpha(x, y, z) = \alpha^/ (x_i^/, y_i^/, z_i^/) \tag{2.30}$$

A scalar is a tensor of zero order. A physical entity is called a vector or a tensor of first order if it has 3 components ξ_i and if these components are connected by the transformation of coordinate law

$$\xi_i^1 = u_{ik} \xi_k \tag{2.31}$$

where $u_{ik} = \mathrm{Cos}\left(u_i^1, u_i\right)$.

This relation can also be written in the matrix form as $\xi_i^1 = u\xi$ where ξ^1, u, ξ contain the elements of the (2.29).

Contravariant vector : Let physical entities has the values $\alpha^1, \alpha^2, \alpha^3$ in the coordinate system x^1, x^2, x^3 and these quantities change to the form $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$ in the coordinate system $\bar{x}^1, \bar{x}^2, \bar{x}^3$. Now if

$$\bar{\alpha}^m = \frac{\partial \bar{x}^m}{\partial x^i} \alpha^i \quad \text{for } m = 1, 2, 3, \ i = 1, 2, 3 \tag{2.32}$$

then the quantities $\alpha^1, \alpha^2, \alpha^3$ are the components of a contravariant vector or a contravariant tensor of the first rank with respect to the transformation

$$\bar{x}^r = F^r(x^1, x^2, x^3) = \bar{x}^r(x^1, x^2, x^3) \text{ for } r = 1, 2, 3 \tag{2.33}$$

in an euclidean space. The curvilinear coordinate space $\left(\bar{x}_1^1, \bar{x}_2^1, \bar{x}^3\right)$ are created by (x^1, x^2, x^3) by transformation of coordinates.

Covariant vector : If a physical entity has the values α_1, α_2, α_3 in the system of coordinate (x_1, x_2, x_3) and these values changes to the form $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)$ in the system of coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and if

$$\bar{\alpha}_m = \frac{\partial x^i}{\partial \bar{x}^m} \alpha_i \tag{2.34}$$

Then α_1, α_2, α_3 are the components of covariant vector or tensor of rank 1.

A tensor in which both contravariant and covariant components are present is called a mixed tensor. As for example

$$\bar{\alpha}_n^m = \frac{\partial \bar{x}^m}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^n} \alpha_j^i \tag{2.35}$$

is a mixed tensor of rank 2. Kronecker delta δ_n^m is a mixed tensor of rank 2. It is defined as

$$\delta_n^m = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2.36}$$

Since earth is an anisotropic and inhomogenous medium, for problems related to anisotropic earth, tensors are used.

2.7 Boundary Value Problems

Solution of boundary value problems is one of the important subjects in mathematical physics. For determining potentials or fields at any point within a closed domain R, bounded by a surface S, it is necessary that the problem satisfy certain boundary conditions or some boundary conditions are imposed on the surface or within the medium to get the necessary solution. In most potential problems, certain arbitrary constants or coefficients appear in the solution. These constants are evaluated applying the boundary conditions. In fact through these boundary conditions geometry of the problem enter into the solution. Detailed discussions and use of boundary conditions are available in Chaps. 6, 7, 8, 9, 11, 12, 13, 15. The boundary conditions are more in electromagnetic methods than in direct current methods. Almost every Maxwell's equation generates a boundary condition(see Chap. 12). Application and nature of boundary conditions are different in different problems. Only some of the approaches used in mathematical geophysics are discussed in this book. Boundary conditions applied in solving problems in complex variables are of different types(see Chap. 9) An important task in solving problems in potential theory is to bring the source and perturbation potentials in the same mathematical form before the boundary conditions are applied. Solved examples in Chaps. 7, 8 and 13 are some of the demonstrations in this direction. For mathematical modelling, we often encounter three types of boundary value problems, viz., Dirichlet's problems, Neumann problems and mixed (Robin /Cauchy) problems.

2.7.1 Dirichlet's Problem

In a closed region R having a closed boundary S, while solving Laplace equation $\nabla^2\phi = 0$ (see Chap. 7) some prescribed values are assigned to the boundaries. When potentials are prescribed on the boundaries, the problems are called Dirichlet's problems and the boundary conditions are Dirichlet's boundary conditions. Potential at any boundary can be zero or a function of distance or a constant. If potential at the said boundary is zero then it is a homogeneous boundary condition. Otherwise the boundary condition is inhomogeneous. Within the boundary, which is well behaved on these regions and takes some prescribed values on the boundary say $\phi\,(x, y, z) = 0$. (Fig. 2.17a), the problems are called Dirichlet's problem. As for example potential at an

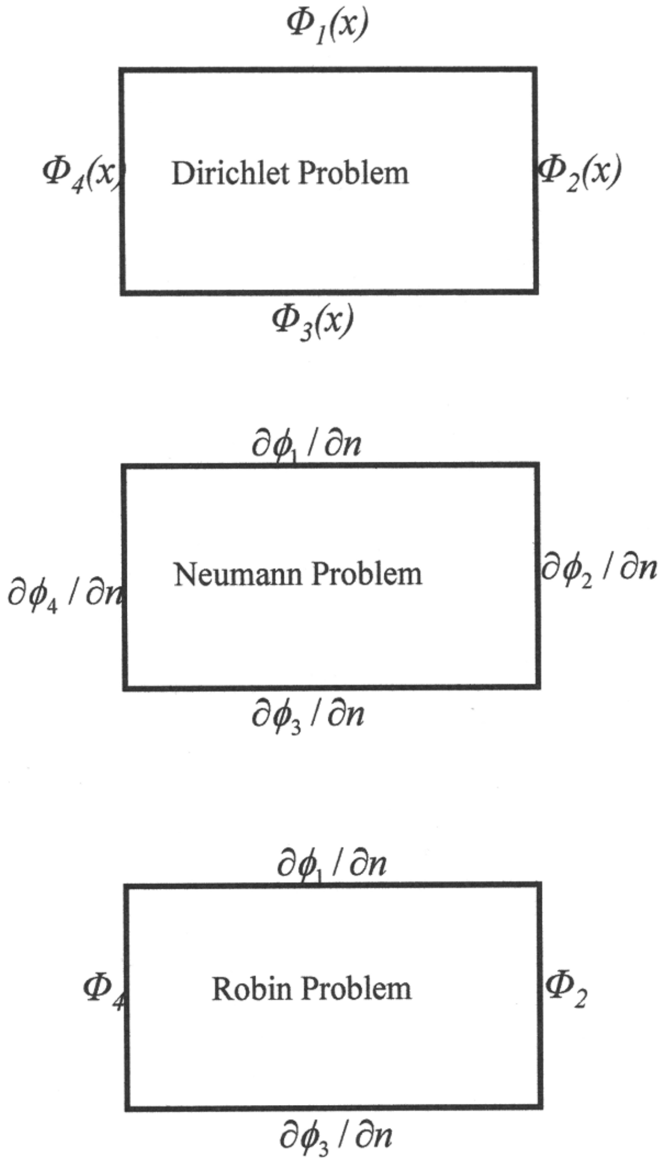


Fig. 2.17a,b,c. Show the direchlet, neumann and mixed boundary conditions

infinite distance from a source is zero. Therefore if we make the working domain to be arbitrarily very large we can prescribe $\phi(x, y, z) = 0$ everywhere on the boundary because potentials die down to zero at infinite distance from the point source following $\frac{1}{r}$ law (see Chaps. 3, 4, 6, 9, 15).

2.7.2 Neumann Problem

For solution of Laplace or Poisson's equation, if the normal derivatives of potentials are prescribed on the boundaries then the problems are called Neumann problems and the boundary conditions are Neumann boundary conditions. Application of Neumann boundary condition is shown in Chaps. 8, 9 and 15. (Fig. 2.17b). Like Dirichlet's boundary conditions, Neumann boundary conditions can be homogenous or inhomogenous.

2.7.3 Mixed Problem

If ϕ is prescribed on certain parts of a boundary and $\frac{\partial \phi}{\partial n}$ is prescribed on rest part of the boundary, then the problems is called a mixed or Robin problem. Most of geophysical problems are mixed problems where ϕ is defined on one part and $\frac{\partial \phi}{\partial n}$ is defined on rest of the boundary. Figure (2.17c) shows a domain of the earth where the top surface is the air-earth boundary. Since the contrast in resistivity at the air-earth boundary is very high, therefore, $\frac{\partial \phi}{\partial n}$, the normal derivative of potential, will always be zero. Other boundaries are pushed away from the working area to force the potential ϕ to be zero.

For solution of the Poisson's equation or Laplace equation, the potentials and their derivatives make the condition $k\frac{\partial \phi}{\partial n} + h\phi = 0$. Application of mixed boundary condition is shown in Chap. 15.

2.8 Dimension of a Problem and its Solvability

For interpretation of geophysical field data, one has to go through the solution of forward and inverse problems. Proper selection of a forward problem is a very important step to begin with. The geophysicists need helps from potential theory for solution of forward problems. An homogeneous and isotropic full space problem is a zero dimensional problem because the physical property does not change in any direction. An homogeneous and isotropic half space has two media with different physical properties and having a sharp boundary between them. An assumed homogeneous earth with an air earth boundary is a homogeneous half space. A layered earth generates an one-dimensional problem because physical property varies only along the vertically downward direction. Therefore all the potential and nonpotential problems related to layered earth are one dimensional problems,.(Fig. 2.18a,b).When a physical property varies along the x and z direction and remain same along the y

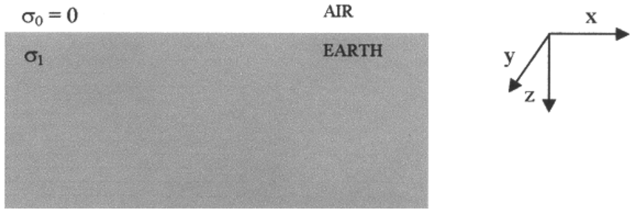


Fig. 2.18a. A sketch for zero dimensional problem

direction, the problem is called a two dimensional problem. Here x, y, z are arbitrary mutually perpendicular directions. An infinitely long cylinder of circular or rectangular or arbitrary shaped cross section placed horizontally at a certain depth from the surface of the earth or exposed on the surface of the earth is an ideal example of a two dimensional problem. Many such earth models with much more complicated geometry are regularly used for interpretation of geophysical data. (Fig. 2.18c) When a physical property varies along the x and z direction and the cylindrical structure has limited length along the y direction, the problem is termed as a two and a half dimensional problem $\left(2\frac{1}{2}\text{D problem}\right)$.

When a physical property varies in all the three directions, the problems are called three-dimensional problems (3-D problem). (Fig. 2.18d) For most of the one-dimensional problems closed form analytical solutions are available. For solution of the two and three dimensional problems finite element, finite difference, integral equation, volume integration

Fourier methods are used. These are numerical methods (Chap. 15). Many of the two dimensional problems have complete analytical solution. Partly numerical and partly analytical techniques are used for solution of some problems (see Chaps. 7, 8, 9).

With the advent of numerical methods and high speed computers, the domain and solvability of the forward problems have increased significantly and it is no longer restricted to bodies of simpler geometries only.

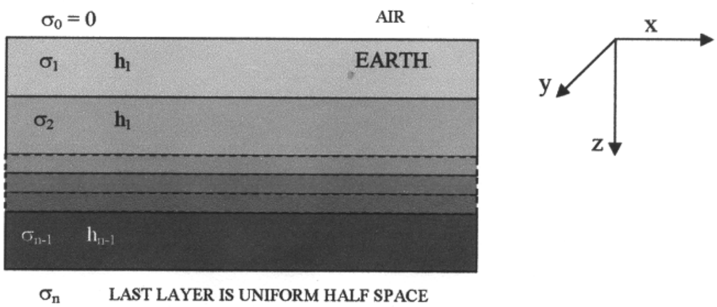


Fig. 2.18b. A sketch for one dimensional problems

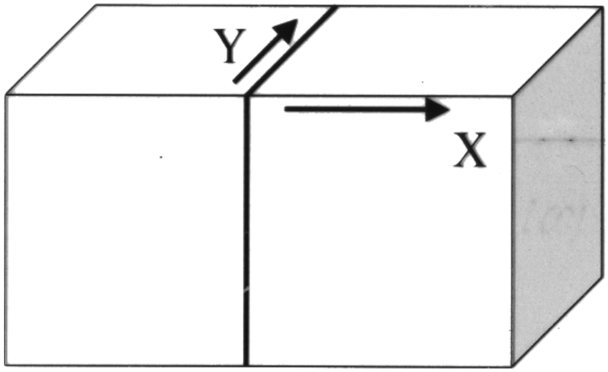


Fig. 2.18c. A sketch for a two dimensional problem

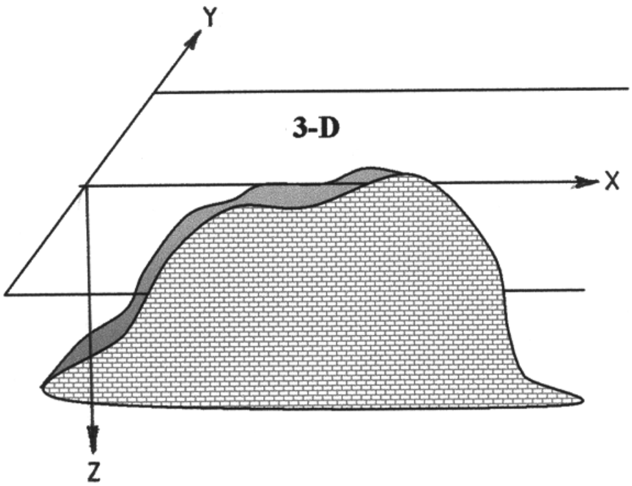


Fig. 2.18d. A three dimensional body where physical property changes in all the three mutually perpendicular directions

2.9 Equations

2.9.1 Differential Equations

A first order ordinary differential equation

$$\frac{dy}{dx} + p(x)y = r(x) \tag{2.37}$$

is a linear differential equation. Important features of this equation is that it is a linear in y and $\frac{dy}{dx}$ where p and r are any given function of x. If the right hand side $r(x) = 0$ for all x in a working region, the equation is said to be homogeneous. Otherwise it is inhomogeneous. An ordinary differential equation may be divided into two large classes, viz., linear and non–linear equations.

A second order differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x) \tag{2.38}$$

is called a homogeneous equation if $r(x) = 0$.

The equation is non linear if it can be written as

$$x\left(\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) + 2\frac{dy}{dx}y = 0. \tag{2.39}$$

An ordinary differential equation of nth order can be written as

$$\frac{d^{ny}}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots.. + P_1\frac{dy}{dx} + P_o(x)y = r(x) \tag{2.40}$$

where $r, P_o, P_1, P_2 \ldots P_n$ are functions of x. In a similar way this equation can be homogeneous and inhomogeneous. Partial differential equations are those where the functions, involved, depend on two or more independent variables. They are used for solving many problems of physics, viz., problems of potential theory, electromagnetic theory, heat conduction and fluid flow theory, solid mechanics etc. We use these partial differential equations for solving boundary value problems of different branches of mathematical physics.

A partial differential equation is linear if it or its partial derivative is of first degree independent variables. Otherwise it is nonlinear. If each term of this type of equation contains dependent variables or one of its derivatives, the equation is said to be homogeneous. Otherwise they are inhomogeneous.

An equation of the form

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \tag{2.41}$$

is a normal form of partial differential equation. The equation is said to be elliptic if $AC-B^2 > 0$, parabolic if $AC-B^2 = 0$ and hyperbolic if $AC-B^2 < 0$. Here A,B,C may be function of x, y, z. As for example Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{2.42}$$

is an elliptic equation. Heat conduction equation

$$\frac{\partial u}{\partial t} = C\frac{\partial^2 u}{\partial x^2} \tag{2.43}$$

is a parabolic equation and wave equation

$$\frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial x^2} \tag{2.44}$$

is a hyperbolic equation.

2.9.2 Integral Equations

The theory of Integral equations deals with the equations in which an unknown function occurs under the integral sign. The subject was developed by V. Volterra (1860–1940), E.I. Fredhom (1866–1927) and D. Hilbert (1862–1943).

If an external force is applied to a linear system and is described by a function f(x) $(a \leq x \leq b)$, then the result or the output is given by the system which can be written as

$$f(x) = \int_a^b G(x, \xi) f(\xi) d\xi \tag{2.45}$$

where $G(x, \xi)$ is a Kernel function or a Green's function which is specified by the given system. Integral equation (IE) has a major role in geophysical forward modeling. Here a certain class of integral equations are introduced. Integral equations appear in a certain class of diffusion and potential problems in geophysics. IE has a major success in handling three dimensional problem both in potential (scalar and vector) and non potential problems. If both the upper and lower limits of an integral are constant then the equations are Fredhom type. If one of the limits is an independent variable, then the equations are Volterra type. An integral equation is said to be linear if all the terms occurring in the equations are linear. These integral equations can be of first, second and third kind as follows :

(i) Fredhom's integral equation of the first kind.

$$\int_a^b K(x, t) f(t) dt = g(x) \tag{2.46}$$

(ii) Fredhom's integral equation of the second kind

$$\int_a^b K(x, t) f(t) dt = g(x) + f(x) \tag{2.47}$$

(iii) Fredhom's integral equation of the third kind

$$\int_a^b K(x, t) f(t) dt = \lambda f(x) \tag{2.48}$$

(iv) Volterra's integral equation of the first kind

$$\int_a^x K(x, t) f(t) dt = g(x) \tag{2.49}$$

(v) Volterra's integral equation of the second kind

$$\int_a^x K(x,t)f(t)dt = g(x) + f(x) \tag{2.50}$$

(vi) Volterra's integral equation of the third kind

$$\int_a^x K(x,t)f(t)dt = \lambda f(x) \tag{2.51}$$

In Fredhom's integral equations, the upper and lower limits are respectively 'b' and 'a' where 'a' and 'b' are constants. In Volterra's integral equation the upper and lower limits are respectively 'x' and 'a' where x is an independent variable and 'a' is a constant. Here f(t) is an unknown function to be determined. K(x,t) and g(x) are known functions. K(x,t) is known as the Kernel function. Equation of the third kind is a homogeneous version of the second kind and it defines an eigen value problem. These integral equations of second kind can be solved either iteratively or using Gauss quadrature method of numerical integration or by series expansion method. Further discussion on integral equation is available in Chap. 15. Eigen value is defined in Chap. 17.

2.10 Domain of Geophysics in Potential Theory

Potential theory as such is a vast subject and is used by physicists, geophysicists, electrical engineers, electrical communication engineers, mathematicians working in fluid dynamics, complex variables, aerodynamics, acoustics, electromagnetic wave theory, heat flow, theory of gravity and magnetics etc. This subject forms the basis of many branches of science and engineering. Only a part of geophysics, controlled by potential theory, includes gravity, magnetic, electrical electromagnetic heat flow and fluid flow methods.. Seismic methods in geophysics and nuclear geophysics are out of bound of potential theory. All potentials of electrochemical and electrokinetic origin strictly do not come under potential theory.

Primary task of a geophysicist is to understand the data collected from the field and interprete them in the form of an acceptable geological model within the limit of finite resolving powers of different geophysical potential fields For proper execution of this task, geophysicists have to frame the earth models and have some ideas about the nature of field response. To have this understanding, geophysicists solve forward problems for different branches of geophysics using potential theory (Scalar and Vector potentials). Therefore potential theory, equipped with different mathematical tools, forms the basis for solution of forward problems. Potential theory is a must to understand the behaviour of the geophysical data. Inverse theory comes next as a tool for interpretation. Advent of inverse theory revolutionized the entire procedure for interpretation of geophysical data. It became possible due to very rapid development of computer science, software technology and numerical methods in mathematical modelling. The phenomenal developments in science and technology in these areas during the last three decades improved the power of vision of geophysicists to look inside the earth. In this book, besides some introduction on gravitational field, magnetostatic field, electrostatic field, direct current flow field and electromagnetic field, we introduced (i) the solution of Laplace equation and its contribution towards solving different type of boundary value problems, (ii) complex variable and its role in solving two dimensional potential problems (iii) Green theorem and its application in solving potential problems (iv) concept of images in potential theory (v) electromagnetic theory and vector potentials and their contribution towards solving the electromagnetic boundary value problems (vi) finite element and finite difference and integral equation methods towards solving two and three dimensional potential problem (vii) Green's function (viii) analytical continuation of potential fields and (ix) inversion of potential field data.

Scientists of other disciplines of physical sciences and technologies may find interest .in some of these topics.

3

Gravitational Potential and Field

In this chapter we included some of the preliminaries of the gravitational potentials and fields, viz, Newton's law of gravitational attraction, gravitational fields and potentials, universal gravitational constant and acceleration due to gravity, the nature of earth's gravitational field, gravitational potentials and fields for bodies of simpler geometries and the basic guiding equations of the gravitational field. A few points about the nature of the earth's gravity field and the basic preliminaries regarding data handling are mentioned. Advanced topics, viz, spherical harmonics Green' theorem, Green's equivalent layers and analytical continuation of potential field are given respectively in Chaps. 7, 10 and 16.

3.1 Introduction

Sir Isaac Newton first published Philosophiae Naturalis Principia Mathematica in 1687. In that he gave ideas both about the law of gravitation as well as the laws of forces. It was realised that (i) gravitational force is always a force of attraction, (ii) It is not only a global force it is an universal force present in the entire universe, (iii) It is one of the weak forces, (iv) It has some relation with mass of a body, (v) principle of superposition is valid for gravitational fields, (vi).centrepetal and centrifugal forces do exist, (vii) movement of planets round different stars and movement of satellites round different planets are controlled by the gravitational force of attraction. Two important physical parameters, viz, 'G' and 'g' came in the fore front for further advancement although mass of a body, its density, has some direct relation with the force of gravitation was realised. Lord Cavendish experimentally determined the value of 'G', the universal gravitational constant, in his laboratory by measuring the force of attraction between two lead balls placed at a certain precise distance and published the value of 'G' in 1798 to be 6.754×10^{-11} $m^3kg^{-1}s^{-2}$ (MKS or SI unit) which was refined later as 6.672×10^{-11} MKS unit. Accelaration due to gravity was first measured by Galileo in his famous experiment by dropping objects from the leaning tower of Pisa. The numerical value of 'g' was found to be around $980\,\mathrm{cm/sec^2}$. In honour of Galileo the unit $1\,\mathrm{cm/sec^2}$, the unit of acceleration due to gravity was termed as 'Gal' or 'gal'. It was understood right from the beginning that the force of gravitation is global and the gravitational force is always a force of attraction and the entire network of billions of stars, planets and satellites are controlled by the force of gravitation. Kepler's three law's i.e., (i) orbit of each planet is an ellipse with sun at one of the foci (ii) orbital radius of the earth sweeps out equal areas in equal interval of time and (iii) the ratio of square of the planet's period of revolution to the cube of the semi major axis of the orbit is a constant for all the planets and could be explained from Newton's law of gravitation.

Next round of researches in this area were centred around accurate evaluation of the absolute value of 'g', the acceleration due to gravity, and G, the universal gravitational constant. Soon it became known to the physicists that the time period of oscillation of a simple pendulum, which executes simple harmonic motion, is connected to the acceleration due to gravity g through the relation $\mathrm{T} = 2\pi\sqrt{l/g}$ where l is the distance between the pivotal point of the hinge to the centre of the mass and T is the time period of oscillation. Although expression looks very simple several stages of developments and generation of compound pendulum was necessary for accurate measurement of 'g'. It could be known, as soon as the value of 'g' is known that 'g' is a latitude dependent quantity and shape of the earth is nearly a spheroid with definite ellipticity. It was known right from the beginning that mass and density of a body are interrelated and they have connection with the gravity field of the earth. Therefore both the mass and the mean density of the earth could be known from 'G' and 'g'.

Much later geophysicists came forward for accurate measurement of minute variation of the value of 'g' due to minor variations of densities of rocks and minerals. Precision of measurement has gone up to such a level that practical unit of measurement of variation of the gravity field was reduced to milligal (10^{-3}). The minute variations of gravity is termed as 'Δg', the gravity anomaly and the precision instruments are named gravimeters. Gravimeters measure the minute variations in 'g' rather than their absolute values. Precision level is heading towards microgal level in 21st century. The ultra sensitive gravimeters are used for geodetic survey, crustal studies and geophysical exploration.In this chapter a brief outline of the gravitational potential and field are given. Some of the topics of gravitational potentials and fields are included in Chaps. 10 and 16.

3.2 Newton's Law of Gravitation

Newton's law of gravitation states that "Every particle in the universe attracts every other particle with a force which is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them." Therefore, Newton's law of gravitation can be stated as

$$\vec{F} \,\alpha\, \frac{m_1\ m_2}{r^2}$$

which can be written as

$$F = G\frac{m_1 m_2}{r^2} \tag{3.1}$$

where F is the force of attraction between the two masses m_1 and m_2 placed at a distance r from each other and G, the constant of proportionality, is the universal gravitational constant. Here m_1 and m_2 are the two masses separated by a distance r. This G is the force of attraction between the two particles of unit mass separated by a unit distance. The acceleration due to gravity is 'g'. Its unit is cm/sec^2 or Gal. The unit Gal is introduced in honour Galileo. The practical unit is milligal $= 10^{-3}$ gal. Assuming the earth as a spherical body of density ρ and radius 'r', the force of attraction between the earth and a body of mass M is given by

$$G\frac{M_e M}{r^2} = Mg \tag{3.2}$$

where M_e is the mass of the earth and is equal to $= \frac{4}{3}\pi r^3 \rho$ and ρ is the mean density of the earth ($\rho = 5.67$ gms/cc approx). Here g is the acceleration due to gravity and is approximately equal to $g = 981\,cm/sec^2$. In C.G.S Unit

$$G = \frac{3g}{4\pi\rho r} = \frac{1}{15,500,000} \quad \text{(approx)} \tag{3.3}$$

$$=6.664 \times 10^{-8}\,cm/g.\,sec^2 (\text{CGS unit})$$

$$6.664 \times 10^{-11}\,m/kg.\,sec^2 (\text{M.K.S. Unit}) \tag{3.4}$$

G is in dyne – cm^2/gm^2 that is equivalent to Newton. Cavendish first experimentally determined the value of G as mentioned.

Gravitational field is a naturally existing conservative, solenidal (in a source free region), irrotational, global and scalar potential field of very large dimension. Two masses separated by a great distance can experience a force of attraction what little be the magnitude of that force. Gravitational force exits between all the celestial bodies. It is a weak force. But the force is always a force of attraction and the direction of the force is along the line joining the centres of two masses m_1 and m_2 (Fig. 3.1) at P (x, y, z) and at Q (ξ, η, ζ) where r is the distance between the two point masses. $\vec{r}$ the directional force vector which is expressed as the distance between the two masses can be written as

$$\vec{r} = \vec{i}(x - \xi) + \vec{j}(y - \eta) + \vec{k}(z - \zeta) \tag{3.5}$$

For an unit point mass m_2, gravitational field due to a point mass is given by

$$\lim_{m_2 \to 1} F = -G\frac{m}{r^2}.\vec{r}$$

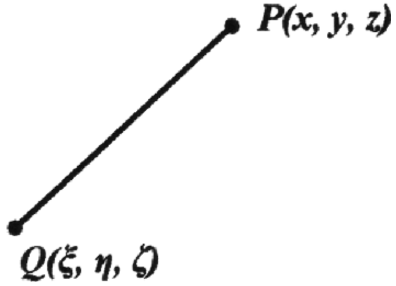


Fig. 3.1. Gravitational attraction at a point P due to a point mass m at a point Q at a distance r from P

where $\vec{r}$ is the direction of the unit vector. Gravitational field is a scalar potential field i.e., the potential is a scalar and the field is a vector. They are related by the relation $\vec{E} = -\text{grad}\phi$. Many authors use negative sign to express,

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\vec{r} \tag{3.6}$$

As mentioned already gravitational field, is always an irrotational field i.e. curl $\vec{g} = 0$.

Absolute values of densities or their variations in crust mantle silicates, originated due to thermal processes inside the earth as well as due to tectonic evolution, are responsible generation of global gravity map. Purpose of investigation fixes the scale of measurement.

3.3 Gravity Field at a Point due to Number of Point Sources

Suppose the masses m_1, m_2, m_3 and m_4 are distributed in a space and their distances from the point of observation P are respectively given by r_1, r_2, r_3, r_4 etc (Fig. 3.2). Let the gravitational attraction at the point P for the masses be g_1, g_2, g_3 and g_4 respectively. Since this will be a vector field, the total field can be determined by resolving the components of forces along the three co-ordinates axes.

Let ξ_i, η_i and ζ_i be the co-ordinates of the ith particle and m_i be its mass. The force acting at the point P due to m_i is $\frac{m_i}{r_i^2}$. The components of the force along the x, y and z directions are respectively given by

$$g_{ix} = \frac{m_i}{r_i^2}x = \frac{m_i}{r_i^2}.\frac{x-\xi}{r_i} \tag{3.7}$$

$$g_{iy} = \frac{m_i}{r_i^2}y = \frac{m_i}{r_i^2}.\frac{y-\eta}{r_i} \tag{3.8}$$

$$g_{iz} = \frac{m_i}{r_i^2}z = \frac{m_i}{r_i^2}.\frac{z-\zeta}{r_i}. \tag{3.9}$$

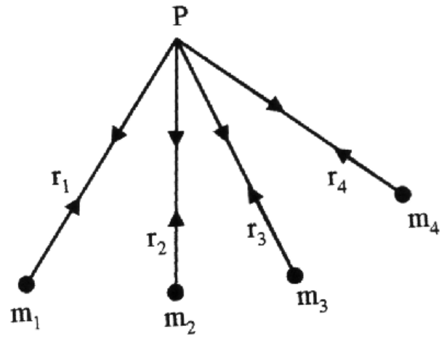


Fig. 3.2. Gravitational attraction at a point P due to masses m_1, m_2, m_3 and m_4

Therefore, the total gravitational field components g_x, g_y, g_z using principle of superposition are given by

$$g_x = \sum_{i=1}^{n} m_i \frac{(x - \xi_1)}{r_i^3}, \quad g_y = \sum_{i=1}^{n} m_i \frac{(y - \eta_i)}{r_i^3}$$

and

$$g_z = \sum_{i=1}^{n} m_i \frac{(z - \zeta_i)}{r_i^3}. \tag{3.10}$$

Here

$$g = \sqrt{g_x^2 + g_y^2 + g_z^2}. \tag{3.11}$$

That gives the expression for gravitational force of attraction at a point due to n number of isolated discrete point masses.

3.4 Gravitational Field for a Large Body

Let us assume an arbitrary shaped large solid three dimensional body of density ρ. Let us take a point of observation P at coordinates (x, y, z) and an elementary mass at Q having coordinates (ξ, η, ζ) within the body (Fig. 3.3). The elementary volume $dv = d\xi d\eta d\zeta$. We can divide the body into number of elementary volumes. The volume density of source is $\underset{\Delta v \to 0}{Lt} \quad \rho = \frac{\Delta m}{\Delta v}$ where ρ is the density of the body. For a small volume dv, its mass is ρdv. The field due to the entire mass along the x, y, and z directions are given by

$$g_x = \iiint\limits_{v} \frac{\rho\,(x - \xi)\, d\xi d\eta d\zeta}{\left[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2\right]^{3/2}} \tag{3.12}$$

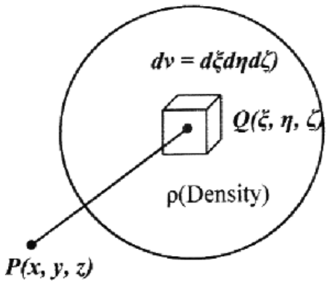


Fig. 3.3. Gravitational attraction at the point P due to a large solid mass

$$g_y = \iiint_v \frac{\rho\,(y-\eta)\,d\xi d\eta d\zeta}{\left[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right]^{3/2}} \tag{3.13}$$

$$g_z = \iiint_v \frac{\rho\,(z-\zeta)\,d\xi d\eta d\zeta}{\left[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right]^{3/2}} \tag{3.14}$$

3.5 Gravitational Field due to a Line Source

Let AB is a line source of mass having linear density λ where $\lambda = \underset{\Delta l \to O}{\mathrm{Lt}} \frac{\Delta m}{\Delta l}$ (Fig. 3.4). The line is put in the xy plane such that the centre of the line is at the origin. Let the line be divided into a number of segments $d\xi$. Thus the field components along the x direction is given by

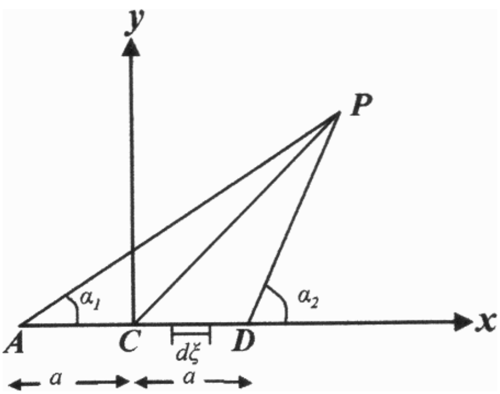


Fig. 3.4. Gravitational field at an external point P due to a line source of finite length

$$F_x = \frac{\lambda d\xi}{(x-\xi)^2 + y^2} \cdot \frac{x-\xi}{\sqrt{(x-\xi)^2 + y^2}} \tag{3.15}$$

and

$$F_y = \frac{\lambda d\xi}{(x-\xi)^2 + y^2} \cdot \frac{y}{\sqrt{(x-\xi)^2 + y^2}}. \tag{3.16}$$

Therefore, the total fields F_x and F_y for the entire linear mass are given by

$$F_x = \lambda \int_{-a}^{a} \frac{(x-\xi)\, d\xi}{\left[(x-\xi)^2 + y^2\right]^{3/2}} \tag{3.17}$$

$$F_y = \lambda \int_{-a}^{a} \frac{y\, d\xi}{\left[(x-\xi)^2 + y^2\right]^{3/2}} \tag{3.18}$$

Let

$$\cos\alpha = \frac{y}{\left\{(x-\xi)^2 + y^2\right\}^{1/2}}$$

and

$$-\sin\alpha\, d\alpha = -\frac{2y\,(x-\xi)\, d\xi}{2\left[(x-\xi)^2 + y^2\right]^{3/2}}.,$$

therefore

$$\begin{aligned} F_x &= \frac{\lambda}{y} \int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha = \frac{\lambda}{y} (\cos\alpha_2 - \cos\alpha_1) \\ &= \frac{\lambda}{y} \left[\frac{y}{\sqrt{(x+a)^2 + y^2}} - \frac{y}{\sqrt{(x-a)^2 + y^2}} \right] \end{aligned} \tag{3.19}$$

Similarly for

$$F_y = \lambda \left[\frac{y d\xi}{\left[(x-\xi)^2 + y^2\right]^{3/2}} \right] \tag{3.20}$$

Let

$$\tan\alpha = \frac{x-\xi}{y}, \text{and } \sec^2\alpha d\alpha = -\frac{d\xi}{y}.$$

since $\sec^3\alpha = \frac{1}{y^3}\left((x-\xi)^2 + y^2\right)^{3/2}$, we get

$$F_y = -\frac{\lambda}{y}\int_{\alpha_2}^{\alpha_1} \cos\alpha d\alpha = \frac{\lambda}{y}\left[\sin\alpha_1 - \sin\alpha_2\right]$$
$$= \frac{\lambda}{y}\left[\frac{x-a}{\sqrt{(x-a)^2+y^2}} - \frac{x+a}{\sqrt{(x+a)^2+y^2}}\right] \tag{3.21}$$

Now

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= \frac{\lambda}{y}\left[2 - 2\cos\alpha_1\cos\alpha_2 - 2\sin\alpha_1\sin\alpha_2\right]^{1/2} \\ &= \frac{\lambda}{y}\left[2 - 2\cos(\alpha_1 - \alpha_2)\right]^{1/2} \\ &= \sqrt{2}\frac{\lambda}{y}\left[2\sin^2\frac{(\alpha_1-\alpha_2)}{2}\right]^{1/2} = 2\frac{\lambda}{y}\sin\left(\frac{\alpha_1-\alpha_2}{2}\right) \\ &= 2\frac{\lambda}{y}\sin\frac{A\hat{P}B}{2} \end{aligned} \tag{3.22}$$

Therefore, the direction of the gravitational force of attraction is along the bisector of the triangle $A\hat{P}B$.

3.6 Gravitational Potential due to a Finite Line Source

Although once field due to a finite line source is known, the gravitational potential at a point is known also. However a separate section is given to highlight a few points of principle.

A line source AB of length L is taken along the z-direction (Fig. 3.5). In a cylindrical co-ordinate, the potential will not depend upon the azimuthal angle ψ but on r and z. Take a small element $d\zeta$. Its mass is $\lambda d\zeta$ where λ is the linear density. Since the gravitational potential is $-G\frac{m}{R}$ where m and r are respectively the mass and distance of the point of observation. We can write

$$\phi = -G\lambda\int_{-l}^{l}\frac{d\zeta}{\left[r^2 + (z-\zeta)^2\right]^{1/2}} \tag{3.23}$$

Let

$$\begin{aligned} z - \zeta &= r\tan\theta, \text{so} \\ -d\zeta &= r\sec^2\theta d\theta \end{aligned}$$

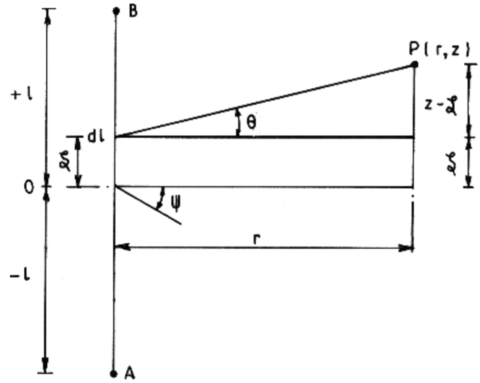


Fig. 3.5. Gravitational potential at a point due to a finite line source

Therefore

$$\phi = G\lambda \ln\left[\sec\theta + \tan\theta\right]_{-l}^{l}$$

$$\phi = G\lambda\left[\ln\left\{\frac{z-l}{r} + \frac{\sqrt{r^2-(z-l)^2}}{r}\right\} - \ln\left\{\frac{z+l}{r} + \frac{\sqrt{r^2+(z+l)^2}}{r}\right\}\right]$$

$$\phi = -G\lambda \ln \frac{(z+l) + \sqrt{r^2 + + (z+l)^2}}{(z-l) + \sqrt{r^2 + + (z-l)^2}} \qquad (3.24)$$

When $\phi =$ Constant

$$K = e^{-\phi/G\lambda} = \frac{z+l+\sqrt{r^2+(z+l)^2}}{z-l+\sqrt{r^2+(z-l)^2}} \qquad (3.25)$$

This is an equation for the equipotential surface and (3.25) can be rewritten with a few steps of algebraic simplification in the form.

$$\frac{(K-1)^2}{(K+1)^2}.\left(\frac{z}{l}\right)^2 + \frac{(K-1)^2}{4K}.\left(\frac{r}{L}\right)^2 = 1. \qquad (3.26)$$

This is an equation of an ellipse. The semi major and minor axes are respectively given by

$$\frac{K+1}{K-1}.l \text{ and } \frac{2\sqrt{K}}{K-1}l.$$

The eccentricity is $e = \sqrt{1-\frac{b^2}{a^2}}$ and the foci are at $\pm l$.

The field components are

$$f_r = -\frac{\partial\phi}{\partial r} = G\lambda.\frac{1}{r}\left[\frac{z+l}{\sqrt{r^2+(z+l)^2}} - \frac{z-l}{\sqrt{r^2+(z-l)^2}}\right] \tag{3.27}$$

$$f_z = -\frac{\partial\phi}{\partial z} = -G\lambda\frac{1}{r}\left[\frac{r}{\sqrt{r^2+(z+l)^2}} - \frac{r}{\sqrt{r^2+(z-l)^2}}\right] \tag{3.28}$$

The total field of $f = \sqrt{f_r^2 + f_z^2}$ describes hyperbolas (Fig. 3.6). For a very long wire where

$$l \to \alpha, g_z = O \text{ and } g_r = -G\lambda\frac{2}{r} \tag{3.29}$$

Here the field is proportional to $\frac{1}{r}$. Hence the potential is

$$\begin{aligned}\phi &= \lambda\int g dr + \text{Constant}\\ &= 2G\lambda \ln r + \text{Constant}\\ &= -2G\lambda \ln\left(\frac{r_0}{r_1}\right)\end{aligned} \tag{3.30}$$

It implies that the potential becomes zero at infinity. Therefore for a line source potential is logarithmic potential and field varies inversely with distance. For a finite line source the equipotentials are ellipses and the eccentricity of the ellipse die down with distance from the source. At infinite distance the eccentricity of the ellipse will be zero and the ellipse will turn into a circle. The field lines will be radial For an infinitely long line source the field lines and equipotential lines will form a square or rectangular grids

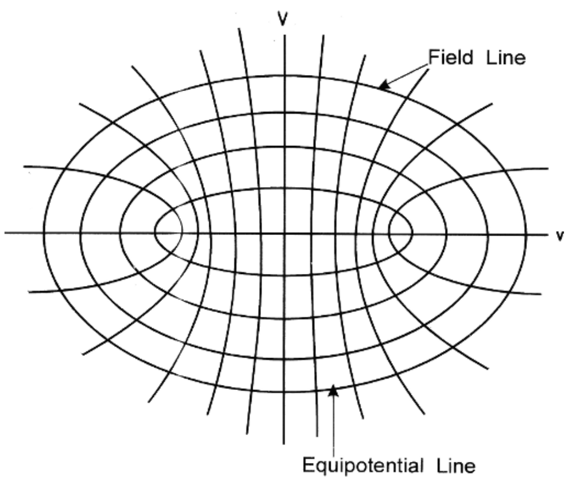


Fig. 3.6. Field lines and equipotential lines due to a line source of finite length

3.7 Gravitational Attraction due to a Buried Cylinder

Vertical component of gravitational attraction at unit mass at the point of observation 'P' due to a small element 'dl' at a distance 'l' from 'Q', the shortest distance of the cylinder from the point of observation is (Fig. 3.7)

$$dg_z = \frac{G\ dm\ Sin\theta}{r^2}.\frac{R}{r} = G\lambda\, dl\ Sin\theta.\frac{R}{r}. \tag{3.31}$$

Here dm is the elementary mass of the thin strip dl, λ is the linear density of the cylinder and is equal to $\pi a^2\lambda$ for unit length where 'a' is the radius of the cylinder and G is the universal gravitational constant. Vertical gravitational attraction due to a cylinder of infinite length is

$$g_z = G\lambda Sin\theta R \int_{-\alpha}^{\alpha} \frac{dl}{(R^2 + l^2)^{3/2}} \tag{3.32}$$

$$= 2\ G\pi a^2\lambda \frac{z}{(x^2 + z^2)}. \tag{3.33}$$

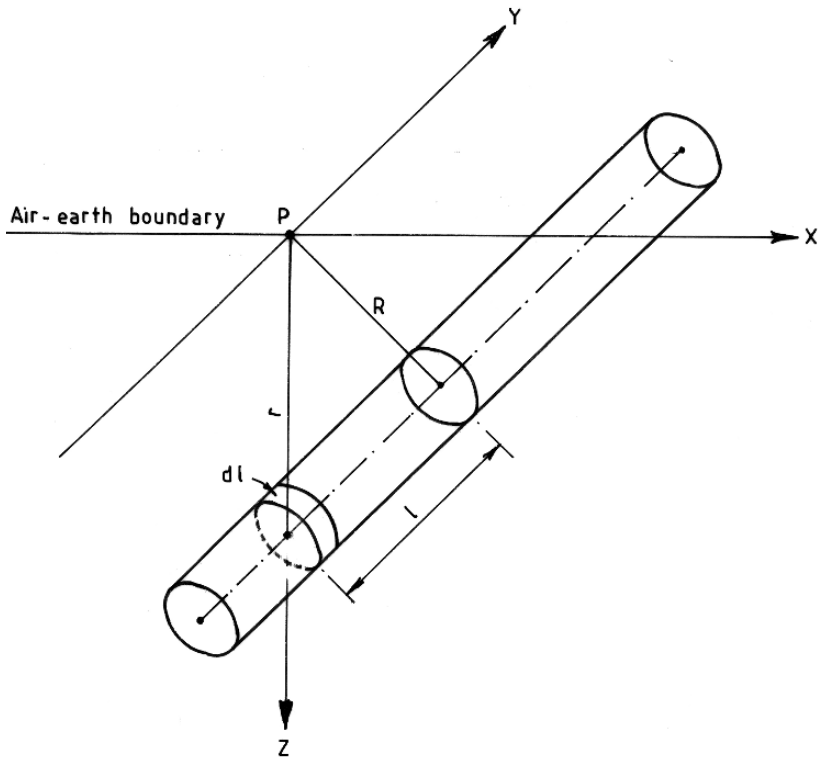


Fig. 3.7. Gravitational anomaly on the surface due to a buried cylindrical body of finite length

3.8 Gravitational Field due to a Plane Sheet

Let us assume a plane sheet in the xy plane and is symmetrical around the axis OP vertical to the plane of the paper (Fig. 3.8).We assume an elementary area ds on the plane sheet. Let

$$\sigma = \lim_{\Delta S \to 0} \frac{\Delta m}{\Delta S}$$

where Δs is the surface area of an infinitesimally small surface area in the plane sheet and Δm be its mass.

For a small area ds, the mass of the area is σds. The gravitational field due to this small element at P is $\frac{\sigma ds}{r^2}$. Vertical component of this field is given by $f_z = \frac{\sigma ds}{r^2} \cos\alpha$ and the component at right angles to z direction is $\frac{\sigma ds}{r^2} \sin\alpha$. Since the point of observation P is symmetrically placed with respect to the plate, the vertical components of the field will get added up. The components perpendicular to the z-direction will get cancelled. The vertical component of the field is

$$f_z = \int\int \frac{\sigma ds \cos\alpha}{r^2} = \int \sigma d\omega = \sigma\omega \tag{3.34}$$

where

$$\frac{ds \cos\theta}{r^2} = d\omega.$$

Here $d\omega$ is the solid angle subtended by the elementary mass at the P, ω is the total solid angle subtended by the plate at the point P. The field at the point P is equal to the density multiplied by the solid angle subtended at the point P. For an infinitely large sheet $\omega = 2\pi$.

$$f_z = 2\pi\sigma \tag{3.35}$$

and. the field is independent of the distance of the point of observation from the plate.

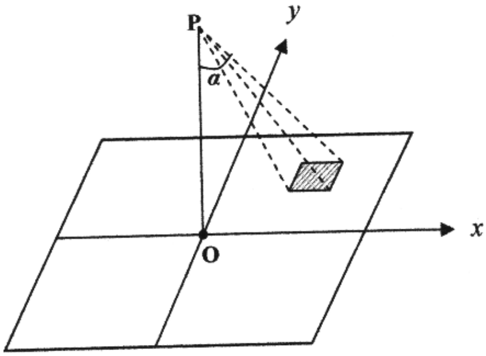


Fig. 3.8. Gravitational field on the vertical axis of a square horizontal plate

3.9 Gravitational Field due to a Circular Plate

In this section, we shall derive the expression for the gravity field due to circular plate. Let us choose the polar co-ordinate (r, ψ). The field at a point P along a vertical line crossing the plane of the plate at right angles for the elementary mass is $\frac{\sigma r dr d\psi}{R^2}$. where σ is surface density (Fig. 3.9). Therefore, the vertical component of the field is

$$\Delta f_z = \frac{\sigma r dr d\psi}{R^2} \cos\alpha. \tag{3.36}$$

Other components will vanish because of symmetry of the problem. Hence

$$\begin{aligned}
f_z &= \int_o^a \int_o^{2\pi} \frac{\sigma r dr}{R^2} d\psi . \frac{z}{R} \left[\because \cos\alpha = \frac{z}{R}, R^2 = r^2 + z^2\right] \\
&= \int_o^a \int_o^{2\pi} \frac{\sigma z r dr d\psi}{(r^2+z^2)^{3/2}} \qquad (3.37) \\
&= 2\pi\sigma z \int_o^a \frac{r dr}{(r^2+z^2)^{3/2}} = 2\pi\sigma z \left[-\frac{1}{\sqrt{r^2+z^2}}\right]_o^a \\
&= 2\pi\sigma z \left[\frac{1}{z} - \frac{1}{\sqrt{z^2+a^2}}\right] = 2\pi\sigma \left[1 - \frac{z}{\sqrt{z^2+a^2}}\right].
\end{aligned}$$

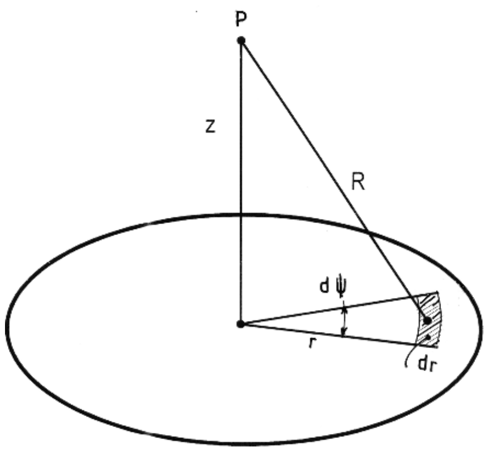


Fig. 3.9. Gravitational field on the vertical axis of a horizontal circular plate

3.10 Gravity Field at a Point Outside on the Axis of a Vertical Cylinder

To compute gravitational field at a point P on the axis of a cylinder at a point outside it, we assume an elementary mass $\sigma\rho d\rho d\psi\, dz$ inside the cylinder (Fig. 3.10). Here R is the radius of the cylinder. h_u and h_d are the depths, from the surface, to the top and bottom planes of a cylinder of length or height H. σ is the volume density of mass.

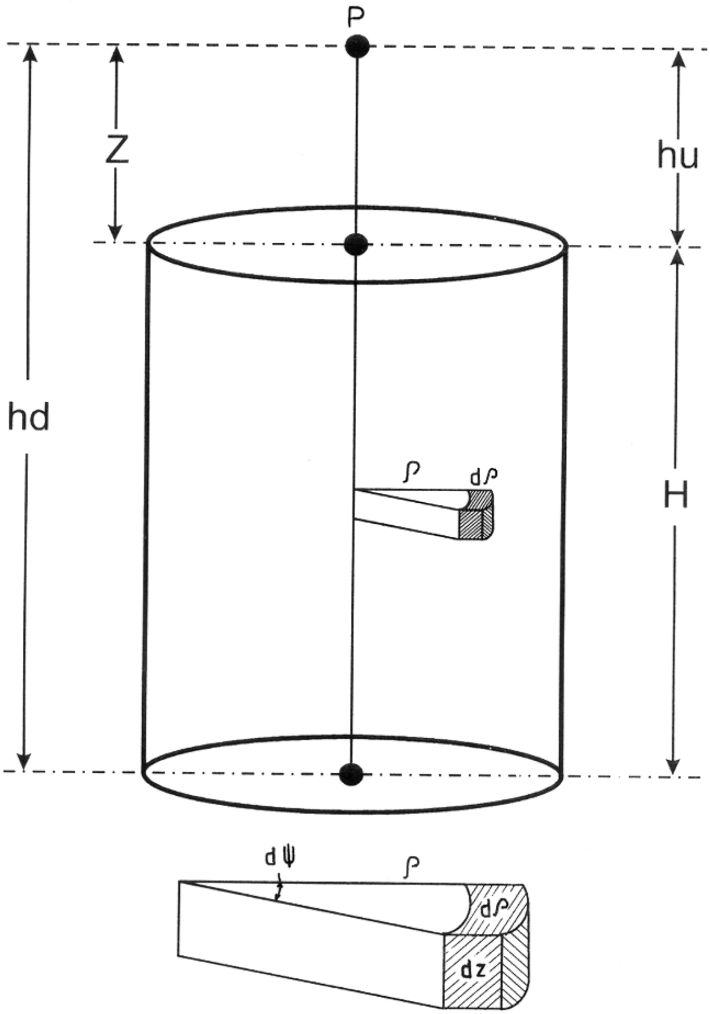


Fig. 3.10. Gravitational field on the axis of a vertical solid cylindrical body at a point outside the body

The gravitational attraction due to an elementary mass at the point P outside the cylinder and on its axis is

$$\mathrm{dg} = \frac{\mathrm{G\ dm}}{\mathrm{r}^2} = \mathrm{G}\frac{\sigma\rho \mathrm{d}\rho \mathrm{d}\psi \mathrm{dz}}{\mathrm{r}^2} \tag{3.38}$$

where $\mathrm{dv} = \rho \mathrm{d}\rho \mathrm{d}\psi \mathrm{dz}$. ρ is the radial distance of the elementary mass from the axis of the cylinder. Since only the vertical component is of interest.

So

$$\begin{aligned}\mathrm{dg_z} &= \mathrm{dg}\cos\theta = \mathrm{dg}\frac{\mathrm{z}}{\mathrm{r}}\\ &= \frac{\mathrm{G}\sigma\rho \mathrm{d}\rho.\mathrm{d}\psi \mathrm{zdz}}{\mathrm{r}^3}.\end{aligned} \tag{3.39}$$

The total gravitational attraction of the cylinder at the point P is given by

$$\Delta \mathrm{g_z} = \mathrm{G}\sigma \int\limits_{\rho=\mathrm{o}}^{\mathrm{R}} \int\limits_{\mathrm{hu}}^{\mathrm{hd}} \int\limits_{\psi=\mathrm{o}}^{2\pi} \frac{\rho \mathrm{d}\rho \mathrm{d}\psi \mathrm{zdz}}{(\rho^2+\mathrm{z}^2)^{3/2}} \tag{3.40}$$

$$\Rightarrow \mathrm{G}\sigma \int\limits_{\mathrm{o}}^{2\pi} \mathrm{d}\psi \int\limits_{\mathrm{hu}}^{\mathrm{hd}} \mathrm{zdz} \int\limits_{\mathrm{o}}^{\mathrm{R}} \frac{\rho \mathrm{d}\rho}{(\rho^2+\mathrm{z}^2)^{3/2}} \tag{3.41}$$

$$\Rightarrow 2\pi \mathrm{G}\sigma \int\limits_{\mathrm{hu}}^{\mathrm{hd}} \mathrm{zdz} \int\limits_{\mathrm{o}}^{\mathrm{R}} \frac{\rho \mathrm{d}\rho}{(\rho^2+\mathrm{z}^2)^{3/2}}$$

$$\Rightarrow 2\pi \mathrm{G}\sigma \int\limits_{\mathrm{hu}}^{\mathrm{hd}} \mathrm{zdz} \left| -\frac{1}{(\rho^2+\mathrm{z}^2)^{1/2}} \right|_{\mathrm{o}}^{\mathrm{R}} \tag{3.42}$$

$$\Rightarrow 2\pi \mathrm{G}\sigma \left[\int\limits_{\mathrm{hu}}^{\mathrm{hd}} \mathrm{dz} - \int\limits_{\mathrm{hu}}^{\mathrm{hd}} \frac{\mathrm{zdz}}{(\mathrm{R}^2+\mathrm{z}^2)} \right]_{\mathrm{o}}^{\mathrm{R}}$$

$$\Rightarrow 2\pi \mathrm{G}\sigma \int\limits_{\mathrm{hu}}^{\mathrm{hd}} \mathrm{zdz} \left(\frac{1}{\mathrm{z}} - \frac{1}{\sqrt{\mathrm{R}^2+\mathrm{z}^2}} \right)$$

$$\Rightarrow 2\pi \mathrm{G}\sigma \left[(\mathrm{h_d}-\mathrm{h_u}) - \sqrt{\mathrm{R}^2+\mathrm{h_d^2}} + \sqrt{\mathrm{R}^2+\mathrm{h_u^2}} \right] \tag{3.43}$$

So

$$\Delta g_z = 2\pi G\sigma \left[H - \sqrt{R^2+h_d^2} + \sqrt{R^2+h_u^2} \right]. \tag{3.44}$$

Case I

If the point of observation is located right on the upper surface of the cylinder. Then $\mathrm{h_u} = \mathrm{O}, \mathrm{h_d} = \mathrm{H}$ and

$$\Delta g_z = 2\pi G\sigma \left[H - \sqrt{R^2 + H^2} + R \right]. \tag{3.45}$$

Case II

When R >> H

$$\Delta g_z = 2\pi G\sigma H. \tag{3.46}$$

This is the expression for Bouguer gravity anomaly due to the plate.

Case III

When R << H

$$\Delta g_z = 2\pi G\sigma R. \tag{3.47}$$

This is the expression for a gravitational field for $H \rightarrow \infty$. i.e., for columnar structures like volcanic pipe or a long cylindrical intrusions like mantle xenoliths etc.

3.11 Gravitational Potential at a Point due to a Spherical Body

Let a small elementary circular shell is assumed in a spherical body at a distance 'a' from the centre of the sphere (Fig. 3.11).The point of observation P is at a distance R from the centre of the sphere and 'r' from the elementary mass.

So

$$dm = \rho a^2 \sin\theta \, d\,\theta \, da \, d\,\psi$$

where ρ is the density of the material of the spherical body (spherical shell or solid sphere). The gravitational potential at a point P is given by

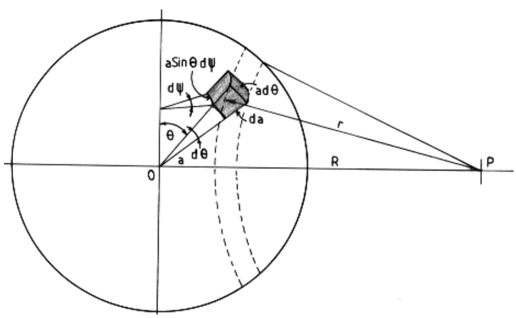


Fig. 3.11. Gravitational potential and field at a point P outside a spherical body (solid or hollow)

$$\phi_p = -G \int_o^{2\pi} \int_o^{\pi} \frac{\rho a^2 \sin\theta \; d\theta \; da \; d\psi}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} \tag{3.48}$$

where

$$r^2 = a^2 + R^2 - 2aR \;\cos\theta$$

and

$$r dr = aR \;\sin\theta \; d\;\theta.$$

So

$$\phi_p = -2\pi G\rho \frac{a}{R} da \int dr. \tag{3.49}$$

Case I

When the point P is outside the sphere

$$\phi_0 = -2\pi G\rho \frac{a}{R} da \int_{R-a}^{R+a} dr$$

$$= -4\pi G\rho \frac{a^2}{R} \; da. \tag{3.50}$$

Since the total mass of the shell $= 4\pi\rho a^2 da$, the potential $\phi_p = -G\frac{m}{R}$ as if the mass of the spherical shell is put at the centre.

Case II

If the point of observation P is inside (Fig. 3.12)

$$\phi_i = -2\pi G\rho \frac{a}{R} da \int_{a-R}^{a+R} dr = -4\pi G\rho \, a da \tag{3.51}$$

$= -G\frac{m}{a}$ where m is the total mass of the shell.

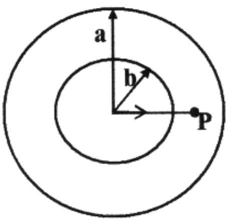


Fig. 3.12. Computation of potential at a point P within a spherical shell of outside radius a and inside radius b

When the outer and inner radii of the shell are respectively 'a' and 'b', the mass is $M = \frac{4}{3}\pi\left(a^3 - b^3\right)\rho$.

Therefore the potential outside is

$$\phi_0 = -\frac{4\pi G\rho}{R}\int_b^a a^2\,da = -\frac{4\pi}{3}\frac{G\rho}{R}\left[a^3 - b^3\right] = -G\frac{M}{R}. \tag{3.52}$$

The potential at internal point

$$\phi_i = -4\pi G\rho\int_b^a a\,da = -2\pi G\rho\left(a^2 - b^2\right). \tag{3.53}$$

Since $a^2 - b^2 =$ Constant, the field inside $\left(\frac{\partial \rho i}{\partial r} = 0\right)$ is zero for the solid sphere. $\phi_i = 2\pi G\rho a^2 =$ Constant. Therefore, the field inside will be zero.

The gravitational potential at any point inside a solid body is determined by the mass internal to the point inside the sphere of radius 'a'. The mass outside does not have any effect on the potential. It shows that the gravitational field at the centre of the earth is zero.

Case III

When the point of observation 'P' is within the spherical shell

For the point P outside, the potential

$$\phi_0 = \frac{4}{3}\pi G\rho\frac{R^3 - b^3}{R} \tag{3.54}$$

and for inside

$$\phi_i = 2\pi G\rho(a^2 - R^2). \tag{3.55}$$

So the total potential ϕ in the material itself is

$$\begin{aligned}\phi_{total} = \phi_0 + \phi_i &= -\frac{4}{3}\pi G\rho\left(\frac{R^3 - b^3}{R}\right) - 2\pi GR\left(a^2 - R^2\right)\\ &= -4\pi G\rho\left[\frac{1}{2}a^2 - \frac{1}{3}\frac{b^3}{R} - \frac{1}{6}R^2\right].\end{aligned} \tag{3.56}$$

The gravity field

$$g_m = -\frac{\varphi_n}{\partial R} = -\frac{4}{3}\pi G\rho\left[-\frac{b^3}{R^2} + R\right] = -\frac{4}{3}\pi G\rho\left[-\frac{R^3 - b^3}{R^2}\right]. \tag{3.57}$$

We can now examine the continuity of the potentials at the boundaries

(i) when $R < b$

$$\phi_p = 2\pi G\rho\left(a^2 - b^2\right) \tag{3.58}$$

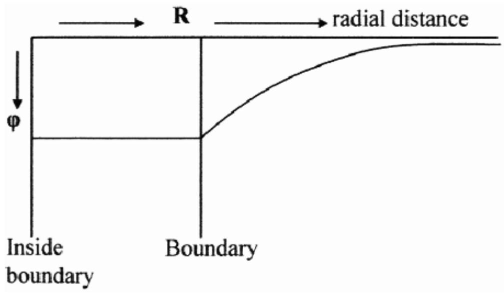


Fig. 3.13a. Potential at a point inside a spherical shell due to mass of the spherical shell

(ii) when $b < R < a$

$$\phi_p = -4\pi G\rho\left(\frac{1}{a^2} - \frac{1}{3}\frac{b^3}{R} - \frac{1}{6}R^2\right) \tag{3.59}$$

(iii) when $R > a$

$$\phi_p = -\frac{4}{3}\pi G\rho\frac{a^3 - b^3}{R^2}. \tag{3.60}$$

When $R = b$, potentials for case (i) and (ii) becomes $2\pi G\rho(a^2 - b^2)$ and when $R = a$, the potentials for case (ii) and (iii) becomes

$$-\frac{4}{3}\pi\rho G\frac{a^3 - b^3}{a}.$$

Therefore, the potential remains same inside the boundary. As soon as the point of observation comes out on the surface, the potential and field intensities decreases with distance as follows (Fig. 3.13a and Fig. 3.13b):

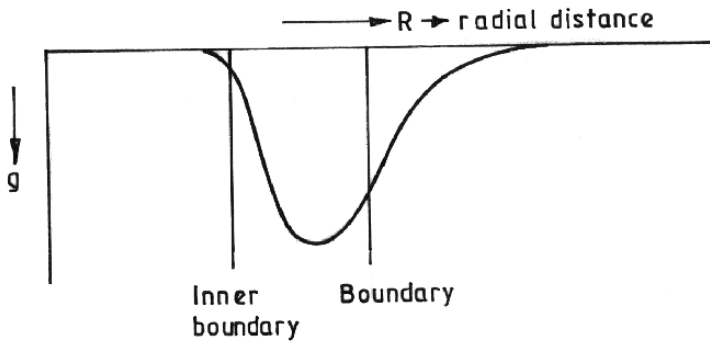


Fig. 3.13b. A curtoon of variation of 'g' both outside and inside the air-earth boundary, $g = 0$ at the centre of the earth as well as in the outer space;maximum value of g is at a certain depth from the surface

(i) for $R < b$

$$g_p = 0$$

(ii) for $b < R < b$

$$g_p = -\frac{4}{3}\pi G\rho \frac{R^3 - b^3}{R^2} \qquad (3.61)$$

Both potential and gravitational field are continuous across the boundary.

3.12 Gravitational Attraction on the Surface due to a Buried Sphere

Gravitational attraction upon unit mass at a point P on the surface due to a buried sphere of radius R, density σ and buried at a depth z is given by

$$G\frac{M}{r^2} = \frac{4}{3}\frac{\pi G R^3 \sigma}{(z^2 + x^2)} \qquad (3.62)$$

where $M(= \frac{4}{3}\pi R^3 \sigma)$ is the mass of the spherical body and $r = \sqrt{z^2 + x^2}$. The vertical component of the gravitational attraction will be equal to (Fig. 3.14)

$$g_z = G\frac{M}{r^2}.\frac{z}{r} = \frac{4}{3}\pi G R^3 \sigma.\frac{3}{(z^2 + r^2)^{3/2}}. \qquad (3.63)$$

Gravitational force will be maximum at the origin i.e., at $z = 0$ and $x = 0$ on the surface. The value will die down symmetrically on both the sides of

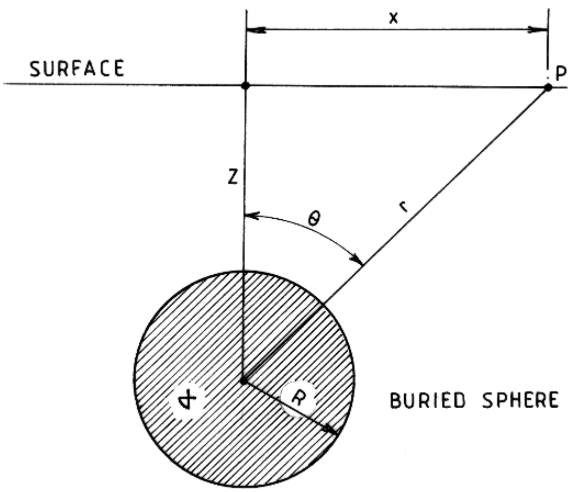


Fig. 3.14. Gravitational anomaly on the surface due to a buried spherical body of radius R

the spherical mass with increasing distance x from the origin. For further studies the readers are referred to the works of Blakely(1996), Talwani and Ewing(1961), Radhakrishnamurthy(1998) Telford et al (1976) and Dobrin and Savit(1988).

3.13 Gravitational Anomaly due to a Body of Trapezoidal Cross Section

Gravitational attraction on the surface at a point P due to a body of rectangular cross section present within the depth extent of z_2 and $z_1 (z_2 > z_1)$ (Fig. 3.15) is given by

$$\Delta g_p = 2G\rho \int_{z_1}^{z_2} \int_{x_1}^{x_2} \frac{z \, dx \, dz}{x^2 + z^2} \tag{3.64}$$

where G is the universal gravitational constant and ρ is the density contrast of the body with the surrounding host rocks

For a two dimensional body of trapezoidal cross-section as shown in the Fig. 3.15.

$$RQ = rd\theta = dx \, \mathrm{Sin}\theta \tag{3.65}$$

The gravitational anomaly due to a two dimensional body of trapezoidal cross section is given by

$$\Delta g_\rho = 2G\rho \int \int \frac{z}{x^2 + z^2} dx \, dz. \tag{3.66}$$

Since $z = r\mathrm{Sin}\theta$, and $\frac{z}{x^2+z^2} = \frac{r \, \mathrm{Sin}\theta}{r^2} = \frac{\mathrm{Sin}\theta}{r}$,
we get

$$\Delta g_\rho = 2G\rho \int \int \frac{\mathrm{Sin}\theta}{r} dx \, dz \tag{3.67}$$

$$= 2G\rho \int_{\theta_1}^{\theta_2} \int_{z_1}^{z_2} d\theta \, dz. \tag{3.68}$$

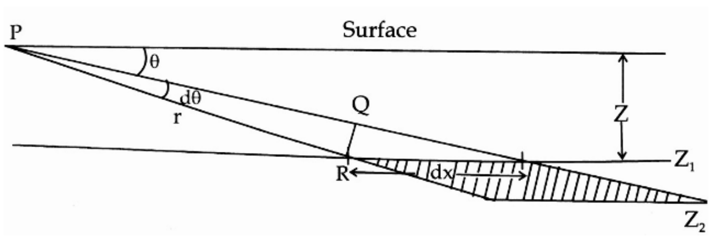


Fig. 3.15. Gravitational anomaly due to a two dimensional body of trapezoidal cross section

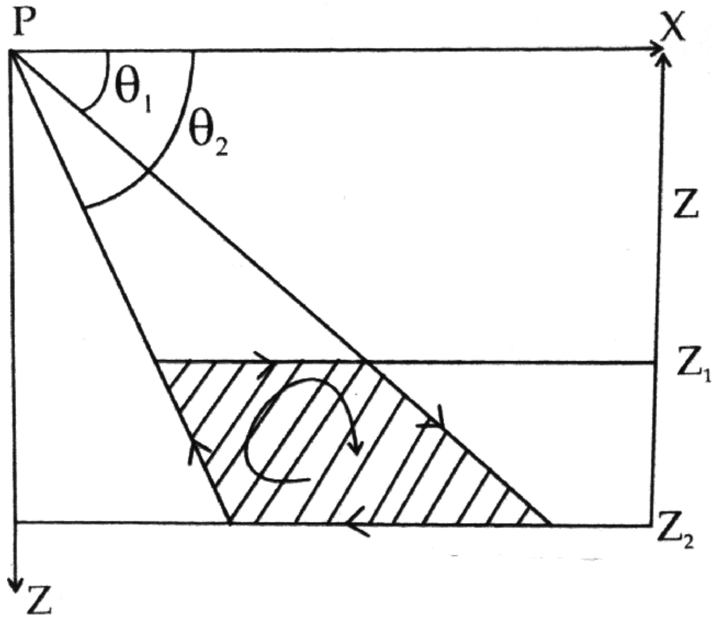


Fig. 3.16. Enlarged view of a trapezoidal cross section of a body and the direction of line integral

Line integral along the trapezoidal cross section (Fig. 3.16) is given by

$$\oint \theta dz = \int_{\theta_1, z_1}^{\theta_2} \theta dz + \int_{z_1, \theta_1}^{z_2} \theta dz + \int_{\theta_2, z_2}^{\theta_1} \theta dz + \int_{z_2, \theta_1}^{z_1} \theta dz \tag{3.69}$$

$$= 0 + \theta_2(z_2 - z_1) + 0 + \theta_1(z_1 - z_2)$$
$$= (z_2 - z_1)(\theta_2 - \theta_1). \tag{3.70}$$

3.13.1 Special Cases

Case I Gravitation Attraction Due to a Two Dimensional Horizontal Slab

Figure 3.17 shows the geometry of the problem. Gravitational anomaly at the point P.

$$\Delta g_\rho = 2G\rho \left[\int_A^B \theta dz + \int_B^C \theta dz + \int_C^D \theta dz + \int_D^A \theta dz \right] \tag{3.71}$$

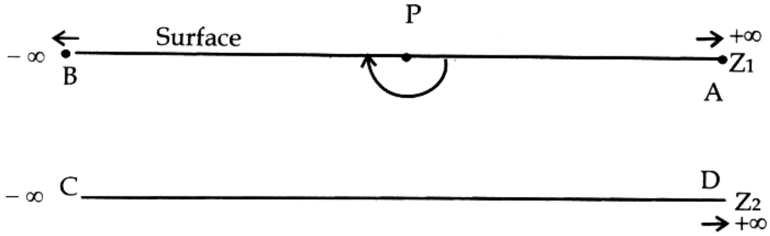


Fig. 3.17. Gravitational attraction at the point P due to a two dimensional horizontal slab

$$= 2G\rho\left[\int_{0,z_1}^{\pi} \theta dz + \int_{\pi,z_1}^{\pi,z_2} \theta dz + \int_{\pi,z_2}^{\pi,z_2} \theta dz + \int_{\pi,z_1}^{\pi,z_2} \theta dz\right] \tag{3.72}$$

$$= 2G\rho[0 + \pi(z_2 - z_1) + 0 + 0]$$

$$= 2\pi G\rho(z_2 - z_1). \tag{3.73}$$

Case II Gravitational Anomaly Due to a Fault with a Small Throw

Figure 3.18 shows the geometry of the figure. The gravitational anomaly at the point P is

$$\Delta g_\rho = 2G\rho\left[\int_{0,z_1}^{\theta} \theta dz + \int_{z_1,\theta}^{z_2} \theta dz + \int_{z_2,\theta}^{0} \theta dz + \int_{z_2,0}^{z_1} \theta dz\right] \tag{3.74}$$

$$= 2G\rho[0 + \theta_2(z_2 - z_1) + 0 + 0] \tag{3.75}$$

$$= 2G\rho\theta h = 2G\rho h\ \tan^{-1}\frac{z}{x}. \tag{3.76}$$

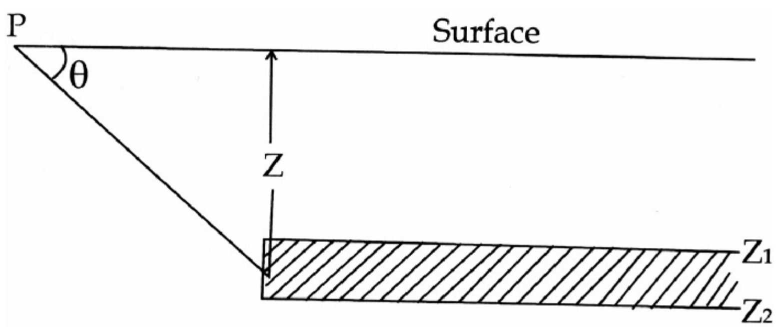


Fig. 3.18. Gravitational anomaly due to a fault with a small throw

Case III Gravitational Anomaly due to a Body of Rectangular Cross Section

Figure 3.19 Show the geometry of the problem. Gravitational anomaly at a point P is given by

$$\Delta g_\rho = 2G\rho\left[\int_{\theta_2,z_1}^{\theta_1}\theta dz + \int_{\theta_1,z_1}^{\theta_2,z_2}\theta dz + \int_{\theta_4,z_2}^{\theta_3}\theta dz + \int_{\theta_3,z_2}^{\theta_2,z_1}\theta dz\right] \tag{3.77}$$

$$= 2G\rho\left[0 + \int_{z_1}^{z_2}\tan^{-1}\frac{z}{x}dz + 0 + \int_{z_2}^{z_2}\tan^{-1}\frac{z}{x_2}dz\right] \tag{3.78}$$

$$= 2G\rho\left[z_2(\theta_4 - \theta_3) + z_1(\theta_2 - \theta_1) - \frac{1}{2}\left(x_1 \ln\frac{x_1^2 + z_2^2}{x_1^2 + z_1^2} + x_2 \ln\frac{x_2^2 + z_1^2}{x_2^2 + z_2^2}\right)\right]. \tag{3.79}$$

Since $\int \tan^{-1}\frac{x}{a}dx = x\ \tan^{-1}\frac{x}{a} - \frac{a}{2}\ln(a^2 + x^2)$

Case IV Gravitational Attraction at a Point on the Surface due to a Thin Plate

Figure 3.20 shows the geometry of the problem. Gravitational anomaly at the point P due to the thin plate of finite length is given by

$$\begin{aligned}\Delta g_\rho &= 2G\rho\left[z_2(\theta_4 - \theta_3) + z_1(\theta_2 - \theta_1)\right]\\ &= 2G\rho\left[\Delta\dot{z}(\theta_1 - \theta_2)\right]\\ &= 2G\rho t\theta\end{aligned} \tag{3.80}$$

where θ is the angle made by the plate at the point of observation P. t $(= \Delta z)$ is the thickness of the plate.

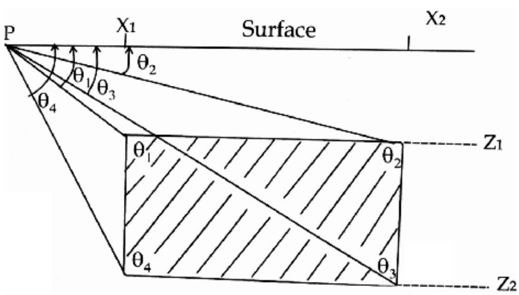


Fig. 3.19. Geometry of the buried prism of rectangular cross section and the point of observation on the surface

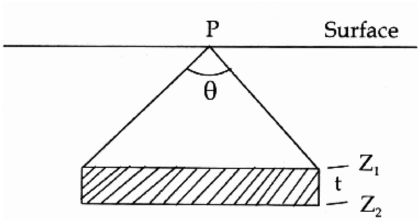


Fig. 3.20. Geometry of a buried thin slab and the point of observation on the surface

Case V Gravitational Attraction at a Point on the Face of a Two Dimensional Ridge

Figure 3.21 shows the geometry of the problem. Gravitational attraction at a point P on the face of a two dimensional ridge is given by

$$\begin{aligned}\Delta g_\rho =&2G\rho\left[\left\{\int_{x_1,z_1}^{\infty}\theta dz+\int_{z_1,\infty}^{z_2}\theta dz+\int_{\infty,z_2}^{x_2}\theta dz+\int_{x_2,z_2}^{x_1,z_1}\theta dz\right\}\right.\\&\left.+\left\{\int_{x_1,z_1}^{x_0,z_0}\theta dz+\int_{x_0,z_0}^{\infty}\theta dz+\int_{z_o,\infty}^{z_1}\theta dz+\int_{0,z_1}^{x_1}\theta dz\right\}\right] \quad (3.81)\\=&2G\rho\left[\{0+2\pi(z_2-z_1)+0+(2\pi-\alpha)(z_1-z_2)\}\right.\\&\left.+\{(\pi-\alpha)(z_o-z_1)+0+0+0\}\right] \quad (3.82)\\=&2G\rho[(z_2-z_1)(2\pi-2\pi+\alpha)+(\pi-\alpha)(z_o-z_1)] \quad (3.83)\\=&2G\rho[\alpha(z_2-z_1-z_o+z_1)+\pi(z_o-z_1)] \quad (3.84)\\=&2G\rho\alpha(z_2-z_o)-2\pi G\rho(z_1-z_o). \quad (3.85)\end{aligned}$$

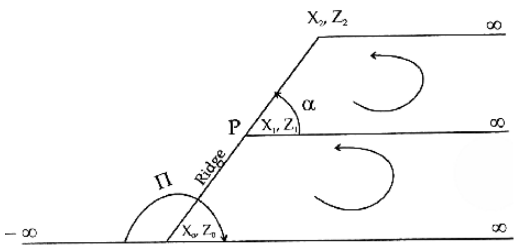


Fig. 3.21. Geometry of a ridge and the point of observation on the ridge surface

Case VI Gravitational Attraction on the Surface due to a Buried Two Dimensional Body of Hexagonal Cross Section.

Figure 3.22 shows the geometry of the problem. Here gravitational attraction at a point P due to an elementary strip of hexagonal cross section is given by

$$\Delta g_\rho = \oint \theta dz. \tag{3.86}$$

In segment BC

$$\begin{aligned} z &= x\tan\theta = (x - a_i)\tan\psi_1 \\ &= x\tan\psi_1 - a_i\tan\psi_i. \end{aligned} \tag{3.87}$$

From the (3.87) we get

$$x = a_i \ \tan\psi_i/(\tan\psi_i - \tan\theta) \tag{3.88}$$

and

$$z = x\tan\theta = \frac{a_i\tan\theta \ \tan\psi_i}{\tan\psi_i - \tan\theta}, \tag{3.89}$$

therefore

$$\Delta g_{BC} = 2G\rho \oint a_i \frac{\tan\theta \ \tan\psi_i}{\tan\psi_i - \tan\theta} d\theta. \tag{3.90}$$

The total gravity anomaly will be

$$\Delta g = \Delta g_{BC} + \Delta g_{CD} + \Delta g_{DE} + \Delta g_{EF} + \Delta g_{FA} + \Delta g_{AB} \tag{3.91}$$

$$\Delta g_{BC} = a_i \sin\phi_i \cos\phi_i \left[\theta_i - \theta_{i+1} + \tan\psi_1 \ln \frac{\cos\theta_i(\tan\theta_i - \tan\psi_i)}{\cos\theta_{i+1}(\tan\theta_{i+1} - \tan\psi_i)}\right] \tag{3.92}$$

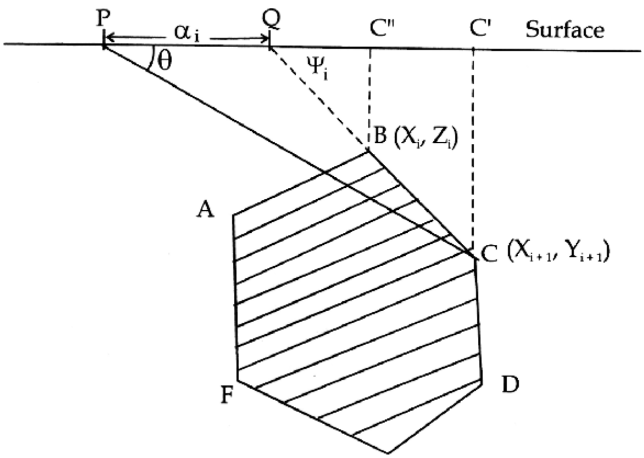


Fig. 3.22. Geometry of a buried two dimensional cylindrical structure of hexagonal cross section and the point of observation on the surface

Since

$$\theta_i = \tan^{-1}\frac{z_i}{x_i}, \theta_{i+1} = \tan^{-1}\frac{z_{i+1}}{x_{i+1}}$$
$$\psi_i = \tan^{-1}\frac{z_{i+1} - z_i}{x_{i+1} - x_i}$$

and

$$a_i = PC^{/} = QC^{/} = x_{i+1} - z_{i+1}\ \mathrm{Cot}\ \psi_i = x_{i+1} + z_{i+1}\frac{z_i - z_{i+1}}{x_{i+1} - x_i}. \tag{3.93}$$

Therefore the final expression for the gravitational anomaly for a prism with n number of face is given by

$$\Delta g_\rho = 2G\rho\sum_{i=1}^{n}\left[\left[\frac{x_i z_{i+1} - z_i x_{i+1}}{(x_{i+1} - x_i)^2 + (z_{i+1} - z_i)^2}\right](x_{i+1} - x_i)(\theta_i - \theta_{i+1}) + (z_{i+1} - z_i)\ln\frac{r_{i+1}}{r_i}\right]. \tag{3.94}$$

3.14 Gravity Field of the Earth

Gravity field of the earth is a global naturally existing field which attracts any mass having a definite density and weight towards the centre of the earth. One can measure this field or it's variations on the surface of the earth, in the air, in the ocean, inside a borehole or inside a mine. Thus the different branches of gravity measurements, viz., aerogravity, marine gravity, borehole gravity and surface gravity have developed. In surface gravity also (i) survey for geodesy, (ii) survey for crustal studies (iii) survey for oil exploration and (iv) survey for mineral exploration have altogether different dimensions. That is why in a square grid gravity survey the distance between the two consecutive gravity stations can be around 100 km for geodetic survey to 10 meters for mineral survey.

Normal gravity field of the earth varies from 978.0327 gal at the equator to 981.2186 gal at the pole (Wilcox 1989). Point to point variation of the gravity field is termed as gravity anomaly Δg and its' unit is milligal.

Gravity data need a few corrections before they can be used for routine interpretation using forward modelling and inversion. The corrections are

(i) Free air correction
(ii) Bouguer correction
(iii) Terrain correction
(iv) Latitude correction
(v) Tidal correction
(vi) Isostatic correctiom
(vii) Drift correction

3.14.1 Free Air Correction

Earth, s normal gravity field varies inversely as the square of the distance. Therefore, hills and valleys bring variations in the earth's gravitational force of attraction. Gravity at a height 'h' at any latitude can be represented by truncated Taylor's series expansion

$$\mathrm{G(R+h) = g(R) + h}\partial g(R)/\partial R + \cdots\cdots\cdots\cdots \tag{3.95}$$

It generates the free air correction as

$$\mathrm{G(R+h) = g(R)} - (0.30855 + .00022\cos 2\psi)\mathrm{h} + 0.073\times 10^{-6}\mathrm{h}^2 - -- \tag{3.96}$$

where h is in meters and the gravity field is in milligal. For routine free air correction 0.3086 h milligal is added to the gravity value. It only depends upon the distance of the point of observation from the centre of the earth.

3.14.2 Bouguer Correction

Bouguer correction accounts for the attraction of the assumed plate at an assumed point in a datum plane from the point of observation on the surface. Through Bouguer correction, all the gravity data are brought back to the same datum plane. Earth materials below the observation point will generate an additional force of attraction at the datum point which was not taken care of in free air correction. The earth materials present below the point of observation and above the assumed datum plane is given an approximate shape of a plate of certain thickness and infinitely large radius. For reducing all the gravity data to the same datum plane the Bouguer correction is always substracted from the gravity data. It's value is 0.04188 ρ where ρ is the density of the slab (see Sect. 3.9)

3.14.3 Terrain Correction

Rugged topography in a hilly terrain with rapid variation in elevation causes an extra correction to be added. Terrain correction is always added both for hills and valleys because the presence of positive and negative masses will always align the force of attraction vector in the same direction. Mass excess in the hill and mass deficiency in the valley will dictate the quantum of correction to be applied. Details are available in Dobrin and Savit (1988).

3.14.4 Latitude Correction

Spheroidal shape of the earth with its equatorial bulge and centrifugal force for rotation of the earth around it's axis and revolution of the earth around the sun in its orbit generates the latitude correction. It's latitude dependence is expressed by the equation(Heiskanen and Vening Meinsz 1958)

$$g(\psi) = 978.049(1 + 0.0052884\mathrm{Sin}^2\psi - 0.0000059\mathrm{Sin}\psi \tag{3.97}$$

where'ψ' is the latitude on the surface of the earth. This is the international gravity formula. The radius of the spheroidal earth is given by

$$R(\psi) = 6378.388(1 - 0.0033670\mathrm{Sin}^2\psi + 0.0000071\mathrm{Sin}^2 2\psi). \tag{3.98}$$

The normal gravity formula for geodetic reference system is as follows

$$g(\psi) = 978.0327(1 + 0.0053024\ \mathrm{Sin}^2\psi - 0.00000071\ \mathrm{Sin}^2 2\psi). \tag{3.99}$$

For routine latitude correction needed for geodetic survey as well as for surveys related with crustal studies the latitude correction used is $\approx 1.307\ \mathrm{Sin}2\psi$.

3.14.5 Tidal Correction

Sensitive gravimeters respond to the position of the sun and moon with respect to the earth and the values of g varies with the tides in the ocean. It may be a fraction of a milligal but it is measurable.

3.14.6 Isostatic Correction

Isostatic correction originated due to the presence of lateral variation of density within the earth, s crust. After the said corrections are applied, the Bouguer anomaly for particular reference plane should have been zero in large scale mapping and keeping aside the local variations in densities .In realities it is was observed that negative Bouguer anomaly exists near the mountains and positive anomaly exits near the oceans.

Two scientists named G.B. Airy and J.H. Pratt proposed two different models for the crust. Airy's model suggests that crust is thick near the mountains and mountains have roots and it is thiner below the ocean bottoms and here the crust has antiroots. The density of the crust is assumed to be the same. Pratt assumed a variable density model and this variation in density has direct relation with the elevation of the ground from the mean sea level. Depth of the ocean floor from the mean sea level causes increase in density. Figures 3.23a.b show the Airy and Pratt's isostatic models.

Isostatic correction is needed only for geodetic survey as well as for surveys related to crustal studies. For routine exploration survey only free air, Bouguer and terrain corrections are necessary. Drift correction is an instrument based correction. Round the clock reading at one point can give the idea about drift correction to be added or subtracted.

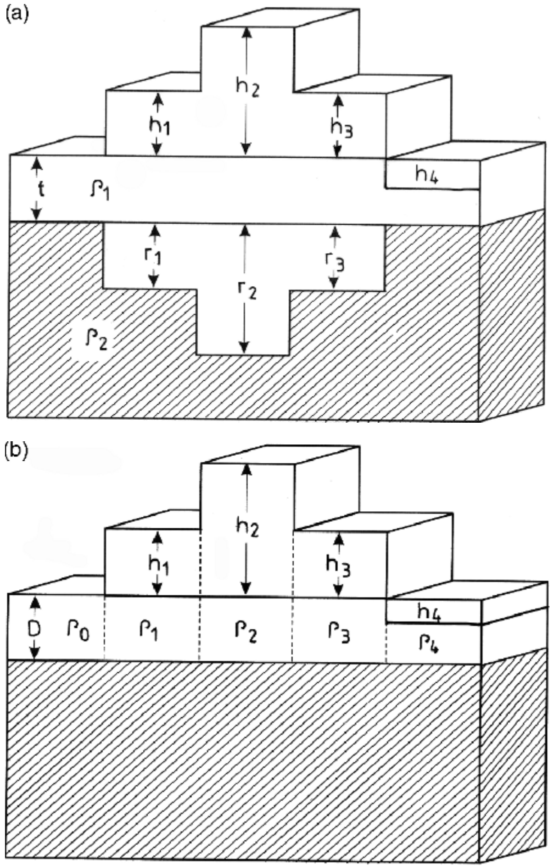


Fig. 3.23. (a) Airy's isostatic model; (b) Pratt's mode

3.15 Units

$$
\begin{aligned}
G &= \mathrm{cm/gmsec^2} \text{ (CGS unit)} \\
&= \mathrm{m/kgsec^2} \text{ (MKS unit)} \\
g &= \mathrm{cm/sec^2} \text{ and gal} \\
\Delta g &= \text{milligal}(10^{-3}\,\text{gal}) \\
\rho &= \text{gms/cc.}
\end{aligned}
$$

3.16 Basic Equation

The basic equations for the gravitational field are

$$\text{(i)} \quad \vec{g} = -\text{grad}\ \phi_g \tag{3.100}$$

where ϕ_g is the scalar gravitational potential.

$$\text{(ii)} \quad \text{Curl } \vec{g} = 0 \tag{3.101}$$

because the force of attraction is along the line joining the two masses.

$$\text{(iii)} \quad \text{div grad } \phi_g = \nabla^2\phi_g = 4\pi Gm \tag{3.102}$$

when the region contains the mass. It satisfies Poisson's equation. In a source free region $\nabla^2\phi_g = 0$ and it satisfies Laplace's equation.

4

Electrostatics

In this chapter a brief introduction on electrostatics is given. Coulomb's law, electrostatic potential, electrical permittivity, electrical fluxes and displacement vector, Gauss's theorem on total normal induction, electrostatic dipole potential and field, Laplace and Poisson's equations,. electrostatic energy, boundary conditions and basic equations in electrostatics are given.

4.1 Introduction

It was known to the people for more than three centuries that if an insulator is rubbed with a cloth it acquires a special property of attracting other objects. It was said that the body got charged. As early as 1785 Coulomb first quantitatively measured the force of attraction or repulsion. It was observed that when a glass rod is rubbed with a silk cloth, it acquires positive charge. Similarly when an amber rod is rubbed with a black cotton cloth negative charge originates. These positive and negative charges are arbitrary convensions. Simply these charges are found to be of opposite polarities. Electrostatics deals with these immovable electric charges of opposite or same signs and the fields created by these charges. The energy spent to rub an amber or a glass rod is partly converted into electrical energy and partly into heat energy due to friction. This electrical energy appears in the form of electrical charges. It is observed that the number of charges in the glass rod is exactly same as that in the silk cloth. It proves the total conservation of charges and energy remain the same. Free energy (which can be easily converted into work) in any thermodynamic system was used to separate these two types of charges present in a body. They originate from an uncharged neutral body. Like charges repel and unlike charges attract. These electric charges remain static and generate electric field. The subject electrostatics is one of the most fundamental subjects in potential theory. Because some of the most fundamental concepts and equations, used in electromagnetic theory, came from electrostatics.

Fig. 4.1. Coulomb force of attraction between a positive and a negative charge separated by a distance r

4.2 Coulomb's Law

Coulomb's law states that two point charges q and q′ separated by a distance 'r' will have (i) a force of attraction or repulsion directly proportional to the product of the two charges and inversely proportional to the square of the distance, (ii)the force of attraction or repulsion will be along the line joining the two charges(Fig. 4.1), (iii) the constant of proportionality k = $1/4\pi \in$, where $\in$ is the electrical permittivity of the medium (Sect. 4.4) (iv) like charges repel and unlike charges attract. The expression for electrostatic force is given by

$$\mathrm{F} = \mathrm{k}\frac{\mathrm{qq}'}{\mathrm{r}^2}. \tag{4.1}$$

This is known as the Coulomb's law of force and k is a constant. Units of q,r and F are respectively in coulomb, meter and Newton,.

4.3 Electrostatic Potential

Potential at a point in an electrostatic field is the amount of work done to bring an unit charge from infinity to that point. Since work done = force x distance, we can write

$$\text{Workdone} = -\int_{\infty}^{R} \vec{F}_r\, d\,\vec{dl}. \tag{4.2}$$

Since

$$\mathrm{F} = \frac{\mathrm{q}}{4\pi \in \mathrm{r}^2} \text{ for an unit charge.} = -\int_{\infty}^{R} \frac{q}{4\pi \in r^2} dr. = \frac{1}{4\pi \in}.\frac{q}{r} \tag{4.3}$$

Therefore the potential at a point P_2 at a distance r from a single charge q at P_1 is given by $\frac{1}{4\pi\in}.\frac{q}{r}$ (Fig. 4.2). The potential $\phi\left(=\frac{1}{4\pi\in}\frac{q}{r}\right)$ has only magnitude and no direction and therefore it is a scalar potential. Potential at a point is independent of the path followed.

The potential at a point P due to number of charges $\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3 \ldots$ situated at distances $\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3$ is given by

$$\phi = \frac{1}{4\pi \in}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots..\right). \tag{4.4}$$

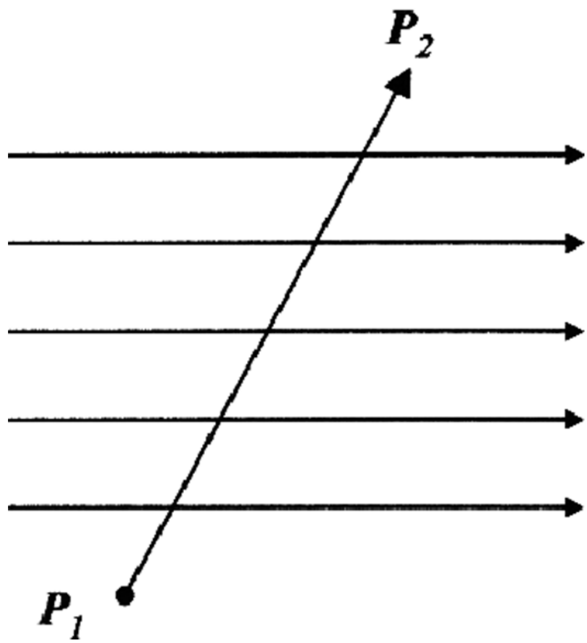


Fig. 4.2. Work done for movement of the unit charge in the field

If there is a continuous distribution of charge throughout a region instead of being located at discrete points, the region can be divided into elements of volume Δv. Then each Δv contains charge $\rho \Delta v$ where ρ is the volume density of charge in the volume element. The potential at a point is given by

$$\phi = \frac{1}{4\pi \in} . \sum_{i=1}^{\infty} \frac{\rho_i \Delta v_i}{r_i} \tag{4.5}$$

where r_i is the distance of the ith volume element from P. When the size of these volume elements become infinitesimally small

$$\phi = \frac{1}{4\pi \in} \int \frac{\rho dv}{r} . \tag{4.6}$$

4.4 Electrical Permittivity and Electrical Force Field

If two opposite faces of an insulator (Fig. 4.3) are charged with potential difference ϕ applying an external electric field, the charges on the two opposite faces will be given by

$$\phi C = q \tag{4.7}$$

where C is the capacitance of a dielectric and ϕ is the voltage across the two faces. Capacitance between the two plates can be defined as the charges needed

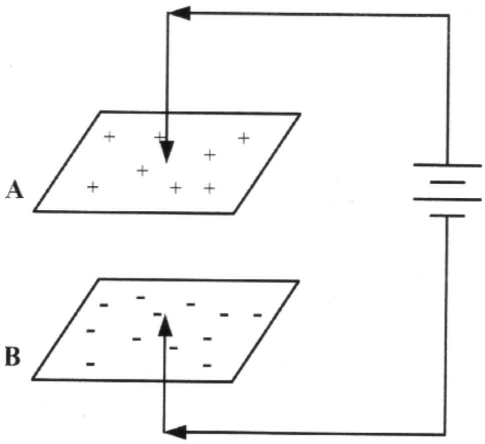


Fig. 4.3. Charging of a dielectric (capacitance) between the two opposite faces

for unit potential difference between the two opposite faces of a capacitor. Unit of capacitance is farad. This name was chosen to honour Micheal Faraday. The unit farad = 1 coul/volt. More practical units are microfarad (1 µf = 10^{-6} f) and picofarad (1pf = 10^{-12}f). Equation (4.7) can be written as,

$$C.\frac{l}{A}.\left(\frac{\phi}{l}\right) = \frac{q}{A} \tag{4.8}$$

Here l is the distance between the two opposite faces of a capacitor.

where $\in = \mathrm{C}\frac{\ell}{\mathrm{A}}$. The capacitance of a body per unit length and unit cross section is termed as electrical permittivity or electrical capacitivity of a medium. Potential per unit width of the capacitor is the field $\vec{\mathrm{E}}$ So $E = \frac{\phi}{l}$ and $\vec{D}\left(= \frac{q}{A}\right)$ is the charge per unit area. It is termed as the dielectric flux density. It is also termed as the displacement vector. Its unit is Coulomb/meter^2. In a vacuum $\in = \in_0 = 8.854 \times 10^{-12}$ farad/meter When an electric field is applied between two opposite faces of a dielectric material, potential generated between the two opposite faces of the dielectric depends upon capacitance of the dielectric. Dielectric constant is given by $\in_k = \in_0 \in_r$ where $\in_0 = \frac{1}{36\pi x 10^9}$ and is the free space electrical permittivity of the medium. $\in_r$ is the relative electrical permittivity of the body with respect to the free space value. The electrostatic field due to a point charge is given by,

$$\vec{E} = \frac{1}{4\pi \in}.\frac{q}{r^3}.\vec{r} \tag{4.9}$$

The field lines for a point source (positive charge) and sink (negative charge) are radial (Fig. 2.3) in a homogeneous and isotropic dielectric and the equipotential surfaces will be spherical. The direction of force is along the line joining the two charges irrespective of polarity of the charges.

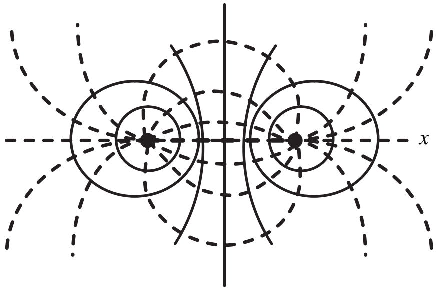


Fig. 4.4. Shows the electrostatic field due to two opposite charges in a homogeneous and isotropic medium

4.5 Electric Flux

When an isolated positive or negative charge is placed in a homogenous and isotropic dielectric medium , field lines originating from the source spread along the radial direction with the charge at the centre (Fig. 2.3). In the case of a negative charge the field lines will converge radially to the negative charge. When a positive and a negative charge is placed in a dielectric medium the field lines will start from a positive charge and will end up in a negative charge as shown in Fig. 4.4. The field lines are the lines of forces or the flux lines. The important properties of these electric flux are (i) these fluxes are independent of the medium, (ii) the magnitude of these fluxes is solely dependent upon the strength of the charge from which the flux lines come out, (iii) the electric flux density must be inversely proportional to the square of the distance if the flux source is covered by a bounded domain say a sphere. The flux lines will be perpendicular to the spherical surface.

4.6 Electric Displacement ψ and the Displacement Vector D

Faraday's famous experiment on movement of electrostatic charges in different spherical shells is as follows: A sphere with charge q is placed within another spherical shell without touching it. The outer sphere is momentarily earthed and when the inner sphere is removed, the charge on the outer shell is found to be exactly the same as that in the inner sphere but of opposite sign. It is true for all sizes of the sphere and for all dielectric constants of the media. There is a displacement of charges from the inner sphere to those in the outer sphere. The amount of displacement depends only upon the magnitude of the charge q. Thus the displacement is in Coulomb i.e., $\Psi = q$. The electric displacement per unit area at any point on a spherical surface of radius 'r' is the electric displacement density $\vec{D}$. It is a vector because there is a definite direction for this displacement. So,

$$\vec{D} = \frac{\Psi}{4\pi r^2} = \frac{q}{4\pi r^2} \tag{4.10}$$

The unit is in Coulomb/meter^2. The displacement per unit area at any point depends upon the direction of the area and it is normal to the surface elements. This displacement is along the direction of the field in a homogeneous and isotropic dielectrics. Therefore we can again write,

$$\vec{D} = \frac{q}{4\pi r^2}.\vec{r} \tag{4.11}$$

$$\vec{D} = \in \vec{E}. \tag{4.12}$$

The vector D is also called displacement vector. We can define the flux $\Psi =$ D.ds where ds is the differential surface elements on the surface S.

In an anisotropic dielectric, the electrical permittivity becomes a tensor and the connecting relation between D and E can be expressed as

$$\begin{aligned} D_x &= \in_{11} E_x + \in_{12} E_y + \in_{13} E_z \\ D_y &= \in_{21} E_x + \in_{22} E_y + \in_{23} E_z \\ D_z &= \in_{31} E_x + \in_{23} E_y + \in_{33} E_z \end{aligned}$$

and in the matrix form

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \in_{11} & \in_{12} & \in_{13} \\ \in_{21} & \in_{22} & \in_{23} \\ \in_{31} & \in_{32} & \in_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{4.13}$$

4.7 Gauss's Theorem

The total normal induction or total displacement of electric flux through any closed surface, which enclosed the charges, is equal to the amount of charge enclosed. From Fig. 4.5, the displacement or electric flux through the elementary surface ds is

$$d\Psi = D \, ds \cos\theta \tag{4.14}$$

where θ is the angle between D and $\vec{n}$. where $\vec{n}$ is normal to the surface ds. The total normal induction through the entire surface is given by

$$\Psi = \oint \vec{D} \, ds \cos\theta \tag{4.15}$$

where the integration is over the whole surface.

Solid angle $d\omega$ is (Fig. 4.6)

$$d\omega = \frac{ds \cos\theta}{r^2} \tag{4.16}$$

and,

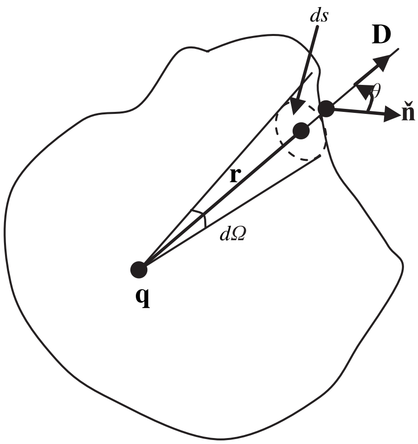


Fig. 4.5. Shows the normal induction through the surface ds when a charge q is bounded by the surface S

$$\psi = \oint Dr^2 d\varpi. \tag{4.17}$$

Substituting the value of D from (4.6), we get

$$\psi = \frac{q}{4\pi} \oint d\omega. \tag{4.18}$$

Since the total solid angle subtended at the point O (occupied by the charge q) by the closed surface is 4π, therefore

$$\psi = q. \tag{4.19}$$

If there be n number of dielectric charge q_i within the enclosed volume, the total flux on the surface will be,

$$\psi = \sum_{i=1}^{n} q_i. \tag{4.20}$$

If the charges are distributed throughout the volume and ρ is the volume density of charge then the total normal induction through the surface is

$$\psi = \int_V \rho\, dv. \tag{4.21}$$

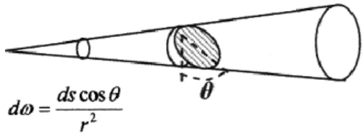


Fig. 4.6. Shows the solid angle subtended at the point o due to the surface ds

Now (4.18) can be written as,

$$\Psi = \oint \mathrm{D.ds}. \tag{4.22}$$

From (4.22), we can write,

$$\oint \vec{\mathrm{D}}.\mathrm{d}\vec{\mathrm{s}} = \int \rho \mathrm{dv}. \tag{4.23}$$

Applying Gauss's divergence theorem, we can write

$$\int_{\mathrm{v}} \mathrm{div}\vec{\mathrm{D}}\mathrm{dv} = \int_{\mathrm{s}} \vec{\mathrm{D}}.\vec{\mathrm{n}}.\mathrm{ds} = \int_{\mathrm{v}} \rho \mathrm{dv} \tag{4.24}$$

$$\mathrm{div}\vec{\mathrm{D}} = \nabla.\vec{\mathrm{D}} = \rho. \tag{4.25}$$

4.8 Field due to an Electrostatic Dipole

Both dipoles and bipoles consist of two poles. The difference lies in the distance between the two poles. In an electrostatic dipole the distance between the two poles is negligibly small in comparison to the distance of the observation point. As a result the potential and field vary inversely as the square and cube of the distance respectively. For bipole fields, the distance between the two charges is comparable to the distance where we measure the field. As a result the potential and field vary inversely as the first and second power of distance.

In this section, we shall develop the expressions for potentials and fields for static field. Let the charges + q and −q separated by a distance 'l' are placed along the z-axis. ql is the moment of the dipole. The point dipole is defined as $\operatorname*{Lim}_{\substack{l\to 0\\ q\to\infty}} \mathrm{ql} = \text{finite}$.

Figure 4.7 shows the location of the dipole. The point of observation P is located at a certain point x, y, z in a cartesian coordinates. The vector l is from +q to −q. The potential at P is given by

$$\phi = \frac{\mathrm{q}}{4\pi \in}\left(\frac{1}{\mathrm{r}_1} - \frac{1}{\mathrm{r}_2}\right) \tag{4.26}$$

Substituting the values of r_1 and r_2, we get

$$\begin{aligned} \phi &= \frac{\mathrm{q}}{4\pi \in}\left[\left(\mathrm{r}^2 + \frac{\mathrm{l}^2}{4} - \mathrm{lr}\cos\theta\right)^{-1/2} - \left(\mathrm{r}^2 + \frac{\mathrm{l}^2}{4} + \mathrm{lr}\cos\theta\right)^{-1/2}\right] \\ &= \frac{\mathrm{q}}{4\pi \in}.\frac{1}{\mathrm{r}}\left[\left(1 + \frac{\mathrm{l}^2}{4\mathrm{r}^2} - \frac{\mathrm{l}}{\mathrm{r}}\cos\theta\right)^{-1/2} - \left(1 + \frac{\mathrm{l}^2}{4\mathrm{r}^2} + \frac{\mathrm{l}}{\mathrm{r}}\cos\theta\right)^{-1/2}\right] \end{aligned} \tag{4.27}$$

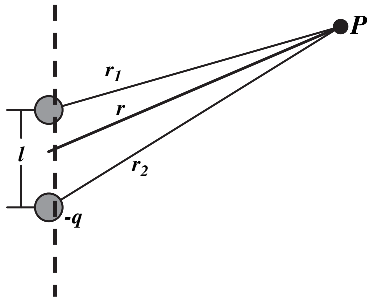


Fig. 4.7. Shows the electrostatic dipole. P is the measuring point of potential and field

$$= \frac{q}{4\pi \in}.\frac{1}{r}\left[1 - \frac{l^2}{8r^2} + \frac{l}{2r}\cos\theta - 1 + \frac{l^2}{8r^2} + \frac{l}{2r}\cos\theta\right] + \ldots. \quad (4.28)$$

$$= \frac{q}{4\pi \in}.\frac{1}{r}\left[\frac{l}{r}\cos\theta + 1^2 \text{and higher order terms}\right]$$

$$= \frac{ql}{4\pi \in}.\frac{\cos\theta}{r^2} = \frac{\vec{P}}{4\pi \in}.\frac{\cos\theta}{r^2} \text{ where } \vec{P} \text{ is the dipole moment.} \quad (4.29)$$

The expression for the dipole field is,

$$\vec{E} = \frac{\vec{P}.\vec{a}_r}{4\pi \in .r^3}. \quad (4.30)$$

The field components in spherical and cartisian coordinates are respectively given by

$$\vec{E}_r = -\frac{\partial\phi}{\partial r} = \frac{2\vec{P}}{4\pi \in}.\frac{\cos\theta}{r^3} \quad (4.31)$$

$$\vec{E}_\theta = -\frac{\partial\phi}{r\partial\theta} = \frac{\vec{P}}{4\pi \in}.\frac{\sin\theta}{r^3} \quad (4.32)$$

$$\vec{E}_\Psi = 0. \quad (4.33)$$

The total field is

$$\frac{\vec{P}}{4\pi \in r^3}\left(\vec{a}_r 2\cos\theta + \vec{a}_\theta.\sin\theta\right)$$
$$= \frac{\vec{P}}{4\pi \in}.\frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}}. \quad (4.34)$$

The components are

$$\vec{E}_x = \frac{\vec{P}}{4\pi \in}.\frac{3xz}{(x^2+y^2+z^2)^{5/2}} = \frac{\vec{P}}{4\pi \in}.\frac{3xz}{r^5}$$

$$\vec{E}_y = \frac{\vec{P}}{4\pi \in}.\frac{3yz}{r^5} \tag{4.35}$$

$$\vec{E}_z = \frac{\vec{P}}{4\pi \in}\left\{\frac{3z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$= \frac{\vec{P}}{4\pi \in}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right). \tag{4.36}$$

For dipoles, the expressions for the potential can also be written as

$$\begin{aligned}\phi &= \frac{1}{4\pi \in}.q\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\\ &= \frac{1}{4\pi \in}.ql\left(\frac{\frac{1}{r_1} - \frac{1}{r_2}}{l}\right)\\ &= \frac{1}{4\pi \in}.\vec{P}.\underset{l\to 0}{\mathrm{Lim}}\ \frac{\partial}{\partial l}\left(\frac{1}{r}\right).\end{aligned}$$

So the differential form of the expression for potential at a point due to a dipole is given by

$$\phi = \frac{\vec{P}}{4\pi \in}.\frac{\partial}{\partial l}\left(\frac{1}{r}\right). \tag{4.37}$$

The potential due to a single pole is $\phi = \frac{q}{4\pi\in}.\frac{1}{r}$. For surface distribution of single poles, the expression for the potential will be

$$\phi = \int\int \frac{\sigma ds}{r} \tag{4.38}$$

where σ is the surface density of charge. For dipoles, the direction of the dipole will be at right angles to the surface. If $\vec{P}$ is the moment of the dipole per unit area, $\vec{P}ds$ is the moment of the dipole for a small area ds. Assume that each dipole is normal to the surface. So the potential at a point p due to the elementary surface ds. (Fig. 4.8) is

$$\phi = \frac{\vec{P}ds}{4\pi \in}.\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \tag{4.39}$$

The total potential due to a surface distribution of dipoles is given by

$$\phi = \int\int \frac{\vec{P}\,ds}{4\pi \in}.\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \tag{4.40}$$

where the direction of the moment is at right angles to the direction of the surface. Therefore

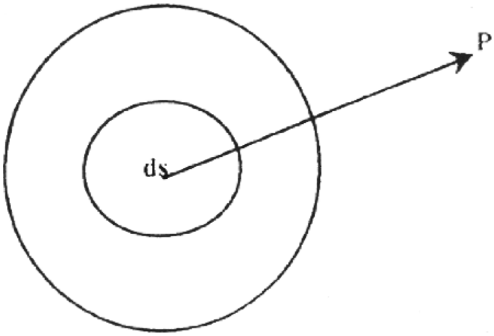


Fig. 4.8. Dipole surface

$$\phi = \int\int \frac{\vec{P}}{4\pi \in}.\frac{ds\cos\theta}{r^2} = \frac{\vec{P}}{4\pi \in}\int\int d\omega = \frac{\vec{P}\omega}{4\pi \in} \tag{4.41}$$

where ω is the solid angle subtented at the point P. $\vec{p}$ is the dipole moment due to the surface S.

4.9 Poisson and Laplace Equations

Starting from equation (4.25),i.e.

$$\nabla \mathrm{D} = \rho,$$

we can write

$$\begin{aligned} \nabla.\varepsilon\vec{\mathrm{E}} &= \rho \\ \Rightarrow \nabla.\vec{\mathrm{E}} &= \frac{\rho}{\in}. \end{aligned} \tag{4.42}$$

Since

$$\mathrm{E} = -\nabla\phi,$$

therefore

$$\begin{aligned} \nabla.\nabla\phi &= -\frac{\rho}{\in} \\ \Rightarrow \nabla^2\phi &= -\frac{\rho}{\in}. \end{aligned} \tag{4.43}$$

This is known as the Poisson's equation. In a free space where there is no electrostatic source, (4.43) reduces to,

$$\nabla^2\phi = 0 \tag{4.44}$$

This is a Laplace equation These equations are of primary importance in scalar potential field theory. In rectangular coordinates, Poisson or Laplace equation is written as,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=-\frac{\rho}{\in}\text{ or }0 \tag{4.45}$$

depending upon whether the source is included in or excluded from the volume.

These are second order partial differential equations and are related to the rate of change of potential in three mutually perpendicular directions. In terms of electric field it can be written as

$$\nabla.\vec{\mathrm{E}}=\frac{\rho}{\in}\text{ or }0. \tag{4.46}$$

4.10 Electrostatic Energy

A capacitor gets charged when a voltage ϕ is established between the two plates. The stored energy can be converted into heat by discharging the capacitor. The amount of energy stored can be calculated from the work done in charging the capacitor.

Since the potential is defined as the work done in moving a unit charge from infinity to a particular point, the work done by moving a small charge Δq through the potential difference of ϕ is $\phi\Delta$q. The voltage ϕ can be expressed as

$$\phi=\frac{q}{C}. \tag{4.47}$$

The work done in increasing charge in a capacitor by an amount dq is

$$\frac{q}{C}dq. \tag{4.48}$$

The total work done in charging the capacitor to q Coulomb is

$$Total\ work=\int_0^q\frac{q}{C}dq=\frac{1}{2}\frac{q^2}{C}. \tag{4.49}$$

Stored energy in a charged capacitor is

$$=\frac{1}{2}\frac{q^2}{c}=\frac{1}{2}\phi q=\frac{1}{2}\phi^2C. \tag{4.50}$$

Since $\vec{\mathrm{E}}=\frac{\phi}{\mathrm{l}}$ i.e., the potential per unit length and $\in=C\frac{l}{A}$, we get the expression of the electrostatic stored energy as

$$\frac{1}{2}\phi^2C=\frac{1}{2}.\overset{2}{E}.\overset{2}{l}.\in.\frac{A}{l}=\frac{1}{2}\in E^2l^3. \tag{4.51}$$

So the stored electrostatic energy per unit volume is

$$\frac{1}{2}\in\mathrm{E}^2. \tag{4.52}$$

4.11 Boundary Conditions

The normal component of $\vec{D}$ is

$$\int_{s} \vec{D}.\vec{n}.ds.$$

Applying Gauss's divergence theorem, we get

$$\int \vec{D}.\vec{n}.ds = \int_{v} div\vec{D}.dv = \int_{v} \rho dv = q \tag{4.53}$$

Here q is the total charge and ρ is the volume density of charge. Let us assume an elementary thin cylinder with negligible thickness with two faces on two sides of the boundary (Fig. 4.9). The normal component of the displacement vectors will go out of this volume for normal induction. Therefore Dn_1 and Dn_2 will be in the opposite direction. From Gauss's law of total normal induction we can write

$$(D_2.n_2 + D_1 n_1)\Delta a = w\Delta a \tag{4.54}$$

where D_1 and D_2 are the displacement vectors, n_1 and n_2 are the normal vectors from the bottom and top surfaces of the cylinder, Δa is the surface area of the cylinder and w is the surface density of charge. Since $q = \int_{v} \rho dv = \rho\Delta l.\Delta a$ (where ρ is the volume density of charge), we get

$$w = \rho\Delta l,$$

and

$$(D_2 - D_1).n = w \tag{4.55}$$

This equation shows that normal component D_n of the vector $\vec{D}$ is discontinuous at an interface due to accumulation of surface charge of density w. On the surface of a conductor, surface charge density dissipates quickly but on a surface of an insulator accumulated charge does not dissipate so quickly. Hence

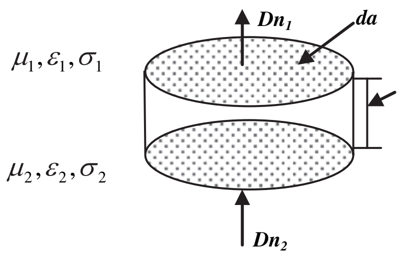


Fig. 4.9. Normal component of the displacement vector at the boundary between the two media of different electrical permittivity

across the interface involving all except the poorest conductors or dielectrics, the normal component of $\vec{D}$ is continuous across the boundary i.e.,

$$\mathrm{Dn}_1 = \mathrm{Dn}_2 \tag{4.56}$$

Since the electrostatic potential is also continuous across the boundary. The boundary conditions generally applied to solve an electrostatic problem are

$$\begin{aligned} &\text{(i)} \quad \phi_1 = \phi_2 \\ &\text{(ii)} \quad \in_1 \left(\frac{\partial \phi}{\partial n}\right)_1 = \in_2 \left(\frac{\partial \phi}{\partial n_1}\right)_2 . \end{aligned} \tag{4.57}$$

4.12 Basic Equations in Electrostatic Field

1. $$\vec{F} = \frac{1}{4\pi \in} . \frac{q_1 q_2}{r^2} \quad \text{Coulomb'slaw} \tag{4.58}$$

2. $$\vec{F} = q\vec{E} \tag{4.59}$$

3. $$\vec{E} = \frac{1}{4\pi \in} . \frac{q}{r^3} . \vec{r} \tag{4.60}$$

4. $$\vec{E} = \lim_{\Delta q \to o} \frac{\Delta F}{\Delta q} \tag{4.61}$$

5. $div\,\vec{E} = \frac{q_\nu}{\in}$ where q_ν is the volume density of charge. or

$$\nabla^2 E = \frac{q_\nu}{\in} \quad \text{Poisson's equation} \tag{4.62}$$

6. $$\vec{E} = -\mathrm{grad}\phi_e . \ \phi_e \text{ is the scalar potential in electrostatics} \tag{4.63}$$

7. $$\nabla^2 \vec{E} = O \text{ Laplace equation is a source free region} \tag{4.64}$$

8. $$\vec{D} = \in \vec{E} \tag{4.65}$$

9. $$\mathrm{div}\,\vec{D} = \rho \tag{4.66}$$

10. $$\mathrm{Curl}\,\vec{\mathrm{E}} = 0 \tag{4.67}$$

11. $$\text{Potential due to a dipole} = \frac{\vec{P}}{4\pi \in}\frac{\cos\theta}{r^2} \tag{4.68}$$

where $\vec{\mathrm{P}}$ is the dipole moment.

12. $$\text{Field due to a dipole} = \frac{\vec{\mathrm{P}}.\vec{\mathrm{a}}_{\mathrm{r}}}{4\pi \in \mathrm{r}^3} \tag{4.69}$$

5

Magnetostatics

In this chapter, some preliminaries of static magnetic fields, viz, Coulombs law, Faradays law of electromagnetic induction, Magnetic Induction, Magnetic field, Lorentz force, Magnetic properties of the rocks, Biot and Savart's law, Ampere's force law, Ampere's circuital law, Magnetic vector potential, Magnetic scalar potential, Magnetomotive force, Magnetostatic energy, Magnetic field due to a dipole source are discussed. Nature of the geomagnetic field are outlined briefly. Besides some of the basics like solenoidal, rotational and irrotational (low frequency approximation) nature of the magnetostatic field, similarities and dissimilarities with other scalar potential fields like gravity, electrostatic and direct current flow field are highlighted. Geomagnetic field along with magnetostatics and time varying magnetic fields made major in roads in the various branches of geophysics. A brief outline of that is given.

5.1 Introduction

Magnetic field originates when a charge moves. Therefore no magnetic field is associated with electrostatics. Magnetic field has link with direct and alternating current flow fields. Magnetism, (i.e., the property of certain metallic objects, to attract or repel some other metallic objects,) was known to the people for the last several hundred years.

The word magnetism came from the word Magnesia an ancient city of Asia Minor. Certain rocks in the vicinity of this city had the property of attracting metallic bodies. It was observed that a needle shaped load stone got deflected along a particular direction irrespective of any arbitrary orientation and it was used by mariners to find out the north-south direction. This kind of movement in the needle is possible when a couple act on it. The presence of a couple is possible, when a field exists in the north-south direction and the needle has north and south polarity at the two ends. Thus the existence of the geomagnetic field was conceptualised.

In 1819 Oersted first observed that there is a close connection between the flow of electric current through a wire and the generation of magnetic field. In 1820 Biot and Savart first experimentally demonstrated the quantitative aspect of strength of the magnetic induction B and magnetic field H. In the same year Ampere proposed his force law i.e., the law for force between the two coils carrying currents.

Magnetic field is a global and naturally occurring field like gravity field and can be measured anywhere on the surface of the earth, in the air, in the ocean bottom and inside a borehole. Both magnetic and gravity fields show local perturbations due to local variations in magnetic susceptibilitiy and density. Geophysicists are interested about these local and global perturbations.

Gravity field generates always a force of attraction but magnetic field, like electrostatic field, can have either the force of attraction or repulsion according to the law "like poles repel and unlike poles attract". Magnetic field has conceptual north and south poles, the way we have positive charge and negative charge in electrostatics. In this particular aspect magnetostatic field has some similarity with the electrostatics field i.e., both the fields satisfy Coulombs law. Electrostatic field, magnetostatic field and gravity field follow inverse square law. The fields vary directly with the product of charges or masses or pole strengths and inversely as the square of the distance. The constants of proportionality are different for different fields. Both in the case of electrostatic field and direct current flow field, we brought the concept of potential and electromotive force using the line integral of force multiplied by distance. Similar concept of magnetomotive force exists where the work is done in the magnetic field and the line integral of the magnetic field times the distance gives magnetomotive force. Magnetostatics has the concept of both scalar and vector potentials as well as rotational and irrotational field. Irrotational nature of the magnetic field comes from the low frequency approximation and in a source free region.

Positive charge and negative charge in the case of electrostatics, source and sink in the case of direct current flow field can generate both bipole and dipole fields depending upon the separation of the two opposite charges or two opposite current sources. Separation between the north pole and south pole can generate bipole and dipole fields in magnetostatics. That way magnetostatic field has some similarities with the electrostatic field and direct current flow field.

Magnetostatic field has significant dissimilarities with the other fields. Magnetostatic field is always a bipole or a dipole field. An isolated north pole or south pole does not exist. A coil carrying current or a thin sheet of magnetic substances with negligible thickness have both north and south poles.

Magnetostatic field is a solenoidal field. Divergence of the magnetic field is always zero because the poles do not stay in isolation. In the case of electrostatic field, direct current flow field, gravity field, heat flow field etc, if the

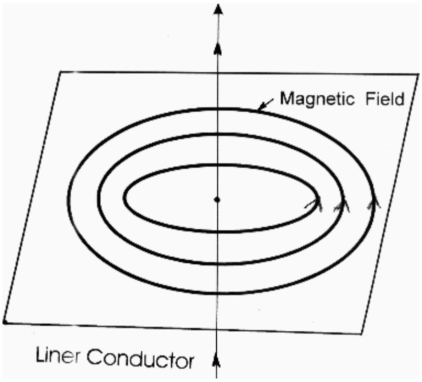


Fig. 5.1. Magnetic field due to a linear conductor carrying current

area under consideration is a source free region, then they become solenoidal field and satisfy Laplace equation. If the region under consideration contains source, then the fields will be nonsolenoidal. These fields then satisfy Poisson's equation. So both the options are there in the said nonmagnetic fields.

Magnetostatic field is a rotational field (Figs. 5.1, 5.2). The curl of a magnetic field is not zero where as curl of gravity, electrostatic, direct current flow, heat flow fields etc. are zero. These fields are irrotational field. Time varying electromagnetic field also is a rotational field. In the absence of any current source magentostatic field can also be an irrotational field. Geomagnetic field, because of its low frequency approximation, becomes an irrotational field and satisfy Laplace equation.

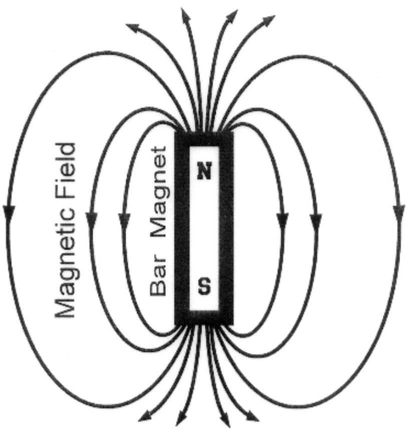


Fig. 5.2. Magnetic field due to a bar magnet

Amongst all the static and stationary fields only magnetostatics has the concept of vector potential. This is also one of the major differences with other fields. These vector potentials received greater attention while solving the electromagnetic boundary value problems.(see Chap. 13) in geophysics.

Magnetic lines of forces are continuous where as the electric field in electrostatics starts from a positive charge and ends in a negative charge; current flow lines in direct current flow field starts from a source and ends in a sink. In this respect also magnetic field has dissimilarities with the electrostatic field and the direct current flow field. The concept of bipole and dipole exists in magnetostatics, electrostatics and DC field. A coil carrying current is called a magnetic dipole. A coil carrying alternating current is termed as an oscillating magnetic dipole. Magnetostatic field, electromagnetic field, gravity field can be measured in the air. Aeromagnetic, aeroelectromagnetic and aerogravity methods are standard geophysical airborne tools. For direct current flow however galvanic contact of both current and potential measuring probes with the ground or any other medium of finite conductivity is necessary. Therefore DC flow field does not have any airborne counterpart.

In the case of Geomagnetic field, div $B = 0$(see in this chapter) leads to div $H = 0$ since $B = \mu H$(see this chapter).Hence $H = -\text{grad}\Phi$ where Φ is a scalar potential. Geomagnetic field also satisfies Laplace equation (see Chap. 7). Since J, the current density in the air is negligible and $\partial D/\partial t$ the displacement vector is zero, curl$H = 0$. Therefore geomagnetic field becomes an irrotational and a scalar potential field in all respect and becomes similar to gravity, electrostatics and DC field.

Different forms of origin of the magnetic fields are shown in the following figures. Figure 5.1 shows the origin of the magnetic field due to a linear conductor(a long straight wire). Here the magnetic field is encircling the current carrying conductor. The magnetic lines of forces are continuous without any break any where showing the rotational nature of the magnetic field. Here magnetic field is at right angles to the direction of flow of current.

Figure 5.2 shows the nature of the magnetic field created due to a bar magnet. It is interesting to note that magnetic field does not originate at the north pole nor it ends in a south pole. Magnetic lines of forces enter in a bar magnet near a point known as south pole and goes out of the magnet from a point known as north pole. These north poles and south poles are fictitious poles and do not exist in reality. These lines of forces are also continuous.

Figure 5.3 shows that magnetic field is created by an electromagnet. If a current carrying coil is wound round a metallic conductor of electricity ,the conductor becomes a magnet and the nature of the magnetic field will be similar to that of a bar magnet.

Figure 5.4 show the nature of magnetic field due to flow of current through a solenoid or a coil with n number of turns. All the magnetic lines of forces will pass through the core of a solenoid.

Figure 5.5 a and b show the nature of the magnetic field due to flow of current through two rectangular coils in opposite directions. Direction of the

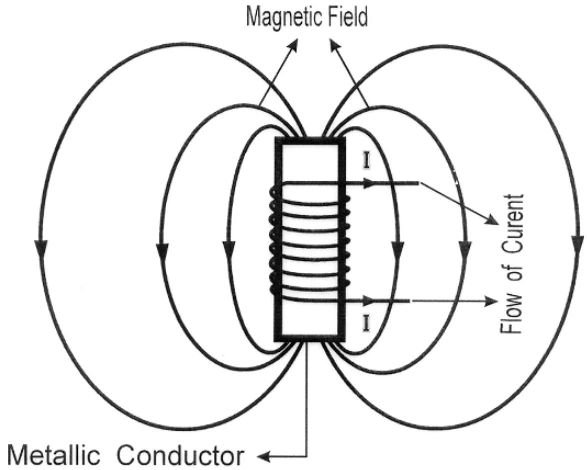


Fig. 5.3. Magnetic field created by an electro magnet

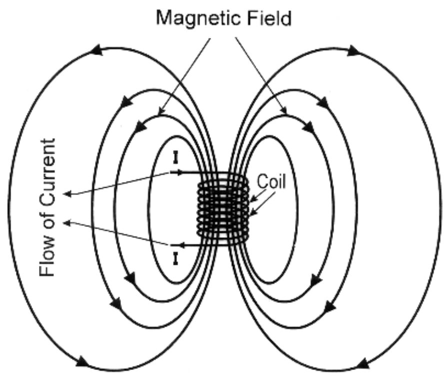


Fig. 5.4. Show the nature of the magnetic field due to a solenoid carrying current

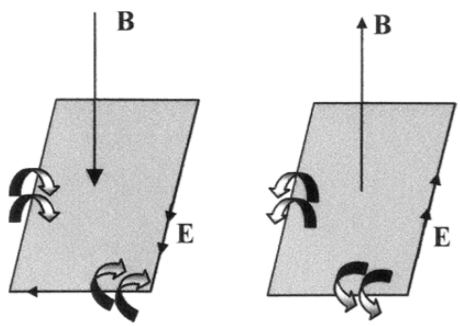


Fig. 5.5. Magnetic fields created due to flow of current in the opposite directions

magnetic field vector will depend upon the direction of flow of current through two rectangular coils.

Figure 5.6 shows a vertical section of the magnetic field due to a circular coil carrying current. The plane of the coil is horizontal but the direction of the magnetic vector is vertical. That is why a horizontal coil carrying current is termed as a vertical magnetic dipole. When alternating current flows through the coil it is termed as an oscillating vertical magnetic dipole. When the plane of the coil is vertical and the direction of the field vector is horizontal the dipole is termed as horizontal magnetic dipole. The direction of the magnetic vector will be at right angles to the plane of the coil.

Figure 5.7 is presented as one of the examples of the nature of distortions of the field lines due to a bar magnet in the presence of another uniform magnetic field. Difference in the orientations of a bar magnet and the external fields can create different patterns of distortions.

Figure 5.8 shows the formation of a couple of force in a magnetic needle in the presence of an external field. This couple acts on the magnetic needle and reorient it along the north south direction as discussed. Mariner compass used was based on this principle.

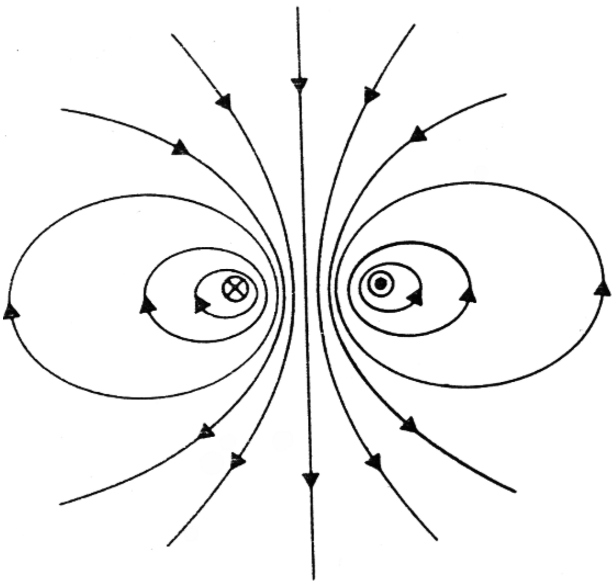


Fig. 5.6. Magnetic field created due to a magnetic dipole (a loop of wire) carrying current

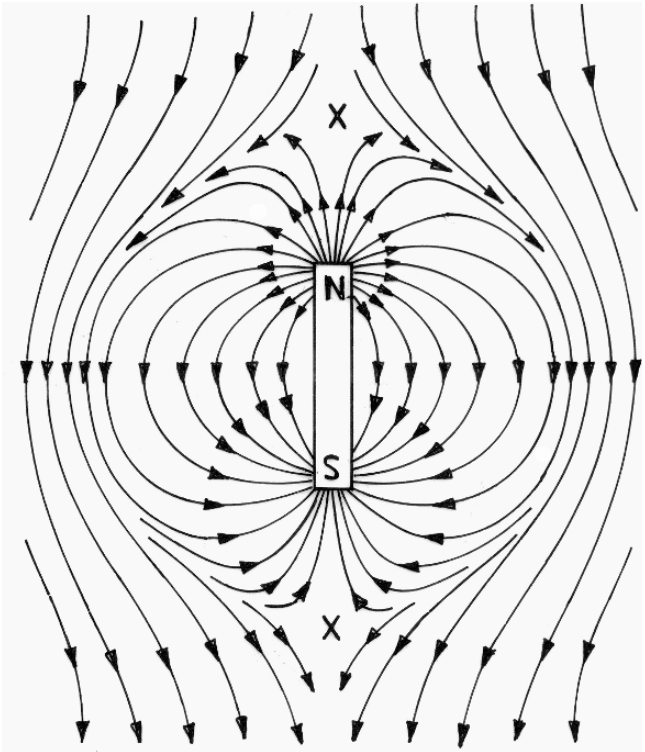


Fig. 5.7. The magnetic field due to a bar magnet in the presence of an external uniform field

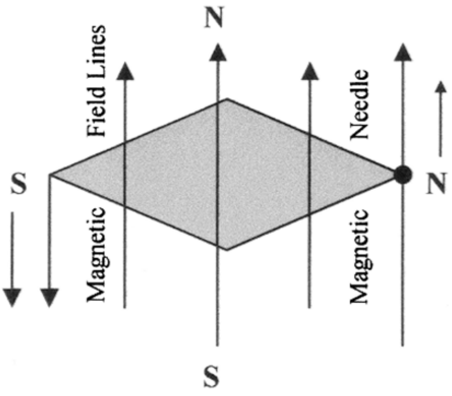


Fig. 5.8. Deflection of the magnetic needle in the presence of a magnetic field

5.2 Coulomb's Law

The concept of magnetic poles, magnetic pole strength 'm', two opposite nature of magnetic poles viz., north pole, south pole, force of attraction and repulsion and inverse square law of force exist in magnetostatics. If the two poles m_1 and m_2 are separated by a distance r then the force of attraction or repulsion is given by

$$\vec{F} = K\frac{m_1 m_2}{r^2} \tag{5.1}$$

where K is the constant of proportionality. This constant of proportionality is given by $\frac{1}{4\pi\mu}$ instead of $\frac{1}{4\pi\in}$ as in the case of electrostatics. μ is termed as magnetic permeability. In a free space or in a non-magnetic rock $\mu = \mu_0$. μ_0 is termed as the free space magnetic permeability and is equal to $4\pi \times 10^{-7}$ henry/meter. The force of attraction or repulsion will be along the line joining the two poles.

Fig. 5.9. Force of attraction between a north and a south pole placed at a distance r

5.3 Magnetic Properties

5.3.1 Magnetic Dipole Moment

The major difference in the nature of the electrostatic and magnetostatic field is (i) the magnetic poles do not exist in isolation. Magnetic poles always exist in dipole form in nature what little be the distance between the two poles. Two poles of pole strength 'm' and separated by a distance l will have magnetic dipole moment

$$\vec{P} = ml.\vec{r} \tag{5.2}$$

The dipole moment is a vector and is directed along the line joining the two poles. $\vec{r}$ is a unit vector along the direction of the line joining the two poles. Fig. (10.a and b) show the orientations magnetic dipoles in a nonmagnet and a magnet.

5.3.2 Intensity of Magnetisation

A magnetic body experiences a force when placed in a magnetic field. The intensity of magnetization is proportional to the external field and direction is along the direction of the external field. Intensity of magnetization is the magnetic moment per unit volume i.e.,

$$I = \frac{\vec{p}}{v}.F \tag{5.3}$$

where v is the volume of the magnetic body.

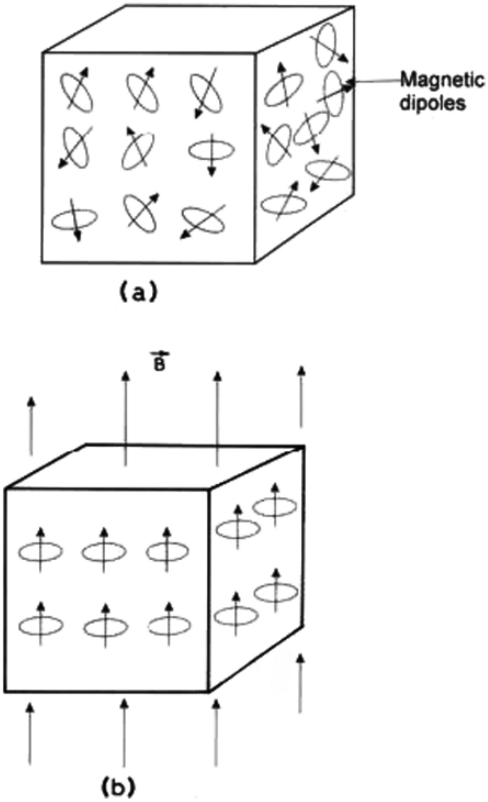


Fig. 5.10. Shows (**a**) random orientation of magnetic dipoles in a nonmagnet (**b**) oriented dipoles in a magnet

5.3.3 Magnetic Susceptibility (Induced Magnetism)

Some substances acquire magnetic properties when they are brought in a magnetic field. This magnetism in a rock or mineral or ore body or metallic alloy are induced magnetism. Magnetism vanishes in some cases when they are withdrawn from a magnetic field. In some cases substances may retain some magnetic field even after they are withdrawn from the inducing field .Third group of substances have their magnetic properties irrespective of their presence in or absence from a magnetic field. Intensity of magnetization is the induced magnetic moment per unit volume. It is directly proportional to the strength of the inducing magnetic field H and is given by

$$\mathrm{I} = \mathrm{KH} \tag{5.4}$$

The constant of proportionality K is termed as magnetic susceptibility Therefore intensity of magnetisation has direct relation with the susceptibility of rocks. It is also a measure of the degree to which a body is magnetised.

Higher the concentration of magnetites or other ferrimagnetic minerals in a rock higher will be the magnetic susceptibility and higher will be the intensity of magnetisation. Some ferromagnetic materials viz.,magnetite, ferrite, permalloy mu metals have higher magnetic susceptibility. Magnetic permeability is connected with the magnetic susceptibility through this relation (K, the susceptibility of a magnetic substance).

$$\mu = 1 + 4\pi K \tag{5.5}$$

5.3.4 Ferromagnetic, Paramagnetic and Diamagnetic Substances

Diamagnetism

Diamagnetic substances are those whose magnetic susceptibilities are very small negative quantities. In the presence of an external inducing magnetic field spinning of orbital electrons are in the opposite direction to the inducing field. Magnetic field within the body vanishes as soon as it is withdrawn from the inducing magnetic field. In diamagnetic substances, the magnetic moment of all atoms is zero. They are generally called non magnetic substances. The common naturally occurring diamagnetic substances are quartz, graphite, gypsum, marbel etc.

Paramagnetism

Paramagnetic substances have very small positive magnetic susceptibility. In the presence of an external magnetic field the spinning electrons in an atom partially align themselves parallel to the inducing field. Paramagnetism is a temperature dependent phenomenon. Therefore the spinning electrons get disordered in the presence of high temperature. Paramagnetic substances also lose their magnetism in the absence of external magnetic field. But these substances have net magnetic moment in the absence of external magnetic field. Pyrites, Zinc Blende and Hematites are paramagnetic substances.

Ferromagnetic Substances

Materials which show strong magnetic effect in the presence or absence of external magnetic field are ferromagnetic substances. Ferromagnetic substances have very high positive magnetic susceptibility. Orbital electrons get quickly aligned towards the direction of the external magnetic field. Magnetic induction will be very high in a ferromagnetic substance. Therefore they distort the external magnetic field considerably. Ferromagnetic substances have three subgroups. They are (i) ferromagnetic (ii) antiferromagnetic and (iii) ferrimagnetic. In ferromagnetic substance the magnetic

dipoles align themselves parallel to one another quickly in the presence of an external magnetic field. In antiferromagnetic substance the dipoles align themselves antiparallel to one another and the net magnetic effects are cancelled.
Ferrimagnetic substances are those where a material has unequal magnetic moment in the opposite direction. Therefore they have a net magnetic moment (5.11). Because of partial cancellation ferrimagnetic substances are less magnetic than ferromagnetic substances.

Metals like Iron, cobalt nickel and metallic alloys like mu metals, permalloy are materials of higher magnetic permeability and magnetites and titanomagnetites are highly permeable ferrimagnetic ore forming minerals and hematite is a paramagnetic substance.

Ferromagnetic substance gets easily and strongly magnetized in the presence of an external magnetic field. Disoriented and nonaligned magnetic dipoles get quickly aligned even in the presence of a relatively weaker magnetic field (Fig. 5.11). Magnetic induction or flux cut will be very high through ferromagnetic substances. Therefore perturbation of magnetic field will be very high in the vicinity of a ferromagnetic substance. Often the sensitive magnetic instruments are made insensitive to measure the strong perturbation field for exploration of magnetite. Ferromagnetic substance had a big role to play towards discovery of the earth's magnetic field. As mentioned load stone or magnetite needles were used by the mariners towards determining the north direction. Another big contribution of the ferromagnetic substances is the use of mu metals or ferrites , highly permeable specially made metallic alloys, in framing a core of a highly sensitive induction coils for measuring very very feeble magnetic field variations. Magnetic field variations of period 10,000 seconds can be detected and measured with a supersensitive induction coils.

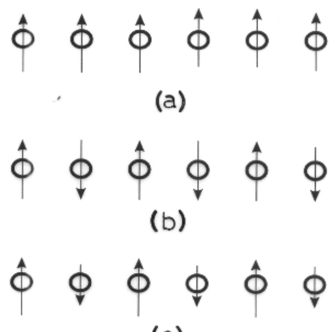


Fig. 5.11. Magnetic dipoles in a (**a**) ferromagnetic (**b**) antiferromagnetic and (**c**) ferrimagnetic substances

5.4 Magnetic Induction B

Magnetic flux B is the number of magnetic lines of forces created by a source of magnetic field. Its unit is Weber. Magnetic flux density or magnetic induction is the number of lines of forces cut per unit area. Its unit is Weber/Square meter or tesla. The direction of B is at right angles to the plane of the loop when the flux cut is maximum. When the loop is in the same plane and is oriented along the direction of the flux, the flux cut will be minimum. Magnetic substances will have higher concentration of flux lines as shown in the Fig. 5.12. Higher the magnetic permeability or magnetic susceptibility higher will be the concentration of magnetic flux lines. Material which get magnetized in the presence of a magnetic field are magnetic substances. Magnetic flux density or magnetic induction is a vector. The magnetic flux can be written as

$$\phi = \int_{s} \vec{B}.\vec{n}.ds \tag{5.6}$$

where B is the magnetic induction vector, n is the direction of the normal to the plane of a coil (Fig. 5.13) or the magnetic substance. ds is a small element of the area got induced. Magnetic induction is connected to the magnetic field strength by the relation

$$B = \mu H \tag{5.7}$$

where μ is the magnetic permeability. So higher the magnetic field intensity and magnetic permeability, higher will be the magnetic induction. High magnetic permeability mu metal cores are used inside a coil to enhance the detectibility of a weak magnetic field as mentioned.

A magnetic body when placed in an external magnetic field H, it gets magnetized and generates field H′ of its own. Magnetic Induction is defined as the total field within this body and is given by

$$\begin{aligned} B = H + H' \quad &= \quad H + 4\pi K \\ = (1 + 4\pi K)H &= \mu H. \end{aligned} \tag{5.8}$$

Variation of B with H is a nonlinear behaviour. It generates a loop known as hysteresis loop. With gradual increase in H, the field strength, the magnetic induction reaches saturation and show no further increase in B.

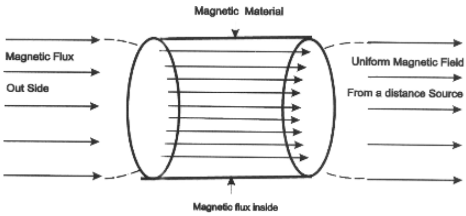


Fig. 5.12. Magnetic induction in a magnetic substances; concentration of flux lines in a higly magnetically permeable substance

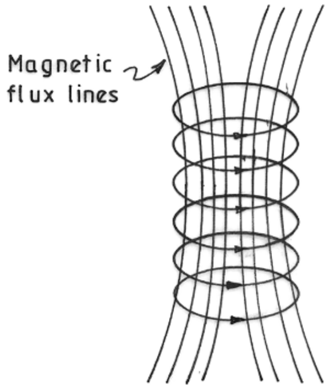


Fig. 5.13. Magnetic flux in a solenoid

That generates the first part of the curve (Fig. 5.14). When the applied magnetic field is reduced to zero, magnetic induction do not reduce to zero but come to a positive magnetic induction 'or' known as residual magnetism. All magnetic substances have positive residual magnetism after the withdrawal of the primary field. If the magnetic field is applied in the negative direction, the magnetic induction will come down to zero. The sector 'oc' is termed as the coercive force, i.e., the field necessary to bring the magnetic induction in the substance to zero. The rest of the loop can be obtained by decreasing and increasing the inducing magnetic field. So B-H curve generates a loop known as hysteresis loop.

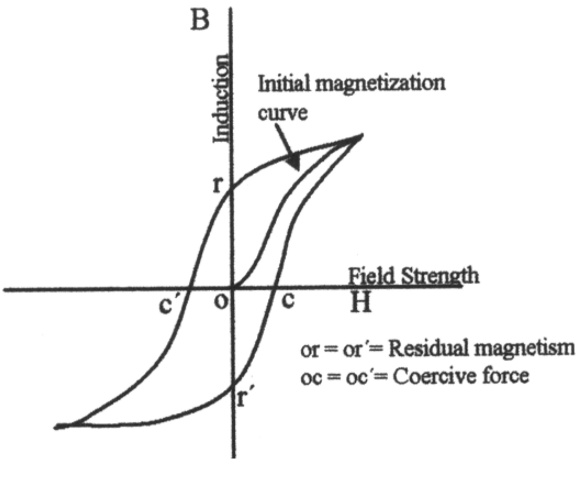


Fig. 5.14. B-H curve: Hysteresis loop

5.5 Magnetic Field Intensity H

From Coulombs law of magnetic force (5.1) we can define the magnetic field strength H as the force on a unit pole of strength m' located at a distance r from the source. The magnetic field due to a point pole of unit pole strength is given by

$$\text{Magnetostatic field} \quad H = \frac{I}{m'} = \frac{1}{4\pi\mu}.\frac{m}{r^3}\,\vec{r} \tag{5.9}$$

It is assumed that m' or the unit pole is not large and near the point of measurement $m' << m$. In electromagnetic unit, it is in oersted i.e., dynes/unit pole. It is shown that the magnetic flux density at a distance r for a wire carrying current I is

$$\vec{B} = \frac{\mu I}{2\pi r} \qquad \text{and} \qquad \vec{H} = \frac{I}{2\pi r} \tag{5.10}$$

The magnetic field strength $\vec{H}$ is thus expressed in terms of current flowing through the wire. The unit of H is ampere/meter or ampere-turns per meter. Equation (5.10) tells that magnetic field is independent of the permeability of a medium. It only depends upon the strength of the current and the distance of the point of observation from the wire carrying current.

5.6 Faraday's Law

Faraday's law of electromagnetic induction states that the voltage induced in a coil, when a bar magnet is brought near by, is proportional to the rate of change of number of lines of forces or the rate of change of flux cut by the coil (Fig. 5.15 a,b,c). If a bar magnet is quickly brought near the coil or quickly withdrawn, the induced voltage will be more. If the relative position of the magnet and coil remains same there will be no voltage even if the coil and the magnet are quite close. One can verify this observation attaching a galvanometer to the ends of a coil and allowing a bar magnet to approach towards the coil or to recede from the coil. This is Faraday's law of electromagnetic induction. The induced e.m.f (electromotive force).

$$\phi = \frac{\partial N}{\partial t} = L\frac{dI}{dt} \tag{5.11}$$

where N is the number of lines of forces and L the constant of proportionality, between the rate of change of current and the flux, is the self inductance. We can write

$$\int_0^T \phi dt = \int LdI = L \int dI = LI \tag{5.12}$$

and magnetic flux

$$\psi = \int_0^T \phi dt = LI. \tag{5.13}$$

From (5.13) we can write $\frac{\psi}{A} = L\frac{I}{A}$ where A is the area of the coil and $\frac{\psi}{A} = B$ and it is the magnetic flux density or magnetic induction.

We can write

$$\frac{\psi}{A} = L\frac{l}{A}\left(\frac{I}{l}\right) \tag{5.14}$$

$$\Rightarrow B = \mu H. \tag{5.15}$$

Here $\mu = L\frac{l}{A}$ i.e. the inductance per unit length and unit cross section is the magnetic permeability and its unit is henry/meter. The unit of inductance is henry. An inductance on a coil generates extra impedance in the circuit.

Figure 5.15(a, b, c) show the generation of current in a single turn of coil or on an n turn solenoid due to movement of the magnet with respect to the coil. Polarity of the bar magnet also changes the direction of the flow of current. Figure 5.16 show the development of current and voltage in a secondary inductive circuit when an alternating current passes through the primary circuit. The currents are generated in the secondary circuit due to flux linkage with the primary circuit. The rate of change of number of lines of forces due to the flow of alternating current in the primary circuit caused the variable flux linkage.

These diagrams in 5.15 a, b, c, 5.16 and 5.17 show the nature of the magnetic field distribution and the close linkage between electricity and magnetism. Faraday's law of electromagnetic induction should come under time varying magnetic field section. Since some of the basics of magnetostatics are explained using this law, it is included in this chapter.

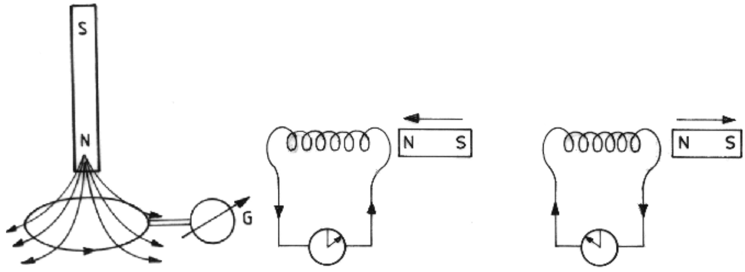


Fig. 5.15(a, b, c). Show the generation of electric current in a coil when a bar magnet is approaching towards or receding from a single term coil or a solenoid

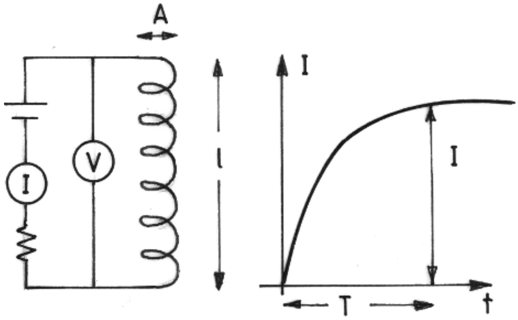


Fig. 5.16. Shows the current build up with time when a direct current is allowed to pass through an inductor

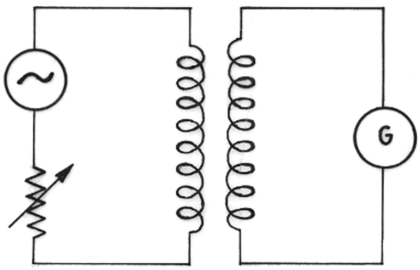


Fig. 5.17. Shows the generation of current in the secondary circuit due to flux linkage when an alternating current flows through the primary circuit

5.7 Biot and Savart's Law

Oersted (1820) discovered that a field is generated surrounding a wire when current is allowed to flow through it. This field is capable of deflecting a magnetic needle. It was established that magnetic field is created when current passes through a wire. So a connecting link between the electricity and magnetium was established. Immediately after this discovery, Biot and Savart experimentally established a relation between the flow of current in a filament of wire and the value of the magnetic induction B. It is given by

$$d\vec{B} = \frac{\mu_o}{4\pi} . \frac{I\,\overrightarrow{dl} \times \vec{r}}{r^3} \tag{5.16}$$

where $\vec{dl}$ is a small filament carrying current I, r is the distance of the point Q, the mid point of the current element dl from the point of observation P. $\vec{r}$ is also the direction of the vector from the point Q to P. $d\vec{B}$ is the elementary magnetic flux, for the current carrying filament $\vec{dl}$. Using the vector product we can write (5.16) as

$$d\vec{B} = \frac{\mu_o}{4\pi} \frac{I\,\overrightarrow{dl}\mathrm{Sin}\theta}{r^2} \tag{5.17}$$

where μ_o is the free space magnetic permeability. Free space magnetic permeability is assumed for vacuum, air and non magnetic materials. In SI unit $\mu_o = 4\pi \times 10^{-7}$ henry/meter. Sin θ is the angle made by the vector $\vec{r}$ with the vector $\overrightarrow{dl}$. Therefore the magnetic induction $\vec{B}$ due to flow of current through the entire wire is given by

$$B = \frac{\mu_o}{4\pi} \oint \frac{I\, dl\, \mathrm{Sin}\theta}{r^2}. \tag{5.18}$$

The direction of $\vec{B}$ is at right angle to the plane containing both $\overrightarrow{dl}$ and $\vec{r}$ (Fig. 5.18). This is the first form of Biot and Savart law.

Since $B = \mu_o H$, the intensity of the magnetic field can be expressed as

$$H = \oint \frac{I\, dl \times r}{4\pi r^2}. \tag{5.19}$$

This is the second form of Biot and savart's law

Since $I = \frac{dq}{dt}$, i.e., the rate of flow of charge and the velocity $\vec{V} = \frac{dl}{dt}$, i.e., dt is the time required for the charge to move a distance dl. Since dq, $\vec{V} = I.dl.$ we can write (5.19) as

$$d\vec{B} = \frac{\mu_o}{4\pi} \cdot \frac{dq(\vec{V} \times r)}{r^3} \tag{5.20}$$

This is the third form of Biot and Savart law.

For a volume distribution of current through a conductor of three dimensional nature (Fig. 5.19), the magnetic field is given by

$$H = \frac{1}{4\pi} \int\int_{v}\int \frac{\vec{J}_v \times \vec{r}}{r^3} dv \tag{5.21}$$

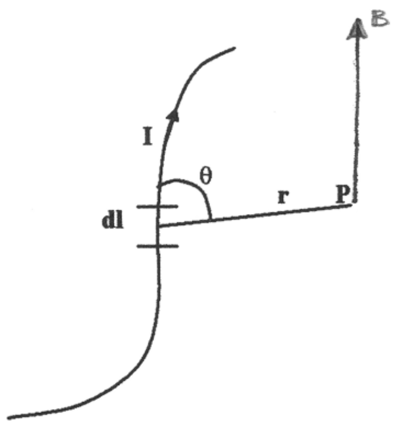


Fig. 5.18. Magnetic field at a point at a distance r from a loop carrying current; the direction of the field is at right angle to the plane containing the current element vector and the position vector r

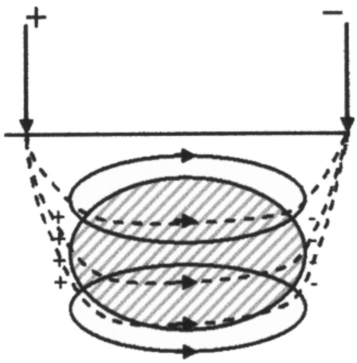


Fig. 5.19. Magnetic field created due to volume distribution of current through a medium of finite conductivity

This is the fourth form of Biot and Savart law and it has direct application in Magnetic Induced Polarisation (MIP) (Seigel, 1974) and Magnetometric Resistivity (MMR) (Edward 1974) methods in geophysics.

5.8 Lorentz Force

Point in a region of space has a magnetic field if a charge (positive or negative) moving in the region experiences a force by virtue of its movement. This force acts at right angles to the force field. This force may be described in terms of a field vector $\vec{\mathrm{B}}$ called the magnetic induction or magnetic flux density. For a stationary charge, the force acting on it, is $\vec{\mathrm{F}} = \mathrm{q}\vec{\mathrm{E}}$ where q is the charge and $\vec{\mathrm{E}}$ is the electric field. When a charge starts moving with a velocity $\vec{V}$ a force known as Lorentz force acts on the charge. This force acts at right angles to the velocity vector $\vec{V}$. This is the magnetic field, which originates due to flow of charge. Therefore the force due to electrostatics and magnetostatics are jointly given as the vector sum as

$$\vec{F} = q\left(\vec{E} + \vec{V}x\vec{B}\right). \tag{5.22}$$

Equation (5.22) shows that electric field and magnetic field are at right angles to each other. The differential form of the force due to the magnetic field is given by

$$\vec{\Delta}F = \rho.\Delta v.\vec{V}x\vec{B} \tag{5.23}$$

where ρ is the volume density of charge and Δv is the elementary volume.

Since

$$\vec{J} = \rho.\vec{V}, \tag{5.24}$$

$$\Delta\vec{\mathrm{F}} = \vec{\mathrm{J}} \times \vec{\mathrm{B}}\ \mathrm{dv} \tag{5.25}$$

and the magnetic force field $\vec{F}$ is

$$\vec{F} = \int_{v} \vec{J} \times \vec{B}.dv. \qquad (5.26)$$

When θ is the angle between the direction of flow of charge and the direction of the magnetic field, the Lorentz force will be

$$F = \rho v B \sin\theta \qquad (5.27)$$

Figure 5.19 shows the direction of the magnetic field at a point P on a plane with respect to the coordinate axis x y z. The value of the magnetic field is

$$B = \frac{\mu_0 \rho}{4\pi r^3}\left(\vec{V} x \vec{r}\right) \qquad (5.28)$$

where μ_0 is the free space magnetic permeability, r is the distance of the point P from the linear conductor.

5.9 Ampere's Force Law

Oersted discovered in 1819 that there is a connection between the electricity and magnetism. He first observed that an wire carrying current is capable of deflecting a magnetic needle. Soon after this discovery Ampere proposed his force law. It states that two complete circuits carrying current are capable of exerting force on one another (Fig. 5.20). He first experimentally demonstrated that the force exerted between the two coils carrying current is given by

$$d\vec{F}_2 = \frac{\mu_o}{4\pi}.I_2 dl_2 \times \left[\frac{I_1\ \overrightarrow{dl_1} \times \vec{R}_{21}}{R_{21}^3}\right] \qquad (5.29)$$

where $d\vec{F}_2$ is the force exerted on the coil 2 due to the flow of current coil 1. Therefore total force acting on the coil 2 due to the flow of current in coil 1 is given by

$$F_2 = \frac{\mu_o}{4\pi}\int_{c} I_2 dl_2 \times \int_{c} \frac{I_1 \overrightarrow{dl}_1 \times \vec{R}_{21}}{R_{21}^3}\ . \qquad (5.30)$$

Applying Biot and Savart law, we can write

$$F_2 = \frac{\mu_o}{4\pi}\int I\overrightarrow{dl}_2 \times \vec{B} \qquad (5.31)$$

where $\vec{B}$ is the magnetic flux density.F_{21} the force in the circuit C_2 is

$$F_{21} = \mu I_2 \oint_{c_2} ds_2 \times H_{21} \qquad (5.32)$$

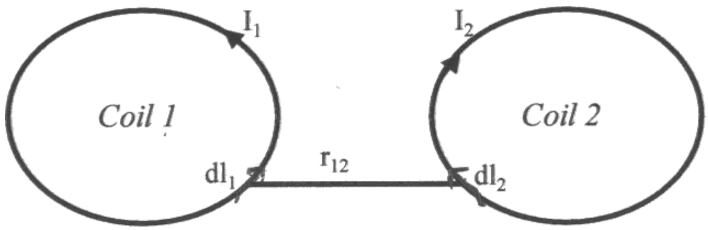


Fig. 5.20. The force of attraction between the two coils

and

$$H_{21} = \frac{I_1}{4\pi} \oint_{c_1} \frac{ds_1 \times \overrightarrow{r_{12}}}{r^2} \tag{5.33}$$

is the magnetic field strength due to the current I_1, in C_1. It is also known as Biot and Savart's law.

5.10 Magnetic Field on the Axis of a Magnetic Dipole

A coil or a loop carrying current is termed as a magnetic dipole. When the same loop will carry alternating current it will be termed as an oscillating magnetic dipole. If the plane of the coil is horizontal but the direction of the magnetic field vector is vertical it is termed as the vertical magnetic dipole. Similarly if the plane of the coil is vertical but the direction of the magnetic field vector through the centre of the circular coil is horizontal it is termed as an horizontal magnetic dipole.

Figure 5.21 show that a circular loop of radius r is in a horizontal plane and flow of current through it is I.

For the small element $\overrightarrow{dl}$ carrying current I, we get

$$\overrightarrow{dl} = r d\psi \overrightarrow{a_\psi} \tag{5.34}$$

where ψ is the azimuthal angle and $\overrightarrow{a_\psi}$ is the unit vector along the azimuthal direction. The distance vector R is the distance between the point of observation P and the mid point of the linear element dl of the coil. Vectorially we can write

$$\vec{R} = -r\overrightarrow{a_\rho} + z\overrightarrow{a_z} \tag{5.35}$$

where r is the radius of the coil and $\vec{a}_\rho$ is the unit vector along the radial direction. z is the vertical distance of the point of observation from the centre of the loop in the plane of the dipole. $\vec{a}_z$ is the unit vector along the vertical z direction. Then the vector product of $\overrightarrow{dl}$ and $\vec{R}$ is given by

$$\overrightarrow{dl} \times \vec{R} = r d\psi \vec{a}_\psi(-r\vec{a}_\rho + z\vec{a}_r) = (r^2\vec{a}_z + rz\vec{a}_\rho)d\psi. \tag{5.36}$$

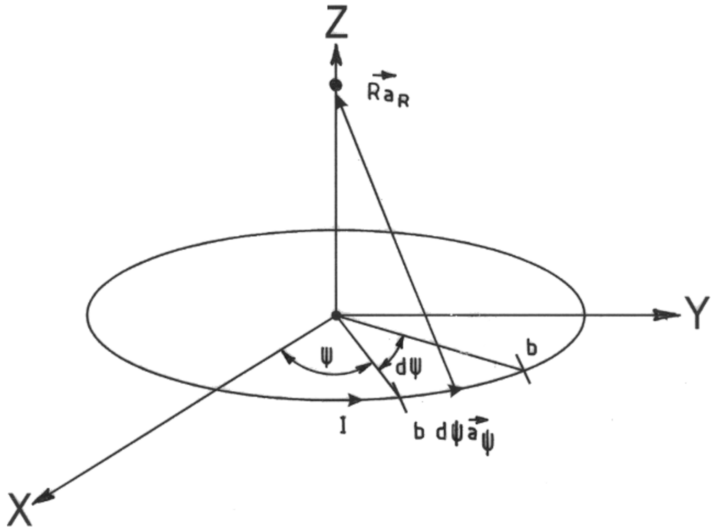


Fig. 5.21. Magnetic field on the axis of a magnetic dipole

Therefore, from Biot and Savart's law, the magnetic flux density can be written as

$$\vec{B} = \frac{\mu_o I r^2}{4\pi}\int_0^{2\pi}\frac{\vec{a}_z d\psi}{(r^2+z^2)^{3/2}} + \frac{\mu_o I\, rz}{4\pi}\int_0^{2\pi}\frac{\vec{a}_\rho d\psi}{(r^2+z^2)^{3/2}}$$

$$= \frac{\mu_o I\, r^2}{2(r^2+z^2)^{3/2}}\vec{a}_z. \tag{5.37}$$

On the axis of the loop, the magnetic flux density has only z-component. Therefore, the magnetic flux at the centre of the loop is

$$\vec{B} = \frac{\mu_o I}{2r}.\vec{z}. \tag{5.38}$$

When the point of observation is in the far zone, i.e. z >> r, the magnetic flux density at the centre of the coil is given by

$$\vec{B} = \frac{\mu_o I r^2}{2z^3}\vec{a}_z. \tag{5.39}$$

That shows that the magnetic field due to a magnetic dipole is inversely proportional to the cube of the distance. Thus it is shown in Chaps. 4, 6 and in this chapter that dipole field dies down inversely as cube of the distance

5.11 Magnetomotive Force (MMF)

The way we defined the potential difference or the electromotive force, as the line integral $\int_b^a \vec{\mathrm{E}}.\mathrm{dl}$ where $\vec{\mathrm{E}}$ is the electric field and dl is the element of length through which current flows, we bring the concept of magnetomotive force in an analogous way.

The line integral

$$\vec{\mathrm{F}} = \int_b^a \vec{\mathrm{H}}.\mathrm{dl} \quad (\mathrm{MMF}) \tag{5.40}$$

is defined as the magnetomotive force between the points a. and b. For a circular path around the wire. $\mathrm{H} = \frac{\mathrm{I}}{2\pi \mathrm{r}}$ (5.10). Thus for a circular path the magnetomotive force

$$\vec{\mathrm{F}} = \oint \vec{\mathrm{H}}.\mathrm{dl} = \mathrm{I} \tag{5.41}$$

This is also known as the Ampere's circuital law or the Ampere's work law. Thus the concept of work is brought here also in the case of magnetostatic field. The important difference between the rotational and irrotation field is, for a closed circular path when the point 'a' coincide with 'b', we get

$$\int_b^a \vec{E}.dl = \phi_a - \phi_b = 0 \tag{5.42}$$

and

$$\int_b^a \vec{\mathrm{H}}.\mathrm{dl} = \mathrm{I}. \tag{5.43}$$

For a toroidal coil with n number of turns in a circular ring of radius R, the magnetomotive force will be

$$\vec{\mathrm{F}} = \mathrm{n\ I}. \tag{5.44}$$

In this case, we can write

$$\mathrm{H} = \frac{\mathrm{F}}{2\pi \mathrm{r}} = \frac{\mathrm{n\ I}}{2\pi \mathrm{r}} = \frac{\mathrm{n\ I}}{\mathrm{l}}. \tag{5.45}$$

Therefore the unit of the magnetic field can also be written as ampere turns/meter.

5.12 Ampere's Law

(i) Ampere's work law, as discussed in the previous section, states that the magnetomotive force around a closed path is equal to the current enclosed by the path, i.e.,

$$\int \vec{\mathrm{H}}.\vec{\mathrm{dl}} = \mathrm{I}\ (\text{amperes}) \tag{5.46}$$

This law can also be presented in the following form

Since Current $\mathrm{I} = \int \vec{\mathrm{J}}.\vec{\mathrm{n}}.\mathrm{ds}$ where $\vec{\mathrm{J}}$ (see Chap. 6) is the current density, applying Stoke's theorem, we can write

$$\int \text{curl}\ \vec{\mathrm{H}}.\mathrm{ds} = \int \vec{\mathrm{H}}.\mathrm{dl} = \int \vec{\mathrm{J}}.\vec{\mathrm{n}}.\mathrm{ds} \tag{5.47}$$

Therefore,

$$\text{curl}\,\vec{\mathrm{H}} = \vec{\mathrm{J}} \tag{5.48}$$

That shows that the magnetostatic field is a rotational field where curl of a magnetic field is nonzero.

5.13 Div B = 0

$$\text{Div}\,\vec{\mathrm{B}} = 0 \tag{5.49}$$

Since magnetic poles cannot be present in isolation, the magnetic field lines always complete a closed circuit (Fig. 15.1). Figures 15.2, 15.3, 15.4 show the nature of the magnetic lines of forces due to a bar magnet, a coil carrying direct current and a long solenoid carrying direct current.

Any magnetic field lines entering a region with or without any source will always go out of the region. Therefore div $\vec{\mathrm{B}}$ will always be zero. From Biot and Savart's law, we get

$$d\vec{B} = \frac{\mu}{4\pi}\frac{Idlx\vec{a}_r}{r^2} \tag{5.50}$$

where

$$\mathrm{r}^2 = (\mathrm{x}-\xi)^2 + (\mathrm{y}-\eta)^2 + (\mathrm{z}-\zeta)^2 \tag{5.51}$$

Here

$$\vec{\mathrm{a}}_\mathrm{r} = \vec{\mathrm{a}}_\mathrm{x}.\frac{\mathrm{x}-\xi}{\mathrm{r}^3} + \vec{\mathrm{a}}_\mathrm{y}\frac{\mathrm{y}-\eta}{\mathrm{r}^3} + \vec{\mathrm{a}}_\mathrm{z}\frac{(\mathrm{z}-\zeta)}{\mathrm{r}^3} \tag{5.52}$$

We write (5.50) as

$$\vec{\mathrm{B}} = \frac{\mu}{4\pi}\int \frac{\vec{\mathrm{dl}}\ \mathrm{x}\ \vec{\mathrm{a}}_\mathrm{r}}{\mathrm{r}^2}.\vec{\mathrm{J}}.\mathrm{ds} \tag{5.53}$$

$$= \frac{\mu}{4\pi}\int \frac{\vec{\mathrm{J}}\ \mathrm{x}\ \vec{\mathrm{a}}_\mathrm{r}}{\mathrm{r}^2}\,\mathrm{dv}. \tag{5.54}$$

From (5.54) we can write

$$\text{div}\,\vec{\mathrm{B}} = \frac{\mu}{4\pi}\int \text{div}\left(\frac{\vec{\mathrm{J}}\ \mathrm{x}\ \vec{\mathrm{a}}_\mathrm{r}}{\mathrm{r}^2}\right). \tag{5.55}$$

Since

$$\text{div}\left(\vec{A} \text{ x } \vec{B}\right) = \vec{B} \text{ curl } \vec{A} - \vec{A} \text{ curl } \vec{B} \tag{5.56}$$

Therefore

$$\text{div}\,\vec{B} = \frac{\mu}{4\pi} \int \left(\frac{\vec{a}_r}{r^2} \text{ curl } \vec{J} - \vec{J} \text{ curl } \frac{\vec{a}_r}{r^2}\right) \text{dv}. \tag{5.57}$$

When a current is flowing along a linear conductor curl $\vec{J} = 0$ and

$$\text{div}\,\vec{B} = -\frac{\mu}{4\pi} \int \left(\vec{J} \text{ curl } \frac{\vec{a}_r}{r^2}\right) \text{dv}. \tag{5.58}$$

Now

$$\text{curl}\left(\frac{\vec{a}_r}{r^2}\right)_x = \frac{\partial}{\partial y}\left(\frac{z-\zeta}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{y-\eta}{r^3}\right) = 0, \tag{5.59}$$

because

$$\frac{\partial}{\partial y}\left(\frac{z-\zeta}{r^3}\right) = 0$$

and

$$\frac{\partial}{\partial z}\left(\frac{y-\eta}{r^3}\right) = 0.$$

Therefore

$$\text{div}\,\vec{B} = 0. \tag{5.60}$$

5.14 Magnetic Vector Potential

Since div $\vec{B} = 0$ always and the divergence of curl of a vector is always zero, we can write

$$\vec{B} = \text{curl}\vec{A} \tag{5.61}$$

where $\vec{A}$ is termed as a vector potential because curl operates on a vector and generates another vector. Since a field is obtained from the spatial derivative of a vector potential, we can write the expression for the vector potential as

$$\text{d}\vec{A} = \frac{\mu \text{Idl}}{4\pi r} \tag{5.62}$$

where $\text{I}\vec{\text{d}}\,\text{l}$ is the current element and r is the distance from the current element where the vector potential is measured. Expression for vector potential for current flow through a complete circuit is given by

$$\text{A} = \int \frac{\mu \text{Idl}}{4\pi r}. \tag{5.63}$$

For a flow of current through a three dimensional conductor, where current is not restricted to flow through a filament or a wire

$$\text{A} = \iiint_v \frac{\mu\, \vec{J}\, \text{dv}}{4\pi r}. \tag{5.64}$$

More information on vector potential are available in Chaps. 12 and 13.

5.15 Magnetic Scalar Potential

In a current free region J (the current density)= 0,we get

$$\text{Curl B} = 0.$$

Once the magnetic induction becomes curl free, it can always be written in terms of a gradient of a scalar potential where $B = -\mu_0 \Delta\phi$ and $H = -\Delta\Phi$ where Φ is the magnetic scalar potential.

Magnetic scalar potential is the work done to bring an unit magnetic pole from infinity to a point at a distance ‘r’ from source of magnetic pole of strength ‘m’.The magnetic potential can be expressed as

$$\phi = \frac{1}{4\pi\mu}\frac{m}{r}. \tag{5.65}$$

Magnetic field will be the space derivative of magnetic potential. Magnetic field $H = -\text{grad}\phi$. We can write

$$\phi = -\int_{\infty}^{r} \vec{H}.dl = \frac{m}{4\pi\mu r} \tag{5.66}$$

where ‘r’ is the distance of the point of observation from the source. Since the magnetic monopoles do not exist in isolation, the field is generally estimated due to a magnetic dipole. Magnetic field at a point P due to a magnetic dipole (Fig. 5.22) is given by

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} = m\left\{\frac{1}{(r^2+l^2-2lr\cos\theta)^{1\backslash 2}} - \frac{1}{(r^2+l^2+2lr\cos\theta)^{1\backslash 2}}\right\}. \tag{5.67}$$

For magnetic dipole r, the distance of the point of observation is much greater than the dipole length $(r \gg l)$. Therefore

$$\phi = \frac{2ml\ \text{Cos}\theta}{r^2} \approx \frac{M\ \text{Cos}\theta}{r^2}. \tag{5.68}$$

(see Chap. 4). Here M is the magnetic dipole moment. The radial and azimuthal component of the magnetic field are respectively given by

$$H_r = -\frac{\partial\phi}{\partial r} = -m\left[\frac{r+l\ \text{Cos}\theta}{(r^2+l^2+2rl\ \text{Cos}\theta)^{3/2}} - \frac{r-l\ \text{Cos}\theta}{(r^2+l^2-2rl\ \text{Cos}\theta)^{3/2}}\right] \tag{5.69}$$

$$H_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = m\left[\frac{l\ \text{Sin}\theta}{(r^2+l^2+2rl\ \text{Cos}\theta)^{3/2}} - \frac{l\ \text{Sin}\theta}{(r^2+l^2-2rl\ \text{Cos}\theta)^{3/2}}\right] \tag{5.70}$$

for $r >> 1$.

After simplification, one gets

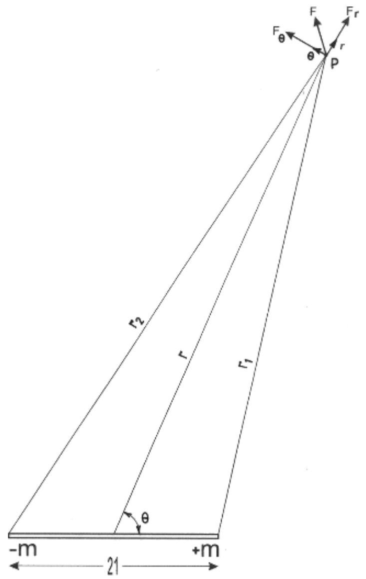


Fig. 5.22. Magnetic field at a point outside due to a magnetic dipole

$$H_r = \frac{2M \ \mathrm{Cos}\theta}{r^2} \tag{5.71}$$

and

$$H_\theta = \frac{M \ \mathrm{Sin}\theta}{r^2}. \tag{5.72}$$

Therefore, the expression for the total field due to a magnetic dipole is

$$H = \left(\frac{2M \ \mathrm{Sin}\theta}{r^3}\right)\vec{r} + \left(\frac{M \ \mathrm{Sin}\theta}{r^2}\right)\vec{\theta}. \tag{5.73}$$

Since the magnetic field is a dipole field, Poisson's equation for magnetic scalar potential is

$$\nabla^2\phi = 4\pi.\nabla.M(r)$$

where M is the magnetic dipole moment.

