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Theory and Applications

2nd Edition

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Preface

Thank you for your interest in the second edition of our book on dissipative systems. The first version of this book has been improved and augmented in several directions (mainly by the first author supported by the second and third authors of the second version). The link between dissipativity and optimal control is now treated in more detail, and many proofs which were not provided in the first edition are now given in their entirety, making the book more self-contained. One difficulty one encounters when facing the literature on dissipative systems is that there are many different definitions of dissipativity and positive real transfer functions (one could say a proliferation), many different versions of the same fundamental mathematical object (like the Kalman-Yakubovich-Popov Lemma), and it is not always an easy task to discover the links between them all. One objective of this book is to present those notions in a single volume and to try, if possible, to present their relationships in a clear way. Novel sections on descriptor (or singular) systems, discrete-time linear and nonlinear systems, some types of nonsmooth systems, viscosity solutions of the KYP Lemma set of equations, time-varying systems, unbounded differential inclusions, evolution variational inequalities, hyperstability, nonlinear H_∞, input-to-state stability, have been added. Conditions under which the Kalman-Yakubovich-Popov Lemma can be stated without assuming the minimality of the realization are provided in a specific section. Some general results (like well-posedness results for various types of evolution problems encountered in the book, definitions, matrix algebra tools, *etc.*) are presented in the Appendix, and many others are presented in the main text when they are needed for the first time. We thank J. Collado and S. Hadd who made us some remarks, and we remain of course open to any comments that may help us continue to improve our book.

Montbonnot, *April* 2006 — Bernard Brogliato

Contents

Notation

- $I\!R$: the set of real numbers; $\mathbb{C}$ the set of complex numbers; $I\!N$: the set of nonnegative integers.
- $I\!R^n$ $(\mathbb{C}^n)$: the set of $n-$dimensional vectors with real (complex) entries.
- A^T: transpose of the matrix $A \in I\!R^{n\times m}$ or $\in \mathbb{C}^{n\times m}$.
- $\bar{A}$: conjugate of the matrix $A \in \mathbb{C}^{n\times m}$.
- $A^\star$: conjugate transpose matrix of the matrix $A \in \mathbb{C}^{n\times m}$.
- $A > 0$ (≥ 0): positive definite (semi positive definite) matrix.
- $\lambda(A)$: an eigenvalue of $A \in I\!R^{n\times m}$.
- $\sigma(A)$ the set of eigenvalues of $A \in I\!R^{n\times m}$ (*i.e.* the spectrum of A).
- $\lambda_{\max}(A)$, $\lambda_{\min}(A)$: the largest and smallest eigenvalue of the matrix A, respectively.
- $\sigma_{\max}(A)$ $(\sigma_{\min}(A))$: largest (smallest) singular value of A.
- $\rho(A)$: the spectral radius of A, *i.e.* $\max\{|\lambda| \ : \ \lambda \in \sigma(A)\}$.
- $\mathrm{tr}(A)$: the trace of the matrix A.
- $A^\dagger$: the Moore-Penrose inverse of the matrix A.
- ODE: Ordinary Differential Equation; PDE: Partial Differential equation.
- BV, LBV, RCLBV: Bounded Variation, Local BV, Right Continuous LBV.
- AC: Absolutely Continuous.
- I_n the $n \times n$ identity matrix, O_n the $n \times n$ zero matrix.
- $\frac{\partial f}{\partial x}(x) \in I\!R^{m\times n}$: the jacobian of the function $f : I\!R^n \to I\!R^m$ at x.
- $\nabla f(x) \in I\!R^{n\times m}$: the euclidean gradient of the function $f : I\!R^n \to I\!R^m$ at x ($\nabla f(x) = \frac{\partial f}{\partial x}^T(x)$).

- A function is said to be smooth if it is infinitely differentiable; C^0 denotes the set of continuous functions; C^r denotes the set of r-times differentiable functions $f(\cdot)$ with $f^{(r)}(\cdot)$ continuous.
- $f(t^+)$: right-limit of the function $f(\cdot)$ at t; ($f(t^-)$: left-limit).
- $||\cdot||$: Euclidean norm in $\mathbb{R}^n$ ($||x|| = \sqrt{x^T x}$ for all $x \in \mathbb{R}^n$).
- $||f||_p$: $\mathcal{L}_p$-norm of a Lebesgue integrable function $f(\cdot)$.
- $\angle H(j\omega)$: the phase of $H(j\omega) \in \mathbb{C}$.
- LTI: Linear Time Invariant (system).
- Ker(A): kernel of $A \in \mathbb{R}^{n\times m}$; Im$(A)$: image of $A \in \mathbb{R}^{n\times m}$.
- dom(f): domain of a function f.
- $\bar{K}$: closure of a domain $K \subseteq \mathbb{R}^n$ ($K = \bar{K}$ if and only if K is closed).
- Int(K): interior of a domain $K \subseteq \mathbb{R}^n$ (Int(K) is always open), *i.e.* the set of interior points of K (points x of K such that there is a neighborhood of x inside K).
- **Re**$[\cdot]$ denotes the real part and **Im**$[\cdot]$ denotes the imaginary part.
- a.e.: almost everywhere (usually in the Lebesgue measure sense).

1 Introduction

Dissipativity theory gives a framework for the design and analysis of control systems using an input-output description based on energy-related considerations. Dissipativity is a notion which can be used in many areas of science, and it allows the control engineer to relate a set of efficient mathematical tools to well known physical phenomena. The insight gained in this way is very useful for a wide range of control problems. In particular the input-output description allows for a modular approach to control systems design and analysis.

The main idea behind this is that many important physical systems have certain input-output properties related to the conservation, dissipation and transport of energy. Before introducing precise mathematical definitions we will somewhat loosely refer to such input-output properties as dissipative properties, and systems with dissipative properties will be termed dissipative systems. When modeling dissipative systems it may be useful to develop the state-space or input-output models so that they reflect the dissipativity of the system, and thereby ensure that the dissipativity of the model is invariant with respect to model parameters, and to the mathematical representation used in the model. The aim of this book is to give a comprehensive presentation of how the energy-based notion of dissipativity can be used to establish the input-output properties of models for dissipative systems. Also it will be shown how these results can be used in controller design. Moreover, it will appear clearly how these results can be generalized to a dissipativity theory where conservation of other physical properties, and even abstract quantities, can be handled.

Models for use in controller design and analysis are usually derived from the basic laws of physics (electrical systems, dynamics, thermodynamics). Then a controller can be designed based on this model. An important problem in controller design is the issue of robustness which relates to how the closed loop system will perform when the physical system differs either in structure or in parameters from the design model. For a system where the basic laws of physics imply dissipative properties, it may make sense to define the model so that it possesses the same dissipative properties regardless of the numerical values of the physical parameters. Then if a controller is designed so that stability relies on the dissipative properties only, the closed-loop system will be stable whatever the values of the physical parameters. Even a change of the system order will be tolerated provided it does not destroy the dissipativity.

Parallel interconnections and feedback interconnections of dissipative systems inherit the dissipative properties of the connected subsystems, and this simplifies analysis by allowing for manipulation of block diagrams, and provides guidelines on how to design control systems. A further indication of the usefulness of dissipativity theory is the fact that the PID controller is a dissipative system, and a fundamental result that will be presented is the fact that the stability of a dissipative system with a PID controller can be established using dissipativity arguments. Note that such arguments rely on the structural properties of the physical system, and are not sensitive to the numerical values used in the design model. The technique of controller design using dissipativity theory can therefore be seen as a powerful generalization of PID controller design.

There is another aspect of dissipativity which is very useful in practical applications. It turns out that dissipativity considerations are helpful as a guide for the choice of a suitable variable for output feedback. This is helpful for selecting where to place sensors for feedback control.

Throughout the book we will treat dissipativity for state space and input-output models, but first we will investigate simple examples which illustrate some of the main ideas to be developed more deeply later.

1.1 Example 1: System with Mass Spring and Damper

Consider a one-dimensional simple mechanical system with a mass, a spring and a damper. The equation of motion is

$$m\ddot{x}(t) + D\dot{x}(t) + Kx(t) = F(t), \quad x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

where m is the mass, D is the damper constant, K is the spring stiffness, x is the position of the mass and F is the force acting on the mass. The energy of the system is

$$V(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2$$

The time derivative of the energy when the system moves is

$$\frac{d}{dt}V(x(t), \dot{x}(t)) = m\ddot{x}(t)\dot{x}(t) + Kx(t)\dot{x}(t)$$

Inserting the equation of motion we get

$$\frac{d}{dt}V(x(t), \dot{x}(t)) = F(t)\dot{x}(t) - D\dot{x}^2(t)$$

Integration of this equation from $t = 0$ to $t = T$ gives

$$V\left[x\left(T\right), \dot{x}\left(T\right)\right] = V\left[x\left(0\right), \dot{x}\left(0\right)\right] + \int_0^T F\left(t\right) \dot{x}\left(t\right) dt - \int_0^T D\dot{x}^2\left(t\right) dt$$

This means that the energy at time $t = T$ is the initial energy plus the energy supplied to the system by the control force F minus the energy dissipated by the damper. Note that if the input force F is zero, and if there is no damping, then the energy $V(\cdot)$ of the system is constant. Here $D \geq 0$ and $V\left[x\left(0\right), \dot{x}\left(0\right)\right] > 0$, and it follows that the integral of the force F and the velocity $v = \dot{x}$ satisfies

$$\int_0^T F\left(t\right) v\left(t\right) dt \geq -V\left[x\left(0\right), \dot{x}\left(0\right)\right] \tag{1.1}$$

The physical interpretation of this inequality is seen from the equivalent inequality

$$-\int_0^T F\left(t\right) v\left(t\right) dt \leq V\left[x\left(0\right), \dot{x}\left(0\right)\right] \tag{1.2}$$

which shows that the energy $-\int_0^T F\left(t\right) \dot{x}\left(t\right) dt$ that can be extracted from the system is less than or equal to the initial energy stored in the system. We will show later that (1.1) implies that the system with input F and output v is passive. The Laplace transform of the equation of motion is

$$\left(ms^2 + Ds + K\right) x(s) = F\left(s\right)$$

which leads to the transfer function

$$\frac{v}{F}(s) = \frac{s}{ms^2 + Ds + K}.$$

It is seen that the transfer function is stable, and that for $s = j\omega$ the phase of the transfer function has absolute value less or equal to 90°, that is,

$$\left|\angle \frac{v}{F}(j\omega)\right| \leq 90^\circ \quad \Rightarrow \quad \mathbf{Re}\left[\frac{v}{F}(j\omega)\right] \geq 0 \tag{1.3}$$

for all $\omega \in [-\infty, +\infty]$. We will see in the following that these properties of the transfer function are consequences of the condition (1.1), and that they are important in controller design.

1.2 Example 2: *RLC* Circuit

Consider a simple electrical system with a resistor R, inductance L and a capacitor C with current i and voltage u. The differential equation for the circuit is

$$L\frac{di}{dt}(t) + Ri(t) + Cx(t) = u(t)$$

where

$$x(t) = \int_0^t i(t')\, dt'$$

The energy stored in the system is

$$V(x, i) = \frac{1}{2}Li^2 + \frac{1}{2}Cx^2$$

The time derivative of the energy when the system evolves is

$$\frac{d}{dt}V(x(t), i(t)) = L\frac{di}{dt}(t)i(t) + Cx(t)i(t)$$

Inserting the differential equation of the circuit we get

$$\frac{d}{dt}V(x(t), i(t)) = u(t)i(t) - Ri^2(t)$$

Integration of this equation from $t = 0$ to $t = T$ gives

$$V[x(T), i(T)] = V[x(0), i(0)] + \int_0^T u(t)\, i(t)\, dt - \int_0^T Ri^2(t)\, dt$$

Similarly to the previous example, this means that the energy at time $t = T$ is the initial energy plus the energy supplied to the system by the voltage u minus the energy dissipated by the resistor. Note that if the input voltage u is zero, and if there is no resistance, then the energy $V(\cdot)$ of the system is constant. Here $R \geq 0$ and $V[x(0), \dot{x}(0)] > 0$, and it follows that the integral of the voltage u and the current i satisfies

$$\int_0^t u(s)\, i(s)\, ds \geq -V[x(0), i(0)] \tag{1.4}$$

The physical interpretation of this inequality is seen from the equivalent inequality

$$-\int_0^t u(s)\, i(s)\, ds \leq V[x(0), i(0)] \tag{1.5}$$

which shows that the energy $-\int_0^t u(s)\, i(s)\, ds$ that can be extracted from the system is less than or equal to the initial energy stored in the system. We will show later that (1.4) implies that the system with input u and output i is passive. The Laplace transform of the differential equation of the circuit is

$$\left(Ls^2 + Rs + C\right) x(s) = u(s)$$

which leads to the transfer function

$$\frac{i}{u}(s) = \frac{s}{Ls^2 + Rs + C}.$$

It is seen that the transfer function is stable, and that, for $s = j\omega$, the phase of the transfer function has absolute value less or equal to 90°, that is,

$$\left| \angle \frac{i}{u}(j\omega) \right| \leq 90^\circ \quad \Rightarrow \quad \mathbf{Re}\left[\frac{i}{u}(j\omega) \right] \geq 0 \tag{1.6}$$

for all $\omega \in [-\infty, +\infty]$. We see that in both examples we arrive at transfer functions that are stable, and that have positive real parts on the $j\omega$ axis. This motivates for further investigations on whether there is some fundamental connection between conditions on the energy flow in equations associated with the control equations (1.1) and (1.4) and the conditions on the transfer functions (1.3) and (1.6). This will be made clear in chapter 2.

1.3 Example 3: A Mass with a PD Controller

Consider the mass m with the external control force u. The equation of motion is

$$m\ddot{x}(t) = u(t)$$

Suppose that a PD controller

$$u = -K_P x - K_D \dot{x}$$

is used. Then the closed loop dynamics is

$$m\ddot{x}(t) + K_D \dot{x}(t) + K_P x(t) = 0$$

A purely mechanical system with the same dynamics as this system is called a mechanical analog. The mechanical analog for this system is a mass m with a spring with stiffness K_P and a damper with damping constant K_D. We see that the proportional action corresponds to the spring force, and that the derivative action corresponds to the damper force. Similarly, as in Example 1, we can define an energy function

$$V(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} K_P x^2$$

which is the total energy of the mechanical analog. In the same way as in Example 1, the derivative action will dissipate the virtual energy that is initially stored in the system, and intuitively, we may accept that the system will converge to the equilibrium $x = 0$, $\dot{x} = 0$. This can also be seen from the Laplace transform

$$\left(ms^2 + K_D s + K_P \right) x(s) = 0$$

which implies that the poles of the system have negative real parts. The point we are trying to make is that, for this system, the stability of the closed loop system with a PD controller can be established using energy arguments. Moreover, it is seen that stability is ensured for any positive gains K_P and K_D independently of the physical parameter m. There are many important results derived from energy considerations in connection with PID control, and this will be investigated in Chapter 2.

1.4 Example 4: Adaptive Control

We consider a simple first order system given by

$$\dot{x}(t) = a^{\star}x(t) + u(t)$$

where the parameter $a^{\star}$ is unknown. An adaptive tracking controller can be designed using the control law

$$u = -Ke - \hat{a}x + \dot{x}_d, \quad e = x - x_d$$

where x_d is the desired trajectory to be tracked, $\hat{a}$ is the estimate of the parameter $a^{\star}$, and K is the feedback gain. The differential equation for the tracking error e is

$$\begin{aligned} \tfrac{de}{dt}(t) &= a^{\star}x(t) + u(t) - \dot{x}_d(t) \\ &= a^{\star}x(t) - Ke(t) - \hat{a}(t)x(t) + \dot{x}_d(t) - \dot{x}_d(t) \\ &= -Ke(t) - \tilde{a}(t)x(t) \end{aligned}$$

where $\tilde{a} = \hat{a} - a^{\star}$ is the estimation error. We now define

$$\psi(t) = -\tilde{a}(t)x(t)$$

which leads to the following description of the tracking error dynamics

$$\frac{de}{dt}(t) + Ke(t) = \psi(t)$$

We define a function V_e which plays the role of an abstract energy function related to the tracking error e:

$$V_e(e) = \frac{1}{2}e^2$$

The time derivative of V_e along the solutions of the system is

$$\dot{V}_e(e(t)) = e(t)\psi(t) - Ke^2(t)$$

Note that this time derivative has a similar structure to that seen in Examples 1 and 2. In particular, the $-Ke^2$ term is a dissipation term, and if we think of

ψ as the input and e as the output, then the $e\psi$ term is the rate of (abstract) energy supplied from the input. We note that this implies that the following inequality holds for the dynamics of the tracking error:

$$\int_0^T e(t)\psi(t)dt \geq -V_e\left[e\left(0\right)\right]$$

To proceed, we define one more energy-like function. Suppose that we are able to select an adaptation law so that there exists an energy-like function $V_a\left(\tilde{a}\right) \geq 0$ with a time derivative

$$\dot{V}_a\left(\tilde{a}(t)\right) = -e(t)\psi(t) \tag{1.7}$$

We note that this implies that the following inequality holds for the adaptation law:

$$\int_0^T \left[-\psi(t)\right] e(t)dt \geq -V_a\left[\tilde{a}(0)\right]$$

Then the sum of the energy functions

$$V\left(e,\tilde{a}\right) = V_e\left(e\right) + V_a\left(\tilde{a}\right)$$

has a time derivative along the solutions of the system given by

$$\dot{V}\left(e(t),\tilde{a}(t)\right) = -Ke^2(t)$$

This means that the energy function $V\left(e,\tilde{a}\right)$ is decreasing as long as $e(\cdot)$ is nonzero, and by invoking additional arguments from Barbalat's Lemma (see Chapter A), we can show that this implies that $e(t)$ tends to zero as $t \to +\infty$. The required adaptation law for (1.7) to hold can be selected as the simple gradient update

$$\frac{d\hat{a}}{dt}(t) = x(t)e(t)$$

and the associated energy-like function is

$$V_a\left(\tilde{a}\right) = \frac{1}{2}\tilde{a}^2$$

Note that the convergence of the adaptive tracking controller was established using energy-like arguments, and that other adaptation laws can be used as long as they satisfy the energy-related requirement (1.7).

2

Positive Real Systems

The notion of *Positive Real* system may be seen as a generalization of the positive definiteness of a matrix to the case of a dynamical system with inputs and outputs. When the input-output relation (or mapping, or operator) is a constant matrix, testing its positive definiteness can be done by simply calculating the eigenvalues and checking that they are positive. When the input-output operator is more complex, testing positive realness becomes much more involved. This is the object of this chapter which is mainly devoted to positive real linear time-invariant systems. They are known as PR transfer functions.

The definition of Positive Real (PR) systems has been motivated by the study of linear electric circuits composed of resistors, inductors and capacitors. The driving point impedance from any point to any other point of such electric circuits is always PR. The result holds also in the sense that any PR transfer function can be realized with an electric circuit using only resistors, inductors and capacitors. The same result holds for any analogous mechanical or hydraulic systems. This idea can be extended to study analogous electric circuits with nonlinear passive components and even magnetic couplings as done by Arimoto [24] to study dissipative nonlinear systems. This leads us to the second interpretation of PR systems: they are systems which dissipate energy. As we shall see later in the book, the notion of *dissipative* systems, which applies to nonlinear systems, is closely linked to PR transfer functions.

This chapter reviews the main results available for PR linear systems. It starts with a short introduction to so-called *passive* systems. It happens that there has been a proliferation of notions and definitions of various kinds of PR or dissipative systems, since the early studies in the 1960s (to name a few: ISP, OSP, VSP, PR, SPR, WSPR, SSPR, MSPR, ESPR; see the index for the meaning of these acronyms). The study of their relationships (are they equivalent, which ones imply which other one?) is not so easy and we bring some elements of answers in this chapter and the next ones. This is why we introduce first in this chapter some basic definitions (passive systems, positive real systems, bounded real transfer functions), their relationships, and then we introduce other refined notions of PR systems. The reason why passive systems are briefly introduced before bounded real and positive real transfer functions, is that this allows one to make the link between an energy-related notion and the frequency domain notions, in a progressive way. This, however, is at the price of postponing a more rigorous and general exposition of passive systems until later in the book.

2.1 Dynamical System State-space Representation

In this book various kinds of evolution, or dynamical systems will be analyzed: linear, time invariant, nonlinear, finite-dimensional, infinite-dimensional, discrete time, non-smooth, "standard" differential inclusions, "unbounded" or "maximal monotone" differential inclusions *etc.* Whatever the system we shall be dealing with, it is of utmost importance to clearly define some basic ingredients:

- A state vector $x(\cdot)$ and a state space X
- A set of admissible inputs $\mathcal{U}$
- A set of outputs $\mathcal{Y}$
- An input/output mappping (or operator) $H : u \mapsto y$
- A state space representation which relates the derivative of $x(\cdot)$ to $x(\cdot)$ and $u(\cdot)$
- An output function which relates the output $y(\cdot)$ to the state $x(\cdot)$ and the input $u(\cdot)$

Such tools (or some of them) are necessary to write down the model, or system, that is under examination. When one works with pure input/output models, one doesn't need to define a state space X; however $\mathcal{U}$ and $\mathcal{Y}$ are crucial. In this book we will essentially deal with systems for which a state space representation has been defined. Then the notion of a (state) solution is central. Given some state space model under the form of an evolution problem (a differential equation or something looking like this), the first step is to provide informations on such solutions: the nature of the solutions (as time-functions, for instance), their uniqueness, their continuity with respect to the initial data and parameters, *etc.* This in turn is related to the set of admissible inputs $\mathcal{U}$. For instance, if the model takes the form of an ordinary differential equation (ODE) $\dot{x}(t) = f(x(t), u(t))$, the usual Carathéodory conditions will be in force to define $\mathcal{U}$ as a set of measurable functions, and $x(\cdot)$ will usually be an absolutely continuous function of time. In certain cases, one may want to extend $\mathcal{U}$ to measures, or even distributions. Then x may also be a measure or a distribution. Since it is difficult (actually impossible) to provide a general well-posedness result for all the systems that will be dealt with in the rest of the book, we will recall the well-posedness conditions progressively as new models are introduced. This will be the case especially for some classes of nonsmooth systems, where solutions may be absolutely continuous, or of local bounded variation.

From a more abstract point of view, one may define a general state-space deterministic model as follows [364, 510, 512]:

There exists a metric space X (the state space), a transition map $\psi : I\!R \times I\!R \times X \times \mathcal{U} \rightarrow X$, and a readout map $r : X \times I\!R^m \rightarrow I\!R^p$, such that:

- (i) The limit $x(t) = \lim_{t_0 \rightarrow -\infty} \psi(t_0, t, 0, u)$ is in X for all $t \in I\!R$ and all $u \in \mathcal{U}$ (then $x(t)$ is the state at time t)
- (ii) (Causality) $\psi(t_0, t_1, x, u_1) = \psi(t_0, t_1, x, u_2)$ for all $t_1 \geq t_0$, all $x \in X$, and all $u_1, u_2 \in \mathcal{U}$ such that $u_1(t) = u_2(t)$ in the interval $t_0 \leq t \leq t_1$
- (iii) (Initial state consistency) $\psi(t_0, t_0, x_0, u) = x_0$ for all $t_0 \in I\!R$, $u \in \mathcal{U}$, and all $x_0 \in X$
- (iv) (Semigroup property) $\psi(t_1, t_2, \psi(t_0, t_1, x_0, u), u) = \psi(t_0, t_2, x_0, u)$ for all $x_0 \in X$, $u \in \mathcal{U}$, whenever $t_0 \leq t_1 \leq t_2$
- (v) (Consistency with input-output relation) The input-output pairs (u, y) are precisely those described *via* $y(t) = r\left(\lim_{t_0 \rightarrow -\infty} \psi(t_0, t, 0, u), u(t)\right)$
- (vi) (Unbiasedness) $\psi(t_0, t, 0, 0) = 0$ whenever $t \geq t_0$ and $r(0, 0) = 0$
- (vii) (Time-invariance) $\psi(t_1 + T, t_2 + T, x_0, u_1) = \psi(t_1, t_2, x_0, u_2)$ for all $T \in I\!R$, all $t_2 \geq t_1$, and all $u_1, u_2 \in \mathcal{U}$ such that $u_2(t) = u_1(t + T)$

Clearly item (vii) will not apply to some classes of time-varying systems, and an extension is needed [512, §6]. There may be some items which do not apply well to differential inclusions where the solution may be replaced by a solution set (for instance the semigroup property may fail). The basic fact that X is a metric space will also require much care when dealing with some classes of systems whose state spaces are not spaces of functions (like descriptor variable systems that involve Schwarz' distributions). In the infinite-dimensional case X may be a Hilbert space (*i.e.* a space of functions) and one may need other definitions, see *e.g.* [39, 507]. An additional item in the above list could be the continuity of the transition map $\psi(\cdot)$ with respect to the initial data x_0. Some nonsmooth systems do not possess such a property, which may be quite useful in some stability results. A general exposition of the notion of a system can be found in [467, Chapter 2]. We now stop our investigations of what a system is since, as we said above, we shall give well-posedness results each time they are needed all through the book.

2.2 Definitions

In this section and the next one, we introduce input-output properties of a system, or operator $H : u \mapsto H(u) = y$. The system is assumed to be well-posed as an input-ouput system, i.e. we may assume that $H : \mathcal{L}_{2,e} \rightarrow \mathcal{L}_{2,e}$ [1].

[1] More details on $\mathcal{L}_p$ spaces can be found in Chapter 4.

Definition 2.1. *A system with input $u(\cdot)$ and output $y(\cdot)$ where $u(t), y(t) \in \mathbb{R}^m$ is passive if there is a constant β such that*

$$\int_0^t y^T(\tau)u(\tau)d\tau \geq \beta \tag{2.1}$$

for all functions $u(\cdot)$ and all $t \geq 0$. If, in addition, there are constants $\delta \geq 0$ and $\epsilon \geq 0$ such that

$$\int_0^t y^T(\tau)u(\tau)d\tau \geq \beta + \delta \int_0^t u^T(\tau)u(\tau)d\tau + \epsilon \int_0^t y^T(\tau)y(\tau)d\tau \tag{2.2}$$

for all functions $u(\cdot)$, and all $t \geq 0$, then the system is input strictly passive (ISP) if $\delta > 0$, output strictly passive (OSP) if $\epsilon > 0$, and very strictly passive (VSP) if $\delta > 0$ and $\epsilon > 0$. ■

Obviously $\beta \leq 0$ as the inequality (2.1) is to be valid for all functions $u(\cdot)$ and in particular the control $u(t) = 0$ for all $t \geq 0$, which gives $0 = \int_0^t y^T(s)u(s)ds \geq \beta$. Thus the definition could equivalently be stated with $\beta \leq 0$. The importance of the form of β in (2.1) will be illustrated in Examples 4.59 and 4.60; see also Section 4.4.2. Notice that $\int_0^t y^T(s)u(s)ds \leq \frac{1}{2}\int_0^t [y^T(s)y(s) + u^T(s)u(s)]ds$ is well defined since both $u(\cdot)$ and $y(\cdot)$ are in $\mathcal{L}_{2,e}$ by assumption.

Theorem 2.2. *Assume that there is a continuous function $V(\cdot) \geq 0$ such that*

$$V(t) - V(0) \leq \int_0^t y(s)^T u(s)ds \tag{2.3}$$

for all functions $u(\cdot)$, for all $t \geq 0$ and all $V(0)$. Then the system with input $u(\cdot)$ and output $y(\cdot)$ is passive. Assume, in addition, that there are constants $\delta \geq 0$ and $\epsilon \geq 0$ such that

$$V(t) - V(0) \leq \int_0^t y^T(s)u(s)ds - \delta \int_0^t u^T(s)u(s)ds - \epsilon \int_0^t y^T(s)y(s)ds \tag{2.4}$$

for all functions $u(\cdot)$, for all $t \geq 0$ and all $V(0)$. Then the system is input strictly passive if there is a $\delta > 0$, it is output strictly passive if there is an $\epsilon > 0$, and very strictly passive if there is a $\delta > 0$ and an $\epsilon > 0$ such that the inequality holds. ■

Proof: It follows from the assumption $V(t) \geq 0$ that

$$\int_0^t y^T(s)u(s)ds \geq -V(0)$$

for all functions $u(\cdot)$ and all $s \geq 0$, so that (2.1) is satisfied with $\beta := -V(0) \leq 0$. Input strict passivity, output strict passivity and very strict passivity are shown in the same way. ■

This indicates that the constant β is related to the initial conditions of the system; see also Example 4.59 for more informations on the role played by β. It is also worth looking at Corollary 3.3 to get more informations on the real nature of the function $V(\cdot)$: $V(\cdot)$ will usually be a function of the state of the system. The reader may have guessed such a fact by looking at the examples of Chapter 1.

Corollary 2.3. *Assume that there exists a continuously differentiable function $V(\cdot) \geq 0$ and a measurable function $d(\cdot)$ such that $\int_0^t d(s)ds \geq 0$ for all $t \geq 0$. Then*

1. *If*
$$\dot{V}(t) \leq y^T(t)u(t) - d(t) \tag{2.5}$$
for all $t \geq 0$ and all functions $u(\cdot)$, the system is passive.
2. *If there exists a $\delta > 0$ such that*
$$\dot{V}(t) \leq y^T(t)u(t) - \delta u^T(t)u(t) - d(t) \tag{2.6}$$
for all $t \geq 0$ and all functions $u(\cdot)$, the system is input strictly passive (ISP).
3. *If there exists a $\epsilon > 0$ such that*
$$\dot{V}(t) \leq y^T(t)u(t) - \epsilon y^T(t)y(t) - d(t) \tag{2.7}$$
for all $t \geq 0$ and all functions $u(\cdot)$, the system is output strictly passive (OSP).
4. *If there exists a $\delta > 0$ and a $\epsilon > 0$ such that*
$$\dot{V}(t) \leq y^T(t)u(t) - \delta u^T(t)u(t) - \epsilon y^T(t)y(t) - d(t) \tag{2.8}$$
for all $t \geq 0$ and all functions $u(\cdot)$, the system is very strictly passive (VSP). ■

If $V(\cdot)$ is the total energy of the system, then $\langle u, y \rangle = \int_0^t y^T(s)u(s)ds$ can be seen as the power supplied to the system from the control, while $d(t)$ can be seen as the power dissipated by the system. This means that the condition $\int_0^t d(s)ds \geq 0$ for all $t \geq 0$ means that the system is dissipating energy. The term $w(u, y) = u^T y$ is called the *supply rate* of the system.

Remark 2.4. All these notions will be examined in much more detail in Chapter 4; see especially Section 4.5.2. Actually the notion of passivity (or dissipativity) has been introduced in various ways in the literature. It is sometimes introduced as a pure input/output property of an operator (*i.e.* the constant β in (2.1) is not related to the state of the system) [125, 499, 500], and serves as a tool to prove some bounded input/bounded output stability results. Willems has, on the contrary, introduced dissipativity as a notion which involves the state space representation of a system, through so-called *storage functions* [510, 511]. We will come back to this subject in Chapter 4. Hill and Moylan started from an intermediate definition, where the constant β is assumed to depend on some initial state x_0 [206–209]. Then, under some controllability assumptions, the link with Willems' definition is made. In this chapter and the next one, we will essentially concentrate on linear time invariant dissipative systems, whose transfer functions are named *positive real* (PR). This is a very important side of passivity theory in Systems and Control theory.