5.16 Poisson's Relation

Magnetic potentials and fields can be estimated from gravitational potential using Poisson's relation. This relation can be expressed as

$$\phi = -\frac{1}{G\rho}\frac{\partial\phi^/}{\partial\lambda} \tag{5.74}$$

where ϕ is the magnetic potential, $\phi^{/}$ the gravitational potential, λ the direction of the magnetic polarisation, I the magnetic polarisation, ρ the density of a medium and G the universal gravitational constant. Horizontal component of the magnetic field is given by

$$Hx = -\frac{\partial \phi}{\partial x} = \frac{I}{G\rho}\frac{\partial}{\partial x}\left(\frac{\partial \phi^{/}}{\partial \lambda}\right). \tag{5.75}$$

If a magnetic body is polarised along the vertical Z direction, the x component of the magnetic field can be written as

$$H_x = -\frac{\partial \phi}{\partial x} = \frac{I}{G\rho}\frac{\partial}{\partial x}\left(\frac{\partial \phi^{/}}{\partial z}\right) \tag{5.76}$$

and the vertical component is

$$H_z = -\frac{\partial \phi}{\partial z} = \frac{I}{G\rho}\left(\frac{\partial^2 \phi^{/}}{\partial z^2}\right) \tag{5.77}$$

here

Φ = Magnetic scalar potential
$\Phi^{'}$ = Gravitational potential
r = Direction of the magnetic polarization
I = Magnetic polarization
ρ = Density of the body
G = Universal Gravitational Constant

5.17 Magnetostatic Energy

Magnetostatic energy in a circuit can be estimated in terms of the amount of work done to establish current I by the electromotive force generated by change in the magnetic flux in a circuit. Magnetic field in a solenoid is given by

$$H = nI/l \tag{5.78}$$

in ampere turns/meter. Here I is the current flowing through the solenoid, l is the length of the solenoid and n is the number of turns in the solenoid.

The voltage generated in a coil is, according to Faraday's law,

$$\phi = -N\frac{d\psi}{dt} \tag{5.79}$$

where ϕ is proportional to $\frac{d\psi}{dt}$. Here $\psi = AB$ where A is the area of the coil and B is the magnetic induction. Therefore

$$\phi = -nA\frac{d\psi}{dt}. \tag{5.80}$$

Since total work done in a circuit for flow of current I due to voltage ϕ is

$$\begin{aligned} w &= -\int_0^t \phi I dt \\ &= -\int_0^t nA\frac{d\psi}{dt} = \int_0^H \mu l A\, H dH. \end{aligned} \tag{5.81}$$

Substituting $B = \mu H, \psi = AB$ and $H = \frac{nI}{l}$, we get

$$w = \mu LA\frac{H^2}{2}. \tag{5.82}$$

Therefore magnetostatic energy per unit volume is

$$\frac{1}{2}\mu H^2. \tag{5.83}$$

Both the concepts of electrostatic and magnetostatic energy are needed to explain the Poynting vector in electromagnetics (see Chap. 12).

5.18 Geomagnetic Field

Detailed research work on the history of development of Geomagnetism and Palaeomagnetism had been published by Merrill and Mcilhenny (1996). The points to be highlighted from their work, in this brief discussion, are as follows: (i) People knew about geomagnetism as early as 6th century B.C., (ii) Earliest magnetic compass came in China as early as 1st century A.D., (iii) First observation on magnetic declination was made in China during 720A.D., (iv) Magnetic Inclination was discovered by George Hartmann in 1544, (v) Henry Gellibrand first discovered the variation of declination of the earth's magnetic field, (vi) In 1546 Gerhard Mercator first realized that earth magnetic pole lies on the surface of the earth and he could fix these poles, (vii) Alexander Von Humbolt first made a global magnetic survey and could establish that intensity of the magnetic field varies with latitude. The field is strongest at the pole and weakest at the equator, (viii) In 1600 Willium Gilbert first proposed that earth as a whole acts like a big magnet (ix) In 1838 Gauss first proposed the mathematical form of the earths magnetic field. He could pin point the position of the geomagnetic poles. These positions are the best fitting dipoles cutting the surface of the earth.

William Gilbert(1540–1603) in his Treatise 'De magnet' first mentioned about the existence of the magnetic field of the earth and that the origin of the magnetic field of the earth lies in the interior of the earth. Geomagnetic stable field of global presence originates due to dynomo current in the earth's core composed most probably of iron, nickel and sulphur. Both iron and nickel are good conductors. The observation regarding the rotation of the magnetic needle(load stone) along a definite direction of a magnetite needle led to the discovery of the earth's magnetic field as mentioned. Even magnetite ores could be unearthed in the sixteenth and seventeenth century recording deflections of magnetite needles. Now it is understood that geomagnetic field originates due to the dynamo current in the core of the earth and it can be approximated as magnetic dipole oriented approximately along the north south direction The axis of the magnetic dipole is tilted about 11.5° with respect to the axis of rotation of the earth. 80% of the earth's relatively stable magnetic fields are of internal origin. It varies very slowly. About 19% of the field is of internal origin of non dipole nature. The magnetic north and south poles are located at $78\frac{1}{2}^\circ$ N, $69\frac{1}{2}^\circ$ W and $78\frac{1}{2}^\circ$ S, $111\frac{1}{2}^\circ$ E. Figure 5.23 shows the angle between the geographical north and the magnetic north. It is termed as declination D. The angle made by the total field with the horizontal is the inclination of the field. Therefore we can write

$$\mathrm{T}^2 = \mathrm{H}^2 + \mathrm{Z}^2 \tag{5.84}$$

where T is the total field, H and Z are respectively the horizontal and vertical component of the magnetic fields. I is the inclination angle made by the total field with the horizontal component, D is the declination made by the magnetic north with that of the geographical north. Here (Fig. 5.23).

$$\begin{aligned} \mathrm{H} &= \mathrm{T}\cos\mathrm{I} \\ \mathrm{Z} &= \mathrm{T}\sin\mathrm{I} \\ \tan\mathrm{I} &= \mathrm{Z}/\mathrm{H} \\ \mathrm{H_X} &= \mathrm{H}\cos\mathrm{D} \\ \mathrm{H_Y} &= \mathrm{H}\sin\mathrm{D} \end{aligned} \tag{5.85}$$

Isogonic, isoclinic and isodynamic maps are respectively the contour maps of equal declination, equal inclination and equal values of H or Z.

About 1% of the earth's quasistatic and time varying magnetic fields are of extra terrestrial origin and get superimposed on the earth's stable magnetic field. Inclination of the Earth's magnetic field is downward throughout the entire northern hemisphere and its inclination is upward throughout the southern hemisphere. Magnetic poles are those where the magnetic field is vertical. Geomagnetic poles are the extension of the magnetic dipole axis on the surface of the earth. Although they are very close to each other but they do not exactly coincide. Magnetic field of dipole and nondipole origin exist upto 30,000 km to 40,000 km from the surface of the earth. The total space above the surface of the earth on both the sides of north and south pole is known as the magnetosphere (Fig. 5.24). Recent space research has revealed

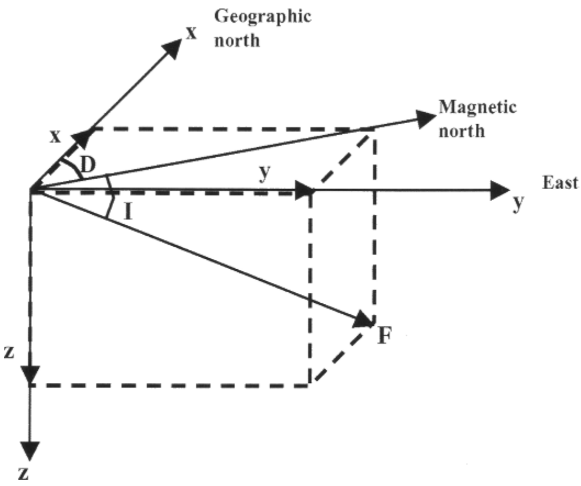


Fig. 5.23. Geomagnetic field showing magnetic lines of forces. It shows the geographic north, magnetic north inclination and declination of the total magnetic field

that magnetosphere get compressed on the sun side due to the interaction of the solar flares composed of near and far ultraviolet rays, hard and soft x-rays, gamma rays, electrons and protons in the form of solar plasma, cosmic rays and particles in the magnetosphere. As a result magnetic field exists only upto 2 to 3 times the radius of the earth during day time, i.e., on the sun side. Magnetosphere extends upto 8 to 10 times the earth's radius on the dark side of the earth and it extends with a tail known as magneto tail. Figure 5.25. Magnetic field ceases to exist in the space above the magnetosphere. This zone is known as the magnetopause. The other important zones of the magnetosphere are (i) Ozonosphere at a height of 23 to 25 km above the surface of the earth which absorbs most of the ultra violet rays (ii) Ionosphere(D,E.F layers) at different heights within the range of 60 to 250 km from the surface of the earth which absorb most of the x rays and gamma rays to get ionized (iii) Van Allen radiation belts are two daughnut shaped conducting zones are

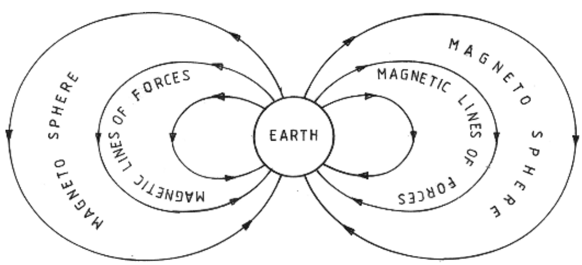


Fig. 5.24. Nature of the assumed Geomagnetic field showing the magnetic lines of forces and magnetosphere

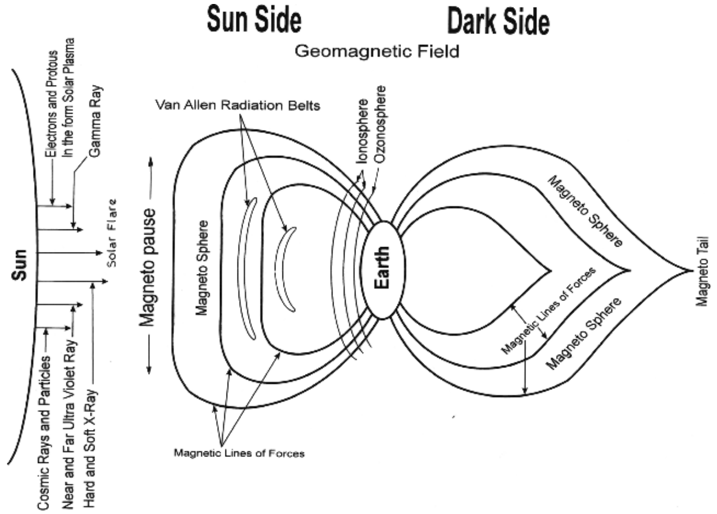


Fig. 5.25. Interaction of the solar flares with the magnetosphere; creation of compressed magnetosphere, Van Allen radiation belts, ionosphere, ozonosphere and magnetopause on the day side and magnetotail on the night side

ionized by the highly penetrating gamma rays and x rays. These radiation belts are at heights of and half times to three times the radius of the earth

Variable solar flares with variable intensities of rays and particles generate time varying magnetic fields. These magnetic fields are superimposed on the permanent and stable geomagnetic field of the earth.

5.18.1 Geomagnetic Field Variations

Major long term variations of the earth's magnetic field are as follows:

(a) Solar Quiet Day Variations(S_q Field)

Solar quiet day variations are those where solar emissions, which are primarily responsible for variation of the magnetic field, are minimum. Geomagnetic field remains more or less stable for a few days at a stretch. These days are known as solar quiet days and variations are known as S_q variations. The variations are periodic over solar quiet days. Their magnitudes are dependent upon the season of the year and latitude and are also stronger in the summer than in the winter. These fields are found to be stronger in higher latitude than in the equatorial zone. Earths main field and the conducting ionosphere constitutes the ionosphere dynamo and creates current in the E-layer. Major part of the S_q variations come from the ionospheric currents and their variations. The other 10% of the S_q variations come from the compression of the magnetosphere happens due to interaction of the solar flares with the magnetosphere.

If sun remains quiet(lower level solar emissions), geomagnetic field also remains quiet. All permanent geomagnetic observatories all over the world are recording these variations. The magnitude of the S_q variations may be of the order of 40 to 60 nanotesla (nT).

(b) L variations

Geomagnetic variations which are associated with lunar time are called L variations. These are much weaker variations and the magnitude is of the order of 2 nT. It is interesting to note that day time L variations are stronger than night time variations.

(c) D and D_{st} variations

These variations are observed due to enhancement in solar flares. Both rays and particles are emitted at an enhanced rate which causes increase in strength of the magnetic field and cause magnetic storms . Magnetic storms originate during the strong and enhanced solar flares. The strength of the magnetic field rises to a certain peak and then decreases gradually.

Magnetic storms have certain periodicity. It is possible to predict the onset of magnetic storms. Disturbance day variations of both horizontal and vertical components of the magnetic field cause auroras in higher latitudes. Enhancement of the magnetic fields creates night time glow in the sky termed as aurora Borealis and aurora Australis respectively in northern and southern polar regions. Time varying magnetic field becomes stronger during solar disturbance day variations and generate geoelectric and geomagnetic pulsations and micropulsations. Besides these variations of the geomagnetic field of outside origin, there are several high frequency components of the geomagnetic field variation.

Those variations are geomagnetic and geoelectric micropulsations, pulsations. There are several classification of these micropulsations namely Pc, Pi, Pp, Pg etc. They do not come under magnetostatics. Electromagnetic theory is applicable for these field signals. Long period variations collected from permanent geomagnetic observatories are used for Geomagnetic Depth Sounding (GDS)(Schmucher 1976) to find out the electrical conductivity of earths mantle and core. Pulsations and micropulsations and spherics are used for magnetotelluric sounding (Cagniard 1953) to study the electritrical conductivity of the earths crust and uppermost mantle. Detailed Spherical Harmonic Analysis (see Chap. 7) show that 99% of the earths magnetic field are of internal origin. 80% of that are of deeper dipole origin. Nineteen percent are of shallower non dipole origin. One percent are of extraterrestrial origin which constitutes the time varying part of the earths natural electromagnetic field discussed in Chap. 13.

For further studies the readers are referred to Keller and Frischknecht (1966), Telford et al (1981), Parasnis (1966), Dobrin and Savit (1988), Blakely (1996), Radhakrishnamurthy (1998), Jackson (1999), Wangsness (1979), Jordon and Balmain (1993), Guru, B. and Hiziroglu (2005).

5.19 Application of Magnetic Field Measurement in Geophysics

Magnetic field has made a major inroad in geophysics It has multifaceted applications in various branches of both solid earth and applied geophysics. It is interesting to note that noise in one branch of geophysics becomes a signal in the other. Different branches of geophysics related to magnetic field measurements are (i) **Ground Magnetic Method**: It is used for (a) mineral exploration, (b) basement mapping for oil exploration (c) mapping the contacts of felsics and mafics (d) mapping volcanics and dyke swarms etc. (ii) **Airborne Magnetic Method**: It is used for(a) mapping the major tectonic settings of a geological terrain (b) quick coverage of accessible and inaccessible areas for mineral exploration (iii) **Geomagnetic Depth Sounding (GDS):** Long period variations of the earth's magnetic field are continuously recorded in permanent geomagnetic observatories all over the world. These geomagnetic signals are interpreted to find out the electrical conductivity of mantle and core of the earth (iv) **Magnetovariational Sounding (MVS):** Here we measure the short and long period variations of the earth's magnetic field to find out the electrical conductivity of the earth's crust and upper mantle (v) **Magnetotelluric Sounding (MT):** In this branch of geophysics we measure both the time varying electric and magnetic components of the earth's natural electromagnetic field and try to find out the electrical conductivity of the earth's crust and upper mantle (vi) **Audiofrequency Magnetotellurics (AMT)** Relatively high frequency components of the earth's natural electric and magnetic field originated due to thunder storm activities in between the earth ionosphere wave guide are measured and used for mapping of shallow structures and mineral exploration. (vii) **Audiofrequency Magnetic Method (AFMAG):** High frequency spherics are recorded for mineral exploration. (viii) **Magnetic Induced Polarisation (MIP):** Here secondary magnetic field, originated due to depolarisation current flow within a polarisable medium, when the primary current is switched of in time domain induced polarization, is measured. (ix) **Magnetometric Resistivity Method (MMR):** In this method primary magnetic field perturbation due to flow of electric current through the ground in the presence of shallow lateral structural heterogeneities are detected (x) **Very Low Frequency Method (VLF):** In this method very low frequency magnetic fields are measured due to primary signals from distant broad casting stations to detect some subsurface structures. Time varying magnetic fields are measured in MT,AMT, AFMAG and VLF. They strictly do not come under Magnetostatics. (xi) In **Palaeomagnetism**, magnetic field fossilized and frozen in geological past, are measured. This information is useful for studying (i) continental drift (ii) rifting of the continents (iii) polar wondering etc.

5.20 Units

The unit of magnetic flux ψ is Weber
1weber = 1 joule/ampere
$= 10^{-5}$ oersted
1Oersted = 1 dyne/unit pole
The unit of B is Weber/meter^2
1weber/meter^2 = 10^4 gauss
The unit of inductance L is henry
1henry = 1joule/(ampere)^2
The unit of magnetic permeability μ is henry/meter
1henry/meter = 1newton/(ampere)^2
The unit of magnetic field intensity H is ampere/meter or ampere turns/meter
Unit of magnetic field intensity is also expressed as tesla
1tesla = 1weber/(meter)^2 = 1 volt second
Practical unit of measurement of magnetic field is in nanotesla or 10^{-9} tesla or in gamma. For most geophysical measurements of the magnetic field, gamma or nanotesla are generally used as units of measurement.
1gamma = 10^{-5} gauss

5.21 Basic Equations in Magnetostatics

1.
$$\mathrm{div}\,\vec{B} = 0 \text{ (Solenoidal field)} \tag{5.86}$$

2.
$$\vec{B} = \mu\vec{H} \tag{5.87}$$

3.
$$\mathrm{Curl}\,\vec{H} = \vec{J} \text{ (Rotational field)} \tag{5.88}$$

4.
$$\mathrm{Curl}\,\vec{H} = 0 \tag{5.89}$$

(Low frequency approximation in Geomagnetic field)

5.
$$\vec{B} = \frac{\mu_o}{4\pi} \int \frac{\vec{I}.\overrightarrow{dl}\ \mathrm{Sin}\theta}{r^2} \text{ (Biot and Savart Law)} \tag{5.90}$$

6.
$$\vec{A} = \iiint_v \frac{\mu\vec{J}dv}{4\pi r} \text{ (Vector potential)} \tag{5.91}$$

7.
$$I = \int \vec{H}.dl \text{ (Ampere's Circuital Law)} \tag{5.92}$$

8.
$$F_{12} = \mu = \frac{I_1 I_2}{4\pi} \oint \oint \frac{dl_1 \times (dl_2 \times r_2)}{r_2^2} \text{ (Ampere's force law)} \tag{5.93}$$

9.
$$F = q(E + \vec{V} \times \vec{B}) \text{ (Lorentz force).} \tag{5.94}$$

10.
$$\vec{H} = \frac{nl}{I} \tag{5.95}$$

11.
$$\vec{B} = \mathrm{Curl}\,\vec{A} \tag{5.96}$$

6

Direct Current Flow Field

In this chapter some parts of the direct current flow field preliminaries used in geophysics are discussed. Topics include the nature of the direct current flow field and the approaches for measurement of potentials, depth of penetration of direct current in a homogenous and isotropic medium, potentials and fields due to a point source, line source, bipole and dipole sources, boundary conditions in direct current flow field. Most of the basic equations of direct current flow field are mentioned. Relatively advanced topics on D.C. boundary value problems are given in Chaps. 7, 8, 9, 11 and 15.

6.1 Introduction

Direct current flow through a medium of finite conductivity or resistivity due to a point or a line source of current generates scalar potential field, where the electric field can be expressed as the negative gradient of the potential. The field and potential at a point created by a point source has similarity with those obtained for electrostatic or gravitational field, i.e., it follows $\frac{1}{\mathrm{r}^2}$ and $\frac{1}{\mathrm{r}}$ laws respectively in a homogeneous and isotropic medium. The current from a point source flows radially outward and the equipotential lines are circular in a plane surface (Fig. 6.1).

It is a man made local field in most of the cases. Quasistatic telluric current flow fields of global presence follow direct current flow field equations. This field can be divergence less or solenoidal in a source free region (Fig. 6.2a, b).

A certain region R in a direct current flow field in the absence of any source or sink satisfies Laplace equation. In the presence of one or more than one source, DC flow field satisfies Poisson's equation. Both bipolar and dipolar fields are generated in a direct current flow domain. It is an irrotational or curl free field. In that respect it is similar to gravity, electrostatic, stream line fluid flow and, heat flow fields. For direct current flow field, principle of superposition and principle of reciprocity are valid. Principle of superposition states that potential at a point due to a number of current sources and sinks

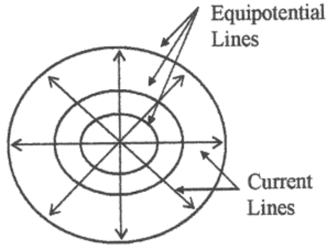


Fig. 6.1. Field lines and equipotential lines due to a point source of current in a homogenous and isotropic medium

will get added and subtracted due to presence of current sources and sinks at the point of observation.

This property generated a series of electrode configurations in direct current flow field. We generally send current through two current electrodes and measure the potential difference through another pair of electrodes known as potential electrodes. There are about 12/13 electrode configurations used regularly by the geophysicists. They are (i) Two electrode (Pole-pole) (ii) Three electrode (Pole-dipole) (iii) Wenner (iv) Schlumberger (v) Collinear dipole-dipole (vi) Unipole system (vii) Seven electrode (Laterolog) (viii) Equatorial dipole (ix) Parallel dipole (x) Perpendicular dipole (xi) Azimuthal dipole (xii) Axial dipole. Different electrode configurations have their different areas of applicability Fig. 6.3 a, b, c, d, e, f and g shows the layout of the different electrode configurations.

The principle of reciprocity states that if we interchange the positions of current and potential electrodes, the potential difference measured between the two potential electrodes in these two cases will remain the same. Theoretically it is true. In actual field practice with the increase in electrode separation some differences between the two sets of measurements are observed because of entry of voltage due to telluric or earth currents and other noises in the measurement. In electrostatics the electric displacement is connected to the electric field and the connecting scalar is $\in$ (4.4); the electrical permittivity is

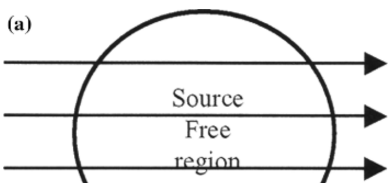


Fig. 6.2a. Source free region in an uniform field

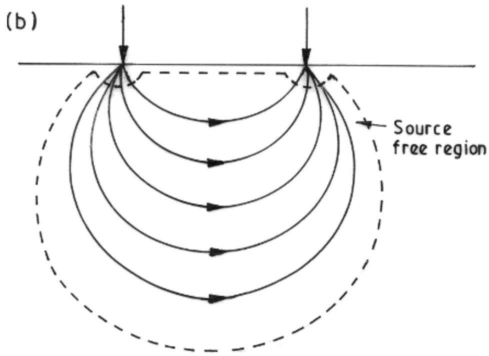


Fig. 6.2b. Source free region in a point source generated bipole field

expressed in farad/meter. In magnetostatics, the magnetic induction is connected to the magnetic field through a scalar μ (5.13) i.e., the magnetic permeability. It is expressed in henry per meter. In direct current flow field, the current density, i.e., the current per unit area is connected to the electric field through a scalar σ (6.9), the electrical conductivity. It is expressed in mho/m or Siemen. These scalars become tensors for an anisotropic medium. These three equations in conjunction with the Maxwell's (12.62 to 12.66) equations form the basis of electromagnetic theory.

In direct current flow field, the physical property, we try to measure is electrical conductivity or electrical resistivity. Electrical resistivity is reciprocal of electrical conductivity. Of all the physical properties of the earth, measured by geophysicists, electrical conductivity is the most sensitive parameter. A little perturbation, in a medium through which current flows, can change the value of the electrical conductivity by several order of magnitude. The ratio of the extreme values of resistivity or conductivity is of the order of 10^{15} or more.

The boundary conditions in direct current flow field are (i) potential must be continuous across the boundary i.e., $\phi_1 = \phi_2$ and (ii) normal component of the current density must be continuous across the boundary,i.e., $J_{n_1} = J_{n_2}$. The boundary value problems also must satisfy (i) Dirichlet (ii) Neumann or (iii) mixed, Robin or Cauchy's boundary conditions.

Since this book includes elaborate treatments on direct current potential and field theory in Chap. 7 and Chap. 8 only a very few points, in preliminary level, are discussed in this chapter. Application of direct current methods in geophysical exploration is beyond the scope of this book and are available in Alpin et al (1966), Keller and Frieschknecht (1966), Bhattacharya and Patra (1968), Koefoed (1979), Zhdanov and Keller (1994). Advanced theories however are given in Chaps. 8 and 15. Some treatments are also available in Chaps. 9 and 11.

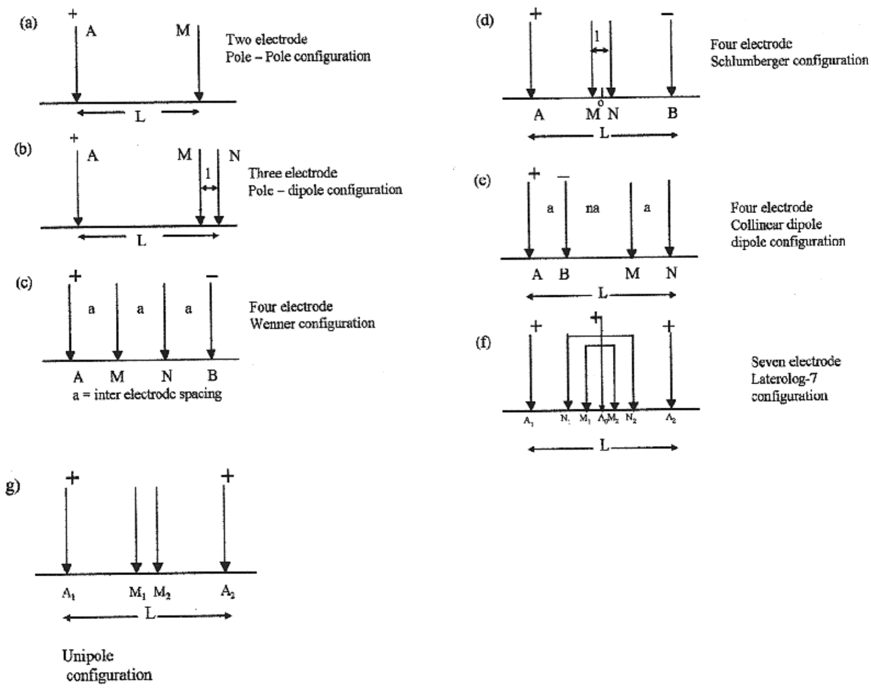


Fig. 6.3. (**a**) Two electrode configuration (normal log arrangement) with one current electrode A and one potential electrode B; the other current electrode for return current path and the other potential electrode are far away from this electrode set up; (**b**) Three electrode configuration (Lateral log arrangement) with one current and two closely spaced potential electrodes, the other return current electrode is placed far away from this set up; (**c**) Four electrode configuration (Wenner electrode arrangement); A and B are two current electrodes and M and N are potential electrodes; these electrodes are equidistant from one another; (**d**) Four electrode configuration (Schlumberger electrode arrangement), A and B are current electrodes, closely spaced M and N are potential electrodes; (**e**) Four electrode configuration (Collinear dipole-dipole configuration) with current dipole AB may have wide separation from potential dipole MN; (**f**) Seven electrode configuration (Latero log arrangement) with central focussing current electrode, two bucking current electrodea A_1 and A_2; two pairs of potential electrodes M_1N_1 and M_2N_2; the return current electrode is far away from this set up; (**g**) Four electrode configuration (Unipole method); here two current electrodes are sources for current focussing; two closely spaced potential electrodes are used to measure pure anomaly; return current electrode is far away from this set up

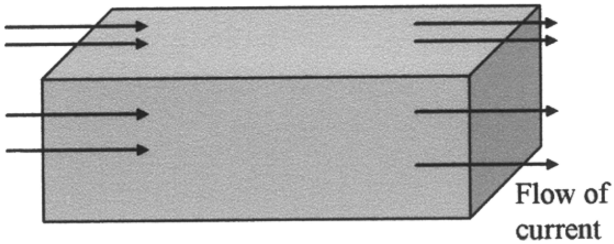


Fig. 6.4. Flow of direct current through two opposite faces of a rectangular parallelepiped

6.2 Direct Current Flow

In direct current flow field, the flow of current is stationary. Let a constant current I flows through a homogeneous and isotropic medium.

We know that current I can be taken as the rate of flow of charge i.e. $I = \frac{dq}{dt}$. At any point in the medium, I cannot be defined but J can be defined. A small area ΔS is chosen normal to the flow of current. The amount of current flows through the face in a time Δt is given by (Fig. 6.4)

$$\Delta I = q_v \frac{\Delta S.\Delta l}{\Delta t} \tag{6.1}$$

where Δl is the distance traveled by the charges and q_v is the volume density of charge. From (6.1) we get

$$\frac{\Delta I}{\Delta S} = q_v \frac{\Delta l}{\Delta t} \tag{6.2}$$

$$\Rightarrow \vec{J} = q_v \vec{v} \tag{6.3}$$

where $\vec{J}$ is the current density and $\vec{v}$ is the velocity. The expression for the current is given by

$$I = \int \vec{J}.\vec{n}.ds \tag{6.4}$$

where $\vec{n}$ is the normal vector.

6.3 Differential form of the Ohm's Law

Ohm's law is defined as temperature remaining same the potential generated between the two points of a conductor has direct proportionality with the current flowing through the ground.

$$\text{So } I = (\phi_1 - \phi_2)\,\frac{1}{R} \tag{6.5}$$

where I is the current flowing through this medium ϕ_1 and ϕ_2 are the potentials at two points in the medium and R, the constant of proportionately is the resistance offered by the ground. Here

$$R = \rho \frac{l}{A} \tag{6.6}$$

where ρ is the specific resistivity, l is the length and A is the cross sectional area for the flow of current. Specific resistivity of a medium is defined as the resistivity offered by the two opposite faces of an unit cube. The unit of resistance is Ohm. The unit of resistivity is Ohm-meter. The reciprocal of resistance is conductance (C). Its unit is mho. The reciprocal of resistivity is conductivity (σ). Its unit is mho/meter. From (6.5 and 6.6), we can write

$$C = \sigma \frac{A}{L} = \sigma \frac{\Delta S}{\Delta l}. \tag{6.7}$$

$$\text{Here } \Delta I = \frac{\Delta S}{\Delta I}.\left(-\Delta\phi\right).\sigma \tag{6.8}$$

$$\Rightarrow \frac{\Delta I}{\Delta S} = -\frac{\Delta\phi}{\Delta l}.\sigma$$

$$\Rightarrow \vec{\mathrm{J}} = \sigma\vec{\mathrm{E}}. \tag{6.9}$$

6.4 Equation of Continuity

Since, the stationary electric fields are conservative, the electric field is expressed as the gradient of a scalar potential (Φ) i.e.,

$$\mathrm{E} = -\nabla\Phi. \tag{6.10}$$

Combining equation (6.9) with equation (6.10), we get

$$\mathrm{J} = -\sigma\nabla\Phi. \tag{6.11}$$

Applying the principle of conservation of charge over a volume, which states that charges can be neither created nor destroyed, although equal amount of positive and negative charges may be simultaneously created. From (6.8) we can write

$$I = \oint_s \vec{J}.\vec{n}.ds \tag{6.12}$$

and this outward flow of positive charge must be balanced by a decrease of positive charge (or an increase of negative charge) within a closed surface. If the charge inside a closed surface is denoted by q_i, then the rate of decrease is – dq_i/dt and the principle of conservation of charge requires

$$I = \oint_s \vec{J}.\vec{n}.ds = -\frac{dq_i}{dt}. \tag{6.13}$$

The negative sign indicates the direction of the current flow. The (6.13) is the continuity equation. By changing the surface integral to a volume integral using divergence theorem, we get

$$\oint_{\mathrm{s}} \mathrm{J.ds} = \int_{\mathrm{vol}} (\nabla.\mathrm{J})\ \mathrm{dv}. \tag{6.14}$$

Writing $\mathrm{q_i} = \int\int\int qdv$, where $\mathrm{q_v}$ is the volume density of charge, we can write

$$\int_{\mathrm{vol}} (\nabla.\mathrm{J})\,\mathrm{dv} = -\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathrm{vol}} \mathrm{q_v dv} \tag{6.15}$$

where $\mathrm{q_v}$ is volume charge density. For outflow of current through a volume, the derivative can be written as a partial derivative and the (6.15) becomes

$$\int_{vol} (\nabla.J)\,dv = \int_{vol} -\frac{\partial q_v}{\partial t} dv \tag{6.16}$$

Since, the expression is true for any volume, however small, it is true for an incremental volume,

$$(\nabla.\mathrm{J})\,\Delta\mathrm{v} = -\frac{\partial \mathrm{q_v}}{\partial \mathrm{t}}\Delta\mathrm{v} \tag{6.17}$$

and the point form of the continuity is

$$(\nabla.\mathrm{J}) = -\frac{\partial \mathrm{q_v}}{\partial \mathrm{t}}. \tag{6.18}$$

$$\nabla.\mathrm{J} = \frac{\partial \mathrm{q}}{\partial \mathrm{t}}\delta\,(\mathrm{x})\,\delta\,(\mathrm{y})\,\delta\,(\mathrm{z})\,. \tag{6.19}$$

6.5 Anisotropy in Electrical Conductivity

For an homogeneous and isotropic medium, the current density J and the electric field E are assumed to be in the same plane. If the electric field ($\mathrm{E_x}$) and conductivity (σ_{xx}) are along the x-direction, then current density along the x direction is $\mathrm{J_{x1}}$. In general, however, not only the field $\mathrm{E_x}$ but also the fields $\mathrm{E_y}$ and $\mathrm{E_z}$ may give rise to current densities in the x-direction in an anisotropic medium. If the additional, current densities are proportional to the fields $\mathrm{E_y}$ and $\mathrm{E_z}$, then σ_{xy} and σ_{xz} are proportionality constants respectively. The total current density in the x-direction is the sum of these three terms. In general, therefore, we can write,

$$\begin{aligned} J_x &= \sigma_{xx}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z \\ J_y &= \sigma_{yx}E_x + \sigma_{yy}E_y + \sigma_{yz}E_z \\ J_z &= \sigma_{zx}E_x + \sigma_{zy}E_y + \sigma_{zz}E_z. \end{aligned} \tag{6.20}$$

The conductivity of a medium (σ) must be a tensor of rank 2, which in Cartesian coordinates will have the nine components:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yx} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{6.21}$$

If the conductivity tensor is symmetric, the off-diagonal terms will have symmetrically equal values i.e., $\sigma_{xy} = \sigma_{yx}$, and so on. If two of the coordinate directions are selected to lie in the direction of maximum and minimum conductivity, (the principal directions of the conductivity tensor), the off-diagonal terms will become zero and σ can be shown as a diagonal matrix.

$$\sigma = \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}. \tag{6.22}$$

In isotropic materials, the three principal values of conductivity are all equal and in effect, conductivity becomes a scalar quantity. In isotropic materials, the electric field vector and the current density vector are collinear, i.e. current flow, is along the direction of applied electric field. In anisotropic media (the equipotential surfaces are no longer normal to the direction of current flow). Here three principal values of the conductivity tensor are not equal. Coincidence of directions occur only when the electric field is directed along one of the principal directions of the tensor conductivity.

6.6 Potential at a Point due to a Point Source

Potential at a point due to a point source of current I at a distance 'r' and in a homogeneous and isotropic medium of resistivity ρ can be derived from the solution of Laplace equation in a spherical coordinate (see Chap. 7) as

$$\phi = \frac{Ip}{4\pi}.\frac{I}{r}. \tag{6.23}$$

Therefore in a homogeneous and isotropic medium, the current lines will be radial and equipotential lines will be circular Fig. 6.1. The nature of current and equipotential lines due to (i) a source and a sink of strength +I and −I (ii) two sources of strength +I and +I are shown in Figs. 6.5 and 6.6 respectively. The nature of the current lines and field lines for bipole and dipole fields for +1 and −1 are respectively given in Figs. 6.5 and 6.7.

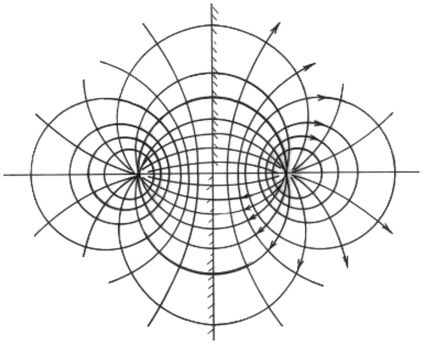


Fig. 6.5. Current lines and equipotential lines due to a current source and a sink

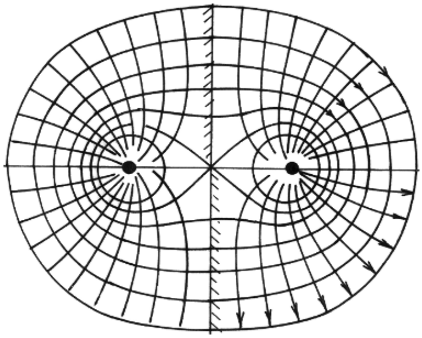


Fig. 6.6. Field lines and equipotential lines due to two sources placed at a certain distence

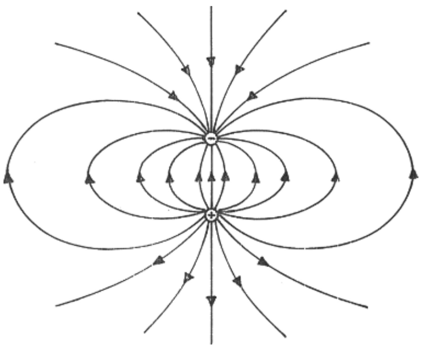


Fig. 6.7. Field lines and current lnes due to two closely spaced source and sink

Potential at a point due to a point source at a distance r on the surface of the earth is

$$\phi = \frac{I\rho}{2\pi}.\frac{1}{r} \tag{6.24}$$

since the solid angle subtended by a point current electrode on air earth boundary is 2π. For Wenner electrode configuration, the potential difference between the two potential electrodes M and N due to two current electrodes placed at A and B is given by (Fig. 6.3 c and d)

$$\Delta\phi = \frac{I\rho}{2\pi}\left[\left(\frac{1}{AM} - \frac{1}{BM}\right) - \left(\frac{1}{BN} - \frac{1}{AN}\right)\right] \tag{6.25}$$

$= \frac{I\rho}{2\pi}.\frac{1}{a}$ where 'a' is the distance between the electrodes. Here for Wenner configuration AM = MN = NB = a. Therefore

$$\rho = 2\pi a\frac{\Delta\phi}{I} \tag{6.26}$$

where $2\pi a$ is the geometric factor for Wenner configuration. Similarly geometric factors for all the electrode configurations cited, can be determined. Numerical value of a geometric factor increases with electrode separation, the farthest distance between the two active electrodes. For an inhomogeneous medium, the resistivity ρ in (6.26) will change to ρ_a, the apparent resistivity. Apparent resistivity is defined as the true resistivity of a fictitious homogenous medium when the response from an inhomogenous earth is same as that from a fictitious homogenous medium. For Schlumberger electrode configuration the expression for the apparent resistivity is

$$\rho_a = (\pi/4)((L^2 - l^2)/l)(\Delta\Phi/I) \tag{6.27}$$

where L, the distance between the two current electrodes, the distance between two potential electrodes is l, I, the current flowing through the medium and $\Delta\Phi$ the potential difference measured between the two potential electrodes.

We defined Geometric factor $K = \frac{2\pi}{\left(\frac{1}{AM}-\frac{1}{BM}\right)-\left(\frac{1}{BN}-\frac{1}{AN}\right)}$ as the exact geometic factor. We brought the idea of approximate geometric factors while discussing the geometric factors for dc dipole configurations. It has a dimension of length for some cases. Geometric factor can be variable in the case of laterolog - 7 configuration and can be negative for certain zones of parallel dipole and wenner gamma or collinear dipole-dipole configurations. In general geomertric factor is mostly an electrode separation dependent quantity.

6.7 Potential for Line Electrode Configuration

Let us consider an infinitely long line electrode through which a current I per unit length is being sent through the half space (Parasnis 1965).

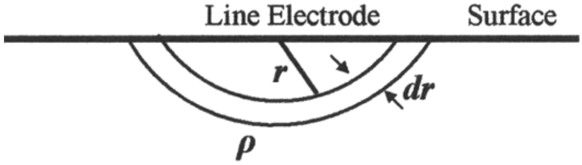


Fig. 6.8. Cylindrical annular space for a long line electrode

Figure 6.8 shows an annular semi cylinder of unit length with the electrode at the axis and having internal and external radii r and r + dr. If ρ is the resistivity of the homogeneous ground; resistance of the annular cylindrical is $dR = \rho\frac{dr}{\pi r}$. Since current is passing out of the cylindrical annular shell, the potential drop across it is

$$d\phi = -IdR = -\frac{IP}{\pi}.\frac{dr}{r}. \tag{6.28}$$

$$\text{Integrating } \phi = -\frac{IP}{\pi}\ln r + C. \tag{6.29}$$

Therefore, potential at a point due to a line electrode is a logarithmic potential. For $\phi = 0$ at $r = \infty$, we have $C = C_\infty$ where C_∞ is an infinite constant. Therefore the potential will be infinite at infinite distances. If $\phi = 0$ at $r = 1$, then $C = 0$. Here potential will be positive for r less than 1 and negative for r greater than 1. Potential will be $-\infty$ as $r \to \infty$. Let r_1 and r_2 be the distances of an observation point from two infinitely long line electrodes. These electrodes are a source and a sink.

The potential from positive and negative electrodes will be of opposite sign. If we choose $C = C\infty$, the total potential at the point will be

$$\phi = \frac{I\rho}{\pi}\ln\frac{r_1}{r_2}. \tag{6.30}$$

The potential difference between the two points P1 and P2 in a field created by two line electrodes is given by (Fig. 6.9)

$$\Delta\phi = \phi_1 - \phi_2 = \frac{I\rho}{\pi}\left\{In\left(\frac{r_2}{r_1}\right) - In\left(\frac{r_4}{r_3}\right)\right\} \tag{6.31}$$

$$= \frac{I\rho}{\pi}In\left(\frac{r_2r_3}{r_1r_4}\right) \tag{6.32}$$

since the infinity constants cancel each other. The potential expressed in (6.32) is finite at all the finite distances and tends to zero when both r_2 and r_1 tends to infinity. Potentialis are also zero when $r_1 = r_2$, i.e., along the vertical plane midway between the source and sink. Surface potential is given by

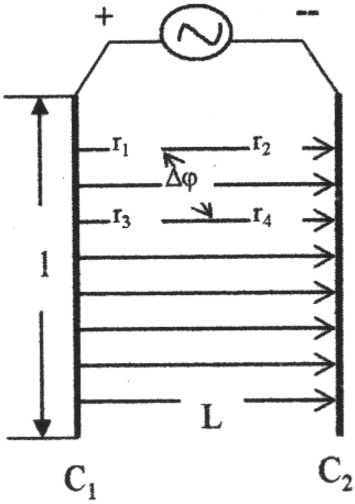


Fig. 6.9. Shows the potential difference measured between the two points of observation P1 and P2 between a line source and a line sink C_1 and C_2 each of length l and are planted on the ground at a distance L

$$\phi = \frac{I\rho}{\pi} \ln \frac{L-x}{L+x} \tag{6.33}$$

where 2L is the distance between the two electrodes and x is measured from the midpoints of C_1C_2.

6.7.1 Potential due to a Finite Line Electrode

A line electrode of length 2b through which current I per unit length is being supplied to a homogeneous ground of resistivity ρ (Fig. 6.10). We determine the potential at a point P whose perpendicular distance from the electrode is x and which is situated on a profile of measurement at a distance y from the centre O of the electrode.

A small element of the line electrode, having a length $d\lambda$, at a distance λ from O can be treated to be a point electrode through which a current I $d\lambda$ is being propagated to the ground. Potential at P will then be

$$d\phi = \frac{I\rho}{2\pi} \frac{d\lambda}{\{x^2 + (\lambda - y)^2\}^{1/2}}. \tag{6.34}$$

Integrating between the limits −b and +b, the potential of the entire line electrode will be

$$\phi = \frac{I\rho}{2\pi} \left\{ \sin h^{-1} \left(\frac{b-y}{x} \right) + \sin h^{-1} \left(\frac{b+y}{x} \right) \right\}. \tag{6.35}$$

Equation (6.35) generates elliptic equipotential lines (Fig. 6.11). Therefore current lines will be hyperbolic.

Integral transform of Roy and Jain (1961) can be stated as follows. Let a point electrode in the vicinity of a two dimensional structure striking in the y direction, produce a potential ϕ (x, y) along a profile in the y-direction. The profile goes through a point P at a distance x from the electrode. Then the transform

$$\phi(x) = \int_{-\infty}^{\infty} \phi(x, y)\, dy \tag{6.36}$$

generates the potential, that would be produced at P by a line electrode parallel to the profile but placed in the position of point electrodes. The total current in the point electrode is the current per unit length of the line electrode. Figure 6.12 shows the variation of potential with distance due to a point and a line electrode.

6.8 Current Flow Inside the Earth

Potential at a point M (Fig. 6.13) in a semi infinite medium of resistivity ρ due to a source +I and sink (−I) on the surface of the earth is given by

$$\phi_m = \frac{\rho_I}{2\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{6.37}$$

where

$$r_1 = \sqrt{(L/2 + x)^2 + y^2 + z^2} \tag{6.38}$$

$$r_2 = \sqrt{(L/2 - x)^2 + y^2 + z^2} \tag{6.39}$$

where L is the electrode separation. Here.

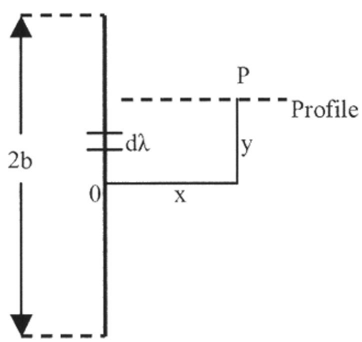


Fig. 6.10. A finite line electrode of length 2b carrying current I

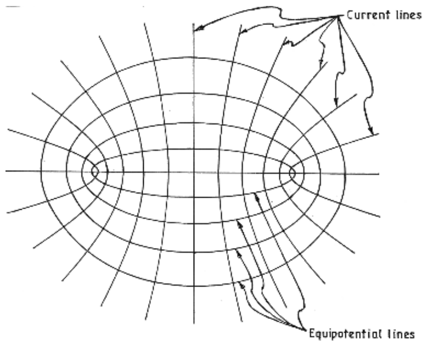


Fig. 6.11. Current lines and equipotential lines due to a line current source of finite length

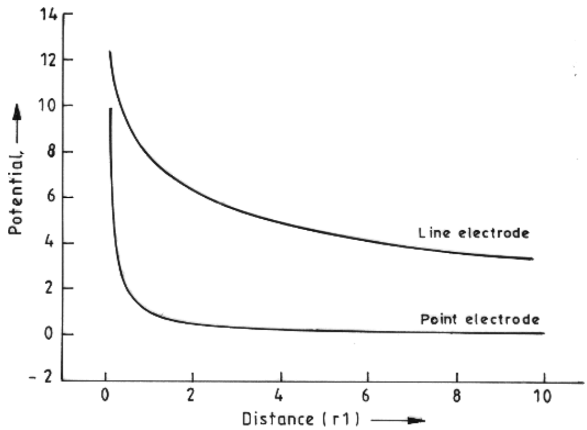


Fig. 6.12. Shows the variation of potential with distance from a point and a line electrode

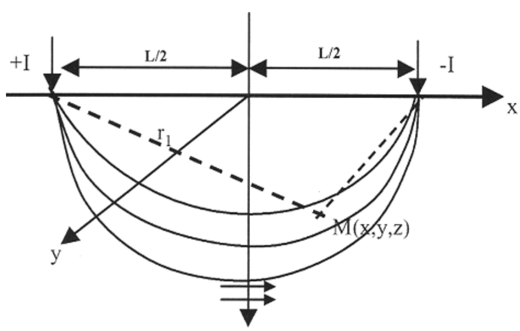


Fig. 6.13. Shows the nature of direct current flow through a homogeneous medium

$$\vec{J}_x = -\frac{1}{\rho}\frac{\partial \phi}{\partial x}$$
$$\vec{J}_y = -\frac{1}{\rho}\frac{\partial \phi}{\partial y} \quad (6.40)$$
$$\vec{J}_z = -\frac{1}{\rho}\frac{\partial \phi}{\partial z}$$

are the current densities along the x, y and z directions when current is flowing through a homogenous medium. Here

$$\vec{J}_x = \frac{I}{2\pi}\left[\frac{\frac{L}{2}+x}{r_1^3} - \frac{\frac{L}{2}-x}{r_2^3}\right] \quad (6.41)$$

$$\vec{J}_y = \frac{I}{2\pi}\left[\frac{y}{r_1^3} - \frac{y}{r_2^3}\right] \quad (6.42)$$

$$\vec{J}_z = \frac{I}{2\pi}\left[\frac{z}{r_1^3} - \frac{z}{r_2^3}\right] \quad (6.43)$$

$$\vec{J} = \sqrt{J_x^2 + J_y^2 + J_z^2}. \quad (6.44)$$

If we bring the point M on the YZ plane then x = 0 and $r_1 = r_2$. J_x reduces to the form.

$$J_x = \frac{IL}{2\pi}\frac{1}{(L/2)^2 + y^2 + z^2}. \quad (6.45)$$

Current density on the surface on the earth at z = 0 is given by

$$J_0 = \frac{I}{\pi}.\frac{4}{L^2} \quad (6.46)$$

and at a depth h is

$$J_h = \frac{Il}{2\pi}.\frac{1}{\sqrt{\left(\frac{L}{2}\right)^2 + h^2}}. \quad (6.47)$$

The ratio of the current density at a certain depth h and that on the surface is given by

$$\frac{J_h}{J_o} = \frac{1}{\left[1 + (2h/L)^2\right]^{3/2}}. \quad (6.48)$$

Figure 6.14 shows the variation of current density with depth in $\frac{J_h}{J_o} vs \frac{h}{L}$ plot. Flow of current upto the depth h is given by

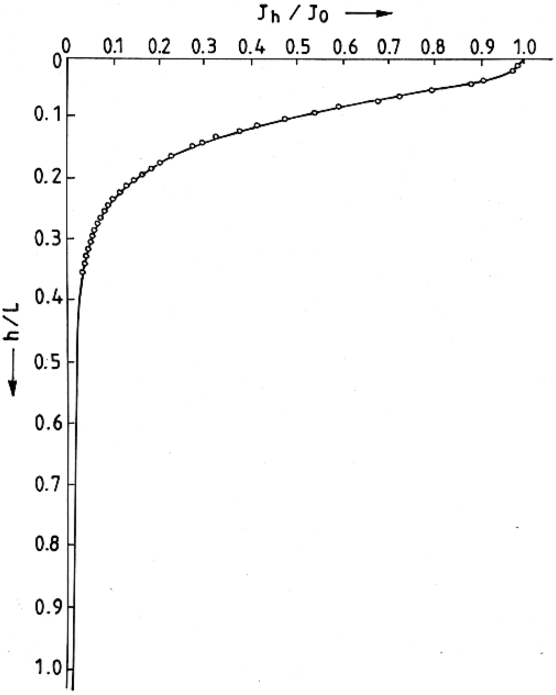


Fig. 6.14. Shows the variation of current density with depth in a xz plane passing through y = 0 at the centre of the electrode configuration

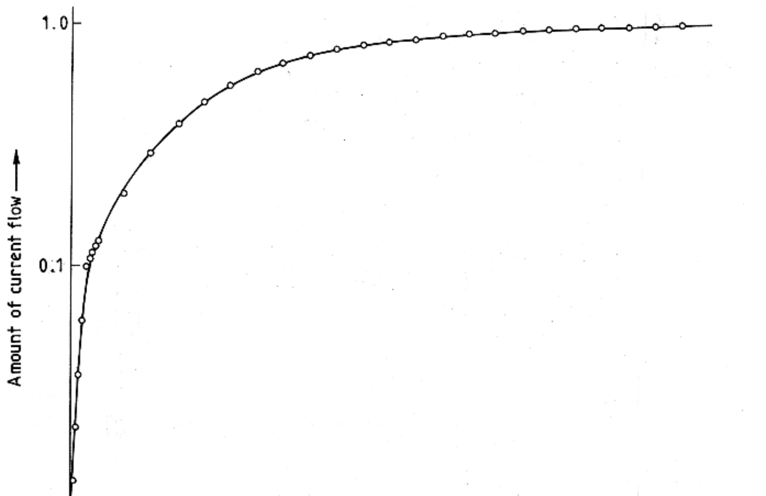


Fig. 6.15. Amount of current flows through the earth with depth and the relation with electrode separation

$$I_h = \frac{IL}{2\pi} \int\limits_{y=-\alpha}^{\alpha} \int\limits_{z=0}^{h} \frac{dydz}{\left[\left(\frac{h}{2}\right)^2 + y^2 + z^2\right]^{3/2}} \tag{6.49}$$

$$= \frac{IL}{\pi} \int\limits_{0}^{h} \frac{dz}{\left(\frac{L}{2}\right)^2 + z^2} = \frac{2I}{\pi} \tan^{-1} \frac{2h}{L} \tag{6.50}$$

$$\frac{I_h}{I} = \frac{2}{\pi} \tan^{-1} \frac{2h}{L}. \tag{6.51}$$

This gives the total amount of current flowing between the surface at any particular depth. Figure 6.15 shows the variation of $\frac{I_n}{I}$ with h/L. It is observed that most of the current is concentrated near the surface.

6.9 Refraction of Current Lines

Direct currents get refracted across a contact of two media of different resistivities and follow 'tan' law unlike 'sine' law for seismic or elastic waves. Two homogeneous and isotropic media of resistivity ρ_1 and ρ_2 are having a horizontal contact Fig. 6.16 of infinite horizontal extent.

Current with a current density J_1 is incident on the horizontal surface at an angle θ_1. J_{X_1} *and* J_{Z_1} are respectively the horizontal and vertical components. This current element is at an angle θ_2 with the vertical.

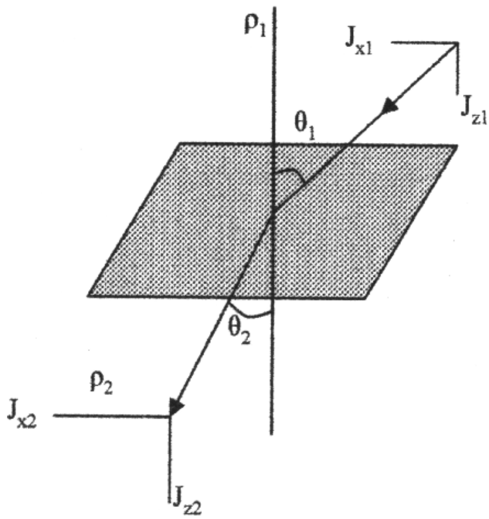


Fig. 6.16. Shows the refraction of the current lines at the boundary between the two media having resistivity ρ_1 and ρ_2

Now for direct current flow field, the potential must be continuous across the boundary i.e., $\varphi_1 = \varphi_2$ and the normal component of the current densities J_{n_1} *and* J_{n_2} must be continuous across the boundary.

In terms of electric field, we can write,

$$E_1 = E_2 \quad \text{and} \quad \sigma_1 E_{n_1} = \sigma_2 E_{n_2}. \tag{6.52}$$

From (6.52), we can write

$$J_{x_1}\rho_1 = J_{x_2}\rho_2 \quad \text{and} \quad J_{z_1} = J_{z_2}. \tag{6.53}$$

From the two equations, we get

$$\rho_1 \left(J_{x_1}/J_{z_1}\right) = \rho_2 \left(J_{x_{21}}/J_{z_2}\right) \quad \Rightarrow \qquad \rho_1 \tan\theta_1 = \rho_2 \tan\theta_2. \tag{6.54}$$

6.10 Dipole Field

Figure 6.7 shows the nature of the direct current flow field for DC dipoles when the distance between the two current electrodes are significantly less in comparison to the distance where we measure the field. The essential differences between a dipole and a bipole field are (i) dipole fields die down at a much faster pace. DC dipole potential varies as $\frac{1}{r^2}$ with distance and field varies as $\frac{1}{r^3}$ with distance. Expression for dipole fields and potential are presented in Chap. 4. Expressions for potentials in dipoles in electrostatic field and direct current flow fields are analogous. (Chap. 4, (4.30) and (4.31)). Only q the charge strength is replaced by current strengths I and $\in$, the electrical permittivity is replaced by electrical conductivity σ.

D.C. dipole fields are being used by the geophysicists primarily to have informarion of the subsurface from a relatively greater depth. Deeper probing is possible by sending more current through the earth and measuring potentials far away from the current dipole.

Direct current dipole-dipole configurations for measuring the electrical resistivity of the earth's crust is known from the works of Alpin et al (1950), Jackson (1966), Keller et al (1966), Anderson and Keller (1966), Zohdy (1969), Alfano (1980).

Important D.C. dipole configurations for deeper probing (sounding) are, (I) equatiorial (ii) azimuthal (iii) parallel (iv) perpendicular and (v) axial. dipoles (Fig. 6.17 a, b, c, d, and e). Important D.C dipole configuration for studying the lateral heterogeneites is the collinear dipole dipole configuration (Fig. 6.18). Figure 6.18 also shows the data plotting points in the pseudosection form.

In bipole-dipole configuration, the length of the current dipole AB may be much larger than the potential dipole MN (Fig. 6.17).

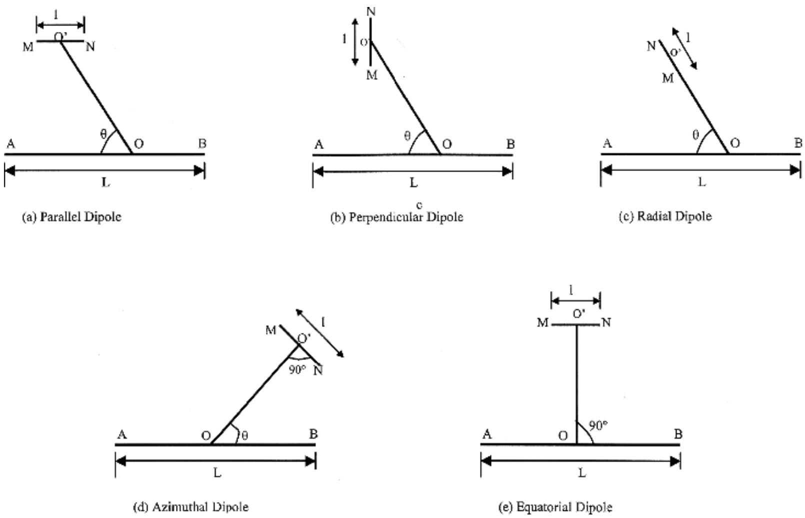


Fig. 6.17. Show the DC dipole configuration; (**a**) Parallel dipole configuration where current dipole and potential dipole are parallel; the line joing the midpoints of the current and potential electrode is making an angle θ not equal to 90°; (**b**) Perpendicular dipole configuration; here potential dipole is at right angles to the current dipoles at all dipole angles; (**c**) Radial dipole configuration: Here potential dipole is in the same line which joins the midpoints of the current and potential electrodes at all dipole angles; (**d**) Azimuthal dipole configuration; here potential dipole is at right angles to the line joining the mid points of the current and potential dipoles; (**e**) Equatorial dipoles; here the current and potential dipoles are parallel and the dipole angle is 90°

For dipole dipole system AB should be nearly equal to MN. Equatorial dipole and azimuthal dipoles are used quite frequently in dipole survey because these data can directly be converted to Schlumberger data and can be interpreted.

The general expression for the geometric factor for all the electrode configurations is

$$K = \frac{2\pi}{\frac{1}{AM} - \frac{1}{BM} - \frac{1}{AN} + \frac{1}{BN}} \tag{6.55}$$

The approximate geometric factors for different bipole-dipole configurations are

$$K_{parallel} = \frac{2\pi r^3}{L} . \frac{1}{3\cos^2\theta - 1} \tag{6.56}$$

$$K_{perpendicular} = \frac{2\pi r^3}{3L} . \frac{1}{\sin\theta \ \cos\theta} \tag{6.57}$$

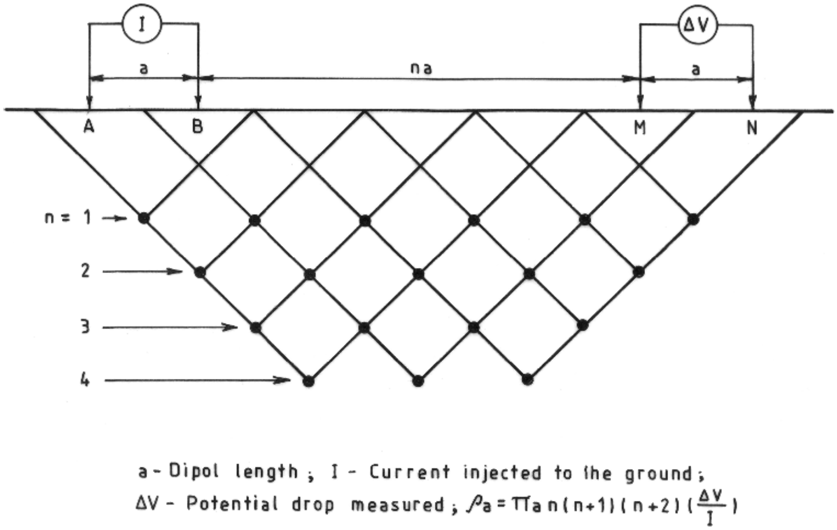


Fig. 6.18. Shows the collinear dipole dipole configuration; here current and potential electrode pairs are in the same line; electrode separations are increased with higher values of n; pseudo net is for plotting data

$$K_{radial} = \frac{\pi r^3}{L\cos\theta}. \tag{6.58}$$

$$K_{azimuthall} = \frac{2\pi r^3}{L\sin\theta}. \tag{6.59}$$

(Aplin 1950, Bhattacharyya and Patra, 1968 and Koefoed, 1979)

Here θ is the dipole angle. L and r are respectively the current dipole length and the dipole separation. The percentage discrepancy

$$\delta = \frac{K_{actual} - K_{approximate}}{K_{actual}} \times 100\%$$

between the actual and approximate geometric factors (6.55 to 6.59) can be very high in some cases. This discrepancy is significant for bipole-dipole system. For dipole -dipole system, when AB $\cong$ MN, this discrepancy δ goes down significantly. Figure 6.19 shows the decrease in the percentage discrepancy δ computed for an homogenous earth model as well as for parallel dipole configuration for MN = 300 meters, $\rho = 1000$ *ohm – meters* for different current dipole lengths and dipole separations and current sent through the ground was assumed to be one ampere.

Discrepancy between the actual and approximate geometric factor reduces down significantly with increasing dipole separation OO' where O and O$'$ are the mid points of the current and potential dipoles.

DC dipole field measurement is essentially an attempt to measure a man made field obtained by a generator driven power at a far of point. Difficult

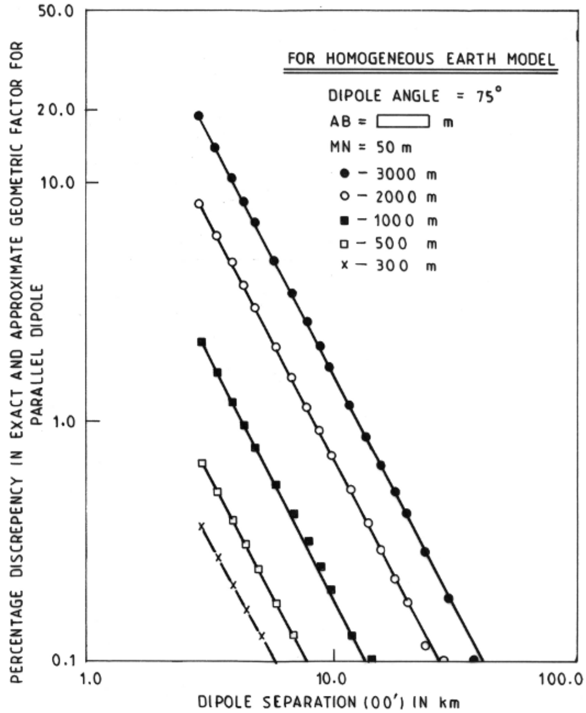


Fig. 6.19. Shows the discrepancy between the exact and approximate geometric factor in the case of parallel dipole for different current dipole lengths and dipole separations (OO'); This figure is for homogenuous and isotropic half space and dipole angle 75°

field logistics to layout several kilometers of field cables for deeper probing in case Schlumberger or Wenner array could be avoided by separating the current and potential dipoles as two independent units. Advent of sophisticated and accurate global positioning system (GPS), distance communication system and mobile telephones significantly reduced the ground hazards in measuring dipole fields and subsequent data analysis specially for earth's crustal.studies.

Cultural noise problem is significantly low in this case in comparison to what one expects for audiofrequency magnetotelluric survey.

Parallel dipole configuration does not work at dipole angle nearly 55° (Keller 1966, $\theta = 53°44'$, Zohdy 1969, $\theta = 54°44'$, Alpin et al 1950, $\theta = 54°44'8''$, Das and Verma, 1980 $\theta = 54°–55°$). At this angle E_x, the parallel current dipole component of the electric field is zero. Therefore, any measurement in the vicinity of this dipole angle is unreliable. It is now realised that not only one should avoid $\theta = 53°$, there is a big zone from $\theta = 35°$ to $\theta = 65°$, where parallel dipole does not work. It is termed as the prohibitive zone (Fig. 6.20) for parallel dipole. Permitted zones for parallel dipole system are $\theta = 0°$ to $35°$ and $65°$ to $90°$. Therefore, the recommended prescription for use of bipole-dipole configurations for various dipole angle are (Fig. 6.21):

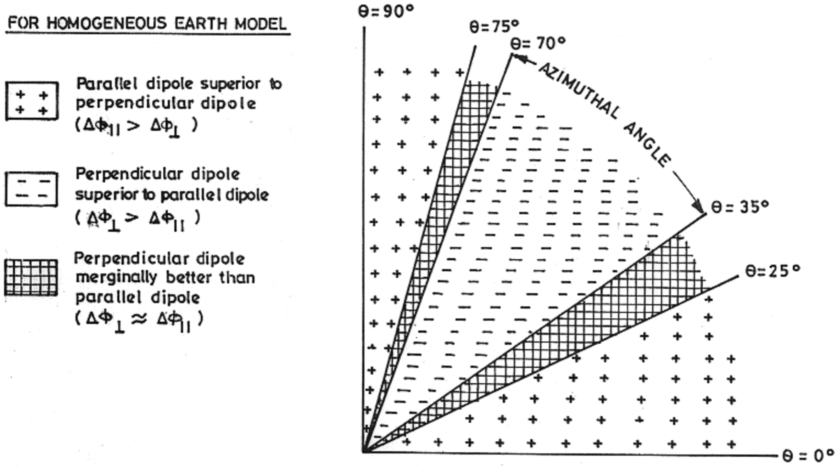


Fig. 6.20. Shows the different working zones for parallel and perpendicular dipoles; Parallel dipole works for dipole angle 0° to 35° and 70° to 90° and perpendicular dipole works best within the dipole angle 35° to 70°

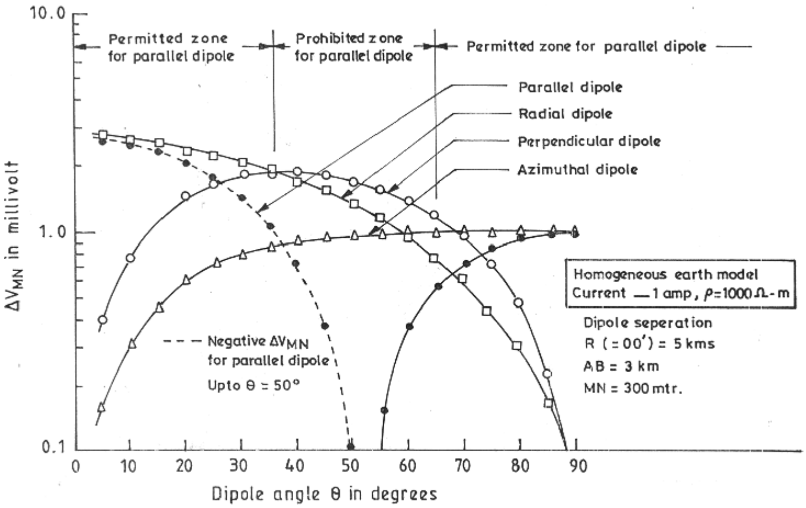


Fig. 6.21. Shows the variations of potentials for different dipole angles and different dipole configurations in a homogeneous and isotropic half space; it is an approximate guideline for choice of DC dipoles for different dipole angles, computation of potentials is made for current dipole length = 3 km, potential dipole length = 300 meter and dipole separation R = OO′ = 5 km

(i) Radial, perpendicular and parallel dipole for $20° \leq \theta \leq 35°$
(ii) Perpendicular, radial and azimuthal for $35° \leq \theta \leq 65°$
(iii) Perpendicular, parallel, redial and azimuthal for $65° \leq \theta \leq 75°$
(iv) Radial and parallel dipoles for $0° \leq \theta \leq 20°$
(v) Azimuthal and parallel for $75° \leq \theta \leq 90°$

Bipole-dipole configuration is preferable than that of dipole-dipole for deep crustal studies. Geometric factors generally used for computation of apparent resistivities in dipole sounding (6.56 to 6.59) should not be used for small ($00' < 4AB$) dipole separation. Combination of parallel and perpendicular dipoles have some logistic advantage in the actual geological ground condition. Once one knows the orientation of the current dipole, orientation of the potential dipoles are also known. In a rugged and dense forest region accurate measurement of dipole angle may be difficult. Measurement of DC dipole field will regain its proper place after significant improvements in field logistics as mentioned and developments in interpretation softwares. Fairly detailed expositions of potential theory related to the direct current flow field are given in Chaps. 7, 8, 9, 11 and 15.

6.11 Basic Equations in Direct Current Flow Field

1. $$\vec{j} = \sigma\vec{E} \tag{6.60}$$

2. $$\vec{E} = -grade\phi \tag{6.61}$$

3. $$\vec{E} = \frac{I}{4\pi\sigma}.\frac{1}{r^3}\vec{r} \tag{6.62}$$

4. $$\vec{\phi} = \frac{I}{4\pi\sigma}.\frac{1}{r}\text{(Point source)} \tag{6.63}$$

5. $$\phi = \frac{\vec{P}Cos\theta}{4\pi\sigma r^2} \text{ for DC dipoles.(Dipole source)} \tag{6.64}$$

6. $$\phi = -\frac{I\rho}{\pi}\ln r \text{ (line source)} \tag{6.65}$$

7. $$div\ \vec{E} = 0\ or\ \nabla^2\phi = 0 \text{ (Laplacian field)} \tag{6.66}$$

8. $$div\ \nabla^2\phi = \rho \text{ (Poissonian field)} \tag{6.67}$$

9. $$Curl\ \vec{E} = 0 \tag{6.68}$$

10. $$div\ j = -\frac{\partial q}{\partial t} \tag{6.69}$$

11. $$J_{n_1} = J_{n_2} \tag{6.70}$$

12. $$\rho_1\ \tan\theta_1 = \rho_2\ \tan\ \theta_2 \tag{6.71}$$

6.12 Units

$\sigma \rightarrow mho/meter(Siemen)$
$\rho \rightarrow Ohm$–meter
$R \rightarrow Ohm$
$\vec{J} \rightarrow Amp/meter^2$
$\phi \rightarrow Volt/Millivolt$
$\vec{E} \rightarrow Volt/meter$

7

Solution of Laplace Equation

In this chapter solutions of Laplace equation in cartisian, cylindrical polar and spherical polar coordinates using the method of separation of variables are discussed in considerable details. The nature of solution of boundary value problems in potential theory is introduced. The nature of Bessel's function , modified Bessel's function, Legendre's Polynomial and Associated Legendre's Polynomial are shown. A brief discussion on Spherical Harmonics is given.

7.1 Equations of Poisson and Laplace

The electric displacement vector is $\vec{\mathrm{D}} = \in \vec{\mathrm{E}}$ { (4.4.)} where $\vec{\mathrm{D}}$ is the electric displacement, $\vec{\mathrm{E}}$ is the electric field and $\in$ is the electrical permittivity of a medium. In addition to the constitutive relation, we use the Gauss's flux theorem of total normal induction on a closed surface due to a charge inside the enclosed volume and it is given by

$$\int_s \vec{D}_n.ds = \int div\vec{D}.dv = q = \int \rho dv \tag{7.1}$$

where ρ is the volume density of charge and dv is the infinitesimal volume. Hence

$$\nabla.\mathrm{D} = \rho \tag{7.2}$$

$$\begin{aligned}
&\Rightarrow \quad \nabla.(\in \mathrm{E}) = \rho \\
&\Rightarrow \quad \nabla.(-\in \nabla\phi) = \rho \\
&\Rightarrow \quad -\in \text{div grad } \phi = \rho
\end{aligned}$$

$$\Rightarrow \nabla^2\phi = -\frac{\rho}{\in} \tag{7.3}$$

$$= 0 \text{ when } \rho = 0.$$

In a source free region (Fig. 6.2 a and b)

$$-\in \nabla^2 \phi = 0. \tag{7.4}$$

For a nonhomogenous but isotropic dielectric (7.4) becomes

$$\frac{\partial}{\partial x}\left(\in \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial \Psi}\left(\in \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial x}\left(\in \frac{\partial \phi}{\partial x}\right) = 0. \tag{7.5}$$

For a homogenous and isotropic dieletric

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{7.6}$$

For three principal axes anisotropy (7.5) will be

$$\frac{\partial}{\partial x}\left(\in_{xx} \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\in_{yy} \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\in_{zz} \frac{\partial \phi}{\partial x}\right) = 0. \tag{7.7}$$

7.2 Laplace Equation in Direct Current Flow Domain

When current is flowing out of a closed region, the flow of charge will be guided by the relation

$$\operatorname{div} \vec{J} = -\frac{\partial \rho}{\partial t} \tag{7.8}$$

where ρ is the volume density of charge in Coulomb/meter^3 and $\vec{J}$ is current density in ampere/meter^2. Since this relation satisfies the law of conservation of charge, it is termed as the equation of continuity. In a source free region

$$\operatorname{div} \vec{J} = 0 \tag{7.9}$$

where $\vec{J} = \sigma \vec{E} = -\sigma grad \phi$, where ϕ is the potential (in volt) and $\vec{E}$ is the electric field in volt/meter. Equation (7.9) can be written as

$$\operatorname{div}(\sigma \text{ grad } \phi) = 0 \tag{7.10}$$

$$\Rightarrow \text{ grad } (\sigma) \text{ grad } \phi + (\sigma) \operatorname{div} \text{grad } \phi = 0. \tag{7.11}$$

For an homogeneous and isotropic medium (7.11) reduces to Laplace equation

$$\nabla^2 \phi = 0. \tag{7.12}$$

Non Laplacian character of (7.11) is demonstrated in Chap. 8.

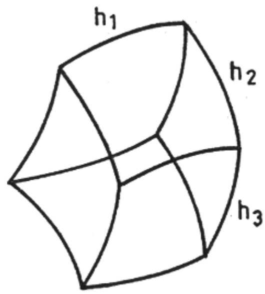


Fig. 7.1. A cuboid with curve sides to represent the curvilinear coordinates

7.3 Laplace Equation in Generalised Curvilinear Coordinates

Laplace equations in cartesian, cylindrical polar and spherical polar coordinates can be expressed from the expression of Laplace equation in generalized curvilinear coordinates (Fig. 7.1).

In orthogonal curvilinear coordinate, the Laplace equation is

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left[\frac{h_1h_2}{h_3}.\frac{\partial\phi}{\partial u_3}\right]\right]. \tag{7.13}$$

Here the value of h_1, h_2, h_3 and u_1, u_2 and u_3 can be expressed as :

(a) In cartesian coordinates (Fig. 7.2)

$$\begin{aligned} u_1 &= x, \quad u_2 = y \quad \text{and} \quad u_3 = z \\ h_1 &= 1, \quad h_2 = 1, \quad \text{and} \quad h_3 = 1 \end{aligned} \tag{7.14}$$

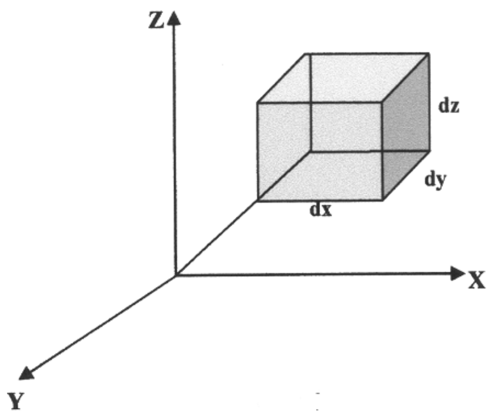


Fig. 7.2. A three dimensional elementary volume in Cartisian coordinatea (x,y,z)

(b) in cylindrical polar coordinates (Fig. 7.3)

$$\begin{aligned} &u_1 = \rho, \quad u_2 = \Psi \quad \text{and} \quad u_3 = z \\ &h_1 = 1, \quad h_2 = \rho \quad \text{and} \quad h_3 = 1 \end{aligned} \tag{7.15}$$

(c) In spherical polar coordinates (Fig. 7.4)

$$\begin{aligned} &u_1 = r, \quad u_2 = \theta \quad \text{and} \quad u_3 = \psi \\ &h_1 = 1, \quad h_2 = r, \quad \text{and} \quad h_3 = r\sin\psi. \end{aligned} \tag{7.16}$$

Therefore, the expressions for the Laplace equation in three coordinate systems are respectively given by

(a)

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{7.17}$$

in Cartesian coordinate, where $\phi = f(x, y, z)$.

(b)

$$\begin{aligned} \nabla^2\phi &= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{\partial}{\partial\Psi}\left(\frac{1}{\rho}\frac{\partial\phi}{\partial\Psi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial\phi}{\partial z}\right)\right] = 0 \\ \Rightarrow &\quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2\phi}{\partial\Psi^2}\right) + \frac{\partial^2\phi}{\partial z^2} = 0 \end{aligned} \tag{7.18}$$

in cylindrical polar coordinates where $\phi = f(\rho, \psi, z.)$.

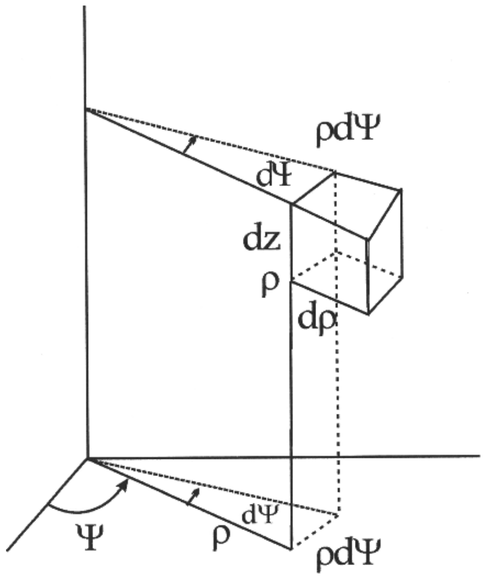


Fig. 7.3. A three dimensional elementary volume in cylindrical polar coordinates (r, ψ, z)

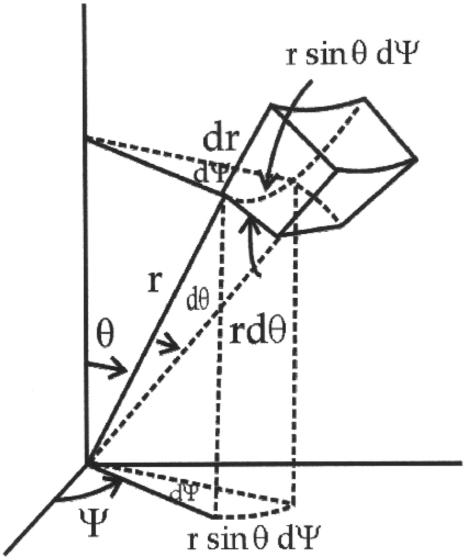


Fig. 7.4. A three dimensional elementary volume in spherical polar coordinates (r, θ, ψ)

(c)

$$\nabla^2\phi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\phi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)\right.$$

$$\left.+\frac{\partial}{\partial\Psi}\left(\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\psi}\right)\right]=0$$

$$\Rightarrow \quad \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{\sin^2\theta}\cdot\frac{\partial^2\phi}{\partial\psi^2}\right]=0 \qquad (7.19)$$

in spherical polar coordinates where $\phi = f(r, \theta, \psi)$. Most of the geophysical problems, dealing with scalar potential field satisfy Laplace equation in a source free region i.e. the region which excludes the source (field exists but not the source) (Fig. 2.5 a and 6.2 a,b). Therefore, the solution of Laplace equation forms a significant part of the potential theory in geophysics. In this chapter we shall deal with the solution of Laplace equation by the method of separation of variable in (i) cartesian (ii) cylindrical polar and (iii) spherical polar coordinates depending upon the nature of the problems. One has to choose the proper coordinate system for solving a particular problem A few simpler problems are included.

7.4 Laplace Equation in Cartesian Coordinates

The solution of the Laplace equation by the method of separation of variables in cartesian coordinate is demonstrated in this section.When potential ϕ is a function of x, y and z where X, Y, Z are independent variables, we can write:

$$\phi = \mathrm{X}\ (\mathrm{x})\ \mathrm{Y}\ (\mathrm{y})\ \mathrm{Z}\ (\mathrm{z}) \tag{7.20}$$

and

$$\frac{\partial \phi}{\partial \mathrm{x}} = \mathrm{YZ}\frac{\partial \mathrm{X}}{\partial \mathrm{x}} \tag{7.21}$$

or,

$$\frac{\partial^2 \phi}{\partial \mathrm{x}^2} = \mathrm{YZ}\frac{\partial^2 \mathrm{X}}{\partial \mathrm{x}^2}. \tag{7.22}$$

Substituting these values in (7.17) we get

$$\nabla^2\phi = \frac{\partial^2 \phi}{\partial \mathrm{x}^2} + \frac{\partial^2 \phi}{\partial \mathrm{y}^2} + \frac{\partial^2 \phi}{\partial \mathrm{z}^2} = 0 \tag{7.23}$$

$$YZ\frac{d^2X}{dx^2} + ZX\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0. \tag{7.24}$$

Dividing the whole (7.24) by XYZ, we get:

$$\frac{1}{\mathrm{X}}.\frac{\mathrm{d}^2\mathrm{X}}{\mathrm{dx}^2} + \frac{1}{\mathrm{Y}}.\frac{\mathrm{d}^2\mathrm{Y}}{\mathrm{dy}^2} + \frac{1}{\mathrm{Z}}.\frac{\mathrm{d}^2\mathrm{Z}}{\mathrm{dz}^2} = 0. \tag{7.25}$$

The sum of these terms will never be zero unless each individual terms are constants and the sum of these constants is zero i.e., if

$$\begin{aligned} \frac{1}{\mathrm{X}}.\frac{\mathrm{d}^2\mathrm{X}}{\mathrm{dx}^2} &= \alpha^2 \\ \frac{1}{\mathrm{Y}}.\frac{\mathrm{d}^2\mathrm{Y}}{\mathrm{dy}^2} &= \beta^2 \\ \frac{1}{Z}.\frac{d^2Z}{dz^2} &= \gamma^2 \end{aligned} \tag{7.26}$$

then

$$\alpha^2 + \beta^2 + \gamma^2 = 0. \tag{7.27}$$

We shall now examine the nature of the expressions for potentials for their dependence on the different axes:

7.4.1 When Potential is a Function of Vertical Axis z, i.e., $\phi = f(z)$

The Laplace equation reduces down to $\frac{\partial^2 \phi}{\partial z^2} = 0$ and the solution is

$$\phi = \mathrm{cz} + \mathrm{d} \tag{7.28}$$

where c and d are two arbitrary constants. Here potential is increasing with z, i.e., higher the value of z, higher will be the potential. One encounters this kind of situation while computing gravitational potentials due to a hypothetical infinite plate. Here

$$\vec{E} = -\frac{\partial \phi}{\partial Z} = c \tag{7.29}$$

i.e., the field is constant at any distance from the plane.

7.4.2 When Potential is a Function of Both x and y, i.e., $\phi = f(x, y)$

Putting $\phi = X\ (x)\ Y\ (y)$

The Laplace equation reduces down to:

$$\frac{1}{X}\cdot\frac{d^2X}{dx^2} + \frac{1}{Y}\cdot\frac{d^2Y}{dy^2} = 0. \tag{7.30}$$

If

$$\frac{1}{X}\cdot\frac{d^2X}{dx^2} = \alpha^2, \quad \text{then } \frac{1}{Y}\cdot\frac{d^2Y}{dy^2} = -\alpha^2. \tag{7.31}$$

And if

$$\frac{1}{X}\cdot\frac{d^2X}{dx^2} = -\beta^2, \text{ then } \frac{1}{Y}\cdot\frac{d^2Y}{dy^2} = \beta^2. \tag{7.32}$$

Therefore, we can write:

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0$$
$$\frac{d^2Y}{dy^2} + \alpha^2 Y = 0 \tag{7.33}$$

The solutions are:

$$X = e^{\alpha x}, e^{-\alpha x}, \cosh \alpha x, \sinh \alpha x \tag{7.34}$$

and

$$Y = e^{i\alpha y}, e^{-i\alpha y}, \cos \alpha y, \sin \alpha y$$

The most general solution of Laplace equation for these two equations are:

$$\phi = \sum_{n=0}^{\infty} \left(a_n\ e^{\alpha_n x} + b_n\ e^{-\alpha_n x}\right)(c_n \cos \alpha_n\ y + d_n \sin \alpha_n\ y) \tag{7.35}$$

and

$$\phi = \sum_{n=o}^{\infty} (a_n \cos \beta_n\ y + b_n \sin \beta_n y)(c_n \cosh \alpha_n x + d_n \sinh \alpha_n x) \tag{7.36}$$

7.4.3 Solution of Boundary Value Problems in Cartisian Coordinates by the Method of Separation of Variables

Let us find out the potential at any point in a two dimensional space when the size of the conductor and the potentials on the boundaries are prescribed.

Problem 1

A rectangular block of a conductor of thickness 'b' is placed in the xy plane. The prescribed values at different boundaries are:

$$x = \infty, \quad \phi = 0$$
$$x = 0, \quad \phi = P\cos\frac{\pi y}{b}'$$
$$y = -\frac{b}{2}, \qquad \phi = 0$$
$$y = +\frac{b}{2}, \qquad \phi = 0$$

Find the potential at any point in the rectangular plate.

Figure (7.5) shows the nature of the problem. Since $\phi = 0$ at $x = \infty$, the general solution of the two dimensional potential problem takes the form:

$$\phi = \sum_{n=0}^{\infty} b_n \, e^{-\alpha_n \, x} \left(c_n \cosh \alpha_n \, y + d_n \text{cin} \alpha_n \, y\right). \tag{7.37}$$

Applying the second boundary condition, we get :

$$P\cos\frac{\pi y}{b} = \sum \left(c'_n \cos\alpha_n y + d'_n \sin\alpha_n y\right) \tag{7.38}$$

where $c'_n = b_n c_n$ and $d'_n = b_n c_n$. These are arbitrary constants to be determined from the boundary conditions. Since the source potential contains 'cos' term, we have to drop the 'sin' terms from the solution. Therefore, the expression for the potential reduces to:

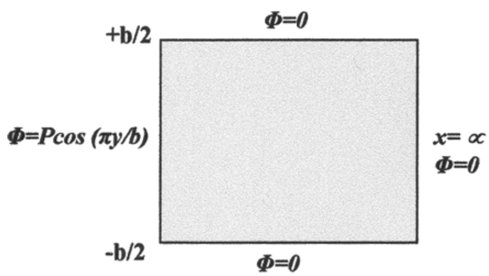


Fig. 7.5. A two dimensional Dirichlet's problem with potentials prescribed in all the boundaries

$$\phi = \sum c'_n e^{-\alpha_n x} \cos \alpha_n y. \tag{7.39}$$

Applying the third boundary condition, we get:

$$0 = \sum c'_n e^{-\alpha_n x} \cos \frac{\alpha_n b}{2}. \tag{7.40}$$

Therefore

$$\cos \frac{\alpha_n b}{2} = 0$$
$$\Rightarrow \alpha_n = \frac{n\pi}{b} \text{ where } n = 1, 3, 5, 7, \ldots\ldots\ldots\ldots.$$

The expression for the potential changes to the form

$$\phi = \sum c'_n e^{-\frac{n\pi}{b}x} \cos^{\frac{n\pi}{b} \cdot y}. \tag{7.41}$$

Applying the second boundary condition, (7.41) becomes:

$$\begin{aligned} P \cos \frac{\pi y}{b} &= c'_n \cos \frac{n\pi y}{b} \\ &= \left(c'_1 \cos \frac{\pi y}{b} + c'_3 \cos \frac{3\pi y}{b} + c'_5 \cos \frac{5\pi y}{b} + \ldots\ldots \right). \end{aligned} \tag{7.42}$$

Equating the coefficients of $\cos \frac{\pi y}{b}$ on both the sides, one gets:

$$P = c'_1 \text{ for } n = 1, \text{ therefore } c'_3 = c'_5 = c'_7 = c'_{9\ldots\ldots\ldots\ldots} = 0.$$

Therefore the final solution of the problem is:

$$\phi = P\, e^{-\frac{\pi x}{b}} \cos \frac{\pi y}{b}. \tag{7.43}$$

Problem 2

A finite rectangular conductor of length ‘a’ and width ‘b’ is placed in the xy plane placing the corner A of the rectangle at the origin. The prescribed potentials at the boundaries are as follows (Fig. 7.6)

$$\Phi = 0 \text{ at } x = 0$$
$$\phi = 0 \text{ at } x = a$$
$$\phi = 0 \text{ at } y = 0$$
$$\phi = f(x), \text{ at } y = b$$

Find the potential at any point on the plate. The solution of the Laplace equation :

$$\frac{\partial^2 \phi}{dx^2} + \frac{\partial^2 \phi}{dy^2} = 0 \tag{7.44}$$

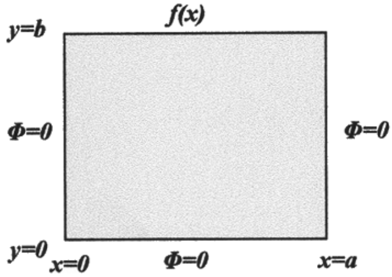


Fig. 7.6. A two dimensional potential problem with potentials prescribed in all the boundaries

are

(i) $e^{\alpha x}, e^{-\alpha x}, \sin \alpha y, \cos \alpha y$
(ii)

$$\sin \alpha x, \cos \alpha x, e^{\alpha y}, e^{-\alpha y} \tag{7.45}$$

and the general solution of the problem is written as :

$$\phi = \sum_{n=o}^{\infty} (A_n \cos \alpha_n x + B_n \sin \alpha_n x) \left(C_n e^{\alpha_n y} + D_n e^{-\alpha_n y}\right). \tag{7.46}$$

Values of these arbitrary constants are determined using the boundary conditions.

Applying the first boundary condition, we get:

$$0 = \Sigma A_n \left(C_n e^{\alpha_n y} + D_n e^{-\alpha_n y}\right). \tag{7.47}$$

The right hand side expression of (7.47) will be zero when $A_n = 0$. Therefore the general expression for the potential reduces to

$$\phi = \sum_{n=0}^{\infty} B_n \sin \alpha_n x \left(C_n e^{\alpha_n y} + D_n e^{-\alpha_n y}\right). \tag{7.48}$$

Applying the second boundary condition, we get:

$$0 = \Sigma \sin \alpha_n a \left(C'_n e^{\alpha_n y} + D'_n e^{-\alpha_n y}\right) \tag{7.49}$$

where $C'_n = B_n C_n$ and $D'_n = B_n D_n$.

Equation (7.49) will be 0 if $\sin \alpha_n a = 0$

$$\Rightarrow \quad \alpha_n a = n\pi$$

$$\Rightarrow \quad \alpha_n = \frac{n\pi}{a}. \tag{7.50}$$

$$\phi = \sum \mathrm{Sin} \frac{n\pi x}{a} \left(C'_n e^{\frac{n\pi y}{a}} + D'_n e^{-\frac{n\pi y}{a}}\right). \tag{7.51}$$

Applying the third boundary condition, we get:

$$0 = \sum \sin \frac{n\pi x}{a} \left(C'_n + D'_n\right). \tag{7.52}$$

Equation (7.52) will be 0 if,

$$\begin{aligned} C'_n + D'_n &= 0 \\ \Rightarrow \quad C'_n &= -D'_n. \end{aligned} \tag{7.53}$$

Hence the expression for the potential becomes

$$\begin{aligned} \phi &= \sum \sin \frac{n\pi x}{a} C'_n \left(e^{\frac{n\pi y}{a}} - e^{\frac{-n\pi y}{a}}\right) \\ &= \sum F'_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \end{aligned} \tag{7.54}$$

where, F'_n is a new constant.

Now applying the fourth boundary condition, we get:

$$f(x) = \sum F'_n \sinh \frac{n\pi b}{a} . \sin \frac{n\pi x}{a}. \tag{7.55}$$

Multiplying both the sides by $\sin \frac{m\pi x}{a}$ and integrating from a to 0, we get:

$$\int_o^a f(x) \sin \frac{m\pi x}{a} dx = \sum \int_o^a F'_n \sin h \frac{n\pi b}{a} . \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} dx. \tag{7.56}$$

Since

$$\begin{aligned} \int_o^a \sin nx \sin mx \, dx \quad &= 0 \quad \text{for } m \neq n \\ &= \frac{a}{2} \quad \text{for } m = n, \end{aligned}$$

therefore, from (7.55), one can write

$$F'_n = \frac{2}{a} . \frac{1}{\sinh \frac{m\pi b}{a}} \int_o^a f(x) \sin \frac{m\pi x}{a} dx. \tag{7.57}$$

Hence, the general solution of the problem is

$$\phi = \sum F_n \sin \frac{m\pi x}{a} \sinh \frac{n\pi y}{a}. \tag{7.58}$$

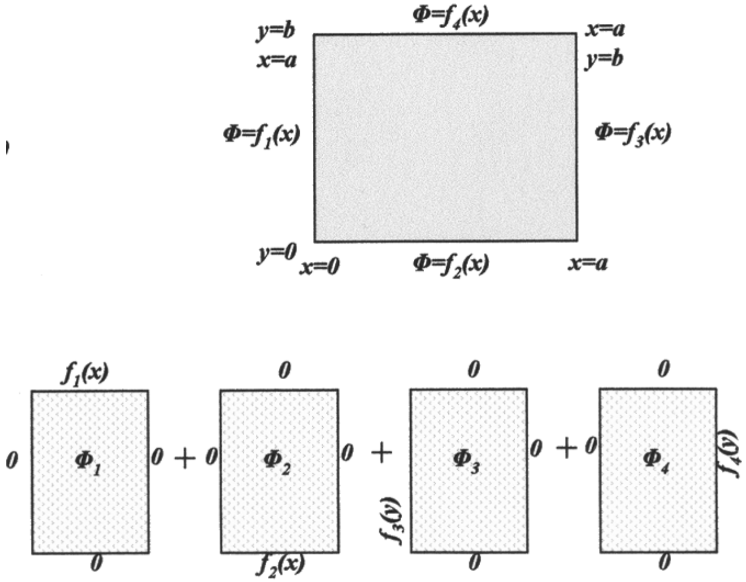


Fig. 7.7. A break up of a two dimensional problem into four parts to make it a easily solvable problem

Problem 3

Find the potential at any point inside a rectangular conductor of length 'a' and width 'b' placed in the xy planes when the following boundary conditions are prescribed, i.e., for (Fig. 7.7)

$$\begin{aligned} x = 0, &\quad \phi = f_1(x) \\ y = 0, &\quad \phi = f_2(x) \\ x = a, &\quad \phi = f_3(x) \\ y = b, &\quad \phi = f_4(x) \end{aligned}$$

This problem can be solved by breaking the problem into four problems similar to that discussed in the previous section, get the potential at a point for four problems and add them up. Since the potentials are scalars and the principle of superposition is valid, we can get

$$\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4. \tag{7.59}$$

7.5 Laplace Equation in Cylindrical Polar Coordinates

Laplace equation in cylindrical coordinate is :

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}.\frac{\partial^2\phi}{\partial\psi^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{7.60}$$

where the coordinates are ρ (along the radial direction), Ψ (along the azimuthal direction) and z (along the vertical direction). Applying the method of separation of variables we can write :

$$\phi = R(\rho)\Psi(\psi)Z(z)$$

where, R, Ψ and Z are respectively the functions of ρ, Ψ and z only. Therefore, from (7.60) we can write

$$\Psi Z\frac{d^2R}{d\rho^2} + \frac{\Psi Z}{\rho}\frac{dR}{d\rho} + \frac{1}{\rho^2}RZ\frac{d^2\Psi}{d\psi^2} + R\psi\frac{d^2Z}{dz^2} = 0. \tag{7.61}$$

Now dividing the equation by RΨZ, we get

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} + \frac{1}{\rho^2}.\frac{1}{\Psi}.\frac{d^2\Psi}{d\psi^2} + \frac{1}{z}.\frac{d^2Z}{dz^2} = 0. \tag{7.62}$$

Let us choose

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \alpha^2. \tag{7.63}$$

Multiplying the (7.62) by ρ^2, we get :

$$\frac{\rho^2}{R}.\frac{d^2R}{d\rho^2} + \frac{\rho}{R}.\frac{dR}{d\rho} + \alpha^2\rho^2 + \frac{1}{\Psi}.\frac{d^2\Psi}{d\psi^2} = 0. \tag{7.64}$$

We, next put

$$\frac{1}{\psi}.\frac{d^2\Psi}{d\psi^2} = -\beta^2 \tag{7.65}$$

and obtain :

$$\frac{\rho^2}{R}.\frac{d^2R}{d\rho^2} + \frac{\rho}{R}.\frac{dR}{d\rho} + \alpha^2\rho^2 - \beta^2 = 0 \tag{7.66}$$

which can be rewritten as

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\alpha^2 - \frac{\beta^2}{\rho^2}\right)R = 0. \tag{7.67}$$

This equation is known as Bessels equation.

Alternately, we can have the second set of equations as follows:

$$\frac{1}{Z}.\frac{d^2Z}{dz^2} = -\alpha^2 \tag{7.68}$$

$$\frac{1}{\psi}.\frac{d^2\Psi}{d\psi^2} = -\beta^2 \tag{7.69}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}.\frac{dR}{d\rho} - \left(\alpha^2 + \frac{\beta^2}{\rho^2}\right)R = 0 \tag{7.70}$$

Equation (7.70) is a modified Bessels equation.

Now let us examine the dependence of potential function on ρ, ψ, z and the corresponding changes in the expressions for potentials.

7.5.1 When Potential is a Function of z ,i.e., $\phi = f(z)$

The Laplace equation takes the form

$$\frac{d^2\phi}{dz^2} = 0, \text{ or } \phi = Az + B \tag{7.71}$$

where A and B are constants. The potential at a point is gradually increasing with z. This is the potential function due to an infinite plate, discussed in the previous section.

7.5.2 When Potential is a Function of Azimuthal Angle Only i.e., $\phi = f(\psi)$

The Laplace equation changes to the form

$$\frac{d^2\phi}{d\psi^2} = 0, \text{ or } \phi = C\psi + D \tag{7.72}$$

where C and D are constants. A circular resistance carrying current can create this type of potential functions.

7.5.3 When the Potential is a Function of Radial Distance, i.e., $\Phi = f(\rho)$

The Laplace equation becomes

$$\frac{\partial^2\phi}{d\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} = 0 \tag{7.73}$$

$$\Rightarrow \quad \frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) = 0$$

$$\rho\frac{\partial\phi}{\partial\rho} = N, \partial\phi = \frac{\partial\rho}{\rho} + N$$

Therefore the solution of this equation is :

$$\phi = M1n\rho + N \tag{7.74}$$

where M and N are constants to be determined from the boundary conditions. Let us take an example.

Problem

Two infinitely long cylinders of radius ‘a’ and ‘b’ are placed co-axially. The potentials at the outer boundary at radius ‘b’ and the inner boundary at radius ‘a’ are respectively 0 and V_o. Find the potential at any point in the

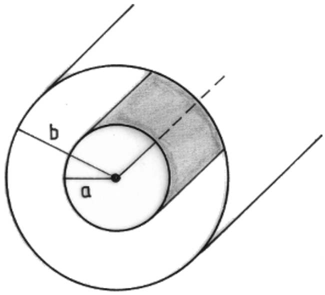


Fig. 7.8. Potential inside a cylinderical shell when the potentials are prescribed in the inner and outer boundaries

annular space between the two cylinders (Fig. 7.8) Applying the boundary conditions, we get:

$$0 = \mathrm{M1n\ b} + \mathrm{N}$$
$$\mathrm{V_o} = \mathrm{M1n\ a} + \mathrm{N} \tag{7.75}$$

Therefore, $\mathrm{V_o} = \mathrm{M1n}\frac{\mathrm{a}}{\mathrm{b}}$,

$$\mathrm{M} = \frac{\mathrm{V_O}}{\ln\left(\frac{\mathrm{a}}{\mathrm{b}}\right)} \text{ and } \mathrm{N} = \frac{\mathrm{V_O}}{\ln\left(\frac{\mathrm{a}}{\mathrm{b}}\right)}.\ln \mathrm{b} \tag{7.76}$$

The potential at any point at a radial distance ρ from the axis of the coaxial cylindrical bodies is given by

$$= \frac{\mathrm{V_O}}{\ln(\mathrm{a/b})}\ \ln\rho - \frac{\mathrm{V_O}\ln \mathrm{b}}{\ln(\mathrm{a/b})}$$
$$= \mathrm{V_O}\frac{\ln(\rho/\mathrm{b})}{\ln(\mathrm{a/b})}. \tag{7.77}$$

7.5.4 When Potential is a Function of Both ρ and ψ, i.e., $\phi = f(\rho, \psi)$

The Laplace equation becomes

$$\frac{\partial^2\phi}{d\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}.\frac{\partial^2\phi}{\partial\psi^2} = 0. \tag{7.78}$$

Applying the method of separation of variables i.e.,

$$\phi = \mathrm{R}(\rho)\Psi(\psi)$$

we get two equations

$$\frac{1}{\psi}.\frac{d^2\Psi}{d\psi^2} = -n^2 \tag{7.79}$$

and

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}.\frac{dR}{d\rho} - \frac{n^2}{\rho^2}R = 0. \tag{7.80}$$

The solution of the (7.79) is

$$\phi = (A\cos n\psi + B\sin n\psi). \tag{7.81}$$

The solution of the (7.80) can be determined as follows:

Multiplying the (7.80) by ρ^2, we get

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} - n^2R = 0. \tag{7.82}$$

Let

$$\theta = \log_e^{\rho}, \text{ then } \rho = e^{\theta}$$

So,

$$\begin{aligned}
\frac{dR}{d\rho} &= \frac{dR}{d\theta}.\frac{d\theta}{d\rho} = \frac{dR}{d\theta}.e^{-\theta} \\
\Rightarrow \frac{d^2R}{d\rho^2} &= \frac{d}{d\rho}\left(\frac{dR}{d\rho}\right) = \frac{d}{d\rho}\left(e^{-\theta}\frac{dR}{d\theta}\right) \\
&= e^{-\theta}\frac{d}{d\theta}\left(e^{-\theta}\frac{dR}{d\theta}\right) \\
&= e^{-\theta}\left(e^{-\theta}\frac{d^2R}{d\theta^2} - e^{-\theta}\frac{dR}{d\theta}\right) \\
&= e^{-2\theta}\left(\frac{d^2R}{d\theta^2} - \frac{dR}{d\theta}\right)
\end{aligned}$$

Substituting these values one gets

$$\frac{d^2R}{d\theta^2} - \frac{dR}{d\theta} + \frac{dR}{d\theta} - n^2R = 0$$

or,

$$\frac{d^2R}{d\theta^2} - n^2R = 0$$

or,

$$\begin{aligned}
R &= A_1e^{n\theta} + A_2e^{-n\theta} \\
&= A_1\rho^n + A_2\rho^{-n}
\end{aligned} \tag{7.83}$$

Therefore, the general solution of Laplace equation when $\Phi = f(\rho, \psi)$, is

$$\phi = \sum_{1}^{\infty}(A_n\cos n\psi + B_n\sin n\psi)\left(C_n\rho^n + D_n\rho^{-n}\right). \tag{7.84}$$

Problem 1

An infinitely long cylinder of dielectric constant $\in_2$ is placed in a medium of dielectric constant $\in_1$ with the axis of the cylinder oriented along the z-direction. The cylinder is placed in an uniform field, i.e., the source and sink are assumed to be at infinity (Fig. 7.9).

Find the potential at any point both inside and outside the cylindrical body.

In a direct current flow field, we assume a cylindrical conductor of electrical conductivity σ_2 is placed in an homogeneous medium of conductivity σ_1. The boundary value problem will essentially remain the same. The field is assumed to be perpendicular to the axis of the cylinder and is assumed to be parallel to the x-axis.

Since

$$E = -\mathrm{grad}\phi = -\frac{\partial\phi}{\partial x} \tag{7.85}$$

where ϕ is the potential function. We get

$$E x.x = -\phi + \text{Constant.} \tag{7.86}$$

Therefore, the source potential is :

$$\phi_o = -E x.x + A$$

where A is a constant and x is the of the point assumed origin. A is dropped while computing the perturbation potential. In an uniform field E_x, in a medium of dielectric constant, $\in_1$ and in the presence of an anomalous body of contrasting physical property $\in_2$, an anomalous or perturbation potential will be generated. It will be added up to the source potential in an uniform field. This perturbation potential will gradually die down with distance of the point of observation from the centre of the cylinder,the anomalous body.

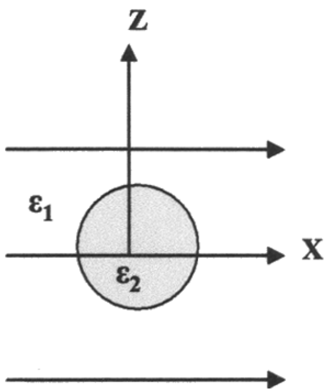


Fig. 7.9. An infinately long cylinder of dielectric constant $\in_2$ is placed in a medium of dielectric constant; $\in_1$ in the presence of an uniform field

Potential at a point both outside and inside a cylinder can be written as:

$$\begin{aligned}\phi_1 &= \phi_o + \phi'(\text{potential outside})\\ \phi_2 &= \phi_o + \phi''(\text{potential inside})\end{aligned} \tag{7.87}$$

Here ϕ' and ϕ'' are the perturbation potentials outside and inside the body. The perturbation potential part also must satisfy Laplace equation.

Therefore

$$\nabla^2\phi' = 0 \text{ and } \nabla^2\phi'' = 0 \tag{7.88}$$

assuming the radius vector is at an angle ψ with the x-axis

$$\phi_o = -E_o x = -E_o \rho \cos\psi \tag{7.89}$$

where Ψ is the angle between the radius vector ρ and the x-axis. Since the source potential has a cos Ψ term, the perturbation potential will also have $\cos\Psi$ terms only. The general expression for the perturbation potential, when it is a function of ρ and Ψ and independent of z, reduces down to

$$\phi' = \sum_{n=o}^{\infty} f_n \cos n\psi, \rho^{-n}(\text{potential outside}) \tag{7.90}$$

Here ρ is the radial distance from the axis of the cylinder. Since the perturbation potential will gradually die down with distance from the centre of the cylinder $D_n\rho^{-n}$ will be the appropriate potential function for outside region. Similarly, the perturbation potential inside the body will be given by $C_n\rho^n$ as the appropriate potential function. Hence

$$\phi'' = \sum_{n=o}^{\infty} g_n \cos n\psi\rho^n. \tag{7.91}$$

Because when ρ tends to zero, $D_n\rho^{-n}$ in (7.84) tends to infinity. Since potential inside a body, when placed in an uniform field, cannot be infinitely high Therefore, ρ^{-n} cannot be a potential function inside the body. Here $f_n = A_nD_n$ and $g_n = A_nC_n$ We can now write down the potentials outside and inside the body respectively as:

$$\phi_1 = -E\rho\cos\psi + \sum_{n=0}^{\infty} f_n \cos n\psi\rho^{-n}$$

and

$$\phi_2 = -E\rho\cos\psi + \sum_{n=0}^{\infty} g_n \cos n\psi\rho^{n}.$$

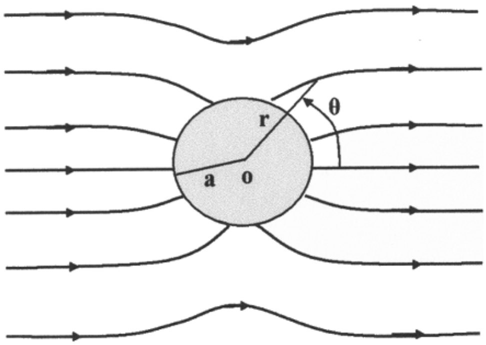


Fig. 7.10. Field line distortions in the presence of a cylinder of contrasting dielectric constant

Applying the boundary conditions :

i)

$$\phi_1 = \phi_2$$

and

ii)

$$\in_1 \left(\frac{\partial \phi}{\partial \rho}\right)_1 = \in_2 \left(\frac{\partial \phi}{\partial \rho}\right)_2 \tag{7.92}$$

at $\rho = \mathrm{a}$. i.e., on the surface of the cylinder of radius 'a', we get

$$\sum \mathrm{f_n} \cos \mathrm{n}\psi, \mathrm{a}^{-\mathrm{n}} = \sum \mathrm{g_n} \cos \mathrm{n}\psi, \mathrm{a}^{\mathrm{n}}$$

and

$$\begin{aligned} &- \in_1 E \cos \psi + \sum (-n)\, f_n \cos n\psi, a^{-(n+1)} \in_1 \\ &= - \in_2 E \cos \psi + \sum n g_n \cos n\psi\ a^{n-1} \in_2 . \end{aligned} \tag{7.93}$$

Since the source potential contains cos Ψ, the perturbation potential will also have the cos ψ term, therefore $\mathrm{n} = 1$. The summation sign vanishes and we obtain, ultimately

$$\mathrm{f_1 a^{-1}} = \mathrm{g_1 a}$$

and

$$- \in_1 \vec{\mathrm{E}} + \mathrm{f_1 a^{-2}} \in_1 = - \in_2 \vec{\mathrm{E}} + \mathrm{g_1} \in_2 . \tag{7.94}$$

From (7.94) the values of the arbitrary constants $\mathrm{f_1}$ and $\mathrm{g_1}$ are obtained respectively as :

$$g_1 = \vec{E}.\frac{\in_2 - \in_1}{\in_2 + \in_1}$$

and

$$f_1 = \vec{E}a^2 \frac{\in_2 - \in_1}{\in_2 + \in_1} \tag{7.95}$$

in terms of the contrast in physical properties, size of the body and strength of the uniform field. Potentials both outside and inside the body can now be written as :

$$\phi_1 = -\vec{E}\rho\cos\psi + \vec{E}a^2.\frac{\in_2 - \in_1}{\in_2 + \in_1}.\cos\psi.\rho^{-1}\text{potential outside} \tag{7.96}$$

and

$$\phi_2 = -\vec{E}\rho\cos\psi + \vec{E}\frac{\in_2 - \in_1}{\in_2 + \in_1}.\cos\psi.\rho.\text{potential inside} \tag{7.97}$$

Equations (7.96) and (7.97) can be written in the form

$$\phi_1 = -\mathrm{E}\left(1 + \mathrm{K}\frac{\mathrm{a}^2}{\rho^2}\right)\rho\cos\psi$$

and

$$\phi_2 = -\mathrm{E}(1+\mathrm{K})\rho\cos\psi \tag{7.98}$$

where $K = \frac{\in_1 - \in_2}{\in_1 + \in_2}$, the reflection factor.

Since the potential is dependent on ρ and ψ, the field inside the body can be written as :

$$\vec{\mathrm{E}}_1 = +\vec{\mathrm{a}}_\rho\left(-\frac{\partial\phi}{\partial\rho}\right) + \vec{\mathrm{a}}_\psi\left(-\frac{\partial\phi}{\rho a\psi}\right). \tag{7.99}$$

Therefore, the fields on both inside and outside the body are respectively given by

$$\vec{\mathrm{E}}_1 = +\vec{\mathrm{a}}_\rho.\vec{\mathrm{E}}\left(1 - \mathrm{K}\frac{\mathrm{a}^2}{\rho^2}\right)\cos\psi - \vec{\mathrm{a}}_\Psi\mathrm{E.K}\frac{\mathrm{a}^2}{\rho^2}\sin\psi$$

$$\Rightarrow \quad \vec{E}_1 = +\vec{a}_\rho.E\cos\psi - EK\frac{a^2}{\rho^2}\left(\vec{a}_\rho\cos\psi - \vec{a}_\psi\sin\psi\right). \tag{7.100}$$

And

$$\begin{aligned}\vec{E}_2 &= -\vec{a}_\rho\vec{E}\left(1+K\right)\cos\psi + \vec{a}_\psi\vec{E}\left(1+K\right)\sin\psi\\ &= \vec{\mathrm{E}}\left(1+\mathrm{K}\right)\left(-\vec{\mathrm{a}}_\rho.\cos\psi + \vec{\mathrm{a}}_\psi\sin\psi\right)\\ &= \mathrm{E}\left(1+\mathrm{K}\right)\vec{\mathrm{a}}_\mathrm{x}.\end{aligned} \tag{7.101}$$

Here $\vec{\mathrm{a}}_\rho, \vec{\mathrm{a}}_\Psi$ are the unit vectors along the radial and azimuthal direction and $\vec{\mathrm{a}}_x$ is the unit vector along the x direction (Fig. 7.10).

Here

$$\vec{\mathrm{a}}_x = -\vec{\mathrm{a}}_\rho\mathrm{Cos}\psi + \vec{\mathrm{a}}_\psi\mathrm{Sin}\psi \tag{7.102}$$

Equation (7.101) shows that the field inside the body is parallel to the external and uniform field. Figure 7.11 shows the nature of distortions in the uniform field and equipotentials due to presence of an infinitely long cylinder of contrasting physical property.

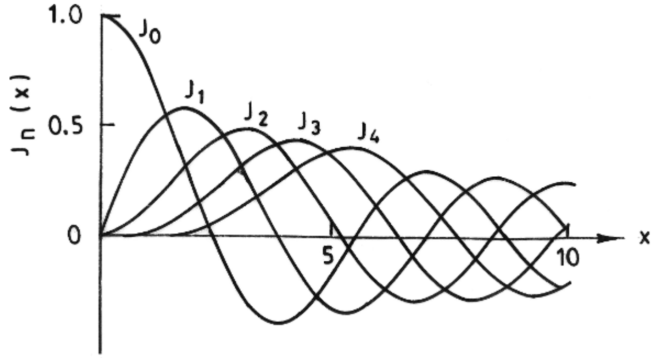


Fig. 7.11. Bessel's function of first kind and 0,1,2,3,4 order

7.5.5 When Potential is a Function of all the Three Coordinates, i.e., $\phi = f(\rho, \psi, z)$

The next problem is to obtain the generalised solution of the Laplace equation in cylindrical coordinates when the potential function (ϕ) is dependent on all the three coordinates ρ, ψ and z.

Laplace's equation in cylindrical coordinates is

$$\frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\cdot\frac{\partial^2\phi}{\partial\psi^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{7.103}$$

Using the method of separation of variables, discussed in the previous section, we have

$$\phi = R(\rho)\Psi(\psi)Z(z)$$

We obtain the three equations

$$\frac{d^2\Psi}{d\psi^2} + n^2\Psi = 0 \tag{7.104}$$

$$\frac{d^2Z}{dz^2} - m^2Z = 0 \tag{7.105}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(m^2 - \frac{n^2}{\rho^2}\right)R = 0 \tag{7.106}$$

where R, Ψ or Z are respectively the functions of ρ, ψ or z only. The solutions of equations 7.104 and 7.105 are discussed in the previous section. Equation 7.106 can be rewritten in the form

$$\frac{d^2R}{d\,(m\rho)^2} + \frac{1}{(m\rho)}\cdot\frac{dR}{d\,(m\rho)} + \left(1 - \frac{n^2}{(m\rho)^2}\right)R = 0. \tag{7.107}$$

Equation 7.107 is of the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \tag{7.108}$$

or,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0. \tag{7.109}$$

It is a Bessel's equation of order n. The standard approach for solution of this type of second order differential equation is to assume a power series. It is known as Frobenius power series.

7.5.6 Bessel Equation and Bessel's Functions

Let us take Y in the power series form as

$$\begin{aligned} Y &= x^P\left(a_o + a_1x + a_2x^2 + \ldots..\right) \\ &= \qquad x^P\sum_O^\infty a_S x_S = \sum_O^\infty a_S x^{P+S}, \end{aligned} \tag{7.110}$$

$$\begin{aligned} \frac{dy}{dx} &= \sum a_S\left(P+S\right)x^{P+S-1} \text{ and} \\ \frac{d^2y}{dx^2} &= \sum a_S\left(P+S\right)\left(P+S-1\right)x^{P+S-2}. \end{aligned} \tag{7.111}$$

Substituting these values in (7.109), we get

$$\begin{aligned} Y = &\sum\left[a_S\left(P+S\right)\left(P+S-1\right) + a_S\left(P+S\right) - a_S n^2\right]x^S \\ &+ \sum a_S x^{S+2} = 0. \end{aligned} \tag{7.112}$$

The following steps are necessary to evaluate the co-efficients a_s:.

i) Equating the co-efficient of x^o, when $S = 0$, we get

$$a_o\left(P^2 - n^2\right) = 0. \tag{7.113}$$

Since a_o is kept arbitrary at this stage and non-zero, therefore

$$\begin{aligned} P^2 - n^2 &= 0 \\ P &= \pm n. \end{aligned}$$

ii) Equating the co-efficient of x^1 when $S = 1$, we get :

$$\begin{aligned} a_1\left[(P+1)P + (P+1) - n^2\right] &= 0 \\ \Rightarrow \quad a_1\left[(P+1)^2 - n^2\right] &= 0. \end{aligned} \tag{7.114}$$

Substituting $P = n$, we get $a_1[(n+1)^2 - n^2] = 0$. Since the second factor cannot be zero even if $n = 0$, therefore $a_1 = 0$.

iii) Equating the co-efficient of x^2 and higher order terms, we get :

$$a_S\left[(P+S)(P+S-1)+(P+S)-n^2\right]+a_{S-2}=0 \quad \text{for } S\geq 2$$

Therefore,

$$a_S=-\frac{a_{S-2}}{(P+S)^2-n^2}. \tag{7.115}$$

Since a_o is arbitrarily chosen to be not equal to zero, therefore $P^2=n^2$ or, $P=\pm n$. Hence the order of equation is either n or $-n$. We then have

$$a_1\left[(n+1)^2-n^2\right]=0.$$

Now, since $\left((n+1)^2-n^2\right)\neq 0$, therefore $a_1=0$. One gets the same result by choosing $P=-n$, i.e., $a_1=0$, for $P=-n$ also. Hence, we write

$$a_S=-\frac{a_{S-2}}{S(S+2n)}. \tag{7.116}$$

Since

$$a_1=0.$$

therefore,

$$a_3=a_5=a_7=\ldots\ldots..=0.$$

With non-zero a_o, one gets

$$\begin{aligned}
a_2&=-\frac{a_o}{2(2+2n)}=-\frac{a_o}{2^2(n+1)}\\
a_4&=-\frac{a_2}{4(4+2n)}=-\frac{a_2}{2^2.2(n+2)}\\
&=+\frac{a_o}{2^4.2.(n+1)(n+2)}\\
a_6&=-\frac{a_4}{6(6+2n)}=-\frac{a_4}{2^3.3.(n+3)}\\
&=+\frac{a_o}{2^6.2.3.(n+1)(n+2)(n+3)}
\end{aligned}$$

and

$$a_{2S}=(-1)^S\frac{a_o}{2^{2S}.s!(n+1)(n+2)\ldots\ldots(n+S)}. \tag{7.117}$$

Now from Frobeneous power series

$$Y=x^P\ \Sigma a_S x^S$$

one gets a_{2S} to be the co-efficient of x^{n+2S} ($\because P=n$).

Therefore,

$$a_{2S}=\frac{(-1)^S.2^n.a_o}{2^{2S+n.}.S!(n+1)(n+2)\ldots\ldots(n+S)} \tag{7.118}$$

where n is an integer. We can write the (7.118) as

$$a_{2S} = \frac{(-1)^S .2^n\Gamma(n+1)}{2^{2S+n}S!\Gamma(n+S+1)}. \tag{7.119}$$

So far a_o was kept arbitrary. Now we are assigning a certain value to a_o i.e.,

$$a_o = \frac{1}{2^n\Gamma(n+1)} \tag{7.120}$$

such that

$$Y = x^n\left[\frac{1}{2^n\Gamma(n+1)} - \frac{\left(\frac{x}{2}\right)^2}{2^n\Gamma(n+1)} + \frac{\left(\frac{x}{2}\right)^4}{2^n.2!\Gamma(n+3)} - \ldots..\right]. \tag{7.121}$$

Since all the terms are defined now, we can write the (7.121) as

$$Y = \sum_{S=O}^{\infty}(-1)^S\frac{\left(\frac{x}{2}\right)^{2S+n}}{S!\Gamma(n+S+1)}. \tag{7.122}$$

Here $\Gamma(n+1)$ etc. are gamma functions.

For many of the physical problems n is put as an integer, therefore we can rewrite the formula as :

$$Y = \sum_{S=O}^{\infty}(-1)^S\frac{\left(\frac{x}{2}\right)^{2S+n}}{S!\,(n+S)!}. \tag{7.123}$$

It is denoted as J_n, the Bessel's function of order n. Hence

$$Y = CJ_n(x) \tag{7.124}$$

We got the solution taking $P = n$. A similar solution can be obtained for $P = -n$.

Therefore the general solution is

$$Y = C\ J_n(x) + D\ J_{-n}(x) \tag{7.125}$$

where n is an integer, it can be very easily shown that

$$\begin{aligned} J_n(x) &= (-1)^n J_{-n}(x) \\ &= F\ J_n(x) \end{aligned}$$

where J_n (x) is the Bessel's function of order n and is given by

$$J_n(x) = \sum_{S=O}^{\infty}\frac{(-1)^S x^{n+2S}}{2^{n+2S}S!\Gamma(n+S+1)}. \tag{7.126}$$

Now let

$$\begin{aligned}
\mathrm{Y} &= \phi\,(\mathrm{x})\,\mathrm{J_n}\,(\mathrm{x})\,, \\
\mathrm{Y}' &= \phi'\mathrm{J_n}\,(\mathrm{x}) + \phi'\mathrm{J_n}(\mathrm{x}) \\
\mathrm{Y}'' &= \phi''\mathrm{J_n} + 2\phi'\mathrm{J}'_\mathrm{n} + \phi J''_\mathrm{n}(\mathrm{x}).
\end{aligned}$$

Substituting the values in the original equation, we have

$$\phi'' J_n + 2\phi' J'_n + \phi J''_n + \frac{\phi'}{x} J_n + \frac{\phi}{x} J'_n + \phi J_n - \frac{n^2}{x^{2.}} \phi J_n = 0. \tag{7.127}$$

We can isolate the part

$$\phi \left[\mathrm{J}''_\mathrm{n} + \frac{1}{\mathrm{x}} \mathrm{J}'_\mathrm{n} + \left(1 - \frac{\mathrm{n}^2}{\mathrm{x}^2}\right) \mathrm{J_n} \right] = 0.$$

This is equal to zero because $\mathrm{J_n}$ is the solution of the differential equations and

Equation 7.127 reduces to the form

$$\phi'' J_n + 2\phi' J'_n + \frac{\phi'}{x} . J_n = 0. \tag{7.128}$$

Rewriting the (7.128), we get

$$\phi'' + \left(\frac{2J'_n}{J_n} + \frac{1}{x} \right) \phi' = 0$$

or

$$\frac{d\phi'}{\phi'} + \left(2\frac{J'_n}{J_n} + \frac{1}{x} \right) \phi' = 0.$$

Integrating, one gets

$$\log \phi' + 2 \log \mathrm{J_n} + \log \mathrm{x} = \log \mathrm{E}$$

or

$$\phi' = \frac{\mathrm{E}}{\mathrm{x}\ \mathrm{J}_\mathrm{n}^2}$$

or

$$\phi = \mathrm{E} \int \frac{\mathrm{dx}}{\mathrm{x}\ \mathrm{J}_\mathrm{n}^2} + \mathrm{G}. \tag{7.129}$$

This part is termed as $\mathrm{Y_n}$. It is

$$\mathrm{Y_n} = \mathrm{E}\ \mathrm{J_n} \int \frac{\mathrm{dx}}{\mathrm{x}\ \mathrm{J}_\mathrm{n}^2} + \mathrm{G}\ \mathrm{J_n}. \tag{7.130}$$

This is the Bessel's function of nth order and second kind. Hence the general solution of the Bessel's equation is

$$\mathrm{Y} = \mathrm{CJ_n(x)} + \mathrm{DY_n(x)} \tag{7.131}$$

where J_n and Y_n are respectively the Bessel's function of the first and second kind and of nth order.

The most general solution of the Laplace equation in cylindrical co-ordinates is

$$\phi = [\mathrm{A}\cos m\psi + \mathrm{B}\sin m\psi]\,[\mathrm{C\ J_n}\,(m\rho) + \mathrm{D\ Y_n}\,(m\rho)]\,\left[Ke^{nZ} + Le^{-nZ}\right] \tag{7.132}$$

where A, B, C, D, K, L are co-efficients, generally determined from the boundary conditions. For some type of boundary value problems, these co-efficients may turn out to be the Kernal functions in Frehdom's integral equations, to be discussed later.

In most of the problems of geophysical interest, the potential generally becomes independent of ψ, when $\phi = f(\rho, Z)$, the Bessel's equation reduces to the form

$$\mathrm{Y} = \mathrm{C\ J_o(x)} + \mathrm{D\ Y_o(x)} \tag{7.133}$$

where J_o and Y_o are the Bessel's function of first and second kind and of order zero.

For $\phi = f(\rho, z)$, the expression for the potential simplifies down to

$$\phi = [C\ J_o\,(m\rho) + DY_o\,(m\rho)]\left[Ke^{mZ} + Le^{-mZ}\right] \tag{7.134}$$

The general expressions for J_o and Y_o are respectively, given by

$$J_o = \sum_{S=o}^{\infty} \frac{x^{2S}}{2^{2S}\,(S!)^2} \tag{7.135}$$

$$J_o = 1 - \frac{x^2}{(2!)^2} + \frac{x^4}{2^4\,(2!)^2} - \frac{x^6}{2^6\,(3!)^2} + \ldots\ldots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots\ldots. \tag{7.136}$$

and

$$Y_0 = \frac{2}{\pi}\left[\ln\frac{x}{2}J_0 + \frac{x^2}{2^2} - \frac{x^4(1+1/2)}{2^4(2!)^2}\ldots\ldots\right] \tag{7.137}$$

$$\Rightarrow\ Y_o = \frac{2}{\pi}\left[\left(\ln\frac{\gamma_x}{2}\right) - \sum_{S=o}^{\infty}(-1)^S\,\frac{x^{2S}}{2^{2S}\,(S!)}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots.\frac{1}{n}\right)\right]$$

where

$$\gamma = \mathrm{Lim}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots. + \frac{1}{n} - \log n\right) \tag{7.138}$$

$= 0.5772157$ and is known as Euler's Constant.

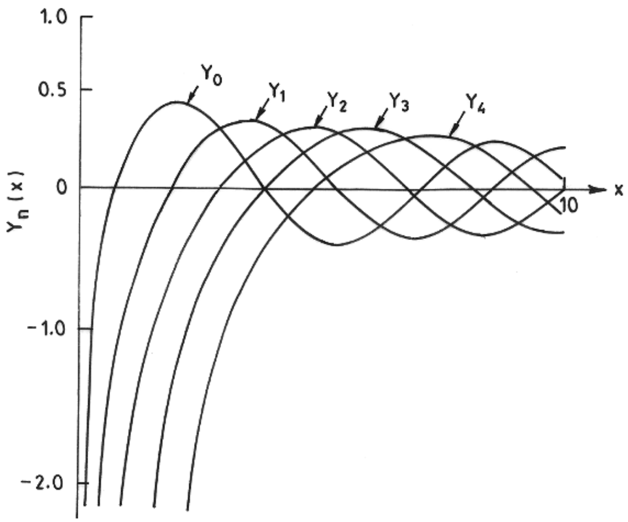


Fig. 7.12. Bessel's function of second kind and 0,1,2,3,4 order

Figures 7.11 and 7.12 shows the behaviour of J_o and Y_o for increasing values of x. Both the functions behave as damped oscillatory functions for larger values of x and the oscillatory characters die down with distance. These points are taken into consideration before choosing them as potential functions. For larger values of x, J_o and Y_o can be computed approximately as

$$J_o(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right)$$

and

$$Y_o(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right) \tag{7.139}$$

as $x \to \infty$.

For large values of x, the oscillatory behaviour vanishes, the potential functions become zero at infinity. In a source free region, where potential function ϕ satisfies Laplace equation, has a finite value at $x = 0$. Therefore for most of the geophysical problem J_n or J_o are treated as more appropriate potential functions. $Y_o = \infty$ at $x = 0$. Therefore near the vicinity of a source Y_o can be taken as a potential function.

7.5.7 Modified Bessel's Functions

If we take

$$\frac{d^2Z}{dz^2} + m^2Z = 0$$

instead of

$$\frac{d^2Z}{dz^2} - m^2Z = 0 \tag{7.140}$$

the Bessel's equation changes to the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{n^2}{x^2}\right) y = 0. \tag{7.141}$$

This equation is called the modified Bessel's equations and is of the form The (7.142) can be rewritten in the form

$$\frac{d^2y}{d\,(ix)^2} + \frac{1}{i^2x}\,\frac{dy}{d\,(ix)} + \left[1 - \frac{n^2}{(ix)^2}\right] y = 0 \tag{7.142}$$

The solution of (7.142) is $J_n(ix)$ and $Y_n(ix)$. We can write $J_n(ix)$ as

$$\begin{aligned} J_n\,(ix) &= \sum_{S=o}^{\infty} \frac{(-1)^S\,(ix)^{n+2S}}{2^{n+2S}S!\Gamma\,(n+S+1)} \\ &= i^n \sum_{S=o}^{\infty} \frac{(-1)^S\,x^{n+2S}}{2^{n+2S}S!\Gamma\,(n+S+1)}. \end{aligned} \tag{7.143}$$

We, therefore, define the modified Bessel's function of the first kind as

$$I_n\,(x) = i^{-n}J_n\,(ix) = \sum_{S=o}^{\infty} \frac{(-1)^S\,.x^{n+2S}}{2^{n+2S}S!\Gamma\,(n+S+1)} \tag{7.144}$$

i.e.

$$I_n(x) = i^{-n}J_n(ix).$$

Modified Bessel's function of the second kind and nth order $K_n(x)$, is

$$K_n\,(x) = i^{-n}J_{-n}\,(ix)\,. \tag{7.145}$$

Therefore, the general solution of the modified Bessel's equation is

$$C_n\ I_n(x) + D_n\ K_n(x). \tag{7.146}$$

When a potential function is independent of ψ, n will be 0, we get the modified Bessel's equation as

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - y = 0 \tag{7.147}$$

and the solution is

$$Y = C_o\ I_o(x) + D_o\ K_o(x)$$

where,

$$I_o\,(x) = \left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \frac{x^6}{2^2.4^2.6^2} + \ldots\ldots\right] \tag{7.148}$$

and

$$\mathrm{K_o}\,(x) = -\,\mathrm{I_o}\ln\left(\frac{\gamma_x}{2}\right) + \frac{x^2}{2^2} + \frac{x^4}{2^4.\,(2!)^2}\left(1+\frac{1}{2}\right) + \frac{x^6}{2^6.\,(3!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots..\right). \tag{7.149}$$

Figures 7.13 and 7.14 show the behaviours of I_o and K_o with x. At larger distances from the source, one can write

$$I_n\,(x) = \left(\frac{1}{2\pi x}\right)^{\frac{1}{2}} e^x \tag{7.150a}$$

and

$$K_n\,(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \tag{7.150b}$$

as $x \to \infty$.

The general solution of Laplace equation

$$\frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{for } \phi = f(\rho, z) \tag{7.151}$$

i.e. when potential is independent of ψ, the azimuthal angle, the expression for the potentials using Bessels functions and modified

Bessel's functions are

$$\phi = \sum_1^{\infty}\left(Ae^{mz} + Be^{-mz}\right)\left[CJ_0(m\rho) + DY_0(m\rho)\right] \tag{7.152}$$

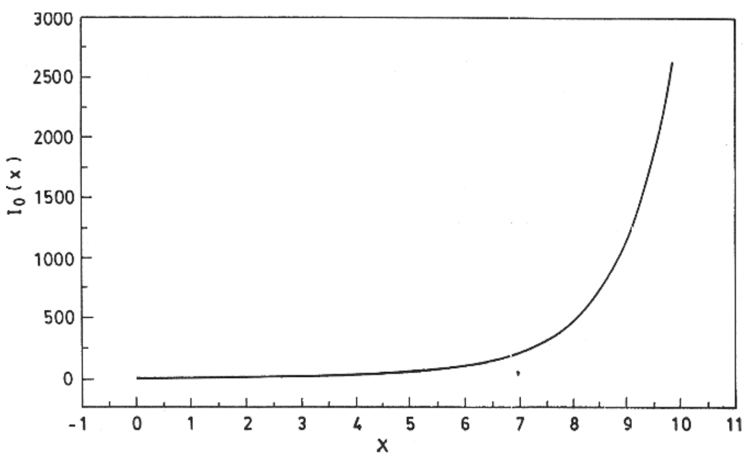


Fig. 7.13. Modified Bessel's function of the first kind and of zero order I_0 and its variation with x

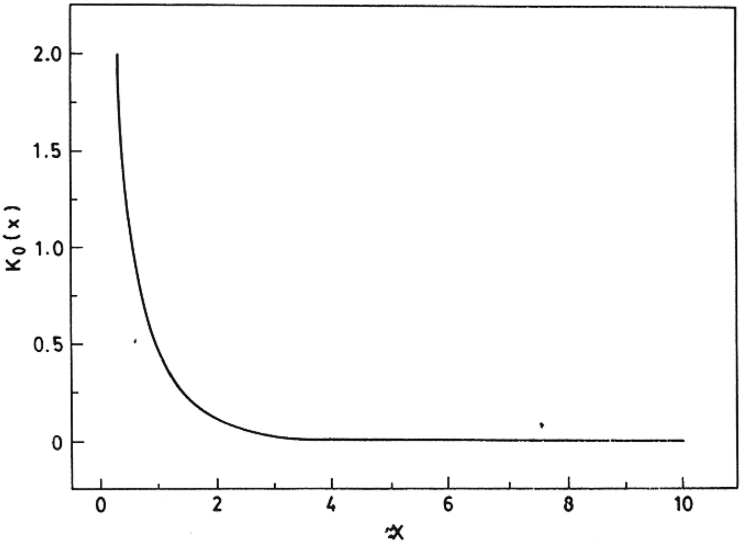


Fig. 7.14. Variation of modified Bessel's function of the second kind and zero order with x

or,

$$\phi = \sum_{1}^{\infty} (A\cos\ mz + B\sin mz)\,[CI_0(m\rho) + DK_0(m\rho)]. \qquad (7.153)$$

Instead of taking the potential functions in the form of A cos mz, B sin mz, we can always express the potential functions in the complex form e^{imz} where the real and imaginary parts can be separated. We now define two new functions of the form

$$H_0^1(m\rho) = J_0(m\rho) + i\ Y(m\rho)$$

and

$$H_0^2(m\rho) = J_0(m\rho) - i\ Y(m\rho) \qquad (7.154)$$

This two functions are called Henkel's functions of the first and second kind. Henkel's functions are also the potential functions and the general solution for the Laplace equation in cylindrical co-ordinates can also be written as

$$\Phi = \sum \left(Ae^{mz} + Be^{-mz}\right)\left[CH_0^1(m\rho) + DH_0^2(m\rho)\right] \qquad (7.155)$$

Here

$$\underset{x\to\infty}{H_0^{(1)}(x)} = \sqrt{\frac{2}{\pi x}}e^{i(x-\pi/4)}$$

and

$$\underset{x\to\infty}{H_0^{(2)}(x)} = \sqrt{\frac{2}{\pi x}}e^{-i(x-\pi/4)}. \qquad (7.156)$$

We find that as

$\rho \to 0$, $J_0 \to 1$, $Y_0 \to \infty$, $I_0 \to 1$, $K_0 \to \infty$ and $H_0 \to \infty$. And for $\rho \to \infty$, $J_0 \to 0$, $Y_0 \to 0$. $I_0 \to \infty$, $K \to 0$ and $H_0 \to 0$. Therefore, remembering the behaviours of these Bessel's function we have to choose the proper potential function dictated by the nature of the boundary value problems. Bessels function of the first kind are generally used. However in problems where modified Bessel's function are needed both first and second kind of are used in the geophysical problems as will be shown in the later chapters. Bessels functions of imaginary and fractional orders are also used in solving geophysical problems(see Chap. 8 and 13).

7.5.8 Some Relation of Bessel's Function

From

$$J_n(x) = \sum_{s=0}^{\alpha} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{\Pi(s)\,\Pi(n+s)} \tag{7.157}$$

we get, taking $J_n^{/}(x) = \frac{d}{dx} J_n(x)$,

$$xJ_n^{/}(x) = \sum_{s=0}^{\alpha} \frac{(-1)^s (n+2S)}{\Pi(s)\,\Pi(n+s)} \cdot \left(\frac{x}{2}\right)^{n+2s} \tag{7.158}$$

$$= n\,J_n + x\sum_{s=1}^{\alpha} \frac{(-1)^s}{\Pi(s-1)\,\Pi(n+s)} \cdot \left(\frac{x}{2}\right)^{n+2s-1}. \tag{7.159}$$

If in (7.159), we put $S = r + 1$, we obtain

$$xJ_n^{/} = nJ_n - x\sum_{r=0}^{\alpha} \frac{(-1)^r}{\Pi(r)\,\Pi(n+1+r)} \left(\frac{x}{2}\right)^{n+r+1} \tag{7.160}$$

$$= nJ_n - xJ_{n+1}. \tag{7.161}$$

In the same way we can prove that

$$x\,J_n^{/} + nJ_n = xJ_{n-1}. \tag{7.162}$$

If we add (7.161) and (7.162) and get

$$2\,J_n^{/} = J_{n-1} - J_{n+1}. \tag{7.163}$$

If we put $n = 0$, we have

$$J_0^{/} = -J_1. \tag{7.164}$$

If we multiply (7.162) by x^{-n-1}, we get

$$x^{-n} J_n^{/} = x^{-n-1} n\,J_n - x^{-n}\,J_{n+1}. \tag{7.165}$$

Hence

$$\frac{d}{dx}\left(x^{-n}\ J_n\right) = -x^{-n}\ J_{n+1}. \tag{7.166}$$

Similarly, it can be proved that

$$\frac{d}{dx}\left(x^n J_n\right) = x^n J_{n-1}. \tag{7.167}$$

If we substract (7.165) from (7.162), we get

$$\frac{2J_n}{x} = J_{n-1} + J_{n+1}. \tag{7.168}$$

Expression for $J_n(x)$ when n is half and odd integer will be as follows.

If we put $n = \frac{1}{2}$, in the general series for $J_n(x)$, we obtain

$$J_{\frac{1}{2}}(x) = \sum_{s=0}^{\alpha} \frac{(-1)^5}{\Pi(s)\,\Pi\left(s+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{2s+\frac{1}{2}}. \tag{7.169}$$

Since

$$\Pi(r) = r\Pi(r-1)$$

and

$$\Pi(s) = S, \text{if } S = 1, 2, 3.$$

We have

$$\Pi\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \text{ and}$$

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\dots\dots\right).$$

Hence

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x. \tag{7.170}$$

if

$$n = -\frac{1}{2}, J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{7.171}$$

From the recurrence formulae, we get for $n = \frac{1}{2}\frac{1}{x}J_{\frac{1}{2}}(x) = J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x)$

$$J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) \tag{7.172}$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right). \tag{7.173}$$

7.6 Solution of Laplace Equation in Spherical Polar Co-ordinates

Lapalce equation $\nabla^2\phi = 0$ in spherical polar co-ordinates where $\phi = f(r, \theta, \Psi)$ is given by (Fig. 7.15)

$$\nabla^2\phi = \frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2} = 0 \tag{7.174}$$

This equation can be solved applying the method of separation of variable choosing

$$\phi = R(r)\Theta(\theta)\Psi(\psi)$$

where R, Θ and Ψ are respectively the functions of r, θ and ψ only.

7.6.1 When Potential is a Function of Radial Distance r i.e., $\phi = f(r)$

When potential is a function of r i.e., $\phi = f(r)$ only and is independent of θ and Ψ. The Laplace equation reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 0 \tag{7.175}$$

$$\Rightarrow\ r^2\frac{\partial\phi}{\partial r} = C_1 \text{ where } C_1 \text{ is a constant.} \tag{7.176}$$

From the (7.176), we can get

$$\phi = C_2 - \frac{C_1}{r} \tag{7.177}$$

where C_2 is another constant. This is the potential at a point at a distance r from the source due to a point source of current. Since the potential will be zero at $r = \infty$. Therefore $C_2 = 0$, and the potential reduces to

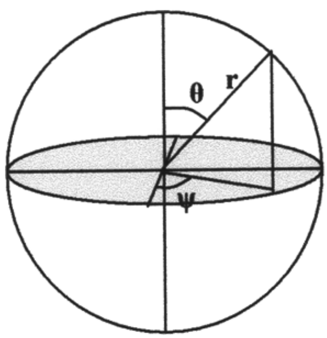


Fig. 7.15. Spherical polar coordinate

$$\phi = -\frac{C_1}{r} \tag{7.178}$$

Since

$$\begin{aligned}\vec{E} &= -\text{grad}\phi = -\frac{C_1}{r^2} \text{ and } I = \int \vec{J}.\vec{n}.ds \\ &= \int \sigma\vec{E}.\vec{n}.ds = -\sigma\int \frac{C_1}{r^2.}ds = -\sigma\frac{C_1}{r^2}4\pi r^2 \\ &= -4\pi\sigma C_1.\text{ Hence } C_1 = -\frac{I}{4\pi\sigma}\end{aligned}$$

and

$$\phi = \frac{I}{4\pi\sigma}, \frac{1}{r} = \frac{I\rho}{4\pi}. = \frac{1}{r}. \tag{7.179}$$

where ρ is the resistivity of the medium and $\rho = .\frac{1}{\sigma}$.

This is the expression for potential at a point at a distance 'r' from a point source in an homogeneous and isotropic full space. The solid angle subtended at the source point is 4π. For an air-earth boundary, when the point electrode is on the surface the potential at a point at a distance 'r' from a point source is

$$\phi = \frac{I\rho}{2\pi}.\frac{1}{r} \tag{7.180}$$

where the solid angle subtended at a point source is 2π on the surface of a homogeneous and isotropic half space.

7.6.2 When Potential is a Function of Polar Angle, i.e., $\phi = f(\theta)$

When potential is independent of r and Ψ and is a function of θ only (Fig. 7.15) the Laplace equation reduces to

$$\frac{1}{\sin\theta}.\frac{\partial}{\partial\theta}\left(\sin\theta - \frac{\partial\phi}{\partial\theta}\right) = 0. \tag{7.181}$$

From (7.181), we get

$$\sin\theta\frac{\partial\phi}{\partial\theta} = C_2. \tag{7.182}$$

Integrating (7.182), we get

$$\phi = C_1 + C_2 \ln\tan\frac{\theta}{2} \tag{7.183}$$

Here the equipotentials form cones at the centre.

7.6.3 When Potential is a Function of Azimuthal Angle i.e., $\varphi = f(\psi)$

Here potential is a function of Ψ only, the Laplace equation is

$$\frac{1}{\mathrm{Sin}^2\theta}\frac{\partial^2\phi}{\partial\psi^2} = 0$$

$$=> \frac{\partial^2\phi}{\partial\psi^2} = 0 \text{ or } \phi = A + B\psi. \tag{7.184}$$

7.6.4 When Potential is a Function of Both the Radial Distance and Polar Angle i.e., $\phi = f(r, \theta)$

And is independent of the azimuthal angle, the Laplace equation is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0 \tag{7.185}$$

$$\Rightarrow \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial\phi}{\partial\theta} = 0. \tag{7.186}$$

Now applying the method of separation of variables, we get $\phi(r, \theta) = R(r)\Theta(\theta)$ where R and Θ are functions of r and θ only. We have

$$\frac{\partial\phi}{\partial r} = \Theta\frac{dR}{dr}; \frac{\partial^2\phi}{\partial r^2} = \Theta\frac{d^2R}{dr^2} \text{ and } \frac{\partial^2\phi}{\partial\theta^2} = R\frac{d^2\Theta}{d\theta^2}.$$

Substituting these values in the Laplace equation, we get

$$\Theta\frac{d^2R}{dr^2} + \frac{2\Theta}{r}\frac{dR}{dr} + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} + \frac{R}{r^2}\mathrm{Cot}\theta\frac{d\Theta}{d\theta} = 0. \tag{7.187}$$

Dividing (7.187) by R and

$$\Theta\frac{1}{R}\frac{d^2R}{dr^2} + \frac{2}{rR}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{r^2\Theta}\mathrm{Cot}\theta\frac{d\Theta}{d\theta} = 0. \tag{7.188}$$

Equation (7.188) can be rewritten as

$$\left[\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr}\right] + \frac{1}{\Theta}\left[\frac{d^2\Theta}{d\theta^2} + \mathrm{Cot}\theta\frac{d\Theta}{d\theta}\right] = 0 \tag{7.189}$$

$$\Rightarrow \left[\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr}\right] = -\frac{1}{\Theta}\left[\frac{d^2\Theta}{d\theta^2} + \mathrm{Cot}\theta\frac{d\Theta}{d\theta}\right] = n^2. \tag{7.190}$$

These two independent equations are function of r and θ respectively and are equal. They are written as

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n^2R = 0 \tag{7.191}$$

and

$$-\frac{1}{\Theta}\left[\frac{d^2\Theta}{d\theta^2}+\mathrm{Cot}\theta\frac{d\Theta}{d\theta}\right]=n^2. \tag{7.192}$$

For solving (7.191), we take

$$R(r)=r^{\alpha},\text{ then}\frac{dR}{dr}=\alpha r^{\alpha-1}\text{ and }\frac{d^2R}{dr^2}=\alpha(\alpha-1)r^{\alpha-2}.$$

Substituting these values in (7.191), we get

$$r^2\left[\alpha(\alpha-1)+r^{\alpha-2}\right]+2r\left[\alpha r^{\alpha-1}\right]-n^2r^{\alpha}=0. \tag{7.193}$$

$$\Rightarrow\quad \alpha(\alpha-1)2\alpha-n^2=0\Rightarrow\alpha^2+\alpha-n^2=0$$

$$\Rightarrow\alpha=\frac{-1\pm\sqrt{1+4n^2}}{2}.$$

Therefore, the two roots are

$$\alpha_1=-\frac{1}{2}+\sqrt{\frac{1}{4}+n^2}\qquad\text{and}\quad\alpha_2=-\frac{1}{2}-\sqrt{\frac{1}{4}+n^2}. \tag{7.194}$$

If $\alpha_1=\alpha$, then $\alpha_2=-(\alpha+1)$, therefore the two solution are $R(r)=r^{\alpha}$ and $r^{-(\alpha+1)}$ taking $\alpha=n$, the general solution of the first equation is

$$R=\left(Ar^n+\frac{B}{r^{n+1}}\right) \tag{7.195}$$

where A and B are arbitrary constants to be determined from the boundary conditions.

The second differential equation is

$$-\frac{1}{\Theta}\left[\frac{d^2\Theta}{d\theta^2}+\mathrm{Cot}\theta\frac{d\Theta}{d\theta}\right]=(n+1) \tag{7.196}$$

$$\Rightarrow\quad\frac{d^2\Theta}{d\theta^2}+\mathrm{Cot}\theta\frac{d\Theta}{d\theta}+n(n+1)\Theta=0$$

$$\Rightarrow\quad\frac{1}{\sin\theta}.\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\sin\theta.n(n+1)\Theta=0 \tag{7.197}$$

$$\Rightarrow\quad\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\sin\theta.n(n+1)\Theta=0.$$

Let $\mu=\cos\theta$, Then $\frac{d}{d\theta}=\frac{d}{d\mu}.\frac{d\mu}{d\theta}=-\mathrm{Sin}\,\theta\frac{d}{d\mu}$

Substituting, these values

$$-\sin\theta\frac{d}{d\mu}\left[\sin\theta(-\sin\theta)\frac{d\Theta}{d\mu}\right]+\sin\theta n(n+1)\Theta=0$$

$$=\frac{d}{d\mu}\left(\sin^2\theta\frac{d\Theta}{d\mu}\right)+n(n+1)=0. \tag{7.198}$$

Since $\sin\theta = \sqrt{1-\mu^2}$, (7.199) changes to the form

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{d\Theta}{d\mu}\right] + n\,(n+1)\,\Theta = 0 \tag{7.199}$$

$$\Rightarrow \quad \left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu^2} + n\,(n+1)\,\Theta = 0. \tag{7.200}$$

This is termed as the Legendre's differential equations. It is written as

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n\,(n+1)\,y = 0. \tag{7.201}$$

7.6.5 Legender's Equation and Legender's polynomial

To solve the (7.201), we use Frobeneous power series and put

$$\begin{aligned} Y &= a_0 + a_1x + a_2x^2 + \\ &= \Sigma\ a_s\ x^s. \end{aligned} \tag{7.202}$$

Substituting this value of Y in (7.202), we get

$$\begin{aligned} &\Sigma a_s s\,(s-1)\,x^{s-2} - \Sigma a_s s\,(s-1)\,x^s - \Sigma 2a_s\ sx^s \\ &+ \Sigma n\,(n+1)\,a_s\ x^s = 0 \end{aligned} \tag{7.203}$$

$$\Rightarrow \quad \Sigma a_s s\,(s-1)\,x^{s-2} + \Sigma[n\,(n+1) - s\,(s+1)]\,a_s x^s = 0. \tag{7.204}$$

If we put S = 2 and equate the co-efficients of x^0, we get

$$a_2 = -\frac{n\,(n+1)}{2}a_0. \tag{7.205}$$

Equating the co-efficient of x^1, we get

$$a_3 = -\frac{(n-1)\,(n+2)}{2.3}.a_1. \tag{7.206}$$

Equating the Co efficient of x^2, we get

$$\begin{aligned} a_4 &= -\frac{\left[n^2+n-6\right]a_2}{3.4} \\ &= \frac{(n-2)\,(n+3)}{3.4}\frac{n\,(n+1)}{2}a_0 \\ &\Rightarrow \frac{(n-2)\,(n+3)\,(n+1)\,(n)}{2.3.4.}a_0. \end{aligned} \tag{7.207}$$

Here a_0 and a_1 are two arbitrary constants is terms of which we can collect the terms and present in the following way

$$y = a_0 \left[1 - \frac{n(n+1)}{\angle 2}x^2 + \frac{n(n-2)(n+1)(n+3)}{\angle 4}x^4 - \ldots\ldots\ldots\ldots\ldots\right]$$
$$+ a_1 x \left[1 - \frac{(n-1)(n+2)}{\angle 3}x^2 + \frac{(n-1)(n+2)(n-3)(n+4)}{\angle 5}x^4 - \ldots\ldots\right]. \tag{7.208}$$

Till now we have not made any restriction on ‘n’ . But for most of the physical problems, n is an integer. When n is even, i.e. 0, 2, 4, ... the first series terminates after a few terms. When n is odd the second series terminates after a few terms. Therefore, we get one polynomial and one infinite series. For $n = 0$,

$$y_0 = a_0 + a_1 x \left[1 + \frac{x^2}{3} + \frac{x^4}{5} + \ldots\ldots\right] \tag{7.209}$$

For $n = 1$

$$y_1 = a_1 x + a_0 \left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \ldots\ldots\right] \tag{7.210}$$

For $n = 2$

$$y_2 = a_0 \left(1 - 3x^2\right) + a_1 x \left[1 - \frac{2}{3}x^2 - \frac{x^4}{5} - \frac{4x^6}{35} + \ldots\ldots\right] \tag{7.211}$$

Therefore the general solution must be constituted of an infinite series and a polynomial. If we take the polynomial part, we can write

$$y_0 = a_0, y_1 = a_1 x, y_2 = a_2 \left(1 - 3x^2\right), y_3 = a_1 x \left(1 - \frac{5x^2}{3}\right). \tag{7.212}$$

The polynomial part of the solution can be written as

$$Y_0 = 1 = P_0(x), Y_1 = x = P_1(x), Y_2 = \frac{1}{2}\left(3x^2 - 1\right) = P_2(x)$$
$$Y_3 = \frac{1}{2}\left(3x^3 - 3x^2\right) = P_3(x). \tag{7.213}$$

These polynomials are known as the Legendre’s polynomial (Fig. 7.16). These polynomials are termed as the Legendre’s function’s of the first kind. The infinite series is the Legendre’s function of the second kind and is denoted by Q.

When

$$n = 0, Q_0(x) = \tan h^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

When

$$n = 1, Q_1(x) = x\, Q_0(x) - 1.$$

When

$$n = 2, Q_2(x) = P_2(x)\, Q_0(x) - \frac{3}{2}x.$$

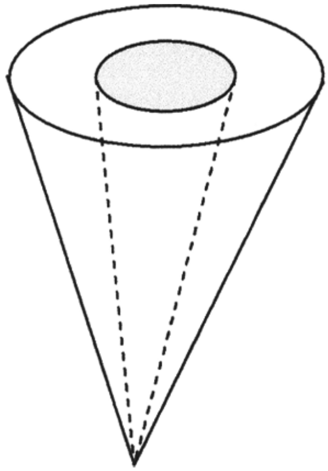


Fig. 7.16. Conical equipotential surface for θ dependence

Similarly one can have the value of Q_n (x) for any value of n. The solution of the Legendre's equation is

$$\sum_{n=0}^{\alpha} \{A_n P_n(x) + B_n Q_n(x)\}. \tag{7.214}$$

Therefore the general solution of the Laplace equation in spherical co-ordinates can be written as

$$\phi = \sum_{0}^{\alpha} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) (C_n P_n(\cos\theta) + D_n Q_n(\cos\theta))$$
$$(E_n \mathrm{Cos}\, m\Psi + F_n\ \mathrm{Sin}\, m\psi) \tag{7.215}$$

Now

$$P_n(1) = 1, P_n(-1) = (-1)^n, P_n(0) = 0,$$

$Q_n(1) = \infty, Q_n(0) = 0, Q_n(-1) = \infty$. In general P_n behaves as a better potential function. Therefore in potential theory involving spherical polar co-ordinates Legendre's polynomials are used in general as potential functions. These polynomials in the Table 7.1 are Legendre's polynomials from 0 to 6th degree. Each of these polynomials satisfy Legendre's differential equations for any value of n. The general expression of the Legendre's polynomial $P_n(x)$ is given by

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \tag{7.216}$$

Table 7.1.

n	$P_n(x)$
0	$P_0(x) = 1$
1	$P_1(x) = x$
2	$P_2(x) = 1/2(3x^2 - 1)$
3	$P_3(x) = 1/2(5x^3 - 3x)$
4	$P_4(x) = 1/8(35x^4 - 30x^2 + 3)$
5	$P_5(x) = 1/8(63x^5 - 70x^3 + 15x)$
6	$P_6(x) = 1/16(231x^6 - 315x^4 + 10x^2 - 5)$

where $N = \frac{n}{2}$ when n is even and $N = (n-1)/2$ when n is odd. Figure (7.17) shows that the Legendre's polynomials are orthogonal to each other. The Radriques' formula for the Legendre's polynomial is

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n \tag{7.217}$$

where n is the degree of the polynomial.

One can prove that (Ramsay 1940),

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n = m \end{cases}. \tag{7.218}$$

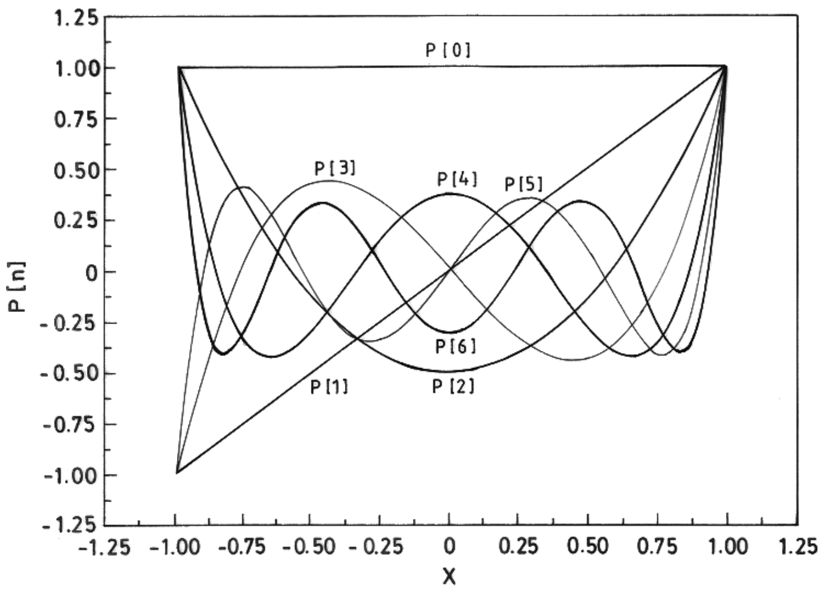


Fig. 7.17. Legendre's polynomial for P(0) to P(6)

The general expression for the Legendre's function of the second kind is given by

$$Q_n(x) = \sum_{r=0}^{r=\infty} \frac{2^n(n+1)!(n+2r)!}{r!(2n+2r+1)!} x^{-h-2r-1} \tag{7.219}$$

$Q_n(x)$ is an infinite series as mentioned and is generally not used as a potential function. So the general solution of the Legendre's equation

$$Y = A\ P_n(x) + BQ_n(x) \tag{7.220}$$

reduces to

$$Y = A\ P_n(x).$$

Problem 1

Find out the nature of distortion of the potential field when a sphere of electrical permittivity $\in_2$ is placed in a medium $\in_1$ in the presence of an uniform field (Fig. 7.18).

A spherical body of electrical permittivity $\in_2$ is placed in a medium of electrical permittivity $\in_1$ in the presence of an uniform field E_0 along the z direction. Here the perturbation potential will be a function of r and θ.

Hence

$$\begin{aligned}\phi &= -E_0\ Z + \text{Constant.}\\ &= -E_0\ r\cos\theta + \text{Constant.}\end{aligned} \tag{7.221}$$

This is the expression for the potential in an uniform field. When we introduce the anomalous body, having different physical property, in the field, the perturbation potential will get added in the vicinity of the body. The perturbation potentials outside and inside the body will be different and are given by

$$\phi_1 = \phi_0 + \phi'$$

and

$$\phi_2 = \phi_0 + \phi'' \tag{7.222}$$

where ϕ_0 is the source potential, ϕ' and ϕ'' are respectively the perturbation potentials outside and inside the body.

In the medium 1, i.e., outside the body, the perturbation potential is

$$\phi' = \sum_{n=1}^{\infty} A_n \frac{1}{r^{n+1}} P_n(\cos\theta). \tag{7.223}$$

Since the potential outside dies down with distance, $\frac{1}{r^{n+1}}$ is the solution and $P_n(\cos\theta)$ is a better potential function. The potential inside is given by

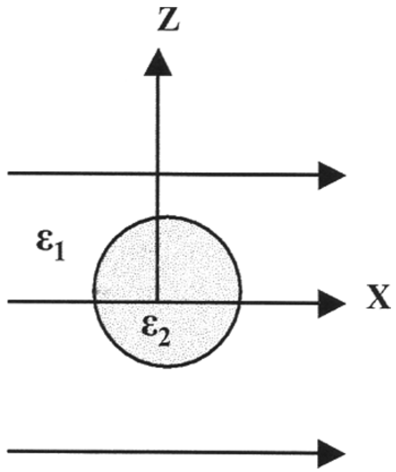


Fig. 7.18. Distorsion in current flow field in the presence of a spherical body

$$\phi'' = \sum_{n=1}^{\infty} B_n r^n P_n(\cos\theta). \tag{7.224}$$

Therefore, potentials both outside and inside are given by

$$\phi_1 = \sum_{n=1}^{\infty} A_n . \frac{1}{r^{n+1}} . P_n(\cos\theta) - E_0 r \cos\theta \quad \text{(Potential outside)} \tag{7.225}$$

and

$$\phi_2 = \sum_{n=1}^{\alpha} B_n . r^n . P_n(\cos\theta) - E_0 r \cos\theta \text{(Potential inside)}. \tag{7.226}$$

Applying the two boundary conditions, i.e.,

i) $\phi_1|_{r=a} = \phi_2|_{r=a}$,
ii) $\in_1 \left(\frac{\partial\phi_1}{\partial r}\right)_1\Big|_{r=a} = \in_2 \left(\frac{\partial\phi}{\partial r}\right)_2\Big|_{r=a}$

where 'a' is the radius of the sphere, we get

$$\sum A_n . \frac{1}{a^{n+1}} P_n(\cos\theta) = \sum B_n a^n P_n(\cos\theta) \tag{7.227}$$

and

$$\begin{aligned} &- \in_1 E_0\ P_1(\cos\theta) + \in_1 \sum A_n . -(n+1) . \frac{1}{a^{n+2}} . P_n(\cos\theta) \\ &\quad = - \in_2 E_0 P_1(\cos\theta) + \in_2 \sum B_n n a^{-(n+1)} P_n(\cos\theta). \end{aligned} \tag{7.228}$$

Since the source potential is E_0 r $\cos\theta$ and $\cos\theta$ can be expressed as $P_1(\cos\theta)$, it became possible to bring the source and perturbation potentials in the same format before the boundary conditions are applied. Here P_1 is the Legendre's polynomial of the first order. Since the source potential is in Legendre's polynomial of first order, the order will remain the same in the perturbation potential also. Therefore n = 1. So the equations (7.227) and (7.228) simplifies down to

$$\frac{A_1}{a^2} = B_1 \, a \tag{7.229}$$

$$-\in_1 \vec{E}_0 - A_1 \frac{2 \in_1}{a^3} = -\in_2 \vec{E}_0 + \in_2 B_1. \tag{7.230}$$

From these two equations, the values of A_1 and B_1, are

$$A_1 = B_1 \, a^3 \tag{7.231}$$

and

$$A_1 = \vec{E}_0 a^3 \frac{\in_2 - \in_1}{\in_2 + 2 \in_1} \tag{7.232}$$

$$B_1 = \vec{E}_0 \frac{\in_2 - \in_1}{\in_2 + 2 \in_1}. \tag{7.233}$$

Substituting the values of A_1 and B_1, in (7.225) and (7.226), we get

$$\phi_1 = -E_0 r \cos\theta + E_0 \frac{\in_2 - \in_1}{\in_2 + 2 \in_1} \frac{a^3}{r^2} \cos\theta (\text{Potential outside}) \tag{7.234}$$

and

$$\begin{aligned} \phi_2 &= E_0 r \cos\theta + E_0 \frac{\in_2 - \in_1}{\in_2 + 2 \in_1} r \cos\theta (\text{Potential inside}). \\ &= -E_0 \frac{3 \in_1}{\in_2 + 2 \in_1} . r \cos\theta. \end{aligned} \tag{7.235}$$

The fields inside the body are

$$E_r = -\left(\frac{\partial \phi_2}{\partial r}\right) = E_0 \frac{3 \in_1}{\in_2 + 2 \in_1} \cos\theta \tag{7.236}$$

and

$$E_\theta = -\left(\frac{\partial \phi_2}{r \partial \theta}\right) = -E_0 \frac{3 \in_1}{\in_2 + 2 \in_1} \sin\theta. \tag{7.237}$$

Therefore, the field inside the body can be written as

$$\begin{aligned} \vec{E} &= (\vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta) \vec{E}_0 \frac{3 \in_1}{\in_2 + 2 \in_1} \\ &= \vec{a}_z . \frac{3 \in_1}{\in_2 + 2 \in_1} \vec{E}_0. \end{aligned} \tag{7.238}$$

Hence the field inside will be parallel to the field outside.

Corollary

Potential at a point on the surface of the earth due to a buried spherical inhomogeneity of conductivity σ_2 is placed in a medium of conductivity σ_1 in the presence of an uniform field in direct current domain.

This is the same problem as given in the previous section. Here the direct current is flowing from a source at infinite distance to generate the uniform field (Fig. 7.19).

The potential will be symmetrical with respect to the polar axis. So the potential will be independent of the azimulhal angle Ψ.

The solution of the problem is

$$\phi = \sum \left[A_n r^n + B_n r^{-(n+1)}\right] Pn(\cos\theta). \tag{7.239}$$

The constants A_n and B_n can be found out from the boundary conditions. $\phi_1 = \phi_2$ and $\frac{1}{\rho_1}\frac{\partial\phi_1}{\partial r} = \frac{1}{\rho_2}\frac{\partial\phi_2}{\partial r}$. at $r = a$ where 'a' is the radius of the sphere.

Since

$$\vec{J} = \frac{\vec{E}}{\rho} = -\frac{1}{\rho_1}\frac{\partial\phi_0}{\partial x} \text{ therefore } \phi_0 = -\vec{J}\rho x \tag{7.240}$$

The potentials outside and inside the body are given by

$$\phi_1 = -J\rho_1 x + \sum_{n=1}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta) - \text{Potential outside}$$

$$\phi_2 = -J\rho_1 x + \sum_{n=1}^{\infty} A_n r^n P_n(\cos\theta) - \text{Potential inside.} \tag{7.241}$$

Applying the boundary condition we get

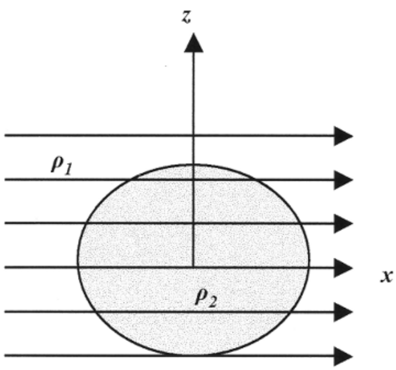


Fig. 7.19. Distortion of the equipotential lines due to the presence of a spherical inhomogenity in an uniform field

$$\sum B_n a^{-(n+1)} P_n(\cos\theta) = \sum A_n a^n P_n(\cos\theta) \tag{7.242}$$

and

$$-J\rho_1 + \sum B_n. -(n+1)a^{-(n+2)}, P_n = -\frac{J\rho_1}{\rho_2}.\rho_1 + \frac{\rho_1}{\rho_2}\sum A_n^n a^{n-1}.P_n(\cos\theta). \tag{7.243}$$

Since the source potential has P_1 terms, (the Legendre's polynomial of the first order) the perturbation potential also will have only P_1 terms. Only A_1 and B_1 will exist and other terms, viz $A_2, A_3 \ldots A_n, B_2, B_3 \ldots B_n$ are all zero. Therefore

$$A_1 = J\rho_1 \frac{\rho_1 - \rho_2}{\rho_1 + 2\rho_2} \text{ and } B_1 = J\rho_1 \frac{\rho_1 - \rho_2}{\rho_1 + 2\rho_2}.a^3 \tag{7.244}$$

Hence the potentials outside and inside are given by

$$\phi_1 = -J\rho_1 r\cos\theta + J\rho_1 \frac{\rho_1 - \rho_2}{\rho_1 + 2\rho_2}\frac{a^3}{r^2}.\cos\theta \tag{7.245}$$

$$\phi_2 = -J\rho_1 r\cos\theta + J\rho_1 \frac{\rho_1 - \rho_2}{\rho_1 + 2\rho_2}.r\cos\theta \tag{7.246}$$

$$= -\vec{J}\frac{3\rho_1\rho_2}{\rho_1 + 2\rho_2} r\cos\theta.$$

Problem 2

Find the potentials at a point on the surface of the earth in the presence of a buried spherical inhomogeneity of resistivity ρ_2 in a half space of resistivity ρ_1 due to point source of current.

A sphere of radius 'a' and resistivity ρ_2 is placed at a depth Z (i.e., the depth to the centre of the sphere from the surface i.e. air earth boundary) in a medium of resistivity ρ_1 (Fig. 7.20). The distances between the current electrode at A and potential electrode at P is R, the electrode separation. The distance from the source to the center of the sphere and from 'O' to 'P' are respectively 'd' and 'r'. $\angle$AOP is the angle subtended by A and P at the center 'O'. We have to determine the angle θ for each move of the current and potential electrodes together or separately.

To solve this problem, we shall first consider the medium to be a homogenous and isotropic free space. A spherical body of resistivity ρ_2 and radius 'a' is embedded in a medium of resistivity ρ_1. Once the problem is solved in full pace, the air earth boundary can be brought.

Considering the effect of image, the potential can be doubled and the solution can be obtained within 5 % error.

The perturbation potential function is

$$\phi = \sum \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\cos\theta) \tag{7.247}$$

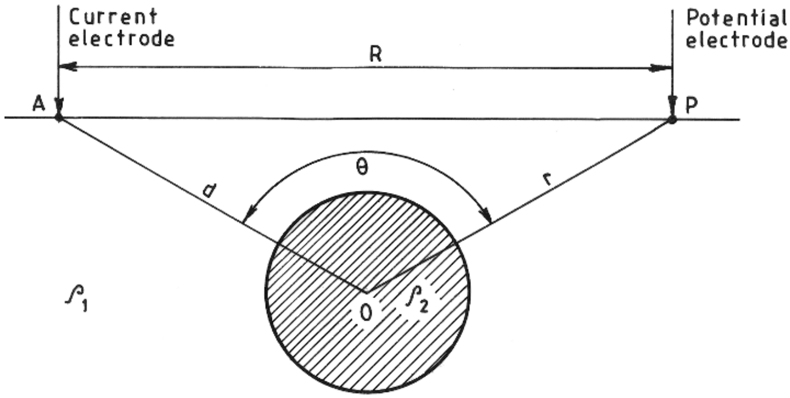


Fig. 7.20. Two electrode response due to a buried spherical body having conductivity contrast with the host rock; Geometry of the problem is shown

in this problem also. Figure (7.20) shows the geometry of the problem and the location of the current and potential electrodes. In a full space the potential at P due to a current electrode at A is given by

$$\phi_0 = \frac{I\rho_1}{4\pi}.\frac{1}{R} \tag{7.248}$$

where $R = AP$. Here $R^2 = r^2 + d^2 - 2rd\cos\theta$. So

$$\begin{aligned} \frac{1}{R} &= \frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} \\ \Rightarrow \frac{1}{R} &= \frac{1}{d\sqrt{1 + \frac{r^2}{d^2} - 2\frac{r}{d}\cos\theta}} \end{aligned} \tag{7.249}$$

Let $u = \frac{r}{d}$ and $x = \cos\theta$, Then

$$\frac{1}{\sqrt{1 + \frac{r^2}{d^2} - 2\frac{r}{d}\cos\theta}} = \frac{1}{\sqrt{1 + u^2 - 2ux}} \tag{7.250}$$

Let $\nu = 2ux - u^2$

So,

$$\frac{1}{\sqrt{1 + \frac{r^2}{d^2} - 2\frac{r}{d}\cos\theta}} = \frac{1}{\sqrt{1-\nu}} = (1-\nu)^{-\frac{1}{2}}$$

After binomial expansion and substituting the values of ν we get

$$= 1 + u\,x + \frac{u^2}{2}\left(3x^2 - 1\right) + \tag{7.251}$$

Since

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right),$$

we can write

$$\frac{1}{\sqrt{1 + \frac{r^2}{d^2} - 2\frac{r}{d}\cos\theta}} = P_0(x)u^0 + P_1(x)u^1 + P_2(x)u^2 + P_3(x)u^3$$
$$= \sum P_n(x)u^n = \sum P_n(x)\left(\frac{r}{d}\right)^n. \tag{7.252}$$

So, the source potential is

$$\phi_o = \frac{\rho_1 I}{4\pi}\frac{1}{d}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r}{d}\right)^2.$$

Therefore the potential inside and outside assuming full space condition is

$$\phi_1 = \frac{I\rho_1}{4\pi}.\frac{1}{d}\sum_{n=0}^{\infty}\left(\frac{r}{d}\right)^n P_n(\cos\theta) + \sum_{n=1}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta)$$
$$\text{(Potential outside)} \tag{7.253}$$

and

$$\phi_2 = \frac{I\rho_1}{4\pi}.\frac{1}{d}\sum_{n=0}^{\infty}\left(\frac{r}{d}\right)^n P_n(\cos\theta) + \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta).$$
$$\text{(Potential inside)} \tag{7.254}$$

Now applying the boundary conditions, at r = a where 'a' is the radius of the sphere.

$$\frac{I\rho_1}{4\pi}.\frac{1}{d}\sum\left(\frac{a}{d}\right)^n P_n(\cos\theta) + \sum B_n a^{-(n+1)} P_n(\cos\theta)$$
$$= \frac{I\rho_1}{4\pi}.\frac{1}{d}\sum\left(\frac{a}{d}\right)^n P_n(\cos\theta) + \sum A_n a^n P_n(\cos\theta). \tag{7.255}$$

From the first boundary conditions, i.e., $\phi_1 = \phi_2$ at $r = a$

$$\frac{1}{\rho_1}.\frac{I\rho_1}{4\pi}\sum\frac{n\ a^{n-1}}{d^{n+1}}.P_n(\cos\theta) + \frac{1}{\rho_1}\sum -(n+1)B_n a^{-(n+2)}.P_n(\cos\theta)$$
$$= \frac{1}{\rho_2}\frac{I\rho_1}{4\pi}\sum\frac{n\ a^{n-1}}{d^{n+1}}P_n(\cos\theta) + \frac{1}{\rho_2}\sum nA_n a^{n-1}P_n(\cos\theta). \tag{7.256}$$

From the second boundary conditions, i.e., $J_{n1} = J_{n2}$ at $r = a$, we write the nth term (these boundary conditions hold good for all the terms) as follows

$$\frac{I}{4\pi}\frac{na^{n-1}}{d^{n+1}} - (n+1)\frac{B_n}{\rho_1}a^{-(n+2)} = \frac{1}{\rho_2}\left[\frac{\rho I}{4\pi}.\frac{na^{n-1}}{d^{n+1}} + nA_na^{n-1}\right]$$

$$\Rightarrow \quad \frac{I}{4\pi}.\frac{na^{n-1}}{d^{n+1}} - \frac{\rho_1}{\rho_2}\frac{I}{4\pi}.\frac{na^{n-1}}{d^{n+1}} = \frac{1}{\rho_1}(n+1)A_na^{n-1} + \frac{1}{\rho_2}n\ A_na^{n-1}$$

$$\Rightarrow \quad \frac{1}{4\pi}.\frac{n\ a^{n-1}}{d^{n+1}}\left[\frac{\rho_2-\rho_1}{\rho_2}\right]$$

$$= A_na^{n-1}\left[\frac{\rho_2 n + \rho_2 + \rho_1 n}{\rho_1\rho_2}\right]$$

$$\Rightarrow \quad A_n = \frac{I\rho_1}{2\pi}.\frac{(\rho_2-\rho_1)}{\rho_2 n + \rho_1 n + \rho_2}.\frac{1}{d^{n+1}} = \rho_n.\frac{I\rho_1}{4\pi}.\frac{1}{d^{n+1}} \tag{7.257}$$

where

$$\rho_n = \frac{(\rho_2-\rho_1)\,n}{(\rho_2 n + \rho_1 n + \rho_2)}.$$

$$B_n = A_na^{2n+1} = \frac{I\rho_1}{4\pi}.\frac{\rho_n}{d^{n+1}}.a^{2n+1}.$$

Therefore

$$\phi_2 = \frac{\rho_1 I}{4\pi R} + \sum \frac{1\rho}{4\pi}.\frac{1}{d^{n+1}}\rho_n r^n P_n(\cos\theta). \tag{7.258}$$

and

$$\phi_1 = \frac{I\rho_1}{4\pi}.\frac{1}{R} + \frac{I\rho_1}{4\pi}\sum_{n=1}^{\infty}\rho_n\frac{a^{2n+1}}{d^{n+1}}.\frac{1}{r^{n+1}}P_n(\cos\theta).$$

(Potential outside) (7.259)

Now if we bring the air earth boundary, we get

$$\phi_1 = \frac{I\rho_1}{2\pi}\left[\frac{1}{R} + \sum_{n=0}^{\infty}\rho_n.\frac{a^{2n+1}}{d^{n+1}}.\frac{1}{r^{n+1}}P_n(\cos\theta)\right]. \tag{7.260}$$

With this approximation the solution can be obtained within 5% error. Once the response for two electrode configuration is obtained (Fig. 7.21), one can compute the response for any other electrode configuration since the principle superposition is valid.