2.3 Interconnections of Passive Systems

A useful result for passive systems is that parallel and feedback interconnections of passive systems are passive, and that certain strict passivity properties are inherited.

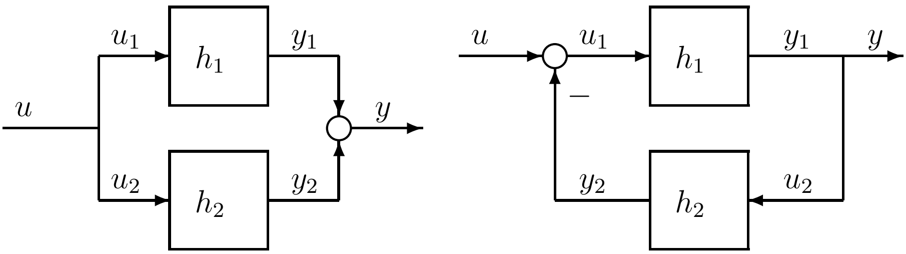

Fig. 2.1. Parallel and feedback interconnections.

To explore this we consider two passive systems with scalar inputs and outputs. Similar results are found for multivariable systems. System 1 has input u_1 and output y_1, and system 2 has input u_2 and output y_2. We make the following assumptions:

1. There are continuous differentiable functions $V_1(t) \geq 0$ and $V_2(t) \geq 0$.
2. There are functions $d_1(\cdot)$ and $d_2(\cdot)$ such that $\int_0^t d_1(s)ds \geq 0$ and $\int_0^t d_2(s)ds \geq 0$ for all $t \geq 0$.
3. There are constants $\delta_1 \geq 0$, $\delta_2 \geq 0$, $\epsilon_1 \geq 0$ and $\epsilon_2 \geq 0$ such that

$$\dot{V}_1(t) = y_1(t)u_1(t) - \delta_1 u_1^2(t) - \epsilon_1 y_1^2 - d_1(t) \tag{2.9}$$

$$\dot{V}_2(t) = y_2(t)u_2(t) - \delta_2 u_2^2(t) - \epsilon_2 y_2^2 - d_2(t) \tag{2.10}$$

Assumption 3 implies that both systems are passive, and that system i is strictly passive in some sense if any of the constants δ_i or ϵ_i are greater than zero. For the parallel interconnection we have $u_1 = u_2 = u$, $y = y_1 + y_2$, and

$$yu = (y_1 + y_2)u = y_1u + y_2u = y_1u_1 + y_2u_2 \tag{2.11}$$

By adding (2.9) (2.10) and (2.11), there exists a $V(\cdot) = V_1(\cdot) + V_2(\cdot) \geq 0$ and a $d_p = d_1 + d_2 + \epsilon_1 y_1^2 + \epsilon_2 y_2^2$ such that $\int_0^t d_p(t')dt' \geq 0$ for all $t \geq 0$, and

$$\dot{V}(t) = y(t)u(t) - \delta u^2(t) - d_p(t) \tag{2.12}$$

where $\delta = \delta_1 + \delta_2 \geq 0$. This means that the parallel interconnection system having input u and output y is passive and strictly passive if $\delta_1 > 0$ or $\delta_2 > 0$. For the feedback interconnection we have $y_1 = u_2 = y$, $u_1 = u - y_2$, and

$$yu = y_1(u_1 + y_2) = y_1u_1 + y_1y_2 = y_1u_1 + u_2y_2 \tag{2.13}$$

Again by adding (2.9) (2.10) and (2.11) we find that there is a $V(\cdot) = V_1(\cdot) + V_2(\cdot) \geq 0$ and a $d_{fb} = d_1 + d_2 + \delta_1 u_1^2$ such that $\int_0^t d_{fb}(s)ds \geq 0$ for all $t \geq 0$ and

$$\dot{V}(t) = y(t)u(t) - \epsilon y^2(t) - d_{fb}(t) \tag{2.14}$$

where $\epsilon = \epsilon_1 + \epsilon_2 + \delta_2$. This means that the feedback interconnection is passive, and in addition output strictly passive if $\epsilon_1 > 0$, $\epsilon_2 > 0$, or $\delta_2 > 0$. By induction it can be shown that any combination of passive systems in parallel or feedback interconnection is passive.

2.4 Linear Systems

Let us now deal with linear invariant systems, whose input-output relationships takes the form of a rational transfer function $H(s)$ (also denoted as $h(s)$), $s \in \mathbb{C}$, and $y(s) = H(s)u(s)$ where $u(s)$ and $y(s)$ are the Laplace transforms of the time-functions $u(\cdot)$ and $y(\cdot)$. Parseval's Theorem is very useful in the study of passive linear systems, as shown next. It is now recalled for the sake of completeness.

Theorem 2.5 (Parseval's Theorem). *Provided that the integrals exist, the following relation holds:*

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)y^*(j\omega)d\omega \tag{2.15}$$

where y^* denotes the complex conjugate of y and $x(j\omega)$ is the Fourier transform of $x(t)$, where $x(t)$ is a complex function of t, Lebesgue integrable. ■

Proof: The result is established as follows: the Fourier transform of the time function $x(t)$ is

$$x(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{2.16}$$

while the inverse Fourier transform is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(j\omega) e^{j\omega t} d\omega \tag{2.17}$$

Insertion of (2.17) in (2.15) gives

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} x(j\omega) e^{j\omega t} d\omega \right] y^*(t) dt \tag{2.18}$$

By changing the order of integration this becomes

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(j\omega) \left[\int_{-\infty}^{\infty} y^*(t) e^{j\omega t} dt \right] d\omega \tag{2.19}$$

Here

$$\int_{-\infty}^{\infty} y^*(t) e^{j\omega t} dt = \left[\int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt \right]^* = y^*(j\omega) \tag{2.20}$$

and the result follows. ■

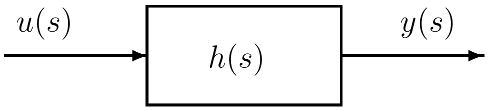

Fig. 2.2. Linear time-invariant system

We will now present important properties of a linear time-invariant passive system, which link the input-output passivity property to frequency-domain conditions, using Parseval's Theorem. These notions will be generalized later in the book, both in the case of LTI and nonlinear systems. Their usefulness will be illustrated through examples of stabilization.

Theorem 2.6. *Given a linear time-invariant linear system with rational transfer function $h(s)$, i.e.*

$$y(s) = h(s) u(s) \tag{2.21}$$

Assume that all the poles of $h(s)$ have real parts less than zero. Then the following assertions hold:

1. *The system is passive* $\Leftrightarrow$ $\mathbf{Re}[h(j\omega)] \geq 0$ *for all* $\omega \in [-\infty, +\infty]$.
2. *The system is input strictly passive (ISP)* $\Leftrightarrow$ *There exists a* $\delta > 0$ *such that* $\mathbf{Re}[h(j\omega)] \geq \delta > 0$ *for all* $\omega \in [-\infty, +\infty]$.
3. *The system is output strictly passive (OSP)* $\Leftrightarrow$ *There exists an* $\epsilon > 0$ *such that*

$$\mathbf{Re}[h(j\omega)] \geq \epsilon \left|h(j\omega)\right|^2$$
$$\Updownarrow$$
$$\left(\mathbf{Re}[h(j\omega)] - \tfrac{1}{2\epsilon}\right)^2 + (\mathbf{Im}[h(j\omega)])^2 \leq \left(\tfrac{1}{2\epsilon}\right)^2$$

■

Remark 2.7. A crucial assumption in Theorem 2.6 is that all the poles have negative real parts. This assures that in Parseval's Theorem as stated in Theorem 2.5, the "integrals exist".

Proof: The proof is based on the use of Parseval's Theorem. In this Theorem the time integration is over $t \in [0, \infty)$. In the definition of passivity there is an integration over $t \in [0, T]$. To be able to use Parseval's Theorem in this proof we introduce the truncated function

$$u_t(\tau) = \begin{cases} u(\tau) & \text{when} \quad \tau \leq t \\ 0 & \text{when} \quad \tau > t \end{cases} \tag{2.22}$$

which is equal to $u(\tau)$ for all τ less than or equal to t, and zero for all τ greater than t. The Fourier transform of $u_T(t)$, which is denoted $u_T(j\omega)$, will be used in Parseval's Theorem. Without loss of generality we will assume that $y(t)$ and $u(t)$ are equal to zero for all $t \leq 0$. Then according to Parseval's Theorem

$$\int_0^t y(\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} y(\tau)u_t(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(j\omega)u_t^*(j\omega)d\omega \tag{2.23}$$

Insertion of $y(j\omega) = h(j\omega)u_T(j\omega)$ gives

$$\int_0^t y(\tau)u(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(j\omega)u_T(j\omega)u_t^*(j\omega)d\omega, \tag{2.24}$$

where

$$h(j\omega)u_t(j\omega)u_t^*(j\omega) = \{\mathbf{Re}[h(j\omega)] + j\mathbf{Im}[h(j\omega)]\}|u_t(j\omega)|^2 \tag{2.25}$$

The left hand side of (2.24) is real, and it follows that the imaginary part on the right hand side is zero. This implies that

$$\int_0^t u(\tau)y(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{Re}[h(j\omega)]|u_t(j\omega)|^2 d\omega \tag{2.26}$$

First, assume that $\mathbf{Re}[h(j\omega)] \geq \delta \geq 0$ for all ω. Then

$$\int_0^t u(\tau)y(\tau)d\tau \geq \frac{\delta}{2\pi}\int_{-\infty}^{\infty} |u_t(j\omega)|^2 d\omega = \delta\int_0^t u^2(\tau)d\tau \tag{2.27}$$

The equality is implied by Parseval's Theorem. It follows that the system is passive, and in addition input strictly passive if $\delta > 0$.

Then, assume that the system is passive. Thus there exists a $\delta \geq 0$ so that

$$\int_0^t y(s)u(s)dsz \geq \delta \int_0^t u^2(s)ds = \frac{\delta}{2\pi}\int_{-\infty}^{\infty} |u_t(j\omega)|^2 d\omega \tag{2.28}$$

for all $u(\cdot)$, where the initial conditions have been selected so that $\beta = 0$. Here $\delta = 0$ for a passive system, while $\delta > 0$ for a strictly passive system. Then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{Re}[h(j\omega)]|u_T(j\omega)|^2 d\omega \geq \frac{\delta}{2\pi}\int_{-\infty}^{\infty} |u_T(j\omega)|^2 d\omega \tag{2.29}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathbf{Re}[h(j\omega)] - \delta)|u_T(j\omega)|^2 d\omega \geq 0 \tag{2.30}$$

If there exists a ω_0 so that $\mathbf{Re}[h(j\omega_0)] < \delta$, then inequality will not hold for all u because the integral on the left hand side can be made arbitrarily small if the control signal is selected to be $u(t) = U\cos\omega_0 t$. The results 1 and 2 follow.

To show result 3 we first assume that the system is output strictly passive, that is, there is an $\epsilon > 0$ such that

$$\int_0^t y(s)u(s)ds \geq \epsilon \int_0^t y^2(s)ds = \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty} |h(j\omega)|^2|u_t(j\omega)|^2 d\omega. \tag{2.31}$$

This gives the inequality (see (2.26))

$$\mathbf{Re}[h(j\omega)] \geq \epsilon\, |h(j\omega)|^2 \tag{2.32}$$

which is equivalent to

$$\epsilon\left[(\mathbf{Re}[h(j\omega)])^2 + (\mathbf{Im}[h(j\omega)])^2\right] - \mathbf{Re}[h(j\omega)] \leq 0 \tag{2.33}$$

and the second inequality follows by straighforward algebra. The converse result is shown similarly as the result for input strict passivity. ■

Note that according to the theorem a passive system will have a transfer function which satisfies

$$|\angle h(j\omega)| \leq 90^\circ \quad \text{for all} \quad \omega \in [-\infty, +\infty] \tag{2.34}$$

In a Nyquist diagram the theorem states that $h(j\omega)$ is in the closed half plane $\mathbf{Re}\,[s] \geq 0$ for passive systems, $h(j\omega)$ is in $\mathbf{Re}\,[s] \geq \delta > 0$ for input strictly passive systems, and for output strictly passive systems $h(j\omega)$ is inside the circle with center in $s = 1/(2\epsilon)$ and radius $1/(2\epsilon)$. This is a circle that crosses the real axis in $s = 0$ and $s = 1/\epsilon$.

Remark 2.8. A transfer function $h(s)$ is rational if it is the fraction of two polynomials in the complex variable s, that is if it can be written in the form

$$h(s) = \frac{Q(s)}{R(s)} \tag{2.35}$$

where $Q(s)$ and $R(s)$ are polynomials in s. An example of a transfer function that is not rational is $h(s) = \tanh s$ which appears in connection with systems described by partial differential equations.

Example 2.9. Note the difference between the condition $\mathbf{Re}[h(j\omega)] > 0$ and the condition for input strict passivity in that there exists a $\delta > 0$ so that $\mathbf{Re}[h(j\omega_0)] \geq \delta > 0$ for all ω. An example of this is

$$h_1(s) = \frac{1}{1 + Ts} \tag{2.36}$$

We find that $\mathbf{Re}[h_1(j\omega)] > 0$ for all ω because

$$h_1(j\omega) = \frac{1}{1 + j\omega T} = \frac{1}{1 + (\omega T)^2} - j\frac{\omega T}{1 + (\omega T)^2} \tag{2.37}$$

However there is no $\delta > 0$ that ensures $\mathbf{Re}[h(j\omega_0)] \geq \delta > 0$ for all $\omega \in [-\infty, +\infty]$. This is seen from the fact that for any $\delta > 0$ we have

$$\mathbf{Re}[h_1(j\omega)] = \frac{1}{1 + (\omega T)^2} < \delta \quad \text{for all} \quad \omega > \sqrt{\frac{1-\delta}{\delta}}\frac{1}{T} \tag{2.38}$$

This implies that $h_1(s)$ is not input strictly passive. We note that for this system

$$|h_1(j\omega)|^2 = \frac{1}{1 + (\omega T)^2} = \mathbf{Re}[h_1(j\omega)] \tag{2.39}$$

which means that the system is output strictly passive with $\epsilon = 1$.

Example 2.10. Consider a system with the transfer function

$$h_2(s) = \frac{s + c}{(s + a)(s + b)} \tag{2.40}$$

where a, b and c are positive constants. We find that

$$\begin{aligned} h_2(j\omega) &= \tfrac{j\omega + c}{(j\omega + a)(j\omega + b)} \\ &= \tfrac{(c + j\omega)(a - j\omega)(b - j\omega)}{(a^2 + \omega^2)(b^2 + \omega^2)} \\ &= \tfrac{abc + \omega^2(a + b - c) + j[\omega(ab - ac - bc) - \omega^3]}{(a^2 + \omega^2)(b^2 + \omega^2)}. \end{aligned}$$

From the above it is clear that

1. If $c \leq a+b$, then $\mathbf{Re}[h_2(j\omega)] > 0$ for all $\omega \in I\!R$. As $\mathbf{Re}[h_2(j\omega)] \to 0$ when $\omega \to \infty$, *the* system is not input strictly passive.
2. If $c > a+b$, then $h_2(s)$ is not passive because $\mathbf{Re}[h_2(j\omega)] < 0$ for $\omega > \sqrt{abc/(c-a-b)}$.

Example 2.11. The systems with transfer functions

$$h_3(s) = 1 + Ts \tag{2.41}$$

$$h_4(s) = \frac{1+T_1 s}{1+T_2 s}, \quad T_1 < T_2 \tag{2.42}$$

are input strictly passive because

$$\mathbf{Re}[h_3(j\omega)] = 1 \tag{2.43}$$

and

$$\mathbf{Re}[h_4(j\omega)] = \frac{1+\omega^2 T_1 T_2}{1+(\omega T_2)^2} \in \left(\frac{T_1}{T_2}, 1\right] \tag{2.44}$$

Moreover $|h_4(j\omega)|^2 \leq 1$, so that

$$\mathbf{Re}[h_4(j\omega)] \geq \frac{T_1}{T_2} \geq \frac{T_1}{T_2} |h_4(j\omega)|^2 \tag{2.45}$$

which shows that the system is output strictly passive with $\epsilon = T_1/T_2$. The reader may verify from a direct calculation of $|h_4(j\omega)|^2$ and some algebra that it is possible to have $\mathbf{Re}[h_4(j\omega)] \geq |h_4(j\omega)|^2$, that is, $\epsilon = 1$. This agrees with the Nyquist plot of $h_4(j\omega)$.

Example 2.12. A dynamic system describing an electrical one-port with resistors, inductors and capacitors is passive if the voltage over the port is input and the current into the port is output, or *vice versa*. In Figure 2.3 different passive one-ports are shown. We consider the voltage over the port to be the input and the current into the port as the output. The resulting transfer functions are admittances, which are the inverses of the impedances. Circuit 1 is a capacitor, circuit 2 is a resistor in parallel with a capacitor, circuit 3 is a resistor in series with a inductor and a capacitor, while circuit 4 is a resistor in series with a parallel connection of an inductor, a capacitor and a resistor. The transfer functions are

$$h_1(s) = Cs \tag{2.46}$$

$$h_2(s) = \frac{1}{R}(1 + RCs) \tag{2.47}$$

$$h_3(s) = \frac{Cs}{1 + RCs + LCs^2} \tag{2.48}$$

$$h_4(s) = \frac{1}{R_1} \frac{1 + \frac{L}{R}s + LCs^2}{1 + (\frac{L}{R_1} + \frac{L}{R})s + LCs^2} \tag{2.49}$$

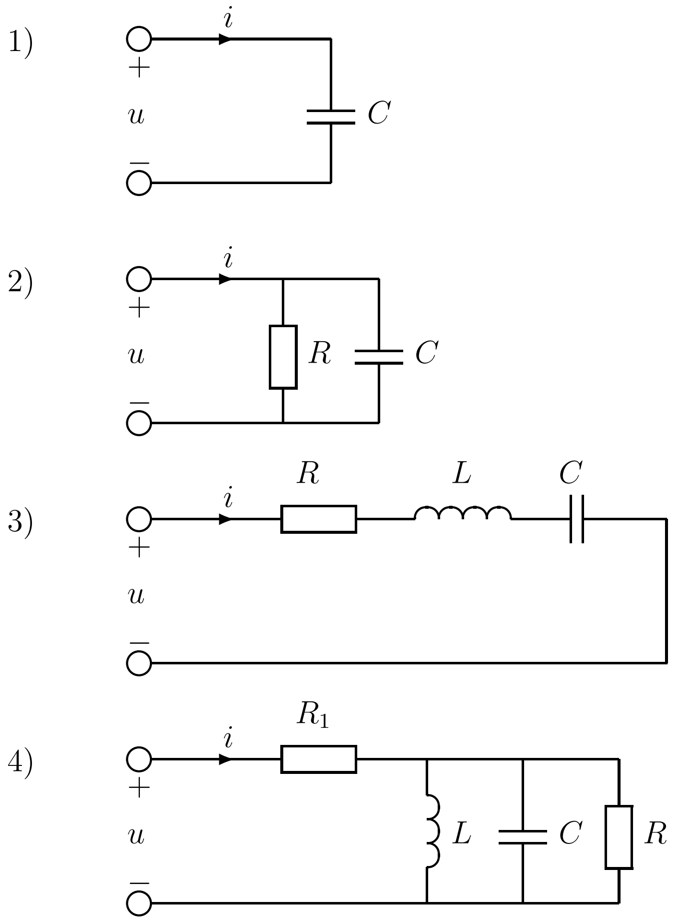

Fig. 2.3. Passive electrical one-ports

Systems 1, 2, 3 and 4 are all passive as the poles have real parts that are strictly less than zero, and in addition $\mathbf{Re}[h_i(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$ and $i \in \{1,2,3,4\}$ (the fact that all the poles are in $\mathbf{Re}[s] < 0$ is important; see Theorem 2.14). It follows that the transfer functions have phases that satisfy $|\angle h_i(j\omega)| \leq 90°$. In addition system 2 is input strictly passive as $\mathbf{Re}[h_2(j\omega)] = 1/R > 0$ for all ω. For system 4 we find that

$$\mathbf{Re}[h_4(j\omega)] = \frac{1}{R_1} \frac{(1-\omega^2 LC)^2 + \omega^2 \frac{L^2}{R_1(R_1+R)}}{(1-\omega^2 LC)^2 + \omega^2 \frac{L^2}{(R_1+R)^2}} \geq \frac{1}{R_1+R} \tag{2.50}$$

which means that system 4 is input strictly passive. ■

So far we have only considered systems where the transfer functions $h(s)$ have poles with negative real parts. There are however passive systems that have transfer functions with poles on the imaginary axis. This is demonstrated in the following example:

Example 2.13. Consider the system $\dot{y}(t) = u(t)$ which is represented in transfer function description by $y(s) = h(s)u(s)$ where $h(s) = \frac{1}{s}$. This means that the transfer function has a pole at the origin, which is on the imaginary axis. For this system $\mathbf{Re}[h(j\omega)] = 0$ for all ω. However, we cannot establish passivity using Theorem 2.6 as this theorem only applies to systems where all the poles have negative real parts. Instead, consider

$$\int_0^t y(s)u(s)ds = \int_0^t y(s)\dot{y}(s)ds \tag{2.51}$$

A change of variables $\dot{y}(t)dt = dy$ gives

$$\int_0^t y(t')u(t')dt' = \int_{y(0)}^{y(t)} y(t')dy = \frac{1}{2}[y(t)^2 - y(0)^2] \geq -\frac{1}{2}y(0)^2 \tag{2.52}$$

and passivity is shown with $\beta = -\frac{1}{2}y(0)^2$. ■

It turns out to be relatively involved to find necessary and sufficient conditions on $h(j\omega)$ for the system to be passive when we allow for poles on the imaginary axis. The conditions are relatively simple and are given in the following Theorem.

Theorem 2.14. *Consider a linear time-invariant system with a rational transfer function $h(s)$. The system is passive if and only if*

1. *$h(s)$ has no poles in $\mathbf{Re}[s] > 0$.*
2. *$\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$ such that $j\omega$ is not a pole of $h(s)$.*
3. *If $j\omega_0$ is a pole of $h(s)$, then it is a simple pole, and the residual in $s = j\omega_0$ is real and greater than zero, that is, $Res_{s=j\omega_0}h(s) = \lim_{s\to j\omega_0}(s - j\omega_0)h(j\omega) > 0$.*

■

The above result is established in Section 2.12. Contrary to Theorem 2.6, poles on the imaginary axis are considered.

Corollary 2.15. *If a system with transfer function $h(s)$ is passive, then $h(s)$ has no poles in $\mathbf{Re}[s] > 0$.* ■

Proposition 2.16. *Consider a rational transfer function*

$$h(s) = \frac{(s+z_1)(s+z_2)\dots}{s(s+p_1)(s+p_2)\dots} \tag{2.53}$$

where $\mathbf{Re}[p_i] > 0$ and $\mathbf{Re}[z_i] > 0$ which means that $h(s)$ has one pole at the origin and the remaining poles in $\mathbf{Re}[s] < 0$, while all the zeros are in $\mathbf{Re}[s] < 0$. Then the system with transfer function $h(s)$ is passive if and only if $\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$. ■

Proof: The residual of the pole on the imaginary axis is

$$\mathrm{Res}_{s=0}h(s) = \frac{z_1 z_2 \dots}{p_1 p_2 \dots} \tag{2.54}$$

Here the constants z_i and p_i are either real and positive, or they appear in complex conjugated pairs where the products $z_i z_i^* = |z_i|^2$ and $p_i p_i^* = |p_i|^2$ are real and positive. It is seen that the residual at the imaginary axis is real and positive. As $h(s)$ has no poles in $\mathbf{Re}\,[s] > 0$ by assumption, it follows that the system is passive if and only if $\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$. ■

Example 2.17. Consider two systems with transfer functions

$$h_1(s) = \frac{s^2 + a^2}{s(s^2 + \omega_0^2)}, \quad a \neq 0, \omega_0 \neq 0 \tag{2.55}$$

$$h_2(s) = \frac{s}{s^2 + \omega_0^2}, \quad \omega_0 \neq 0 \tag{2.56}$$

where all the poles are on the imaginary axis. Thus condition 1 in Theorem 2.14 is satisfied. Moreover,

$$h_1(j\omega) = -j\frac{a^2 - \omega^2}{\omega(\omega_0^2 - \omega^2)} \tag{2.57}$$

$$h_2(j\omega) = j\frac{\omega}{\omega_0^2 - \omega^2} \tag{2.58}$$

so that condition 2 also holds in view of $\mathbf{Re}[h_1(j\omega)] = \mathbf{Re}[h_2(j\omega)] = 0$ for all ω so that $j\omega$ is not a pole in $h\,(s)$. We now calculate the residual, and find that

$$\mathrm{Res}_{s=0}h_1(s) = \frac{a^2}{\omega_0^2} \tag{2.59}$$

$$\mathrm{Res}_{s=\pm j\omega_0}h_1(s) = \frac{\omega_0^2 - a^2}{2\omega_0^2} \tag{2.60}$$

$$\mathrm{Res}_{s=\pm j\omega_0}h_2(s) = \frac{1}{2} \tag{2.61}$$

We see that, according to Theorem 2.14, the system with transfer function $h_2(s)$ is passive, while $h_1(s)$ is passive whenever $a < \omega_0$.

Example 2.18. Consider a system with transfer function

$$h(s) = -\frac{1}{s} \tag{2.62}$$

The transfer function has no poles in $\mathbf{Re}\,[s] > 0$, and $\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \neq 0$. However, $Res_{s=0}h(s) = -1$, and Theorem 2.14 shows that the system is not passive This result agrees with the observation

$$\int_0^t y(s)u(s)ds = -\int_{y(0)}^{y(t)} y(s)dy = \frac{1}{2}[y(0)^2 - y(t)^2] \tag{2.63}$$

where the right hand side has no lower bound as $y(t)$ can be arbitrarily large.

2.5 Passivity of the PID Controllers

Proposition 2.19. *Assume that $0 \le T_d < T_i$ and $0 \le \alpha \le 1$. Then the PID controller*

$$h_r(s) = K_p \frac{1 + T_i s}{T_i s} \frac{1 + T_d s}{1 + \alpha T_d s} \tag{2.64}$$

is passive. ■

This follows from Proposition 2.16.

Proposition 2.20. *Consider a PID controller with transfer function*

$$h_r(s) = K_p \beta \frac{1 + T_i s}{1 + \beta T_i s} \frac{1 + T_d s}{1 + \alpha T_d s} \tag{2.65}$$

where $0 \le T_d < T_i$, $1 \le \beta < \infty$ and $0 < \alpha \le 1$. Then the controller is passive and, in addition, the transfer function gain has an upper bound $|h_r(j\omega)| \le \frac{K_p \beta}{\alpha}$ and the real part of the transfer function is bounded away from zero according to $\mathbf{Re}\,[h_r(j\omega)] \ge K_p$ for all ω. ■

It follows from Bode diagram techniques that

$$|h_r(j\omega)| \le K_p \beta \cdot 1 \cdot \frac{1}{\alpha} = \frac{K_p \beta}{\alpha} \tag{2.66}$$

The result on the $\mathbf{Re}\,[h_r(j\omega)]$ can be established using Nyquist diagram, or by direct calculation of $\mathbf{Re}\,[h_r(j\omega)]$. ■

2.6 Stability of a Passive Feedback Interconnection

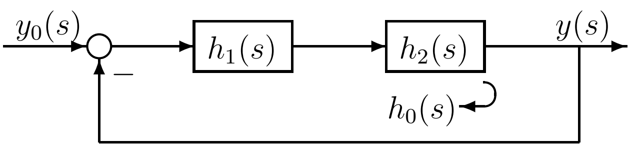

Fig. 2.4. Interconnection of a passive system $h_1(s)$ and a strictly passive system $h_2(s)$

Consider a feedback loop with loop transfer function $h_0(s) = h_1(s) h_2(s)$ as shown in Figure 2.4. If h_1 is passive and h_2 is strictly passive, then the phases of the transfer functions satisfy

$$|\angle h_1(j\omega)| \le 90^\circ \quad \text{and} \quad |\angle h_2(j\omega)| < 90^\circ \tag{2.67}$$

It follows that the phase of the loop transfer function $h_0(s)$ is bounded by

$$|\angle h_0(j\omega)| < 180^\circ \tag{2.68}$$

As h_1 and h_2 are passive, it is clear that $h_0(s)$ has no poles in $\mathbf{Re}\,[s] > 0$. Then according to standard Bode-Nyquist stability theory the system is asymptotically stable and BIBO stable [2]. The same result is obtained if instead h_1 is strictly passive and h_2 is passive.

We note that, in view of Proposition 2.20, a PID controller with limited integral action is strictly stable. This implies that

- A passive linear system with a PID controller with limited integral action is BIBO stable.

For an important class of systems passivity or strict passivity is a structural property which is not dependent on the numerical values of the parameters of the system. Then passivity considerations may be used to establish stability even if there are large uncertainties or large variations in the system parameters. This is often referred to as robust stability. When it comes to performance it is possible to use any linear design technique to obtain high performance for the nominal parameters of the system. The resulting system will have high performance under nominal conditions, and in addition robust stability under large parameter variations.

2.7 Mechanical Analogs for PD Controllers

In this section we will study how PD controllers for position control can be represented by mechanical analogs when the input to the system is force and the output is position. Note that when force is input and position is output, then the physical system is not passive. We have a passive physical system if the force is the input and the velocity is the output, and then a PD controller from position corresponds to PI controller from velocity. For this reason we might have referred to the controllers in this section as PI controllers for velocity control.

We consider a mass m with position $x(\cdot)$ and velocity $v(\cdot) = \dot{x}(\cdot)$. The dynamics is given by $m\ddot{x}(t) = u(t)$ where the force u is the input. The desired position is $x_d(\cdot)$, while the desired velocity is $v_d(\cdot) = \dot{x}_d(\cdot)$. A PD controller $u = K_p(1 + T_d s)\,[x_d(s) - x(s)]$ is used. The control law can be written as

$$u(t) = K_p(x_d(t) - x(t)) + D(v_d(t) - v(t)) \tag{2.69}$$

where $D = K_p T_d$. The mechanical analog appears from the observation that this control force is the force that results if the mass m with position x is connected to the position x_d with a parallel interconnection of a spring with stiffness K_p and a damper with coefficient D as shown in Figure 2.5.

[2] Bounded Input-Bounded Output.

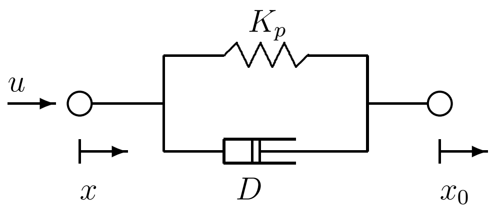

Fig. 2.5. Mechanical analog of PD controller with feedback from position

If the desired velocity is not available, and the desired position is not smooth a PD controller of the type

$$u(s) = K_p x_d(s) - K_p(1 + T_d s)x(s), \quad s \in \mathbb{C}$$

can be used. Then the control law is

$$u(t) = K_p(x_d(t) - x(t)) - Dv(t) \tag{2.70}$$

This is the force that results if the mass m is connected to the position x_d with a spring of stiffness K_p and a damper with coefficient D as shown in Figure 2.6.