7.6.6 When Potential is a Function of all the Three Coordinates Viz, Radial Distance, Polar Angle and Azimuthal Angle,i.e., $\phi = f(r, \theta, \psi)$

When potential is a function of all the three coordinates, i.e. $= f(r, \theta, \psi)$, the Laplace equation in spherical polar coordinate (7.174) can be solved applying the method of separation of variables in the form

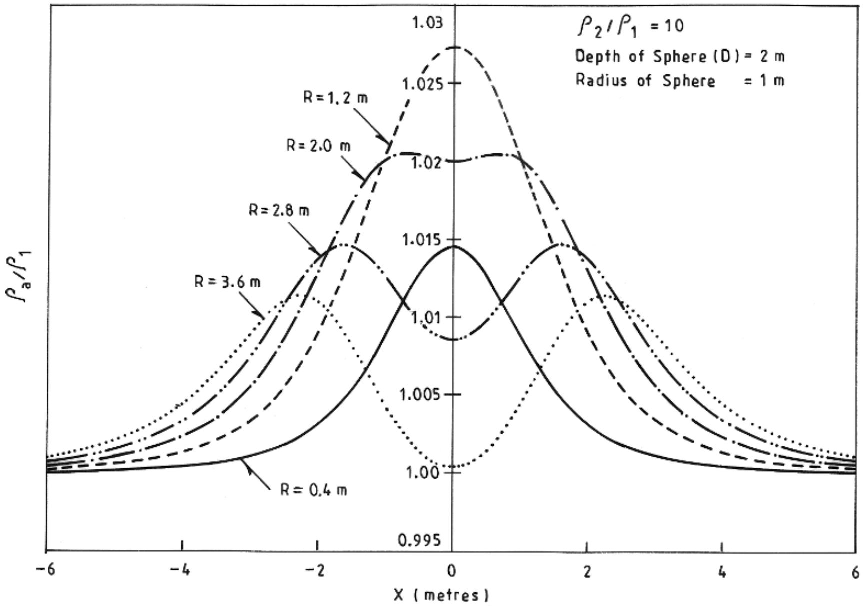


Fig. 7.21. Two electrode apparent resistivity anomaly on the surface due to a buried spherical inhomogeneity

$$\phi = R(r)\Theta(\theta)\Psi(\psi) \tag{7.261}$$

It can be rewritten as

$$\phi = R(r)S(\theta, \psi). \tag{7.262}$$

Keeping the surface components together. Equation (7.174) changes to the form

$$\frac{1}{R}\frac{d}{dR}\left[r^2\frac{dR}{dr}\right] = -\frac{1}{S\mathrm{Sin}\theta}\frac{\partial}{\partial\theta}\left[\mathrm{Sin}\theta\frac{\partial S}{\partial\theta}\right] - \frac{1}{S\mathrm{Sin}^2\theta}\frac{\partial^2 S}{\partial\Psi^2}. \tag{7.263}$$

Equation (7.174) breaks into two parts as discussed earlier

$$\frac{d}{dR}\left[r^2\frac{dR}{dr}\right] - Rn(n+1) = 0 \tag{7.264}$$

$$\frac{1}{\mathrm{Sin}\theta}\frac{\partial}{\partial\theta}\left[\mathrm{Sin}\theta\frac{\partial S}{\partial\theta}\right] + \frac{1}{\mathrm{Sin}^2\theta}\frac{\partial^2 S}{\partial\phi^2} + Sn(n+1) = 0. \tag{7.265}$$

The solutions of (7.191) are in the form r^n and $r^{-(n+1)}$ as discussed earlier.

Now

$$S(\theta, \Psi) = \Theta(\theta)\Psi_{(\psi)}. \tag{7.266}$$

Therefore (7.263) can be rewritten in the form

$$\frac{\text{Sin}\theta}{\Theta}\frac{d}{d\theta}\left[\text{Sin}\theta\frac{\partial\Theta}{d\theta}\right]+\text{Sin}^2\theta n(n+1)=-\frac{1}{\psi}\frac{d^2\psi}{d\psi^2}. \tag{7.267}$$

Both sides of this (7.267) are equated to a constant m^2. We get two differential equations as follows :

$$\frac{d^2\psi}{d\psi^2}+m^2\psi=0. \tag{7.268}$$

and

$$\frac{d}{d\theta}\left[\text{Sin}\theta\frac{d\Theta}{d\theta}\right]+\Theta\left[n(n+1)\text{Sin}\theta-\left(\frac{m^2}{\text{Sin}\theta}\right)\right]=0 \tag{7.269}$$

The solutions of (7.268) are $\sin m\Psi$ and $\cos\ m\psi$.

Equation (7.269) is termed as the Associated Legandre's equation. The solution of this equation are $P_n^m(\text{Cos}\,\theta)$ and $Q_n^m(\text{Cos}\,\theta)$. Here $P_n^m(\text{Cos}\,\theta)$ and $Q_n^m(\text{Cos}\,\theta)$ are known as the Associated Legendre's function of the first kind and second kind. It has already been mentioned that $Q_n^m(\text{Cos}\,\theta)$ is an infinite series and is generally not considered for solution of any kind of potential problem. Therefore, the most general solution of Laplace equation in spherical polar coordinate is

$$\begin{aligned}\phi(r,\theta,\Psi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}&\left[\left(A_n^m r^n+B_n^m\frac{1}{r^{n+1}}\right)\right.\\&\left.(C_n^m\text{Cos } m\psi+D_n^m\text{Sin } m\psi)\right]P_n^m(\text{Cos}\theta)\end{aligned} \tag{7.270}$$

This series expansion of the general solution of the Laplace equation in spherical polar coordinate is termed as spherical harmonic expansion as discussed in the next section. Here n is the degree and m is the order of the Associated Legendre's polynomial. When $m = 0$, Associated Legendre Polynomial changes to Legendre Polynomial. The coefficients $A_n^m, B_n^m, C_n^m, D_n^m$, are generally determined either from detailed spherical harmonic analysis or by applying suitable boundary conditions depending upon the nature of the problem.

7.6.7 Associated Legendre Polynomial

If a Legendre Polynomial in spherical polar coordinate depends both on colatitude and longitude, the polynomial is termed as Associated Legendre Polynomial. Associated Legendre Polynomial is denoted by $P_n^m(\theta)$ or $P_n, m(\theta)$ where n is the degree and m is the order of the polynomial. If we differentiate the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \tag{7.271}$$

m times, with respect to x, we get

$$(1-x^2)\frac{d^2\xi}{dx^2}-2x(m+1)\frac{d\xi}{dx}+(n-m)(n+m+1)\xi=0 \tag{7.272}$$

where

$$\xi = \frac{d^m y}{dx^m} = \frac{d^m P_n(x)}{dx^m} \tag{7.273}$$

If we take

$$\eta = \xi(1 - x^2)^{m/2}$$

then it can be shown that (7.272) changes to the form

$$(1 - x^2)\frac{d^2\eta}{dx^2} - 2x\frac{d\eta}{dx} + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]\eta = 0 \tag{7.274}$$

This is the Associated Legendre equation. It is observed that

$$\eta = (1 - x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \tag{7.275}$$

This η is the Associated Legendre Polynomial and is denoted by $P_n^m(x)$ or $P_n^m(\theta)$. Therefore

$$P_n^m(x) = (1 - x^2)^{m/2}\frac{d^m}{dx^m}P_n(x). \tag{7.276}$$

7.7 Spherical Harmonics

A function is said to be harmonic if the function and its first derivative are continuous within the domain and it satisfies Laplace equation $\nabla^2\phi = 0$. Many such functions exist which satisfy these conditions and many of those functions can be expressed in series forms. Harmonic functions which satisfy Laplace equation in spherical polar coordinates (r, θ, ψ) are called spherical harmonics where the potential function is dependent upon the radial direction, latitude or colatitude and longitude or the azimuthal direction. Harmonics which are functions of r, θ, ψ are called spherical solid harmonics in a three dimensional space. Harmonics on the surface, which are function of θ and ψ are called spherical surface harmonics. If ϕ is a spherical harmonics of degree n, then it satisfies the following two conditions

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{7.277}$$

and

$$x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + z\frac{\partial\phi}{\partial z} = n\phi. \tag{7.278}$$

A spherical surface harmonics is the set of values of solid spherical harmonics takes on the surface of unit radius with its origin at the centre. Equation (7.174) is the Laplace equation in spherical polar coordinate. Now if the potential $\phi = H_n = r^n S_n$ where H_n is the solid spherical harmonics and S_n is the surface spherical harmonics. S_n is a function of ϕ and Ψ alone and of degree n. From (7.264) and (7.265). it can be shown that 'r' dependence part of the Laplace equation can be written in the form

$$\frac{\partial^2(r\Psi)}{\partial r^2} = r\frac{\partial^2(r^{n+1}S_n)}{\partial r^2} = n(n+1)r^n S_n \tag{7.279}$$

After removing the r^n part, the radial component, the surface component of the Laplace equation in spherical coordinate changes to the form.

$$\frac{1}{\mathrm{Sin}^2\psi}\frac{\partial^2 S_n}{\partial\theta^2} + \frac{1}{\mathrm{Sin}\psi}\frac{\partial}{\partial\psi}(\mathrm{Sin}\psi\frac{\partial S_n}{\partial\psi}) + n(n+1)S_n = 0. \tag{7.280}$$

Every spherical surface harmonics of degree n satisfies this equation and every solution of this equation is a surface harmonics of degree n (Macmillan, 1958).

7.7.1 Zonal, Sectoral and Tesseral Harmonics

A spherical harmonic which can be expressed as a function of r and θ and independent of Ψ the longitude, is a zonal harmonic (Fig. 7.22). Legendre polynomials $P_n(\theta)$ of degree n are independent of Ψ and are zonal harmonics.

The guiding equation for zonal harmonics is

$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + n(n+1)P_n = 0 \tag{7.281}$$

The zonal harmonics from degree 0 to 6 in the form suitable for hand computations are given as

$$\begin{aligned}
P_0 &= 1\\
P_1 &= \mu\\
P_2 &= \frac{3}{2}\mu^2 - \frac{1}{2}\\
P_3 &= \frac{5}{2}\mu^3 - \frac{3}{2}\mu\\
P_4 &= \frac{7}{2}.\frac{5}{2}.\mu^4 - \frac{5}{2}.\frac{3}{2}\mu^2 + \frac{3}{2}.\frac{1}{4}\\
P_5 &= \frac{9}{2}.\frac{7}{4}.\mu^5 - \frac{7}{2}.\frac{5}{2}.\mu^3 + \frac{5}{2}.\frac{3}{4}.\mu\\
P_6 &= \frac{11}{2}.\frac{9}{4}.\frac{7}{6}\mu^6 - \frac{9}{2}.\frac{7}{4}.\frac{5}{2}\mu^4 + \frac{7}{2}.\frac{5}{2}.\frac{3}{4}.\mu^2 - \frac{5}{2}.\frac{3}{4}.\frac{1}{6}
\end{aligned} \tag{7.282}$$

A few points about zonal harmonics are :

(a) Zonal harmonics are orthogonal (Fig. 7.17)
(b) Zeros of zonal harmonics are real and they all lie between 0 and 1.
(c) A recurrence relation of the Legendre polynomial or zonal harmonics is

$$P_{n+1} - 2\mu P_n + P'_{n+1} = P_n \tag{7.283}$$

where primed P' is the first derivative of P.

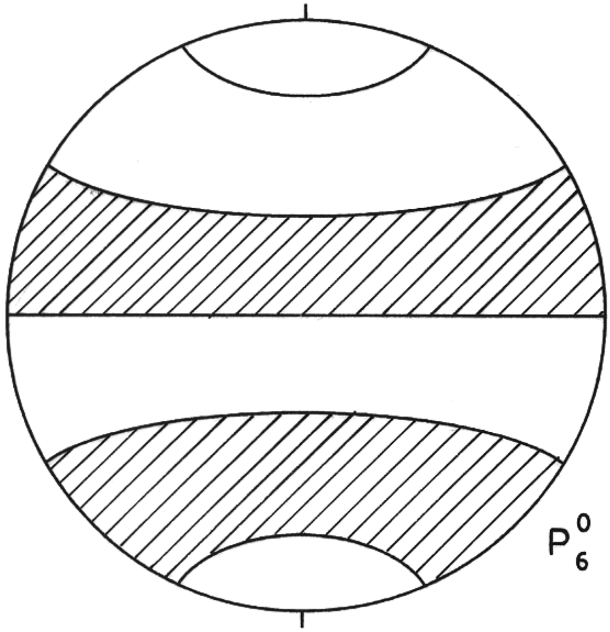


Fig. 7.22. Nature of the zonal harmonics

(d) The general formula for Legendre polynomial or zonal harmonics is

$$P_n(\text{Cos}\theta) = \sum_{J=0}^{n}(-1)^J \frac{(2n - 2J - 1)}{[n - 2j][n - 2j - 1][2j]^{n-2j}} \tag{7.284}$$

(e) Legendre polynomial or zonal harmonics are better potential functions.
(f) On the surface, P_n vanishes along the n circles of latitude θ. One of which is the equator.

Legendre polynomial is used for analysing a set of data collected on the surface of a spherical earth when the function is dependent only on the latitude or colatitude. If the surface harmonics is dependent both on θ and ψ, then more powerful orthogonal polynomials are used. They are Associated Legendre Polynomial and can be denoted by

$$P_{n,m}(\theta) = \text{Sin}^m\theta \frac{\partial^m}{\partial(\text{Cos}\theta)^m} P_o(\text{Cos}\theta) \tag{7.285}$$

where m is the degree and n is the order of the Legendre Polynomial. Associated Legendre polynomials are more powerful in the sense that Legendre Polynomial is only a special case of Associated Legendre Polynomial. Because when m, the order is zero, i.e., when the dependence on longitude vanishes ALP changes to LP as mentioned.

The general solution of surface harmonics is

$$\phi(\theta, \psi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\left(C_{n,m} \mathrm{Cos}\ m\psi + D_{n,m} \mathrm{Sin}\ m\psi \right) P_{n,m}(\theta) \right] \tag{7.286}$$

The surface harmonics $\phi(\theta, \Psi)$ is represented by an infinite sum of Associated Legendre Polynomials, sines and cosines. Here $P_{n,m}(\theta)\mathrm{Cos}\ m\Psi$ and $P_{n,m}(\theta)\mathrm{Sin}\ m\psi$ are the components of the surface spherical harmonics. If $m = 0$, the surface harmonics depends only on the colatitude and is called a zonal harmonics as mentioned already. If $n = m$, it depends on the longitude ψ, it is then termed as the sectoral harmonics. If $m > 0$ and $n > m$, then the harmonics is termed as tesseral harmonics(Figs. 7.23, 7.24).Tesseral harmonics are those which becomes zero both in latitude and longitude. In a spherical polar coordinate system we express the harmonic function of degree n or order k as

$$\phi(\theta, \psi) = r^n e^{ik\theta}, \mathrm{Sin}^k \psi P_n^k \tag{7.287}$$

where P_n^k is the Legendre Polynomial of degree n, and order k, $\mathrm{Sin}^k\Psi$ shows the longitudinal dependence, $e^{ik\theta}$ is the dependence on colatitude and r^n is the radial dependence part. If we remove r^n, the surface harmonics part can be written as

$$\phi(\theta, \psi) = e^{ik\theta} \mathrm{Sin}^k \psi P_n^k. \tag{7.288}$$

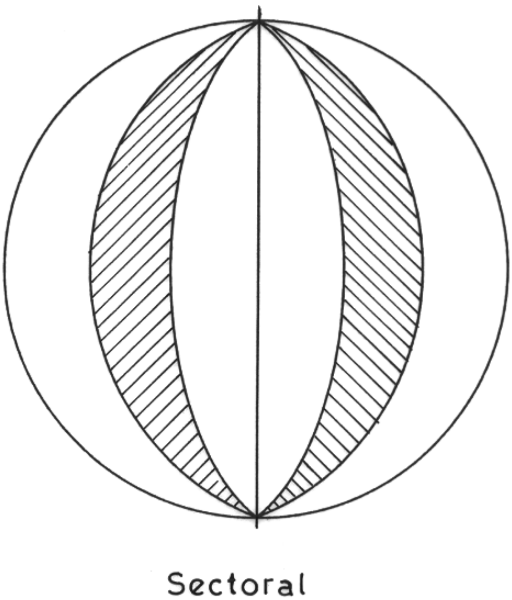


Fig. 7.23. Nature of the sectoral harmonics

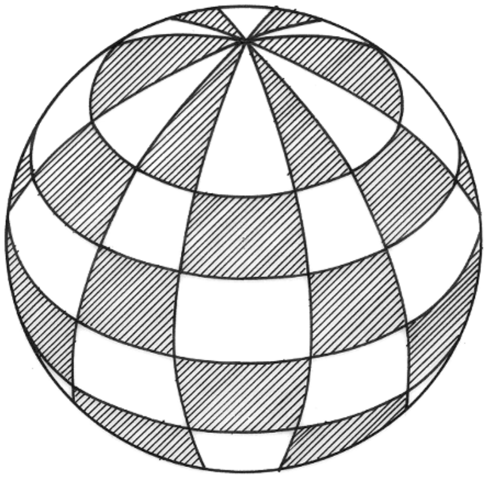


Fig. 7.24. Nature of the tesseral harmonics

The real and imaginary parts of (7.288) is given by

$$\left.\begin{array}{l}\text{Real part of } \phi(\theta,\psi) = \mathrm{Sin}^k\psi P_n^k \mathrm{Cos}\ k\theta \\ \text{Imaginary part of } \phi(\theta,\psi) = \mathrm{Sin}^k\psi P_n^k \mathrm{Sin}\ k\theta\end{array}\right\} \tag{7.289}$$

These real and imaginary parts are the two Tesseral harmonics.

Fairly detailed description on the procedures for determining the spherical harmonic components are available in Blakely (1996). Materials for further studies on spherical harmonics are available in Macmillan (1958), Kellog (1953) and Ramsay (1940).

8

Direct Current Field Related Potential Problems

In this chapter the basic nature of DC resistivity boundary value problems mostly in one dimensional domain are demonstrated. Nature of the surface and subsurface kernels for an N-layered earth in DC field domain, nature of boundary value problem with cylindrical symmetry needed in borehole geophysics, nature of boundary value problems where resistivity or conductivity varies continuously along the vertical or horizontal direction in the presence of both Laplacian and nonlaplacian terms are discussed. The last two problems are on the potential problems for dipping contacts and layered anisotropic beds.

8.1 Layered Earth Problem in a Direct Current Domain

Direct current field related potentials in a layered earth model can be evaluated as follows. Figure 8.1 shows the geometry of the problem. We assume the earth to be horizontally layered and the number of layers is N. ρ_1, ρ_2, $\rho_3, \ldots\ldots..\rho_N$, are the resistivities of the first, second, third and Nth layer and h_1, h_2, h_3 are their corresponding thicknesses. Thickness of the last layer is assumed to be infinity. Potential at a point due to a point source of Current I will have cylindrical symmetry, over a layered earth therefore, the Laplace equation in cylindrical coordinates are chosen. Since potential is independent of the azimuthal angle, therefore, the Laplace equation in cylindrical coordinates reduce down to

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{8.1}$$

and the general solution of the equation from (7.153) is

$$\phi = \int_0^\infty \left(A\,(m)\,e^{-mz} + B\,(m)\,e^{mz}\right) J_o\,(mr)\,dm. \tag{8.2}$$

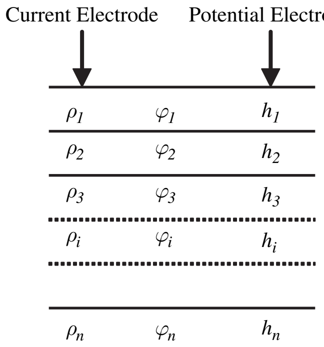


Fig. 8.1. N – Layered earth model

The potential at a point due to a point source at a distance r from the point source of Current I on the surface of the earth (from 7.181) is given by

$$\phi_0 = \frac{I\rho}{2\pi} \cdot \frac{1}{r}. \tag{8.3}$$

While dealing with the air earth boundary, we often use the term half space problem because the electrical resistivity of the air is infinity. The earth has finite resistivity. The solid angle subtended at the current source is 2π. For surface geophysics, when measurements are taken on the surface of the earth, $\frac{I\rho}{2\pi}$ becomes the multiplication factor. In well logging or borehole geophysics when electrode is placed inside a borehole, potential at a point at a distance r from the source is $\frac{I\rho}{4\pi}.\frac{1}{r}$ as shown in the next section. Equation (7.180).

For surface geophysics, the potential due to a point source on the surface of an N-layered earth is

$$\phi_0 = \frac{I\rho}{2\pi} \frac{1}{\left(r^2 + z^2\right)^{1/2}} \tag{8.4}$$

where r and z are respectively the radial and vertical distances from the source and $R = \sqrt{r^2 + z^2}$. Potentials in the different layers can, therefore, be written as

$$\begin{gathered}
\phi_1 = \phi_o + \phi_1' \\
\phi_2 = \phi_o + \phi_2' \\
\dots\dots\dots\dots \\
\dots\dots\dots\dots \\
\underline{\phi_i = \phi_o + \phi_i} \\
\phi_N = \phi_o + \phi_{N'}'
\end{gathered}$$

where ϕ_o is the source potential and ϕ_1', ϕ_2', $\phi_3' \ldots\ldots \phi_N'$ are the perturbation potentials in different media. Potential at the i^{th} media can be written as

$$\phi_i = \frac{I\rho_1}{2\pi}.\frac{1}{(r^2+z^2)^{1/2}} + \int_0^\infty \left(A_i(m)\,e^{-mz} + B_i(m)\,e^{mz}\right) J_o(mr)dm \tag{8.5}$$

where A_i and B_i are the arbitrary constants determined from the boundary conditions. At the air earth boundary, the current density vector $\vec{J}\left(=\sigma\vec{E}\right)$ cannot cut across the air earth boundary because resistivity of air is infinitely high. Therefore, $\frac{1}{\rho_1}\frac{\partial\phi_1}{\partial z} = 0$ at $z = 0$ where ρ_1 and ϕ_1 are respectively the resistivity and potential of the first medium.

$$\begin{aligned}\left(\frac{\partial\phi}{\partial z}\right)_{z=o} &= \left\{\frac{I\rho_1 z}{2\pi\left(r^2+z^2\right)^{1/2}} + \int_o^\infty \left(-A_1e^{-mz} + B_1^{mz}\right) J_o(mr)\,mdm\right\}_{z=o} \\ &= \int_0^\infty (B_1 - A_1)\,J_o(mr)\,mdm = 0 \end{aligned}\tag{8.6}$$

Since the relation is true for any value of r,

$$B_1 = A_1. \tag{8.7}$$

Therefore, expression for potential in the first medium is

$$\phi_1 = \frac{I\rho_1}{2\pi}.\frac{1}{(r^2+z^2)^{1/2}} + \int_0^\infty A_1(m)\left(e^{mz} + e^{-mz}\right) J_o(mr)\,dm. \tag{8.8}$$

Last layer thickness h_N is infinitely high.

In this layer the perturbation potential decays down with depth. Therefore, e^{-mz} will be the appropriate potential function and the expression for potential in the last layer is

$$\phi_N = \frac{I\rho_1}{2\pi}.\frac{1}{(r^2+z^2)^{1/2}} + \int_0^\infty e^{-mz} J_0(mr)\,dm. \tag{8.9}$$

To get the expressions for the unknown functions $A_1(m), A_2(m)\ldots A_N(m)$, it is necessary to apply the boundary conditions (i) $\phi_i = \phi_{i+1}$ and (ii) $J_{n_i} = J_{n_{i+1}}$ at each boundary. Before applying the boundary conditions, it is necessary to bring the source and perturbation potential in the same format. This is an important task in these boundary value problems. This point is highlighted for all the problems discussed in this volume. In this problem, we use the Weber Lipschitz identity to bring the source potential and perturbation potential in the same format, i.e.,

$$\frac{1}{r} = \int_0^\infty e^{-mz} J_o(mr)\,dm. \tag{8.10}$$

Substituting $q = \frac{I\rho}{2\pi}$,

we can write the expression for potentials in different media as,

$$\phi_1 = q\int_0^\infty e^{-mz}\ J_o(mr)\,dm + \int_0^\infty A_1\left(e^{-mz} + e^{mz}\right) J_o(mr)\,dm$$

$$\phi_2 = q\int_0^\infty e^{-mz} J_o(mr)\,dm + \int_0^\alpha \left(A_2\ e^{-mz} + B_2\ e^{mz}\right) J_o(mr)\,dm$$

....

$$\phi_i = q\int_0^\infty e^{-mz} J_o(mr)\,dm + \int_0^\alpha \left(A_i\ e^{-mz} + B_i\ e^{mz}\right) J_o(mr)\,dm$$

....

$$\phi_N = q\int_0^\infty e^{-mz} J_o(mr)\,dm + \int_0^\alpha A_N\ e^{-mz} J_o(mr)\,dm \tag{8.11}$$

Applying the boundary conditions at each boundary i.e.,

$$\phi_i = \phi_{i+1}$$

and

$$\frac{1}{\rho_i}\left(\frac{\partial\phi_i}{\partial z}\right) = \frac{1}{\rho_{i+1}}\left(\frac{\partial\phi_{i+1}}{\partial z}\right)\ \text{at } z = hi, \tag{8.12}$$

one gets

$$\begin{aligned}
&A_1\left(e^{-mh_1} + e^{mh_1}\right) - A_2 e^{-mh_1} - B_2 e^{mh_1} = 0\\
&A_1\rho_2\left(e^{mh_1} - e^{-mh_1}\right) - A_2\rho_1 e^{-mh_1} + B_2\rho_1 e^{mh_1} = q\left(\rho_1 - \rho_2\right) e^{-mh_1}\\
&A_i e^{-mh_i} + B_i\ e^{mh_i} - A_{i+1}\ e^{-mh_i} - B_{i+1}\ e^{mhi} = 0\\
&\rho_{i+1}\left(-A_i\ e^{-mh_i} + B_i\ e^{mh_i}\right) + \rho_i\left(A_{i+1}\ e^{-mh_i} - B_{i+1}\ e^{mh_i}\right)\\
&= q\left(\rho_{i+1} - \rho_i\right) e^{-mh_i}\\
&A_{N-1}\ e^{-mh_{N-1}} + B_{N-1}\ e^{-mh_{N-1}} - A_N\ e^{-mh_{N-1}} = 0\\
&- A_{N-1}\rho_N\ e^{-mh_{N-1}} + B_{N-1}\rho_N\ e^{mh_{N-1}} + A_N\rho_{N-1}\ e^{-mh_{N-1}}\\
&= q\left(\rho_N - \rho_{N-1}\right) e^{mh_{N-1}}
\end{aligned} \tag{8.13}$$

From this system of equations, it is possible to find out potential at a point in any medium for an N-layered earth due to a point source. The factors A_is and B_is are to be determined from the boundary conditions. They are termed as the kernel functions because they carry information about all the 2N-1 layer parameters. Here ρ_1 to ρ_N are the layer resistivities and h_1 to h_{N-1} the thicknesses. Thickness of the Nth layer is infinitely high.

8.1.1 Cramer's Rule

Cramer's rule for evaluation of determinants states that if we have a set of linear equations

$$\begin{aligned}
&\mathrm{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{1n}x_n = k_1}\\
&\mathrm{a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{2n}x_n = k_2}\\
&\mathrm{a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{nn}x_n = k_n}
\end{aligned} \tag{8.14}$$

and if determinant of the coefficients is not equal to zero, i.e., if

$$|\mathrm{a}| = \begin{vmatrix} \mathrm{a_{11}} & \mathrm{a_{12}} & \ldots\ldots.. & \mathrm{a_{1n}} \\ \mathrm{a_{21}} & \mathrm{a_{22}} & \ldots\ldots.. & \mathrm{a_{2n}} \\ \ldots\ldots & \ldots & \ldots\ldots & \ldots \\ \mathrm{a_{n1}} & \mathrm{a_{n2}} & \ldots\ldots & \mathrm{a_{nn}} \end{vmatrix} \neq 0 \tag{8.15}$$

then

$$x_1 = \frac{|D_1|}{|a|}, x_2 = \frac{|D_2|}{|a|} \ldots\ldots\ldots\ldots\ldots x_n = \frac{|D_n|}{|a|} \tag{8.16}$$

where $\mathrm{D_r}$ is the determinant formed by replacing the elements $\mathrm{a_{1r}, a_{2r}, \ldots\ldots\ldots}$ $\mathrm{a_{nr}}$ of the column of $|\mathrm{a}|$ by $\mathrm{k_1, k_2, k_3 \ldots\ldots\ldots k_n}$.

8.1.2 Two Layered Earth Model

For a two layered earth problem, resistivity and thickness of the first layer are ρ_1 and $\mathrm{h_1}$. Resistivity of the second layer is $\rho_2.\mathrm{h_2}$, the thickness of the second layer is infinitely high. Equations for potentials in the two media can easily be obtained from (8.11) and (8.13). Only the kernels $\mathrm{A_1}$ and $\mathrm{A_2}$ are to be evaluated and they can be expressed as the ratio of the two determinants. Let the denominator of the quotient be $\mathrm{D_2}$, thus

$$\mathrm{D_2} = \begin{vmatrix} \mathrm{x^{h_1} + x^{-h_1}} & \mathrm{-x^{h_1}} \\ \rho_2\left(\mathrm{-x^{h_1} + x^{-h_1}}\right) & \rho_1\mathrm{x^{h_1}} \end{vmatrix} \tag{8.17}$$

obtained from (8.13) where $\mathrm{u = x^2 = e^{-2m}}$.

Let

$$\mathrm{D_2 = P_2(x) + H_2(x)}$$

Then

$$\begin{aligned}
P_2(x) &= \begin{vmatrix} x^{h_1} & -x^{h_1} \\ -\rho_2 x^{h_1} & \rho_1 x^{h_1} \end{vmatrix}\\
&= (\rho_1 - \rho_2)x^{h_1} = -(\rho_2 - \rho_1)u^{h_1}\\
&= (\rho_2 + \rho_1)\mathrm{k_1 u}^{h_1} = -(\rho_2 + \rho_1)\mathrm{P_2(u)}
\end{aligned} \tag{8.18}$$

where $\mathrm{P_2(u) = k_1 u}^{h_1}$ and

$$k_1 = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}.$$

k_1 is termed as the reflection factor. Similarly

$$H_2(x) = \begin{vmatrix} x^{-h_1} & -x^{h_1} \\ \rho_2 x^{-h_1} & \rho_1 x^{h_1} \end{vmatrix}$$
$$= (\rho_2 + \rho_1) = (\rho_2 + \rho_1) H_2(u) \tag{8.19}$$

where $H_2(u) = 1.H_2(u)$ is introduced to generate the Kernel function in the form of a recurrence formula for an N-layered earth. For a two-layer earth D_2 becomes

$$D_2 = (\rho_2 + \rho_1)\left[H_2(u) - P_2(u)\right]. \tag{8.20}$$

Surface Kernel A_1

For a two-layer earth, the numerator of the quotient of the (8.13) obtained from (8.11) is given by

$$N_{21} = \begin{vmatrix} 0 & -x^{h_1} \\ (\rho_2 - \rho_1)\, x^{h_1} & \rho_1 x^{h_1} \end{vmatrix}. \tag{8.21}$$

By adding the second column of the determinant with the first column, we get

$$N_{21} = -P_2(x) = (\rho_2 + \rho_1) P_2(u) \tag{8.22}$$

Therefore the Kernel A_1 for a two layer earth is

$$\frac{N_{21}}{D_2} = q \frac{P_2(u)}{H_2(u) - P_2(u)}. \tag{8.23}$$

This is a surface kernel because for surface geophysics where the measurements are taken on the surface of the earth, we are mostly interested in A_1.

Subsurface Kernel A_2

For the Kernel A_2, the numerator N_{22} from (8.13) is given by

$$N_{22} = \begin{bmatrix} x^{h_1} + x^{-h_1} & 0 \\ \rho_2\left(-x^{h_1} + x^{-h_1}\right) & (\rho_2 - \rho_1)\, x^{h_1} \end{bmatrix}. \tag{8.24}$$

Let

$$N_{22} = (N_{22})_1 + (N_{22})_2$$

where

$$(N_{22})_1 = \begin{bmatrix} x^{h_1} & 0 \\ -\rho_2 x^{h_1} & (\rho_2 - \rho_1)\, x^{h_1} \end{bmatrix}$$

and

$$(\mathrm{N}_{22})_2 = \begin{bmatrix} \mathrm{x}^{-\mathrm{h}_1} & 0 \\ -\rho_2 \mathrm{x}^{-\mathrm{h}_1} & (\rho_2 - \rho_1)\,\mathrm{x}^{\mathrm{h}_1} \end{bmatrix}. \tag{8.25}$$

By adding the first column of (N_{22}), to its second column it can be shown that $(\mathrm{N}_{22})_1 = -\mathrm{P}_2(\mathrm{x}) = (\rho_2 + \rho_1)\mathrm{P}_2(\mathrm{u})$ and

$$(\mathrm{N}_{22})_2 = (\rho_2 + \rho_1)\,\mathrm{k}_1 = (\rho_2 + \rho_1)\,[1 + \mathrm{k}_1 - \mathrm{H}_2\,(\mathrm{u})]$$

where $\mathrm{H}_2(\mathrm{u}) = 1$. Therefore, we can write down the expression for subsurface Kernel A_2 as

$$\mathrm{A}_2 = \frac{\mathrm{N}_{22}}{\mathrm{D}_2} = \mathrm{q}\left[\frac{1 + \mathrm{k}_1}{\mathrm{H}_2\,(\mathrm{u}) - \rho_2\,(\mathrm{u})} - 1\right]. \tag{8.26}$$

8.1.3 Three Layered Earth Model

For a three layered earth, the kernels to be determined from the (8.13) are A_1, A_2, B_2 and A_3. Only A_1 is the surface kernel. The denominator D_3 of (8.13) can be written as

$$\mathrm{D}_3 = \begin{bmatrix} \mathrm{x}^{\mathrm{h}_1} + \mathrm{x}^{-\mathrm{h}_1} & -\mathrm{x}^{\mathrm{h}_1} & -\mathrm{x}^{-\mathrm{h}_1} & 0 \\ \rho_2\left(-\mathrm{x}^{\mathrm{h}_1} + \mathrm{x}^{-\mathrm{h}_1}\right) & \rho_1 \mathrm{x}^{\mathrm{h}_1} & -\rho_1 \mathrm{x}^{-\mathrm{h}_1} & 0 \\ 0 & \mathrm{x}^{\mathrm{h}_2} & \mathrm{x}^{-\mathrm{h}_2} & -\mathrm{x}^{\mathrm{h}_2} \\ 0 & -\rho_3 \mathrm{x}^{\mathrm{h}_2} & \rho_3 \mathrm{x}^{-\mathrm{h}_2} & \rho_2 \mathrm{x}^{\mathrm{h}_2} \end{bmatrix}. \tag{8.27}$$

Let $\mathrm{D}_3 = \mathrm{P}_3(\mathrm{x}) + \mathrm{H}_3(\mathrm{x})$
where

$$\mathrm{P}_3\,(\mathrm{x}) = \begin{bmatrix} \mathrm{x}^{\mathrm{h}_1} & -\mathrm{x}^{\mathrm{h}_1} & -\mathrm{x}^{-\mathrm{h}_1} & 0 \\ -\rho_2 \mathrm{x}^{\mathrm{h}_1} & +\rho_1 \mathrm{x}^{\mathrm{h}_1} & -\rho_1 \mathrm{x}^{-\mathrm{h}_1} & 0 \\ 0 & \mathrm{x}^{\mathrm{h}_2} & \mathrm{x}^{-\mathrm{h}_2} & -\mathrm{x}^{\mathrm{h}_2} \\ 0 & -\rho_3 \mathrm{x}^{\mathrm{h}_2} & \rho_3 \mathrm{x}^{-\mathrm{h}_2} & \rho_2 \mathrm{x}^{\mathrm{h}_1} \end{bmatrix} \tag{8.28}$$

and

$$\mathrm{H}_3\,(\mathrm{x}) = \begin{bmatrix} \mathrm{x}^{-\mathrm{h}_1} & -\mathrm{x}^{\mathrm{h}_1} & -\mathrm{x}^{-\mathrm{h}_1} & 0 \\ \rho_2 \mathrm{x}^{-\mathrm{h}_1} & \rho_1 \mathrm{x}^{\mathrm{h}_1} & -\rho_1 \mathrm{x}^{-\mathrm{h}_1} & 0 \\ 0 & \mathrm{x}^{\mathrm{h}_2} & \mathrm{x}^{-\mathrm{h}_2} & -\mathrm{x}^{\mathrm{h}_2} \\ 0 & -\rho_3 \mathrm{x}^{\mathrm{h}_2} & \rho_3 \mathrm{x}^{-\mathrm{h}_2} & \rho_2 \mathrm{x}^{\mathrm{h}_2} \end{bmatrix}. \tag{8.29}$$

Let us now consider the determinant P_3 (x). Multiplying the third row by ρ_2 and adding it to the fourth row, we get

$$P_3\,(x) = \begin{bmatrix} & P_2\,(x) & -x^{-h_1} & 0 \\ & & -\rho_1 x^{-h_1} & 0 \\ 0 & x^{h_2} & x^{-h_2} & -x^{h_2} \\ 0 & -(\rho_3 - \rho_2)\,x^{h_2} & (\rho_3 + \rho_2)\,x^{-h_2} & 0 \end{bmatrix}. \tag{8.30}$$

Developing it with respect to the last column, we get

$$P_3\,(x) = -x^{h_2} \begin{vmatrix} & & -x^{-h_1} \\ & P_2\,(x) & \rho_1 x^{-h_1} \\ 0 & (\rho_3 - \rho_2)\,x^{h_2} & (\rho_3 + \rho_2)\,x^{-h_2} \end{vmatrix}. \tag{8.31}$$

Developing next with respect to the last row, we get

$$\begin{aligned}
P_3(x) &= (\rho_3+\rho_2)\,P_2(x) + (\rho_3-\rho_2)\,x^{2h_2}\begin{vmatrix} x^{h_1} & -x^{-h_1} \\ -\rho_2 x^{h_1} & -\rho_1 x^{-h_1}\end{vmatrix} \\
&= (\rho_3+\rho_2)\left[P_2(x) - k_2 x^{2h_2} H_2\left(x^{-1}\right)\right] \qquad (8.32)
\end{aligned}$$

where

$$H_2\left(x^{-1}\right) = \begin{vmatrix} x^{h_1} & -x^{-h_1} \\ \rho_2 x^{h_1} & \rho_1 x^{-h_1}\end{vmatrix}. \qquad (8.33)$$

It differs from the determinant $H_2\,(x)$ only in the power of x. Here k_2, the reflection factor at the boundary between the second and third layer is equal to

$$k_2 = (\rho_3-\rho_2)/(\rho_3+\rho_2).$$

Therefore,

$$\begin{aligned}
P_3(x) &= -\,(\rho_2+\rho_1)\,(\rho_3+\rho_2)\left[P_2(u) + k_2 u^{h_2} H_2\left(u^{-1}\right)\right] \qquad (8.34) \\
&= -\,(\rho_2+\rho_1)(\rho_3+\rho_2)P_3(u)
\end{aligned}$$

where

$$\begin{aligned}
P_3(u) &= P_2(u) + k_2 u^{h_2} H_2\left(u^{-1}\right) \\
&= k_1 u^{h_1} + k_2 u^{h_2}. \qquad (8.35)
\end{aligned}$$

Similarly

$$\begin{aligned}
H_3(x) &= x^{h_2}\begin{vmatrix} & -x^{-h_1} \\ H_2(x) & -\rho_1 x^{-h_1} \\ 0 - (\rho_3-\rho_2)\,x^{h_2} & (\rho_3+\rho_2)\,x^{-h_2}\end{vmatrix} \\
&= (\rho_3+\rho_2)\,H_2(x) + (\rho_3-\rho_2)\,x^{2h_2}\begin{vmatrix} x^{-h_1} & -x^{-h_1} \\ \rho_2 x^{-h_1} & -\rho_1 x^{-h_1}\end{vmatrix} \\
&= (\rho_3+\rho_2)\left[H_2(x) - k_2 x^{2h_2} P_2\left(x^{-1}\right)\right] \qquad (8.36)
\end{aligned}$$

where

$$\begin{aligned}
P_2\left(x^{-1}\right) &= \begin{vmatrix} x^{-h_1} & -x^{-h_1} \\ -\rho_2 x^{-h_1} & \rho_1 x^{-h_1}\end{vmatrix} \\
&= (\rho_2+\rho_1)(\rho_3+\rho_2)H_3(u). \qquad (8.37)
\end{aligned}$$

Here

$$\begin{aligned}
H_3(u) &= H_2(u) + k_2 u^{h_2} P_2\left(u^{-1}\right) \\
&= 1 + k_1 k_2 u^{h_2-h_1} \qquad (8.38)
\end{aligned}$$

and

$$D_3 = (\rho_2+\rho_1)\,(\rho_3+\rho_2)\,[H_3(u) - P_3(u)]\,. \qquad (8.39)$$

Surface Kernel A_1

For the Kernel A_1, in a three layered earth model the numerator of the quotient from (8.13) is given by

$$N_{31} = \begin{vmatrix} 0 & -x^{-h_1} & -x^{-h_1} & 0 \\ (\rho_2 - \rho_1)\, x^{h_1} & \rho_1 x^{h_1} & -\rho_1 x^{-h_1} & 0 \\ 0 & x^{h_2} & x^{-h_2} & -x^{h_2} \\ (\rho_3 - \rho_2)\, x^{h_2} & -\rho_3 x^{h_2} & \rho_3 x^{-h_2} & \rho_2 x^{h_2} \end{vmatrix} \tag{8.40}$$

The value of this determinant is

$$N_{31} = -P_3(x) = (\rho_2 + \rho_1)(\rho_3 + \rho_2) P_3(u)$$

Therefore

$$A_1 = \frac{N_{31}}{D_3} = q\frac{P_3\,(u)}{H_3\,(u) - P_3\,(u)}. \tag{8.41}$$

Subsurface Kernel A_2

The numerator for the kernel A_2 may be written from (8.13) as

$$\begin{aligned} N_{32} &= \begin{vmatrix} x^{h_1} + x^{-h_1} & 0 & -x^{-h_1} & 0 \\ \rho_2 \left(-x^{h_1} + x^{-h_1}\right) & (\rho_2 - \rho_2)\, x^{h_1} & -\rho_1 x^{-h_1} & 0 \\ 0 & 0 & x^{-h_2} & -x^{h_2} \\ 0 & (\rho_3 - \rho_2)\, x^{h_2} & \rho_3 x^{-h_1} & \rho_2 x^{h_1} \end{vmatrix} \\ &= (N_{32})_1 + (N_{32})_2 \end{aligned} \tag{8.42}$$

where

$$(N_{32})_1 = \begin{vmatrix} x^{h_1} & 0 & -x^{-h_1} & 0 \\ -\rho_2 x^{h_1} & (\rho_2 - \rho_1)\, x^{h_1} & -\rho_1 x^{-h_1} & 0 \\ 0 & 0 & x^{-h_2} & -x^{h_2} \\ 0 & (\rho_3 - \rho_2)\, x^{h_2} & \rho_3 x^{-h_2} & \rho_2 x^{h_1} \end{vmatrix} \tag{8.43}$$

and

$$(N_{32})_2 = \begin{vmatrix} x^{-h_1} & 0 & -x^{-h_1} & 0 \\ \rho_2 x^{-h_1} & (\rho_2 - \rho_1)\, x^{h_1} & -\rho_1 x^{-h_1} & 0 \\ 0 & -x^{h_2} & x^{-h_2} & -x^{h_2} \\ 0 & \rho_3 x^{h_2} & \rho_3 x^{-h_2} & \rho_3 x^{h_2} \end{vmatrix}. \tag{8.44}$$

$(N_{32})_1$ can be shown to be equal to

$$= -P_3(x) = (\rho_2 + \rho_1)(\rho_3 + \rho_2) P_3(u). \tag{8.45}$$

$$(N_{32})_2 = x^{h_2}\begin{vmatrix} & & -x^{-h_1} \\ & (N_{22})_2 & -\rho_1 x^{-h_1} \\ 0 & (P_3-\rho_2)\,x^{h_2} & (\rho_3+\rho_2)\,x^{-h_2}\end{vmatrix}$$

$$\begin{aligned}
&= (\rho_3+\rho_2)\,(\mathrm{N}_{22})_2 - (\rho_3-\rho_2)\,\mathrm{x}^{2\mathrm{h}_2}\begin{vmatrix} \mathrm{x}^{-\mathrm{h}_1} & -\mathrm{x}^{-\mathrm{h}_1} \\ \rho_2\mathrm{x}^{-\mathrm{h}_1} & -\rho_1\mathrm{x}^{-\mathrm{h}_1}\end{vmatrix} \\
&= (\rho_3+\rho_2)\left[(\mathrm{N}_{22})_2 + \mathrm{k}_2\mathrm{x}^{2\mathrm{h}_2}\mathrm{P}_2\left(\mathrm{x}^{-1}\right)\right] \\
&= (\rho_2+\rho_1)\,(\rho_3+\rho_2)\left[1+\mathrm{k}_1 - \mathrm{H}_2\,(\mathrm{u}) - \mathrm{k}_2\mathrm{u}^{\mathrm{h}_2} - \mathrm{P}_2\left(\mathrm{u}^{-1}\right)\right] \\
&= (\rho_2+\rho_1)\,(\rho_3+\rho_2)\,[1+\mathrm{k}_1 - \mathrm{H}_3\,(\mathrm{u})] \\
&= (\rho_2+\rho_1)\,(\rho_3+\rho_2)\,[(1+\mathrm{k}_1)\,\mathrm{B}_{23}\,(\mathrm{u}) - \mathrm{H}_3\,(\mathrm{u})] \qquad (8.46)
\end{aligned}$$

where

$$\mathrm{B}_{23}(\mathrm{u}) = 1.$$

Therefore

$$\mathrm{N}_{32} = (\rho_2+\rho_1)\,(\rho_3+\rho_2)\,[(1+\mathrm{k}_1)\,\mathrm{B}_{23}\,(\mathrm{u}) - \mathrm{H}_3\,(\mathrm{u}) + \mathrm{P}_3\,(\mathrm{u})] \qquad (8.47)$$

and the kernel A_2 can be written as

$$\mathrm{A}_2 = \mathrm{q}\left[\frac{(1+\mathrm{k}_1)\,\mathrm{B}_{23}\,(\mathrm{u})}{\mathrm{H}_3\,(\mathrm{u}) - \mathrm{P}_3\,(\mathrm{u})} - 1\right]. \qquad (8.48)$$

Subsurface Kernel $\mathbf{B_2}$

The determinant N_{33} for the numerator of the Kernel B_2 is

$$\begin{aligned}
\mathrm{N}_{33} &= \begin{vmatrix} \mathrm{x}^{\mathrm{h}_1}+\mathrm{x}^{-\mathrm{h}_1} & -\mathrm{x}^{\mathrm{h}_1} & 0 & 0 \\ \rho_2\left(-\mathrm{x}^{\mathrm{h}_1}+\mathrm{x}^{-\mathrm{h}_1}\right) & \rho_1\mathrm{x}^{\mathrm{h}_1} & (\rho_2-\rho_1)\,\mathrm{x}^{\mathrm{h}_1} & 0 \\ 0 & \mathrm{x}^{\mathrm{h}_2} & 0 & -\mathrm{x}^{\mathrm{h}_2} \\ 0 & -\rho_3\mathrm{x}^{\mathrm{h}_2} & (\rho_3-\rho_2)\,\mathrm{x}^{\mathrm{h}_2} & \rho_2\mathrm{x}^{\mathrm{h}_2}\end{vmatrix} \qquad (8.49) \\
&= (\mathrm{N}_{33})_1 + (\mathrm{N}_{33})_2
\end{aligned}$$

where

$$(\mathrm{N}_{33})_1 = \begin{vmatrix} \mathrm{x}^{\mathrm{h}_1} & -\mathrm{x}^{\mathrm{h}_1} & 0 & 0 \\ -\rho_2\mathrm{x}^{\mathrm{h}_1} & \rho_1\mathrm{x}^{\mathrm{h}_1} & (\rho_2-\rho_1)\,\mathrm{x}^{\mathrm{h}_1} & 0 \\ 0 & \mathrm{x}^{\mathrm{h}_2} & 0 & -\mathrm{x}^{\mathrm{h}_2} \\ 0 & -\rho_3\mathrm{x}^{\mathrm{h}_2} & (\rho_3-\rho_2)\,\mathrm{x}^{\mathrm{h}_2} & \rho_2\mathrm{x}^{\mathrm{h}_2}\end{vmatrix} = 0 \qquad (8.50)$$

$$\begin{aligned}
(\mathrm{N}_{33})_2 &= \begin{vmatrix} \mathrm{x}^{-\mathrm{h}_1} & -\mathrm{x}^{\mathrm{h}_1} & -\mathrm{x}^{\mathrm{h}_1} & 0 \\ \rho_2\mathrm{x}^{-\mathrm{h}_1} & \rho_1\mathrm{x}^{\mathrm{h}_1} & \rho_3\mathrm{x}^{\mathrm{h}_1} & 0 \\ 0 & \mathrm{x}^{\mathrm{h}_2} & 0 & -\mathrm{x}^{\mathrm{h}_2} \\ 0 & -\rho_3\mathrm{x}^{\mathrm{h}_2} & 0 & \rho_2\mathrm{x}^{\mathrm{h}_2}\end{vmatrix} \\
&= (\rho_2+\rho_1)\,(\rho_3+\rho_2)\,(1+\mathrm{k}_1)\,\mathrm{k}_2\mathrm{u}^{\mathrm{h}_2} \qquad (8.51)
\end{aligned}$$

where $\frac{2\rho_2}{\rho_2+\rho_1} = 1+\mathrm{k}_1$. If we take

$$D_{23}(u) = k_2 u^{h_2},$$

we can write

$$N_{33} = (\rho_2 + \rho_1)(\rho_3 + \rho_2)(1 + k_1)D_{23}(u)$$

and

$$B_2 = q\,(1 + k_1)\,\frac{D_{23}\,(u)}{H_3\,(u) - P_3\,(u)}. \tag{8.52}$$

Kernel A_3

The numerator of this kernel N_{34} is

$$N_{34} = \begin{vmatrix} x^{h_1} + x^{-h_1} & -x^{h_1} & -x^{-h_1} & 0 \\ \rho_2\left(-x^{h_1} + x^{-h_1}\right) & \rho_1 x^{h_1} & -\rho_1 x^{-h_1} & (\rho_3 - \rho_1)\,x^{h_1} \\ 0 & x^{h_2} & x^{-h_2} & 0 \\ 0 & -\rho_3 x^{h_2} & \rho_3 x^{-h_2} & (\rho_3 - \rho_2)\,x^{h_2} \end{vmatrix}$$
$$= (\rho_2 + \rho_1)\,(\rho_3 + \rho_2)\,[(1 + k_1)\,(1 + k_2) - H_3\,(u) + P_3\,(u)] \tag{8.53}$$

Hence

$$A_3 = \frac{q\,(1 + k_1)\,(1 + k_2)}{H_3\,(u) - \rho_3\,(u)} - 1. \tag{8.54}$$

8.1.4 General Expressions for the Surface and Subsurface Kernels for an N-Layered Earth

After deriving the expressions for the surface and subsurface kernels upto a three layered earth, we write down the recurrence formulae for the kernels for an N-layered earth and write down the expressions for the potentials in different media.

(i) Surface Kernel A_1 for an N-layered earth can be Written as

$$A_1 = q\frac{P_N\,(u)}{H_N\,(u) - P_N\,(u)} \tag{8.55}$$

where

$$P_N\,(u) = P_{N-1}\,(u) + k_{N-1}u^{h_{N-1}}H_{N-1}\left(u^{-1}\right)$$

and

$$H_N\,(u) = H_{N-1}\,(u) + k_{N-1}u^{h_{N-1}}P_{N-1}\left(u^{-1}\right) \tag{8.56}$$

where

$$P_2(u) = k_1 u^{h1}$$

and

$$H_2(u) = 1.$$

(ii) Kernels A_2 and B_2 of the Second Layer

The expressions for the Kernels A_2 and B_2 are respectively given by

$$A_2 = q\left[\frac{(1+k_1)\,B_{2N}(u)}{H_N(u) - P_N(u)} - 1\right] \tag{8.57}$$

and

$$B_2 = q\left[\frac{(1+k_1)\,D_{2N}(u)}{H_N(u) - P_N(u)}\right] \tag{8.58}$$

where

$$\begin{aligned} B_{22}(u) &= 1 \qquad & D_{22}(u) &= O \\ B_{23}(u) &= 1 \qquad & D_{23}(u) &= k_2 u^{h2} \end{aligned}$$

$$\begin{aligned} B_{24}(u) &= B_{23}(u) + k_3 u^{h3} D_{23}(u^{-1}) \\ D_{24}(u) &= B_{23}(u) + k_3 u^{h3} B_{23}(u^{-1}) \end{aligned} \tag{8.59}$$

$$\begin{aligned} B_{2N}(u) &= B_{2(N-1)}(u) + k_{N-1} u^{h_{N-1}} D_{2(N-1)}\left(u^{-1}\right) \\ D_{2N}(u) &= D_{2(N-1)}(u) + k_{N-1} u^{h_{N-1}} B_{2(N-1)}\left(u^{-1}\right). \end{aligned} \tag{8.60}$$

(iii) The Expressions for the Kernels A_i and B_i for ith Layer can be Written as

$$A_i = q\left[\frac{\left\{\prod_{s=1}^{i-1}(1+k_S)\right\} B_{iN}}{H_N(u) - P_N(u)} - 1\right] \tag{8.61}$$

and

$$B_i = q\left[\frac{\left\{\prod_{s-1}^{i-1}(1+k_S)\right\} D_{iN}}{H_N(u) - P_N(u)}\right] \tag{8.62}$$

where

$$\prod_{S=1}^{i-1}(1+k_s) = (1+k_1)(1+k_2)(1+k_3)\ldots\ldots(1+k_{i-1})$$

and

$$k_s = \frac{\rho_{s+1} - \rho_s}{\rho_{s+1} + \rho_s}.$$

B_{iN} (u) and D_{iN} (u) are related through the recurrence relations given as

$$
\begin{aligned}
B_{ii}(u) &= 1, \qquad D_{ii}(u) = O, \\
B_{i(i+1)}(u) &= B_{ii}(u) + k_1 u^{h_1} D_{ii}\left(u^{-1}\right) \\
D_{i(i+1)}(u) &= D_{ii}(u) + k_1 u^{h_i} B_{ii}\left(u^{-1}\right)
\end{aligned} \tag{8.63}
$$

and

$$
\begin{aligned}
B_{iN}(u) &= B_{i(N-1)}(u) + k_{N-1} u^{h_{N-1}} D_{i(N-1)}\left(u^{-1}\right) \\
D_{iN}(u) &= D_{i(N-1)}(u) + k_{N-1} u^{h_{N-1}} B_{i(N-1)}\left(u^{-1}\right).
\end{aligned} \tag{8.64}
$$

(iv) The Kernel Function A_N for the Nth Layer is given by

$$
A_N = q \left[\frac{\left\{ \prod_{s=o}^{N-1} (1 + k_S) \right\} B_{NN}(u)}{H_N(u) - P_N(u)} - 1 \right] \tag{8.65}
$$

where $B_{NN}(u) = 1$, it should be noted that $B_{N(N-1)}(u) = 1$. For the kernels B_{iN} and D_{iN} where i stands for the ith layer and N stands for the total number of layers.

8.1.5 Kernels in Different Layers for a Five Layered Earth

In this section we are presenting the values of the kernels P_N (u), H_N (u), B_{iN} (u) and D_{iN} (u) for a five layered earth model. One can just write down the expressions of the surface and subsurface kernels for any number of layers.

Let ρ_1, ρ_2, ρ_3, ρ_4, ρ_5 are the restivities of the five layers and h_1, h_2, h_3, h_4 and h_5 are the thicknesses of the five layer where the last layer thickness is infinitely large. From the general expressions for the kernels for an N-layered earth, we can write the expressions for these surface and subsurface kernels as follows:

A. Surface Kernels

(i) For the first layer (i = 1)

(a)

$$
\begin{aligned}
P_2(u) &= k_1 u^{h_1} \\
H_2(u) &= 1
\end{aligned} \tag{8.66}
$$

(b)

$$
\begin{aligned}
P_3(u) &= P_2(u) + k_2 (u)^{h_2} H_2\left(u^{-1}\right) \\
&= k_1 u^{h_1} + k_2 u^{h_2} \\
H_3(u) &= H_2(u) + k_2 (u)^{h_2} P_2\left(u^{-1}\right) \\
&= 1 + k_1 k_2 u^{h_2 - h_1}
\end{aligned} \tag{8.67}
$$

(c)

$$
\begin{aligned}
P_4(u) &= P_3(u) + k_3 u^{h_3} H_3\left(u^{-1}\right) \\
&= k_1 u^{h_1} + k_2 u^{h_2} + k_3 u^{h_3} + k_1 k_2 k_3 u^{h_3 - h_2 + h_1} \\
\mathrm{H}_4(\mathrm{u}) &= \mathrm{H}_3(\mathrm{u}) + \mathrm{k}_3 \mathrm{u}^{\mathrm{h}_3} \mathrm{P}_3\left(\mathrm{u}^{-1}\right) \\
&= 1 + \mathrm{k}_1 \mathrm{k}_2 \mathrm{u}^{\mathrm{h}_2 - \mathrm{h}_1} + \mathrm{k}_1 \mathrm{k}_2 \mathrm{u}^{\mathrm{h}_3 - \mathrm{h}_1} + \mathrm{k}_2 \mathrm{k}_3 \mathrm{u}^{\mathrm{h}_3 - \mathrm{h}_2}
\end{aligned}
\tag{8.68}
$$

(d)

$$
\begin{aligned}
P_5(u) =& P_4(u) + k_4 u^{h_4} H_4\left(u^{-1}\right) \\
&= \mathrm{k}_1 \mathrm{u}^{\mathrm{h}_1} + \mathrm{k}_2 \mathrm{u}^{\mathrm{h}_2} + \mathrm{k}_3 \mathrm{u}^{\mathrm{h}_3} + \mathrm{k}_4 \mathrm{u}^{\mathrm{h}_4} + \mathrm{k}_1 \mathrm{k}_2 \mathrm{u}^{\mathrm{h}_3 - \mathrm{h}_2 + \mathrm{h}_1} \\
&\quad + \mathrm{k}_1 \mathrm{k}_2 \mathrm{k}_4 \mathrm{u}^{\mathrm{h}_4 - \mathrm{h}_2 + \mathrm{h}_1} + \mathrm{k}_1 \mathrm{k}_3 \mathrm{k}_4 \mathrm{u}^{\mathrm{h}_4 - \mathrm{h}_3 + \mathrm{h}_1} \\
&\quad + \mathrm{k}_2 \mathrm{k}_3 \mathrm{k}_4 \mathrm{u}^{\mathrm{h}_4 - \mathrm{h}_3 + \mathrm{h}_2}
\end{aligned}
\tag{8.69}
$$

$$
\begin{aligned}
\mathrm{H}_5(\mathrm{u}) =& \mathrm{H}_4(\mathrm{u}) + \mathrm{k}_4 \mathrm{u}^{\mathrm{h}_4} \mathrm{P}_4\left(\mathrm{u}^{-1}\right) \\
&= 1 + \mathrm{k}_1 \mathrm{k}_2 \mathrm{u}^{\mathrm{h}_2 - \mathrm{h}_1} + \mathrm{k}_1 \mathrm{k}_3 \mathrm{u}^{\mathrm{h}_3 - \mathrm{h}_1} + \mathrm{k}_1 \mathrm{k}_4 \mathrm{u}^{\mathrm{h}_4 - \mathrm{h}_1} \\
&\quad + \mathrm{k}_2 \mathrm{k}_3 \mathrm{u}^{\mathrm{h}_3 - \mathrm{h}_2} + \mathrm{k}_2 \mathrm{k}_4 \mathrm{u}^{\mathrm{h}_4 - \mathrm{h}_2} + \mathrm{k}_3 \mathrm{k}_4 \mathrm{u}^{\mathrm{h}_4 - \mathrm{h}_3} \\
&\quad + \mathrm{k}_1 \mathrm{k}_2 \mathrm{k}_3 \mathrm{k}_4 \mathrm{u}^{\mathrm{h}_4 - \mathrm{h}_3 + \mathrm{h}_2 - \mathrm{h}_1}
\end{aligned}
\tag{8.70}
$$

B. Subsurface Kernels

(i) Kernels for the second layer (i = 2) are

(a)

$$
\begin{aligned}
\mathrm{B}_{22}(\mathrm{u}) &= 1 \\
\mathrm{D}_{22}(\mathrm{u}) &= 0,
\end{aligned}
\tag{8.71}
$$

(b)

$$
\begin{aligned}
\mathrm{B}_{23}(\mathrm{u}) &= \mathrm{B}_{22}(\mathrm{u}) + \mathrm{k}_2 \mathrm{u}^{\mathrm{h}_2} \mathrm{D}_{22}\left(\mathrm{u}^{-1}\right) = 1 \\
\mathrm{D}_{23}(\mathrm{u}) &= \mathrm{D}_{22}(\mathrm{u}) + \mathrm{k}_2 \mathrm{u}^{\mathrm{h}_2} \mathrm{B}_{22}\left(\mathrm{u}^{-1}\right) = \mathrm{k}_2 \mathrm{u}^{\mathrm{h}_2}.
\end{aligned}
\tag{8.72}
$$

(c)

$$
\begin{aligned}
\mathrm{B}_{24}(\mathrm{u}) &= \mathrm{B}_{23}(\mathrm{u}) + \mathrm{k}_3 \mathrm{u}^{\mathrm{h}_3} \mathrm{D}_{23}\left(\mathrm{u}^{-1}\right) \\
&= 1 + \mathrm{k}_2 \mathrm{k}_3 \mathrm{u}^{\mathrm{h}_3 - \mathrm{h}_2}. \\
\mathrm{D}_{24}(\mathrm{u}) &= \mathrm{D}_{23}(\mathrm{u}) + \mathrm{k}_3 \mathrm{u}^{\mathrm{h}_3} \mathrm{B}_{23}\left(\mathrm{u}^{-1}\right) \\
&= \mathrm{k}_2 \mathrm{u}^{\mathrm{h}_2} + \mathrm{k}_3 \mathrm{u}^{\mathrm{h}_3}.
\end{aligned}
\tag{8.73}
$$

(d)

$$\begin{aligned}
B_{25}(u) &= B_{24}(u) + k_4u^4D_{24}\left(u^{-1}\right) \\
&= 1 + k_2k_3u^{h_3-h_2} + k_2k_4u^{h_4-h_2} + k_3k_4u^{h_4-h_3}. \\
D_{25}(u) &= B_{24}(u) + k_4u^{h_4}B_{24}\left(u^{-1}\right) \\
&= k_2u^{h_2} + k_3u^{h_3} + k_4u^{h_4} + k_2k_3k_4u^{h_4-h_3+h_2}.
\end{aligned} \tag{8.74}$$

(ii) Kernels for the third layer (i = 3) are
(a)

$$\begin{aligned}
B_{33}(u) &= 1 \\
D_{33}(u) &= 0.
\end{aligned} \tag{8.75}$$

(b)

$$\begin{aligned}
B_{34}(u) &= B_{33}(u) + k_3u^{h_3}B_{33}\left(u^{-1}\right) = 1 \\
D_{34}(u) &= D_{33}(u) + k_3u^{h_3}B_{33}\left(u^{-1}\right) = k_3u^{h_3}.
\end{aligned} \tag{8.76}$$

(c)

$$\begin{aligned}
B_{35}(u) &= B_{34}(u) + k_4u^{h_4}D_{34}\left(u^{-1}\right) = 1 + k_3k_4u^{h_4-h_3} \\
D_{35}(u) &= B_{34}(u) + k_4u^{h_4}B_{34}\left(u^{-1}\right) = k_3u^{h_3} + k_4u^{h_4}.
\end{aligned} \tag{8.77}$$

(iii) Kernels for the fourth layer (i = 4)
(a)

$$B_{44}(u) = 1 \tag{8.78}$$
$$D_{44}(u) = O,$$

(b)

$$\begin{aligned}
B_{45}(u) &= B_{44}(u) + k_4u^{h_4}D_{44}\left(u^{-1}\right) = 1 \\
D_{45}(u) &= D_{44}(u) + k_4u^{h_4}B_{44}\left(u^{-1}\right) = k_4u^{h_4}..
\end{aligned} \tag{8.79}$$

(v) Kernels for the fifth layer (i = 5)
(a)

$$\begin{aligned}
B_{55}(u) &= 1 \\
D_{55}(u) &= O.
\end{aligned} \tag{8.80}$$

8.1.6 Potentials in Different Media

Substituting the values of surface and subsurface kernels, we can write the expressions for the potentials in different media as

$$\phi_1 = q\int_0^\infty e^{-mz}J_o(mr)\,dm + q\int_0^\infty \frac{P_N(u)}{H_N(u)-P_N(u)}$$
$$\left(e^{-mz}+e^{mz}\right)J_o(mr)\,dm \tag{8.81}$$

$$\phi_2 = q\int_0^\infty e^{-mz}J_o(mr)\,dm + q\int_0^\infty \left[\frac{(1+k_1)\,B_{2N}}{H_N(u)-\rho_N(u)} - 1\right]$$
$$e^{-mz}J_o(mr)\,dm + q\int_0^\infty \left[\frac{(1+k_1)\,D_{2N}(u)}{H_N(u)-P_N(u)}\right] e^{mz}J_o(mr)\,dm$$
$$q\int_0^\infty e^{-mz}J_o(mr)\,dm + q(1+k_1)\int_0^\infty \left[E_2(u)e^{-mz} + F_2(u)e^{mz}\right]J_o(mr)\,dm \tag{8.82}$$

$$\phi_i = q\int_0^\infty e^{-mz}J_o(mr)\,dm + q\left[\prod_{s=1}^{i}(1+k_s)\right]$$
$$\int_0^\infty \left[E_i(u)\,e^{-mz} + F_i(u)\,e^{mz}\right]J_o(mr)\,dm \tag{8.83}$$

$$\phi_N = q\int_0^\infty e^{-mz}J_o(mr)\,dm + q\left[\prod_{s=1}^{N-1}(1+k_s)\right]$$
$$\int_0^\infty E_{NN}e^{-mz}J_o(mr)\,dm \tag{8.84}$$

where

$$E_i(u) = \frac{B_{iN}(u)}{H_N(u)-P_N(u)}$$
$$F_i(u) = \frac{D_{iN}(u)}{H_N(u)-P_N(u)} \tag{8.85}$$

and

$$E_{NN} = \frac{B_{NN}(u)}{H_N(u)-P_N(u)}.$$

Kernel A_1 is regularly used by the geophysicists to compute the apparent resistivity for an one-dimensional N-layered earth problem. Apparent resistivity is defined as the resistivity of a fictitious homogeneous medium, which generates the same potential in the potential electrodes as one gets for an inhomogeneous earth. Apparent resistivity is expressed as $\rho_a = K\frac{\Delta\phi}{I}$, where K is the geometric factor, I is the current and $\Delta\phi$ is the potential difference between the two potential electrodes created by the direct current flow field. For two electrode system, i.e. for one current and one potential electrodes $\Delta\phi = \phi$, because the other potential electrode is kept away by about 10 time the distance between current and potential electrodes. Once the expression for potential for an N layered earth for two-electrode system is obtained, one can compute potential for any other electrode configuration. Potential at a point due to different sources and sinks can be added or subtracted as the case may be (mentioned in Chap. 7). One can get deeper and deeper information about the subsurface by gradual increase in electrode separation. Further details about D.C. resistivity sounding, electrode configuration, concept of apparent resistivity is outside the scope of this book. The readers may consult Keller and Frisehknect (1966), Bhattacharya and Patra (1968), Koefoed(1979) and Zhdanov and Keller (1994).

The constants in potential functions, which are determined applying boundary conditions, are kernel functions because they contain information about all the layer parameters. The expressions of the kernels are shown in (8.66) to (8.85). There may be some applications of the subsurface kernels in surface to borehole measurements. Researchers in Potential theory may be interested to know in detail about the behaviours of these subsurface kernels and their dependence on resistivities at the boundaries and layer thicknesses. Both potentials and normal components of current densities will be continuous across the boundaries. However refraction, reflection and transmission of current lines may be studied with greater details for academic interest. One gets potentials at all the subsurface nodes in finite element and finite difference formulation of 2D/3D structures (Chap. 15). Therefore a comparison can be made between the subsurface potentials obtained analytically and numerically. The researchers generally calibrate their finite element and finite difference source codes against the responses obtained from an analytical solution. In this sector, the subsurface kernels may have some use. Behaviours of the surface kernels can also be studied for electromagnetics and electromagnetic transients.

Deriving recurrence relation of kernel functions for an N-layered earth in one form proposed by Mooney et al (1960) is demonstrated here. There is one more approach for deriving the kernel function in the recurrence form as available in Zhdanov and Keller (1994).

8.2 Potential due to a Point Source in a Borehole with Cylindrical Coaxial Boundaries

Borehole of radius ‘r’ contains drilling mud of resistivity ρ_1. This cylindrical mud column in the borehole (Fig. 8.2) is surrounded by a flushed zone of resistivity ρ_2. It is termed as the flushed zone because most of the pore fluids are flushed out by the mud filtrate due to high pressure maintained in the borehole. Radius of the outer boundary of the flushed zone is r_2. The flushed zone is surrounded by uncontaminated zone of resistivity ρ_3. Radii of the inner and outer boundaries are respectively given by r_2 and infinity. Thickness of the bed is also assumed to be infinitely high so that the effects of shoulder beds are negligibly small. Effect of the transitional invaded zone in between the flushed zone and the uncontaminated zone is discussed in the next section where non-laplacian equation will be involved in generating potential functions. Here the potential ϕ is a function of r and z. In a source free region the potential satisfies Laplace equation,

$$\nabla^2\phi = 0 \tag{8.86}$$

i.e.,

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0$$

because the potential will have the axial symmetry.

Applying the method of separation of variables i.e. $\phi = R(r)Z(z)$, we get

$$\begin{aligned}\frac{\partial\phi}{\partial r} &= \frac{dR}{dr}.Z\\ \frac{\partial^2\phi}{\partial r^2} &= Z\frac{d^2R}{dr^2}\\ \frac{d^2\phi}{\partial z^2} &= R\frac{d^2Z}{dz^2}\end{aligned} \tag{8.87}$$

and (8.87) changes to the form

$$Z\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}.Z + R\frac{d^2z}{dz^2} = 0 \tag{8.88}$$

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{Rr}\frac{dR}{dr} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{8.89}$$

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{Rr}\frac{dR}{dr} = -\frac{1}{Z}\frac{d^2Z}{dz} = m^2. \tag{8.90}$$

We get,

$$\frac{d^2Z}{dz^2} + m^2Z = 0. \tag{8.91}$$

The solutions of this equation are sin mz and cos mz.

The solution of the second equation is

$$I_0(mr) \text{ and } K_0(mr) \tag{8.92}$$

where $I_0(mr)$ and $K_0(mr)$ are respectively the modified Bessel's function of first and second kind and of zero order. Since inside a borehole, the potential will be independent of sign of Z, i.e. with respect to the source point, the potential above or below the source point will be same provided the distance from the source remains the same. Therefore cos mz will be the proper potential function and not sin mz. The general solution of the potential function is

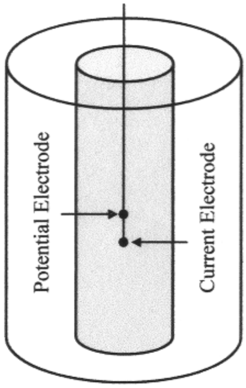


Fig. 8.2. Bore hole D.C. resistivity model with a current and a potential electrode inside a borehole and with cylindrical coaxial boundaries

$$\phi = \int_0^\infty \mathrm{A}(\mathrm{m})\mathrm{J}_0(\mathrm{mr}) \cos \mathrm{mz}\, \mathrm{dm} + \int_0^\infty \mathrm{B}(\mathrm{m})\, \mathrm{K}_0(\mathrm{m}) \cos \mathrm{mz}\, \mathrm{dm} \tag{8.93}$$

where A(m) and B(m) are arbitrary kernel functions.

For this problem, the Weber Lipschitz's integral can be written as

$$\frac{1}{\sqrt{\mathrm{r}^2 + \mathrm{z}^2}} = \frac{2}{\pi} \int_0^\infty \mathrm{K}_0(\mathrm{mr}) \cos \mathrm{mz}\, \mathrm{dm}. \tag{8.94}$$

Therefore, the potential due to a point source is

$$\begin{aligned}\phi_0 &= \frac{\rho_1 \mathrm{I}}{4\pi} . \frac{1}{\mathrm{R}} = \frac{\rho_1 \mathrm{I}}{4\pi} \frac{1}{\sqrt{\mathrm{r}^2 + \mathrm{z}^2}} = \frac{\rho_1 \mathrm{I}}{4\pi} . \frac{2}{\pi} \int_0^\infty \mathrm{K}_0(\mathrm{mr}) \cos \mathrm{mz}\, \mathrm{dm} \\ &= \frac{\rho_1 \mathrm{I}}{2\pi^2} \int_0^\infty \mathrm{K}_0(\mathrm{mz}) \cos \mathrm{mz}\, \mathrm{dm}. \end{aligned} \tag{8.95}$$

Since $\mathrm{I}_0(\mathrm{mr})$ will be the better potential function within the borehole or medium 1. Therefore the potential in medium 1 is

$$\phi_1 = \phi_0 + \phi^{/}$$

where ϕ_0 is the source potential and $\phi^{/}$ is the perturbation potential.

Here

$$\Rightarrow \phi_1 = \frac{\rho_1 \mathrm{I}}{2\pi^2} \left[\int_0^\infty \mathrm{K}_0(\mathrm{mz}) \cos \mathrm{mz}\, \mathrm{dm} + \int_0^\infty \mathrm{C}_0(\mathrm{m}) \mathrm{I}_0(\mathrm{mr}) \cos \mathrm{mz}\, \mathrm{dm} \right] \tag{8.96}$$

where

$$C_0(m) = \frac{2\pi^2}{\rho_2 I} A_0(m).$$

$$\phi_2 = \frac{\rho_2 I}{2\pi^2} \cdot \frac{2\pi^2}{\rho_2 I} \left[\int_0^\infty A_2(m)\, I_0(mr) \cos mz\ dm + \int_0^\infty B_2(m)\, K_0(mr) \cos mz\ dm \right.$$

$$\Rightarrow \phi_2 = \frac{\rho_2 I}{2\pi^2} \int_0^\infty C_2(m) I_0(mr) \cos mz\ dm$$

$$+ \int_0^\infty D_2(m)\, K_0(mr) \cos mz\ dm \qquad (8.97)$$

where

$$C_2(m) = \frac{2\pi^2}{\rho_2 I} A_2(m)$$

$$D_2(m) = \frac{2\pi^2}{\rho_2 I} B_2(m).$$

$$\phi_3 = \frac{\rho_3 I}{2\pi^2} \cdot \frac{2\pi^2}{\rho_3 I} \left[\int_0^\infty B_3(m)\, K_0(mr) \cos mz\ dm \right]$$

$$\Rightarrow \phi_3 = \frac{\rho_3 I}{2\pi} \cdot \int_0^\infty D_3(m)\, K_0(mr) \cos mz\ dm \qquad (8.98)$$

where

$$D_3(m) = \frac{2\pi^2}{\rho_3 I} B_3(m).$$

$$\frac{\partial \phi_2}{\partial r} = \frac{\rho_2 I}{2\pi^2} \left[\int_0^\infty C_2(m)\, I_0^{/}(mr)\, m\ \cos mz\ dm \right.$$

$$+ \int_0^\infty D_2(m)\, K_0^{/}(mr)\, m\ \cos\ mz\ dm$$

$$\Rightarrow \frac{\partial \phi_2}{\partial r} = \frac{\rho_2 I}{2\pi^2} \left[\int_0^\infty C_2(m)\, I_1(mr) - K_1(mr)\, D_2(m) \right] m \cos\ mz\ dm. \qquad (8.99)$$

$$\phi_3 = \frac{\rho_3 I}{2\pi^2}\int_0^\infty D_3(m)\,K_0(mr)\cos\, mz\, dm. \tag{8.100}$$

$$\frac{\partial\phi_3}{\partial r} = \frac{\rho_3 I}{2\pi^2}\int_0^\infty D_3(m)\,K_0^{/}(mr)\,m\cos\, mz\, dm$$

$$= -\frac{\rho_3 I}{2\pi^2}\int_0^\infty D_3(m)\,K_1(mr)\,m\,\cos\, mz\, dm. \tag{8.101}$$

The boundary conditions are:

(i)

$$\phi_1\,|_{r=1} = \phi_2\,|_{r=1}$$
$$\phi_2\,|_{r=r_1} = \phi_3\,|_{r=r_2}\,.$$

The normal component of the current density should be continuous across the boundaries. i. e.,

$$\frac{1}{\rho_1}\left|\frac{\partial\phi_1}{\partial r}\right|_{r=r_1} = \frac{1}{\rho_2}\left|\frac{\partial\phi_1}{\partial r}\right|_{r=r_1}$$
$$\frac{1}{\rho_2}\left|\frac{\partial\phi}{\partial r}\right|_{r=r_2} = \frac{1}{\rho_3}\left|\frac{\partial\phi}{\partial r}\right|_{r=r_2}. \tag{8.102}$$

Assuming the radius of the cylinder to be unity, we put $\phi_1 = \phi_2$ at $r = 1$ and get $\frac{\rho_1 I}{2\pi^2}\left[\int_0^\infty K_0(mr)\cos\, mz\, dm + \int_0^\infty C_1(m)\,I_0(mr)\cos\, mz\, dm\right]$

$$= \frac{\rho_2 I}{2\pi^2}\left[\int_0^\infty C_2(m)\,I_0(mr)\cos\, mz\, dm + \int_0^\infty D_2(m)\,K_0(mr)\cos\, mz\, dm\right.$$

$$\Rightarrow \int_0^\infty \{\rho_1\,[C_1(m)\,I_0(mr) + K_0(mr)] - \rho_2\,[C_2\,(m)\,I_0(mr)$$
$$+D_2(m)\,K_0(mr)]\}\cos mz\; dm = 0. \tag{8.103}$$

Similarly at

$$r = r_2, \qquad \phi_2 = \phi_3.$$

Therefore

$$\frac{\rho_2 I}{2\pi^2}\left[\int_0^\infty C_2(m) I_0(mr_2)\cos mz\ dm + \int_0^\infty D_2(m) K_0(mr_2)\cos mz\ dm\right]$$

$$= \frac{\rho_3 I}{2\pi^2}\left[\int_0^\infty D_3(m) K_0(mr_2)\cos mz\ dm\right]$$

$$\Rightarrow \int_0^\infty \{\rho_2\left[C_2(m)\, I_0(mr_2) + D_2(m)\, K_0(mr_2)\right]$$

$$= \rho_3 D_3(m)\, K_0(mr_2)\}\cos mz\ dm. \tag{8.104}$$

From (8.103) and (8.104), the expressions within the bracket are zero, i.e.,

$$\rho_2 C_2(m)\, I_0(mr_2) + \rho_2 D_2(m)\, K_0(mr_2) - \rho_3 D_3(m)\, K_0(mr_2) = 0.$$

To apply the other boundary condition, i.e., the normal components of the current densities are equal at the boundaries i.e.

$$J_1 = J_2 \text{ at } \quad r = r_1 \quad \text{and} \quad J_2 = J_3 \text{ at } \quad r = r_2$$

$\frac{\partial\phi_1}{\partial r}$, $\frac{\partial\phi_2}{\partial r}$ and $\frac{\partial\phi_3}{\partial r}$ are to be determined. Since $\vec{J} = -\sigma\nabla\phi$,

$$\frac{\partial\phi_1}{\partial r} = \frac{\rho_1 I}{2\pi^2}\left[\int_0^\infty C_1\ (m)\, I_0^{/}(mr)\cos mz\ m\ dm + \int_0^\infty K_0^{/}(mr)\cos mz\ m\ dm.\right.$$

Since

$$I_0^{/}(x) = I_1(x)$$

$$K_0^{/}(x) = -K_1(x)$$

therefore,

$$\frac{\partial\phi_1}{\partial r} = \frac{\rho_1 I}{2\pi^2}\left[\int_0^\infty \{C_1(m)\, I_1(mr) - K_1(mr)\}\, m\cos mz\ dm\right] \tag{8.105}$$

$$\frac{\partial\phi_2}{\partial r} = \frac{\rho_2 I}{2\pi^2}\left[\int_0^\infty C_2(m)\, I_0^{/}(mr)\, m\cos mz\ dm\right.$$

$$\left. + \int_0^\infty D_2(m)\, K_0^{/}(mr)\, m\cos mz\ dm\right]$$

$$\Rightarrow \frac{\partial\phi_2}{\partial r} = \frac{\rho_2 I}{2\pi^2}\left[\int_0^\infty \{C_2(m)\, I_1(mr) - K_1(mr)\, D_2(m)\, m\cos mz\ dm\}\right]. \tag{8.106}$$

Since

$$\phi_3 = \frac{\rho_3 I}{2\pi^2} \int_0^\infty D_3(m) K_0(mr) \cos mz \, dm \tag{8.107}$$

So

$$\begin{aligned} \frac{\partial \phi_3}{\partial r} &= \frac{\rho_3 I}{2\pi^2} \int_0^\infty D_3(m) K_0^{/}(mr)\, m \cos mz \, dm \\ &= -\frac{\rho_3 I}{2\pi^2} \int_0^\infty D_3(m) K_1(mr)\, m \cos mz \, dm. \end{aligned} \tag{8.108}$$

Using the boundary conditions, i.e., at $r = r_1$ $\frac{1}{\rho_1}\frac{\partial \phi_1}{\partial r} = \frac{1}{\rho_2}\frac{\partial \phi_2}{\partial r}$, we get

$$\begin{aligned} &\int_0^\infty [C_1(m) I_1(mr_1) - K_1(mr_1)] \cos mz.m.dm \\ &= \int_0^\infty [C_2(m) I_1(mr_1) + D_2(m) K_1(mr_1)]\, m \cos mz \, dm \\ &\Rightarrow C_1(m) I_1(mr_1) - K_1(mr_1) - C_2(m) I_1(mr_1) + D_2(m) K_1(mr_1) = 0. \end{aligned} \tag{8.109}$$

Similarly at

$$r = r_2, \frac{1}{\rho_2}\frac{\partial \phi_2}{\partial r} = \frac{1}{\rho_3}\frac{\partial \phi_3}{\partial r},$$

hence

$$\begin{aligned} &\int_0^\infty \{C_2(m) I(mr_2) - D_2(m) K_1(mr_2)\} \cos mz \; m \; dm \\ &= -\int_0^\infty D_3(m) K_1(mr_2) \cos mz \; m \; dm \\ &\Rightarrow C_2(m) I_1(mr_2) - D_2(m) K_1(mr_2) + D_3(m) K_1(mr_2) = 0. \end{aligned} \tag{8.110}$$

The four simultaneous equations with four unknowns are given by

$$\rho_1 C_1(m) I_0(mr_1) + \rho_1 K_0(mr_1) - \rho_2 I_0(mr_1) C_2(m) - \rho_2 D_2(m) K_0(mr_1) = 0 \tag{8.111}$$

$$\rho_2 C_2(m) I_0(mr_2) + \rho_2 D_2(m) K_0(mr_2) - \rho_3 D_3(m) K_0(mr_2) = 0 \tag{8.112}$$

$$C_1(m) I_1(mr_2) - K_1(mr_2) - C_2(m) I_1(mr_2) + D_2(m) K_1(mr_2) = 0 \tag{8.113}$$

$$C_2(m) I_1(mr_2) - D_2(m) K_1(mr_2) + D_3(m) K_1(mr_2) = 0. \tag{8.114}$$

In borehole geophysics, we are mostly interested to find out potential on axis of the borehole, i.e. in medium 1. Therefore for computation of potential in borehole, we need to evaluate the kernel C_1 (m) only. However the other kernels C_2 (m), D_2 (m) and D_3 (m) which are of academic interest in Potential Theory can also be evaluated from these sets of equations. Equations 8.111, 8.112, 8.113 and 8.114 can be rewritten in the matrix form and

C_1 (m) can be evaluated using Cramer's rule.

Let

$$C_1(m) = \frac{N}{\Delta},$$

where

$$N = \begin{bmatrix} -\rho_1 K_1(mr) & -\rho_2 I_0(mr) & -\rho_2 K_0(mr) & 0 \\ 0 & \rho_2 I_0(mr_2) & \rho_2 K_0(mr_2) & -\rho_3 K_0(mr_2) \\ K_1(mr) & -I_1(mr) & K_1(mr_1) & 0 \\ 0 & I_1(mr_2) & -K_1(mr_2) & K_1(mr_2) \end{bmatrix}. \tag{8.115}$$

Determinant of the numerator N of C_1 (m) is

$$\begin{aligned} N = {} & (\rho_3 - \rho_2)(\rho_3 - \rho_2) K_0(mr_1) K_0(mr_2) [I_1(mr_2) K_1(mr_2) \\ & - I_1(mr_1) K_1(mr_2)] + (\rho_3 - \rho_2) \rho_3 K_0(mr_2) K_1(mr_2) \\ & + \frac{(\rho_2 - \rho_1)\rho_2}{mr_2} K_0(mr_1) K_1(mr). \end{aligned} \tag{8.116}$$

The denominator Δ is given by

$$\Delta = \begin{bmatrix} \rho_1 I_0(mr_1) & -\rho_2 I_0(mr_1) & -\rho_1 K_0(mr) & 0 \\ 0 & \rho_2 I_0(mr_2) & \rho_2 K_0(mr_2) & -\rho_3 K_0(mr_2) \\ I_1(mr_1) & -I_1(mr_1) & K_1(mr_1) & 0 \\ 0 & I_1(mr_2) & -K_1(mr_2) & K_1(mr_2) \end{bmatrix}. \tag{8.117}$$

After a few steps algebraic simplification while calculating the determinant from (8.117), we get.

$$\begin{aligned} \Delta = {} & [I_0(mr_1) K_1(mr_2) + I_1(mr_2) K_0(mr_1)] \\ & + [I_1(mr_1) K_0(mr_2) + (\mu_2 - 1)(\mu_1 - 1)(mr_2)] \\ & + I(mr_2) K_0(mr_2) + (mr_2)(\mu_2 - 1) \\ & + I_1(mr_1) K_0(mr_1)(mr_1)(\mu_1 - 1) + 1 \end{aligned} \tag{8.118}$$

where $\mu_1 = \frac{\rho_2}{\rho_1}$ and $\mu_2 = \frac{\rho_3}{\rho_2}$. Thus the value of Co (m) $= N/\Delta$ can be computed.

The expression for the potential in a borehole is given by

$$\phi_1 = \frac{\rho_0 I}{2\pi^2} \left[\int_0^\infty K_0(mr) \cos mz \, dm + \int_0^\infty C_0(m) I_0(mr) \cos mz \, dm \right]. \tag{8.119}$$

From Weber – Lipschitz integral, we get

$$\frac{2}{\pi}\int_0^\infty K_0(mr)\cos mz\, dm = \frac{1}{\sqrt{r^2+z^2}}. \tag{8.120}$$

At any point on the z axis where r = 0, we get

$$I_0(mr) = I_0(0) = 1.$$

Therefore,

$$\begin{aligned}\phi_1\,|_{r=0} &= \frac{\rho_1 I}{2\pi^2}\left[\int_0^\infty K_0(mr)\cos mz\, dm\right.\\ &\quad \left. + \int_0^\infty C_0(m)\, I_0(mr)\cos mz\, dm\right.\\ &= \frac{\rho_1 I}{2\pi^2}\left[\frac{\pi}{2Z} + \int_0^\infty C_0(m)\cos mz\, dm\right].\end{aligned} \tag{8.121}$$

This is the expression for potential at a point on the z axis due to a point electrode. Since apparent resistivity of a medium is expressed as $\rho_a = K\frac{\Delta\phi}{I}$. where K is the geometric factor, $\Delta\phi$ is the potential difference between the two potential electrodes and I is the current flowing through the ground. For a two electrode configuration the geometric factor $K = 4\pi L$, where L is the electrode separation and $\Delta\phi = \phi$, i.e. the potential measured at the potential electrode M (Fig. 8.3).

The other potential electrode is theoretically at infinitely and practically is at a distance about ten times the distance between the current and potential electrodes. In practice the other electrodes are on the surface. The apparent resistivity can be expressed as

$$\begin{aligned}\rho_a &= \frac{\rho_1}{2\pi^2}.4\pi L\left[\frac{\pi}{2Z} + \int_0^\infty C_0(m)\cos mz\, dm\right]\\ &\Rightarrow \frac{2\rho_1 L}{\pi}.\frac{\pi}{2Z}\left[\frac{2Z}{\pi} + \int_0^\infty C_0(m)\cos mz\, dm\right]\\ &\Rightarrow \frac{\rho_1 L}{Z}\left[1 + \frac{2Z}{\pi} + \int_0^\infty C_0(m)\cos mz\, dm\right].\end{aligned}$$

Hence

$$\frac{\rho_a}{\rho_1} = 1 + \frac{2L}{\pi}\int_0^\infty C_0(m)\cos\ mL\, dm. \tag{8.122}$$

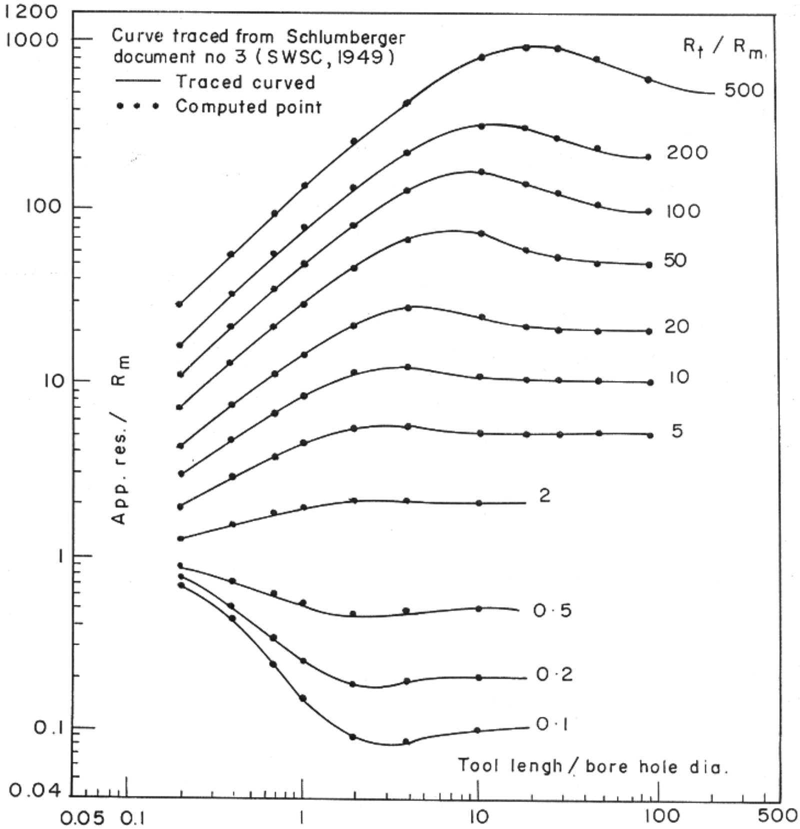


Fig. 8.3. Computed apparent resistivity curve in two electrode configuration bore hole environment

8.3 Potential for a Transitional Earth

8.3.1 Potential for a Medium Where Physical Property Varies Continuously with Distance

This section deals with the boundary value problems where resistivity of a medium is assumed to vary continuously with depth or radial distance. The place where physical property of the earth changes continuously with depth, the nature of the boundary value problem changes because of the inclusion of nonlaplacian terms. The resistivity ρ is assumed to be a function of depth Z. There are some zones in the subsurface where resistivity varies continuously with depth. As for example, in a hard rock area, weathered granite overlies hard and compact granite. It may be a transitional zone where resistivity increases continuously with depth. In a borehole the mud filtrate invades through the porous and permeable zone because of high borehole pressure and an invaded zone is formed surrounding the borehole wall. It is always a transition zone, where resistivity varies continuously along the radial direction.

For a transitional earth ρ is either a function of z or r or both r, z. In this section treatments for both $\rho = f(z)$ and f(r) are given. When $\rho = f(z)$, the starting equation in a source free region and for an isotropic earth is

$$\begin{aligned}
&\operatorname{div} \vec{J} = 0. \\
&\Rightarrow \quad \operatorname{div}(\sigma \operatorname{grad} \phi) = 0 \\
&\Rightarrow \sigma \operatorname{div} \operatorname{grad} \phi + \operatorname{grad} \sigma \operatorname{grad} \phi = 0 \\
&\Rightarrow \nabla^2 \phi + \frac{1}{\sigma} \operatorname{grad} \sigma \operatorname{grad} \phi = 0.
\end{aligned} \tag{8.123}$$

If we take cylindrical co-ordinates with z axis downward for $\phi = f(r, z)$, (8.123) can be written as

$$\sigma \nabla^2 \phi + \frac{\partial \sigma}{\partial r} . \frac{\partial \phi}{\partial r} + \frac{\partial \sigma}{\partial z} . \frac{\partial \phi}{\partial z} = 0. \tag{8.124}$$

If we assume that $\frac{\partial \sigma}{\partial r} = 0$, i, e., the lateral variation in conductivity or resistivity is absent (8.124) changes to the form

$$\nabla^2 \phi + \frac{1}{\sigma} \frac{\partial \sigma}{\partial z} . \frac{\partial \phi}{\partial z} = 0. \tag{8.125}$$

Here conductivity varies continuously with depth (Fig. 8.4). Therefore, we can write (8.125) as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{\sigma} \frac{\partial \sigma}{\partial z} . \frac{\partial \phi}{\partial z} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{8.126}$$

Using the method of separation of variables, we obtain

$$\phi = R\ (r) Z(z)$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - m^2 R = 0 \tag{8.127}$$

$$\frac{d^2 Z}{dz^2} + \frac{1}{\sigma} \frac{d\sigma}{dz} . \frac{dZ}{dz} - m^2 Z = 0. \tag{8.128}$$

To solve this problem, we need to know the value of $\frac{d\sigma}{dz}$. Let the solution of the second (8.128) be Z(m,z). The solution of the first equation is $J_0(mr)$ and $Y_0(mr)$.Since $J_0(mr)$ is a better behaved potential function at $r = 0$, the expression for the potential is

$$\phi = \int_0^{\alpha} A\,(m) Z\,(mz)\, J_0\,(mr)\, dm. \tag{8.129}$$

Now applying the first boundary condition i.e.

$$\frac{\partial \phi}{\partial z} = 0 \ at\ z = 0,$$

we get

$$\frac{\partial\phi}{\partial z} = \int_0^{\alpha} A\,(m)\,Z'\,(mz)\,J_0\,(mr)\,dm \tag{8.130}$$

where

$$Z' = \frac{dZ\,(m,z)}{dz}.$$

From this boundary condition, we can write

$$\int_0^{\alpha} A\,(m)Z'\,(m,0)\,J_0\,(mr)\,dm = 0 \quad \text{at } z = 0. \tag{8.131}$$

Since $\int_0^{\alpha} mJ_0(mr)dm = 0$ always from the theory of Bessel's function, (Watson 1966), therefore

$$A\,(m)\,Z'\,(m,0) = A\,(m)\,Z'\,(m,0) = Cm.m$$

where Cm is a constant.

Hence

$$A\,(m) = \frac{Cm.m}{Z'\,(m,0)} \tag{8.132}$$

and

$$\begin{aligned}\phi\,(r,z) &= \int_0^{\alpha} \frac{Cm.m}{Z'\,(m,0)} Z\,(mz)\,J_0\,(mr)dm \\ &= C\int_0^{\alpha} k\,(mz)\,J_0\,(mr)dm.\end{aligned} \tag{8.133}$$

Here k(m,z) is said to be the kernel of this integral.

For a homogeneous earth of resistivity ρ_0,

$$\phi\,(r,Z) = \frac{\rho_0 I}{2\pi\sqrt{r^2+z^2}} = \frac{I\rho_0}{2\pi}\int_0^{\alpha} e^{-mZ} J_0\,(mr)\,dm. \tag{8.134}$$

(Weber Lipschitz identity)

e^{-mZ} is the appropriate potential function because potential vanishes to zero at infinite depth.

Therefore

$$\phi\,(r,z) = C\int_0^{\alpha} e^{-mZ} J_0\,(mr)\,dm \tag{8.135}$$

where $C = \frac{I\rho_0}{2\pi}$.

Comparing this equation with the present problem, we get

$$\phi(r,z) = \frac{I\rho_0}{2\pi}\int_0^{\alpha} k(m,0)\, J_0(mr)\, dm \tag{8.136}$$

where $\rho_0 = \rho$ right at the surface.

Since in surface geophysics, we take measurements on the surface,

$$\phi(r,0) = \frac{I\rho_0}{2\pi}\int_0^{\infty} k(m,0)\, J_0(m,r)\, dm \tag{8.137}$$

where

$$k(m,0) = \frac{m\, Z(m,0)}{Z'(m,0)}. \tag{8.138}$$

The exact value of ϕ will depend upon Z. Three such cases, are discussed in this section.

Problem 1

Expressions for the potential when the resistivity of the earth decreases, exponentially with depth (Fig. 8.4).

The (8.128) changes to the form

$$\frac{d^2Z}{dz^2} + \beta\frac{dZ}{dz} - m^2Z = 0 \tag{8.139}$$

This is a differential equation of a damped oscillatory circuit. The solution is

$$Z = e^{\frac{-\beta+\sqrt{\beta^2+4\,m^2}}{2}Z}$$

or

$$e^{\frac{-\beta-\sqrt{\beta^2+4\,m^2}}{2}Z}. \tag{8.140}$$

The second value of Z satisfies the proper potential function, hence

Fig. 8.4. A medium where electrical conductivity is varying continuously wit depth

$$Z(m, z) = e^{\frac{-\beta - \sqrt{\beta^2 - 4m^2}}{2} Z} \tag{8.141}$$

or

$$Z'(m, z) = \frac{-\beta - \sqrt{\beta^2 - 4\,m^2}}{2} . Z(m, z) .$$

Since $k(m, z) = \frac{m\ Z(m,z)}{Z'(m,z)}$

$$\phi(r, 0) = \int_0^{\alpha} k(m, 0) J_0(mr)\, dm \tag{8.142}$$

where

$$k(m, o) = m\left[\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} - m^2}\right].$$

Problem 2

When electrical conductivity of the medium varies linearly with depth (Fig. 8.4). Let

$$\sigma = \sigma_0 + \sigma_1 Z = \sigma_0\left(1 + \frac{z}{a}\right)$$

Here σ_0 is the conductivity at the surface. Therefore,

$$\frac{1}{a} = \frac{1}{\sigma_0} . \frac{d\sigma}{dz} = \frac{\sigma_1}{\sigma_0} \tag{8.143}$$

and (8.128) takes the form

$$\frac{d^2 Z}{dz^2} + \frac{\sigma_1}{\sigma_0 + \sigma_1 z} \frac{dZ}{dz} - m^2 Z = 0 \tag{8.144}$$

σ_1 has the unit mho/m and a has the unit of length. If we put

$$\sigma_0 + \sigma_1 Z = \sigma_1 \xi(z) \tag{8.145}$$

or

$$\xi = \frac{\sigma_0}{\sigma_1} + z = z + a$$

Then the (8.144) changes to the form

$$\frac{d^2 Z}{d\xi^2} + \frac{1}{\xi} \frac{dZ}{d\xi} - m^2 Z = 0 \tag{8.146}$$

This is the modified Bessel's equation of zero order and the solutions are $I_0(m\xi)$ and $K_0(m\xi)$. Since $I_0(m\xi)$ is not an appropriate potential function because $I_0 \to \infty$ for $Z \to \infty$, $K_0(m\xi)$ is a better potential function. Therefore the solution of the (8.146) is

$$Z = A\ K_0(m\xi) = A\ K_0(m(z + a)) \tag{8.147}$$

Since

$$k\,(m, 0) = \frac{K\,(ma)}{K_0'\,(ma)}$$

Therefore

$$\phi\,(r, 0) = \frac{1}{2\pi\sigma_0} \int_0^\infty \frac{K_0\,(ma)}{K_0'\,(ma)} J_0\,(mr)\,dm. \tag{8.148}$$

Put $ma = \lambda$

Then

$$\phi\,(r, 0) = \frac{I\rho_0}{2\pi r} \frac{r}{a} \int_0^\infty \frac{K_0\,(\lambda)}{K_0'\,(a)} J_0 \left(\lambda \frac{r}{a}\right) d\lambda. \tag{8.149}$$

Here $\frac{r}{a}$ is dimensionless and λ is the integration variable. For an homogeneous earth

$$\phi_0\,(r, 0) = \frac{I\rho_0}{2\pi r}.$$

Therefore

$$\frac{\phi\,(r, 0)}{\phi_0\,(r, 0)} = \frac{r}{a} \int_0^\infty \frac{K_0\,(\lambda)}{K_0'\,(\lambda)} J_0 \left(\frac{\lambda r}{a}\right) d\lambda \tag{8.150}$$

Problem 3

When the electrical resistivity varies linearly with depth (Fig. 8.4)

Let

$$\rho = \rho_0 + \rho_1\,z = \rho_0(1 + z/b) \tag{8.151}$$

The equation (8.128) takes the form

$$\frac{d^2\,Z}{dz^2} - \frac{\rho_1}{\rho_0 + \rho_1 z} \frac{dZ}{dz} - m^2 Z = 0. \tag{8.152}$$

If we put $\rho_0 + \rho_1 Z = \rho_1 \xi$, $\xi = b + z$, the (8.152) changes to the form

$$\frac{d^2\,Z}{d\xi^2} - \frac{1}{\xi}\frac{dZ}{d\xi} - m^2\,Z = 0. \tag{8.153}$$

Substituting $Z = \xi\eta$, we get

$$\frac{d^2\eta}{d\xi^2} + \frac{1}{\xi}\frac{d\eta}{d\xi} - \left(m^2 + \frac{1}{\xi^2}\right)\eta = 0. \tag{8.154}$$

The solutions of this modified Bessel's equation in $m\xi$ are $I_1(m\xi)$ and $K_1(m\xi)$. Since $K_1(m\xi)$ is a better potential function.

therefore

$$Z = C(z+b)K_1[m(z+b)]. \tag{8.155}$$

Where C is constant and

$$Z'(m,0) = K_1(mb) + mbK_1'(mb)$$

and

$$\begin{aligned}\phi(r,0) &= \frac{I\rho_0}{2\pi}\int_0^\infty \frac{mb\,K_1(mb)}{K_1(mb)+mb\,(K_1'(mb))}J_0(mr)dm \\ &= \frac{I\rho_0}{2\pi}\int_0^\infty \frac{mb\,K_1(mb)}{K_0(mb)}J_0(mr)dm.\end{aligned} \tag{8.156}$$

Here

$$K_0(mb) = K_1(mb) + mb\,K_1'(mb)\ (\text{Watson}, 1966)$$

$$\frac{I\rho_0}{2\pi}\int_0^\infty \frac{K_0'(mb)}{K_0(mb)}J_0(mr)\,dm \tag{8.157}$$

Putting $\lambda = mb$, we get

$$\phi(r,0) = \frac{I\rho_0}{2\pi r}\frac{r}{b}\int_0^\infty \frac{K_0'(\lambda)}{K_0(\lambda)}J_0\left(\lambda\frac{r}{b}\right)d\lambda. \tag{8.158}$$

Problem 4 An Alternative Approach for Solution of ϕ (r,z) When $\rho = f(z)$

When ρ the resistivity of the earth is a function of z only, the governing equation in the Laplacian field is $\operatorname{div}\vec{J} = 0$. It can be written as

$$\nabla.\left(\frac{\nabla\phi(r,z)}{\rho(z)}\right) = 0. \tag{8.159}$$

Since

$$\operatorname{div}\vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0 \tag{8.160}$$

we can write

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\frac{1}{\rho(z)}\vec{E}_x\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho(z)}\vec{E}_y\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\vec{E}_z\right) = 0 \\ &\Rightarrow \frac{1}{\rho(z)}\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right] + \frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\right)\frac{\partial\phi}{\partial z} = 0.\end{aligned}$$

Since

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2} \tag{8.161}$$

when ϕ is independent of the azimuthal angle, hence (8.161) can be written as

$$\frac{1}{\rho(z)}\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}\right)+\frac{1}{\partial z}\left(\frac{1}{\rho(z)}\right)\frac{\partial\phi}{\partial z}=0 \tag{8.162}$$

$$\Rightarrow \quad \frac{1}{\rho(z)}\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}\right)-\frac{1}{(\rho(z))^2}\frac{\partial\rho}{\partial z}\frac{\partial\phi}{\partial z}=0$$

$$\Rightarrow \quad \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}-\frac{\rho'(z)}{\rho(z)}.\frac{\partial\phi}{\partial z}=0. \tag{8.163}$$

Applying the method of separation of variables and assuming $\phi(r_1 z)=R(r)Z(z)$, we get

$$\frac{\partial^2\phi}{\partial r^2}=\frac{\partial^2 R}{\partial r^2}Z(z)\,\frac{\partial^2\phi}{\partial z^2}=R(r)\,\frac{\partial^2 Z}{\partial z^2}$$

$$\frac{\partial\phi}{\partial z}=R(r)\,\frac{\partial R}{\partial z}. \tag{8.164}$$

Therefore

$$\frac{d^2R}{dr^2}Z(z)+\frac{1}{r}\frac{dR}{dr}Z(z)+\frac{d^2Z}{dz^2}R(r)-\frac{\rho'(z)}{\rho(z)}\frac{dZ}{dz}R(r)=0$$

$$\frac{d^2R}{dr^2}Z(z)+\frac{1}{r}\frac{dR}{dr}Z(z)=-\frac{d^2Z}{dz^2}R(r)+\frac{\rho'(z)}{\rho(z)}.\frac{dZ}{dz}R(r)$$

$$\Rightarrow \quad \frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}+\frac{dR}{dr}\right)=-\left(\frac{d^2Z}{dz^2}-\frac{\rho'}{\rho}\frac{dZ}{dz}\right)\frac{1}{Z}=-\lambda^2. \tag{8.165}$$

We get

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\lambda^2R=0$$

$$\Rightarrow \quad r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\lambda^2r^2R=0. \tag{8.166}$$

The second equation is

$$-\left(\frac{d^2Z}{dz^2}-\frac{\rho'}{\rho}\frac{dZ}{dz}\right)+\lambda^2Z=0$$

$$\Rightarrow \quad -\frac{d^2Z}{dz^2}+\frac{\rho'}{\rho}\frac{dZ}{dz}+\lambda^2h=0$$

$$\Rightarrow \quad Z''-\frac{\rho'}{\rho}Z'-\lambda^2h=0. \tag{8.167}$$

Therefore, the general solution of the equation

$$\phi(r,z)=\int_0^\infty A(\lambda)\,Z(z,\lambda)J_0(\lambda r)\,dx. \tag{8.168}$$

Since $\frac{\partial\phi}{\partial z}$ at the air-earth boundary vanishes, we have.

$$\frac{\partial\phi}{\partial z}=\int_0^\infty A(\lambda)\,J_0(\lambda,r)Z'(z,\lambda)\,d\lambda=0 \tag{8.169}$$

where

$$\mathrm{Z}'(\mathrm{z},\lambda)=\frac{\mathrm{dZ}}{\mathrm{d}\lambda}.$$

We get

$$\mathrm{A}(\lambda)=\frac{\mathrm{B}(\lambda)}{\mathrm{Z}'(0,\lambda)}.$$

Since

$$\int_0^\infty \lambda\mathrm{J}_0(\lambda\mathrm{r})\,\mathrm{d}\lambda=0,\phi(r_1,z)=B\int_0^\infty \lambda J_0(\lambda r)\,k(z,\lambda)\,d\lambda \tag{8.170}$$

where

$$\kappa(\mathrm{z},\lambda)=\frac{\mathrm{Z}(\mathrm{z},\lambda)}{\mathrm{Z}'(\mathrm{z},\lambda)}.$$

From homogeneous earth analogy, we get

$$\mathrm{B}=\frac{\mathrm{I}\rho_0}{2\pi}$$

The solution of the equation is

$$\phi(\mathrm{r},\mathrm{z})=\frac{\mathrm{I}\rho_0}{2\pi}\int_0^\infty \lambda\frac{Z(\mathrm{z},\lambda)}{\mathrm{Z}'(0,\lambda)}.\mathrm{J}_0(\lambda,\mathrm{r})\,\mathrm{d}\lambda. \tag{8.171}$$

8.3.2 Potential for a Layered Earth with a Sandwitched Transitional Layer

In this section, a three layer earth is assumed in which the second layer is a transitional layer where $\sigma=\mathrm{f}(\mathrm{Z})$ (Fig. 8.5). The conductivity of the first and second layer are respectively σ_1 and σ_3. Thickness of the first and second layer are respectively h_1 and h_2; thickness of the third layer is infinitely high. Laplace equation is satisfied in the first and third layer. For the second layer the non Laplacian governing equation is (Mallick and Roy, 1968).

$$\nabla^2\phi + \frac{1}{\sigma}\frac{\partial\sigma}{\partial z}.\frac{\partial\phi}{\partial z} = 0. \tag{8.172}$$

In the transitional layer, the conductivity is assumed to vary linearly with depth; so we get

$$\sigma = \sigma_1 + \frac{\sigma_2 - \sigma_1}{h_2 - h_1}(Z - h_1), \tag{8.173}$$

$\sigma = \sigma_1$ at $Z = h_1$ and σ_2 at $Z = h$.

The guiding equations for solving the potential problems for the first, second and third layers are respectively given by

$$\begin{aligned} \nabla^2\phi_1 &= 0 \\ \nabla^2\phi + \frac{1}{\sigma}\frac{\partial\sigma}{\partial z}.\frac{\partial\phi}{\partial z} &= 0 \\ \nabla^2\phi_2 &= 0 \end{aligned} \tag{8.174}$$

where ϕ_1, ϕ and ϕ_2 are the potentials in the three regions

Here

$$\phi_1 = q\int_0^\infty e^{-mz}J_0(mr)\,dm + \int_0^\infty A(m)\left[e^{-mz} + e^{mz}\right]J_0(mr)\,dm \tag{8.175}$$

$$\phi_2 = \int_0^\infty D(m)\,e^{-mz}J_0(mr)dm \tag{8.176}$$

In the transitional zone

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial Z^2} + \frac{\alpha}{\sigma_1 + \alpha(z - h_1)}\frac{\partial\phi}{\partial Z} = 0 \tag{8.177}$$

where

$$\alpha = \frac{(\sigma_2 - \sigma_1)}{(h_2 - h_1)}.$$

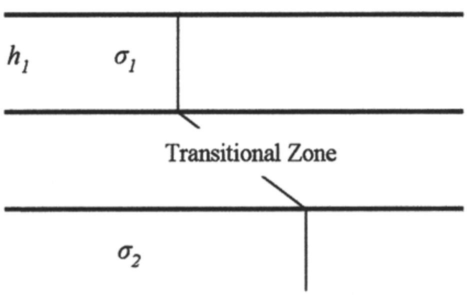


Fig. 8.5. A model of a three layer earth with middle layer as a transitional layer

Applying the method of separation of variables $\phi = R(r)\,Z\,(z)$, one gets,

$$\frac{dR^2}{dr^2} + \frac{1}{r}\frac{dR}{dr} + m^2 R = 0 \tag{8.178}$$

and

$$\frac{d^2Z}{dz^2} + \frac{\alpha}{\sigma_1 + \alpha\,(Z - h_1)}\frac{dZ}{dz} - m^2 Z = 0. \tag{8.179}$$

In the (8.179), we substitute

$$\tau = \sigma_1 + \alpha(Z - h_1).$$

One gets

$$\frac{d\tau}{dz} = \alpha.$$

Therefore

$$\frac{dZ}{dz} = \frac{dZ}{d\tau}.\frac{d\tau}{dz} = \alpha\frac{dZ}{d\tau}$$

and (8.179) changes to the form

$$\frac{d^2 Z}{d\tau^2} + \frac{1}{\tau}\frac{dZ}{d\tau} - \frac{m^2}{\alpha^2}Z = 0. \tag{8.180}$$

The solution of (8.180) is

$$\phi = \int_0^\infty (B\,(m)\,I_o\,(m\tau/\alpha) + C\,(m)\,K_o\,(m\tau/\alpha))J_o(mr)dm. \tag{8.181}$$

Now the constants A (m), B (m), C (m) and D (m) are to be evaluated from the boundary conditions.

$$\phi_1 = \phi \text{ and } \frac{\partial\phi_1}{\partial Z} = \frac{\partial\phi}{\partial Z} \text{ at } Z = h_1$$

$$\phi_2 = \phi \text{ and } \frac{\partial\phi}{\partial Z} = \frac{\partial\phi_2}{\partial Z} \text{ at } Z = h_2.$$

Applying the boundary conditions, we get

$$qe^{-mh_1} + A\,(m)\left[e^{-mh_1} + e^{mh_1}\right] = B\,(m)\,I_0\left[\frac{m\tau_1}{\alpha}\right] + C\,(m)\,K_0\left[\frac{m\tau_1}{\alpha}\right] \tag{8.182}$$

$$-qe^{-mh_1} + A\,(m)\left[-e^{-mh_1} + e^{mh_1}\right] = B\,(m)\,I_1\left[\frac{m\tau_1}{\alpha}\right] - C\,(m)\,K_1\left[\frac{m\tau_1}{\alpha}\right] \tag{8.183}$$

$$D\,(m)\,e^{-mh_2} = B\,(m)\,I_0\left[\frac{m\tau_2}{\alpha}\right] + C\,(m)\,K_0\left[\frac{m\tau_2}{\alpha}\right] \tag{8.184}$$

$$-D\,(m)\,e^{-mh_2} = B\,(m)\,I_1\left[\frac{m\tau_2}{\alpha}\right] - C\,(m)\,K_1\left[\frac{m\tau_2}{\alpha}\right] \tag{8.185}$$

For surface geophysics, we are interested in A (m). It can be shown that

$$A(m) = \frac{\frac{\gamma-1}{\gamma+1}e^{-2mh_1}}{1 - \frac{\gamma-1}{\gamma+1}e^{-2mh_1}} \tag{8.186}$$

where

$$\gamma = \frac{K_0\left[\frac{m\sigma_1}{\alpha}\right] + uI_0\left[\frac{m\sigma_1}{\alpha}\right]}{K_1\left[\frac{m\sigma_1}{\alpha}\right] - uI_1\left[\frac{m\sigma_1}{\alpha}\right]}$$

here

$$u = -\frac{K_0\left[\frac{m\sigma_2}{\alpha}\right] + K_1\left[\frac{m\sigma_2}{\alpha}\right]}{I_0\left[\frac{m\sigma_2}{\alpha}\right] + I_1\left[\frac{m\sigma_2}{\alpha}\right]}.$$

Substituting the value of A (m) and putting z = 0, one gets the value of potential at a point on the surface of the earth as

$$\phi_1(r,0) = \frac{I}{2\pi\sigma_1}\left\{\frac{1}{r} + 2\int_0^\infty \frac{\frac{\gamma-1}{\gamma+1}e^{-2mh_1}}{1 - \frac{\gamma-1}{\gamma+1}e^{-2mh_1}} J_0(mr)dm\right\}. \tag{8.187}$$

8.3.3 Potential with Media Having Coaxial Cylindrical Symmetry with a Transitional Layer in Between

Figure (8.6) explains the geometry of the problem. In borehole geophysics an one dimensional problem will have cylindrical boundaries. The maximum numbers of layers generally created are five. They include (i) borehole mud, (ii) mud cake, (iii) flushed zone, (iv) invaded zone and (v) uncontaminated zone. Figure (8.6) explains the presence of these different zones. However the readers should consult any text book on borehole geophysics to understand these well logging terminologies. In borehole geometry, these layers are coaxial cylinders. The effect of mud cake is negligible in the normal (two electrode) or lateral (three electrode) log response curves. Therefore, the problem is presented as a four layer problem. However, it can be extended to any number of layers. (Dutta, 1993, Roy and Dutta 1994).This problem is presented to show that a differential equation, not having an easy solution, can be handled using Frobeneous power series.

The problem is framed as a boundary value problem with a point source of current on the borehole axis having cylindrical coaxial layered media with infinite bed thickness. Invaded zone is present as one of those layers as a transitional zone where nonlaplacian term appears. Potentials satisfy Laplace equation in all the homogenous and isotropic media where resistivities are assumed to be constant.

Potential in the medium 1 (i.e. in the borehole) is given by

$$\phi_1 = \frac{R_m I}{2\pi^2 r_m}\int_0^\infty K_0(mr)\cos\ mz\ dm + \int_0^\infty C_1(m)\, I_0(mr)\cos\ mz\ dm. \tag{8.188}$$

(Dakhnov, 1962) (Sect. 8.2).

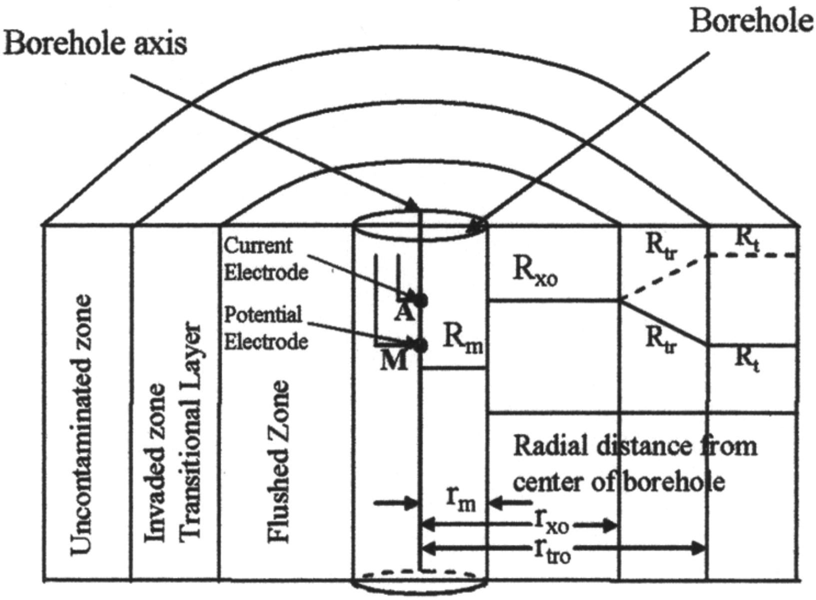


Fig. 8.6. A cross section of a borehole with transitional invaded zone

The first integral is a potential due to a point source. This expression is obtained after applying the Weber Lipschitz identity. The second integral is the perturbation potential. I_0 (mr) and K_0 (mr) are respectively the modified Bessel functions of the first and second kind of order zero. Here the parameter R_m is the resistivity of the borehole mud and r_m is the radius of the borehole. m is the integration variable, z is a point of observation in the assumed borehole, r is the radial distance from the axis of the borehole. C_1 (m) is the kernel function to be evaluated applying suitable boundary conditions. Potentials in the second and fourth media are given by

$$\phi_2 = \int_0^\infty C_2\,(m)\,K_0\,(mr)\cos\ mz\ dm + \int_0^\infty C_3\,(m)\,I_0\,(mr)\cos\ mz\ dm \qquad (8.189)$$

and

$$\phi_4 = \int_0^\infty C_6\,(m)\,K_0\,(mr)\cos\ mz\ dm. \qquad (8.190)$$

For ϕ_4 there is no integral involving I_0 (mr) which tend to become infinite for large values of the argument r.

Potential in the transitional invaded zone is derived from,

$$\phi^2\phi_3 + \frac{1}{\sigma_{tr}}\mathrm{grad}\,\sigma_{tr}\ \mathrm{grad}\,\phi_3 = 0 \qquad (8.191)$$

where, σ_{tr} is the conductivity of the transition zone and grad σ_{tr} is non zero. Simplification of the differential equation by a suitable substitution, like previous problems, was not possible in this case. Two different situations can exist for the transition zone i.e., i) when $R_t < R_{x0}$ and ii) $R_t > R_{x0}$. Here R_t is the resistivity ofthe uncontaminated zone or zone 4 and R_{x0} is the resistivity of the flushed zone or zone 2. Resistivities of these two zones are assumed to be fixed. For these two cases, two different modes of solution are presented. In the first case resistivity in the transition zone is varying linearly with radial distance and in the second case conductivity has a linear gradient with radial distance as shown in Part I and Part II.

Part 1: Potential Function in Transition Zone where $R_t < R_{x0}$

Since the potential is independent of the azimuthal angle, (8.191) reduces to the form, taking $\sigma_{tr} = 1/R_{tr}$.

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_3}{\partial r} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{R_{tr}}\frac{\partial R_{tr}}{\partial r}\frac{\partial\phi_3}{\partial r} = 0 \tag{8.192}$$

where, R_{tr} is the resistivity of the transition zone.

Applying the method of separation of variables, i.e., $\phi_3 = R\,(r)\,Z\,(z)$, (8.192) reduces to

$$\frac{d^2Z}{dz^2} + m^2Z = 0 \tag{8.193}$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{1}{R_{tr}}\frac{dR_{tr}}{dr}\frac{dR}{dr} - m^2R = 0. \tag{8.194}$$

Equation (8.194) can be modified assuming linear transition in the invaded zone, i.e.,

$$R_{tr} = R_{x0} + \frac{R_t - R_{x0}}{r_{tr} - r_{x0}}\,(r - r_{x0}) \tag{8.195}$$

where, R_{tr}, the resistivity of the transition zone or invaded zone is a function of radial distance r. r_{tr} and r_{x0} are the radial distances of the boundaries between (i) uncontaminated zone and invaded zone and (ii) invaded zone and flushed zone, we write

$$R_{tr} = a_1 + \alpha r$$

where, $a_1 = R_{x0} - \alpha r_{x0}$ and $\alpha = \frac{R_t - R_{x0}}{r_{tr} - r_{x0}}$.

After a few steps of algebraic simplifications, (8.194) reduces to the form

$$\frac{d^2R}{dr^2} + \frac{a}{r\,(a + r)}\frac{dR}{dr} - m^2R = 0. \tag{8.196}$$

Here a $(= a_1/\alpha)$ is also a constant. Equation (8.195) is solved by Frobenious extended power series assuming

$$R = \sum_{p=0}^{\infty} A_p r^{p+q} \tag{8.197}$$

(Kreyszig, 1985; Ayres, 1972). In this case (8.194), at $r = 0$, the coefficient of dR/dr is not analytic and the extended power series method is considered for solution.

Equation (8.195) is rewritten as

$$r^2R'' + ar\,R'' + a\,R' - m^2r^2R - a\,m^2r\,R = 0. \tag{8.198}$$

Substituting the values of R'', R' and R obtained from (8.197) and (8.198), where R'' and R' are the second and first derivatives of R with respect to r, one gets,

$$\begin{aligned}&\sum_{p=0}^{\infty}(p+q)\,(p+q-1)\,A_p r^{p+q} + a\sum_{p=0}^{\infty}(p+q)\,(p+q-1)\,A_p r^{(p+q-1)}\\ &\quad + a\sum_{p=0}^{\infty}(p+q)\,A_p r^{(p+q-1)} - m^2\sum_{p=0}^{\infty}A_p r^{(p+q+2)} - am^2\sum_{p=0}^{\infty}A_p r^{(p+q+1)} = 0.\end{aligned} \tag{8.199}$$

The normal procedure is to equate the smallest power of r to zero and taking $p = 0$, we get

$$Aq(q-1)A_0 + aq\,A_0 = 0. \tag{8.200}$$

Since a and A_0 are not zero, hence q has distinct double roots and both of them are zero. In order to find out the generalised form of the expression for the coefficient A_p, we equate the coefficients of r^{q+r-1} to obtain

$$\begin{aligned}&(q+p-1)(q+p-2)A_{p-1} + a(q+p-1)(q+p)A_p + a(q+p)A_p\\ &- m^2A_{p-3} - am^2A_{p-2} = 0.\end{aligned} \tag{8.201}$$

From (8.201), one gets

$$A_p = \frac{1}{a\,(q+p)^2}\left[m^2A_{p-3} + am^2A_{p-2} - (q+p-1)\,(q+p-2)\,A_{p-1}\right]. \tag{8.202}$$

Since (8.196) can be solved by taking the solution in the form of a Frobenious extended power series. One of the solutions will be the (8.197) provided we can determine the coefficient A_p. Since the indicial equation has two distinct roots, (8.194) must have a second solution.

$$R_2 = \frac{\partial R_1}{\partial q}$$

(Ayres 1972, Kreyszig 1985)

$$= \frac{\partial}{\partial q}\left[r^q \sum_{p=0}^{\infty} A_p r^p\right]$$

$$= r^q \ln r \sum_{p=0}^{\infty} A_p r^p + r^q \sum_{p=0}^{\infty} \frac{\partial A_p}{\partial q} r^p. \tag{8.203}$$

Therefore, $q = 0$, the two solutions are

$$T_1(m, r) = R_1 \left.\right|_{q=0} = \sum_{p=0}^{\infty} A_p r^p \tag{8.204}$$

$$T_2(m, r) = R_2 \left.\right|_{q=0} = \ln\ R_1 + \sum_{p=0}^{\infty} K_p r^p \tag{8.205}$$

where

$$K_p = \frac{\partial A_p}{\partial q}.$$

(Ayres 1972, Kreyszig 1985)

The generalised expression for K_p can be shown, after a few steps of simple differentiation and algebraic simplification. Taking $A_0 = 1$, we get

$$K_1 = \frac{\partial A_1}{\partial q} = \frac{\partial}{\partial q}\left[-\frac{q(q-1)}{a(q+1)^2}\right]$$

$$= -\frac{1}{a}\frac{(q+1)\left(2q^2-1\right) - 2q(q-1)(q+1)}{(q+1)^4}. \tag{8.206}$$

Therefore at $q = 0$

$$K_1 = 1/a$$

and

$$K_2 = \frac{\partial A_2}{\partial q}$$

$$= -\frac{2}{a(q+2)^3}\left[am^2 + \frac{q(q-1)}{a(q+1)}\right] + \frac{1}{a(q+2)^2}$$

$$\left[-\frac{1}{a}\frac{(q+1)\left(3q^2-2q\right) - q^2(q-1)}{(q+1)^2}\right]. \tag{8.207}$$

At $q = 0$

$$K_2 = -\frac{m^2}{4}$$

$$K_3 = \frac{\partial A_3}{\partial q}$$

$$= -\frac{2}{a(q+3)^3}\left[am^2 A_1 + m^2 A_0 - (q+2)(q+1)A_2\right]$$

$$+\frac{1}{a\,(q+3)^2}\left[am^2K_1-(q+2)\,(q+1)\,K_2-(2q+3)\,A_2\right]$$
$$K_3=-\frac{1}{a}\frac{m^2}{27}+\frac{1}{a}\frac{m^2}{12}. \tag{8.208}$$

Generalised expression for the term K_p for $p \geq 4$ can be formed taking $q=0$, as

$$K_p=-\frac{2}{ap^3}C_p+\left[m^2K_{p-3}+am^2K_{p-2}-(p-1)\,(p-2)\,K_{p-1}\right.$$
$$\left.-(2p-3)\,A_{p-1}\right]/ap^2 \tag{8.209}$$

where,

$$C_p=\left[m^2A_{p-3}+am^2A_{p-2}-(p-1)\,(p-2)\,A_{p-1}\right].$$

Part 2: Potential Function an Borehole Transitional Earth Model zone where $R_t > R_{X0}$

In this parametric setup where $R_t > R_{X0}$ the value of **a** in the (8.199) may assume a zero value for certain value of r, and the coefficient of dR/dr becomes zero. To avoid this algebraic singularity, conductivity is considered to vary linearly in this model ($R_t > R_{X0}$) in case of transitional invaded zone.

Potential expression in the transition zone (8.194) can be expressed as

$$\frac{\partial^2\phi_3}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_3}{\partial r}+\frac{1}{\sigma_{tr}}\frac{\partial\sigma_{tr}}{\partial r}\frac{\partial\phi_3}{\partial r}+\frac{\partial^2\phi_3}{\partial z^2}=0 \tag{8.210}$$

which is independent of the azimuthal angle. Applying the method of separation of variables, (8.210) reduces to

$$\frac{d^2Z}{dz^2}+m^2z=0 \tag{8.211}$$
$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\frac{1}{\sigma_{tr}}\frac{d\sigma_{tr}}{dr}\frac{dR}{dr}-m^2R=0. \tag{8.212}$$

Equations (8.194) and (8.212) are same as there is no heterogeneity in the z direction. Assuming linear transition in electrical conductivity in the transition zone i.e.,

$$\sigma_{tr}=\sigma_{x0}+\frac{\sigma_t-\sigma_{x0}}{r_{tr}-r_{x0}}\,(r-r_{x0}) \tag{8.213}$$

where, σ_{tr} is the conductivity of the transition zone and a function of radial distance. σ_t and σ_{x0} are the conductivity of the uncontaminated and the flushed zones respectively.

We write $\sigma_{tr}=a_1+\sigma r$

where,

$$a_1=\sigma_{x0}-\sigma r_{x0}$$

and

$$\sigma = \frac{\sigma_t - \sigma_{x0}}{r_{tr} - r_{x0}}.$$

With algebraic simplifications (taking $a = a_1/\sigma_t$) one gets,

$$\frac{d^2R}{dr^2} + \frac{a + 2r}{r\,(a + r)}\frac{dR}{dr} - m^2R = 0 \tag{8.214}$$

$$\Rightarrow \quad r^2\frac{d^2R}{dr^2} + ar\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + a\frac{dR}{dr} - m^2r^2R - m^2arR = 0$$

$$\Rightarrow r^2R'' + ar\ R'' + 2r\ R' + aR' - m^2ar\ R = 0. \tag{8.215}$$

Here also we assume the solution in an extended power series form, as at $r = 0$ the coefficient of dR/dr is not analytic,

$$R = \sum_{p=0}^{\infty} A_p r^{p+q}.$$

Substituting the values of $R''R'$ and R in the (8.214) one gets,

$$\sum_{p=0}^{\infty}(p+q)\,(p+q-1)\,A_p r^{p+q} + a\sum_{p=0}^{\infty}(p+q)\,(p+q-1)\,A_p r^{(p+q-1)}$$
$$+\,2\sum_{p=0}^{\infty}(p+q)\,A_p r^{(p+q)} + a\sum_{p=0}^{\infty}(p+q)\,A_p r^{(p+q-1)} - m^2\sum_{p=0}^{\infty}A_p r^{(p+q+2)}$$
$$-\,am^2\sum_{0}^{\infty}A_p r^{(p+q+1)} = 0. \tag{8.216}$$

Equating the coefficient of r^{q-1}, taking $p = 0$ the above equation reduces to $a\,q^2A_0 = 0$

As a and A_0 cannot be zero, hence $q = 0$. It implies that q has distinct double roots.

Putting the value of q in (8.215) and equating the coefficient of r^{p-1}, one gets the generating expression of A_P

i.e.,

$$A_p = \frac{1}{ap^2}\left[m^2A_{p-3} + am^2A_{p-2} - (p-1)\,pA_{p-1}\right] \tag{8.217}$$

We can write the solutions of the differential (8.215) as, (taking $q = 0$)

$$T_1\,(m, r) = r_1 = R\,|_{q=0}$$
$$T_2\,(m, r) = R_2 = R_1 \ln r + \sum_{p=0}^{\infty} K_p r^p. \tag{8.218}$$

Since the indicial equation has distinct double roots, we have two solutions of the differential equation. First solution is T_1 (m, r) provided we can determine the coefficient A_p. Second solution of differential equation is T_2 (m, r) and the coefficient K_p has to be determined. Generalised expression of the coefficient term can be found out with simple mathematical steps.

From (8.215), we get

$$R_2' = R_1' \ln r + R_1/r + \sum_{p=1}^{\infty} p\, K_p r^{p-1} \tag{8.219}$$

$$R_2'' = R_1'' \ln r + 2R_1'/r - R_1/r^2 + \sum_{p=0}^{\infty} p\,(p-1)\, K_p r^{p-2} \tag{8.220}$$

Putting these values in the differential (8.219 and 8.220) and after some mathematical steps, terms for K can be found out after equating the coefficients of different powers of r.

Equating the coefficients of r^0, r^1, r^2 and r^3, we respectively get

(i)

$$K_1 = -A_0/a \tag{8.221}$$

(ii)

$$K_2 = -\left(2K_1 + 4aA_2\right)/a_2^2 \tag{8.222}$$

(iii)

$$K_3 = \left(am^2K_1 - 6K_2 - 5A_2 - 6aA_3\right)/a_3^2 \tag{8.223}$$

(iv)

$$K_4 = (m^2K_1 + am^2K_2 - 4(4-1)K_3 - 7A_3 - 8aA_4)/a4^2. \tag{8.224}$$

The generalised solution for potential of the transitional invaded zone

$$K_p = (m^2K_{p-3} + am^2K_{p-2} - p(p-1)K_{p-1} - (2p-1)A_{p-1} - 2ap\ A_p)/ap^2 \tag{8.225}$$

(for both the models) can be written as,

$$\phi_3 = \int_0^{\infty} \left[C_4'(m)\, T_1(m,r) + C_5'(m)\, T_2(m,r)\right]\left[F\cos mz + G\sin mz\right] dm \tag{8.226}$$

Here T_1 (m, r) and T_2 (m, r) are the two solutions of the second order differential equation. T_1 (m, r) and T_2 (m, r) the two solutions, are entirely different for the two different cases (i.e., $R_t > R_{x0}$ and $R_t < R_{x0}$). This can be easily verified from the (8.197) and (8.215). C_4' (m), C_5' (m), F and G are arbitrary constants generally determined applying the boundary conditions. In this problem, the potential is independent of the sign of z therefore sin mz term is neglected.

Equation (8.226) reduces to the form

$$\phi_3 = \int_0^\infty C_4(m)T_1(m, r)\cos mz\ dm + \int_0^\infty C_5(m)T_2(m, r)\cos mz\ dm \quad (8.227)$$

where $C_4(m) = C_4'(m)F$ and $C_5(m) = C_5'(m)G$

The general expressions for the potential in different media are

$$\phi_1 = C\int_0^\infty K_0(mr)\cos\ mz\ dm + \int_0^\infty C_1(m)I_0(mr)\cos\ mz\ dm \quad (8.228)$$

where

$$C = \frac{R_m I}{2\pi^2 r_m}.$$

$$\phi_2 = \int_0^\infty C_2(m)I_0(mr)\cos\ mz\ dm + \int_0^\infty C_3(m)K_0(mr)\cos\ mz\ dm \quad (8.229)$$

$$\phi_3 = \int_0^\infty C_4(m)T_1(m, r)\cos\ mz\ dm + \int_0^\infty C_5(m)\ T_2(m, r)\cos\ mz\ dm \quad (8.230)$$

$$\phi_4 = \int_0^\infty C_6(m)K_0(mr)\cos\ mz\ dm. \quad (8.231)$$

Boundary conditions are as follows: (i) $\phi_1 = \phi_2$ and $J_1 = J_2$ at $r = r_m$ (ii) $\phi_2 = \phi_3$ and $J_2 = J_3$ at $r = r_{x0}$ and $\phi_3 = \phi_4$ and $J_3 = J_4$ at $r = r_{tr}$ where J_1, J_2, J_3 and J_4 are the normal component of the current densities at the interfaces between the four different media. On both the sides of the transition zone, $R_{x0} = R_{tr}$ at $r = r_{x0}$ and $R_{tr} = R_t$ at $r = r_{tr}$. Therefore the boundary conditions $J_2 = J_3$ and $J_3 = J_4$ reduce to $\phi_2' = \phi_3'$ and $\phi_3' = \phi_4'$, where ϕ_1', ϕ_2', ϕ_3' and ϕ_4' are the derivatives of the potentials with respect to r at the respective boundaries, $r = r_{x0}$ and $r = r_{tr}$. Applying the boundary conditions to (8.227), (8.228), (8.229), (8.230), (8.231) and arranging them in matrix form, one gets

$$\begin{bmatrix} I_0(mr_m) & I_0(mr_m) & K_0(mr_m) & 0 & 0 & 0 \\ 0 & I_0(mr_{x0}) & K_0(mr_{x0}) & T_1(m, r_{x0}) & -T_2(m, r_{x0}) & 0 \\ 0 & 0 & 0 & T_1(m, r_{tr})\,m & T_2(m, r_{tr}) & K_0(m\,r_{tr}) \\ I_1(mr_m)/R_m & -I_1(mr_m)/R_{x0} & K_1(mr_m)/R_{x0} & 0 & 0 & 0 \\ 0 & I_0(mr_{x0})\,m & -K_1(mr_{x0})\,m & -T_1'(m, r_{x0}) & -T_2'(m, r_{x0}) & 0 \\ 0 & 0 & 0 & T_1'(m, r_{x0}) & T_1'(m, r_{x0}) & K_1(m\,r_{tr})\,m \end{bmatrix}$$

$$\begin{bmatrix} C_1(m) \\ C_2(m) \\ C_3(m) \\ C_4(m) \\ C_5(m) \\ C_6(m) \end{bmatrix} = \begin{bmatrix} -CK_0(mr_m) \\ 0 \\ 0 \\ CK_1(mr_m)/R \\ 0 \\ 0 \end{bmatrix} \tag{8.232}$$

using the Bessel function identity $I_0'(x) = I_1(x)$ and $K_0'(x) = -K_1(x)$

$$\frac{d\ T_1(m, r)}{dr} = T_1'(m, r)\,; \frac{d\ T_2(m, r)}{dr} = T_2'(m, r)\,.$$

Above matrix can be written in the form

$$X\ C_v = Y \tag{8.233}$$

where, C_v is the column vector of kernel functions to be determined. Since potential is measured in the borehole, evaluation of C_1 (m) is only required from (8.227).

Potential ϕ_1 is given by, taking mz = x

$$\phi_1 = C \int_0^\infty K_0\left(\frac{x}{z} r\right) \cos x \frac{dx}{z} + \frac{1}{z} \int_0^\infty C_1 \frac{x}{z} I_0\left(\frac{x}{z} r\right) \cos x\ dx \tag{8.234}$$

Applying the Weber Lipschitz identity, the first integral reduces to $\frac{\pi}{2z}$ at r = 0 i.e., along the borehole axis. The second integral (I_2) reduces to the form,

$$I_2 = \int_0^\infty C_1\left(\frac{x}{z}\right) \cos x\ dx \tag{8.235}$$

Since $I_0\left(\frac{x}{z} r\right)$ at r = 0 is 1.0. The infinite integral I_2 is determined using Gauss quadrature method. The kernel function C_1 (x/z) is determined using cubic spline interpolation.

Expression of the potential along the borehole axis reduces to the form

$$\phi_1 = \frac{R_m I}{2\pi^2 r_m}\left[\frac{\pi}{2z}\right] + \frac{1}{z}.I_2. \tag{8.236}$$

The expression for the apparent resistivity (R_a) for a two electrode system can be written as

$$R_a = 4\pi z \phi_1 / I \tag{8.237}$$

(Dakhnov, 1962)

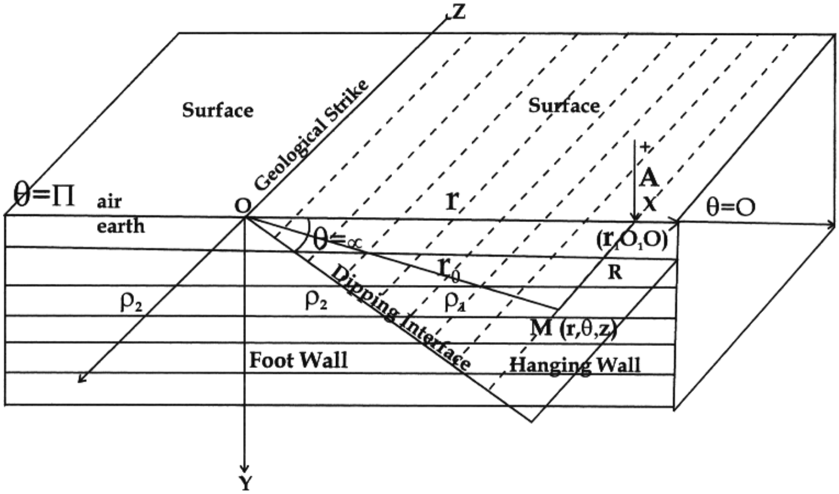


Fig. 8.7. A model of a dipping interface between the two media of resistivities ρ_2 and ρ_1; A (r,o,o) is the location of the current electrode and M is the pont of observation

Hence the apparent resistivity for two electrode system can be expressed as,

$$\frac{R_a}{R_m} = \frac{1}{r_m} + \frac{4\pi}{R_m z} I_2. \tag{8.238}$$

Taking derivatives of potential expression with respect to z, one gets the expression for apparent resistivity for three electrode system (lateral configuration) (Fig. 8.7) as

$$\frac{R_a}{R_m} = \frac{1}{r_m} + \frac{4\pi}{R_m} \int_0^\infty C_1 \left(x/z\right) x \sin x \, dx \tag{8.239}$$

In this section, an elaborate treatment on how to handle nonlaplacian terms along with the laplacian terms in a second order differential equation is shown. It is demonstrated that any second order differential equation, not solvable in closed form with suitable substitution as demonstrated, can be solved using Frobenious power series.

8.4 Geoelectrical Potential for a Dipping Interface

Dipping bed or dipping contact problem was solved by Skalskaya (1948), Maeda (1955), Van Nostrand and Cook (1955), De Gery and Kunetz (1956), Dakhnov (1962) and Others. Figure 8.7 show the geometry of the problem. Here cylindrical system of coordinate (r, θ, z) was chosen. z-axis is taken along the geological strike direction,i.e., along the line of contact between two bodies having different physical properties. Dipping contact is having a dip $\theta = \alpha$. A point current electrode is placed on the surface and at the point A (r,0,0). Because the origin is chosen at the point O(0,0,0). The Y axis is assumed vertically downward from the origin O. Right hand portion of the homogenous formation of resistivity ρ_1 is termed as the hanging wall and left portions of the homogenous formation may be termed as the foot wall. These are geological terms. On the hanging wall side $\theta = 0$ and on the foot wall side $\theta = \pi$ Coordinate of the point of observation M is (r, θ, z). The distances between O and M is r, O and A is r_0 and A and M is R.Since resistivity of the air is infinitely high, $\frac{\partial \phi}{\partial n} = 0$ both for $\theta = 0$ and $\theta = \pi$.

In Chap. 7 we have discussed two forms of the solutions of Laplace equation in cylindrical coordinatrs in terms of Bessel's functions and modified Bessel's functions. Application of both the forms are demonstrated in this chapter. In this problem, the third form of solution of Laplace equation by the method of separation of variables using modified Bessel's function of imaginary order is demonstrated.

In this problem Laplace equation is valid at all points except at the point A. In this problem the Laplace equation is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{8.240}$$

The third set of equations used in the present problem are

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda^2 \tag{8.241}$$

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -i^2s^2 (\text{i.e. n becomes i s}) \tag{8.242}$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\lambda^2 + \frac{(is)^2}{r^2}\right)R = 0 \tag{8.243}$$

The solution of this third set of equations are respectively $\cos\lambda z$, $\sin\lambda z$, $\cosh s\theta$, $\sinh s\theta$ and $I_{is}(\lambda r)$ and $K_{is}(\lambda r)$. $I_{is}(\lambda r)$ and $K_{is}(\lambda r)$ are the modified Bessel's function of the first and second kind and of imaginary order. In this problem the potential is independent of the sign of z, therfore $\cos\lambda z$ will be the appropriate potential function and not $\sin\lambda z$. When $r \to \infty, \phi \to 0$. Since $I_{is}(\lambda r)$ approaches high values when $r \to \infty$, therefore $I_{is}(\lambda r)$ cannot be the proper potential function and $K_{is}(\lambda r)$ is the suitable potential function.

Therefore, the expression for the perturbation potential can be written as

$$\phi = \int\limits_{\lambda=0}^{\infty}\int\limits_{s=0}^{\infty} \cos\lambda z d\lambda \left(L(s,\lambda)\cosh(s\theta) + M(s,\lambda)\sinh(s\theta)K_{is}(\lambda r)\right) ds. \tag{8.244}$$

To find the potential at M $(\mathrm{r}, \theta, \mathrm{z})$, We have

$$\phi_1 = \frac{1\rho_1}{2\pi} \cdot \frac{1}{AM} \cdot + \int_{\lambda=0}^{\infty} \int_{s=0}^{\infty} \cos \lambda z d\lambda \left(A(s,\lambda) \cos hs\theta \right.$$
$$\left. + B(s,\lambda) \sin h(s\theta) K_{is}(\lambda r) ds \right. \tag{8.245}$$

$$\phi_2 = \frac{1\rho_1}{2\pi} \cdot \frac{1}{AM} \cdot + \int_{\lambda=0}^{\infty} \int_{s=0}^{\infty} \cos \lambda z d\lambda \left(C(s,\lambda) \cos hs\theta \right.$$
$$\left. + D(s,\lambda), \sin hs\theta \right) K_{is}(\lambda r) ds. \tag{8.246}$$

Now four constants in these equations can be determined applying four boundary conditions, i.e.,

$$\begin{aligned} \phi_1 &= \phi_2 && \text{at } \theta = \alpha \\ \frac{1}{\rho_1} \frac{\partial \phi_1}{\partial \theta} &= \frac{1}{\rho_2} \frac{\partial \phi_2}{\partial \theta} && \text{at } \theta = \alpha \\ \frac{\partial \phi_1}{\partial \theta} &= 0 && \text{at } \theta = 0 \\ \frac{\partial \phi_1}{\partial \theta} &= 0 && \text{at } \theta = \pi \end{aligned}$$

The last two boundary conditions originate because current cannot flow across the air-earth boundary. For this type of boundary value problems ,as has already been discussed, one has to express the source potential in the same format as the perturbation potential before the boundary condition is applied. We have to express $\frac{1}{\mathrm{AM}}$ or $\frac{1}{\mathrm{r}}$ in the form of $\mathrm{K_{is}}(\lambda \mathrm{r})$. Skalskaya (1948) solved this problem. She has shown that $\frac{1}{\mathrm{AM}}$ can be expressed as

$$\frac{1}{AM} = \frac{1}{r} = \frac{1}{\sqrt{r_0^2 + r^2 - 2r_0 r \cos\theta + z^2}}$$
$$= \frac{4}{\pi^2} \int_0^{\infty} \cos \lambda z d\lambda \int_0^{\infty} \cos hs(\pi - \theta) K_{is}(\lambda r_0) K_{is}(\lambda r) ds \tag{8.247}$$

Let $\frac{\mathrm{I}\rho}{2\pi} = \mathrm{q}$. Now, applying the second set of boundary conditions $\frac{\mathrm{q}}{\mathrm{AM}}$ terms will cancel out from both the sides. Now $\sinh \mathrm{s}\theta = 0$ and $\cosh \mathrm{s}\theta = 1$ for $\theta = 0$. Therefore, we can write down

$$\phi_1 = \frac{\mathrm{q}}{\mathrm{AM}} + \int_0^{\infty} \cos \lambda \mathrm{z}\, d\lambda \int_0^{\infty} \mathrm{L}(\mathrm{S}_1 \lambda) \cos \mathrm{S}\theta \mathrm{K_{is}}(\lambda \mathrm{r})\, \mathrm{dS} \tag{8.248}$$

and

$$\phi_2 = \frac{\mathrm{q}}{\mathrm{AM}} + \int_0^{\infty} \cos\lambda \mathrm{z}\, d\lambda \int_0^{\infty} \mathrm{M}(\mathrm{S}_1 \lambda) \cos \mathrm{h}\ \mathrm{S}(\pi - \theta)\, \mathrm{K_{is}}(\lambda \mathrm{r})\, \mathrm{dS} \tag{8.249}$$

Two boundary conditions are used to get two equations

$$\mathrm{L}(\mathrm{S}_1\lambda)\cosh \mathrm{S}\alpha = \mathrm{M}(\mathrm{S}_1\lambda)\cosh \mathrm{S}(\pi-\alpha) \tag{8.250}$$

$$\begin{aligned}&\frac{1}{\rho}L\left(S_1\lambda\right)\sinh S\alpha - \frac{1}{\rho_2}M\left(S_1\lambda\right)\sinh S\left(\pi-\alpha\right)\\ &= \left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\mathrm{q}.\frac{4}{\pi^2}\sinh \mathrm{S}\left(\pi-\alpha\right)\mathrm{K}_{\mathrm{is}}\left(\lambda \mathrm{r}_0\right)\end{aligned} \tag{8.251}$$

From these two equations L and M can be determined. After a few steps of algebraic simplifications we get

$$\begin{aligned}L\left(S,\lambda\right) &= q\frac{4}{\pi^2}.\frac{K_{12}\sinh 2S\left(\pi-\alpha\right)}{\sinh S\pi - K_{is}\sinh S\left(\pi-2\alpha\right)}\\ &= \mathrm{q}.\frac{4}{\pi^2}.\mathrm{A}\left(\mathrm{s}\right)\end{aligned} \tag{8.252}$$

and

$$\begin{aligned}M\left(S,\lambda\right) &= q.\frac{4}{\pi^2}.\frac{\sinh S\pi + \sinh S\left(\pi-2\alpha\right)}{\sinh S\pi - K_{12}\sinh S\left(\pi-2\alpha\right)}\\ &= \mathrm{q}.\frac{4}{\pi^2}.\mathrm{B}\left(\mathrm{s}\right)\end{aligned} \tag{8.253}$$

Therefore the expressions for the potentials are

$$\begin{aligned}\phi_1 = \frac{\mathrm{I}\rho_1}{2\pi}\Bigg[\frac{1}{\mathrm{R}} + \frac{4}{\pi^2}\int_0^\infty \cos\lambda \mathrm{z}\ \mathrm{d}\lambda\int_0^\infty \mathrm{A}\left(\mathrm{S}\right)\cosh \mathrm{S}\theta\\ \mathrm{K}_{\mathrm{is}}\left(\lambda \mathrm{r}_0\right)\mathrm{K}_{\mathrm{is}}\left(\lambda \mathrm{r}\right)\mathrm{ds}\Bigg]\end{aligned} \tag{8.254}$$

$$\begin{aligned}\phi_2 = \frac{\mathrm{I}\rho_1}{2\pi}\Bigg[\frac{1}{\mathrm{R}} + \frac{4}{\pi^2}\int_0^\infty \cos\lambda \mathrm{z}\ \mathrm{d}\lambda\int_0^\infty \mathrm{B}\left(\mathrm{S}\right)\cosh \mathrm{S}\left(\pi-\theta\right)\\ \mathrm{K}_{\mathrm{is}}\left(\lambda \mathrm{r}_0\right)\mathrm{K}_{\mathrm{is}}\left(\lambda \mathrm{r}\right)\mathrm{ds}\Bigg].\end{aligned} \tag{8.255}$$

To get potential on the surface, one has to put $\theta = 0$ and get.

$$A\left(S\right) = \frac{K_{12}\sinh 2S\left(\pi-\alpha\right)}{\sinh S\pi - K_{12}\sinh S\left(\pi-2\alpha\right)} \tag{8.256}$$

where $\mathrm{K}_{12} = (\rho_2-\rho_1)/(\rho_2+\rho_1)$, the reflection factor

8.5 Geoelectrical Potentials for an Anisotropic Medium

8.5.1 General Nature of the Basic Equations

In an anisotropic medium

$$\vec{J}_x = \frac{1}{\rho_x}\vec{E}_x, \vec{J}_y = \frac{\vec{E}_y}{\rho_y} \text{ and } J_z = \frac{\vec{E}_z}{\rho_z} \tag{8.257}$$

where $\rho x, \rho y$ and ρz and Ex,Ey and Ez are the resistivities and fields along the principal axes of anisotropy x,y,z. The equation of continuity $\mathrm{div}\vec{J} = 0$ in a source free region may be written as

$$\frac{\partial}{\partial x}\left(\frac{E_x}{\rho_x}\right) + \frac{\partial}{\partial y}\left(\frac{E_y}{\rho_y}\right) + \frac{\partial}{\partial z}\left(\frac{E_z}{\rho_z}\right) = 0 \tag{8.258}$$

For a homogeneous but anisotropic medium. This equation reduces to

$$\frac{1}{\rho_x}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{\rho_y}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{\rho_z}\frac{\partial^2\phi}{\partial z^2} = 0 \tag{8.259}$$

For a homogeneous and isotropic medium, (8.259) reduces to

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \text{ since } \rho_x = \rho_y = \rho_z \tag{8.260}$$

For an anisotropic medium let us choose a new sets of coordinates, such that $\xi = x\sqrt{\rho_x}, \eta = y\sqrt{\rho_y} and \zeta = z\sqrt{\rho}_z$ and (8.259) reduces to

$$\frac{\partial^2\phi}{\partial \xi^2} + \frac{\partial^2\phi}{\partial \eta^2} + \frac{\partial^2\phi}{\partial \zeta^2} = 0 \tag{8.261}$$

The solution of the equation is

$$\phi = \frac{C}{\left(\xi^2 + \eta^2 + \zeta^2\right)^{1/2}} \tag{8.262}$$

where C is the constant of integration.

$$\Rightarrow \quad \frac{C}{\left(\rho_x x^2 + \rho_y y^2 + \rho_z z^2\right)} \tag{8.263}$$

The expression for the equipotential surface is given by

$$\rho_x x^2 + \rho_y y^2 + \rho_z z^2 = R^2 \tag{8.264}$$

Equation (8.264) is an equation of an ellipsoid. The axes of the ellipsoid coincide with the principal axes of anisotropy. The current densities are given by

$$J_x = -\frac{1}{\rho_x}\frac{\partial \phi}{\partial x} = \frac{Cx}{\left(\rho_x x^2 + \rho_y y^2 + \rho_z z^2\right)^{3/2}} \tag{8.265}$$

$$J_y = -\frac{1}{\rho_y}\frac{\partial \phi}{\partial y} = \frac{Cy}{\left(\rho_x x^2 + \rho_y y^2 + \rho_z z^2\right)^{3/2}} \tag{8.266}$$

$$J_z = -\frac{1}{\rho_z}\frac{\partial \phi}{\partial z} = \frac{Cz}{\left(\rho_x x^2 + \rho_y y^2 + \rho_z z^2\right)^{3/2}} \tag{8.267}$$

Here current lines are straight lines spreading out radially from the source similar to that happens in an isotropic medium. The electric lines of force in an anisotropic medium form a family of curvilinear trajectories orthogonal to the equipotential surfaces. They do not coincide with the direction of the current lines except along the principal axes.

In geological sedimentary rocks the anisotropies are along and at right angles to the plane of stratification. Two resistivities are horizontal resistivity ρ_l (parallel to the plane of stratification) and vertical and the transverse resistivity ρ_t (perpendicular to the plane of stratification). If the plane of stratification is closer as the xy plane then (8.259) reduces to

$$\frac{1}{\rho_l}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + \frac{1}{\rho_t}\frac{\partial^2 \phi}{\partial z^2} = 0. \tag{8.268}$$

The expression for equipotential surface is given by

$$x^2 + y^2 + (\rho_t/\rho_l)z^2 = \text{constant} \tag{8.269}$$

i.e., the equipotential surfaces are ellipsoid of revolution around the z-axis. For an anisotropic medium, two more parameters are defined. They are

$$\lambda = \sqrt{\rho_t/\rho_l} \quad \text{and} \quad \rho_m\sqrt{\rho_t\rho_l} \tag{8.270}$$

where λ is called the coefficient of anisotropy and ρ_m is the root mean square resistivity of the media. From (8.269), we get

$$\rho_m = \lambda\rho_l = \frac{1}{\lambda}\rho_t \tag{8.271}$$

The solution of (8.268) may now be written as

$$\phi = \frac{C}{\rho_l^{1/2}\left(x^2 + y^2 + \lambda^2 z^2\right)^{1/2}}. \tag{8.272}$$

The current densities are

$$\vec{J}_x = -\frac{Cx}{\rho_l^{3/2}\left(x^2+y^2+\lambda^2 Z^2\right)^{3/2}} \tag{8.273}$$

$$\vec{J}_y = -\frac{Cy}{\rho_l^{3/2}\left(x^2+y^2+\lambda^2 z^2\right)^{3/2}} \tag{8.274}$$

$$\vec{J}_z = -\frac{Cz}{\rho_l^{3/2}\left(x^2+y^2+\lambda^2 z^2\right)^{3/2}} \tag{8.275}$$

such that

$$J = \left(J_x^2+J_y^2+J_z^2\right)^{1/2} = -\frac{C\left(x^2+y^2+z^2\right)^{1/2}}{\rho_l^{3/2}\left(x^2+y^2+\lambda^2 z^2\right)^{3/2}}. \tag{8.276}$$

In order to find the constant of integration C (8.276), we construct around the point P, a sphere of radius R and calculate the total current flowing out through this spherical surface. This is equal to the total current flowing through the point P.

Thus

$$I = \int_s J.ds = \int_0^{2\pi} JR^2 \sin\theta \, d\theta \, d\psi.$$

Now

$$x^2 + y^2 = R^2 \sin^2\theta$$

and

$$Z^2 = R^2\cos^2\theta$$

and (8.276) becomes

$$J = \frac{C}{\rho_l^{3/2} R^2\left(\sin^2\theta + \lambda^2\cos^2\theta\right)^{3/2}} = \frac{C}{\rho_s^{3/2} R^2\left[1+\left(\lambda^2-1\right)\cos^2\theta\right]^{3/2}}. \tag{8.277}$$

We can write

$$J = \frac{I\lambda\left(x^2+y^2+z^2\right)^{1/2}}{4\pi\left(x^2+y^2+\lambda^2 z^2\right)^{3/2}} = \frac{I\lambda}{4\pi R^2\left[1+\left(\lambda^2-1\right)\cos^2\theta\right]^{3/2}}$$

$$I = \frac{C}{\rho_l^{3/2}}\int_0^{2\pi} d\psi \int_0^{\pi}\frac{\sin\theta d\theta}{\left[1+\left(\lambda^2-1\right)\cos^2\theta\right]^{3/2}} \tag{8.278}$$

$$= \frac{2\pi C}{\rho_l^{3/2}}.\frac{2}{\lambda} = \frac{4\pi C}{\lambda\rho_s^{3/2}}. \tag{8.279}$$

Hence the expressions for the potentials and the current densities become

$$\phi = \frac{I\lambda\rho_S}{4\pi\left(x^2 + y^2 + \lambda^2 z^2\right)^{1/2}} = \frac{I\rho_m}{4\pi R\left[1 + \left(\lambda^2 - 1\right)\cos^2\theta\right]^{1/2}} \tag{8.280}$$

and

$$J = \frac{I\lambda\left(x^2 + y^2 + z^2\right)^{1/2}}{4\pi\left(x^2 + y^2 + \lambda^2 z^2\right)^{3/2}} = \frac{I\lambda}{4\pi R^2\left[1 + \left(\lambda^2 - 1\right)\cos^2\theta\right]^{3/2}}. \tag{8.281}$$

8.5.2 General Solution of Laplace Equation for an Anisotropic Earth

Equation (8.259) is the guiding equation for solving problems for an isotropic earth. When resistivity of a medium varies along the longitudinal and traverse direction, (8.259) changes to the form

$$\frac{1}{\rho_1}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{1}{\rho_t}\frac{\partial^2\phi}{\partial z^2} = 0. \tag{8.282}$$

Applying the method of separation of variables $\phi = R(r)Z(z)$, we get

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{dR}{dr} - \lambda^2 R = 0 \tag{8.283}$$

and

$$\frac{\partial^2 Z}{\partial z^2} - m^2\lambda^2 Z = 0. \tag{8.284}$$

Solving these two-equations (8.283) and (8.284) and substituting the values, we get the expression for the perturbation potential as

$$\phi = \int_0^\infty A(\lambda)\, e^{-m\lambda z} J_0(\lambda r)\, d\lambda + q. \tag{8.285}$$

The source potential q, can be written for anisotropic earth as

$$q = \frac{I\rho_m}{2\pi}\int_0^\infty e^{-m\lambda z} J_0(\lambda r)\, d\lambda \tag{8.286}$$

using Weber Lipschitz identity where

$$\rho_m = \sqrt{\rho_t\rho_l}.$$

For a two layer earth, potentials in the three regions following Wait (1982) and Negi and Saraf (1989), we can write

$$\phi_0 = \frac{I\rho_0}{4\pi R} + \int_0^\infty A_0(\lambda)\, e^{\lambda z} J_0(\lambda r)\, d\lambda \qquad \text{(in the air)} \tag{8.287}$$

$$\phi_1 = \int_0^\infty \left[A_1(\lambda)\, e^{-m_1\lambda z} + B_1(\lambda)\, e^{m\lambda z}\right] J_0(\lambda r)\, d\lambda \quad \text{(in the first layer)} \tag{8.288}$$

and

$$\phi_2 = \int_0^\infty A_2(\lambda)\, e^{-m_2\lambda z} J_0(\lambda r)\, d\lambda \qquad \text{(in the lower half space)} \tag{8.289}$$

where $\frac{I\rho_0}{4\pi R}$ is the source term

where

$$R = \sqrt{r^2 + (z + z_0)^2}, \quad m_1 = \sqrt{\rho_{t1}/\rho_{l1}} \tag{8.290}$$

and

$$m_2 = \sqrt{\rho_{t2}/\rho_{l2}}.$$

To evaluate A_0, A_1, B_1, A_2 the boundary conditions are applied. The boundary conditions are

(i) $\phi = \phi_1$ at $z = 0$
(ii) $\phi = \phi_2$ at $z = h$
(iii) $J_0 = J_1$ at $z = 0$
(iv) $J_1 = J_2$ at $z = h$

where J_0, J_1 and J_2 are the current densities in the respective media. Here for anisotropic earth

$$\frac{1}{\rho_0}\frac{\partial\phi_0}{\partial z} = \frac{1}{\rho_{t1}}\frac{\partial\phi_1}{\partial z} \text{ at } z = 0 \quad \text{and} \quad \frac{1}{\rho_{t1}}\frac{\partial\phi_1}{\partial z} = \frac{1}{\rho_{t2}}\frac{\partial\phi_2}{\partial z} \text{ at } z = h$$

Using the four boundary conditions,

$A_0(\lambda)$ can be written as

$$A_0(\lambda) = -\frac{I\rho_0}{4\pi}.e^{-\lambda z_0}\frac{\rho_0 - \rho_{1n}(P_1/P_2)}{\rho_0 + \rho_{1n}(P_1/P_2)} \tag{8.291}$$

where

$$\rho_{ln} = (\rho_{l1}/\rho_{t1})^{1/2}, \rho_{2n} = (\rho_{l2}/\rho_{t2})^{1/2}$$
$$P_1 = \left(1 - R_1 e^{-2\lambda h_1}\right), \quad P_2 = \left(1 + R_1 e^{-2\lambda h_1}\right)$$

where

$$m_1 h = h_1$$

and

$$R_1 = \frac{\rho_{1n} - \rho_{2n}}{\rho_{1n} + \rho_{2n}}.$$

When the current source is at the air earth interface the potential in the first medium

$$\phi_0 = \frac{I\rho_0}{2\pi} \int_0^\infty [1 + B_0(\lambda)]\, e^{-\lambda z} J_0(\lambda r)\, d\lambda \tag{8.292}$$

where

$$B_0(\lambda) = -\frac{4\pi}{I\rho_0} A_0(\lambda) = \frac{2\rho_{1n}(P_1/P_2)}{\rho_0 + (P_{1n}P_1/P_2)} - 1 \tag{8.293}$$

$\rho_0 >> \rho_{1n}$ and ρ_{2n} (8.292) reduces to

$$\phi_0 = \frac{I\rho_{1n}}{2\pi} \int_0^\infty \frac{1 - R_1 e^{-2\lambda h_1}}{1 + R_1 e^{-2\lambda h_1}} e^{-hz}\ J_0(\lambda r)\, d\lambda. \tag{8.294}$$

On the surface of the earth at $z = 0$

$$\phi_0(r_1 0) = \frac{I\rho_{1n}}{2\pi r}\left[1 - 2R_1 r \int_0^\infty \frac{e^{-2\lambda h_1} J_0(\lambda r)}{1 + Re^{-2\lambda h_1}} d\lambda\right]. \tag{8.295}$$

9

Complex Variables and Conformal Transformation in Potential Theory

In this chapter, we have shown (i) how the complex variables can be used to solve some problems in potential theory, (ii) how real and imaginary quantities together can represent field lines and equipotential lines and satisfy Laplace equation, (iii) how analytic function and Cauchy Rieman's equations in complex variables can be used for solving certain kind of two dimensional problems. Schwarz Cristoffel method of conformal transformation can be used for solving two dimensional geoelectrical problems. Three types of problems are given where S-C transformations are used. These boundary value problems are (i) where closed form solution is possible (ii) where closed form solution is not possible and one has to use numerical methods and (iii) where closed form solution is possible using elliptic integrals and elliptic functions. For the benefit of the students a brief introduction on elliptic integrals and elliptic functions are appended.

9.1 Definition of Analytic Function

If E is a certain point set in a complex plane ($z = x + iy$), if for every z, there exists one or more complex number w, if a function of the complex variable z with its value equal to w defined in E or briefly if $w = f\ (z)$ and if $z = x + iy$ and $w = u + iv$, then we have u and v as functions of x and y (Fig. 9.1 a,b). The function w is said to be continuous at a point $z_0 = (x_0 + iy_{0+})$ if u (x, y) tends to $u_0(x_0, y_0)$ and v (x, y) tends to $v_0(x_0, y_0)$ when x tends to x_0 and y tends to y_0. Such a function is said to be analytic in a particular domain if the differential coefficient of f (z) i.e.,

$$f'(z) = \operatorname*{Lim}_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{9.1}$$

exists for all paths joining z to z_0. The necessary and sufficient conditions for differentiability are

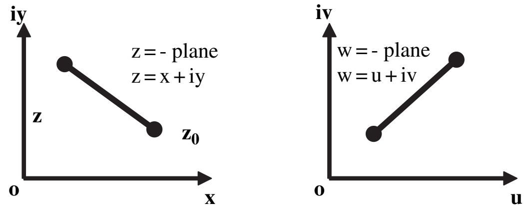


Fig. 9.1 a,b. Complex Z and W plane; Point by point mapping in the Z and W plane

$$\frac{\partial \mathrm{u}}{\partial \mathrm{x}} = \frac{\partial \mathrm{v}}{\partial \mathrm{y}}; \frac{\partial \mathrm{u}}{\partial \mathrm{y}} = \frac{\partial \mathrm{v}}{\partial \mathrm{x}}; \tag{9.2}$$

and these partial derivatives are continuous. When these conditions are satisfied f (z) is said to be analytic at the point z_0.

9.2 Complex Functions and their Derivatives

We can define w to be a function of the complex variable z if for each value of z, belonging to a prescribed set S, there will be corresponding one or more values of w. Such a definition proves to be too general to be particularly useful for physical application. So we shall confine our attention to a much more limited class of functions. Those which are single valued continuous and possess a single class of function are generally chosen.

In defining the derivative of a complex function, we shall have to introduce the concept of limit. Let f (z) be defined as single valued at all points in the neighbourhood of z_0 except possibly at z_0 itself. Then we say that f (z) approaches the limit w_0. This is written as

$$\underset{\mathrm{z}-\mathrm{z}_0}{\mathrm{Lim}} \rightarrow \mathrm{f(z)} = \mathrm{w}_0. \tag{9.3}$$

If f (z) can be made arbitrarily close to w_0 then in the neighbourhood of $\mathrm{z}_0, \mathrm{z} - \mathrm{z}_0$ is taken sufficiently small. Arithmatically this is expressed as follows: Corresponding to each preassigned positive numeric ε, no matter how small, there exists a positive number δ such that $|f(z) - w_0| < \varepsilon whenever 0 < |z - z_0| < \delta$, where δ is infinitesimally small. For a function to be continuous at a point i.e., for f (z) to be continuous at z_0, the basic requirement is f (z_0) must exist at z_0 i.e,

$$\underset{\mathrm{z}-\mathrm{z}_0}{\mathrm{Lim}} \rightarrow \mathrm{f(z)} = \mathrm{f(z_0)}.$$

This definition is also valid where z_0 lies on the boundary of a closed region. Let us put

$$\mathrm{w} = \mathrm{f(z)} = \mathrm{u(x,y)} + \mathrm{i}\nu\mathrm{(x,y)}. \tag{9.4}$$

Two complex numbers are equal if and only if their real and imaginary parts are separately equal,

we get

$$\mathrm{Lim}_{\substack{x\to 0\\ y\to 0}} \mathrm{u(x,y)} = \mathrm{u(x_0,y_0)} \tag{9.5}$$

$$\mathrm{Lim}_{\substack{x\to 0\\ y\to 0}} v\,(x,y) = v\,(x_0,y_0) \tag{9.6}$$

and their path of movements lie entirely within the region of definition.

The derivative of f (z) with respect to z is given by

$$f'(z) = \frac{df(z)}{dz} = \underset{\Delta z\to 0}{Lim}\frac{f(z+dz)-f(z)}{\Delta z} = \underset{\Delta z\to 0}{Lim}\frac{\Delta f}{\Delta z}. \tag{9.7}$$

Since Δz $=$ Δx $+$ iΔy, we see that there are an infinite number of paths in the z plane along which Δz can approach zero. A unique value of the derivative regardless of the mode of approach, a set of necessary conditions for the existence of a unique derivative at a point is obtained (Fig. 9.2). Certainly if the value of the limit obtained by first setting Δy $= 0$ and then permitting Δx to approach zero, the derivative is not independent of the path. If we first put Δx $= 0$, (9.7) becomes

$$\frac{\mathrm{df}}{\mathrm{dz}} = \underset{\Delta \mathrm{y}\to 0}{\mathrm{Lim}}\frac{\mathrm{f\,(z+i\Delta y)-f\,(z)}}{\mathrm{i\Delta y}} \tag{9.8}$$

$$= \underset{\Delta y\to 0}{Lim}\frac{u\,(x,y+\Delta y)-u\,(x,y)}{i\Delta y} + \frac{i\,[v\,(x,y+\Delta y)-v\,(x,y)]}{i\Delta y} \tag{9.9}$$

$$= -\mathrm{i}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}}. \tag{9.10}$$

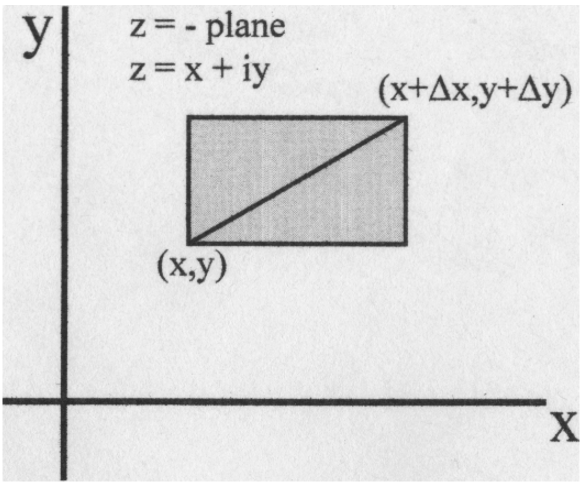


Fig. 9.2. Movements in the z-plane for determining the derivatives (Cauchy Reimann Condition)

Similarly, putting $\Delta y \to 0$ and taking the limit

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{f(z+\Delta x, y) - f(z)}{i\Delta x} \tag{9.11}$$

$$\Rightarrow \quad \underset{\Delta x \to 0}{Lim} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{i[v(x+\Delta x, y) - v(x,y)]}{\Delta x} \tag{9.12}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

These two expressions for the derivatives are equal if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{9.13}$$

$$\Rightarrow \quad \frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial \nu}{\partial y} + i\frac{\partial v}{\partial x}. \tag{9.14}$$

These pairs of first order differential equations are Cauchy-Riemann equations.

A function f (z) of complex variable z is said to be analytic at the point z_0, if it is single valued and possesses a derivative not only at z_0 but at every point in a neighbourhood of z_0. Otherwise the point z_0 is a singular point of the analytic function. If f (z) is analytic at every point of a domain D, then, differentiating the first equation with respect to x and second equation with respect to y, we have,

$$\frac{\partial^2 \nu}{\partial x^2} = \frac{\partial^2 \nu}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}.$$

Therefore

$$\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2} = 0. \tag{9.15}$$

Similarly, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{9.16}$$

Both the real and imaginary parts of a complex functions are analytic functions, and they satisfy Laplace equation in two dimensions.

$\nu(x, y) = C_1$ is a family of curves where C_1 is constant.

$$m_1 = \frac{\partial y}{\partial x} = +(\partial u/\partial x)/(\partial u/\partial y) \tag{9.17}$$

$$m_1 = \frac{\partial y}{\partial x} = +(\partial u/\partial x)/(\partial u/\partial y). \tag{9.18}$$

Similarly $u(x, y) = C_2$, where C_2 is another constant

$$.m_2 = \frac{\partial y}{\partial x} = +(\partial \nu/\partial x)/(\partial \nu/\partial y). \tag{9.19}$$

Now taking the products of m_1 and m_2, we have

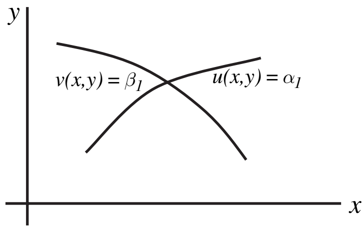


Fig. 9.3. Orthogonal property of u and v functions

$$\mathrm{m_1.m_2} = \frac{\partial \mathrm{u}/\partial \mathrm{x}.\partial \mathrm{v}/\partial \mathrm{x}}{\partial \mathrm{u}/\partial \mathrm{y}.\partial \mathrm{v}/\partial \mathrm{y}} = -1. \tag{9.20}$$

These two sets of tangents are orthogonal (Fig. 9.3). Therefore, if u (x, y) are the equipotential lines then v, (x, y) will be the field lines. Thus it is proved that if u and v are analytic functions in a complex plane then they satisfy Laplace equation and the slopes of the tangents are mutually orthogonal. Hence u and v can be used to denote equipotential and field lines.

9.3 Conformal Mapping

Equations (9.14) are normally known as Cauchy Riemann equations. Geometrical interpretation of derivative $\left|\frac{\mathrm{dw}}{\mathrm{dz}}\right| = |\mathrm{f}'(\mathrm{z})|$ is a measure of elongation of an element in the z-plane when it is transferred to w-plane. Arg $\{\mathrm{f}'(\mathrm{z})\}$ is interpreted as rotation of an element dz with respect to an element dw. Arg stands for argument in a complex quantity. It is represented in the form of an angle similar to phase angles in an electromagnetics .

If we draw two curves through a point $\mathrm{z_0}$ in the z plane and draw two tangents at $\mathrm{z_0}$ and map the two curves in the w plane by a function $\mathrm{w} = \mathrm{f(z)}$ which is analytic in a region such that $\mathrm{f}'(\mathrm{z})$ does not vanish (since otherwise the mapping will not be one to one) and draw a tangent to each of the curves at the point of intersection, the angle between the two tangents remain invariant under mapping. This property of mapping is called conformal mapping in the domain of analyticity (Fig. 9.4 a,b). If the sense of angle is preserved together with its magnitude, it is called conformal mapping of the first kind and if the sense is preserved keeping its magnitude constant, it is called the conformal mapping of the second kind.

A complex potential, which is an analytic function, must have a singularity at infinity otherwise it will reduce to a constant. Physical interpretation of the singularity can be given. By definition, the singularities are points where a function ceases to be analytic. Such points are precisely the points where physical sources which give rise to potentials are located. Singular points in a complex plane may be poles, zeros, essential singularities or branch points Spiegel (1964).