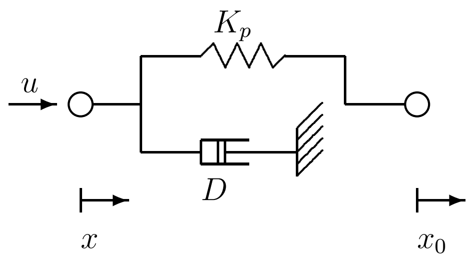

Fig. 2.6. Mechanical analog of a PD controller without desired velocity input

If the velocity is not measured the following PD controller can be used

$$u(s) = K_p \frac{1 + T_d s}{1 + \alpha T_d s}[x_d(s) - x(s)] \tag{2.71}$$

where $0 \leq \alpha \leq 1$ is the filter parameter. We will now demonstrate that this transfer function appears by connecting the mass m with position x to a spring with stiffness K_1 in series with a parallel interconnection of a spring with stiffness K and a damper with coefficient D as shown in Figure 2.7.

To find the expression for K_1 and K we let x_1 be the position of the connection point between the spring K_1 and the parallel interconnection. Then the force is $u = K_1(x_1 - x)$, which implies that $x_1(s) = x(s) + u(s)/K_1$. As

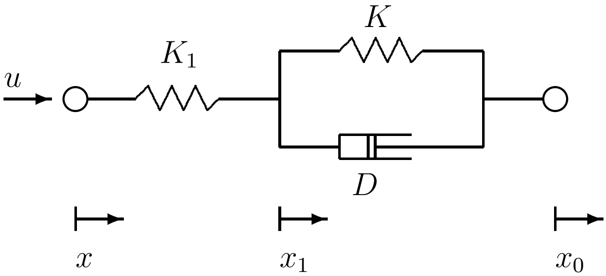

Fig. 2.7. Mechanical analog of a PD controller without velocity measurement

there is no mass in the point x_1 there must be a force of equal magnitude in the opposite direction from the parallel interconnection, so that

$$u(s) = K[x_d(s) - x_1(s)] + D[v_d(s) - v_1(s)] = (K + Ds)[x_d(s) - x_1(s)] \quad (2.72)$$

Insertion of $x_1(s)$ gives

$$u(s) = (K + Ds)[x_d(s) - x(s) - \frac{1}{K_1}u(s)] \quad (2.73)$$

We solve for $u(s)$ and the result is

$$\begin{aligned} u(s) &= K_1 \tfrac{K+Ds}{K_1+K+Ds}[x_d(s) - x(s)] \\ &= \tfrac{K_1 K}{K_1+K} \tfrac{1+\frac{D}{K}s}{1+\frac{K}{K_1+K}\frac{D}{K}s}[x_d(s) - x(s)] \end{aligned}$$

We see that this is a PD controller without velocity measurement where

$$\begin{cases} K_p = \tfrac{K_1 K}{K_1+K} \\ T_d = \tfrac{D}{K} \\ \alpha = \tfrac{K}{K_1+K} \in [0, 1) \end{cases}$$

2.8 Multivariable Linear Systems

Theorem 2.21. *Consider a linear time-invariant system*

$$y(s) = H(s)u(s) \quad (2.74)$$

with a rational transfer function matrix $H(s) \in \mathbb{C}^{m \times m}$, input $u(t) \in I\!R^m$ and input $y(t) \in I\!R^m$. Assume that all the poles of $H(s)$ are in $\mathbf{Re}[s] < 0$. *Then,*

1. *The system is passive $\Leftrightarrow \lambda_{min}[H(j\omega)+H^*(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$.*

2. *The system is input strictly passive* $\Leftrightarrow$ *There is a* $\delta > 0$ *so that* $\lambda_{min}[H(j\omega) + H^*(j\omega)] \geq \delta > 0$ *for all* $\omega \in [-\infty, +\infty]$.

■

Remark 2.22. Similarly to Theorem 2.6, a crucial assumption in Theorem 2.21 is that the poles have negative real parts, *i.e.* there is no pole on the imaginary axis.

Proof: Let $A \in \mathbb{C}^{m\times m}$ be some Hermitian matrix with eigenvalues $\lambda_i(A)$. Let $x \in \mathbb{C}^m$ be an arbitrary vector with complex entries. It is well-known from linear algebra that x^*Ax is real, and that $x^*Ax \geq \lambda_{min}(A)|x|^2$. From Parseval's Theorem we have

$$\begin{aligned}\int_0^\infty y^T(s)u_t(s)ds &= \sum_{i=1}^m \int_0^\infty y_i(s)(u_i)_t(s)ds \\ &= \sum_{i=1}^m \tfrac{1}{2\pi}\int_{-\infty}^\infty y_i^*(j\omega)(u_i)_t(j\omega)d\omega \\ &= \tfrac{1}{2\pi}\int_{-\infty}^\infty y^*(j\omega)u_t(j\omega)d\omega\end{aligned}$$

where we recall that $u_t(\cdot)$ is a truncated function and that s in the integrand is a dumb integration variable (not to be confused with the Laplace transform!). This leads to

$$\begin{aligned}\int_0^t y^T(s)u(s)ds &= \int_0^\infty y^T(s)u_t(s)ds \\ &= \tfrac{1}{2\pi}\int_{-\infty}^\infty y^*(j\omega)u_t(j\omega)d\omega \\ &= \tfrac{1}{4\pi}\int_{-\infty}^\infty [u_T^*(j\omega)y(j\omega) + y^*(j\omega)u_t(j\omega)]d\omega \\ &= \tfrac{1}{4\pi}\int_{-\infty}^\infty u_t^*(j\omega)[H(j\omega) + H^*(j\omega)]u_t(j\omega)d\omega\end{aligned}$$

Because $H(j\omega) + H^*(j\omega)$ is Hermitian we find that

$$\int_0^t y^T(\tau)u(\tau)d\tau \geq \frac{1}{4\pi}\int_{-\infty}^\infty \lambda_{\min}[H(j\omega) + H^*(j\omega)]|u_t(j\omega)|^2 d\omega \tag{2.75}$$

The result can be established along the lines of Theorem 2.6. ■

2.9 The Scattering Formulation

By a change of variables an alternative description can be established where passivity corresponds to small gain. We will introduce this idea with an example from linear circuit theory. Consider a linear time-invariant system describing an electrical one-port with voltage e, current i and impedance $z(s)$ so that

$$e(s) = z(s)i(s) \tag{2.76}$$

Define the wave variables

$$a = e + z_0 i \quad \text{and} \quad b = e - z_0 i \tag{2.77}$$

where z_0 is a positive constant. The Laplace transform is

$$\begin{aligned} a(s) &= [z(s) + z_0]i(s) \\ b(s) &= [z(s) - z_0]i(s) \end{aligned}$$

Combining the two equations we get

$$b(s) = g(s)a(s) \tag{2.78}$$

where

$$g(s) = \frac{z(s) - z_0}{z_0 + z(s)} = \frac{\frac{z(s)}{z_0} - 1}{1 + \frac{z(s)}{z_0}} \tag{2.79}$$

is the *scattering function* of the system. The terms wave variable and scattering function originate from the description of transmission lines where a can be seen as the incident wave and b can be seen as the reflected wave.

If the electrical circuit has only passive elements, that is, if the circuit is an interconnection of resistors, capacitors and inductors, the passivity inequality satisfies

$$\int_0^t e(\tau)i(\tau)d\tau \geq 0 \tag{2.80}$$

where it is assumed that the initial energy stored in the circuit is zero. We note that

$$a^2 - b^2 = (e + z_0 i)^2 - (e - z_0 i)^2 = 4z_0 ei \tag{2.81}$$

which implies

$$\int_0^t b^2(\tau)d\tau = \int_0^t a^2(\tau)d\tau - 4z_0 \int_0^t e(\tau)i(\tau)d\tau \tag{2.82}$$

From this it is seen that passivity of the system with input i and output e corresponds to small gain for the system with input a and output b in the sense that

$$\int_0^t b^2(\tau)d\tau \leq \int_0^t a^2(\tau)d\tau \tag{2.83}$$

This small gain condition can be interpreted loosely in the sense that the energy content b^2 of the reflected wave is smaller than the energy a^2 of the incident wave. For the general linear time-invariant system

$$y(s) = h(s)u(s) \tag{2.84}$$

introduce the wave variables

$$a = y + u \quad \text{and} \quad b = y - u \tag{2.85}$$

where, as above, a is the incident wave and b is the reflected wave. As for electrical circuits it will usually be necessary to include a constant z_0 so that $a = y + z_0 u$ $b = y - z_0 u$ so that the physical units agree. We tacitly suppose that this is done by letting $z_0 = 1$ with the appropriate physical unit. The scattering function is defined by

$$g(s) \stackrel{\Delta}{=} \frac{b}{a}(s) = \frac{y-u}{y+u}(s) = \frac{h(s)-1}{1+h(s)} \tag{2.86}$$

Theorem 2.23. *Consider a system with rational transfer function $h(s)$ with no poles in $\mathbf{Re}[s] \geq 0$, and scattering function $g(s)$ given by (2.86). Then*

1. *The system is passive if and only if $|g(j\omega)| \leq 1$ for all $\omega \in [-\infty, +\infty]$.*
2. *The system is input strictly passive, and there is a γ so that $|h(j\omega)| \leq \gamma$ for all $\omega \in [-\infty, +\infty]$ if and only if there is a $\gamma' \in (0, 1)$ so that $|g(j\omega)|^2 \leq 1 - \gamma'$.* ■

Proof: Consider the following computation

$$\begin{aligned} |g(j\omega)|^2 &= \tfrac{|h(j\omega)-1|^2}{|h(j\omega)+1|^2} \\ &= \tfrac{|h(j\omega)|^2 - 2\mathbf{Re}[h(j\omega)]+1}{|h(j\omega)|^2+2\mathbf{Re}[h(j\omega)]+1} \\ &= 1 - \tfrac{4\mathbf{Re}[h(j\omega)]}{|h(j\omega)+1|^2} \end{aligned} \tag{2.87}$$

It is seen that $|g(j\omega)| \leq 1$ if and only if $\mathbf{Re}[h(j\omega)] \geq 0$. Result 1 then follows as the necessary and sufficient condition for the system to be passive is that $\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$. Concerning the second result, we show the "if" part. Assume that there is a δ so that $\mathbf{Re}[h(j\omega)] \geq \delta > 0$ and a γ so that $|h(j\omega)| \leq \gamma$ for all $\omega \in [-\infty, +\infty]$. Then

$$|g(j\omega)|^2 \geq 1 - \frac{4\delta}{(\gamma+1)^2} \tag{2.88}$$

and the result follows with $0 < \gamma' < \min\left(1, \frac{4\delta}{(\gamma+1)^2}\right)$. Next assume that $g(j\omega)|^2 \leq 1 - \gamma'$ for all ω. Then

$$4\mathbf{Re}\left[h(j\omega)\right] \geq \gamma' \left(|h(j\omega)|^2 + 2\mathbf{Re}[h(j\omega)] + 1\right) \tag{2.89}$$

and strict passivity follows from

$$\mathbf{Re}\left[h(j\omega)\right] \geq \frac{\gamma'}{4-2\gamma'} > 0 \tag{2.90}$$

Finite gain of $h(j\omega)$ follows from

$$\gamma'|h(j\omega)|^2 - (4-2\gamma')\,\mathbf{Re}[h(j\omega)] + \gamma' \leq 0 \tag{2.91}$$

which in view of the general result $|h(j\omega)| > \mathbf{Re}[h(j\omega)]$ gives the inequality

$$|h(j\omega)|^2 - \frac{(4-2\gamma')}{\gamma'}\,|h(j\omega)| + 1 \leq 0 \tag{2.92}$$

This implies that

$$|h\,(j\omega)| \leq \frac{(4-2\gamma')}{\gamma'} \tag{2.93}$$

■

We shall come back on the relationships between passivity and bounded realness in the framework of dissipative systems and H_∞ theory; see Section 5.9. A comment on the input-output change in (2.85): the association of the new system with transfer function $g(s)$ merely corresponds to writing down $uy = \frac{1}{4}(a+b)(a-b) = \frac{1}{4}(a^2-b^2)$. Thus if $\int_0^t u(s)y(s)ds \geq 0$ one gets $\int_0^t a^2(s)ds \geq \int_0^t b^2(s)ds$: the $\mathcal{L}_2$-norm of the new output $b(t)$ is bounded by the $\mathcal{L}_2$-norm of the new intput $a(t)$.

2.10 Impedance Matching

In this section we will briefly review the concept of impedance matching. Again an electrical one-port is studied. The one-port has a voltage source e, serial impedance z_0, output voltage v and current i. The circuit is coupled to the load which is a passive one-port with driving point impedance $z_L(s)$ as shown in Figure 2.8.

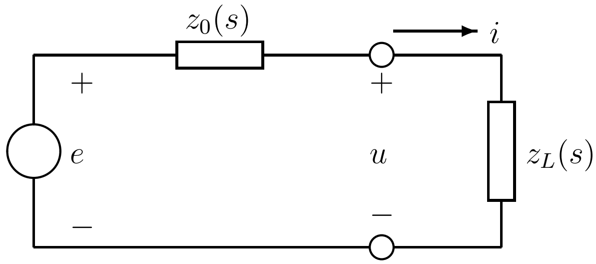

Fig. 2.8. Impedance matching

The following problem will be addressed: suppose $z_0\,(s)$ is given and that $e\,(t) = E\sin\omega_e t$. Select $z_L\,(s)$ so that the power dissipated in z_L is maximized.

The current is given by

$$i(s) = \frac{e(s)}{z_0(s) + z_L(s)} \tag{2.94}$$

while the voltage over z_L is

$$u(s) = z_L(s)i(s) \tag{2.95}$$

The power dissipated in z_L is therefore

$$\begin{aligned} P(\omega_e) &= \tfrac{1}{2}\mathbf{Re}[u_L(j\omega_e)i^*(j\omega_e)] \\ &= \tfrac{1}{2}\mathbf{Re}[z_L(j\omega_e)]i(j\omega_e)i^*(j\omega_e) \\ &= \tfrac{1}{2}\tfrac{\mathbf{Re}[z_L(j\omega_e)]}{[z_0(j\omega_e)+z_L(j\omega_e)]^*[z_0(j\omega_e)+z_L(j\omega_e)]}E^2 \end{aligned}$$

where $(\cdot)^*$ denotes the complex conjugate. Denote

$$z_0(j\omega_e) = \alpha_0 + j\beta_0 \quad \text{or} \quad z_L(j\omega_e) = \alpha_L + j\beta_L \tag{2.96}$$

This gives

$$P = \frac{1}{2}\frac{\alpha_L E^2}{(\alpha_0+\alpha_L)^2 + (\beta_0+\beta_L)^2} \tag{2.97}$$

We see that if $\alpha_L = 0$, then $P = 0$, whereas for nonzero α_L then $|\beta_L| \to \infty$, gives $P \to 0$. A maximum for P would be expected somewhere between these extremes. Differentiation with respect to β_L gives

$$\frac{\partial P}{\partial \beta_L} = \frac{E^2}{2}\frac{-2\alpha_L(\beta_0+\beta_L)}{[(\alpha_0+\alpha_L)^2+(\beta_0+\beta_L)^2]^2} \tag{2.98}$$

which implies that the maximum *of* P appears for $\beta_L = -\beta_0$. Differentiation with respect to α_L with $\beta_L = -\beta_0$ gives

$$\frac{\partial P}{\partial a} = \frac{E^2}{2}\frac{\alpha_0^2 - {\alpha_L}^2}{[(\alpha_0+\alpha_L)^2+(\beta_0+\beta_L)^2]^2} \tag{2.99}$$

and it is seen that the maximum is found for $\alpha_L = \alpha_0$. This means that the maximum power dissipation in z_L is achieved with

$$z_L(j\omega_e) = z_0^*(j\omega_e) \tag{2.100}$$

This particular selection of $z_L(j\omega_e)$ is called impedance matching. If the voltage source $e(t)$ is not simply a sinusoid but a signal with a arbitrary spectrum, then it is not possible to find a passive impedance $z_L(s)$ which satisfies the impedance matching condition or a general series impedance $z_0(j\omega)$. This is because the two impedances are required to have the same absolute values, while the phase have opposite signs. This cannot be achieved for one particular $z_L(s)$ for all frequencies.

However, if $z_0(j\omega) = z_0$ is a real constant, then impedance matching at all frequencies is achieved with $z_L = z_0$. We now assume that z_0 is a real constant, and define the wave variables to be

$$a = u + z_0 i \quad \text{or} \quad b = u - z_0 i \tag{2.101}$$

Then it follows that

$$a = e \tag{2.102}$$

for the system in Figure 2.8. A physical interpretation of the incident wave a is as follows: let u be the input voltage to the one-port and let i be the current into the port. Consider the extended one-port where a serial impedance z_0 is connected in to the one-port as shown in Figure 2.9. Then a is the input voltage of the extended one-port.

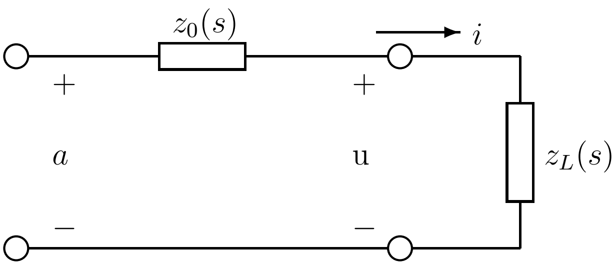

Fig. 2.9. Extended one-port with a serial impedance z_0

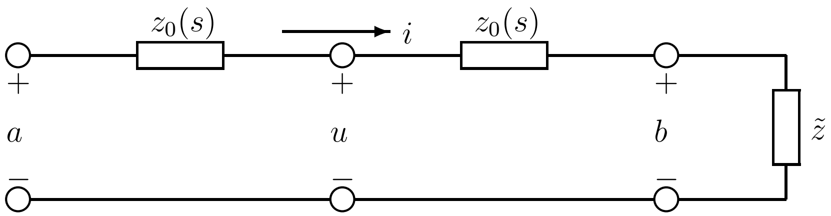

Fig. 2.10. Physical interpretation of the reflected wave b where $\tilde{z} = z_L(s) - z_0(s)$

The physical interpretation of the reflected wave b is shown in figure 2.10. We clearly see that if $z_L = z_0$, then

$$u = z_0 i \quad \Rightarrow \quad b = 0 \tag{2.103}$$

This shows that if impedance matching is used with z_0 being constant, then the scattering function is

$$g(s) = \frac{b(s)}{a(s)} = 0 \tag{2.104}$$

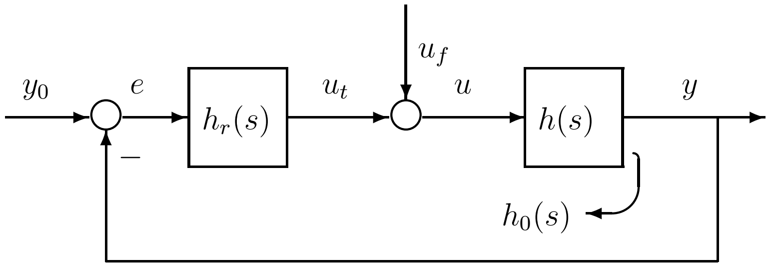

Fig. 2.11. Feedback interconnection of two passive systems

2.11 Feedback Loop

A feedback interconnection of two passive linear time-invariant systems is shown in Figure 2.11 where signals are given by

$$y(s) = h(s)u(s), \qquad u_t(s) = h_r(s)e(s) \tag{2.105}$$

$$u(t) = u_f(t) + u_t(t), \qquad e(t) = y_0(t) - y(t) \tag{2.106}$$

We can think of $h(s)$ as describing the plant to be controlled, and $h_r(s)$ as describing the feedback controller. Here u_t is the feedback control and u_f is the feedforward control. We assume that the plant $h(s)$ and that the feedback controller $h_r(s)$ are strictly passive with finite gain. Then, as shown in Section 2.6 we have $\angle|h_0(j\omega)| < 180°$ where $h_0(s) := h(s)h_r(s)$ is the loop transfer function, and the system is BIBO stable.

A change of variables in now introduced to bring the system into a scattering formulation. The new variables are

$$a \triangleq y + u \quad \text{and} \quad b \triangleq y - u$$

for the plant and

$$a_r \triangleq u_t + e \quad \text{and} \quad b_r \triangleq u_t - e$$

for the feedback controller. In addition input variables

$$a_0 \triangleq y_0 + u_f \quad \text{and} \quad b_0 \triangleq y_0 - u_f$$

are defined. We find that

$$a_r = u_t + y_0 - y = u - u_f + y_0 - y = b_0 - b \tag{2.107}$$

and

$$b_r = u_t - y_0 + y = u - u_f - y_0 + y = a - a_0 \tag{2.108}$$

The associated scattering functions are

$$g(s) \triangleq \frac{h(s) - 1}{1 + h(s)} \quad \text{and} \quad g_r(s) \triangleq \frac{h_r(s) - 1}{1 + h_r(s)}$$

Now, $h(s)$ and $h_r(s)$ are passive by assumption, and as a consequence they cannot have poles in $\mathbf{Re}[s] > 0$. Then it follows that $g(s)$ and $g_r(s)$ cannot have poles in $\mathbf{Re}[s] > 0$ because $1 + h(s)$ is the characteristic equations for $h(s)$ with a unity negative feedback, which obviously is a stable system. Similar arguments apply for $1 + h(s)$. The system can then be represented as in Figure 2.12 where

$$b(s) = g(s)a(s), \qquad b_r(s) = g_r(s)a_r(s) \tag{2.109}$$

$$a(t) = b_r(t) + a_0(t), \qquad a_r(t) = b_0(t) - b(t) \tag{2.110}$$

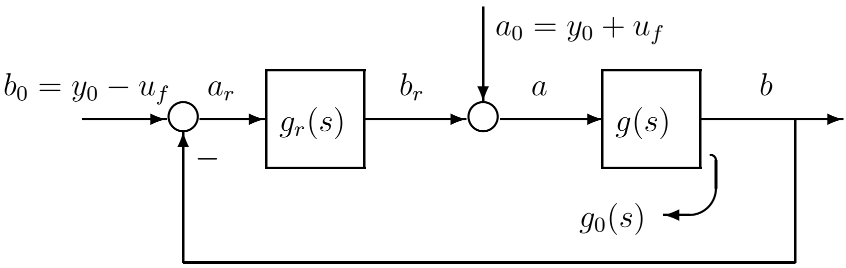

Fig. 2.12. Equivalent system

In the passivity setting, stability was ensured when two passive systems were interconnected in a feedback structure because the loop transfer function $h_0(j\omega)$ had a phase limitation so that $\angle h_0(j\omega) > -180°$. We would now like to check if there is an interpretation for the scattering formulation that is equally simple. This indeed turns out to be the case. We introduce the loop transfer function

$$g_0(s) \triangleq g(s)g_r(s) \tag{2.111}$$

of the scattering formulation. The function $g_0(s)$ cannot have poles in $\mathbf{Re}[s] > 0$ as $g(s)$ and $g_r(s)$ have no poles in $\mathbf{Re}[s] > 0$ by assumption. Then we have from Theorem 2.23:

1. $|g(j\omega)| \leq 1$ for all $\omega \in [-\infty, +\infty]$ because $h(s)$ is passive.
2. $|g_r(j\omega)| < 1$ for all $\omega \in [-\infty, +\infty]$ because $h_r(s)$ is strictly passive with finite gain.

As a consequence of this,

$$|g_0(j\omega)| < 1 \tag{2.112}$$

for all $\omega \in [-\infty, +\infty]$, and according to the Nyquist stability criterion the system is BIBO stable.

2.12 Bounded Real and Positive Real Transfer Functions

Bounded real and positive real are two important properties of transfer functions related to passive systems that are linear and time-invariant. We will in this section show that a linear time-invariant system is passive if and only if the transfer function of the system is positive real. To do this we first show that a linear time-invariant system is passive if and only if the scattering function, which is the transfer function of the wave variables, is bounded real. Then we show that the scattering function is bounded real if and only if the transfer function of the system is positive real. We will also discuss different aspects of these results for rational and irrational transfer functions.

We consider a linear time-invariant system $y(s) = h(s)u(s)$ with input u and output y. The incident wave is denoted $a \overset{\Delta}{=} y + u$, and the reflected wave is denoted $b \overset{\Delta}{=} y - u$. The scattering function $g(s)$ is given by

$$g(s) = \frac{h(s) - 1}{1 + h(s)} \tag{2.113}$$

and satisfies $b(s) = g(s)a(s)$. We note that

$$u(t)y(t) = \frac{1}{4}[a^2(t) - b^2(t)] \tag{2.114}$$

For linear time-invariant systems the properties of the system do not depend on the initial conditions, as opposed to nonlinear systems. We therefore assume for simplicity that the initial conditions are selected so that the energy function $V(t)$ is zero for initial time, that is $V(0) = 0$. The passivity inequality is then

$$0 \leq V(t) = \int_0^t u(s)y(s)ds = \frac{1}{4}\int_0^t [a^2(s) - b^2(s)]ds \tag{2.115}$$

The properties bounded real and positive real will be defined for functions that are analytic in the open right half plane $\mathbf{Re}[s] > 0$. We recall that a function $f(s)$ is *analytic* in a domain only if it is defined and infinitely differentiable for all points in the domain. A point where $f(s)$ ceases to be analytic is called a *singular point*, and we say that $f(s)$ has a *singularity* at this point. If $f(s)$ is rational, then $f(s)$ has a finite number of singularities, and the singularities are called *poles*. The poles are the roots of the denominator polynomial $R(s)$ if $f(s) = Q(s)/R(s)$, and a pole is said to be simple pole if it is not a multiple root in $R(s)$.

Definition 2.24. *A function $g(s)$ is said to be bounded real if*

1. *$g(s)$ is analytic in* $\mathbf{Re}[s] > 0$.
2. *$g(s)$ is real for real and positive s.*
3. *$|g(s)| \leq 1$ for all* $\mathbf{Re}[s] > 0$.

This definition extends to matrix functions $G(s)$ as follows:

Definition 2.25. *A transfer matrix $G(s) \in \mathbb{C}^{m\times m}$ is bounded real if all elements of $G(s)$ are analytic for $\mathbf{Re}[s] \geq 0$ and the $H_\infty-$norm satisfies $||G(s)||_\infty \leq 1$ where we recall that $||G(s)||_\infty = \sup_{\omega\in I\!R} \sigma_{\max}(G(j\omega))$. Equivalently the second condition can be replaced by: $I_m - G^T(-j\omega)G(j\omega) \geq 0$ for all $\omega \in I\!R$. Strict Bounded Realness holds when the ≥ 0 inequalities are replaced by > 0 inequalities.*

■

Theorem 2.26. *Consider a linear time-invariant system described by $y(s) = h(s)u(s)$, and the associated scattering function $a = y + u$, $b = y - u$ and $b(s) = g(s)a(s)$ where*

$$g(s) = \frac{h(s) - 1}{1 + h(s)} \tag{2.116}$$

which satisfies $b(s) = g(s)a(s)$ $a = y + u$ and $b = y - u$. Then the system $y(s) = h(s)u(s)$ is passive if and only if $g(s)$ is bounded real. ■

Proof: Assume that $y(s) = h(s)u(s)$ is passive. Then (2.115) implies that

$$\int_0^t a^2(\tau)d\tau \geq \int_0^t b^2(\tau)d\tau \tag{2.117}$$

for all $t \geq 0$. It follows that $g(s)$ cannot have any singularities in $\mathbf{Re}[s] > 0$ as this would result in exponential growth in $b(t)$ for any small input $a(t)$. Thus, $g(s)$ must satisfy condition 1 in the definition of bounded real.

Let σ_0 be an arbitrary real and positive constant, and let $a(t) = e^{\sigma_0 t}\mathbf{1}(t)$ where $\mathbf{1}(t)$ is the unit step function. Then the Laplace transform of $a(t)$ is $a(s) = \frac{1}{s-\sigma_0}$, while $b(s) = \frac{g(s)}{s-\sigma_0}$. Suppose that the system is not initially excited so that the inverse Laplace transform for rational $g(s)$ gives

$$b(t) = \sum_{i=1}^{n} \left(\mathrm{Res}_{s=s_i} \frac{g(s)}{s-\sigma_0}\right) e^{s_i t} + \left(\mathrm{Res}_{s=\sigma_0} \frac{g(s)}{s-\sigma_0}\right) e^{\sigma_0 t}$$

where s_i are the poles of $g(s)$ that satisfy $\mathbf{Re}\,[s_i] < 0$, and $\mathrm{Res}_{s=\sigma_0} \frac{g(s)}{s-\sigma_0} = g(\sigma_0)$. When $t \to \infty$ the term including $e^{\sigma_0 t}$ will dominate the terms including

$e^{s_i t}$, and $b(t)$ will tend to $g(\sigma_0)e^{\sigma_0 t}$. The same limit for $b(t)$ will also be found for irrational $g(s)$. As $a(t)$ is real, it follows that $g(\sigma_0)$ is real, and it follows that $g(s)$ must satisfy condition 2 in the definition of bounded realness.