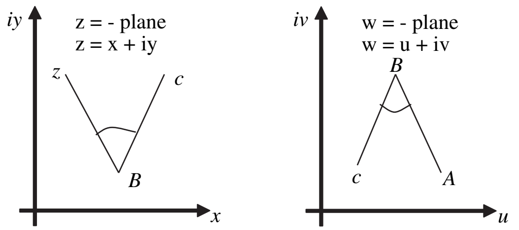


Fig. 9.4 a,b. Show the movement of a point in the Z plane and the corresponding movement in the W plane where the angle of movement is preserved

The mapping is said to be one to one over the finite domain of the z-plane if there exists an inverse transformation function $z = f^{-1}(w)$ which will map the w-plane onto the z-plane. Let $w(z) = u(x, y) + i\nu(x, y)$. We can find the values at x and y if the Jacobians is non zero.
i.e.

$$J\left|\frac{u, \nu}{x, y}\right| = \begin{vmatrix} \dfrac{\partial u}{\partial x}, & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial \nu}{\partial x}, & \dfrac{\partial \nu}{\partial y} \end{vmatrix} \neq 0. \tag{9.21}$$

On making use of the Cauchy Riemann conditions into the (9.20), the Jacobian can be shown to be

$$J\left|\frac{u, \nu}{x, y}\right| = \left|f'(z)\right|^2. \tag{9.22}$$

This means that inverse transformation function exists if $f'(z) \neq 0$.

Now assuming that an inverse transformation function exists and can be found, let us take the transformation of a complex potential function.

Let $\phi(z)$ be a complex potential in the z-plane

$$\phi(z) = U(x, y) + iV(x, y). \tag{9.23}$$

We shall replace z by w using inverse mapping function

$$\phi\left(f^{-1}(w)\right) = U(u, v) + iV(u, \nu). \tag{9.24}$$

It can be shown that $U(u, \nu)$ and $V(u, \nu)$ satisfy Laplace equation.

We know

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{9.25}$$

Now

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial \nu}{\partial x}\frac{\partial}{\partial u} \tag{9.26}$$

and

$$i\frac{\partial}{\partial y} = i\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + i\frac{\partial \nu}{\partial y}\frac{\partial}{\partial \nu}. \tag{9.27}$$

By adding and substracting (9.26) and (9.27) and using 9.22 we obtain

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)=f'(z)\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial \nu}\right) \tag{9.28}$$

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)=f'(z)\left(\frac{\partial}{\partial u}-i\frac{\partial}{\partial \nu}\right). \tag{9.29}$$

Hence from (9.28) and (9.29) is get

$$\left(\frac{\partial^2}{\partial x^2}+i\frac{\partial^2}{\partial y^2}\right)u=\left|f'(z)\right|^2\left(\frac{\partial^2 U}{\partial u^2}+i\frac{\partial^2 U}{\partial \nu^2}\right). \tag{9.30}$$

From (9.30) we get

$$\left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}+\mathrm{i}\frac{\partial^2 \mathrm{u}}{\partial \mathrm{y}^2}\right)=0=\left|\mathrm{f}'(\mathrm{z})\right|^2\left(\frac{\partial^2 \mathrm{U}}{\partial \mathrm{u}^2}+\mathrm{i}\frac{\partial^2 \mathrm{U}}{\partial \nu^2}\right). \tag{9.31}$$

Since $\mathrm{f}'(\mathrm{z}) \neq 0$,

$\frac{\partial^2 \mathrm{U}}{\partial \mathrm{u}^2}+\frac{\partial^2 \mathrm{V}}{\partial \nu^2}=0$ and from (9.14) it can be shown that

$$\frac{\partial^2 \mathrm{U}}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{U}}{\partial \mathrm{y}^2}=0 \text{ and } \frac{\partial^2 \mathrm{V}}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{V}}{\partial \mathrm{y}^2}=0. \tag{9.32}$$

Hence both $\mathrm{U}(\mathrm{u}, \nu)$ and $\mathrm{V}(\mathrm{u}, \nu)$ will satisfy Laplace's equation. This is an alternative approach to prove that analytic functions in two-dimensional medium in a complex plane satisfy Laplace equations.

9.4 Transformations

The set of equations

$$\begin{aligned} \mathrm{u} &= \mathrm{u}(\mathrm{x}, \mathrm{y}) \\ \nu &= \mathrm{v}(\mathrm{x}, \mathrm{y}) \end{aligned} \tag{9.33}$$

defines in general a transformation or mapping which establishes a correspondence between points in the uν and xy planes. These (9.32) are called transformation equations. If to each point of the uv plane there corresponds one and only one point in the xy plane and conversely, we speak of one to one transformation or mapping, in such a case a set of points in the xy plane is mapped onto a set of points in the uv plane. The corresponding set of points is often called images of each other. When u and v are real and imaginary parts of an analytical function, a complex variable $\mathrm{w}=\mathrm{u}+\mathrm{iv}=\mathrm{f}(\mathrm{z})=\mathrm{f}(\mathrm{x}+\mathrm{iy})$ and the transformation will be one to one in the region where $\mathrm{f}'(\mathrm{z}) \neq 0$.

A mapping in the plane is said to be angle preserving or conformal if it preserves angle between oriented curves in magnitude as well as in sense i.e., the image of any two intersecting oriented curves taken with their corresponding orientation make the same angle of intersection with the curve both in magnitude and direction.

Corollary 1 – If f (z) is analytic and $f'(z) \neq 0$ in a region R, then the mapping w = f (z) is conformal at all points of R.

Corollary 2 – The mapping defined by an analytic function f (z) is conformal except a points where the derivative $f'(z)$ is zero.

9.4.1 Simple Transformations

In this section we choose two simple mathematical relations in a complex domain to demonstrate that these relations can show the nature of field lines and equipotential lines in a two dimensional problem..

Problem 1

In this problem, we chose a simple transformation z = k coshw (Pipes 1953) and show how orthogonal current lines and equipotential lines in a square grid in $w(= u + iv)$ plane transforms to ellipses and hyperbolas in the $z(= x + iy)$ plane to represent equipotential lines and current lines for a finite line source.

Let

$$z = k \cosh w = k \cosh(u + iv) \tag{9.34}$$

where k is a real constant. (Fig. 9.5 a,b) To study this transformation, we must determine the curve u = const and v = const. Expanding $\cosh(u + iv)$ into its real and imaginary parts, we obtain

$$x + iy = k(\cosh u \cos v + i \sinh u \sin v) \tag{9.35}$$

$$x = k \cosh u \cos v \tag{9.36}$$

$$y = k \ \sinh \ u \ \sin \ v$$

This may be written in the form

$$\cos \ v = \frac{x}{k \ \cosh \ u} \tag{9.37}$$

$$\sin \ v = \frac{y}{k \ \sinh \ u}$$

Therefore, on squaring these equations and adding them, we have

$$\frac{x^2}{k^2 \ \cosh^2 u} + \frac{y^2}{k^2 \ \sinh^2 u} = 1. \tag{9.38}$$

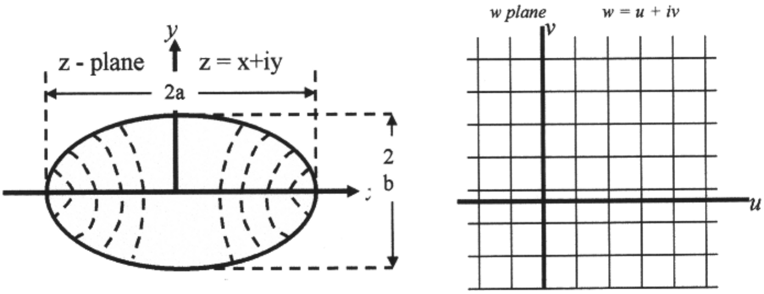


Fig. 9.5 a,b. Simple cosine hyperbolic transformation changes ellipse and hyperbola in the z plane to square grid in the w – plane

If we put

$$a = k\ \cosh\ u \qquad (9.39)$$
$$b = k\ \sinh\ u,$$

the (9.38) can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (9.40)$$

This is the equation of an ellipse with its centre is at the origin and having a major axis of length 2a and a minor axis of length 2b. Hence the curve u = constant are a family of confocal ellipses.

To obtain the curve u = constant, we write in the form

$$\cos\ h\ u = \frac{x}{k \cos u} \text{ and } \sinh\ u = \frac{y}{k\ \sin\ \nu} \qquad (9.41)$$

$$\cos\ h^2 u - \sinh^2\ u = \frac{x^2}{k^2\ \cos^2\ u} - \frac{y^2}{k^2\ \sin^2\ \nu} = 1. \qquad (9.42)$$

If we set $a' = k\ \cos\ \nu$ and $b' = k\ \sin\ \nu$,
we have

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1. \quad \text{This is an equation of a hyperbola.} \qquad (9.43)$$

For a direct current flow field the equipotential lines are elliptical and field lines and current flow lines are hyperbolic due to a line source of finite length.

Case 2

Problem 2

A simple transformation of a line source and a sink in a z plane maps the field lines and equipotential lines in a w-plane.

Let

$$w = A \ln \frac{z - a}{z + a}. \tag{9.44}$$

We have seen that the equation

$$w = -\frac{q}{2\pi k} \ln z = u + i\nu \tag{9.45}$$

gives the appropriate transformation to study the electric field in the region surrounding a charged circular cylinder with its centre at the origin and having a charge Q per unit length (Fig. 9.6a and b). In this case the real part of the transformation is

$$u = -\frac{q}{2\pi k}$$

ln r and the imaginary part is

$$\nu = \frac{-q}{2\pi k} \ln \theta. \tag{9.46}$$

Let us now consider a field produced by a line charge of +q per unit length at $z = a$ and another line charge of $-q$ per unit length at $z = -a$. The fields produced by both the line charges gets added up and is given by

$$w = +\frac{q}{2\pi k} \ln (z - a) - \ln (z + a) = -\frac{q}{2\pi k} \ln \frac{z - a}{z + a}. \tag{9.47}$$

Equation (9.47) represents the proper transformation to determine the field and equipotentials of two line charges.

Let

$$A = -\frac{q}{2\pi k}, \tag{9.48}$$

we then have

$$u + i\nu = A \ln \frac{z - a}{z + a}. \tag{9.49}$$

If we now let the distance of the point P (Fig. 9.6 a,b) from the point $z = a$ and $z = -a$ be r_1 and r_2 respectively, we have

$$z - a = r_1 e^{i\theta_1} \tag{9.50}$$

$$z + a = r_2 e^{i\theta_2} \tag{9.51}$$

where $\theta_1 = \arg(z - a)$ and $\theta_2 = \arg(z + a)$.

Hence $u + i\nu = A\,[\ln (z - a) - \ln (z + a)]$

$$= A\,[\ln r_1 + i\theta_1 - \ln r_2 - i\theta_2]\,. \tag{9.52}$$

Therefore,

$$u = A \ln \frac{r_1}{r_2} \tag{9.53}$$

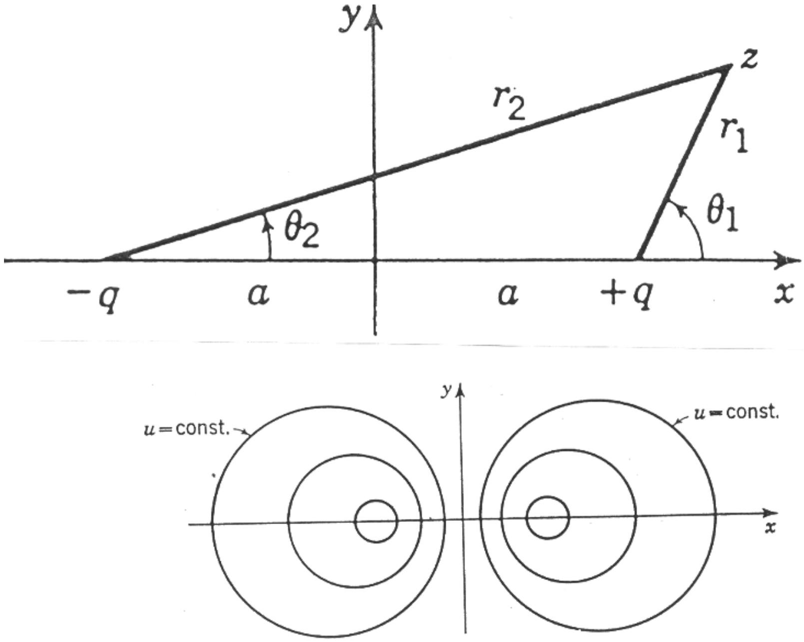


Fig. 9.6 a,b. Simple logarithmic transformation simulates the fields and equipotentials for a line source and a sink

and

$$\nu = A(\theta_1 - \theta_2). \tag{9.54}$$

These are the curves for u = constant and ν = constant.

If we put

$$\ln \frac{r_1}{r_2} = u/A \tag{9.55}$$

then

$$\frac{r_1}{r_2} = e^{u/A} = \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} = e^{2u/k} = K \tag{9.56}$$

Therefore

$$y^2 + \left[x - \frac{a\,(1+K)}{(1-K)}\right]^2 = \frac{4a^2K}{(1-K)^2}. \tag{9.57}$$

We thus see that the curves u = constant are a family of circles with eccentric gradual shifting of centres along the line joining the source and sink.

$$x = a\,(1+k)\,/\,(1-k) \;\text{ and } r = \frac{2a\sqrt{k}}{1-k} \tag{9.58}$$

These eccentric circles are the equipotentials due to a line source and a line sink in a homogeneous and isotropic medium.

9.5 Schwarz Christoffel Transformation

9.5.1 Introduction

Schwarz and Christoffel independently proposed that a suitable transformation function or mapping function can be determined for transferring the problem in the z plane to the w plane such that the problem is solved in the w plane and at the end we can return back to the z-plane again and present the final answer. In case it is necessary one may have to go for successive transformation depending upon the nature of the problem. As for example in certain cases one may have to go from w plane to t plane ,from t plane to t' plane, get the solution in the t' plane and finally come back to z plane to present the final answer.

9.5.2 Schwarz-Christoffel Transformation of the Interior of a Polygon

The problem to find a mapping function to map the interior of a n-sided polygon in the z-plane onto the upper half of the w-plane such that the boundary of the polygon goes over to the real axis of the w-plane. The upper half of the w-plane goes to inside of a polygon. Let $w = a_1, a_2, a_3, \ldots\ldots a_n$ be images on the real axis of the vertices of a polygon $z = z_1, z_2, z_3, \ldots\ldots z_n$. We shall assume that $a_1 < a_2 < a_3 \ldots\ldots a_n$ (Fig. 9.7 a,b). The mapping function should be such that (i) A segment of real axis of the w-plane bounded by two images of the vertices say, a_k, a_{k+1} must go over the corresponding sides of the polygon joining the points z_k and z_{k+1}.

This requires that the argument of the mapping function $\frac{dz}{dw} = f'(w)$ must remain constant for $a_k < w < a_{k+1}$.

(ii) As we cross an image of the vertices, say a_{k+1}, the argument of the mapping function must change discontinuously by an amount equal to the exterior angle of the polygon at $z = z_{k+1}$ say $\theta = \pi\alpha_{k+1} (-1 < \alpha_{k+1} < 1)$ such that $\sum\limits_{k=1}^{n} \alpha_k = 2$ where $(-1 < \alpha_k < 1)$. Note that $w = a_1, a_2, a_3, \ldots\ldots a_n$ must be branch points of the mapping function because at these points the angle between the tangents of the real axis does not remain constant during transformation.

(iii) The upper half of the w-plane and the interior must have one to one relation. Considering the mapping function of the following from

$$\frac{dz}{dw} = A\,(w - a_1)^{-\alpha_1}\,(w - a_2)^{-\alpha_2}\,(w - a_3)^{-\alpha_3} \ldots\ldots\ldots\ldots (w - a_n)^{-\alpha_n}, \tag{9.59}$$

The argument of $f'(w)$ will be given by

$$\text{Arg } f'(w) = \text{Arg } A - \alpha_1 \text{ Arg}(w - a_1) - \alpha_2 \text{Arg}(w - a_2) \ldots\ldots\ldots \alpha_n \text{ Arg}(w - a_n) \tag{9.60}$$

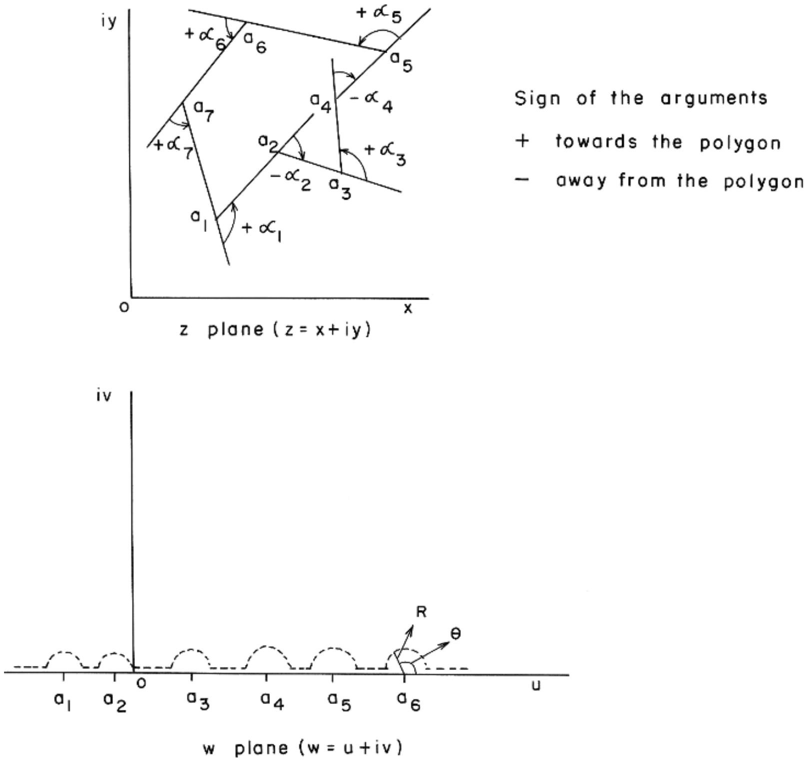


Fig. 9.7 a,b. Show the conformal transformation of a complex geometry of a polygon on to the real axis of the w-plane

Thus the net change in the argument is $\pi\alpha_1$. The argument remains constant while $a_1 < w < a_2$, but as we cross a_2 there is a further increase in the argument by πa_2. Thus there is an increase in the argument of f' (z) at each of the image vertices by an amount equal to say $\pi\alpha_k$ at the kth vertice. Total increase in the argument after crossing a_n is

$$\sum_{k=1}^{n} \pi\alpha_k = 2\pi = \text{Arg } A \tag{9.61}$$

Since the argument of f' (w) remains constant for any interval on the real axis of the w-plane, that interval should be mapped onto a side of the polygon. For examples, the interval $(a_{k+1} - a_k)$ must be mapped onto the kth side of length 1_k. This gives us n relations.

$$1_k = |A| \int_{a_k}^{a_{k+1}} \left| f' \begin{pmatrix} w \\ w \end{pmatrix} \right| dw \begin{pmatrix} w_{k+1} \\ w_k \end{pmatrix} (k = 1, 2, 3, \ldots n) \tag{9.62}$$

These n equations are used in determining n unknown quantities a_1, a_2, $a_3 \ldots\ldots a_n$ images of the vertices of the polygon.

The mapping function f (z) is analytic in the upper half plane and the real axis except at the images of the vertices $a_1, a_2, a_3 \ldots\ldots..a_n$. These points are branch points in mapping the real axis of the w-plane. We shall have to avoid these branch points by following an infinitesimally small semicircle with the centre at these branch points.

9.5.3 Determination of Unknown Constants

Integration (9.58), the mapping function may be written as follows

$$Z = A\int_{w_0}^{w} (w - a_1)^{-\alpha_1} (w - a_2)^{-\alpha_2} \ldots\ldots\ldots (w - a_n)^{-\alpha_n} \, dw. \tag{9.63}$$

Hence the n + 2 unknowns are A, $w_0, a_1, a_2, \ldots\ldots..a_n.\alpha_1, \alpha_2, \alpha_3, \alpha_n$ are easily given by external angles of polygon. Hence we do not consider them as unknowns. These constants can be determined applying suitable boundary conditions as demonstrated in different problems presented in this chapter.

9.5.4 S-C Transformation Theorem

This is one of the most powerful transformation techniques in a complex domain. It transforms the interior of a polygon in the z-plane on to the upper half of another plane, say the w_1 plane, in such a manner that the side of a polygon in the z-plane are transformed to the real axis of the w-plane. (Fig. 9.7 a,b) Schwarz and Christoffel independently have shown that given the required polygon, a certain differential equation may be written which when integrated gives directly the desired transformation. Consider the expression

$$\frac{dz}{dw} = A (w - a_1)^{\phi_1} (w - a_2)^{\phi_2} \ldots\ldots\ldots (w - a_n)^{\phi_n} \tag{9.64}$$

where A is a complex constant; $a_1, a_2, \ldots\ldots a_n$ and $\phi_1, \phi_2, \ldots\ldots \phi_n$ are real numbers and their arguments. Since the argument of a product of a complex number is equal to sum of the individual factors, we have

$$\begin{aligned}\frac{dz}{dw} &= \arg A + \phi_1 \arg (w - a_1) + \phi_2 \arg (w - a_2) \\ &\quad + \ldots\ldots + \phi_n \arg(w - a_n).\end{aligned} \tag{9.65}$$

The real numbers $a_1, a_2, \ldots\ldots a_n$ are plotted on the real axis of the w-plane (Fig. 9.7b). If w is a real number then the argument of $N_r = w - a_r$ is

$$\arg (w - a_r) = \begin{bmatrix} 0 \text{ if } w > a_r \\ \pi \text{ if } w < a_r \end{bmatrix}. \tag{9.66}$$

Let us suppose that the w traverses the real axis of the w-plane from left to right. Then $(w - a_r)$ will be positive if w is greater than a_r and it will be negative when w is less than a_r.

Let $\theta_r = \arg \frac{dz}{dw}$ when $a_r < w < a_{r+1}$.

We obtain

$$\theta_r = \arg A + (\phi_{r+1} + \phi_{r+2} + \ldots\ldots\ldots + \phi_n)\pi$$
$$\theta_{r+1} = \arg A + (\phi_{r+2} + \phi_{r+3} + \ldots\ldots\ldots + \phi_n)\pi. \tag{9.67}$$

Hence

$$\theta_{r+1} - \theta_r = -\pi\phi_{r+1}$$

Now

$$\arg \frac{dz}{dw} = \arg \frac{dx + i\,dy}{du} = \tan^{-1} \frac{dy}{dx}. \tag{9.68}$$

We see that this is the angle the element dz in the z-plane rotates into the mapping of dw in the w plane by the S–C transformation. As we move along the sides of the polygon in the z-plane, the corresponding movement in the w-plane for one to one correspondence will be along the real axis.

When the point w passes from the left of a_{r+1} to the right of a_{r+1} in the w-plane, the direction of the point z in the z-plane is suddenly changed by an angle of $-\pi\phi_{r+1}$ measured mathematically in the positive sense. If we imagine the broken line to form the closed polygon then the angle α_{r+1} measured between the two adjacent sides of the polygon is called an interior angle.

We, then have

$$\alpha_{r+1} - \pi\phi_{r+1} = \pi \tag{9.69}$$

Hence

$$\phi_{r+1} = \frac{\alpha_{r+1}}{\pi} - 1. \tag{9.70}$$

Substituting these values, we have

$$\frac{dz}{dw} = A\,(w - a_1)^{\frac{\alpha_1}{\pi} - 1} (w - a_2)^{\frac{\alpha_2}{\pi} - 1} (w - a_3)^{\frac{\alpha_3}{\pi} - 1}$$
$$= A \prod_{r=1}^{r=n} (w - a_r)^{\frac{\alpha_r}{\pi} - 1}. \tag{9.71}$$

Integrating the expression with respect to w, we have

$$z = A \int \prod_{r=1}^{n} (w - a_r)^{\frac{\alpha_r}{\pi} - 1} \, dw + B \tag{9.72}$$

where B is an arbitrary constant. This transformation transforms the real axis of the w-plane onto a polygon in the z-plane. The angles α_r are the interior angle of the polygon. The modulus of the constant A determines the size of the polygon and the argument of the constant A determines the orientation of the polygon. The constant B determines the location of the polygon.

9.6 Geophysical Problems on S-C Transformation

Two dimensional boundary value problems, which satisfy Laplace equation, are presented in this section with considerable details for only four problems. These four problems can be classified in three categories i.e., (i) problems where closed form solutions are available (ii) problems where closed form solutions are not possible and numerical integration was required for its solution (iii) problems where closed form solutions can be obtained using elliptic integrals and elliptic functions.

9.6.1 Problem 1 Conformal Transformation for a Substratum of Finite Thickness

In this problem two plane parallel boundaries are assumed. The region in between those boundaries constitutes the interior of a polygon with one angle only as we move from one point of the polygon to the other. This region is assumed to have a finite resistivity. Resistivities outside are infinitely high. In geophysical model simulation resistivity of the crystalline basement is assumed to be infinitely high in comparison to sediments of finite resistivity and thickness. Above the air earth boundary, resistivity of the air is infinitely high (Roy 1967).

Figure 9.8 a,b show the geometry of the problem in the Z plane and its transformation on the real axis of the w plane. Here the thickness and resistivity of the top layer is 'h' and 'ρ' respectively. The point source of current I is located at (0, h). For every problems we have to fix the origin on the z-plane. Here the origin is fixed at a depth 'h' from the surface and on top of the basement. Now the boundary conditions are dv/dy = 0 at y = 0 and h. The Schwarz Christoffel method of conformal transformation, for transforming the geometry of the z-plane on to the real axis of the w-plane are bounded by the following equation

$$\frac{dz}{dw} = A\,(w-a)^{-\alpha/\pi}\,(w-b)^{-\beta/\pi}\,(w-c)^{-\gamma/\pi}\,(w-d)^{-\delta/\pi} \tag{9.73}$$

where 'a', 'b', 'c', 'd' are the values of w at the corners of the polygon and 'A' is a constant as discussed. As we move from A to C, we cross the corner BB$'$ only as there are no other corners. Hence the (9.73) reduces to the form

$$\frac{dz}{dw} = A\,(w-a)^{-\alpha/\pi}\,. \tag{9.74}$$

Now as we move across BB$'$, our movement has turned through an angle of 180^0 or π because we started moving in the opposite direction. Therefore $\alpha = \pi$. Since we move towards the polygon (shown by the dotted lines) the sign of the argument 'α' will be + or positive. Therefore (9.74) reduce, to the form

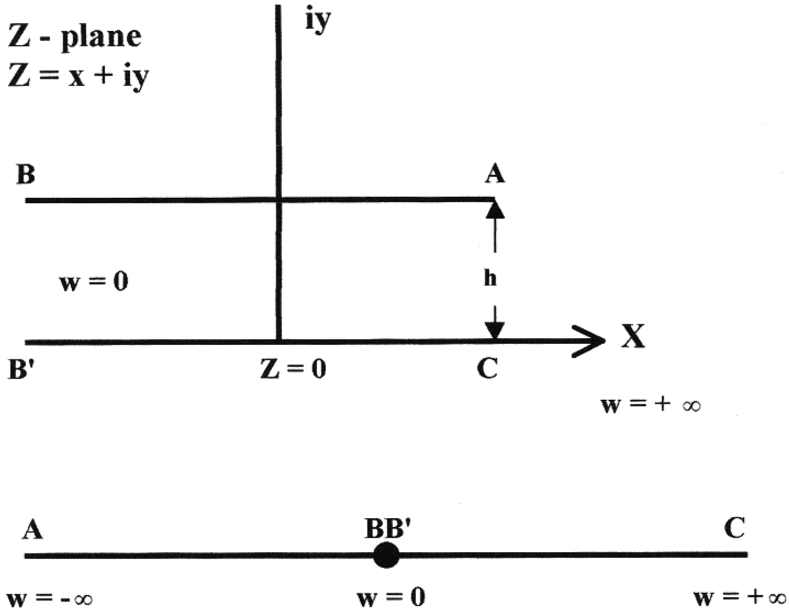


Fig. 9.8. (**a**) S-C transformation of a horizontal overburden of uniform thichness over a resistive basement in the Z plane; (**b**) Map on to the real axis of the W plane

$$\frac{dz}{dw} = A\,(w-0)^{-\pi/\pi} \qquad \text{because } w = 0 \text{ at } BB' \tag{9.75}$$

$$dz = A\frac{dw}{w}$$

$$z = A\ln w + C_1 \tag{9.76}$$

This is the transformation function which maps the $z = x + iy$ plane onto the real axis of the $w = u + i\nu$ plane. Now to solve the problem completely we have to determine 'A' and 'C_1'. The constant 'A' can be determined as follows. As we move from A' to C' over a large semicircle from π to 0, the movement in the z-plane is – 'ih', where 'h' is the thickness of the substratum and the vertical axis is the imaginary axis.

Let $w = Re^{i\theta}$. As $R \to \infty$ the (9.75) reduces to the form

$$-ih = A\int_{\pi}^{0} \frac{i\,Re^{i\theta}d\theta}{Re^{i\theta}} = A\int_{\pi}^{0} id\theta \tag{9.77}$$

or

$$A = h/\pi$$

We can choose the origin in such a way that the constant of integration becomes zero. If we choose $z = 0$ at $w = 1$. Then $C_1 = 0$. And our origin is fixed at $w = 1$. Therefore,

$$z = \frac{h}{\pi} \ln w \tag{9.78}$$

and

$$u = e^{\pi x/h}. \cos(\pi y/h) \tag{9.79}$$

$$\nu = e^{\pi x/h}. \sin(\pi y/h) . \tag{9.80}$$

Position of the current source I is $u = -1$, and potential in the w-plane ϕ_w is given by

$$\phi_w = \frac{I\rho}{\pi} \ln \frac{1}{r} \tag{9.81}$$

where r is the distance between the position of the source and the point of measurement. In the w-plane

$$r = \{(u + 1)^2 + \nu^2\}^{1/2}$$

therefore

$$\phi_w = -\frac{I\rho}{2\pi} \ln \left[(u + 1)^2 + \nu^2\right] .$$

In the z plane

$$\phi_z = -\frac{I\rho}{2\pi} \ln \left[e^{2\pi x/h} + 2.e^{\pi x/h} \cos(\pi y/h) + 1\right] . \tag{9.82}$$

For $y = h$

$$\phi_z = -\frac{I\rho}{2\pi} \ln \left[e^{2\pi x/h} + 2.e^{\pi x/h} + 1\right] \tag{9.83}$$

$$= \frac{I\rho}{\pi} \ln \left[e^{\pi x/h} - 1\right] . \tag{9.84}$$

9.6.2 Problem 2 Telluric Field over a Vertical Basement Fault

The geometry of the problem is shown is Fig. (9.9 a, b,). The problem was solved independently by Kunetz and de Gery (1956), Berdichevsku, M.N. (1950), Li. Y. Shu (1963). Here the basement is assumed to be vertically faulted. The thicknesses of the sedimentary layer on the downthrow and upthrow sides are 'H' and 'h' respectively. ρ_1, the resistivity of the sedimentary column, is assumed to be unity. ρ_2, the resistivity of the basement, is considered to be infinitely high ($\rho_2 = \infty$). Since the areas adjacent to the fault planes are suitable sites for oil accumulation, telluric current method is used as a reconnaissance tool for location of these fault planes.

Telluric currents or earth currents originate due to interaction of the earth's natural electromagnetic field with the earth's crust. Earth's natural electromagnetic field originates due to interaction of the solar flares with the magnetosphere of the earth. Telluric current sources are assumed to be

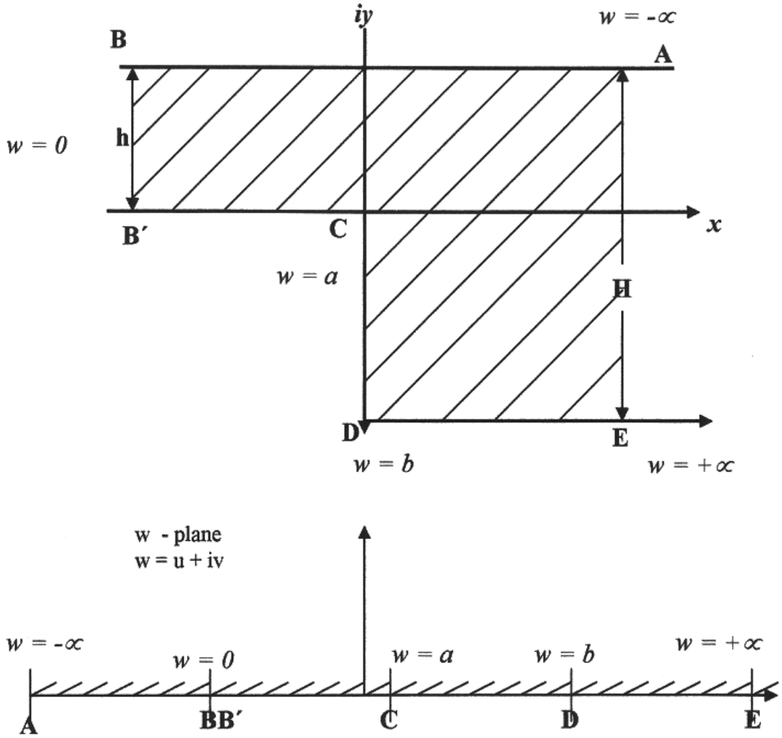


Fig. 9.9. (**a**) S - C Transformation of a vertical fault type of structure in the plane; (**b**) Map on to the real axis of the W plane

infinitely long line current sources placed at infinity such that uniform field is created.

The Schwarz-Christoffel transformation function, which maps the structure in the z-plane onto the real axis of the w-plane, is given by

$$\frac{\mathrm{dz}}{\mathrm{dw}} = \mathrm{A}\ \mathrm{w}^{-1}\,(\mathrm{w}-\mathrm{a})^{1/2}\,(\mathrm{w}-\mathrm{b})^{-1/2} \tag{9.85}$$

because the corners crossed are at BB', C and D; the corresponding angles are π, $-\pi/2$ and $+\pi/2$. The change in angles at C and D are $\pi/2$ in both cases. At the point C the movement is away from the polygon and at the point D the movement is towards the polygon. Hence there will be change in sign at C and D. This step is important to get the starting integral correctly before SC transformation starts.

Here

$$\mathrm{z} = \mathrm{A}\int \frac{(\mathrm{w}-\mathrm{a})^{1/2}}{\mathrm{w}\,(\mathrm{w}-\mathrm{b})^{1/2}}\mathrm{dw} + \mathrm{C}_1 \tag{9.86}$$

where A is a constant to be determined. Applying the first boundary condition, i.e. we integrate the (9.86) over a large semicircle of radius R($\rightarrow \infty$) as w changes from $-\infty$ to $+\infty$ and θ varies from 0 to π. The movement in the z-plane is – iH. Therefore, we have

$$-\mathrm{iH} = \mathrm{A}\int_{\pi}^{0} \frac{\left(\mathrm{Re}^{\mathrm{i}\theta} - \mathrm{a}\right)^{y_2}}{\mathrm{Re}^{\mathrm{i}\theta}\left(\mathrm{Re}^{\mathrm{i}\theta} - \mathrm{b}\right)^{1/2}} \mathrm{i\ Re}^{\mathrm{i}\theta}\,\mathrm{d}\theta. \tag{9.87}$$

Since R is infinitely large, (9.87) reduces to the form

$$-iH = A\int_{\pi}^{o} \frac{\left(\mathrm{Re}^{i\theta}\right)^{1/2}}{\mathrm{Re}^{i\theta}\left(\mathrm{Re}^{i\theta}\right)^{1/2}} i\mathrm{Re}^{i\theta} d\theta$$

or

$$\mathrm{A} = \frac{\mathrm{H}}{\pi}. \tag{9.88}$$

We next apply the second boundary condition i.e. we integrate (9.86) over an infinitesimal semicircle around BB' with $w = re^{i\theta}$ where $r \rightarrow 0$ and θ varies from π to 0. The movement in the z-plane is –ih. Hence

$$\begin{aligned} -ih &= \frac{H}{\pi}\int_{\pi}^{0} \frac{\left(\mathrm{Re}^{i\theta} - a\right)^{1/2}}{\mathrm{Re}^{i\theta}\left(\mathrm{Re}^{i\theta} - b\right)^{1/2}} i\mathrm{Re}^{i\theta} d\theta \\ &= \frac{\mathrm{H}}{\pi}\sqrt{\frac{\mathrm{a}}{\mathrm{b}}}\int_{\pi}^{0} \mathrm{i}\,\mathrm{d}\theta \\ &= \frac{\mathrm{h}}{\pi}\sqrt{\frac{\mathrm{a}}{\mathrm{b}}}. \end{aligned} \tag{9.89}$$

Since in this problem one of the values of a or b can be chosen arbitrarily. We choose the value of a = 1, therefore

$$\sqrt{\mathrm{b}} = \frac{\mathrm{H}}{\mathrm{h}}. \tag{9.90}$$

The integral (9.86) can be solved with the substitution of

$$\mathrm{t} = \sqrt{\frac{\mathrm{w} - 1}{\mathrm{w} - \mathrm{b}}}$$

since a = 1, we can write

$$\mathrm{z} = \frac{\mathrm{H}}{\pi}\int \frac{(\mathrm{w} - 1)^{1/2}\,\mathrm{dw}}{\mathrm{w}\,(\mathrm{w} - \mathrm{b})^{1/2}} = \frac{\mathrm{H}}{\pi}\int \frac{2\mathrm{t}^2\,(1 - \mathrm{b})\,\mathrm{dt}}{(\mathrm{t}^2 - 1)\left(\mathrm{bt}^2 - 1\right)} \tag{9.91}$$

because

$$w = \frac{bt^2 - 1}{t^2 - 1}$$

and

$$dw = \frac{2t\,(1 - b)}{\left(t^2 - 1\right)^2}.$$

The integral can be solved by the well known method of partial fraction and it reduces to the form

$$z = \frac{H}{\pi}\left[\frac{1}{\sqrt{b}}\ln\frac{\sqrt{b}t - 1}{\sqrt{b}t + 1} + \ln\frac{1 + t}{1 - t}\right] + C_1 \tag{9.92}$$

$$z = \frac{H}{\pi}\left[\frac{1}{\sqrt{b}}\ln\frac{\sqrt{b}\sqrt{w-1} - \sqrt{w-b}}{\sqrt{b}\sqrt{w-1} + \sqrt{w-b}} + \ln\frac{\sqrt{w-b} + \sqrt{w-1}}{\sqrt{w-b} - \sqrt{w-1}}\right] + C_1. \tag{9.93}$$

Equation (9.93) is the required mapping function for transformation of the geometry from the z-plane to the w-plane. Now in order to determine the value of C_1, we fix the origin in the z-plane at $w = 1$. Fixing the origin within the prescribed geometry of the z-plane is at our disposal. We can fix the origin at any point we want. Now if $z = 0$ at $w = 1$, we get from (9.93).

$$0 = \frac{H}{\pi}\left[\frac{1}{\sqrt{b}}\ln(-1) + \ln(1)\right] + C_1$$

or

$$0 = \frac{H}{\pi}\left[\frac{1}{\sqrt{b}}.i\pi\right] + C_1$$

$$C_1 = -i\frac{H}{\sqrt{b}} = -ih. \tag{9.94}$$

Computation of Telluric Field

In order to compute the telluric field, the source and sink are assumed to be at infinity. The source is at $z = -\infty$, at $w = 0$ and the sink is at $z = +\infty$, and $w = -\infty$, the expression for the telluric field in given by

$$\phi = \frac{I\rho_1}{\pi}\ln\frac{1}{w}. \tag{9.95}$$

Since the source and sink are assumed to be the infinitely long line electrodes, therefore

$$\frac{d\phi}{dw} = -\frac{I\rho_1}{\pi}\frac{1}{w} \tag{9.96}$$

The expression for the telluric field in the z-plane is

$$\frac{d\phi}{dz} = \frac{d\phi}{dw}\frac{dw}{dz}$$

or

$$\frac{d\phi}{dx} = \frac{d\phi}{du}\frac{du}{dx}. \tag{9.97}$$

Since the measurements are taken on the surface and on the real axis of the w plane $\nu = 0$, therefore

$$\frac{dw}{dz} = \frac{du}{dx}. \tag{9.98}$$

Again for a particular geometry of the structure $y = \text{const} \equiv ih$ and $dz = dx$. Therefore telluric field can be computed from the (9.97) where the expressions for $\frac{d\phi}{du}$ *and* $\frac{du}{dx}$ are known.

9.6.3 Problem 3 Telluric Field and Apparent Resistivity Over an Anticline

This is also a similar type of boundary value problem as discussed in the previous section i.e., in connection with flow of telluric currents over basement structure. Important points to be highlighted are: (i) close form solution of this problem is not possible, therefore one has to use numerical methods for solution of a part of this problem (ii) movements in the complex plane from the tip of the anticline and the trajectory of movement are demonstrated .

A two dimensional model of an anticline is shown in (Fig. 9.11 a,b) ECABD is the infinitely resistive basement and D' E' is the earth surface. The domain delineated by the polygonal boundary ECABD $D'E'$ is filled with a medium of finite resistivity ($\rho = 1$). Telluric currents, far away from the structure, are assumed to be horizontal current sheets confined to the channel bounded by the surface and the basement. The thickness of the overburden, away from the structure, is assumed to be unity. (Roy and Naidu 1970).

Potential distribution in a homogeneous and isotropic medium and in a source free region is given by the Laplace's equation

$$\frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial y^2} = 0. \tag{9.99}$$

The method of Schwarz-Christoffel transformation is used for conformal mapping. The transformation maps the complex geometry of the problem onto a simple geometry consisting of whole of the positive w-plane and the boundary along the u-axis while keeping the Laplace's equation and the boundary conditions invariant. Since ECABD $D'E'$ is a five cornered polygon, there will be five terms of the type $(w - a_n)^{-a_n/\pi}$ where $n = 1 \ldots\ldots 5$, out of which three may be chosen arbitrarily. The choices are as follows (9.10 a,b)

$a_1 = \pm\infty$ (Corresponding to EE')
$a_2 = -k$ (Corresponding to C)
$a_3 = 0$ (Corresponding to A
$a_4 = 1$ (Corresponding to B)
$a_5 = 1$ (Corresponding to DD')

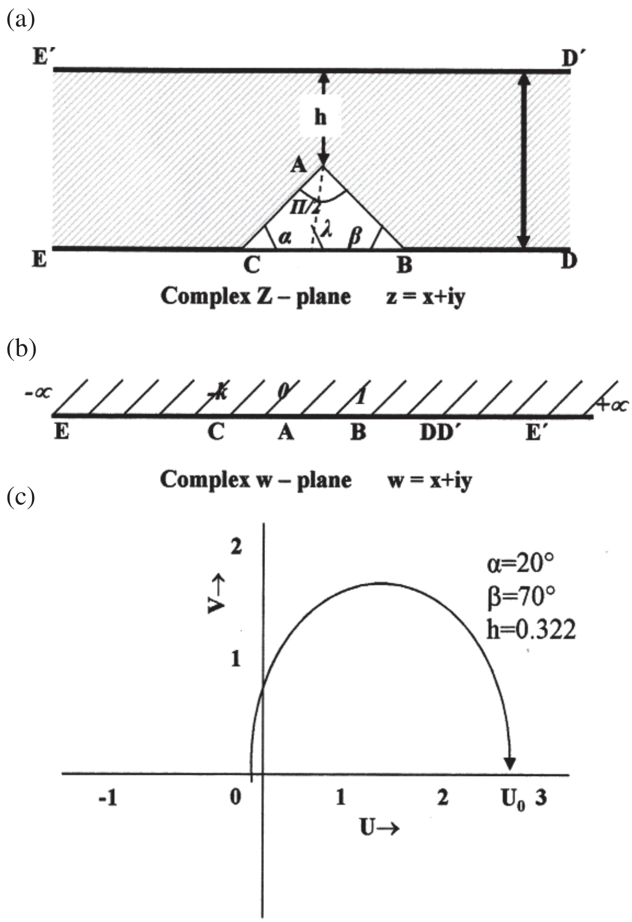


Fig. 9.10. (**a**) Map of the basement asymmetric anticline in the Z plane; (**b**) its map onto the real axis of the W plane; (**c**) trajectory in the W plane of vertical movement from the tip of the anticle A to the epicentre of A on the surface in the Z plane

and the arguments α_1, α_2, α_3, α_4, α_5 are given as follows

$\alpha_5 = \pi$ (Corresponding to EE′)
$\alpha_2 = \alpha$ (Corresponding to C)
$\alpha_3 = -(\alpha + \beta)$ (Corresponding to A)
$\alpha_4 = \beta$ (Corresponding to B)
$\alpha_5 = \pi$ (Corresponding to DD′)

Hence the Schwarz – Christoffel transformation function for the present problem may be expressed in the differential form as

$$\frac{dz}{dw} = A_1 w^{\frac{\alpha+\beta}{\pi}} (w+k)^{-\alpha/\pi} (w-l)^{-\beta/\pi} (w-1)^{-1} \tag{9.100}$$

which can be rewritten as

$$z = A \int_0^w \frac{w^{\frac{\alpha+\beta}{\pi}} dw}{(w+k)^{\alpha/\pi} (w-l)^{\beta/\pi} (w-l)} + C. \tag{9.101}$$

The term corresponding to the point EE′ does not enter into the differential equation because the value of w at E and E′ are $\pm\infty$. The constant of integration is $C = 0$ because the origin in the z-plane is fixed.

The evaluations of the unknowns $A_1, \ldots k$ are attempted using the following boundary conditions. Equation (9.101) is integrated at a point A along a semiinfinite circle in the upper half of the w-plane. Substituting $w = Re^{i\theta}$ and $R \to \infty$.

One gets

$$z = A_1 \int_\pi^0 \frac{\left(Re^{i\theta}\right)^{\frac{\alpha+\beta}{\pi}} i\, Re^{i\theta} d\theta}{\left(Re^{i\theta}+k\right)^{\alpha/\pi} \left(Re^{i\theta}-l\right)^{\beta/\pi} \left(Re^{i\theta}-l\right)} \tag{9.102}$$

By moving in an infinitely large semicircle from to 0 to π along the upper half of the w-plane, the corresponding movement in the z-plane is from E to E′, i.e. 'iH' where 'H' is the thickness of the overburden (assumed to be unity).

So $z = iH = i$

As $R \to \infty$ the integral reduces to

$$i = A_1 \int_\pi^0 i\, d\theta \text{ or } A_1 = -\frac{1}{\pi}. \tag{9.103}$$

Next the (9.101) is integrated along an infinitesimal semicircle around DD′ in the upper half of the w-plane. Substituting $w = 1 + re^{i\theta}$ as $r \to 0$, one gets

$$z = A_1 \int_\pi^0 \frac{\left(1+re^{i\theta}\right)^{\frac{\alpha+\beta}{\pi}} ire^{i\theta} d\theta}{(1+re^{i\theta}+k)^{\alpha/\pi} (1+re^{i\theta}-\ell)^{\beta/\pi} (1+re^{i\theta}-1)}$$

$$i = A_1 \int^0 \frac{id\theta}{(1+k)^{\alpha/\pi} (1-\ell)^{\beta/\pi}}$$

because movement in the z-plane is from D to D′ which is 'iH' or simply 'i'. Therefore

$$i = \frac{1}{\pi} \frac{1}{(1+k)^{\alpha/\pi} (1-1)^{\beta/\pi}} - i\pi$$

Or

$$k = (1 - \ell)^{\beta/\alpha} - 1. \tag{9.104}$$

One more equation is required to solve for 'k' and '1' uniquely. The following procedure is adopted. The inverse of the (9.98) is written as

$$\frac{dw}{dz} = A_1^{-1}\, w^{\frac{-(\alpha+\beta)}{\pi}} (w + k)^{\alpha/\pi} (w - \ell)^{\beta/\pi} (w - 1)\,. \tag{9.105}$$

A solution of the differential equation in a closed form is not possible except when both α/π and β/π can be expressed as ratio of two integers. Hence (9.105) has to be solved numerically.

Since

$$\frac{dw}{dz} = \frac{dv}{dy} - i\frac{du}{dy}$$

One gets

$$\frac{dv}{dy} = +\mathrm{Real}\left\{\frac{1}{A_1} w^{\frac{-(\alpha+\beta)}{\pi}} (w + k)^{\alpha/\pi} (w - \ell)^{\beta/\pi} (w - 1)\right\} \tag{9.106}$$

$$\frac{du}{dy} = +\mathrm{Imag}\left\{\frac{1}{A_1} w^{\frac{-(\alpha+\beta)}{\pi}} (w + k)^{\alpha/\pi} (w - \ell)^{\beta/\pi} (w - 1)\right\}. \tag{9.107}$$

Values of 'k' and '1' are selected and the system of (9.106) and (9.107) are integrated from $y = 0$ to $y = h$ keeping x constant (here $x = 0$). While integrating numerically the point in the w-plane describes a trajectory which commences at the origin and ends somewhere on the positive part of the real axis (Fig. 9.11c). This point represents the projection of the vertex of the triangle, it is termed as 'epicentre' U_0. Of course, the initial condition is $u = v = 0$ when $y = 0$. But at this point the integrands (9.106) and (9.107) are singular. They have an algebraic singularity and as a result a numerical solution cannot be initiated at this point. This difficulty is avoided by obtaining an asymptotic solution. Instead of starting $y = 0$, it is necessary to start at $y = \partial y$ where $\partial y << 1$ as follows.

$$\frac{dz}{dw} = A_1 w^{\frac{\alpha+\beta}{\pi}} (w + k)^{-\alpha/\pi} (w - \ell)^{-\beta/\pi} (w - 1)^{-1} \tag{9.108}$$

$$\frac{dz}{dw_{w=0}} = A_1 w^{\frac{\alpha+\beta}{\pi}} (k)^{-\alpha/\pi} (\ell)^{-\beta/\pi} (-1)^{-1}\,. \tag{9.109}$$

Therefore

$$z = -A_1 (k)^{-\alpha/\pi} (-\ell)^{-\beta/\pi} \frac{w^{\left(1+\frac{\alpha+\beta}{\pi}\right)}}{\left(1 + \frac{\alpha+\beta}{\pi}\right)} \tag{9.110}$$

$$= -A_{1.} (k)^{-\alpha/\pi} (\ell)^{-\beta/\pi} \frac{(u + iv)^{\left(1+\frac{\alpha+\beta}{\pi}\right)}}{\left(1 + \frac{\alpha+\beta}{\pi}\right)}. \tag{9.111}$$

Hence

$$x = -A_1 . (k)^{-\alpha/\pi} (\ell)^{-\beta/\pi} \frac{\left[\sqrt{u^2+v^2}\right]^{\left(1+\frac{\alpha+\beta}{\pi}\right)}}{\left(1+\frac{\alpha+\beta}{\pi}\right)} \cos\left\{\tan^{-1}\frac{v}{u}x\left(1+\frac{\alpha+\beta}{\pi}\right)-\beta\right\} \tag{9.112}$$

$$y = -A_1 . (k)^{-\alpha/\pi} (\ell)^{-\beta/\pi} \frac{\left[\sqrt{u^2+v^2}\right]^{\left(1+\frac{\alpha+\beta}{\pi}\right)}}{\left(1+\frac{\alpha+\beta}{\pi}\right)} \sin\left\{\tan^{-1}\frac{v}{u}x\left(1+\frac{\alpha+\beta}{\pi}\right)-\beta\right\}. \tag{9.113}$$

Since $x = 0$ through out the path

$$\cos\left\{\tan^{-1}\frac{v}{u}x\left(1+\frac{\alpha+\beta}{\pi}\right)-\beta\right\} = 0 \tag{9.114}$$

or

$$\left\{\tan^{-1}\frac{v}{u}x\left(1+\frac{\alpha+\beta}{\pi}\right)-\beta\right\} = \pi/2.$$

Hence

$$\frac{v}{u} = \tan\left(\frac{\pi/2+\beta}{1+\frac{\alpha+\beta}{\pi}}\right). \tag{9.115}$$

Either v or u may be chosen arbitrarily such that it is very nearly equal to zero. Then

$$v_0 = \frac{v}{u}u_0 \tan\left(\frac{\pi/2+\beta}{1+\frac{\alpha+\beta}{\pi}}\right). \tag{9.116}$$

$$y_0 = -A_1 . (k)^{-\alpha/\pi} (\ell)^{-\beta/\pi} \frac{\left[\sqrt{u_0^2+v_0^2}\right]^{\left(1+\frac{\alpha+\beta}{\pi}\right)}}{\left(1+\frac{\alpha+\beta}{\pi}\right)}. \tag{9.117}$$

It is now possible to integrate the (9.116) and (9.117) numerically. The projection of the epicentre is given by

$$u_0 = -\int_0^h \mathrm{Im\ ag}\left\{\frac{1}{A_1}w^{\frac{-(\alpha+\beta)}{\pi}}(w+k)^{\alpha/\pi}(w-\ell)^{\beta/\pi}(w-1)\right\}dy. \tag{9.118}$$

and

$$0 = -\int_0^h \mathrm{Real}\left\{\frac{1}{A_1}w^{\frac{-(\alpha+\beta)}{\pi}}(w+k)^{\alpha/\pi}(w-\ell)^{\beta/\pi}(w-1)\right\}dy. \tag{9.119}$$

In principle, (9.118) and (9.119) can now be used to solve for ‘k’ and ‘l’ uniquely. For solution of the problem values of ‘k’ and ‘l’ satisfying (9.118) and (9.119) are selected and ‘h’, for which (9.104) is satisfied, is determined. By varying ‘k’ ‘l’ , ‘h’ can be varied; however, this method is not particularly convenient if it is desired to vary h in regular steps. It may be noted that (9.118) and (9.119) are non-linear and can be solved only numerically. The Runge-Kutta method of order four was used. Figure 9.11c shows the trajectory of the path in the W-plane as the point moves from A (Fig. 9.11a) to the epicentre of the point A on the surface. It is needed for point to point mapping.

Computation of Telluric Field and Apparent Resistivity

Having transformed the complex geometry of the problem into a simple one, the field problem in the w-plane is solved first and then it is transferred onto the z-plane. The telluric field in the z-plane may be looked upon as that due to a point source and a point sink placed at $\pm\infty$ respectively. After transformation the point source at $+\infty$ is mapped onto DD' while the sink is still at infinity. The boundary conditions in the w-plane are: Potential gradient across the real axis is zero and the potential goes to zero on the semicircle with infinite radius in the upper half of the w-plane. The potential due to a point source, satisfying the above boundary conditions and the Laplace's equation is

$$\phi = \frac{I}{\pi} \ln \left[\frac{1}{w-1} \right]. \tag{9.120}$$

The gradient is transferred to the z-plane and the field equation is

$$E = -\frac{d\phi}{dz} = -\frac{d\phi}{dw}\frac{dw}{dz} = \frac{I}{\pi} A_1^{-1} (w)^{\frac{-(\alpha+\beta)}{\pi}} (w+k)^{\alpha/\pi} (w-\ell)^{\beta/\pi}. \tag{9.121}$$

Since the telluric field is measured on the earth's surface (where $y = 0$ and $v = 0$), (9.121) may further be simplified to

$$E = -\frac{d\phi}{dx} = \frac{I}{\pi} A_1^{-1} (u)^{\frac{-(\alpha+\beta)}{\pi}} (u+k)^{\alpha/\pi} (u-\ell)^{\beta/\pi}. \tag{9.122}$$

To determine the telluric field at a given point on the x-axis, the map of x on the u axis is computed and then (9.122) is evaluated taking point by point mapping. From (9.120) we can write the telluric field as

$$E = -\frac{d\phi}{dw} = \frac{I}{\pi} \left[\frac{1}{w-1} \right]. \tag{9.123}$$

The (9.105) changes to

$$\frac{dx}{du} = A_1 (u)^{\frac{-\alpha+\beta}{\pi}\alpha/\pi} (u-\ell)^{\beta/\pi} (u-1)^{-1} \tag{9.124}$$

for the purpose of mapping on the real axis. Taking its inverse the equation is

$$\frac{du}{dx} = A_1^{-1}\,(u)^{\frac{-(\alpha+\beta)}{\pi}}\,(u+k)^{\alpha/\pi}\,(u-\ell)^{\beta/\pi}\,(u-1)^{-1} \qquad (9.125)$$

which may be considered as a non-linear differential equation connecting u and x. By integrating (9.118) and (9.119) numerically with the initial condition, viz. $u = u_0$ at $x = 0$ which was determined earlier, a map of any point on the real x-axis is obtained. Figure 9.10d shows the telluric field response over an anticlinal structure.

9.6.4 Problem 4 Telluric Field Over a Faulted Basement (Horst)

In this section a problem is presented where closed form solution is obtained and conformal transformation is used along with elliptic integrals and elliptic functions. For the benefit of the readers a brief outline of the elliptic integrals and elliptic functions are given in Sect. 9.7. (Roy 1973).

A two dimensional model of the horst is shown in (Fig. 9.12 a, b). In this section a rectangular basement structure is chosen. As in the previous case it is assumed that the resistivity 'ρ' and the thickness 'H' of the sedimentary layer are both unity. 'h' and 'd' are respectively the throw of the fault and half width of the horst. In the z-plane the source and sink are assumed to be at $\pm\infty$ so that in the w-plane they are mapped at DE ($w = \frac{1}{k_1}$) and at $D'E'(w = -1/k_1)$ respectively.

Since the basement is assumed to be of infinite resistivity, the problem reduces to a Neumann problem, i.e., the potential should satisfy Laplace's

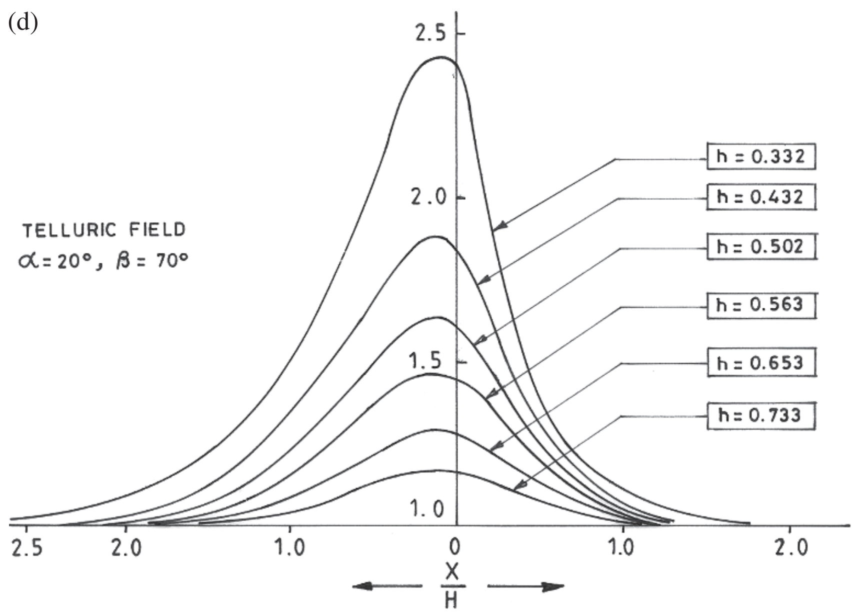


Fig. 9.11. (**d**)Telluric field anomaly over an asymmetric anticline

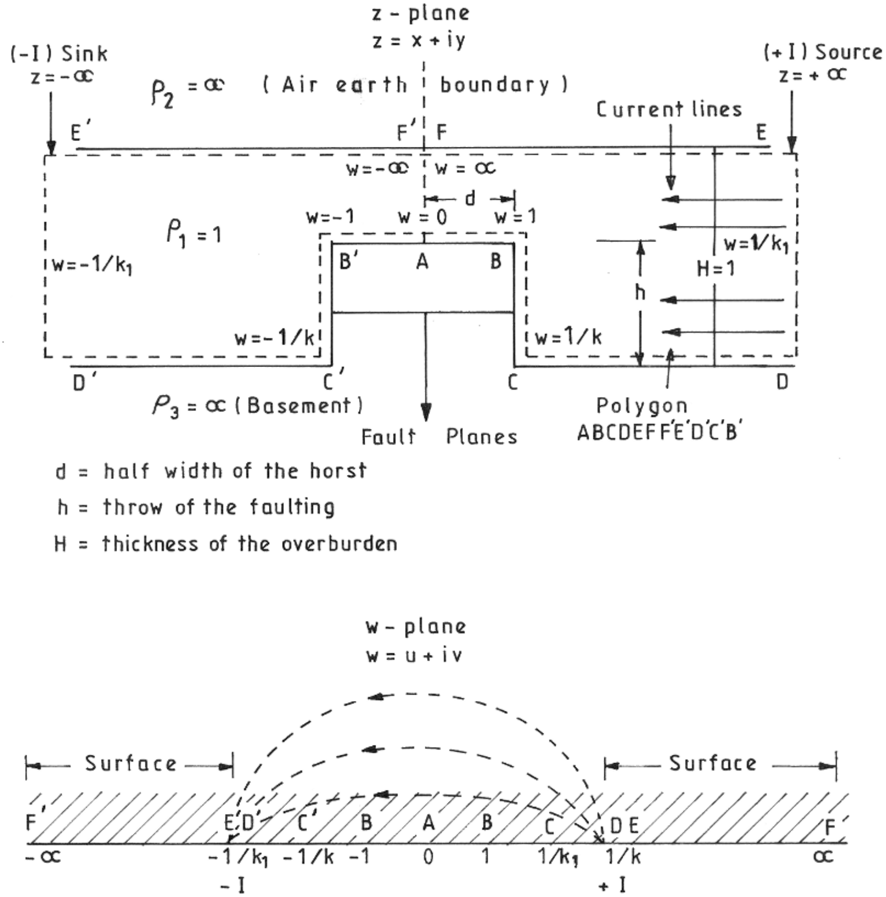


Fig. 9.12 a, b. Show the faulted basement with horst type of structure in the Z-plane and its map on the real axis of the W-plane; map of surface and subsurface onto the real axis of the w-plane ; flow of direct current from infinity; long line source and sink placed at infinite distance away in the z-plane and are respectively mapped at DE and E'D' on the real axis

equation with the condition that the normal gradient $\frac{\partial \phi}{\partial n} = 0$ throughout the entire boundary FEDCBA B′C′D′E′F′. A two dimensional potential distribution in a homogeneous isotropic and in a source free region is given by the Laplace's equation.

Closed form mathematical solution can be obtained using conformal transformation and elliptic integrals and function. These problems satisfy Laplace equation

$$\frac{\partial^2 \phi\,(x, y)}{\partial x^2} + \frac{\partial^2 \phi\,(x, y)}{\partial y^2} = 0. \tag{9.126}$$

Using the method of Schwarz – Christoffel transformation. The transformation function for the present problem may be expressed in the differential form as

$$\begin{aligned}\frac{\mathrm{dz}}{\mathrm{dw}} &= \mathrm{A}\,(\mathrm{w}-1)^{1/2}(\mathrm{w}-\mathrm{a})^{-1/2}(\mathrm{w}-\mathrm{b})^{-1}(\mathrm{w}+1)^{1/2}(\mathrm{w}+\mathrm{a})^{-1/2}(\mathrm{w}+\mathrm{b})^{-1}\\ &= \mathrm{A}\frac{(\mathrm{w}^2-1)^{1/2}}{(\mathrm{w}^2-\mathrm{a}^2)^{1/2}(\mathrm{w}^2-\mathrm{b}^2)} \qquad (9.127)\end{aligned}$$

$$z = A\int_0^w \frac{(w^2-1)^{1/2}\,dw}{(w^2-a^2)^{1/2}(w^2-b^2)} + C_1. \qquad (9.128)$$

Here $C_1 = 0$, because the origin in the z-plane is fixed. Equation (9.128) is in the form of an elliptic integral. The above integral is brought into a standard form by substituting $a = 1/k$ and $b = 1/k_1$ (Kober 1957) and Byrd Friedman (1954). In the next section, an introductory level discussion on elliptic integrals and elliptic functions are given. For better background, the readers will have to read the standard text books on elliptic integrals and functions. Equation (9.128) therefore reduces to

or

$$\mathrm{z} = \mathrm{C}\int_0^{\mathrm{w}} \frac{(1-\mathrm{w}^2)^{1/2}\,\mathrm{dw}}{(1-\mathrm{k}^2\mathrm{w}^2)^{1/2}(1-\mathrm{k}_1^2\mathrm{w}^2)} \qquad (9.129)$$

where $C = -k_1^2\frac{A}{k}$ is a new constant. Hence the values of C, k and k_1 are to be known before solving the boundary value problem. In order to evaluate the constant C the following boundary condition is applied. It is noted from (9.129) that at the point $w = 1/k_1$ an algebraic singularity is present. Therefore, (9.129) is integrated around an infinitesimal semicircle from $w = \frac{1}{k_1} + \varepsilon$ to $w = \frac{1}{k_1} - \varepsilon$ where $\varepsilon \to 0$ and thus the singular point is avoided. However, the approach is different from that shown in the previous problem. Since the shift in the z-plane is 'iH' the (9.129) can be written as :

$$\begin{aligned}\mathrm{iH} &= \mathrm{C}\frac{(1-1/\mathrm{k}_1^2)^{1/2}}{(1-\mathrm{k}^2/\mathrm{k}_1^2)^{1/2}(1+\mathrm{k}_1\mathrm{w})}\left\{\int_0^{\mathrm{w}}\frac{\mathrm{dw}}{(1-\mathrm{k}_1\mathrm{w})}\right\}_{1/\mathrm{K}_1-\varepsilon}^{1/\mathrm{k}_1+\varepsilon}\\ &= \mathrm{C}\frac{(1-1/\mathrm{k}_1^2)^{1/2}}{(1-\mathrm{k}^2/\mathrm{k}_1^2)^{1/2}(1+\mathrm{k}_1\mathrm{w})}\left\{-\frac{1}{\mathrm{k}_1}\log(\mathrm{k}_1\mathrm{w}-1)\right\}_{1/\mathrm{K}_1-\varepsilon}^{1/\mathrm{k}_1+\varepsilon}. \qquad (9.130)\end{aligned}$$

Now as w passes through the singular point ($w = 1/k_1$), the expression $(k_1w - 1)$ changes its sign, i.e., $\log(k_1w-1)$ increases by an amount $-i\pi$. Therefore, the change in the value of $-\left(\frac{1}{k_1}\right)\log(k_1w-1)$ from $\frac{1}{k_1} - \varepsilon$ to $\frac{1}{k_1} + \varepsilon$ is $i\frac{\pi}{k_1}$.

Hence

$$\mathrm{iH} = \frac{\mathrm{C}\,(1-\mathrm{k}_1^2)^{1/2}}{2\,(\mathrm{k}^2-\mathrm{k}_1^2)^{1/2}}.\mathrm{i}\frac{\pi}{\mathrm{k}_1}.$$

And

$$C = \frac{2H}{\pi}\frac{\left(k^2 - k_1^2\right) k_1}{\left(1 + k_1^2\right)^{1/2}}. \tag{9.131}$$

Substituting $k_1 = k\ \mathrm{sn}\alpha$ one gets

or

$$\begin{aligned} C &= \frac{2H}{\pi}\frac{\left(k^2 - k^2\mathrm{sn}^2\alpha\right)^{1/2}}{\left(1 - k^2\mathrm{sn}^2\alpha\right)^{1/2}} k\ \mathrm{sn}\alpha \\ &= \frac{2H}{\pi}\frac{k^2\mathrm{sn}\alpha\,\mathrm{cn}\alpha}{\mathrm{dn}\alpha}. \end{aligned} \tag{9.132}$$

where $\mathrm{sn}\alpha$, $\mathrm{cn}\alpha$ and $\mathrm{dn}\alpha$ are the Jacobian elliptic functions. In order to evaluate the constants k and k_1 (or k and α since $k_1 = k\mathrm{sn}\alpha$), it is necessary to integrate (9.129) and then to apply suitable boundary conditions.

Equation (9.129) can be rewritten in the form

$$z = C\int_0^w \frac{\left(1 - w^2\right) dw}{\left(1 - k^2w^2\right)^{1/2}\left(1 - k_1^2w^2\right)\left(1 - w^2\right)^{1/2}}. \tag{9.133}$$

The form of the integral suggests that it is an elliptic integral of the third kind which can be separated into two parts:

$$I_1 = C\int_0^w \frac{dw}{\left(1 - k^2w^2\right)^{1/2}\left(1 - k_1^2w^2\right)\left(1 - w^2\right)^{1/2}} \tag{9.134}$$

and

$$I_2 = C\int_0^w \frac{w^2 dw}{\left(1 - k^2w^2\right)^{1/2}\left(1 - k_1^2w^2\right)\left(1 - w^2\right)^{1/2}} \tag{9.135}$$

Here $I_1 = C\pi\ (w, k, k_1)$, i.e. the standard Legendre's from of elliptic integrals of the third kind. However, it is difficult to express I_2 in the Legendre's form. In order to avoid this difficulty both the integrals are transformed into the Jacobian form by substituting

$$w = \mathrm{sn}\lambda$$

therefore

$$dw = \mathrm{cn}\ \lambda\ \mathrm{dn}\ \lambda\ d\lambda.$$

Since

$$\mathrm{dn}\ \lambda = (1 - k^2\mathrm{sn}^2\lambda)^{1/2}$$

and

$$\mathrm{cn}\ \lambda = (1 - \mathrm{sn}^2\lambda)^{1/2}$$

The integral I_1 can be rewritten as

$$I_1 = C\int_0^{\lambda} \frac{\left(1-k^2\mathrm{sn}^2\lambda\right)^{1/2}\left(1-\mathrm{sn}^2\lambda\right)^{1/2}d\lambda}{\left(1-k_1^2\mathrm{sn}^2\lambda\right)^{1/2}\left(1-k^2\mathrm{sn}^2\lambda\right)^{1/2}\left(1-\mathrm{sn}^2\lambda\right)^{1/2}} \tag{9.136}$$

$$= C\int_0^{\lambda}\frac{d\lambda}{\left(1-k_1^2\mathrm{sn}^2\lambda\right)}$$

$$= C\left\{\lambda + k^2\mathrm{sn}^2\alpha\int_0^{\lambda}\frac{\mathrm{sn}^2\lambda d\lambda}{\left(1-k_1^2\mathrm{sn}^2\lambda\right)}\right\} \qquad \text{where } k_1 = k\,\mathrm{sn}\alpha$$

$$= C\left\{\lambda + \frac{\mathrm{sn}\,\alpha}{\mathrm{cn}\alpha\,\mathrm{dn}\alpha}\pi\left(\lambda,\alpha\right)\right\} \tag{9.137}$$

where $\pi(\lambda,\alpha)$ is the Jacobian form of the elliptic integral of the third kind.

$$I_2 = C\int_0^{\lambda}\frac{\left(\mathrm{sn}^2\lambda\right)\left(1-\mathrm{sn}^2\lambda\right)^{1/2}\left(1-k^2\mathrm{sn}^2\lambda\right)^{1/2}d\lambda}{\left(1-k^2\mathrm{sn}^2\lambda\right)^{1/2}\left(1-k_1^2\mathrm{sn}^2\lambda\right)^{1/2}\left(1-\mathrm{sn}^2\lambda\right)}$$

$$= C\int_0^{\lambda}\frac{\mathrm{sn}^2\lambda d\lambda}{\left(1-k_1^2\mathrm{sn}^2\lambda\right)}$$

$$= C\frac{\pi\left(\lambda,\alpha\right)}{k^2\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}. \tag{9.138}$$

Hence

$$z = C\left\{\lambda + \frac{\mathrm{sn}\alpha}{\mathrm{cn}\alpha\,\mathrm{dn}\alpha}\pi\left(\lambda,\alpha\right) - \frac{\pi\left(\lambda,\alpha\right)}{k^2\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}\right\}. \tag{9.139}$$

Equation (9.139) is the function required for mapping the z-plane onto the real axis of the w-plane.

Now at $w = 1/k, z = d - ih$ as shown in Fig. 11 b, c. Therefore

$$\lambda = \mathrm{sn}^{-1}w = \int_0^{1/k}\frac{dw}{\left(1-w^2\right)^{1/2}\left(1-k^2w^2\right)^{1/2}}$$

$$= K + iK'. \tag{9.140}$$

Here K and K′ are the complete elliptic integrals of the first kind from the definition. Hence (9.139) can be rewritten as

$$d - ih = C\left\{\left(K+iK'\right) + \frac{\mathrm{sn}\alpha}{\mathrm{cn}\alpha\,\mathrm{dn}\alpha}\pi\left(K+iK',\alpha\right) - \frac{\pi\left(K+iK',\alpha\right)}{k^2\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}\right\}. \tag{9.141}$$

From the well-known relation between the Jacobi's elliptic integral of the third kind and his theta and zeta functions one can write

$$\pi\left(\mathrm{K}+\mathrm{i}k',\alpha\right)=\frac{1}{2}\log\frac{\Theta\left(\mathrm{K}+\mathrm{i}k'-\alpha\right)}{\Theta\left(\mathrm{K}+\mathrm{i}\mathrm{K}'+\alpha\right)}+\left(\mathrm{K}+\mathrm{i}\mathrm{K}'\right)\mathrm{Z}\left(\alpha\right) \tag{9.142}$$

which simplifies to

$$KZ\left(\alpha\right)+i\left(\frac{\pi\alpha}{2K}+K'Z\left(\alpha\right)\right) \tag{9.143}$$

where z (α) is the Jacobi's zeta function. On substituting the value of C and $\pi(\mathrm{K}+\mathrm{i}\mathrm{K}',\alpha)$ (9.139) becomes

$$\mathrm{d}-\mathrm{i}\mathrm{h}=\frac{2\mathrm{H}}{\pi}\mathrm{k}^2\frac{\mathrm{cn}\alpha\,\mathrm{sn}\alpha}{\mathrm{dn}\alpha}\left[\left(\mathrm{K}+\mathrm{i}\mathrm{K}'\right)+\frac{\mathrm{sn}\alpha}{\mathrm{cn}\alpha\,\mathrm{dn}\alpha}\right] \tag{9.144}$$

$$\left\{\mathrm{K}\,\mathrm{Z}\left(\alpha\right)+\mathrm{i}\left(\frac{\pi\alpha}{2\mathrm{K}}+\mathrm{K}'\mathrm{Z}\left(\alpha\right)\right)\right\}-\frac{1}{\mathrm{k}^2\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}$$
$$\left.\left\{\mathrm{K}\,\mathrm{Z}\left(\alpha\right)+\mathrm{i}\left(\frac{\pi\alpha}{2\mathrm{K}}+\mathrm{K}'\mathrm{Z}\left(\alpha\right)\right)\right\}\right]. \tag{9.145}$$

Separating the real and imaginary parts one gets the following two equations

$$\frac{\mathrm{d}}{\mathrm{H}}=\frac{2\mathrm{K}}{\pi\,\mathrm{dn}\alpha}\left[\mathrm{k}^2\mathrm{cn}\alpha\mathrm{sn}\alpha+\frac{\mathrm{k}^2\mathrm{sn}^2\alpha}{\mathrm{dn}\alpha}\mathrm{Z}\left(\alpha\right)-\frac{\mathrm{Z}\left(\alpha\right)}{\mathrm{dn}\alpha}\right] \tag{9.146}$$

$$\frac{\mathrm{h}}{\mathrm{H}}=\frac{2}{\pi\mathrm{dn}\alpha}\left[\frac{1}{\mathrm{dn}\alpha}\left(\frac{\pi\alpha}{2\mathrm{K}}+\mathrm{K}'\mathrm{Z}\left(\alpha\right)\right)-\frac{\mathrm{k}^2\mathrm{sn}^2\alpha}{\mathrm{dn}\alpha}\left(\frac{\pi\alpha}{2\mathrm{K}}+\mathrm{K}'\mathrm{Z}\left(\alpha\right)\right)-\mathrm{K}'\mathrm{k}^2\mathrm{cn}\alpha\,\mathrm{sn}\alpha\right]. \tag{9.147}$$

Mention may be made that all the factors, e.g., K, K$'$, snα, cnα, dnα, Z (α) are dependent on the values of k and α. Only two equations are available and there are several unknowns. It is not possible as yet to determine the values of k. and k_1 (or k snα) from known values of d/H and h/H which are fixed by the geometry of the structure. To avoid this difficulty the process is reversed. In that, a series of values of k and α are assumed and the corresponding values of d/H and h/H are determined. The values of k and α can be established for a given set of d/H and h/H ratios. For chosen values of k and α, the value of C can be determined as

$$\mathrm{C}=\frac{2\mathrm{H}}{\pi}\frac{\mathrm{k}^2\mathrm{sn}\alpha\,\mathrm{cn}\alpha}{\mathrm{dn}\alpha}. \tag{9.148}$$

Mapping the desired region onto the upper half of the w-plane along the entire u-axis, the problems can now be solved. In order to plot the telluric field point by point on the surface, the values of u for different values of x must be known. For this purpose, the inverse transformation function (indicated in the previous problem), from (9.127) can be written as:

$$\frac{\mathrm{dw}}{\mathrm{dz}}=\frac{1}{\mathrm{C}}\frac{\left(1-\mathrm{k}^2\mathrm{w}^2\right)^{1/2}\left(1-\mathrm{k}_1^2\mathrm{w}^2\right)}{\left(1-\mathrm{w}^2\right)^{1/2}}. \tag{9.149}$$

Since

$$\frac{\mathrm{dw}}{\mathrm{dz}}=\frac{\mathrm{du}}{\mathrm{dx}}+\mathrm{i}\frac{\mathrm{dv}}{\mathrm{dx}},$$

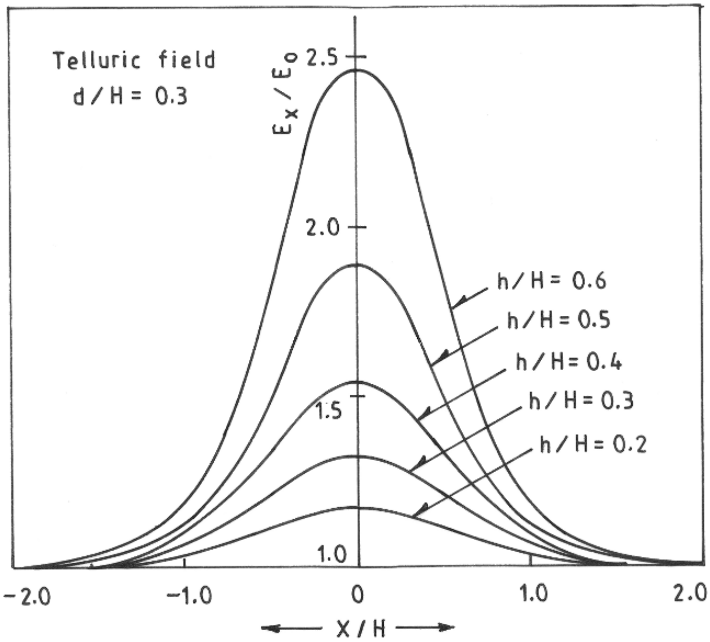


Fig. 9.13. Telluric field anomaly over a horst type of structure

the (9.149) reduces to

$$\frac{du}{dx} = \frac{1}{C}\left(1 - k^2u^2\right)^{1/2}\left(1 - k_1^2u^2\right)\left(1 - u^2\right)^{-1/2} \tag{9.150}$$

because v = 0 on the u-axis which corresponds to the air earth boundary in the z-plane (for $\infty > u > 1/k_1$).

Equation (9.150) is a non-linear differential equation and, in principle, may be numerically integrated. Although it is necessary to know the values of both 'u' and 'x' at the staring point, in order to determine the telluric field as explained earlier, the source and the sink is placed at $\pm\infty$ in the z-plane such that they are at $w = 1/k_1$ and $w = -1/k_1$ respectively in the w-plane. The potential distribution is given by

$$\phi = -\frac{I}{\pi}\log\left(w - 1/k_1\right) + \frac{I}{\pi}\log\left(w + 1/k_1\right). \tag{9.151}$$

Therefore, the telluric field on the surface (Fig. 9.13) is given by

$$\begin{aligned} E &= -\frac{d\phi}{dx} = -\frac{d\phi}{du}.\frac{du}{dx} \\ &= -\frac{I}{\pi}.\frac{1}{C}\frac{2k_1\left(1 - k^2u^2\right)^{1/2}}{\left(1 - u^2\right)^{1/2}}. \end{aligned} \tag{9.152}$$

In this section one problem is presented where conformal transformation is used along with the elliptic integrals and elliptic functions. Researchers on electrical communication, electrical power engineering and applied mathematics need this kind of solution of boundary value problems.

Analytical solution results can be compared with those obtained numerically for calibration of the finite element or finite difference code.

9.7 Elliptic Integrals and Elliptic Functions

Let us consider the integral

$$\lambda = \int_0^{w} \frac{dw}{(1 - w^2)^{1/2}(1 - k^2w^2)^{1/2}}. \tag{9.153}$$

The constant k is called the modulus of λ. In actual physical problems the value of k is found to vary in between zero and unity. For this reason k can be, and often is, designed by sin θ where θ is called the modulus angle. This integral is called the elliptic integral of the first kind. Such integrals were called elliptic because they were first encountered in the determination of length of arc of an ellipse. The form of the integral (9.153) is known as Jacobi's notation of the elliptic integral.

9.7.1 Legendre's Equation

Second notation is that of Legendre. It can be obtained from that of Jacobi by putting

$$w = \sin\phi \tag{9.154}$$

where ϕ is called the amplitude angle. This angle may be obtained from θ the modulus angle. We get

$$\lambda = \int_0^{\phi} \frac{d\phi}{\left(1 - k^2\sin^2\phi\right)^{1/2}}. \tag{9.155}$$

The integral in (9.155) is the elliptic integral of the first kind. In Legendre's notation $\phi = \mathrm{am}\alpha$ signifying that ϕ is the amplitude of α.

9.7.2 Complete Integrals

If the upper limit of the integral is made unity the integral is said to be complete. The complete elliptic integral of the first kind is always denoted by K and hence

$$K = \int_0^1 \frac{dw}{(1-w^2)^{1/2}(1-k^2w^2)^{1/2}}. \tag{9.156}$$

Equation (9.156) gives k in Jacobi's notation. In Legendre's notation since $w = \sin\phi$, the limits of integration are from zero to $\pi/2$. Hence in Legendre's notation

$$K = \int_0^{\pi/2} \frac{d\phi}{\left(1-k^2\sin^2\phi\right)^{1/2}}. \tag{9.157}$$

It may sometimes be necessary to state the modulus of K. We then write K (k) for the complete elliptic integral of the first kind of modulus k.

The Complementary Modulus

We now need to define a complementary modulus k' related to the modulus k by the equation $k^2 + k'^2 = 1$ or

$$k' = (1-k^2)^{1/2} \tag{9.158}$$

From (9.158), when k is written as $\sin\theta, k'$ must be written as $\cos\theta$. The complete elliptic integral of the 1st kind to modulus k' must be entirely different from K which has the modulus k.

$$K'(k) = K(k'). \tag{9.159}$$

Hence K and K' are related through their modulus and K, K' must be a real number.

By this definition K' is given by the integral

$$K' = \int_0^1 \frac{dw}{(1-w^2)^{1/2}(1-k'^2w^2)^{1/2}}. \tag{9.160}$$

The next thing to find is the equation defining K' in terms of the normal modulus k. In (9.160), let

$$\begin{aligned} t &= \frac{1}{(1-k'^2w^2)^{1/2}} \\ w^2 &= \frac{1}{k'^2}\left(1-\frac{1}{t^2}\right) \\ &= \frac{t^2-1}{k'^2t^2} \end{aligned} \tag{9.161}$$

Therefore,

$$1-\mathrm{w}^2=\frac{\mathrm{k}'^2\mathrm{t}^2-\mathrm{t}^2+1}{\mathrm{k}'^2\mathrm{t}^2}$$
$$=\frac{1-\mathrm{t}^2\left(1-\mathrm{k}'^2\right)}{\mathrm{k}'^2\mathrm{t}^2}=\frac{1-\mathrm{k}^2\mathrm{t}^2}{\mathrm{k}'^2\mathrm{t}^2}$$

and hence

$$\left(1-\mathrm{w}^2\right)^{1/2}=\frac{\left(1-\mathrm{k}^{2'}\mathrm{t}^2\right)^{1/2}}{\mathrm{k}'\mathrm{t}} \tag{9.162}$$

Furthermore from (9.161)

$$\mathrm{w}=\frac{\left(\mathrm{t}^2-1\right)^{1/2}}{\mathrm{k}'\mathrm{t}}$$

Therefore

$$dw=\frac{t^2\left(t^2-1\right)^{-1/2}\left(t^2-1\right)^{1/2}}{k't^2}dt \tag{9.163}$$
$$\Rightarrow\quad\frac{\mathrm{t}^2-\left(\mathrm{t}^2-1\right)}{\mathrm{k}'\mathrm{t}^2\left(\mathrm{t}^2-1\right)^{1/2}}$$
$$\Rightarrow\quad\frac{\mathrm{dt}}{\mathrm{k}'\mathrm{t}^2\left(\mathrm{t}^2-1\right)^{1/2}}.$$

Substituting these values, we get

$$\mathrm{k}'=\int\frac{\mathrm{dt}}{\mathrm{k}'\mathrm{t}^2\left(\mathrm{t}^2-1\right)^{1/2}}\frac{\mathrm{k}'\mathrm{t}}{\left(1-\mathrm{k}'^2\mathrm{t}^2\right)^{1/2}}$$

with new limits. To find the new limits note that when $\mathrm{w}=0,\mathrm{t}=1$ and when $\mathrm{w}=1$

$$\mathrm{t}=\frac{1}{\left(1-\mathrm{k}'^2\right)^{1/2}}=\frac{1}{\mathrm{k}}.$$

Therefore, simplifying the integrand and adding the new limits, we get

$$\mathrm{k}'=\int_1^{1/\mathrm{k}}\frac{\mathrm{dt}}{\left(\mathrm{t}^2-1\right)^{/2}\left(1-\mathrm{k}'^2\mathrm{t}^2\right)^{1/2}}$$
$$\mathrm{k}'=\int_1^{1/\mathrm{k}}\frac{\mathrm{dt}}{\mathrm{i}\left(1-\mathrm{t}^2\right)^{/2}\left(1-\mathrm{k}'^2\mathrm{t}^2\right)^{1/2}} \tag{9.164}$$

and this is the required relation. Notice that the integral is the same as that for k but the limits are changed and the integral is imaginary. Combining the integrands of K and K′ one gets the total integration from the lower limit zero to the upper limit 1/k and it is the complex number

$$\mathrm{K}+\mathrm{i}\mathrm{K}'=\int_1^{1/\mathrm{k}}\frac{\mathrm{dw}}{\left(1-\mathrm{w}^2\right)^{1/2}\left(1-\mathrm{k}'^2\mathrm{w}^2\right)^{1/2}}. \tag{9.165}$$

9.7.3 Elliptic Functions

The elliptic functions of interest to engineers are Jacobi's elliptic functions.

$$\lambda = \int_0^{\omega} \frac{dw}{(1-w^2)^{1/2}} = \sin^{-1} w. \tag{9.166}$$

Thus the effect of getting rid of k is to make λ a function only of w and more over an elementary function of w. we get

$$w = \sin\lambda.$$

It can be shown that Legendre's form of the integral can be obtained by putting

$$\begin{aligned} w &= \sin\phi \\ &= \sin\ \mathrm{am}\ \lambda \\ &= \mathrm{sn}\ \lambda \\ &= \mathrm{sn}\ \lambda\ (k) \\ &= \mathrm{sn}(\lambda, 0) = \sin\lambda. \end{aligned} \tag{9.167}$$

Now

$$\begin{aligned} (1-w^2)^{1/2} &= \cos\phi = \cos\ \mathrm{am}\ \lambda \\ &= \mathrm{cn}\lambda = \mathrm{cn}(\lambda, k). \end{aligned} \tag{9.168}$$

Since $\sin^2\phi + \cos^2\phi = 1$, it can be deduced that

$$\mathrm{sn}^2\lambda + \mathrm{cn}^2\lambda = 1 \tag{9.169}$$

$$\begin{aligned} \mathrm{dn}\ \lambda &= (1-k^2w^2)^{1/2} \\ &= (1-k^2\ \sin^2\phi)^{1/2} \\ &= (1-k^2\ \mathrm{sn}^2\lambda)^{1/2}. \end{aligned} \tag{9.170}$$

The three functions $\mathrm{sn}\lambda$, $\mathrm{cn}\lambda$, $\mathrm{dn}\lambda$ are the principal Jacobian elliptic functions

$$\mathrm{tn}\ \lambda = \frac{\mathrm{sn}\ \lambda}{\mathrm{cn}\ \lambda}. \tag{9.171}$$

We have defined

$$K = \int_0^{\pi/2} \frac{d\phi}{(1-k^2\sin^2\phi)^{1/2}}.$$

If the upper limit of the integral is some value of ϕ less than $\pi/_2$, the integral is incomplete and is then written as F (ϕ, k). Thus

$$F(\phi, k) = \int_0^{\phi} \frac{d\phi}{\left(1 - k^2 \sin^2 \phi\right)^{1/2}}. \tag{9.172}$$

The integral is frequently written in terms of the amplitude angle and the modular angle as F (ϕ, θ) and rather less frequently in terms of the amplitude and modulus as F (w, k). Here

$$F(w, k) = F(\pi/2, k) \equiv K(k) = K \tag{9.173}$$

$$F(w, k) = F(\phi, k) = \lambda \tag{9.174}$$

We have

$$\lambda = sn^{-1}(w, k) \tag{9.175}$$

Therefore

$$sn^{-1}(w, k) = F(w, k)$$

$$I = \int_0^{w} \frac{\left(1 - k^2 w^2\right)^{1/2}}{\left(1 - w^2\right)^{1/2}}\, dw. \tag{9.176}$$

This is the Jacobi's form of the elliptic integral of the second kind and it is denoted by E (w, k) where, as before, w is the argument or amplitude and k is the modulus. Legendre's form is given by

$$E(\phi, k) = \int_0^{\phi} \left(1 - k^2 \sin^2 \phi\right) d\phi \tag{9.177}$$

Another form is obtained by introducing the variable u where ϕ is the amplitude of u.

$\sin\phi = \mathrm{snu}$
and by differentiating both sides
$\mathrm{cn}\phi\ d\phi = \mathrm{cnu\ dnu\ du}$.

$$d\phi = \frac{\mathrm{cnu\ dnu\ du}}{\left(1 - sn^2 u\right)^{1/2}} = \mathrm{dnu\ du}. \tag{9.178}$$

The integral (9.177) can be written as

$$E(u, k) = \int_0^{u} \left(1 - k^2 sn^2 u\right)^{1/2}\ \mathrm{dnu\ du}$$

$$= \int_0^{u} dn^2 u\ du. \tag{9.179}$$

It defines the elliptic integral of the second kind in terms of elliptic functions.

Complete elliptic integrals of the second kind is expressed as

$$E(1,k) \equiv E(\pi/2,k) \equiv E(k) \equiv E \tag{9.180}$$

9.7.4 Jacobi's Zeta Function

$$Z(\phi) = E(\phi) = Z(\phi)\frac{E}{K} \tag{9.181}$$

all to modulus k.

9.7.5 Jacobi's Theta Function

More important still are Jacobi's Theta functions. His original theta function is denoted by Θ and widely used and they are indeed almost indispensable.

$$Z(\phi) = \frac{\Theta'(\phi)}{\Theta(\phi)}$$

a relation given by

$$\int_0^\phi Z(\phi)\,d\phi = \log\Theta(\phi) + C \tag{9.182}$$

where series for theta function is

$$\Theta(\phi) = 1 + 2\sum_{m=1}^{\infty}(-1)^m .q^{m^2}\cos\frac{m\pi\phi}{k} \tag{9.183}$$

where $q = e^{-\pi\frac{K'}{K}}$.

Elliptic Integral of the Third kind is

$$\pi(w,k_1,k) = \int_0^w \frac{d\phi}{(1-k_1^2w^2)(1-k^2w^2)^{1/2}(1-w^2)^{1/2}}. \tag{9.184}$$

Now putting $w = \sin\phi$ giving

$$\pi(\phi,k_1,k) = \int_0^\phi \frac{d\phi}{\left(1-k_1^2\sin^2\phi\right)\left(1-k^2\sin^2\phi\right)^{1/2}}. \tag{9.185}$$

If we put $w = snu$

$dw = cnu\ dnu\ du$

$$\pi(u,k_1,k) = \int_0^u \frac{du}{1-k_1^2 sn^2 n}. \tag{9.186}$$

All these are three forms of Legendre's elliptic integral .

9.7.6 Jacobi's Elliptic Integral of the Third Kind

Jacobi adopted not merely a different form but an entirely different way of presentation of elliptic integrals of the third kind.

Putting

$$k_1 = k \text{ sn } \alpha \text{ to mod } k,$$

the integrand is

$$= \frac{du}{1 - k^2 \text{ sn}^2\alpha \text{ sn}^2 u}. \tag{9.187}$$

With this substitution Jacobi defined his elliptic integral of the third kind as

$$\pi(u, \alpha) = k^2 \text{ sn}\alpha \text{ cn } \alpha \text{ dn } \alpha \int_0^u \frac{\text{sn}^2 \text{ u du}}{1 - k^2 \text{sn}^2\alpha \text{ sn}^2 u} \tag{9.188}$$

(H.E.I.E.P. Handbook of Elliptic Integrals for Engineers and Physicists 400.01).

Jacobi's definition of the elliptic integral can be expressed in terms of his zeta function and also in terms of his theta functions

$$\text{sn}(u + \alpha) + \text{sn}(u - \alpha) = \frac{2 \text{ snu cn}\alpha \text{ du}\alpha}{1 - k^2 \text{ sn}^2 u \text{ sn}^2 \alpha}.$$
$$\text{(H.E.I.E.P.123.02)} \tag{9.189}$$

And hence substituting these values

$$\pi(u, \alpha) = \frac{1}{2} k^2 \text{ sn}\alpha \int_0^u \text{snu}\{\text{sn}(u + \alpha) + \text{sn}(u - \alpha)\} \text{ du}. \tag{9.190}$$

But from (H.E.I.E.P. 142.02)

$$k_2 \text{ sn}\alpha \text{ snu sn}(u + \alpha) = -Z(u + \alpha) + Z(u) + Z(\alpha)$$

and

$$k_2 \text{ sn}\alpha \text{ snu sn}(u - \alpha) = Z(u - \alpha) - Z(u) + Z(\alpha).$$

Hence

$$\pi(u, \alpha) = \frac{1}{2} \int_0^u \{Z(u - \alpha) - Z(u + \alpha) + 2Z(\alpha)\} \text{ du}. \tag{9.191}$$

This gives a definition of Jacobi's elliptic integral of the third kind in terms of this zeta function.

Since

$$\int_0^u Z(u) \text{ du} = \log \Theta(u) + C, \tag{9.192}$$

we get

$$\pi(\mathrm{u},\alpha) = \frac{1}{2}\{\log\Theta(\mathrm{u}-\alpha) - \log\Theta(\mathrm{u}+\alpha)\} + \int_0^{\mathrm{u}} \mathrm{Z}(\alpha)\ \mathrm{du}$$
$$= \frac{1}{2}\log\frac{\Theta(\mathrm{u}-\alpha)}{\Theta(\mathrm{u}+\alpha)} + \mathrm{uz}(\alpha). \tag{9.193}$$

It gives yet another definition of the elliptic integral of the third kind in terms of his zeta and theta functions.

Real and Imaginary parts of $\pi(\mathrm{K}+\mathrm{iK}',\alpha)$

$$\pi\left(\mathrm{K}+\mathrm{iK}',\alpha\right) = \frac{1}{2}\left\{\log\Theta\left(\mathrm{K}-\alpha+\mathrm{iK}'\right) - \log\Theta\left(\mathrm{K}+\theta+\mathrm{iK}'\right)\right\} + \left(\mathrm{K}+\mathrm{iK}'\right)\mathrm{Z}(\alpha). \tag{9.194}$$

From H.E.I.E.P. 141.01

$$\mathrm{Z}(\mathrm{K}-\alpha+\mathrm{iK}') - \mathrm{Z}(\mathrm{K}-\alpha) + \mathrm{cs}(\mathrm{K}-\alpha)\ \mathrm{du}(\mathrm{K}-\alpha) - \mathrm{i}\pi/2\mathrm{K}. \tag{9.195}$$

Integrating both sides of this equation with respect to $(\mathrm{K}-\alpha)$, the integration of the zeta functions can be taken.

The integration of csu dnu can be found as

$$\int \mathrm{cs\ u\ dn\ u\ du} = \int \frac{\mathrm{csu\ dnu}}{\mathrm{snu}}\ \mathrm{du}\ (\text{H.E.J.E.P. } 120.02)$$
$$= \int \frac{\mathrm{dsnu}}{\mathrm{snu}} = \log\ \mathrm{sn\ u}. \tag{9.196}$$

The complete integration therefore produces

$$\log\Theta(\mathrm{K}-\alpha+\mathrm{iK}') + \mathrm{C}_1$$
$$= \log\Theta(\mathrm{K}-\alpha) + \mathrm{C}_2 + \log\ \mathrm{sn}(\mathrm{K}-\alpha) - \mathrm{i}\pi(\mathrm{K}-\alpha)/2\mathrm{K}. \tag{9.197}$$

Similarly

$$\log\Theta(\mathrm{K}+\alpha+\mathrm{iK}') + \mathrm{C}_1$$
$$= \log\Theta(\mathrm{K}+\alpha) + \mathrm{C}_2 + \log\ \mathrm{sn}(\mathrm{K}+\alpha) - \mathrm{i}\pi(\mathrm{K}+\alpha)/2\mathrm{K}. \tag{9.198}$$

By subtraction

$$\log\Theta(\mathrm{K}-\alpha+\mathrm{iK}') - \log\Theta(\mathrm{K}+\alpha+\mathrm{iK}')$$
$$= \log\frac{\Theta(\mathrm{K}-\alpha)}{\Theta(\mathrm{K}+\alpha)} + \log\frac{\mathrm{sn}(\mathrm{K}-\alpha)}{\mathrm{sn}(\mathrm{K}+\alpha)} + \frac{\mathrm{i}\pi}{2\mathrm{K}}(2\alpha). \tag{9.199}$$

From H.E.I.E.P. 1051.03

$$= \log \frac{\Theta\,(K-\alpha)}{\Theta\,(K+\alpha)} + \log \frac{\Theta_1\,(-\alpha)}{\Theta_1\,(\alpha)} = \log 1 = 0.$$

Similarly

$$= \log \frac{\mathrm{sn}\,(K-\alpha)}{\mathrm{sn}\,(K+\alpha)} = \log 1 = 0.$$

Hence $\log\Theta(K-\alpha+iK') - \log\Theta(K+\alpha+iK') = i\pi\alpha/K$.

And

$$\pi(K - iK', \alpha) = (K + iK')\ Z(\alpha) + i\pi\alpha/2K. \tag{9.200}$$

Necessary formulae for deriving the expressions are available in 'Handbook of elliptic integrals for engineers and physicists, by Byrd, P.F. and Friedman, M.D. (1954) and 'Elements of the theory of elliptic and associated functions with applications by Dutta, M., Debnath, L. (1965). All the values of $\mathrm{sn}\alpha, \mathrm{cn}\alpha, \mathrm{dn}\alpha, Z(\alpha)$ are read from the 'Jacobian elliptic function tables' by Milne-Thomson Dover New York (1950).

10

Green's Theorem in Potential Theory

In this chapter Green's first second and third identities are defined. Using Green's theorem one can arrive at Poisson's equation. Gauss' theorem of total normal induction in gravity field, estimation of mass of a subsurface body from gravitational potential are given. It could be shown that the basic formula of analytical continuation of potential field can be derived from Green's theorem. Two dimensional nature of the Green's identities are shown Theory of Green's equivalent layer which explains the ambiguity in interpretation of gravitational potentials is discussed. Application of Green's theorem for deriving Green's function and analytical continuation are respectively given in Chaps. 14 and 16. Nature of the vector Green's theorem is shown.

10.1 Green's First Identity

Let a region R includes the Vol. V enclosed by the surface S. Let ϕ (x, y, z) and ψ (x, y, z) are two scalar functions and we assume that both ψ and ϕ are continuous and have non-zero first and second derivatives(Fig. 10.1).

We can define a vector in the form

$$\vec{F} = \phi \operatorname{grad} \psi \tag{10.1}$$

Since

$$\operatorname{div} \left(a\vec{A} \right) = a \operatorname{div} \vec{A} + \vec{A} \operatorname{grad} a \tag{10.2}$$

where a and $\vec{A}$ are respectively a scalar and a vector. Applying divergence operation on both the sides of (10.1), we get,

$$\operatorname{div} \vec{F} = \operatorname{grad} \phi \operatorname{grad} \psi + \phi \operatorname{div} \operatorname{grad} \psi. \tag{10.3}$$

Integrating both the sides, we get

$$\int_{\vartheta} \operatorname{div} \vec{F} d\nu = \int_{\vartheta} (\operatorname{grad} \phi \operatorname{grad} \psi) \, d\nu + \int_{\vartheta} \phi \nabla^2 \psi dv. \tag{10.4}$$

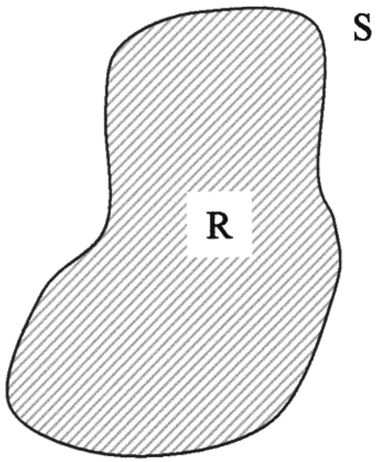


Fig. 10.1. A region R is having Vol. V and surrounded by the surface S

From Gausses divergence theorem , we get

$$\int_{v} \operatorname{div} \vec{F}\, dv = \int_{s} F_n ds = \int \phi \frac{\partial \psi}{\partial n} ds. \tag{10.5}$$

This is known as the Green's Theorem in non-symmetrical form. It is also known as the Green's first identity.

If we write

$$\vec{F} = \psi \operatorname{grad} \phi \tag{10.6}$$

we get a complementary equation which can be written in the form as

$$\int_{s} \psi \frac{\partial \phi}{\partial n} ds = \int_{v} (\operatorname{grad} \psi \operatorname{grad} \phi)\, dv + \int \psi \nabla^2 \phi dv. \tag{10.7}$$

Subtracting (10.7) from (10.5) , we get

$$\int_{v} \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) dv = \int_{s} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) ds \tag{10.8}$$

This is known as the Green's second identity or Green's theorem in symmetrical form.

10.2 Harmonic Function

Harmonic function is defined as the function which is continuous within a finite region R, it has non zero first and second derivatives and it satisfies Laplace equation within the region.

10.3 Corollaries of Green's Theorem

Some of the Corollaries of Green's theorem are as follows:
Cor. 1 If ϕ and ψ are both harmonic and continuous within the region R, then

$$\int \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds = 0 \tag{10.9}$$

since both $\nabla^2 \phi = \nabla^2 \psi = 0$ and ϕ and ψ satisfy Laplace equation.
Cor. 2 If ϕ is harmonic and continuously differentiable in a closed region then the integral of the normal derivative of ϕ over the boundary vanishes.

$$\text{If we put } \psi = 1, \qquad \text{then} \quad \int \frac{\partial \phi}{\partial n} ds = 0. \tag{10.10}$$

This region does not include any source. The surface integral of the normal derivative of a harmonic function over any closed surface is zero.
Cor. 3 If a function ϕ is harmonic in a closed sphere of radius 'a' with the centre at the point C, then ϕ_c is equal to the average of its values on the boundary surface. This is known as the mean value theorem or the average value theorem in potential theory. If this theorem is applied in the case of bodies of simpler geometrics, say, in the case of a sphere (Fig. 10.2) then for a sphere, the normal is along the radial direction, $\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial r}$. Elementary area $ds = r^2 \sin\theta \, d\theta \, d\psi$, where θ and ψ are respectively the polar and azimuthal angles.

Hence

$$\begin{aligned} \int \frac{\partial \phi}{\partial n} ds &= \int \frac{\partial \phi}{\partial n} r^2 \sin\theta \, d\theta \, d\psi \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{\partial \phi}{\partial r} r^2 \sin\theta \, d\theta \, d\psi = 0 \end{aligned} \tag{10.11}$$

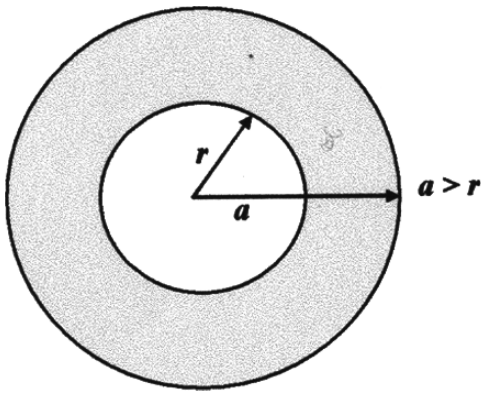


Fig. 10.2. A sphere of outer and inner radii 'a' and 'r' respectively

Multiplying both the sides of (10.11) by dr and integrating with in the region, we get

$$= \int_0^r \mathrm{d}r \int_0^{2\pi}\int_0^{\pi} \frac{\partial\phi}{\partial r}\, r^2 \ \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\psi = 0$$

$$\Rightarrow \int_0^{2\pi}\int_0^{\pi} (\phi_r - \phi_0) \ r^2 \ \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\psi = 0$$

$$\Rightarrow \int_0^{2\pi}\int_0^{\pi} \phi_r \ r^2 \ \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\psi - 4\pi r^2 \phi\,(c) = 0$$

$$\Rightarrow \phi_c = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} \phi_r \ r^2 \ \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\phi$$

$$= \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} \phi_r \ \mathrm{d}s$$

$$= \frac{1}{4\pi r^2}\int\int \phi_s \ \mathrm{d}s. \tag{10.12}$$

This gives the value of the potential at the centre which is the average of its potential on the surface i.e., the mean value theorem is

$$\phi_c = \frac{1}{S}\oint_s \phi_s \ \mathrm{d}s \tag{10.13}$$

Cor. 4 If a function ϕ is harmonic in a closed sphere, then ϕ_c at the centre is equal to the average of its value through out the sphere. This is the second average value or mean value theorem.

From (10.12), we can write

$$\int_0^a \phi_c 4\pi r^2 \ \mathrm{d}r = \int_0^a\int_s \phi \ \mathrm{d}s\mathrm{d}r = \int_\nu \phi \ \mathrm{d}\nu \tag{10.14}$$

where a is the radius of the sphere. Therefore

$$\Rightarrow \phi_c = \frac{1}{\nu}\int_\nu \phi \ \mathrm{d}\nu. \tag{10.15}$$

Here the sphere considered is a solid sphere and ν is the volume of the sphere.
Cor. 5 If ϕ is a harmonic function and not constant in a closed region, then ϕ cannot have maximum or minimum inside the region.

Cor. 6 A maximum or a minimum value of a harmonic function occurs only at a boundary of the region.
Cor. 7 If a function is harmonic in a region and is constants on the surface, then it is constant throughout the region.
Cor. 8 Two functions ϕ and ψ which are harmonic in a region and are equal at every point in the boundary are equal at every point in the region.
Cor. 9 If a solution of Laplace equation is found and has prescribed values on the boundary, then the solution is unique. This is known as the uniqueness theorem in potential theory.
Cor. 10 If a function is harmonic in a closed region and its normal derivatives vanish in the boundary, then the function is constant throughout the region.
Cor. 11 If two functions are harmonic in a closed region and have the same normal derivative at the boundary, then they differ by a constant.

10.4 Regular Function

The space outside the closed volume (Fig. 10.1) is called the infinite region where $\mathrm{r} \to \infty$. If there be any function ϕ such that $\lim\limits_{\mathrm{r}\to\infty} \mathrm{r}\phi = \text{finite}$ or $\lim\limits_{\mathrm{r}\to\infty} \mathrm{r}\,\mathrm{grad}\,\phi = \text{finite}$ then ϕ is called a regular function at infinity. A potential function is a regular function provided the source does not exist in the region.

Cor. 12 A function is harmonic in an infinite region if it has continuous second derivative, satisfies Laplace equation and is regular at infinity.

With this definition of the harmonic and regular function, the theorem, which we get is valid for infinite region. For an infinite region, the value of a harmonic function is uniquely determined by the values of the normal derivatives at the boundary.
Cor. 13 If ϕ and ψ are harmonic functions within a closed surface S and ψ has a single pole on S so that

$$\psi = \frac{1}{\mathrm{r}} + \rho$$

where ρ is harmonic, then

$$\phi(x, y, z) = \frac{1}{4\pi} \int\limits_{\mathrm{s}} \left[\psi \frac{\partial \psi}{\partial n} - \phi \frac{\partial \phi}{\partial n}\right] ds. \tag{10.16}$$

Cor. 14 If ϕ and ψ are harmonic within a closed surface and ϕ and ψ have single poles at ρ_1 and ρ_2 respectively and

$$\phi = \frac{1}{\mathrm{r}_1} + \rho_1 \text{ and } \psi = \frac{1}{\mathrm{r}_2} + \rho_2$$

then

$$\frac{1}{4\pi}\int_s \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) ds = \phi(\rho_1) - \psi(\rho_2) \tag{10.17}$$

where $\phi(\rho_2)$ is the value of the function ϕ at the point ρ_2.

10.5 Green's Formula

Let a finite region R is bounded by the surface S. The point Q may be within the volume or outside the region. The point P also may be within or outside the region (Fig. 10.3). The coordinates of P and Q are respectively (x, y, z) and (ξ, η, ζ). Here P is the observation point. Then the value of $\frac{1}{r}$, which behaves as a potential function, is given by

$$\frac{1}{r} = \frac{1}{\left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{1/2}} \tag{10.18}$$

(a) When the point P is outside
Let us take $\frac{1}{r}$ as a harmonic function and ψ as any other function. From Green's theorem, we get

$$\int_v \frac{1}{r}\nabla^2\psi \, dv = \int_s \left[\frac{1}{r}\frac{d\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds. \tag{10.19}$$

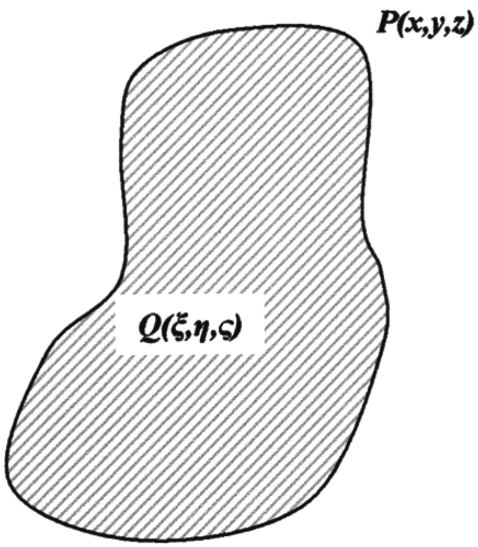


Fig. 10.3. Observation point P is outside the region R

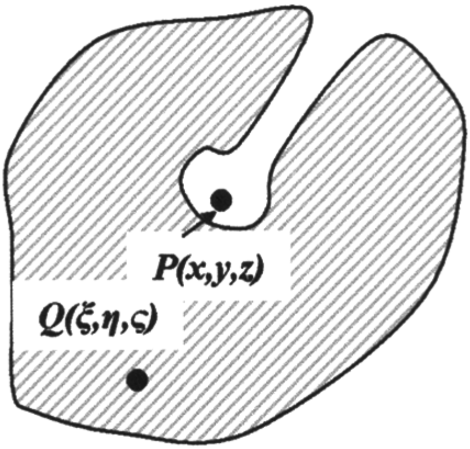


Fig. 10.4. Observation point P is inside the region R

(b) When the point P is inside the body, 'r' may or may not be harmonic strictly. We can isolate the point with a small semicircle (Fig. 10.4). In the rest of the region $\frac{1}{r}$ is harmonic. For this region, the normal is always outside the region. The boundary, which demarcates the region, and the boundary, which isolates the point P, should be taken into consideration separately.
Using $\nabla^2\left(\frac{1}{r}\right) = 0$, we get

$$\int\limits_{v} \frac{1}{r}\nabla^2\psi \, dv = \int\limits_{s} \left[\frac{1}{r}\frac{\partial\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds + \int\limits_{s'} \left[\frac{1}{r}\frac{d\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds. \tag{10.20}$$

Let

$$I_1 = \int\limits_{s} \left[\frac{1}{r}\frac{\partial\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds$$

$$I_2 = \int\limits_{s'} \left[\frac{1}{r}\frac{\partial\psi}{\partial n}\right] ds$$

and

$$I_3 = \int\limits_{s'} \left[\psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds.$$

Let us first evaluate the second integral, which entered into the (10.20) due to the origin of the second surface, which isolates the point P. The circle which isolates the point P is of radius 'a'. Therefore

$$I_2 = \int_{s'} \frac{1}{a}\left(-\frac{\partial\psi}{\partial r}\right) a^2 \ \sin\theta \ d\theta \ d\psi. \tag{10.21}$$

Here $-\frac{\partial\psi}{dr}$ is the normal towards the centre because the movement is in the clockwise direction as indicated by the arrows. Therefore

$$I_2 = a \int_0^{2\pi}\int_0^{\pi} -\left(\frac{\partial\psi}{\partial r}\right)\sin\theta \ d\theta \ d\psi. \tag{10.22}$$

Now $\underset{a\to 0}{\text{Lim}}\ I_2 \to 0$.
In the limit $r \to 0$

$$I_3 = \int_s \left[-\psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] \ ds. \tag{10.23}$$

Since $\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = -\frac{1}{r^2} = -\frac{1}{a^2}$ when $r \to a$, (10.23) reduces to

$$I_3 = -\int_0^{2\pi}\int_0^{\pi} \psi \ \sin \ \theta \ d\theta \ d\psi. \tag{10.24}$$

Now taking the limit $a \to 0$, the integral reduces to $4\pi\psi_\rho$ when the point P is inside. Green's theorem changes to the form

$$\psi_p = -\frac{1}{4\pi}\int_v \frac{1}{r}\nabla^2\psi \ dv + \frac{1}{4\pi}\int_s \left[\frac{1}{r}\frac{\partial\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds \tag{10.25}$$

This is the expression for the potential at a point when the point P is inside the region R.

(c) When the point is on the boundary

When the point P is right over the boundary, the function $\frac{1}{r}$ is not strictly harmonic. Approaching in a similar way, we get the expressions for the potentials as

$$\psi_p = -\frac{1}{2\pi}\int_v \frac{1}{r}\nabla^2\psi \ dv + \frac{1}{2\pi}\int_s \left(\frac{1}{r}\frac{\partial\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right) ds \tag{10.26}$$

because the solid angle subtended at the point P is 2π and not 4π.

If ψ is a potential function which is harmonic within the region then $\nabla^2\psi = 0$ and

$$\psi_p = \frac{1}{2\pi}\int_s \left[\frac{1}{r}\frac{\partial\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] \ ds. \tag{10.27}$$

Now we can summarise the Green's formulae for potential as follows.

(a) When the point P is outside

$$\int_s \left[\frac{1}{r}\frac{\partial \psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds = 0. \tag{10.28}$$

(b) When the point P is inside, then

$$\psi_p = \frac{1}{4\pi}\int_s \left[\frac{1}{r}\frac{\partial \psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds. \tag{10.29}$$

This is the Green's third formula.

(c) When the point P is on the boundary

$$\psi_p = \frac{1}{2\pi}\int_s \left[\frac{1}{r}\frac{\partial \psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds. \tag{10.30}$$

Green's first and second identities are also known as Green's formulae. The second identity i.e., the Green's symmetrical formula is more frequently used. Only the first and second derivatives of ϕ and ψ enter in the surface integrals and they are the normal derivatives. ϕ and ψ have continuous second derivatives in the interior of the region V (entire volume). ϕ, ψ, $\frac{\partial \phi}{\partial n}$ and $\frac{\partial \psi}{\partial n}$ remains continuous in the closed region v + s, i.e. volume plus surface.

The second derivatives of ϕ and ψ are piecewise continuous in the volume V. Green's theorem is valid for each of the subregions into which the V is divided by the surface of discontinuity. By addition of these formulae for each subregions, we can obtain the theorem for the entire region.

10.6 Some Special Cases in Green's Formula

(a) when $\psi = 1$, then

$$\int\int_v\int \nabla^2\phi \, d\nu = \int_s \frac{\partial \phi}{\partial n}\, ds \tag{10.31}$$

(b) if $\phi = \psi$, then

$$\int\int_v\int (\nabla\phi)^2 \, d\nu = \int\int_s \phi\frac{\partial \phi}{\partial n} ds - \int\int_v\int \phi\nabla^2\phi d\nu \tag{10.32}$$

(c) if ϕ is a regular harmonic function in v, then $\nabla^2\phi = 0$, and one gets

$$\int\int_v\int (\nabla\phi)^2 \, d\nu = \int\int \phi\frac{\partial \phi}{\partial n} ds. \tag{10.33}$$

(d) If ϕ and ψ are both harmonic functions inside the closed surface S, then

$$\int\int_s \left(\phi\frac{\partial \psi}{\partial n} - \psi\frac{\partial \phi}{\partial n}\right) ds = 0. \tag{10.34}$$

10.7 Poisson's Equation from Green's Theorem

Let $\phi(\xi, \eta, \zeta)$ is a function at a coordinate ξ, η, ζ which is continuous in a volume and is bounded by a closed surface S. Its first derivative is also continuous. From Green's theorem, we can write

$$\int_{v} \nabla^2 \phi \; d\nu = \int_{s} \frac{\partial \phi}{\partial n} ds \tag{10.35}$$

for continuous and finite distribution of matters. Here ϕ is a potential function for continuous and finite distribution of matters. From Gauss's theorem, we can write

$$\int \frac{\partial \phi}{\partial n} \; ds = -4\pi \; M \text{ or } \int \nabla^2 \phi \; d\nu = -4\pi \; M. \tag{10.36}$$

Here M stands for the distribution of mass and is equal to $M = \int_{v} \sigma d\nu$. $\sigma(\xi, \eta, \zeta)$ is the density of the matters distributed in the volume We can rewrite (10.36) in the form

$$\int_{v} \left(\nabla^2 \phi + 4\pi\sigma\right) d\nu = 0 \tag{10.37}$$

It reduces to

$$\nabla^2 \phi = -4\pi\sigma \tag{10.38}$$

This is the Poisson equation and it is valid for any kind of distribution of matters.

10.8 Gauss's Theorem of Total Normal Induction in Gravity Field

Let ϕ be the gravitational potential due to certain distribution of masses both inside and outside the domain R. Let ψ, the other scalar potential function is assumed to be constant both outside and inside the region S. We can write from Green's second identity

$$\int_{v} \nabla^2 \phi \, dv = \int_{s} \frac{\partial \phi}{\partial n} ds \tag{10.39}$$

where ϕ is a harmonic function. Since ψ is assumed to be constant, its derivative with respect to the direction normal to the surface is zero. Let ϕ_{in} and ϕ_{out} are respectively the potential both inside and outside S. ϕ_{out} obeys Laplace equation is a source free region and ϕ_{in} obeys Poisson's equation because the masses are included within the domain.

$$\int\int_{v}\int \nabla^2 \phi_{out} dv = 0 \tag{10.40}$$

and

$$\int\int_{\mathrm{v}}\int \nabla^2 \phi_{\mathrm{out}}\,\mathrm{dv} = -4\pi \mathrm{GM} \tag{10.41}$$

Here M is the total mass included by the surface. It is important to note these Green's formulae is independent of the sizes and shapes of the distribution of masses and sizes and shapes of the boundaries. Hence

$$\int\int_{\mathrm{v}}\int \nabla^2 \phi \mathrm{dv} = \int\int \frac{\partial \phi}{\partial \mathrm{n}} \mathrm{ds} = -4\pi \mathrm{GM} \tag{10.42}$$

This is Gauss' law of total normal induction. It states that the total normal gravitation flux on a closed bounded surface is equal to 4πG times the total M of one body or multiple bodies inside the closed domain. Here G is the universal gravitational constant.

10.9 Estimation of Mass in Gravity Field

Since Gauss' law of total normal induction is valid in gravity field also, we can assume that the anomalous masses which are generating gravity anomaly because of density contrast are relatively nearer to the surface and we estimate the total normal induction on the surface of a sphere of infinite radius. We divide this sphere into two hemispheres and the central horizontal plane which cuts the sphere into two parts.The first part represent the surface of the earth, where we seek the mass to be estimated. And the second part is upper hemisphere of infinite radius.We can now write the total normal induction as

$$\int_{Plane} \frac{\partial \phi}{\partial n} ds + \int_{Hemisphere} \frac{\partial \phi}{\partial n} ds = -4\pi GM \tag{10.43}$$

We can divide the total normal induction equally in upper and lower hemisphere and each sector will have -2πGM where M is the total anomalous mass. This induction is independent of the total number of bodies present and their sizes and shapes. Therefore, on the surface, we can write the expression for the integral on gravity anomaly as

$$\int_{surface} \Delta g ds = 2\pi GM \tag{10.44}$$

where Δg is the gravity anomaly and M is the mass excess due to density contrast with the host rock. Surface integration is carried over the plane of observation.

10.10 Green's Theorem for Analytical Continuation

By analytical continuation we mean potential measured in one plane or level can be transferred to another plane or level mathematically or analytically (see Chap. 16). If P is any point outside the domain with surface S and $\phi^{/} = \frac{1}{r}$ where r is the distance of P from any volume element dv at Q inside the domain bounded by S (Fig. 10.4). Since $\phi^{/}$ is an harmonic function, it satisfies Laplace equation $\nabla^2\phi^{/} = 0$ at all points throughout V. Inside the volume V_1 the potential satisfies Poisson's equation. Therefore we can write $\nabla^2\phi = -4\pi GM$, where G is the universal gravitational constant and M is the total mass. From Green's second identity or symmetrical form of Green's theorem, we can write

$$4\pi \int\int_{V}\int \frac{G\rho\ dv}{r} = \int\int_{S} \left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right] ds \tag{10.45}$$

$$\Rightarrow \phi_P = \frac{1}{4\pi}\int_{S} \left[\phi \frac{\partial}{rn}\left(\frac{1}{r}\right) - \frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)\right] ds \tag{10.46}$$

where

$$\phi_P = \int \frac{Gdm}{r} \tag{10.47}$$

is the potential at P due to mass distribution inside S. One can estimate the potential at any point outside a closed surface S if the potential ϕ and its normal derivative $\frac{\partial\phi}{\partial n}$ are known at all points on the surface. The potential ϕ is the combined potential due to the masses inside and outside S. ϕ is the potential due to the masses enclosed by the surface only.

Now on the surface of a hemisphere

$$\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) = 0 \text{ and } (1/r)\left(\frac{\partial\phi}{\partial n}\right) = 0$$

because the hemisphere radius is infinitely high. Hence (10.47) becomes

$$\phi_P = \frac{1}{4\pi}\int_{\text{Plane}} \left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right] ds^{/} \tag{10.48}$$

Figure 10.5 shows that the images of mass distribution are within the enclosed volume of the upper hemisphere. These images also produce the potential on the boundary surface of the two hemisphere. If ϕ_1 is the potential due to the distribution of images in the upper hemisphere, the potential and its normal derivatives must be equal on the plane which divides the two hemispheres. Since the normals n and n_1 are in the opposite direction, we can write

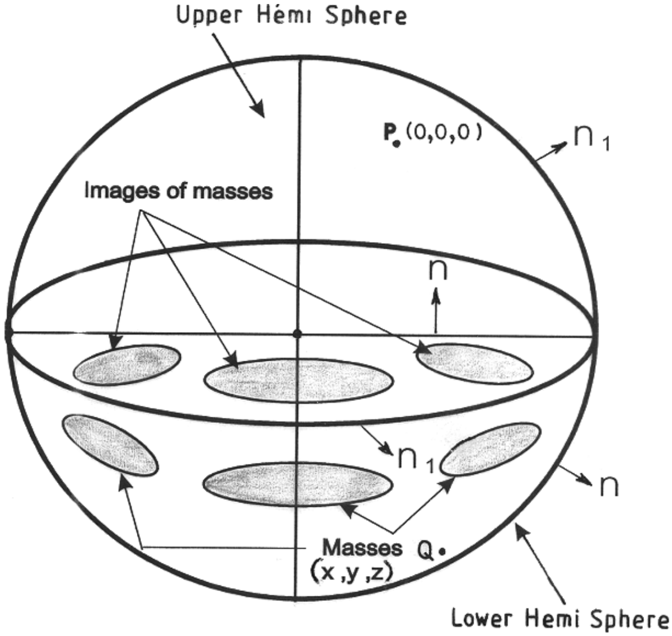


Fig. 10.5. Distribution of masses in the lower half space and their images in the upper half space

$$\phi = \phi_1 \text{ and } \frac{\partial \phi}{\partial \mathrm{n}} = \frac{\partial \phi_1}{\partial \mathrm{n}_1}$$

on the plane. For upper hemisphere

$$\int \left[\phi_1 \frac{\partial}{\partial \mathrm{n}_1} \left(\frac{1}{\mathrm{r}} \right) - \frac{1}{\mathrm{r}} \frac{\partial \phi_1}{\partial \mathrm{n}_1} \right] \mathrm{ds} = 0 \tag{10.49}$$

with no masses inside. Thus

$$\int_{\text{Plane}} \phi \frac{\partial}{\mathrm{rn}} \left(\frac{1}{\mathrm{r}} \right) \mathrm{ds} = - \int_{\text{Plane}} \frac{1}{\mathrm{r}} \frac{\partial \phi}{\partial \mathrm{n}} \mathrm{ds}. \tag{10.50}$$

Therefore (10.49) can be rewritten as

$$\phi_{\mathrm{P}} = -\frac{1}{2\pi} \int_{\text{Plane}} \frac{1}{\mathrm{r}} \frac{\partial \phi}{\partial \mathrm{n}} \mathrm{ds}. \tag{10.51}$$

Taking z axis as positive vertically downward and normal to the plane we get

$$\phi_{\mathrm{P}} = -\frac{1}{2\pi} \int_{\mathrm{s}} \frac{1}{\mathrm{r}} \frac{\partial \phi}{\partial \mathrm{n}} \mathrm{ds} \tag{10.52}$$

This is the upward continuation integral. This integral is valid for the upper hemisphere where (10.52) is valid. The plane of observation at any height in the upper hemisphere does not contain any mass responsible for the potential ϕ. Further details on the topic is discussed in Chap. 16.

10.11 Green's Theorem for Two Dimensional Problems

For a two-dimensional problem, the potential is logarithmic

$$\phi = -2G_{m_\ell}\,\ell n\left(\frac{C}{r}\right) \tag{10.53}$$

This is the expression for gravitational potential due to a long line source of mass m_ℓ. C is any arbitrary constant. For two-dimensional problems

(a) the Laplace and other connecting equations are

$$\text{(i)}\qquad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{10.54}$$

$$\text{(ii)}\qquad \nabla^2\left(\ell n\frac{1}{r}\right)=0. \tag{10.55}$$

(b) Gauss's divergence theorem is

$$\text{div}\vec{F}=\underset{\Delta s\to 0}{\text{Lim}}\oint\frac{\vec{F}.\vec{dl}}{\Delta S} \tag{10.56}$$

where as for three dimensional problem it is

$$\text{div}\vec{F}=\underset{\Delta \nu\to 0}{\text{Lim}}\int_s\frac{\vec{F}.ds}{\Delta\,\nu}. \tag{10.57}$$

(c) Green's second identity can be obtained from

$$\vec{F}=\phi\ \text{grad}\,\psi$$

where both ϕ and ψ are functions of x and y. It is given by

$$\int_s\left(\phi\nabla^2\psi-\psi\nabla^2\phi\right)ds=\int_\ell\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)d\ell. \tag{10.58}$$

(d) The mean value theorem is the value of the logarithmic potential at the centre of the circle and it is the average of potentials at the circumference.
(e) For Green's formula in two dimensions $\phi=\ell n\left(\frac{1}{r}\right)$ and ψ is any function. Here

$$r=\sqrt{(x-\xi)^2+(y-\eta)^2}.$$

If the point P is outside (Fig. 10.4) the closed surface, $\ell n\left(\frac{1}{r}\right)$ is continuous and harmonic. When the point P is inside, we shall have to draw a small circle surrounding the point P to avoid singularity as it is done in the 3-D case. Green's formula for the 2-D case is given by

$$\begin{aligned}\psi_p &= \frac{1}{2\pi}\oint_{\ell}\left[\ell n\left(\frac{1}{r}\right)\frac{\partial\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\ell n\frac{1}{r}\right)\right]d\ell \\ &= 2\pi\oint_{\ell}\left(\psi\frac{\partial}{\partial n}\ell n\, r - \ell n\, r\frac{\partial\psi}{\partial n}\right)d\ell. \qquad (10.59)\end{aligned}$$

When the point P is outside, we get

$$0 = \frac{1}{2\pi}\oint_{\ell}\left(\psi\frac{\partial}{\partial n}\ell n\, r - \ell n\, r\frac{\partial\psi}{\partial n}\right)d\ell. \qquad (10.60)$$

When the point P is on the boundary of an area in two dimensions

$$\psi_p = \frac{1}{\pi}\oint_{\ell}\left\{\psi\frac{\partial}{\partial n}\ell n\, r - \ell n\, r\frac{\partial\psi}{\partial n}\right\}d\ell. \qquad (10.61)$$

If the point P lies on the boundary C, the point P is excluded from the region of integration by enclosing it in a small circle of radius 'a' and the 'a' is made to approach zero. The factor $\frac{1}{2\pi}$ enters here instead of $\frac{1}{4\pi}$, since 2π is the circumference of an unit circle. Every potential possesses continuous partial derivatives.

10.12 Three to Two Dimensional Conversion

For three to two dimensional conversions of the potential problems the following factors are to be changed i.e.

	3-D	2-D	
(i)	Volume $\rightarrow$	area	
(ii)	Surface $\rightarrow$	curve lines	
(iii)	Sphere $\rightarrow$	circle	
(iv)	$\frac{1}{r} \rightarrow$	$\ell n\left(\frac{1}{r}\right)$	(10.62)
(v)	$4\pi \rightarrow$	2π	
(vi)	$2\pi \rightarrow$	π	
(vii)	$\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$ or	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}$	

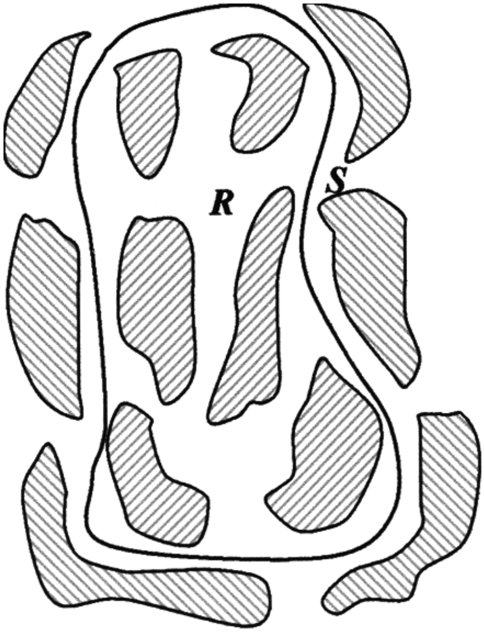


Fig. 10.6. Distribution of masses both inside and outside the region R and surface S

10.13 Green's Equivalent Layers

For Poisson's field when the source is included

$$\nabla^2\phi = -4\pi\rho \tag{10.63}$$

where ρ is the volume density of mass and $\nabla^2\phi = 0$ for Laplace field in a source free region. Let us assume a certain region R has the volume V and is bounded by the surface S. Some masses are present inside and some are outside the domain boundaries as shown in the Fig. 10.6. Potential at a point P due to the distribution of mass using Green's theorem is

$$\phi_p = \underbrace{\frac{1}{4\pi}\int_s\int\left[\frac{1}{r}\cdot\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]ds}_{I_1\downarrow} + \underbrace{\int\int_v\int\frac{\rho dv}{r}d\nu}_{I_2\downarrow} \tag{10.64}$$

I_2 indicates the components of potential at a point P due to all the sources included in the volume V. The surface S is drawn to separate the volume V that contains all the sources inside and the sources outside the region.

$$\mathrm{I}_1 = \underset{\mathrm{s}}{\int\int} \frac{1}{\mathrm{r}}\left(\frac{1}{4\pi}\cdot\frac{\partial\phi}{\partial \mathrm{n}}\right)\mathrm{ds} - \underset{\mathrm{s}}{\int\int} \frac{\phi}{4\pi}\cdot\frac{\partial}{\partial \mathrm{n}}\left(\frac{1}{\mathrm{r}}\right)\mathrm{ds} \tag{10.65}$$

$\sigma_s \downarrow$ $\downarrow$

Surface distribution of masses Double layer distribution

The first integral represents the potentials due to simple surface distribution σ_s. The second integral represents the potential due to surface distribution of double layer dipoles of charge or mass distribution of moment m. Hence we can show the redistribution of masses as shown in the Fig. 10.7. We can divide the total normal induction equally in upper and lower hemisphere and each sector will have -2πGM where M is the total anomalous mass. This induction is independent of the total number of bodies present and their sizes and shapes. Therefore, on the surface, we can write the expression for the integral on gravity anomaly as

$$\int_{surface} \Delta g ds = 4\pi GM \tag{10.66}$$

where Δg is the gravity anomaly and M is the mass excess due to density contrast with the host rock. Surface integration is carried over the plane of observation.

$$\int\int\int \frac{\rho d\nu}{\mathrm{r}} + \int\int \frac{\sigma_s ds}{\mathrm{r}} +$$

$\downarrow$ $\downarrow$

Volume distribution of masses — Surface distribution of masses

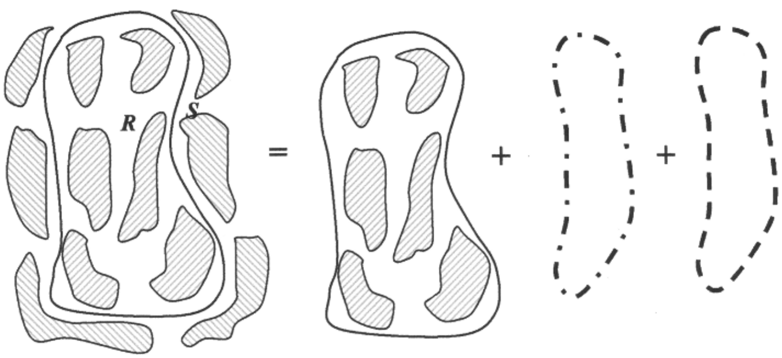


Fig. 10.7. Mass distribution in the region both inside and outside the region can be changed to mass distribution inside,surface distribution of masses and dipolar distribution

$$\underset{\substack{\downarrow \\ \text{Double layer} \\ \text{distribution of masses}}}{\int\int \frac{\mathrm{m}}{4\pi}\frac{\partial}{\partial \mathrm{n}}\left(\frac{1}{\mathrm{r}}\right)\mathrm{ds}}$$

The sources outside the surface can be redistributed and taken on the bounding surface S as a simple surface distribution and a double layer distribution. In a source free region where $\nabla^2\phi = 0$, the potential is

$$\phi_{\mathrm{p}} = \frac{1}{4\pi}\int_{\mathrm{s}}\int\left[\frac{1}{\mathrm{r}}\frac{\partial\phi}{\partial\mathrm{n}} - \phi\frac{\partial}{\partial\mathrm{n}}\left(\frac{1}{\mathrm{r}}\right)\right]\mathrm{ds} \tag{10.67}$$

This is known as the Green's third formula. From this equation, we can see that the value of a harmonic function in the interior region of volume V, where it is regular, is determined. We know the value of the function and its normal derivatives on the boundary, We assumed also that ϕ and $\frac{\partial\phi}{\partial\mathrm{n}}$ are continuous on approaching the boundary S.

Green's third formula states that: every regular harmonic function can be represented as the sum of the potentials due to a surface distribution and a double layer distribution on the surface.

Green's theorem of equivalent layer states that

(a) the effect of matter lying outside of any closed surface S may be replaced at all the interior points by the superposition of a single layer and a double layer on S.
(b) The effect of matter lying within a closed surface may be replaced at all the exterior points by the superposition of a single layer and a double layer.
(c) The matter contained outside (or inside) any closed equipotential surface S in a given field can be spread over the surface with a surface density - $\frac{1}{4\pi}\frac{\partial\psi}{\partial\mathrm{n}}$ at a point on the surface without altering the potential at any point in the field inside (or outside) S. Some parts of the matter are also distributed in the form of double layer with dipole moment $\frac{\mathrm{m}}{4\pi}$. distribution and is knows as Green's equivalent layers and it explains the ambiguity in the potential fields i.e., many different sets of distribution of masses can generate the same type of gravity responses on the surface. Ambiguity do exist in other scalar and vector potential fields also.

10.14 Unique Surface Distribution

In a region bounded by S (Fig. 10.8), if ϕ_0 is harmonic outside and ϕ_{i} is harmonic inside, then one can write from Green's theorem

$$\phi_i = \frac{1}{4\pi}\oint\left[\frac{1}{r}\frac{\partial\phi_i}{\partial n_0} - \phi_i\frac{\partial}{\partial n_i}\left(\frac{1}{r}\right)\right]ds \tag{10.68}$$

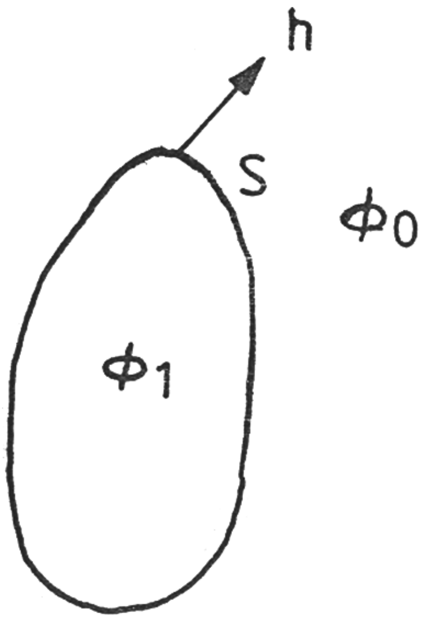


Fig. 10.8. Unique surface distribution when the potential is harmonic both inside and outside

when the observation point is also outside, one gets

$$0 = \frac{1}{4\pi} \oint \left[\frac{1}{r} \frac{\partial \phi_i}{\partial n_0} - \phi_i \frac{\partial}{\partial n_0} \left(\frac{1}{r} \right) \right] ds \tag{10.69}$$

The two formulae are for potential inside and outside when the observation point is inside. When the observation point is outside, One gets

$$0 = \frac{1}{4\pi} \oint \left[\frac{1}{r} \frac{\partial \phi_0}{\partial n} - \phi_0 \frac{\partial}{\partial n_i} \left(\frac{1}{r} \right) \right] \; ds \rightarrow \text{inside} \tag{10.70}$$

and

$$\phi_0 = \frac{1}{4\pi} \oint \left[\frac{1}{r} \frac{\partial \phi_0}{\partial n} - \phi_0 \frac{\partial}{\partial n_0} \left(\frac{1}{r} \right) \right] \; ds \rightarrow \text{outside} \tag{10.71}$$

Hence we get

$$\phi_i = \frac{1}{4\pi} \oint \frac{1}{r} \left(\frac{\partial \phi_i}{\partial n_0} + \frac{\partial \phi_0}{\partial n_i} \right) \, ds - \frac{1}{4\pi} \oint \left[\phi_i \frac{\partial}{\partial n_0} \left(\frac{1}{r} \right) + \phi_0 \frac{\partial}{\partial n_i} \left(\frac{1}{r} \right) \right] \, ds \tag{10.72}$$

Now $\frac{\partial}{\partial n_i} = -\frac{\partial}{\partial n_0}$ and on the surface $\phi_i = \phi_0$

Therefore,

$$\phi_i = \frac{1}{4\pi}\oint\limits_s \frac{1}{r}\left(\frac{\partial\phi_i}{\partial n_0} - \frac{\partial\phi_0}{\partial n_0}\right)ds \tag{10.73}$$

$$\phi_0 = \frac{1}{4\pi}\oint\limits_s \frac{1}{r}\left(\frac{\partial\phi_i}{\partial n_0} - \frac{\partial\phi_0}{\partial n_0}\right)ds \tag{10.74}$$

by adding the terms. Both values in (10.73) and (10.74) are exactly equal. If ϕ_0 be the potential at any point outside a closed surface S due to masses inside the surface and ϕ_i is the potential at points inside the surface due to masses outside and if $\phi_0 = \phi_i$ on the surface then the expression $\frac{1}{4\pi}\int\left(\frac{\partial\phi_i}{\partial n} - \frac{\partial\phi_0}{\partial n}\right)\frac{ds}{r}$ has the value ϕ_0 at the external point and the value ϕ_i at the internal points. Here r denotes the distance from the point source at any point on the surface. n is the direction of the normal.

If in a closed surface S there are certain distribution of mass and if ϕ_i is harmonic inside and ϕ_0 is harmonic outside and if $\phi_i = \phi_0$ on the surface, then there exists one and only one surface distribution for which ϕ_i is the internal potential and ϕ_0 is the external potential.

10.15 Vector Green's Theorem

Tai (1992) first demonstrated that if we take

$$\vec{F} = \vec{P} \times \nabla \times \vec{Q} \tag{10.75}$$

where $\vec{P}$ and $\vec{Q}$ are two vectors. Applying the vector identity we get

$$\nabla.\vec{F} = (\nabla \times \vec{P}).(\nabla \times \vec{Q}) - \vec{P}.\nabla \times \nabla \times \vec{Q} \tag{10.76}$$

Substituting these values in Gauss's divergence theorem we get

$$\begin{aligned}&\int\int\limits_v\int \left[(\nabla \times \vec{P}) \bullet (\nabla \times \vec{Q}) - \vec{P} \bullet \nabla \times \nabla \times \vec{Q}\right]dv = \\ &\int\limits_s\int (\vec{P} \times \nabla \times \vec{Q})d\vec{s} = \oint\limits_s\int \widehat{n}.(\vec{P} \times \nabla \times \vec{Q})ds\end{aligned} \tag{10.77}$$

where $\vec{n}$ denotes the outward normal unit vector on the surface.

By interchanging the role of $\vec{P}$ and $\vec{Q}$, the way it was done for scalar Green's function, and taking the difference of the two resultant equations, we get the vector Green's theorem of second symmetrical formula as

$$\int\int_{\mathrm{v}}\int(\vec{\mathrm{P}}.\nabla\times\nabla\times\vec{\mathrm{Q}}-\vec{\mathrm{Q}}.\nabla\times\nabla\times\vec{\mathrm{P}})\mathrm{dv}$$

$$=\int\int_{\mathrm{s}}(\vec{\mathrm{Q}}\times\nabla\times\vec{\mathrm{Q}}-\vec{\mathrm{P}}\times\nabla\times\vec{\mathrm{Q}})\ \vec{\mathrm{d}}\mathrm{s}$$

$$-\int\int_{\mathrm{s}}(\vec{\mathrm{n}}.(\vec{\mathrm{Q}}\times\nabla\times\vec{\mathrm{P}}-\vec{\mathrm{P}}\times\nabla\times\vec{\mathrm{Q}})\ \mathrm{ds}. \tag{10.78}$$

11

Electrical Images in Potential Theory

In this chapter a brief idea about the concept of electrical image and its use in computation of perturbation potential in direct current flow field for layered earth structures in bore hole geophysics are demonstrated. Necessary formulae for computation of potentials across a horizontally stratified 3 layered bed are given for two, three and seven(laterolog-7) electrode configurations. This approach for solution of the boundary value problems is restricted to problems with simpler geometries.

11.1 Introduction

Many of the potential problems in direct current potential domains can be solved using the concept of images. The analogy is drawn from optics. It is shown that using more than one reflector one can get several series of images, the way one gets the optical images in between the two mirrors. The only condition required to satisfy is potentials must be continuous across the boundary (or reflector). This technique for computation of potential is only valid for plane boundaries (Dakhnov (1962), Keller and Frischknecht (1966). Concept of images with spherical boundaries are available in. (Ramsay 1940), Macmillan (1958).

11.2 Computation of Potential Using Images (Two Media)

The potential at a point at a distance ‘r’ from a point current source embedded in an infinite and homogeneous medium of resistivity ‘ρ’ is given by

$$\phi = \frac{I\rho}{4\pi}\frac{1}{r} \tag{11.1}$$

where I is the current flowing through the medium. In a semiinfinite medium when the current electrode is on the surface,

$$\phi = \frac{I\rho}{2\pi}\frac{1}{r}. \tag{11.2}$$

For two homogeneous and isotropic semiinfinite media of resistivity ρ_1 and ρ_2 (Fig. 11.1) the potential at a point P due to current source I is

$$\phi = \frac{\rho_1}{4\pi}\left(\frac{I}{r} + \frac{I'}{r'}\right). \tag{11.3}$$

Here I' is termed as the electrical image of I at P. P and P′ are at a same distance from the boundary of the two media of resistivity ρ_1 and ρ_2. Some analogy can be drawn from the theory of optics. This perturbation term, originated from the concept of image and has to satisfy the basic boundary conditions in direct current electrical method i.e., (i) the potential on both the media must be same on the boundary and (ii) the normal component of the current density is continuous across boundary.

These boundary conditions are $\phi_1 = \phi_2$ and

$$J_1\left(= \frac{1}{\rho_1}\frac{\partial\phi}{\partial n}\right)_1 = J_2\left(= \frac{1}{\rho_2}\frac{\partial\phi}{\partial n}\right)_2. \tag{11.4}$$

Potential in a medium 2 due to a current source in a medium 1 is given by

$$\phi_2 = \frac{\rho_2 I''}{4\pi r''} \tag{11.5}$$

where I″ is the reduced current strength and r″ is the distance of the point of observation in the medium 2 from the source I in the medium 1.

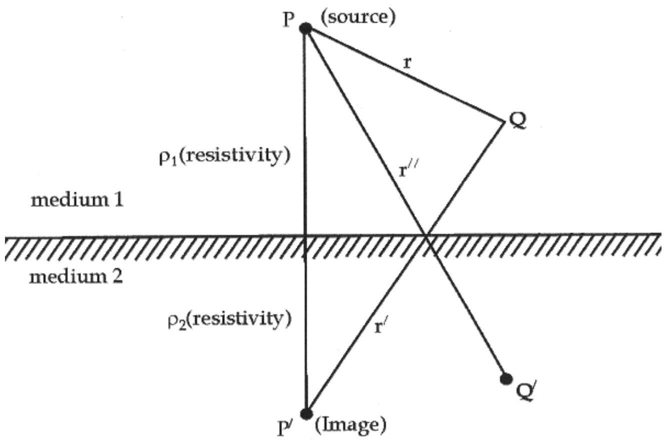


Fig. 11.1. Electrical Image for a plane single boundary between two semi infinite media of resistivity ρ_1 and ρ_2 respectively

In order to determine the strengths of the fictitious current, we shall apply the boundary conditions stated above. At the boundary since $r = r'$ (Fig. 11.1), we get

$$\rho_1(I + I') = \rho_2 I''. \tag{11.6}$$

Applying the second boundary condition, we get

$$\begin{aligned}\left(\frac{1}{\rho_1}\frac{\partial\phi_1}{\partial n}\right)_1 &= \frac{1}{4\pi}\left[\frac{\partial}{\partial r}\left(\frac{I}{r}\right)\frac{\partial r}{\partial n} + \frac{\partial}{\partial r'}\left(\frac{I'}{r'}\right)\frac{\partial r'}{\partial n}\right] \\ &= -\frac{1}{4\pi}\left(\frac{I}{r^2}\frac{dr}{dn} + \frac{I'}{r'^2}\frac{dr'}{dn}\right)\end{aligned} \tag{11.7}$$

and

$$\frac{1}{\rho_2}\frac{\partial\phi_2}{\partial n} = \frac{1}{4\pi}\frac{\partial}{\partial r}\left(\frac{I''}{r''}\right)\frac{dr}{dn} = -\frac{I''}{4\pi r''^2}\frac{dr''}{dn}. \tag{11.8}$$

At the boundary $r = r'$ and $\frac{dr}{dn} = -\frac{dr'}{dn}$. Equating (11.7) and (11.8), we get

$$I - I' = I''. \tag{11.9}$$

Equations (11.6) and (11.9) yields

$$I' = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} I = K_{12} I \tag{11.10}$$

and

$$I'' = \left(1 - \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} I\right) = \frac{2\rho_2}{\rho_1 + \rho_2} I = (1 - K_{12})\, I. \tag{11.11}$$

K_{12} and $(1-K_{12})$ are termed respectively as reflection factor and transmission factor. Potentials in the first and second media are respectively given by

$$\phi_1 = \frac{\rho_1 I}{4\pi}\left(\frac{1}{r} + \frac{K_{12}}{r'}\right) \tag{11.12}$$

$$\phi_2 = \frac{\rho_2 I}{4\pi}(1 - K_{12}). \tag{11.13}$$

We can now compare electrical images with optical images. Current source should be replaced by a source of light and boundary between the two media should be replaced by a mirror of reflection coefficient K_{12} and transmission coefficient $(1-K_{12})$. If the light source is seen from medium 1, one can measure the intensity of light due to the source and the reflected light intensity I' and coming from the image point P'. If the light source is viewed from medium 2, one will see the light with reduced intensity $(1 - K_{12})$ I.

11.3 Computation of Potential Using Images (for Three Media)

A simple approach for computing potential field using the theory of electrical images is discussed. The use of images, has major application in plane boundary problems. Concept of image can also be applied for spherical boundary problem. The potential field can be found out very easily for a medium with two plane and parallel boundaries using this method. The problem is one of determining the potential function for three regions designated as medium 1,2 and 3 having resistivities ρ_1, ρ_2 and ρ_3. The regions are separated by two plane parallel boundaries P and Q (Fig. 11.2). The general set up and the position of the different series of current images and their strengths, when the current source is situated in medium 1 at a point A, are shown In the Fig. 11.2. Here H is the thickness of medium 2, the target bed, Z, is the distance of the current electrode from the interface P, between medium 1 and 2 ; D, is the distance between the current and potential electrodes. K_{ij} is the reflection factor for medium i and j, where $K_{ij} = (\rho_j - \rho_I)/(\rho_j + \rho_I)$, i and j varies between 1 2 and 3 and I is the current strength. For determining the potential at a point, we have to first find out the potential at a point due to the source irrespective of whether the source and the observation point (s) are in the same or in different medium (or media). Next the contributions from series of images are computed and are algebraically added up.

Distribution of image current sources in a medium with two parallel plane boundaries; the original current source A [I] in medium 1 are shown in the Fig. (11.2). These series of images originated to satisfy the two boundary conditions at the interfaces, i.e.,

1) $\phi = \phi'$
2) $J_n = J'_n$

where ϕ and ϕ' are the potentials and J_n and J'_n are the normal components of the current densities on both the sides of the interface. More than one series of images are generated for any current source placed in any medium.

Any current source placed in medium 1 at a distance z from the interface P has to satisfy the boundary conditions at P. By introducing the image source ${A_1}^{(2)}$ of strength IK_{12} the boundary condition at P is satisfied. The potential function in medium 2 will be the same as it would be in a fully infinite and homogeneous medium with a source of reduced intensity $I(1 - K_{12})$.

Now, to satisfy the boundary condition at Q, another fictitious current source ${A_1}^{(3)}$ in medium 3 at a distance 2 (H + Z) from the current source at A has to be introduced. The strength of this source will be $1(1 - K_{12})K_{23}$.

Addition of this second fictitious source indicates that the boundary conditions at P is no longer valid. A third image at the point ${A_1}^{(1)}$ in the first medium is needed to satisfy the boundary conditions. The strength of the image source will be $I\ (1 - K_{12})K_{21}K_{23}$ and located at a distance $2(2H + Z)$ from the image ${A_1}^{(3)}$ in the third medium and 2H from the original current

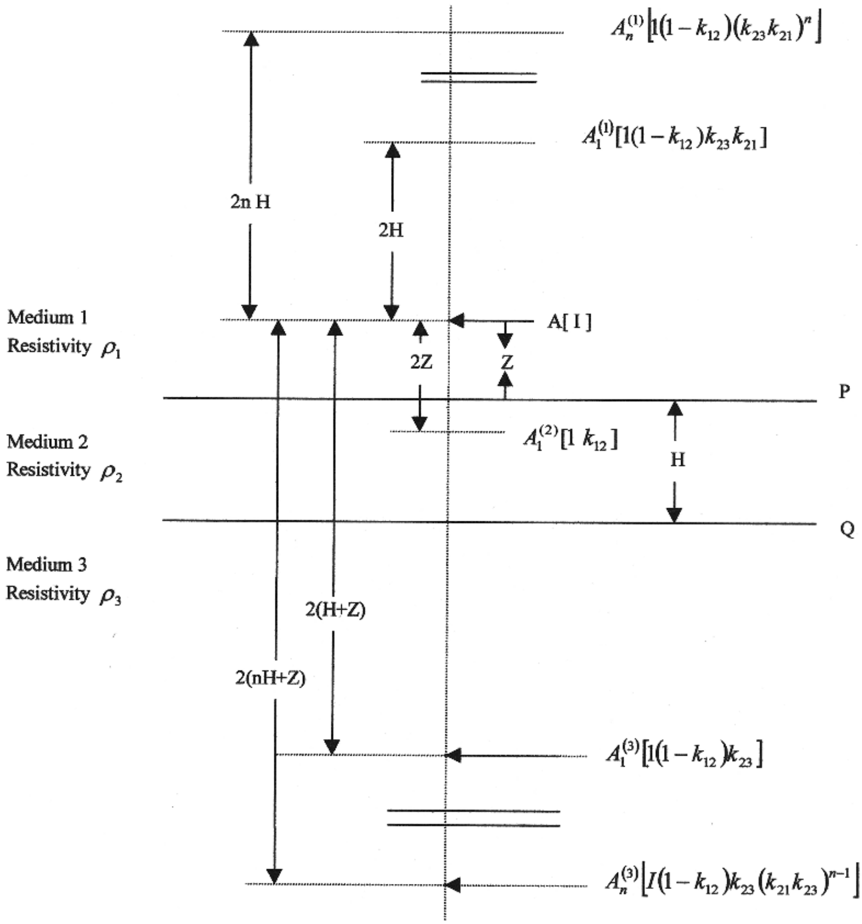


Fig. 11.2. Shows the formation of images in a layered medium

source A. When the addition of third image $A_1{}^{(1)}$ satisfy the boundary conditions at P, the boundary conditions at the interface Q becomes invalid. To restore this, another image source $A_2{}^{(3)}$ of strength $I(1 - K_{12})K_{23}\ K_{21}\ K_{23}$ has to be introduced in the third medium at a distance $2(2H + Z)$ from the original current source at A.

The process of developing image sources will continue infinitely for balancing out the proceeding boundary condition. This process will create two infinite series of image sources one in medium 1 and other in medium 3. In medium 1, the generalized expression of the image current source are $I(1 - K_{12})(K_{21}\ K_{23})^n$ and are located at a distance 2nH from the original current source at A and in medium 3, the expression of image current sources are $I(1 - K_{12})K_{23}(K_{21}\ K_{23})^{n-1}$ at a distance of $2(nH + Z)$ from the source current. Depending on the position of the electrodes current image sources are used for calculation of potentials. For example if potential current electrodes are in medium 1, the images in medium 1 are not considered.

Proceeding in a same way, it can be shown that if we have a current source in medium 2, then there will be four series of images. Considering a point source electrode at a distance Z from the interface P, the generalized expressions for image sources are described below.

a) First series of images in the medium 1 with strength $IK_{21}(K_{21}\ K_{23})^{n-1}$ at a distance $2(n-1)H+Z$ from the current source.
b) Second series of images in the medium 1 with strength $I(K_{21}\ K_{23})^{n}$ at a distance 2nH from the current source.
c) First series of images in the medium 3 with strength $IK_{23}(K_{21}\ K_{23})^{n-1}$ at a distance $2(nH-Z)$ from the original current source.
d) Second series of images in the medium 3 with strength $I(K_{21}\ K_{23})^{n}$

Proceeding in the similar way, expression for image sources when the current electrode is in medium 3 can be found out. In the following section a set of general expressions of potentials are given considering one current and one potential electrode. They cover all possible combination that can appear considering two electrodes and three media with two parallel boundaries.

11.4 General Expressions for Potentials Using Images

To generate these expressions for potentials we have taken a 3 layered earth model with upper shoulder bed resistivity ρ_1 (medium 1), target bed resistivity ρ_2 (medium 2) and lower shoulder bed resistivity ρ_3 (medium 3) ; H is the target bed thickness, distance of current electrode z is calculated from the interface between medium 1 and medium 2, when the current electrode is in the medium 1. The distance z is also calculated from the interface of the medium 2 and medium 3 when the current electrode is in the medium 3, D is the electrode separation, K_{12}, K_{23}, K_{21} are the reflection coefficients where $K_{12} = (\rho_2 - \rho_1)/(\rho_2 - \rho_1)$ and $K_{23} = (\rho_3 - \rho_2)/(\rho_3 + \rho_2)$ and I is the current sent. In some expressions $\pm$ sign is given. This is due to the relative positions of the electrodes in a particular medium. Basic equation for calculation of potential measured at the borehole axis ($r = 0$) due to one point source current electrode can be written as follows.

a) Expressions of potential at a point when both the electrodes are in medium 1

$$\phi_1^1 = \frac{\rho_1 I}{4\pi}\left[\frac{1}{D} + \frac{K_{12}}{(2Z \pm D)} + (1 - K_{12})\,K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z \pm D)}\right] \quad (11.14)$$

b) Expression of potential at a point when the current electrode is in medium 1 and the potential electrode is in medium 2

$$\phi_2^1 = \frac{\rho_2 I}{4\pi}(1 - K_{12})\left[\sum_{n=0}^{\infty}\frac{(K_{12}\ K_{23})\,n}{(2nH + D)} + K_{23}\sum_{n-1}^{\infty}\frac{(K_{21}\ K_{23})}{(2nH + 2Z - D)}\right] \tag{11.15}$$

c) Expression of potential at a point when the current electrode is in medium 1 and the potential electrode is in medium 3

$$\phi_3^1 = \frac{\rho_2 I}{4\pi}(1 - K_{12})(1 - K_{23})\sum_{n=0}^{\infty}\frac{(K_{21}\ K_{23})^n}{2nH + D} \tag{11.16}$$

d) Expression of potential at a point when the current electrode is in medium 2 and the potential electrode is in medium 1

$$\phi_1^2 = \frac{\rho_1 I}{4\pi}(1 - K_{21})\left[\frac{1}{D} + \sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^n}{(2nH + D)}\right.$$
$$\left. + K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{\{2(n-1)H + 2(H - Z) + D\}}\right] \tag{11.17}$$

e) Expression of potential at a point when both the electrodes are medium 2

$$\phi_2^2 = \frac{\rho_2{}^I}{4\pi}\left[\frac{1}{D} + K_{21}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{\{2(n-1)H + 2Z \pm D\}}\right.$$
$$+ K_{23}\sum_{n-1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{\{2(n-1)H + 2(H - Z) \mp D\}}$$
$$\left. + \sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^n}{2nH \pm D} + \sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^n}{(2nH \mp D)}\right] \tag{11.18}$$

f) Expression of potential at a point when the current electrode is in medium 2 and the potential electrode is in medium 3

$$\phi_3^2 = \frac{\rho_3 I}{4\pi}(1 - K_{23})\left[\frac{1}{D} + \sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^n}{(2nH + D)}\right.$$
$$\left. + K_{21}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{\{2(n-1)H + 2Z + D\}}\right] \tag{11.19}$$

g) Expression of potential at a point when the current electrode is in medium 3 and the potential electrode is in medium 1

$$\phi_1^3 = \frac{\rho_3 I}{4\pi}(1 + K_{12})(1 + K_{23})\sum_{n=0}^{\infty}\frac{(K_{21}\ K_{23})^n}{(2nH + D)} \tag{11.20}$$

h) Expression of potential at a point when the current electrode is in medium 3 and the potential electrode is in medium 2

$$\phi_2^3 = \frac{\rho_2 I}{4\pi}(1+K_{23})\left[\sum_{n=0}^{\infty}\frac{(K_{21}\ K_{23})^n}{(2nH+D)} - K_{12}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{(2nH+2Z-D)}\right] \tag{11.21}$$

i) Expressions of potential at a point when both the electrodes are in medium 3

$$\phi_3^3 = \frac{\rho_3 I}{4\pi}\left[\frac{1}{D} - \frac{K_{23}}{2Z\pm D} - (1-K_{23})K_{12}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{(2nH+2Z\pm D)}\right] \tag{11.22}$$

11.5 Expressions for Potentials for Two Electrode Configuration

For full space with two plane parallel boundaries there will be in total five cases for two electrode configurations (Fig. 11.3) when target bed thickness is greater than tool length. If target bed thickness is less than tool length a special case arises (Fig. 11.3). In any set up the total number of tool positions with respect to target bed are five. Expressions for Potential for different cases are given below. For two-electrode configuration, the return current electrode and other potential electrode are theoretically assumed at infinite distances. In reality, they are kept away from the electrode system AM. Tool configuration is shown in the Fig. 11.3 and AM (=D) is the electrode separation. Expressions for potentials for five cases are

Cases for thick bed (H > L)

Case 1:

$$\phi_M \frac{\rho_1 I}{4\pi}\left[\frac{1}{D} + \frac{K_{12}}{2Z-D} + (1-K_{12})K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{(2nH+2Z-D)}\right] \tag{11.23}$$

Case 2:

$$\phi_M = \frac{\rho_2 I}{4\pi}(1-K_{12})\left[\sum_{n=0}^{\infty}\frac{(K_{21}K_{23})^n}{(2nH+D)} + K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{(2nH+2Z-D)}\right] \tag{11.24}$$

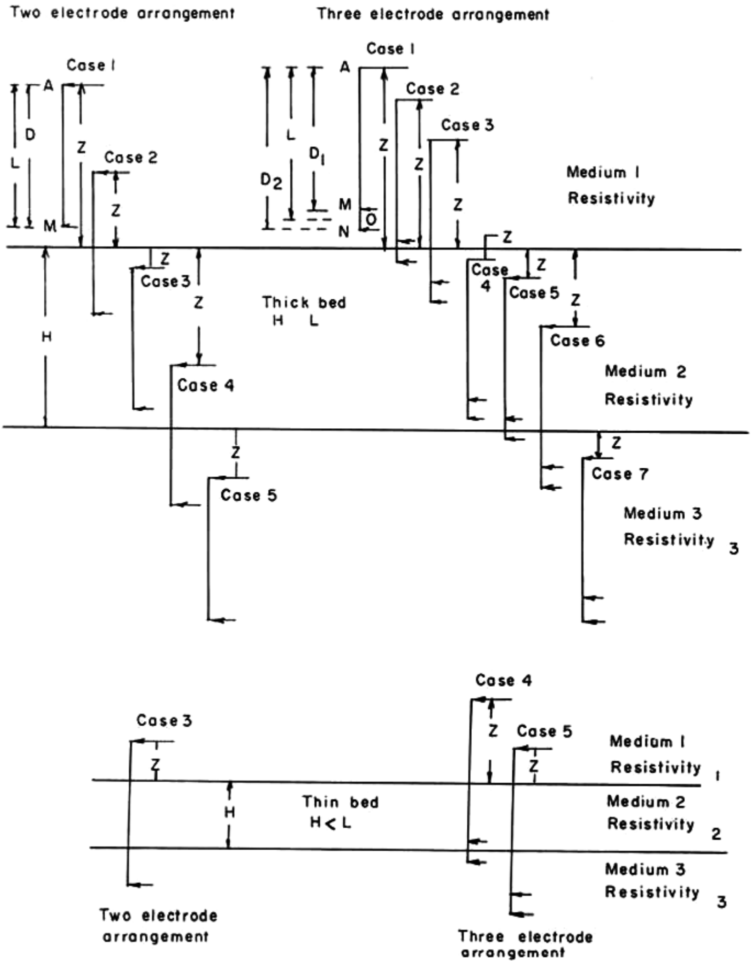


Fig. 11.3. Three media with two plane parallel boundaries; first and third media are semiinfinite media; second medium has finite thickness; two electrode (AM) and three electrode (AMN) set up are approaching and crossing the two boundaries; 5 and 7 set of images are respectively formed as shown; thin bed case, i.e., bed thickness less than the electrode separation are presented

Case 3

$$\phi_M = \frac{\rho_2 I}{4\pi}\left[\frac{1}{D} + K_{21}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{\{2(n-1)H + 2Z + D\}}\right.$$
$$\left. + K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{\{2(n-1)H + 2(H-Z) - D\}}\right.$$

$$+\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n}}{(2nH+D)}+\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n}}{(2nH-D)}\Bigg] \tag{11.25}$$

Relative tool positions for normal and lateral electrode

Case 4:

$$\phi_M=\frac{\rho_3 I}{4\pi}(1-K_{23})\left[\frac{1}{D}+\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n}}{(2nH+D)}+K_{21}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{\{2\,(n-1)\,H+2Z+D\}}\right] \tag{11.26}$$

Case 5:

$$\phi_M=\frac{\rho_3 I}{4\pi}\left[\frac{1}{D}-\frac{K_{23}}{(2Z+D)}-(1-K_{23})\,K_{12}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{(2nH+2Z+D)}\right] \tag{11.27}$$

Special case for thin bed (H < L):

Case 3:

$$\phi_M=\frac{\rho_3 I}{4\pi}(1-K_{12})\,(1-K_{23})\sum_{n=0}^{\infty}\frac{(K_{21}\ K_{23})^{n}}{(2nH+D)} \tag{11.28}$$

Apparent resistivity (ρ_a) at each point of the run is computed using the formula

$$\rho_a=4\pi.AM.\frac{\phi_M}{I} \tag{11.29}$$

where, AM is the tool length and I is the current intensity.

11.6 Expressions for Potentials for Three Electrode Configuration

In case of lateral or three-electrode arrangement, the total cases in any setup are seven. Both thick bed (H > L) and thin bed (H < L) cases are considered. The different positions including the special cases for thin bed are shown in Fig. 11.3 Bed thickness less than potential probe separation (MN) is not considered, as the separation is very less compared to tool length. Tool configuration is shown in Fig. 11.3 where A is the current electrode and M, N are the two potential electrodes. $AM(=D_1)$, $AN(=D_2)$ and MN are distances between different electrodes. O is the mid point between M, N and AO is considered the tool length. Return current electrode is considered at an infinite distance. Expressions of potentials at different positions are given in the following section.

Cases for Thick Bed (H > L):

Case 1:

$$\phi_M = \frac{\rho_1 I}{4\pi}\left[\frac{1}{D_1} + \frac{K_{12}}{(2Z - D_1)} + (1 - K_{12})K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z - D_1)}\right] \quad (11.30)$$

$$\phi_N = \frac{\rho_1 I}{4\pi}\left[\frac{1}{D_2} + \frac{K_{12}}{(2Z - D_2)} + (1 - K_{12})\,K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z - D_2)}\right] \quad (11.31)$$

Case 2:

$$\phi_M = \frac{\rho_1 I}{4\pi}\left[\frac{1}{D_1} + \frac{K_{12}}{(2Z - D_1)} + (1 - K_{12})\,K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z - D_1)}\right] \quad (11.32)$$

$$\phi_N = \frac{\rho_2 I}{4\pi}(1 - K_{12})\left[\sum_{n=0}^{\infty}\frac{(K_{21}\ K_{23})^{n}}{(2nH + D_2)} + K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{(2nH + 2Z - D_2)}\right] \quad (11.33)$$

Case 3:

$$\phi_M = \frac{\rho_1 I}{4\pi}\left(\frac{\rho_2 I}{4\pi}\right)(1 - K_{12})\left[\sum_{n=0}^{\infty}\frac{(K_{21}K_{23})^{n}}{(2nH + D_1)} + K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{(2nH + 2Z - D_1)}\right] \quad (11.34)$$

$$\phi_N = \frac{\rho_2 I}{4\pi}(1 - K_{12})\left[\sum_{n=0}^{\infty}\frac{(K_{21}\ K_{23})^{n}}{(2nH + D_2)} + K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{(2nH + 2Z - D_2)}\right] \quad (11.35)$$

Case 4:

$$\phi_M = \frac{\rho_2 I}{4\pi}\left[\frac{1}{D_1} + K_{21}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{\{2\,(n-1)\,H + 2Z + D_1\}}\right.$$

$$+ K_{23} \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{\{2\,(n-1)\,H + 2\,(H-Z) - D_1\}}$$

$$+ \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n}}{(2nH + D_1)} + \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n}}{(2nH - D_1)} \tag{11.36}$$

$$\phi_N = \frac{\rho_2 I}{4\pi}\left[\frac{1}{D_2} + K_{21} \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n-1}}{\{2\,(n-1)\,H + 2Z + D_2\}}\right.$$

$$+ K_{23} \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n-1}}{\{2\,(n-1)\,H + 2\,(H-Z) - D_2\}}$$

$$\left. + \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n}}{(2nH - D_2)} + \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n}}{(2nH - D_2)}\right] \tag{11.37}$$

Case 5:

$$\phi_M = \frac{\rho_2 I}{4\pi}\left[\frac{1}{D_1} + K_{21} \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n-1}}{\{2\,(n-1)\,H + 2Z + D_1\}}\right.$$

$$+ K_{23} \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n-1}}{\{2\,(n-1)\,H + 2\,(H-Z) - D_1\}}$$

$$\left. + \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n}}{(2nH + D_1)} + \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n}}{2nH - D_1}\right] \tag{11.38}$$

$$\phi_N = \frac{\rho_3 I}{4\pi}\,(1 - K_{23})\left[\frac{1}{D_2} + \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n}}{(2nH + D_2)}\right.$$

$$+ K_{21} \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})\ n-1}{\{2\,(n-1)\,H + 2Z + D_2\}} \tag{11.39}$$

Case 6:

$$\phi_M = \frac{\rho_3 I}{4\pi}\,(1 - K_{23})\left[\frac{1}{D_1} + \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n}}{(2nH + D_1)} + K_{21} \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{\{2\,(n-1)\,H + 2Z + D_2\}}\right] \tag{11.40}$$

$$\phi_N = \frac{\rho_3 I}{4\pi}\,(1 - K_{23})\left[\frac{1}{D_2} + \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n}}{(2nH + D_2)} + K_{21} \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n-1}}{\{2\,(n-1)\,H + 2Z + D_2\}}\right] \tag{11.41}$$

Case 7:

$$\phi_M = \frac{\rho_3 I}{4\pi}\left[\frac{1}{D_1} - \frac{K_{23}}{(2Z + D_1)} - (1 - K_{23})\,K_{12}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{(2nH + 2Z + D_1)}\right] \quad (11.42)$$

$$\phi_N = \frac{\rho_3 I}{4\pi}\left[\frac{1}{D_2} - \frac{K_{23}}{(2Z + D_2)} - (1 - K_{23})\,K_{12}\sum_{n=1}^{\infty}\frac{(K_{21}\ K_{23})^{n-1}}{(2nH + 2Z + D_2)}\right] \quad (11.43)$$

Special Cases for Thin Bed (H < L)

Case 4:

$$\phi_M = \frac{\rho_2 I}{4\pi}(1 - K_{12})\left[\sum_{n=0}^{\infty}\frac{(K_{21}\ K_{23})^{n}}{(2nH + D_1)} + K_{23}\sum_{n=1}^{\infty}\frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z - D_1)}\right] \quad (11.44)$$

$$\phi_N = \frac{\rho I}{4\pi}(1 - K_{12})(1 - K_{23})\sum_{n=0}^{\infty}\frac{(K_{21}K_{23})^{n}}{(2nH + D_2)} \quad (11.45)$$

Case 5:

$$\phi_M = \frac{\rho_3 I}{4\pi}(1 - K_{12})(1 - K_{23})\sum_{n=0}^{\infty}\frac{(K_{21}K_{23})^{n}}{(2nH + D_1)} \quad (11.46)$$

$$\phi_N = \frac{\rho_3 I}{4\pi}(1 - K_{12})(1 - K_{23})\sum_{n=0}^{\infty}\frac{(K_{21}K_{23})^{n}}{(2nH + D_2)} \quad (11.47)$$

Apparent resistivity (ρ_a) is calculated point by point using the relation

$$\rho_a = 4\pi.\frac{AM.AN}{MN}.\frac{\Delta\phi}{I} \quad (11.48)$$

where, $\Delta\phi = \phi_M - \phi_N$ and I is the current intensity.

11.7 Expression for Potentials for Seven Electrode Configurations

Direct current response across the horizontal beds of contrasting resistivity in an otherwise homogeneous and isotropic medium using vertically moving seven electrode system in the absence of any borehole is computed to highlight some points of principle. Seven electrode system assumed in this model differs from Schlumberger laterolog system on one point i.e., the bucking current electrodes are not shorted to check the variation of I_1/I_0 and I_2/I_0. The potential electrodes M_1, M_4 and M_2, M_3 are shorted the way it is done in LL7 (Laterolog -7) Schlumberger Log Interpretation Principle (1972). Differences in potential between M_1, M_2 and M_3, M_4 has been brought to the same level by equating the potentials developed. To get the maximum focusing all the four potentials are equated. Theory of electrical images are used for computation of potentials. This derivation is based on four points.

(a) Current through the central electrode $\hat{A}_0$ can always be kept constant.
(b) It is possible to bring all the four potential electrodes in a same equipotential line by adjusting bucking currents. Bucking currents are the currents I_1 and I_2 sent by two bucking or guard electrodes A_1 and A_2 (Fig. 11.3). To avoid any current flow through the potential electrodes, they are instantaneously brought to the same potential by adjusting the bucking currents.
(c) By equating potentials developed in the four potential electrodes M_1, M_2, M_3 and M_4, we get four nontrivial equations for $\phi_{M1} = \phi_{M2}$, $\phi_{M2} = \phi_{M3}$, $\phi_{M3} = \phi_{M4}$ and $\phi_{M4} = \phi_{M1}$ to solve for either two (I_1 and I_2) or three ((I_1, I_2 and I_0) unknowns for fixed or variable I_0. Here two unknowns I_1 and I_2 are determined from four equations using the generalized inverse.
(d) Variable geometric factor is considered (Roy 1977) for computation of apparent resistivity. A seven electrode system with a central focusing current electrode $\hat{A}_0$ and a pair of guard or bucking current electrodes $\hat{A}_1$ and $\hat{A}_2$ are considered. Two pairs of shorted potential electrodes M_1M_4 and M_2M_3 are taken and mathematical shorting is done equating $\phi_{M1} = \phi_{M2}$ and $\phi_{M3} = \phi_{M4}$. That ensures focusing of currents, i.e., the currents are forced to flow in a particular direction in the form of a beam of current. 0_1 and 0_2 are respectively the mid points of M_1M_2 and M_3M_4.

$\hat{A}_1\hat{A}_2(= L)$ is the electrode separation. Most of the computations were done with $0_10_2 = 0.4L$. Return current and potential electrodes are assumed to be far away from the electrode system. Bucking current $\hat{A}_1$ and $\hat{A}_2$ are allowed to remain open (not short circuited) and I_1 and I_2 are adjusted till the potentials at M_1, M_2, M_3 and M_4 are made the same. Focussing current I_0 is kept fixed. Apparent resistivity at each point is computed using the geometric factor for laterolog system, i.e.,

$$\rho_a = \frac{4\pi}{\left[\dfrac{1}{A_0M_1} + \dfrac{I_1/I_0}{A_1M_1} + \dfrac{I_2/I_0}{A_2M_1}\right]} \frac{\phi_{M_1}}{I_0} \tag{11.49}$$

In total there will be 36 cases for computation of potential for different positions of the seven electrode system. 15 of them are for thick beds (H > L) where H is the bed thickness and L is the electrode separation. Rest 21 cases are for thin beds (not shown). The expressions for potentials for thick beds

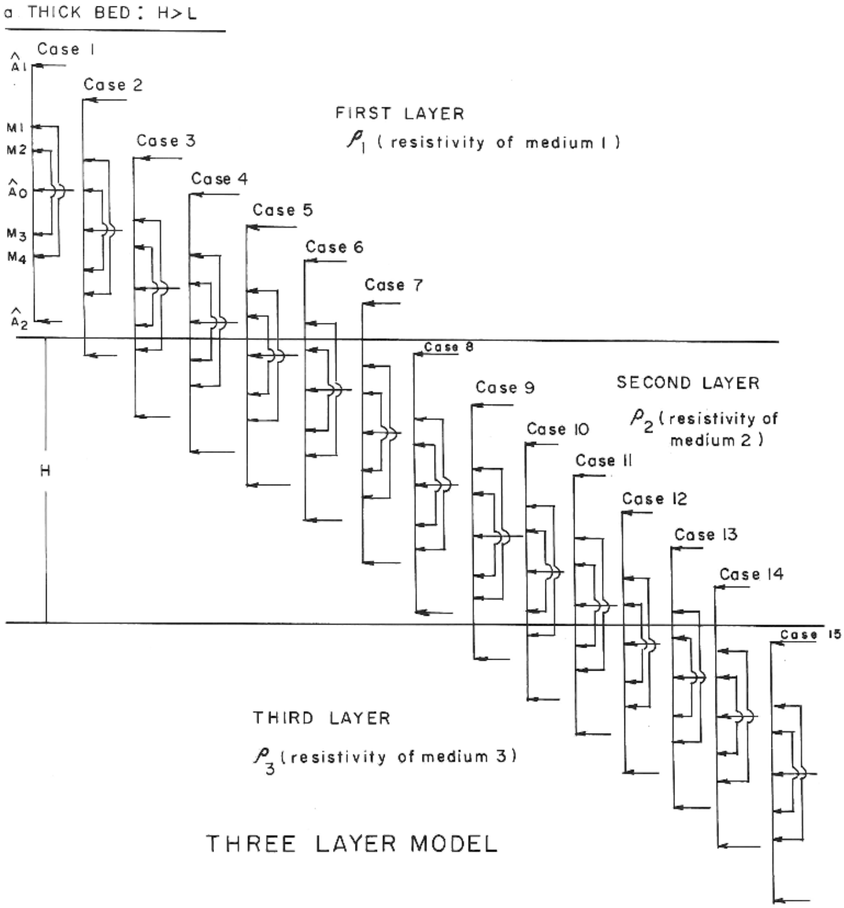


Fig. 11.4. Seven electrode laterolog – 7 is crossing the boundaries; 15 cases of electrode positions and formations of images are presented

and only for case 1 (Fig. 11.4) are presented here as a sample. Since the derivation of these expressions are quite simple and straight forward (Keller and Frischknecht 1966 and Dakhnov 1962), the rest of the exercise is omitted from the text.

Here $X_{11}, X_{12}, \ldots\ldots X_{43}$ are the coefficients of I_0, I_1 and I_2. These expressions will differ for different cases. From equation (11.54) one can write

$$\phi_{M1} - \phi_{M2} = 0 = Y_{11}I_0 + Y_{12}I_1 + Y_{13}I_2 \tag{11.50}$$
$$\phi_{M2} - \phi_{M3} = 0 = Y_{21}I_0 + Y_{22}I_1 + Y_{23}I_2 \tag{11.51}$$
$$\phi_{M3} - \phi_{M4} = 0 = Y_{31}I_0 + Y_{32}I_1 + Y_{33}I_2 \tag{11.52}$$
$$\phi_{M4} - \phi_{M1} = 0 = Y_{41}I_0 + Y_{42}I_1 + Y_{13}I_2 \tag{11.53}$$

where,

$$Y_{11} = (X_{11} - X_{21}), Y_{12} = (X_{12} - X_{22}) \tag{11.54}$$
$$Y_{13} = (X_{13} - X_{23}), Y_{21} = (X_{21} - X_{31}) \tag{11.55}$$
$$Y_{22} = (X_{22} - X_{32}), Y_{23} = (X_{23} - X_{33}) \tag{11.56}$$
$$Y_{31} = (X_{31} - X_{41}), Y_{32} = (X_{32} - X_{42}) \tag{11.57}$$
$$Y_{33} = (X_{33} - X_{43}), Y_{41} = (X_{41} - X_{11}) \tag{11.58}$$
$$Y_{42} = (X_{42} - X_{12}), Y_{43} = (X_{43} - X_{13}) \tag{11.59}$$

Equation (11.50) to (11.53) can be written in a matrix form as

$$\begin{bmatrix} Y_{12} & Y_{13} \\ Y_{22} & Y_{23} \\ Y_{32} & Y_{33} \\ Y_{42} & Y_{43} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = - \begin{bmatrix} Y_{11} \\ Y_{21} \\ Y_{31} \\ Y_{41} \end{bmatrix} I_0 \tag{11.60}$$

In a matrix notation equation (11.65) can be written as

$$Y_1 = Y' I_0 \tag{11.61}$$

Since all the four equation (11.55) to (11.58) are non trivial, the solution of equation (11.66) for estimation of bucking currents can be obtained using the least square estimator of a rectangular matrix (Draper and Smith, 1966) as

$$I = (Y^T Y)^{-1} (Y^T I_0) I_0 \tag{11.62}$$

Here Y^T is the transpose of Y.

With the help of these 9 equations (11.55) to (11.60) one can easily generate the potential expressions associated with four potential electrodes in this seven electrode system. Potential generated at any position will be the algebraic sum of potential developed due to the current flowing at different current electrodes. As an example we can give expression for potential for case 1 for thick bed ($h > L$).

Thick Bed Case: 1

Using the list of symbols given in the next section, we can write the expressions for potentials at all the four potential electrodes as

$$\begin{aligned} \phi_{M_1} = & \frac{\rho_1 I_1}{4\pi} \left[E_1 + K_{12} 0_1 + (1 - K_{12}) K_{23} A_1\right] \\ & + \frac{\rho_1 I_0}{4\pi} \left[E_5 + K_{12} 0_7 + (1 - K_{12}) K_{23} A_7\right] \\ & + \frac{\rho_1 I_2}{4\pi} \left[E_9 + K_{12} 0_9 + (1 - K_{12}) K_{23} A_9\right] \end{aligned} \tag{11.63}$$

$$\begin{aligned}\phi_{M_2} =& \frac{\rho_1 I_1}{4\pi}\left[E_2 + K_{12}0_2 + (1 - K_{12})\, K_{23}A_2\right] \\ &+ \frac{\rho_1 I_0}{4\pi}\left[E_6 + K_{12}0_8 + (1 - K_{12})\, K_{23}A_8\right] \\ &+ \frac{\rho_1 I_2}{4\pi}\left[E_{10} + K_{12}0_{10} + (1 - K_{12})\, K_{23}A_{10}\right] \end{aligned} \tag{11.64}$$

$$\begin{aligned}\phi_{M_3} =& \frac{\rho_1 I_1}{4\pi}\left[E_3 + K_{12}0_3 + (1 - K_{12})\, K_{23}A_3\right] \\ &+ \frac{\rho_1 I_0}{4\pi}\left[E_7 + K_{12}0_5 + (1 - K_{12})\, K_{23}A_5\right] \\ &+ \frac{\rho_1 I_2}{4\pi}\left[E_{11} + K_{12}0_{11} + (1 - K_{12})\, K_{23}A_{11}\right] \end{aligned} \tag{11.65}$$

$$\begin{aligned}\phi_{M_4} =& \frac{\rho_1 I_1}{4\pi}\left[E_4 + K_{12}0_4 + (1 - K_{12})\, K_{23}A_4\right] \\ &+ \frac{\rho_1 I_0}{4\pi}\left[E_8 + K_{12}0_6 + (1 - K_{12})\, K_{23}A_6\right] \\ &+ \frac{\rho_1 I_2}{4\pi}\left[E_{12} + K_{12}0_{12} + (1 - K_{12})\, K_{23}A_{12}\right] \end{aligned} \tag{11.66}$$

Above expressions for potentials (11.50) to (11.53), can be written in a short form after separating out the terms of I_1 and I_0 and I_2 as

$$\begin{aligned} \phi_{M1} &= X_{11}I_0 + X_{12}I_1 + X_{13}I_2 \\ \phi_{M2} &= X_{21}I_0 + X_{22}I_1 + X_{23}I_2 \\ \phi_{M3} &= X_{31}I_0 + X_{32}I_1 + X_{33}I_2 \\ \phi_{M4} &= X_{41}I_0 + X_{42}I_1 + X_{43}I_2 \end{aligned} \tag{11.67}$$

Computation for two electrode configuration for thick and thin bed ($H > L$) are done following Roy and Rathi (1981) and Dutta (1993).

List of Symbols

$$\begin{aligned} &E_1 = \frac{1}{D_{11}}; \quad E_2 = \frac{1}{D_{12}}; \quad E_3 = \frac{1}{D_{13}}; \quad E_4 = \frac{1}{D_{14}}; \quad E_5 = \frac{1}{D_{21}}; \\ &E_6 = \frac{1}{D_{22}}; \quad E_7 = \frac{1}{D_{11}}; \\ &E_8 = \frac{1}{D_{24}}; \quad E_9 = \frac{1}{D_{31}}; \quad E_{10} = \frac{1}{D_{32}}; \quad E_{11} = \frac{1}{D_{33}}; \quad E_{12} = \frac{1}{D_{34}} \\ &0_1 = \frac{1}{2Z_1 - D_{11}}; \quad 0_2 = \frac{1}{2Z_1 - D_{12}}; \quad 0_3 = \frac{1}{2Z_1 - D_{13}}; \end{aligned}$$

$$0_4 = \frac{1}{2Z_1 - D_{14}};$$
$$0_5 = \frac{1}{2Z_2 - D_{23}}; \quad 0_6 = \frac{1}{2Z_2 - D_{24}}; \quad 0_7 = \frac{1}{2Z_2 + D_{21}};$$
$$0_8 = \frac{1}{2Z_1 + D_{22}};$$
$$0_9 = \frac{1}{2Z_3 + D_{31}}; \quad 0_{10} = \frac{1}{2Z_3 - D_{32}}; \quad 0_{11} = \frac{1}{2Z_3 + D_{33}};$$
$$0_{12} = \frac{1}{2Z_3 + D_{34}} \quad ;$$
$$A_1 = \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z_1 - D_{11})}; \quad A_2 = \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z_1 - D_{12})};$$
$$A_3 = \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z_1 - D_{13})}$$
$$A_4 = \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n-1}}{(2nH + 2Z_1 - D_{14})}; A_5 = \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z_2 - D_{23})};$$
$$A_6 = \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z_1 - D_{24})}$$
$$A_7 = \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z_2 + D_{21})}; \quad A_8 = \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z_2 - D_{22})}$$
$$A_9 = \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z_3 + D_{31})}; \quad A_{10} = \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n-1}}{(2nH + 2Z_3 + D_{32})};$$
$$A_{11} = \sum_{n=1}^{\infty} \frac{(K_{21}\ K_{23})^{n-1}}{(2nH + 2Z_3 + D_{33})}; \quad A_{12} = \sum_{n=1}^{\infty} \frac{(K_{21}K_{23})^{n-1}}{(2nH + 2Z_3 + D_{34})}$$

where

K_{12}, K_{21}, K_{23} are the reflection factors.

Z_1, Z_2, Z_3 are the distances of the current electrodes from the interfaces (Fig. 11.5).

D_{11}, D_{12}, D_{13}, D_{14}, D_{21}, D_{22}, D_{23}, D_{24}, D_{31}, D_{32}, D_{33}, D_{34} are the distances between different current and potential electrodes (Fig. 11.5).

H is the target bed thickness. The nature of seven electrode responses across a three media formation is given in Fig. 11.4.

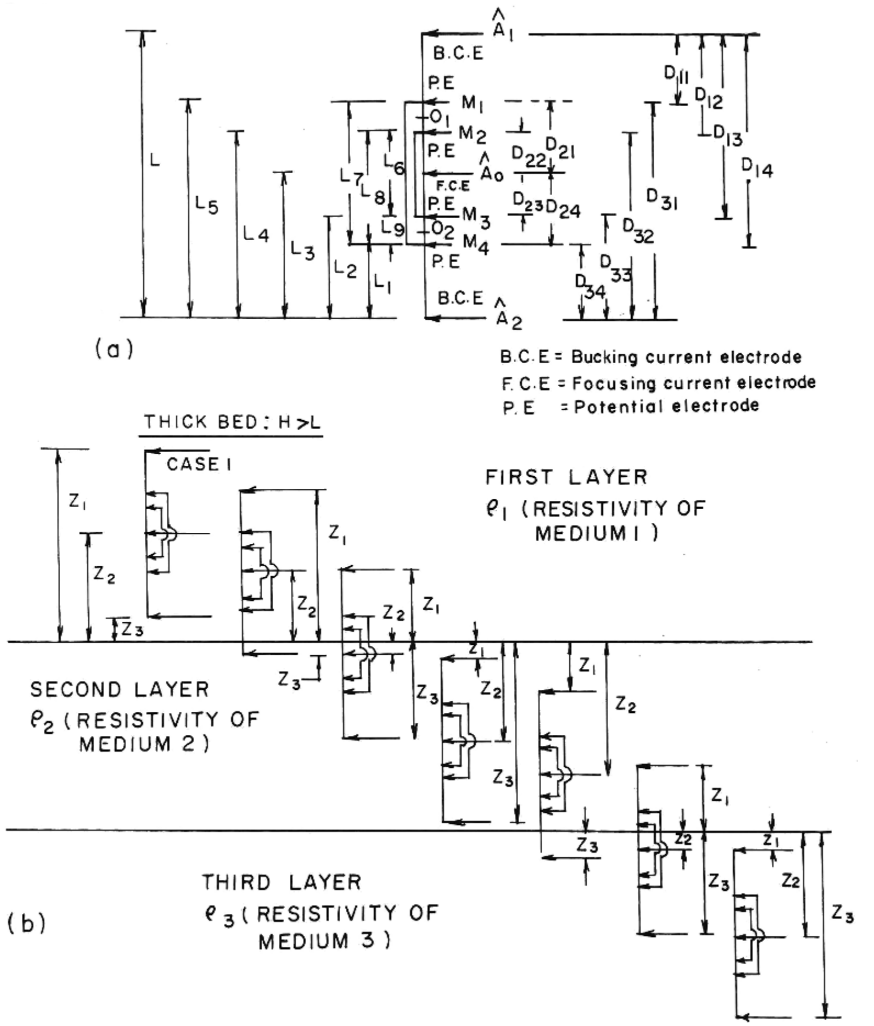


Fig. 11.5. Different positions of the current and potential electrodes with respect to the boundaries in a seven electrode laterolog seven system for computation of potentials

12

Electromagnetic Theory (Vector Potentials)

In this chapter basics of electromagnetics are introduced. Nature of a wavelet, characteristics of waves, its amplitude and phase and their characteristic equations, elliptic polarization, mutual inductance and their complex nature, nature of wave propagation, attenuation of waves, skin depth, propagation constant, Maxwell's electromagnetic equations, Helmholtz electromagnetic wave equations, Hertz and Fitzerald vector potentials, boundary conditions in electromagnetics and poynting vector are discussed.

12.1 Introduction

In this chapter and in the next we shall discuss a few important aspects of electromagnetic theory or the theory of time varying electromagnetic fields. The treatments are based on vector potentials. Therefore the mathematical treatments are essentially based on Maxwell's electromagnetic equations and Helmholtz electromagnetic wave equations. The treatments have typical geophysical orientation.

Magnetic field originates due to flow of charge or flow of current. When this current becomes time varying in a source, electromagnetic field originates and electromagnetic waves start propagating. These electromagnetic waves are transverse waves, i.e., the direction of vibration of a particle in a medium, through which the wave passes, is at right angles to the direction of propagation (Fig. 12.1).

Electromagnetic waves are similar to light waves and shear elastic waves so far as the nature of propagation is concerned. Electromagnetic waves can travel through any medium like air, gas, vacuum but not through a perfect conductor. Through vacuum EM waves travel at a speed of light. In the very broad spectrum of electromagnetic waves starting from gamma rays to radio waves and beyond, visible light waves occupy only a narrow band VIBGYOR. Velocity of electromagnetic waves get significantly retarded by several order

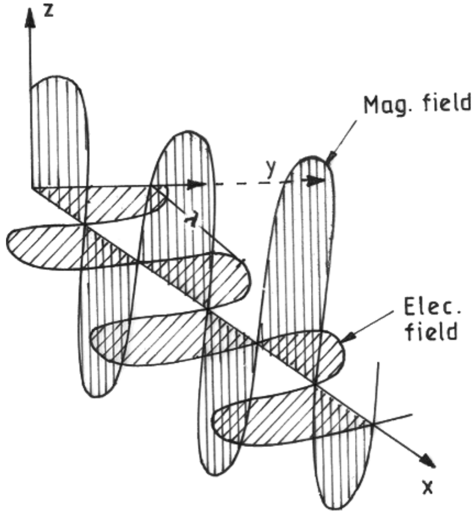


Fig. 12.1. Transverse nature of electromagnetic wavelets orthogonal to the direction of propagation

of magnitude when it travels through a medium of finite electrical conductivity. Even then the velocity of electromagnetic waves remain significantly more than velocity of sound and elastic longitudinal P waves. Any wire loop or a linear conductor carrying alternating or time varying current can become a source of electromagnetic waves. As early as 1912 Faraday has demonstrated through his famous experiment on electromagnetic induction that when a bar magnet approaches a coil (circular or any other shape), current flows through the coil and this current deflects a galvanometer (Chap. 5). Magnitude of induced e.m.f. is proportional to the rate of change of number of lines of forces. Magnitude of current depends on speed of forward and backward movement to and from a coil. That is how electricity and magnetism are attached to each other both in magnetostatics and in time varying electromagnetic fields. So whenever time varying magnetic field is cut by a conductor, i.e. a body of finite electrical conductivity, induced currents or eddy currents are generated within these conductors. All the important terminologies, viz. wavelength, velocity, frequency, time period, amplitude, phase, wave number are used to characterise an electromagnetic or time varying field. Magnitude of these induced currents or eddy currents are very much dependent upon the frequency of the electromagnetic signals and the conductivity of the medium through which an electromagnetic wave travels. So higher the conductivity and/or frequency higher will be the eddy currents and higher will be the attenuation of EM signals according to the Lenz's law of electromagnetic induction. So electromagnetic waves travel through vacuum almost unattenuated. it travels through air with very little attenuation but gets attenuated significantly when it travels through a conductor. According to Lenz's law, the eddy current opposes the incoming electromagnetic waves. So higher the conductivity and/or frequency of the em signal, stronger will be the eddy currents, stronger will be the force of opposition, greater will be the attenuation.

Electromagnetic waves are used in many walks of life such as information technology, space technology, communication technology, defence, sports, navigation etc. The details are beyond the scope of this book.

Scientists and engineers of many disciplines viz. electrical communication, electrical power, aerospace engineering, physics, geophysics etc use electromagnetic waves. The role of electromagnetic waves in geophysics is very much special in the sense that geophysicists deal with the earth, a spherical solid body of radius nearly 6370 km and the upper atmosphere upto the height of magnetosphere/ magnetopause level, 3 to 6 times the radius of the earth, in the outer space and the space above. Electrical Communication Engineers and space technologists deal with chips, microchips, semi conductors, dielectrics or machines for very long distance propagation, reflection, refraction, diffraction and scattering of electromagnetic waves.

Geophysicists probably use the widest frequency band of the electromagnetic waves. Their necessity forced them to do so. To study the earth from electrical conductivity point of view, one needs very large depth of penetration of the electromagnetic waves. Pacific or Atlantic Ocean with 4 to 5 km of saline water cover, absorbs all the high frequency electromagnetic signals propagated from the upper atmosphere towards the earth. So the starting frequency of the signal is cycles/12 hours over ocean surface. Cycles/day cycles/week, cycles/month, cycles/year, cycles/11 year are the frequency ranges for ocean bottom electromagnetic studies. To have larger depth of penetration of electromagnetic waves geophysicists went toward lower and lower frequencies or longer and longer periods. In exploration geophysics, the requirement for depth of penetration is considerably less for shallow ground water and mineral exploration problems. For oil exploration problem maximum depth of investigation should be of the order of 5 to 6 kms from the surface and that too in sedimentary rocks. For solid earth geophysics, when we try to reach the centre of the earth, one needs very very long period magnetic signals available only from the permanent geomagnetic observatories. It was observed that resistivity of sedimentary rock is of the order of 1 to 30 ohm-m. Therefore 4 to 5 kms of sedimentary rocks as in Cambay basin of western India, Bengal basin in eastern India can considerably absorb the high frequency signals and attenuate the longer period signals. Hence for deeper probing, one should avoid soft rocks and should try to take measurements over hard rock exposures. Scientists search for granite windows to see beyond Moharovicic discontinuity, lithosphere – asthenosphere boundary, olivine-spinel transition zone and beyond. Highly resistive granites allow the high frequency signals to penetrate deep inside.

There are other problems beyond upper crustal level. Depth of penetration is controlled by skin depth (discussed later). Shallow magma chambers in a high heat flow areas, lower crustal conductors originated due to accumulation of fluids, presence of continuous phase grain boundary graphites will absorb the relatively higher frequency signals. Electrical conductivity of the earth's crust mantle silicates have very strong dependence on temperature. Since temperature is increasing with depth and it reaches about 1200° C to 1300° C at the lithosphere asthenosphere boundary, an insulator at room temperature can become a conductor at 1300° C/1400° C. Conducting asthenosphere absorbs electromagnetic signals. Longer and longer period electromagnetic signals with poorer and poorer resolving power enter deep inside the earth. Therefore, magnetovariational sounding (MVS), magnetotelluric sounding (MTS) and long period geomagnetic depth sounding (GDS) are used to study deep inside the earth. Fortunately the energy level of the earth's natural electromagnetic field is very high (of the order of 10^{23} ergs) and the energy level increases with period of the em signals coming from the magnetosphere level and above. This naturally gifted property of the em signals coming from the outer space with a very wide band of frequencies make the magnetotelluric (MTS), magnetovariational(MVS) and geomagnetic depth(GDS) sounding possible.

For deeper probing, these earth's natural electromagnetic fields are used. These natural signals which originate due to interaction of the solar flares with the magnetosphere, tidal oscillation of the ionosphere in the magnetosphere, ring current, diurnal and seasonal variations of the time varying extraterrestrial electromagnetic fields, thunderstorm activities in between the earth ionosphere waveguide have a very wide range of frequencies. Due to phenomenal advancement in instrumentation, and development of softwares for data processing and interpretation, it is now possible to collect data over a very wide band of frequencies specially for very very low frequencies. Four to five decades of research in instrumentation made these measurements possible.

For shallow electromagnetics, used for groundwater and mineral exploration, man made low power units having frequencies in the range of 300 C/s to 5000 C/s are available. Signals within the frequency range of 20,000 Hz are used for induction logging. Signals of GigaHertz and MegaHertz frequencies are used in electromagnetic propagation tool (EPT) in borehole geophysics. EPT is used to demarcate the oil-water contact at the borehole wall. Here half to one inch of penetration of EM signal is good enough. Therefore, the frequency could be raised to the level which are used by communication engineers. One naturally gifted properly which helped the geophysicists is the significant difference in dielectric constant of water (80) and hydrocarbons (5). At very high frequencies, displacement current dominates over the conduction current. Since displacement current components have direct relation with the dielectric constant, it become possible to detect the oil-water contact in a borehole within the 1 to 2 inches of depth of penetration along the borehole wall. Inside the coal mines also one uses MHz to GHz signals to estimate the thickness of the coal columns. A promising very high frequency tool came up during the last two decades is known as 'Ground Penetrating

Radar'(GPR). Its operating frequency is in gegahertz range(GHz) and its resolving power is comparable to that of seismic P waves. It can see only 5 to 10 meters from the ground surface. Successive reduction in frequency increases the depth of investigation. It can map very finer details of the shallow subsurface. Besides these three uses of the high frequency signals, most of the geophysical EM jobs are in the audiofrequency ranges where displacement currents are negligible.

Conduction current and diffusion of electromagnetic waves through the air and earth are important for geophysicists. Most of the geophysical work are done in the conduction zone where the electrical conductivity rather than the dielectric constant control the electromagnetic wave propagation. Since electromagnetic waves can travel through the air; even an antenna in the air can excite the ground. That brought the air borne electromagnetics, an important branch of electromagnetics, in geophysics. Air borne geoelectromagnetics is a very big branch of geophysics, where both time domain and frequency domain measurements are possible and are done. The other name of time domain electromagnetics is electromagnetic transients. It is a very important branch of geophysics for probing upto the upper crust of the earth. Fortunately the time domain decay of an em signal is much faster than the time domain decay of induced polarization signals. That is why it became possible to isolate the em transients from IP in time domain measurements. Since the behaviour of the induced polarization signal of electrochemical origin has some similarity with the presence of capacitance in a resistive circuit, variation of impedances and their phases of these capacitors with frequency are studied in frequency domain and in spectral induced polarization, measurement of IP phase or variation of decay during discharging of a capacitor in a time domain IP are studied.

For direct current flow one needs galvanic contact of current electrodes with the ground and potentials are measured through a pair of potential electrodes. In electromagnetics transmitting and receiving dipoles are used for one type of measurement. Current can be sent through a large loop insulated from the ground or a grounded loop or line source and the response can be measured in receiver loops. For earth's natural electromagnetic field, the magnetic signals are received in induction coils or in magnetometers (flux gate, proton procession, SQUIDS etc.) and electrical signals are measured by two pairs of non polarisable electrodes planted on the ground. A coil carrying current is termed as magnetic dipole. When alternating current passes through it, the coil becomes an oscillating magnetic dipole(Chap. 5 and 13). The measured signals in electromagnetics are complex quantities. Therefore real and imaginary components or amplitude and phase are measured. EM signals are measured in many different ways. The design and nature of transmitters and receivers are purpose and problem dependent. Only some basic theories, needed for the graduate students to start with, are discussed in this chapter and Chap. 13.

12.2 Elementary Wavelet

In a continuous medium, any disturbance in the form of an electrical pulse or sound pulse will continue to propagate waves. These waves are functions of space and time and can be expressed as

$$\phi = f(x, t). \tag{12.1}$$

This disturbance will move forward or backward with a certain velocity c. ϕ changes to

$$\phi = f(x - ct) \text{ for a forward moving wave and} \tag{12.2}$$
$$\phi = f(x + ct) \text{ for a backward moving wave} \tag{12.3}$$

where t is time.
Any time varying and propagating harmonic signal can be expressed as

$$\phi = a \cos \omega x \text{ or } a \sin \omega x. \tag{12.4}$$

Here a is the amplitude and ω ($= 2\pi n$, n is the frequency of the time varying signal) is the angular frequency and x is the displacement. Many different ways, the same (12.4) can be written. The same expression for a time varying field can be written as

$$\phi = a \cos \omega (x - ct). \tag{12.5}$$

For a harmonic wave equation, we can write

$$x = x + \frac{2\pi}{\omega} \tag{12.6}$$

where $2\pi = 360°$. Figure 12.2 shows a harmonic wave where wavelength λ, amplitude a, phases $0°, 90°, 180°, 270°$ and $360°$ are shown. In the phase domain plot, the vector Ex_1 comes back to the original position after rotation for an angle 2π. Amplitude of the wave is the length of the vector Ex_1 and it is the radius of the circle presented in Fig. 12.3. Figure 12.2 shows the two signals of same amplitude and frequency but difference in phase. Phase is an angle and it is measured either in degrees or in radian or milliradian.The angle shown between the vectors Ex_1 and Ex_2 is the phase angle and it is the phase difference between two sinusoidal signals shown in Fig. 12.2. For a particular signal phase repeats after the lapse of time period T.
Therefore (12.5) can be written as

$$\phi = a \cos \omega \left(x + \frac{2\pi}{\omega} - ct \right). \tag{12.7}$$

Frequency n ($\omega = 2\pi n$) of a time varying signal is the number of wavelets of length λ cross a particular point in the space domain per second. Here λ is the wave length and T is the time period of the signal and $n = \frac{1}{T}$ i.e., frequency

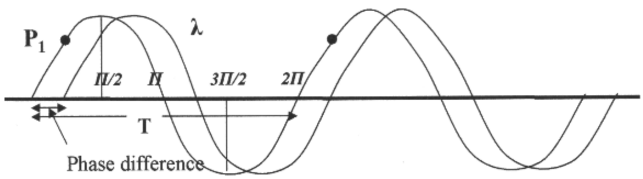


Fig. 12.2. Two sine wavelets of same frequency and same amplitude with a dieffer-ence of phase

and time period have reciprocal relation. $\omega T = 2\pi$, or, $T = \frac{2\pi}{\omega}$ i.e., after a complete time period T, the phase will rotate by 2π and the wave will repeat itself. That is why (12.5) could be written as (12.7) using the relation (12.6). Equation (12.7) can be written as

$$\begin{aligned} \phi &= a\cos[\omega(x - ct) + 2\pi] \qquad (12.8)\\ &= a\cos\omega(x - ct). \end{aligned}$$

After traveling a distance $\frac{2\pi}{\omega}$, the wave profile repeat itself.

Since $c = n\lambda$; $\lambda = \frac{c}{n}$; $\omega T = 2\pi$ and $T = \frac{1}{n}$, therefore $n = \frac{\omega}{2\pi}$. Hence $\lambda = \frac{2\pi c}{\omega}$ or $\frac{2\pi c}{\lambda} = \omega$, and

$$\phi = a\cos\frac{2\pi c}{\lambda}(x - ct)\,. \qquad (12.9)$$

Wave number $k = \frac{1}{\lambda}$, and $\frac{\lambda}{c} = T = \frac{1}{n}$, therefore,

$$\begin{aligned} \omega(x - ct) &= \frac{2\pi c}{T}\left(\frac{x}{c} - t\right) = \frac{2\pi c}{T}\left(\frac{x}{n\lambda} - t\right)\\ &= \frac{2\pi c}{t}\left(\frac{xT}{\lambda} - t\right) = 2\pi c\left(\frac{x}{\lambda} - \frac{t}{T}\right) \qquad (12.10) \end{aligned}$$

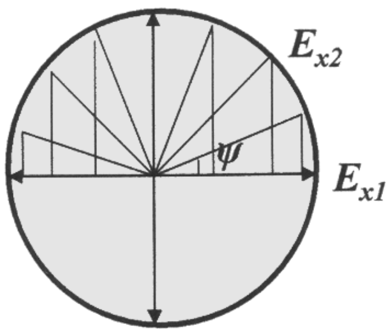


Fig. 12.3. Representation of two electric vector with a phase difference; projection of these vectors on the abscissa executes simple harmonic motion

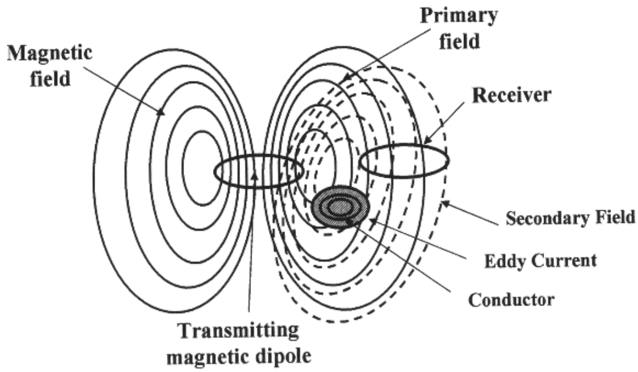


Fig. 12.4. Primary electromagnetic field due to a transmitting magnetic dipole; generation of eddy currents and secondary field in the presence of a conductor

$$\Rightarrow \phi = a\cos 2\pi c\left(\frac{x}{\lambda} - \frac{t}{T}\right) \tag{12.11}$$

$$\Rightarrow a\cos 2\pi c\left(\frac{x}{\lambda} - nt\right) \tag{12.12}$$

$$\Rightarrow a\cos 2\pi c(kx - nt). \tag{12.13}$$

If we insert a phase component, we get

$$\phi = a\cos[2\pi c(kx - nt) + \in] \tag{12.14}$$

where $\in$ is the change in phase.

From (12.5) we get, after differentiating twice,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{12.15}$$

This is the one dimensional wave equation. For two and three dimensional cases the expressions for the waves are respectively given by

$$\phi = f(lx + my - ct) \tag{12.16}$$

$$\phi = f(lx + my + pz - ct) \tag{12.17}$$

where l, m and p are direction cosines.

12.3 Elliptic Polarisation of Electromagnetic Waves

When two electromagnetic waves differing in amplitude and phase interact, they get elliptically polarised. These sources of the electromagnetic waves can be two independent sources or it may be due to interaction of primary and secondary fields. When an electromagnetic wave propagates from a source, it generates a primary field. When these time varying signals start propagating through a conductor, eddy currents are generated within it due to rate of change of number of lines of forces in any section. These eddy currents start generating electromagnetic waves of different amplitude and phase but same frequency. This is a secondary field (Fig. 12.4). These two primary and secondary signals interact and get elliptically polarized. Let the two signals be

$$\mathrm{E_x = E_{x_0} \cos \omega t} \tag{12.18}$$

$$\mathrm{E_y = E_{y_0} \cos (\omega t - \phi)} \tag{12.19}$$

where $\mathrm{E_{xo}, E_{y0}}$ are the amplitudes and ϕ is the phase difference between the two signals $\mathrm{E_x}$ and $\mathrm{E_y}$. From (12.18) and (12.19), we can write

$$\frac{E_y}{E_{y0}} = \frac{E_x}{E_{x0}} \cos \phi + \sqrt{1 - \frac{{E_x}^2}{{E_{x_0}}^2}} \sin \phi$$

$$\left(\frac{\mathrm{E_y}}{\mathrm{E_{y_0}}} - \frac{\mathrm{E_x}}{\mathrm{E_{x_0}}} \cos \phi \right)^2 = \left(1 - \frac{\mathrm{E_x^2}}{\mathrm{E_{x_0}^2}} \right) \sin^2 \phi$$

$$\Rightarrow \quad \frac{E_y^2}{E_{y_a}^2} + \frac{E_x^2}{E_{y_a}^2} \cos^2 \varphi - \frac{2 E_y E_x}{E_{y_a} E_{x_a}} \frac{\cos \varphi}{\sin^2 \varphi} = 1 \tag{12.20}$$

$$\Rightarrow \quad \mathrm{LE^2{}_x + NE_y^2 - 2ME_xE_y = 1} \tag{12.21}$$

when $\mathrm{M^2 - LN < 1}$. Then (12.21) becomes an equation of an ellipse. The major and minor axes of the ellipse are respectively given by

$$\mathrm{a} = \sqrt{\frac{2}{\mathrm{L + N - \sqrt{4M^2 + (L - N)^2}}}} \tag{12.22}$$

$$\mathrm{b} = \sqrt{\frac{2}{\mathrm{L + N + \sqrt{4M^2 + (L - N)^2}}}}. \tag{12.23}$$

Here

$$\mathrm{M} = \frac{\cos \phi}{\mathrm{E_{y_0} E_{x_0}} \sin^2 \phi} \tag{12.24}$$

$$\mathrm{L} = \frac{1}{\mathrm{{E_{x_0}}^2} \sin^2 \phi} \tag{12.25}$$

$$\mathrm{N} = \frac{1}{\mathrm{{E_{y_0}}^2} \sin^2 \phi}. \tag{12.26}$$

So

$$\begin{aligned} \mathrm{M^2 - LN} &= \frac{\cos^2 \phi}{\mathrm{{E_{y_0}}^2 {E_{x_0}}^2} \sin^4 \phi} - \frac{1}{\mathrm{{E_{x_0}}^2 {E_{y_0}}^2} \sin^4 \phi} \\ &= \frac{1}{{E_{y_0}}^2 {E_{x_0}}^2 \sin^4 \phi} \left(\cos^2 \varphi - 1 \right). \end{aligned} \tag{12.27}$$

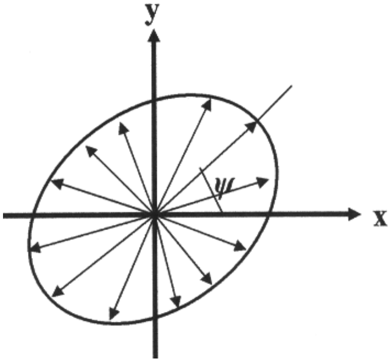


Fig. 12.5. Elliptic polarisation due to interaction of two electromagnetic waves having different amplitude and phase

Since $\cos^2\phi < 1$, therefore $M^2 - LN < 1$, therefore the field is elliptically polarised (Fig. 12.5). The angle of tilt of the ellipse is given by

$$\tan 2\psi = \frac{2M}{L - N}. \tag{12.28}$$

Hence when an alternating current is sent through the ground, an elliptically polarised field is established. It has no longer constant direction and magnitude but traces ellipse in the xy/xz/yz plane. This property generated the dip angle or tilt angle method of electromagnetic prospecting in geophysics.

12.4 Mutual Inductance

The Fig. 12.6 shows an arrangement of two coils for developing the induced e.m.f.
The magnetic field inside the longer solenoid is uniform and has the magnitude

$$\vec{B} = \frac{\mu}{4\pi}\frac{N_1 I_1}{l} \tag{12.29}$$

where N_1 is the number of turns and l is its length in coil 1; I_1 is the current flowing through it. Cross sectional area of the coil 1 is S; the flux is its magnitude time S. If the coil 2 has N_2 turns, this flux links the coil N_2 times. Therefore, emf in coil 2 is given by

$$\varepsilon_2 = N_2 S \frac{\partial \vec{B}}{\partial t} \tag{12.30}$$

Equation (12.30) can be written as

$$\varepsilon_2 = \frac{\mu}{4\pi} S \frac{N_2 N_1}{l}\frac{dI_1}{dt}. \tag{12.31}$$

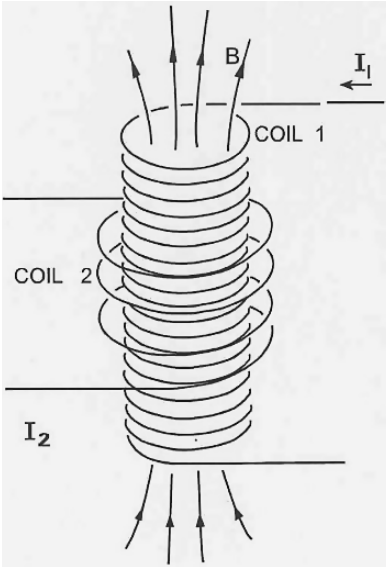


Fig. 12.6. Mutual induction between the two coils carrying currents I_1 and I_2

Here emf in coil 2 is proportional to the rate of change of current in coil 1. The constant of proportionality which is basically a geometric factor of the two coils is termed as the mutual inductance and is written as

$$\varepsilon_2 = M_{21}\frac{dI_1}{dt}. \tag{12.32}$$

If current I_2 passes through coil 2, then the emf in coil 1 will be

$$\varepsilon_1 = M_{12}\frac{dI_2}{dt} \tag{12.33}$$

because field will be proportional to I_2 and the flux linkage through the coil 1 would be $\frac{dI_2}{dt}$.

12.4.1 Mutual Inductance Between any Two Arbitrary Coils

The general expression for the induced emf in the coil 1 is $\varepsilon = -\frac{\partial}{\partial t}\int \vec{B}\vec{n}.ds$ where $\vec{B}$ is the magnetic induction and the integral is taken over the surface bounded by the coil 1. Applying Stoke's theorem one can write $\int \vec{B}.\vec{n}.ds = \oint \vec{A}.dl$, where $\vec{A}$ is a vector potential and $\vec{B} = \mathrm{curl}\,\vec{A}.dl_1$ is the element of the circuit in coil. The line integral is taken around the circuit 1. The emf in coil 1, can be written as

$$\varepsilon_1 = -\frac{d}{dt}\oint \vec{A}.dl_1. \tag{12.34}$$

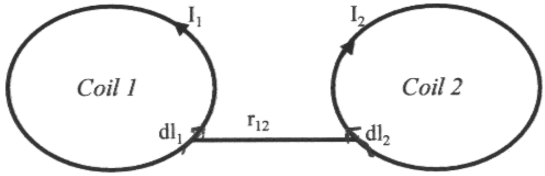


Fig. 12.7. Mutual inductance between the two noncoaxial arbitrary coils separated by a certain distance

Let the vector potential in circuit 1 comes from the current in circuit 2. The line integral around the circuit 2 can be written as (Fig. 12.7)

$$A = \frac{\mu}{4\pi}\oint \frac{I_2 dl_2}{r_{12}} \tag{12.35}$$

where I_2 is the current in the circuit 2 and r_{12} is the distance from the element of the circuit dl_2 to the point of measurement on the circuit 1 at which we are evaluating the vector potential. Hence the emf in the circuit 1 appears as a double line integral.

$$\varepsilon_1 = -\frac{\mu}{4\pi}\frac{l}{dt}\oint_1 \oint_2 \frac{I_2 dl_2 dl_1}{r_{12}}. \tag{12.36}$$

In this equation, the integrals are all taken with respect to the stationary circuits. The variable quantity I_2 does not depend upon the variable of integration. We can, therefore, write

$$\varepsilon_1 = m_{12}\frac{dI_2}{dt}. \tag{12.37}$$

Here

$$m_{12} = -\frac{\mu}{4\pi}\oint_1 \oint_2 \frac{dl_1 dl_2}{r_{12}} \tag{12.38}$$

and m_{12} is the mutual inductance.

If there are two currents in the two coils simultaneously, the magnetic flux linking the two coils will be sum of the two fluxes linking separately. The emf in either coil will therefore be proportional not only to the change of current in the other coil but also to the change of current in the coil itself. Therefore, the total emf is

$$\varepsilon_2 = m_{21}\frac{dI_1}{dt} + m_{22}\frac{dI_2}{dt} \tag{12.39}$$

and

$$\varepsilon_1 = m_{12}\frac{dI_2}{dt} + m_{11}\frac{dI_1}{dt} \tag{12.40}$$

The coefficients m_{22} and m_{11} are self inductances of the coils. The self inductance exits for one coil. Here $m_{11} = -L_1$ and $m_{22} = -L_2$ where L_1 and L_2

are the self inductances of the coil 1 and 2 respectively. Since emf generated due to the rate of change of number of lines of forces oppose the changes of current, according to Lenz's law, the self inductances are negative.

12.4.2 Simple Mutual Inductance Model in Geophysics

Grant and West (1965) in a simple mutual inductance model, demonstrated the very basic nature of the electromagnetic response one gets for geophysical problems due to vertical magnetic dipoles used as a transmitter and a receiver. Here conducting earth is replaced by a vertical buried coil (Fig. 12.8)

Let us suppose that an alternating current $I_o e^{i\omega t}$ is made to flow in the transmitting coil. This current generates an alternating magnetic field in the surrounding environment which in turn induces an emf both in the conductor as well as in the receiving coil. These emf's are governed by the Faraday's law of electromagnetic induction, i.e.,

$$E_j = -M_{ij}\frac{\partial I_1}{\partial t} \tag{12.41}$$

where E_j is the emf induced in one circuit by the current I_1 flowing in another circuit and M_{ij} is the mutual inductance. The emf induced in the receiver by the primary field is, therefore,

$$\begin{aligned}\varepsilon_2 &= -M_{02}\frac{d}{dt}I_0 e^{i\omega t}\\ &= -i\omega M_{02} I_0 e^{i\omega t}.\end{aligned} \tag{12.42}$$

Here M_{02} is the mutual inductance between the transmitter and the receiver coil. The emf induced in the vertical coil is

$$\varepsilon_1 = -i\omega M_{01} I_0 e^{i\omega t} \tag{12.43}$$

where M_{01} is the mutual inductance between the transmitting and the vertical coil representing the subsurface. To this emf ε_1, we must add $\varepsilon_1{}^{+}$, the sum of the voltage drop across the resistance and self inductance of the vertical coil carrying current $I_1 e^{i\omega t}$. Thus

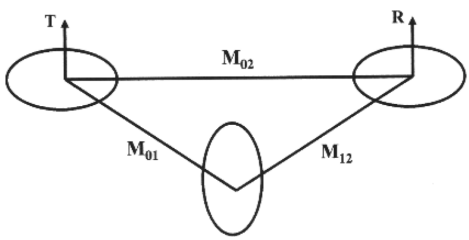


Fig. 12.8. A simple mutual inductance model to represent electromagnetic Induction in the earth (Grant and West 1985)

$$\varepsilon_1{}^{+} = -\mathrm{RI}_1 e^{i\omega t} - \mathrm{L}\frac{d}{dt}\mathrm{I}_1 e^{i\omega t} = -(\mathrm{R} + i\omega\mathrm{L})\mathrm{I}_1 e^{i\omega t}. \tag{12.44}$$

To find the current I_1, we observe that around any closed loop, the total emf must vanish, i.e., $\varepsilon_1 + \varepsilon_1{}' = 0$ and therefore

$$\mathrm{I}_1 e^{i\omega t} = -i\omega \frac{\mathrm{M}_{01}}{(\mathrm{R} + i\omega\mathrm{L})} \mathrm{I}_0 e^{i\omega t} \tag{12.45}$$

$$I_1 e^{i\omega t} = -\frac{M_{01}}{L}\left[\frac{i\omega L\,(R - i\omega L)}{R^2 + \omega^2 L^2}\right] I_0 e^{i\omega t}. \tag{12.46}$$

This is the expression for the eddy currents induced in the underground circuit. The secondary magnetic field generated by this induced current to the receiving coil in the circuit is given by

$$\varepsilon_2^{(s)} = -i\omega \mathrm{M}_{12}\mathrm{I}_1 e^{i\omega t}. \tag{12.47}$$

Here M_{12} in the mutual inductance between the underground circuit and the receiving coil. The ratio of secondary emf and primary emf, i.e., $\varepsilon_2^{(s)}/\varepsilon_2^{(p)}$ is the em response function for the vertical coil and it is given by

$$\begin{aligned}
\frac{\varepsilon_2^{(s)}}{\varepsilon_2^{(p)}} &= \frac{-i\omega \mathrm{M}_{12}\mathrm{I}_1 e^{i\omega t}}{-i\omega \mathrm{M}_{02}\mathrm{I}_1 e^{i\omega t}} \\
&= \frac{M_{01}M_{12}}{M_{02}}\left[\frac{i\left(\frac{\omega L}{R}\right)\left(1 - \frac{i\omega L}{R}\right)}{1 + \left(\frac{\omega L}{R}\right)^2}\right] \\
&= -\frac{\mathrm{M}_{01}\mathrm{M}_{12}}{\mathrm{M}_{02}}\left(\frac{\alpha^2 + i\alpha}{1 + \alpha^2}\right)
\end{aligned} \tag{12.48}$$

where

$$\alpha = \frac{\omega \mathrm{L}}{\mathrm{R}}$$

The electromagnetic response is a complex quantity (Fig. 12.9). The response can be a measure either in the form of real and imaginary components or in the form of amplitude and phase. The real part of the response is also termed as the in phase component i.e., the component which is in phase with the primary field. The imaginary component, which is as real as the real component, is termed as the out of phase or quadrature component. This component will be 90° out of phase with the primary field. The secondary field generated by eddy currents in a medium of finite conductivity will always differ with the primary field both in amplitude and phase. The real component has the contributions both from the primary and secondary fields where as the quadrature component or the phase angles come from the secondary field. Thus the basic nature of the electromagnetic response in the presence of a medium of finite conductivity can be explained using the concept of mutual inductance.

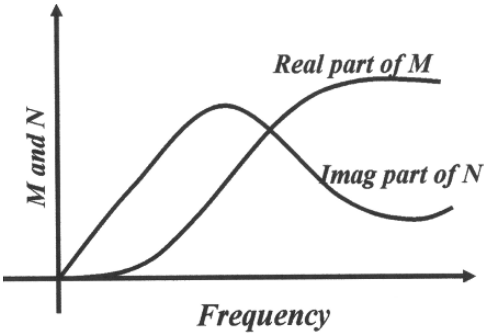


Fig. 12.9. Variation of the real and imaginary components of the electromagnetic field in a simple mutual inductance model

12.5 Maxwell's Equations

From the basic principles of electrostatics, magnetostatics and direct current flow fields , we know

$$\begin{aligned} \vec{D} &= \varepsilon\vec{E} \\ \vec{B} &= \mu\vec{H} \\ \vec{J} &= \sigma\vec{E} \end{aligned} \tag{12.49}$$

Here the basic electromagnetic vectors and scalars are given by (i) $\vec{E}$ (electric field) in volt/meter (ii) $\vec{D}$ (electric displacement) in coulomb/meter^2 (iii) $\vec{H}$ (magnetic field) in ampere/meter or ampere turn/m (iv) $\vec{B}$ (magnetic induction) in Weber/meter^2 (v) $\vec{J}$ (current density) in ampere/meter^2 (vi) ε (electrical permittivity) in farad/meter (vii) μ (magnetic permeability) in henry/meter (viii) σ (electrical conductivity) in mho/meter.
According to Ampere's law

$$\mathrm{I} = \overrightarrow{\int_{l}} \vec{H}.dl = \int_{s} \mathrm{curl}\,\vec{H}.n.ds \tag{12.50}$$

where I is the current, H is the magnetic field and dl is the element of length in a conductor carrying current.
Since

$$\mathrm{I} = \int \vec{J}.\vec{n}.ds,$$

we can write, applying Stoke's theorem,

$$\mathrm{curl}\,\vec{H} = \vec{J}. \tag{12.51}$$

The electromotive force generated in a conductor is due to the rate of change of number of lines of forces (or magnetic flux) and is equal to

$$\varepsilon = -\frac{d\vec{N}}{dt}. \tag{12.52}$$

Since emf ε can be written as $\oint \vec{E}.dl$ and magnetic flux as $N = \int \vec{B}.\vec{n}.ds$, we, therefore, can write

$$\oint E.dl = -\frac{\partial}{\partial t}\int \vec{B}.\vec{n}.ds.$$

From Stoke's theorem we write

$$\Rightarrow \int_s \operatorname{curl}\vec{E}.ds = -\frac{\partial}{\partial t}\int_s \vec{B}.\vec{n}.ds \tag{12.53}$$

$$\Rightarrow \operatorname{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t}. \tag{12.54}$$

This is the Maxwell's first equation.

Current flow in a particular medium is the rate of flow of charge. Since the current flowing out of a particular medium is the current loss inside, we can write

$$\int \vec{J}.\vec{n}.ds = -\frac{dq}{dt}. \tag{12.55}$$

Applying the divergence theorem we can write

$$\int \operatorname{div}\vec{J}d\nu = -\frac{d}{dt}\int \rho d\nu \tag{12.56}$$

where ρ is the volume density of charge. We can write

$$div\,\vec{J} = -\frac{\partial\rho}{dt}. \tag{12.57}$$

This is known as the equation of continuity. For uniform flow of current with the source situated outside the region

$$\operatorname{div}\vec{J} = 0. \tag{12.58}$$

Equation (12.57) is treated as the Maxwell's fifth equation.

Since $div\,\vec{D} = +\rho$ from (4.26), bringing the time derivatives on both the sides, we get

$$\frac{\partial}{\partial t}\left(div\,\vec{D}\right) = \frac{\partial\rho}{\partial t}. \tag{12.59}$$

It changes to

$$div\left(\frac{\partial\vec{D}}{\partial t}\right) = \frac{\partial\rho}{\partial t}. \tag{12.60}$$

The equation of continuity changes to the form

$$\operatorname{div}\vec{J} = -\operatorname{div}\left(\frac{\partial\vec{D}}{\partial t}\right) \tag{12.61}$$

$$\Rightarrow \operatorname{div}\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) = 0$$

$$\Rightarrow \operatorname{div}\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) = \operatorname{div}\operatorname{curl}\vec{H}.$$

Because divergence of curl of a vector is always zero.
Hence

$$\operatorname{curl}\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}. \tag{12.62}$$

This is Maxwell's second electromagnetic equation.

Since $\vec{D} = \frac{Q}{A}(Coulomb/meter^2)$, $\frac{\partial\vec{D}}{\partial t}$ is termed as the displacement current. The rate of change of dielectric vector has the same dimension as $\vec{J}$. So $\frac{\partial\vec{D}}{\partial t}$ will generate the magnetic field. Therefore the five Maxwell's electromagnetic equations are

$$\text{i)} \qquad \operatorname{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \tag{12.63}$$

$$\text{(ii)} \qquad \operatorname{curl}\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}, \tag{12.64}$$

$$\text{(iii)} \qquad \operatorname{div}\vec{B} = 0, \tag{12.65}$$

$$\text{(iv)} \qquad div\,\vec{D} = \rho \text{ and} \tag{12.66}$$

$$\text{(v)} \qquad div\,\vec{J} = -\frac{\partial\rho}{\partial t} \tag{12.67}$$

where (12.65) (12.66) and (12.67) are respectively taken from Chaps. 4, 5 and 6. The respective equations are (5.49), (4.26), (6.18) and (12.57).

If there are impressed electric and magnetic currents the first two Maxwell's equations change to the form

$$\operatorname{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t} - \vec{M}_i \tag{12.68}$$

$$\operatorname{curl}\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} + \vec{J}_i. \tag{12.69}$$

Using (12.49), we can write down the Maxwell's (12.63) and (12.64) as

$$\operatorname{curl}\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \tag{12.70}$$

and

$$\operatorname{curl}\vec{H} = \sigma\vec{E} + \in\frac{\partial\vec{E}}{\partial t}. \tag{12.71}$$

12.5.1 Integral form of Maxwell's Equations

Integral form of Maxwell's equations are

(a)

$$\iiint \operatorname{curl} \mathrm{E}\, dv = -\iiint \frac{\partial \vec{B}}{\partial t}\, dv, \tag{12.72}$$

(b)

$$\iiint \operatorname{curl} \vec{H}\, dv = \iiint \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) dv, \tag{12.73}$$

(c)

$$\iiint \operatorname{div} \vec{J} = -\iiint \frac{\partial \rho}{\partial t} dv, \tag{12.74}$$

(d)

$$\iiint \operatorname{div} \vec{B} = 0 \quad \text{and} \tag{12.75}$$

(e)

$$\iiint \operatorname{div} \vec{D} = \iiint \rho\, dv \tag{12.76}$$

12.6 Helmholtz Electromagnetic Wave Equations

From equations (12.49) and (12.63) we can write

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\vec{E} &= -\mu\frac{\partial}{\partial t}\operatorname{curl}\vec{H}\\ &= -\mu\frac{\partial}{\partial t}\left[\sigma\vec{E} + \in \frac{\partial \vec{E}}{\partial t}\right]\end{aligned}$$

from (12.71).
Hence

$$\operatorname{grad}\operatorname{div}\vec{E} - \nabla^2\vec{E} = -\mu\sigma\frac{\partial \vec{E}}{\partial t} - \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \tag{12.77}$$

$$\Rightarrow \nabla^2\vec{E} = \mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} - \operatorname{grad}\operatorname{div}\vec{E} \tag{12.78}$$

$$\Rightarrow \nabla^2\vec{E} = \mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu \in \frac{\partial^2 \mathrm{E}}{\partial t^2} \tag{12.79}$$

where $\operatorname{div}\vec{E} = 0$ in a source free region (12.58) and is equal to $\frac{\rho}{\in}$ where the region contains the source. Equation (12.79) is the Helmholtz electromagnetic wave equation.

Similarly it can be proved that

$$\nabla^2\vec{H} = \mu\sigma\frac{\partial\vec{H}}{\partial t} + \mu \in \frac{\partial^2 H}{\partial t^2}. \tag{12.80}$$

Since $\vec{E}$ and $\vec{H}$ are electric and magnetic field vectors. They have three components along the three mutually perpendicular coordinate axes. These wave equations are valid for each component i.e.,

$$\nabla^2 H_x = \mu\sigma\frac{\partial H_x}{\partial t} + \mu\varepsilon\frac{\partial^2 H_x}{\partial t^2}. \tag{12.81}$$

For very high frequency, where the displacement current dominates over the conduction current. Therefore

$$\nabla^2\vec{H}_x = \mu \in \frac{\partial^2 H_x}{\partial t^2}. \tag{12.82}$$

At lower frequencies in the audio range, conduction current dominates over the displacement current and the Helmholtz equation changes to the form

$$\nabla^2 H_x = \mu\sigma\frac{\partial H_x}{\partial t}. \tag{12.83}$$

Since both $\vec{E}$ and $\vec{H}$ are vectors and having three components each, theoretically one has to determine six components to define the electromagnetic field totally. In actual practice, for different types of source excitation, there will be some zero and non zero electric and magnetic vectors as shown in the next chapter. We try to solve for the non zero $\vec{E}$ and $\vec{H}$ components.

Alternatively if we express the electromagnetic field in terms of a vector and a scalar potentials, the number of components to be determined will be 4 i.e., three components for one vector and one scalar. These vectors are termed as vector potentials. One of the options for solving the electromagnetic boundary value problems is to use these vector and scalars (please see Chap. 13). If B, H and E are expressed respectively as curl A, curl A' and curl A'', these A, A', A'' are termed as vector potentials because curl operates on a vector and generates another vector. A brief introduction about the definition and mathematical expressions for the vector potential is given in Chap. 5.

Since $\operatorname{div}\vec{B} = 0$ always because the monopoles in magnetostatics do not exist, we can write $\operatorname{div}\vec{H} = 0$, because $\vec{B} = \mu\vec{H}$. Since divergence of a curl of a vector is always zero, we can write

$$\vec{H} = \operatorname{curl}\vec{A} \tag{12.84}$$

where $\vec{A}$ is a vector potential. $\operatorname{div}\vec{E} = 0$ in a source free region. From (12.70), we get

$$\mathrm{curl}\,\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\Rightarrow -\mu\mathrm{curl}\,\frac{\partial\vec{A}}{\partial t} \tag{12.85}$$

$$\Rightarrow \mathrm{curl}\left(\vec{E} + \mu\frac{\partial\vec{A}}{\partial t}\right) = 0. \tag{12.86}$$

If curl of a vector is zero, then that vector can always be expressed in terms of gradient of a scalar potential.

So

$$\vec{E} + \mu\frac{\partial\vec{A}}{\partial t} = -\mathrm{grad}\,\phi. \tag{12.87}$$

Since $\vec{H} = \mathrm{curl}\,\vec{A}$, we can write

$$\mathrm{curl}\,\mathrm{curl}\,\vec{A} = -\mu\sigma\frac{\partial\vec{A}}{\partial t} - \sigma\,\mathrm{grad}\,\phi$$

$$-\mu \in \frac{\partial^2 A}{\partial t^2} - \in \mathrm{grad}\,\frac{\partial\phi}{\partial t} = \mathrm{grad}\,\mathrm{div}\vec{A} - \nabla^2\vec{A}. \tag{12.88}$$

Since the vector potential is quite arbitrary, we can choose

$$\mathrm{grad}\,\mathrm{div}\vec{A} = -\sigma\ \mathrm{grad}\,\phi - \varepsilon\,\mathrm{grad}\,\frac{\partial\phi}{\partial t} \tag{12.89}$$

which leads to

$$\nabla^2\vec{A} = \mu\sigma\frac{\partial\vec{A}}{\partial t} + \mu \in \frac{\partial^2\vec{A}}{\partial t^2}. \tag{12.90}$$

And

$$\mathrm{div}\,\vec{A} = -\sigma\phi - \varepsilon\frac{\partial\phi}{\partial t}. \tag{12.91}$$

From (12.87), we get

$$\mathrm{div}\,\vec{E} = -\mu\frac{\partial}{\partial t}\mathrm{div}\,\vec{A} - \nabla^2\phi = 0 \tag{12.92}$$

and

$$\nabla^2\phi = \mu\sigma\frac{\partial\phi}{\partial t} + \mu \in \frac{\partial^2\phi}{\partial t^2}. \tag{12.93}$$

Therefore, both vector and scalar potentials satisfy Helmholtz wave equations. The connecting links between a vector and a scalar potential with electric and magnetic fields are given by

$$\vec{H} = \mathrm{curl}\,\vec{A} \tag{12.94}$$

$$\vec{E} = -\mu\frac{\partial\vec{A}}{\partial t} - \mathrm{grad}\,\phi. \tag{12.95}$$

If electromagnetic field is harmonically varying field, i.e., $E = Ee^{i\omega t}$ and $H = He^{i\omega t}$, the Helmholtz wave equation changes to the form

$$\begin{aligned}\nabla^2\vec{E} &= i\omega\mu\sigma\vec{E} + i^2\omega^2\mu \in \vec{E}\\ &= i\omega\mu\,(\sigma + i\omega \in)\,\vec{E}\\ &= \gamma^2\vec{E}.\end{aligned}$$

Here $\gamma\left(= \sqrt{i\omega\mu\,(\sigma + i\omega \in)}\right)$ is termed as the propagation constant. Helmholtz equations can be written shortly as

$$\begin{aligned}\nabla^2\vec{E} &= \gamma^2\vec{E}\\ \nabla^2\vec{H} &= \gamma^2\vec{H}\\ \nabla^2\vec{A} &= \gamma^2\vec{A}\\ \nabla^2\phi &= \gamma^2\phi.\end{aligned} \tag{12.96}$$

For audio frequency range the displacement current is negligible. Therefore $\gamma = \sqrt{i\omega\mu\sigma}$. In the megahertz range $\gamma = \sqrt{i\omega\mu\,(\sigma + i\omega \in}$ and in the gigahertz range $\gamma = i\omega\sqrt{\mu \in}$.

12.7 Hertz and Fitzerald Vectors

Hertz defined a vector potential $\vec{A}$ in the form

$$\vec{A} = \sigma\vec{\Pi} + \varepsilon\frac{\partial\vec{\Pi}}{\partial t} \tag{12.97}$$

where $\vec{\Pi}$ is termed as the Hertz vector. Taking divergence on both the sides, we get

$$\begin{aligned}\sigma\mathrm{div}\,\vec{\Pi} + \varepsilon\frac{\partial}{\partial t}\mathrm{div}\,\vec{\Pi} &= \mathrm{div}\,\vec{A}\\ &= -\sigma\phi - \varepsilon\frac{\partial\phi}{\partial t}\end{aligned} \tag{12.98}$$

from (12.87). From (12.98) we get

$$\phi = -\mathrm{div}\,\vec{\Pi}. \tag{12.99}$$

Equation (12.99) connects the Hertz vector with the scalar potential ϕ. Because the divergence operates on a vector and generates a scalar.

From (12.93) and (12.94) we can write

$$\begin{aligned}H &= \sigma\,\mathrm{curl}\,\vec{\Pi} + \varepsilon\,\mathrm{curl}\,\frac{\partial\vec{\Pi}}{\partial t}\\ &= (\sigma + i\omega \in)\,\mathrm{curl}\,\vec{\Pi}\end{aligned} \tag{12.100}$$

and

$$\begin{aligned}\vec{E} &= -\mu\sigma\frac{\partial\vec{\Pi}}{\partial t} - \mu \in \frac{\partial^2\Pi}{\partial t^2} + \text{grad}\,\text{div}\vec{\Pi} \\ &= -\nabla^2\vec{\Pi} + \text{grad}\,\text{div}\vec{\Pi} \\ &= \text{curl}\,\text{curl}\,\vec{\Pi}. \end{aligned} \tag{12.101}$$

From (12.88 and 12.95), we write

$$\nabla^2\text{div}\,\vec{\Pi} = \mu\sigma\frac{\partial}{\partial t}\text{div}\,\vec{\Pi} + \mu\varepsilon\frac{\partial^2}{\partial t^2}\text{div}\,\vec{\Pi}. \tag{12.102}$$

Here

$$\nabla^2\vec{\Pi} = \mu\sigma\frac{\partial\vec{\Pi}}{\partial t} + \mu \in \frac{\partial^2\vec{\Pi}}{\partial t^2}. \tag{12.103}$$

Therefore Hertz vector satisfies Helmoholtz wave equation. Equations (12.100) and (12.101) are the connecting links between the Hertz vector and the electric and magnetic fields.
Similarly, assuming $\vec{E} = \text{curl}\,\vec{A}'$ where A' is the electric vector potential and following the same procedure it can be shown that

$$\text{curl}\left(\vec{H} - \sigma A' - \varepsilon\frac{\partial\vec{A}}{\partial t}\right) = 0 \tag{12.104}$$

or

$$\vec{H} - \sigma A' + \varepsilon\frac{\partial A'}{\partial t} + \text{grad}\,\psi \tag{12.105}$$

where $\vec{H}$ is the magnetic scalar potential. The vector potential F is, known as Fitzerald vector which connects the E and H field through these two equations

$$\vec{H} = \text{curl}\,\text{curl}\,\vec{F} \tag{12.106a}$$

and

$$\vec{E} = -i\omega\mu\,\text{curl}\,\vec{F} \tag{12.106b}$$

and will also satisfy the Helmholtz wave equation

$$\nabla^2\vec{F} = \mu\sigma\frac{\partial\vec{F}}{\partial t} + \mu \in \frac{\partial^2 F}{\partial t^2}. \tag{12.107}$$

Many of the electromagnetic boundary value problems can be solved using Hertz or Fitzerald vectors. In that case the number of equations to be solved is only three. Once a solution is obtained in $\vec{\Pi}$ and $\vec{F}$ one can move to $\vec{E}$ and $\vec{H}$ fields through these two pairs of connecting links i.e., (12.100), (12.101) and (12.106a and b).

12.8 Boundary Conditions in Electromagnetics

12.8.1 Normal Component of the Magnetic Induction B is Continuous Across the Boundary in a Conductor

From Maxwell's equation, we know that $\mathrm{div}\vec{B} = 0$. From divergence theorem we can write

$$\int \mathrm{div}\,\vec{B}.\mathrm{d}\nu = \int \vec{B}.\vec{n}.\mathrm{ds}. = 0. \tag{12.108}$$

Now assuming a small box of area Δa and thickness Δl across the boundary shown in the Fig. 12.10 we take the integration over the volume with an approximation

$$(B_2.n_2 + B_1.n_1)\Delta a + \text{Contribution from the wall} = 0.$$

Since the contribution from the walls is directly proportional to Δl, its contribution will be zero when $\Delta l = 0$.
Therefore

$$B_2.n_2 + B_i.n_1 = 0.$$

Since the normal vectors n_2 and n_1 on two opposite sides of the boundary are in the opposite direction, i.e., $n_2 = -n_1$, therefore

$$(B_2 - B_1).n = 0. \tag{12.109}$$

In other words the normal components of the magnetic induction will be continuous across the boundary.

12.8.2 Normal Component of the Electric Displacement is Continuous Across the Boundary

Since $\mathrm{div}D = q_v$,, we can write Gauss's divergence theorem

$$\int_v \mathrm{div}\,\vec{D}\,\mathrm{d}\nu = \int_s \vec{D}.\vec{n}.\mathrm{ds}. = \int_v q_v \mathrm{dv} = q \tag{12.110}$$

where q is the total charge and q_v is the volume density of charge. Therefore, we get as in the previous case (Fig. 12.11)

$$\left(\vec{D}_2.n_2 + \vec{D}_1.n_1\right)\Delta a = w\Delta a \tag{12.111}$$

where w is the surface density of charge and

$$q = \int_v q_v \mathrm{dv} = q_v.\Delta l.\Delta a. \tag{12.112}$$

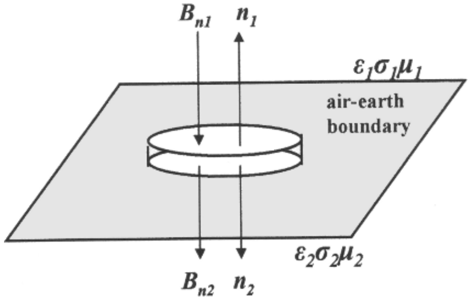


Fig. 12.10. Normal component of the magnetic flux across the boundary with different values of $\varepsilon_{\pm}, \sigma_{\pm}, \mu$

Since $n_2 = -n_1$ as shown in previous case, we can write

$$\left(\vec{D}_2 - \vec{D}_1\right).n = \omega \tag{12.113}$$

This equation shows that normal component D_n of the vector $\vec{D}$ is discontinuous across the boundary due to accumulation of surface charge density w. At the surface of a conductor the surface charge density dissipates very quickly. Hence across the interface involving all but the poorest conductors, normal D is continuous across the boundary
Therefore

$$(D_2.n_2 + D_1.n_1) = 0 \tag{12.114}$$

$$(D_2.n_2 - D_1.n_1) = 0 \tag{12.115}$$

$$\Rightarrow (D_2 - D_1).n = 0$$

$$\Rightarrow D_{n_2} = D_{n_1} \tag{12.116}$$

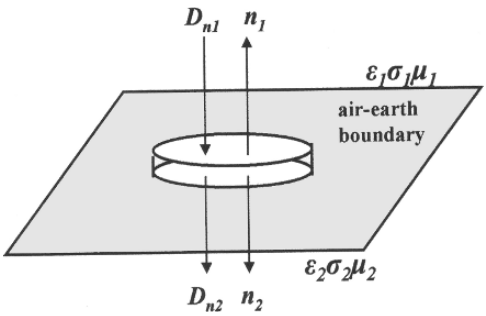


Fig. 12.11. Normal component of the electric displacement across the boundary with different values of ε, σ, μ

12.8.3 Tangential Component of E is Continuous Across the Boundary

From Maxwell's equation $\mathrm{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$,
we can write

$$\int_s \left(\vec{\nabla} \times \vec{E}\right) \mathrm{nda} - \int \frac{\partial B}{\partial t}.\mathrm{nda} = 0 \tag{12.117}$$

where n is the unit vector normal to the surface S. Applying the Stoke's theorem we can write

$$\oint \vec{E}.\mathrm{dl} = \int \mathrm{curl}\,\vec{E}.\mathrm{n.da}. \tag{12.118}$$

Therefore

$$-\int \frac{\partial \vec{B}}{\partial t}.\mathrm{n.da} = \oint \vec{E}.\mathrm{dl} = \vec{E}_2.\Delta l + \vec{E}_1 \left(-\Delta l\right)$$

+contribution from the end where $\vec{E}_2.\Delta l$ and $\vec{E}_1.\Delta l$ are the tangential components of $\vec{E}$ in the medium 1 and 2 (Fig. 12.12). If we reduce the thickness of the rectangular loop to zero such that $\vec{E}_2.\Delta t = 0$. That brings the line segments on the surface. The equation reduces to

$$-\frac{\partial B}{\partial t}.\mathrm{n}\Delta t.\Delta l = \left(E_2 - E_1\right).\Delta l \tag{12.119}$$

In the limit when the thickness of the strip $\Delta t = 0$, the left hand side is zero, so that across the boundary

$$E_{t_1} = E_{t_2} \text{ or } n \times \left(E_2 - E_1\right) = 0 \tag{12.120}$$

where n is the unit vector normal to the boundary surface. In other words, the tangential component of the electric field is continuous across the boundary.

12.8.4 Tangential Component of H is Continuous Across the Boundary

From Maxwell's equation

$$\mathrm{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

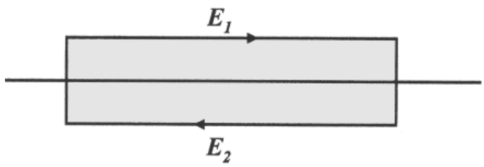


Fig. 12.12. Tangential component of the electric vector is continuous across the boundary with different values of ε, σ and μ

we can write

$$\oint_{l} \mathrm{H.dl} - \int \frac{\partial \vec{\mathrm{D}}}{\partial \mathrm{t}}.\mathrm{n.da} = \int \vec{\mathrm{J}}.\mathrm{n.da} \tag{12.121}$$

The contour of integration is the same as that shown in the previous case. By using the component of H tangential to the boundary surface (Fig. 12.13) and by proceeding to the limit of a contour of negligible height Δl, we can write

$$\mathrm{nx}\,(\mathrm{H}_2 - \mathrm{H}_1) = \overset{\mathrm{Lim}}{\Delta \mathrm{t} \to 0} \left(\frac{\partial \vec{\mathrm{D}}}{\partial \mathrm{t}} + \vec{\mathrm{J}} \right) \; \Delta \mathrm{t} = 0 \tag{12.122}$$

for finite current density and bounded nature of D and its derivatives. Therefore, the boundary condition for media of finite conductivity will be

$$\mathrm{nx}(\mathrm{H}_1 - \mathrm{H}_2) = 0 \tag{12.123}$$

in the absence of any kind of surface currents. Hence, the tangential component of the magnetic field in continuous across the boundary.

12.8.5 Normal Component of the Current Density is Continuous Across the Boundary

Currents entering and leaving an elementary box across the boundary spanning two conductive media consists partly of normal components and partly of tangential components. As the thickness of the box across the boundary tends to zero, the current crossing the interface may be computed either as

$$\mathrm{I} = \mathrm{J}_2.\mathrm{n}\Delta \mathrm{a} \quad \text{or as} \quad \mathrm{I} = \mathrm{J}_1.\mathrm{n}.\Delta \mathrm{a}$$

Thus the normal component of J, i.e., $\mathrm{J_n}$ must be continuous across the boundary. Hence

$$(\mathrm{J}_2 - \mathrm{J}_1).\mathrm{n} = 0 \tag{12.124}$$

or

$$\mathrm{J}_{\mathrm{n}_1} = \mathrm{J}_{\mathrm{n}_2}$$

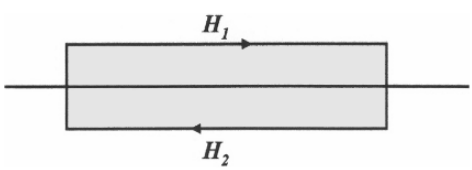


Fig. 12.13. Tangential component of the magnetic field is continuous across the boundary of two media having different values of ε, σ, and μ

12.8.6 Scalar Potentials are Continuous Across the Boundary

In electrostatics and magnetostatics problems $\vec{E} = -\text{grad}\,\phi$ and $H = -\text{grad}\,\phi^*$ where ϕ and ϕ^* are the scalar potentials. In the absence of any source potentials ϕ and ϕ^* must be continuous across a boundary surface (Fig. 12.14). For the work required to carry a small electric charge or a magnetic pole from infinity to two adjacent points located on opposite sides of the surface with negligible distance between them must be the same, i.e.

$$\phi_1 = \phi_2$$

and

$$\phi_1^* = \phi_2^*.$$

Thus it is seen that each and every Maxwell's equation generate one boundary condition in electromagnetics. Integral form of Maxwell's equations for generating the boundary conditions are

(f)

$$\int\int (\vec{n} \times \vec{E})\text{ds} = -\int\int\int \frac{\partial \vec{B}}{\partial t}\text{div}, \tag{12.125}$$

(g)

$$\int\int (\vec{n} \times \vec{H})\text{ds} = \int\int\int \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\text{dv}, \tag{12.126}$$

(h)

$$\int\int (\vec{n}.\vec{J})\text{ds} = -\int\int\int \frac{\partial \rho}{\partial t}\text{dv}, \tag{12.127}$$

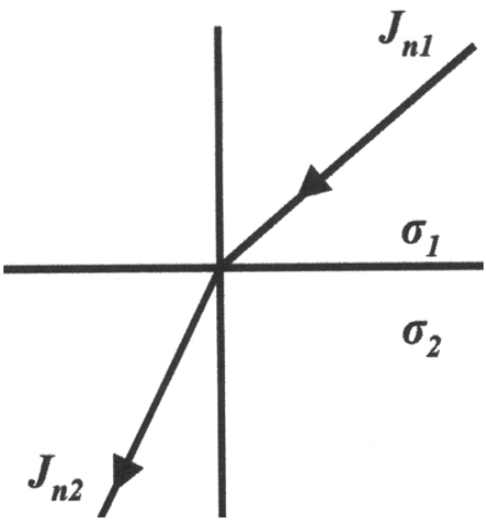


Fig. 12.14. Normal component of current density J is across the boundary

(i)

$$\int\int(\vec{n}.\vec{B})ds = 0 \text{ and} \tag{12.128}$$

(j)

$$\int\int(\vec{n}.\vec{D})ds = \int\int\int \rho dv \tag{12.129}$$

where $\vec{n}$ is the vector normal to the surface. (12.130)

12.9 Poynting Vector

As electromagnetic waves travel through space from their source to the distant receiving points, there is a transfer of energy from the source to the receiver. There exists a simple and direct relation between the rate of this energy transfer and amplitudes of this electric and magnetic field strengths of the electromagnetic waves. This relation can be obtained from the Maxwell's equation as follows. From the second Maxwell's (12.62), we can write

$$\vec{J} = \text{curl}\,\vec{H} - \varepsilon\frac{\partial\vec{E}}{\partial t}. \tag{12.131}$$

This expression has the dimension of current density on both the sides. Multiplying both the sides of the (12.131) by $\vec{E}$, the modified equation has the dimension of power per unit volume i.e.,

$$\vec{E}.\vec{J} = \vec{E}.\vec{\nabla}\times\vec{H} - \in \vec{E}\frac{\partial\vec{E}}{\partial t} \tag{12.132}$$

And

$$\nabla.\vec{E}\times\vec{H} = \vec{H}.\vec{\nabla}\times\vec{E} - \vec{E}.\vec{\nabla}\times\vec{H}. \tag{12.133}$$

Therefore

$$\vec{E}.\vec{J} = \vec{H}.\nabla\times\vec{E} - \nabla.\vec{E}\times\vec{H} - \in \vec{E}.\frac{\partial\vec{E}}{\partial t}. \tag{12.134}$$

From Maxwell's first equation we get

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}. \tag{12.135}$$

Therefore

$$\vec{E}.\vec{J} = -\mu\vec{H}\frac{\partial\vec{H}}{\partial t} = - \in \vec{E}\frac{\partial\vec{E}}{\partial t} - \nabla.\vec{E}\times\vec{H}. \tag{12.136}$$

Since

$$\vec{H}.\frac{\partial\vec{H}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\overset{-2}{H}$$

and

$$\vec{E}\frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\overset{-2}{E}$$

Therefore

$$\vec{E}.\vec{J} = -\frac{\mu}{2}\frac{\partial}{\partial t}\overset{-2}{H} - \frac{\in}{2}\frac{\partial}{\partial t}\overset{-2}{E} - \vec{\nabla}.\vec{E}\times\vec{H}. \tag{12.137}$$

Integrating over a volume v,

$$\int_v \vec{E}.\vec{J}dv = -\frac{\partial}{\partial t}\int_v \left(\frac{\mu}{2}H^2 + \frac{\in}{2}E^2\right) dv - \oint \vec{E}\times\vec{H}\ ds$$
$$-\int_v \nabla.\vec{E}\times\vec{H}\ dv. \tag{12.138}$$

Using Gauss's divergence theorem, the last term of (12.138) changes from volume integral to surface integral over the surfaces and encompassing the volume v i.e.,

$$\int_v \nabla.\vec{E}\times\vec{H}\ dv = \oint \vec{E}\times\vec{H}\ ds. \tag{12.139}$$

We can write (12.139) as

$$\int_v \vec{E}.\vec{J}\ dv = -\frac{\partial}{\partial t}\int_v \left(\frac{\mu}{2}H^2 + \frac{\in}{2}E^2\right) dv - \oint \vec{E}\times\vec{H}\ ds. \tag{12.140}$$

As per basic definitions in electrical engineering, a conductor carrying current I and having a voltage drop E per unit length will have a power loss of $\vec{E}\vec{I}$ watt per unit length. The power dissipated per unit volume is $\frac{\vec{E}\vec{I}}{A} = \vec{E}.\vec{J}$. For a homogeneous and isotropic medium $\vec{E}$ and $\vec{J}$ will be in the same direction. In general for an inhomogeneous and anisotropic medium. $\vec{E}$ and $\vec{J}$ may not be in the same direction. Even the power dissipated will be the product of $\vec{J}$ and the component of $\vec{E}$ along the direction of $\vec{J}$. Therefore the total power dissipated in a volume will be

$$\int_v \vec{E}.\vec{J}.dv. \tag{12.141}$$

The expressions $\frac{1}{2} \in E^2$ and $\frac{1}{2}\mu H^2$ are respectively the stored electrical and magnetic energy per unit volume of the electric field.

The integral shows the dissipation of the total electrical and magnetic energy. The rate of energy dissipation in the volume V must be equal to the rate at which the stored energy in the volume is decreasing. The term $-\oint \vec{E}\times\ \vec{H}$ ds represents the rate of flow of energy outward through the surface enclosed in the volume V. The integral of $\vec{E}\times\vec{H}$ is a measure of the energy out flow through that surface. It is seen that the vector is

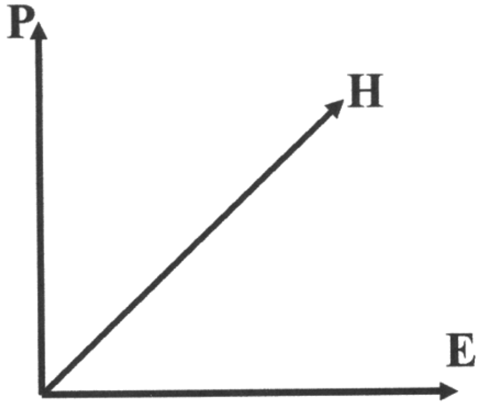


Fig. 12.15. Direction of propagation of a poynting vector

$$\vec{P} = \vec{E} \times \vec{H}. \tag{12.142}$$

The dimension of watt per square meter. It is a Poynting vector in electromagnetics (Fig. 12.15). The Poynting vector $\vec{P}$ is directed at right angles to the plane which contain $\vec{E}$ and $\vec{H}$. It is a cross product of E and H and the energy flows at right angles to the plane which contains $\vec{E}$ and $\vec{H}$.
Now

$$\nabla . \left(\vec{E} \times \vec{H}\right) = \vec{H}. \left(-\mu \frac{\partial H}{\partial t}\right) - \vec{E} \left(- \in \frac{\partial \vec{E}}{\partial t}\right). \tag{12.143}$$

In a perfect dielectric where the conductivity is equal to zero, we get

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

or

$$\vec{\nabla} \times \vec{H} = - \in \frac{\partial \vec{D}}{\partial t}.$$

So

$$\nabla . \left(\vec{E} \times \vec{H}\right) = -\frac{\partial}{\partial t} \left(\frac{1}{2}\mu H^2 + \frac{1}{2} \in E^2\right). \tag{12.144}$$

Here $\frac{1}{2}\mu H^2$ is the energy density associated with the magnetic field and $\frac{1}{2} \in E^2$ is the energy density associated with the electric field. We therefore, can write

$$\nabla . \left(\vec{E} \times \vec{H}\right) = +\frac{\partial}{\partial t} \left[\frac{1}{2}\mu H^2 + \frac{1}{2} \in E^2\right] = 0 \tag{12.145}$$

$$\Rightarrow \vec{\nabla}.\vec{P} + \frac{\partial}{\partial t} \left[\frac{1}{2}\mu H^2 + \frac{1}{2} \in E^2\right] = 0. \tag{12.146}$$

This is the equation for the conservation of energy or the equations of continuity in electromagnetic field. This equation is similar to the Maxwell's fifth equation of continuity which is also valid for direct current, heat, fluid flow fields, i.e.,

$$\operatorname{div}\vec{J} + \frac{\partial q_v}{\partial t} = 0. \tag{12.147}$$

For a medium of finite conductivity, we have

$$\begin{aligned}
\nabla.\left(\vec{E}\times\vec{H}\right) &= \vec{H}.\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}.\left(\vec{\nabla}\times\vec{H}\right)\\
&= -\mu\vec{H}\frac{\partial\vec{H}}{\partial t} - \vec{E}.\left[\sigma\vec{E} + \in\frac{\partial\vec{E}}{\partial t}\right]\\
&= -\frac{\partial}{\partial t}\left[\frac{1}{2}\mu H^2 + \frac{1}{2}\in E^2\right] - \sigma E^2. \qquad (12.148)
\end{aligned}$$

Equation (12.148) can be written as

$$\begin{aligned}
\iint_s \left(\vec{E}\times\vec{H}\right) ds = \iint \vec{P}\, ds = -\frac{\partial}{\partial t}\iiint\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\in H^2\right) dv\\
-\iiint_v \sigma E^2 dv.
\end{aligned}$$

The last term on the right hand side is interpreted as the energy consumed as heat in the volume for a medium of finite conductivity.

13

Electromagnetic Wave Propagation Problems Related to Geophysics

In this chapter a basic guideline for solving electromagnetic boundary value problems using Helmholtz electromagnetic wave equation and the method of separation of variables are discussed for simplest problems. The use of vector potentials and their relation with E and H fields and application of boundary conditions are demonstrated.

Structure of the electromagnetic boundary value problems for (i) plane wave electromagnetics, (ii) oscillating vertical electric dipole in an homogenous full space, (iii) oscillating vertical magnetic dipole over an homogenous earth, (iv) oscillating horizontal magnetic dipole over an homogenous earth, (v) line source carrying alternating current over an homogenous full space, (vi) line source over an homogenous earth (vii) electromagnetic response for a buried conducting horizontal cylinder in an uniform vertical field and (viii) electromagnetic response for a buried sphere in an uniform vertical field are discussed. Principle of electrodynamic similitude is included
.

13.1 Plane Wave Propagation

A source of electromagnetic wave at infinite distance generates plane waves. Electromagnetic waves generated due to interactions of the solar emissions with the magnetosphere come to the surface of the earth as plane or nearly plane waves. The electric and magnetic vectors are mutually orthogonal in a plane of a propagating wave front. All vibrating particles in a wave front are in the same phase. Magnitude of electric and magnetic vectors remain invariant in a plane wave front. The direction of propagation of electromagnetic waves will be at right angles to the plane of the wave front.

In a rectangular coordinate system, the Helmholtz electromagnetic wave equations are given by

$$\nabla^2 \vec{E}_x = \gamma^2 \vec{E}_x$$
$$\nabla^2 \vec{E}_y = \gamma^2 \vec{E}_y \tag{13.1}$$
$$\nabla^2 \vec{E}_z = \gamma^2 \vec{E}_z$$

where γ is the propagation constant. When a wave is propagating along the z direction, as shown in Fig. 13.1, $E_z = 0$ and $H_z = 0$. For an uniform plane wave, there will be no variation of electric and magnetic vectors with respect to x or y i.e., $\frac{\partial}{\partial x} = 0$ and $\frac{\partial}{\partial y} = 0$. Therefore, the Helmoholtz wave equation reduces to

$$\frac{\partial^2 E_x}{\partial z^2} = \gamma^2 E_x \tag{13.2}$$

or

$$\frac{\partial^2 E_y}{\partial z^2} = \gamma^2 E_y. \tag{13.3}$$

If electric field is in only one direction, then it is a linearly polarized uniform plane waves in the x and y direction. In a source free region

$$\text{curl } \vec{E}_x = -i\omega\mu\vec{H}_y \tag{13.4}$$

$$\Rightarrow \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_x}{\partial z} = -i\omega\mu\vec{H}_y. \tag{13.5}$$

If $\vec{E}_x \neq 0$, then $\vec{E}_y = 0$ for linearly polarized field and $H_x = 0$, $H_z = 0$ with $H_y \neq 0$. Hence

$$\frac{\partial E_x}{\partial z} = -i\omega\mu H_y$$
$$\Rightarrow H_y = -\frac{1}{i\omega\mu}\frac{\partial E_x}{\partial z} \tag{13.6}$$

Similarly in other cases, where E_y is present, the governing equations are

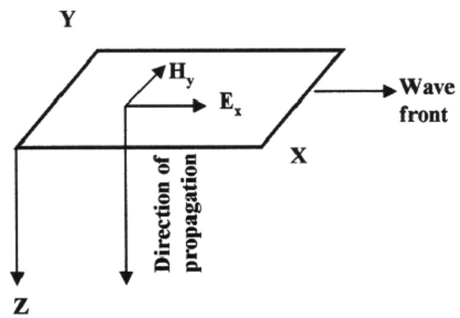


Fig. 13.1. Propagation of plane electromagnetic waves

$$\frac{\partial^2 E_y}{\partial z^2} = \gamma^2 E_y \tag{13.7}$$

$$H_x = \frac{1}{i\omega\mu}\frac{\partial E_y}{\partial z} \tag{13.8}$$

These two sets of equations for plane electromagnetic waves, reflect the transverse nature of electromagnetic waves. A general solution to the (13.2) is given by

$$\vec{E}_x = Ae^{\gamma z} + Be^{-\gamma z} \tag{13.9}$$

and from (13.5)

$$\vec{H}_y = -\frac{\gamma}{i\omega\mu}\left(Ae^{\gamma z} - e^{-\gamma z}\right) \tag{13.10}$$

where γ is the propagation constant and $\gamma^2 = i\omega\mu(\sigma + i\omega \in)$.
Let $\gamma = \alpha + i\beta$ then

$$\alpha^2 + \beta^2 + 2i\alpha\beta = i\omega\mu(\sigma + i\omega \in). \tag{13.11}$$

Equating the real and imaginary parts, we obtain

$$\alpha^2 - \beta^2 = -\omega^2\mu \in$$

and

$$2\alpha\beta = \omega\mu\sigma. \tag{13.12}$$

Thus α and β can be expressed as

$$\alpha = \omega\left[\frac{\mu \in}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2 \in^2}} - 1\right)\right]^{1/2} \tag{13.13}$$

and

$$\beta = \omega\left[\frac{\mu \in}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2 \in^2}} + 1\right)\right]^{1/2}. \tag{13.14}$$

Case I
For

$$\frac{\sigma^2}{\omega^2 \in^2} << 1$$

i.e., when the displacement current dominates over the conduction current, we get

$$\alpha = \omega\left[\frac{\mu \in}{2}.\frac{1}{2}.\frac{\sigma^2}{\omega^2 \in^2}\right]^{1/2} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\in}} \tag{13.15}$$

and

$$\begin{aligned}\beta &= \omega\sqrt{\mu \in}\left(1 + \frac{1}{8}\frac{\sigma^2}{\omega^2 \in^2}\right)\\ &= \omega\sqrt{\omega \in}.\end{aligned} \tag{13.16}$$

For $\alpha = 0$, $\beta = \omega\sqrt{\mu \in}$. At a very high frequency i.e., in the megahertz or gigahertz range, displacement current dominates over conduction current and conduction current becomes negligibly small.

Case II
For

$$\frac{\sigma^2}{\omega^2 \in^2} << 1$$

we get

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}. \tag{13.17}$$

When conduction current dominates over the displacement current at lower frequencies, the attenuation factor becomes a function of electrical conductivity σ, angular frequency ω and magnetic permeability μ.

13.1.1 Advancing Electromagnetic Wave

The expression for an advancing electromagnetic wave is given by

$$\vec{E}_x = Ae^{\alpha z}.e^{i\beta z} + Be^{-\alpha z}.e^{i\beta z}$$

$$\vec{E}_x = \overset{-}{E}_{xo}\, e^{i\phi_x}.e^{\alpha z}.e^{i(\omega t+\beta z)} + \overset{+}{E}_{xo}\, e^{i\phi_x}.e^{-\alpha z}.e^{i(\omega t-\beta z)} \tag{13.18}$$

where $\overset{+}{E}_{xo}$ and $\overset{-}{E}_{xo}$ are respectively the forward and backward propagating waves. ϕ_x is the initial phase. $t = 0$, is the initial instant of time. The real part of this expression is given by

$$\begin{aligned} E_{x_{real}} &= \overset{-}{E}_{xo}e^{\alpha z}.\cos(\omega t + \beta z + \phi_x) \\ &+ \overset{+}{E}_{xo}e^{-\alpha z}\cos(\omega t - \beta z + \phi_x)\,. \end{aligned} \tag{13.19}$$

In an infinite space, there is no reflected component, and only the forward propagating wave will be present.
Therefore, we can write

$$\vec{E}_x = E^{+}_{xo}e^{-\alpha z}\cos(\omega t - \beta z + \phi_x) \tag{13.20}$$

and

$$\vec{H}_y = \frac{\beta}{\omega\mu}E^{+}_{xo}e^{-\alpha z}\cos(\omega t - \beta z + \phi_x)\,. \tag{13.21}$$

Thus the ratio gives

$$\frac{E_x}{H_y} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{\omega\sqrt{\mu \in}} = \sqrt{\frac{\mu}{\in}}. \tag{13.22}$$

Since E_x is in volt/meter and H_y is in Ampere/meter. The unit of $\sqrt{\frac{\mu}{\in}}$ is ohm. It is termed as the intrinsic impedance of the medium. Taking μ as the free space magnetic permeability $\mu_0 = 4\pi \times 10^{-7} = 1.257 \times 10^{6}$ henry/meter and the electrical permittivity as the free space electrical permittivity $\in_0 = 8.854 \times 10^{-12}$ farad/meter, we get

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\in}} = 377\,\text{ohms}. \tag{13.23}$$

This is the resistance experienced by an electromagnetic wave during its propagation. If electrical conductivity of a medium is zero, there will be no attenuation of propagating electromagnetic wave. It can travel through vacuum and does not need any medium for its propagation. In a vacuum or in air it travels with a velocity of light. In a conductive medium (say earth), the velocity goes down by several order of magnitude and the velocity is controlled by $\sqrt{\frac{\mu}{\in}}$ of the medium. This particular observation that the ratio of the electric and the transverse magnetic field gives a measure of impedance of a medium was known to the electromagnetic research community long before Cagniard (1953) Tikhnov (1950) proposed the magnetotelluric theory in geophysics.

13.1.2 Plane Wave Incidence on the Surface of the Earth

When a wave impinges normally on the surface of the earth, the wave front will be parallel to the air-earth boundary.
For a linearly polarized EM field

$$\vec{E}_x = \vec{a}_x \vec{E}_x \tag{13.24}$$

where $\vec{E}_x$ is in the x direction and $\vec{H}_y$ is in the y direction. The wave equation for this problem is given by

$$\nabla^2 \vec{E}_x = \gamma^2 \vec{E}_x. \tag{13.25}$$

For a plane wave propagating along the z-direction, $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$, therefore the wave equation reduces to

$$\frac{\partial^2 E_x}{\partial z^2} = \gamma^2 E_x \tag{13.26}$$

and its solution is

$$\vec{E}_x = A^{-\gamma z} + Be^{\gamma z}. \tag{13.27}$$

We can write

$$E_{xo} = A_0 e^{-\gamma_0 z} + B_0 e^{\gamma_0 z} \tag{13.28}$$

for air

$$E_{x1} = A_1 e^{-\gamma_1 z} + B_1 e^{\gamma z} \tag{13.29}$$

for earth or medium of finite conductivity.

Similarly the corresponding magnetic fields are

$$H_{yo} = \frac{1}{\eta_0}\left(A_0 e^{-\gamma_0 z} - B_0 e^{\gamma_0 z}\right) \tag{13.30}$$

$$\Rightarrow H_{y1} = \frac{1}{\eta_1} A_1 e^{-\gamma_1 z} \tag{13.31}$$

where η_0 and η_1 are respectively $\sqrt{\frac{i\omega\mu}{\sigma_0}}$ and $\sqrt{\frac{i\omega\mu}{\sigma_1}}$.

The second term for both electric and magnetic fields will not exist in the absence of any reflecting boundary ($B_1 = 0$).

Now at $z = 0$, i.e., on the surface of the earth we have

$$E_x = E_{x0}$$

and

$$E_{x_0} = A_0\left(e^{-\gamma_0 z} + \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} e^{\gamma_0 z}\right). \tag{13.32}$$

In the second medium

$$E_{x1} = A_0 \frac{2\eta_1}{\eta_0 + \eta_1} e^{-\gamma_1 z}. \tag{13.33}$$

Here the reflection and the transmission coefficients are respectively given by

(i)

$$\text{reflection coefficient} = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} \tag{13.34}$$

and

(ii)

$$\text{transmission coefficient} = \frac{2\eta_1}{\eta_1 + \eta_0}. \tag{13.35}$$

On the surface, $\left|\frac{E_x}{H_y}\right|_{z=0} = \eta_1$ ohm. It is termed as the surface impedance of the medium. It is independent of the amplitude of the incident wave. In the low frequency regime, conduction current dominates over displacement current. The propagation constant γ_1 reduces to

$$\gamma_1 = \sqrt{i\omega\mu\sigma_1} \quad \text{and} \quad \eta_1 = \sqrt{\frac{i\omega\mu}{\sigma_1}}. \tag{13.36}$$

The surface impedance

$$Z = \eta_1 = \sqrt{\frac{\omega\mu_1}{\sigma_1}}.\sqrt{i} = \sqrt{\frac{\omega\mu_1}{\sigma_1}}.e^{-i\pi/4}. \tag{13.37}$$

Therefore

$$Z^2 = \frac{\omega\mu}{\sigma} = \frac{8\pi^2 \times 10^{-7}}{T\sigma} \tag{13.38}$$

and

$$\rho = \frac{T\left|Z^2\right|}{8\pi^2 \times 10^7}. \tag{13.39}$$

Here ρ is resistivity of the medium and is reciprocal of σ.

$\frac{\pi}{4}$ is the argument of Z. This suggests that the electric and magnetic fields are at a phase difference of 45^0 on the surface for a plane wave incidence over an homogeneous earth. Moreover resistivity is proportional to the product of the square of the impedance 'Z' and time period of the EM signal 'T'. This is the starting point of magnetotelluric sounding.

13.2 Skin Depth

When an electromagnetic wave propagates through a medium of finite conductivity, its amplitude attenuates (Fig. 13.2) and phase rotates constantly. Depth where amplitude of an electromagnetic signal reduces to $\frac{1}{e}$ th of its original value on the surface is termed as skin depth of a signal in that medium. In electromagnetics , skin depth is also termed as the depth of penetration of electromagnetic signals.

On surface of the earth, we can write the expression for electric field as

$$E_{x1} = E_0\ e^{-\gamma_1 z} = E_0 e^{-\sqrt{i\omega\mu\sigma}z} \tag{13.40}$$

$$\Rightarrow E_1 = E_0 e^{-z/\delta}.e^{iz/\delta}.e^{i\pi/4} \tag{13.41}$$

where $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$. This δ is termed as the skin depth. It is a function of electrical conductivity, magnetic permeability and the frequency of the electromagnetic signal. At a depth δ, the amplitude of the signal becomes $\frac{E_0}{e}$. So the expression for the EM signal at a certain depth Z is given by

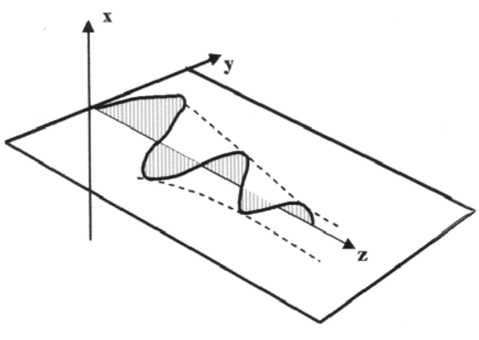


Fig. 13.2. Decay of electromagnetic wave amplitude for its propagation through a medium of finite conductivity

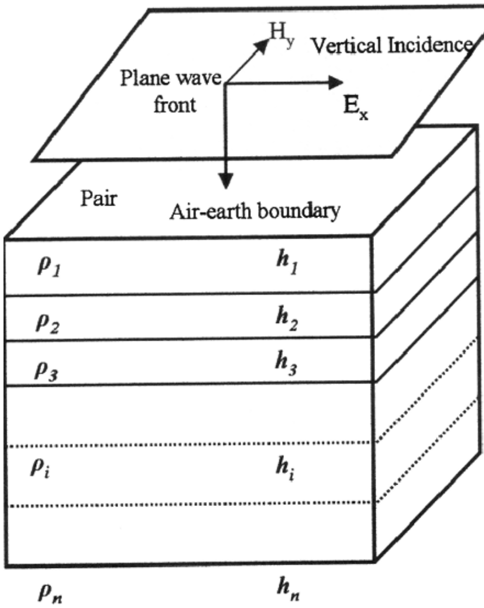


Fig. 13.3. Magnetotelluric sounding over a layered earth; plane wavefront is approaching the surface

$$E_1 = E_0 e^{-z/\delta} \cos(\omega t + \psi - z/\delta + \pi/4)\,. \tag{13.42}$$

From theoretical one dimensional magnetotelluric modeling, it is observed that an EM signal can see a target for certain cases when it is beyond the skin depth (Roy and Singh (1993)). Brian Spies (1989) also reported that an EM signal can see a target when it is below the skin depth level. In general it is treated as an estimate of the depth of penetration of the electromagnetic signal. An EM signal amplitude decreases continuously with continuous rotation of phase. Figure (13.3) shows the nature of the reduction of an EM signal amplitude with depth.

13.3 Perturbation Centroid Frequency

In an homogeneous and isotropic medium of finite conductivity if a certain perturbation is inserted at a certain depth in the form of a thin layer having different conductivity, an EM signal of a certain frequency will see that perturbation to the maximum extent. That is termed as the perturbation centroid frequency or the depth of investigation or the depth which contributes maximum towards the signal of a certain frequency measured on the surface. Perturbation centroid frequency related with depth of investigation does not have any straight forward relation with skin depth.

13.4 Magnetotelluric Response for a Layered Earth Model

For a homogeneous, earth, the wave equation is given by

$$\nabla^2 E = \mu\sigma \frac{\partial E}{\partial t} + \mu \in \frac{\partial^2 E}{\partial t^2}. \tag{13.43}$$

Component of the wave equation (13.43) changes to the form

$$\frac{\partial^2 \mathrm{E_x}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{E_x}}{\partial \mathrm{y}^2} + \frac{\partial^2 \mathrm{E_x}}{\partial \mathrm{z}^2} = \mu\sigma \frac{\partial \mathrm{E_x}}{\partial \mathrm{t}} + \mu \in \frac{\partial^2 \mathrm{E_x}}{\partial \mathrm{t}^2}. \tag{13.44}$$

For a plane electromagnetic wave propagating along the z direction (for a harmonically varying field) is given by

$$\frac{\partial^2 \mathrm{E_x}}{\partial \mathrm{z}^2} = \gamma^2 \mathrm{E_x} \qquad \text{where} \quad \gamma^2 = \mathrm{i}\omega\mu(\sigma + \mathrm{i}\omega \in). \tag{13.45}$$

From (13.45), we get

$$\mathrm{E_x} = \mathrm{A}e^{\gamma z} + \mathrm{B}e^{-\gamma z}. \tag{13.46}$$

For a homogeneous earth $\mathrm{E_x}$ vanishes at $Z \to \infty$. This condition puts

$$\mathrm{A} = 0 \text{ and} \qquad \mathrm{E_x} = \mathrm{B}e^{-\gamma z}. \tag{13.47}$$

Here $\mathrm{E_x}$ is time varying ($\mathrm{E_x}(e^{\mathrm{i}\omega t})$) and generates a time variant orthogonal magnetic field in the y direction. The vertical component $\mathrm{H_z} = 0$. From Maxwell's equation,

$$\mathrm{curl}\ \vec{\mathrm{E}}_{\mathrm{y}} = -\mathrm{i}\omega\mu\vec{\mathrm{H}}_{\mathrm{y}}. \tag{13.48}$$

For an uniform field

$$\frac{\partial \mathrm{E_x}}{\partial \mathrm{z}} = -\mathrm{i}\omega\mu \mathrm{H_y} \tag{13.49}$$

$$\mathrm{H_y} = -\frac{\gamma}{\mathrm{i}\omega\mu} \mathrm{B}e^{-\gamma z}. \tag{13.50}$$

Therefore Cagniard impedance for a homogeneous ground is

$$Z = \frac{\mathrm{E_x}}{\mathrm{H_y}} = \frac{\mathrm{i}\omega\mu}{\gamma}. \tag{13.51}$$

At low frequency $\gamma = \sqrt{\mathrm{i}\omega\mu\sigma}$, $Z = \sqrt{\omega\mu\rho}\, e^{\mathrm{i}\pi/4}$ and

$$\rho = \frac{1}{\omega\mu} \left| \frac{\mathrm{E_x}}{\mathrm{H_y}} \right|^2 \tag{13.52}$$

or

$$\rho = 0.2\mathrm{T}\left|\frac{\mathrm{E_x}}{\mathrm{H_y}}\right|^2 . \tag{13.53}$$

For a two layer earth model with layer resistivities ρ_1 and ρ_2 and thickness h_1, the electrical and magnetic components can be written as

$$\mathrm{E_x} = \mathrm{A}e^{\gamma x} + \mathrm{B}e^{-\gamma x} \tag{13.54}$$

$$\mathrm{H_y} = -\frac{\gamma}{i\omega\mu}\left(\mathrm{A}e^{+\gamma x} - \mathrm{B}e^{-\gamma x}\right). \tag{13.55}$$

The ratio of $\mathrm{E_x}$ and $\mathrm{H_y}$ yields

$$\frac{E_x}{H_y} = -\frac{i\omega\mu}{\gamma}\left(\frac{Ae^{\gamma z} + Be^{-\gamma z}}{Ae^{\gamma z} - Be^{-\gamma z}}\right) \tag{13.56}$$

Dividing the numerator and the denominator by $\sqrt{\mathrm{AB}}$, substituting

$$\sqrt{\frac{\mathrm{A}}{\mathrm{B}}} = \exp\left(\ln\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}\right).$$

We can rewrite the relation as

$$\begin{aligned}\frac{\mathrm{E_x}}{\mathrm{H_y}} &= -\frac{i\omega\mu}{\gamma}\frac{\exp\left(\gamma z + \ln\sqrt{\frac{A}{B}}\right) + \exp\left(-\gamma z - \ln\sqrt{\frac{A}{B}}\right)}{\exp\left(\gamma z + \ln\sqrt{\frac{A}{B}}\right) - \exp\left(-\gamma z - \ln\sqrt{\frac{A}{B}}\right)}\\ &= -\frac{i\omega\mu}{\gamma}\mathrm{Coth}\left(\gamma z + \ln\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}\right).\end{aligned} \tag{13.57}$$

We determine the ratio of wave impedances at two different depths to eliminate A and B. For this purpose we evaluate Z_2 at a depth z_2 with reference to Z_1 at a depth z_1 such that z_1 and z_2 are in the same medium. The impedance Z_1 at a depth z_1 is given as

$$Z_1 = -\frac{i\omega\mu}{\gamma}\mathrm{Coth}\left(\gamma z_1 + \ln\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}\right). \tag{13.58}$$

From (13.58) we get

$$\ln\sqrt{\frac{\mathrm{A}}{\mathrm{B}}} = -\left\{\mathrm{Coth}^{-1}\left(\frac{Z_1\gamma}{i\omega\mu}\right) + \gamma z_1\right\}. \tag{13.59}$$

The impedance Z_2 at a depth z_2 is

$$Z_2 = -\frac{i\omega\mu}{\gamma}\left[\mathrm{Coth}\left\{\gamma(z_2 - z_1) - \mathrm{Coth}^{-1}\left(\frac{\gamma z_1}{i\omega\mu}\right)\right\}\right]. \tag{13.60}$$

This equation is valid as long as Z_1 and Z_2 are in same medium. If we put $z_2 = 0$, then $z_2 - z_1 = h_1$, i.e., thickness of the top layer, then we have Z_2 at the ground surface $z = 0$.

$$Z_2\,(z_2 = 0) = \frac{i\omega\mu}{\gamma}\left[\operatorname{Coth}\left\{\gamma h_1 + \operatorname{Coth}^{-1}\right\}\left(\frac{\gamma z_1}{i\omega\mu}\right)\right]. \tag{13.61}$$

If we now let $h_1 \to \infty$, the geologic structure assumes a homogeneous ground and the (13.60) reduces to

$$Z_2 = \frac{i\omega\mu}{\gamma}. \tag{13.62}$$

For a two layer earth, we write Z_2 at a depth $z = h_1$ but in the second medium and equate it with Z_1 at the same depth. Thus we get

$$\begin{aligned} Z_2 &= \frac{i\omega\mu}{\gamma_2}\operatorname{Coth}\left[\gamma^2\infty + \operatorname{Coth}^{-1}\frac{\gamma_2 Z_1}{i\omega\mu}\right] \\ &= \frac{i\omega\mu}{\gamma_2}\frac{\gamma_2 Z_1}{i\omega\mu} = Z_1. \end{aligned} \tag{13.63}$$

In the first medium, we have

$$Z_0 = \frac{i\omega\mu}{\gamma_1}\operatorname{Coth}\left[\gamma_1 h_1 + \operatorname{Coth}^{-1}\frac{\gamma_1 Z_1}{i\omega\mu}\right]. \tag{13.64}$$

Substituting the value of Z_1 from (13.62) to (13.63) we get

$$Z_0 = \frac{i\omega\mu}{\gamma_1}\operatorname{Coth}\left[\gamma_1 h_1 + \operatorname{Coth}^{-1}\left(\frac{\gamma_1}{\gamma_2}\right)\right]. \tag{13.65}$$

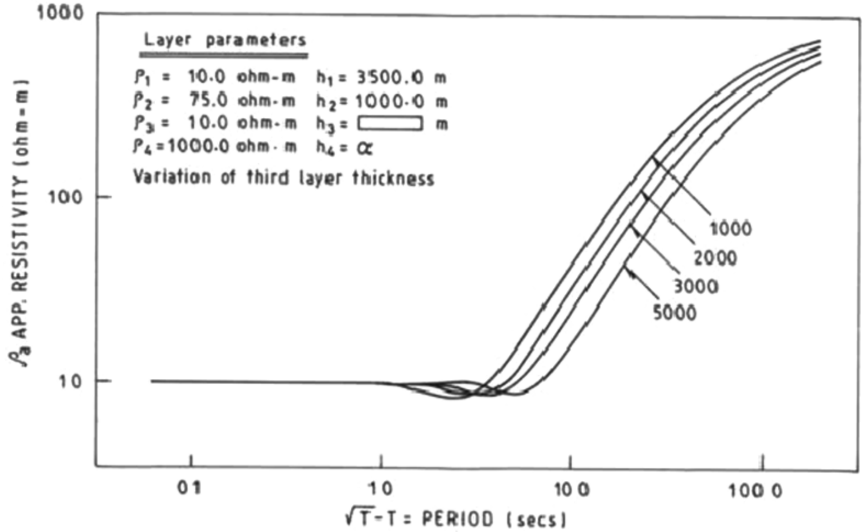
In a similar manner, we can write the impedance for a three layer earth on the ground surface as

$$Z_0 = \frac{i\omega\mu}{\gamma_1}\operatorname{Coth}\left[\gamma_1 h_1 + \operatorname{Coth}^{-1}\left\{\frac{\gamma_1}{\gamma_2}\operatorname{Coth}\left(\gamma_2 h_2 + \operatorname{Coth}^{-1}\frac{\gamma_2}{\gamma_1}\right)\right\}\right]. \tag{13.66}$$

The general expression for impedance for an N-layered earth is (from (13.65)) (Fig. 13.3)

$$\begin{aligned} Z_0 = \frac{i\omega\mu}{\gamma_1}&\operatorname{Coth}\left[\gamma_1 h_1 - \operatorname{Coth}^{-1}\left\{\frac{\gamma_1}{\gamma_2}\operatorname{Coth}\left(\gamma_2 h_2 + \operatorname{Coth}^{-1}\right.\right.\right. \\ &\left.\left.\left.\left(\frac{\gamma_2}{\gamma_3}\cdots\cdots\cdots + \operatorname{Coth}^{-1}\left(\frac{\gamma_{n-2}}{\gamma_{n-1}}\operatorname{Coth}\left(\gamma_{n-2} h_{n-1} + \operatorname{Coth}^{-1}\frac{\gamma_{n-1}}{\gamma_{n-2}}\right)\right)\right)\right)\right]. \end{aligned} \tag{13.67}$$

Equation (13.66) is one of the approaches for computation of one dimensional magnetotelluric response for an N-layered earth (Fig. 13.4 and Fig. 13.5).



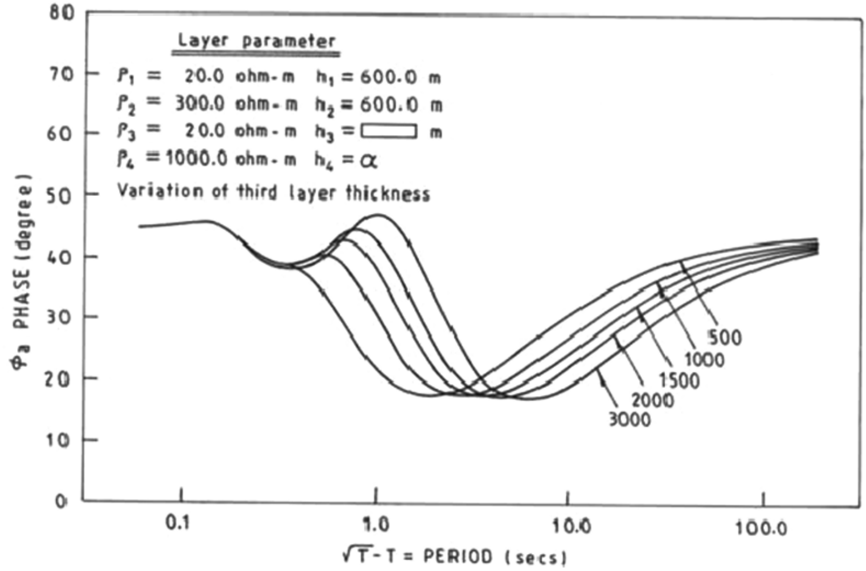


Fig. 13.4. Magnetotelluric apparent resistivity and phase variations with period; theoretical model parameter values are given in the diagrams

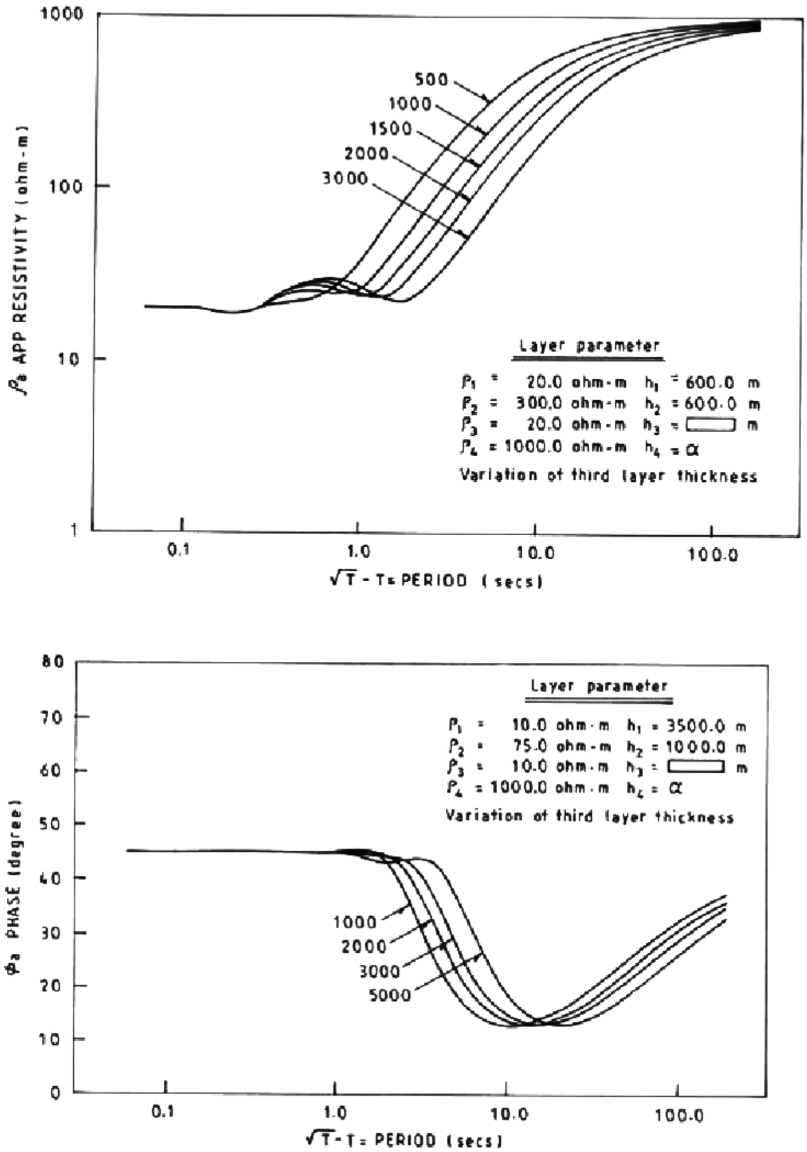


Fig. 13.5. Magnetotelluric apparent resistivity and phase variations with period;model parameters are given in the diagram

13.5 Electromagnetic Field due to a Vertical Oscillating Electric Dipole

A small oscillating electric dipole is assumed in an infinite full space. The electromagnetic waves will propagate in all the directions as if a point source is located at the centre (Fig. 13.6). Therefore, overall potential distribution will have spherical symmetry and the Laplacian operator ∇^2 will be in spherical coordinate and it will be a function of 'r' the radial direction only.

Let as assume that the Hertz vector $\vec{\Pi}$ has only z-component and is expressed as

$$\vec{\Pi}_z = \vec{a}_z \vec{\Pi}_z. \tag{13.68}$$

The wave equation $\nabla^2 \vec{\Pi}_z = \gamma^2 \Pi_z$ reduces to the form

$$\begin{aligned}
&\frac{\partial^2 \Pi_z}{\partial r^2} + \frac{2}{r}\frac{\partial \Pi_z}{\partial r} = \gamma^2 \Pi_z \\
&\Rightarrow \frac{\partial^2}{\partial r^2}\left(r\vec{\Pi}_z\right) = \gamma^2\left(r\vec{\Pi}_z\right) \qquad (13.69)\\
&\Rightarrow r\vec{\Pi}_z = ae^{\gamma r} + be^{-\gamma r} \\
&\Rightarrow \vec{\Pi}_z = \frac{ae^{\gamma r}}{r} + \frac{be^{-\gamma r}}{r}. \qquad (13.70)
\end{aligned}$$

In an infinite and homogenous medium, there is no possibility for a wave to be reflected back. The potential will die down with distance. Therefore the term $\frac{a\,e^{\gamma r}}{r}$ cannot be a potential function. The correct term is

$$\vec{\Pi}_z = \frac{be^{-\gamma r}}{r}. \tag{13.71}$$

The vector potential for an electromagnetic field changes to $\vec{\Pi}_z = \frac{b}{r}$ for $\gamma = 0$ i.e., for zero frequency. Vector potential in electromagnetics changes to scalar potential in direct current field for zero frequency. Frequency dependent Hertz vector potential die at a faster pace with distance than the frequency independent scalar potential due to a DC point source. At this moment the value of 'b' is arbitrary.

The connecting relations between Hertz vector and the magnetic and electric fields can be written as

$$\vec{E}_r = -\gamma^2 \vec{\Pi}_r + \frac{\partial}{\partial r}\left(div \vec{\Pi}_z\right) \tag{13.72}$$

$$\vec{E}_\theta = -\gamma^2 \vec{\Pi}_\theta + \frac{\partial}{r\partial\theta}\left(\frac{\partial \vec{\Pi}_z}{\partial z}\right) \tag{13.73}$$

$$\vec{E}_\psi = -\gamma^2 \vec{\Pi}_\psi + \frac{1}{\sin\theta}\frac{\partial}{\partial\psi}\left(\frac{\partial \vec{\Pi}_z}{\partial z}\right). \tag{13.74}$$

Now

$$\vec{\Pi}_r = \vec{\Pi}_z \cos\theta \tag{13.75}$$
$$\vec{\Pi}_\theta = -\vec{\Pi}_z \sin\theta$$

and

$$\vec{\Pi}_\psi = 0$$

For oscillating vertical electric dipole, $\vec{\Pi}_\psi = 0$ and $\frac{\partial}{\partial\psi} = 0$ where ψ is an azimuthal angle. Therefore,

$$\vec{H}_r = (\sigma + i\omega \in) \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\vec{\Pi}_\psi\right) - \frac{\partial}{\partial\psi}\vec{\Pi}_\theta\right] = 0 \tag{13.76}$$

$$\vec{H}_\theta = (\sigma + i\omega \in) \frac{1}{\sin\theta}\left[\frac{\partial\vec{\Pi}_r}{\partial\psi} - \sin\theta\frac{\partial\left(r\vec{\Pi}_\psi\right)}{\partial r}\right] = 0 \tag{13.77}$$

$$\vec{H}_\phi = (\sigma + i\omega \in) \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\vec{\Pi}_\theta\right) - \frac{\partial\vec{\Pi}_r}{\partial\theta}\right] \tag{13.78}$$

Since H_r and H_θ are zero, therefore E_r and E_θ are not zero because electromagnetic waves are transverse in nature. The magnetic component which exists is $H\psi$.
Now

$$\vec{\Pi}_z = \frac{be^{-\gamma r}}{r}$$

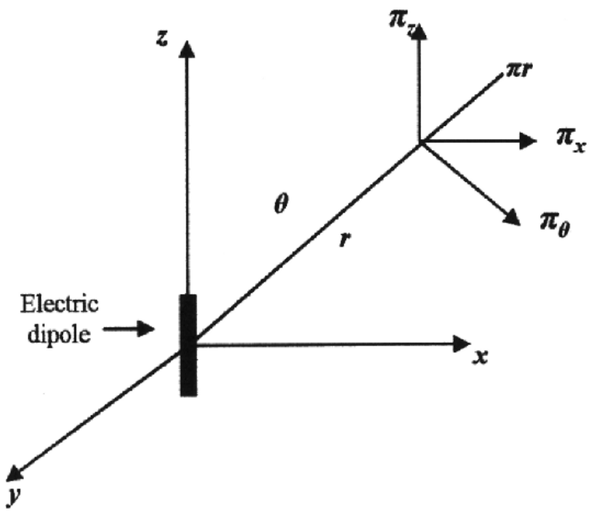


Fig. 13.6. Vertical oscillating electric dipole in a homogeneous full space

and

$$\frac{\partial \vec{\Pi}_z}{\partial z} = \left(-\frac{be^{-\gamma r}}{r^2} - \frac{b\gamma}{r}e^{-\gamma r}\right). \tag{13.79}$$

Hence

$$\begin{aligned}
-\frac{\gamma^2 be^{-\gamma r}}{r}\cos\theta\vec{E}_r &= -\frac{r^2 be^{-\gamma r}}{r}\cos\theta + \frac{\partial}{\partial r}\left(-\frac{b}{r^2}e^{-\gamma r} - \frac{b\gamma}{r} - e^{-\gamma r}\right)\cos\theta \\
&= be^{-\gamma r}\cos\theta\left[-\frac{\gamma^2}{r} + \frac{\gamma}{r^2} + \frac{\gamma^2}{r}\frac{2}{r^3} + \frac{\gamma}{r^2}\right] \\
&= \frac{2b\cos\theta}{r^3}\left(1+\gamma r\right)e^{-\gamma r}.
\end{aligned} \tag{13.80}$$

Exactly in the same way, the value of E_θ can be found out as

$$\begin{aligned}
\vec{E}_\theta &= \frac{-\gamma^2 be^{-\gamma r}}{r}\sin\theta + \frac{\partial}{r\partial\theta}\left(\frac{\partial\vec{\Pi}_z}{\partial z}\right) \\
&= \frac{b\sin\theta}{r^3}\left(1+\gamma r+\gamma^2 r^2\right)e^{-\gamma r}
\end{aligned} \tag{13.81}$$

and

$$\begin{aligned}
\vec{H}_\psi &= (\sigma + i\omega\in)\frac{1}{r}\left[\frac{\partial}{\partial r}\left(-\frac{\gamma be^{-\gamma r}}{r}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{be^{-\gamma r}}{r}\cos\theta\right)\right] \\
&= (\sigma + \omega\in)\frac{b\sin\theta}{r^2}\left(1+\gamma r\right)e^{-\gamma r}.
\end{aligned} \tag{13.82}$$

To assign a certain value to ‘b’ which is quite arbitrary so far, we seek analogy from the electrostatic and DC conduction case. In the electrostatic and direct current flow field cases, the potential at a point at a distance ‘r’ from a dipole of charge +q and −q (and +I and −I as the case may be) separated by a distance dz are

$$\phi = \frac{qdz}{4\pi\varepsilon}\frac{\cos\theta}{r^2} \quad \text{(electrostatic case)} \tag{13.83}$$

and

$$\phi = \frac{Idl}{4\pi\sigma}\frac{\cos\theta}{r^2}\text{(DC conduction case)} \tag{13.84}$$

$$\vec{E}_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{qdz}{4\pi\varepsilon}\left[\frac{\sin\theta}{r^3}\right]\text{(electrostatic case)} \tag{13.85}$$

and

$$\vec{E}_\theta = \frac{Idl}{4\pi\sigma}\left[\frac{\sin\theta}{r^3}\right]\text{(DC conduction case).} \tag{13.86}$$

Therefore, we can write the value of $\vec{\Pi}_z$ in case of an oscillating electric dipole as

$$\vec{\Pi}_z = \frac{I\, l}{4\pi\,(\sigma + i\omega \in)} . \frac{e^{-\gamma r}}{r} \tag{13.87}$$

where $b = \frac{I\, l}{4\pi(\sigma + i\omega\in)}$. I is the current strength and l is the length of the current dipole i.e., the distance between +I and −I. In order to accommodate the time varying part, σ was replaced by $(\sigma + i\omega \in)$. We can now write the existing components of the electric and magnetic fields as

$$\vec{E}_r = \frac{I\, l}{4\pi\,(\sigma + i\omega \in)} . \frac{2\cos\theta}{r^3}\,(1 + \gamma r)\, e^{-\gamma r} \tag{13.88}$$

$$\vec{E}_\theta = \frac{I\, l}{4\pi\,(\sigma + i\omega \in)} . \frac{\sin\theta}{r^3}\left(1 + \gamma r + \gamma^2 r^2\right) e^{-\gamma r} \tag{13.89}$$

$$\vec{E}_\psi = \frac{I\, l}{4\pi} . \frac{\sin\theta}{r^2}\,(1 + \gamma r)\, e^{-\gamma r}. \tag{13.90}$$

Special cases
Case 1 when $|\gamma r| << 1$
This zone is known as the near zone or static zone.
Here

$$\vec{E}_r = \frac{I\, l}{4\pi\,(\sigma + i\omega \in)} . \frac{2\cos\theta}{r^3} \tag{13.91}$$

$$\vec{E}_\theta = \frac{I\, l}{4\pi\,(\sigma + i\omega \in)} . \frac{\sin\theta}{r^3} \tag{13.92}$$

and

$$\vec{H}_\psi = \frac{I\, l}{4\pi} . \frac{\sin\theta}{r^2}. \tag{13.93}$$

Case II when $|\gamma r| >> 1$
This is the far zone or the radiation zone. In this case

$$\vec{E}_r = \frac{I\, l\gamma}{4\pi\,(\sigma + i\omega \in)} . \frac{2\cos\theta}{r^2} e^{-\gamma r} \tag{13.94}$$

$$\vec{E}_\theta = \frac{I\, l\gamma}{4\pi\,(\sigma + i\omega \in)} . \frac{\sin\theta}{r} e^{-\gamma r} \tag{13.95}$$

and

$$\vec{H}_\psi = \frac{I\, l\gamma}{4\pi} . \frac{\sin\theta}{r} . e^{-\gamma r}. \tag{13.96}$$

In this case, the radial component $\vec{E}_r$ is decreasing rapidly with distance, E_θ and H_ψ will form a transverse wave at a great distance when $\vec{E}_r$ has become negligible. Since E_θ and H_ψ are in the same plane, the resultant effect will be a plane transverse wave propagating outward.

Here

$$\frac{\vec{E}_\theta}{\vec{H}_\psi} = \frac{\gamma^2}{\sigma + i\omega \in}.\frac{1}{\gamma} = \frac{\gamma}{\sigma + i\omega \in} = \sqrt{\frac{i\omega\mu}{\sigma + i\omega \in}} = \eta \tag{13.97}$$

is the impedance of the medium. We can express E_θ as

$$\begin{aligned}\vec{E}_\theta &= \frac{I\,l}{4\pi}.\omega\mu.\frac{\sin\theta}{r}e^{-(\alpha+i\beta)^r}.e^{i\omega t}.e^{i\pi/2}\\ &= \frac{I\,l}{4\pi}.\omega\mu.\frac{\sin\theta}{r}e^{-\alpha r}.e^{i(\omega t-\beta r+\pi/2)}.\end{aligned} \tag{13.98}$$

Since

$$\begin{aligned}\gamma &= \alpha + i\beta = \sqrt{\alpha^2+\beta^2}.e^{i\tan -1\left(\frac{\beta}{\alpha}\right)}\\ H_\psi &= \frac{I\,l}{4\pi}\sqrt{\alpha^2+\beta^2}.\frac{\sin\theta}{r}.e^{-\alpha r}.e^{i(\omega t-\beta+\pi/2)}.\end{aligned} \tag{13.99}$$

If electrical conductivity of the medium is zero, then

$$\beta = \omega\sqrt{\mu \in},$$

therefore

$$\vec{E}_\theta = \frac{Il}{4\pi}.\frac{\omega\mu\sin\theta}{r}\cos\left(\omega t - \omega\sqrt{\mu \in}.r + \pi/2\right) \tag{13.100}$$

and

$$\vec{H}_\psi = \frac{I\,l}{4\pi}.\omega\sqrt{\mu \in}.\frac{\sin\theta}{r}\cos\left(\omega t - \omega\sqrt{\mu \in}.r + \pi/2\right). \tag{13.101}$$

(Figs. 13.7 and 13.8). Since there is no phase difference, it is a linearly polarized uniform plane wave and

$$\frac{\vec{E}_\theta}{\vec{H}_\psi} = \frac{\omega\mu}{\omega\sqrt{\mu \in}} = \sqrt{\frac{\mu}{\in}} \tag{13.102}$$

gives intrinsic impedances of the medium.

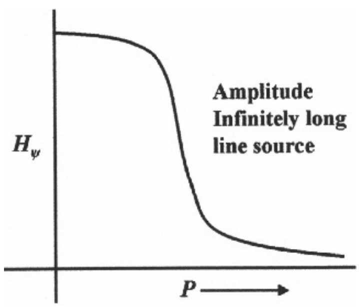


Fig. 13.7. Variation of amplitude of the magnetic field H_ψ with electromagnetic parameter $P(= l^2\omega\mu\sigma)$; l is the distance

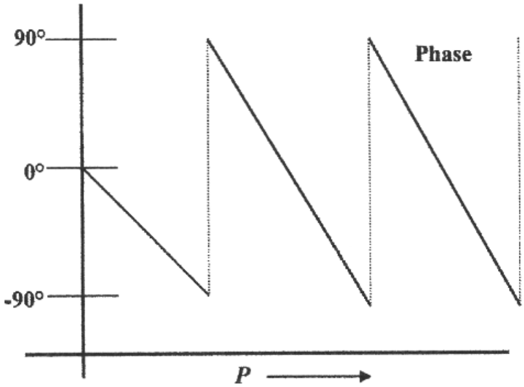


Fig. 13.8. Variation of phase of H_ψ

13.6 Electromagnetic Field due to an Oscillating Vertical Magnetic Dipole Placed on the Surface of the Earth

A coil carrying current is a magnetic dipole. It generates the magnetic field around the coil (Fig. 13.9). When an alternating current is sent through the same coil, it becomes an oscillating magnetic dipole and electromagnetic waves start propagating in all the directions from this magnetic dipole. It is termed as the vertical oscillating magnetic dipole when the plane of the coil is horizontal and the magnetic field vector is vertical. When the plane of the coil is vertical and the magnetic field vector is at right angles to the plane of coil i.e., horizontal, it is termed as the horizontal magnetic dipole (See next section)

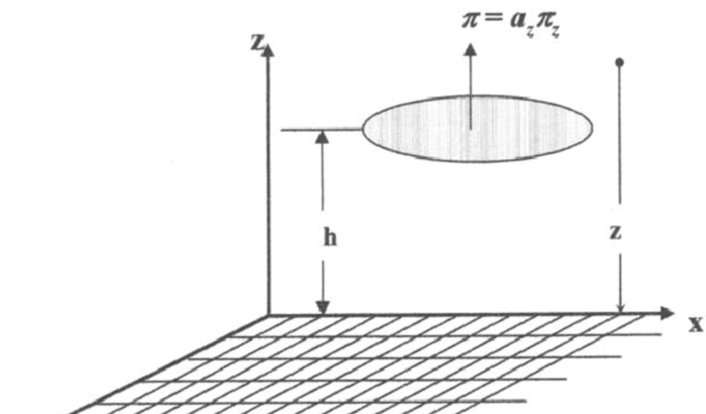


Fig. 13.9. Vertical oscillating magnetic dipole on the surface of a homogenous earth

In this problem, both the coil and the point of observation are initially assumed to be in the air. They are finally brought down to the surface of the earth to have a simplified solution to the problem. However, researchers can introduce further complexities in the problem and arrive at a more general solution taking this solution as a starting point. The simplest problems are presented in this section. Many problems in this series have already been solved and they are available in published literatures. The problem due to an oscillating magnetic dipole z is solved using Fitzerald vector potential $\vec{F}$. From Sect. (13.2) we have seen that the expression for vector potential at a distance r due to an oscillating electric dipole is $\vec{\Pi} = \frac{be^{-\gamma r}}{r}$. Similarly we have written the expression for the source potential for vertical magnetic dipole as

$$\vec{F} = m.\frac{e^{-\gamma r}}{r} \tag{13.103}$$

where m is the moment of the dipole. For electric dipole

$$\vec{m} = \vec{a}_l \frac{Il}{4\pi(\sigma + i\omega \in)} \tag{13.104}$$

and for magnetic dipole

$$\vec{m} = \vec{a}_n \frac{IS}{4\pi} \tag{13.105}$$

where S is the surface area of the current carrying coil. $\vec{a}_l$ and $\vec{a}_n$ are respectively the unit vectors. Our problem is to find out the fields at any point either outside or on the surface of the earth. Moment of the vertical magnetic dipole is directed normal to the boundary. The boundary plane is the air earth boundary and the oscillating dipole is placed at a height ‘h’ from the surface. The magnetic dipoles are taken along the z direction and are represented by the Fitzerald vector $\vec{F}$. Basic structures of this type of boundary value problems are more or less the same with some problem dependent variations in finer details.

This type of boundary value problems start with electromagnetic wave equations. The Laplacian operator, to be used, varies from source to source of the electromagnetic waves. The problem can be solved either in $\vec{E}$ and $\vec{H}$ field domain as done in Sect. 13.1 or in vector potential domain as shown in Sect. 13.2 to 13.8. The EM wave partial differential equations are solved using the method of separation of variables. Once potential problems are solved in a vector potential domain, one can obtain the electric and magnetic fields using the appropriate relations between vector potentials and $\vec{E}$ and $\vec{H}$ fields ((13.72) to (13.74) and (13.76) to (13.77)). These relations are connectors between $\vec{E}$ and $\vec{H}$ field with $\vec{\Pi}$ or $\vec{F}$ or $\vec{A}$ and ϕ, as discussed in the previous chapter. In the $(\vec{A}, \phi)$ formulation ϕ is a scalar potential. At this stage one has to find out the vanishing and non-vanishing components of the $\vec{E}$ and $\vec{H}$ fields.Physics and geometry of the problems help in determining the zero and non-zero components of the EM fields. The source potential and the perturbation potentials are determined. They are brought to the same form before the boundary conditions are applied. This step is more or less the same as those used for potential field problems in direct current problem. Bringing source potential and perturbation potential in the same form may turn out to be a fairly complex mathematical problem. These arbitrary coefficients, obtained while computing the perturbation potentials by solving the wave equation by method of separation of variables, are determined applying boundary conditions. The final expressions for $\vec{E}$ and $\vec{H}$ fields are obtained either analytically or numerically depending upon the level of complexities in mathematics.

The Fitzerald vector $\vec{F}$ is directed along the z direction.

$$\vec{F} = \vec{a}_z.\frac{IS}{4\pi}.\frac{e^{-\gamma r}}{r}. \tag{13.106}$$

In view of the cylindrical symmetry, the field

$$\vec{F}_z = f(\rho, z). \tag{13.107}$$

Therefore, the Helmholtz wave equation

$$\nabla^2\vec{F}_z = \gamma^2\vec{F}_z \tag{13.108}$$

reduces to

$$\frac{\partial^2 F_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_z}{\partial \rho} + \frac{\partial^2 F_z}{\partial z^2} = \gamma^2 F_z \tag{13.109}$$

In this problem $\vec{F}_\psi = 0$ and $\frac{\partial}{\partial \psi} = 0$, where ψ is the azimuthal angle.
Now applying the method of separation of variables

$$\vec{F} = R(\rho)Z(z) \tag{13.110}$$

we get

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho R}.\frac{dR}{d\rho} = -\frac{1}{Z}\frac{d^2Z}{dz^2} + \gamma^2. \tag{13.111}$$

If we substitute

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 + n^2 \tag{13.112}$$

(13.111) reduces to

$$\frac{1}{R}.\frac{d^2R}{d\rho^2} + \frac{1}{\rho R}.\frac{dR}{d\rho} + n^2 = 0. \tag{13.113}$$

The solutions of the first and second equations are respectively given by

$$(i)e^{\pm\sqrt{\gamma^2 + n^2}Z}$$

and

$$(ii)J_0(n\rho) \text{ and } Y_0(n_\rho) \tag{13.114}$$

where J_0 and Y_0 are the Bessel's function of first and second kind and of zero order. The solutions of the vector potentials in the two media are

$$\vec{F}_0 = \int_0^{\alpha} \left(A_0 e^{-Z\sqrt{\gamma^2+n^2}} + B_0 e^{+Z\sqrt{\gamma^2+n^2}} \right) J_0(n\rho) dn \tag{13.115}$$

and

$$\vec{F}_1 = \int_0^{\alpha} B_1 e^{+z\sqrt{\gamma^2+n^2}} J_0 (n\rho)\, dn. \tag{13.116}$$

Since Y_0 tends to infinity as $r \to 0, Y_0$ cannot be present in the solution of the problem. $\vec{F}_0$ in the first medium, will contain the source potential. Therefore

$$\vec{F}_0 = \frac{me^{-\gamma_0 r}}{r} + \int_0^{\alpha} A_n e^{-Z\sqrt{\gamma_0^2+n^2}} J_0 (n\rho)\, dn. \tag{13.117}$$

B_n cannot be present in the expression because the field must vanish at infinite distance. In the second medium Z is assumed to be negative. The vector potential should die down with depth in the absence of any kind of reflector. Therefore

$$\vec{F}_1 = \int_0^{\alpha} B_n e^{+Z\sqrt{\gamma_1^2+n^2}} J_0(n\rho) dn. \tag{13.118}$$

The connecting link between the Fitzerald vector and the electric and magnetic fields are respectively given by

$$\vec{E} = -i\omega\mu \text{ curl } \vec{F} \tag{13.119}$$

$$\begin{aligned} \vec{H} &= -\gamma^2\vec{F} + \text{grad div}\vec{F} \\ &= \text{curlcurl}\vec{F} \qquad (13.120) \\ &= -\frac{1}{i\omega\mu} \text{ curl } \vec{E}. \qquad (13.121) \end{aligned}$$

In cylindrical coordinate system

$$\begin{aligned} \text{curl}\vec{F} =& \vec{a}_\rho \left(\frac{1}{\rho}\frac{\partial \vec{F}_z}{\partial \rho} - \frac{\partial \vec{F}_\psi}{\partial z} \right) \\ &+ \vec{a}_\psi \left(\frac{\partial \vec{F}_\rho}{\partial z} - \frac{\partial \vec{F}_z}{\partial \rho} \right) \\ &+ \vec{a}_z \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\psi) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \psi} \right). \end{aligned} \tag{13.122}$$

In this problem $\frac{\partial}{\partial \phi} = 0, \vec{E}_p = \vec{E}_z = \vec{H}_\psi = 0$ and

$$\vec{E}_{\psi} = i\omega\mu\frac{\partial F_z}{\partial \rho} \tag{13.123}$$

$$\vec{H}_{\rho} = \frac{1}{i\omega\mu}\frac{\partial \vec{E}_{\psi}}{\partial Z} = \frac{\partial}{\partial Z}\frac{\partial \vec{F}}{\partial \rho} \tag{13.124}$$

$$\vec{H}z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \vec{F}_z}{\partial \rho}\right). \tag{13.125}$$

Boundary conditions are

$$\begin{aligned}(E_{\psi})_0 &= (E_{\psi})_1\\ (H_{\rho})_0 &= (H_{\rho})_1\end{aligned} \tag{13.126}$$

Therefore,

$$i\omega\mu_0\left(\frac{\partial F}{\partial \rho}\right)_0 = i\omega\mu_1\left(\frac{\partial F}{\partial \rho}\right)_1$$

and

$$\left(\frac{\partial}{\partial Z}\cdot\frac{\partial F}{\partial \rho}\right)_0 = \left(\frac{\partial}{\partial Z}\frac{\partial F}{\partial \rho}\right)_1. \tag{13.127}$$

If a function is continuous across the boundary

$$\begin{aligned}\mu_0 F_0 &= \mu_1 F_1\\ \frac{\partial F_0}{\partial Z} &= \frac{\partial F_1}{\partial Z}.\end{aligned} \tag{13.128}$$

These are the two sets of boundary conditions in this problem.
Weber Lipschitz integral

$$\frac{1}{r} = \frac{1}{\sqrt{\rho^2 + z^2}} = \int_0^{\alpha} e^{-nZ}J_0(n\rho)dn \tag{13.129}$$

has already been used in direct current field problem to convert the potential function $\frac{1}{r}$ in the form of an integral involving Bessel's function to bring parity between the source potential and the perturbation potential. This is an essential step for all electrical and electromagnetic boundary value problems. For a time varying electromagnetic field Sommerfeld suggested the formula

$$\frac{e^{-\gamma\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} = \int_0^{\alpha} e^{-Z\sqrt{\gamma^2+n^2}}.\frac{n}{\sqrt{n^2+\gamma^2}}.J_0\,(n\rho)\,dn. \tag{13.130}$$

For $\gamma = 0$, i.e. for zero frequency, Sommerfeld's formula reduces down to the Weber's formula. 'r' is always positive irrespective of the position of the point of observation with respect to the ground and the transmitting antenna i.e.,

$$r^2 = \rho^2 + (h - z)^2$$

or

$$\rho^2 + (z - h)^2$$

is always positive whether h is greater or less than z.

Sommerfeld formula for this problem is

$$\frac{e^{-\gamma\sqrt{\rho^2+(z-h)^2}}}{\sqrt{\rho^2 + (z-h)^2}} = \int_0^{\alpha} e^{-|z-h|\sqrt{\gamma^2+n^2}}.\frac{n}{\sqrt{n^2+\gamma^2}}. \tag{13.131}$$

The vector potentials in the two media now be written as

$$\begin{aligned}\vec{F}_0 =& m\int_0^{\alpha} e^{-|z-h|\sqrt{\gamma_0^2+n^2}}.\frac{n}{\sqrt{\gamma_0^2+n^2}}..J_0(n\rho)dn\\ &+\int_0^{\alpha} A_n e^{-Z\sqrt{\gamma_0^2+n^2}} J_0(n\rho)dn\end{aligned} \tag{13.132}$$

and

$$\vec{F}_1 = \int_0^{\alpha} B_n e^{Z\sqrt{\gamma_1^2+n^2}}.J_0(n\rho)dn. \tag{13.133}$$

Applying the boundary conditions at $z = 0$, we get

$$\begin{aligned}&\mu_0 m\int_0^{\alpha} \frac{n}{\sqrt{n^2+\gamma_0^2}} e^{-h\sqrt{\gamma_0^2+n^2}} J_0(n\rho)dn\\ &+\mu_0\int_0^{\alpha} A_n J_0(n\rho)\ dn = \mu_1\int_0^{\alpha} B_n J_0(n\rho)dn\end{aligned} \tag{13.134}$$

$$\Rightarrow \mu_0 m.\frac{n}{n^2+\gamma_0^2}\sqrt{\gamma_0^2+n^2}.e^{-h\sqrt{\gamma_0^2+n^2}} - \mu_0 A_n\sqrt{\gamma_0^2+n^2} = \mu_1.B_n\sqrt{\gamma_1^2+n^2}. \tag{13.135}$$

From these two equations, we get the value of A_n as

$$A_n = m.\frac{n}{\sqrt{\gamma_0^2+n^2}}.\frac{\sqrt{\gamma_0^2+n^2}-\sqrt{\gamma_1^2+n^2}}{\sqrt{\gamma_0^2+n^2}+\sqrt{\gamma_1^2+n^2}} e^{-n\sqrt{\gamma_0^2+n^2}}. \tag{13.136}$$

Now let

$$\begin{aligned}n^2 + \gamma_0^2 &= n_0^2\\ n^2 + \gamma_1^2 &= n_1^2.\end{aligned}$$

Then the (14.134, 14.135) simplify down to

$$\mu_0 m \frac{n}{n_0} e^{-n_0 h} + \mu_0 A_n = \mu_1 B_n \tag{13.137}$$

$$m \frac{n}{n_0} . e^{-n_0 h} - n_0 A_n = n_1 B_n . \tag{13.138}$$

From these two equations, the values of A_n and B_n comes out to be

$$A_n = m . \frac{n}{n_0} . \frac{P n_0 - n_1}{P n_0 + n_1} e^{-n_0 h} \tag{13.139}$$

$$B_n = m . \frac{2 n_1}{P n_0 + n_1} . e^{-n_0 h} \tag{13.140}$$

where

$$\rho = \frac{\mu}{\mu_0} .$$

Therefore the vector potentials in the two media are

$$\begin{aligned} \vec{F}_0 = & m \int_0^{\alpha} \frac{n}{\sqrt{n^2 + \gamma_0^2}} . e^{-|z-h|\sqrt{\gamma_0^2 + n^2}} J_0(n\rho)\, dn \\ & + m \int_0^{\alpha} \frac{n}{n_0} \frac{P n_0 - n_1}{P n_0 + n_1} . e^{-n_0(z+h)} J_0(n\rho) dn \end{aligned} \tag{13.141}$$

and

$$\vec{F}_1 = m \int_0^{\alpha} \frac{2n}{P n_0 + n_1} . e^{-n_0 h + n_1 z} J_0(n\rho)\, dn . \tag{13.142}$$

The $\vec{F}_0$ can be rewritten as

$$\begin{aligned} \vec{F}_0 = & m \int_0^{\alpha} \frac{n}{n_0} . e^{-|h-z| n_0} . J_0(n\rho)\, dn \\ & + m \int_0^{\alpha} \frac{n}{n_0} e^{-n_0(z+h)} J_0(n\rho)\, dn \\ & - m \int_0^{\alpha} \frac{n}{n_0} . \frac{2 n_1}{P n_0 + n_1} . e^{-n_0(z+h)} . J_0(n\rho)\, dn . \end{aligned} \tag{13.143}$$

This is equivalent to

$$\frac{m\ e^{-\gamma_0 r}}{r} + \frac{m\ e^{-\gamma_0 r_1}}{r_1} - P' \tag{13.144}$$

where

$$r = \sqrt{\rho^2 + (z-h)^2}$$

$$r_1 = \sqrt{\rho^2 + (z+h)^2}.$$

Expression (13.144) stands for a source, an image of the source plus a perturbation term. In electrostatics and D.C. conduction case one gets only one image in medium 2 for a source in medium 1 (Fig. 13.1).
Now the coil is brought down to the surface i.e., when we put $h = 0$

$$\vec{F}_0 = 2m \int_0^{\alpha} \frac{n}{n_0} e^{-n_0 z} J_0(n\rho)\,dn$$

$$- m \int_0^{\alpha} \frac{n}{n_0} \cdot \frac{2n_1}{Pn_0 + n_1} . e^{-n_0 z} J_0(n\rho)\,dn \tag{13.145}$$

$$= 2m . \frac{n}{n_0} \int_0^{\alpha} \frac{n_0}{n_0 + n_1} . e^{-n_0 z} J_0(n\rho)\,dn. \tag{13.146}$$

Since $p = 1$, for $\mu_0 = \mu_1$ and $1 - \frac{n_1}{Pn_0+n_1} = \frac{Pn_0}{Pn_0+n_1}$.
This is the expression for the vector potential due to an oscillating magnetic dipole placed on the surface of the earth. For $\gamma = 0$, the vector potential reduces to

$$\vec{F}_0 = m \int_0^{\alpha} e^{-nz} J_0(n\rho)\,dn = \frac{m}{\sqrt{\rho^2 + z^2}} = \frac{m}{r}. \tag{13.147}$$

It is a DC potential at a distance r.

The vector potential inside the earth is

$$\vec{F}_1 = 2m \int_0^{\alpha} \frac{n}{n_0 + n_1} e^{-n_1 z} J_0(n\rho)\,dn. \tag{13.148}$$

The existing non-zero electric and magnetic fields are $\vec{E}_\psi, \vec{H}_\rho$ and $\vec{H}_z$. The expressions for these vectors be written as

$$\vec{E}_\psi = i\omega\mu_0 2m \frac{\partial I}{\partial \rho} \tag{13.149}$$

$$\vec{E}_p = \frac{\partial^2}{\partial\rho\partial Z} . 2m\ I \tag{13.150}$$

$$\vec{H}_z = 2m . \frac{1}{\rho} . \frac{\partial}{\partial\rho} . \rho \frac{\partial I}{\partial \rho} \tag{13.151}$$

where I is the integral $\left[\int_0^{\alpha} \frac{n}{n+n_1} J_0(n\rho)\,dn\right]$.
When both the antenna as well as the point of observation is on the surface,the expressions for $\vec{E}_\psi, \vec{H}_\rho and \vec{H}_z$ are

$$\vec{E}_{\psi} = i\omega\mu_0 2m\frac{\partial}{\partial\rho}\left[\int_0^{\alpha}\frac{n}{n+n_1}J_0(n\rho)\,dn\right] \tag{13.152}$$

$$\vec{H}_{\rho} = -2m\frac{\partial}{\partial\rho}\int_0^{\infty}\frac{n^2}{n+n_1}J_0(n\rho)\,dn \tag{13.153}$$

$$\vec{H}_z = -2m.\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\int_0^{\alpha}\frac{n}{n+n_1}J_0(n\rho)\,dn \tag{13.154}$$

The integral

$$\begin{aligned}\int_0^{a}\frac{n}{n+n_1}J_0(n\rho)dn &= \int_0^{a}\frac{n(n-n_1)}{n^2-n_1^2}.J_0(n\rho)\,dn\\ &= -\frac{1}{\gamma_1^2}\int_0^{\alpha}n^2J_0(n\rho)\,dn + \frac{1}{\gamma_1^2}\int_0^{\alpha}nn_1J_0(n\rho)\,dn.\end{aligned} \tag{13.155}$$

These integrals can be calculated using the Weber and Sommerfeld formula. If we differentiate Weber's formula twice we get

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = \int_0^{\alpha}n^2e^{-nz}.J_0(n\rho)\,dn. \tag{13.156}$$

At $z = 0$, eqn. (13.156) reduces to

$$\left.\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\right|_{z=0} = \int_0^{\alpha}n^2J_0(n\rho)\,dn. \tag{13.157}$$

Now $\frac{\partial}{\partial z}.\frac{1}{r} = -\frac{1}{r^2}\frac{dr}{dz} = -\frac{z}{r^3}$ and

$$\frac{\partial^2}{\partial z^2}.\frac{1}{r} = \left(\frac{-1}{r^3}\right) - z\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right)$$

because

$$r = \sqrt{\rho^2+z^2} \text{ and } \frac{dr}{dz} = \frac{1}{2}\left(\rho^2+z^2\right)^{-1/2}.2z = \frac{z}{r}.$$

We get

$$\left.\frac{\partial^2}{\partial z^2}.\frac{1}{r}\right|_{z=0} = -\frac{1}{\rho^3}$$

and

$$\int_0^\alpha n^2 J_0(n\rho)\,dn = \frac{1}{r^3} \tag{13.158}$$

on the surface of the earth.
If we differentiate Sommerfeld formula twice, we get,

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\left(\frac{e^{-\gamma r}}{r}\right) &= \int_0^\alpha n n_1 e^{-n_1 z} J_0(n\rho)dn \\ \Rightarrow \frac{\partial^2}{\partial z^2}\left(\frac{e^{-\gamma r}}{r}\right)_{z=0} &= \int_0^\alpha n n_1 J_0(n\rho)dn.\end{aligned} \tag{13.159}$$

Now

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{e^{-\gamma r}}{r}\right) &= e^{-\gamma r}\left(-\frac{z}{r^3} - \frac{\gamma z}{r^2}\right) \\ &= -z\left(\frac{1}{r^3} + \frac{\gamma}{r^2}\right)e^{-\gamma r} = Zf(r)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\left(\frac{e^{-\gamma r}}{r}\right) &= f(r) + z\frac{\partial}{\partial z}f(r) \\ &\Rightarrow \frac{\partial^2}{\partial z^2}\left(\frac{e^{-\gamma r}}{r}\right)_{z=0} = f(r)\,.\end{aligned} \tag{13.160}$$

Therefore, the final expression for E_ψ and H_z will come in the form

$$\vec{E}_\psi = i\omega\mu_0 . \frac{-2m}{\gamma_1^2\rho^4}\left[3 - e^{\gamma\rho}\left(3 + 3\gamma\rho + \gamma^2\rho^2\right)\right] \tag{13.161}$$

and

$$H_z = \frac{2m}{\gamma_1^2\rho^3}\left[9 - e^{\gamma\rho}\left(9 + 9\gamma\rho + 4\gamma^2\rho^2 + \gamma^3\rho^3\right)\right]. \tag{13.162}$$

13.7 Electromagnetic Field due to an Oscillating Horizontal Magnetic Dipole Placed on the Surface of the Earth

Sommerfeld first suggested that if the vector potential for horizontal magnetic dipole is taken as

$$\vec{\Pi} = \vec{a}_x \vec{\Pi}_x,$$

It will lead to absurd result i.e., $\gamma_0 = \gamma_1$, and the vector potential must be taken in the form as

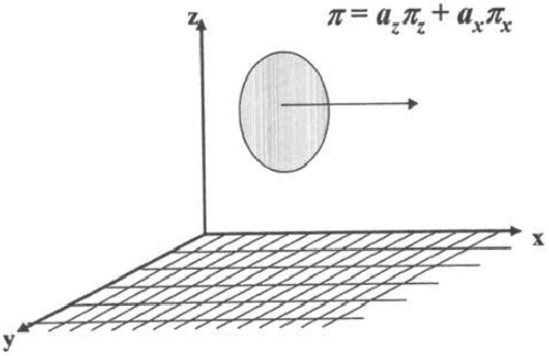


Fig. 13.10. Horizontal oscillating magnetic dipole on the surface of the earth

$$\vec{\Pi} = \vec{a}_x\vec{\Pi}_x + \vec{a}_z\vec{\Pi}_z \tag{13.163}$$

i.e. $\vec{\Pi}$ must have two components $\vec{\Pi}_x$ and $\vec{\Pi}_z$.
In this problem the plane of symmetry is the X - Z plane (Fig. 13.10) and the axis of symmetry does not exist. Therefore the wave equation is

$$\frac{\partial^2\vec{\Pi}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\vec{\Pi}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\vec{\Pi}}{\partial\psi^2} + \frac{\partial^2\vec{\Pi}}{\partial z^2} = \gamma^2\vec{\Pi} \tag{13.164}$$

in cylindrical polar coordinate and $\vec{\Pi} = f(\rho, \psi, z)$. Using the method of separation of variables we can write

$$\vec{\Pi} = R(\rho)\,\Psi(\psi)\,Z(z) \tag{13.165}$$

Now let

$$\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2} = -m^2 \tag{13.166}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = n^2 + \gamma^2 \tag{13.167}$$

Substituting in (13.164) we get

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} + \left(n^2 - \frac{m^2}{\rho^2}\right)R = 0 \tag{13.168}$$

The solutions are

(a) $\cos m\psi \quad \sin\ m\psi$
(b) $e^{\pm z\sqrt{n^2+\gamma^2}}$
(c) $J_m(n\rho), Y_m(n\rho)$ (13.169)

Since x z plane is a plane of symmetry, therefore $\cos m\psi$ is a solution and $\sin m\psi$ can not be a solution. As a result the expression for the vector potential is

$$\vec{\Pi} = \sum_{m=0}^{\alpha} \int_0^{\alpha} \cos m\psi \left(f_n e^{-z\sqrt{n^2+\gamma^2}} + g_n e^{z\sqrt{n^2+\gamma^2}} \right) J_m(n\rho)\, dn. \tag{13.170}$$

Because $Y_m(n,\rho)$ is not a good potential function as discussed in Chap. 7. It is assumed that $\vec{\Pi}_x$ behave in a similar way as $\vec{\Pi}_z$ behaved in the case of a vertical oscillating electromagnetic dipole (Sect. 13.3). So the vector potential will be independent of the azimuthal angle ψ and m will be zero. The general solution for $\vec{\Pi}_x$ will be

$$\vec{\Pi}_x = \int_0^{\alpha} \left(f_n e^{-z\sqrt{n^2+\gamma^2}} + g_n e^{z\sqrt{n^2+\gamma^2}} \right) J_0(n\rho)\, dn. \tag{13.171}$$

Since

$$\vec{E}_x = -\gamma^2 \vec{\Pi}_x + \frac{\partial}{\partial x}\left(\frac{\partial \vec{\Pi}_x}{\partial x} + \frac{\partial \vec{\Pi}_z}{\partial z} \right) \tag{13.172}$$

We can write the x component of the electric field in the two media as follows.

$$\vec{E}_{0x} = -\gamma_0^2 \vec{\Pi}_{0x} + \frac{\partial}{\partial x}\left(\frac{\partial \vec{\Pi}_{0x}}{\partial x} + \frac{\partial \vec{\Pi}_{0z}}{\partial z} \right) \tag{13.173}$$

$$\vec{E}_{1x} = -\gamma_1^2 \vec{\Pi}_{1x} + \frac{\partial}{\partial x}\left(\frac{\partial \vec{\Pi}_{1x}}{\partial x} + \frac{\partial \vec{\Pi}_{1z}}{\partial z} \right). \tag{13.174}$$

Since the tangential component of the electric field is continuous across the boundary we get

$$\vec{E}_{0x} = \vec{E}_{1x} \text{ at } z = 0,$$

$$\gamma_0^2 \vec{\Pi}_{0x} = \gamma_1^2 \vec{\Pi}_{1x} \tag{13.175}$$

$$\frac{\partial \vec{\Pi}_{0x}}{\partial x} + \frac{\partial \vec{\Pi}_{0z}}{\partial z} = \frac{\partial \vec{\Pi}_{1x}}{\partial x} + \frac{\partial \vec{\Pi}_{1z}}{\partial z}. \tag{13.176}$$

Since

$$\frac{\partial \vec{\Pi}_x}{\partial x} = \frac{\partial \vec{\Pi}_x}{\partial \rho}.\frac{\partial \rho}{\partial x} = \frac{\partial \vec{\Pi}_x}{\partial \rho} \cos \psi_1 \tag{13.177}$$

(13.176) can be written as

$$\frac{\partial \vec{\Pi}_{0x}}{\partial \rho} \cos\psi + \frac{\partial \vec{\Pi}_{0z}}{\partial z} = \frac{\partial \vec{\Pi}_{1x}}{\partial \rho} \cos\psi + \frac{\partial \vec{\Pi}_{1z}}{\partial z} \tag{13.178}$$

In order to apply the boundary condition, we have to equate the coefficient of $\cos\psi$ on both the sides. Therefore $\vec{\Pi}_z$ must have a $\cos\psi$ term. Since the source potential has $\cos\psi$ terms, the perturbation potential will also have the $\cos\psi$ terms. Therefore $m = 1$, for the z component of the perturbation potential and it will be

$$\vec{\Pi}_z = \cos\psi \int_0^{\alpha} \left(p_n e^{-z\sqrt{n^2+\gamma^2}} + q_n e^{z\sqrt{n^2+\gamma^2}}\right) J_1(n\rho)\, dn. \tag{13.179}$$

Since the source potential is $\frac{\mathrm{m}\ \mathrm{e}^{-\gamma r}}{\mathrm{r}}$ where $\mathrm{r} = (\mathrm{z}-\mathrm{h})^2 + \rho^2$ or $(\mathrm{h}-\mathrm{z})^2 + \rho^2$ depending upon whether z is greater or less than h. The vector potentials in the first and second media can be written following the procedure laid down in the previous problem as

$$\begin{aligned}\vec{\Pi}_{0x} =& m\int_0^{\alpha} \frac{n}{n_0} e^{-|h-z|n_0}/J_0(n\rho)\, dn \\ &+ \int_0^{\alpha} \mathrm{f_n}\ \mathrm{e}^{-\mathrm{n_0 z}} \mathrm{J_0}(\mathrm{n}\rho)\, \mathrm{dn}\end{aligned} \tag{13.180}$$

$$\vec{\Pi}_{1\mathrm{x}} = \int_0^{\alpha} \mathrm{g_n} \mathrm{e}^{\mathrm{n_1 z}} \mathrm{J_0}(\mathrm{n}\rho)\, \mathrm{dn} \tag{13.181}$$

Here

$$\mathrm{n}_0^2 = \mathrm{n}^2 + \gamma_0^2$$

and

$$\mathrm{n}_1^2 = \mathrm{n}^2 + \gamma_1^2. \tag{13.182}$$

The z component vector potentials are

$$\vec{\Pi}_{0z} = \cos\psi \int_0^{\alpha} p_n e^{-n_0 z} J_1(n\rho)\, dn \tag{13.183}$$

$$\vec{\Pi}_{1z} = \cos\psi \int_0^{\alpha} q_n e^{n_1 z} J_1(n\rho)\, dn. \tag{13.184}$$

In this problem

$$\vec{\mathrm{E}}_{\mathrm{x}} = -\gamma^2 \vec{\Pi}_{\mathrm{x}} + \frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial \vec{\Pi}_{\mathrm{x}}}{\partial \mathrm{x}} + \frac{\partial \vec{\Pi}_{\mathrm{z}}}{\partial \mathrm{z}}\right) \tag{13.185}$$

$$\vec{\mathrm{E}}_{\mathrm{y}} = \frac{\partial}{\partial \mathrm{y}}\left(\frac{\partial \vec{\Pi}_{\mathrm{x}}}{\partial \mathrm{x}} + \frac{\partial \vec{\Pi}_{\mathrm{z}}}{\partial \mathrm{z}}\right) \tag{13.186}$$

$$\vec{E}_z = -\gamma^2\vec{\Pi}_z + \frac{\partial}{\partial z}\left(\frac{\partial\vec{\Pi}_x}{\partial x} + \frac{\partial\vec{\Pi}_z}{\partial z}\right). \tag{13.187}$$

$$\begin{aligned} H_x &= (\sigma + i\omega \in)\mathrm{curl}\Pi_x \\ &= (\sigma + i\omega \in)\left(\frac{\partial\Pi_z}{\partial y} - \frac{\partial\Pi_y}{\partial z}\right) \end{aligned} \tag{13.188}$$

$$H_y = (\sigma + i\omega \in)\left(\frac{\partial\Pi_x}{\partial z} - \frac{\partial\Pi_z}{\partial x}\right) \tag{13.189}$$

$$H_z = (\sigma + i\omega \in)\frac{\partial\Pi_x}{\partial y}. \tag{13.190}$$

The boundary conditions are

$$\gamma_0^2\vec{\Pi}_{0x} = \gamma_1^2\vec{\Pi}_{1x} \tag{13.191}$$

$$\frac{\partial\vec{\Pi}_{0x}}{\partial x} + \frac{\partial\vec{\Pi}_{0z}}{\partial z} = \frac{\partial\vec{\Pi}_{1x}}{\partial x} + \frac{\partial\vec{\Pi}_{1x}}{\partial z} \tag{13.192}$$

$$\gamma_0^2\vec{\Pi}_{oz} = \gamma_1^2\vec{\Pi}_{1z} \tag{13.193}$$

$$\frac{\gamma_0^2}{\mu_0}\left(\frac{\partial\Pi_{0x}}{\partial z} - \frac{\partial\Pi_{0z}}{\partial x}\right) = \frac{\gamma_1^2}{\mu_1}\left(\frac{\partial\Pi_{1x}}{\partial z} - \frac{\partial\Pi_{1z}}{\partial x}\right). \tag{13.194}$$

Applying these boundary conditions, we get

$$m.\frac{n}{n_0}e^{-n_0h} + f_n = g_n \tag{13.195}$$

$$\gamma_0^2 m.\frac{n}{n_0}e^{-hn_0} + \gamma_0^2 f_n = \gamma_1^2 g_n \tag{13.196}$$

$$\gamma_0^2 m.\frac{n}{n_0}.n_0e^{-hn_0} - \gamma_0^2.n_0f_n = \gamma_1^2 n_1g_n. \tag{13.197}$$

Using these boundary conditions, f_n and g_n can be obtained as

$$f_n = m.\frac{n}{n_0}.\frac{n_0 - n_1}{n_0 + n_1}e^{-n_0h} \tag{13.198}$$

$$g_n = 2m.\frac{n}{n_0 + n_1}.\frac{\gamma_0^2}{\gamma_1^2}.e^{-n_0h}. \tag{13.199}$$

Using the other boundary conditions i.e.,

$$\frac{\partial\vec{\Pi}_{0x}}{\partial x} + \frac{\partial\vec{\Pi}_{0z}}{\partial z} = \frac{\partial\vec{\Pi}_{1x}}{\partial x} + \frac{\partial\vec{\Pi}_{1z}}{\partial z}$$

and

$$\gamma_0^2\vec{\Pi}_{0z} = \gamma_1^2\vec{\Pi}_{1z},$$

we get

$$m.\frac{n}{n_0}e^{-n_0h}.n + f_n - n_0p = ng + n_1.q \tag{13.200}$$

and

$$\gamma_0^2 p = \gamma_1^2 g \tag{13.201}$$

and

$$p = m.\frac{2n^2}{n_0 + n_1}.\frac{\gamma_0^2 - \gamma_1^2}{n_0\gamma_1^2 + n_1\gamma_0^2}.e^{-n_0 h}$$
$$q = m.\frac{2n^2}{n_0 + n_1}.\frac{\gamma_0^2}{\gamma_1^2}.\frac{\gamma_0^2 - \gamma_1^2}{n_0\gamma_1^2 + n_1\gamma_0^2}e^{-n_0 h}. \tag{13.202}$$

Once all the constants are determined applying the boundary conditions, the expression for the vector potential can be determined.
When we bring the horizontal magnetic dipole on the surface of the earth i.e., $h = 0$
We get

$$p = 2m\frac{n}{n_0 + n_1}.\frac{\gamma_0^2}{\gamma_1^2} \tag{13.203}$$

and

$$q = m.\frac{2n^2}{n_0 + n_1}.\frac{\gamma_0^2}{\gamma_1^2}\frac{\gamma_0^2 - \gamma_1^2}{n_0\gamma_1^2 + n_1\gamma_0^2}. \tag{13.204}$$

Therefore

$$\vec{\Pi}_{1x} = 2m\int_0^{\alpha}\frac{n}{n_0 + n_1}.\frac{\gamma_0^2}{\gamma_1^2}.e^{n_1 z}J_0(n\rho)dn \tag{13.205}$$

and

$$\vec{\Pi}_{1z} = 2m\cos\psi\int_0^{\alpha}\frac{n^2}{n_0 + n_1}.\frac{\gamma_0^2}{\gamma_1^2}.\frac{\gamma_0^2 - \gamma_1^2}{n_0\gamma_1^2 + n_1\gamma_0^2}e^{-n_1 z}J_1(n\rho)\,dn \tag{13.206}$$

since

$$m = \frac{I.dx}{4\pi(\sigma + i\omega \in)} = \frac{Idx}{4\pi}.\frac{i\omega\mu}{\gamma_0^2}. \tag{13.207}$$

γ_0 is negligibly small in comparison to γ_1 (i.e. conductivity of the air in negligible in comparison to that of the earth) (13.205) and 13.206). We get

$$\vec{\Pi}_{1x} = \frac{Idx}{2\pi}.\frac{i\omega\mu}{\gamma_1^2}\int_0^{\alpha}\frac{n}{n + n_1}e^{n_1 z}J_0(n\rho)\,dn \tag{13.208}$$

$$\vec{\Pi}_{1z} = -\frac{Idx}{2\pi}.\frac{i\omega\mu}{\gamma_1^2}, \cos\psi\int_0^{\alpha}\frac{n}{n + n_1}e^{n,z}J_1(n\rho)\,dn \tag{13.209}$$

since

$$\frac{\partial}{\partial\rho}(J_0(n\rho) = -nJ_1(n\rho)$$

The expression for H_z is

$$H_z = -\frac{\gamma^2}{i\omega\mu}\frac{\partial\vec{\Pi}_x}{\partial y} = -\frac{\gamma^2}{i\omega\mu}\sin\psi\frac{\partial\Pi_x}{\partial\rho} \tag{13.210}$$

$$= -\frac{\gamma^2}{i\omega\mu}\sin\psi\frac{Idx}{2\pi}.\frac{i\omega\mu}{\gamma^2}\frac{\partial}{\partial\rho}\left[\frac{1}{\gamma^2\rho^3} - \frac{1}{\gamma^2}e^{-\gamma\rho}\left(\frac{1}{\rho^3}+\frac{\gamma}{\rho^2}\right)\right]$$

$$\Rightarrow \quad \vec{H}_z = -\frac{Idx}{2\pi}\sin\psi\frac{1}{\gamma^2\rho^4}\left(3 - e^{-\gamma\rho}\left(3+3\gamma\rho+\gamma^2\rho^2\right)\right). \tag{13.211}$$

When $\psi = 0, \vec{H}_z = 0$ i.e., there cannot be any vertical magnetic field along the horizontal axis of the vertical magnetic dipole. H_z is maximum when $\psi = \pi/2$ i.e., along the Y axis we get

$$H_z\,|_{\max} = -\frac{Idx}{2\pi}.\frac{1}{\gamma^2\rho^4}\left[3 - e^{-\gamma\rho}\left(3+3\gamma\rho+\gamma^2\rho^2\right)\right]. \tag{13.212}$$

The electric field is

$$\vec{E}_x = -\gamma^2\vec{\Pi}_x + \frac{\partial}{\partial x}\left(\frac{\partial\vec{\Pi}_x}{\partial x} + \frac{\partial\vec{\Pi}_z}{\partial z}\right). \tag{13.213}$$

Since

$$\frac{\partial}{\partial\rho}\left(J_0\left(n\rho\right)dn\right) = -n\ J_1\left(n\rho\right)$$

We can write

$$\left(\frac{\partial\vec{\Pi}_x}{\partial x}\right)_{z=0} = \frac{-\vec{I}dx}{2\pi}.\frac{i\omega\mu}{\gamma^2}.\cos\psi\int_0^\alpha -\frac{n}{n+n_1}J_1\left(n\rho\right)dn$$

$$= \frac{Idx}{2\pi}.\frac{i\omega\mu}{\gamma^2}.\cos\psi\left[\frac{1}{\gamma^2\rho^3} - \frac{1}{\gamma^2}e^{-\gamma\rho}\left(\frac{1}{\rho^3}+\frac{\gamma}{\rho^2}\right)\right]. \tag{13.214}$$

Similarly

$$\left(\frac{\partial\Pi_z}{\partial z}\right)_{z=0} = -\frac{I\vec{d}x}{2\pi}.\frac{i\omega\mu}{\gamma^2}.\cos\psi\int_0^\alpha \frac{nn_1}{n+n_1}J_1\left(n\rho\right)dn. \tag{13.215}$$

Adding (13.214) and (13.215), the integral part reduces to

$$\int_0^\alpha \frac{n^2+nn_1}{n+n_1}J_1\left(n\rho\right)dn = \int_0^\alpha n\ J_1\left(n\rho\right)dn = -\frac{\partial}{\partial\rho}\int_0^\alpha J_0\left(n\rho\right)dn. \tag{13.216}$$

Therefore

$$\vec{E}_x = -\gamma^2 \vec{\Pi}_x + \frac{\partial}{\partial \rho}\left(\frac{\partial \vec{\Pi}_x}{\partial x} + \frac{\partial \vec{\Pi}_z}{\partial z}\right)\cos\psi \tag{13.217}$$

$$\vec{E}_z = -\frac{\gamma^2 \mathrm{Idx}}{2\pi}.\frac{i\omega\mu}{\gamma^2}.\left(\frac{1}{\gamma^2\rho^3} - \frac{1}{\gamma^2}e^{-\gamma\rho}\left(\frac{1}{\rho^3} - \frac{\gamma}{\rho^2}\right)\right) + \frac{\partial^2}{\partial\rho^2}\int_0^{\alpha} J_0(n\rho)\,dn\cos^2\psi \tag{13.218}$$

Now taking

$$\vec{E}_x = -\frac{Idx}{2\pi}.\frac{i\omega\mu}{\gamma^2}.\left\{\left[\frac{1}{\gamma^2\rho^3} - \frac{1}{\gamma^2}e^{-\gamma\rho}\left(\frac{1}{\rho^3} - \frac{\gamma}{\rho^2}\right)\right] + \frac{\partial^2}{\partial\rho^2}\left(\frac{1}{\rho}\right)\right] \tag{13.219}$$

where

$$\frac{1}{\rho} = \int_0^{\alpha} J_0(n\rho)\,dn.$$

Equation (13.219) simplifies down to

$$\vec{E}_x = \frac{\mathrm{Idx}}{2\pi}.\frac{i\omega\mu}{\gamma^2\rho^3}\left[2 - \frac{3x^2}{\rho^2} - e^{-\gamma\rho}(1+\gamma\rho)\right] \tag{13.220}$$

$$\vec{E}_y = -\frac{\mathrm{Idx}}{2\pi}.\frac{i\omega\mu}{\gamma^2}, \frac{3\sin\psi\cos\psi}{\rho^3}. \tag{13.221}$$

We can express the electrical components in $\vec{E}_\psi$ and $\vec{E}_\rho$ in the form

$$\vec{E}_\rho = -\frac{Idx}{2\pi}.\frac{i\omega\mu}{\gamma^2}.\frac{1}{\rho^3}\left[\left[\left\{2 - \frac{3x^2}{\rho^2} - e^{-\gamma\rho}(1+\gamma\rho)\right\}\right]\cos\psi + \sin^2\psi\cos\psi\right]. \tag{13.222}$$

For $\psi = 90^0$, i.e., in the broad side position there will be no radial component of the electric field and the magnetic field will be maximum.
Now

$$\vec{E}_\rho = E_y\cos\psi + E_x\sin\psi$$
$$= -\frac{Idx}{2\pi}.\frac{i\omega\mu}{\gamma^2\rho^3}\left[-\left\{2 - 3\cos\phi - e^{-\gamma\rho}(1+\gamma\rho)\right\}\sin\psi + 3\cos^2\psi\sin\psi\right]. \tag{13.223}$$

$$E_\psi = E_y\cos\psi - E_x\sin\psi. \tag{13.224}$$

When $\psi = 90^0$

$$\vec{E}_\psi = \frac{Idx}{2\pi}.\frac{i\omega\mu}{\gamma^2}.\frac{1}{\rho^3}.\left[2 - e^{-\gamma\rho}\left(1+\gamma\rho\right)\right], \tag{13.225}$$

$$\vec{H}_z = \frac{Idx}{2\pi}.\frac{i\omega\mu}{\gamma^2}.\frac{1}{\rho^4}.\left[3 - e^{-\gamma\rho}\left(3+3\gamma\rho+\gamma^2\rho^2\right)\right] \tag{13.226}$$

and

$$\vec{E}_\rho = 0. \tag{13.227}$$

Thus both $\vec{E}_\psi$ and $\vec{H}_z$ have maximum values at the broad side position.
Case I: When $\gamma\rho << 1$

$$\vec{H}_z = \frac{Idx}{2\pi}.\frac{3}{\rho^2}, E\psi = \frac{Idx}{2\pi\sigma}.\frac{1}{\rho^3} \quad \text{(Static Case).} \tag{13.228}$$

Case II: When $\gamma\rho >> 1$ in the radiation zone

$$\vec{H}_z = -\frac{Idx}{2\pi}.\frac{1}{\gamma^2\rho^4}.\gamma^2\rho^2.e^{-\gamma\rho} = -\frac{Idx}{2\pi}.\frac{1}{\rho^2}e^{-\gamma\rho} \tag{13.229}$$

$$\vec{E}_\rho = \frac{Idx}{2\pi}.\frac{1}{\gamma^2\rho^4}.\gamma^2\rho^2.e^{-\gamma\rho} = -\frac{Idx}{2\pi}.\frac{1}{\rho^2}e^{-\gamma\rho}$$

$$\vec{E}_\psi = -\frac{Idx\gamma\rho}{2\pi\gamma^2}.\frac{e^{-\gamma\rho}}{\rho^3} - \frac{Idx}{2\pi\gamma}.\frac{e^{-\gamma\rho}}{\rho^2} = -\frac{Idx}{2\pi}.\frac{1}{a+ib}.\frac{e^{-a\rho}}{\rho^2}.e^{-ib\rho}. \tag{13.230}$$

$$\mathrm{Re}E_\psi = -\frac{Idx}{2\pi}.\frac{1}{\sqrt{a^2+b^2}}.\frac{e^{-\alpha\rho}}{\rho^2}\cos\left(\omega t - b\rho - \tan^{-1}\frac{b}{a}\right). \tag{13.231}$$

Thus finally we can define the impedance Z, which is given by

$$Z = \left|\frac{E_\psi}{H_z}\right| = \frac{1}{\sqrt{\omega\mu\rho}}. \tag{13.232}$$

Thus by measuring $\vec{E}_\psi$ and $\vec{H}_z$ we can measure the conductivity of the earth.

13.8 Electromagnetic Field due to a Long Line Cable Placed in an Infinite and Homogenous Medium

An infinitely long line cable is placed along the Z–direction (Fig 13.11). Therefore the vector potential will also be in the Z-direction, i.e.,

$$\vec{\Pi} = \vec{a}_z\vec{\Pi}_z. \tag{13.233}$$

For an infinitely long line cable,

$$\frac{\partial}{\partial z} = 0, \frac{\partial}{\partial\psi} = 0 \tag{13.234}$$

where ψ is the azimuthal angle.