Let $s_0 = \sigma_0 + j\omega_0$ be an arbitrary point in $\mathbf{Re}[s] > 0$, and let the input be $a(t) = \mathbf{Re}[e^{s_0 t}\mathbf{1}(t)]$. Then $b(t) \to \mathbf{Re}[g(s_0)e^{s_0 t}]$ as $t \to \infty$ and the power

$$P(t) := \frac{1}{4}[a^2(t) - b^2(t)] \tag{2.118}$$

will tend to

$$P(t) = \frac{1}{4}[e^{2\sigma_0 t}\cos^2\omega_0 t - |g(s_0)|^2 e^{2\sigma_0 t}\cos^2(\omega_0 t + \phi)]$$

where $\phi = \arg[g(s_0)]$. This can be rewritten using $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$, and the result is

$$\begin{aligned} 8P(t) &= (1+\cos 2\omega_0 t)e^{2\sigma_0 t} - |g(s_0)|^2[1+\cos(2\omega_0 t + 2\phi)]e^{2\sigma_0 t} \\ &= [1-|g(s_0)|^2]e^{2\sigma_0 t} + \mathbf{Re}[\left(1-g(s_0)^2\right)e^{2s_0 t}] \end{aligned}$$

In this expression s_0 and σ_0 are constants, and we can integrate $P(t)$ to get the energy function $V(T)$:

$$\begin{aligned} V(t) &= \textstyle\int_{-\infty}^{t} P(s)ds \\ &= \tfrac{1}{16\sigma_0}[1-|g(s_0)|^2]e^{2\sigma_0 t} + \tfrac{1}{16}\mathbf{Re}\{\tfrac{1}{s_0}[1-g(s_0)^2]e^{2s_0 t}\} \end{aligned}$$

First it is assumed that $\omega_0 \neq 0$. Then $\mathbf{Re}\{\frac{1}{s_0}[1-g(s_0)^2]e^{2s_0 t}\}$ will be a sinusoidal function which becomes zero for certain values of t. For such values of t the condition $V(t) \geq 0$ implies that

$$\frac{1}{16\sigma_0}[1-|g(s_0)|^2]e^{2\sigma_0 t} \geq 0$$

which implies that

$$1 - |g(s_0)|^2 \geq 0$$

Next it is assumed that $\omega_0 = 0$ such that $s_0 = \sigma_0$ is real. Then $g(s_0)$ will be real, and the two terms in $V(t)$ become equal. This gives

$$0 \leq V(t) = \frac{1}{8\sigma_0}[1-g^2(s_0)]e^{2\sigma_0 t}$$

and with this it is established that for all s_0 in $\mathbf{Re}[s] > 0$ we have

$$1 - |g(s_0)|^2 \geq 0 \Rightarrow |g(s_0)| \leq 1$$

To show the converse we assume that $g(s)$ is bounded real and consider

$$g(j\omega) = \lim_{\substack{\sigma \to 0 \\ \sigma > 0}} g(\sigma + j\omega) \tag{2.119}$$

Because $g(s)$ is bounded and analytic for all $\mathbf{Re}\,[s] > 0$ it follows that this limit exists for all ω, and moreover

$$|g(j\omega)| \leq 1$$

Then it follows from Parseval's Theorem that with a_T being the truncated version of a we have

$$\begin{aligned} 0 &\leq \tfrac{1}{8\pi} \textstyle\int_{-\infty}^{\infty} |a_t(j\omega)|^2 \left(1 - |g(j\omega)|^2\right) d\omega \\ &= \tfrac{1}{4} \textstyle\int_0^t [a^2(s) - b^2(s)] ds \\ &= \textstyle\int_0^t u(s)y(s) ds \end{aligned}$$

which shows that the system must be passive. ■

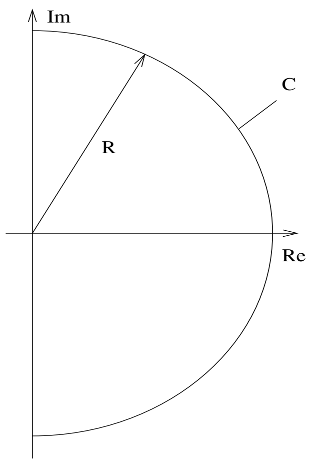

Fig. 2.13. Contour in the right half plane.

Define the contour C which encloses the right half plane as shown in Figure 2.13. The maximum modulus theorem is as follows. Let $f(s)$ be a function that is analytic inside the contour C. Let M be the upper bound on $|f(s)|$ on C. Then $|f(s)| \leq M$ inside the contour, and equality is achieved at some point inside C if and only if $f(s)$ is a constant. This means that if $g(s)$ is bounded real, and $|g(s)| = 1$ for some point in $\mathbf{Re}[s] > 0$, then $|g(s)|$ achieves its maximum inside the contour C, and it follows that $g(s)$ is a constant in $\mathbf{Re}[s] \geq 0$. Because $g(s)$ is real for real $s > 0$, this means that $g(s) = 1$ for all

s in $\mathbf{Re}[s] \geq 0$. In view of this $[1 - g(s)]^{-1}$ has singularities in $\mathbf{Re}[s] > 0$ if and only if $g(s) = 1$ for all s in $\mathbf{Re}[s] \geq 0$.

If $g(s)$ is assumed to be a rational function the maximum modulus theorem can be used to reformulate the condition on $|g\,(s)|$ to be a condition on $|g\,(j\omega)|$. The reason for this is that a rational transfer function satisfying $|g(j\omega)| \leq 1$ for all ω will also satisfy

$$\lim_{\omega \to \infty} |g(j\omega)| = \lim_{|s| \to \infty} |g(s)| \tag{2.120}$$

Therefore, for a sufficiently large contour C we have that $|g(j\omega)| \leq 1$ implies $|g(s)| \leq 1$ for all $\mathbf{Re}[s] > 0$ whenever $g(s)$ is rational. This leads to the following result:

Theorem 2.27. *A rational function $g(s)$ is bounded real if and only if*

1. *$g(s)$ has no poles in $\mathbf{Re}[s] \geq 0$.*
2. *$|g(j\omega)| \leq 1$ for all $\omega \in [-\infty, +\infty]$.*

■

Let us now state a new definition.

Definition 2.28. *A transfer function $h(s)$ is said to be positive real (PR) if*

1. *$h(s)$ is analytic in $\mathbf{Re}[s] > 0$*
2. *$h(s)$ is real for positive real s*
3. *$\mathbf{Re}[h(s)] \geq 0$ for all $\mathbf{Re}[s] > 0$*

The last condition above is illustrated in Figure 2.14 where the Nyquist plot of a PR transfer function $H(s)$ is shown. The notion of positive realness extends to multivariable systems:

Definition 2.29. *The transfer matrix $H(s) \in \mathbb{C}^{m \times m}$ is positive real if:*

- *$H(s)$ has no pole in $\mathbf{Re}[s] > 0$*
- *$H(s)$ is real for all positive real s*
- *$H(s) + H^\star(s) \geq 0$ for all $\mathbf{Re}[s] > 0$* ■

An interesting characterization of multivariable PR transfer functions is as follows:

Theorem 2.30. *Let the transfer matrix $H(s) = C(sI_n - A)^{-1} + D \in \mathbb{C}^{m \times m}$, where the matrices A, B, C, and D are real, and every eigenvalue of A has a negative real part. Then $H(s)$ is positive real if and only if $y^\star[H^\star(j\omega) + H(j\omega)]y = y^\star \Pi(j\omega) y \geq 0$ for all $\omega \in \mathbb{R}$ and all $y \in \mathbb{C}^m$.* ■

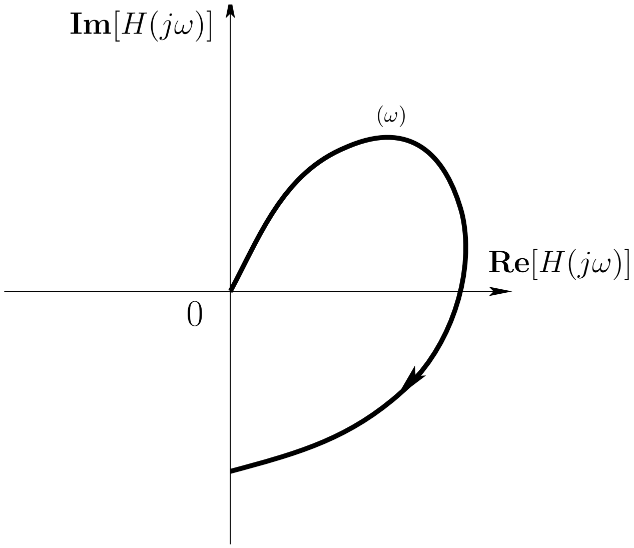

Fig. 2.14. Positive real transfer function

This result was proved in [8, p.53]. The rational matrix $\Pi(s) = c(sI_n - A)^{-1}B - B^T(sI_n + A^T)^{-1}C^T + D + D^T$ is known as the Popov function of the system. It is a rational spectral function, since it satisfies $\Pi(s) = \Pi^T(-s)$. The introduction of the spectral function $\Pi(s)$ allows us to state a result on which we shall come back in Section 3.3. Let $\Lambda : \mathcal{L}_{2,e} \to \mathcal{L}_{2,e}$ be a rational input-output operator $u(\cdot) \mapsto y(\cdot) = \Lambda(u(\cdot))$. Assume that the kernel of Λ has a minimal realization (A, B, C, D). In other words, the operator is represented in the Laplace transform space by a transfer matrix $H(s) = C(sI_n - A)^{-1}B + D$, where (A, B) is controllable and (A, C) is observable. The rational matrix $\Pi(s)$ is the spectral function associated to Λ.

Proposition 2.31. *The rational operator Λ is non-negative, i.e.*

$$\int_0^t u(\tau)\Lambda(u(\tau))d\tau \geq 0$$

for all $u \in \mathcal{L}_{2,e}$, if and only if its associated spectral function $\Pi(s)$ is non-negative. ■

Proof: We assume that $u(t) = 0$ for all $t < 0$ and that the system is causal. Let the output $y(\cdot)$ be given as

$$y(t) = Du(t) + \int_0^t Ce^{A(t-\tau}Bu(\tau)d\tau \tag{2.121}$$

Let $U(s)$ and $Y(s)$ denote the Laplace transforms of $u(\cdot)$ and $y(\cdot)$, respectively. Let us assume that $\Pi(s)$ has no pole on the imaginary axis. From Parseval's Theorem one has

$$\int_{-\infty}^{+\infty}[y^T(t)u(t)+u^T(t)y(t)]dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}[Y^\star(j\omega)U(j\omega)+U^\star(j\omega)Y(j\omega)]d\omega \tag{2.122}$$

One also has $Y(s)=\big(D+C(sI_n-A)^{-1}B\big)\,U(s)$. Therefore

$$\int_{-\infty}^{+\infty}[y^T(t)u(t)+u^T(t)y(t)]dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}U^\star(j\omega)\Pi(j\omega)U(j\omega)d\omega. \tag{2.123}$$

It follows that:

- $\Pi(j\omega)\geq 0$ for all $\omega\in I\!R$ implies that $\int_{-\infty}^{+\infty}[y^T(t)u(t)+u^T(t)y(t)]dt\geq 0$ for all admissible $u(\cdot)$.
- Reciprocally, given a couple (ω_0,U_0) that satisfies $U_0^T\Pi(j\omega_0)U_0<0$, there exists by continuity an interval Ω_0 such that $U_0^T\Pi(j\omega)U_0<0$ for all $\omega\in\Omega_0$. Consequently the inverse Fourier transform $v_0(\cdot)$ of the function

$$U(j\omega)=\begin{cases}U_0 \text{ if } \omega\in\Omega_0\\ \\ 0 \quad \text{if } \omega\notin\Omega_0\end{cases} \tag{2.124}$$

makes the quadratic form $\frac{1}{2\pi}\int_{\Omega_0}U_0^T\Pi(j\omega)U_0 d\omega<0$. Therefore positivity of Λ and of its spectral function are equivalent properties.

If $\Pi(s)$ has poles on the imaginary axis, then Parseval's Theorem can be used under the form

$$\int_{-\infty}^{+\infty}e^{-2at}[y^T(t)u(t){+}u^T(t)y(t)]dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}U^\star(a{+}j\omega)S(a{+}j\omega)U(a{+}j\omega)d\omega \tag{2.125}$$

which is satisfied for all real a, provided the line $a+j\omega$ does not contain any pole of $\Pi(s)$. ∎

Remark 2.32. It is implicit in the proof of Proposition 2.31 that the initial data on $y(\cdot)$ and $u(\cdot)$ and their derivatives, up to the required orders, are zero. Consequently, the positivity of the operator $\Lambda(\cdot)$, when associated to a state space representation (A,B,C,D), is characterized with the initial state $x(0)=0$. Later on in Chapter 4, we shall give a definition of dissipativity, which generalizes that of positivity for a rational operator such as $\Lambda(\cdot)$, and which precisely applies with $x(0)=0$; see Definition 4.22.

It is sometimes taken as a definition that a spectral function $\Pi(s)$ is nonnegative if there exists a PR function $H(s)$ such that $\Pi(s)=H(s)+H^T(-s)$

[145, Definition 6.2]. We shall make use of Proposition 2.31 in Section 5.10 on hyperstability. Notice that Proposition 2.31 does not imply the stability of the above mentioned operator (provided one has associated a state space realization to this operator). The stability is in fact obtained if one makes further assumptions like the observability and controllability. We shall come back on these points in the next chapters on dissipative systems and their stability, via the Kalman-Yakubovich-Popov Lemma; see Remark 3.32.

The next theorem links bounded realness with positive realness.

Theorem 2.33. *Consider the linear time-invariant system $y(s) = h(s)u(s)$, and the scattering formulation $a = y + u$, $b = y - u$ and $b(s) = g(s)a(s)$ where*

$$g(s) = \frac{h(s) - 1}{1 + h(s)} \tag{2.126}$$

Assume that $g(s) \neq 1$ for all $\mathbf{Re}[s] > 0$. Then $h(s)$ is positive real if and only if $g(s)$ is bounded real. ■

Proof: Assume that $g(s)$ is bounded real and that $g(s) \neq 1$ for all $\mathbf{Re}[s] > 0$. Then $[1 - g(s)]^{-1}$ exists for all s in $\mathbf{Re}[s] > 0$. From (2.126) we find that

$$h(s) = \frac{1 + g(s)}{1 - g(s)} \tag{2.127}$$

where $h(s)$ is analytic in $\mathbf{Re}[s] > 0$ as $g(s)$ is analytic in $\mathbf{Re}[s] > 0$, and $[1 - g(s)]^{-1}$ is nonsingular by assumption in $\mathbf{Re}[s] > 0$. To show that $\mathbf{Re}[h(s)] \geq 0$ for all $\mathbf{Re}[s] > 0$ the following computation is used:

$$\begin{aligned} 2\mathbf{Re}[h(s)] &= h^*(s) + h(s) \\ &= \tfrac{1+g^*(s)}{1-g^*(s)} + \tfrac{1+g(s)}{1-g(s)} \\ &= 2\tfrac{1-g^*(s)g(s)}{[1-g^*(s)][1-g(s)]} \end{aligned} \tag{2.128}$$

We see that $\mathbf{Re}[h(s)] \geq 0$ for all $\mathbf{Re}[s] > 0$ whenever $g(s)$ is bounded real.

Next assume that $h(s)$ is positive real. Then $h(s)$ is analytic in $\mathbf{Re}[s] > 0$, and $[1 + h(s)]$ is nonsingular in $\mathbf{Re}[s] > 0$ as $\mathbf{Re}[h(s)] \geq 0$ in $\mathbf{Re}[s] > 0$. It follows that $g(s)$ is analytic in $\mathbf{Re}[s] > 0$. From (2.128) it is seen that $|g(s)| \leq 1$ in $\mathbf{Re}[s] > 0$; it follows that $g(s)$ is bounded real. ■

It is noteworthy that Theorem 2.33 extends to multivariable systems:

Theorem 2.34. *Let $H(s) \in \mathbb{C}^{m \times m}$ be a square transfer function, with det$(H(s) + H(-s)) \neq 0$ for $\mathbf{Re}[s] \geq 0$, and $H(j\infty) + H(^T(j\infty) \geq 0$. Then the bounded realness of $G(s) = (G(s) - I_m)(G(s) + I_m)^{-1}$ implies that $H(s)$ is positive real.* ■

From Theorem 2.26 and Theorem 2.33 it follows that:

Corollary 2.35. *A system with transfer function $h(s)$ is passive if and only if the transfer function $h(s)$ is positive real.*

Example 2.36. A fundamental result in electrical circuit theory is that if the transfer function $h(s)$ is rational and positive real, then there exists an electrical one-port built from resistors, capacitors and inductors so that $h(s)$ is the impedance of the one-port [126, p. 815]. If e is the voltage over the one-port and i is the current entering the one-port, then $e(s) = h(s)i(s)$. The system with input i and output e must be passive because the total stored energy of the circuit must satisfy

$$\dot{V}(t) = e(t)i(t) - g(t) \tag{2.129}$$

where $g(t)$ is the dissipated energy.