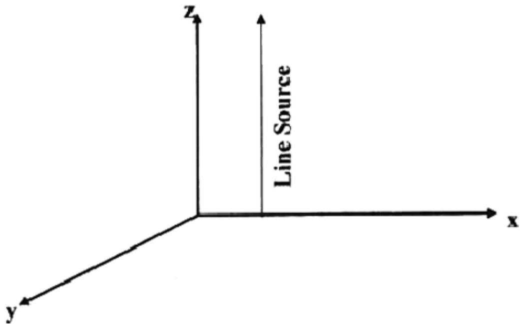


Fig. 13.11. An infinitely long line electrode is place vertically in an homogeneous and isotropic full space

The wave equation reduces to

$$\nabla^2\vec{\Pi}_z = \gamma^2\vec{\Pi}_z \tag{13.235}$$

$$\Rightarrow \frac{\partial^2\Pi_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Pi_z}{\partial\rho} - \gamma^2\Pi_z = 0. \tag{13.236}$$

The solution of the (13.236) is $I_0(\gamma\rho)$ and $K_0(\gamma\rho)$ where I_0 and K_0 are the modified Bessel's function of zero order and first and second kind. Since $I_0(\gamma\rho)$ tends to be infinity as $\rho \to \alpha$, therefore the appropriate expression for the potential is

$$\vec{\Pi}_z = bK_0(\gamma\rho). \tag{13.237}$$

In this problem, the electric field is

$$\vec{E}_z = -\gamma^2\vec{\Pi}_z + \text{grad}\frac{\partial\vec{\Pi}_z}{\partial z} = -\gamma^2\vec{\Pi}_z + \frac{\partial^2\Pi_z}{\partial z^2}. \tag{13.238}$$

Here

$$\vec{E}_\rho = 0 \text{ and } \vec{E}_\psi = 0. \tag{13.239}$$

For an infinitely long cable, there is no variation of $\vec{E}_z$ along the Z–axis i.e. $\frac{\partial}{\partial z} = 0$. Therefore

$$\vec{E}_z = -\gamma^2\vec{\Pi}_z. \tag{13.240}$$

The magnetic field

$$\vec{H} = (\sigma + i\omega \in)\,\text{curl}\vec{\Pi}. \tag{13.241}$$

In this problem

$$\vec{H}_\rho = 0 \tag{13.242}$$

$$\vec{H}_z = 0 \tag{13.243}$$

and

$$\vec{H}_\psi = -(\sigma + i\omega \in) \frac{\partial \vec{\Pi}_z}{\partial \rho}. \tag{13.244}$$

We can now write

$$\vec{E}_z = -\gamma^2 b\ K_0(\gamma\rho) \tag{13.245}$$

$$\vec{H}_\psi = -(\sigma + i\omega \in) b \frac{\partial}{\partial \rho} K_0(\gamma\rho). \tag{13.246}$$

The value of 'b' which is still arbitrary can be determined from the magnetostatic analog :

The magnetic field at a point due to a small current element is

$$H_\psi = \frac{Idz \sin\theta}{4\pi r^2} = \frac{Idz}{4\pi} . \frac{1}{\rho^2 + z^2} . \frac{\rho}{r}. \tag{13.247}$$

For an infinitely long line cable

$$H_\psi = \int_{-\alpha}^{\alpha} \frac{Idz}{4\pi} . \frac{\rho}{(\rho^2 + z^2)^{3/2}} = \frac{I}{4\pi} . \frac{2}{\rho} = \frac{I}{2\pi\rho}. \tag{13.248}$$

Now

$$E_z = -\gamma^2 b\ K_0(\gamma\rho) \tag{13.249}$$

$$\begin{aligned} H_\psi &= -(\sigma + i\omega \in) b\gamma K_0'(\gamma\rho) \\ &= (\sigma + i\omega \in) b\gamma k_1(\gamma\rho). \end{aligned} \tag{13.250}$$

We know

$$K_0(z) = -\ln \frac{zc}{2} I_0(z) + \sum_{m=1}^{\infty} \frac{z^{2m}}{2^{2m}(m!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots \frac{1}{m}\right) \tag{13.251}$$

where c is a constant and has some definite value.

$$K_0'(z) = -\frac{1}{z} I_0(z) - \ln \frac{zc}{2} I_0'(z) + \frac{\partial}{\partial z} \sum_{m=1}^{\infty} \frac{z^{2m}}{2^{2m}(m!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots \frac{1}{m}\right). \tag{13.252}$$

Now the value of

$$I_0(z) = \sum \frac{z^{2m}}{2^{2m}(m!)^2} = 1 + \frac{z^2}{2^2} + \frac{z^4}{2^4(2!)^2} + \ldots\ldots. \tag{13.253}$$

Since

$$\underset{z \to 0}{\text{Lim}} K_0'(z) = -\frac{1}{z}, \tag{13.254}$$

We can write

$$H_\psi = \frac{\sigma b\gamma}{\gamma\rho} = \frac{\sigma b}{\rho} = \frac{I}{2\pi\rho} \quad \text{when } \omega \to 0. \tag{13.255}$$

Therefore

$$\text{'b' is} = \frac{I}{2\pi\sigma}. \tag{13.256}$$

In a time varying field

$$b = \frac{1}{2\pi\left(\sigma + i\omega \in\right)}. \tag{13.257}$$

Therefore

$$\vec{\Pi}_z = bK_0\left(\gamma\rho\right) = \frac{1}{2\pi\left(\sigma + i\omega \in\right)}.K_0\left(\gamma\rho\right) \tag{13.258}$$

$$= \frac{I}{2\pi}.\frac{i\omega\mu}{\gamma^2}.K_0\left(\gamma\rho\right). \tag{13.259}$$

The existing electric and magnetic field components are

$$\vec{E}_z = -i\omega\mu\frac{I}{2\pi}K_0\left(\gamma\rho\right) \tag{13.260}$$

$$\vec{H}_\psi = -\frac{I}{2\pi}.\frac{\partial}{\partial\rho}K_0\left(\gamma\rho\right). \tag{13.261}$$

All other field components are zero i.e.,

$$E\rho = E\psi = 0 \ H\rho = H_z = 0. \tag{13.262}$$

In the radiation zone, where $\gamma\rho >> 1$

$$K_0(z) \to K_1(z) \to \sqrt{\frac{\pi}{2z}}e^{-z}. \tag{13.263}$$

the asymptotic values of E_z and H_ψ are given as

$$\vec{E}_z = -i\omega\mu.\frac{I}{2\pi}\sqrt{\frac{\pi}{2\gamma\rho}}.e^{-\gamma\rho} \tag{13.264}$$

and

$$\vec{H}_\psi = -\frac{I\gamma}{2\pi}.K_1\left(\gamma\rho\right)$$

$$= \frac{I\gamma}{2\pi}\sqrt{\frac{\pi}{2\gamma\rho}}e^{-\gamma\rho}. \tag{13.265}$$

Now substituting $\gamma = \alpha + i\beta$, we can separate the real and imaginary parts

$$\frac{1}{\sqrt{\gamma}} = \frac{1}{\sqrt{\alpha + i\beta}} = \frac{1}{4\sqrt{\alpha^2 + \beta^2}}.e^{-\frac{i}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)}.$$

Therefore the expressions for E_z and H_ψ can be written as

$$\vec{E}_z = \frac{\mu I\omega}{2\pi}.e^{\frac{-i\pi}{2}}.\sqrt{\frac{\pi}{2\rho}}.\frac{1}{4\sqrt{\alpha^2+\beta^2}}.e^{-\frac{i}{2}\tan^{-1}\frac{\beta}{\alpha}}.e^{-(\alpha+i\beta)\rho} \tag{13.266}$$

$$\vec{H}_\psi = \frac{I}{2\pi}\sqrt{\alpha^2+\beta^2}e^{i\tan^{-1}\frac{\beta}{\alpha}}.\sqrt{\frac{\pi}{2\rho}}.\frac{1}{\sqrt[4]{\alpha^2+\beta^2}}.e^{-\frac{i}{2}\tan^{-1}\frac{\beta}{\alpha}}.e^{-(\alpha+i\beta)\rho}. \tag{13.267}$$

Real components of the electric and magnetic fields are given by

$$Re\vec{E}_z = \frac{\mu I\omega}{\sqrt{8\pi}}.\frac{e^{-\alpha\rho}}{\sqrt{\rho}}.\frac{1}{\sqrt[4]{\alpha^2+\beta^2}}\cos\left(\omega t - \frac{1}{2}\tan^{-1}\frac{\beta}{\alpha} - \frac{\pi}{2} - \beta\rho\right) \tag{13.268}$$

$$Re\vec{H}_\psi = \frac{I}{\sqrt{8\pi}}.\frac{e^{-\alpha\rho}}{\sqrt{\rho}}.\frac{1}{\sqrt[4]{\alpha^2+\beta^2}}\cos\left(\omega t - \beta\rho + \frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right). \tag{13.269}$$

For negligible displacement current

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}},$$

hence

$$\left|\frac{E_z}{H_\psi}\right| = \mu\omega.\frac{1}{\sqrt{\alpha^2+\beta^2}} = \sqrt{\frac{\omega\mu}{\sigma}} = Z. \tag{13.270}$$

So we can measure conductivity of the ground by measuring E_Z and H_ψ. Since $\gamma = \sqrt{i\omega\mu\sigma}$,

$$E_z = -i\omega\mu\frac{I}{2\pi}K_0\left(P\sqrt{i}\right) \tag{13.271}$$

and

$$H_\psi = -\frac{I}{2\pi}.\frac{\partial}{\partial\rho}\left[K_0\left(P\sqrt{i}\right)\right] \tag{13.272}$$

where $P = \sqrt{\omega\mu\sigma}$. Here $K_0\left(P\sqrt{i}\right)$ and $K_0'\left(P\sqrt{i}\right)$ have both real and imaginary components as

$$K_0\left(P\sqrt{i}\right) = \ker P + \text{keip}\ P = K_0\left(Pe^{i\pi/4}\right) \tag{13.273}$$

and

$$K_0'\left(p\sqrt{i}\right) = \ker'\ p + \text{kei}'p. \tag{13.274}$$

The phase varies from - $\pi/2$ to $+\pi/2$

13.9 Electromagnetic Field due to a Long Cable on the Surface of a Homogeneous Earth

An infinitely long cable is placed along the x direction (Fig. 13.12) therefore the starting wave equation is

$$\nabla^2\vec{\Pi}_x = \gamma^2\vec{\Pi}_x \tag{13.275}$$

where $\vec{\Pi}_x$ is the vector potential.

In this problem $\vec{\Pi}_y = 0$ and $\vec{H}_z = 0$. Since there is no variation along the x – direction, the Helmholtz equation reduces to the form in Cartisian coordinate as

$$\frac{\partial^2\vec{\Pi}_x}{\partial z^2} + \frac{\partial^2\vec{\Pi}_x}{\partial y^2} = \gamma^2\Pi_x. \tag{13.276}$$

Applying the method of separation of variables, i.e.

$$\vec{\Pi}_x = Y\,(y)\,Z\,(z)$$

we get

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -n^2. \tag{13.277}$$

The solution are e^{iny}, e^{-iny}. If we take

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 + n^2 \tag{13.278}$$

The solution are

$$Z = e^{\pm\sqrt[z]{\gamma^2+n^2}}.$$

Therefore the expression for the perturbation potential is

$$\vec{\Pi}_x = \int_0^{\alpha}\left(f_n e^{-z\sqrt{\gamma^2+n^2}} g_n e^{z\sqrt{\gamma^2+n^2}}\right) e^{-iny}\,dn. \tag{13.279}$$

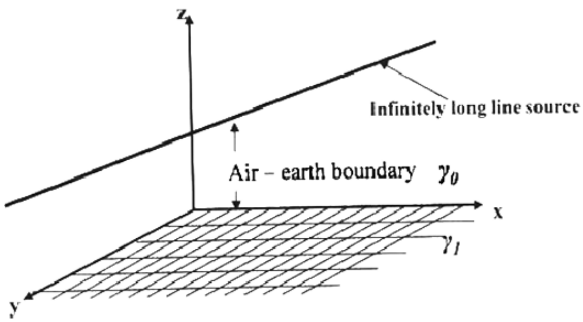


Fig. 13.12. An infinitely long line electrode carrying alternating current on the surface of the earth

The expressions for the vector potentials in the first and second media are respectively given by

$$\vec{\Pi}_{0x} = \vec{\Pi}_0 + \int_0^{\alpha} f_n e^{-Z\sqrt{n^2+\gamma_0^2}}.e^{-iny}\, dn \tag{13.280}$$

[in the first medium] and

$$\vec{\Pi}_{1x} = \int_0^{\alpha} g_n e^{Z\sqrt{n^2+\gamma_1^2}} e^{-iny}\, dn \tag{13.281}$$

[in the second medium] .

Here the source potential $\vec{\Pi}_0$ is

$$\vec{\Pi}_0 = \frac{I}{2\pi\left(\sigma + i\omega \in\right)}.K_0\left(\gamma\rho\right) = \frac{i\omega\mu I}{2\pi\gamma_0^2}K_0\left(\gamma\rho\right). \tag{13.282}$$

Substituting the value of Π_0 the source potential, the (13.280) becomes

$$\vec{\Pi}_{0x} = \frac{i\omega\mu_0 I}{2\pi\gamma_0^2}K_0\left(\gamma_0\rho\right) + \int_0^{\alpha} f_n e^{-\sqrt{n^2+z^2}.Z} e^{-iny} dn \tag{13.283}$$

and the vector potential in the second medium is

$$\vec{\Pi}_{1x} = \int_0^{\alpha} g_n e^{z\sqrt{n^2+\gamma_1^2}} e^{-iny}\, dn. \tag{13.284}$$

Now

$$\begin{aligned}\vec{E}_x &= -\gamma^2\vec{\Pi}_x + \text{grad div}\vec{\Pi}_x \\ &= -\gamma^2\vec{\Pi}_x.\end{aligned} \tag{13.285}$$

Since $\frac{\partial}{\partial x} = 0$

$$Hy = \frac{\gamma^2}{i\omega\mu}\text{curl}\Pi_x = \frac{-\gamma^2}{i\omega\mu}.\frac{\partial\vec{\Pi}_x}{\partial z} \tag{13.286}$$

$$\vec{H}_z = -\frac{\gamma^2}{i\omega\mu}\frac{\partial\vec{\Pi}_x}{\partial y} \tag{13.287}$$

because $\vec{\Pi}_y$ and $\vec{\Pi}_z$ are also zero.

The Sommerfeld formula

$$\frac{e^{-\gamma\sqrt{\rho^2+z^2}}}{\sqrt{\rho^2+z^2}} = \int_0^{\alpha} \frac{n}{\sqrt{n^2+\gamma^2}} e^{-nz} J_0\left(n\rho\right)\, dn \tag{13.288}$$

is used in time varying field to bring the source potential in the same format as perturbation potential.

It changes to the Webers formula for $\gamma = 0$ i.e.,

$$\frac{1}{\sqrt{\rho^2 + z^2}} = \int_0^{\alpha} e^{-nz} J_0(n\rho)\, dn = \frac{2}{\pi} \int_0^{\alpha} K_0(\gamma\rho)\; e^{inz}\; dn. \qquad (13.289)$$

In this problem

$$\frac{1}{\sqrt{n^2 + \gamma_0^2}} = \frac{1}{\pi} \int_{-\alpha}^{\alpha} K_0(\gamma_0 y)\, e^{-iny} dy. \qquad (13.290)$$

Here the changes are (i) from ρ to γ_0, (ii) from n to y and (iii) from z to n. The Sommerfeld formula can change to the form

$$\frac{e^{-z\sqrt{n^2+\gamma_0^2}}}{\sqrt{n^2 + \gamma_0^2}} = \frac{1}{\pi} \int_{-\alpha}^{\alpha} K_0\left(\gamma_0 \sqrt{y^2 + z^2}\right)\; e^{iny}\; dy. \qquad (13.291)$$

Now let

$$K_0(\gamma_0\rho) = K_0\left(\gamma_0 \sqrt{y^2 + z^2}\right) = \int_{-\infty}^{\infty} A_n\; e^{-iny} dn_y,$$

then

$$A_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} K_0\left(\gamma_0 \sqrt{y^2 + z^2}\right)\; e^{iny}.dy. \qquad (13.292)$$

Applying the Fourier Transform, we shall have

$$A_n = \frac{1}{2\pi}.\pi.\frac{e^{-\sqrt{n^2+\gamma_0^2}}}{\sqrt{n^2 + \gamma_0^2}}. \qquad (13.293)$$

Therefore

$$K_0\left(\gamma_0 \sqrt{y^2 + z^2}\right) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-z\sqrt{n^2+\gamma_0^2}}}{\sqrt{n^2 + \gamma_0^2}}\; e^{-iny}.dn. \qquad (13.294)$$

The vector potential $\vec{\Pi}_{0x}$ is

$$\begin{aligned}\vec{\Pi}_{0x} =& P \int_{-\infty}^{\infty} \frac{e^{-|z-h|\sqrt{n^2+\gamma_0^2}}}{\sqrt{n^2 + \gamma_0^2}}\; e^{-iny}\, dn \\ &+ \int_0^{\infty} f_n\; e^{-z\sqrt{n^2+\gamma_0^2}}.e^{-iny}\; dn, \quad \text{where } P = \frac{i\omega\mu_0 I}{2\pi\gamma_0^2}. \end{aligned} \qquad (13.295)$$

Applying the boundary conditions

$$\gamma_0^2 \vec{\Pi}_{0x} = \gamma_1^2 \vec{\Pi}_{1x} \tag{13.296}$$

$$\frac{\gamma_0^2}{\mu_0} . \frac{\partial \vec{\Pi}_{0x}}{\partial z} = \frac{\gamma_1^2}{\mu_1} . \frac{\partial \vec{\pi}_{1x}}{\partial z} \tag{13.297}$$

and substituting

$$P = \frac{i\varpi\mu_0 I}{2\pi\gamma_0^2},$$

we get

$$\frac{\gamma_0^2 . P e^{-h\sqrt{n^2+\gamma_0^2}}}{\sqrt{n^2+\gamma_0^2}} + \gamma_0^2 f = \gamma_1^2 g \tag{13.298}$$

$$\frac{\gamma_0^2 P e^{h\sqrt{n^2+\gamma_0^2}} . \sqrt{n^2+\gamma_0^2}}{\sqrt{n^2+\gamma_0^2}} + \gamma_0^2 \left(-1\right) . \sqrt{n^2+\gamma_0^2} . f_n = \gamma_1^2 \sqrt{n^2+\gamma_1^2} . g_n . \tag{13.299}$$

Substituting

$$n_0^2 = n^2 + \gamma_0^2$$
$$n_1^2 = n^2 + \gamma_1^2,$$

the values of f_n and g_n are found out to be equal to

$$f_n = P . \frac{n_0 - n_1}{n_0 \left(n_0 + n_1\right)} e^{-n_0 h} \tag{13.300}$$

$$g_n = 2P . \frac{\gamma_0^2}{\gamma_1^2} . e^{-n_0 h} . \frac{1}{n_0 + n_1} . \tag{13.301}$$

When the line source is brought on the surface of the earth. i.e., when h = 0, the vector potential on the surface of the earth is given by

$$\vec{\Pi}_{0x} = P \int_0^\infty \frac{e^{-n_0 z}}{n_0 + n_1} e^{-iny} dn + P \int_0^\infty \frac{n_0 - n_1}{n_0 \left(n_0 + n_1\right)} e^{-n_0 z} e^{-iny} dn$$
$$= 2P \int_0^\infty \frac{e^{-n_0 z}}{n_0 + n_1} e^{-iny} dn. \tag{13.302}$$

Therefore, the electric and magnetic fields are

$$\vec{E}_{0x} = -2P\gamma_0^2 \int_0^\infty \frac{e^{-n_0 z}}{n_0 + n_1} . e^{-iny} dn \tag{13.303}$$

$$\vec{H}_{0y} = -2P\frac{\gamma_0^2}{i\omega\mu}\int_0^\infty \frac{n_0 e^{-n_0 z}}{n_0 + n_1}.e^{-iny}dn \tag{13.304}$$

$$\vec{H}_{0z} = -2P\frac{\gamma_0^2}{i\omega\mu}\int_0^\infty \frac{(-in)\ e^{-n_0 z}}{n_0 + n_1}.e^{-iny}dn. \tag{13.305}$$

On the surface of the earth at z = 0 and substituting the value of P, we get

$$\vec{\mathrm{E}}_{\mathrm{x}} = \frac{\mathrm{i}\omega\mu\mathrm{I}}{\pi}\int_0^\infty \frac{\mathrm{e}^{-\mathrm{iny}}}{\mathrm{n}_0 + \mathrm{n}_1}.\mathrm{dn} \tag{13.306}$$

$$\vec{\mathrm{H}}_{\mathrm{y}} = -\frac{\mu\mathrm{I}}{\pi}\int_0^\infty \frac{\mathrm{n}_0\mathrm{e}^{-\mathrm{iny}}}{\mathrm{n}_0 + \mathrm{n}_1}.\mathrm{dn} \tag{13.307}$$

$$.\vec{\mathrm{H}}_{\mathrm{z}} = -\frac{\mu\mathrm{I}}{\pi}\int_0^\infty \frac{-\mathrm{in}}{\mathrm{n}_0 + \mathrm{n}_1}.\mathrm{e}^{-\mathrm{iny}}\ \mathrm{dn}.. \tag{13.308}$$

Since

$$\mathrm{K}_0\left(\gamma_0\sqrt{\mathrm{y}^2 + \mathrm{z}^2}\right) = \int_0^\infty \frac{\mathrm{e}^{-\mathrm{n}_0\mathrm{z}}}{\mathrm{n}_0}.\mathrm{e}^{-\mathrm{iny}}\mathrm{dn} \tag{13.309}$$

we get

$$\vec{\mathrm{E}}_{\mathrm{x}} = -\frac{\mathrm{i}\omega\mu\mathrm{I}}{\pi}\int_0^\infty \frac{(\mathrm{n}_1 - \mathrm{n}_0)\,\mathrm{e}^{-\mathrm{iny}}}{(\mathrm{n}_1^2 - \mathrm{n}_0^2)}\mathrm{dn} \tag{13.310}$$

$$= -\frac{\mathrm{i}\omega\mu\mathrm{I}}{\pi\,(\gamma_1^2 - \gamma_0^2)}\left[\int_0^\infty \mathrm{n}_1\mathrm{e}^{-\mathrm{iny}}\mathrm{dn} - \int_0^\infty \mathrm{n}_0\mathrm{e}^{-\mathrm{iny}}\mathrm{dn}\right]. \tag{13.311}$$

If we differentiate with respect to z twice, we shall get

$$\frac{\partial^2}{\partial \mathrm{z}^2}\left[\mathrm{K}_0\left(\gamma_0\sqrt{\mathrm{y}^2 + \mathrm{z}^2}\right)\right] = \int_0^\infty \mathrm{n}_0\mathrm{e}^{-\mathrm{n}_0\mathrm{z}}.\mathrm{e}^{-\mathrm{iny}}\ \mathrm{dn}. \tag{13.312}$$

Now,

$$z\overset{Lim}{\longrightarrow}0 = \int_0^\infty n_0 e^{-iny}dn$$

and

$$\mathrm{K_0}\left(\gamma_1\sqrt{\mathrm{y}^2+\mathrm{z}^2}\right)=\int_0^\infty \frac{\mathrm{e}^{-\mathrm{n_1 z}}}{\mathrm{n_1}}.\mathrm{e}^{-\mathrm{iny}}\mathrm{dn} \tag{13.313}$$

$$\frac{\partial^2}{\partial \mathrm{z}^2}\left(\mathrm{K_0}\left(\gamma_1\sqrt{\mathrm{y}^2+\mathrm{z}^2}\right)\right)_{\mathrm{z}=0}=\int_0^\infty \mathrm{n_1 e}^{-\mathrm{iny}}\ \mathrm{dn}.$$

Therefore, the expression for $\vec{\mathrm{E}}_\mathrm{x}$ can be written as

$$\begin{aligned}\vec{E}_x=-\frac{i\omega\mu I}{\pi\left(\gamma_1^2-\gamma_0^2\right)}&\left[\underset{z\to 0}{Lt}\frac{\partial^2}{\partial z^2}K_0\left(\gamma_1\sqrt{y^2+z^2}\right)\right.\\&\left.-\underset{z\to 0}{Lt}\frac{\partial^2}{\partial z^2}K_0\left(\gamma_0\sqrt{y^2+z^2}\right)\right].\end{aligned} \tag{13.314}$$

Differentiating $\mathrm{K_0}\left(\gamma_0\sqrt{\mathrm{y}^2+\mathrm{z}^2}\right)$ with respect to z, we get

$$\frac{\partial}{\partial z}K_0\left(\gamma_0\sqrt{y^2+z^2}\right)=\left(\frac{\gamma_0}{\sqrt{y^2+z^2}}\right)K_0'\left(\gamma_0\sqrt{y^2+z^2}\right) \tag{13.315}$$

$$\begin{aligned}\frac{\partial^2}{\partial z^2}K_0\left(\gamma_0\sqrt{y^2+z^2}\right)=&\frac{\gamma_0^2 z^2}{y^2+z^2}K_0''\left(\gamma_0\sqrt{y^2+z^2}\right)\\&\frac{+\gamma_0\sqrt{y^2+z^2}+\frac{z^2}{\sqrt{y^2+z^2}}}{y^2+z^2}K_0'\left(\gamma_0\sqrt{y^2+z^2}\right).\end{aligned}$$

At

$$z=0,\frac{\partial^2}{\partial z^2}K_0\left(\gamma_0\sqrt{y^2+z^2}\right)=\frac{\gamma_0}{y}K_0'\left(\gamma_0 y\right). \tag{13.316}$$

Therefore ,

$$\vec{E}_x=\frac{-i\omega\mu I}{\pi\left(\gamma_1^2-\gamma_0^2\right)}\left[\frac{\gamma_1}{y}K_0'\left(\gamma_1 y\right)-\frac{\gamma_0}{y}K_0'\left(\gamma_0 y\right)\right]. \tag{13.317}$$

Since

$$\begin{aligned}&K_0'(x)=-K_1(x)\\&\quad=\vec{\mathrm{E}}_\mathrm{x}-\frac{\mathrm{i}\omega\mu\mathrm{I}}{\pi\left(\gamma_1^2-\gamma_1^2\right)}.\frac{1}{\mathrm{y}^2}\left[\gamma_0\mathrm{yK_1}\left(\gamma_0\mathrm{y}\right)-\gamma_1\mathrm{yK_1}\left(\gamma_1\mathrm{y}\right)\right].\end{aligned} \tag{13.318}$$

When $\gamma_0\mathrm{y} << 1$

$$\underset{\mathrm{x}\to 0}{\mathrm{Lt}}\mathrm{K_1}\left(\mathrm{x}\right)=1$$

Here, $\mathrm{x}=\gamma_0\mathrm{y}$

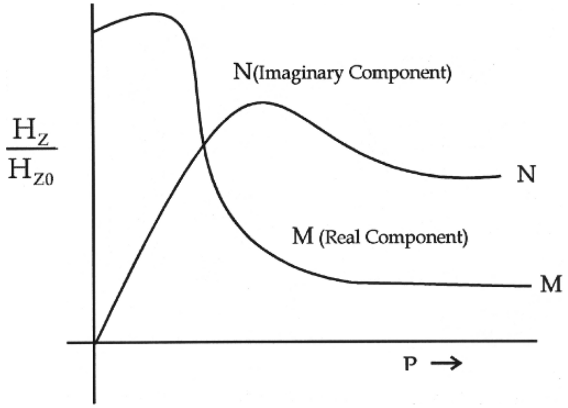


Fig. 13.13. Variation of amplitude and phase with the electromagnetic parameter P

$$\vec{E}_x = -\frac{i\omega\mu I}{\pi\gamma_1^2 y^2}\left[1 - \gamma_1 y K_1(\gamma_1 y)\right]. \tag{13.319}$$

Since

$$\vec{E}_x^0 = \frac{-i\omega\mu I}{\pi},$$

$$\frac{E_x}{E_x^0} = \frac{1}{\gamma_1^2 y^2}\left[1 - \gamma_1 y\ K_1(\gamma_1 y)\right]. \tag{13.320}$$

The general expressions for the electric field and the magnetic field are given by

$$\vec{E}_x = -\frac{i\omega\mu I}{\pi\left(\gamma_1^2 - \gamma_0^2\right)} \cdot \frac{\gamma_1}{y^2}\left[\gamma_0 y K_1(\gamma_0 y) - \gamma_1 y K_1(\gamma_1 y)\right] \tag{13.321}$$

and

$$\vec{H}_z = -\frac{I}{\pi\left(\gamma_1^2 - \gamma_0^2\right)} \cdot \frac{1}{y^3}\left[2\gamma_0 y / K_1(\gamma_0 y) + \gamma_0^2 y^2 K_0(\gamma_0 y) - 2\gamma_1 y K_1(\gamma_1 y) - \gamma_1^2 y^2 K_0(\gamma_1 y)\right]. \tag{13.322}$$

When $\gamma_0 \to 0$

$$H_z = -\frac{I}{\pi\gamma_1^2} \cdot \frac{1}{y^3}\left[2 - 2\gamma_1 y K_1(\gamma_1 y) - \gamma_1^2 y^2 K_0(\gamma_1 y)\right] \tag{13.323}$$

$$H_z^0 = -\frac{I}{2\pi} \cdot \frac{\partial}{\partial y} K_0(\gamma_1 y) \approx \frac{1}{2\pi y}. \tag{13.324}$$

Then

$$\frac{H_z}{H_z^0} = \frac{2}{\gamma_1^2 y^2}\left[2 - 2\gamma_1 y K_1(\gamma_1 y) - \gamma^2 y^2 K_0(\gamma_1 y)\right]. \tag{13.325}$$

Since

$$\gamma_1 = \sqrt{i\omega\mu\sigma},$$

Therefore

$$\gamma_1 y = y\sqrt{\omega\mu\sigma}.\sqrt{i} = P\sqrt{i}$$
$$\frac{H_z}{H_z^0} = \frac{2}{P^2}\left[2 - P\sqrt{i}K_1\left(P\sqrt{i}\right) - P^2 i K_0\left(P\sqrt{i}\right)\right]. \quad (13.326)$$

13.10 Electromagnetic Induction due to an Infinite Cylinder in an Uniform Field

An infinitely long cylinder has its axis horizontal and is directed along the z direction. The field is vertical due to an oscillating magnetic dipole of larger radius such that one gets uniform vertical field at the center of the loop (Fig. 13.14) The starting Helmholtz equation is

$$\nabla^2\vec{A} = \gamma^2\vec{A} \quad (13.327)$$

where $\vec{A}$ is a vector potential. Here the field is assumed to be harmonic, i.e. $H = He^{i\omega t}$. Since the cylinder is infinitely long, $\frac{\partial}{\partial z} = 0$.

From Maxwell's equation, we can write

$$\operatorname{curl}\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} = (\sigma + i\omega \in)\vec{E}. \quad (13.328)$$

We define

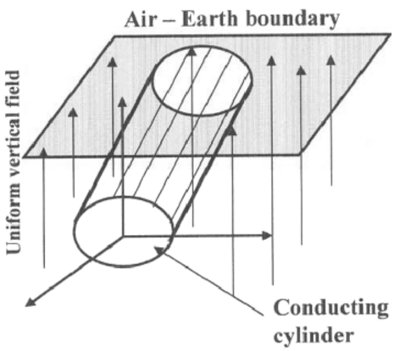


Fig. 13.14. An infinitely conducting cylinder buried inside a earth in the presence of a vertical uniform field

$$\vec{\mathrm{H}} = \mathrm{curl}\,\vec{\mathrm{A}} \tag{13.329}$$

$$\Rightarrow curl\ curl\,\vec{A} = (\sigma + i\omega \in)\,\vec{E} \tag{13.330}$$

$$\Rightarrow grad\ div\,\vec{A} - \nabla^2\vec{A} = (\sigma + i\omega \in)\,\vec{E} \tag{13.331}$$

$$\Rightarrow grad\ div\,\vec{A} - i\omega\mu\,(\sigma + i\omega \in)\,\vec{A} = (\sigma + i\omega \in)\,\vec{E} \tag{13.332}$$

$$\Rightarrow \vec{E} = -i\omega\mu\vec{A} + \frac{1}{\sigma + i\omega \in} grad\ div\,\vec{A}. \tag{13.333}$$

We can write the magnetic field components from (13.329) as

$$\vec{\mathrm{H}}_{\mathrm{r}} = \frac{1}{\mathrm{r}}\frac{\partial\vec{\mathrm{A}}_{\mathrm{z}}}{\partial\psi} - \frac{\partial\vec{\mathrm{A}}_{\psi}}{\partial\mathrm{z}} \tag{13.334}$$

$$\vec{\mathrm{H}}\psi = \frac{\partial\vec{\mathrm{A}}_{\mathrm{r}}}{\partial\mathrm{z}} - \frac{\partial\vec{\mathrm{A}}_{\mathrm{z}}}{\partial\mathrm{r}} \tag{13.335}$$

$$\vec{\mathrm{H}}_{\mathrm{z}} = \frac{1}{\mathrm{r}}\frac{\partial}{\partial\mathrm{r}}\left(\mathrm{r}\vec{\mathrm{A}}_{\psi}\right) - \frac{1}{\mathrm{r}}\frac{\partial\mathrm{A}_{\mathrm{r}}}{\partial\psi}. \tag{13.336}$$

Since $\frac{\partial}{\partial \mathrm{z}} = 0$ and $\mathrm{A}_{\psi} = 0$ and $\mathrm{A}_{\mathrm{r}} = 0$, we get

$$\vec{\mathrm{H}}_{\mathrm{r}} = \frac{1}{\mathrm{r}}\frac{\partial\vec{\mathrm{A}}_{\mathrm{z}}}{\partial\psi} \tag{13.337}$$

$$\vec{\mathrm{H}}_{\psi} = -\frac{\partial\vec{\mathrm{A}}_{\mathrm{z}}}{\partial\mathrm{r}} \tag{13.338}$$

$$\vec{\mathrm{H}}_{\mathrm{z}} = 0. \tag{13.339}$$

The wave equation can be written in the form

$$\frac{\partial^2\vec{A}_z}{\partial r^2} + \frac{1}{r}\frac{\partial\vec{A}_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2\vec{A}_z}{\partial\psi^2} = \gamma^2\vec{A}_z$$

which is independent of the variation along the z-direction.

Applying the method of separation of variables we have

$$\mathrm{A} = \mathrm{R}\Psi \tag{13.340}$$

$$\frac{d^2\Psi}{d\psi^2} + n^2\Psi = 0 \tag{13.341}$$

and

$$\frac{\mathrm{d}^2\mathrm{R}}{\mathrm{dr}^2} + \frac{1}{\mathrm{r}}\frac{\mathrm{dR}}{\mathrm{dr}} - \left(\gamma^2 + \frac{\mathrm{n}^2}{\mathrm{r}^2}\right)\mathrm{R} = 0. \tag{13.342}$$

The solutions are

$$\text{(i)}\quad \cos \mathrm{m}\psi, \quad \sin \mathrm{m}\psi$$

and

$$\text{(ii)} \quad \mathrm{I_n}(\gamma \mathrm{r}), \quad \mathrm{K_n}(\gamma \mathrm{r}). \tag{13.343}$$

The vector potentials outside and inside the body are given by

$$\mathrm{A_0} = \mathrm{Hr}\sin\psi + \sum\left(\mathrm{c_n}\cos \mathrm{n}\psi + \mathrm{d_n}\sin \mathrm{n}\psi\right)\mathrm{K_n}\left(\gamma \mathrm{r}\right) \tag{13.344}$$

and

$$\mathrm{A_i} = \sum\left(\mathrm{f_n}\cos \mathrm{n}\psi + \mathrm{g_n}\sin \mathrm{n}\psi\right)\mathrm{I_n}\left(\gamma \mathrm{r}\right). \tag{13.345}$$

At the boundary of the cylinder i.e., at

$$\mathrm{r} = \mathrm{a}, \vec{\mathrm{H}}\psi_0 = \vec{\mathrm{H}}\psi_1 \tag{13.346}$$

$$\vec{\mathrm{E}}_{\mathrm{z}_0} = \vec{\mathrm{E}}_{\mathrm{z}_1} \tag{13.347}$$

$$\frac{\partial \mathrm{A_0}}{\partial \mathrm{r}} = \frac{\partial \mathrm{A_1}}{\partial \mathrm{r}} \tag{13.348}$$

$$\mu_0 \mathrm{A_0} = \mu_1 \mathrm{A_1}. \tag{13.349}$$

Applying the first boundary condition, we get

$$\begin{aligned}\mu_0 \mathrm{Ha}\sin\phi + \mu_0 \sum(\mathrm{c_n}\cos \mathrm{n}\psi + \mathrm{d_n}\sin \mathrm{n}\psi)\mathrm{K_n}(\gamma_0 \mathrm{a})\\ = \mu_1 \sum(\mathrm{f_n}\cos \mathrm{n}\psi + \mathrm{g_n}\sin \mathrm{n}\psi)\mathrm{I_n}(\gamma_1 \mathrm{a}).\end{aligned} \tag{13.350}$$

Equating the coefficients of sines and cosines we get,

$$\mu_0 \mathrm{C_n}\ \mathrm{K_n}(\gamma_0 \mathrm{a}) = \mu_1 \mathrm{f_n}\mathrm{I_n}(\gamma, \mathrm{a}). \tag{13.351}$$

Since the source field does not contain any cosine term, the secondary field also should not contain any cosine terms. Therefore

$$\mathrm{c_n} = \mathrm{f_n} = 0. \tag{13.352}$$

Similarly

$$\mathrm{d_n} = \mathrm{g_n} = 0 \quad \text{except for } \mathrm{n} = 1 \tag{13.353}$$

and

$$\mu_0 \mathrm{Ha} + \mu_0 \mathrm{d_1}\ \mathrm{K_1}\left(\gamma_0 \mathrm{a}\right) = \mu_1 \mathrm{g_1}\mathrm{I_1}\left(\gamma_1 \mathrm{a}\right). \tag{13.354}$$

From the second boundary condition

$$H + d_1\gamma_0 K_1'\left(\gamma_0 a\right) = g_1\gamma_1 I_1\left(\gamma_1 a\right). \tag{13.355}$$

From these two equations

$$d_1 = H\frac{\mu_1 I_1\left(\gamma_1 a\right) - \mu_0\gamma_0 a I_1\left(\gamma_0 a\right)}{\mu_0\gamma_1 k_1\left(\gamma_0 a\right) I_1\left(\gamma_1 a\right) - \mu_1\gamma_0 I_1\left(\gamma_1 a\right)\left(\gamma_0 a\right)}. \tag{13.356}$$

The vector potential outside the cylinder is given by

$$\vec{A}_0 = \vec{H}_r \sin\phi + d_1 \sin\phi K_1(\gamma_0 r) \tag{13.357}$$

Source field perturbation field
When

$$\gamma_0 \to 0, \mathrm{K}_1(\gamma_0 \mathrm{a}) \approx \frac{1}{\gamma_0 \mathrm{a}} \text{ and } \mathrm{K}_1'(\gamma_0 \mathrm{a}) \approx -\frac{1}{(\gamma_0 \mathrm{a})^2} \tag{13.358}$$

we get

$$d_1 = H\gamma_0 a^2 \frac{nI_1(\gamma_1 a) - \gamma a I_1'(\gamma_0 a)}{\gamma_1 a I_1'(\gamma_1 a) + nI_1(\gamma_1 a)} \tag{13.359}$$

where $\mathrm{n} = \mu/\mu_0$.
Since

$$\mathrm{I}_1^{/}(\mathrm{x}) = \mathrm{I}_0(\mathrm{x}) - \frac{1}{\mathrm{x}}\mathrm{I}_1(\mathrm{x}) \tag{13.360}$$

we can write

$$\mathrm{A}_0 = \mathrm{H}\sin\psi\left[\mathrm{r} + \frac{\mathrm{a}^2}{\mathrm{r}}\mathrm{T}\right] \tag{13.361}$$

where

$$\mathrm{T} = \frac{(\mathrm{n}+1)\,\mathrm{I}_1(\gamma_1 \mathrm{a}) - \gamma_1 \mathrm{a}\mathrm{I}_0(\gamma_1 \mathrm{a})}{\gamma_1 \mathrm{a}\ \mathrm{I}_0(\gamma_1 \mathrm{a}) + (\mathrm{n}-1)\,\mathrm{I}_1(\gamma_1 \mathrm{a})}. \tag{13.362}$$

Here

$$\vec{H}_r = \frac{1}{r}\frac{\partial \vec{A}_z}{\partial \psi} = \vec{H}\left(1 + \frac{a^2}{r^2}T\right)\cos\psi. \tag{13.363}$$

This is the field in the radial direction. Field in the azimuthal direction is

$$\vec{H}\psi = -\frac{\partial \vec{A}_z}{\partial r} = -H\left(1 - \frac{a^2}{r^2}T\right)\sin\psi. \tag{13.364}$$

Since we measure only the magnetic field in the horizontal and vertical directions using Cosψ, we get

$$\begin{aligned}
\mathrm{H}_\mathrm{x} =& \mathrm{H}_\mathrm{r}\cos\psi - \mathrm{H}_\phi \sin\psi \\
=& H\cos^2\psi + H\frac{a^2}{r^2}T\cos^2\psi + H\sin^2\psi - H\frac{a^2}{r^2}T\sin^2\psi \\
=& \mathrm{H} + \frac{Ha^2}{r^2}T\left(\cos^2\psi - \sin^2\psi\right) \\
=& \mathrm{H}\left(1 + \mathrm{Ta}^2\frac{\mathrm{h}^2 - \mathrm{y}^2}{(\mathrm{h}^2+\mathrm{y}^2)^2}\right).
\end{aligned} \tag{13.365}$$

Here y is the distance of the point of measurement from the projection of the top of the body to the surface and h is the depth of the body.

$$\begin{aligned}
\mathrm{H_y} =& \mathrm{H_r} \sin\psi + \mathrm{H_\phi} \cos\psi \\
=& \mathrm{H}\cos\psi\sin\psi + H.\frac{a^2}{r^2}T\cos\psi\sin\psi + H\frac{a^2}{r^2}T\sin\psi - H\sin\phi\cos\psi \\
=& 2\mathrm{Ha}^2.\mathrm{T}.\frac{\mathrm{hy}}{\left(\mathrm{h}^2+\mathrm{y}^2\right)^2}.
\end{aligned} \tag{13.366}$$

In the horizontal direction normal and secondary field is present. Hence the electromagnetic anomalies are

$$\frac{\partial \mathrm{H_x}}{\mathrm{H_x}} = \mathrm{Ta}^2\frac{\mathrm{h}^2-\mathrm{y}^2}{\left(\mathrm{h}^2+\mathrm{y}^2\right)^2}. \tag{13.367}$$

The anomaly in the horizontal direction is

$$\frac{\partial \mathrm{H_y}}{\mathrm{H_y}} = \mathrm{Ta}^2\frac{\mathrm{hy}}{\left(\mathrm{h}^2+\mathrm{y}^2\right)}. \tag{13.368}$$

Except the T factor, the anomaly generally depends upon the geometry of the body. We can compute the em anomaly along the profile at a constant frequency and compare with that obtained in the field and get the depth of the body. From multiple frequency observation some idea about the conductivity of the body can be made.

Now at $\mathrm{Y}=0$, $\frac{\partial \mathrm{H_x}}{\mathrm{H_x}}$ = maximum

By measuring the vertical component of the anomaly we can pin point the location of the top of the conductive body. By measuring the horizontal component and locating the position of $\left(\frac{\mathrm{dH_y}}{\mathrm{H_y}}\right)_{\max}$ from the center of the body, we can measure the depth of the body since distance of the point of inflection of the horizontal component from that of the vertical component is $\frac{\mathrm{h}}{\sqrt{3}}$ (Fig. 13.15).

13.10.1 Effect of Change in Frequency on the Response Parameter

The value of T is

$$\mathrm{T} = \frac{(\mathrm{n}+1)\,\mathrm{I}_1(\gamma_1\mathrm{a}) - \gamma_1\mathrm{a}\;\mathrm{I}_0(\gamma_1\mathrm{a})}{\gamma_1\mathrm{a}\;\mathrm{I}_0(\gamma_1\mathrm{a}) + (\mathrm{n}-1)\,\mathrm{I}_1(\gamma_1\mathrm{a})}. \tag{13.369}$$

For non magnetic body i.e. for $\mathrm{n} = \frac{\mu}{\mu_0} = 1$,

$$\mathrm{T} = \frac{2}{\gamma_1\mathrm{a}}.\frac{\mathrm{I}(\gamma_1\mathrm{a})}{\mathrm{I}_0(\gamma_1\mathrm{a})} - 1. \tag{13.370}$$

Since the frequency chosen is generally small

$$\gamma = \sqrt{i\omega\mu\left(\sigma + i\omega\in\right)} \approx \sqrt{i\omega\mu\sigma}.$$

Therefore

$$\mathrm{T} = \frac{2}{\mathrm{a}\sqrt{\mathrm{i}\omega\mu\sigma}} \cdot \frac{\mathrm{I}_1\left(\mathrm{a}\sqrt{\mathrm{i}\omega\mu\sigma}\right)}{\mathrm{I}_0\left(\mathrm{a}\sqrt{\mathrm{i}\omega\mu\sigma}\right)} - 1. \tag{13.371}$$

If we write $P = a\sqrt{\omega\mu\sigma}$ as the response parameter,

$$\begin{aligned}\mathrm{T} =& \frac{2}{\rho\sqrt{i}} \cdot \frac{I_1\left(P\sqrt{i}\right)}{I_0\left(P\sqrt{i}\right)} - 1. \\ =& \mathrm{M} - \mathrm{iN} = \mathrm{A}\ \mathrm{e}^{-\mathrm{i}\phi}.\end{aligned} \tag{13.372}$$

Since T is a complex function, it may be written as

$$|\mathrm{T}| = \sqrt{\mathrm{M}^2 + \mathrm{N}^2} \text{ and } \phi = \tan^{-1}\frac{\mathrm{N}}{\mathrm{M}}.$$

Since P, the response parameter, is a combined effect of the radius 'a' conductivity "σ", frequency "ω" and magnetic permeability "μ", the response does not give any idea about the conductivity of the body at a single frequency. Frequency spectrum for M and N reveals (Fig. 13.16) that sensitivity characteristics M reaches the zone of saturation where as N reaches zero. Therefore, the operation frequency should be in the zone where the slope of M and N or A and ϕ are maximum.

Now

$$\mathrm{T} = \frac{2}{\alpha} \cdot \frac{\mathrm{I}_1(\alpha)}{\mathrm{I}_0(\alpha)} - 1. \tag{13.373}$$

For small values of α.

$$\mathrm{I}_0(\alpha) = 1 + \left(\frac{1}{2}\alpha\right)^2 + \left(\frac{1}{2}\alpha\right)^4 \cdot \frac{1}{2^2} + \tag{13.374}$$

and

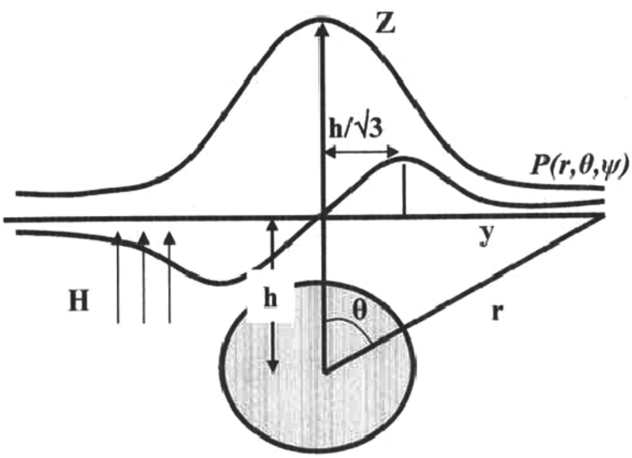


Fig. 13.15. Variation of horizontal and vertical magnetic field over a buried cylindrical inhomogeneity

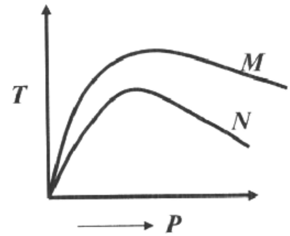


Fig. 13.16. Variation of real and imaginary part of T with electromagnetic parameter P

$$\mathrm{I}_1(\alpha) = \frac{1}{2}\alpha + \left(\frac{1}{2}\alpha\right)^3 \frac{1}{2} + \tag{13.375}$$

Therefore

$$|\mathrm{T}| \approx \frac{1}{8}\alpha^2 = \frac{1}{8}\mathrm{a}^2\omega\mu\sigma \tag{13.376}$$

This relation (13.376) holds good unless ω is very large.

13.11 Electromagnetic Response due to a Sphere in the Field of a Vertically Oscillating Magnetic Dipole

In this problem we shall discuss on the electromagnetic field in the presence of a spherical body of contrasting electrical conductivity embedded in a host rock. The source field is assumed to have originated from an oscillating vertical magnetic dipole. Vector potential approach is adapted here to solve the boundary value problem. Figure 13.17 shows the geometry of the problem.

Let

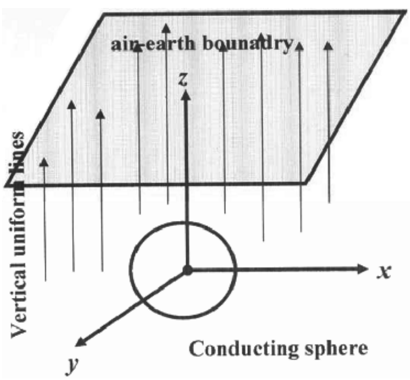


Fig. 13.17. A buried conducting sphere in the presence of a vertical uniform magnetic field

$$\vec{\mathrm{H}} = \operatorname{curl} \vec{\mathrm{A}}. \tag{13.377}$$

We can write

$$\vec{\mathrm{H}}_{\mathrm{r}} = \frac{1}{\mathrm{r}^2 \sin\theta} \left[\left(\frac{\partial}{\partial\theta} \mathrm{r} \sin\theta \vec{\mathrm{A}}_{\psi} \right) - \frac{\partial}{\partial\psi} \left(\mathrm{r}\vec{\mathrm{A}}_{\theta} \right) \right] \tag{13.378}$$

$$\vec{\mathrm{H}}_{\theta} = \frac{1}{\mathrm{r}\sin\theta} \left[\frac{\partial}{\partial\psi} \vec{\mathrm{A}}_{\mathrm{r}} - \frac{\partial}{\partial \mathrm{r}} \left(\mathrm{r} \sin \vec{\mathrm{A}}_{\psi} \right) \right] \tag{13.379}$$

and

$$\vec{\mathrm{H}}_{\psi} = \frac{1}{\mathrm{r}} \left[\frac{\partial}{\partial \mathrm{r}} \left(\mathrm{r}\vec{\mathrm{A}}_{\theta} \right) - \frac{\partial}{\partial\theta} \left(\vec{\mathrm{A}}_{\mathrm{r}} \right) \right]. \tag{13.380}$$

In this problem, the perturbation field will be independent of the azimuthal angle; therefore $\frac{\partial}{\partial\psi} = 0$. The vanishing and existing component of the vector potential and magnetic field are respectively given by $\vec{\mathrm{A}}_{\mathrm{r}} = 0$, $\vec{\mathrm{A}}_{\theta} = 0$, and $\vec{\mathrm{H}}_{\mathrm{r}} \neq 0$, $\vec{\mathrm{H}}_{\theta} \neq 0$, $\vec{\mathrm{H}}_{\psi} = 0$. Since $\vec{\mathrm{A}}_{\mathrm{r}}$ and $\vec{\mathrm{A}}_{\theta}$ are both zero, therefore $\mathrm{H}_{\psi} = 0$. In this problem

$$H_r = H \cos\theta \tag{13.381}$$

$$\vec{H}_{\theta} = -\vec{H} \sin\theta \tag{13.382}$$

$$\vec{H} = \vec{k}\vec{H}_0 \tag{13.383}$$

where $\vec{\mathrm{k}}$ is the unit vector along $\vec{\mathrm{z}}$ direction and

$$\begin{aligned} \vec{r} &= \vec{i}x + \vec{j}y + \vec{k}z \\ &= \vec{\mathrm{i}}(\mathrm{r}\sin\theta\cos\psi) + \vec{\mathrm{j}}(\mathrm{r}\sin\theta\sin\psi) + \vec{\mathrm{k}}(\mathrm{r}\cos\theta). \end{aligned} \tag{13.384}$$

Therefore

$$\frac{\mathrm{d}\vec{\mathrm{r}}}{\mathrm{dr}} = \vec{\mathrm{i}}(\sin\theta\cos\psi) + \vec{\mathrm{j}}(\sin\theta\sin\psi) + \vec{\mathrm{k}}(\cos\theta), \tag{13.385}$$

and

$$\left| \frac{\mathrm{d}\vec{\mathrm{r}}}{\mathrm{dr}} \right| = 1. \tag{13.386}$$

Since

$$\vec{e_r} = \vec{i}\sin\theta\cos\psi + \vec{j}\sin\theta\sin\psi + \vec{k}\cos\theta, \tag{13.387}$$

$$\frac{\partial \mathrm{r}}{\partial\theta} = \vec{\mathrm{i}}(\mathrm{r}\cos\theta\cos\psi) + \vec{\mathrm{j}}(\mathrm{r}\cos\theta\sin\psi) + \vec{\mathrm{k}}(-\mathrm{r}\sin\theta), \tag{13.388}$$

$$\left| \frac{\partial \mathrm{r}}{\partial\theta} \right| = \mathrm{r},$$

and

$$\vec{\mathrm{e}_{\theta}} = \vec{\mathrm{i}}\cos\theta\cos\psi + \vec{\mathrm{j}}\cos\theta\sin\psi - \vec{\mathrm{k}}\sin\theta. \tag{13.389}$$

For the $\vec{e}_\psi$ component along the azimuthal angle, we have

$$\frac{\partial r}{\partial \psi} = \vec{i}\,(-r\sin\theta\sin\psi) + \vec{j}\,(r\sin\theta\cos\psi)\,, \tag{13.390}$$

$$\left|\frac{\partial r}{\partial \psi}\right| = r\sin\theta \tag{13.391}$$

and

$$\vec{e}_\psi = -\vec{i}\sin\psi + \vec{j}\cos\psi. \tag{13.392}$$

We can write down these three equations in the matrix form as

$$\begin{bmatrix} \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \\ \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = \begin{bmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\psi \end{bmatrix}. \tag{13.393}$$

We can solve for $\vec{i}, \vec{j}$ and $\vec{k}$ using the Cramer's rule for evaluation of the matrix.

Here the denominator is

$$D = \begin{bmatrix} \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \\ \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \end{bmatrix} = 1 \tag{13.394}$$

and

$$D_3 = \begin{bmatrix} \sin\theta\cos\psi & \sin\theta\sin\psi & \vec{e}_r \\ \cos\theta\cos\psi & \cos\theta\sin\psi & \vec{e}_\theta \\ -\sin\theta & \cos\psi & \vec{e}_\psi \end{bmatrix} \tag{13.395}$$

$$= \vec{e}_r\cos\theta - \vec{e}_\theta\sin\theta. \tag{13.396}$$

Here

$$\vec{k} = \frac{|D_3|}{|D|} = \vec{e}_r\cos\theta - \vec{e}_\theta\sin\theta \tag{13.397}$$

and the source vector

$$\vec{H} = \vec{k}\vec{H}_0 = (\vec{e}_r\cos\theta - \vec{e}_\theta\sin\theta)\,H_0 = \vec{e}_r\,(H_{r\cos\theta}) + \vec{e}_\theta\,(-H_0\sin\theta)\,. \tag{13.398}$$

Thus $\vec{H}_r = \vec{H}_0\cos\theta$ and $\vec{H}_\theta = -\vec{H}_0\sin\theta$

Let us assume that the source vector has only the ψ component. We can write

$$\begin{aligned} \mathrm{curl}\vec{A} =& \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(r\sin\theta\vec{A}_\psi\right) - \frac{\partial}{\partial\psi}\left(r\vec{A}_\theta\right)\right]\vec{e}_r \\ &+ \left[\frac{\partial}{\partial\psi}\vec{A}_r - \frac{\partial}{\partial r}\left(r\sin\theta\vec{A}_\psi\right)\right]\vec{r}\,\vec{e}_\theta \\ &+ \left[\frac{\partial}{\partial r}\left(r\vec{A}_\theta\right) - \frac{\partial}{\partial\theta}\left(\vec{A}_r\right)\right] r\sin\theta\vec{e}_\psi. \end{aligned} \tag{13.399}$$

Equation (13.399) can be written as

$$\begin{aligned}\text{curl } \vec{A} = &\left[\frac{\partial}{\partial\theta}\left\{r\sin\theta\left(\frac{1}{2}H_0 r\sin\theta\right)\right\}\vec{e}_r\right.\\&\left.-\frac{\partial}{\partial r}\left\{r\sin\theta\left(\frac{1}{2}H_0 r\sin\theta\right)\right\}^{r}\vec{e}_\theta\right]\\=&\frac{1}{r^2\sin\theta}\left[\frac{1}{2}r^2 H_0\frac{\partial}{\partial\theta}\left(\sin^2\theta\right)\vec{e}_r\right.\\&\left.-\frac{1}{2}H_0\sin^2\theta\frac{\partial}{\partial r}\left(r^2\right)\vec{e}_\theta\right].\end{aligned} \tag{13.400}$$

$$\begin{aligned}\Rightarrow\text{curl } \vec{A} = &\frac{1}{r^2\sin\theta}\left[\frac{1}{2}r^2 H_0 2\sin\theta\cos\theta\vec{e}_r\right.\\&\left.-\frac{1}{2}H_0^r\sin^2\theta\frac{\partial}{\partial r}\left(r^2\right)\vec{e}_\theta\right]\\=&H_0\cos\theta\vec{e}_r - H_0\sin\theta\vec{e}_\theta.\end{aligned} \tag{13.401}$$

Therefore, the source vector potential $\vec{A}_\psi$

$$= \frac{1}{2}H_0 r\sin\theta\vec{e}_\psi.e^{i\omega t}. \tag{13.402}$$

An uniform alternating primary field can therefore be obtained from the vector potential $\vec{A}$.

For this problem, we can write the vector potential

$$\vec{A} = \vec{A}\left(r,\theta\right)\vec{e}_\psi \tag{13.403}$$

which changes to

$$\vec{A} = -\vec{A}\sin\psi\vec{i} + \vec{A}\cos\psi.\vec{j}.$$

If we now operate upon the components in a rectangular coordinate with the operator ∇^2 and recombine them, it can be shown that Helmholtz equation changes to the form.

$$\left(\nabla^2\vec{A} - \frac{\vec{A}}{\sin^2\theta}\right) = \gamma^2\vec{A} \tag{13.404}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\vec{A}}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\vec{A}}{\partial\theta}\right) - \frac{\vec{A}}{\sin^2\theta} = \gamma^2\vec{A}. \tag{13.405}$$

Now substituting

$$\vec{A} = r^{-1/2}R\left(r\right)\Theta\left(\theta\right) \tag{13.406}$$

and using the method of separation of variable we get

$$\left(1-\mu^2\right)\frac{\partial^2\Theta}{\partial\mu^2} - 2\mu\frac{\partial\Theta}{\partial\mu} + \left\{n\left(n+1\right) - \frac{1}{1-\mu^2}\right\}\Theta = 0 \tag{13.407}$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left[r^2 + \frac{n(n+1) + \frac{1}{4}}{r^2}\right] R = 0. \tag{13.408}$$

The (13.407) is a second order differential equation whose solutions are in Legendre's polynomial $P'_n(\cos\theta)$ and $Q'_n(\cos\theta)$. $P'_n(\cos\theta)$ is a better behaved potential function.

The solution of (13.408) will come in terms of the modified Bessel's function of fractional order in the form

$$I_{n+\frac{1}{2}}(\gamma r) \qquad \text{and } K_{n+\frac{1}{2}}(\gamma r)\,. \tag{13.409}$$

Therefore, the general solution of the vector potential outside the sphere is

$$\vec{A}_0 = \frac{1}{2}\vec{H} r \sin\theta + \sum r^{-\frac{1}{2}} dn I_{n+\frac{1}{2}}(\gamma_0 r) P'_n(\cos\theta) \tag{13.410}$$

$\rightarrow$ outside the sphere.

$$\vec{A} = \sum r^{-\frac{1}{2}} C_n I_{n+\frac{1}{2}}(\gamma r) P'_n(\cos\theta) \tag{13.411}$$

$\rightarrow$ inside the sphere.

Since the source potential has sin θ term, the perturbation potential will also have sin θ term only. Since $P'_1(\cos\theta) = \sin\theta$, therefore $d_n = 0$, except for $n = 1$.

Hence

$$A_0 = \frac{1}{2}\vec{H} r \sin\theta + r^{-1/2} d_1 I_{3/2}(\gamma_0 r) \sin\theta \tag{13.412}$$

and

$$A = r^{-1/2} c_1 I_{3/2}(\gamma r) \sin\theta. \tag{13.413}$$

Applying the boundary conditions

$$\frac{\partial}{\partial r}(r\ A_0) = \frac{\partial}{\partial r}(rA)$$
$$\mu_0 A_0 = \mu_1 A_1 \qquad \text{at } r = a \tag{13.414}$$

where 'a' is the radius of the sphere.

We get

$$\frac{1}{2} H a \mu_0 + \mu_0 d_1 a^{-1/2} I_{3/2}(r_0 a) = \mu_1 a^{-1/2} c_1 I_{3/2}(\gamma a) \tag{13.415}$$

$$Ha - d_1 a^{-2} = c_1 \left[-\frac{1}{2} a^{-3/2} I_{3/2}(\gamma a) + a^{-1/2} \gamma I'_{3/2}(\gamma a)\right]. \tag{13.416}$$

From the (13.415) and (13.416), we get

$$d_1 = -\frac{1}{2} H a^3 \left[\frac{\gamma a I'_{3/2}(\gamma a) - \left(2\gamma - \frac{1}{2}\right) I_{3/2}(\gamma_0 a)}{\gamma a I'_{3/2}(\gamma a) + \left(\gamma + \frac{1}{2}\right) I_{3/2}(\gamma_0 a)}\right]. \tag{13.417}$$

It is assumed that $\mu = \mu_0$.

Since

$$I'_{3/2}(\alpha) = I_{1/2}(\alpha) - \frac{3}{2\alpha} I_{3/2}(\alpha)$$

$$I_{1/2}(\alpha) = \left(\frac{2}{\pi\alpha}\right)^{1/2} \sin h\alpha$$

and

$$I_{3/2}(\alpha) = \left(\frac{2}{\pi\alpha}\right)^{1/2} \left(\cosh\alpha - \frac{1}{2}\sin h\alpha\right),$$

d_1 can be written in the form

$$\begin{aligned} d_1 &= -\frac{1}{2}.Ha^3 \frac{a \sin h\alpha - (2\gamma+1)\left(\cosh\alpha - \frac{1}{\alpha}\sin h\alpha\right)}{\alpha \sin h\alpha + (\gamma - 1)\left(\cosh\alpha - \frac{1}{\alpha}\sinh\alpha\right)} \\ &= -\frac{1}{2}Ha^3 \frac{\alpha^2 - (2\gamma+1)(\alpha\coth\alpha - 1)}{\alpha^2 + (\gamma - 1)(\alpha^2\coth\alpha - 1)} \\ &= -\frac{1}{2}Ha^3 S. \end{aligned} \tag{13.418}$$

Hence

$$\vec{A}_0 = \frac{1}{2}Hr\sin\theta - \frac{1}{2}H\frac{a^3}{r^2}S\sin\theta \tag{13.419}$$

$$\vec{H}_r = H\cos\theta - H\frac{a^3}{r^2}S\cos\theta \tag{13.420}$$

(normal field) ↓ (perturbation field) ↓

$$\vec{H}_\theta = -H\sin\theta - \frac{1}{2}\frac{Ha^3}{r^2}.S\sin\theta \tag{13.421}$$

(normal field) (perturbation field)

If we take $M = -\frac{1}{2}Ha^3S$, the radial and angular components are

$$\vec{H}_r = \frac{2\vec{M}\cos\theta}{r^3},$$

$$\vec{H}_\theta = \frac{\vec{M}\sin\theta}{r^3} \tag{13.422}$$

The effect of a sphere is as if a dipole is oriented along the Z-axis and it is placed at the centre. We can express the vertical and horizontal components as

$$\vec{H}_z = \vec{H}_r\cos\theta - \vec{H}_\theta\sin\theta \tag{13.423}$$

$$= H - \frac{1}{2}Ha^3S\left[-\frac{1}{r^3} + \frac{3h^2}{r^5}\right] \tag{13.424}$$

where

$$r^2 = y^2 + h^2 \tag{13.425}$$

and

$$\vec{H}_y = \vec{H}_r \sin\theta + \vec{H}_\theta \cos\theta = -\frac{1}{2} Ha^3 S. \frac{3hy}{r^5}. \tag{13.426}$$

Here the vertical field is only the normal component and it will be maximum at $y = 0$ and the horizontal component will be maximum i.e., $\left(\frac{\partial H_y}{H}\right)$ will be maximum at $y = \pm\frac{h}{\sqrt{2}}$ (Fig. 13.18). Hence the anomaly field depends upon the S and geometry of the body.

Here $S = M + iN$ where M and N are respectively the in phase and quadrature components.

Now for smaller values of 'α'

$$S = -\frac{3\alpha(\coth\alpha - 1) - \alpha^2}{\alpha^2} \tag{13.427}$$

$$= 1 + \frac{3}{\alpha^2} - \frac{3\coth\alpha}{\alpha}. \tag{13.428}$$

Since

$$\coth\alpha = 1 + \frac{\alpha}{3} - \frac{\alpha^3}{15} + \ldots.., \tag{13.429}$$

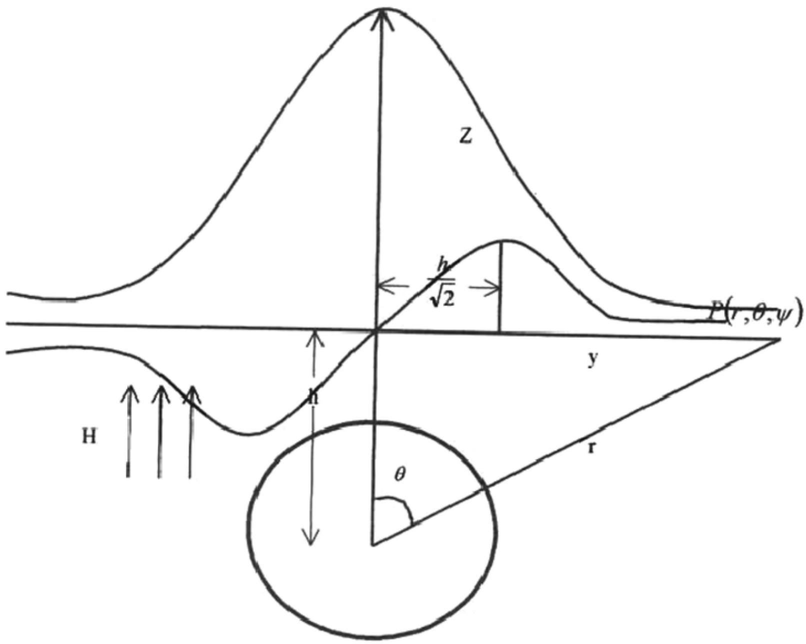


Fig. 13.18. Variation of horizontal and vertical magnetic field over a buried sphereical inhomogenities

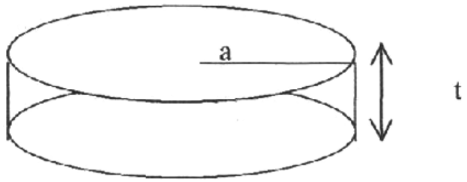


Fig. 13.19. A conducting plate of radius a and thickness t

the approximate value of S

$$S \approx \frac{\alpha^2}{15} = \frac{1}{15}\alpha^2\omega\mu\sigma.$$

For a spherical body, the multiplication factor to $\alpha^2\omega\mu\sigma$ is $\frac{1}{15}$ instead of $\frac{1}{8}$ in the case of an infinitely long cylinder. For a circular plate of finite diameter and thickness the response factor is $\frac{2}{15}$ at $\omega\mu\sigma$ where a is the radius of the plate and t is the thickness (Fig. 13.19).

The frequency response for all bodies (conductive or resistive) can be written as $|F| = k\ L^2\omega\mu\sigma$ where k is the multiplication factor.

13.12 Principle of Electrodynamic Similitude

From Maxwell equations $curl\,\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$, and $curl\,\vec{H} = \sigma\vec{E} + \in \frac{\partial E}{\partial t}$, div $\left(\mu\vec{H}\right) = 0, div\left(\in\vec{E}\right) = 0$ where there is no source in the region. Let l_0 and t_0 represent the unit of length and time. $\in_0, \mu_0, e_0$ and h_0 are the corresponding units of electrical permittivity, magnetic permeability, electric field and magnetic field. We can then write $l = l_0 l', \quad t = t_0 t', \quad E = e_0 E' and H = h_0 H'$. Curl has the dimension of $1/length$. We can then write from Maxwell's first equation.

$$\frac{e_0}{l_0}.Curl' E' = -\frac{\mu_0}{t_0}.\mu'.h_0.\frac{dH'}{dt'} \tag{13.430}$$

Here curl is a pure number.

From Maxwell's second equation we get,

$$\frac{h_0}{l_0}.Curl'\,H' = \sigma_0\sigma' e_0 E' + \frac{\in_0\in'}{t_0}e_0\frac{\partial E'}{\partial t}. \tag{13.431}$$

we can write from (13.430) and (13.431)

$$.Curl' E' = -\frac{\mu_0 l_0}{t_0}.\frac{h_0}{e_0}.\mu'\frac{\partial H'}{\partial t'} \tag{13.432}$$

and

$$curl\,H' = -\frac{\sigma_0 l_0 e_0}{h_0}.\sigma' E' + \frac{\in_0\,l_0}{t_0}.\frac{e_0}{h_0}.\in'\frac{\partial E'}{\partial t}. \tag{13.433}$$

From these two equations, we can write

$$\frac{\mu_0 l_0}{t_0}.\frac{h_0}{e_0}. = \quad \text{constant } \mathrm{k}_1 \tag{13.434}$$

$$\frac{\mu_0 l_0}{t_0}.\frac{h_0}{e_0}. = \quad \text{constant} = \mathrm{k}_2$$

and

$$\frac{\in_0 l_0 e_0}{t_0 h_0} = \quad \text{constant } = \mathrm{k}_3. \tag{13.435}$$

Since the Maxwell's equations are valid, these factors will be constants. From these three equations, we can write

$$\frac{\mu_0.l_0.h_0}{t_0 e_0}. \quad \frac{\in_0 l_0.e_0}{t_0 h_0} = \mathrm{k}_1\mathrm{x}\ \mathrm{k}_2 = \mathrm{K}_1 \tag{13.436}$$

$$\frac{\mu_0.l_0.h_0}{t_0 e_0}. \quad \frac{\in_0 l_0 e_0}{t_0 h_0} = \mathrm{k}_1\mathrm{x}\ \mathrm{k}_3 = \mathrm{K}_2 \tag{13.437}$$

Thus $\frac{\mu_0\sigma_0 l_0^2}{t_0}. \ = \text{constant and} \frac{\mu_0 \in_0 l_0^2}{t_0^2} = \text{constant.}$

From these two equations, we can write

1. $$\mu_0 f_0 \sigma_0 l_0^2 = \ \text{constant} \tag{13.438}$$

   and

2. $$\mu_0 f_0^2 \in_0 l_0^2 = \ \text{constant} \tag{13.439}$$

These are two basic equations of the electrodynamics similitude. Usually the displacement current will be significant when the frequency is of the order of 10^6. That is in the megahertz range frequency, both the equations must be satisfied for any kind of model simulation. For geophysics, where the operating frequency is in the audio frequency range, the displacement current is negligible and only one equations must be satisfied for simulation of models i.e., $\mu_0\sigma_0^2 f_0 h_0^2 = \text{constant}$. Thus $l^2 w\mu\sigma$ is the electromagnetic response parameter and is used in geophysical exploration. If M stands for the model and F stands for the fields data, then

$$\mu_M \sigma_M w_M L_M^2 = \mu_F \sigma_F w_F L_F^2 \tag{13.440}$$

where μ, σ, w, and L are respectively the magnetic permeability, electrical conductivity, angular frequency and linear dimension of the model. When we simulate the electgromagnetic model for non magnetic materials,

$$\mu_M = \mu_F = \mu_0$$

and the (13.440) reduces to

$$\mu_M^2 w_M \sigma_M = L_F^2 w_F \sigma_M \tag{13.441}$$

Once we set up the model for a particular set of $L^2\omega\mu\sigma$ = constant, we can utilise the response function by suitably multiplying it by a constant. In otherwords, we know responses due to a (i) sphere $\left(\frac{1}{15}a^2 w\mu\sigma\right)$ (ii) cylinder $\left(\frac{1}{8}a^2\omega\mu\sigma\right)$ and (iii) plate $\left(\frac{2}{15}atw\mu\sigma\right)$, we have to find out the suitable multiplication factor for bodies of other geometrical shapes for the relation $kl^2 w\mu\sigma$.

14

Green's Function

In this chapter we briefly discussed about some of the basic natures of Green's function. It is a mathematical tool having multifaceted application in various branches of applied mathematics and mathematical physics. Some of the properties and applications of Green's function in potential theory are given. Dyadics and dyadic Green's functions are defined. The nature of scalar and tensor Green's function are shown. Application of Green's function in mathematical modeling is given in Chap. 15.

14.1 Introduction

Green's function is a mathematical tool used to solve potential and nonpotential boundary value problems in scalar and vector potential field domain. It can represent a potential function which is harmonic and regular maintaining some differences in their properties with potential function at the boundaries. In the vector potential field domain it can represent a field or a vector potential. Green function can be a tensor in the vector potential field domain and both scalar and tensor Green's function can be the kernel function in the Fredhom's or Volterra,s integral equations. In the presence of a boundary surface nearby, image of a source appears in the Green function's formula. Green's functions are always associated with two points say $\mathrm{G}(\mathrm{r}, \mathrm{r}_0)$ where r and r_0 are respectively the distances of the observation point and source point from an assumed origin in the space domain (Fig. 14.1).

These points may be within a domain R or one of the points may be within a domain and the other point may be on the surface S which binds the domain. This surface may be at finite distance from the source or it may be at infinite distance. So Green's functions are generally associated with the boundary conditions and/or initial conditions. It is used as a mathematical tool for solution of elliptic, parabolic and hyperbolic differential equations (see Chap. 2) with homogeneous and inhomogeneous boundary conditions.

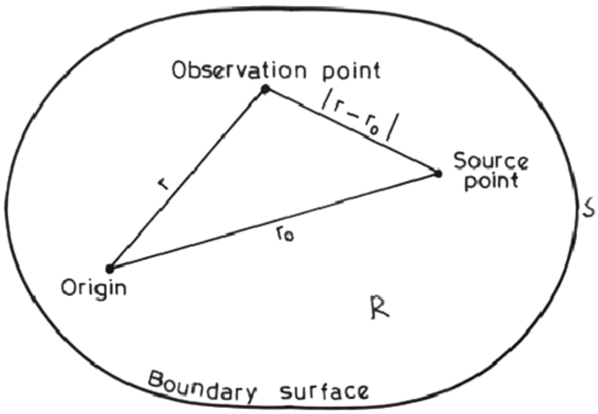


Fig. 14.1. A green's function domain with source point, observation point, boundary surface and origin

It is often used in some cases for solution of Poisson and Laplace equation with homogeneous and inhomogeneous Dirichlet, Neumann and mixed (Robin or Cauchy) boundary conditions. Once Green's function is evaluated for homogeneous equation (say Laplace equation) with homogeneous $\left(say\ \phi = 0 \text{ or } \frac{\partial\phi}{\partial \mathrm{n}} = 0 \text{ on the boundary}\right)$ boundary conditions it can be used for inhomogeneous equations with inhomogeneous boundary conditions in some cases.

The form, application and method of determining Green's function vary from problem to problem in time and space domain. It depends upon the field where it is applied as well as on the guiding equation. It vanishes on the surface of a bounded region. It can represent a potential at a point due to an unit source. It can reduce the unknowns to be determined in a potential problem Green's function is symmetric and the principle of reciprocity is valid. Some of the discontinuities in a potential function at boundaries are removed in Green's function domain. Green's function also can have discontinuity on the boundary. Green's function can be used as a mathematical tool for solution of problems related to heat conduction, electromagnetic wave propagation, em transients initial value problems, impulse response problems etc. The procedural details for evaluation of the Green's function differ in different topics of mathematical physics. So far as geophysics is concerned the major application of Green's function lies in solution of boundary value problems using Integral Equation method where non dyadic form appears in direct current flow field or any other scalar potential field and dyadic form appears in electromagnetic field.. For solution of Laplace's equation, we have other options like series solution of harmonic function, method of separation of variable , conformal transformation etc. We need not go for Greens function for all types of problems specially where evaluation of Green's function may invite tougher mathematics. For time varying EM fields, the Green's function appear in the form of a dyadic function generally since the scalar is replaced by a vector and dot product appears in place of algebraic product. Dot product with a vector source appears only if G, the Green's function is in the form of a dyadic. Since in vector potential domain both the field and the potential are vectors and Green,s function can represent both the fields and vector potentials. In vector potential domain both scalar and tensor Green's functions coexist. Green's theorem has a major role in evaluating Green's function in potential theory. Vector Green's function and tensor algebra have contribution towards deriving dyadic Green's function obtained from Helmholtz electromagnetic wave equation. It is dyadic for a vector source and nondyadic for a scalar source.

The way some similarities exist in operations between a matrix inverse and an operator inverse, an identity matrix and an idem factor or an identity operator in operator domain, some such similarities do exist between a nine component second order tensor and a dyadic. A few simple examples of determining Green's function are given.

This topic is briefly introduced in this chapter. Further details are available in Lanczos (1941, 1997), Morse and Feshbach (1953), Blakely (1996), Tai (1971), Stackgold (1968, 1979), Roach (1970), Sneider (2001), Macmillan (1958), Sobolev (1981), Ramsay (1959), Barton(1989), Van Bladel (1968), Hohmann (1971, 1975, 1983, 1988).

14.2 Delta Function

Dirac delta function was introduced by Paul Dirac. It states that a function 'r' is assumed to vanish everywhere outside the point at $r = r_0$. At the point $r = r_0$, the value of the function $\delta(x, r)$ becomes infinitely high such that the total area or volume under the curve is unity. It can be expressed as

$$\int \delta(r - r_0)\ dr = 1. \tag{14.1}$$

One can write

$$\begin{aligned} \delta_{ij} &= 1 \text{ for } i = j \\ \delta_{ij} &= 0 \text{ for } i \neq j \end{aligned} \tag{14.2}$$

where δ_{ij} are the values of an identity matrix and it is known as Kronecker delta, i.e., $I_{ij} = \delta_{ij}$, where I is the identity matrix. For a multidimensional space, we have

$$\delta(r - r_0) = \delta(x - x_0)\ \delta(y - y_0)\ \delta(z - z_0) \tag{14.3}$$

where x, y, z are the three coordinates in an Euclidian space and the coordinates of r and r_0 are respectively (x, y, z) and (x_0, y_0, z_0).

In the integral form, we have

$$\int_v \delta(r - r_0) f(r) dv = \int \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) f(x_0 y_0 z_0) dxdydz \tag{14.4}$$

14.3 Operators

Solution of a boundary value problem is an important area in mathematical physics. It initiates the mathematical formulation of forward problems which is a basic ingredient for solving an inverse problem needed for interpretation of geophysical data. Many of the forward problems are based on elliptic, parabolic and hyperbolic type of differential equations. In some cases these equations can be ordinary differential equations. These equations can be written as

$$\mathrm{L}\Phi = \mathrm{f} \tag{14.5}$$

where L is the operator. It can be a linear or nonlinear differential operator. Φ is the unknown function to be determined and f is a known function. Equation (14.5) can be a first, second or higher order ordinary or partial differential homogeneous or inhomogeneous equations with homogeneous or inhomogeneous boundary conditions. The differential operators are

$$\mathrm{L} = \frac{\mathrm{d}}{\mathrm{dx}} \tag{14.6}$$

$$\mathrm{L} = \frac{\mathrm{d}^2}{\mathrm{dx}^2} + \frac{\mathrm{d}}{\mathrm{dx}} \tag{14.7}$$

$$\mathrm{L} = \frac{\partial^2}{\partial \mathrm{x}^2} + \frac{\partial^2}{\partial \mathrm{y}^2} + \frac{\partial^2}{\partial \mathrm{z}^2} \tag{14.8}$$

Equation (14.8) is also known as Laplace or Poisson operator for second order partial differential equations and is denoted by ∇^2 and Δ. Using the same operator we can write Laplace, Poisson and Helmholtz equations as

$$\mathrm{L}\phi = \Delta\phi = \nabla^2\phi = 0 \tag{14.9}$$

$$\mathrm{L}\phi = \Delta\phi = \nabla^2\phi = \mathrm{f} \tag{14.10}$$

$$\mathrm{L}\phi = \Delta\phi = \nabla^2\vec{\mathrm{H}} = \gamma^2\mathrm{H} \tag{14.11}$$

One of the methods for solving the partial differential equation is to go for searching an inverse operator L^{-1} where $\mathrm{L}^{-1}\mathrm{L} = \mathrm{LL}^{-1} = \mathrm{I}$ where I is the identity operator. Since L is the differential operator, L^{-1} is termed as an inverse integral operator. Nature of these integral operators takes the form of Fredhom or Volterra's integral equation. The kernel of this integral is termed as the Green's function for the operator L. Assuming L to be a linear or linear differential operator, we get

$$\mathrm{L}^{-1}\phi(\mathrm{x}) = \int \mathrm{G}^{/}(\mathrm{x}, \mathrm{x}_0)\phi(\mathrm{x}_0)\mathrm{dx}_0. \tag{14.12}$$

From the relation $\mathrm{L}^{-1}\mathrm{L} = \mathrm{LL}^{-1} = \mathrm{I}$, we can write

$$\begin{aligned}\phi(\mathrm{x}) &= \mathrm{I}\phi(\mathrm{x}) = \mathrm{LL}^{-1}\phi(\mathrm{x}) = \mathrm{L}\textstyle\int \mathrm{G}^{/}(\mathrm{x}, \mathrm{x}_0) \quad \phi(\mathrm{x}_0)\mathrm{dx}_0 \\ &\Rightarrow \textstyle\int \mathrm{G}(\mathrm{x}, \mathrm{x}_0) \quad \phi(\mathrm{x}_0)\ \mathrm{dx}_0\end{aligned} \tag{14.13}$$

where $\mathrm{LG}^{/}(\mathrm{x}, \mathrm{x}_0) = \mathrm{G}(\mathrm{x}, \mathrm{x}_0)$

14.4 Adjoint and Self Adjoint Operator

In a matrix domain we can have a set of linear equations which can be expressed in a matrix form as

$$\underset{n\times m}{A}\ \underset{m\times 1}{x} = \underset{n\times 1}{b} \tag{14.14}$$

for an $n \times m$ system where n and m are respectively the number of rows and number of columns. Here A is a rectangular matrix, x is a column vector of unknowns and b is a column vector of known quantities or parameters. Its adjoint system (Lanczos 1941) is

$$\underset{m\times n}{A^T}\ \underset{n\times 1}{y} = \underset{m\times 1}{C} \tag{14.15}$$

where A^T is the transpose of A. A^T is termed as the adjoint operator of A. If $A^T = A = A^{-1}$, the matrix system is termed as a self adjoint matrix and the operator A^T is termed as the self adjoint operator. For a square symmetric matrix we get the condition $A = A^T = A^{-1}$.

For a linear or linear differential operator L, we define L^* as the complex conjugate transpose of L. Taking into account the similarities in the behaviours of a matrix and that of a linear or linear differential operator, we define the adjoint operator as

$$(\psi, L\Phi) = (L^*\psi, \Phi) \tag{14.16}$$

where $L = \frac{d}{dx}$, the differential operator. If $L = L^*$, the operator is a self adjoint operator.

14.5 Definition of a Green's Function

Green's function is an inverse integral operator in a self adjoint system. It is a response due to a source of unit strength or an unit impulse response. It becomes a kernel function in Fredhom's or Volterra's integral equations.

The Green's function is derived to find the effect of Delta function source at a field point. It's form depends upon whether the point is in free space or there is a surface in the vicinity. Let us take a linear differential equation written in the general form as

$$L(r)\ \phi(r) = f(r) \tag{14.17}$$

where $L(r)$ is a linear, self adjoint differential operator, $\phi(r)$ is an unknown function to be determined and $f(r)$ is a known inhomogeneous or nonzero term. The solution of this equation is

$$\phi(r) = L^{-1}\ f(r) \tag{14.18}$$

where L^{-1} is the inverse of the differential operator and is termed as the inverse integral operator. It is possible to define L^{-1} f(r) as

$$L^{-1}\ f(r) = \int G(r, r_0)\ f(r_0)\ dr_0 \tag{14.19}$$

where $G(r, r_0)$ is the kernel function associated with the differential operator L. As mentioned Green's function is a two point function and depends upon the position of observation point and source point in a space domain The self adjoint operator generates the solution of a linear differential equation using Green's function. Green's function is also defined as the response of a linear system to a delta function input i.e., Green's function of a system is the impulse response due to Dirac delta type of excitation i.e.,

$$LG(r, r_0) = \delta(r - r_0). \tag{14.20}$$

The response of the input $\delta(r - r_0)$ is given by $G(r, r_0)$ and the source function can be written as the combined effect of this delta function with another factor $f(r_0)$. This shows that the solution of LU = f is given by the superposition of the Green's function $G(r, r_0)$ with the factor $f(r_0)$. Thus we can write

$$f(r) = \int_{-\infty}^{\infty} \delta(r - r_0)\ f(r_0)\ dr_0. \tag{14.21}$$

It shows that

$$\phi(r) = \int\ G(r, r_0)\ f(r_0)\ dr_0. \tag{14.22}$$

Using Dirac delta function as an identity operator I and using the properties of the Dirac delta function

$$\int_{-\infty}^{\infty} \quad \delta(r_0)\ dr_0 = 1 \tag{14.23}$$

we can rewrite equation (14.22) as

$$L\phi(r) = L\int_{-\infty}^{\infty} \quad G(r, r_0)\ f(r_0)\ dr \tag{14.24}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \quad LG(r, r_0)\ f(r_0)\ dr_0 \\ &= \int_{-\infty}^{\infty} \quad \delta(r - r_0)\ f(r_0)\ dr_0 \\ &= f(r). \end{aligned} \tag{14.25}$$

If $\phi(x, y)$ is a function of two variables x and y and L is a partial differential operator, the Green's function $G(x, y, x_0, y_0)$ satisfies the equation

$$LG = \delta(x - x_0)\ (y - y_0) \tag{14.26}$$

where $\delta(x - x_0)\ (y - y_0)$ represent the Diract delta function in two dimensions. Let $G(x, y, x_0, y_0)$ satisfy certain homogeneous boundary conditions on a boundary in the x,y plane, then

$$L\phi = f(x, y) \tag{14.27}$$

satisfy the same boundary conditions and can be expressed as

$$\phi(x, y) = \int\int G(x, y, x_0, y_0)\ f(x_0, y_0)\ dx_0\ dy_0. \tag{14.28}$$

14.6 Free Space Green's Function

For Poisson's problem $\Delta\phi$ or $\nabla^2\phi = f(x)$ with $\phi(x) = 0$ at $x \to \infty$, we get $\Delta G(x, y) = \delta(x-y)$ where $G(x, y) = 0$ as $|x-y| \to \infty$. For a radially symmetric space in a spherical coordinate system (see Chaps. 7 and 13)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = \frac{1}{4\pi r}\delta(r). \tag{14.29}$$

The solution is

$$G = \frac{A}{r} + B \quad \text{for } r > 0. \tag{14.30}$$

Here A and B are constants. Since $G \to 0$ as $r \to \infty$, $B = 0$. The outward normal to the surface is

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r} = \frac{A}{R^2}. \tag{14.31}$$

On the surface at $r = R$, one gets

$$\int_0^{\pi}\int_0^{2\pi}\left.\frac{\partial G}{\partial n}\right|_{r=R} R^2\ \sin\theta\, d\theta\, d\Psi = 1 \tag{14.32}$$

where $R^2\ \sin\theta\ d\theta\ d\Psi$ is an elementary area on a spherical surface (see Chaps. 2, 3, 7, 13).
Thus,

$$A\int_0^{\pi}\int_0^{2\pi}\quad \sin\theta\ d\theta\ d\Psi = 1 \tag{14.33}$$

$$\Rightarrow A = \frac{1}{4\pi}$$

$$\phi = \frac{1}{4\pi r} = \frac{1}{4\pi}\frac{1}{[r - r_0]}. \tag{14.34}$$

14.7 Green's Function is a Potential due to a Charge of Unit Strength in Electrostatics

Potential at a point due to a number of electrostatic charges distributed over an entire space is obtained as a single algebraic superposition of potentials produced at a point by each charge (see Chap. 4) . If $q_1, q_2, q_3, \ldots.. q_n$ are located at distances $r_1, r_2, r_3, \ldots.. r_n$ respectively from the point P, the potential at P is given by

$$\phi = \frac{1}{4\pi \in}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots.. \frac{q_n}{r_n}\right) = \frac{1}{4\pi \in}\sum_{i=1}^{n}\frac{q_i}{r_i} \tag{14.35}$$

where $\in$ is the electrical permittivity.

If the charges are distributed continuously throughout a region, rather than located at discrete number of points, the regions can be divided into elements of volume Δv each containing charge $\rho\Delta v$, where ρ is the volume density of charge, the potential is then given by

$$= \frac{1}{4\pi \in}\sum_{i=1}^{i=n}\frac{\rho_i \Delta v}{r_i} \tag{14.36}$$

Equation (14.36) can be expressed in the integral form as

$$\phi = \frac{1}{4\pi \in}\int_v \frac{\rho dv}{r} \tag{14.37}$$

The integration is performed throughout the volume where ρ has certain value. Equation (14.37) can be written as

$$\phi = \int_v \rho \, G dv \tag{14.38}$$

where

$$G = \frac{1}{4\pi \in r}. \tag{14.39}$$

The function G is a potential at a point of an unit point charge and is referred to as electrostatic Green's function for an unbounded region.

If $G(r, r_0)$ is the field at the observer's point r caused by the unit point source at the point r_0, then the field at 'r' caused by the source distribution $\rho(r_0)$ is the integral of $G\rho$ over the whole range of r_0 occupied by the sources as shown in (14.38). Here G is the Green's function. The inhomogeneous source vector P is

$$P = \sum_{x_0 y_0 z_0} \rho(x_0, y_0, z_0)\ \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \tag{14.40}$$

where

$$P(x_0, y_0, z_0) = \iiint\limits_{V} \rho(r_0)\delta(r - r_0)\delta x_0 \delta y_0 \delta z_0 \tag{14.41}$$

Potential function in a three-dimensional space, in terms of Green's function, can be written as

$$\phi(x, y, z) = \iiint\limits_{V} G(x, y, z/x_0, y_0, z_0)\rho(x_0, y_0, z_0)dx_0, dy_0, dz_0 \tag{14.42}$$

where G becomes a kernel function in an integral equation.

14.8 Green's Function can Reduce the Number of unknowns to be Determined in a Potential Problem

We can use Green's function to find the solution of Laplace equation $\nabla^2\phi = 0$ inside a required domain R bounded by a closed surface S (Fig. 14.1) when the value of ϕ and $\frac{\partial\phi}{\partial n}$ are known over the surface. From Green's third formula (see Chap. 10), we get

$$\phi_\rho = \frac{1}{4\pi}\int\limits_{s}\left(\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right) ds \tag{14.43}$$

giving the value of ϕ at any point inside S in terms of the values Φ and $\frac{\partial \phi}{\partial n}$ on the surface. It is well known that one gets unique solution of Laplace equation in a bounded domain if ϕ and $\frac{\partial\phi}{\partial n}$ have prescribed values on the boundary. One needs these two values at the boundary to determine ϕ inside. Green,s function can be used to determine potential at a point on the boundary. Let ϕ be the solution of Laplace equation inside the boundary S which takes the value $\left(-\frac{1}{r}\right)$ on S. Here r is the distance of the point P from the surface (Fig. 14.2). Let $G = \Psi + \frac{1}{r}$. This G is the Green's function for the point P and the surface S. By definition, G vanishes at the boundary. One can frame a Green's function $G = \Psi + \frac{1}{r}$ where Ψ is the solution of the Laplace equation inside the S. When both ϕ and Ψ are harmonic we get

$$\int\left(\phi\frac{\partial\Psi}{\partial n} - \Psi\frac{\partial\phi}{\partial n}\right)ds = 0 \tag{14.44}$$

Adding this equation (14.44) to the Green's third formula, we get

$$\begin{aligned}\phi_\rho &= \frac{1}{4\pi}\int\left\{\left(\Psi + \frac{1}{r}\right)\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\Psi + \frac{1}{r}\right)\right\} \\ &= \frac{1}{4\pi}\int\left(G\frac{\partial G}{\partial n} - \phi\frac{\partial G}{\partial n}\right)\end{aligned} \tag{14.45}$$

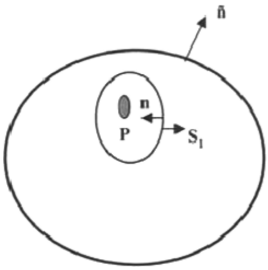


Fig. 14.2. When the point P is inside the domain R; and infinitesimally small circle surround the point P to avoid singularity

where G vanishes at the boundary. Therefore, we get

$$\phi_\rho = -\frac{1}{4\pi}\int \phi \frac{\partial G}{\partial n} ds. \tag{14.46}$$

This is the expression for the potential at any point inside S in terms of Green's function G. The properties of the Green's function are G is harmonic and its value vanishes on the boundary. In this section we have shown that we can avoid determining both ϕ and $\frac{\partial \phi}{\partial n}$ on the boundary and can determine only Green's function to find out the potential at any point inside the domain.

14.9 Green's Function has Some Relation with the Concept of Image in Potential Theory

When a source point P and an observation point Q are within the region S, we can write $G(P, Q) = \frac{1}{r} + \Psi(P, Q)$ where $\Psi(P, Q) = -\frac{1}{r}$ such that $G\,(P, Q) = 0$ on the surface as mentioned earlier. Potential at Q is $\frac{1}{r}$ due to a unit source of charge at P. The potential at Q (ξ, η, ζ) due to the image point P' of P is $-\frac{1}{r^{/}}$. Green's function for an infinite plane is (Figs. 14.3, 14.4)

$$G(P, Q) = \frac{1}{r} - \frac{1}{r^{/}}. \tag{14.47}$$

When the observation point is on the surface i.e., when $r = r'$, $G(P, Q) = 0$. If the Green's function is known, the Dirichlet's problem can be solved. In equation

$$4\pi\phi_\rho = -\int \phi(P, Q)\frac{\partial G(P, Q)}{\partial n} ds. \tag{14.48}$$

We have

$$r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 \tag{14.49}$$

and

$$r'^2 = (\xi' - x)^2 + (\eta' - y)^2 + (\zeta^{/} - z)^2. \tag{14.50}$$

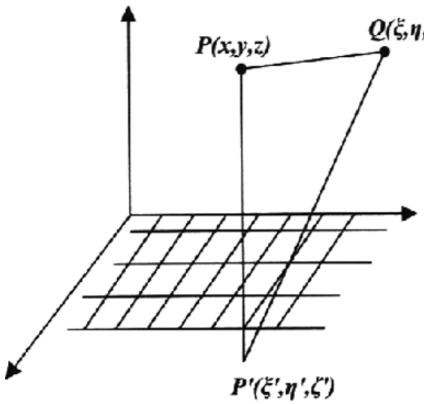


Fig. 14.3. Concept of image in analytical continuation

Here

$$\frac{\partial G}{\partial n} = \left[\frac{\partial}{\partial z}\left[\frac{1}{r}\right] - \frac{\partial}{\partial z}\left[\frac{1}{r}\right]\right] = \frac{\zeta - z}{r^3} - \frac{\zeta + z}{r/^3} \tag{14.51}$$

$$\left(\frac{\partial G}{\partial n}\right)_{z=0} = -\frac{2z}{r^3}. \tag{14.52}$$

Therefore, potential at any point is

$$4\pi\phi_\rho = 2z\int_\infty \frac{\phi(Q)}{r^3}ds$$

$$\Rightarrow \quad \phi_\rho = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\phi(\xi_0,\eta_0)d\xi_0 d\eta_0}{\left\{(\xi_0 - x)^2 + (\eta_0 - y)^2 + z^2\right\}^{1/2}}. \tag{14.53}$$

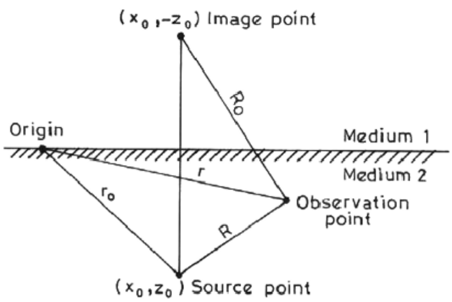


Fig. 14.4. Shows the concept of image in Green's Function domain in presence of a boundary between the two media

This is the simplest example of application of Green's function in Potential theory. Equation (14.53) shows that if a potential is prescribed on the boundary, we can find out potential at any point in the region. Thus one gets a scalar potential field upward analytical continuation formula from Green's function.

Figure 14.4 shows the position of the observation point in the presence of a source and its image and a plane horizontal boundary between a medium 1 and 2. Computation of potential at the observation point as shown in chapter 11 can also done in Green's function domain.

Problems 1 and 2 in Sect. 14.14 are some of the examples.

14.10 Reciprocity Relation of Green's Function

In this section we show that the principle of reciprocity is valid for Green's function and the symmetry exists in the behaviour of the Green's function.

If P and Q are the two points inside a region bounded by a surface S and G (P, Q) shows the value of the Green's function at Q for point P and surface S, then $G(P, Q) = G(Q, P)$ (Fig. 14.5).

Applying Green's theorem (10.8) to the region bounded by S, S_1 and S_2 where S_1 and S_2 are the surfaces of the spheres of infinitesimal small radii r_1 and r_2 having their centers at P and Q. We put

$$\phi = G' = \frac{1}{r} + \Psi$$

and

$$\phi' = G'' = \frac{1}{r'} + \Psi'$$

where r and r′ are the distances of the points of observation from the points P and Q. Here G′ and G″ are respectively the Green's functions for P and S as well as for Q and S. We have from Green's theorem

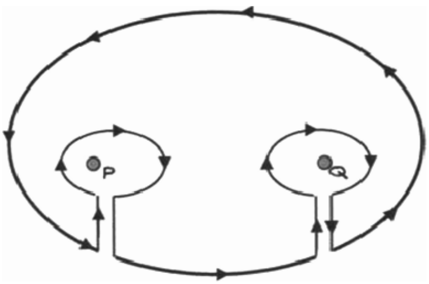


Fig. 14.5. Reciprocity Relation of Green's Function

$$\int_{s}\left(G'\frac{\partial G''}{\partial n}-G''\frac{\partial G'}{\partial n}\right)ds+\int_{s_1}\left(G'\frac{\partial G''}{\partial n}-G''\frac{\partial G'}{\partial n}\right)ds_1$$
$$\int_{s_2}\left(G'\frac{\partial G''}{\partial n}-G''\frac{\partial G'}{\partial n}\right)ds_2=\int_{v}(G'\nabla^2G''-G''\nabla^2G')dv \tag{14.54}$$

Since Ψ and Ψ' are the solutions of Laplace equation and since P and Q have been excluded from the region of volume of integration, therefore $\nabla^2G'=0, \nabla^2G''=0$. Otherwise also Green's function is a harmonic function. The right hand side of equation (14.54) is zero. By definition G' and G'' vanish on the boundary S, so that the first integral on the left is zero. In the second integral on the left, we put $ds_1 = r_1 d\omega$ where $d\omega$ is the element of the solid angle. Hence

$$\int_{s_1} G'\frac{\partial G''}{\partial n}\, ds_1=\int\left(\frac{1}{r_1}+\Psi\right)\frac{\partial G''}{\partial n}r_1^2 d\omega \tag{14.55}$$

tends to zero, when $r_1 \to 0$ and Ψ is assumed to be finite. Again

$$-\int_{s_1} G'\frac{\partial G''}{\partial n}ds_1=-\int\left(\frac{1}{r_1^2}+\Psi\frac{\partial \Psi}{\partial n}\right)r_1^2\, d\omega \tag{14.56}$$

tends to zero as $r_1 \to 0$. The integral reduces to $-4\pi G'_p$ where G'_p denotes the value of the Green's function at P for Q and S, i.e. G (Q, P). Similarly the third integral reduces to G (P, Q) which is equal to G (Q, P). Thus the principal of reciprocity is valid for Green's function i.e., $G(P,Q) = G(Q,P)$. It is also mentioned as the symmetrical property of the Green's function.

14.11 Green's Function as a Kernel Function in an Integral Equation

In this section we shall show how Green's function appears in an integral equation in electrostatics or direct current flow field. This derivation is given by Eskola (1979, 1992), Eskola and Hongisto (1981).

In direct current flow field, the govering equations are (i) Laplace equation $\nabla^2\phi = 0$ and Poisson's equation $\nabla^2\phi = \rho/\in$. The boundary conditions are

$$\phi_1=\phi_2 \text{ and } \frac{1}{\rho_1}\left(\frac{\partial\phi}{\partial n}\right)_1=\frac{1}{\rho_2}\left(\frac{\partial\phi}{\partial n}\right)_2 \tag{14.57}$$

on the surface S and $\phi_2 = \phi_3$ and $\frac{1}{\rho_2}\left(\frac{\partial\phi}{\partial n}\right)_2=\frac{1}{\rho_3}\left(\frac{\partial\phi}{\partial n}\right)_3$ at the boundary A (Fig. 14.6). The product $R\phi$ is bounded as ϕ is a regular function. The regularity condition i.e., RG is bounded as R tends to infinity. (See Chap. 10).

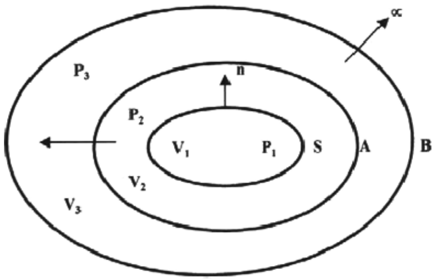


Fig. 14.6. Bounded domain with the outer boundary goes to infinity

The boundary value problems based on Laplace or Poisson's equation can be converted into an integral equation using Green's theorem and Green's function. In Green's function domain the basic equations are

$$\nabla^2 G(r, r_0) = -\delta(r - r_0) \tag{14.58}$$

where δ is the dirac delta function. The solution of which satisfy the boundary conditions.

$$\begin{aligned} G_2 &= G_3 \\ \frac{1}{\rho_2}\frac{\partial G_2}{\partial n} &= \frac{1}{\rho_3}\frac{\partial G_3}{\partial n} \end{aligned} \tag{14.59}$$

on the surface A (Fig. 14.6).

Green's function from these two equations can be written as sum of the two terms.

$$G = G_0 + G_s. \tag{14.60}$$

Here

$$G_0(r, r_0) = 1/(4\pi(r - r_0)) \tag{14.61}$$

is a free space or whole space Green's function. In

$$\nabla^2 G = -\delta(r - r_0). \tag{14.62}$$

G_0 is a singular function at $r = r_0$. This function is regular at infinity and together with its derivatives will automatically be continuous on A. The function G_s is a nonsingular function and is a solution of the Laplace equation $\nabla^2 G_s = 0$ that satisfies boundary conditions (14.59) on the surface A. The method of deducing an integral equation representation.

$$\Phi(r) = \int G(r, r_0)\rho(r_0)dv_0 \tag{14.63}$$

from the differential equation and the boundary condition is based on the application of Green second identity. Green's formulae are valid for two scalar functions as discussed in Chap. 10 which together with their first and second derivatives are continuous in a closed region and on the surface. Singularities at points and along lines can occur and the problem is then treated by suitably isolating the singularities. Substituting ϕ and G into Green's second identity and applying it to region v_2 and v_1 interior to the boundary A (Fig. 14.6). We get

$$\int_{v} \left(G_2\nabla^2\phi_2 - \phi_2\nabla^2 G_2\right) dv = \int_{A} \left(G_2\frac{\partial\phi_2}{\partial n} - \phi_2\frac{\partial G_2}{\partial n}\right) \tag{14.64}$$

where r_0 is in v_2 and r is in $v_1 + v_2$ and the operations are performed with respect to the variable R. Substituting Poisson's equation $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$ and $\nabla^2 G = -\delta(r - r_0)$, the results in the previous equation acquires the form

$$\phi_2 = \frac{1}{\epsilon_0}\int_{v} G_2\rho dv + \int_{A} \left(G_2\frac{\partial\phi_2}{\partial n} - \phi_2\frac{\partial G_2}{\partial n}\right) dA. \tag{14.65}$$

Applying Green's second identity to the region v_3 exterior to the surface A and keeping r_0 in v_2, the volume integral on the left hand side of Green's identity vanishes because ϕ and G both satisfies Laplace equation in v_3. The integral over the outermost surface B, approaches zero as B recedes to infinity. Green's second identity thus takes the form

$$0 = \int_{A} \left(G_3\frac{\partial\phi_3}{\partial n} - \phi_3\frac{\partial G_3}{\partial n}\right) ds \tag{14.66}$$

Since ϕ satisfies boundary conditions (14.59), we can eliminate unspecified values of ϕ and its normal derivatives on the boundary.

Multiplying equation (14.65) by $\frac{\rho_2}{\rho_3}$ (Escola 1992) and adding it to equation (14.66), we get

$$\begin{aligned}\phi =& \frac{1}{\epsilon_0}\int_{v} G_2\rho dv + \int_{A} \left[\left(G_2\frac{\partial\phi_2}{\partial n} - \phi_2\frac{\partial G_2}{\partial n}\right) - \frac{\rho_2}{\rho_3}G_3\frac{\partial\phi_3}{\partial n}\right. \\ &\left. + \frac{\rho_2}{\rho_3}\phi_3\frac{\partial G_3}{\partial n}\right].\end{aligned} \tag{14.67}$$

Substituting the boundary conditions (14.59) , the surface integrals of (14.65) vanishes. Interchanging the variables r as r_0 and using the reciprocity property of Green's function $G(r, r_0) = G(r_0, r)$. The potential ϕ reduces to

$$\phi(r) = \frac{1}{\epsilon_0}\int_{v} G(r, r_o)\rho(r_0)dv_0 \tag{14.68}$$

where the integration is around the volume v_1 (Fig. 14.6) and the surface S_1. ρ, the charge density in electrostatics and resistivity in direct current flow field is around the source at r_0.

Thus we have seen in this section that Green's function appears as a kernel in an integral equation.

14.12 Poisson's Equation and Green's Function

Let

$$L\phi = f(r) \tag{14.69}$$

in the domain R.

$$\phi = k(r) \text{ on the boundary S.} \tag{14.70}$$

L is the Laplacian operator Δ or ∇^2. For an n dimensional problem $G(r, r')$ is the Green's function for the source point at r and the observation point at r' where $r = (r_1, r_2, \ldots r_n)$ and $r'(r_1', r_2', r_3' \ldots r_n')$. Hence

$$LG(r, r') = \delta(r_1 - r_1')\ \delta(r_1 - r_2')\ \delta(r_3 - r_3') \ldots . + \delta(r_n - r_n') \tag{14.71}$$

for an n dimensional hyperspace. This equation is valid for all the values of r and r' in the domain R and $G(r, r') = 0$ on the boundary for all r' on S and all r in the domain R (Fig. 14.1). The n dimensional Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2} + \frac{\partial^2}{\partial r_3^2} \ldots + \frac{\partial^2}{\partial r_n^2}. \tag{14.72}$$

Since

$$\nabla(\Psi\Delta\phi - \phi\nabla\Psi) = \nabla\Psi.\nabla\phi + \Psi\nabla^2\phi - \nabla\Psi.\nabla\phi - \phi\nabla^2\Psi \tag{14.73}$$

when both Ψ and ϕ are harmonic and scalar potential functions (see Chap. 10), we can write

$$\Psi\nabla^2\phi = \phi\nabla^2\Psi + \nabla(\Psi\nabla\phi - \phi\nabla\Psi) \tag{14.74}$$

Integrating and applying Gauss's divergence theorem, we can write

$$\iiint_v \Psi\nabla^2\phi dv = \iiint_v \phi\nabla^2\Psi dv + \iint_s \left(\Psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\Psi}{\partial n}\right) ds \tag{14.75}$$

where dv and ds are respectively the elements of volume and surface. Equation (14.75) can be written as

$$\iiint_v G(r, r')f(r)dv_r = \iiint_v k(r)\delta(r - r')dv_r - \iint_s k(f)\frac{\partial G(r, r')}{\partial n_r} ds_r \tag{14.76}$$

Since $\Psi = G(r, r')$ is zero on the surface.
Hence

$$\phi(r) = \int\int_{v}\int G(r, r')f(r)dv + \int_{v}\int k(r)\frac{\partial G(r, r')}{\partial n}ds \tag{14.77}$$

For a three dimensional Euclidean space

$$\nabla^2 G(r, r^{/}) = \delta(r_1 - r_1^{/})\delta(r_2 - r_2^{/})\delta(r_3 - r_3^{/}) \tag{14.78}$$

We can then summerise on the conditions for application of Green's function for solution of Poisson's equation as

(i)

$$\nabla^2 \phi = f(r) \text{ within the domain R} \tag{14.79}$$

(ii)

$$G(r, r^{/}) = 0 \text{ on the surface S} \tag{14.80}$$

(iii)

$$\nabla^2 G(r, r^{/}) = \delta(r - r^{/}) \tag{14.81}$$

(iv)

$$\phi = f(r) \text{ on the surface S} \tag{14.82}$$

(v)

$$G(r, r^{/}) < 0 \text{ within the domain R}$$

(vi)

$$\phi(x) = \int\int_{v}\int G(r, r^{/})f(r)dv + \int\int_{s}\int k(r)\frac{\partial G(r, r^{/})}{\partial n}ds \tag{14.83}$$

(vii) G is harmonic inside the domain and outside except on the surface where it becomes zero. G approaches zero also at infinite distance from the source.

(viii) It is also a regular function at infinite distance from the source. In this section it can be shown that Green's function can be used for solution of Poisson's equations satisfying the Dirichlet, Neumann and mixed boundary conditions.

14.13 Problem 1

A highly conductive body of conductivity σ_2 and of arbitrary shape is assumed in a homogenous half space of conductivity σ_1, where $\sigma_2 \gg \sigma_1$. The conductive body is charged with direct current I. What will be the potential at any point in a medium or on the surface?

Highly conductive body has become a source of current as soon as it is charged with direct current and the outer surface of the body becomes an equipotential surface. The charged body in a medium 2 (earth)will have an

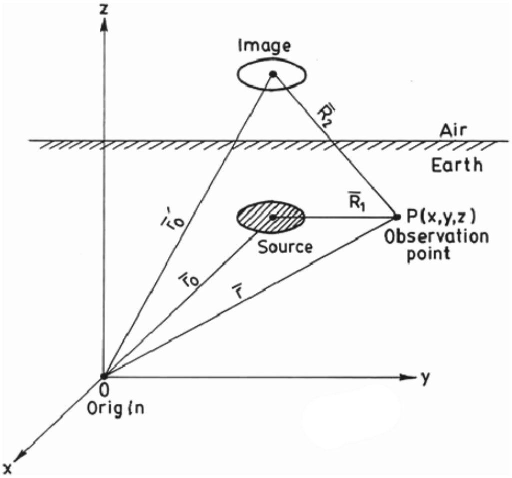


Fig. 14.7. Source and its image for solution of a problem in Green's function domain

image in medium 1(air). Figure 14.7 shows the geometry of the conductive body, the source, its image in the air, the observation point and the assumed origin of the problem. The Green's function for this problems is

$$G(r, r_0) = \frac{1}{|\bar{R}_1|} + \frac{1}{|\bar{R}_2|} = \frac{1}{|(\bar{r} - \bar{r}_0)|} + \frac{1}{\left|\left(\bar{r} - \bar{r}_0'\right)\right|} \tag{14.84}$$

where

$$\bar{R}_1 = \bar{r} - \bar{r}_0$$

and

$$\bar{R}_2 = \bar{r} - \bar{r}_0'.$$

The potential at a point P(x, y, z) due to the stationary current is given by

$$\phi(\bar{r}) = \phi_0(\bar{r}) + \frac{1}{4\pi \in_0} \int_s G(r, r_0)\rho ds \tag{14.85}$$

where ρ is the volume density of charge, $G(r, r_0)$ is the Green's function of the problem, $\in_0$ is the free space electrical permittivity and $\phi_0(\bar{r})$ is the free space potential in the absence of any kind of inhomogeneity in half space. The integration is done over the surface S of the conductor. The boundary conditions for this problem are

(i)

$$\rho = D_{n_1} - D_{n_2} \tag{14.86}$$

(see Chap. 4)

where D_{n_1} and D_{n_2} are the normal components of the displacement currents. Since $\vec{D} = \in \vec{E}$, we can write

$$\rho = \in_0 (J_{n_1}/\sigma_1 - J_{n_2}/\sigma_2) \tag{14.87}$$

where J_{n_1} and J_{n_2} are the normal components of the current density vectors and they are continuous across the boundary i.e., $J_{n_1} = J_{n_2}$. From equation (14.87), we get

$$\rho = \in_0 \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) J_n \tag{14.88}$$

Thus the equation (14.88) becomes after substituting the value of ρ

$$\phi(\bar{r}) = \phi_0(\bar{r}) + \frac{1}{4\pi \in_0} . \in_0 \int G(r, r_0) \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) J_n ds \tag{14.89}$$

Equation (14.89) can be rewritten as

$$\phi(\bar{r}) = \phi_0(\bar{r}) + \frac{1}{4\pi} \int G(r, r_0) Q\ J_n\ ds \tag{14.90}$$

where $Q = \frac{1}{\sigma_1} - \frac{1}{\sigma_2} = \frac{1}{\sigma}$ since $\frac{1}{\sigma_2} \approx 0$ for $\sigma_2 \gg \sigma_1$.
Thus :

$$\phi(\bar{r}) = \phi_0(\bar{r}) + \frac{1}{4\pi\sigma_1} \int G(r, r_0) J_n\ ds \tag{14.91}$$

$\phi_0(\bar{r})$ is the potential at a point P(x, y, z) due to an homgeneous half space for a source at a distance $\bar{r}$. Green's function $G(r, r_0)$ will vary from problem to problem. Substituting the value of $G(r, r_0)$ for this problem from (14.84) one will get an integral to be solved analytically or numerically using Gauss quadrature or Simpson's rule or Weddles rule. In case analytical solutions of the integrals are not possible one has to choose one of the said numerical tools. Author prefers 7 point Gauss Quadrature for getting a reasonably accurate answer. It is a simplest example to show how Green's function can be used for solving problems in potential theory.

14.14 Problem 2

An homogeneous half space is divided into two compartments of resistivity ρ_1 and ρ_2. A vertical wall separates the two compartments. (Fig. 14.8). Source of current is in the medium 1. The source point S_1 has images at S_2 in medium 2 and S_0 in medium 0 of resistivity ρ_0 (air). The image S_2 will have an image S_3 in medium 0. Therefore, one vertical contact in the subsurface generates three images. For solution of this problem in the Green's function domain the guiding equations are

(i)

$$\nabla^2 G_1 = -\delta(\bar{r}' - \bar{r}'_0) \tag{14.92}$$

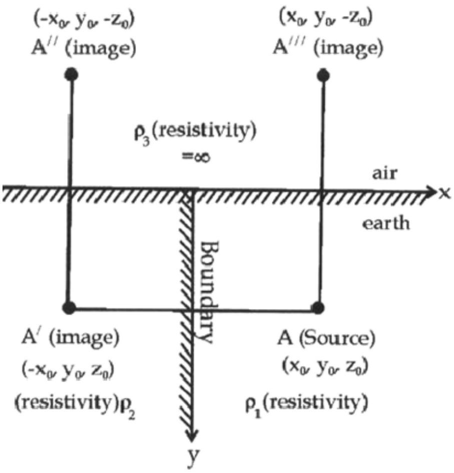


Fig. 14.8. A source in a medium of resistivity ρ_1 faces an air-earth boundary along the vertical z direction and vertical boundary between the two media of resistivities ρ_1 and ρ_2 along the horizontal × direction

(ii)

$$\nabla^2 G_2 = 0 \tag{14.93}$$

and the boundary conditions are

(i)

$$G_1 = G_2 \tag{14.94}$$

(ii)

$$\frac{1}{\rho_1}\frac{\partial G_1}{\partial n} = \frac{1}{\rho_2}\frac{\partial G_2}{\partial n} \tag{14.95}$$

On the vertical contact

$$\frac{\partial G_1}{\partial n} = \frac{\partial G_2}{\partial n} = 0 \tag{14.96}$$

On the ground surface

$$J_{n_1} = J_{n_0} = 0 \tag{14.97}$$

The Green's function in the two media are

$$G_1 = \frac{1}{4\pi}\left\{\frac{1}{|\bar{r}' - \bar{r}_0^1|} + \frac{1}{|\bar{r}' - \bar{r}_0^4|} + \frac{k}{|\bar{r}' - \bar{r}_0^2|} + \frac{k}{|\bar{r}' - \bar{r}_0^3|}\right\} \tag{14.98}$$

$$G_2 = \frac{1+k}{4\pi}\left\{\frac{1}{|\bar{r}' - \bar{r}_0^1|} + \frac{1}{|\bar{r}' - \bar{r}_0^4|}\right\}. \tag{14.99}$$

Here the coordinates of the source and images are (Fig. 14.8)

$$\begin{aligned}\bar{r}_0^1 &= (x_0, y_0, z_0)\\ \bar{r}_0^2 &= (-x_0, y_0, z_0)\\ \bar{r}_0^3 &= (-x_0, y_0, -z_0)\\ \bar{r}_0^4 &= (x_0, y_0, -z_0)\end{aligned} \tag{14.100}$$

and k is the reflection factor (see Chaps. 8 and 11)

$$k = (\rho_2 - \rho_1)/(\rho_2 + \rho_1).$$

14.15 Problem 3

A perfectly conducting body is placed in an homogeneous and isotropic earth. Find the potential at any point on the surface or at any point in the half space. A conducting homogeneous and isotropic earth is assumed as an half space over which a non conducting half space filled with air ($\rho_{air} = \infty$). A perfectly conducting body is placed in a homogeneous earth. A direct current source of strength I is applied directly to the body. In the internal region V_i = constant, i.e., $\phi_i = \phi_0$. In the external region V_e satisfies Laplace equation in all the regions of uniform conductivity, i.e., $\nabla^2\phi_e = 0$. Potentials satisfies all the boundary conditions on all the surfaces of conductivity discontinuity i.e.,

(i)

$$\phi_1 = \phi_2 \tag{14.101}$$

(ii)

$$\sigma_1 \frac{\partial \phi_1}{\partial n} = \sigma_1 \frac{\partial \phi_2}{\partial n} \tag{14.102}$$

(iii) $R\phi$ and RG are regular functions in the potential and Green's function domain at infinity.

(iv) Both potential and Green's function are harmonic functions both in the internal and external region. On the boundary ϕ is a continuous function but G is a singular function.

Starting equation of this problem in the Green's function domain is

$$\nabla^2 G(\bar{r}, \bar{r}_0) = -4\pi\delta(r - r_0) \tag{14.103}$$

It satisfy the boundary conditions specified above.

Applying Green's second identity (see Chap. 10), we get

$$\begin{aligned}&\iiint_v \left[\left(\phi_e(\bar{r}_1)\nabla^2 G(\bar{r}, r_0) - G(\bar{r}, \bar{r}_0)\right) \nabla^2 \phi_e(\bar{r}_0)\right] dv\\ &\quad= \iint_s \left[\phi_e(\bar{r}) \frac{\partial}{\partial n} G(\bar{r}, \bar{r}_0) - G(\bar{r}, r_0) \frac{\partial}{\partial n} \phi_e(\bar{r})\right] ds\end{aligned} . \tag{14.104}$$

Since the potential must be continuous across the surface of the conductor, the first integral can be written as

$$\frac{1}{4\pi}\int\int_{s}\phi_e(r_0)\frac{\partial}{\partial n}G(\bar{r},\bar{r}_0)dS_0 \tag{14.105}$$

$$\Rightarrow \frac{\phi_0}{4\pi}\int\int\frac{\partial}{\partial n}G(\bar{r},\bar{r}_0)ds_0 \tag{14.106}$$

$$\Rightarrow \frac{\phi_0}{4\pi}\int\int\int \text{div grad } G(\bar{r},\bar{r}_0)ds_0 \tag{14.107}$$

applying Gauss's divergence theorem. Therefore (14.107) can be rewritten as

$$= -\frac{\phi_0}{4\pi}\int\int_{v}\int\nabla^2 G(\bar{r},\bar{r}_0) = 0. \tag{14.108}$$

Since G is harmonic in the external region.

$$\phi_e(\bar{r}) = -\frac{1}{4\pi}\int\int_{s}G(\bar{r},\bar{r}_0)\frac{\partial}{\partial n}\phi_e(\bar{r}_0)ds_0. \tag{14.109}$$

Since $\vec{J} = \sigma\vec{E}$ and $I = \int\vec{J}.\vec{n}.ds$ (see Chap. 6)

We can write

$$I = \int_{s}\int\sigma_e(\bar{r}_0)\frac{\partial}{\partial n}.\phi_e(r_0). \tag{14.110}$$

14.16 Dyadics

Dyadic Green's function are also called tensor Green's function because it has nine components similar to that of a 3×3 second order tensor. Tai (1971) gave a detailed account about the properties of dyadics and the dyadic Green's function.

Dyad means a group of two quantities and a dyadic is a group of two vectors i.e.,

$$\vec{\vec{D}} = \vec{A}.\vec{B}. \tag{14.111}$$

Use of dyadic Green's function in solving geophysical boundary value problems using integral equation method are discussed in detail by Hohmann (1971, 1975, 1983, 1988), Weidelt (1975), Raiche (1975). Ting and Hohmann (1981) Das and Verma (1981), Beasley and Ward (1986), Meyer (1976).

A vector can be written in the cartisian coordinate as

$$\vec{D} = \sum_{1}^{3} D_i\vec{x}_i \tag{14.112}$$

where D_is are scalar components along the x, y, z directions and represent the three unit vectors along the three directions for $i = 1, 2, 3$. For three distinct vectors along the three directions, we can write

$$\overrightharpoon{D}_j = \sum_{i=1}^{3} \vec{D}_{ij}\vec{X}_i, \text{for } j = 1, 2, 3 \tag{14.113}$$

and a dyadic function is denoted by $\vec{D}$ and is defined as denoted by

$$\vec{\vec{D}} = \sum_{j=1}^{3} \vec{D}_j X_j \tag{14.114}$$

where $\vec{D}_j$ are the three vector components of $\vec{\vec{D}}$ for $j = 1, 2, 3$. We can write

$$\vec{\vec{D}}\,\vec{\vec{D}} = \sum_i \sum_j D_{ij}\vec{X}_i\vec{X}_j \tag{14.115}$$

where $\vec{D}_{ij}$ are the nine scalar components of $\vec{\vec{D}}$ and $\vec{x}_i\vec{x}_j$ are the nine pairs of the unit dyadics. Each of these dyadics are formed by a pair of two unit vectors in mutually orthogonal direction (Tai – 1971).

Therefore, we can write as

$$\vec{\vec{D}} = \begin{bmatrix} A_xB_x\vec{x}\vec{x} & A_xB_y\vec{x}\vec{y} & A_xB_z\vec{x}\vec{z} \\ A_yB_x\vec{y}\vec{x} & A_yB_y\vec{y}\vec{y} & A_yB_z\vec{y}\overleftarrow{z} \\ A_zB_x\vec{z}\vec{x} & A_zB_y\vec{z}\vec{y} & A_zB_z\vec{z}\vec{z} \end{bmatrix}. \tag{14.116}$$

Important Properties of the dyadic Green's function are

(i)

$$\vec{\vec{I}} = \vec{x}_i\vec{x}_i + \vec{x}_j\vec{x}_j + \vec{x}_k\vec{x}_k \tag{14.117}$$

where $\vec{\vec{I}}$ is the unit dyadic or idem factor. Here

$$\vec{\vec{D}}\,\vec{\vec{D}}^{-1} = I$$
$$\vec{\vec{D}}^{-1}\vec{\vec{D}} = \mathrm{I} \tag{14.118}$$

where $\vec{\vec{D}}^{-1}$ is the reciprocal of the dyadic $\vec{\vec{D}}$.

(ii) Dyadic is defined by two vector $\vec{\vec{D}} = \vec{A}\vec{B}$ as mentioned where functions, i.e. $\vec{A}$ and $\vec{B}$ are respectively defined as the anterior and posterior vectors and its transpose is given by

$$\vec{\vec{D}}^{T} = \vec{B}\vec{A} \tag{14.119}$$

Green's function is dyadic for a vector source and is nondyadic for a scalar source. For vector source also the Green's function formula is derived to find the effect of the Dirac delta function. In a similar way, as the scalar Green's function, its form depends upon whether the point is in the free space or there is a surface is the vicinity.

For a time varying vector source, the electric and magnetic potentials are vector potentials say which satisfy the Helmholtz electromagnetic wave equation as

$$\nabla^2\vec{A} - \gamma^2\vec{A} = -\vec{J} \tag{14.120}$$

where γ is the propagation constant. For a delta function source $\vec{J}$ is replaced by $(u_x+u_y+u_z)\delta(r-r_o)$ where $(u_x+u_y+u_z)$ are the unit vectors along the x, y and z directions. For a spherically symmetric free space the vector potential is

$$(u_x + u_y + u_z) = \frac{e^{-\gamma|r-r_o|}}{4\pi\,|r - r_o|}. \tag{14.121}$$

This is considered as the vector Green's function i.e.,

$$\vec{G}(r, r_o) = \frac{e^{-\gamma|r-r_o|}}{4\pi\,|r - r_o|} \tag{14.122}$$

where the vector potential is derived for a spherically symmetric free space (Chap. 13).

For a general current distributions $\vec{J}$, the solution is obtained by a superposition integral using the conditions that x, y and z components of the current densities $J_xJ_yJ_z$ respectively are associated with the unit vectors $u_xu_yu_z$ in the Green's function domain. This rule is readily taken into account by introducing the dyadic Green's function which is the solution of

$$\nabla^2\overrightarrow{\vec{G}}(\vec{r},\vec{r}_o) - \gamma^2\overrightarrow{\vec{G}}(\vec{r},\vec{r}_o) = -\overrightarrow{\vec{I}}\,\delta(r - r_o) \tag{14.123}$$

where $\overrightarrow{\vec{G}}(\vec{r}-\vec{r}_o)$ is the dyadic Green's function, $\overrightarrow{\vec{I}}$ is the idem factor or unit dyadic $(u_xv_x+u_yv_y+u_zv_z)$. For a spherically symmetric free space the solution of (14.122) is

$$\overrightarrow{\vec{G}}(\vec{r},\vec{r}_o) = \overrightarrow{\vec{I}}\,\frac{e^{-\gamma|r-r_o|}}{4\pi\,|r - r_o|}. \tag{14.124}$$

and the vector potential in dyadic space is given by

$$\vec{A}(r) = \int\!\!\int_{v}\!\!\int \overrightarrow{\vec{G}}(r, r_o).\vec{J}(r_o)dv_o \tag{14.125}$$

We can write the three components of the dyadic Green's function as

$$\vec{\nabla} \times \vec{\nabla} \times \overrightarrow{\vec{G}}_o^{(x)} (r, r_o) + \gamma^2 \overrightarrow{\vec{G}}_o^{(x)} (r, r_o) = \delta(r, r_o)\vec{x} \tag{14.126}$$

$$\vec{\nabla} \times \vec{\nabla} \times \overrightarrow{\vec{G}}_o^{(y)} (r, r_o) + \gamma^2 \overrightarrow{\vec{G}}_o^{(y)} (r, r_o) = \delta(r, r_o)\vec{y} \tag{14.127}$$

$$\vec{\nabla} \times \vec{\nabla} \times \overrightarrow{\vec{G}}_o^{(z)} (r, r_o) + \gamma^2 \overrightarrow{\vec{G}}_o^{(z)} (r, r_o) = \delta(r, r_o)\vec{z} \tag{14.128}$$

The solutions of these equations are

$$\overrightarrow{\vec{G}}_o^{(x)} (r, r_o) = \left(\overrightarrow{\vec{I}} - \frac{1}{\gamma^2} \nabla\nabla \right) G_o(r, r_o)\vec{x} \tag{14.129}$$

$$\overrightarrow{\vec{G}}_o^{(y)} (r, r_o) = \left(\overrightarrow{\vec{I}} - \frac{1}{\gamma^2} \nabla\nabla \right) G_o(r, r_o)\vec{y} \tag{14.130}$$

$$\overrightarrow{\vec{G}}_o^{(z)} (r, r_o) = \left(\overrightarrow{\vec{I}} - \frac{1}{\gamma^2} \nabla\nabla \right) G_o(r, r_o)\vec{z} \tag{14.131}$$

Van Bladel (1968), Tai (1971) Hohmonn (1971, 1975).

Dyadic free space Green's functions includes all the three components as

$$\overrightarrow{\vec{G}}_o(r, r_o) = G_o^{(x)}(r, r_o)\vec{x} + G_o^{(y)}(r, r_o)\vec{y} + G_o^{(z)}(r, r_o)\vec{z} \tag{14.132}$$

and $G_o(r, r_o)$ is the nondyadic Green's function.

15

Numerical Methods in Potential Theory

In this chapter, application of numerical methods in solution of boundary value potential or field problems in geophysics are discussed. Finite element, finite difference and Integral equation methods for solution of mostly two and three dimensional boundary value problems are discussed.

A few geoelectrical problems are presented where these numerical approaches are used. Two dimensional finite difference modelling in direct current flow field domain both for surface and borehole geophysics (problems with cylindrical symmetry) are demonstrated in considerable details. Some of the essential differences in these two problems with difference in geometry are highlighted. A structure of the low frequency plane wave electromagnetic finite difference three dimensional modelling (magnetotellurics) is presented briefly. Two dimensional finite element modelling in direct current flow field in surface geophysics using Rayleigh Ritz energy minimisation method are discussed in detail. A brief mention is made about the nature of a 3D problem. Two dimensional finite element modelling in magnetotellurics using Galerkins method is given. Procedure for using advanced level nodes using isoparametric elements and Galerkin's method is outlined. Integral equation method for solution of a three dimensional electromagnetic boundary value problem is demonstrated for low frequency electromagnetics.

15.1 Introduction

For interpretation of geophysical data, forward problems must be solved before entering into an inverse problem. Forward problems are mathematically manageable only for models of simpler geometries viz. (i) one dimensional layered earth problems, (ii) a sphere or a cylinder in a homogeneous and isotropic half space or full space, and (iii) problems with anisotropic half space or problems where physical property vary continuously with distance. Some two dimensional problems with simpler geometries are solved analytically. Since the subsurface of the earth has a complex geometry, solution of any realistic inverse problem demands solution of the forward problems for similar type of subsurface structure. Thus numerical methods entered in potential theory with all its well known tools, viz, finite difference, finite element, integral equation, volume integral, boundary integral, hybrids (mixture) and thin sheet methods.

With rapid advancement in computation facilities, software technology, numerical methods in applied mathematics , solvability of geophysical forward problems has increased immensely. Because one can insert any amount of complications in the models and still get a solution. That has revolutionised the interpretation of geophysical data. The only note of caution is one must use these tools and softwares only after proper calibration. Analytical solution of a forward problem for a subsurface map of simpler geometries and its numerical solutions must match with minimum allowable discrepancies. Because numerical methods are approximate methods always. Application of these approaches for solving geophysical problems became possible and are being used extensively these days for two and three dimensional geophysical problems. Finite difference, finite element and integral equation methods are well-established subjects because the contributions came from different branches of physical sciences and technologies.

Foundation of finite element method (FEM) was laid down by Zienkiewicz, O.C. (1971), Zienkiewicz and Taylor (1989), Bathe (1977), Kardestuncer (1987), Reddy (1993), Krishnamurthy (1991). FEM was intro duced in the electrical methods in geophysics by Coggon (1971), Silvester and Haslam (1972), Reddy et al (1977), Kisak and Silvester (1975), Rodi (1976), Kaikkonen (1977, 1986), Pridmore (1978), Pridmore et al (1981), Queralt et al (1991), Wannamaker et al (1987), Xu and Zhao (1987) and others.

Finite difference modelling in geoelectrical problems became a developed subject with the contributions from Jepson (1969). Yee (1966), Jones and Price (1969, 1970), Jones and Pascoe (1972), Stoyer and Greenfield (1976), Mufti (1976, 1978, 1980), Brewitt, Taylor and Weaver (1978), Dey and Morrison (1979), Zhdanov and Keller (1994), Mundry (1984), Madden and Mackie (1989), Mackie et al (1993) , Roy and Dutta (1994) and others.

Integral equation method (IEM) developed in geophysics through the contributions from Hohmann (1971, 1975, 1983, 1988), Weidelt (1975), Raiche (1975), Ting and Hohmann (1981), Stodt et al (1981), Wannamaker et al (1984), Wannamaker (1991), Beasley and Ward (1986), Eloranta (1984, 1986, 1988), Escola (1992) and others.

Thin sheet modelling grew as a topic in mathematical modelling with the contributions from Lajoie and West (1976), Vassuer and Weidelt (1977), Green and Weaver (1978), Dawson and Weaver (1975), Hanneson and West (1984), Ranganayaki and Madden (1981).

Hybrid technique developed through the research of Lee et al, (1991), Best et al (1985), Tarlowskii et al (1984), Gupta et al (1984) and others.

Finite difference (FDM) and finite element (FEM) methods are based on differential equations. Depending upon the subject area the starting equations are chosen. As for example for geoelectrical and electromagnetic boundary value problems Poisson's and Helmholtz electromagnetic wave equations are the starting points. Every branch of science and engineering where these mathematical tools are used has their own starting equations. Integral equation method (IEM),as the name suggests, is based on integral equation and has great success in handling three dimensional problems in geophysics. In all the three important numerical methods, the domains are discretized and ultimately the solution of the problem depends upon the efficiency of the matrix solver. In FDM and FEM, the entire domains are taken into consideration for solution of the boundary value problems. Therefore size of a matrix becomes very large for three dimensional problems and they are symmetric or asymmetric sparse matrics with most of the elements being zeros. In IEM only the anomalous zones are taken into consideration. Therefore the size of the matrix is considerably small but solid. That gave upper hand to IEM in handling 3D problems. But FDM and FEM have better complication handling capability than IEM.

Mathematical modelling is one of the important areas of geophysics because it has direct link with understanding the nature of geophysical data. That will lead to imaging interior of the earth. The steps involved in mathematical modeling are (i) choice of the basic mathematical equation (ii) discretization of the domain and mathematical formulation (iii) impostion of the boundary conditions (iv) Computation of the response on the simulated air earth boundary or inside the earth using a matrix solver. (v) Comparison of the model responses obtained by these numerical methods with those obtained using analytical method for models with simpler geometries; (vi) refining the numerical tool till the discrepancy between the numerical and analytical results are minimum (vii) testing the numerical tool, thus developed, several times with many simpler models for proper calibration and subsequent use in mathematical modelling.

In this chapter, finite difference, finite element and integral equation methods are demonstrated for solving direct current resistivity and magnetotelluric (plane wave electromagnetic) problems in considerable details. These subjects are well developed. The students have to read all the books and research papers cited here.

15.2 Finite Difference Formulation/Direct Current Domain (Surface Geophysics)

15.2.1 Introduction

In finite difference method the first step is to choose the basic equation depending upon the problem to be solved. As for example we start with Poisson's equation in direct current flow field. Similarly different branches of science and engineering have their own starting equations. In this finite difference geoelectrical problem the next step is to change the differential equation to a difference equation. Since it is an approximation, the very basis on which the numerical method stands, an attempt is being made to minimise the error accumulated in this approximation as much as possible. In direct current flow field the boundary conditions needed and applied are (i) Dirichlet boundary condition (ii) Neumann boundary condition and (iii) Mixed boundary condition. Most of the geophysical problems are mixed boundary value problems where Dirichlet conditions are satisfied on some boundaries and Neumann conditions are satisfied on the other.

For a two dimensional direct current flow field problem xz plane is assumed to be a half space where the domain goes to infinity both in the $x(=\pm\infty)$ and $z(=+\infty)$, positive downward, directions(Fig. 15.1). For a two dimensional problem, the physical property along the y direction remains invariant. So any xz plane will have identical texture from physical property point of view. On the surface of the earth or air earth boundary $z = 0$. Since the upper half space, filled with air, is an infinitely high resistive zone ,air earth boundary becomes a boundary of infinite resistivity contrast and Neumann boundary condition is satisfied at this boundary.

The whole 2D domain is discretized into number of cells or elements (Fig. 15.1) of generally rectangular or square shapes in finite difference domain. The intersection points of the grid lines are called nodes (Figs. 15.1 and 15.2). Finite difference equations are generated for each nodes to develop a matrix equation The discretized half space is generally divided into two parts viz, the working zone and the far zone. In the working area the geological models are simulated. Dense mesh or grids of much smaller dimensions are chosen so that in each cell the assumed linear or nonlinear variation of potential is not a severe approximation. In DC flow field domain, variation of potential follow 1/r law, which is depicted in several connecting elements or cells. In general the law is higher the potential gradient smaller should be the

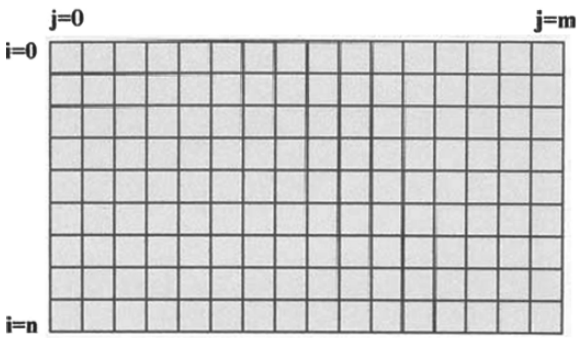


Fig. 15.1. A simplest discretized domain showing the cells, nodes and boundaries

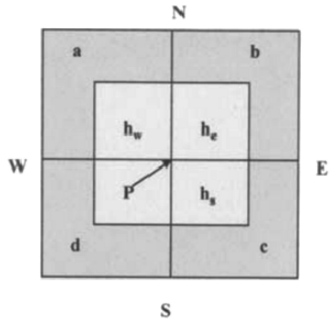


Fig. 15.2. Enlarged view of a finite difference cell with a node at the centre

grid size and finer should be the mesh. As one moves away from the source, gradient of potential diminishes, mesh can also be coarser. The general rule is finer the mesh more accurate will be the finite difference solution, larger will be the matrix size, higher will be memory allocation in a computer, higher will be the computation time and more accurately the differential equation is changed to a difference equation. Depending upon the computational infrastructure available one has to make a suitable compromise on the degree of refinement of the mesh. With the refinement in matrix, the size of the matrix increases at an alarming rate. To reduce a matrix size often blocks are made of 4 or 9 or 16 cells having same physical property. For surface of complicated geometry it is advisable to have finer mesh keeping in mind the limited resolving power of the geoelectrical potentials in mapping the subsurface. Some grid lines must pass through the assumed geological target such that it's effect enters into the mathematical solution. The topic of expanding grid is discussed in the Sect. 15.3. In the far zone or non working zone the finite difference cells are made coarser and coarser to make an expanding grid such that the domain boundaries are pushed back to infinite distances from the source such that potentials at those boundaries become zero and Dirichlet's boundary condition is satisfied in all the three boundaries (Fig. 15.3). These boundary conditions will enter into the system matrix modifying some of the boundary equations.

Difference equations are assembled to generate a matrix equation of large size. These matrices are sparse matrices with nonzero main diagonal and four nonzero side diagonals with two nonzero diagonals each in upper and lower triangle in the matrix. For a 3D problem, the number of nonzero side diagonals will be 3 each in upper and lower triangles. Therefore one generally gets penta diagonal and hepta diagonal matrices for 2D and 3D problems respectively. Sparsity in a matrix is a guiding equation dependent subject and cannot be generalised in a few words. But sparsity remarkably reduces computer storage and computation time because most of the elements are zeros. This matrix equation is solved using one of the well known and widely used matrix solvers, viz., Gauss elimination, Gauss Seidel iteration, Cholesky's decomposition,

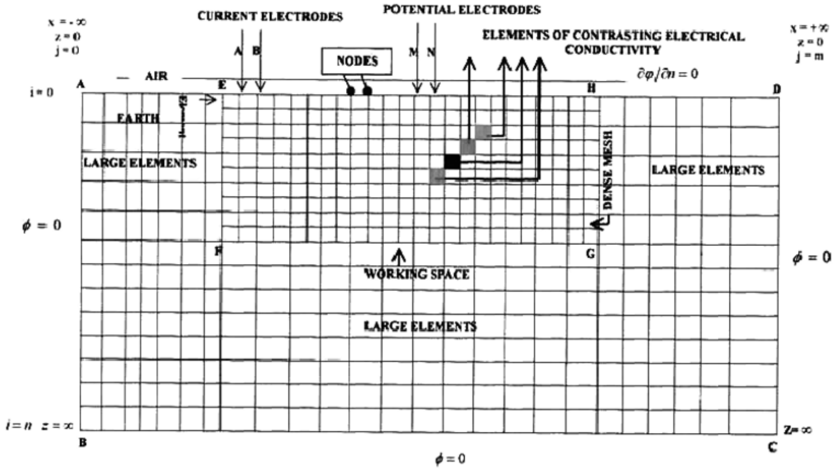


Fig. 15.3. Shows finite difference mesh and dipole-dipole electrode configuration measurements; working zone and far zone., domain boundaries; cells of different physical properties and nodes

conjugate gradient minimisation, successive over relaxation (SOR), successive line over relaxation (SLOR), LU decomposition with backward forward substitution etc. This is a big area of mathematics and is not discussed in this book. The solutions of these equations generate potentials at each nodal point per one source point. Since the principle of superposition is valid in direct current flow field, potentials at the nodal points can be computed for any number of sources and sinks. Each cell can be assigned different electrical conductivity, hence it can handle problems of any degree of complication in subsurface geometry.

15.2.2 Formulation of the Problem

Derivation of finite difference equation in a Cartesian coordinate for surface measurement is presented in this section. Flow of steady current in a non-uniform medium can be given by

$$-\nabla.\vec{J} = \frac{\partial Q}{\partial t}. \tag{15.1}$$

This is the continuity equation for time invariant current density $\vec{J}$ and volume density of charge Q. The equation can be written as

$$-\nabla.\left[\sigma\vec{E}\right] = \frac{\partial Q}{\partial t} \tag{15.2}$$

or

$$-\nabla.\left[\frac{1}{\rho\left(x,y,z\right)}.\nabla\phi(xyz)\right] = \frac{\partial Q}{\partial t}. \tag{15.3}$$

Here ρ and ϕ are respectively the resistivity and scalar potential. Simply we can write the (15.2) taking $\frac{\partial Q}{\partial t} = q(x, y, z)$ as

$$\nabla . [\sigma \nabla \phi] + q = 0 \quad . \tag{15.4}$$

If we consider this equation in a two dimensional domain assuming there is no variation in conductivity in the y direction, i.e.,

$$\frac{\partial}{\partial y} [\sigma (x, y, z)] = 0, \tag{15.5}$$

we can write (15.4) as

$$\frac{\partial}{\partial x} \left[\sigma (x, y) \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial z} \left[\sigma (x, y) \frac{\partial \phi}{\partial z} \right] + q (x, z) = 0 \tag{15.6}$$

where q is the current density. Hence for an element p, (Fig. 15.2) the value of q is I / area (a b c d), or

$$\begin{aligned} \mathrm{q} &= \frac{I}{\frac{1}{2} (h_N + h_S) . \frac{1}{2} (h_E + h_W)} \\ &= \frac{4I}{(h_N + h_S) (h_E + h_W)} . \end{aligned} \tag{15.7}$$

I is the strength of the source. Here in D.C. resisitivity finite difference modelling, when intervals of the grids are same, $h_N = h_S = h_E = h_N = h$, and $\mathrm{q} = 4I/2h.2h = \frac{I}{h^2}$. This only happens when the elements are in the subsurface. When the source is on the surface, $h_N = 0$ and hence $\mathrm{q} = \frac{2I}{h^2}$. The effective strength of the source is doubled when it is placed on the surface of the earth. In practice, the grid size for discretization may not be equal and hence the generalised equation for q is

$$q = 4I/ (h_E + h_W) hs. \tag{15.8}$$

15.2.3 Boundary Conditions

The numerical solution is obtained applying the following boundary conditions:

(a) $\varphi (x, y, z)$ must be continuous across each element boundary of the contrasting physical property distribution of $\sigma (x, y)$.
(b) The normal component of J $\left(= -\sigma \partial \varphi / \partial n\right)$ must be continuous across each boundary.
(c) $\phi_{i,j} (= f (x, z))$ along the subsurface boundaries AB, BC and CD are zeros as the domain boundaries are pushed far away from the working area. Dirichlets boundary conditions are satisfied.
(d) $\frac{\partial \phi}{\partial z} = 0$ on the air earth boundary AD . Here the Neumann boundary condition is satisfied (Fig. 15.3).

(e) $q_{ij} = 0$ every where except at the location of the current electrodes. Here source and sink of +q and −q are inserted. These two are the only nonzero elements in column vector B in the matrix equation Ax = B.
(f) The grid is chosen to be rectangular with arbitrary, irregular spacing of the nodes in the x- and z-directions respectively. The nodes in the x-direction are labelled $\mathrm{j} = 0, 1, 2, 3, \ldots\ldots\ldots\ldots\ldots \mathrm{m}$ and those in the z-direction, $\mathrm{i} = 0, 1, 2, 3, \ldots\ldots\ldots \mathrm{n}$ (Fig. 15.1).
(g) The left and right edges at infinite distances in the heterogeneous half space are simulated by the lines j = 0 and j = m respectively. .
It is possible to choose appropriate boundary conditions at infinite distance from the source and can be brought nearer with finite choice of m and n.
(h) The Dirichlet's boundary condition for all $\mathrm{J} = 0, 1, 2, 3 \ldots .\mathrm{n}$ with i = 0 at $\mathrm{x} = \pm\infty$ and $\mathrm{z} = \infty$ is done by extending the meshes far enough away from the sources and the conductivity inhomogeneities such that the potential distribution approaches asymptotically to zero.
(i) Electrical conductivity distribution $\sigma(x, z)$ is assigned in each cell depending upon the nature of the problem.
(j) ϕ_{ij}, the potentials on any surface and subsurface node (i, j) are available with regular or irregular grid spacing in the x- and z-direction.
(k) The Neumann boundary condition is satisfied on the surface.

15.2.4 Structure of the FD Boundary Value Problem

From the Fig. (15.4) [part of the grid system] we can generate the approximate relation valid at the point P (i, j) and can write as

$$\left(\frac{d\phi}{dx}\right)_{i,j} = \frac{\phi_{i,j+h\frac{E}{2}} - \phi_{i,j-h\frac{h}{2}}}{(h_E + h_w)/2}. \tag{15.9}$$

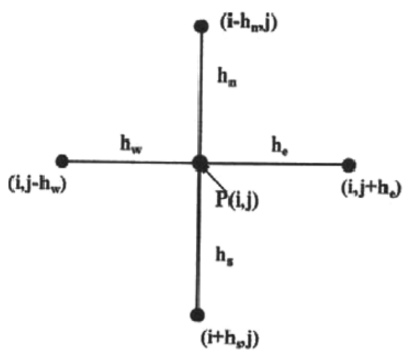


Fig. 15.4. Surrounding area of a node P_{ij} with adjacent nodes

Therefore, for the point $i, j + h\frac{E}{2}$, we can write

$$\left(\frac{d\phi}{dx}\right)_{i,j+\frac{hE}{2}} = \frac{\phi_{i,j+\frac{hE}{2}} - \phi_{i,j-\frac{hE}{2}}}{\left(h_E + h_w\right)/2}. \tag{15.10}$$

Hence

$$\left(\sigma\frac{\partial\phi}{\partial x}\right)_{i,j+hE/2} = \left(\sigma_{i,j+hE/2}\right).h_E^{-1}\left(\phi_{i,j+hE} - \phi_{i,j}\right). \tag{15.11}$$

Now $\left[\frac{\partial}{\partial x}\left(\sigma\frac{\partial\phi}{\partial x}\right)\right]$ will be from (15.9)

$$\left[\frac{\partial}{\partial x}\left(\sigma\frac{\partial\phi}{\partial x}\right)\right]_{i,j} = \frac{2}{h_E + h_w}\left[\left(\sigma\frac{\partial\phi}{\partial x}\right)_{i,j+hE/2} - \left(\sigma\frac{\partial\phi}{\partial x}\right)_{i,j-hW/2}\right]. \tag{15.12}$$

Putting the values from (15.10), we get

$$\left[\frac{\partial}{\partial x}\left(\sigma\frac{\partial\phi}{\partial x}\right)\right]_{i,j} = \frac{2}{h_E + h_w}\left[\frac{\sigma_{i,j+hE/2}}{h_E}\left(\phi_{i,j+hE} - \phi_{i,j}\right) - \frac{\sigma_{i,j-hw/2}}{hw}\left(\phi_{i,j} - \phi_{i,j-hw}\right)\right]. \tag{15.13}$$

Hence (15.6) can be written as

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\sigma\frac{\partial\phi}{\partial x}\right)_{i,j} + \frac{\partial}{\partial z}\left(\sigma\frac{\partial\phi}{\partial z}\right)_{i,j} + q\left(x,z\right) = 0 \\
&\Rightarrow \quad \frac{2}{h_E + h_w}\left[\frac{\sigma_{i,j+hE/2}}{h_E}\left(\phi_{i,j+hE} - \phi_{i,j}\right) - \frac{\sigma_{i,j-hw/2}}{hw}\left(\phi_{i,j} - \phi_{i,j-hw}\right)\right] \\
&+ \frac{2}{h_N + h_S}\left[\frac{\sigma_{i,j+hS/2}}{h_S}\left(\phi_{i,j+hS} - \phi_{i,j}\right) - \frac{\sigma_{i,j-hN/2}}{h_N}\left(\phi_{i,j} - \phi_{i,j-hN/2,j}\right)\right] \\
&+ q_{ii,j} = 0.
\end{aligned} \tag{15.14}$$

We can rewrite this equation separating the terms of φ as,

$$\alpha_E\phi_{i,j+hE} + \alpha_w\phi_{i,j-hW} + \alpha_N\phi_{i-hN,j} + \alpha_S\phi_{i+hS,j} - \alpha_P\phi_{ij} + q_{ij} = 0 \tag{15.15}$$

where

$$\begin{aligned}
\alpha_E &= 2\left(\sigma_{i,j+hE/2}\right)\left[h_E\left(h_E + h_W\right)\right]^{-1} \\
\alpha_W &= 2\left(\sigma_{i,j+hW/2}\right)\left[h_W\left(h_E + h_W\right)\right]^{-1} \\
\alpha_N &= 2\left(\sigma_{i,j+hN/2}\right)\left[h_N\left(h_N + h_S\right)\right]^{-1} \\
\alpha_S &= 2\left(\sigma_{i,j+hS/2}\right)\left[h_S\left(h_N + h_S\right)\right]^{-1} \\
\alpha_p &= \alpha_E + \alpha_W + \alpha_N + \alpha_S.
\end{aligned} \tag{15.16}$$

Upto this point, we have calculated the values inside the medium.

If we consider the ground surface element, the equation changes in the following way. Here we consider a fictitious row of elements above the surface of the ground such that

$$h_N = h_S$$

and $\sigma_{1-hN,j} = \sigma_{1+hS,j}$ as i = 1 represent the ground surface j = 1, 2, 3 . . . . .n. Hence

$$\alpha_N = \alpha_S \tag{15.17}$$

for the element (1, j). We must have $\frac{\partial \phi}{\partial z} = 0$ [boundary condition]. This is only possible if

$$\phi_{1+hS,j} = \phi_{1-hN,j}. \tag{15.18}$$

Putting (15.15) and (15.17) in (15.14) are have

$$\alpha_E \phi_{i,j+hE} + \alpha_w \phi_{i,j-hW} + \alpha_S \phi_{i-hS,j} + \alpha_p \phi_{i,j} + q_{ij} = 0. \tag{15.19}$$

From the entire set of equations obtained for both the surface and subsurface nodes, one gets a matrix equation of the form AΦ = B (Fig. 15.5)

$$\mathbf{A} \qquad \mathbf{\Phi} = \mathbf{B} \tag{15.20}$$

Since potential is computed in a two dimensional xz plane, the source and the sink extend to infinite distance along the y direction. Therefore these sources are line sources. For converting these potential due to a line source and sink to potentials due to a point source and sink, integral transform must be used.

15.2.5 Inverse Fourier Cosine Transform

For two-dimensional earth models, the potentials computed are for line source when it remains independent of one co-ordinate axis. The resistivity distribution is assumed to be a function of only two co-ordinates (x, z). Since three dimensional point source is used, the problem cannot be treated purely as three-dimensional. It is necessary to remove the source variation along y direction by fourier transformation in order to solve the two-dimensional problem. Assuming symmetry along $y = 0$, the potential variations may be transformed by the application of inverse fourier cosine transformation. We define

$$\phi\left(x, y, z\right) = \frac{2}{\pi} \int_0^\infty \phi\left(x, \lambda, z\right) Cos(\lambda y) d\lambda \tag{15.21}$$

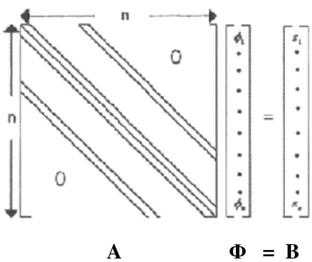


Fig. 15.5. Curtoon of a system matrix in finite difference problem using Poisson's equation and showing sparsity

On the y-axis or when y = 0, we get

$$\varphi(x, o, z) = \frac{2}{\pi} \int_{0}^{\infty} \phi(x, \lambda, z) d\lambda \tag{15.22}$$

where λ is the integration variable. Many authors (Mwenifumbo (1980), Sasaki (1982), Dey and Morrison (1976), Pridmore (1978)) have used 5 to 7 values of integration variables λ between 0 and some finite value to bring down the upper limit of integration form ∞. Behaviour of this integral is studied in greater detail. It was found that better results are obtained if the $\lambda\,s$ have a gaussian distribution between the limits of integral. Therefore the principle of gauss quadrature integration is followed for evaluating (15.21). Using this procedure the effect of different sets of λ values on the accuracy in computing potentials with distance from the source in the working area is examined in detail.

It is observed that, within fixed limits of integration, higher the number of λ values, greater will be the distance of the point from the source till the analytical and computed values of potential have a considerable agreement. For example, the distance will be 50 units, 8 units and 2.5 units for sets with 11, 7 and 3 λ values respectively.

15.2.6 Calibration

The discrepancy between numerical and analytical solutions must be minimised comparing the responses for the bodies of simpler geometries. At this stage the program source code will be ready for operation.

Figure 15.6 shows a two electrode apparent resistivity profile over a vertical dyke of higher resistivity obtained using finite difference source code. Values are compared with those obtained using analytical formulae based on image theory (see Chap. 11).

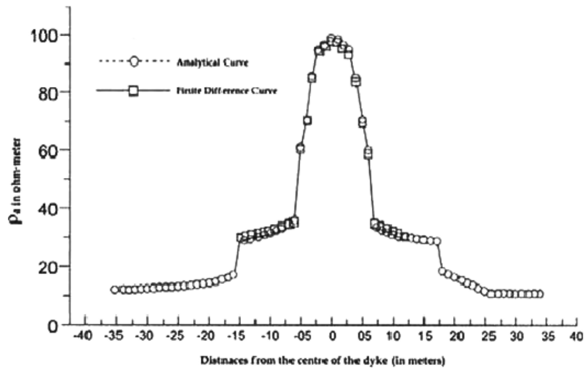


Fig. 15.6. Resistivity two electrode profile across a vertical dyke; comparision between the finite difference model and that obtained with those obtained from analytical solution

15.3 Finite Difference Formulation Domain with Cylindrical Symmetry DC Field Borehole Geophysics

15.3.1 Introduction

In this section finite difference modelling in DC field for a domain of cylindrically symmetric structure is given highlighting some difference in the boundary conditions, geometry of the model and grid system used. In this model the domain extends from r = 0 to r = ∞ and z = ±∞ where r is the radial distance from the axis of the cylinder and z is the vertical distance of both upward and downward boundaries. Discretization of one half of the r-z plane for solution of the boundary value problem is only needed. That reduces the computer storage space considerably. In this section a few more points on finite difference modelling are discussed. Figure (15.7) shows the geometry of a problem in a zone of cylindrical symmetry. In this problem the Neumann boundary condition is satisfied on the axis of a cylindrical domaim. Dirichlet boundary conditions will be met at r = ∞ and z = ±∞. The model is set up in a cylindrical co-ordinate system. A point source, located in a radially symmetric environment, generate the potential field.

The proposed medium, with certain structures, is discretized and divided into rectangular blocks by using vertical and horizontal grid lines, whose

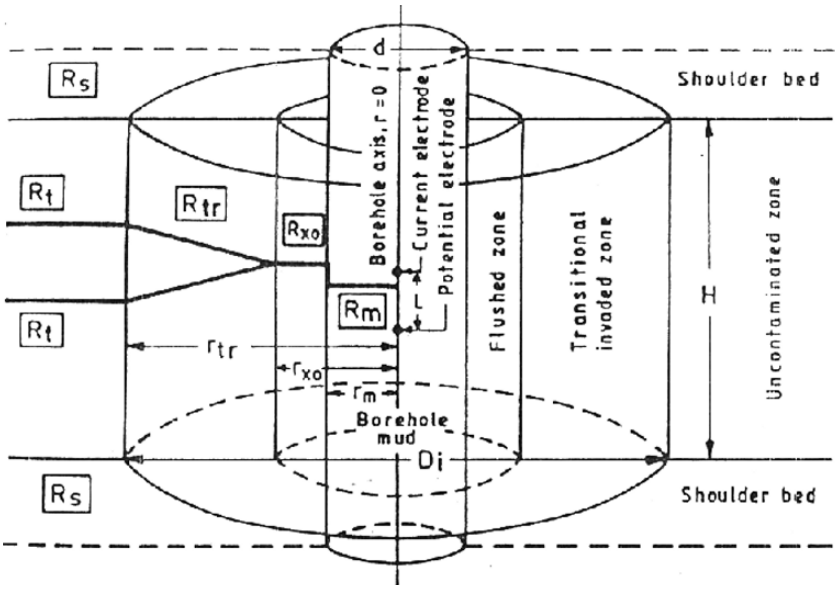


Fig. 15.7. Borehole model with coaxial cylindrical symmetry; coaxial cylindrical zones respectively show the borehole mud, flushed zone, invaded zone, uncontaminated zone and shoulder beds on top and bottom (Anon 1972)

mutual separations increased exponentially, both in the vertical and radial directions. Radial variation of resistivity in one of the coaxial cylindrical shells is simulated in FD modelling.

15.3.2 Formulation of the Problem

In a cylindrical co-ordinate system with radial symmetry, the starting Poisson's equation is written as

$$\frac{\partial}{\partial r}\left(\sigma\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\sigma\frac{\partial\varphi}{\partial z}\right)+\frac{1}{r}\left(\sigma\frac{\partial\varphi}{\partial r}\right)+q=0 \tag{15.23}$$

where, $\sigma = \sigma$ (r, z) and $\varphi = \varphi$ (r, z). This relation represents an elliptic second order differential equation and defines the electric potential due to a current source in a medium. The quantity q must be interpreted as a variable current density in a typical 3-dimensional model, which is reduced to 2-dimension in the cross section considering radial symmetry.

15.3.3 Boundary Conditions

For finite difference modeling, the infinite medium is made finite by placing an artificial boundary. Figure (15.7) shows one such boundary in vertical cross section of an earth model. The medium is discretized by dividing it into number of rectangular cells with vertical and radial grids. The intersection points of the grid are called pivotal or nodal points.

Pivotal points which lie on the axis of symmetry or the borehole axis i.e., the boundary through the points $\mathrm{P}_{(1,1)}$ and $\mathrm{P}_{(\mathrm{i}_{\max'}\,1)}$ should follow the boundary conditions i.e.,

$$\frac{\partial\varphi\,(r,z)}{\partial r}\,|_{r=0}\ =0 \tag{15.24}$$

2) and $\varphi(\mathrm{r,z}) = 0$ when $\mathrm{r} \to \infty$ and $\mathrm{z} \to \pm\infty$

15.3.4 Grid Generation for Discretization

The domain has been discretized using vertical and radial grids. As the model is axially symmetric, only one half of the vertical section, i.e., $\mathrm{r} \geq 0$ is considered. It is necessary that the grids to be finely spaced near the current source (s) as the variation of potential around the source is maximum. As one moves away from the current source in any direction, the change in potential gradually diminishes. Hence at points far from the current source, the grids may be much coarser. It is convenient and justified to increase the grid spacing exponentially with distance from the current source. Co-ordinate of a node can be conveniently denoted by (i, j). The node, corresponds to ith row and jth column can be denoted by $\mathrm{P}_{(\mathrm{i,j})}$. j = 1 corresponds to borehole axis , $\mathrm{r} = 0, \mathrm{i} = \mathrm{i}_{\max}$ corresponds to the upper and lower boundaries in the

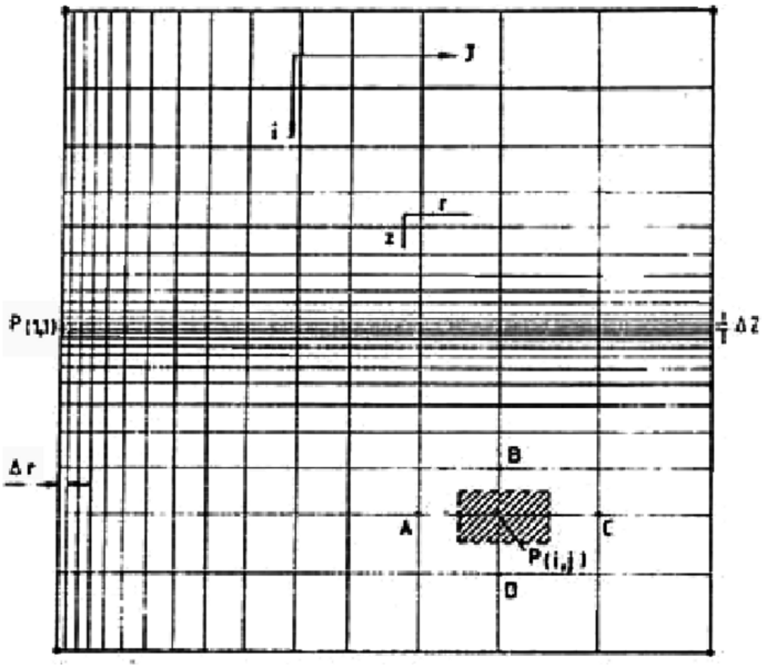


Fig. 15.8. Expanding grid for finite difference modelling; $P_{1,1}$,the location of source point

vertical plane and $j = j_{max}$ corresponds to the radial boundary. The current point source is located at a point $P_{(1,1)}$ (Fig. 15.8).

Location of any pivotal point $P_{(i,j)}$ can be expressed in an expanding grid system as (Mufti 1976)

$$
\begin{aligned}
r(i,j) &= \frac{\Delta r \left(a^{j-1} - 1\right)}{a - 1} \qquad \text{for } j = 1 \text{ to } j_{max} \\
z(i,j) &= \frac{\Delta z \left(b^{i-1} - 1\right)}{b - 1} \qquad \text{for } i = 1 \text{ to } i_{max}
\end{aligned}
\tag{15.25}
$$

where, a and b are the expansion ratios and Δr and Δz are the smallest spacing in the radial and vertical directions respectively. The choice of grid expansion can be determined by trial and error on the basis of desired accuracy and computational efficiency.

15.3.5 Finite Difference Equations

We can consider arbitrarily chosen element $P_{(i,j)}$ from the grid system shown in Fig. (15.9). It's four immediate surrounding neighbourhood pivotal points are A, B, C and D denoted by $P_{(i,j-1)}$, $P_{(i-1,j)}$, $P_{(i,j+1)}$ and $P_{(i+1,j)}$ respectively.

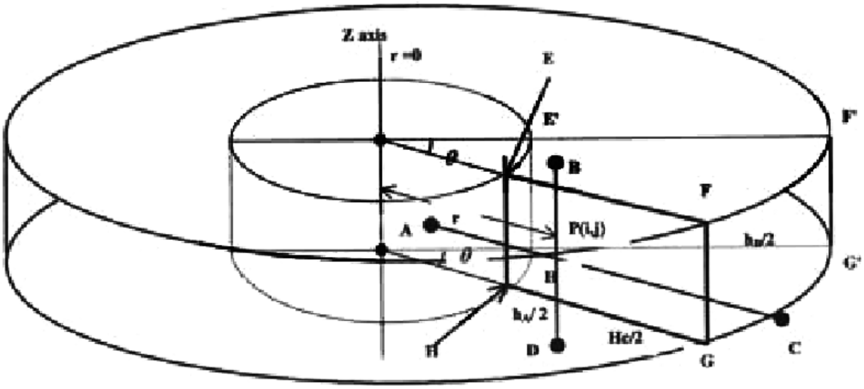


Fig. 15.9. Enlarged view of a rectangular cell in a ring element

Distance from $P_{(i,j)}$ to these neighbours can be denoted as $h_{A'}$ $h_{B'}$ $h_{C'}$ $h_{D'}$. $P_{(i,1)}$ points fall on the borehole axis and are used as locations for current and potential electrodes on the basis of the required position of which $P_{(1,1)}$ is the position of the current electrode. Figure 15.9 is a vertical section of a three dimensional model. It only represents a slice of the earth and must be rotated through 360 degrees to represent the actual setup. The 3-dimensional space is divided into circular ring elements, which have rectangular cross sections. Figure 15.9 shows a ring disc volume element associated to the nodal point $P_{i,j}$.. Because of axial symmetry, these ring elements can be viewed as rectangular element in the section plane, which reduces the model to 2-D. Figure (15.9) shows a node $P_{(i,j)}$ along with it's four neighbours. The radial distance of $P_{(i,j)}$ from the borehole axis (r = 0) is r. Point $P_{(i,j)}$ represents the rectangular element EFGH, covering an area of $(h_A + h_C)/2 \times (h_B + h_D)/2$ in the vertical section. $\varphi_{i,j}$ at the point $P_{(i,j)}$ in terms of its neighbouring nodal points can be derived. On account of the central difference formula, the following approximations are valid at the point P. The difference equation can be written as

$$\left(\frac{\partial \varphi}{\partial r}\right)_{i,j} = \frac{\varphi_{i,j+h_C/2} - \varphi_{i,j-h_A/2}}{h_A + h_C}. \tag{15.26}$$

Therefore, for the point $(i, j + h_C/2)$, it can be expressed as

$$\left(\frac{\partial \varphi}{\partial r}\right)_{i,j+h_C/2} = \frac{\varphi_{i,j+h_C} - \varphi_{i,j}}{h_C}. \tag{15.27}$$

Hence,

$$\left(\sigma \frac{\partial \varphi}{\partial r}\right)_{i,j+h_C/2} = \frac{\sigma_{i,j+h_C/2}}{h_C}\left(\varphi_{i,j+h_C} - \varphi_{i,j}\right). \tag{15.28}$$

Similarly for the point $(i, j - h_A/2)$, one gets,

$$\left(\sigma\frac{\partial\varphi}{\partial r}\right)_{i,j-h_A/2} = \frac{\sigma_{i,j-h_A/2}}{h_A}\left(\sigma_{i,j} - \varphi_{i,j-h_A}\right). \tag{15.29}$$

Here, $\sigma_{i,j+h_C/2}$ *and* $\sigma_{i,j-h_A/2}$ can be expressed as

$$\sigma_{i,j+h_C/2} = \frac{\sigma_{i,j} + \sigma_{i,j+h_C}}{h_C} \tag{15.30}$$

and

$$\sigma_{i,j-h_A/2} = \frac{\sigma_{i,j} + \sigma_{i,j-h_A}}{h_A} \tag{15.31}$$

where, $\sigma_{i,j'}\sigma_{i,j+h_C}$ *and* $\sigma_{i,j-h_A}$ are the conductivity values assumed at the pivotal points $P_{(i,j)'}P_{(i,j+h_C)}$ or $P_{(i,j+1)'}$ and $P_{(i,j-h_A)}$ or $P_{(i,j-1)}$. Other values of σ associated with the subsequent equations can be calculated in the same way. For discretization of the earth model, conductivity value is assigned to each element or cell. Since conductivity and resistivity have reciprocal relation ($\sigma = 1/\rho$), therefore, the assigned value at the nodal resistivities can be changed to nodal conductivities.

Applying central difference formula once, one gets

$$\left[\frac{\partial}{\partial r}\left(\sigma\frac{\partial\varphi}{\partial r}\right)\right]_{i,j} = \frac{2}{h_A + h_C}\left[\left(\sigma\frac{\partial\varphi}{\partial r}\right)_{i,j+h_C/2} - \left(\sigma\frac{\partial\varphi}{\partial r}\right)_{i,j-h_A/2}\right]. \tag{15.32}$$

Considering (15.28) to (15.31), (15.32) can be expressed as

$$\begin{aligned}\left[\frac{\partial}{\partial r}\left(\sigma\frac{\partial\varphi}{\partial r}\right)\right]_{i,j} =& \frac{2}{h_A + h_C}\left[\frac{\sigma_{i,j+h_C/2}}{h_C}\left(\varphi_{i,j+h_C} - \varphi_{i,j}\right)\right.\\ &\left. - \frac{\alpha_{i,j-h_A/2}}{h_A}\left(\varphi_{i,j} - \varphi_{i,j-h_A}\right)\right].\end{aligned} \tag{15.33}$$

Proceeding in the same fashion, generalized self adjoint finite difference approximation of the differential equation (15.33) can be represented as

$$\begin{aligned}&\frac{2}{h_A + h_C}\left[\frac{\sigma_{i,j+h_C/2}}{h_C}\left(\varphi_{i,j+h_C} - \varphi_{i,j}\right) - \frac{\sigma_{i,j-h_A/2}}{h_A}\left(\varphi_{i,j} - \varphi_{i,j-h_A}\right)\right] +\\ &\frac{2}{h_B + h_D}\left[\frac{\sigma_{i+h_D/2,j}}{h_D}\left(\varphi_{i+h_{D'}j} - \varphi_{i,j}\right) - \frac{\sigma_{i,j-h_B/2,j}}{h_B}\left(\varphi_{i,j} - \varphi_{i,h_{B'}j}\right)\right] +\\ &\frac{\sigma_{i,j}}{r_{i,j}\left(h_A + h_C\right)}\left[\varphi_{i,j+h_C} - \varphi_{i,j-h_A}\right] + q_{i,j} = 0\end{aligned} \tag{15.34}$$

where, $\varphi_{i,j-h_A'}\varphi_{i-h_B'j'}\varphi_{i,j+h_C} and \varphi_{i+h_D'j}$ are the potential function associated to the $P_{i,j-1'}P_{i-1,j'}P_{i,j+1}$ and $P_{i+1,j}$ nodal points.

From (15.34) after separating out the co-efficient of ϕ, one gets

$$-\alpha_C\varphi_{i,j+h_C} - \alpha_A\varphi_{i,j-h_A} - \alpha_D\varphi_{i+h_D'J} - \alpha_B\varphi_{i,h'_Bj} + \alpha_P\varphi_{i,j} = q_{ij} \tag{15.35}$$

where,

$$
\begin{aligned}
\alpha_C &= \frac{2}{(h_A + h_C)\,h_C}\left[\sigma_{i,j+h_C/2}\right] + \frac{\sigma_{i,j}}{(h_A + h_C)\,r_{i,j}} \\
\alpha_A &= \frac{2}{(h_A + h_C)\,h_A}\left[\sigma_{i,j-h_A/2}\right] - \frac{\sigma_{i,j}}{(h_A + h_C)\,r_{i,j}} \\
\alpha_B &= \frac{2}{(h_B + h_D)\,h_B}\left[\sigma_{i-h_B/2,j}\right] \\
\alpha_D &= \frac{2}{(h_B + h_D)\,h_D}\left[\sigma_{i-h_D/2,j}\right] \\
\alpha_{\mathrm{P}} &= \alpha_{\mathrm{A}} + \alpha_{\mathrm{B}} + \alpha_{\mathrm{C}} + \alpha_{\mathrm{D}}.
\end{aligned} \tag{15.36}
$$

Above (15.35) gives the potential relation for an arbitrary inside pivotal point $\mathrm{P_{i,j}}$ in the discretized domain.

In (15.23) the term $\frac{1}{r}\left(\sigma\frac{\partial\varphi}{\partial r}\right)$, contains a division by r. That leads to mathematical indeterminacy as $r \rightarrow 0$ on the borehole axis. This algebraic singularity is avoided assuming [Mufti (1978, 1980)].

$$
\lim_{r} \rightarrow 0\frac{1}{r}\frac{\partial\varphi}{\partial r} = \left.\frac{\partial^2\varphi}{\partial r^2}\right|_{r=0} \tag{15.37}
$$

From the (15.23), the generalized expression for evaluating the potential along the axis of symmetry changes to

$$
\frac{\partial}{\partial r}\left(\sigma\frac{\partial\varphi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\sigma\frac{\partial\varphi}{\partial z}\right) + \sigma\frac{\partial^2\varphi}{\partial r^2} + q = 0. \tag{15.38}
$$

Finite difference equivalent of the (15.38) can be written as

$$
\begin{aligned}
&\frac{2}{h_A + h_C}\left[\frac{\sigma_{i,j+h_C/2}}{h_C}\left(\varphi_{i,j+h_C} - \varphi_{i,j}\right) - \frac{\sigma_{i,j-h_A/2}}{h_A}\left(\varphi_{i,j} - \varphi_{i,j-h_A}\right)\right] + \\
&\frac{2}{h_B + h_D}\left[\frac{\sigma_{i,j+h_D/2,j}}{h_D}\left(\varphi_{i+h_D{}'j} - \varphi_{i,j}\right) - \frac{\sigma_{i-h_B/2,j}}{h_B}\left(\varphi_{i,j} - \varphi_{i-h_B{}'j}\right)\right] + \\
&\frac{2\sigma_{i,j}}{h_A + h_C}\left[\frac{1}{h_C}\left(\varphi_{i,j+h_C} - \varphi_{i,j}\right) - \frac{1}{h_A}\left(\varphi_{i,j} - \varphi_{i,j-h_A}\right)\right] + q_{i,j} = 0.
\end{aligned} \tag{15.39}
$$

The nodal points, located on the borehole axis, have the value $r = 0$, and corresponds to the first column of nodes i.e., for $j = 1$. Hence the co-ordinates of the pivotal points will be $\mathrm{P_{(i,1)}}$. For calculation of potential on the axis of symmetry, a fictitious column of element parallel to the axis of symmetry at the left of (i, 1) is considered and introduced, such that

$$
\mathrm{h_A} = \mathrm{h_C} \tag{15.40}
$$

and

$$
\sigma_{i,1+h_C} = \sigma_{i,1-h_A}. \tag{15.41}
$$

Elements fall on this axis (i, 1) should satisfy the Neumann boundary condition

$$\left.\frac{\partial \varphi}{\partial r}\right|_{r=0} = 0$$

and this is possible only if

$$\varphi_{i,1+h_C} = \varphi_{i,1-h_A}. \tag{15.42}$$

Putting all these assumptions described above in (15.39) and separating out the coefficient of φ, one gets

$$-\alpha_C\ \varphi_{i,1+h_C} - \alpha_D\ \varphi_{i+h_D{}'1} - \alpha_B\ \varphi_{i-h_B{}'1} + \alpha_P\ \varphi_{i,1} = q_{i,1} \tag{15.43}$$

$$\begin{aligned}
\alpha_C =& \frac{2}{h_C^2}\left[\sigma_{i,1+h_C/2} + 2\sigma_{i,j}\right] \\
\alpha_B =& \frac{2}{(h_B + h_D)\, h_B}\left[\sigma_{i-h_B/2,1}\right] \\
\alpha_D =& \frac{2}{(h_B + h_D)\, h_D}\left[\sigma_{i+h_D/2,1}\right] \mathrm{\alpha_P} = \mathrm{\alpha_B} + \mathrm{\alpha_C} + \mathrm{\alpha_D}.
\end{aligned} \tag{15.44}$$

Equations (15.34) to (15.43) are the main finite difference approximations, which can be solved for the potential expression φ (r, z). In practical problems, a linear equation should be formed using the relations mentioned. Potential at each node must be expressed in terms of its immediate neighbour and set of coefficients $\mathrm{\alpha_{A'}}$, $\mathrm{\alpha_{B'}}$, $\mathrm{\alpha_{C'}}$, $\mathrm{\alpha_D}$ and $\mathrm{\alpha_P}$ are evaluated. Systematic ordering of the grid points develops a sparse conductivity coefficient matrix, which is solved for evaluation of potential field at a desired location using a suitable matrix solver.

15.3.6 Current Density Factor q at the Source

In this two dimensional borehole d.c. resistivity forward problem involving radial symmetry, the source is assumed to be a point source and located along the axis of symmetry which coincides with the borehole axis. It is also stated that in the expanding grid system the source is considered at the point $\mathrm{P}_{(1,1)}$ from where the grid expands exponentially on both the sides in the vertical (z) and radial (r) directions. The volume associated with the pivotal point is a cylinder of radius $\mathrm{h_C}/2$ and height $(\mathrm{h_B} + \mathrm{h_D})/2$. Figure 15.10 shows such a volume element associated to any of pivotal points that lie on the axis of symmetry. The volume of this cylinder can be written as $\pi\left(\frac{\mathrm{h_C}}{2}\right)^2\left(\frac{\mathrm{h_B}+\mathrm{h_D}}{2}\right)$. Borehole axis is a boundary and Neumann boundary condition is satisfied here. The approximation is made considering a fictitious grid parallel to this axis of symmetry which has already been stated in the previous section. Let a current of constant strength I be emitted from the

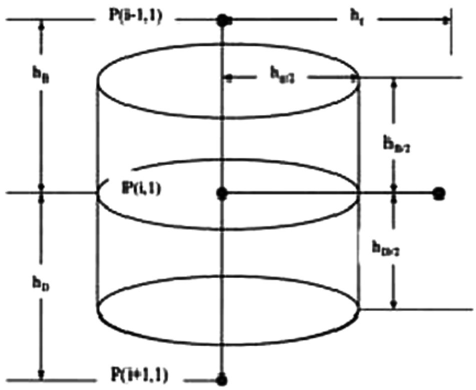


Fig. 15.10. An elementary volume near the point current electrode. On the axis of a borehole

point source $P_{(1,1)}$. The volume density of current q can then be easily calculated by dividing the strength of current at any instance by the volume of the cylinder.

Hence

$$q = \frac{I}{\pi\left(\frac{h_C}{2}\right)^2\left(\frac{h_B + h_D}{2}\right)} \tag{15.45}$$

$$= \frac{8I}{\pi h_C^2\,(h_B + h_D)} \tag{15.46}$$

In the (15.35) and (15.43), the current density factor q only comes in the expression at the current source and sink nodal positions with + and − signs. For other positions of the nodal points, including those occupied by potential electrodes, the factor q is zero. Here sources and sinks are considered on the axis.

15.3.7 Evaluation of the Potential

The finite difference equation (15.35) and (15.43) constructs a large set of linear equations considering each pivotal points of the discretized earth model. Finally this large set of linear equations are arranged in a matrix form

$$A\,x = b \tag{15.47}$$

where, A is the conductivity coefficient banded sparse matrix, b is the column vector where only non zero elements are from the locations of the source and sink.

The conductivity co-efficient matrix A is a pentadiagonal, asymmetric and banded in nature, where four off diagonal and the main diagonal entries have

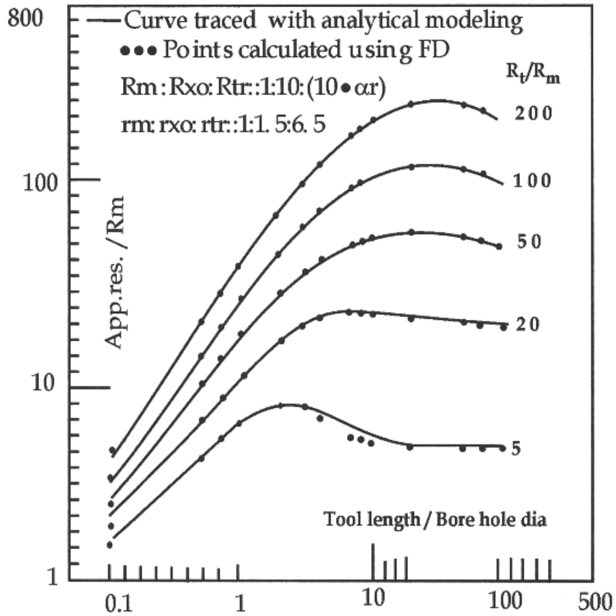


Fig. 15.11. Comparison of analytical and finite difference modelling results

nonzero values. The remaining entries are all zeros. Direct solution of the matrix equation can be done using one of the matrix solvers. For a thick bed, Fig. 15.11 shows the apparent resistivity curves in the presence of flushed zone, invaded zone and uncontaminated zone. Finite difference curves are compared with those available in Schlumberger Well Surveying Corporation Document (1972).

15.4 Finite Difference Formulation Plane Wave Electromagnetics Magnetotellurics

In the last two Sects. 15.2 and 15.3, finite difference formulations both for surface and borehole geophysics are demonstrated for direct current flow field domain in considerable details where the guiding equations were Poisson's equation.

In this section we have given a brief structure of the finite difference formulations for plane wave electromagnetics using Maxwell's equations and Helmholtz electromagnetic wave equations as guiding equations. Boundary conditions are considerably different in electromagnetics. Details of electromagnetic boundary conditions are discussed in the Chap. 12.

In Magnetotellurics (MT) two sets of field data in frequency domain are obtained after continuous time domain recording of electric and magnetic fields in two mutually perpendicular north-south and east-west directions. The ratio of the electric field to the transverse magnetic field in frequency domain are obtained after processing and fourier transforming two sets of time domain E and H fields. One gets impedances, viz, $Z_{xy}(= E_x/H_y)$ and $Z_{yx}(E_y/H_x)$. Madden and Nelson (1960) proposed that magnetotelluric impedance Z is actually a 2×2 MT tensor where Z_{xy} and Z_{yx} are off diagonal elements and Z_{xx} and Z_{yy} are diagonal elements. In 1967 Swift brought the concepts of mathematical rotation of the MT impedance tensor, optimum rotation angle and TE(Transverse Electric) and TM(Transverse Magnetic) mode in 2-D MT. Swift rotation (1967) equation is given by, Vozoff(1972),

$$Z' = MZM^T \tag{15.48}$$

where,

$$Z' = \begin{bmatrix} Z'_{xx} Z'_{xy} \\ Z'_{yx} Z'_{yy} \end{bmatrix}. \tag{15.49}$$

The primed components denote the components of the impedance tensor Z after rotation. Here

$$M = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{ and} \tag{15.50}$$

M^T is the transpose of M .

For a two dimensional (2-D) structures (see Chap. 2) as well as for optimum rotation angle θ, the angle between the geographic north and the strike direction of the 2-D structure, one gets $Z_{xx} = Z_{yy} = 0$ and $\left({Z'_{xy}}^2 + {Z'_{yx}}^2\right) =$ maximum. The rotated apparent resistivity and phases are given by

$$\rho'_{axy} = 0.2T \left|Z'_{xy}\right|^2 \text{ and } \varphi'_{xy} = \tan^{-1}\left(\frac{\mathrm{Im}\left(Z'_{xy}\right)}{\mathrm{Re}\left(Z'_{xy}\right)}\right) \tag{15.51}$$

$$\rho'_{ayx} = 0.2T \left|Z'_{yx}\right|^2 \text{ and } \varphi'_{yx} = \tan^{-1}\left(\frac{\mathrm{Im}\left(Z'_{yx}\right)}{\mathrm{Re}\left(Z'_{yx}\right)}\right). \tag{15.52}$$

Here the rotated primed parameters are the E and H polarization parameters. They are ρ_{TE}, ϕ_{TE} and ρ_{TM}, ϕ_{TM}.

At optimum rotation angle, Helmholtz wave equation $\nabla^2 \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \gamma^2 \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}$ decouples into two separate independent sets of equations. TE or E_{II} stands for transverse electric or E-polarization mode where electric field is parallel to the geological strike. TM or $E_\perp$ stands for transverse magnetic or H-polarization mode (Table 15.1). Here magnetic field is parallel to the strike and electric field is perpendicular to the strike Figs. (15.12) and (15.13 a, b) respectively show the strike direction in an ideal 2D model

Table 15.1. Break up of Helmholtz electromagnetic wave equation for E and H polarisations

E-polarization equations	H-polarization equations
i) $\mathbf{E} = \mathbf{E}(0, \mathrm{E_y}, 0), \mathbf{H} = \mathbf{H}(\mathrm{H_x}, 0, \mathrm{Hz})$	i) $\mathbf{E} = \mathbf{E}(\mathrm{E_x}, 0, \mathrm{Ez}), \mathbf{H} = \mathbf{H}(0, \mathrm{H_y}, 0)$
$\frac{\partial \mathrm{H_x}}{\partial \mathrm{z}} - \frac{\partial \mathrm{H_z}}{\partial \mathrm{x}} = \sigma \mathrm{E_y}$	$-\frac{\partial \mathrm{H_y}}{\partial \mathrm{z}} = \sigma \mathrm{E_x}$
$-\frac{\partial \mathrm{E_y}}{\partial \mathrm{z}} = \mathrm{i}\omega\mu \mathrm{H_x}$	$\frac{\partial \mathrm{H_y}}{\partial x} = \sigma \mathrm{E_z}$
$\frac{\partial \mathrm{E_y}}{\partial \mathrm{x}} = \mathrm{i}\omega\mu \mathrm{H_z}$	$\frac{\partial \mathrm{E_x}}{\partial \mathrm{z}} - \frac{\partial \mathrm{E_z}}{\partial \mathrm{x}} = \mathrm{i}\omega\mu \mathrm{H_y}$
$-\frac{1}{\mathrm{i}\omega\mu}\frac{\partial^2 \mathrm{E_y}}{\partial \mathrm{z}^2} - \frac{1}{\mathrm{i}\omega\mu}\frac{\partial^2 \mathrm{E_y}}{\partial \mathrm{x}^2} = \sigma \mathrm{E_y}$	$-\frac{\partial^2 \mathrm{H_y}}{\partial \mathrm{z}^2} - \frac{\partial^2 \mathrm{H_y}}{\partial \mathrm{x}^2} = \mathrm{i}\omega\mu\sigma \mathrm{H_y}$
$\frac{\partial^2 \mathrm{E_y}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{E_y}}{\partial \mathrm{z}^2} = \gamma^2 \mathrm{E_y}$	$\frac{\partial^2 \mathrm{H_y}}{\partial \mathrm{z}^2} + \frac{\partial^2 \mathrm{H_y}}{\partial \mathrm{x}^2} = \gamma^2 \mathrm{H_y}$
$\mathrm{H_x} = -\frac{1}{\mathrm{i}\omega\mu}\frac{\partial \mathrm{E_y}}{\partial \mathrm{z}}$	$\mathrm{E_x} = -\frac{1}{\sigma}\frac{\partial \mathrm{H_y}}{\partial \mathrm{z}}$
$\mathrm{H_z} = \frac{1}{\mathrm{i}\omega\mu}\frac{\partial \mathrm{E_y}}{\partial \mathrm{x}}$	$\mathrm{E_z} = \frac{1}{\sigma}\frac{\partial \mathrm{H_y}}{\partial \mathrm{x}}$

and the directions of electric and magnetic vectors in E-polarisation and H-polarisation.

Since the low frequencies are involved in MT exploration, conduction currents dominate over displacement currents. The integral forms of Maxwell's equations in mks units can be written as

$$\oint H.dl = \int\int J.dS = \int\int \sigma E.dS \tag{15.53}$$

$$\oint E.dl = \int\int i\mu\omega H.dS, \tag{15.54}$$

where in general, σ and μ are scalars for a homogeneous and isotropic medium and tensors for inhomogenous and anisotropic medium (Chap. 2). These Maxwell's equations and Helmholtz wave equations given in Chaps. 12 and 13

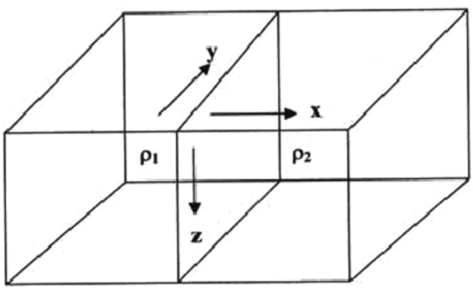


Fig. 15.12. A model of a two dimensional earth with a strike along the y direction; ρ_1 and ρ_2 are the resistivities of the two blocks

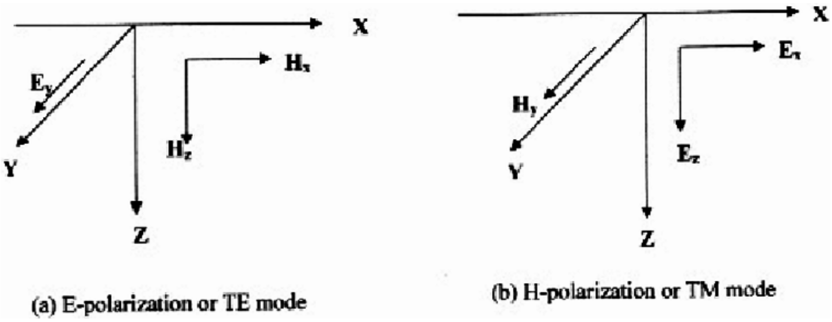


Fig. 15.13. Directions of electric and magnetic fields for E and H polarizations

and in Table 15.1 are used to generate the finite difference modelling structures. We can define a difference scheme such that (15.53) or (15.54) are exactly satisfied. Finite difference scheme in Magnetotellurics proposed by Mackie et al (1993) are as follows (Fig. 15.14 a,b)

a) The x, y, z components of (15.53) are

$$\begin{aligned}
&\{[H_z(i,j+1,k)-H_z(i,j,k)]-[H_y(i,j,k+l)\\
&\qquad -H_y(i,j,k)]\}\,L=J_x(i,j,k)\,L^2\\
&\{[H_x(i,j,k+1)-H_x(i,j,k)]-[H_z(i+1,j,k)\\
&\qquad -H_z(i,j,k)]\}\,L=J_y(i,j,k)\,L^2\\
&\{[H_y(i+1,j,k)-H_y(i,j,k)]-[H_x(i,j+1,k)\\
&\qquad -H_x(i,j,k)]\}\,L=J_y(i,j,k)\,L^2\\
&\{[E_z(i,j,k)-E_z(i,j-1,k)]-E_y(i,j,k)\\
&\qquad -E_y(i,j,k-1)\}\,L=i\omega<\mu_{xx}>H_{xx}(i,j,k)\,L^2
\end{aligned} \tag{15.55}$$

(b)

$$\begin{aligned}
E_x(i,j,k)&=\frac{[\rho_{xx}(i,j,k)+\rho_{xx}(i-1,j,k)]}{2}J_x(i,j,k)\,,\\
E_y(i,j,k)&=\frac{[\rho_{yy}(i,j,k)+\rho_{yy}(i,j-1,k)]}{2}J_y(i,j,k)\,,\\
E_z(i,j,k)&=\frac{[\rho_{zz}(i,j,k)+\rho_{zz}(i,j,k-1)]}{2}J_z(i,j,k)\,.
\end{aligned} \tag{15.56}$$

(c)

$$\begin{aligned}
&\{[E_x(i,j,k)-E_x(i,j,k-1)]-E_z(i,j,k)\\
&\qquad -E_z(i-1,j,k)\}\,L=i\omega<\mu_{yy}>H_{yy}(i,j,k)\,L^2\\
&\{[E_y(i,j,k)-E_y(i-1,j,k)]-E_x(i,j,k)\\
&\qquad -E_x(i,j-1,k)\}\,L=i\omega<\mu_{zz}>H_{zz}(i,j,k)\,L^2
\end{aligned} \tag{15.57}$$

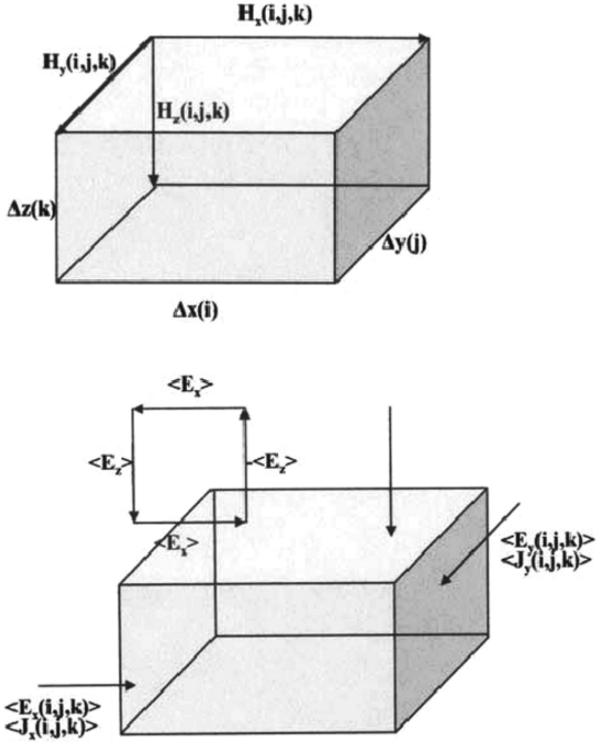


Fig. 15.14. a,b Magnetotelluric finite difference rectangular parallelepiped cells showing the direction of electric and Magnetic Fields (Mackie ,Madden and Wannamaker 1993)

where we define the permeabilities as :

$$< \mu_{xx} > = \frac{\mu_{xx}\,(i, j-1, k-1) + \mu_{xx}\,(i, j, k-1) + \mu_{xx}\,(i, j-1, k) + \mu_{xx}\,(i, j, k)}{4}$$

$$< \mu_{yy} > = \frac{\mu_{yy}\,(i-1, j, k-1) + \mu_{yy}\,(i, j, k-1) + \mu_{yy}\,(i-1, j, k) + \mu_{yy}\,(i, j, k)}{4} \tag{15.58}$$

$$< \mu_{zz} > = \frac{\mu_{zz}\,(i-1, j-1, k) + \mu_{zz}\,(i-1, j, k) + \mu_{zz}\,(i, j-1, k) + \mu_{zz}\,(i, j, k)}{4}$$

Here ρ and μ the resistivity and magnetic permeability are chosen to be tensors. E is defined as the average along a contour and H is defined as the average across the surface enclosed by the contours Once the basic equations for TE and TM mode are available and the guide lines to prepare the difference equations from Maxwell's equations are obtained one can frame the finite difference formulation program .

15.4.1 Boundary Conditions

In TE mode, the boundary conditions are not simple as E_x, H_y and H_z are all continuous across the air – earth interface. Consequently, the horizontal H field is not constant everywhere above the ground in the vicinity of a lateral inhomogeneity. Thus, it is necessary to consider air layers above the ground . An upward extension of the model cross section upto the air – ionosphere boundary is necessary in an attempt to get numerical solutions for E polarization problem. This requires an interface, where lateral changes in conductivity do not exist. Since we need $H_z = 0$ and E_x is constant, and $\partial E_x/\partial y = i\omega\mu_o H_z$, ten air layers are generally introduced(Wannamaker 1987). They are the rows next to the air – earth interface and are of the same height as the rows just below the surface under the assumption that electric current flow across the air – earth interface is negligible. These 10 air layers above the surface of the earth with exponential increased thickness in some cases are chosen such that the geometric perturbations in the longest wavelength H field gets damped. On top of the air layer an arbitrary electric field Eo is assumed. Since all the fields at different depths get normalised, the arbitrary choice is permissible. The lower half space below the air earth boundary are chosen in such a way such that Eo = 0 at the bottom. Generally Eo is chosen as 1 above the air layers. For TM mode modelling air layers are not necessary. Ho also is assumed to be 1(Ho = 1). The boundary values on the side faces of the model are chosen from the values obtained from 1D on the surface of the earth. At the bottom of the half space Ho also is assumed to be zero. Once the boundary conditions are imposed one gets a series of equations which can be clubbed together to get a matrix

$$\mathrm{Ax} = \mathrm{b}, \tag{15.59}$$

where b contains terms associated with the known boundary values and source field. This system of equations is sparse, symmetric and complex (all elements are real except for the diagonal elements). Once this system has been solved for the **H** fields, the **E** fields can be determined by application of curl **H** = J = σE neglecting the displacement current as mentioned. The values of the electric and magnetic field on all the nodes in the model are obtained.It is interesting to note that the column vector in this case has mostly nonzero elements.The matrix A is symmetric in this case. In the DC domain, the matrix A is asymmetric These considerable difference in boundary conditions and formulation do exist between direct current and electromagnetic field domain. Choice of matrix solver may be different. Otherwise step by step structure of the FD formulation in both the cases are more or less the same.

15.5 Finite Element Formulation Direct Current Resistivity Domain

15.5.1 Introduction

The basic concept in the finite element method is that a continuous space domain, is assumed to be composed of a set of piecewise continuous functions defined over a finite number of subdomains or elements. The piecewise continuous space called elements and any function say potential or field or stress or strain are defined using values of continuous quantity at a finite number of point in the solution domain assuming linear or nonlinear variations in polynomials. Discretization of the space domain, elements, nodes, boundary conditions, use of matrix solver are more or less same as those we discussed in Sect. 15.2 and 15.3. But the solution of the problem in finite element (FEM) domain is considerably different from what we have seen in Sect. 15.2 and 15.3. The steps involved in formulating a problem in the finite element domain may be summarised as follows:

1) The solution domain is made finite and divided into a finite number of elements, each having suitable physical property assigned. These elements may be one, two, or three-dimensional according to the problem being considered. The shape of the elements can be one of the many different forms (Zienkiewicz, 1971) viz., triangle, quadrilateral, rectangle, square, tetrahedron, cube, parallelepiped etc with straight or curved boundaries. 2) The elements are interconnected at common nodal points situated at the element boundaries or vertices. A parameter for the unknown potentials to be determined is assigned to each of these elements, and the potentials will be obtained from these nodal points. 3) A polynomial function is chosen to define the behaviour of potential field within the element, in terms of the nodal values. The interpolating polynomial may be linear, quadratic, or cubic. 4) The approximating polynomial functions are then substituted for the true solution into the equation describing the potential or field behaviour. 5) The summation of all the elemental equations gives an approximation to the equation for the continious potential function. A system of equations is obtained from which the nodal potentials may be obtained. 6) One of the approaches viz, Rayleigh Ritz energy minimization method based on variational calculus or the Galerkin's weights are chosen. 7) Natural coordinates with isoparametric elements are also used to bring in curved boundaries. In the present finite element approximation, the cross-section of a 2-D structure is represented by a number of triangular elements. Inside each of these elements a linear behaviour of potential field is assumed. The nodes of the elements are situated at the vertices of the triangles to which these variables are assigned. The approximation of a two dimensional field variable ϕ, within an element, e, may be written in terms of the element unknown parameters ϕ^e as,

$$\phi^e = \left[N_\beta \ \phi_\beta \right] \tag{15.60}$$

where $\beta = i, j, k, \ldots\ldots\ldots r$, r is the number of unknown parameters and $N_\beta, \beta = i, r$ are the element shape functions.

The element shape functions cannot be chosen arbitrarily because these functions have to satisfy the continuity and convergence requirements of the method. More details concerning the types of element shape functions are given in Kardestuncer (1987).

One can derive the finite element form of the governing differential equation of a problem in different ways. In the present finite element analysis the variational approach is used where a functional is minimised. In the next sections, GalerKin's method of finite element analysis without and with use of the isoparametric elements are demonstrated.

In this Rayleigh-Ritz approach, a physical problem, governed by a differential equation, may be equivalently expressed as an extremum problem by the method of calculus of variations. For the steady state field problems, the field equation to be solved is the quasi harmonic equation expressed in general terms as

$$\frac{\partial}{\partial x}\left(\sigma_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma_y \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\sigma_z \frac{\partial \phi}{\partial z}\right) = -\nabla . J_s \tag{15.61}$$

It is shown, by means of the calculus of variations, that a physical problem defined by the above differential equation is identical to that of finding a function ϕ, that minimizes a functional ψ. The same boundary conditions applied to the differential equation are applicable to this integral equation

$$\psi = \int \left[\sigma_z \left(\frac{\partial \phi}{\partial x}\right)^2 + \sigma_y \left(\frac{\partial \phi}{\partial y}\right)^2 + \sigma_z \left(\frac{\partial \phi}{\partial z}\right)^2 - 2\phi\nabla . J_s\right] d\nu. \tag{15.62}$$

Minimization of this functional with respect to the unknown function ϕ, gives the potential ϕ, which also satisfies the differential equation (Coggon 1971). In this resistivity problem, this integral is associated with power dissipation within the conducting medium. The minimum condition thus requires that the potential distribution within the medium is such that the power dissipated (Joule heat in this case) is minimum Reddy (1986). A brief derivation of the power integral expressed in terms of potential and suitable for direct applications of the finite element method is given. Inverse fourier cosine transform discussed in Sect. 15.2 regarding transformation of line source potential to point source potential is applicable here also

15.5.2 Derivation of the Functional from Power Considerations

Coggon (1971), Mwinifumbo (1980), Reddy (1986), Silvester and Haslam (1972), Zienkiewicz and Taylor (1987) have presented the mathematical treatments of the approaches based on the variational calculus. This mathematical treatment is given by Mwinifumbo (1980) The power dissipated per volume ψ_c in a conducting medium with resistivity, ρ and current density, J_c, is

$$\psi_c = \rho J_c^2. \tag{15.63}$$

The relations $J_c = \sigma E$, and $E = -\nabla\phi$ allow the equation to be written as

$$\psi_c = \sigma(\nabla\phi)^2 \tag{15.64}$$

If there is a current source within the volume, the power dissipated per unit volume by the source is

$$\psi_s = -J_s.\nabla\phi. \tag{15.65}$$

The power supplied by the source to the field is equal to the total power dissipated as Joule heat, that is,

$$\sigma(\nabla\phi)^2 + J_s.\nabla\phi = 0. \tag{15.66}$$

We wish to minimize the power dissipated within a volume where there is a source, but the function describing the dissipated power and the power supplied by the source is zero. This problem is handled by a method similar to the Lagrangian multiplier method. Instead of minimizing the Joule heat directly (which does not have the source terms), we construct a new function as follows

$$\sigma(\nabla\phi)^2 + J_s.\nabla\phi = 0 \tag{15.67}$$

which in essence is just the negative of the power supplied to the unit volume. The integral form of power dissipation within the solution domain is then given as

$$\psi = \int_v \left[\sigma(\nabla\phi)^2 + 2J_s.\nabla\phi\right] dv. \tag{15.68}$$

Using the identity

$$\nabla.(\phi J_s) = J_s.\nabla\phi + \phi\nabla.J_s \tag{15.69}$$

the first term on the right hand side of (15.69) may be written as

$$\int_v (J_s.\nabla\phi)\, dv = \int_v [\nabla.(\phi J_s) - \phi\nabla.J_s] dv. \tag{15.70}$$

Applying the divergence theorem to the first term on the right hand side of this equation, we have

$$\int_v (J_s.\nabla\phi)\, dv = \int_v \phi J_s.nds - \int_v \phi\nabla.J_s dv \tag{15.71}$$

since there are no sources on the surface of the solution domain, the surface integral vanishes, then

$$\int_v (J_s.\nabla\phi)\, dv = -\int_v \phi\nabla.J_s dv. \tag{15.72}$$

Equation (15.69) then becomes:

$$\psi = \int_{v} \left[\sigma (\nabla\phi)^2 - 2\phi\nabla.J_s\right] dv. \tag{15.73}$$

The current density in a conducting medium is distributed in such a way that the power dissipated (ohmic power) is minimum. This minimization principle may be written as

$$\Delta\psi = \Delta \int_{v} \left[\sigma (\nabla\phi)^2 - 2\phi\nabla.J_s\right] dv = 0 \tag{15.74}$$

where $\Delta\psi$ is the variation of the total power ψ with change of the function ϕ (x, y, z). This equation is known as the variational equation. In the present study we will be modeling two-dimensional structures in the presence of 3-D sources. If we apply a Fourier cosine transform along the Y direction (strike direction of the 2-D structure being considered), then the variational equation reduces to

$$\Delta\psi = \int_{A} \Delta \left[\sigma \left(\phi_x^2 + \lambda^2\phi^2 + \phi_z^2\right) - 2\phi I\delta(x)\,\delta(z)\right] dA = 0. \tag{15.75}$$

Integration is now over area instead of a volume. By using the variational method, it is shown below that the variation of the power leads to Poisson's differential equation. This is the basis for most methods of calculating potential distributions and is known as Ray leigh - Ritz energy minimisation method.

15.5.3 Equivalence between Poisson's Equation and the Minimization of Power

Integration and taking the variation in the above equation may be interchanged as follows,

$$\Delta\psi = \int_{A} \Delta \left[\sigma \left(\phi_x^2 + \lambda^2\phi^2 + \phi_z^2\right) - 2\phi I\delta(x)\,\delta(z)\right] dA = 0. \tag{15.76}$$

Differentiation and the operation of variation may also be interchanged, thus enabling the (15.76) to be written as

$$\Delta\psi = \int_{A} \left[\sigma \left(\phi_x.\frac{\partial(\Delta\phi)}{\partial x} + \lambda^2\phi(\Delta\phi) + \phi_z.\frac{\partial(\Delta\phi)}{\partial z}\right) - I(\Delta\phi)\,\delta(x)\,\delta(z)\right] dA = 0. \tag{15.77}$$

Integrating the first term under the integral in (15.77) in the square brackets yields

$$\int_A \frac{\partial \phi}{\partial x} . \frac{\partial \Delta \phi}{\partial x} dA = \int_A \left[\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} . \Delta \phi \right) \right] dA - \int_A \left[\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) \Delta \phi \right] dA. \quad (15.78)$$

Applying the divergence theorem to the first integral on the right hand side of (15.78), we get

$$\int_A \left[\frac{\partial \phi}{\partial x} \frac{\partial (\Delta \phi)}{\partial x} \right] dA = \int_L \left[l_x \frac{\partial \phi}{\partial x} \Delta \phi \right] dL - \int_A \left[\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) \Delta \phi \right] dA \quad (15.79)$$

where l_x is the direction cosine of the normal to the exterior boundary with respect to the X-axis, and L is the bounding contour of the solution domain area. A performing similar operations on the third term of the square brackets of (15.77) gives

$$\int_A \left[\frac{\partial \phi}{\partial z} \frac{\partial (\Delta \phi)}{\partial z} \right] dA = \int_L l_z \frac{\partial \phi}{\partial z} \Delta \phi dL - \int_A \left[\frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial z} \right) \Delta \phi \right] dA. \quad (15.80)$$

On substitution of the expression (15.79) and (15.80) into the variational equation we get

$$\begin{aligned} \Delta \psi = & \int_A \left[\sigma \left(\frac{\partial^2 \phi}{\partial x^2} + \lambda^2 \phi + \frac{\partial^2 \phi}{\partial x^2} \right) - I \delta (x) \delta (z) \right] (\Delta \phi) dA \\ & + \int_L \left[\sigma \left(l_x \frac{\partial \phi}{\partial x} + l_z + \frac{\partial \phi}{\partial z} \right) \right] (\Delta \phi) dL = 0 \end{aligned} \quad (15.81)$$

When the boundary values are given, $\Delta \phi = 0$ on L, the line integral vanishes. When the boundary values are not given, the variation of ϕ on L is in general nonzero. Since the variation of ϕ within the solution domain is not necessarily zero, $\Delta \psi = 0$ occurs only if the square bracketed terms within the integral vanishes. This requirement yields the following equations.

$$\begin{aligned} & \sigma \left[\frac{\partial^2 \phi}{\partial x^2} - \lambda^2 \phi + \frac{\partial^2 \phi}{\partial z^2} \right] + I \delta (x) \delta (z)] = 0 \\ & \sigma \left[l_x \frac{\partial \phi}{\partial x} + l_z \frac{\partial \phi}{\partial z} \right] = 0 \end{aligned} \quad (15.82)$$

The first part of (15.82) is Poisson's equation with a Fourier cosine transformation applied along the Y direction and the second part is the natural boundary condition (homogeneous Neumann boundary condition).

15.5.4 Finite Element Formulation

This formulation is based on the work of Mwinifumbo (1980), Zienkiewicz and Taylor (1989), Reddy (1986), Krishnamurthy (1991) Roy and Jaiswal

(2001) (Fig. 15.5). In the present analysis a linear polynomial and triangular elements are used, that is, the function ϕ is taken to vary linearly over the triangular element with nodes at the vertices (Fig. 15.15).

Let a given triangular element with nodes i, j, k and coordinates of the three nodes being (x_i, z_I), (x_j, z_j) and (x_k, z_k), have the following nodal values of the potential function ϕ; ϕ_i, ϕ_j, ϕ_k.

The interpolating polynomial is

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 z \tag{15.83}$$

with the nodal conditions

$$\begin{aligned} \phi &= \phi_I \text{ at } x = x_i,\ z = z_i \\ \phi &= \phi_j \text{ at } x = x_j, z = z_j \\ \phi &= \phi_k \text{ at } x = x_k, z = z_k. \end{aligned} \tag{15.84}$$

Substituting of these nodal values into (15.83), one gets a system of equations.

$$\begin{aligned} \phi_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 z_i \\ \phi_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 z_j \\ \phi_k &= \alpha_1 + \alpha_2 x_k + \alpha_3 z_k \end{aligned} \tag{15.85}$$

which yield

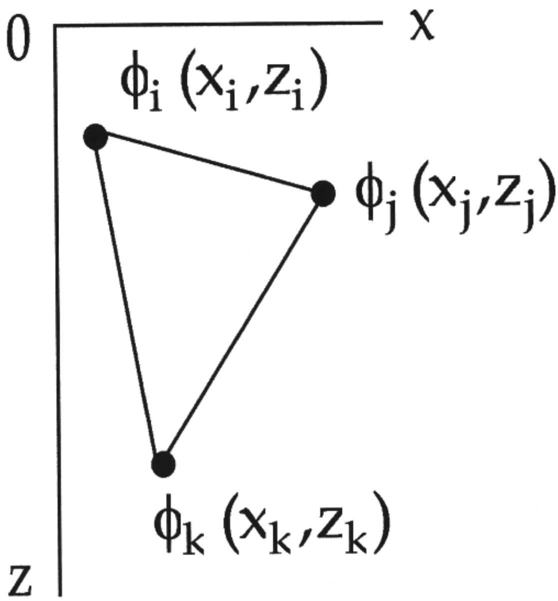


Fig. 15.15. A triangular finite element showing the coordinates of the nodes

$$\alpha_1 = \frac{1}{2A}\left[(x_j z_k - x_k z_j)\,\phi_i + (x_k z_i - x_i z_k)\,\phi_j + (x_i z_j - x_j z_i)\,\phi_k\right]$$
$$\alpha_2 = \frac{1}{2A}\left[(z_j - z_k)\,\phi_i + (z_k - z_i)\,\phi_j + (z_i - z_j)\,\phi_k\right]$$
$$\alpha_3 = \frac{1}{2A}\left[(x_k - x_j)\,\phi_i + (x_j - x_k)\,\phi_j + (x_j - x_i)\,\phi_k\right] \tag{15.86}$$

where 2A is equal to

$$2A = \begin{vmatrix} 1 & x_i & z_i \\ 1 & x_j & z_j \\ 1 & x_k & z_k \end{vmatrix} \tag{15.87}$$

A being the area of the triangle.

When the value of α_1, α_2 and α_3 are substituted into the expression for the interpolating polynomial, the following element equation is obtained after the terms have been rearranged i.e.,

$$\phi^e = N_i\phi_i + N_j\phi_j + N_k\phi_k \tag{15.88}$$

where ϕ^e is the potential over the element, e, and N_i, N_j and N_k are the three shape functions associated with each node of the triangular element and are defined as follows:

$$N_i = \frac{1}{2A}\left(a_i + b_i x + c_i z\right)$$
$$N_j = \frac{1}{2A}\left(a_j + b_j x + c_j z\right) \tag{15.89}$$

and

$$N_k = \frac{1}{2A}\left(a_k + b_k x + c_k z\right)$$

where,

$$a_i = x_j z_k - x_k z_j$$
$$b_i = z_j - z_k \tag{15.90}$$
$$c_i = x_k - x_j$$
$$a_j = x_k z_i - x_j z_k$$
$$b_j = z_k - z_i \tag{15.91}$$
$$c_j = x_i - x_k$$

and

$$a_k = x_i z_j - x_j z_i$$
$$b_k = z_i - z_j \tag{15.92}$$
$$c_k = x_j - x_i.$$

The potential function ϕ is thus a function of a set of shape functions which are linear in x and z. This means that the gradients in either x or z direction will be constant. These are determined as follows:

$$\begin{aligned}\frac{\partial\phi^e}{\partial x} &= \frac{\partial N_i}{\partial x}\phi_i + \frac{\partial N_j}{\partial x}\phi_j + \frac{\partial N_k}{\partial x}\phi_k\\ \frac{\partial\phi^e}{\partial z} &= \frac{\partial N_i}{\partial z}\phi_i + \frac{\partial N_j}{\partial z}\phi_j + \frac{\partial N_k}{\partial z}\phi_k\end{aligned} \tag{15.93}$$

The partial derivatives of the shape functions with respect to x and z axes are evaluated as follows:

$$\begin{aligned}\frac{\partial N_\beta}{\partial x} &= \frac{1}{2A}b_\beta\\ \frac{\partial N_\beta}{\partial z} &= \frac{1}{2A}c_\beta\end{aligned} \tag{15.94}$$

$\beta = i, j, k.$

Therefore, (15.93) reduce to

$$\begin{aligned}\frac{\partial\phi^e}{\partial x} &= \frac{1}{2A}\left(b_i\phi_i + b_j\phi_j + b_k\phi_k\right)\\ \frac{\partial\phi^e}{\partial z} &= \frac{1}{2A}\left(c_i\phi_i + c_j\phi_j + c_k\phi_k\right).\end{aligned} \tag{15.95}$$

The parameters b_i, b_j, b_k, c_i, c_j, c_k are constants (they are fixed once the nodal coordinates are specified) and ϕ_i, ϕ_j and ϕ_k are independent of the space coordinates. Figure 15.16 shows Dirichlet and Neuman boundaries in a FEM problem with triangular elements as shown in Sect. 15.2.

15.5.5 Minimisation of the Power

When we minimize the power defined by (15.68), it is only required to derive that the necessary conditions for a typical element. Influence of other elements will follow an identical pattern. To obtain the general equation, we simply add contributions of all the elements.

The power dissipated in an element, e, defined by nodes i, j, and k is given as (if there are no sources within the element).

$$\psi^e = \int\int_e \sigma\left[\left(\frac{\partial\phi^e}{\partial x}\right)^2 + \lambda^2\phi^{e2} + \left(\frac{\partial\phi^e}{\partial z}\right)^2\right]\mathrm{d}x\mathrm{d}z. \tag{15.96}$$

On substitution of the nodal values and their derivatives (15.96) becomes

$$\begin{aligned}\psi^e = \iint_e \frac{\sigma}{4A^2}\Big[&\left(b_i\phi_i + b_j\phi_j + b_k\phi_k\right)^2 + \lambda^2\left(N_i\phi_i + N_j\phi_j + N_k\phi_k\right)\\ &+ \left(c_i\phi_i + c_j\phi_j + c_k\phi_k\right)^2\Big]\,dx\,dz.\end{aligned} \tag{15.97}$$

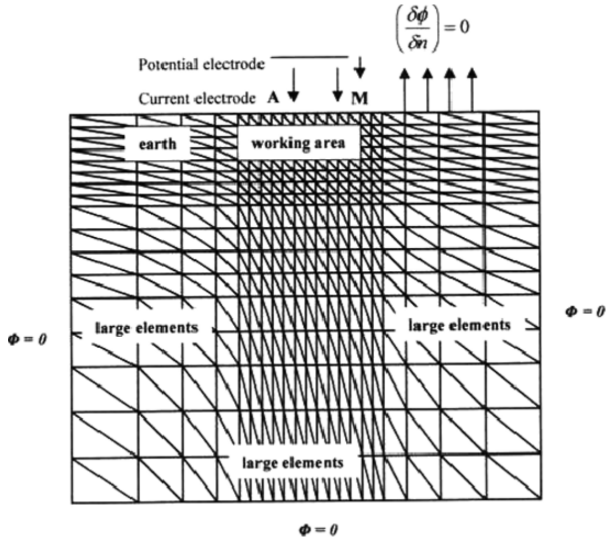


Fig. 15.16. A finite triangular element mesh showing the working and adjoining areas, Dirichlet and Neumann conditions at the boundaries

To obtain the minimum of the power within an element, the differentials of ψ^e with respect to the nodal parameters ϕ_i, ϕ_j, ϕ_k are evaluated first. To illustrate the procedure of obtaining these differentials, (15.96) is differentiated with respect to the nodal values

$$\phi.\frac{\partial\psi^e}{\partial\phi_i} = \frac{\sigma}{2A}\left[\left(b_i^2 + c_i^2\right)\phi_i + \left(b_2 b_j + c_2 c_j\right)\phi_j + \left(b_j b_k + c_k c_i\right)\phi_k\right]\iint_e dxdz$$
$$+ 2\sigma x^2 \iint_e \left[N_i^2 K\phi_i + N_i N_j K_j +\right]. \tag{15.98}$$

The integrals in the above equation are taken over the area of the element. Integration is straight forward when the triangular area coordinates, L_1, L_2, L_3 (Zienkiewicz (1971)) are used. These coordinates specify the position of any point within the triangle by giving the distance measured perpendicularly from each side to the point, distances being expressed as fractions of the triangle altitude.

The area coordinates span the range of numerical values from 0 to 1 in any triangle. For a triangular element with corners numbered 1, 2 and 3, these area coordinates are related to the x and z coordinates by

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} (x_2 z_3 - x_3 z_2) & (z_2 - z_3) & (x_3 - x_2) \\ (x_3 z_1 - x_1 z_3) & (z_3 - z_1) & (x_1 - x_3) \\ (x_1 z_2 - x_2 z_1) & (z_1 - z_2) & (x_2 - x_1) \end{bmatrix}\begin{bmatrix} 1 \\ x \\ z \end{bmatrix}. \tag{15.99}$$

Using these coordinate variables, which are also the shape functions for the linear element; that is

$$\mathrm{N_i = L_1}$$
$$\mathrm{N_j = L_2}$$
$$\mathrm{N_k = L_3}$$

We have

$$\int_{\mathrm{A}} \mathrm{L_1^a L_2^b L_3^c dA} = \frac{\mathrm{a!b!c!}}{\mathrm{(a+b+c+2)!}} \mathrm{2A} \tag{15.100}$$

where a, b, c are the powers of the coordinates Using the above equation the integrals in (15.98) are evaluated as

$$\int\int_{\mathrm{e}} \mathrm{dxdz = A}$$
$$\int\int_{e} \left[N_i^2 \phi_i + N_i N_j \phi_j + N_i N_k \phi_k\right] dxdz = \frac{A}{6}\phi_i + \frac{A}{12}\phi_j + \frac{A}{12}\phi_R. \tag{15.101}$$

Equation (15.98) reduces to

$$\frac{\partial \psi_e}{\partial \phi_i} = \frac{\sigma}{2A}\left(b_i + t_i j \phi_j + t_i k \phi_k\right) + \frac{\lambda^2 \sigma A}{6}\left(2\phi_i + \phi_j + \phi_k\right) \tag{15.102}$$

where

$$\begin{aligned} \mathrm{t_{ii}} &= \mathrm{(z_j - z_k)^2 + (x_k - x_j)^2} \\ \mathrm{t_{ij}} &= \mathrm{(z_j - z_k)(z_k - z_j) + (x_k - x_i)(x_i - x_k)} \\ \mathrm{t_{ik}} &= \mathrm{(z_j - z_k)(z_i - z_j) + (x_k - x_i)(x_j - x_i).} \end{aligned} \tag{15.103}$$

For the whole element i, j, k the differentiation of the power integral is with respect to the parameters ϕ_i, ϕ_j, ϕ_k. In matrix notation, the elemental system of equations may be written as

$$\left[\frac{\partial \psi}{\partial \phi}\right]^{\mathrm{e}} = \begin{bmatrix} \frac{\partial \psi^{\mathrm{e}}}{\partial \phi_{\mathrm{i}}} \\ \frac{\partial \psi^{\mathrm{e}}}{\partial \phi_{\mathrm{j}}} \\ \frac{\partial \psi^{\mathrm{e}}}{\partial \phi_{\mathrm{k}}} \end{bmatrix}. \tag{15.104}$$

The coefficients of ϕ's for each element form two symmetric (3×3) element matrices

$$\frac{\sigma}{2A}\begin{bmatrix} \left(b_i^2 + c_i^2\right) & (b_i b_j + c_i c_j) & (b_i b_k + c_i c_k) \\ (b_i b_j + c_i c_j) & \left(b_j^2 + c_j^2\right) & (b_j b_k + c_j c_k) \\ (b_i b_k + c_i c_k) & (b_j b_k + c_j c_k) & \left(b_k^2 + c_k^2\right) \end{bmatrix} and \frac{\lambda^2 \sigma A}{6}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \tag{15.105}$$

The final equations for minimising total power within the solution domain are assembled by adding the contributions of each element to a typical differential and equating the result to zero. Thus

$$\frac{\partial \psi}{\partial \phi_i} = \sum_{\beta=1}^{N} \left[\frac{\partial \psi_\beta}{\partial \phi_i} \right] = 0 \tag{15.106}$$

where N is the number of elements sharing the node i. It should be noted that for a particular node i only the values of ϕ at nodes connected to it will appear and that the coefficients will involve only contributions from the elements adjacent to the node being considered. This results into a typical narrow banded coefficient matrix for a set of simultaneous equations. If the potential function ϕ is to be determined at M number of nodes, then a set of M linear simultaneous equations result. The required values $\phi_1, \phi_2 \ldots . \phi_M$ are obtained upon solving these equations. In matrix form, the equations may be written as

$$[\mathrm{T}]\{\phi\} = \{\mathrm{S}\} \tag{15.107}$$

where [T] is the matrix of coefficients of the equation, ϕ is the column vector of M unknown potential values and S is the column vector containing source terms extracted from the right-hand side of (15.107). The matrix [T] is a large, symmetric, diagonally dominant, penta diagonal and sparse matrix.

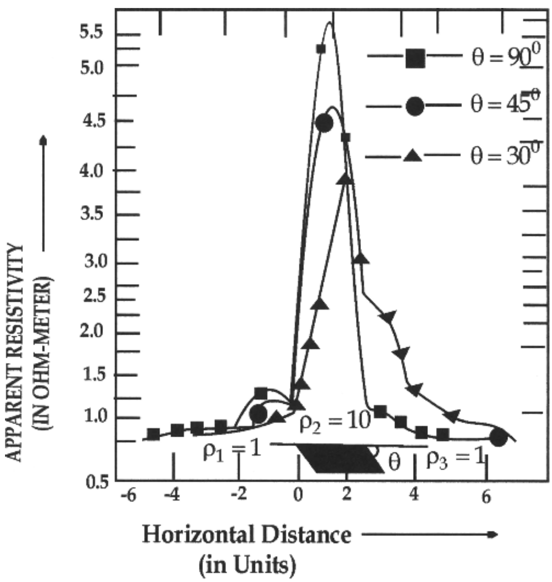


Fig. 15.17. Two electrode profile over a dipping dyke

That is, most of the terms in any one row are zeros and the non-zero terms are restricted to a band about the diagonal. Only non-zero terms need to be stored in along with appropriate locations in the matrix. That reduces considerable amount of storage space needed on the computer. Since the matrix is symmetrical about the diagonal, only one half of the non-zero terms need to be stored . Integral transform to change the line source response to point source response, implementation of Dirichlet and Neumann boundary conditions and generation of expanding grids are more or less the same as discussed in Sect. 15.2. Figure (15.17) show the theoretically computed two electrode profiles across a dipping dyke using FEM source code.

15.6 3D Model

Three Dimensional space is divided into discrete tetrahedral elements. The space consists of a working volume with small tetrahedrons and the outer volume with large and exponentially expanding elements intended for application of the Dirichlet's boundary conditions. The process of division is two fold. The whole volume is first divided into octahedral elements. These octahedral elements are divided into five tetrahedrons each. The division of these octahedral elements leads to two different arrangements of the tetrahedrons (Fig. 15.18a,b). In order to generate the stiffness matrix, a suitable polynomial approximation is chosen to fit the potential variation within the tetrahedral elements. Standard linear polynomial is given by

$$\varphi = \alpha_1 + \alpha_2 \mathrm{x} + \alpha_3 \mathrm{y} + \alpha_4 \mathrm{z} \tag{15.108}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are constants. On this basis we prepare the element stiffness and hence the global stiffness matrix as follows: Consider a tetrahedral element having the coordinates $(\mathrm{x_I},\ \mathrm{y_I},\ \mathrm{z_I})$, $(\mathrm{x_j},\ \mathrm{y_j},\ \mathrm{z_j})$, $(\mathrm{x_k},\ \mathrm{y_k},\ \mathrm{z_k})$ and (x_p, y_p, z_p) with potential values ϕ_i, ϕ_j, ϕ_m and ϕ_p at i^{th}, j^{th}, m^{th} and p^{th} node respectively.

The nodal conditions are

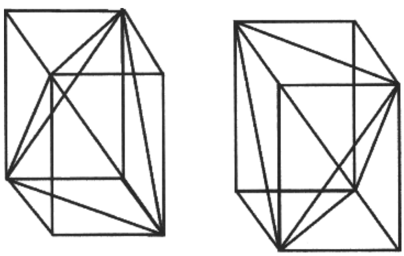


Fig. 15.18. Tetrahedral elements for a 3D Finite element problem

$$\begin{aligned}
\phi &= \phi_i \text{ at } x = x_i, y = y_i, z = z_i \\
\phi &= \phi_j \text{ at } x = x_j, y = y_j, z = z_j \\
\phi &= \phi_m \text{ at } x = x_m, y = y_m, z = z_m \\
\phi &= \phi_p \text{ at } x = x_p, y = y_p, z = z_p.
\end{aligned} \tag{15.109}$$

Writing in matrix form

$$\begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \\ \phi_p \end{bmatrix} = \begin{bmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_m & y_m & z_m \\ 1 & x_p & y_p & z_p \end{bmatrix} \times \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \tag{15.110}$$

Or

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \times \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \\ \phi_p \end{bmatrix}. \tag{15.111}$$

The power dissipation in an element (with no sources within) is given by

$$\psi^e = \int_v o[(\frac{\partial \phi^e}{\partial x}) + (\frac{\partial \phi^e}{\partial y})^2 + (\frac{\partial \phi^e}{\partial z})^2]dv. \tag{15.112}$$

To obtain minimum power within an element, the derivatives of ψ^e with respect to ϕ_i, ϕ_j, ϕ_m and ϕ_p are equated to zero.

$$\left[\frac{\partial \psi}{\partial \phi}\right]^e = \begin{bmatrix} \frac{\partial \psi^e}{\partial \phi_i} \\ \frac{\partial \psi^e}{\partial \phi_i} \\ \frac{\partial \psi^e}{\partial \phi_m} \\ \frac{\partial \psi^e}{\partial \phi_p} \end{bmatrix}. \tag{15.113}$$

Alternatively the element stiffness matrix is

$$\left[\frac{\partial \psi}{\partial \phi}\right]^e = V \times \begin{bmatrix}
a_{21}a_{21} + a_{31}a_{31} + a_{41}a_{41} & a_{21}a_{22} + a_{31}a_{32} + a_{41}a_{42} & a_{21}a_{23} + a_{31}a_{33} + a_{41}a_{43} & a_{21}a_{24} + a_{31}a_{34} + a_{41}a_{44} \\
a_{21}a_{22} + a_{31}a_{32} + a_{41}a_{42} & a_{22}a_{22} + a_{32}a_{32} + a_{42}a_{42} & a_{22}a_{23} + a_{32}a_{33} + a_{42}a_{43} & a_{22}a_{24} + a_{32}a_{34} + a_{42}a_{44} \\
a_{21}a_{23} + a_{31}a_{33} + a_{41}a_{43} & a_{22}a_{23} + a_{32}a_{33} + a_{42}a_{43} & a_{23}a_{23} + a_{33}a_{33} + a_{43}a_{43} & a_{23}a_{24} + a_{33}a_{34} + a_{43}a_{44} \\
a_{21}a_{24} + a_{31}a_{34} + a_{41}a_{44} & a_{22}a_{24} + a_{32}a_{34} + a_{42}a_{44} & a_{23}a_{24} + a_{33}a_{34} + a_{43}a_{44} & a_{24}a_{24} + a_{34}a_{34} + a_{44}a_{44}
\end{bmatrix} \tag{15.114}$$

The element stiffness matrix is mapped into the global stiffness matrix element by element. Assembly of all the equations gives

$$[A].[\phi] = [S]. \tag{15.115}$$

For a 3-D problem with tetrahedral elements,the matrix A will be a hepta diagonal,diagonally dominant sparse symmetric matrix.Suitable matrix solver may be used to obtain ϕ at all the nodes.

15.7 Finite Element Formulation Galerkin's Approach Magnetotellurics

15.7.1 Introduction

Coggon (1971), Silvester and Heslam (1972), Rodi (1975), Pelton et al (1978), Wannamaker (1984, 1987), Kikkonen (1977) did mathematical formulaton on finite element method for geo-electric and geoelectromagnetic problems. In this section we present a brief description of the finite element method for TE and TM mode magnetotellurics (Plane wave normal incidence electromagnetics). The finite element approach for solution of the Helmoltz wave equation is presented.

Let us consider TE mode MT in which $E_x = E_z = H_y = \partial/\partial y = 0$. For harmonic fields in the case of plane wave electromagnetics, we get

$$\partial E_y/\partial z = i\omega\mu\ H_x \tag{15.116}$$

$$\partial E_y/\partial z = -i\omega\mu\ H_z \tag{15.117}$$

and

$$\partial H_x/\partial z - \partial H_z/\partial x = i\omega\varepsilon\ E_y. \tag{15.118}$$

We get

$$\partial/\partial x(1/i\omega\mu\ \partial E_y/\partial x) + \partial/\partial z(1/i\omega\mu\ \partial E_y/\partial z) - i\omega \in\ E_y = 0. \tag{15.119}$$

For TM mode, we get

$$E_y = H_x = H_z = \partial/\partial y = 0$$

and

$$\partial H_y/\partial z = -i\omega \in E_x \tag{15.120}$$

$$\partial H_y/\partial x = i\omega \in\ E_z \tag{15.121}$$

and

$$\partial E_x/\partial z - \partial E_z/\partial x = -i\omega\mu\ H_y. \tag{15.122}$$

Substituting (15.120) and (15.121) in (15.122), we obtain

$$\partial/\partial x(1/i\omega \in \partial H_y/\partial x) + \partial/\partial z(1/i\omega \in\ \ \partial H_y/\partial z) - i\omega\mu\ H_y = 0. \tag{15.123}$$

The general expression for two dimensional Helmoltz equation is

$$\partial/\partial x(1/k\ \partial f/\partial x) + \partial/\partial y(1/k\ \partial f/\partial y) + p(f) = S. \tag{15.124}$$

15.7.2 Finite Element Formulation for Helmholtz Wave Equations

This finite element formulation is based on the work of Wannamaker (1984, 1986 and 1987) The finite element formulation is constructed as follows:

(a) The region is divided into finite number of sub-domains selected here as triangular elements. These elements are connected at common node points and collectively form the shape of the region.
(b) The continuous unknown function 'f' is approximated over each element by polynomials selected here as linear polynomials. These polynomials are defined using the noded values of the continuous function 'f'. The value of a continuous function 'f' at each nodal point is denoted as a variable which is to be determined.
(c) The equations for behaviours of field over each element are derived from the Helmholtz equation using linear polynomials.
(d) The regions of application of Neumann and Dirichlet boundary conditions are established.
(e) Element equations are converted into element matrix equations.
(f) The matrix element equations are assembled to form the global matrix equations.
(g) The boundary conditions are introduced.
(h) The system of linear equations are solved.

One important aspect of the finite element method is the design of the discretized domain i.e. the construction of finite element mesh. The construction of the mesh is problem dependent. The working domain is discretized with finite elements of different 2-D or 3-D shapes depending upon the dimension of the problem. The size of the mesh must be variable and near the discontinuities the mesh size should be finer. The area where there is no inhomogeneity the mesh can be coarser. For two dimensional bodies triangular, rectangular, hexagonal meshes can be used. For 3-D bodies cubical parallelopiped, tetrahedral elements can be used. Depending upon the nature of complexity complicated isoparametric elements with 8 nodes, 20 nodes, 32 nodes cubic elements can be used as shown in the next section. The connecting points of all the elements are nodes. In the discretized domain we try to find out the fields or potentials at these nodal points.

In the following section we present the basics of finite element formulation for magnetotelluric boundary value problems using triangular elements. Initial part of the formulation is same as that outlined in the previous section. In this section the Galerkins methods is used. So the formulation takes a different path. The elements are triangular, the simplest elements for a two dimensional problem. The guiding equations are electromagnetic wave equations and Maxwell's electromagnetic equations. The boundary conditions are mentioned in the Sect. 15.4.

We assume an arbitrary triangular element (e) within the finite domain with nodes at the vertices of the triangles (Fig. 15.19).

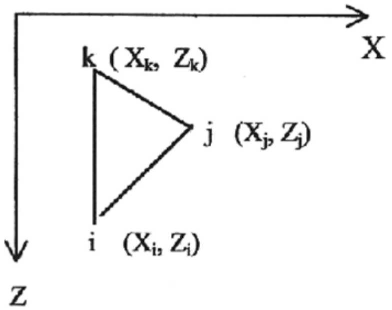


Fig. 15.19. Triangular finite element for MT field formulation

With this type of domain discretization we allowed the functional to vary linearly over each element. The plane passing through the nodal values of 'f' attached with each element (e) can be described by the equation:

$$f^e(x, y) = a_1^{(e)} + a_2^{(e)} X + a_3^{(e)}\ Z. \tag{15.125}$$

At the three nodes the values of the assumed polynomials are:

$$f_i^{(e)}(x, y) = \alpha_1^{(e)} + \alpha_2^{(e)} X_i + \alpha_3^{(e)}\ Z_i \tag{15.126}$$

$$f_j^{(e)}(x, y) = \alpha_1^{(e)} + \alpha_2^{(e)} X_j + \alpha_3^{(e)} Z_j \tag{15.127}$$

$$f_k^{(e)}(x, y) = \alpha_1^{(e)} + \alpha_2^{(e)} X_k + \alpha_3^{(e)}\ Z_k.$$

We solve these equations for α_1, α_2 and α_3 and insert these values in (15.125) to get

$$f^e = N_i^{(e)} f_i^{(e)} + N_j^{(e)} f_j^{(e)} + N_k^{(e)} f_k^{(e)} \tag{15.128}$$

in which

$$N_i^{(e)} = 1/2\Delta(a_i + b_i\ X + c_i Z) \tag{15.129}$$

where

$$a_i = x_j z_k - x_k z_j \tag{15.130}$$

$$b_i = z_{j-} z_k$$

$$c_i = x_k - x_j \tag{15.131}$$

and Δ is the area of the triangular element (e). Like wise the terms $N_j^{(e)}$ and $N_k^{(e)}$ are obtained through a cyclic permutations of the subscripts i, j and k. The functions $N_i^{(e)}$, $N_j^{(e)}$ and $N_k^{(e)}$ are called the shape functions, interpolation function or the basis function.

Equation (15.128) can be expressed as the matrix equation in the form

$$f^{(e)} = N_{(e)} f_{(e)}^t. \tag{15.132}$$

In which

$$\mathrm{N}_{(\mathrm{e})} = [\mathrm{N}_{\mathrm{i}}^{(\mathrm{e})}\ \mathrm{N}_{\mathrm{j}}^{(\mathrm{e})}\ \mathrm{N}_{\mathrm{k}}^{(\mathrm{e})}] \tag{15.133}$$

and

$$\mathrm{f}_{\mathrm{e}}^{\mathrm{T}} = [\mathrm{f}_{\mathrm{i}}^{(\mathrm{e})}\ \mathrm{f}_{\mathrm{j}}^{(\mathrm{e})}\ \mathrm{f}_{\mathrm{k}}^{(\mathrm{e})}]. \tag{15.134}$$

If the domain contains M triangular elements, the complete representation of the unknown function 'f' over the whole domain is given by

$$f(x,z) = \sum_{e=1}^{M} f^{e}(x,z) = \sum_{e=1}^{M} N_e f_e^T. \tag{15.135}$$

This is the function 'f' defined over whole domain. We shall derive the element equations.

15.7.3 Element Equations

The domain equation may be written in the concise form as

$$\mathrm{L\ f = s} \tag{15.136}$$

In which

$$L = \frac{\partial}{\partial x}\left(\frac{1}{k}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{k}\frac{\partial}{\partial y}\right) + P \tag{15.137}$$

Inserting the approximate value of f(x, z) given by (15.136) and (15.137) we get

$$\mathrm{L}(\sum \mathrm{N}_{(\mathrm{e})}\ \mathrm{f}_{(\mathrm{e})}^{\mathrm{T}}) - \mathrm{s} = \boldsymbol{\varepsilon} \tag{15.138}$$

in which $\boldsymbol{\varepsilon}$ is the residual error to be minimized. Our way to accomplish this objective is to use the inner product or dot-product between the error vector and the function. The dot product becomes zero when they are orthogonal or

$$< N_j^e, \varepsilon >= \int\int_{(e)} N_h^{(e)} \varepsilon dx dz = 0 \tag{15.139}$$

for each of the basis function $\mathrm{N}_{\mathrm{h}}^{(\mathrm{e})}$. This integral mathematically states that the basis function must be orthogonal to the error over the element (e). This is Galerkin's method.

Using (15.136) to (15.139), we can write

$$\int\int_{(e)} [N_n^e[\frac{\partial}{\partial x}(\frac{1}{k}\frac{\partial f^e}{\partial x})] + Pf^{(e)} - s] dx dz = 0 \tag{15.140}$$

in which (e) is the triangular region and $\mathrm{n = i, j}$ and k.

Applying integration by parts, we get

$$\iint_{(e)} N_n^{(e)} \frac{\partial}{\partial x}(\frac{1}{k}\frac{\partial f^{(e)}}{\partial x})dxdz = -\iint \frac{1}{k}\frac{\partial f^{(e)}}{\partial x}\frac{\partial N_n^{(e)}}{\partial x}dxdz + \oint \frac{1}{k}\frac{\partial f^{(e)}}{\partial x}n_x N_n^{(e)} dl \tag{15.141a}$$

in which n_x is the x-component of the unit normal to the boundary, dl is a differential arc length along the boundary. When we treat the second term in (15.140) in the same manner, the (15.140) takes the form

$$\iint_{(e)} [-\frac{1}{k}(\frac{\partial f^{(e)}}{\partial x}\frac{\partial N_n}{\partial x} - \frac{\partial f^{(e)}}{\partial z}\frac{\partial N_n^{(e)}}{\partial z}) + N_n(Pf^{(e)} - s)]dxdz$$

$$+ \oint \frac{1}{k}\frac{\partial f^{(e)}}{\partial n}N_n^{(e)} dl = 0 \tag{15.141b}$$

in which

$$\frac{\partial f^{(e)}}{\partial n} = \frac{\partial f^{(e)}}{\partial x}n_x + \frac{\partial f^{(e)}}{\partial z}n_z. \tag{15.142}$$

With the surface integral in (15.141b) and selected boundary conditions, we can write (15.141b) using (15.140,15.141a)

$$\iint -\frac{1}{k}(\frac{\partial N_{(e)}}{\partial x}f^T\frac{\partial N_n^{(e)}}{\partial x} + \frac{\partial N_{(e)}}{\partial z}f^T\frac{\partial N_n^{(e)}}{\partial z})dxdz + \iint_{(e)} PN_{(e)}f_{(e)}^T N_n^{(e)} dxdz$$

$$- \iint_{(e)} SN_n^{(e)} dxdz = 0 \tag{15.143}$$

in which $n = i, j, k$. We can rewrite (15.143) in the matrix form as

$$(K_{(e)} + P_{(e)})f_{(e)}^T = S_{(e)}^T \tag{15.144}$$

in which the matrices $K_{(e)}$, $P_{(e)}$ and vector $S_{(e)}^T$ have the entities

$$K_{ij} = -\iint_{(e)} \frac{1}{k}(\frac{\partial N_i}{\partial x}\cdot\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z})dxdz \tag{15.145}$$

$$P_{ij} = \iint_{(e)} PN_iN_j dxdz \tag{15.146}$$

and

$$S_i = \iint_e SN_i dxdz. \tag{15.147}$$

Assuming K and P to be constant within the element e and using (15.144) to (15.146) and the integral

$$\iint\limits_{(e)} N_i^a N_j^b N_k^c dxdz = \frac{2a!b!c!\Delta}{(a+b+c+2)!} \tag{15.148}$$

in which Δ is the area of the element, we may write (15.134) to (15.145) as

$$-\frac{1}{4k\Delta}\begin{bmatrix} b_i^2+c_i^2 & b_ib_j+c_ic_j & b_ib_k+c_ic_k \\ & b_j^2+c_j^2 & b_jb_k+c_jc_k \\ & & b_k^2+c_k^2 \end{bmatrix}$$
$$+\frac{P\Delta}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} f_{(e)}^T + S_{(e)}^T = 0. \tag{15.149}$$

It is interesting to note that (15.149) is applicable for MT, DC resistivity, EM etc. , but source term will be different in different problems. Imposition of boundary condition will also be different for different problems. For magnetotelluric problem $\mathrm{S}_{(\mathrm{e})}^{\mathrm{T}}$ is identically zero.

Equation (15.149) is known as element matrix equation for the triangular element (e). For each element in the domain an equation of the form (15.148) can be derived. These equations can be assembled (summed) into a single matrix equation. Details of the assemblage of the global matrix equation are discussed in the Sect. 15.5 and given in Zienkiewicz (1971). The global matrix equation can be written as

$$\mathrm{Gf} = \mathrm{S} \tag{15.150}$$

where G is a N×N symmetric, sparse, banded and diagonally dominant matrix and N is the total number of nodal points in the entire discretized domain. The vector f is a column vector of N unknown values of the function f (x, z) at each node of the model. The vector S is a column vector that contains the source information given by adding the contribution of all the source terms

$$S_{(e)} = \frac{\alpha I}{360}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{15.151}$$

where α is the angle subtended by element e. I is the current strength. It is applicable for DC resistivity and not for MT. For the global matrix, all the boundary conditions demanded by the problem must be introduced first and then the matrix equation is solved using one of the suitable matrix solver viz., (i) Gauss Elimination (ii) Gauss-Siedel Iteration (iii) Cholesky's Decomposition (iv) Conjugate Gradient Minimization etc. Thus the unknown value of 'f' i.e. E or H in this case of MT and potential in the case of DC resistivity at all the nodal points will be obtained. Boundary conditions for MT and DC resistivity are considerably different as discussed already.

15.8 Finite Element Formulation Galerkin's Approach Isoparametric Elements Magnetotellurics

15.8.1 Introduction

In this section a brief presentation on the structure of finite element formulation using isoparametric elements and Galerkin's method is given. Since Galerkin's method is discussed in the previous section with all it's essential details, some of the steps will be avoided while demonstrating the finite element formulation using eight noded isoparametric elements (Murthy 2000). Basic structure of formulation is based on the work of Wannamaker et al (1987). All the details about mesh generation, boundary conditions in TE and TM mode Plane wave electromagnetics , governing differential equations are given in Sects. 15.4 and 15.7. A few points about isoparametric elements and natural coordinates are added here.

To simulate a complicated and sharp curvatures of a boundary of a body of irregular shape, one needs numerous small elements straight edges to reduce the difference in shape of the actual and simulated body. Isoparametric elements can significantly reduce the number of elements to be taken because the elements can take care of curved boundaries of a body very effectively. These elements are typically meant for bodies of arbitrary shapes. In isoparametric domain the coordinates are called natural coordinates and they are nondimensional. These elements are called isoparametric because the number of nodes in an element in cartisian coordinate and natural coordinates are same.

For any variable function Φ within a triangular parent element (simplest 2D element) the prescribed values at the nodes can be obtained assuming a linear polynomial function within the element as

$$\Phi = \alpha_1 + \alpha_2 \xi + \alpha_3 \zeta. \qquad (15.152)$$

In this Fig. (15.20)

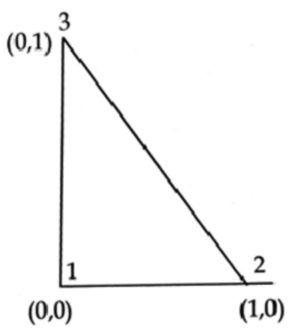


Fig. 15.20. A simplest triangular element with natural coordinate

Since

$$\begin{aligned}
\Phi &= \Phi_1 \text{ at node } 1(\xi = 0, \zeta = 0),\\
\Phi &= \Phi_2 \text{ at node } 2(\xi = 1, \zeta = 0)\\
\Phi &= \Phi_3 \text{ at node } 3(\xi = 0, \zeta = 1),
\end{aligned} \tag{15.153}$$

from (15.153), we get

$$\begin{aligned}
\alpha_1 &= \Phi_1\\
\alpha_1 + \alpha_2 &= \Phi_2\\
\alpha_1 + \alpha_3 &= \Phi_3.
\end{aligned} \tag{15.154}$$

The polynomial coefficients from these equation are

$$\begin{aligned}
\alpha_1 &= \Phi_1\\
\alpha_2 &= \Phi_2 - \Phi_1\\
\alpha_3 &= \Phi_3 - \Phi_1.
\end{aligned} \tag{15.155}$$

Substituting these values, the (15.152) can be written as

$$\Phi = \mathrm{N}_1\Phi_1 + \mathrm{N}_2\Phi_2 + \mathrm{N}_3\Phi_3 \tag{15.156}$$

where Φ_1, Φ_2 and Φ_3 are nodal potentials or fields and N_1, N_2 and N_3 are element shape functions. They are connected to the natural coordinates in isoparametric domain as

$$\begin{aligned}
\mathrm{N}_1 &= 1 - \xi - \zeta\\
\mathrm{N}_2 &= \xi\\
\mathrm{N}_3 &= \zeta
\end{aligned} \tag{15.157}$$

From (15.154), (15.155), (15.156) and (15.157) we get where

$$\mathrm{N}_1 + \mathrm{N}_2 + \mathrm{N}_3 = 1 \tag{15.158}$$

$$\begin{aligned}
\mathrm{N}_\mathrm{i} &= 1\\
\mathrm{N}_\mathrm{j} &= 0\\
\mathrm{N}_\mathrm{k} &= 0.
\end{aligned}$$

Thus we can write the connecting relationship between the cartisian coordinates and the natural coordinates as

$$\begin{bmatrix}\mathrm{X}\\ \mathrm{Z}\end{bmatrix} = \sum_{i=1}^{3} N_i(\xi, \zeta) \begin{bmatrix}\mathrm{X_i}\\ \mathrm{Z_i}\end{bmatrix} \tag{15.159}$$

where $\mathrm{X_i}$ and $\mathrm{Z_i}$ are the cartisian coordinates of the ith element. The connecting link between the fields at the centre of a triangle and the nodal values are

$$\begin{bmatrix} \mathrm{E_x} \\ \mathrm{E_z} \end{bmatrix} = \sum_{i=1}^{3} N_i(\xi, \zeta) \begin{bmatrix} \mathrm{E_{X_i}} \\ \mathrm{E_{Z_i}} \end{bmatrix}. \tag{15.160}$$

The cartisian derivatives of the shape function $\mathrm{N_i}$ in terms of the natural coordinates can be written as

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \xi} \tag{15.161}$$

$$\frac{\partial N_i}{\partial \zeta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \zeta} \tag{15.162}$$

which in matrix form can be written as

$$\begin{bmatrix} \frac{\partial \mathrm{N_i}}{\partial \xi} \\ \frac{\partial \mathrm{N_i}}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathrm{x}}{\partial \xi} & \frac{\partial \mathrm{z}}{\partial \xi} \\ \frac{\partial \mathrm{x}}{\partial \zeta} & \frac{\partial \mathrm{z}}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathrm{N_i}}{\partial \mathrm{x}} \\ \frac{\partial \mathrm{N_i}}{\partial \mathrm{z}} \end{bmatrix} = [\mathrm{J}] \begin{bmatrix} \frac{\partial \mathrm{N_i}}{\partial \mathrm{x}} \\ \frac{\partial \mathrm{N_i}}{\partial \mathrm{z}} \end{bmatrix}. \tag{15.163}$$

where J is the Jacobian matrix and can be evaluated using

$$|\mathrm{J}| = \sum_{\mathrm{i}=1}^{3} \begin{bmatrix} \frac{\partial \mathrm{N_i}}{\partial \xi}\mathrm{x_i} & \frac{\partial \mathrm{N_i}}{\partial \xi}\mathrm{z_i} \\ \frac{\partial \mathrm{N_i}}{\partial \zeta}\mathrm{x_i} & \frac{\partial \mathrm{N_i}}{\partial \zeta}\mathrm{z_i} \end{bmatrix}. \tag{15.164}$$

The derivatives of shape function in Cartisian coordinate can be obtained in terms of natural coordinate and can be written as

$$\begin{bmatrix} \frac{\partial \mathrm{N_i}}{\partial \mathrm{x}} \\ \frac{\partial \mathrm{N_i}}{\partial \zeta} \end{bmatrix} = [\mathrm{J}]^{-1} \begin{bmatrix} \frac{\partial \mathrm{N_i}}{\partial \xi} \\ \frac{\partial \mathrm{N_i}}{\partial \zeta} \end{bmatrix} \tag{15.165}$$

where the determinant of the Jacobian matrix must be non zero.

15.8.2 Finite Element Formulation

Figure (15.21) shows the typical mesh generated using the quadrilateral finite elements. A homogeneous external region is included in the mesh to facilitate inclusion of boundary conditions. The resistivity of this region is held fixed at the average value of the apparent resistivity data. The nodal field values at the bottom and sides of the mesh are set equal to the analytical values obtained for the homogeneous space. Vertical element dimensions may be increased approximately exponentially downward from the air-earth interface because of exponential decay of the fields. Along the horizontal element boundaries, we extended the mesh from fine to coarse as we go away from the working zone. The nodes in the y-direction are indexed by $\mathrm{i} = 1, 2, 3 \ldots \mathrm{M}$ and the nodes in the z-directions are $\mathrm{j} = 1, 2, 3 \ldots \mathrm{N}$.

Taking the x-axis parallel to strike, y-axis in the horizontal and z-axis positive downward, for the TE mode 2-D geometries, $\mathrm{E_y} = \mathrm{E_z} = \mathrm{H_x} = \frac{\partial}{\partial x} = 0$.

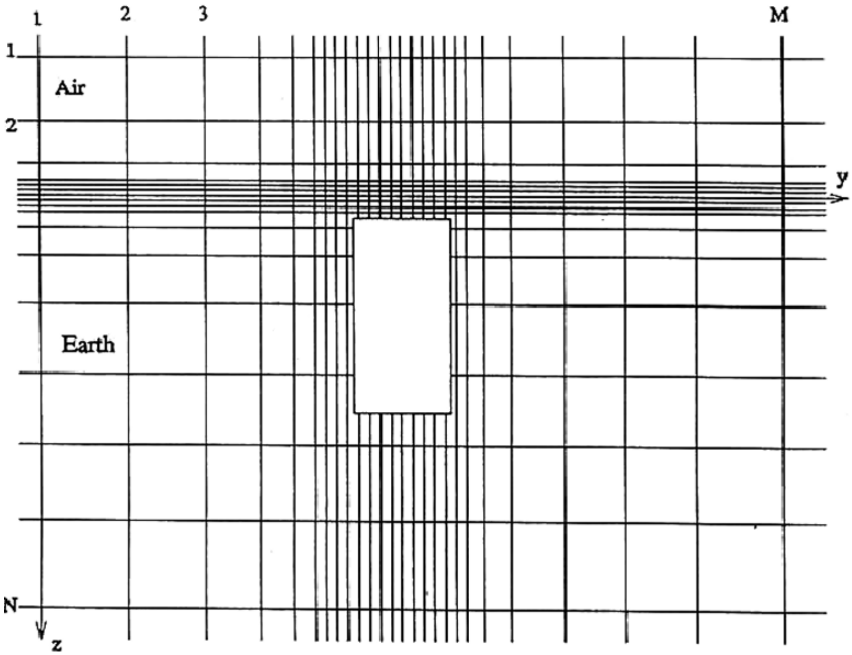


Fig. 15.21. A finite element mesh with quadrilateral boundaries;model shows the air-earth boundary and air layers on top of it,the anomalous body and the host rocks;expanding grid is shown both upward,downward ans lateral directions from the centre of the body

From Maxwell's equations we can write

$$\frac{\partial E_{xs}}{\partial z} = \hat{z} H_{ys} \tag{15.166}$$

$$\frac{\partial E_{xs}}{\partial y} = \hat{z} H_{zs} \tag{15.167}$$

$$\frac{\partial H_{xs}}{\partial z} = \hat{y} E_{ys} + \Delta \hat{y} E_{yp} \tag{15.168}$$

$$\frac{\partial H_{xs}}{\partial y} = \hat{y} E_{zs} + \Delta \hat{y} E_{zp} \tag{15.169}$$

and

$$\frac{\partial H_{zs}}{\partial y} - \frac{\partial H_{ys}}{\partial z} = \hat{y} E_{xs} + \Delta \hat{y} E_{xp} \tag{15.170}$$

(Chaps. 12 and 13 and Table 15.1)) where $\hat{y} = \sigma + i\omega \in$ is the admittivity, $\hat{z} = i\omega\mu_o$ is the impedivity, $\Delta\hat{y} = \hat{y}_i\,\tilde{}\,\hat{y}_j$. where $\hat{y}_i$ and $\hat{y}_j$ indicates the admittivity difference between the layered host and its 2D inhomogeneity. Subscripts p and s refer to primary (layered earth) and secondary field components. Substituting (15.166) and (15.167) into (15.168) , the TE mode Helmholtz equation is

$$\frac{\partial}{\partial y}\left(\frac{1}{\hat{z}}\frac{\partial E_{xs}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\partial z}\frac{\partial E_{xs}}{\partial z}\right)-\hat{y}E_{xs}=\Delta\hat{y}E_{xp}. \tag{15.171}$$

Similarly, Maxwell's equations for the secondary components of the TM mode, $E_x = H_y = H_z = \partial/\partial x = 0$. Here the guiding equations are

$$\frac{\partial H_{xs}}{\partial z}=\hat{y}E_{ys}+\Delta\hat{y}E_{yp} \tag{15.172}$$

$$\frac{\partial H_{xs}}{\partial y}=\hat{y}E_{zs}+\Delta\hat{y}E_{zp} \tag{15.173}$$

and

$$\frac{\partial E_{zs}}{\partial y}-\frac{\partial E_{ys}}{\partial z}=-\hat{z}H_{xs}. \tag{15.174}$$

Substituting (15.172) and (15.173) into (15.174) and rearranging, the TM mode Helmholtz equation one gets

$$\frac{\partial}{\partial y}\left(\frac{1}{\hat{y}}\frac{\partial H_{xs}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\hat{y}}\frac{\partial H_{xs}}{\partial z}\right)-\hat{z}H_{xs}=\Delta\frac{\Delta k^2}{\hat{y}}H_{xp}+\frac{\partial}{\partial z}\left(\frac{\Delta\hat{y}}{\hat{y}}\right)E_{yp} \tag{15.175}$$

where $\Delta k^2 = -\Delta\widehat{y}\,\widehat{z}$ and E_{zp} is zero in the magnetotellurics.

The (15.171) and (15.175) can be written in more general form as

$$\frac{\partial}{\partial y}\left(\frac{1}{q}\frac{\partial \hat{f}^e}{\partial y}+\frac{\partial}{\partial z}\frac{\partial \hat{f}^e}{\partial z}\right)+p\widehat{f}^{\,e}-s=0 \tag{15.176}$$

where, $q = \hat{z}, p = -\hat{y}, s = \Delta\widehat{y}\,E_{xp}$ for TE-mode.

$$q=\widehat{y}\,,p=-\widehat{z}\,,s=\tfrac{\Delta k^2}{\hat{y}}H_{xp}-\tfrac{\partial}{\partial z}\left(\tfrac{\Delta\hat{y}}{\hat{y}}\right)E_{yp}\text{ for TM-mode.}$$

The(15.176) can be written as

$$\frac{\partial}{\partial y}(F1)+\frac{\partial}{\partial z}(F2)+p\hat{f}^e-s=0 \tag{15.177}$$

where,

$$\mathrm{F1}=\left(\tfrac{1}{q}\tfrac{\partial\widehat{f}^{\,e}}{\partial y}\right) \tag{15.178}$$

$$\mathrm{F2}=\left(\tfrac{1}{q}\tfrac{\partial\widehat{f}^{\,e}}{\partial z}\right).$$

The element equations for finite element solution,

$$L\widehat{f}=s \tag{15.179}$$

where L is the Helmholtz operator, $\hat{f}$ is the unknown secondary electric or magnetic field parallel or perpendicular to the strike depending upon TE or TM mode formulation and s is the source function. We assume that '$\hat{f}$' is a piecewise linear functions over quadrilateral sub region 'e' of the domain and we can write

$$\hat{f} = \sum_{e=1}^{m} \hat{f}^e \tag{15.180}$$

where, m is total number of quadrilateral sub domains and,

$$\hat{f}^e = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2. \tag{15.181}$$

For node 'i'

$$\hat{f}^e = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 x_i y_i + \alpha_5 {x_i}^2 + \alpha_6 {y_i}^2 + \alpha_7 {x_i}^2 y_i + \alpha_8 x_i {y_i}^2. \tag{15.182}$$

Having defined the form of the approximation over these domains, the error approximation (ε) can be obtained after substituting (15.182) into (15.179) i.e.,

$$\mathrm{L}\hat{f} - s \equiv \varepsilon. \tag{15.183}$$

As discussed in Sect. 15.7, by proper choice of the weights, the error can be minimised to zero. These weights are Galerkin's weights. One gets

$$\int\int_e W^e \left[L\hat{f} - s\right] dydz = 0. \tag{15.184}$$

Mathematically, this states that the error of approximation be orthogonal to the weight functions 'W^e' over each sub domain 'e' i.e.,

$$\int_{\Omega^e} W \left[\frac{\partial}{\partial y}(F_1) + \frac{\partial}{\partial z}(F_2) + p\hat{f} - s\right] dydz = 0. \tag{15.185}$$

Integrating first two terms in (15.185) by parts, we get

$$\frac{\partial}{\partial y}(WF_1) = \frac{\partial W}{\partial y} F_1 + W \frac{\partial F_1}{\partial y} \tag{15.186}$$

$$\frac{\partial}{\partial z}(WF_2) = \frac{\partial W}{\partial z} F_2 + W \frac{\partial F_2}{\partial z}. \tag{15.187}$$

Equation (15.186 and 15.187) can be written as

$$\begin{aligned} -W\frac{\partial F_1}{\partial y} &= \frac{\partial W}{\partial y} F_1 - \frac{\partial}{\partial y}(WF_1) \\ -W\frac{\partial F_2}{\partial z} &= \frac{\partial W}{\partial z} F_2 - \frac{\partial}{\partial z}(WF_2). \end{aligned} \tag{15.188}$$

From Stokes theorem,

$$\int\limits_{\Omega^e} \frac{\partial}{\partial y}(WF_1)\,dydz = \oint\limits_{r^e} WF_1 n_y ds$$
$$\int\limits_{\Omega^e} \frac{\partial}{\partial z}(WF_2)\,dydz = \oint\limits_{r^e} WF_2 n_z ds \tag{15.189}$$

where, n_y and n_z are the components (i.e., direction cosines) of the unit normal vector

$$\begin{aligned}\hat{n} &= n_y i + n_z j \\ &= \cos(\alpha)\, i + \sin(\alpha) j\end{aligned} \tag{15.190}$$

on the boundary Γ^e, and ds is the arc length of an infinitesimal line element along the boundary. Using (15.188) and (15.189) in (15.185)

$$\int\limits_{\Omega^e} \left[\frac{\partial W}{\partial y}(F_1) + \frac{\partial W}{\partial z}(F_2) + Wp\hat{f} - Ws\right] dydz \tag{15.191}$$
$$- \oint\limits_{\Gamma^e} W\,[n_y(F_1) + n_z\,(F_2)]ds = 0.$$

Equation (15.191) can be written as

$$\int\limits_{\Omega^e} \left[\frac{\partial W}{\partial y}\left(\frac{1}{q}\frac{\partial \hat{f}^e}{\partial y}\right) + \frac{\partial W}{\partial z}\left(\frac{1}{q}\frac{\partial \hat{f}^e}{\partial z}\right) + Wp\hat{f}^e - Ws\right] dydz \tag{15.192}$$
$$- \oint\limits_{\Gamma^e} W\left[n_y\left(\frac{1}{q}\frac{\partial \hat{f}^e}{\partial y}\right) + n_z\left(\frac{1}{q}\frac{\partial \hat{f}^e}{\partial z}\right)\right] ds = 0$$

which is equal to

$$\int\limits_{\Omega^E} \left(-\frac{1}{q}(\frac{\partial \hat{f}^e}{\partial y}\frac{\partial W}{\partial y} + \frac{\partial \widehat{f}^e}{\partial z}\frac{\partial W}{\partial z}) + W\left[p\widehat{f}^e - s\right]\right) dydz$$
$$+ \oint\limits_{\Gamma^e} \frac{1}{q}\frac{\partial \hat{f}^e}{\partial \eta} W ds \equiv 0. \tag{15.193}$$

But, in Garlerkin's technique, the weights are equivalent to the approximate (or shape) functions.

The (15.193) further can be written as

$$\int\limits_{\Omega^e} \left(-\frac{1}{q}(\frac{\partial \hat{f}^e}{\partial y}\frac{\partial N_n^e}{\partial y} + \frac{\partial \widehat{f}^e}{\partial z}\frac{\partial N_n^e}{\partial z}) + N_n^e\left[p\widehat{f}^e - s\right]\right) dydz$$
$$+ \oint\limits_{\Gamma^e} \frac{1}{q}\frac{\partial \hat{f}^e}{\partial \eta} N_n^e ds \equiv 0 \tag{15.194}$$

where, $\frac{\partial \widehat{f}^{e}}{\partial \eta}$ is the normal derivative of the basis function at the element boundary, which is equivalent to

$$\frac{\partial \widehat{f}^{e}}{\partial \eta} = \frac{\partial \widehat{f}^{e}}{\partial y} n_y + \frac{\partial \widehat{f}^{e}}{\partial z} n_z \tag{15.195}$$

where n_y and n_z are the y and z component of the unit vector normal to the boundary. ds is the differential arc length along the boundary Ω^e of the quadrilateral element. Since internal element boundaries would be traversed twice in opposite directions during integration, the surface integral term of (15.193) is henceforth dropped from consideration which for the external boundaries the terms are either zero for Neumann boundary conditions or need not be evaluated for Dirichlet boundary conditions.

The (15.194) can be written as

$$\left(Q^e + P^e\right) \hat{f}^e = S^e \tag{15.196}$$

where

$$Q_{ij}^e = -\iint\limits_{e} \frac{1}{q} \left(\frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} + \frac{\partial N_j^e}{\partial z} \frac{\partial N_j^e}{\partial z} \right) dydz \tag{15.197}$$

$$P_{ij}^e = \iint\limits_{e} p N_i^e N_j^e dydz \tag{15.198}$$

$$S_i^e = \iint\limits_{e} s N_i^e dydz. \tag{15.199}$$

15.8.3 Shape Functions Using Natural Coordinates (ξ, η)

Isoparametric formulation makes it possible to generate elements that are non-rectangular (or) non-quradrilateral and have curved boundaries. These shapes have obvious usage in grading a mesh from coarse to fine in modelling arbitrary shapes, and curved boundaries. In formulating isoparametric elements, natural coordinate system (ξ, η) may be used. Secondary fields are expressed in terms of natural coordinates, but must be differentiated with respect to global coordinate y and z.

A non-rectangular region cannot be represented by using rectangular element; however, it can be represented by quadrilateral elements. Since, the interpolation function are easily derivable for a rectangular element, and it is easy to evaluate integrals over rectangular geometries, we transform the finite element integrals defined over quadrilaterals to rectangles. Therefore, numerical integration schemes, such as Gauss–Legendre scheme, require that the integral be evaluated on a specific domain or with respect to specific coordinate system. The transformation of the geometry and the variable coefficients of the differential equation from the problem coordinates (x, y) to the natural coordinates (ξ, η) results in algebraically complex expression and this precludes analytical (i.e., exact) evaluation of the integrals. Thus, the transformation of a given integral expression is defined over the element Ω^e, to one of the domain $\frown{\Omega}$ and must be such as to facilitate numerical integration. Each element of the finite element mesh is transformed to $\frown{\Omega}$, only for the purposes of numerically evaluating the integrals. The element Ω^e, is called a master element.

The transformation between Ω^e and $\frown{\Omega}$ i.e., between (x, y) and (ξ, η) is accomplished by a coordinate transformation of the form.

$$y = \sum_{i=1}^{8} N_i y_i \tag{15.200}$$

$$z = \sum_{i=1}^{8} N_i z_i$$

for eight noded isoparametric element, axes ξ and η pass through mid points of opposite sides as shown in (Fig. 15.22). Axes ξ and η need be orthogonal, and neither need be parallel to the y-axis nor the z-axis. Side of the element are at $\xi = \pm 1$ and $\eta = \pm 1$. The interpolation (or shape) functions following Sect. 15.8.1 can be worked out to be

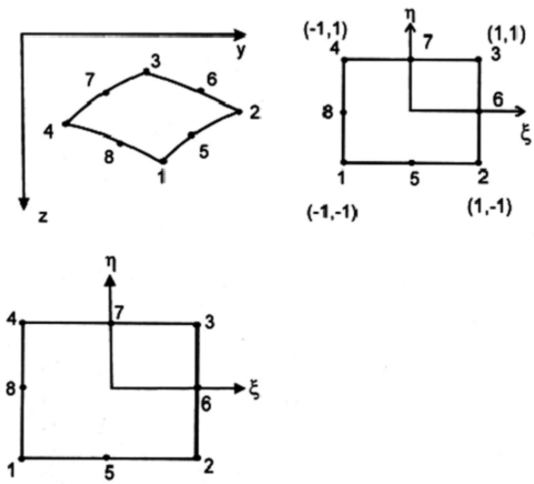


Fig. 15.22. Advanced level eight nodded isoparametric element

For node 1: $\frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1)$

For node 2: $\frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$

For node 3: $\frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$

For node 4: $\frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1)$

For node 5: $\frac{1}{2}\left(1-\xi^2\right)(1-\eta)$

For node 6: $\frac{1}{2}(1+\xi)\left(1-\eta^2\right)$

For node 7: $\frac{1}{2}\left(1-\xi^2\right)(1+\eta)$

For node 8: $\frac{1}{2}(1-\xi)\left(1-\eta^2\right)$

15.8.4 Coordinate Transformation

The transformation of a quadrilateral element of a finite element mesh to the master element $\widehat{\Omega}$ is solely for the purpose of numerically evaluating the integrals (15.197), (15.198), (15.199). The resulting algebraic equations of the finite element formulations are always among the nodal values of the physical domain. Different element of the finite element mesh can be generated from the same master element by assigning the global coordinates of the elements. With the help of an appropriate master element, any arbitrary element of a mesh can be generated. However, the transformation of a master element should be such that there are no spurious gaps between elements and no element overlaps.

When a typical element of the finite element mesh is transformed to its master element for the purpose of numerically evaluating integrals, the integrand must also be expressed in terms of coordinates (ξ,η) of the master element. In the (15.194), the integrand i.e., the expression in the square brackets under the integral) and their derivatives are functions of the global coordinates y and z.. We must rewrite it in terms of (ξ) and (η) using transformation (15.160 to 15.165).

Therefore, we must relate $\frac{\partial N_i^e}{\partial y}$ and $\frac{\partial N_i^e}{\partial z}$ to $\frac{\partial N_i^e}{\partial \xi}$ and $\frac{\partial N_i^e}{\partial \eta}$.

The function N_i^e can be expressed in terms of the local coordinates ξ and η by means of (15.194). Hence, by chain rule of partial differentiation, we have

$$\begin{aligned}\frac{\partial N_i^e}{\partial \xi} &= \frac{\partial N_i^e}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial N_i^e}{\partial z}\frac{\partial z}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} &= \frac{\partial N_i^e}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial N_i^e}{\partial z}\frac{\partial z}{\partial \eta}.\end{aligned} \tag{15.201}$$

In matrix notation

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{pmatrix} \begin{pmatrix} \frac{\partial N_i^e}{\partial y} \\ \frac{\partial N_i^e}{\partial z} \end{pmatrix} \tag{15.202}$$

which gives the relation between the derivatives N_i^e with respect to the global and local coordinates. The (15.202) can be written as

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{pmatrix} = [J] \begin{pmatrix} \frac{\partial N_i^e}{\partial y} \\ \frac{\partial N_i^e}{\partial z} \end{pmatrix} \tag{15.203}$$

where, $[J]$ is known as Jacobian matrix which is equal to

$$[J] = \begin{pmatrix} \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{pmatrix}. \tag{15.204}$$

To find the cartesian derivatives of N_i^e, we get

$$\begin{pmatrix} \frac{\partial N_i^e}{\partial y} \\ \frac{\partial N_i^e}{\partial z} \end{pmatrix} = [J]^{-1} \begin{pmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{pmatrix} \tag{15.205}$$

$$= \frac{1}{|J|} \begin{pmatrix} \frac{\partial z}{\partial \eta} & -\frac{\partial z}{\partial \xi} \\ -\frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{pmatrix}$$

where,

$$|J| = \frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \xi}. \tag{15.206}$$

This requires that the Jacobian matrix $[J]$ be non-singular.

The (15.206) can be written as

$$\begin{aligned} \frac{\partial N_i^e}{\partial y} &= \frac{\partial N_i^e}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i^e}{\partial \eta}\frac{\partial \eta}{\partial y} \\ \frac{\partial N_i^e}{\partial z} &= \frac{\partial N_i^e}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_i^e}{\partial \eta}\frac{\partial \eta}{\partial z}. \end{aligned} \tag{15.207}$$

From (15.202), the (15.204) will be

$$[J] = \begin{pmatrix} \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} x_i \\ \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} x_i \end{pmatrix}. \tag{15.208}$$

Equation (15.206) can be restructured as

$$[J] = \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & \frac{\partial N_5}{\partial \xi} & \frac{\partial N_6}{\partial \xi} & \frac{\partial N_7}{\partial \xi} & \frac{\partial N_{81}}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & \frac{\partial N_5}{\partial \eta} & \frac{\partial N_6}{\partial \eta} & \frac{\partial N_7}{\partial \eta} & \frac{\partial N_{81}}{\partial \eta} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \\ x_7 & y_7 \\ x_8 & y_8 \end{pmatrix}. \tag{15.209}$$

The element area dA = dydz in the element Ω^e is transformed to

$$\mathrm{dA} = \mathrm{dydz} = |J|\, \mathrm{d}\xi d\eta \tag{15.210}$$

in the master element $\widehat{\Omega}$.

Equation (15.197), (15.198) and (15.199) can be written as

$$Q_{ij}^e = -\int_{-1}^{1} \frac{1}{q}\left(\frac{\partial N_i^e}{\partial \xi}\frac{\partial N_j^e}{\partial \xi}\right)|J|\, d\xi d\eta - \int_{-1}^{1} \frac{1}{q}\left(\frac{\partial N_i^e}{\partial \xi}\frac{\partial N_j^e}{\partial \xi}\right)|J|\, d\xi d\eta \tag{15.211}$$

$$P_{ij}^e = pN_i^e N_j^e\, |J|\, d\xi d\eta \tag{15.212}$$

and

$$S_{ij}^e = sN_i^e\, |J|\, d\xi d\eta \tag{15.213}$$

the (15.209) can be written as

$$Q_{ij}^e = Q1_{ij}^e + Q2_{ij}^e \tag{15.214}$$

where

$$Q1_{ij}^e =$$

$$\begin{pmatrix}
\frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_2}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_3}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_4}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_5}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_6}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_7}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_8}{\partial \xi} \\
\frac{\partial N_2}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi}\frac{\partial N_2}{\partial \xi} & \frac{\partial N_2}{\partial \xi}\frac{\partial N_3}{\partial \xi} & \frac{\partial N_2}{\partial \xi}\frac{\partial N_4}{\partial \xi} & \frac{\partial N_2}{\partial \xi}\frac{\partial N_5}{\partial \xi} & \frac{\partial N_2}{\partial \xi}\frac{\partial N_6}{\partial \xi} & \frac{\partial N_2}{\partial \xi}\frac{\partial N_7}{\partial \xi} & \frac{\partial N_2}{\partial \xi}\frac{\partial N_8}{\partial \xi} \\
\frac{\partial N_3}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_3}{\partial \xi}\frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi}\frac{\partial N_3}{\partial \xi} & \frac{\partial N_3}{\partial \xi}\frac{\partial N_4}{\partial \xi} & \frac{\partial N_3}{\partial \xi}\frac{\partial N_5}{\partial \xi} & \frac{\partial N_3}{\partial \xi}\frac{\partial N_6}{\partial \xi} & \frac{\partial N_3}{\partial \xi}\frac{\partial N_7}{\partial \xi} & \frac{\partial N_3}{\partial \xi}\frac{\partial N_8}{\partial \xi} \\
\frac{\partial N_4}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_4}{\partial \xi}\frac{\partial N_2}{\partial \xi} & \frac{\partial N_4}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_4}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_4}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_4}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_5}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_5}{\partial \xi}\frac{\partial N_2}{\partial \xi} & \frac{\partial N_5}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_5}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_5}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_5}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_6}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_6}{\partial \xi}\frac{\partial N_2}{\partial \xi} & \frac{\partial N_6}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_6}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_6}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_6}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_7}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_7}{\partial \xi}\frac{\partial N_2}{\partial \xi} & \frac{\partial N_7}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_7}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_7}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_7}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_8}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_8}{\partial \xi}\frac{\partial N_2}{\partial \xi} & \frac{\partial N_8}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_8}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_8}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_8}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi}
\end{pmatrix} \tag{15.215}$$

and $Q2^e_{ij} =$

$$\begin{pmatrix}
\frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_2}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_3}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_4}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_2}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_3}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_4}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_5}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_6}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_7}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} \\
\frac{\partial N_8}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \xi}\frac{\partial N_1}{\partial \xi}
\end{pmatrix} \tag{15.216}$$

For i = 1 to 8, $\partial N_i/\partial \xi$ is

$$\begin{aligned}
\frac{\partial N_1}{\partial \xi} &= \frac{(1-\eta)(2\xi+\eta)}{4} \\
\frac{\partial N_2}{\partial \xi} &= \frac{(1-\eta)(2\xi-\eta)}{4} \\
\frac{\partial N_3}{\partial \xi} &= \frac{(1+\eta)(2\xi+\eta)}{4} \\
\frac{\partial N_4}{\partial \xi} &= \frac{(1+\eta)(\eta-2\xi)}{4} \\
\frac{\partial N_5}{\partial \xi} &= -\xi(1-\eta) \\
\frac{\partial N_6}{\partial \xi} &= \frac{1-\eta^2}{2} \\
\frac{\partial N_7}{\partial \xi} &= -\xi(1+\eta) \\
\frac{\partial N_8}{\partial \xi} &= -\frac{1-\eta^2}{2}.
\end{aligned} \tag{15.217}$$

Similarly, for i = 1 to 8, $\partial N_i/\partial \eta$ is

$$\begin{aligned}
\frac{\partial N_1}{\partial \eta} &= \frac{(1-\xi)(2\eta+\xi)}{4} \\
\frac{\partial N_2}{\partial \eta} &= \frac{(1+\xi)(\xi-2\eta)}{4} \\
\frac{\partial N_3}{\partial \eta} &= \frac{(1+\xi)(2\eta+\xi)}{4} \\
\frac{\partial N_4}{\partial \eta} &= \frac{(1-\xi)(2\eta-\xi)}{4} \\
\frac{\partial N_5}{\partial \eta} &= -\frac{1-\xi^2}{2} \\
\frac{\partial N_6}{\partial \eta} &= -\eta(1+\xi) \\
\frac{\partial N_7}{\partial \eta} &= -\frac{1-\xi^2}{2}
\end{aligned} \tag{15.218}$$

where, q, p and s depend upon whether TE or TM modes are being considered for evaluation. All the elements of the matrices are to be obtained using numerical integration using Gauss Legender Quadrature . Gauss weights for one two and three dimensional problems are available in standard mathematical handbooks.One chooses 3,5,7,9 point Gauss Quadrature depending upon the accuracy needed. The steps include: generation of element matrices Q^e, P^e and S^e as defined in (15.197), (15.198) and (15.199) respectively. These element matrices are assembled to form the global matrix, before imposition of boundary conditions. The nodal field variables are obtained by solving the finite element equations.

15.9 Integral Equation Method

15.9.1 Introduction

An equation, where an unknown function to be determined, remains within the integral sign, is an integral equation. The integral equations are defined by Fredhom and Volterra. Fredhom and Volterra's integral equations of the first, second and third kind are given by the (2.46, 2.47, 2.48, 2.49, 2.50, 2.51) Often the Kernel functions of these integral equations are Green's function. If the limits of these integral are finite, the integrals are of Fredhom type. For an infinite or undefined limits of the integral, the integral equations are of Volterra's type. Geophysical Boundary value problems solved using integral equations generally appear in the form of Fredhom's integral equation of the second kind.

$$\mathrm{f(r)} - \lambda \int_{\mathrm{v}} \mathrm{G(r, r_0) f(r_0) dv_0} = \mathrm{g(r)} \tag{15.219}$$

where $\mathrm{G(r, r_0)}$ is the Green's function with observation points and source points are respectively at r and $\mathrm{r_0}$; g(r) is the known function and f(r) is the unknown function to be determined. r and r_0 are respectively 'the distances of the points of observation and the source from the origin. If the Green's function is singular in a region of integration, then the equation is a singular integral equation otherwise it is non-singular and continuous function. If the equation can be solved for a certain value of λ, then the problem is said to be an eigen value problem. In general it is easier to handle the integral equation of the second kind. The most important component regarding the solutions of the integral equation is the solution of these Green's functions. Since Green's function appears under the integral equation, the integrals are solved using numerical methods, viz. Gauss quadrature, Simpson's rule, Weddles rule etc. For simpler cases analytical solution of the integral is possible. In the case of a discretized domain in IE, each element of the matrix equations becomes an integral containing Green's function. In integral equation method, distortion in the field due to anomaly causing body are of interest. The anomaly causing body of contrasting physical property in a half space or a layered half space is replaced by the scattering current while formulation of the problem. In IEM, these scattering currents are of interest to geophysicists. Therefore, the volume of integration is restricted to the anomaly causing body. As a result the size of the matrix in formulation of an integral equation is considerably smaller in comparison to those encountered while formulating the problems using finite difference and finite element methods. Both FDM and FEM are differential equation based methods and entire space outside the target body are taken into consideration in the discretized domain. As a result IEM becomes a very powerful tool for solving three dimensional boundary value problems. In IEM, the matrices are solid but of much smaller size. In FDM and FEM, the matrices are sparse but of large dimension. For modeling subsurface target body of complicated geometry, FDM and FEM have slight edge over IEM with the gradual advancement in computation facilities. Mathematics may become quite tougher in IEM in comparison to what we face for handling FDM and FEM problems. FDM is well known for its inherent simplicity. In electromagnetic boundary value problems, both scalar and tensor Green's function appear in the solution. Tensor Green's function are known as dyadic Green's functions having 9 components. Therefore, the integral equation is changed to matrix equation and these equations are solved using the method of moments by judicious choice of the basis function and weighting function. Green's function becomes a tensor because the direction of the source dipole and the observation dipoles are in the different directions. Since the behaviours of the dyadics is similar to that of a 3×3 tensor having nine components, the dyadic Green's function are termed as tensor Green's function.

15.9.2 Formulation of an Electromagnetic Boundary Value Problem

This subject is developed by Hohmann (1971, 1975, 1983, 1988), Wannamaker (1984a, 1984b, 1991), Meyer (1976), Weidelt (1975), Raichi (1975), Newman et al (1986), Newman et al (1987), Ting and Hohmann (1981). Sanfilipo and Hohmann (1985) and Sanfilipo et al (1985 Fig. (15.23) shows the geometry of the problem. An anomalous body with a contrast in electrical conductivity from the rest of the layered half space is assumed. The problem is to determine the electrical and magnetic field due to the anomaly causing body by replacing it with the scattering currents. These scattering currents are in excess of conduction and displacement currents. These scattering currents are assumed in an otherwise homogeneous and isotropic layered half space. Scattering current flows only through the conductivity inhomogeneity. Computation of the anomalous fields is restricted to the volume occupied by the target. Both primary and scattered secondary fields are harmonic fields. In Maxwell's equation

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \tag{15.220}$$

$\vec{J}$ is divided into two parts, i.e., $\vec{J} = \vec{J}_c + \vec{J}_s$ where $\vec{J}_c$ is the conduction current because for most of the geophysical problems, displacement current is negligible. Displacement current becomes significant in megahertz and gigahertz range. J_c is connected with the scalar conductivity σ. J_s is the source current which flows through the host rock. The first step in this scattering problem is to replace the inhomogeneity by $\vec{J}_s$. This J_s is the current density due to the flow of current through the inhomogeneity. The electrical conductivity of the anomalous region is σ_s. The total electric field are field due to half space without any inhomogeneity and that due to scattering current which is superimposed over the conduction current. Therefore

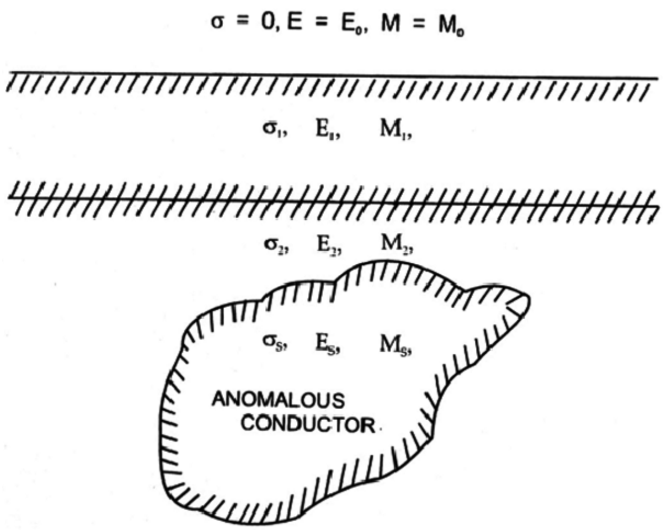


Fig. 15.23. An anomalous conductor in a layered half space (Mayer 1976)

$$\vec{E} = \vec{E}_s(\vec{r}) + \vec{E}_i(\vec{r}). \tag{15.221}$$

$\vec{E}_i(\vec{r})$ is the electric field, that would exist at the field point or observation point, in the absence of the anomalous body. $\vec{E}_s(\vec{r})$ is the scattered field at $\vec{r}$ due to anomaly causing body. $\vec{E}_s(\vec{r})$ is a function of $\vec{J}_s$ which exists in the entire anomalous region. The relationship can be written as

$$\vec{E}_s(\vec{r}) = \int_{v_a} \vec{\vec{G}}(\vec{r}, \vec{r}_0)\vec{J}_s(\vec{r}_0)dv_0 \tag{15.222}$$

Morse and Feshback (1953), Tai (1971), Hohman (1971), Van Bladel (1968), Eskola (1992). Here v_a is the anomalous volume for integration, $\vec{r}_0$ is the potential vector for the source point. $\vec{\vec{G}}(\vec{r}, \vec{r}_0)$ is the tensor Green's function, because the direction of the field may or may not be in the same direction as the current elements in the medium. $\vec{\vec{G}}(\vec{r}, \vec{r}_0)$ is the scattered electric field at $\vec{r}$ due to unit current density at r_0. $\vec{E}_s(\vec{r})$ have some contribution due to the magnetic current also. Since free space magnetic permeability μ_0 is assumed for the entire half space with no contrast anywhere, therefore the magnetic current is absent and is not included in the solution. $\vec{J}_s$, the anomalous current density is just the conduction current density in the anomalous body minus the conduction current in the surrounding host rock. Therefore

$$\vec{J}_s(\vec{r}_0) = (\sigma_s^E - \sigma_2^E)\vec{E}_s(\vec{r}_0) \tag{15.223}$$

where σ_s^E and σ_2^E are the anomalous and true electrical conductivity in the anomalous zone neglecting displacement current. Therefore we can write (15.222) as

$$\vec{E}_s(\vec{r}) = \int_{v_a} \vec{\vec{G}}(\vec{r}, \vec{r}_0)(\sigma_s - \sigma_2)\vec{E}(\vec{r}_0)dv_0. \tag{15.224}$$

Assuming σ_s and σ_2 to be constant, $(\sigma_s - \sigma_2)$ can be taken out of the sign of integration. Equation (15.224) can be rewritten in the form

$$\vec{E}_s(\vec{r}) = (\sigma_s - \sigma_2)\int_{v_a} \vec{\vec{G}}(\vec{r}, \vec{r}_0)E(r_0)dv_0. \tag{15.225}$$

Using (15.221), we can write

$$\vec{E}(\vec{r}) = \vec{E}_i(r) + (\sigma_s - \sigma_2)\int_{v_a} G(\vec{r}, \vec{r}_0)\vec{E}(\vec{r}_0)dv_0. \tag{15.226}$$

Equation (15.226) is the vector Fredhom's integral equation of the second kind. $\vec{E}_i(r)$, the electric field induced in the free space can be computed. $\vec{\vec{G}}(\vec{r}, \vec{r}_0)$, the Green's tensor, which depends upon the geometry of the problem, can be calculated (Hohmann, 1975, Weidelt, 1975, Raichi, 1975, Wannamaker 1984, Meyer, 1976). Equation (15.198) is used to calculate $\vec{E}_i(\vec{r})$. Equation (15.197) is used to calculate $\vec{E}_s(r)$. Method of moments (Harrington, 1968) can be based for solving (15.226) by proper choice of basis function and weighting functions (Hohmann 1988). Volume V_a of the anomalous conductor is divided into N smaller volumes (Fig. 15.24.).

The electrical field within each of the smaller volumes are assumed to be constant. The integral equation at the centre of each of these smaller cubical volumes (Fig. 15.24) is

$$\vec{E}^m = \vec{E}_i^m + \sum_{n=1}^{N} (\sigma_s - \sigma_2) \int_{v_a} \vec{\vec{G}}(\vec{r}, \vec{r}_0) \vec{E}_n dv_0. \tag{15.227}$$

where $\vec{E}^m$ is the electric field of the centre of the mth cell. $\vec{r}_m$ is the position vector for the mth cell. It is the distance of the centre of the mth cell from the origin.

Since $\vec{E}^n$, the electric field in the nth cell (Fig. 15.25) is assumed to be constant throughout the anomalous conductor, $\vec{E}^n$ can be taken outside the sign of integration. Therefore (15.227) can be written as

$$\vec{E}^m = \vec{E}_i^m + \sum_{n}^{N} (\sigma_s - \sigma_2) \int_{v_a} \vec{\vec{G}}(\vec{r}_m, \vec{r}_0) dv_0 . \vec{E}^n. \tag{15.228}$$

Now let

$$(\sigma_s - \sigma_2) \int_{v_a} \vec{\vec{G}}(\vec{r}_m, \vec{r}_0) dv_0 = \vec{\vec{\Gamma}}_{mn}. \tag{15.229}$$

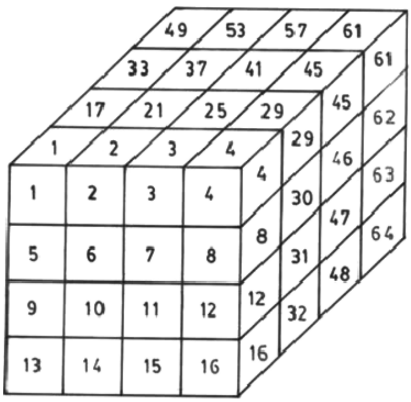


Fig. 15.24. Division of the anomaly causing body into cubic cells

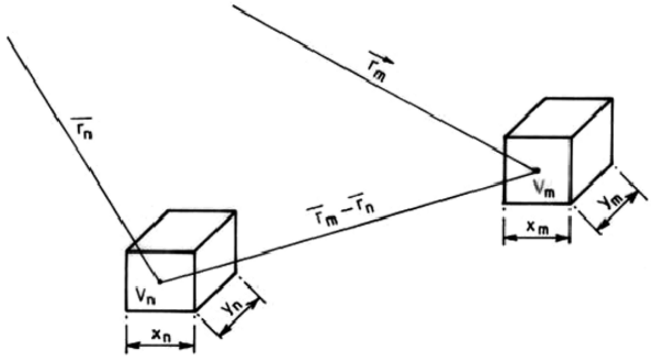


Fig. 15.25. Interrelation between the position vectors of two cubical elements inside the body; position vectors are with respect to the assumed origin

Therefore, (15.228) becomes

$$\vec{\mathrm{E}}^{\mathrm{m}} = \vec{\mathrm{E}}_{\mathrm{i}}^{\mathrm{m}} + \sum \vec{\vec{\Gamma}}_{\mathrm{mn}}(\vec{\mathrm{r}}_{\mathrm{m}}, \vec{\mathrm{r}}_0).\vec{\mathrm{E}}^{\mathrm{n}}. \tag{15.230}$$

Equation (15.230) can be written as

$$\vec{\mathrm{E}}^{\mathrm{m}} = \sum \vec{\vec{\mathrm{K}}}_{\mathrm{mn}} \vec{\mathrm{E}}_{\mathrm{n}}. \tag{15.231}$$

Here

$$\vec{\vec{\mathrm{K}}}_{\mathrm{mn}} = \vec{\vec{\Gamma}}_{\mathrm{mn}} - \vec{\vec{\delta}}_{\mathrm{mn}} \tag{15.232}$$

where,

$$\vec{\vec{\delta}}_{\mathrm{mn}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for m } \neq \mathrm{n} = \vec{\vec{0}}. \tag{15.233}$$

It is termed as the null dyadic and

$$\vec{\vec{\delta}}_{\mathrm{mn}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for m } \neq \mathrm{n}. \tag{15.234}$$

It is termed as the identity dyadic or idem factor.

In (15.231), there are N possible values of the superscript m and there are three components in each vector. So (15.231) represents 3N equations in 3N unknowns $\vec{\mathrm{E}}^{\mathrm{n}}$. Matrix elements are composed of the elements of the dyadic Green's function. $\vec{\mathrm{E}}_{\mathrm{i}}^{\mathrm{m}}$ are calculated for each particular source geometry. $\vec{\mathrm{E}}_{\mathrm{i}}$ is calculated in the absence of any inhomogeneity. In the matrix form we can write (15.231) as

$$[\mathrm{E}] = [\mathrm{K}]^{-1}[\mathrm{E}_{\mathrm{i}}] \tag{15.235}$$

where $[K]^{-1}$ is a $3N \times 3N$ matrix containing the information about the subsurface geometry and the conductivity distribution. $[E_i]$ is a $3N \times 1$ column vector containing information about the source and its effect on the host in the absence of the target body. Actual value of the electric field in the inhomogeneity is $\vec{E}$ is also $3N \times 1$ column vector. These values of $\vec{E}$ can then be used to calculate $\vec{E}_s(r)$ any where in the host rock and in the air. Elaborate treatments on computations of dyadic Green's tensors are available in Tai (1971), Weidelt (1975), Raiche (1975), Meyer (1976), Beasely and Ward (1986), Wannamaker (1984).

16

Analytical Continuation of Potential Field

In this chapter the techniques for analytical continuation of potential field are discussed. The topics covered, related to upward and downward continuation of potential fields, are (i) harmonic analysis (ii) Taylor's series expansion and finite difference scheme (iii)Green's theorem, Green's function and integral equation (iv) Peter's areal average (v) Lagrange interpolation formula and integral equation (vi) Green's theorem in electromagnetic fields.

16.1 Introduction

Analytical continuation of potential field is a process of finding a potential field on any plane from the measured values of that field on any other plane (Fig. 16.1). If a field is measured on the ground surface within a certain area, mathematical technique of analytical continuation allows one to find out the field on any other plane above or below the level of the plane where the measurements were done. Continuation of the field above the level of measurement is known as upward continuation. Continuation below the level of measurement is known as downward continuation. These basic properties of mathematical physics were used by the geophysicists for interpretation of field data by upward or downward continuation. Upward continuation allows the data to be smooth and as such there is no problem of instability in mathematical continuation.

Downward continuation, on the other hand, sharpens the geophysical anomalies in potential field and may invite instability at different levels of continuation. Upward and downward continuation of potential fields started in geophysics for interpretation of gravity and magnetic field data. Later application of analytical continuation found its place in interpretation of self potential, telluric current and electromagnetic data. Aeromagnetic or aerogravity data can be compared with upward continued gravity and magnetic potentials recorded at a particular area. Even for downward continuation, a few units of upward continuation is necessary. Analytical continuation of potential field

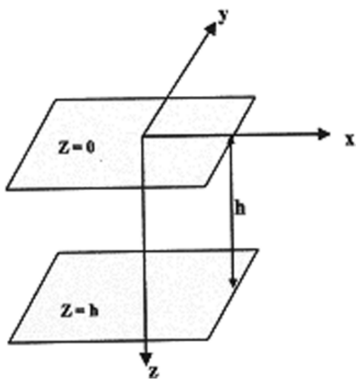


Fig. 16.1. Shows two horizontal planes at a difference of height 'h'; potentials measured in one plane can be continued to the other plane

also comes under the broad umbrella of inversion of potential field data discussed in the next chapter.

The idea of analytical continuation of potential field in geophysics started coming from the third decade of the last century. Authors developed this subject are Evjen (1936), Tsuboi and Fuchida (1937), Tsuboi (1938), Hughes (1942), Bullard and Cooper (1948), Peters (1949), Henderson and Zeitz (1949), Trejo (1954), Roy (1960, 1962, 1963, 1966a, 1966b, 1967, 1968, 1969), (1961) and Huestis and Parker (1979).

Different mathematical tools used for analytical continuation are Taylor's series expansion, and finite difference approximation solution of Laplace equation, harmonic analysis, Green's theorem, Fourier sine transform, relaxation, Lagrange interpolation, integral equation and spatial averaging. The subject originally came forward for handling gravity and magnetic data. Roy(papers cited above) have shown that the same technique can be extended in the fields of telluric current, self potential, direct current and electromagnetic field of geophysical interest. Huestis and Parker (1979) have proposed the use of Bachus-Gilbert inversion approach (1968, 1970) for analytical continuation of potential field. Analytical continuation of gravity and magnetic data is used widely by geophysicists in oil and mineral industries.

In this chapter a few approaches of analytical continuation are presented the way the different authors have proposed.

16.2 Downward Continuation by Harmonic Analysis of Gravity Field

Tsuboi and Fuchida (1937) first suggested the harmonic analysis approach for downward continuation of the gravity field. The general solution of Laplace equation in Cartisian coordinate is given by

$$\phi\,(x, y, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \begin{matrix}\cos mx \; \cos ny \\ \sin mx \; \sin my\end{matrix}\, e^{\sqrt{m^2+n^2}}\; z \tag{16.1}$$

and

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = g\,(x, y)\,. \tag{16.2}$$

Here ϕ is the gravitational potential and g is the vertical component of the gravity field.

From (16.1) we get

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \sqrt{m^2+n^2}.\begin{matrix}\cos \; mx \; \cos ny \\ \sin mx \;\; \sin ny\end{matrix} \tag{16.3}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} \begin{matrix}\cos mx \; \cos ny \\ \sin mx \; \sin ny\end{matrix} \tag{16.4}$$

where

$$B_{mn} = A_{mn}\sqrt{m^2+n^2}.$$

And

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=d} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} \begin{matrix}\cos mx \; \cos ny \\ \sin mx \; \sin ny\end{matrix}$$

where

$$C_{mn} = B_{mn}\; e^{\sqrt{m^2+n^2}.d}. \tag{16.5}$$

Equation (16.5) gives the downward continued gravity field.

16.3 Taylor's Series Expansion and Finite Difference Approach for Downward Continuation

16.3.1 Approach A

Bullard and Coopper's approach

Bullard and Cooper (1948) suggested the finite difference approach and used Taylor's series expansion to obtain downward continued gravity values in a two dimensional square grid of separation 'a' in a xz plane (Fig. 16.2). One can write

$$g_1 = g_0 + a\frac{\partial g}{\partial x} + \frac{1}{2!}a^2\frac{\partial^2 g}{\partial x^2} + \tag{16.6}$$

$$g_2 = g_0 - a\frac{\partial g}{\partial x} + \frac{1}{2!}a^2\frac{\partial^2 g}{\partial x^2} - \tag{16.7}$$

$$g_3 = g_0 + a\frac{\partial g}{\partial z} + \frac{1}{2!}a^2\frac{\partial^2 g}{\partial z^2} + \tag{16.8}$$

$$g_4 = g_0 - a\frac{\partial g}{\partial z} + \frac{1}{2!}a^2\frac{\partial^2 g}{\partial z^2} + \tag{16.9}$$

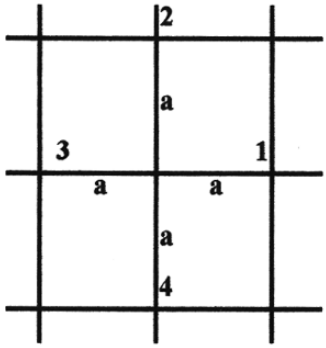


Fig. 16.2. Finite difference cell

where g_1, g_2, g_3, g_4 and g_0 are the gravity values at the points 1,2,3,4 and 0. Adding equations (16.6) , (16.7) and (16.8) ,(16.9). we get

$$\begin{aligned} g_1 + g_2 + g_3 + g_4 \equiv & 4g(0) + \frac{a^2}{2!}\left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial z^2}\right] \\ & + \frac{a^4}{4!}\left[\frac{\partial^4 g}{\partial x^4} + \frac{\partial^4 g}{\partial z^4}\right]. \end{aligned} \tag{16.10}$$

For two dimensional bodies, the gravity field in a source free region satisfy Laplace's equation.

Hence

$$4g(0) = g_1 + g_2 + g_3 + g_4 \tag{16.11}$$

16.3.2 Approach B

For downward continuation of two dimensional fields, we can write

$$\begin{aligned} g(+h) &= g(0) + h\left.\frac{\partial g}{\partial g}\right|_{z=0} + \frac{h^2}{2!}\left.\frac{\partial^2 g}{\partial z^2}\right|_{z=0} + \\ &= \sum_{n=0}^{\infty} \frac{(h)^n}{n!}\left.\frac{\partial^n g}{\partial z^n}\right|_{z=0}. \end{aligned} \tag{16.12}$$

The general solution of the Laplace equation in cylindrical polar coordinate is

$$g(r, \theta, z) = \sum_{n=0}^{\infty}\sum_{k=1}^{k} e^{\mu_k z}\left(A_{k\,m}\cos n\theta + B_{k\,m}\sin n\theta\right) J_n(\mu_k r) \tag{16.13}$$

where $J_n(\mu_k\ r)$ is the Bessel's function of n order and first kind.

A_{km} and B_{km} are the kernels to be determined from the boundary condition.

For azimuthal independence of the potential field, (16.13) reduces to

$$\bar{g}(r) = \sum_{k=1}^{k} A_k e^{\mu_k z} J_0(\mu_k r) \tag{16.14}$$

and

$$g(0) = \sum_{k=1}^{k} A_k \; e^{\mu_k z} \tag{16.15}$$

since

$$J_0(\mu_k r) = 1 \; for \; r = 0$$

$$\left.\frac{\partial g}{\partial z}\right|_{z=0} = \sum_{k=1}^{k} \mu_k A_k \tag{16.16}$$

$$\left.\frac{\partial^n g}{\partial z^n}\right|_{z=0} = \sum \mu_{k^n} A_k \tag{16.17}$$

Hence

$$g(+h) = \sum_{n=1}^{\infty} \frac{h^n}{n!} \sum_{k=1}^{n} \mu_k^n \; A_k. \tag{16.18}$$

Choosing maximum number of k = 3 and minimum number of n = 11, we get

$$\bar{g}(1) = A_1 J_0(\mu_1) + A_2 J_0(\mu_2) + A_3 J_0(\mu_3) \tag{16.19}$$

$$\bar{g}(2) = A_1 J_0\left(\sqrt{2}\mu_1\right) + A_2 J_0\left(\sqrt{2}\mu_2\right) + A_3 J_0\left(\sqrt{2}\mu_3\right). \tag{16.20}$$

For $r = 0, S, S\sqrt{2}$,

$$g(+h) = \sum_{n=0}^{11} \frac{n}{n!} \sum_{k=1}^{3} \mu_k^n A_k. \tag{16.21}$$

16.3.3 An Example of Analytical Continuation Based on Synthetic Data

Roy (1966) computed downward continued gravity data based on computed gravity values due to an infinitely long buried cylinder buried at a depth H = h.The analytical continuation depths are h-2Δh, h-Δh, h, h + Δh, h + 2Δh. The depths chosen were 0.50, 0.75, 1.00, 1.25, 1.50, 1.75 and 2.00. The depth interval is 0.25 unit.The grid interval was also 0.25 unitsThe working formula for computation of the gravity field is given in Chap. 3 (Sect. 3.7), (3.33). Figures 16.2 and (16.11) are the guidelines for downward continuation of gravity field.

Figure 16.3 shows that the central peak of the gravitational anomaly increases at a very faster pace with gradual increase in depth of continuation. Figure 16.4 show that beyond a certain depth, the gravitational anomaly explodes i.e., heading towards an infinitely high value. The point of maximum gradient or inflection gives estimated depth of the body.

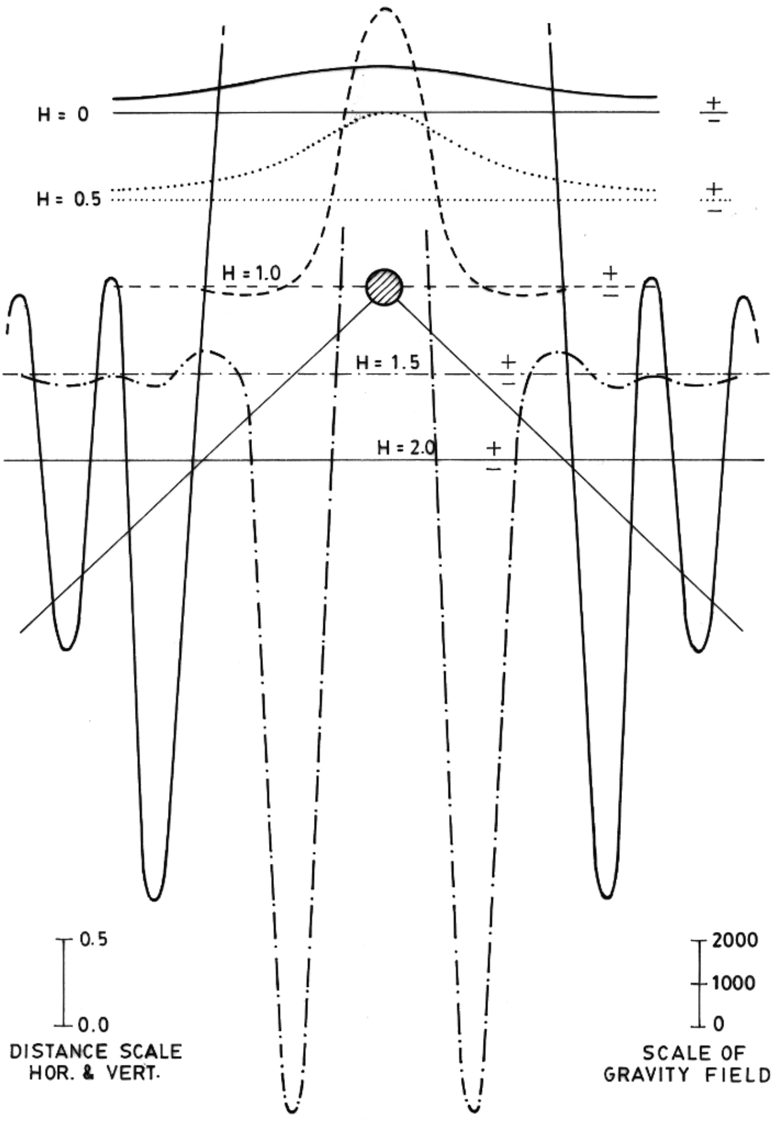


Fig. 16.3. Analytical continuation of synthetic gravity data generated due to an infinitely long buried cylinder (Roy 1966)

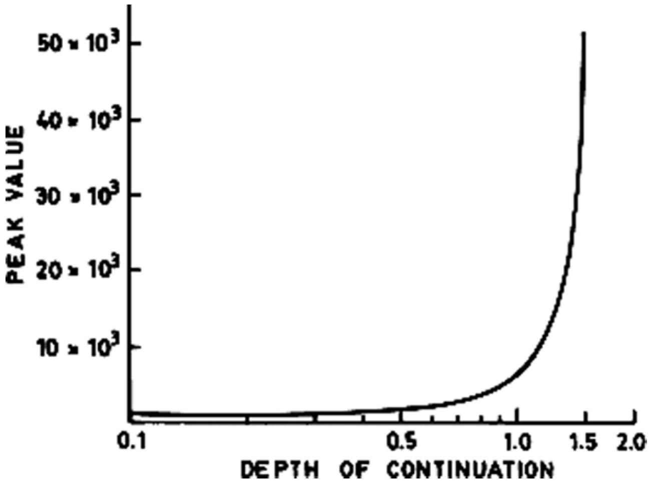


Fig. 16.4. Gradual increase in amplitude of the peak value of the gravitational anomaly in different stages of downward continuation; point of inflection of the curve gives the depth estimation of the causative body (Roy 1966)

16.4 Green's Theorem and Integral Equations for Analytical Continuation

Upward continuation integral based on the Green's Theorem is presented in this section Roy (1962), Blakely (1996). From Green's third formula (Chap. 10, (10.29)), the potential ϕ_p inside a source free region R bounded by a surface S (Fig. 16.5) is given by

$$\phi_\rho = \frac{1}{4\pi} \iint_S \left[\frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) \right] ds \tag{16.22}$$

where n is the direction outward normal to S and r is the distance between the point and the surface element ds. It is known from the uniqueness theorem that either ϕ or $\frac{\partial \phi}{\partial n}$ alone specified on S, should completely determine the potential distribution inside S (Chap. 7.14). It should, therefore, be possible to eliminate either ϕ or $\frac{\partial \phi}{\partial n}$ from (16.22). Assuming both ϕ and G' as harmonic, where G' is another scalar function, we can write from Green's Second identity in symmetric form as

$$0 = \iint_S \left[\phi \frac{\partial G'}{\partial n} - G' \frac{\partial \phi}{\partial n} \right] ds \tag{16.23}$$

Multiplying (16.23) by $-\frac{1}{4\pi}$ and adding it to (16.22) we get

$$\phi_\rho = \frac{1}{4\pi} \iint_S \left[\left(G' + \frac{1}{r} \right) \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left(G' + \frac{1}{r} \right) \right] ds \tag{16.24}$$

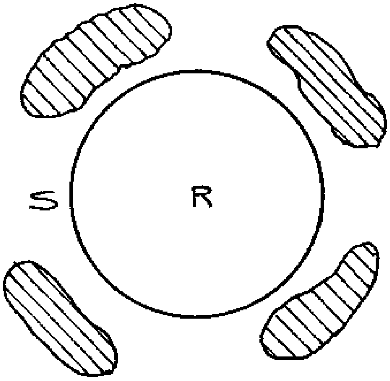


Fig. 16.5. Source free region R bounded by the surface S; all the source (masses) are outside S

If G′ is so chosen that in addition to its being a solution of Laplace equation in R, it also has a normal derivative at any point on S and is equal to the negative of the normal derivative or $\frac{1}{r}$ at the same point, then the (16.24) simply reduces to

$$\phi_\rho = \frac{1}{4\pi} \int_S \int \left[\left(G' + \frac{1}{r} \right) \frac{\partial \phi}{\partial n} \right] ds. \tag{16.25}$$

In (16.25) ϕ is eliminated at the cost of G′. It is a Green's function. G′ will depend upon the nature of the surface S. Usefulness of (16.25) is depended upon getting a suitable value of G′ for specific cases. The surface S of Fig. 16.6 consists of two portions, i.e., a flat ground surface at $z = 0$ and an hemispherical surface of infinite radius. All the sources are below the ground surface and are, therefore, outside R as required. The surface integration in (16.25) now reduces simply to an integration over the plane $z = 0$ (ground surface) because $\frac{\partial \phi}{\partial n} = 0$ at all points on the infinite hemisphere. The outward drawn normal becomes identical with the conventional positive direction of z. With the surface S defined like this, G′ is obviously given by $\frac{1}{r}$, where

$$r = \left[x^2 + y^2 + (z + h)^2 \right]^{1/2}$$

and

$$r' = \left[x^2 + y^2 + (z - h)^2 \right]^{1/2} \tag{16.26}$$

as may be verified by differentiating $\left(\frac{1}{r}\right)$ and $\left(\frac{1}{r'}\right)$ with respect to z and then putting $z = 0$. Thus (16.24) becomes

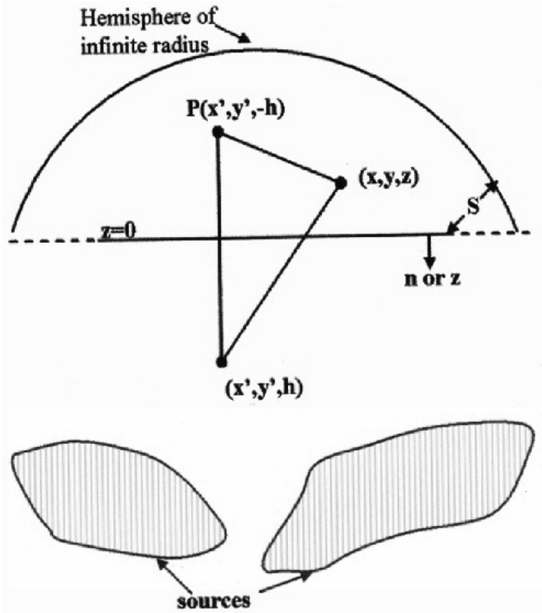


Fig. 16.6. Shows source free region R bounded by a horizontal surface S and hemisphere of infinite radius

$$\phi_\rho = \frac{1}{4\pi} \int\limits_{z=0}\int \left[\frac{1}{r} + \frac{1}{r'}\right]_{z=0} \left.\frac{\partial\phi}{\partial z}\right|_{z=0} dx\ dy \tag{16.27}$$

$$= \frac{1}{2\pi} \int\limits_{z=0}\int \frac{g(x, y, 0)\, dx\ dy}{(x^2 + y^2 + h^2)^{3/2}}. \tag{16.28}$$

Gravity field at the point P (Fig. 16.6) is given by

$$\begin{aligned} g(0, 0, -h) &= \left.\frac{\partial\phi_\rho}{\partial z}\right|_{z=-h} \\ &= \frac{1}{2\pi}\int\int g(x, y, 0)\left[\frac{\partial}{\partial z}\left(\frac{1}{R}\right)\right]_{z=-h} \\ &= \frac{h}{2\pi}\int\int \frac{g(x, y, 0)\, dx\ dy}{(x^2 + y^2 + h^2)^{3/2}} \end{aligned} \tag{16.29}$$

where $R^2 = x^2 + y^2 + z^2$.

So far as gravity is concerned, elimination of ϕ_p from (16.22) is suitable because $\left(\partial\phi_p/\partial n\right)$ ultimately turns out to be g, the vertical component of gravitational attraction due to anomalous masses. In many geophysical measurements other components may also be measured. For such cases, it may become necessary to eliminate $\frac{\partial\phi}{\partial n}$ from (16.22) and one finally gets

$$\phi(0,0,-h) = \frac{h}{2\pi} \int\limits_{z=0}\int \frac{\phi(x,y,0)\,dx\,dy}{(x^2+y^2+z^2)^{3/2}}. \tag{16.30}$$

Equation (16.30) is valid for any measurable solution of Laplace's equation. Equation (16.28) is same as (16.30) written for 'g'.

For potential or field at any point on an upper plane, one may thus replace the real sources by an imaginary laminar distribution on a lower parallel plane with a surface density of $g(x,y,0)/2\pi G$ Here G is the gravitational constant. It can be proved that the areal density $g(x,y,0)/2\pi G$, when integrated over the entire plane, yields the anomalous mass causing the gravity anomaly.

16.5 Analytical Continuation using Integral Equation and Taking Areal Averages

Peter's (1949) proposed the following techniques for upward and downward continuation of potential field. Peter presented his theory using magnetic field, here his formulation is shown in terms of the gravity field.

16.5.1 Upward Continuation of Potential Field

We have seen from (16.30) that gravitational field at any height h can be written in terms of the surface values of the gravity field as

$$g(\alpha,\beta,-h) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{g(x,y,0)\,h\,dx\,dy}{\left\{(x-\alpha)^2+(y-\beta)^2+h^2\right\}^{3/2}}. \tag{16.31}$$

Negative sign is for upward continued values.
Let

$$x-\alpha = r\ \cos\theta$$

and

$$y-\beta = r\ \sin\theta \tag{16.32}$$

then

$$g(\alpha,\beta,-h) = \int\limits_{0}^{\infty-} \frac{g(r)\,h\ r\ dr}{(r^2+h^2)^{3/2}} \tag{16.33}$$

where $\bar{g}(r)$ is the average value of gravity at a point within the radial distance 'r'. Equation 16.33 can now be written as

$$
\begin{aligned}
g(\alpha,\beta,-h) \approx{}& \frac{g(0)+\bar{g}(r_1)}{2}\int_0^{r_1}\frac{h\ r\ d\ r}{\left(r^2+h^2\right)^{3/2}} \\
&+\frac{\bar{g}(r_1)+\bar{g}(r_2)}{2}\int_{r_1}^{r_2}\frac{h\ r\ d\ r}{\left(r^2+h^2\right)^{3/2}}+\frac{\bar{g}(r_2)+\bar{g}(r_3)}{2}\int_{r_2}^{r_3}\frac{h\ r\ d\ r}{\left(r^2+h^2\right)^{3/2}} \\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.
\end{aligned} \tag{16.34}
$$

Since

$$
\int\frac{h\ r\ d\ r}{\left(r^2+h^2\right)^{3/2}} \approx -\frac{h}{\left(r^2+h^2\right)^{1/2}} \tag{16.35}
$$

we can write

$$
\begin{aligned}
g(\alpha,\beta,-h) ={}& \frac{g(0)}{2}\left[1-\frac{h}{\left(h^2+r_1^2\right)^{1/2}}\right] \\
&+\frac{\bar{g}(r_1)}{2}\left[1-\frac{h}{\left(h^2+r_2^2\right)^{1/2}}\right] \\
&+\frac{\bar{g}(r_2)\,h}{2}\left[\frac{1}{\left(h^2+r_1^2\right)^{1/2}}-\frac{1}{\left(h^2+r_3^2\right)^{1/2}}\right] \\
&+\frac{\bar{g}(r_3)\,h}{2}\left[\frac{1}{\left(h^2+r_2^2\right)^{1/2}}-\frac{1}{\left(h^2+r_4^2\right)^{1/2}}\right] \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+\frac{\bar{g}(r_n)\,.h}{2}\left[\frac{1}{\left(h^2+r_{n-1}^2\right)^{1/2}}-\frac{1}{\left(h^2+r_{n+1}^2\right)^{1/2}}\right]. \\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{16.36}
$$

The (16.36) is used to determine the upward continued potentials. The values of r_1, r_2, r_3.............r_n and the coefficients of $\bar{g}(r)$ for $h = 1$ and $h = 2$ are given in Table 16.1. Depending upon the need, coefficients for any value of the h_n can be determined.

Radii of the Peter's circles for upward continuation can approximately be written as

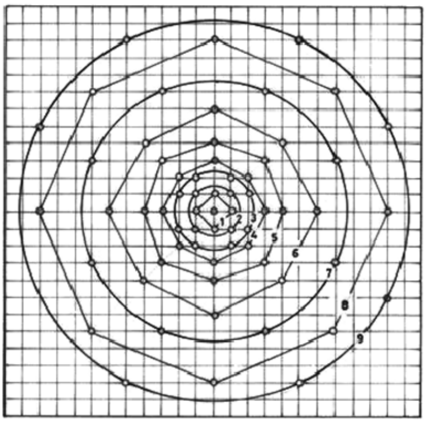
(i) $\sqrt{2} = \sqrt{1^2+1^2}$
(ii) $\sqrt{5} = \sqrt{2^2+1^2}$
(iii) $\sqrt{8.5} \approx \sqrt{2^2+2^2} \approx \sqrt{3^2}$
(iv) $\sqrt{17} = \sqrt{4^2+1^2}$
(v) $\sqrt{34} \approx \sqrt{6^2} \approx \sqrt{4^2+4^2}$
(vi) $\sqrt{58} \approx \sqrt{6^2+5^2}$

Table 16.1. Coefficients of $\bar{g}\,(r)$ for Different Level of Continuation

	Radius	Coefficient of g (r) for h = 1	Coefficient of g (r) for h = 2
0	0	0.1464	0.0528
1	1	0.2113	0.0918
2	$\sqrt{2}$	0.1494	0.1139
3	$\sqrt{5}$	0.1265	0.1254
4	$\sqrt{8.5}$	0.0863	0.1151
5	$\sqrt{17}$	0.0777	0.1206
6	$\sqrt{34}$	0.0528	0.0912
7	$\sqrt{58}$	0.0345	0.0637
8	$\sqrt{99}$	0.0206	0.0320
9	$\sqrt{125}$	0.0945	0.1866

(vii) $\sqrt{99} \approx \sqrt{10^2} \approx \sqrt{7^2 + 7^2}$
(viii) $\sqrt{125} \approx \sqrt{10^2 + 5^2}$

Gravity data used for interpretation are available in the form of contour maps where the gravity values are taken at the grid points. Figure 16.7 shows the grid point and Peter's circles. If the intensity is to be calculated one unit above the plane z = 0, for h = 1 and r = 1, then the term multiplying $g\,(r_1)$ is $\frac{1}{2}\left[1 - 1/\sqrt{2}\,\right] = 0.1464$. Similarly a circle with radius $\sqrt{2}$ about 0, four values lie on this circle and $\bar{g}\,(r_2)$ is one fourth the sum of the four values $\left(r_2 = \sqrt{2}\right)$. The term multiplying $\bar{g}\,(r_1)$ is $\frac{1}{2}\left[1 - 1/\sqrt{3}\right] = 0.2113$. The values of r_3 can be taken as $\sqrt{5}$.



Circle No.	Average Radius
0	0
1	1
2	$\sqrt{2}$
3	$\sqrt{5}$
4	$\sqrt{8.5}$
5	$\sqrt{17}$
6	$\sqrt{34}$
7	$\sqrt{58}$
8	$\sqrt{99}$
9	$\sqrt{125}$

Fig. 16.7. Peter's circle for areal averaging

The circle of radius r_3 passes through eight points on the corner of an octagon. The values at these eight point are used to compute averages.

16.5.2 Downward Continuation of Potential Field (Peters Approach)

Using Taylors series expansion, one can write

$$g\,(+h) = g\,(0) + h\left.\frac{\partial g}{\partial z}\right|_{z=0} + \frac{h^2}{2!}\left.\frac{\partial^2 g}{\partial z^2}\right|_{z=0} + \ldots\ldots\ldots\ldots.. \tag{16.37}$$

and

$$g\,(-h) = g\,(0) - h\left.\frac{\partial g}{\partial z}\right|_{z=0} + \frac{h^2}{4!}\left.\frac{\partial^4 g}{\partial z^4}\right|_{z=0} + \ldots\ldots\ldots\ldots..\,. \tag{16.38}$$

Adding (16.37) and (16.38) we get

$$\begin{aligned} g\,(0,0,h) =&2\left[g\,(0,0,0) + \frac{h^2}{2!}\left.\frac{\partial^2 g}{\partial z^2}\right|_{z=0} + \frac{h^4}{4!}\left.\frac{\partial^2 g}{\partial z^4}\right|_{z=0}\right. \\ &\left. + \text{- -}\right] - g\,(0,0,-h)\,. \end{aligned} \tag{16.39}$$

Since in a source free region the potential satisfies Laplace equation, we can write

$$\frac{\partial^2 g}{\partial z^2} = -\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right). \tag{16.40}$$

In a cylindrical polar coordinate we can write

$$\begin{aligned} g\,(0,0,h) =&g\,(0,0,0) - \frac{h^2}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{g}\,(r) \\ &+ \frac{h^4}{4!}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 \bar{g}\,(r) - \frac{h^6}{6!}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^3 \bar{g}\,(r) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,. \end{aligned} \tag{16.41}$$

It can be shown from Green's Theorem that $\bar{g}\,(r)$ is an even function and it may be written in the form

$$\bar{g}\,(r) = b_0 + b_2 r^2 + b_4 r^4 + b_6 r^6 + \ldots\ldots\ldots\ldots\,. \tag{16.42}$$

If we carry out the operation as indicated in (16.41), we can find

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{g}\,(r) = 4b_2 \tag{16.43}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2\bar{g}\,(r) = 6.4b_4 \tag{16.44}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^3\bar{g}\,(r) = 2304b_6. \tag{16.45}$$

The (16.39) for the continuation down ward becomes

$$\mathrm{g}\,(0,0,\mathrm{h}) \cong 2\left[\mathrm{b}_0 - 2\mathrm{b}_2\mathrm{h}^2 + \frac{8}{3}\mathrm{b}_4\mathrm{h}^4 - \frac{16}{5}\mathrm{b}_6\mathrm{h}^6 + \ldots\ldots\right] - \mathrm{g}\,(0,0,\mathrm{h})\,. \quad (16.46)$$

The values of $\mathrm{b}_0, \mathrm{b}_2$ and b_4 are obtained by least squares solution of the abridged form of (16.42)

$$\mathrm{g}\,(0,0,\mathrm{h}) \cong \mathrm{b}_0 + \mathrm{b}_2\mathrm{r}^2 + \mathrm{b}_4\mathrm{r}^4. \quad (16.47)$$

Using the average values of $\bar{\mathrm{g}}\,(\mathrm{r})$ around the circles of radius 0, 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8.5}$, $\sqrt{17}$, $\sqrt{34}$, $\sqrt{58}$, $\sqrt{99}$, the values of $\mathrm{b}_0, \mathrm{b}_2, \mathrm{b}_4$, obtained by the least squares method as (Peters 1949).

$$\begin{aligned}
\mathrm{b}_0 =& 0.2471\bar{\mathrm{g}}\,(0) + 0.2351\bar{\mathrm{g}}\,(1) + 0.2234\bar{\mathrm{g}}\left(\sqrt{2}\right) \\
& + 0.1874\bar{\mathrm{g}}\left(\sqrt{5}\right) + 0.1521\bar{\mathrm{g}}\left(\sqrt{8.5}\right) + 0.0717\bar{\mathrm{g}}\left(\sqrt{17}\right) \\
& - 0.0449\bar{\mathrm{g}}\left(\sqrt{34}\right) - 0.1095\bar{\mathrm{g}}\left(\sqrt{58}\right) + 0.0500\bar{\mathrm{g}}\left(\sqrt{99}\right) \qquad (16.48) \\
\mathrm{b}_2 =& - 0.0119\bar{\mathrm{g}}\,(0) - 0.0105\bar{\mathrm{g}}\,(1) - 0.0091\bar{\mathrm{g}}\left(\sqrt{2}\right) \\
& - 0.0053\bar{\mathrm{g}}\left(\sqrt{5}\right) - 0.0011\bar{\mathrm{g}}\left(\sqrt{8.5}\right) + 0.0077\bar{\mathrm{g}}\left(\sqrt{17}\right) \\
& + 0.0192\bar{\mathrm{g}}\left(\sqrt{34}\right) + 0.0218\bar{\mathrm{g}}\left(\sqrt{58}\right) - 0.0108\bar{\mathrm{g}}\left(\sqrt{99}\right) \qquad (16.49) \\
\mathrm{b}_4 =& - 0.00010\bar{\mathrm{g}}\,(0) + 0.00009\bar{\mathrm{g}}\,(1) + 0.0007\bar{\mathrm{g}}\left(\sqrt{2}\right) \\
& + 0.00004\bar{\mathrm{g}}\left(\sqrt{5}\right) - 0.\bar{\mathrm{g}}\left(\sqrt{8.5}\right) - 0.00009\bar{\mathrm{g}}\left(\sqrt{17}\right) \\
& - 0.00020\bar{\mathrm{g}}\left(\sqrt{34}\right) - 0.00020\bar{\mathrm{g}}\left(\sqrt{58}\right) + 0.00020\bar{\mathrm{g}}\left(\sqrt{99}\right) \qquad (16.50)
\end{aligned}$$

These values are substituted in (16.47) to get the coefficients for the circles. By combining the coefficients for continuation upward g (0, 0, −h), given in

Table 16.2. Values of the coefficients for downward continuation

	a_0	a_2	a_4
$\bar{\mathrm{g}}\,(0)$	0.2473	−0.0122	0.0001
$\bar{\mathrm{g}}\,(\mathrm{S})$	0.2353	−0.0107	0.00009
$\bar{\mathrm{g}}\,(\mathrm{S}\sqrt{2})$	0.2236	−0.0093	0.00007
$\bar{\mathrm{g}}\,(\mathrm{S}\sqrt{5})$	0.1895	−0.0055	0.00004
$\bar{\mathrm{g}}\,(\mathrm{S}\sqrt{8.5})$	0.1521	−0.0013	−0.000007
$\bar{\mathrm{g}}\,(\mathrm{S}\sqrt{17})$	0.0714	0.0075	−0.0009
$\bar{\mathrm{g}}\,(\mathrm{S}\sqrt{34})$	0.460	0.0190	−0.00020
$\bar{\mathrm{g}}\,(\mathrm{S}\sqrt{58})$	0.1124	0.0215	−0.0021
$\bar{\mathrm{g}}\,(\mathrm{S}\sqrt{99})$	−0.0432	−0.0116	+0.00020

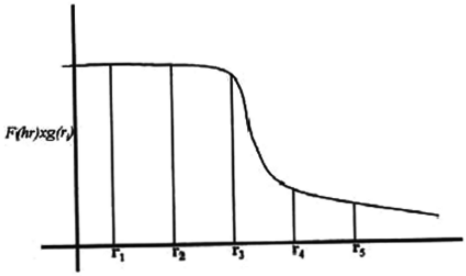


Fig. 16.8. Variation of f(hr) with radial distance

Table 16.1, the coefficients for continuation downward can be obtained for $h = 1$ and $h = 2$. These coefficients are given in Table 16.2 and are used along with the values given (Fig. 16.8).

Trejo (1954) suggested that one can get better results for downward continuation if Peters upward continuation program and three dimensional finite difference program is used together for downward continuation. His working relation is (Fig. 16.9)

$$\begin{aligned} g(0,0,h) =& 6g(0,0,0) - g(h,0,0) - g(-h,0,0) \\ & - g(0,h,0) - g(0,-h,0) - g(0,0-h). \end{aligned} \tag{16.51}$$

This relation gives the value of g at a depth h in terms of the measured values of g at a height $-h$, and at the five points on the surface $z = 0$.

$$g(0,0,h) = -\int_0^\infty \frac{\bar{g}(r)\,h\,r\,dr}{(r^2+h^2)^{3/2}} \tag{16.52}$$

is valid only for negative non-zero values of h. At $h = 0$, the expression on the right side of the expression is not well defined. Baranov (1953) argued that if this expression could be made to behave properly at $h = 0$, such

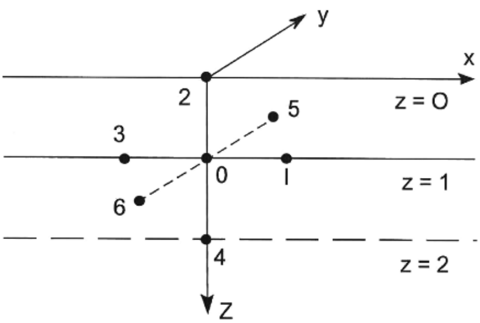


Fig. 16.9. Finite difference grid points at three different levels

that it yields the observed g (0,0) at that point, then one could extrapolate g (0, h) backwards, so to say to positive value of h and achieve downward continuation. Baranov was able to do this by breaking up the range of integration in to 10 intervals expressing $\bar{g}(r)$ in suitable polynomials and then determining the constants of the polynomial in terms of $\bar{g}(r)$ at $r = 0, 1, \sqrt{2}, \sqrt{5}, \sqrt{10}, \sqrt{17}, \sqrt{25}, \sqrt{40}, \sqrt{68}, 10$.

16.6 Upward and Downward Continuation using Integral Equation and Lagrange Interpolation Formula

Henderson and Zeitz (1949) independently suggested the approach for upward continuation which is also based on the integral which connects the gravity or magnetic fields at different levels, i.e.,

$$g(\alpha, \beta, -h) = \frac{h}{2\pi} \int_{-\infty}^{\infty} \frac{g(x, y, 0)\, dxdy}{\left[(x-\alpha)^2 + (y-\beta)^2 + h^2\right]^{3/2}}. \tag{16.53}$$

This equation is valid for upward continuation. In cylindrical coordinate it is

$$g(\alpha, \beta, -h) = \int_{0}^{\infty} \frac{\bar{g}(r)\, h\, r\, dr}{(h^2 + z^2)^{3/2}}. \tag{16.54}$$

Equation (16.54) can be written as

$$\begin{aligned} &= \frac{1}{2} \sum_{i=1}^{n-1} \bar{g}(r)\, f(r_i, h)\, (r_{i+1} - r_{i-1}) \\ &\quad + \frac{1}{2} g(r_n)\, f(r_n, h)\, (r_{n+1} - r_{n-1}) \end{aligned} \tag{16.55}$$

where

$$f(r_i, h) = \frac{h\, r_i}{(h^2 + r_i^2)^{3/2}}. \tag{16.56}$$

Using equation (16.53), $g(0, 0, -h)$, the upward continued values of the gravitational field can be calculated using the Lagrange interpolation polynomial as

$$g(0, 0, -h) = \sum_{m=0}^{n} \frac{(-1)^m\, h\, (h+s)\, (h+2s) \ldots\ldots\ldots (h+ns)}{S^n\, (h+ms)\, (n-m)!m!} g(-ms). \tag{16.57}$$

Here S is the grid spacing and n is the highest value of m.

Henderson and Zeitz (1949) used Lagrange's interpolation formula for downward continuation. If the anomaly on the plane of observation and upward continuation at 5 heights each of one grid spacing apart are known, the gravity field at a depth 'h' can be determined from the formula

$$\mathrm{g}\,(0,0,-\mathrm{h}) = \sum_{\mathrm{n}=0}^{\mathrm{n}} \frac{(-1)^{\mathrm{m}}\,\mathrm{h}\,(\mathrm{h}+\mathrm{a})\,(\mathrm{h}+2\mathrm{a})\ldots\ldots\ldots(\mathrm{h}+\mathrm{na})}{\mathrm{a}^{\mathrm{n}}\,(\mathrm{h}+\mathrm{ma})\,(\mathrm{n}-\mathrm{m})!\mathrm{n}!}. \qquad (16.58)$$

Equation (16.57) and (16.58) are same. The sign of 'h' will dictate whether the continuation is upward or downward. For downward continuation the maximum value of n is taken as 5. That is 5 units of upward continuation is necessary for downward continuation from general solution of Laplace equation and Taylor's series expansion.

16.7 Downward Continuation of Telluric Current Data

Earth currents or telluric currents are continuously flowing through the conducting portion of the upper crust. These currents are induced on the earth by electromagnetic waves which are propagated from the magnetosphere towards the surface of the earth. These currents are flowing through the subsurface as if the currents are generated by two infinitely long line electrodes placed at infinite distance and this electrode pair is continuously rotating to generate elliptically or circularly polarized telluric fields.

For telluric current flow it is assumed that a perfectly insulating basement is overlain by a homogeneous conducting layer of sediments through which the telluric current flows. Since $\left(\frac{\partial\phi}{\partial z}\right)$, the derivative of telluric potential normal to the ground surface at $z = 0$, is zero everywhere, it follows that

$$\frac{\partial^3\phi}{\partial z^3} = -\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial\phi}{\partial z}\right] = 0 \qquad (16.59)$$

on the ground surface. Similarly it can be shown that all the higher odd derivatives are also zero on the ground surface. The Taylor's series expansion of (16.39), simplifies to

$$\phi\,(\pm\mathrm{h}) = \phi\,(0) + \left.\frac{\partial^2\phi}{\partial z^2}\right|_{z=0}\frac{\mathrm{h}}{2!} + \left.\frac{\partial^4\phi}{\partial z^4}\right|_{z=0}\frac{\mathrm{h}^4}{4!} + \qquad (16.60)$$

where ϕ is the DC telluric potential. Although telluric field is a time varying field, the frequencies are very low in general. Therefore both static and dynamic theories are used with equal validity to analyse earth's natural electromagnetic field signals. Static theory is used to study the electrotelluric fields and potentials, dynamic theory is used in magnetotellurics although the frequencies of the signals may be of the same order. Laplace equation is used to study electrotellurics,

Helmholtz electromagnetic wave equation $\nabla^2 H = \gamma^2 H$ is used in magnetotellurics neglecting the displacement current component in the propagation constant γ (i.e. $\gamma = \sqrt{i\omega\mu\sigma}$).

The continuation in this case is thus independent of the sign of h and can be carried out very easily. One can directly apply the finite difference form of Laplace's equation to the measured data in order to obtain downward continued potentials step by step. Specifically for two dimensional problem.

$\phi(0,1) = \phi(4) = \phi(2) = 2\phi(0) - [\phi(1) + \phi(3)]/2$, for the first unit of depth of continuation and

$$\phi(0,n) = \phi(4) = 4\phi(0) - \phi(1) - \phi(2) - \phi(3) \tag{16.61}$$

will be the working formula for subsequent units of depth of continuation. For three dimensional case

$$\phi(0,1) \equiv \phi(4) = \phi(2) = 3\phi(0) - [\phi(1) + \phi(3) + \phi(5) + \phi(6)]/2 \tag{16.62}$$

for the first units of the depth of continuations and

$$\phi(0,\mathrm{n}) \equiv \phi(4) = 6\phi(0) - [\phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(5)] \tag{16.63}$$

for the subsequent units.

After having obtained the downward continued potentials of a sufficiently large number of levels, one can draw the equipotential contours in suitable section. In the same vertical section one can draw the streamlines which are orthogonal to the equipotentials. One of these stream lines correspond to the basement surface, since the top of the basement also happens to be a flow surface where $\frac{\partial\phi}{\partial \mathrm{n}} = 0$. Without any other information, it is not possible to decide which of the streamlines actually represent the basement topography. If the depth to the top of the basement is known aprior or is determined by magnetotelluric sounding at one point, one can find out the entire topography of the basement surface by choosing that flow surface which passes through the known points.

16.8 Upward and Downward Continuation of Electromagnetic Field Data

Roy (1966, 1968) prescribed the methodology for upward continuation of electromagnetic field starting from the Helmholtz electromagnetic wave equation

$$\nabla^2\vec{\psi} = \gamma^2\vec{\psi} \tag{16.64}$$

where $\vec{\psi}$ is the complex electromagnetic field potential and γ is the propagation constant $\left(= \sqrt{\mathrm{i}\omega\mu\,(\sigma + \mathrm{i}\omega\varepsilon)}\right)$. Ψ can be expressed as $\vec{\psi} = \vec{\psi}_R + i\vec{\psi}_I$. Here R and I respectively represent real and imaginary components. ω, μ, σ, and ε, and $\sqrt{\mathrm{i}}$ are standard notation, in electromagnetics and are available in Chaps. (12 and 13). It can be shown that the value of $\vec{\psi}$ at any interior point $P(x', y', z')$

(see Chap. 10) is given by those on the surface S according to the Green's formula

$$\Psi\left(x', y', z'\right) = \frac{1}{4\pi} \int\!\!\int_{s} \left[\frac{\partial \Psi}{\partial n}\frac{e^{-\gamma r}}{r} - \Psi \frac{\partial}{\partial n}\left(\frac{e^{-\gamma r}}{r}\right)\right] ds. \qquad (16.65)$$

Based on two important assumptions the Helmholtz electromagnetic wave equation reduces to Laplace equation. These assumptions are

(a) the frequencies of the electromagnetic signals with which geophysicists work are very much on the lower side to achieve reasonable skin depth or the depth of penetration of the em signals. At those frequencies, conduction currents dominate over the displacement current. Therefore the propagation constant changes to the form $\gamma = \sqrt{i\omega\mu\sigma}$ neglecting the contributions from the displacement currents.
(b) The second assumption is electrical conductivity of the basement is more than two order of magnitude less than that of the sedimentary overburden. Therefore conductivity of the basement is assumed to be zero. Since $\gamma = 0$ for $\sigma = 0$ and Helmoholtz equation reduces to Laplace equation. Therefore the formula applicable for static or stationery cases are also applicable for special cases in electromagnetics.

Let $r = \left[\left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2\right]^{1/2}$ is the distance of P from a surface dS at (x, y, z). Since, by the electromagnetic uniqueness theorem, $\Psi\left(x', y', z'\right)$ is completely determined once Ψ is defined on the surface.

If is now possible to eliminate $\frac{\partial \Psi}{\partial n}$. Let us now consider Green's second identity

$$\int\!\!\int\!\!\int \left(\Psi \nabla^2 \phi - \phi \nabla^2 \Psi\right) dv = \int\!\!\int_{s} \left(\Psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \Psi}{\partial n}\right) ds \qquad (16.66)$$

where ϕ and Ψ are two scalar functions of position that have continuous first and second derivative (Chap. 10) throughout the region R and on the surface S. Now if both ϕ and Ψ are harmonic then (16.66) changes to the form

$$0 = \int\!\!\int_{s} \left(\Psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \Psi}{\partial n}\right) ds. \qquad (16.67)$$

This is the reciprocity theorem, valid in the same form for two electromagnetic fields as it is for two static and stationary fields. Multiplying (16.66) by $\left(\frac{1}{4\pi}\right)$ and adding to (16.67), we get

$$\psi\left(x', y', z'\right) = \frac{1}{4\pi} \int\!\!\int_{s} \left[\frac{\partial \psi}{\partial n}\left\{\frac{e^{-\gamma r}}{r} - \phi\right\} - \psi \frac{\partial}{\partial n}\left\{\frac{e^{-\gamma r}}{r} - \phi\right\}\right]. \qquad (16.68)$$

If ϕ is chosen so that, in addition to satisfying (16.68), it also assumes the value of $\left(\frac{e^{-\gamma r}}{r}\right)$ on the boundary S,(16.68) reduces to

$$\Psi\left(x', y', z'\right) = \frac{1}{4\pi} \int\limits_{s}\int \Psi \frac{\partial}{\partial n}\left\{\frac{e^{-\gamma r}}{r} - \phi\right\} ds. \tag{16.69}$$

Thus $\frac{\partial \Psi}{\partial n}$ has been eliminated from the (16.69) at the cost of introducing Green's function as already mentioned. The homogeneous and isotropic volume v can now be identified with geometry as shown in (Fig. 16.6).

Let 'r' denote the radial distance of the region of air ($z < 0$) bounded by a horizontal plane at $z = 0$ (ground surface) and an infinite hemisphere above it (Fig. 16.10). For this geometry equation (16.69) can be rewritten as

$$\Psi\left(x', y', z' = -h\right) = -\frac{1}{4\pi} \int \int_{z=0} \Psi \frac{\partial}{\partial z}\left\{\frac{e^{-\gamma r}}{r} - \phi\right\} dx\, dy. \tag{16.70}$$

Since the integral does not vary on the infinite hemisphere, the required function ϕ, in this case, is obviously

$$\phi = \frac{e^{-\gamma r'}}{r'} \tag{16.71}$$

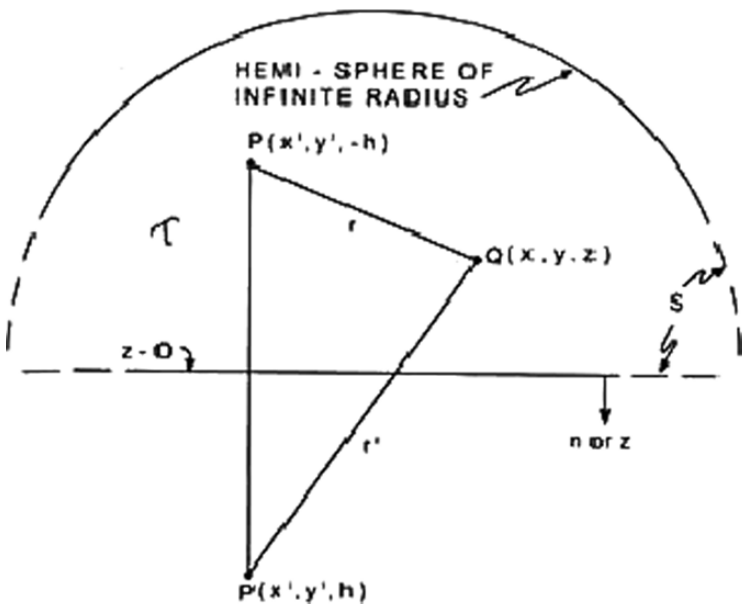


Fig. 16.10. Geometry of the hemispherical space

where

$$\mathrm{r} = \left[(\mathrm{x} - \mathrm{x}')^2 + (\mathrm{y} - \mathrm{y})^2 + (\mathrm{z} + \mathrm{h})^2 \right]^{1/2} \tag{16.72}$$

and

$$\mathrm{r}' = \left[(\mathrm{x} - \mathrm{x}')^2 + (\mathrm{y} - \mathrm{y}')^2 + (\mathrm{z} - \mathrm{h})^2 \right]^{1/2}.$$

Substitution of (16.71 and 16.72) in (16.70) and differentiation yield

$$\Psi\left(\mathrm{x}', \mathrm{y}', -\mathrm{h}\right) = \frac{\mathrm{h}}{2\pi} \int \int_{\mathrm{z}=0} \left\{ \left. \frac{1}{\mathrm{R}} - \mathrm{i}\gamma \right] \mathrm{e}^{-\gamma\mathrm{R}} . \frac{\Psi\left(\mathrm{x}, \mathrm{y}, 0\right)}{\mathrm{R}^2} \right\} \mathrm{dx}\,\mathrm{dy} \tag{16.73}$$

where

$$\mathrm{R} = \left[(\mathrm{x} - \mathrm{x}')^2 + (\mathrm{y} - \mathrm{y}')^2 + \mathrm{h}^2 \right]^{1/2}. \tag{16.74}$$

For static and stationary fields, $\gamma = 0$ and (16.73) reduces to the well known upward continuation integral. It is also known as the Poisson's integral.

Since γ, the propagation constant can be written as $\gamma = \alpha + \mathrm{i}\beta$ where the values of α and β are given in (13.13) and (13.14) We can write

$$\exp\left(\mathrm{i}\gamma\mathrm{R}\right) = \exp\left(-\beta\mathrm{r}\right), \left(\cos\left(\alpha\beta\right) + \mathrm{i}\mathrm{Sin}\left(\alpha\mathrm{R}\right)\right). \tag{16.75}$$

The real and imaginary parts of the electromagnetic upward continuation, integral (16.73) are therefore, given as

$$\begin{aligned} \Psi_{\mathrm{R}}\left(\mathrm{x}', \mathrm{y}', -\mathrm{h}\right) = & \frac{\mathrm{h}}{2\pi} \int \int_{\mathrm{z}=0} \mathrm{e}^{-\beta\mathrm{R}} \left[\Psi_{\mathrm{R}}\left(\mathrm{x}, \mathrm{y}, 0\right) \left\{ \left(\beta + \frac{1}{\mathrm{R}}\right) \cos\left(\alpha\mathrm{R}\right) \right. \right. \\ & \left. + \alpha \sin\left(\alpha\mathrm{R}\right) \right\} - \Psi_{\mathrm{i}}\left(\mathrm{x}, \mathrm{y}, 0\right) \left\{ \left(\beta + \frac{1}{\mathrm{R}}\right) \sin\left(\alpha\mathrm{R}\right) \right. \\ & \left. \left. - \alpha \cos\left(\alpha\mathrm{R}\right) \right] \frac{1}{\mathrm{R}^2} \mathrm{dx}\,\mathrm{dy} \end{aligned} \tag{16.76}$$

$$\begin{aligned} \Psi_{\mathrm{l}}\left(\mathrm{x}', \mathrm{y}', -\mathrm{h}\right) = & \frac{\mathrm{h}}{2\pi} \int \int_{\mathrm{z}=0} \mathrm{e}^{-\beta\mathrm{R}} \left[\left. \left(\beta + \frac{1}{\mathrm{R}}\right) \sin\alpha\mathrm{R} - \alpha \cos\left(\alpha\mathrm{R}\right) \right\} \right. \\ & \left. + \Psi_{\mathrm{l}}\left(\mathrm{x}, \mathrm{y}, 0\right) \left\{ \left(\beta + \frac{1}{\mathrm{R}}\right) \cos\alpha\mathrm{R} + \alpha \sin\left(\alpha\mathrm{R}\right) \right\} \right] \\ & \times \frac{1}{\mathrm{R}^2} \mathrm{dx}\,\mathrm{dy} \oint \int \mathrm{dy}\,\mathrm{dx}. \end{aligned} \tag{16.77}$$

In air, $\sigma = \beta = 0$, and $\alpha = \mathrm{w/c} = \frac{2\pi}{\lambda}$, where c is the velocity of propagation and λ is the wave length. Considering the origin to be located on the surface vertically below p, one has

$$\Psi_R(0,0,-h) = \frac{h}{2\pi}\int\int_{z=0} e^{-\beta R}\left[\Psi_R(x,y,0)\left\{\frac{\cos\left(\frac{2\pi R}{\lambda}\right)}{R} + \frac{2\pi}{\lambda}\sin\left(\frac{2\pi R}{\lambda}\right)\right\}\right. \tag{16.78}$$

$$\left. -\Psi_I(x,y,0) = \left\{\frac{\sin\left(\frac{2\pi R}{\lambda}\right)}{R} - \frac{2\pi}{\lambda}\cos\left(\frac{2\pi R}{\lambda}\right)\right\}\right]\frac{1}{R^2}dx\,dy \tag{16.79}$$

$$\Psi_i(0,0,-h) = \frac{h}{2\pi}\int\int_{z=0}\left[\Psi_R(x,y,0)\left\{\frac{\sin\left(\frac{2\pi R}{\lambda}\right)}{R} - \frac{2\pi}{\lambda}\cos\left(\frac{2\pi R}{\lambda}\right)\right\}\right. \tag{16.80}$$

$$\left. +\Psi_I(x,y,0) = \frac{h}{2\pi}\int\int_{z=0}\left[\Psi_R(x,y,0)\left\{\frac{\cos\left(\frac{2\pi R}{\lambda}\right)}{R} + \frac{2\pi}{\lambda}\sin\left(\frac{2\pi R}{\lambda}\right)\right\}\right]\frac{1}{R^2}dx\,dy\right.$$

where $R = (x^2 + y^2 + h^2)^{1/2}$. For static field where $\gamma = 0$, $\lambda = \infty$, $\Psi_I(x, y, 0)$, (16.79) and (16.80) reduces to the static upward continuation integral.

16.9 Downward Continuation of Electromagnetic Field

Roy's (1969) formulation for downward continuation is presented here. If the xy plane at $z = 0$ (z positive downward) represents the air earth boundary, it is known that both tangential components of the magnetic field H_x and H_y and the normal component H_z will be continuous across the boundary provided $\mu_{air} = \mu_{earth} = \mu_{vacuum}$ where μ is the magnetic permeability of the medium. Horizontal derivatives of all orders of H_x, H_y and H_z are also continuous across the boundary. Since these derivatives are determined solely from the field values of $z = 0$. If we combine these back ground with the relation.

$$\operatorname{div}\vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{16.81}$$

on either side of $z = 0$ (air earth boundary), it follows that $\frac{\partial H_z}{\partial z}$ is also continuous across this plane, Higher order vertical derivatives are discontinuous across the air earth boundary since they satisfy Helmholtz electromagnetic wave equations

$$\nabla^2\vec{H}_a = \gamma^2\vec{H}_a \qquad \text{(in air)} \tag{16.82}$$

$$\nabla^2\vec{H}_e = \gamma^2\vec{H}_e \qquad \text{(in earth)} \tag{16.83}$$

where $\gamma_a^2 = \mu_0\varepsilon\omega^2$ and $\gamma_e^2 = i\omega\mu_0(\sigma + i\omega\varepsilon)$, μ_0 is the magnetic permeability of the vacuum. The subscripts 'a' and 'e' relate to quantities in air and in earth immediately on two sides of the interface at $z = 0$. Since

$$\frac{\partial^2 \vec{H}_{za}}{\partial x^2} = \frac{\partial^2 \vec{H}_{ze}}{\partial x^2} \tag{16.84}$$

$$\frac{\partial^2 \vec{H}_{za}}{\partial y^2} = \frac{\partial^2 \vec{H}_{ze}}{\partial y^2} \tag{16.85}$$

and

$$H_{za} = H_{ze},$$

it follows from (16.82) and (16.83) that

$$\frac{\partial^2 H_{ze}}{\partial z^2} = \left(\gamma_e^2 - \gamma_a^2\right) H_{za} + \frac{\partial^2 H_{za}}{\partial z^2} \quad \text{at } z = 0. \tag{16.86}$$

Replacing H_{Z_a} in (16.84) and (16.85) in turn by $\frac{\partial H_z}{\partial z}, \frac{\partial^2 H_z}{\partial z^2}$ and so on and using the fact that

$$\frac{\partial^3 H_{za}}{\partial x^2 \partial z} = \frac{\partial^3 H_{ze}}{\partial x^2 \partial z}, \frac{\partial^3 H_{za}}{\partial y^2 \partial z} = \frac{\partial^3 H_{ze}}{\partial y^2 \partial z}$$

and

$$\frac{\partial H_{za}}{\partial z} = \frac{\partial H_{ze}}{\partial z} \tag{16.87}$$

for higher order derivatives , one can show that

$$\begin{aligned}
\frac{\partial^3 H_{ze}}{\partial z^3} &= \left(\gamma_e^2 - \gamma_0^2\right) \frac{\partial H_{ze}}{\partial z} + \frac{\partial^3 H_{za}}{\partial z^3} \quad \text{at } z = 0 \\
\frac{\partial^4 H_{ze}}{\partial z^4} &= \left(\gamma_e^2 - \gamma_0^2\right)^2 H_{za} + 2\left(\gamma_e^2 - \gamma_a^2\right) \frac{\partial^2 z_a}{\partial z^2} + \frac{\partial^4 z_a}{\partial z^4} \text{ at } z = 0 \\
\frac{\partial^5 H_{ze}}{\partial x^5} &= \left(\gamma_e^2 - \gamma_a^2\right)^2 \frac{\partial H_{za}}{\partial z} + 2\left(\gamma_e^2 - \gamma_a^2\right) \frac{\partial^3 H_{za}}{\partial z^3} + \frac{\partial^5 Z_{za}}{\partial z^5} \text{ at } z = 0
\end{aligned} \tag{16.88}$$

and so on .

In general, for $n = 1, 2, 3$

$$\frac{\partial^{2n} H_{ze}}{\partial z^{2n}} = \left[\left(\gamma_a^2 - \gamma_e^2\right) + \frac{\partial^2}{\partial z^2}\right]^n H_{za} \quad \text{at } z = 0 \tag{16.89}$$

and so on. On the ground, for $n = 1, 2, 3$

$$\frac{\partial^{2n+1} H_{ze}}{\partial z^{2n+1}} = \left[\left(\gamma_e^2 - \gamma_a^2\right) + \frac{\partial^2}{\partial z^2}\right]^n \frac{\partial H_{za}}{\partial z}, \quad \text{at } z = 0 \tag{16.90}$$

where the superscript n indicates that the operation within the brackets has to be carried out n terms successively.
$\frac{\partial H_z}{\partial z}$ is continuous across the boundary $\frac{\partial H_x}{\partial z}$ and $\frac{\partial H_y}{\partial z}$ are not because

$$\frac{\partial H_{za}}{\partial y} - \frac{\partial H_{ya}}{\partial z} = i\omega\varepsilon_a E_{xa} \quad \text{(in air)} \tag{16.91}$$

and

$$\frac{\partial H_{ze}}{\partial y} - \frac{\partial H_{ye}}{\partial z} = (\sigma + i\omega\varepsilon_e)\, E_{xe}(\text{in earth}), \tag{16.92}$$

since $E_{xe} = E_{xa}$ and $\frac{\partial H_{za}}{\partial y} = \frac{\partial H_{ze}}{\partial y}$ at $z = 0$.

We have

$$\frac{\partial H_{ye}}{\partial z} = \frac{\partial H_{za}}{\partial y} + \frac{\sigma + i\omega\varepsilon}{\omega\varepsilon_a}\left[\frac{\partial H_{za}}{\partial y} - \frac{\partial H_{ya}}{\partial z}\right] \text{at } z = 0. \tag{16.93}$$

Replacing H_y for H_z in (16.92) and (16.93) yields

$$\frac{\partial^2 H_{ye}}{\partial z^2} = \left(\gamma_e^2 - \gamma_a^2\right) H_{ya} + \frac{\partial^2 H_a}{\partial z^2} \quad \text{at } z = 0. \tag{16.94}$$

It can also be shown that

$$\frac{\partial^3 H_{ye}}{\partial z^3} = \left(\gamma_e^2 - \gamma_a^2 + \frac{\partial^2}{\partial z^2}\right)\left[\frac{\partial H_{xa}}{\partial y} + \frac{\sigma + i\omega\varepsilon}{\omega\varepsilon_a}\left\{\frac{\partial H_{za}}{\partial y} - \frac{\partial H_{ya}}{\partial z}\right\}\right] \tag{16.95}$$

at $z = 0$

where the operations indicated in the first parentheses are to be carried out over the quantity in the squared brackets. In general again for $n = 1, 2, 3, \ldots..$

$$\frac{\partial^{2n} H_{ye}}{\partial z^{2n}} = \left(\gamma_1^2 - \gamma_a^2 + \frac{\partial^2}{\partial z^2}\right)^n H_{ya} \quad \text{at } z = 0 \tag{16.96}$$

$$\frac{\partial^{2n} H_{ye}}{\partial z^{2n+1}} = \left(\gamma_1^2 - \gamma_a^2 + \frac{\partial^2}{\partial z^2}\right)^n \left[\frac{\partial H_{za}}{\partial y} + \frac{\sigma + i\omega\varepsilon}{\omega\varepsilon_a}\left\{\frac{\partial H_{za}}{\partial y} - \frac{\partial H_{ya}}{\partial z}\right\}\right]$$
$$\text{at } z = 0. \tag{16.97}$$

With the superscript n having the same meaning as in expression (16.84) (16.85) and (16.86). Similar expressions can be derived for the vertical derivatives of H_{xe}. In case it is the electric field components that are observed, one has (Stratton, 1941, p 483).

$$E_{xe} = E_{xa}, E_{ye} = E_{ya}, E_{ze} = \left(\frac{\gamma_a^2}{\gamma_e^2}\right).E_{za} \tag{16.98}$$

at $z = 0$. From the divergence relation

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{16.99}$$

which holds on both the sides of $z = 0$, it is obvious that $\frac{\partial E_z}{\partial z}$, like $\frac{\partial H_z}{\partial z}$ is continuous across the boundary. However as in the magnetic case, $\frac{\partial E_x}{\partial z}$ and $\frac{\partial E_y}{\partial z}$ are not continuous. Since

$$\frac{\partial E_{xa}}{\partial z} - \frac{\partial E_{za}}{\partial x} = -i\omega\mu_0\, H_{y0}(\text{in air}) \tag{16.100}$$

$$\frac{\partial E_{xe}}{\partial z} - \frac{\partial E_{ze}}{\partial x} = -i\omega\mu_0\, H_{ye}(\text{in earth}) \tag{16.101}$$

and $H_{ye} = H_{ya}$ at $z = 0$. We get

$$\frac{\partial E_{xe}}{\partial z} - \frac{\partial E_{xa}}{\partial z} = -\left(1 - \frac{\gamma_a^2}{\gamma_e^2}\right)\frac{\partial E_{za}}{\partial x} \qquad \text{at } z = 0 \tag{16.102}$$

$$\frac{\partial E_{ye}}{\partial z} = \frac{\partial E_{ya}}{\partial z} = -\left(1 - \frac{\gamma_a^2}{\gamma_e^2}\right)\frac{\partial E_{za}}{\partial y} \qquad \text{at } z = 0 \tag{16.103}$$

The higher derivatives are

$$\frac{\partial^2 E_{ze}}{\partial z^2} = \frac{\gamma_a^2}{\gamma_e^2}\left[\left(\gamma_a^2 - \gamma_e^2\right)E_{za} + \frac{\partial^2 E_{za}}{\partial z^2}\right] \text{at } z = 0 \tag{16.104}$$

$$\frac{\partial^3 E_{ze}}{\partial z^3} = \left(\gamma_e^2 - \gamma_a^2\right)\frac{\partial E_{za}}{\partial z^3} + \frac{\partial^3 E_{za}}{\partial z^3}\text{at } z = 0 \tag{16.105}$$

$$\frac{\partial^4 E_{ze}}{\partial z^4} = \frac{\gamma_a^2}{\gamma_e^2}\left[\left(\gamma_e^2 - \gamma_a^2\right) + \frac{\partial^2}{\partial z^2}\right]^2 E_{za} \quad \text{at } z = 0 \tag{16.106}$$

$$\frac{\partial^{2n} E_{ze}}{\partial z^{2n}} = \frac{\gamma_a^2}{\gamma_e^2}\left[\left(\gamma_e^2 - \gamma_a^2\right) + \frac{\partial^2}{\partial z^2}\right]^2 E_{za}\text{at } z = 0 \tag{16.107}$$

$$\frac{\partial^{2n+1} E_{ze}}{\partial z^{2n+1}} = \left[\left(\gamma_e^2 - \gamma_a^2\right) + \frac{\partial^2}{\partial z^2}\right]^2 \frac{\partial E_{za}}{\partial z}\text{at } z = 0 \tag{16.108}$$

$$\frac{\partial^2 E_{xe}}{\partial z^2} = \left(\gamma_e^2 - \gamma_a^2\right)E_{xa} + \frac{\partial^2 E_{xa}}{\partial z^2} \quad \text{at } z = 0 \tag{16.109}$$

$$\frac{\partial^3 E_{xe}}{\partial z^3} = \left(\gamma_e^2 - \gamma_a^2\right)\frac{\partial^2}{\partial z^2}\left[\frac{\partial E_{xa}}{\partial z} - \left(1 - \frac{\gamma_a^2}{\gamma_e^2}\right)\frac{\partial E_{xa}}{\partial x}\right] \text{at } z = 0 \tag{16.110}$$

$$\frac{\partial^{2n} E_{xe}}{\partial z^{2n}} = \left[\left(\gamma_e^2 - \gamma_a^2\right) + \frac{\partial^2}{\partial z^2}\right]^n E_{xa} \text{ at } z = 0 \tag{16.111}$$

$$\frac{\partial^{2n+1} E_{xe}}{\partial z^{2n}} = \left[\left(\gamma_e^2 - \gamma_a^2\right) + \frac{\partial^2}{\partial z^2}\right]^n E_{xa}\text{at } z = 0. \tag{16.112}$$

Equation (16.110) and (16.111) give the necessary guide lines to find out the higher vertical derivatives for E_{ye}

16.9.1 Downward Continuation of H_z

The downward continued values of H_{ze} (h) can be written in terms of values on the $z = 0$ boundary in the form of Taylor expansion as follows

$$H_{ze}(h) = H_{ze}(0) + h\frac{\partial H_{ze}}{\partial z}\bigg|_{z=0} + \frac{h^2}{2!}\frac{\partial^2 H_{ze}}{\partial z^2}\bigg|_{z=0} + \frac{h^3}{3!}\frac{\partial^3 H_{ze}}{\partial z^3}\bigg|_{z=0} + \frac{h^4}{4!}\frac{\partial^4 H_{ze}}{\partial z^4}\bigg|_{z=0} + ----------. \tag{16.113}$$

Substituting (16.111) and (16.112) upto the fourth order of vertical derivatives, one can write

$$\mathrm{H_{ze}}\,(\mathrm{h}) = [\mathrm{H}]\,.$$

Substituting (16.111) (16.112) (16.113) upto the fourth order of vertical derivatives, one can write

$$\begin{aligned}\mathrm{H_{ze}}\,(\mathrm{h}) = &\left[\mathrm{H_{za}}\,(0)) + \mathrm{h}\frac{\partial \mathrm{H_{xa}}(0)}{\partial \mathrm{z}} + \frac{\mathrm{h}^2}{2!}\frac{\partial^2 \mathrm{H_{za}}(0)}{\partial \mathrm{z}^2} + \frac{\mathrm{h}^3}{3!}\frac{\partial^3 \mathrm{H_{za}}(0)}{\partial \mathrm{z}^3} + \cdots\right.\\ &\left.+\frac{\mathrm{h}^4}{4!}\frac{\partial^4 \mathrm{H_{za}}(0)}{\partial \mathrm{z}^4}\right] + \left[\frac{\mathrm{h}^2}{2!}\left(\gamma_e^2 - \gamma_a^2\right)\mathrm{H_{za}}\,(0) + \frac{\mathrm{h}^4}{4!}\left(\gamma_e^2 - \gamma_a^2\right)^2 \mathrm{H_{za}}\,(0)\right.\\ &\left.+\frac{\mathrm{h}^3}{3!}\left(\gamma_e^2 - \gamma_a^2\right)\frac{\partial \mathrm{H_{za}}\,(0)}{\partial \mathrm{z}^2}\right]. \end{aligned} \tag{16.114}$$

If $\mathrm{H_{xa}}$ and $\mathrm{H_{ya}}$ are also observed, one can use a finite difference approximation of (16.112) and (16.113) to $\mathrm{H_{za}}\,(0)/\mathrm{dz}$. Thus it is possible to determine $\mathrm{H_{ze}}\,(\mathrm{h})$.

Alternately one can use Peter's (1949) static field approximation by using the average values of $\mathrm{H_{za}}(0)$ on circles around the point of continuation using suitable coefficients discussed in Sect. (16.5). That involves the development of a numerical scheme for evaluation of the electromagnetic upward continuation integrals (16.80).

17

Inversion of Potential Field Data

In this chapter a few basic points about inversion of geophysical data are given. Structures of different approaches for inversion, viz, Singular Value Decomposition, Least Squares including Ridge Regression, and Weighted Ridge Regression, Minimum Norm Algorithm, Bachus Gilbert Inversion, Stochastic Inversion, Occam's Inversion Global Optimization Techniques including Monte Carlo Inversion, Simulated Annealing and Genetic Algorithm, Artificial Neural Network and Joint inversion are discussed.

17.1 Introduction

The task of retrieving complete information about model parameters from a complete and precise set of data is inversion. In geophysics, these models are earth models and the task is to establish a link between a data space and model space (Fig. 17.1).

If we have a set of data collected from the field, we try to say about the earth model with those finite data set. How many different ways can one try

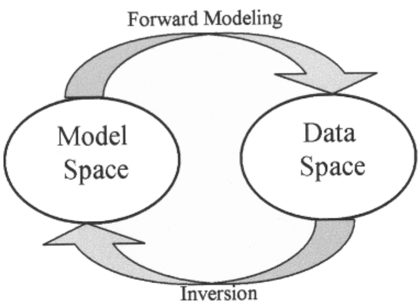


Fig. 17.1. Connecting link between a data and the corresponding model space in a forward and an inverse problem

to travel within a Data space (or domain) and a model space (or domain), how good the flow of information is from a data space to a model space, what are the difficulties can one encounter, how many different ways can one try to overcome those difficulties, how much information can one really gather and what are their limitations, what are the precautions can one take on the way as one moves through the multidimensional hyperspace (Figs. 17.2 and 17.3) are some of the questions to be addressed briefly. Inverse theory is applicable to all the branches of science and engineering. Inverse problems are also termed as identification problems or optimization problems. It is a part of information theory and communication theory. Inverse Theory is the most scientific and accurate mathematical tool to be used judiciously for interpretation of geophysical or other scientific data.

Some sense of movement, distance and projection of a structure from different angles are involved in an inverse problem. Data and model spaces are assumed to be n and m-dimensional abstract spaces (Figs. 17.2 and 17.3) where information from the data space are transmitted to the model space and vice versa through some connecting links. We shall discuss on these links in the text. Figure 17.2 show the distance between the starting and the end point in an m dimensional hyperspace. This sense of distance and movement are present both in the data and model space. These distances are $d^{obs} - d^{predicted}$ in the data space and $m^{true} - m^{prior}$ in the model space. We try to minimize these distances in both the spaces. It means that the model with which we started our experiment has certain co-ordinates in the abstract space. That m-dimensional co-ordinate point moves towards the actual answer as the inverse process progresses. This is the sense of movement in an inverse problem. These movements can be continuous in small discrete steps or it can be by random jumps in the entire parameter space. What is really meant by this movement? Here comes the concept of forward modeling. Figure 17.3 shows the different initial choices of the model parameters at different places in the m-domain but all the movement from the starting points are towards the actual answer. An initial and judicious choice of a model is the starting point of an inverse problem. For existence of an inverse problem, the forward problem must exist. Therefore, solution of a forward problem is the starting

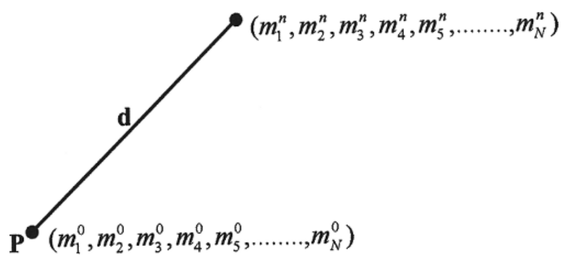


Fig. 17.2. Movement of an assumed model and the sense of distance in an N dimensional model space

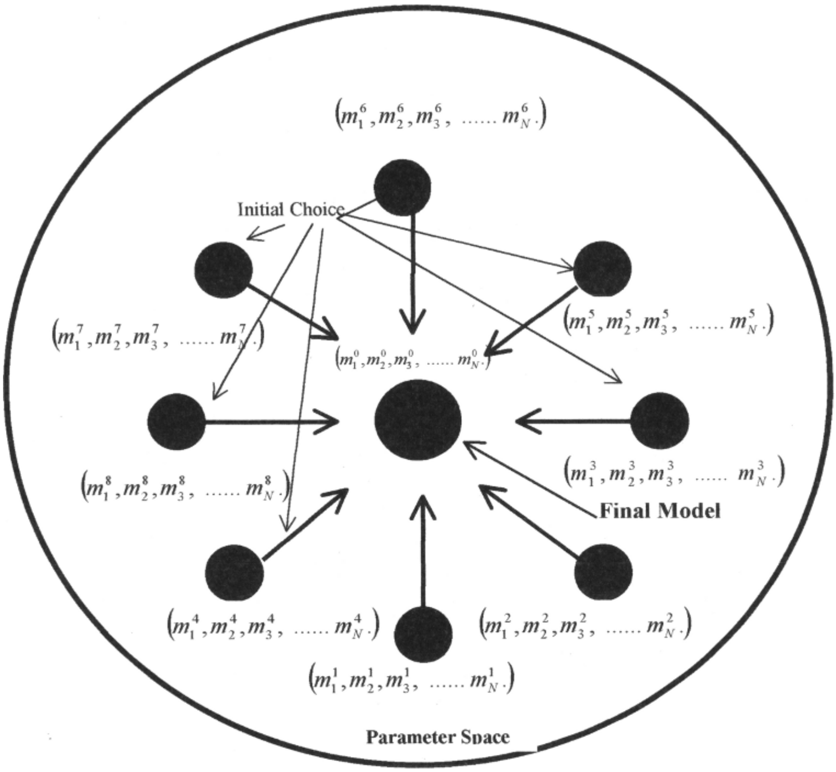


Fig. 17.3. Different initial choices and in M dimensional parameter space and their movements towards the actual answer

point of an inverse problem. Geophysicists collect a set of data on the surface of the earth or in the air or inside a bore hole or at the ocean bottom.What can we say about the earth with these limited noisy data? That's how, 'norm', convergence, metric space, inner product space, Hilbert space n-dimensional vector space entered in an inverse problem. Figure 17.2 shows the distance between the actual answer and the starting point.

Generally gravity / magnetic / D.C. resistively / electromagnetic / S.P. / I.P / Seismic reflection and refraction / earthquake Seismology / heat flow data are collected. Interpreters of data have to guess judiciously at this stage on what kind of subsurface structure can generate this kind of data distribution. Interpreters must have some insight about the nature of distribution of data for different type of subsurface structures as well as for different types of potential and non potential fields. Judicious choice of an initial model reduces the distance between m^{prior} and m^{true} where these points are locations of a priori or initial choice of the model parameters in an abstract space. To achieve this insight one has to solve mathematical boundary value problems for different types of naturally occurring or artificial man made fields and examine the nature of responses responsible for different types of sub surface structures. These boundary value problems are forward problems and can be one / two / or three-dimensional problems. With increase in complexity in an assumed earth model, the mathematical complications in the solution of a forward problem increases. Advent of numerical methods, viz., finite difference, finite element, integral equation, volume integral, transmission line, hybrids, increased the horizon of solvability of the problems of complex subsurface geometry and that increased the horizon of applicability of inverse problems. For the last seven decades, geophysicists solved numerous forward boundary value problems needed for interpretation of geophysical data. Hence inversion of geophysical data grew as a subject at a faster pace. The data we generate by solution of boundary value problem are noise free synthetic data obtained from a synthetic model. These synthetic data are called $d^{\text{Predicted}}$ or d^{Pre} and a synthetic model is m^{Prior}. In geophysics d^{Observed} or d^{Obvs} are generally the field data collected on the surface of the earth or in the air or in ocean surface or in a borehole. These are experimental data in other branch of science and engineering and are contaminated with noise. $(d^{\text{Obs}} - d^{\text{Pre}})$ and $(m^{\text{true}} - m^{\text{Prior}})$ are the two distances or norms we were talking about and we try to minimize these distances in the data space and model space by dual minimization simultaneously. Inverse theory is based on a few basic concepts(Parker 1977) viz., (i) existence (ii) construction (iii) approximations (iv) stability and (v) nonuniqueness.

Existence : For existence of an inverse problem, forward problem must exist The solution of a boundary value problem is either available or it is to be solved. Solutions of forward models in analytical form are available (already solved) for simpler sub surface geometries. With the introduction of realistic touches in earth models, a boundary value problem becomes mathematically unmanageable and one has to solve the problems numerically.

So the forward problem must be solved or at least must be available before construction of an inverse problem. Since the model space and data space are respectively the Hilbert space and Euclidean space, the concept of projection of models from different angles appear. With limited data and with limited resolving power of many of the potential problems we can only see a projection of the model from a particular angle. That invites a serious problem of non uniqueness to be taken up.

Construction : Construction of an inverse problem can be done in many ways. It is centered around (i) examination of data and to take decision on application of regularization (ii) judicious assumption of an initial model based on the nature of the data (iii) solution of the forward problem (iv) comparison of field data with the synthetic model data or predicted data (v) estimation of the discrepancies $(d^{\text{obs}} - d^{\text{pre}})$ in data space and $(m^{\text{est}} - m^{\text{prior}})$ in the model space quantitatively in the form of squared residuals or chi square errors or residual variance or energy function or cost function or error function etc. (vi) choose a particular type of inversion approach based either on linearised inversion approach or random walk technique for global optimization approach

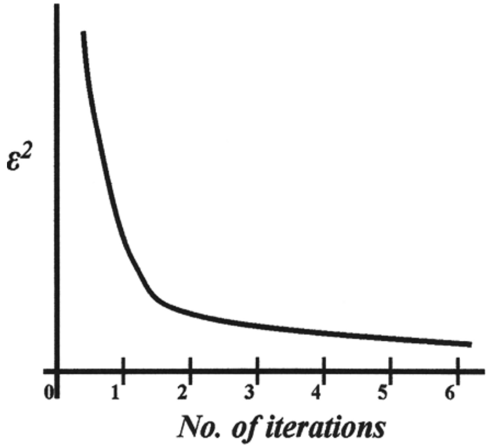


Fig. 17.4. Convergence of an inverse problem in successive iterations

(vi) obtain the convergence of an iterative solutions (Fig. 17.4). (vii) find out the uncertainly level in parameter estimation (ix) study the level of non uniqueness and find out the way to overcome it if possible (x) mix some additional information from different branches of geophysics if available and try to go for joint inversion.

Multi domain geophysical data based on different physical properties can be used either for joint inversion or these additional information can be used to constrain the model to be estimated. Some of the very well known approaches for construction of an inverse problems are (i) Singular value decomposition (ii) Least squares estimator (iii) Ridge regression and weighted ridge regression estimators (iv) Minimum norm estimator for an underdetermined problem (v) Method of steepest descent (vi) Conjugate gradient minimization (vii) Bachus Gilbert inversion (viii) Stochastic inversion. (ix) Occam's inversion. (x) Monte Carlo inversion (xi) Simulated Annealing (xii) Genetic Algorithm (xiii) Neural network etc. This is a broad outline about the construction of an inverse problem. There are many other approaches not listed here.

Figure 17.5 shows a schematic diagram regarding construction of some of the inversion procedures. Procedural details are different for different inversion approaches. Figure 17.6 shows the paths for convergence and divergence in the parameter space starting from the initial choice.

Approximation is a major component in framing an inverse problem. Approximation enters into the system through many channels. As for examples most of the geophysical problems are non-linear problems. We linearise the non-linear problems by truncating higher order terms of the Taylor's series expansion and introduce approximation to enter into the domain of generalized linear inverse problems. The effect of this truncation of higher order terms may be less for certain problems. They are called weakly non-linear or linearisable problems. For certain other cases the effect of linearization may

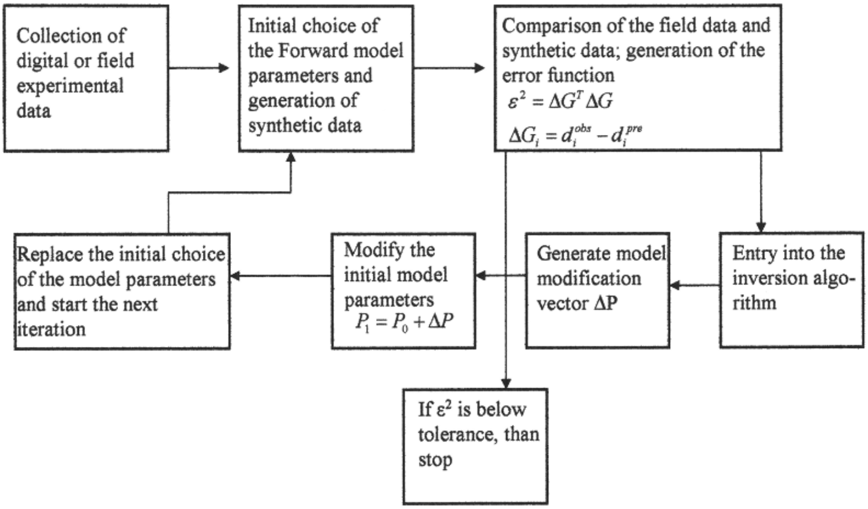


Fig. 17.5. A cartoon of a flow chart for an inverse problem

be severe. They are called strongly non-linear problems. Some of these non-linear problems can be linearised at the maximum likelihood point (discussed later) whereas some other problems cannot be linearised at all. Therefore, for strongly nonlinear problems one should use one of the global optimization approaches. The next major approximation enters through the choice of

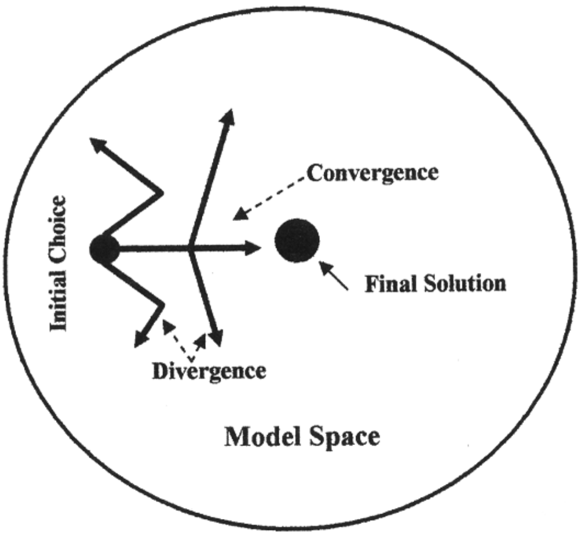


Fig. 17.6. Convergence and divergence of an inverse problem as the initial model starts moving towards the actual answer

forward problems. Earth structure is much more complicated than whatever forward model we choose to initiate an inverse process. The approximation in very severe for one dimensional modeling and inversion. Only for a few cases we get perfect horizontal layers inside the earth. Although numerical approaches for forward modeling viz. finite element and finite difference can take care of a part of the degree of complications inside the earth in 2-D/3-D forward problems, it cannot be a complete substitute for an earth model. The interpreter must have a reasonable knowledge about the geology of the area for proper selection of an earth model. A model widely off from the actual earth's subsurface will generate wrong answer even if one gets reasonably good convergence of an inverse problem. Approximations also enter through data inadequacy, data smoothing and applications of several constraints. Some kinds of regularization, viz, Tikhnov's regularization (Tikhnov and Arsenin 1977), Occam's rajor (Constable et al 1987), minimum structure algorithm (Smith and Booker, 1991) are necessary. Data smoothing is very common before inversion unless the data are of very high quality.

Stability of an inverse problem means smooth and trouble free movement of all the parameters of a particular model towards the actual answer and should reach nearer to the destination (Fig. 17.6). For movement from the initial choice point in the Hilbert space, the inverse problem may start diverging out. It means the discrepancy between the field and predicted data and the distance between model values and the actual answer will also start increasing, i.e., sum of the squared residual will start increasing instead of decreasing. Larger the number of parameters in a model space greater will be the probability for divergence. In the parameter space all the parameters must move towards the actual answer. In that case the inverse problem is stable. There are a few reasons for an inverse problem to be unstable. These are (i) data inadequacy (ii) data inaccuracy (iii) poor initial choice of the model parameters (iv) generation of ill conditioned matrix with several zeros or very small eigen values for the problems dealing with generalized inverse (v) unconstrained optimization. Presence of several local minimum pockets will not allow the system to converge. The data space and model space are connected by a linear or a linear differential operator. Since most of the geophysical problems are nonlinear, linear differential operators are generally used and they construct the sensitivity matrix. If the matrix thus generated is an well conditioned matrix, then with adequate good quality data one can generate a well posed problem.

In an well posed problem if one gives a small perturbation in the model parameters, small perturbation in the data space will result and vice versa. On the other hand if a small perturbation in the data space causes large perturbation in the model space, then the problem is an ill posed problem. Most of the geophysical problems are ill posed problems. Zero and small eigen values of the sensitivity matrix bring this instability. To improve stability of an inverse problem quite a few steps are taken viz. (i) an attempt is being made to make the problem overdetermined. An over determined problems are those where the number of data points (n) are more than the number of unknown parameters (m). n greater than m does not guarantee that the problem will be an overdetermined problems. Data must be capable of seeing the entire earth model within its power of detectibility. For most of the geophysical problems it becomes possible to collect more data than the number of parameters to be resolved from the model.

For certain geophysical problems, it may not be possible to collect as many data as practicable and the problem becomes underdetermined problem i.e., the number of data points is less than the number of unknowns parameters ($n < m$). Theoretically or mathematically when $n = m$, we have an even determined problem and one should be able to determine m completely. But presence of noise in field data makes the system unstable and the problem diverges. (ii) detailed eigen value analysis is done to study the rank deficiency of the sensitivity matrix and to weed out the zero and small eigen values (iii) Marquardt Levenberg coefficients or parameter variance covariance values are added to the diagonal elements of the sensitivity matrix to ensure stability (iv) Positivity or non negativity constraints are imposed for not allowing any of the parameter to be negative when it is known that the parameters to be determined are definitely positive quantities. Constraints on upper and lower bounds of a particular parameter can also be introduced. Constraints on movement in the parameter space is also introduced to ensure better stability (v) introduction of a prior assumption from reliable other geophysical or geological data may reduce the number of unknown parameters to be determined and thereby improve the stability condition and quality of inversion to a certain extent.

Nonuniquenesses are very important aspects of an inverse problem to be taken care of. If we have or assume a model we can generate a unique response for a particular type of field. But the reverse is not true, i.e., if we have a set of data, infinitely many models may satisfy this set of data It is possible to overcome this hopeless situation to a great extent and that is why the inverse theory survived the test of time. The causes for the non uniqueness in an inverse problem are as follows:

Nonuniqueness is due to (i) principle of ambiguity present in the potential theory (ii) data inadequacy (iii) data inaccuracy, (iv) different form of error present in the data (v) noise (vi) bad initial choice of the model parameters, (vii) different types of constraints imposed, (viii) presence of several local minima pockets in the parameter space (Figs. 17.7 a, b) (ix) simplification of the forward model, (x) linearization of the nonlinear problem, (xi) different approaches are adapted for solution of inverse problems, (xii) different types of corrections applied to the data, (xiii) various types of smoothing applied to the data. (xiv) different forward problem soft wares used to interpret same data.

Green's theorem of equivalent layers in potential theory discussed in Chap. 10 is an example of non uniqueness present in theory. This theorem shows that one can get same gravitational field for different type of mass

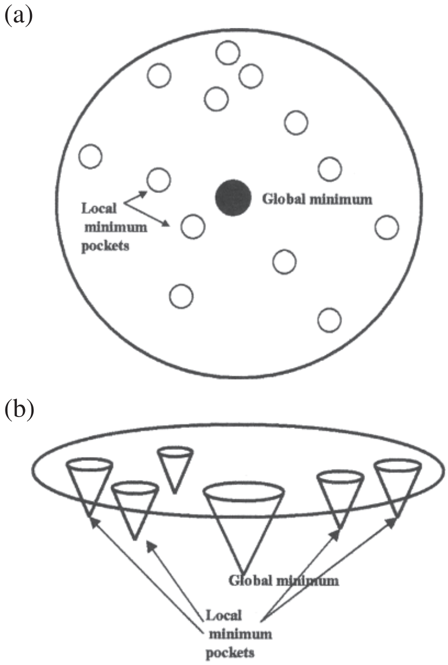


Fig. 17.7. a, **b**. Local minima pockets in the parameters space; these are model traps, prevent the models from further movement towards global minimum

distribution. Principle of equivalence in electrical and electromagnetic theory tells that for different combinations of layer resistivity and thickness one gets $T(= h\rho)$ and $S(= \frac{h}{\rho})$ equivalence where ρ is the resistivity and h is the thickness of a layer in an one dimensional layered earth problem.

Since retrieval of a model from a set of data is to see the different projections of the model in the Hilbert space from different angles, data inadequacy will increase the non uniqueness due to partial view of the model from different projection angles.

Noise in the field data is an inevitable process. Noise will be there in the data collected from the field. Higher noise level can completely vitiate inverse process. Opacity increases in the observation power of the data. So the model space can be either empty (null information) or infinite dimensional (Bachus and Gilbert 1967).

Varied initial choices and presence of many local minima pockets bring non uniqueness in an inverse problem. From different points in the space when the initially assumed models start moving, they get trapped at different local minima and generate different answers. Therefore judicious initial choice of the model parameters is essential to reduce the starting distance between m^{Prior} and m^{true} where m^{true} is the true model parameter. In that case this probability to get trapped in local minimum reduces. Interpreter should choose the initial model with available known geology of the area. Difference in the choice of forward model will generate different answers. It is interesting to note that the interpreter often over parameterises the model to make it an underdetermined problem and get interpretable encouraging results. Even if the initial forward model is totally a wrong choice and does not have any closeness with the local geology, one may get both convergence and stability of an inverse problem and get an answer. This is a note of caution. The convergence i.e., the reductions of the discrepancy between the field data and the model data in successive iteration does not guarantee any acceptable result. Interpreter should have some knowledge about the local geology of the area to get a meaningful results.

A class of inverse problems which use differential operators are generally made linear by truncating higher order terms of the Taylor's series expansion of nonlinear problems. This approximation may be too severe for a certain class of strongly nonlinear problems. This linearisation, coupled with limited resolving power of the potential problems, inadequate and inaccurate data generate non uniqueness. Interpreter dependent factors are the (i) nature of smoothness used in data processing (ii) choice of a particular approach for inversion (iii) different way of discretisation to generate data. (iv) use of different softwares.

Even with all these problems of non uniqueness if we have good quality adequate data, one can get a cluster of models in the parameter space or M-space near the actual answer. If several earth models obtained using different approaches of inversion have some common feature then earth must have that property (Bachus and Gilbert 1967). Thus inverse theory survives with considerable success while imaging extremely complicated, inhomogeneous and anisotropic earth even in this high level of non uniqueness.

17.2 Wellposed and Illposed Problems

The concept of well posed and ill posed problems were introduced by J. Hadamard (1902, 1932) in an attempt to classify the different types of differential equations along with their boundary conditions.

A solution is said to be well posed for solution of the equation

$$\underset{n\times m}{\mathrm{A}}\ \underset{m\times 1}{\mathrm{z}} = \underset{n\times 1}{\mathrm{u}} \tag{17.1}$$

where the initial choice of model parameter z is in the model space M and u is in the data space D. These spaces are M and N dimensional abstract spaces. These spaces are metric spaces, Euclidean space, pre-Hilbert space, Hilbert space, normed inner product space etc(Sect. 17.4).

A problem of determining a solution z in the space M from the initial data u in the D space is well posed on these spaces (D and M) if the following condition are satisfied:

a. For every element $u \in D$, there exists a solution in the M space
b. The solution is unique
c. The solution is stable on the spaces D and M

Problems that do not satisfy these conditions are termed illposed problems. Most of the geophysical problems are illposed because the data are noisy and as a result there can be infinitely many solutions which will satisfy all the data.

17.3 Tikhnov's Regularisation

If a small perturbation in the data generates a small perturbation in the model parameters and vice versa then the inverse problem is a stable but not necessarily well posed .If a minor change in the data invites major change in the model the problem is definitely an ill posed problem. Regularisation in the data may be necessary to an ill posed problem and to get an approximate solution which is stable. It may be based on providing supplementary information from completely independent sources. An attempt is being made to construct an approximate solution of (17.1), that are stable under small changes in the data space through use of a regularization operator. With noisy data set u_δ, the approximate model set is $Z_\alpha = R(u_{\delta,}\alpha)$, obtained with the aid of regularizing operator $R(u,\alpha)$ where $\alpha = \alpha(\delta, u_\delta)$. Here δ is the noise level. α is the regularizing parameter. Every regularizing parameter defines a stable method of approximate construction of a solution of (17.1). The choice of the regularizing parameter(s) should be consistent with the accuracy of assessment of δ the noise level in the data $(\alpha = \alpha(\delta)$. This regularizing parameter α is chosen in such a way that when $\delta \to 0$ and $u_\delta \to u$ then Z tends towards Z_T or $Z_{true.}$. Thus the problem of finding an approximate solution of (17.1), is centered around getting a stable solution under minor perturbation in the data space and determining the regularization parameter α from additional information related to the problem. This method of constructing the approximate solution is called the regularization method. This is the basic philosophy of Tikhnov's Regularisation(Tikhnov and Arsenin 1977).

All kinds of data smoothing, Occam's rajor, generation of minimum structure algorithm, introduction of different kinds of constraints, application of data and model variance-covariance matrices for bringing stability in an inversion algorithm are the members of the regularization club.

17.4 Abstract Spaces

17.4.1 N–Dimensional Vector Space

A vector is a collection of n real numbers $a_1 \ldots\ldots\ldots a_n$ in a definite order where n is any integer. This vector is $a = (a_1 \ldots\ldots\ldots\ldots a_n)$ where

$a_1 \ldots\ldots\ldots a_n$ are the components of the vector 'a'. Two vectors 'a' and 'b' are equal if $a_i = b_i$ for $i = 1 \ldots\ldots\ldots n$.

These vectors in an n-dimensional space fulfill certain algebraic laws. These are

(i)
$$(a + b) = (b + a)(\text{associativity law of addition}) \tag{17.2}$$

(ii)
$$(a + b) + c = a + (b + c)(\text{distributive law}) \tag{17.3}$$

(iii)
$$\lambda(a + b) = \lambda a + \lambda b(\text{multiplication law}) \tag{17.4}$$

where λ is a scalar.

(iv)
$$\lambda(\mu a) = (\lambda\mu)\, a(\text{associativity law of multiplication}) \tag{17.5}$$

(v)
$$0a = \theta(\text{concept of null vector}). \tag{17.6}$$

where θ is a zero or null vector. A real space in space domain can have 3 dimensions. We often bring the concept of 4th dimension bringing time or frequency into consideration. But when we try to conceive of a space for a set of geophysical data collected on the surface of the earth, we think of an n dimensional data space and an m dimensional parameter space. 'n' will depend upon the number of data collected and 'm' depends on the parameters that can be retrieved from the information collected. Variation of the physical properties and their dimensions in the space domain generate m number of model parameters. These data and model parameters generate D and M vector spaces.

17.4.2 Norm of a Vector

Length or norm of the vector 'a' in 'R' is the arithmetic square root of sum of the square of the components. The norm of a vector 'a' is denoted by $||a||$ and is given by

$$||a|| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots\ldots a_n^2} \tag{17.7}$$

If coordinate of a point P is (x, y, z) in a three dimensional space, its distance from the origin (0, 0, 0) is $d = \sqrt{x^2 + y^2 + z^2}$. Therefore (17.7) is an n dimensional generalisation of 3 dimensional real space concept of distance or norm. The properties of norm are

(i)
$$||a|| \geq 0 \text{for all a and} ||a|| = 0 \tag{17.8}$$

only if $a = \theta$.

(ii)
$$||\lambda a|| = |\lambda|||a|| \tag{17.9}$$

(iii)
$$||a + b|| \leq ||a|| + ||b||. \tag{17.10}$$

An n–dimensional vector space R in which norm of a vector is defined is called an n–dimensional Euclidean space. A vector with an infinite set of components in an infinite sequence of real numbers $(a_1, a_2, \ldots .a_\infty)$ is an infinite dimensional Euclidean space.

17.4.3 Metric Space

A metric space is an n dimensional abstract space where a non-negative real number called distance is defined between any two elements of a_1 and a_2 and is denoted by $d(a_1, a_2)$. It satisfies the following conditions, i.e.,

(i)
$$d(a_1, a_2) = 0 \text{ if } a_1 = a_2 \tag{17.11}$$

(ii)
$$d(a_1, a_2) = d(a_2, a_1) \tag{17.12}$$

(iii) for any three elements of the abstract space.
$$d(a_1, a_2) \leq d(a_1, a_3) + d(a_3, a_2) \tag{17.13}$$

(iv) $d(a_m, a_n) \to 0$ for n, m $\to \infty$

17.4.4 Linear System

A set of vectors S forms a linear system if the concept of additions and multiplications are satisfied as shown in (17.2 and 17.4).

17.4.5 Normed Space

A linear system 'a' is called a normed space if for every element a_n in S, a real number called norm is defined. Normed space will have the following properties.

(i)
$$||a|| \geq 0 \text{ for any } a \in S \tag{17.14}$$

(ii)
$$||a|| = 0 \text{ for } a = \theta \tag{17.15}$$

(iii)
$$||\lambda a|| = |\lambda| ||a|| \tag{17.16}$$

for any $a \in S$ and for any scalar number λ.

(iv)
$$||a + b|| \leq ||a|| + ||b|| \text{ or any } a, b \in S \tag{17.17}$$

(v)

$$d(a, b) = ||a - b||, \tag{17.18}$$

the distance in a normed space.

(vi)

$$||a|| = ||a - \theta|| = d(a, \theta), \tag{17.19}$$

i.e., the norm of any element is its distance from the origin.

17.4.6 Linear Dependence and Independence

Elements $a_1, a_2 \ldots\ldots a_n \in S$ are called linearly independent, if for a linear combination of these elements, the relation $\Sigma\lambda_k a_k = \theta$ is satisfied only for $\lambda_1 = \lambda_2 = \lambda_3 \ldots\ldots \lambda_n = 0$. Otherwise the elements are called linearly dependent. Elements $a_1, a_2, \ldots..a_n$ are linearly dependent if and only if, at least one of them can be expressed in the form of a linear combination of the others.

If an elements $a \in S$ can be expressed in the form of a linear combination of linearity independent elements a_1, a_2, $a_3 \ldots..a_n$, and $a = \sum_{k=1}^{n} \lambda_k a_k$, then λ_k will have unique values. These normed space are finite dimensional.

17.4.7 Inner Product Space

Suppose $a = (a_1, a_2, \ldots.a_n)$ and $b = (b_1, b_2, b_3 \ldots.b_n)$ are vectors in the space R. The dot product or inner product of the two vectors are defined by (a, b). It is defined to be the scalars obtained by multiplying corresponding components and adding the resulting products.

$$u.v = \sum_{k=1}^{n} u_k.v_k \tag{17.20}$$

17.4.8 Hilbert Space

Let the space S be a real linear system and suppose that in any two of its elements a, b there exists a real number denoted by (a, b) and also possess the following properties

(i)

$$(a, b) = (b, a) \tag{17.21}$$

(ii)

$$(a + b, c) = (a, c) + (b, c) \tag{17.22}$$

(iii)

$$(\lambda a, b) = \lambda(a, b) \tag{17.23}$$

(iv)

$$(a, a) = 0 \text{if and only if } x = \theta \tag{17.24}$$

(v)

$$||a|| = \sqrt{(a, 0)}. \tag{17.25}$$

A complete normed real space in which the norm is generated by the inter product or scalar product is called a Hilbert space.

Abstract spaces in Inverse Theory, viz. data space and model space follow the properties of these Eucledean space, Metric space, normed linear space, Inner product space and Hilbert space. Further details are beyond the scope of this volume.

17.5 Some Properties of a Matrix

17.5.1 Rank of a Matrix

Many of the physical problems in their final stages of formulation reduce to a matrix equation in the form

$$Ax = B \tag{17.26}$$

where A is an $n \times m$ known matrix, B is a known vector of n elements and x is an unknown vector of m elements. We have to consider whether there exists a vector x which satisfy this equation and whether the set of simultaneous equations which satisfy (17.26) is consistent. The answer is provided by rank of the matrix 'A'. 'n' elements $f_1, f_2, f_3 \ldots\ldots\ldots\ldots f_n$ of a linear space R are linearly dependent if there exists $C_j \in F$ not all equal to zero, such that

$$\sum_{j=1}^{n} C_j f_j = 0 \tag{17.27}$$

otherwise they are linearly independent. The rank of a matrix is equal to the number of linearly independent equations and the number of nonzero eigen values present in the system. The order of the largest non-zero minor is also the rank. Theoretically, if rank of a matrix is equal to the number of unknown parameters m, the solution should be unique if it exists. For practical problems that is not possible because the data are noisy.

If the rank of the matrix is k and the system of equations is consistent, then we shall have six possible cases, i.e.,

(i) $n = m$ and $k = m$

$\rightarrow$ Even determined problem

(ii) $n = m$ and $k < m$

(iii) $n > m$ and $k = m$

$\rightarrow$ Over determined problem

(iv) $n > m$ and $k < m$

(v) $n < m$ and $k = n$

$\rightarrow$ Underdetermined problem

(vi) $n < m$ and $k < n$

where n is the number of data points in D-space and m is the number of parameters in M-space. k will always be either equal to or less than m or n which ever is less. $n > m$ does not guarantee that the problem has become an overdetermined problem. The capability of data to see a target completely must be taken into consideration. Similarly one can make an overdetermined problem underdetermined by overparameterising the models. Sometimes it gives more realistic result specially in one dimensional problems.

17.5.2 Eigen Values and Eigen Vectors

Determine a scalar quantity λ and a vector U which satisfies (17.2). This equation can be expressed in the polynomial form as

$$\lambda^{n} + C_{n-1}\,\lambda^{n-1} + \ldots\ldots\ldots + C_1\,\lambda = 0 \tag{17.28}$$

The n roots of this polynomial

$$\lambda_1, \lambda_2, \lambda_3 \ldots\ldots\ldots \lambda_n$$

are the eigen values of the system matrix. They satisfy the equations

$$Au_i = \lambda_i\, u_i \tag{17.29}$$

where u_i is called the eigen vector. The set of eigen values form a diagonal matrix and is called the spectrum of A. The largest absolute value of λ is called the spectral radius. Every eigen value has a eigen vector. It is a column vector. Even a zero eigen value has a non zero column eigen vector.

An $n \times n$ square matrix has at least one and at the most n distinct eigen values. We can write (17.29) as

$$\begin{aligned}
&a_{11}\,x_1 + \ldots\ldots\ldots\ldots\ldots + a_{1\,n}\,x_n = \lambda_1\,x_1\\
&a_{21}\,x_1 + \ldots\ldots\ldots\ldots\ldots + a_{2n}\,x_n = \lambda_2\,x_2\\
&`\\
&`\\
&a_{n1}\,x_1 + a_{n2}\,x_2 \ldots\ldots\ldots\ldots\ldots + a_{nn}\,x_n = \lambda_n\,x_n.
\end{aligned} \tag{17.30}$$

By transferring the right hand column to the left hand side we get

$$\begin{aligned}
&(a_{11} - \lambda_1)\,x_1 && + a_{12}\,\lambda_2 + \ldots\ldots\ldots\ldots\ldots + a_{1n}\,x_n = 0\\
&a_{21}\,x_1 && + (a_{22} - \lambda_2)\,x_2 + \ldots\ldots\ldots\ldots\ldots + a_{2n}\,x_n = 0\\
&`\\
&`\\
&a_{n1}\,x_1 && + a_{n2}\,x_2 \ldots\ldots\ldots\ldots\ldots + (a_{nn} - \lambda_n)\,x_n = 0
\end{aligned} \tag{17.31}$$

Number of non zero eigen values dictates the quality of the matrix, for starting an inverse problem.

17.5.3 Properties of the Eigen Values

1. The sum of the diagonal elements of the matrix is equal to the trace of the matrix and is equal to the sum of the eigen values i.e.,

$$\text{Trace} \quad A = \sum_{i=1}^{n} \lambda_1. \tag{17.32}$$

2. The product of the eigen values is equal to the determinant of the matrix A.

$$\text{Det } A = \prod_{i=1}^{n} \lambda_i. \tag{17.33}$$

3. A matrix has same eigen values as its transpose .
4. For a real matrix each eigen value is either real or one of a complex conjugate pair of eigen values.
5. A real symmetric matrix has real eigen values only.
6. The eigen values of triangular matrix are equal to its diagonal matrix .
7. If the rows and the corresponding columns are changed, the eigen values remain unchanged .
8. The eigen value of a matrix remain unchanged if a row is multiplied by f and the corresponding columns is multiplied by 1/f
9. The rank of a matrix is equal to the number of non-zero eigen values
10. The eigen value of the pth power of a matrix is equal to the pth power of the eigen values of the matrix .
11. Eigen value matrix is a diagonal matrix

$$\lambda = \text{diag} \quad (\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots\ldots\ldots \lambda_n).$$

12. Eigen vector matrix is an orthogonal matrix, i.e., $UU^T = I$ and $U^TU = I$, where I is the identity matrix .
13. Eigen vectors corresponding to two distinct eigen values of a symmetric matrix are orthogonal, i.e.

$$U_i^T U_j = 0 \quad \text{for} \quad i \neq j. \tag{17.34}$$

14. 'n' equations can be combined into a matrix equation

$$AU = UA \tag{17.35}$$

where U is an orthogonal matrix. Therefore

$$A = U\lambda U^T \tag{17.36}$$

15. For a rectangular matrix A will be equal to

$$A = U\lambda V^T \tag{17.37}$$

where U is an eigen vector matrix in the n-space and V is the eigen vector matrix in the m space in an n x m system, discussed in the Sect. 17.7.

17.6 Lagrange Multiplier

Lagrange multiplier is a mathematical tool, used for maximizing or minimizing a function in the presence of a constraint equation.
Let $\varphi(x, y)$ be a mathematical function of x and y. We want to maximize or minimize this function in the presence of another function of two variables say $\psi(x, y) = 0$. The second equation is a constraint equation.

Since maximum or minimum of a function can be obtained when the total differential of $\varphi(x, y)$ is made zero, we can write

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0. \tag{17.38}$$

The total differential of the constrained equation is

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0. \tag{17.39}$$

By multiplying the (17.39) with an unknown multiplier λ and adding this equation with (17.38), we get

$$\left(\frac{\partial \phi}{\partial x} + \lambda \frac{\partial \psi}{\partial x} \right) dx + \left(\frac{\partial \phi}{\partial y} + \lambda \frac{\partial \psi}{\partial y} \right) dy = 0. \tag{17.40}$$

From (17.40), λ can be determined. Once λ is known, one can get the extreme value i.e., the maximum or minimum value of $\varphi(x, y)$ from the set of equations

$$\frac{\partial \phi}{\partial x} + \lambda \frac{\partial \psi}{\partial x} = 0 \tag{17.41}$$

$$\frac{\partial \phi}{\partial y} + \lambda \frac{\partial \psi}{\partial y} = 0 \tag{17.42}$$

$$\psi(x, y) = 0. \tag{17.43}$$

Lagrange multiplier is used in (i) generalized linear inverse problem described by Bachus Gilbert (1968, 1970),(ii) minimum norm algorithm for an underdetermined problems proposed by Menke (1984), (iii) Occam's Inversion discussed by Constable et al (1987), Degroot Hedlin Constable (1990) and (iv) Reduced Basis Occam Inversion (REBOCC) proposed by Siripunvaraporn and Egbert (2000).

17.7 Singular Value Decomposition (SVD)

For most of the geophysical problems, we can connect the data space with model space using Fredhom's integral equation of the first kind.

$$\mathrm{d_i} = \int_0^\infty \mathrm{G_i}\,(\mathrm{r})\,\mathrm{m}\,(\mathrm{r})\,\mathrm{dr} \tag{17.44}$$

where $\mathrm{d_i}$ is the data in the data space and m (r) is the model at a radial distance r, $\mathrm{G_i}$ (r) is the Kernel function. This function is also called the Frechet kernel or the Green's function. This continuous Fredhoms integral can be written in the discrete form as

$$\mathrm{d_i} = \sum \mathrm{G_{ij}}\,(\mathrm{r})\,\mathrm{m_j}\,(\mathrm{r})\,. \tag{17.45}$$

Equation (17.45) can be written in the matrix form as

$$\mathrm{d} = \mathrm{Gm} \tag{17.46}$$

where d is a $\mathrm{n} \times 1$ column vector of n data points. m is the $\mathrm{m} \times 1$ column vector of model parameters and G is a rectangular matrix and linear differential operator or linear operator which connects the data to the model space. Here

$$\mathrm{d} = \{\mathrm{d_1}, \mathrm{d_2}, \mathrm{d_3} \ldots\ldots \mathrm{d_N}\}^{\mathrm{T}} \quad \text{for } \mathrm{i} = 1, \ldots.\mathrm{N} \quad \text{and} \tag{17.47}$$

$$\mathrm{m} = \{\mathrm{m_1}, \mathrm{m_2}, \mathrm{m_3} \ldots\ldots \mathrm{m_M}\}^{\mathrm{T}} \quad \text{for } \mathrm{j} = 1 \text{ to } \mathrm{m} \tag{17.48}$$

and T is the transpose. T changes the column vector to a row vector.

Let

$$\mathrm{d} = \mathrm{Gm}. \tag{17.49}$$

It is taken to another plane where the connecting equation between the data and the model space is given by

$$\mathrm{d}' = \mathrm{G}'\ \mathrm{m}'. \tag{17.50}$$

The connecting link is established by multiplying both the sides of the (17.19) by u where u is the eigen vector in the n-space or data space. Since eigen vector matrix is an orthogonal matrix we can write

$$\mathrm{d}' = \mathrm{u^T d} \tag{17.51}$$

$$\mathrm{m}' = \mathrm{u^T m}. \tag{17.52}$$

From (17.51) and (17.52) we can write

$$\mathrm{ud}' = \mathrm{uGm}' \tag{17.53}$$

$$\Rightarrow \quad \mathrm{u^T ud}' = \mathrm{u^T uGm}' \tag{17.54}$$

$$\Rightarrow \quad \mathrm{d}' = \mathrm{G}'\mathrm{m}' \tag{17.55}$$

$$\text{since} \quad \mathrm{Gu} = \mathrm{u}\lambda \tag{17.56}$$

where λ and u are respectively the eigen value and eigen vector matrix. Hence

$$u^T G u = u^T u \lambda = \lambda. \tag{17.57}$$

Therefore

$$d' = \lambda m'. \tag{17.58}$$

So a square matrix could be changed to diagonal eigen value matrix in the transformed plane. This is the principal axis transformation. We can write

$$m' = \lambda^{-1} d' \tag{17.59}$$

$$\begin{aligned}
&\Rightarrow \quad u^T m = \lambda^{-1} u^T d \\
&\Rightarrow \quad u u^T m = u \lambda^{-1} u^T d \\
&\Rightarrow \quad m = u \lambda^{-1} u^T d \\
&= G^{-1} d.
\end{aligned} \tag{17.60}$$

This is the generalized inverse of a square matrix. For a rectangular system $(n \times m)$

$$d = u d', \qquad \text{or} \qquad d' = u^T d \tag{17.61}$$

and

$$m = \nu m' \qquad \text{or } m' = \nu^T m \tag{17.62}$$

Therefore,

$$\begin{aligned}
&d = G m \\
&\Rightarrow \quad u d' = G \nu m' \\
&\Rightarrow \quad u^T u d' = u^T G \nu m' \\
&\Rightarrow \quad d' = u^T G \nu m' \\
&= G' m'
\end{aligned} \tag{17.63}$$

where ν is the eigen vector matrix for the m space or the parameter space. In an $n \times m$ system u and ν are respectively the eigen vector matrices for the n and m spaces respectively.

Since $G\nu = u\lambda$ (17.63) in an $n \times m$ system (Lanczos 1941) we can write

$$u^T G \nu = u^T u \lambda = \lambda \tag{17.64}$$

$$\Rightarrow \quad u u^T G \nu = u \lambda \tag{17.65}$$

$$\begin{aligned}
&\Rightarrow \quad G \nu \nu^T = u \lambda \nu^T \\
&\Rightarrow \quad G = u \lambda \nu^T.
\end{aligned} \tag{17.66}$$

Since both u and ν are orthogonal matrices hence the generalized inverse is

$$G^{-1} = \nu \lambda^{-1} u^T. \tag{17.67}$$

It is used for inversion of geophysical data.

For an arbitrary n x m system we can have two matrix equations

$$\begin{array}{ccc} \mathrm{A} & \mathrm{y} & = \mathrm{b} \\ \mathrm{n} \times \mathrm{m} & \mathrm{m} \times 1 & \mathrm{n} \times 1 \end{array} \tag{17.68}$$

and

$$\begin{array}{ccc} \mathrm{A}^{\mathrm{T}} & \mathrm{x} & = \mathrm{c} \\ \mathrm{m} \times \mathrm{n} & \mathrm{n} \times 1 & \mathrm{m} \times 1 \end{array} \tag{17.69}$$

where (17.69) is the self adjoint of (17.68) (Lanczos 1941). In (17.68) A has n rows and m columns and transforms the column vector y of m components to column vector b of n components. The matrix A^{T} of (17.69) has m rows and n columns. The vector x and c are in a reciprocity relation to the vectors y and b. x and b are the vectors in the n space and y and c are the vectors in the m space. Lanczos (1941) has shown that combining these two equations and transferring to a larger n + m space we get the matrix

$$\mathrm{Sz} = \mathrm{a}. \tag{17.70}$$

Figure 17.8 shows the n + m system and the locations of A, A^{T}, y, b, x and c. The eigen value equation of this $(\mathrm{n} + \mathrm{m}) \times (\mathrm{n} + \mathrm{m})$ square matrix is

$$\mathrm{S}\omega = \lambda\omega \tag{17.71}$$

This eigen value matrix equation can disintegrate into a pair of eigen value equations for an $\mathrm{n} \times \mathrm{m}$ system. These are

$$\begin{aligned} A\nu &= \lambda u \\ A^T u &= \lambda v \end{aligned} . \tag{17.72}$$

These equations are termed as shifted eigen value equations where u and ν are respectively the eigen vectors in the n and m space respectively. λ, is the diagonal matrix of eigen values. Here u and ν on the right hand side has shifted their positions. If we postmultiply the first equation by A^{T} and second equation by A, we get

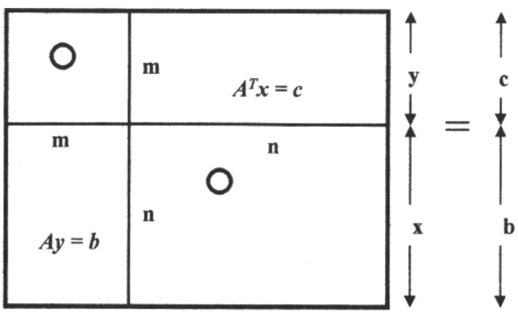


Fig. 17.8. Matrix in a n + m space; shifted eigen value problem (Lanczos 1941)

$$\begin{aligned} AA^T u &= \lambda^2 u \\ A^T A\nu &= \lambda^2 \nu \end{aligned}. \tag{17.73}$$

Equation (17.73) is a regular eigen value problem. Here AA^T and A^TA are square symmetric matrices in the n and m spaces respectively. It is interesting to note that significant eigen values for both the $n \times n$ and $m \times m$ matrices will be same. In other words, $n \times n$ matrix will have only m significant eigen values and they are equal to those of $m \times m$ system. $(n - m)$ eigen values for AA^T are trivial and sum of the eigen values for both the matrices will be exactly same.

Most of the geophysical inverse problems are nonlinear. We linearise the nonlinear problem by truncating higher order terms of the Taylor's series expansion. Let G (P, x) the written as

$$G(P, x) = G\left(P^0, x\right) + \sum_{i=1}^{M} \frac{\partial G}{\partial P} \Delta P + \text{higher order terms} \tag{17.74}$$

G is termed as the gross earth functional, P is a vector of unknown values i.e., the parameters to be determined and x is a column vector of n known quantities. x varies from subject to subject. As for example for dc resistivity sounding x stands for electrode separation, in magnetotellurics x stands for periods of the MT signal, in electromagnetic frequency sounding x stands for frequency etc. Since G (P^0, x) is an initial choice of the model parameters or a priori model. We can write (17.74) as

$$\Delta G = A\Delta\ P \tag{17.75}$$

where Δ P is the difference between the actual and the initial choice model parameters. i.e

$$\Delta P = \left| \begin{array}{c} \left(m^1 - m^{\text{Prior}^1}\right) \\ \left(m^2 - m^{\text{Prior}^2}\right) \\ . \\ . \\ . \\ . \\ \left(m^m - m^{\text{Prior}^m}\right) \\ mx1 \end{array} \right|. \tag{17.76}$$

Here ΔP is a $(m \times 1)$ column vector in the m or model space and changes in successive iterations. m is the number of parameters to be modified.

Column vector, Δ G is the difference between d^{Observed} i.e., the field data and d^{Pre} or $d^{\text{Predicted}}$, the synthetic data $.d^{\text{pre}}$ are obtained from computation of the forward model. The column vector Δ G is

$$\Delta G = \begin{vmatrix} \left(d^{Obs^1} - d^{Pr}e^1\right) \\ \left(d^{Obs^2} - d^{Pr}e^2\right) \\ . \\ . \\ . \\ . \\ \left(d^{Obs^n} - d^{Pr\,e^n}\right) \\ n \times 1 \end{vmatrix}. \tag{17.77}$$

It is an n × 1 column vector in the D-space or data space. The connecting link between the two spaces are obtained from the n × m rectangular matrix $\underset{n\times m}{A}$. A is termed as the sensitivity matrix or derivative matrix or the linear differential operator in a linearisable problems. Here A is

$$A = \begin{vmatrix} \left(\frac{\partial G}{\partial P}\right)_1 & \cdots\cdots\cdots & \left(\frac{\partial G}{\partial P_m}\right)_1 \\ . & & \\ . & & \\ . & & \\ \left(\frac{\partial G}{\partial P_1}\right)_n & \cdots\cdots\cdots & \left(\frac{\partial G}{\partial P_m}\right)_n \end{vmatrix} \tag{17.78}$$

where derivatives of the gross earth functionals with respect to all the parameters must exist for existence of the sensitivity matrix and therefore for existence of an inverse problem. Fortunately for most of the geophysical problems the frechet derivatives exist. For an over determined problem from (17.75) we can write

$$\Delta P = A^{-g}\Delta G \tag{17.79}$$

where A^{-g} is the generalized inverse. We can write (17.79) from (17.67) as.

$$\Delta P = \nu\lambda^{-1}u^T\Delta G \tag{17.80}$$

Equation (17.80) is the basic equation for inversion of geophysical data using singular value decomposition (Lanczos, 1941, Inman et al 1973, Glenn et al 1973). Here ΔP is the model modification (m × 1) vector. v is the parameter eigen vector in the M-space. u is the data eigen vector in the n-space and λ is the eigen value which remains the same both in data and model space. The matrix $\underset{n\times n}{AA^T}$ is a square symmetric matrix having the eigen values λ^2s along the diagonal and eigen vector u. The matrix $\underset{m\times m}{A^TA}$ is the square symmetric matrix having the same eigenvalues λ^2s and the eigen vector v. The trace

$$\text{Trace} \qquad AA^T = Trace\ A^TA = \sum_{i=1}^{n}\lambda_i^2 = \sum_{i=1}^{m}\lambda_i^2. \tag{17.81}$$

Equation (17.80) with all non zero eigen values of the system matrix is given by

$$\underset{m \times 1}{\Delta P} = \underset{m \times m}{v} \; \underset{m \times m}{\lambda^{-1}} \; \underset{m \times n}{u^T} \; \underset{n \times 1}{\Delta G} \tag{17.82}$$

If $k < m$, then eliminating zero eigen values we get

$$\Delta P = \underset{m \times k}{v} \; \underset{k \times k}{\lambda^{-1}} \; \underset{k \times n}{u^T} \; \underset{n \times 1}{\Delta G}. \tag{17.83}$$

Eliminating very small eigen values, which bring instability we get

$$\Delta P = \underset{m \times q}{v} \; \underset{q \times q}{\lambda^{-1}} \; \underset{q \times n}{u^T} \; \underset{n \times 1}{\Delta G} \tag{17.84}$$

where $q(q < k < m)$ is the number of significant eigen values. Although u is an $n \times n$ square matrix. We take only the rows for the number of non-zero eigen values. Higher the value of k better will be the quality of inversion.

Both data and parameter eigen vector matrices u and v are orthogonal matrices. Therefore we get

$$uu^T = I_n \tag{17.85}$$

and

$$vv^T = I_m \tag{17.86}$$

Equation (17.85) is known as Information density matrix and (17.86) is known as Resolution matrix. Both are theoretically identity matrices. While dealing with field data one may get diagonally dominant matrices. Information density matrix will indicate qualitatively about the number of parameters which are contributing towards the total signals. Resolution matrix give a qualitative indication about the resolution of the parameters. In other words when the values are unity, one gets best resolution. Diagonal elements will not deviate significantly from unity for proper resolution.

17.8 Least Squares Estimator

In least squares estimator problem, let $x_{11}, x_{12}, x_{13}, \ldots\ldots\ldots x_{1m}$ are independent variables and y_1, y_2, y_3 are dependent variables such that

$$y_1 = a_0 + a_1x_{11} + a_2x_{12} + a_3x_{13} + \ldots\ldots\ldots + a_nx_{1n} \tag{17.87}$$

$$y_2 = a_0 + a_1x_{21} + a_2x_{22} + a_3x_{23} + \ldots\ldots\ldots + a_nx_{2n}. \tag{17.88}$$

For fitting in a format of the type

$$Y = a_0 + a_1x_1 + a_2x_2 + a_3x_3 + \ldots\ldots\ldots\ldots + a_nx_{n.} \tag{17.89}$$

The method of least squares say that the best representative curve is the one for which the sum of the square of the residual is minimum, i.e.,

$$f = \sum_{i=1}^{n} \left(y - (a_0 + a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n)\right)^2 \tag{17.90}$$

is minimum.

The conditions for f (a_0, a_1, a_2, ...a_n) to be minimum are

$$\frac{\partial f}{\partial a_0} = -2 \sum_{0}^{m} (y - a_0 - a_1 x_1 - a_2 x_2 \ldots\ldots a_n x_n) = 0, \tag{17.91}$$

$$\frac{\partial \mathrm{f}}{\partial \mathrm{a}_1} = -2 \sum_{0}^{\mathrm{m}} (\mathrm{y} - \mathrm{a}_0 - \mathrm{a}_1 \mathrm{x}_1 - \mathrm{a}_2 \mathrm{x}_2 \ldots\ldots \mathrm{a}_\mathrm{n} \mathrm{x}_\mathrm{n})\, \mathrm{x}_1 = 0, \tag{17.92}$$

$$\frac{\partial f}{\partial a_2} = -2 \sum_{0}^{m} (y - a_0 - a_1 x_1 - a_2 x_2 \ldots\ldots a_n x_n)\, x_2 = 0, \tag{17.93}$$

....

$$\frac{\partial f}{\partial a_n} = -2 \sum_{0}^{m} (y - a_0 - a_1 x_1 - a_2 x_2 \ldots\ldots a_n x_n)\, x_n = 0. \tag{17.94}$$

There are n + 1 linear equations for (n + 1) unknowns a_0 ....a_n.

These sets of equations in (17.91) to (17.94) can be written in the matrix form as

$$\begin{bmatrix} m & \sum x_1 & \sum x_2 & --- & \sum x_n \\ \sum x_1 & \sum x_1^2 & \sum x_1 x_2 & --- & \sum x_1 x_n \\ \sum x_2 & \sum x_2 x_1 & \sum x_2^2 & --- & \sum x_2 x_n \\ / & / & / & --- & / \\ / & / & / & --- & / \\ / & / & / & --- & / \\ \sum x_n & \sum x_n x_1 & \sum x_n x_2 & --- & \sum x_n^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ / \\ / \\ / \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum y x_1 \\ \sum y x_2 \\ / \\ / \\ / \\ \sum y x_n \end{bmatrix}. \tag{17.95}$$

Equation (17.94) can be written in the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 & --- & 1 \\ x_{11} & x_{12} & x_{13} & x_{14} & --- & x_{1m} \\ x_{21} & x_{22} & x_{23} & x_{24} & --- & x_{2m} \\ / & / & / & / & --- & / \\ / & / & / & / & --- & / \\ / & / & / & / & --- & / \\ x_{n1} & x_{n2} & x_{n3} & x_{n4} & --- & x_{nm} \end{bmatrix} \begin{bmatrix} 1 & x_{11} & --- & x_{n1} \\ 1 & x_{12} & --- & x_{n2} \\ 1 & x_{13} & --- & x_{n3} \\ 1 & / & --- & / \\ 1 & / & --- & / \\ 1 & x_{1m} & --- & x_{nm} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ / \\ / \\ / \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum y x_1 \\ \sum y x_2 \\ / \\ / \\ / \\ \sum y x_n \end{bmatrix} \tag{17.96}$$

Equation (17.95) can be written in the matrix form

$$\mathrm{X}^\mathrm{T}\,\mathrm{X}\,\mathrm{A} = \mathrm{X}^\mathrm{T}\mathrm{Y} \tag{17.97}$$

$$\Rightarrow \mathrm{A} = (\mathrm{X}^\mathrm{T}\mathrm{X})^{-1}\,\mathrm{X}^\mathrm{T}\mathrm{Y}. \tag{17.98}$$

This is the mathematical expression for the least squares estimator.

Alternatively, we can say that to find the vector A which minimizes the sum of the squared residuals ε^2. We write

$$\begin{aligned}\varepsilon^2 &= \varepsilon^{\mathrm{T}}\varepsilon = (\mathrm{AX} - \mathrm{Y})^{\mathrm{T}}(\mathrm{AX} - \mathrm{Y}) \\ &= \Delta\mathrm{G}^{\mathrm{T}}\Delta\mathrm{G} \quad \text{where } \Delta\mathrm{G} = \mathrm{XA} - \mathrm{Y} \qquad (17.99) \\ &= \mathrm{A}^{\mathrm{T}}\mathrm{X}^{\mathrm{T}}\mathrm{A} - \mathrm{A}^{\mathrm{T}}\mathrm{X}^{\mathrm{T}}\mathrm{Y} - \mathrm{Y}^{\mathrm{T}}\mathrm{XA} + \mathrm{Y}^{\mathrm{T}}\mathrm{Y}. \qquad (17.100)\end{aligned}$$

Differentiating the expressions with respect to A^{T} and setting the result equal to zero, we get

$$\left[X^T X\right] A = X^T Y \qquad (17.101)$$

$$\Rightarrow \mathrm{A} = (\mathrm{X}^{\mathrm{T}}\mathrm{X})^{-1}\mathrm{X}^{\mathrm{T}}\mathrm{Y}. \qquad (17.102)$$

17.9 Ridge Regression Estimator

The expression for the least square estimator can be obtained from the sum of the squared residuals. $(\mathrm{X}^{\mathrm{T}}\mathrm{X})^{-1}\mathrm{X}^{\mathrm{T}}$ is also a generalized inverse of a rectangular matrix. Hoerl and Kennard (1970a, 1970b) show that the rectangular matrix $\mathrm{X}^{\mathrm{T}}X$ becomes nearly singular quite often because of the presence of zero and very small eigen values in $\mathrm{X}^{\mathrm{T}}\mathrm{X}$. Marquardt (1963, 1770) Hoerl and Kennard independently have shown that considerable amount of stability in the solution can be obtained by adding a numerical coefficient to the diagonal element of the $(\mathrm{X}^{\mathrm{T}}\mathrm{X})$ matrix. This coefficient is known as Marquardt's coefficient or Marquardt – Levenberg coefficient. In effect the Marquardt's coefficient is added to all the eigen values. It reduces the instability considerably due to the presence of zero and very small eigen values. So the least squares estimator

$$\Delta\mathrm{P} = (\mathrm{X}^{\mathrm{T}}\mathrm{X})^{-1}\mathrm{X}^{\mathrm{T}}\ \Delta\mathrm{G} \qquad (17.103)$$

changes to the form

$$\Delta\mathrm{P}^* = (\mathrm{X}^{\mathrm{T}}\mathrm{X} + \mathrm{KI})^{-1}\mathrm{X}^{\mathrm{T}}\Delta\mathrm{G} \qquad (17.104)$$

where K is the Marquardt's coefficient, I is the identity matrix. $\Delta\mathrm{P}$ is the model modification vector X is the sensitivity matrix. $\Delta\mathrm{P}^*$ is the model modification vector with a reduced rate. Equation (17.104) is known as the Ridge Regression Estimator or Damped Least Squares Estimator. It is called the damped least squares because the amplitude of the model modification goes down i.e., $\Delta\mathrm{P}^* < \Delta\mathrm{P}$ (Fig 17.9). Ridge Regression Estimator is much more stable than Least Squares estimator. It has both the qualities of Newton-Rhapson method and gradient method. Newton Rhapson method converges very fast if the starting value is close to the actual answer. The system diverges when the initial guess is away from the real answer. In the gradient method, however, the convergence is possible even if the initial guess is considerably

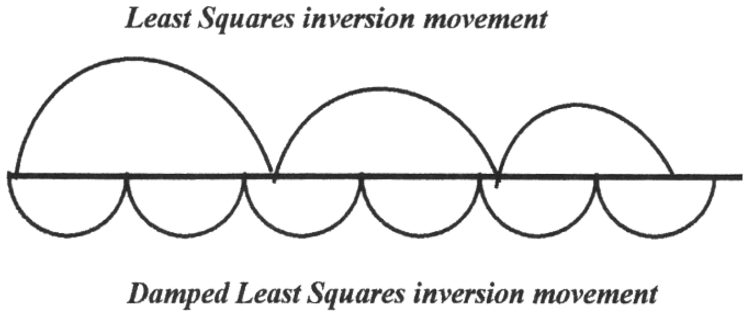


Fig. 17.9. Movements in the least square and damped least square iterative process

away from the actual answer. But convergence is very slow near the actual answer. Ridge regression has qualities of both the approaches i.e., it converges very fast near the actual answer and it's radius of convergence is reasonably high. It means even if the initial guess is poor i.e., the distance between the m^{Prior} and m^{true} is high, ridge regression can drag the model towards the actual answer. Larger the number of parameters, lesser will be the radius of convergence. Data inadequacy and data inaccuracy has direct relation with the radius of convergence. Choice of the value of K is dependent upon the interpreter. Starting value of K can be anything between 10.0, 1.00, 0.01, 0.001 as suggested by Marquardt (1963). But as the iterative solution converges, the value of K must be successively lowered down till its value becomes negligible. Many interpreters used variance – covariance values instead of a pure number as Marquardt's coefficient (Tarantola 1987, Menke, 1984).

17.10 Weighted Ridge Regression

In most of the scientific work we see that some of the experimental data in any experiment are less reliable than the others. This is quite common in geophysical field data analysis. It means that the data variances are not all equal. In other words the matrix Var (ε) (Variance (ε) is not in the form of $I\sigma^2$ where I is the identity matrix and σ^2 is the variance (square of the standard deviation) in the data. But Var (ε) is diagonally dominated matrix with unequal diagonal elements. It happens in some problems that the off diagonal elements of Var (ε) are not zero, i.e., the observations are correlated. When either or both of these occur, the general least squares estimator (17.104) is not valid and it is necessary to change the procedure for obtaining the estimator. Draper and Smith (1968) suggested that one has to transfer the observation

$$Y = X\beta + \varepsilon \tag{17.105}$$

to another variable Z in a different plane which do satisfy the basic conditions of linear regression and one can write

$$Z = Q\beta + f \tag{17.106}$$

where $E(f) = 0$ and Var $(f) = I\sigma^2$. The variables Y and X in the original plane will change to another set of variables Z and Q such that the $E(\varepsilon) = 0$ and $\mathrm{Var}(\varepsilon) = W\sigma^2$ changes to $E(f) = 0$ and $\mathrm{Var}(f) = I\sigma^2$. Here W is the weight attached to each data. For transformation from one plane to the other, the following procedure is adapted. It is possible to find an unique non singular symmetric matrix P such that

$$P^T P = PP = P^2 = W. \tag{17.107}$$

For transformation from one plane to the other we premultiply both the sides of the regression equation by P^{-1}. Let $f = P^{-1}(\varepsilon)$ such that $E(f) = 0$ then $E(ff^T) = \mathrm{var}\,(f)$ when the mathematical expectations are taken separately for every term in the square $n \times n$ matrix ff^T. We get

$$\mathrm{var}\,(f) = E(ff^T) = E(\phi^{-1}\varepsilon\varepsilon^T\phi^{-1}). \tag{17.108}$$

Since

$$(\phi^{-1})^T = \phi^{-1}$$

we can then write

$$P^{-1}E(\varepsilon\varepsilon^T)\ P^{-1} \Rightarrow P^{-1}\ \mathrm{var}\,(\varepsilon)\ P^{-1}$$

$$\Rightarrow \quad P^{-1}\ W\sigma^2\ P^{-1} \quad \Rightarrow\ P^{-1}P^2\ P^{-1}\ \sigma^2 = P^{-1}\ PPP^{-1}\ \sigma^2 \tag{17.109}$$

$$= I\sigma^2. \tag{17.110}$$

Thus if we premultiply (17.105) by P^{-1}, We obtain a new model in a new plane i.e.,

$$P^{-1}Y = P^{-1}\ X\beta + P^{-1}\varepsilon \tag{17.111}$$

Equation (17.111) is written as

$$Z = Q\beta + f. \tag{17.112}$$

It is now clear that if we apply the basic least squares theory to the (17.112), since E $(f) = 0$ and $\mathrm{var}\,(f) = I\sigma^2$, we get the normal equation as

$$Q^T Q\ b = Q^T\ Z \tag{17.113}$$

$$\Rightarrow \quad X^T\ W^{-1}\ X\ b = X^T\ W^{-1}Y \tag{17.114}$$

$$\Rightarrow \quad b = (X^T\ W^{-1}X)^{-1}\ X^T\ W^{-1}Y. \tag{17.115}$$

This is the basic formulation of the weighted least squares inverse. The variance – covariance matrix is

$$\mathrm{var}\,(b) = (Q^T Q)^{-1}\sigma^2 = (X^T\ W^{-1}X)^{-1}\ \sigma^2 \tag{17.116}$$

and the sum of the squared residual is

$$
\begin{aligned}
f^T f &= \varepsilon^T\ W^{-1}\ \varepsilon = (y - x\beta)^T\ W^{-1}(y - x\beta) \\
&= \Delta G^T\ W^{-1}\ \Delta G^T.
\end{aligned} \tag{17.117}
$$

When the observations are independent and the errors are uncorrelated, all the covariance terms will be zero and the variance – covariance matrix will simply be a diagonal matrix.

Therefore the weighted least squares estimator can be written as

$$
\overset{\wedge}{\Delta P} = \left(X^T W^{-1} X\right)^{-1} X^T W^{-1} \Delta G \tag{17.118}
$$

and weighted ridge regressions estimator can be written as

$$
\overset{\wedge}{\Delta P}^{*} = \left(X^T W^{-1} X + K\ I\right)^{-1} X^T W^{-1} \Delta G. \tag{17.119}
$$

Here the data are weighted by the inverse square root of the data variance-covariance matrix. (Inman, 1975).

The covariance matrix is given by

$$
V = \sigma^2 \left(X^T W^{-1} A\right)^{-1} \tag{17.120}
$$

where

$$
\sigma^2 = \frac{\Delta G^T W^{-1} \Delta G}{n - m}. \tag{17.121}
$$

Here σ^2 is termed as the residual variance and n and m are respectively the number of data points and number of parameters to be retrieved. The parameter standard error or the parameter uncertainly is defined as the square root of the diagonal elements of the parameter variance-covariance matrix. Thus $\sqrt{V_{11}}, \sqrt{V_{22}} \ldots\ldots\ldots$ etc. the parameter uncertainties are added to the retrieved model parameters i.e., the estimated parameters will now be written as

$$
\left(m_1 \pm \sqrt{V_{11}}\right), \quad \left(m_2 \pm \sqrt{V_{22}}\right), \quad \ldots\ldots\text{etc.}
$$

Data variance covariance matrix W is assumed to be a diagonal variance matrix with zero or negligible off diagonal covariance part.Reciprocals of the standard deviations are data weights and is given by

$$
W = (1/\sigma_1, 1/\sigma_2, 1/\sigma_3, 1/\sigma_4, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 1/\sigma_N)
$$

where variance is square of standard deviation.

17.11 Minimum Norm Algorithm for an Under Determined Problem

17.11.1 Norm

Norm is a measure of distance or length i.e., the distance between d^{Obs} and d^{Pre} or m^{Prior} and m^{true}. For each observations, one defines a predictable error or misfit

$$e_i = d_i^{Obs} - d_i^{Pre}.$$

The best line is the one with smallest over all error E.

$$E = \sum_{i=1}^{n} e_1^2 = e^T e.$$

This exactly is the squared Euclidean length. In the parameter domain $L = \sum_{i=1}^{n} m_i^2 = m^T m$. Here E and L are respectively the square of sum of the lengths in the data and parameter domains. Here $m_i = m_i^{true} - m_i^{Prior}$ for the ith parameter. The norm is indicated by double bar i.e., $E = \|e\|$. The L_1, L_2, L_3- - - - - - - - - - - -L_α norms are defined as

$$L_{1\ norm} = \|e\|_1 = \left[\sum_{i=1}^{n} |e|^1\right] \tag{17.122}$$

$$L_{2\ norm} = \|e\|_2 = \left[\sum_{i=1}^{n} |e_i|^2\right]^{1/2} \tag{17.123}$$

$$L_{n\ norm} = \|e\|_n = \left[\sum_{i=1}^{n} |e_i|^n\right]^{1/n} \tag{17.124}$$

$$L_{\infty\ norm} = \|e\|_\infty = \max |e_i| \tag{17.125}$$

Successive higher norms give larger weightage to the largest element of e. Therefore L_∞ norm is the highest value of e_i.

17.11.2 Minimum Norm Estimator

William Menke (1984) discussed this algorithm for solving underdetermined problems using Lagrange multipliers. Underdetermined problems are those where the number of data points are less than the number of unknown parameters to be determined. Underdetermined problems have generally infinite number of solutions. It is difficult to choose the correct one out of many. That is why some additional information known as a priori information are necessary to interprete an underdetermined inverse problem. Lagrange multipliers maximizes or minimizes a certain function using a constraint equation.

Here in this problem we have to find a model $m^{estimated}$ or m^{est} that minimizes

$$L = m^T m = \sum m_e^2 \tag{17.126}$$

subject to the constraint equation that $e = d - Gm = 0$
where e is the error. The inverse problem is $d = Gm$. This problem can be solved using Lagrange multiplier. We can minimize the function

$$\phi\,(\mathrm{m}) = \mathrm{L} + \sum_{\mathrm{i}=1}^{\mathrm{N}} \lambda_{\mathrm{i}} \mathrm{e}_{\mathrm{i}} \tag{17.127}$$

$$\phi\,(m) = L + \sum_{i=1}^{N} \lambda_i e_i = \sum_{i=1}^{M} m_i^2 + \sum_{i=1}^{N} \lambda_i \left[d_i - \sum_{J=1}^{M} G_i m_i \right] \tag{17.128}$$

where λ_{I} is the Lagrange multiplier, G_{ij} is the frechet kernel. Taking the derivations with respect to m_{q}, we get

$$\frac{\partial \phi}{\partial m_q} = \sum_{i=1}^{M} 2 \frac{\partial m_i}{\partial m_q} . m_i - \sum_{i=1}^{N} \lambda_i \sum_{j=1}^{M} \frac{\partial m_i}{\partial m_q} \tag{17.129}$$

$$= 2\mathrm{m}_{\mathrm{q}} - \sum_{\mathrm{i}=1}^{\mathrm{N}} \lambda_{\mathrm{i}} \mathrm{G}_{\mathrm{iq}}. \tag{17.130}$$

Taking

$$\frac{\partial \phi}{\partial m_q} = 0,$$

we get

$$2\mathrm{m} = \mathrm{G}^{\mathrm{T}} \lambda \tag{17.131}$$

in matrix notation. Taking the constraint equation $\mathrm{Gm} = \mathrm{d}$, and substituting this equation in (17.101), we get

$$\mathrm{d} = \mathrm{Gm} = \mathrm{G}(\mathrm{G}^{\mathrm{T}} \lambda / 2). \tag{17.132}$$

The matrix GG^{T} is an $\mathrm{N} \times \mathrm{N}$ matrix. If its inverse exists then we get the value of the Lagrange multiplier as

$$\lambda = 2 \left(\mathrm{GG}^{\mathrm{T}} \right)^{-1} \mathrm{d}. \tag{17.133}$$

The expression for the minimum norm estimator for an under determined problem is

$$\begin{aligned} \mathrm{m}^{\mathrm{est}} &= \mathrm{G}^{\mathrm{T}} \left(\mathrm{G}^{\mathrm{T}} \mathrm{G} \right)^{-1} \mathrm{d} \\ &= \mathrm{G}^{-\mathrm{g}} \mathrm{d}. \end{aligned} \tag{17.134}$$

$\mathrm{G}^{-\mathrm{g}}$ is a generalized inverse for an underdetermined problem. Whatever prescriptions are made for least squares estimator for an over determined problem regarding use of Marquardt's coefficient to handle zero and very small eigen values of the system matrix are also valid for under determined problems. The least squares and minimum norm generalized inverses are respectively given by

$$\mathrm{G}^{-\mathrm{g}} = \left(\mathrm{G}^{\mathrm{T}} \mathrm{G} \right)^{-1} \mathrm{G}^{\mathrm{T}} \quad \text{(Least Squares)} \tag{17.135}$$

and

$$G^{-g} = G^T \left(G\ G^T\right)^{-1} \quad \text{(Minimum norm)} \tag{17.136}$$

Generalized inverse gives $m^{estimated}$ or $m^{est} = G^{-g}d^{obs}$ from $d = Gm$. Therefore we can write $d^{Pre} = G\ m^{est} = G(G^{-g}d^{obs}) = (GG^{-g})d^{obs}$.

$$= N\ d^{obs}. \tag{17.137}$$

The $n \times n$ square matrix $N = GG^{-g}$ is called the data resolution matrix. The diagonal elements of this matrix give a quantitative idea about the data importance.

Data importance $n = \mathrm{diag}(N)$.

Similarly model resolution matrix

$$m^{est} = G^{-g}d^{obs} = G^{-g}\left[G\ m^{true}\right] \tag{17.138}$$

$$= \left[G^{-g}G\right] m^{true}$$

$$= Rm^{true}. \tag{17.139}$$

Here R is $m \times m$ model resolution matrix. If $R = 1$, then each model is well determined. Like N, it is also a diagonally dominant matrix.

Information density matrix $uu^T = I_n$ and resolution matrix $vv^T = I_n$ has some similarity with N and R respectively. But the values do not come exactly same. $uu^T = I_n$ and $N = GG^{-g}$ are in data space, $vv^T = I_m$ and $R = G^{-g}G$ are in the model space.

Dirichlet's spread functions are defined as

$$\text{Spread} = (N) = \|N - I\| = \sum_{i=1}^{N}\sum_{j=1}^{N}\left[N_{ij} - I_{ij}\right]^2 \tag{17.140}$$

$$\text{Spread} = (R) = \|N - I\| = \sum_{i=1}^{M}\sum_{j=1}^{M}\left[R_{ij} - I_{ij}\right]^2. \tag{17.141}$$

The spread qualitatively indicates the quality of inversion. Larger the spread worse will be the inverted model. n and m are respectively the number of data points and number of parameters. I is the identity matrix therefore $I_{ij} = 0$.

17.12 Bachus – Gilbert Inversion

17.12.1 Introduction

Bachus – Gilbert theory of inversion is also a member of the linear inverse theory. But the approach is significantly different from those we have discussed so far. Here we describe and apply a method of finding the resolving power of a finite set of data when these data are used to construct an earth model.

Bachus – Gilbert's theory say that if we can construct an inverse problem such that when we try to find out the physical property of the earth at a depth z_0 say $(m)_{Z0}$, then the mathematical system should pickup maximum information from that depth z_0 and very little information from above and below that depth. In other words if we can construct a kernel which have dirac delta type behaviour, then only it will be possible to pick up information only from one depth. With finite number of data, it is not possible to get such a sharp dirac delta type kernel. B-G named this kernel as an averaging kernel $A(z, z_0)$. It not only picks up information from that depth z_0 but also carries some information from depths above and below. This averaging kernel has certain width or spread. Lower the spread, sharper will be the pick of the averaging kernel. B-G made the mathematical formulation in such a way that they tried to minimize the spread using Lagrange multiplier and using the constraint equation. i.e., the area under the averaging kernel in equal to unity. So the deltaness criterion of the averaging kernel and minimization of the spread function go together at all depths and the model parameter $\langle m\rangle_{Z0}$ is determined. Gradual increase of the B-G spread function with depth will roughly tell about the depth upto which the earth model could be estimated with a certain degree of certainty using a finite number of data.

17.12.2 B-G Formulation

The important advantages of the B-G method are (i) layered earth approximation is not required. The physical property say resistivity or density is a function of the depth z only i.e. $\rho = f(z)$ at the point of measurement. Therefore data collected over a complicated Archean-Proterozoic geological terrain can also be inverted using B-G approach (ii) depth of exploration of the D.C resistivity, electromagnetic and magnetotelluric sounding data can be estimated from B-G spread function.

B-G assumed that the earth functionals are linear and they are fre'che't differentiable. Therefore the linear earth functionals $g_1, g_2, \ldots\ldots g_N$ (field data) can be connected to the earth model parameter through the Fredhom's integral of the first kind i.e.

$$g_i(m) = \int_{Z_0}^{Z} G_i(z)\, m(z)\, dz \tag{17.142}$$

where $G_i(z)$ is a known function of z because g is a known linear functional. So the inverse problem is what can we say about m(z) at $z = z_0$ when all we know is a set of $g_1, g_2, \ldots\ldots g_N$ collected on the surface of the earth. We try to compute $\langle m\rangle_{Z0}$ which is the average value of m taken within a short interval $z_0 \pm \Delta z_0$. These local averages are the model value at a particular depth z_0 within the resolving length $z_0 + \Delta z_0$ to $z_0 - \Delta z_0$. B-G defined the local average using the averaging kernel in the form

$$\langle m\rangle_{z0} = \int_{z0}^{z\,max} A(z_0, z)\, m(z)\, dz \tag{17.143}$$

where averaging kernel assumed to be unimodular and it is

$$\int_{z}^{z\,max} A(z_0, z)\, dz = 1. \tag{17.144}$$

The averaging kernel A is distributed in z i.e., at depths where we shall try to compute m(z), we shall have to compute the averaging kernel to examine the quality of inversion based on the strength of the data. Data quality and adequacy improves the quality of A. Theoretically the nature of A should be unimodular, for practical problems when the data contains noise, A can show multimodular nature and spread may increase very fast. Ideally $A(z_0, z) = \delta(z - z_0)$ where δ is the Dirac delta distribution. With only a finite number of field data, available for computing $\langle m\rangle_{Z0}$, we should not expect the local average to be so localized.

We assume that the average $\langle m\rangle_{Z0}$, whatever it turns out to be, depends only on $g_1(m)$, $g_2(m) \ldots\ldots g_N(m)$.

Since $\langle m\rangle_{Z0}$ and $g_1(m)$, $g_2(m) \ldots\ldots g_N(m)$ have linear relation, we can write

$$\langle m\rangle_{Z0} = \sum_{i=1}^{N} a_i(z_0)\, g_i(m) \tag{17.145}$$

These coefficients depends on the depth z_0. At each depth we have to find out these coefficients $a_i(z_0)$ to estimate $\langle m\rangle_{Z0}$ and the additional constraint is

$$A(z_0, z) = \sum_{i=1}^{N} a_i(z_0)\, G_i(z)\,. \tag{17.146}$$

So we have to determine the $a_i(z_0)$ at all depths such that it satisfies (17.143) (17.144) (17.145). B-G chose a function $K(z_0, z)$ which vanishes at $z = z_0$ and increases on both the sides of z_0. B-G chose the function

$$K = 12 \int_{0}^{\infty} (z - z_0)^2 A^2(z, z_0)\, dz. \tag{17.147}$$

Here the factor 12 in (17.147) is used so that the spread becomes equal to width of the averaging kernel. We can write the value of K from (17.147) and (17.146) in the form

$$K = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \left\{ 12 \int_{0}^{\infty} (z - z_0)^2 G_i(z)\, G_j(z)\, dz \right\} \tag{17.148}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i\ a_j\ S_{ij}. \tag{17.149}$$

In the matrix form

$$\mathrm{K} = \mathrm{a}^{\mathrm{t}}\ \mathrm{S}\ \mathrm{a} \tag{17.150}$$

where $\mathrm{S} = \mathrm{S}_{\mathrm{ij}}$.

K is interpreted as the spread around z_0 (Oldenburg 1979). The matrix S is called the spread matrix. The elements of matrix S can be written in simpler computational form as

$$\mathrm{S}_{\mathrm{ij}} = 12\int_{o}^{\infty} (\mathrm{z}-\mathrm{z}_0)^2 \mathrm{G}_{\mathrm{i}}(\mathrm{z})\,\mathrm{G}_{\mathrm{j}}(\mathrm{z})\,\mathrm{dz} \tag{17.151}$$

$$= \left[12\int_{0}^{\infty} \mathrm{z}^2 \mathrm{G}_{\mathrm{i}}(\mathrm{z})\,\mathrm{G}_{\mathrm{j}}(\mathrm{z})\,\mathrm{dz}\right] - 2\mathrm{z}_0\left[12\int_{0}^{\infty} \mathrm{z}\ \mathrm{G}_{\mathrm{i}}(\mathrm{z})\,\mathrm{G}_{\mathrm{j}}(\mathrm{z})\,\mathrm{dz}\right] + \mathrm{z}_0^2\left[12\int_{o}^{\infty} \mathrm{G}_{\mathrm{i}}(\mathrm{z})\,\mathrm{G}_{\mathrm{j}}(\mathrm{z})\,\mathrm{dz}\right]. \tag{17.152}$$

The construction of the averaging Kernel requires computation of averaging coefficients $\mathrm{a}_{\mathrm{j}}(\mathrm{z}_0), (\mathrm{i} = 1, 2, \ldots\ldots\ldots \mathrm{N})$ around the depth of interest. These coefficients are determined minimizing the spread using Lagrange multiplier and using the constraint i.e., the area under the averaging kernel is equal to unity. We can write (17.148) in the form

$$\sum_{\mathrm{i}=1}^{\mathrm{N}} \int_{0}^{\infty} \mathrm{a}_{\mathrm{i}}(\mathrm{z}_{\mathrm{o}})\,\mathrm{G}_{\mathrm{i}}(\mathrm{z})\,\mathrm{dz} = 1 \tag{17.153}$$

$$\Rightarrow \quad \sum_{i=1}^{N} a_i u_i = 1 \tag{17.154}$$

where

$$u_i = \int_{0}^{\infty} G_i(z)\,dz. \tag{17.155}$$

And in the matrix form we can write

$$\Rightarrow \quad \mathrm{a}^{\mathrm{t}}\mathrm{u} = 1. \tag{17.156}$$

The minimization of ‘$\mathrm{a}^{\mathrm{t}}\ \mathrm{S}\ \mathrm{a}$’ under the constraint $\mathrm{a}^{\mathrm{t}}\mathrm{u} = 1$ using Lagrange multiplier technique solves the problem of determining coefficients of the averaging kernel. Applying this technique, the condition for minimization of the spread with respect to a is

$$\frac{\partial}{\partial a_i}\left[\sum_{i=1}^{N}\sum_{J=1}^{N} a_i\ a_j\ S_{ij} - \lambda'\left(\sum a_i u_i - 1\right)\right] = 0 \tag{17.157}$$

where λ' is the Lagrange multiplier

and when $i = j, a_i \ a_j = a_i^2$ and $\frac{\partial}{\partial a_i} = 2a_i$. Therefore

$$2\sum_{j=1}^{N} a_i \ S_{iJ} - \lambda' u_i = 0 \tag{17.158}$$

where

$$\lambda = \frac{\lambda'}{2}.$$

In the matrix form

$$\begin{aligned} S \ a - \lambda \ u &= 0 \\ \Rightarrow \quad a &= \lambda \ S^{-1} u. \end{aligned} \tag{17.159}$$

Since

$$\begin{aligned} a^t u &= 1 \\ \Rightarrow \quad \lambda \ u^t \ S^{-1} u &= 1 \\ \Rightarrow \quad \lambda &= \frac{1}{u^t S^{-1} u}. \end{aligned} \tag{17.160}$$

Hence

$$a = S^{-1} \ u/(u^t \ S^{-1} \ u). \tag{17.161}$$

Equations (17.159) and (17.161) shows respectively the values of the Lagrange multipliers and the coefficients of the averaging kernel which minimizes the spread. The corresponding model estimate is given by

$$\begin{aligned} \langle m \rangle_{z0} &= a^t d \\ &= \frac{u^t \ S^{-1} d}{u^t S^{-1} u} \end{aligned} \tag{17.162}$$

and the averaging kernel is determined as

$$A\left(z, z_0\right) = \frac{a^t \ G\left(z\right)}{u^t S^{-1} u}. \tag{17.163}$$

Thus all the parameters of B-G inversion are obtained. But one problem is yet to be solved i.e. how to determine the fre'che't kernel G (z). Oldenburg (1978 and 1979) have shown the procedure for determination of G (z) for d.c resistivity and magnetotelluric inversion applying B-G method. In the presence of noise, the spread matrix 'S' is replaced by W in the form.

$$W = (1 - \alpha) S + \alpha \ \mathrm{Cov} \ (m). \tag{17.164}$$

Here error in the model estimate is related to the error in the data. The variance in the estimated model is considered as a measure of error. Covariance matrix is constructed and a fraction of that is taken in the modified spread. α is a fraction. It varies from 0 to 1. For very good quality data $\alpha = 0$, and W = S. With increase in noise α increases. $\alpha = 1$ for totally noisy data. The averaging coefficients are obtained by minimizing a^t W a instead of a^t S a. The coefficients thus obtained are the functions of α. The procedure for minimizing W is the same as that for S. A trade off between resolution and the variance-covariance is made for optimum choice of α.

In B-G inversion no initial choice of the model parameter is necessary. Therefore it can be used for any kind of complicated geology. Any physical property is assumed to be just a function of depth. One can see the strength of the data and the quality of inversion from the nature of the averaging kernel. One can directly see the depth of investigation of the data. Beyond a certain depth, the B-G spread function explodes i.e., the spreads start increasing at a very faster rate. That dictates the depth upto which the data could see the subsurface.

17.13 Stochastic Inversion

17.13.1 Introduction

In this inversion approach most of the basic ingradients of inversion are viewed from the angle of probility density functions although it is basically a dual minimization approach. Instead of minimizing the error function in successive iteration we maximize the probability density function assuming Gaussian distribution for linear and linearisable problems. Some of the ideas of Tarantola (1987) are described very briefly. Four classes of problems (Fig. 17.10) viz, linear, linearisable, nonlinear but linearisable at the maximum

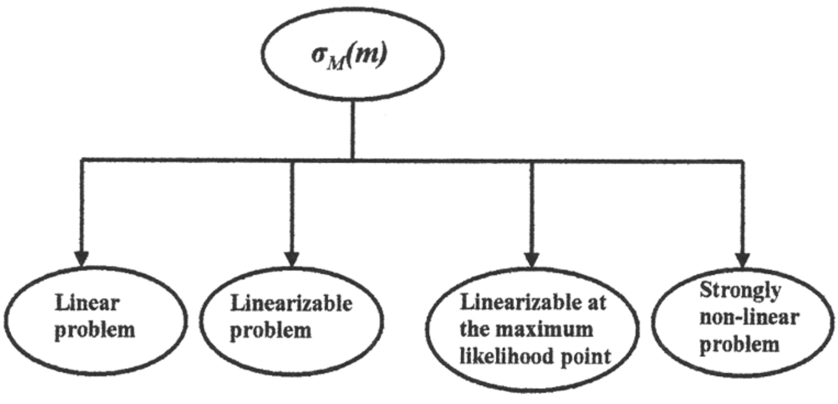


Fig. 17.10. Four classes of an inverse problems (**a**) linear (**b**) non linear but linearisable (**c**) nonlinear but linearisable at the maximum likelihood point (**d**) nonlinear and not linearisable

likelihood point and strongly nonlinear problems and three classes of probability density functions viz, joint probability density function, conditional probability density function and marginal probability density function are used (Freund and Walpole,1987). Since most of the geophysical problems are nonlinear we work mostly with last three classes of problems. It has already been mentioned that one linearises a nonlinear problem by truncating higher order terms of the Taylor's series expansion. The quantum of error present in this approximation dictates the degree of nonlinearity of a problem. Higher the degree of nonlinearity more will be the deviation of the nature of the probability density function from the Gaussian nature. In the stochastic domain the existence of an inverse problem is connected with the existence of marginal probability density function. Here there are more of mixing of information in the (D $\times$ M) parameter space collected from the data (D) and Model (M) spaces.

Errors due to modeling, instrumentation is expressed in the form of joint probability density – functions $\theta(\mathrm{d},\mathrm{m})$ and $\rho(\mathrm{d},\mathrm{m})$. Here 'd' stands for data and m stands for model. $\rho(\mathrm{d},\mathrm{m})$ is a mixture of information from $\rho(\mathrm{d})$ and $\rho(\mathrm{m})$ where $\rho(\mathrm{d})$ is the probability density function due to the data error $(\mathrm{d}^{\mathrm{obs}} - \mathrm{d}^{\mathrm{Pre}})$ and $\rho(\mathrm{m})$ is the probability density function due to modeling error $(\mathrm{m}^{\mathrm{Prior}} - \mathrm{m}^{\mathrm{true}})$. These probabilities are described in the form of Gaussion distribution. These probabilities are combined to form a joint probability density function. $\sigma(\mathrm{d},\mathrm{m})$. It is the starting point of stochastic inversion. From $\sigma(\mathrm{d},\mathrm{m})$, we find out the marginal probability density functions $\sigma_{\mathrm{M}}(\mathrm{m})$ in the model space and $\sigma_{\mathrm{D}}(\mathrm{d})$ in the data apace. Since geophysicists are mostly interested to get models from a set of data, therefore, we are interested in $\sigma_{\mathrm{M}}(\mathrm{m})$, the marginal probability density function in the model space. It extracts all information about the model from the joint a posteriori probability density function $\sigma(\mathrm{d},\mathrm{m})$. Figure 17.11 is a curtoon of stochastic inversion scheme.

Error due to modeling will always exist. We shall never be able to choose an earth model which will exactly match with the reality. In general the chosen model will always be much simpler than the real earth where the data are collected. So there will always be some difference between $\mathrm{d}^{\mathrm{Pre}}$ and $\mathrm{d}^{\mathrm{Obs}}$. Even when we get $\mathrm{d}^{\mathrm{Obs}} \approx \mathrm{d}^{\mathrm{Pre}}$ in an iterative process there is no guarantee that we obtained all the subsurface features in the earth model and in fact we retrieve a much simpler model. Error due to instrumentation was severe in earlier days. With sophistication in instrumentation in modern day digital electronics, error due to instrumentation has significantly gone down. Tarantola expressed these uncertainties in modeling in the form of Gaussian probability density function

$$\begin{aligned}\theta\left(d\left|m\right.\right) = \left((2\pi)^{ND}\det\left(C_T\left(m\right)\right)\right)^{-1/2} * \\ \exp\left\{-\frac{1}{2}\left(d-g\left(m\right)\right)^t C_T^{-1}\left(m\right)\left(d-g\left(m\right)\right)\right\}.\end{aligned} \tag{17.165}$$

Here $\mathrm{d}^{\mathrm{Pre}} = \mathrm{d}^{\mathrm{Cal}} = \mathrm{g}\ (\mathrm{m})$. For linear inverse problem we have shown already that we linearise at the reference point as

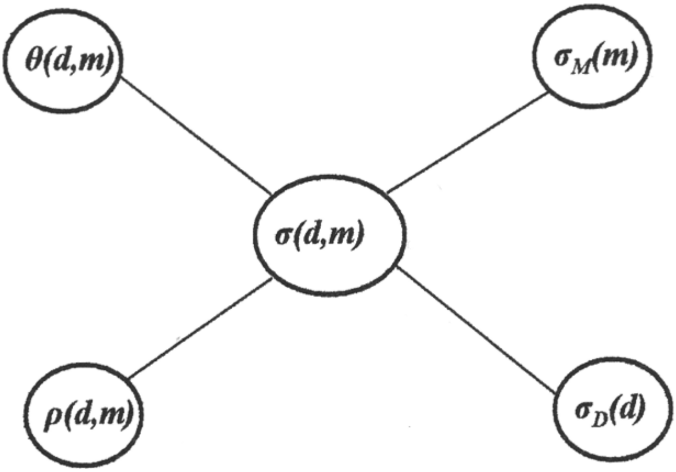


Fig. 17.11. Conjunction of the state of information and extraction of model parameters

$$\mathrm{g}\,(\mathrm{m}) = \mathrm{g}\,(\mathrm{m}_{\mathrm{ref}}) + \mathrm{G}_{\mathrm{ref}}\,(\mathrm{m} - \mathrm{m}_{\mathrm{ref}}) \tag{17.166}$$

to start the inverse process where

$$\mathrm{G}^{\mathrm{i}\alpha}_{\mathrm{ref}} = \left|\frac{\partial \mathrm{g}^{\mathrm{i}}}{\partial \mathrm{m}^{\infty}}\right|_{\mathrm{m}_{\mathrm{ref}}}. \tag{17.167}$$

Here $\mathrm{G}^{\alpha}_{\mathrm{ref}}$ is the derivative at the reference point with respect to each unknown parameters and each known parameters. It is often difficult to quantitatively assess the error introduced due to linearisation of a non linear problem. Tarantola used the joint probability density function $\theta(\mathrm{d}, \mathrm{m})$ over the space $\mathrm{D} \times \mathrm{M}$ to describe information about the resolution of the forward model. Similarly two more probability density functions are defined viz.,

$$\begin{aligned}\rho_D\,(d) = &\left((2\pi)^{ND} \det C\left(d^{Obs}\right)\right)^{-1/2} * \\ &\exp\left\{\frac{1}{2}\left(d - d^{Obs}\right)^t C\left(d^{Obs}\right)^{-1}\left(d - d^{Obs}\right)\right\}.\end{aligned} \tag{17.168}$$

Here $\mathrm{d}^{\mathrm{Pre}}(\mathrm{d})$ is the actual input in the instrument and $\mathrm{d}^{\mathrm{Obs}}$ is what we have collected from the field. By apriori information, we mean information, which is obtained independently of the results of measurements. The probability density function representing the a priori information will be denoted by $\rho_{\mathrm{M}}(\mathrm{m})$ and is equal to

$$\begin{aligned}\rho_M\,(m) = &\left((2\pi)^{NM} \det\left(C_M\right)\right)^{-1/2} * \\ &\exp\left\{-\frac{1}{2}\left(m - m^{\mathrm{Pr}\ ior}\right)^t C_M^{-1}\left(m - m_{\mathrm{Pr}\ ior}\right)\right\}\end{aligned} \tag{17.169}$$

Here we define the a priori information on model parameters as information independent of the observations or data. The information we have in both model parameters (m) and data (d) can then by described in the $\mathrm{D} \times \mathrm{M}$ space by the joint probability density function

$$\rho\left(d, m\right) = \rho_D\left(d\right) \rho_M\left(m\right). \tag{17.170}$$

$\mathrm{C_D}$ and $\mathrm{C_M}$ are the data and model covariance matrices (Draper and Smith 1984).

17.13.2 Conjunction of the State of Information

The probability density function $\rho(\mathrm{d}, \mathrm{m})$ defined in the $\mathrm{D} \times \mathrm{M}$ space represents information both from data and a prior information about the model. These two states of information combine with information about the error in modeling to produce the a posteriori state of information $\sigma(\mathrm{d}, \mathrm{m})$ in the $\mathrm{D} \times \mathrm{M}$ space and is given by

$$\sigma(\mathrm{d}, \mathrm{m}) = \rho\ (\mathrm{d}, \mathrm{m})\ \theta\ (\mathrm{d}, \mathrm{m}) \tag{17.171}$$

$\sigma(\mathrm{d}, \mathrm{m})$ is the first step for entering into the arena of inversion. Once the a posteriori joint probability density function in the $\mathrm{D} \times \mathrm{M}$ space is obtained, the a posterior information in the model space is given by the marginal probability density function $\sigma_{\mathrm{M}}\left(\mathrm{m}\right) = \int\limits_{\mathrm{D}} \mathrm{d}\ \mathrm{d}\ \sigma\left(\mathrm{d}, \mathrm{m}\right)$ while the posteriori information in the data space is $\sigma_{\mathrm{D}}\left(\mathrm{d}\right) = \int\limits_{\mathrm{M}} \mathrm{d}\ \mathrm{m}\ \sigma\left(\mathrm{d}, \mathrm{m}\right)$. The marginal probability density functions $\sigma_{\mathrm{M}}(\mathrm{m})$ contain all information about the model parameters. Geophysicists are mostly interested in $\sigma_{\mathrm{M}}(\mathrm{m})$ (Fig. 17.11).

17.13.3 Maximum Likelyhood Point

When the problems do not have small number of parameters and the computation of $\sigma_M\left(m\right)$ at any point 'm' is expensive. One should find out ways to reach m_{ML} maximizing $\sigma_M\left(m\right)$. This is the maximum likely hood point.If $\sigma_M\left(m\right)$ is a differential function of 'm' the maximum likely hood point can be obtained using the gradient of $\sigma_M\left(m\right)$ i.e., $\partial\sigma_M/\partial m^{\alpha}$. $\sigma_M\left(m\right)$ is in general, an arbitrary complicated function of m. There is no guarantee that the maximum likelyhood function is unique or that a given point which is locally maximum is the absolute maximum.

The general expression for $\sigma_M\left(m\right)$ is

$$\begin{aligned} &\sigma_M\left(m\right) = \text{Const.} \\ &\exp\left[-\frac{1}{2}\left[\left(g\left(m\right) - d_{obs}\right)^t C_D^{-1} - \begin{pmatrix} g\left(m\right) - d_{obs}) \\ +(m - m_{prior})^t C_M^{-1}(m - m_{prior}) \end{pmatrix}\right]\right]. \end{aligned} \tag{17.172}$$

A. When the Problem is Linear

Instead of writing $d = g(m)$, we write d = Gm where G is the linear operator acting from the model space into the data space. This gives σ_M (m) = Const. exp(−S(m)) where

$$S(m) = \frac{1}{2}\begin{bmatrix} (G_m - d_{obs})^t C_D^{-1}(Gm - d_{obs}) \\ + (m - m_{prior})^t C_M^{-1}(m - m_{prior}) \end{bmatrix}. \tag{17.173}$$

B. When the Problem is Non Linear

$$\sigma_M(m) = \text{const.}\exp\begin{bmatrix} -\frac{1}{2}(g(m) - d_{obs})^t C_D^{-1}(g(m) - d_{obs}) \\ + (m - m_{prior})^t C_M^{-1}(m - m_{prior}) \end{bmatrix}. \tag{17.174}$$

Since g(m) is not a linear function, for weakly nonlinear problems g(m) = g$(m_{prior}) + G_o(m - m_{prior})$ where G_o represent the derivative operator with element

$$G_o^{i\alpha} = \left[\frac{\partial g^i}{\partial m^\alpha}\right]_{m\,prior}. \tag{17.175}$$

The aposteriori marginal probability density function will be approximately linear to describe it adequately by its central estimator and the estimators of dispersion. Among the central estimator, the maximum likelyhood point value m_{ML} is chosen.

$$m_{ML} : \sigma_M(m_{ML})\,\text{max} \tag{17.176}$$

The maximum likelyhood point is given by

$$m_{mL} = m_{prior} + \left[G_o^t C_D^{-1} G_o + C_M^{-1}\right]^{-1} G_o^t C_D^{-1}(d_{obs} - g(m_{prior})) \tag{17.177}$$

and the a posteriori covariance operator is approximately given by

$$C_M' \approx \left[G_o^t C_D^{-1} G_o + C_M^{-1}\right]^{-1}. \tag{17.178}$$

For strongly nonlinear problems:

d = g(m) can be linearised only around the true maximum likelyhood point m_{ML}

where

$$\text{g(m)} \approx g(m_{ML}) + G_\alpha(m - m_{ML}). \tag{17.179}$$

Here G_α represents the derivative operator with element

$$G_\alpha^{i\alpha} = \left[\frac{\partial g^i}{\partial m^\alpha}\right]. \tag{17.180}$$

The point m_{ML} to be obtained by iterative minimisation of S(m)

$$S(m) = \frac{1}{2}\begin{bmatrix} (g(m) - d_{obs})^t C_D^{-1} (g(m) - d_{obs}) \\ + (m - m_{prior})^t C_M^{-1} (m - m_{prior}) \end{bmatrix} \tag{17.181}$$

where $m_{n+1} = m_n + \delta m_n$. If m_α is the point where we want to stop the iteration, then $m_\alpha = m_{ML}$. The posterior covariance operator can then be written as

$$C'_M = \left[G_\alpha^t C_D^{-1} G_\alpha + C_M^{-1}\right]^{-1}. \tag{17.182}$$

(iii) When the forward problem cannot be linearised a posterior probability density remains far from Gaussian and global optimization methods should be used.

17.14 Occam's Inversion

Constable et al (1987) and Degroot-Hedlin and Constable (1990) developed this approach for inversion for minimum structure by dual minimization of roughness factors and data misfit using Lagrange multiplier. Roughness factors are defined by Constable et al (1987) as

$$\mathrm{R}_1 = \int (\partial \mathrm{m}/\partial \mathrm{z})^2 \, \mathrm{dz} \tag{17.183}$$

and

$$\mathrm{R}_2 = \int \left(\partial^2 \ \mathrm{m}/\partial \mathrm{z}^2\right) \mathrm{dz} \tag{17.184}$$

where m(z) is a quantitative value of a parameter at a particular depth z. In the case of dc resistivity it is an apparent resistivity. In the case of magnetotellurics it is either an apparent resistivity or a phase. The mathematical function for a particular geophysical tool can be written as

$$\mathrm{d_j} = \mathrm{g_j(m)}, \quad \text{for } \mathrm{j} = 1, 2, \ldots..\mathrm{M} \tag{17.185}$$

where $\mathrm{g_j}$ is a subject dependent functional for the forward problem associated with the jth datum. R_1 and R_2 are respectively the two roughness functions based on first and second derivatives of the model parameters with respect to depth z. As mentioned, d = Gm for a linear problem and d = g(m) for a non linear but linearisable problem. If σ_i is the standard deviation of the ith data collected in the field, the weights in the weighted least squares are $1/\sigma_1$, $1/\sigma_{2,}$, $1/\sigma_3 \ldots\ldots\ldots 1/\sigma_\mathrm{N}$ (Sect. 17.6). For dc resistivity sounding where the number of observation are at the most 3 or 4 per electrode separation, 1%,2% or 5% standard deviation for all data depending upon their quality (Inman 1975) are assumed; even variable Ws are permissible. For MT standard deviations are available for all apparent resistivities and phases after processing.

For linear problem the parameter estimator for Occam inversion is given by

$$\mathrm{m} = \left[\lambda \partial^\mathrm{T} \partial + (\mathrm{WG})^\mathrm{T} \mathrm{WG}\right]^{-1} (\mathrm{WG})^\mathrm{T} \ \mathrm{Wd} \tag{17.186}$$

where λ^{-1} is the Lagrange multiplier used to maximize or minimize a function using a constraint equation (Sects. 17.7, 17.8). $\partial(m)$ is the roughness factor, and ∂^T is its transpose. W,G and d are respectively the weights, linear operator and data respectively. For a nonlinear problem as discussed already,

$$\mathrm{g}(\mathrm{m}_{\mathrm{i}+}\Delta\mathrm{m}_{\mathrm{i}}) = \mathrm{g}(\mathrm{m}_{\mathrm{i}}) + \mathrm{J}\ \Delta\mathrm{m}_{\mathrm{i}} + \varepsilon \tag{17.187}$$

where J is the linear differential operator obtained truncating higher order terms of the Taylor's series expansion. Elements of J forms the sensitivity matrix in linearised inversion. The parameter estimator for non linear problems is

$$\mathrm{m} = \left[\lambda\partial^{\mathrm{T}}\partial + (\mathrm{WJ}_1)^{\mathrm{T}}\mathrm{WJ}_1\right]^{-1}(\mathrm{WJ}_1)^{\mathrm{T}}\ \mathrm{Wd}. \tag{17.188}$$

Equations (17.188) and (17.186) are more or less the same where linear operator G is replaced by linear differential operator J. The roughness matrix R_1 may be replaced by R_2 or it may be kept as R_1.

These roughness factors are defined by Constable and his coworkers as

$$\mathrm{R}_1 = \sum_{\mathrm{i}=1}^{\mathrm{N}}(\mathrm{m}_{\mathrm{i}} - \mathrm{m}_{\mathrm{i}-1})^2 \tag{17.189}$$

and

$$\mathrm{R}_2 = \sum_{\mathrm{i}=2}^{\mathrm{N}}(\mathrm{m}_{\mathrm{i}-1} - 2\mathrm{m}_{\mathrm{i}} + \mathrm{m}_{\mathrm{i}})^2. \tag{17.190}$$

This roughness matrix can be written as

$$\mathrm{R}_1 = \|\partial\mathrm{m}\|^2 \tag{17.191}$$

where ∂ is a $\mathrm{N}\times\mathrm{N}$ and is given by

$$\partial = \begin{bmatrix} 0 & & & & & \bigcirc \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ \bigcirc & & & & -1 & 1 \end{bmatrix}. \tag{17.192}$$

This dual minimization of roughness and data misfit in the modified weighted least squares domain to achieve a smooth and minimum structure is Occam's inversion developed by Constable and his coworkers and is widely used by geophysicists to generate 2D models. For further details the readers are requested to go through the original papers.

17.15 Global Optimization

17.15.1 Introduction

In global optimization one searches the entire parameter space by random walk for solution of an inverse problem. These approaches are very much unlike the local search tools we have discussed earlier. Global search tools do not have any problem for getting trapped in the local minima pockets. Because the movements in the parameter space is by random jumping and not through any systematic mathematical procedures. Once the forward problem is solved inverse problem can be solved by random jump in model space and trial and error. That eliminates the stability problem of a sensitivity matrix and local minima problem . These inversion approaches move towards the global minimum instead of a local minimum and hence they are classified as global optimization tools. These global optimization tools choose a model by random walk, compute the forward problem, compute the discrepancy between the field data and model data in the form of an error function or cost function or energy function, move to the next model by random jump and follow the same procedure till you reach a model where the discrepancy is minimum between the field data and the model data.

The members in this family of random walk are (i) Monte Carlo inversion (ii) Simulated Annealing and (iii) Genetic Algorithm. Neural Network does not come strictly under the random walk techniques. Monte Carlo method is an unguided random walk technique without any artificial intelligence. Simulated Annealing and Genetic Algorithm are guided random walk tools having artificial intelligence. Neural network on the otherhand is a tool with artificial intelligence but with a partially random approach at the beginning regarding selection of weights. SA is designed bringing some analogy from the chemical thermodynamics and chemical annealing process of metals. Genetic Algorithm mimics the biological processes and goes through the survival for the fittest test. Neural Network imitates the behaviours and functioning of the neurons in the brain.That is why these networks are trained to do some specific jobs.

These tools are very powerful in the sense that MC, SA, GA search the entire parameter space by random jumping. Since there is no mathematics for inversion, there is (i) no problem of finding out the analytical derivatives or frechet derivatives (ii) there is no stability problem in inversion. But problem of non uniqueness remains because of the finite resolving power of the scalar and vector potentials and the parameters generated out of these potentials can be collected from the field free from any noise. (iii) both linear and nonlinear problems can be handled with equal ease (iv) these optimization tools become more powerful to handle 2-D/3-D problems.

Since several thousands computations of forward problem are involved, an efficient and minimum time run algorithm for forward problem computation is essential to use these tools. Otherwise the computation time requirement will be prohibitive. Monte Carlo inversion needs more than one million forward problem computations. The same process will continue for 10,000 times in Simulated Annealing and more in Genetic Algorithm because GA works with a population of initial models. Artificial Neural Network(ANN) is strictly not a member of the global optimization tools. ANN tools are trained several times to do a particular type of job by adjustment of weights in the hidden layers through backward propagation of information from output to input.

17.15.2 Monte Carlo Inversion

Monte Carlo inversion is an unguided random walk technique and without any artificial intelligence component in it. In Monte Carlo inversion, the models are chosen randomly in the parameter space, the forward models are computed, the model data (d^{pre}) are computed and are compared with the field data(d^{obs}), if the constraints imposed in this problem are satisfied, the cost function or error function is within the prescribed limit then only that particular model is allowed to pass through the mesh. This model is one of the successful model. The next model is chosen by the process of random walk and the same procedures are repeated. This way out of one million trial runs 10 models may trickle through the Monte Carlo mesh.

All realistic, judicious and acceptable constraints must be used to reduce the search space and computation time. Since most of the physical parameters or properties are positive quantities, positivity constraint can be used by not allowing any parameter to be negative. Each parameter is assigned an upper and lower bound once the number of parameters to be retrieved is decided. Dimensionality of the model (i.e. 1D, 2D or 3D) will dictate the number of parameters to be retrieved. Apriori information obtained from other geophysical methods or local geology or any other sources of reliable information can be incorporated in the system. That is how the Monte Carlo mesh is prepared. With so many constraints, very few models in one or two million trials will cross all the barriers and trickle down as the prospective solutions. For each parameter the upper and lower bounds are fixed i.e., $m_i^{max} \leq m_i \leq m_i^{min}$ is set for all the parameters. Using the random number generator a random numbers is taken to perturb the old model and the new model will be

$$m_i^{new} =\leq m_i^{min} + \alpha \left(m_i^{max} - m_i^{min} \right) \tag{17.193}$$

where $0 \leq \alpha \leq 1$. Larger the value of

$$N = \frac{m_i^{max} - m_i^{min}}{\Delta m_i} \tag{17.194}$$

finer will be the searching, better will be resolution and higher will be the computation time. The number of successful models to be taken is generally predefined. Probability density based on the error functions of each model should be tested and the model showing the highest probability should be taken as the most probable model. If several of peaks of the probability densities of the successful models are of the same order, then the mean and standard deviations of the best three models should be taken. It is very unlikely that all the twenty successful models will shows the same probability density of the error function. If that happens then more models should be allowed to trickle through the several hurdles. If they cross successfully, they become successful models. Interpreter decides how many successful models will he choose for further statistical analysis and decide to stop the inversion process. One main advantage in the Monte Carle method is its inherent simplicity. Press (1968), Wiggins (1969) used this approach for inversion of seismic data successfully.

17.15.3 Simulated Annealing

Simulated annealing is one of the approaches for global optimization. Like Monte Carlo inversion it is one of the random walk techniques, but with certain artificial intelligence. Kirkpatrick (1983) first proposed this approach of global optimization. It was later developed by Van Larhoven and Aarts (1983), Rothman (1985, 1986), Geman and Geman (1984), Oldenburg (1978), Sen and Stoffa (1995). This approach has certain similarity with the annealing process in chemical thermodynamics. The annealing is done to join two metallic plates or to seal the cracks in a metallic plate. When a metal is heated above its melting point and the molten matter is allowed to cool down slowly, it gets crystallized and the annealing is done properly without leaving any zone of weakness.

In simulated annealing we bring the concept of temperature and we call it a controlling parameter although temperature has nothing to do with an inverse problem. The energy function in thermodynamics is replaced by error function. This error function is also termed as cost function or energy function in global optimization. Gibb's probability density function in thermodynamics is

$$\mathrm{P}\left(\mathrm{E_j}\right) = \frac{\exp\left(-\frac{\mathrm{E_i}}{\mathrm{KT}}\right)}{\sum \exp\left(-\frac{\mathrm{E_i}}{\mathrm{KT}}\right)} \tag{17.195}$$

where K is the Boltzmon constant. In global optimization K is assumed to be unity because it does not have any role in iterative inversion process.

Difference between the observed field data $\mathrm{d^{obvs}}$ and synthetic data created from the model i.e., $\mathrm{d^{Pre}}$ generates the error function, defined as

$$\mathrm{E}\ (\mathrm{m}) = (\mathrm{d^{Obs}} - \mathrm{d^{Pre}})^{\mathrm{T}}(\mathrm{C_D^{-1}})(\mathrm{d^{Obs}} - \mathrm{d^{Pre}}) \tag{17.196}$$

where $\mathrm{C_D}$ is the data covariance operator, discussed in the section of weighted ridge regression. In simulated annealing the inverse problem starts with a very high initial temperature or the controlling parameter. In successive iteration, the temperature is lowered down slowly to make the Gibb's probability density function more and more sensitive. Many approaches of Simulated Annealing are now available in the literature, viz. (i) Metropolis algorithm (ii) Heat bath algorithm (iii) Fast Simulating Annealing (FSA) and (iv) Very Fast Simulated Annealing (VFSA). In different approaches the cooling schedules are different. Cooling schedules are the schemes for lowering down the temperature. The different formats for lowering down the temperature are as follows :

$$\text{(i)}\mathrm{T_K} = \mathrm{T_0}(0.99)^{\mathrm{K}} \text{ or } \mathrm{T_0}(0.98)^{\mathrm{K}} \text{ (ii) } \mathrm{T} = \frac{\mathrm{T_0}}{\mathrm{K}} \text{ (iii) } T = \frac{T_0}{\ln K} \tag{17.197}$$

etc. The number of model parameters to be taken depends upon the structure of a problem. For finite element and finite difference 2-D/3-D forward problems, a few elements are clubbed together to form one parameter, because of prohibitive computation time. Too much of coarse structuring of a model will eliminate many essential details of a subsurface tomography. So a judicious compromise is made between the computation time and the finer details to be retrieved from the data. For 1-D earth models, number of data points can be more than the number of parameters to make it an overdetermined problem. Each model parameter is assigned upper and lower bounds and the range $\left(m_i^{max} - m_i^{min}\right)$ is discretised M times. Higher the value of M finer will be the model resolution.

Metropolis Algorithm

Metropolis (1953) first proposed this algorithm in Simulated Annealing (SA). In that a small perturbation is given to model m_i to obtain a new model m_{i+1} such that error functions for the two models are respectively E (m_i) and E (m_{i+1}). Difference in energy level $\Delta E_i = (E(m_{i+1}) - E\ (m_i))$ is computed. If $\Delta E_i < 0$, the new model is accepted. If $\Delta E_i \geq 0$, the new model is accepted after examining the probability function $P = \exp(-\Delta E_i/T)$ where T is the temperature (controlling parameter) in an optimization problem. This is the basic point of the Metropolis algorithm. This step is repeated several times at each temperature till the temperature comes to zero or very low value. In Metropolis algorithm every parameter has finite probability of acceptance. Geman and Geman (1984) has shown that one heads towards the convergence of the problem and as the temperature or the controlling parameter is lowered down the probability peaks become sharper and sharper. The perturbations which reduces ΔE_i are accepted.

Heat Bath Algorithm

In heat bath algorithm, once number of parameters in an inverse problem is guessed, discretization in the parameter domain is completed choosing the upper and lower bounds for each parameter as discussed in the previous section. One value of each parameter is chosen at random using one of the random number generators within the respective upper and lower bounds of each parameter. Keeping the values of the second to the last parameter fixed, random search is made for all possible values of the first parameter. Search literally means, for each value of the first parameter and the randomly chosen other parameter values, the forward problems are solved, error functions and Gibbs probability density are computed and the model with highest probability is chosen as the preferable first parameter. Keeping now the first parameter fixed at the chosen value and 3rd to last parameters are fixed at the first randomly chosen values, the second parameter is varied from upper to the lower bounds by random choice and the parameter with highest probable value of m_2 is chosen. This process will continue till the nth parameter m_N is seen from the upper to the lower bounds by random choice. That completes only one iteration. Figure 17.12 shows the Similated Annealing tree for the 1st and 2nd iteration for one dimensional three layer DC resistivity problems. The second

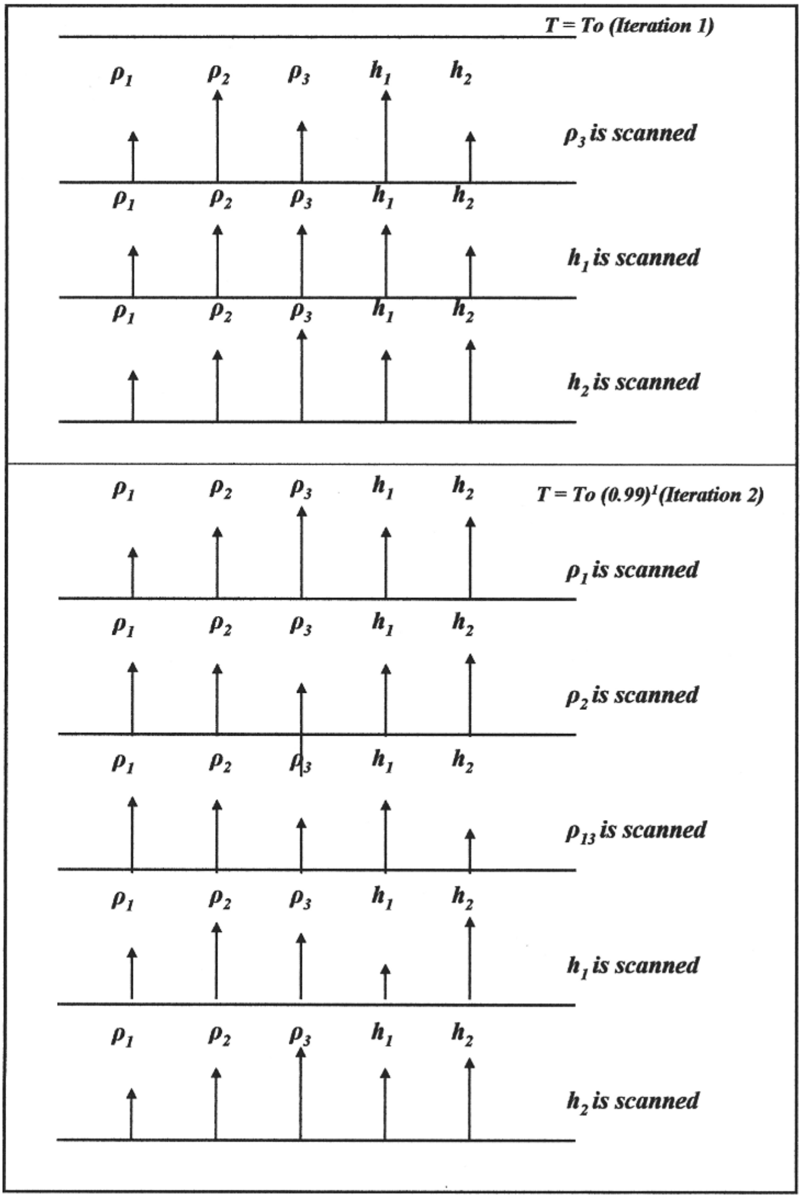


Fig. 17.12. Simulated annealing tree in heat bath algorithm

iteration starts with lower temperature with the second set of upper and lower bounds used for m_1 to m_N, found in the first iteration. This way the temperature is lowered slowly till the complete convergence of the inverse problem is achieved. In heat bath algorithm there is no rejection of models. 2000 to

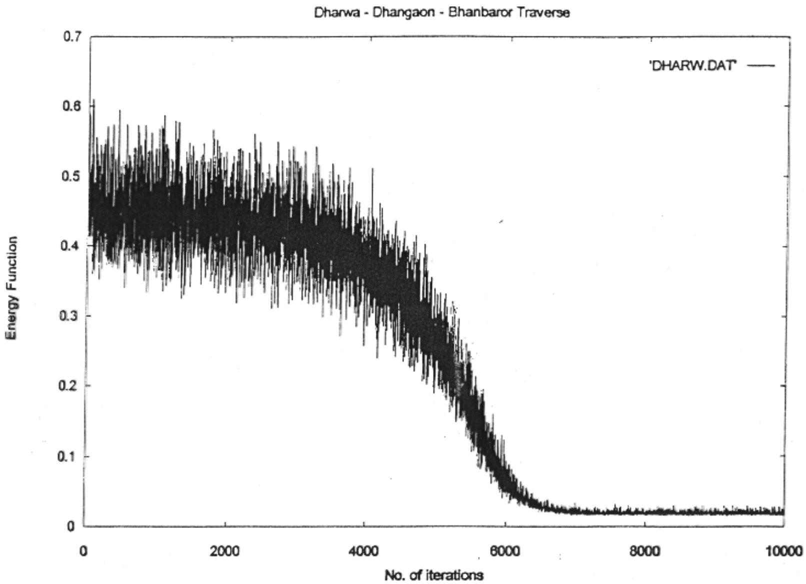


Fig. 17.13. Nature of convergence in VFSA inversion

10,000 iterations are needed for convergence. Because of random jump, the probability, of a solution, getting trapped in a local minimum pocket, is zero. Once a forward problem is solved, inverse problem is solved automatically. In global search, there is no mathematics in an inverse problem so stability is guaranteed. However computation time required to handle 10,000 iterations remains as a problem. Geman and Geman (1984) showed that a necessary and sufficient condition for convergence to the global minimum level for Simulated Annealing is given by the following cooling schedule

$$\mathrm{T}\,(\mathrm{K}) = \frac{\mathrm{T}_0}{\ln \mathrm{k}} \tag{17.198}$$

where T (K) is the temperature at iteration K.

Fast Simulated Annealing (FSA)

Szu and Hartley (1987) proposed a new algorithm known as Fast Simulated Annealing. This algorithm uses Cauchy like distribution which shows very sharp peak at lower temperature (Sen and Stoffa 1995). This Cauchy like distribution is a function of temperature and is given by

$$\mathrm{f}\,(\Delta \mathrm{m_i})\,\alpha \frac{\mathrm{T}}{\left(\Delta \mathrm{m_i^2} + \mathrm{T}^2\right)^{1/2}} \tag{17.199}$$

where T is the temperature or the controlling parameter to be lowered as per any specific cooling schedule. $\Delta \mathrm{m_i}$ is the perturbation in the ith parameter model with respect to the present model. Szu and Hartely showed that for choice of the perturbed model generation program, the cooling schedule for convergence is chosen as

$$\mathrm{T}(\mathrm{K}) = \frac{\mathrm{T}_0}{\mathrm{K}} \tag{17.200}$$

where K is the number of iteration. The nature of Cauchy type distribution is very much different from the Gaussian type distribution. At lower temperature, Cauchy type peaks can be used to find out the highest probability values with greater sensitivity.

Very Fast Simulated Annealing (VFSA)

Ingber (1989, 1993) did several modifications of Simulated Annealing and proposed a new algorithm known as Very Fast Simulated Annealing (VFSA) Ingber proposed a new probability distribution for model generation such that a slow cooling schedule is no longer required. The special features of this algorithm are (i) each parameter can have different types of discretization and different degree of perturbation (ii) the temperature or the controlling parameter may be different for different parameters (iii) new probability distribution avoids Cauchy type distribution but avoids slow cooling and quick convergence of an inverse problem. Thus it became a very effective tool to handle geophysical inverse problems and is used by many geophysicists to solve 2D/3D inverse problems. A brief sketch of the Ingber's (1993) VFSA model is

$$\mathrm{m}_{\mathrm{i}}^{\mathrm{k}+1} = \mathrm{m}^{\mathrm{k}} + \mathrm{y}_{\mathrm{i}}\left(\mathrm{m}_{\mathrm{i}}^{\max} - \mathrm{m}_{\mathrm{i}}^{\min}\right) \tag{17.201}$$

where $\mathrm{m}_{\mathrm{i}}^{\mathrm{k}}$ is the ith model parameters after the iteration K and this value is within the upper and lower bounds i.e., $\mathrm{m}_{\mathrm{i}}^{\min} \le \mathrm{m}_{\mathrm{i}}^{\mathrm{k}} \le \mathrm{m}_{\mathrm{i}}^{\max}$.$\mathrm{m}_{\mathrm{i}}^{\mathrm{k}+1}$ is the model parameter after $(\mathrm{K}+1)$ iteration and it is also within the upper and lower bounds i.e. $\mathrm{m}_{\mathrm{i}}^{\mathrm{k}+1}$ is also within $\mathrm{m}_{\mathrm{i}}^{\min} \le \mathrm{m}_{\mathrm{i}}^{\mathrm{k}+1} \le \mathrm{m}_{\mathrm{i}}^{\max}$. The special parameter Y_{i} is defined by (Ingber 1993) as

$$\begin{aligned}\mathrm{g}_{\mathrm{T}}(\mathrm{y}) &= \prod_{\mathrm{i}=1}^{\mathrm{NM}} \frac{1}{2\left(|\mathrm{Y}_{\mathrm{i}}| + \mathrm{T}_{\mathrm{i}}\right) \ln\left(1 + \frac{1}{\mathrm{T}_{\mathrm{i}}}\right)} \\ &= \prod_{\mathrm{i}=1}^{\mathrm{NM}} \mathrm{g}_{\mathrm{T}_{\mathrm{i}}}(\mathrm{y}_{\mathrm{i}}).\end{aligned} \tag{17.202}$$

Equation (17.202) has the following cumulative probability as

$$\mathrm{G}_{\mathrm{T}_{\mathrm{i}}}(\mathrm{y}_{\mathrm{i}}) = \frac{1}{2} + \frac{\mathrm{Sgn}(\mathrm{y}_{\mathrm{i}}) \ln\left(1 + \frac{|\mathrm{Y}_{\mathrm{i}}|}{\mathrm{T}_{\mathrm{i}}}\right)}{\ln\left(1 + \frac{1}{\mathrm{T}_{\mathrm{i}}}\right)}. \tag{17.203}$$

Thus a random number u_{i} drawn from an uniform distribution U (0, 1) can be mapped into the above distribution with the following formulae

$$Y_i = \mathrm{Sgn}\left(u_i - \frac{1}{2}\right) T_i \left[\left(1 + \frac{1}{T_i}\right)^{|2u_i - 1|} - 1\right]. \tag{17.204}$$

Distribution of global minimum can be statistically obtained using the cooling schedule

$$T_i(k) = T_{0_i} \exp\left(-C_i K^{1/NM}\right). \tag{17.205}$$

T_i is the controlling parameter for the ith model parameter and T_{0_i} the initial controlling parameter in the model. The value of u_i randomly varies within $+1$ to -1. Sgn denotes the sign of the expression. So model perturbation has some relation with the probability distribution of the respective parameters. Generally Metropolis criterion is used here also for selection of the models automatically. In global search, there is no mathematics in an inverse problem so stability is guaranteed. However computation time required to handle 10,000 iterations may be more as mentioned. Geman and Geman (1984) has shown the conditions for convergence of the model. The temperature is used to decide on the acceptance criterion. VFSA also has the flexibility of changing the cooling schedule. Figure (17.13) shows the nature of convergence in VFSA applied to two dimensional DC resistivity problem. Convergence in an inverse problem starts after several thousand iterations are over.

17.15.4 Genetic Algorithm

Introduction

Genetic Algorithm is one of the powerful tools for global optimization. It is also based on the principle of random walk in the parameter space. GA has artificial intelligence like SA and can handle strongly nonlinear problems. GA was discovered by Holland (1977) and was brought to the present stage of development by Goldberg (1989) and Davis (1991). The procedure for genetic algorithm has some similarity with the biological evolution and survival for the fittest. Like Monte Carlo inversion, and Simulated Annealing, GA does not need any derivatives or curvature information. Therefore once the forward problem is solved, inverse problem can be solved automatically because the entire exercise is to choose some models, compute synthetic data, compare d^{Pre} with d^{Obs}, compute the cost function or error function and go on choosing the better and better models through certain guide lines in deciding on the acceptance criterion.

Unlike other approaches, GA works with a population of models. Larger the number models to start with more efficient will be the global search in the parameter space. The basic guidelines for genetic algorithms are divided into the following nine steps viz.,

(i) Selection of the even number (N) models from a large set of models.
(ii) Randomly generate N/2 number of pairs.

(iii) Discretize the of model parameters in one, two or three dimensional domain. In that process a continuous domain gets discretized.
(iv) Discretize each model parameter fixing the upper and lower bounds and ρ_i be the number of models parameters for the parameter m_i. Here m_i is the i th model parameter and ρ_i is the number of discretized values between the upper and lower bounds. Larger the value of ρ_i finer will be model resolution. ρ_i may be same or different for different parameters of the model.
(v) Binary coding is done for each model parameter from upper and lower bounds in O'S and 1'S and bit strings or chromosomes in GA language are generated.
(vi) Generation of Concatenated strings.
(vii) Generation of randomly chosen pairs and crossovers with reasonably high crossover probability are the next steps.
(viii) Mutation with reasonably low mutation probability is done next to retain certain diversity in the models or to remove any bias.
(ix) Model updating or survival for the fittest test is done next. This is the last step where the best models are chosen from the population. The entire genetic operation is completed at this stage and only first iteration of the Genetic algorithm is over at this stage. With the updated models the second iteration starts. The entire operation continues till the complete convergence is obtained upto a specified limit.

Selection

Since GA starts with a population of models instead of a single model, an arbitrarily 200 or 300 models are randomly generated using the random number generator and the energy function or cost function or error function

$$E(m) = \frac{1}{NS}\left[d^{Obs} - d^{\mathrm{Pr}\,e}\right]^t \left[d^{Obs} - d^{\mathrm{Pr}\,e}\right] \tag{17.206}$$

is computed where d^{Obs} and d^{pre} as defined earlier are observed and predicted field data. t is the transpose and NS is the number of data points. Gibbs probability distribution

$$P(m_{ij}) = \frac{\exp(-Em_{ij})/T}{\sum\limits_{j=1}^{m} \exp(-Em_{ij})/T} \tag{17.207}$$

is computed. T the temperature or the controlling parameter is also used these days for model search the way it is used in simulated annealing. In case we decide to work with 50 initial models, we select 50 models with relatively higher probabilities from the arbitrary population of 300 models. For one dimensional problems, 10 to 20 models are good enough to start with the Genetic process. For two dimensional problem, initial population of 50 to 100 should be appropriate. For three dimensional problem 100 to 1000 models should be chosen to start with. One should remember that larger the number of initial, models, higher will be the computation time and more efficient will be the genetic algorithm process.

Discretization

Double discretization of the model space is done in two stages. In the first stage, the discrete set of model parameters are assumed for an earth model, where the physical property might be varying continuously. The second set of discretization comes from fixing the upper and lower bounds for the model parameters i.e.
where

$$\mathrm{m_i} = \mathrm{m_i^{min}} + \mathrm{J}\ \Delta\ \mathrm{m_i} \tag{17.208}$$

where J varies from 0 to $\mathrm{N_i}$ and

$$\Delta \mathrm{m_i} = \frac{\mathrm{m_i^{max}} - \mathrm{m_i^{min}}}{\mathrm{N_i}}. \tag{17.209}$$

So for a finite number of models N having M parameters the number of model will be

$$\mathrm{N} = \prod_{\mathrm{i}=1}^{\mathrm{M}} \mathrm{N_i}. \tag{17.210}$$

Parameter Coding

Q number of models are binary coded with 0's and 1's. Binary coding is done to represent a model parameter, such that all the bits become zero at the minimum value of the parameter and all the bits become one at the maximum value as shown in the Fig. (17.14). Depending upon the level of resolution needed and the discretization done, a parameter can be a 3, 4, 5 ...... 9 bit string for one dimensional problem. for two /three dimensional problem, the parameter can be of 15 to 20 bit string or even more. These 'bits' are called 'genes' which can take a value of '0' or '1' called alleles. For a 7 bit string 2^7-1 discrete values of the model parameters will be obtained. For two dimensional problems 15 bit strings have been used with success. These 7 bit, 9 bit or 15 bit strings are called chromosomes. One chromosome is generated for one parameter. They are attached one after the other to obtain binary coded concatenated strings. Larger the number of bits in a chromosome, longer will be the concatenated string, higher will be the computation time. For Two/three dimensional problems in finite difference or finite element forward problem, the concatenated strings will be very long because each element or a group of elements will form one parameter.

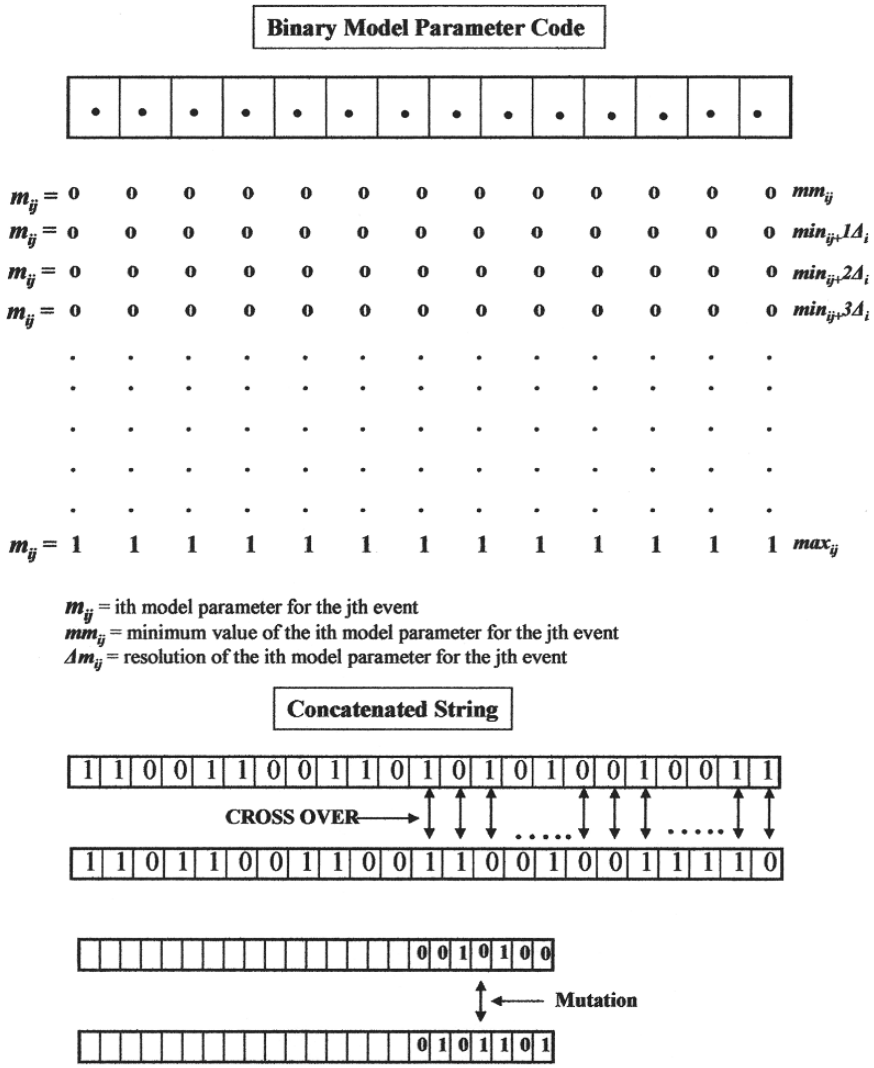


Fig. 17.14. Genetic Algorithm structure: (**a**) Parameter coding and preparation of concatenated string (**b**) selection of pairs(**c**) crossover (**d**) mutation

Crossover

These binary coded strings (Fig. 17.14) are randomly paired to generate Q/2 pairs of models. At this stage the crossover starts at randomly chosen crossover point with high crossover probability. Crossover can be a single point or a multipoint crossover. The exchange of bits or 'genes' take place to all the bits on the right of the crossover point (Fig. 17.14). The bits, left to the crossover point, remain unaltered. For multipoint crossover also, the crossover, is done with randomly selected crossover points for each chromosome and for each parameter. But the crossover or floor crossing of the bits take place only to the bits right of the crossover points. For two and three dimensional problems multipoint crossover is advisable. For one dimensional problems single point crossover works well. Genetic Algorithm is a very active area of research. Therefore these procedures are also changing.

Mutation

After crossover is over, just to give little diversity and to remove any kind of bias, mutation is done with low mutation probability. Here floor crossing of one or two bits is done and allowed the system to converge. (Fig. 17.14). High mutation probability will delay the convergence and may lead to Monte Carlo type inversion. Mutation probability is always kept low.

Model update

This is the last genetic operation. This operation is also termed as the survival for the fittest test. Three or four different approaches are available in the literature regarding the model update. In this chapter Stochastic Remainder Selection Without Replacement (SRSWR) as proposed by Goldberg (1989) and as it is used by the author for inversion of one or two dimensional problem is discussed .

Once $\frac{Q}{2}$ pairs of offspring models are generated from $\frac{Q}{2}$ pairs of parents. Each of these modes had to pass through the survival for the fittest test. The stronger models will survive and strengthen their position. The weaker models are eliminated from the contest.

Each model has its own objective function or error function. The probabilities of these new generation of populations are multiplied by the number models present. Some of these fractional probabilities have integer numbers. It can be 1 or 2 or 3. Some of the members do not have integer number. In the selection for next generation of new models, all the models having integer number will go first. If the model no.1 has the multiplied probability 2.5 (say), then the model no.1 will go twice. If it is 3.4, then it will go thrice. Once all the models having multiplied probability greater than 1 are taken care of, the fractional part of the multiplied probabilities are taken. Higher fractional parts are given preference to get a berth in the next generation models with which the next generation genetic process will start. If a particular model had multiplied probability 2.8 (say), then if gets two berths for two integer numbers and one berth for higher fractional number. So this model gets three berths to start the next genetic process. These are offspring models. Parents models are destroyed after the crossover is completed.

If we start with 50 initial models, there will be 50 berths in the model update. Who will occupy these berths and how. The berths are given to the models with higher multiplied probabilities. The stronger models may occupy

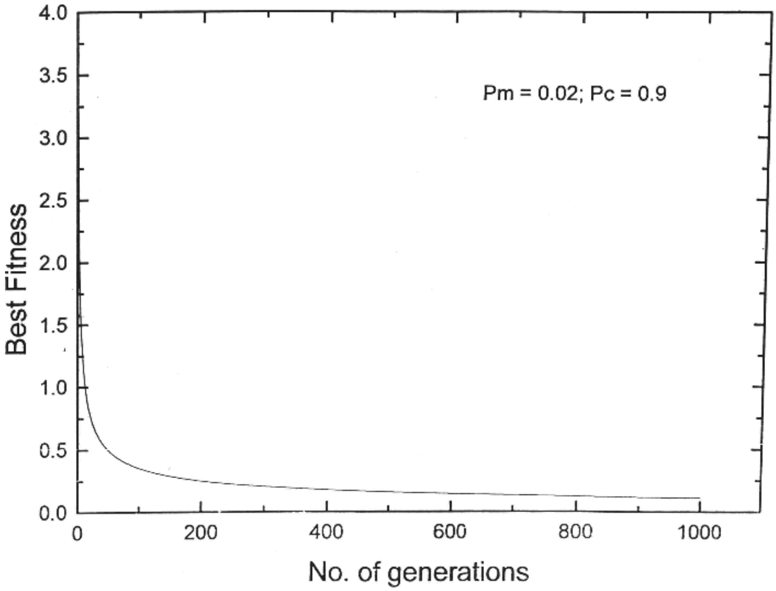


Fig. 17.15. Nature of convergence in two dimensional Genetic algorithm optimization

3 or 4 berths. Weaker model have to vacate their berths. Weaker model means, the models with less probability or higher error function or higher discrepancy between the observed and predicted data.

Model updating continues this way in successive iterations till one gets the convergence (Fig. 17.15). Ultimately the stronger models only will occupy all the berths. Since the parameter values change within a model in successive crossover, therefore near convergence the parameter values come quite chose to each other in different models near global minimum point. The number of models also reduces in successive iteration. Self learning process or the artificial intelligence takes the model search in the direction of high probability density models and makes it a powerful tool.

17.16 Neural Network

17.16.1 Introduction

Neural Network as the name suggests, mimics the functioning of brain. Artificial Neural Network (ANN) originated to simulate the complex behaviour of the brain initially. Later it proved to be a versatile tool to be used in many branches of science and engineering viz., pattern recognition, global optimization, noise reduction and classification, data compression etc.

These are some of the areas where artificial neural network can be and is being used. This subject as such is vast and it is outside the scope of this book. Here only a few basic points regarding the use of neural network for global optimization of geophysical data will be highlighted very briefly.

There are millions of structure elements within the brain called neurons. Millions of synapses are formed to connect these neurons so that the brain becomes a parallel computing system. Synapses are also elements of structural units, which are responsible for the interaction between the neurons. These neurons with variable synaptic connection functions as information processing units in the human brain. In neural network, the structure is made with artificial neuron interconnected to each other. These artificial neural networks are designed to function the way the brain does a particular job. Human brain is an immensely superior system. ANN networks are trained to a particular job and it tries to do that job only. Hopfields (1975) procedure is discussed briefly.

A Neural network can be defined as follows: A neural network is a system composed of many simple processing elements operating in parallel whose function is determined by network structure, strengths of the connecting links. The processing is performed at the nodes or neurons. In global optimization problem, the structure of ANN consists of an input layer, one or more hidden layers and an output layer. The number of hidden layers can vary from 1 to n where the value of n will be problem dependent. For geophysical inverse problems the input layer will contain the data and the output layer contains all the models parameters obtained. Each of the nodes of the input layer is connected with each of the nodes of the first hidden layer. Nodes of the input layers are not connected to each other.

Each of the nodes of the output layer is connected to each of the nodes of the nth hidden layer. That is how a neural net is constructed. The number of data points of an inverse problem will dictate the number of nodes in the input layer and the number of parameters to be retrieved from a set of data will be equal to the number of nodes in the output layer. How many hidden layers and how many nodes, required in each hidden layers, are decided through repeated experiment while trying to solve a particular problem in a particular field. Number of nodes in a hidden layer and number of hidden layers in a problem are highly variable parameters. Each of these connections between the nodes is assigned a certain weightage. These weights are initially selected at random. Then the learning process or the training process starts. It starts with known synthetic models for which both inputs and outputs are known. The message about the discrepancy between actual output and the outputs obtained with randomly chosen synaptic weights are back propagated through preceptors and the weights are changed in successive iterations till the actual output and computed output becomes more or less the same with already prescribed minimum error level. This learning process is done several times with noise free synthetic data and then with data mixed with Gaussian noise.

Say for one-dimensional direct current resistivity or magnetotelluric problems, the neural net is trained for 2 to n hidden layers. The same program trained for different models can be kept separately and used accordingly. The artificial neural network acquires knowledge through this learning process that involves modification of the connecting weights. The acquired knowledge is stored in the form of modified weights. Information processing occurs at the nodes simultaneously and the weights are gradually modified till the error function or cost function is modified. Once a neural net is trained for a known system, it is now ready to take the field data or experimental data to find out the model. Neural net can function only for which it is trained. A net trained for some other models or for some other purpose will generate wrong results. Therefore the interpreter has to choose the properly trained package. Choice of the forward model (ID or 2D or 3D) will dictate the proper choice of the package. Except random choice of the synaptic weights at the beginning of the training or learning process, there is nothing random is subsequent stages. The functioning of the neural network is similar to gradient minimization in an inverse problem.

17.16.2 Optimization Problem

The procedure is as follows:

Let the desired output values at different nodes are $m_1^{true} \ldots\ldots\ldots\ldots\ldots\ldots m_M^{true}$ for a known system where as the actual computed model parameters for the known synthetic data are $m_1^{\mathrm{Pr}\,e} \ldots\ldots\ldots\ldots\ldots\ldots m_M^{\mathrm{Pr}\,e}$ which passed through the randon weights. The discrepancy between the actual and observed values of the model parameters generate the error function

$$\varepsilon_M = \sum_{i=1}^{M} \left(m_i^{true} - m_i^{\mathrm{Pr}\,e}\right)^2. \tag{17.211}$$

One obtains this $m_1^{\mathrm{Pr}\,e}$ from all the connecting paths from the hidden layer (s). Multi channel input comes from the hidden layers to the output layer to generate a single output at a particular node. Since neural network is a parallel processing system, the same process continues at all the nodes with different interactions with the nodes or neurons of the hidden layer because the interacting links and the synaptic weights are different. The processing occur at the say j th node where the signals d_j^h are coming from the different input nodes. Each of these input signals are multiplied by their weights to generate the output function.

$$m_j = \sum_{i=1}^{n} W_{ji} d_i^h. \tag{17.212}$$

These output values m_j pass through a nonlinear activation function, generally a sigmoidal functions to produce the output $\mathrm{M_j}$. Here

$$M_j = f_i\left(\sum W_{ji} d_i^h\right). \tag{17.213}$$

Superscript 'h' stands for the hidden layer. If input signals (data) are directly connected with the output signals (model) then d_j^h will be only d_j. Generally one or more than one hidden layer (s) is / are necessary to generate the desired output by gradual and systematic changes of weights. The output $O_i = f_i\left(M_j\right) =$

$$f_i\left(\sum W_{ji} d_i^h\right). \tag{17.214}$$

This function maps the total input from all the nodes of a hidden layer to generate the output, i.e., once the j th parameter output m_1^{Pre} is obtained, it is compared with the true output m_1^{true}. This error signal $\left(O_j^{true} - O_j^{obs}\right)$ is back propagated through perceptrons. Perceptron is an active layer, which provides the learning process mechanism to the artificial neural network. The learning process starts with the adjustment of weights at all sectors. The error signals are then transmitted back ward from the output layer to each node or neuron in the intermediate layer that contribute directly to the output layer and the contribution comes from each node of the intermediate layer. The error signal is also proportional to the fraction of the contribution made by each node. This process repeats in each subsequent hidden layer till the back propagation of information reaches the first hidden layer.

Briefly the procedures for training the network are as follows: (a) apply the input synthetic data as a data vector in the input layer, (b) compare the correct output with the output obtained with random selection of weights in all the synaptic connections between neurons when the net is getting trained with a synthetic model where both the inputs and outputs are known, (c) the errors in each sector is computed, (d) one has to find out at this stage whether the weights should be increased or decreased to reduce the error and the quantum of change to be made in each weight in successive iteration, (f) these corrections are then applied to each weights, (g) the whole process is repeated till the training vector reaches a state where difference between the true output and the observed output is minimized. The rate of change also gets minimized at this stage, (h) the training is partially completed for one model say a simple two layered earth, (i) separate neural nets be preferably trained for 1D, 2D and 3D models, (j) the nets should be trained several times for 2 to n layered earth models with synthetic data for 1D models. Once the net is trained for synthetic data, Gaussian and random noises are added to the input data and the network is trained again to take care of noise in the field or experimental data. (l) Thus the training process is complete. So far as the geophysical inverse problem goes, the field data are inserted at the input layer and the model parameters are taken from the output layer.

Theory: Let the input and output vectors are respectively D and M where D is the data space and M is in the parameter space. Here M = f(D) where D stands for n real variables i.e.,number of data points and M contains m real variables i.e., model parameters. The input vector (Fig. 17.16)

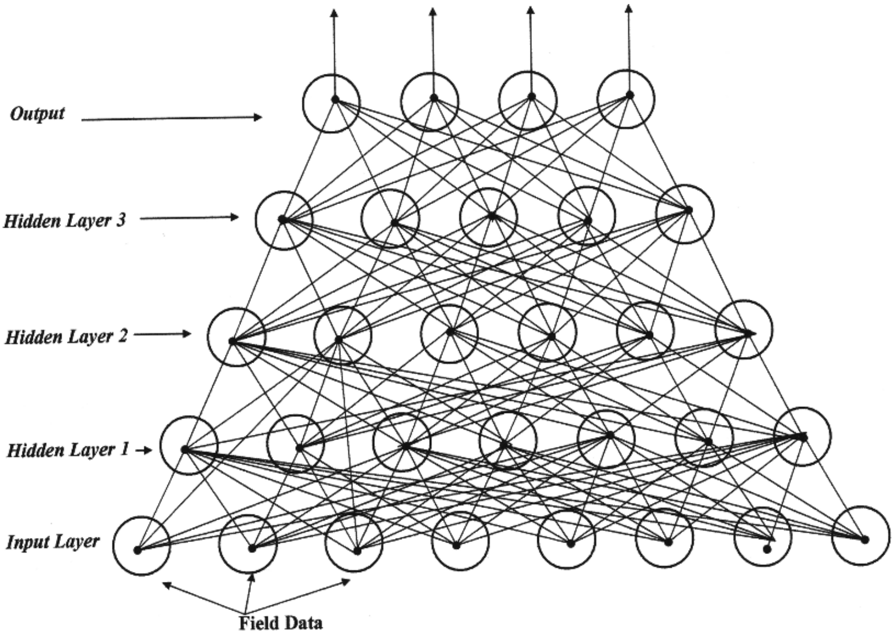


Fig. 17.16. Neural Network Optimisation Problem with input, output and hidden layers

$D_N = (D_{y_1}, D_{y_2} - - - - - - - - - - - - - - - - D_{y_n})$ is applied to the input layer. This information gets distributed in all the wings of the hidden layer. A particular node in the hidden layer will have the input.

$$D_N = \sum_{i=1}^{n} W_{ji} D_i^h + Z_j^h \tag{17.215}$$

where W_{ji}^h is the weight on the connections from the i th input unit and Z_j^h is the threshold value. Superscript h stands for the hidden layer.

$$K_{y_j} = f_j^h \left(X_{yi}^h \right) \tag{17.216}$$

for Y th input node to J th output node for the hidden layer h. Therefore, the equation for the output nodes are

$$X_{yg}^0 = \sum_{j=1}^{L} W_{gj}^0 k_{y_i} + Z_g^0 \tag{17.217}$$

and

$$m_{yg}^0 = f_g^0 \left(x_{yg}^0 \right) \tag{17.218}$$

where the superscript ‘O’ stands for the outputs layer, L is the number of output nodes and information comes from the g th hidden node from the last hidden layer to y th node at the output layer. W's are the weights connecting the g th node of the hidden layer to y th node of the output layer. They are chosen randomly to start with. For geophysical inverse problem, we train the net with a known model for which both input and output are known because the mathematical solution of the forward problem is known. Therefore the output values are m^{true} and the output values obtained from the random choice of the weights are $m^{\Pr\ edicted}$ or $m^{\Pr\ e}$. The error is

$$\varepsilon_m = \sum_{g=1}^{M} \left(m_{yg}^{true} - m_{yg}^{\Pr\ e}\right)^2 = \sum_{g=1}^{M} \delta_{yg}^2. \tag{17.219}$$

The back propagation or errors start at this stage and the signal reaches to all the connecting links in the neural net and the whole iterative process is continued to minimize the rate of change of weights in all the sectors till the true output and the observed output becomes more or less the same. The negative gradient of the error can be written as

$$-gradient\ \varepsilon_m = \left(m_{yg}^{true} - m_{yg}^{\Pr\ e}\right) f_g^0\left(X_{yg}^0\right) k_{yj}. \tag{17.220}$$

The magnitude of the weight change is proportional to the negative gradient. The weights in the output layers are modified as

$$W_{gj}^0\left(e+1\right) = W_{gj}^0\left(e\right) + \delta W_{gj}^0\left(e\right) \tag{17.221}$$

where

$$\delta W_{gi}^0(e) = \alpha(m_{yg}^{true} - m_{yg}^{\Pr\ e}) f_g^0(x_{yg}^0) k_{yj}. \tag{17.222}$$

Here the factor α is the learning rate parameter and its value varies from 0 to l. Generally nonlinear sigmoidal output functions are used where

$$f_g^0\left(x_{yg}\right) = \left(1 + e^{-x_{ig}}\right)^{-1} \tag{17.223}$$

e is the iteration number.

The weight adjustment is continued till the error is minimum. The same net is tested mixing gaussian noise.

Once the learning process is over, the field data can be inserted as an input and the outputs are obtained as model parameters. So far ANN could be used successfully for geophysical 1D models. This tool is gradually getting established for solution of 2-D/3-D inverse problems.

17.17 Joint Inversion

Joint inversion is a topic of inverse theory and it is an art of retrieving maximum possible information about the earth's structure from more than one set of data collected on the surface of the earth / in the air/ in a borehole. The data may originate from the same or different types of excitation as well as for the same or different types of physical properties of the earth. The inverse theory in geophysics, applied for a single set of data are also applicable for joint inversion. It is now more or less an established fact that joint inversion improves the resolution of the subsurface imaging because of greater information content in the several sets of data Joint inversion can be done with several sets of data together.

Different approaches do exist in joint inversion technology. The entire domain of electromagnetics deal with complex quantities. Therefore, all observations have either real and quadrature components or amplitude and phase. In Magnetotellurics we get apparent resistivities and phases. Apparent resistivities and phases are jointly inverted always for any kind of interpretation of MT data. Here we get two sets of data from the same excitation.In the case of MT/GDS/Gravity/Magnetic fields we deal with naturally existing fields. Joint inversion can be between (i) Magnetotellurics and DC resistivity Harinarayana (1992), Sasaki (1989), Dobroka et al (2001), Vozoff and Jupp (1975), Jupp and Vozoff (1977b) (ii) Magnetotellurics and Seismic Refraction Dobroka et al (1991) (iii) MT and Seismic Reflection (iv) Magnetotellurics and Geomagnetic Depth Sounding Ritz and Vassel (1986) (v) Gravity and Magnetics Zehen and Pous (1993), Gallardo-Degado et al (2003) and Bosch and Mcgaughey (2001), (v) Magnetotelluric and Magneto variational sounding (vi) Surface waves and body waves in seismics (vii) different surface waves (viii) Dc resistivity, MT, TEM, EMAP Meju et al., (1999) (ix) Gravity and Seismics (Ursin et al, 2003) (x) Magnetics and Seismics (xi) Resistivity and Induced Polarisation Roy and Rathi (1988), Roy et al (1995), Pelton et al (1978) and Bhattacharyya et al (2003), (xii) Electromagnetics and Magnetics Benech et al (2002), Seguin (1975) (xiii) Transient Electromagnetics and Magnetotellurics (Meju (1996), (xiv) Electromagnetics and DC resistivity Verma and Sharma (1993), Sharma and Kaikkonnen (1999), Raiche et al (1999), (xv) Seismic Reflection and Refraction Habro et al (2003), Zhang et al (1998), (xvi) Seismic and DC resistivity Dobroka (1991), (xvii) local and regional teleseismics (Federica et al (2003), (xviii) VLF and VLF_R Kaikkonen and Sharma (1998) and (xix) tomographic inversion. Therefore all possible combination of joint inversions are possible. Although there are some apprehensions in different group of scientists regarding feasibility and credibility of joint inversion based on data originated from different physical properties. The important question raised is whether one is allowed to frame a large sensitivity matrix based on different sets of data of different physics origin.In reality it is found that it works. Specially when we go for global optimization, the sensitivity matrix does not exist and the technology is free from this issue of mixing of information in the sensitivity matrix.

Joint inversion can be done in three to four different ways.First approach is the preparation of the sensitivity matrix for different sets of data from same or different fields together. Since the range and nature of variation of different sets of data may be widely different, regularization of the sensitivity matrix is of primary importance (Jupp and Vozoff (1975)) Different sets of data may have different regularization parameters before they are taken in the same sensitivity matrix. Second approach is centered around inclusion of global optimization techniques where sensitivity matrix is not required at all. Different sets of data taken for joint inversion may or may not go through the same global optimization technology. One set of data may go through Very Fast Simulated Annealing, The second set may go through Genetic Algorithm and the third set may go through the path of Artificial Neural Network or Monte carlo Inversion. Since the binding force of joint inversion is missing here therefore the freedoms of the interpreters are more here.

The third approach is the interactive inversion where two or three sets of data are inverted one after another in one iteration. Second iteration starts with the modified values of all the sectors. Here the data are collected from the same excitation and one set of data are genetically connected with the other. Different sectors gradually head towards convergence. Interactive inversion also remains free from controversy on the issue on whether we are permitted to bring seismic, em and gravity data under one umbrella and mix up.

A series of workers have worked on joint inversion with the sole intension of resolving the subsurface with higher degree of clarity. Almost all possible combinations of using geophysical tools for mapping the subsurface are being tried to assess quantitatively or at least semiquantitatively the improvements in clarity of images of subsurface with some additional finer details.

Hering et al (1995) did joint inversion of both dispersive Love and Rayleigh waves on one hand and joint inversion of dipole-dipole, Schlumberger and two electrode sounding data on the other and could achieve better resolution and finer details of the subsurface. Li and Oldenburg (2000) did joint inversion of surface and borehole three component magnetic data and could prepare a better 3D magnetic susceptibility map of the subsurface with better depth control. Li-Yun Fu(2004) gave an example of interactive inversion scheme with surface and borehole seismic data for better preparation of an acoustic impedance model. Gallardo-Delgado et al (2003) did a joint inversion of 3D gravity and magnetic data and have shown that ambiguity in gravity and magnetic interpretation gets reduced considerably. Dobroka et al (2001) have claimed better resolution of the subsurface by joint inversion of magnetotelluric and DC resistivity data. Vozoff and Jupp (1975)and Vozoff (1975), Verma and Sharma (1993) have shown that joint inversion of magnetotelluric and direct current resistivity data can resolve thin bed with greater degree of clarity. Even the problem of equivalence and suppression in geoelectrical modeling can be handled more efficiently by joint inversion of MT and DC resistivity data. Benech et al (2002) have shown that even by joint inversion of electromagnetic and magnetic dada one can prepare the in depth magnetic susceptivility map of the area. Raiche et al (1985) have shown that by joint inversion of transient electromagnetics and DC resistivity, layer parameters can be estimated with better degree of clarity. Joint error function. for seismic refraction travel time and magnetotelluric apparent resistivity and phase is given by

$$\varepsilon = \left[\frac{1}{ns} \sum_{i=1}^{nt} \left(\frac{t_i^o - t_i^c}{t_i^o} \right)^2 + \frac{1}{nt} \sum_{i=1}^{nt} \left(\frac{\varphi_i^o - \varphi_i^c}{\varphi_i^o} \right)^2 + \frac{1}{nt} \sum_{i=1}^{nt} \left(\frac{\rho_i^o - \rho_i^c}{\rho_i^o} \right)^2 \right] \tag{17.224}$$

t_i^o and t_i^c are the observed and computed travel time, φ_i^o and φ_i^c are the observed and computed phases, while ρ_i^o and ρ_i^c are the observed and computed apparent resistivities, respectively, ns is number of seismic stations and nt is the numbers of time periods at which MT field recording is made.

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List of Symbols

A. Symbols used to Represent more than one Physical Entity in Different Chapters

ρ	(i) Density of fluid (ii) Volume density of solid (iii) Volume density of charge (iv) Specific resistivity of a medium (v) Resistivity (vi) Radius Vector, (vii) Distance along the radial direction
σ	(i) Surface density of mass (ii) Electrical conductivity
G	(i) Universal Gravitational Constant (ii) Greens Function (iii) Linear Operator
g	(i) Accelaration due to Gravity, (ii) Gravity Field (iii) Nonlinear Operator
λ	(i) Linear Density of Mass (ii) Wave Length (iii) Integration Variable (iv) Lagrange Multiplier (v) Eigen Value (vi) Direction of Magnetic polarisation
F	(i) Field (ii) Vector Potential (iii) Lorentz force (iii) Magnetomotive Force
∇	(**i**) Differential Operator (ii) Gradient
$\vec{A}$	(i) Vector (ii) Vector Potential
a	(i) Scalar (ii) radial distance
T	(i) Time period (ii) Temparature (iii) Tensor (iv) Conductivity Matrix
I	(i) Current (ii) Modified Bessel's Function ($I_{n,0}$) (iii) Identity Matrix (iv) Identity Operator (v) Idem Factor (vi)Inclination of the Magnetic Field

J	(i) Current density (ii)Bessel's Function($J_{n,0}$) (iii) Jacobian Matrix
K	(i) Modified Bessel's function($K_{n,0}$) (ii) Reflection Factor (iii) Complete Elliptic Integral of the first kind (iv) Magnetic Susceptibility(v) Geometric Factor
k	(i) Reflection factor (ii) Modulus of Elliptic function and Integral (iii) Wave Number
L	(i) Differenrial Operator (ii) Self Inductance
$\vec{P}$	(i) Dipole Moment (ii) Poynting Vector
D	(i) Displacement Vector (ii) Declination of the Magnetic Field
M	(i) Mass (ii) Magnetic Moment
C	(i) Conductance (ii) Capacitance
m	(i) Mass (ii) Magnetic Pole Strength (iii) Model Parameters in a parameter space
ω	(i) Angular Frequency (ii) Solid Angle
θ	(i) Angle (ii) One axis in spherical Polar Coordinates(polar angle)
ψ	(i) Azimuthal Angle (ii) Electrostatic Flux (iii) Magnetic Flux
B	(i) Magnetic Induction (ii) vector
q	(i) Charge (ii) Volume Density of Charge
R	(i) Resistance (ii) Distance

B. Symbols used to Represent one Parameter

ϕ	*Potential*
b	Scalars
$\vec{i}, \vec{j}, \vec{k}$	*UnitVectors*
V	Velocity
V	Volume
$\nabla.$	Divergence of a vector
$\nabla\times$	Curl of a vector
σ_{ij}	Electrical conductivity tensor

$\sigma(d, m)$	Aposteriori Joint Probability Density Function
σ_M (m)	Marginal Probability Density Function
σ_{xy}	Electrical conductivity in an anisotropic medium
$\ddot{E}$	Electric Field
E_x, E_y and E_z	Electric field Components along the x,y and z directions In Cartisian Coordinate System
E_r, E_ψ and E_z	Electric Field Components along the r, ψ and z directions in Cylindrical Polar Coordinate System
E_R, E_θ, and $E\psi$	Electric Field Componentss along the R,θ and ψ Directions in Spherical Polar Coordinates
$\vec{H}$	Magnetic Field
H_x, H_y and H_z	Magnetic Field in the x,y and z directions in cartisian coordinates
H_r, H_ψ, and H_z	Magnetic Field along the r,ψ and z directions In cylindrical polar coordinate system
H_R, H_θ, and H_ψ	Magnetic Field along R,θ and ψ directions in Spherical Polar Coordinate System.
$\vec{a}_{R,}\vec{a}_{\theta,}\vec{a}_\varphi$	Unit Vectors in the three Mutually
	Orthogonal Directions in Spherical Polar Coordinates
$\vec{a}_{r,}\vec{a}_{\varphi,}\vec{a}_z$	Unit vectors in the three Mutually
	Orthogonal Directions in Cylindrical Polar Coordinate System
$\vec{F}_R, \vec{F}_\theta, \vec{F}_\psi$	Field components in spherical coordinates
$\vec{F}_\rho, \vec{F}_\psi, \vec{F}_z$	Field components in cylindrical coordinates
R and r	Distance
γ	Propagation Constant
γ_0	Free Space Propagation Constant
τ_{ij}	Tensors
$\xi_i^/$	Tensor
$\in_{xy}$	Electrical Permittivity – Tensor
q	Volume density of charge
q_v	Volume density of charge
μ	Magnetic Permeability
μ_0	Free Space Magnetic Permeability
R	Resistance

$\in$	Electrical Permittivity
$\in_0$	Free Space Electrical Permittivity
J_n	Bessel's function of the first kind and nth order
J_o	Bessel's function of the first kind of zero order
Y_n	Bessel's function of the second kind and nth order
Y_o	Bessel's function of the second kind and zero order
I_n	Modified Bessel's function of the first kind and nth order
I_o	Modified Bessel's function of the first kind and zero order
K_n	Modified Bessel's function of the second kind and nth order
K_o	Modified Bessel's function of the second kind and zero order
P_n	Legandre's Polynomial of nth order
Q_n	Infinite series of nth order
H_n	Henkel's function of nth order
H_o	Henkel's function of 0^{th} order
P_n^m	Associated Legendre's Polynomial of the first kind and nth order
Q_n^m	Associated Legendre's function of the second kind and nth order.
$A_1, A_2 \ldots A_N$	Kernel Functions
$B_1, B_2, \ldots B_N - 1$	Kernel Functions
k_{12}, k_{23} etc	Reflection factors
ρ_a	Apparent Resistivity
K_1 and K_1^1	Modified Bessel's function of the second kind and first order and its derivative
$\pi(w, k, k_1)$	Legendre form of elliptic integral of the third kind
$\pi(\lambda, \alpha)$	Jacobian form of the elliptic integral of the third kind
$Sn\alpha$, $Cn\alpha$ and $dn\alpha$	Elliptic Functions
$\Theta(K + iK^{/} - \alpha)$	Jacobi's theta function
$z(\phi)$	Jacobi's zeta function
c	Velocity of electromagnetic wave
f,n	Frequency
δ	Skin depth
$\vec{\pi}$	Hertz vector

$\vec{F}$	Fitzerald vector
∇^2	Second Order Differential Operator
I_1	Modified Bessel's function of the first kind and first order
K_1	Modified Bessel's function of the second kind and first order
$I_{n+\frac{1}{2}}$	Modified Bessel's function of first kind and fractional order
$K_{n+\frac{1}{2}}$	Modified Bessel's function of the second kind and fractional order.
$G(r, r_0)$	Green's function
L	Operator
$\overrightarrow{\vec{G}}$	Dyadic Green's function
S	Source matrix
ϕ	Column vector of potentials
ϕ (x,y,z)	Potential due to a line source
N	Shape function
$f^c(x, y)$	Polynomial function
W_i	Galerkin's weights
ξ, η	Natural coordinates
[J]	Jacobian matrix
U,u	Eigen vectors
V,v	Eigen vectors
d	data
A	Matrix
A^T	Transpose of the Matrix A
$\langle m \rangle_{z_o}$	Model value at a depth z_o
m^{max}	Maximum value of the parameter
m^{min}	Minimum value of the parameter

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