Example 2.37. The transfer function

$$h(s) = \frac{1}{\tanh s} \tag{2.130}$$

is irrational, and positive realness of this transfer function cannot be established from conditions on the frequency response $h(j\omega)$. We note that $\tanh s = \sinh s / \cosh s$, where $\sinh s = \frac{1}{2}(e^s - e^{-s})$ and $\cosh s = \frac{1}{2}(e^s + e^{-s})$. First we investigate if $h(s)$ is analytic in the right half plane. The singularities are given by

$$\sinh s = 0 \Rightarrow e^s - e^{-s} = 0 \Rightarrow e^s(1 - e^{-2s}) = 0$$

Here $|e^s| \geq 1$ for $\mathbf{Re}[s] > 0$, while

$$e^s(1 - e^{-2s}) = 0 \Rightarrow e^{-2s} = 1$$

Therefore the singularities are found to be

$$s_k = jk\pi, \quad k \in \{0, \pm 1, \pm 2 \ldots\} \tag{2.131}$$

which are on the imaginary axis. This means that $h(s)$ is analytic in $\mathbf{Re}[s] > 0$. Obviously, $h(s)$ is real for real $s > 0$. Finally we check if $\mathbf{Re}\,[h(s)]$ is positive in $\mathbf{Re}[s] > 0$. Let $s = \sigma + j\omega$. Then

$$\begin{aligned}\cosh s &= \tfrac{1}{2}[e^{\sigma}(\cos\omega + j\sin\omega) + e^{-\sigma}(\cos\omega - j\sin\omega)] \\ &= \cosh\sigma\cos\omega + j\sinh\sigma\sin\omega\end{aligned}$$

while

$$\sinh s = \sinh\sigma\cos\omega + j\cosh\sigma\sin\omega \tag{2.132}$$

This gives

$$\mathbf{Re}[h(s)] = \frac{\cosh\sigma\sinh\sigma}{|\sinh s|^2} > 0, \quad \mathbf{Re}\,[s] > 0 \tag{2.133}$$

where it is used that $\sigma = \mathbf{Re}\,[s]$, and the positive realness of $h(s)$ has been established. ■

Consider a linear system represented by a rational function $H(s)$ of the complex variable $s = \sigma + j\omega$:

$$H(s) = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \tag{2.134}$$

where $a_i, b_i \in I\!R$ are the system parameters n is the order of the system and $r = n - m$ is the relative degree. For rational transfer functions it is possible to find conditions on the frequency response $h(j\omega)$ for the transfer function to be positive real. The result is presented in the following theorem:

Theorem 2.38. *A rational function $h(s)$ is positive real if and only if*

1. *$h(s)$ has no poles in $\mathbf{Re}[s] > 0$.*
2. *$\mathbf{Re}[h(j\omega)] \geq 0$ for all $\omega \in [-\infty, +\infty]$ such that $j\omega$ is not a pole in $h(s)$.*
3. *If $s = j\omega_0$ is a pole in $h(s)$, then it is a simple pole, and if ω_0 is finite, then the residual*

$$\mathrm{Res}_{s=j\omega_0} h(s) = \lim_{s\to j\omega_0} (s - j\omega_0)h(s)$$

is real and positive. If ω_0 is infinite, then the limit

$$R_\infty := \lim_{\omega\to\infty} \frac{h(j\omega)}{j\omega}$$

is real and positive. ■

Proof: The proof can be established by showing that conditions 2 and 3 in this Theorem are equivalent to the condition

$$\mathbf{Re}[h(s)] \geq 0 \tag{2.135}$$

for all $\mathbf{Re}[s] > 0$ for $h(s)$ with no poles in $\mathbf{Re}[s] > 0$.

First assume that conditions 2 and 3 hold. We use a contour C as shown in Figure 2.15 which goes from $-j\Omega$ to $j\Omega$ along the $j\omega$ axis with small semicircular indentations into the right half plane around points $j\omega_0$ that are poles of $h(s)$. The contour C is closed with a semicircle into the right half plane. On the part of C that is on the imaginary axis $\mathbf{Re}[h(s)] \geq 0$ by assumption. On the small indentations

$$h(s) \approx \frac{\mathrm{Res}_{s=j\omega_0} h(s)}{s - j\omega_0} \tag{2.136}$$

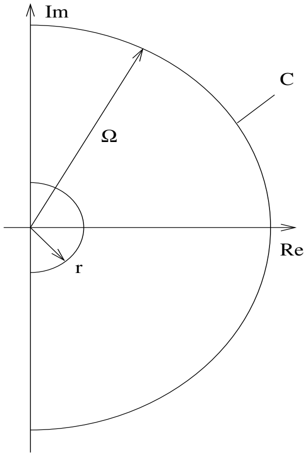

Fig. 2.15. Contour C of $h(s)$ in the right half plane.

As $\mathbf{Re}[s] \geq 0$ on the small semi-circles and $\mathrm{Res}_{s=j\omega_0} h(s)$ is real and positive according to condition 3, it follows that $\mathbf{Re}[h(s)] \geq 0$ on these semi-circles. On the large semi-circle into the right half plane with radius Ω we also have $\mathbf{Re}[h(s)] \geq 0$ and the value is a constant equal to $\lim_{\omega\to\infty} \mathbf{Re}[h(j\omega)]$, unless $h(s)$ has a pole at infinity at the $j\omega$ axis, in which case $h(s) \approx sR_\infty$ on the large semi-circle. Thus we may conclude that $\mathbf{Re}[h(s)] \geq 0$ on C. Define the function

$$f(s) = e^{-\mathbf{Re}[h(s)]}$$

Then $|f(s)| \leq 1$ on C, and in view of the maximum modulus theorem, $|f(s)| \leq 1$ for all $s \in \mathbf{Re}[s] > 0$. It follows that $\mathbf{Re}[h(s)] \geq 0$ in $\mathbf{Re}[s] > 0$, and the result is shown.

Next assume that $\mathbf{Re}[h(s)] \geq 0$ for all $\mathbf{Re}[s] > 0$. Then condition 2 follows because

$$h(j\omega) = \lim_{\substack{\sigma\to 0\\ \sigma>0}} h(\sigma + j\omega)$$

exists for all ω such that $j\omega$ is not a pole in $h(s)$. To show condition 3 we assume that ω_0 is a pole of multiplicity m for $h(s)$. On the small indentation with radius r into the right half plane we have $s - j\omega_0 = re^{j\theta}$ where $-\pi/2 \leq \theta \leq \pi/2$. Then

$$h(s) \approx \frac{\mathrm{Res}_{s=j\omega_0} h(s)}{r^m e^{jm\theta}} = \frac{\mathrm{Res}_{s=j\omega_0} h(s)}{r^m} e^{-jm\theta} \tag{2.137}$$

Clearly, here it is necessary that $m = 1$ to achieve $\mathbf{Re}[h(s)] \geq 0$ because the term $e^{-jm\theta}$ gives an angle from $-m\pi/2$ to $m\pi/2$ in the complex plane. Moreover, it is necessary that $\mathrm{Res}_{s=j\omega_0} h(s)$ is positive and real because $e^{-jm\theta}$ gives an angle from $-\pi/2$ to $\pi/2$ when $m = 1$. The result follows. ∎

The foregoing theorem extends to multivariable systems:

Theorem 2.39. *The rational function $H(s) \in \mathbb{C}^{m \times m}$ is positive real if and only if:*

- *$H(s)$ has no poles in $\mathbf{Re}[s] > 0$*
- *$H(j\omega) + H^{\star}(j\omega) \geq 0$ for all positive real ω such that $j\omega$ is not a pole of $H(\cdot)$*
- *If $i\omega_0$, finite or infinite, is a pole of $H(\cdot)$, it is a simple pole and the corresponding residual K_0 is a semi positive definite Hermitian matrix.*

■

2.13 Examples

2.13.1 Mechanical Resonances

Motor and Load with Elastic Transmission

An interesting and important type of system is a motor that is connected to a load with an elastic transmission. The motor has moment of inertia J_m, the load has moment of inertia J_L, while the transmission has spring constant K and damper coefficient D. The dynamics of the motor is given by

$$J_m \ddot{\theta}_m(t) = T_m(t) - T_L(t) \tag{2.138}$$

where $\theta_m(\cdot)$ is the motor angle, $T_m(\cdot)$ is the motor torque, which is considered to be the control variable, and $T_L(\cdot)$ is the torque from the transmission. The dynamics of the load is

$$J_L \ddot{\theta}_L(t) = T_L(t) \tag{2.139}$$

The transmission torque is given by

$$T_L = -D\left(\dot{\theta}_L - \dot{\theta}_m\right) - K\left(\theta_L - \theta_m\right) \tag{2.140}$$

The load dynamics can then be written in Laplace transform form as

$$\left(J_L s^2 + Ds + K\right)\theta_L(s) = (Ds + K)\,\theta_m(s) \tag{2.141}$$

which gives

$$\frac{\theta_L}{\theta_m}(s) = \frac{1 + 2Z\frac{s}{\Omega_1}}{1 + 2Z\frac{s}{\Omega_1} + \frac{s^2}{\Omega_1^2}} \tag{2.142}$$

where

$$\Omega_1^2 = \frac{K}{J_L} \tag{2.143}$$

and

$$\frac{2Z}{\Omega_1} = \frac{D}{K} \tag{2.144}$$

By adding the dynamics of the motor and the load we get

$$J_m \ddot{\theta}_m(t) + J_L \ddot{\theta}_L(t) = T_m(t) \tag{2.145}$$

which leads to

$$J_m s^2 \theta_m(s) + J_L s^2 \frac{1 + 2Z\frac{s}{\Omega_1}}{1 + 2Z\frac{s}{\Omega_1} + \frac{s^2}{\Omega_1^2}} \theta_m(s) = T_m(s) \tag{2.146}$$

and from this

$$\frac{\theta_m}{T_m}(s) = \frac{1 + 2Z\frac{s}{\Omega_1} + \frac{s^2}{\Omega_1^2}}{Js^2(1 + 2\zeta\frac{s}{\omega_1} + \frac{s^2}{\omega_1^2})} \tag{2.147}$$

where

$$J = J_m + J_L \tag{2.148}$$

is the total inertia of motor and load, and the resonant frequency ω_1 is given by

$$\omega_1^2 = \frac{1}{1 - \frac{J_L}{J}} \Omega_1^2 = \frac{J}{J_m} \Omega_1^2 \tag{2.149}$$

while the relative damping is given by

$$\zeta = \sqrt{\frac{J}{J_m}} Z \tag{2.150}$$

We note that the parameters ω_1 and ζ depend on both motor and load parameters, while the parameters Ω_1 and Z depend only on the load.

The main observation in this development is the fact that $\Omega_1 < \omega_1$. This means that the transfer function $\theta_m(s)/T_m(s)$ has a complex conjugated pair of zeros with resonant frequency Ω_1, and a pair of poles at the somewhat higher resonant frequency ω_1. The frequency response is shown in Figure 2.16 when $K = 20$, $J_m = 20$, $J_L = 15$ and $D = 0.5$. Note that the elasticity does not give any negative phase contribution.

By multiplying the transfer functions $\theta_L(s)/\theta_m(s)$ and $\theta_m(s)/T_m(s)$ the transfer function

$$\frac{\theta_L}{T_m}(s) = \frac{1 + 2Z\frac{s}{\Omega_1}}{Js^2(1 + 2\zeta\frac{s}{\omega_1} + \frac{s^2}{\omega_1^2})} \tag{2.151}$$

is found from the motor torque to the load angle.

The resulting frequency response is shown in Figure 2.17. In this case the elasticity results in a negative phase contribution for frequencies above ω_1.

Example 2.40. Typically the gear is selected so that $J_m = J_L$. This gives

$$\Omega_1 = \frac{1}{\sqrt{2}} \omega_1 = 0.707\omega_1 \tag{2.152}$$

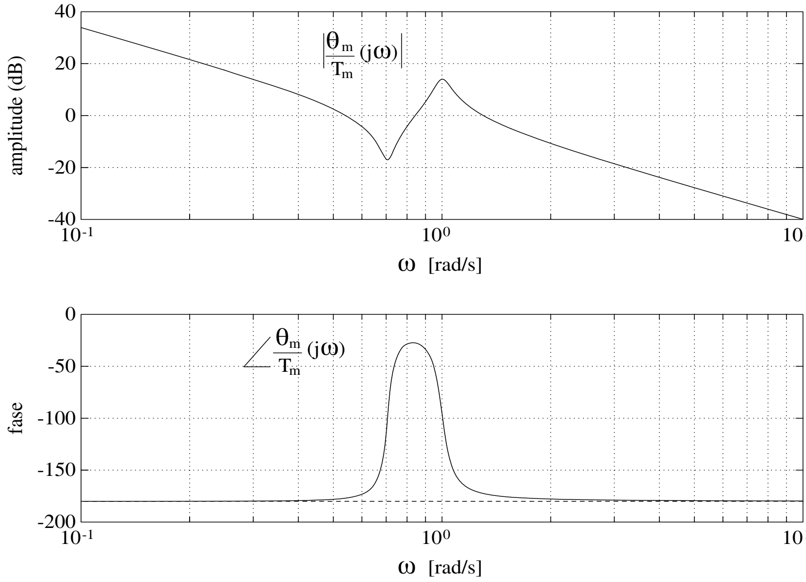

Fig. 2.16. Frequency response of $\theta_m(s)/T_m(s)$

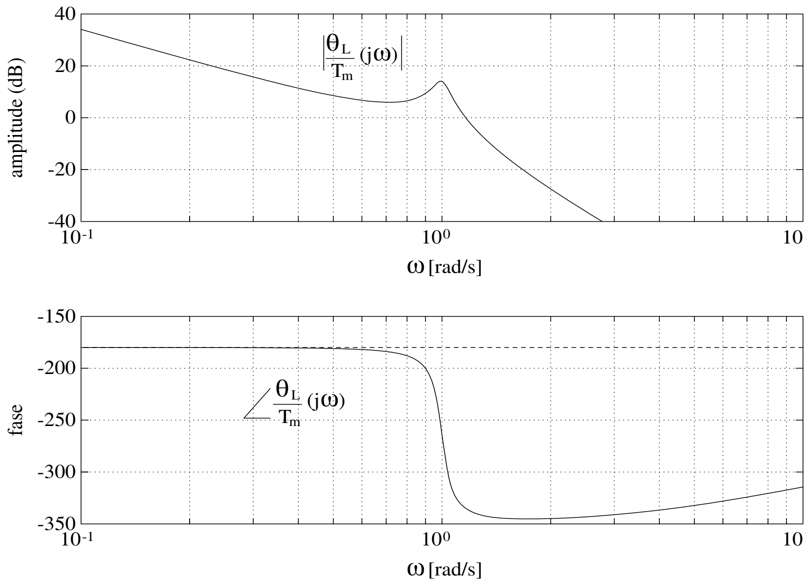

Fig. 2.17. Frequency response of $\theta_L(s)/\theta_m(s)$

Example 2.41. Let $Z = 0.1$ and $J_m = J_L$. In this case

$$\frac{\theta_L}{T_m}(s) = \frac{1 + \frac{s}{3.535\omega_1}}{Js^2(1 + 2\zeta\frac{s}{\omega_1} + \frac{s^2}{\omega_1^2})} \tag{2.153}$$

Passivity Inequality

The total energy of motor and load is given by

$$V(\omega_m, \omega_L, \theta_L, \theta_m) = \frac{1}{2}J_m\omega_m^2 + \frac{1}{2}J_L\omega_L^2 + \frac{1}{2}K[\theta_L - \theta_m]^2 \tag{2.154}$$

where $\omega_m(t) = \dot{\theta}_m(t)$ and $\omega_L(t) = \dot{\theta}_L(t)$. The rate of change of the total energy is equal to the power supplied from the control torque $T_m(t)$ minus the power dissipated in the system. This is written

$$\dot{V}(t) = \omega_m(t)T_m(t) - D[\omega_L(t) - \omega_m(t)]^2 \tag{2.155}$$

We see that the power dissipated in the system is $D[\omega_L(t) - \omega_m(t)]^2$ which is the power loss in the damper. Clearly the energy function $V(t) \geq 0$ and the power loss satisfies $D[\Delta\omega(t)]^2 \geq 0$,. It follows that

$$\int_0^t \omega_m(s)T_m(s)ds = V(t) - V(0) + \int_0^t D[\Delta\omega(s)]^2 ds \geq -V(0) \tag{2.156}$$

which implies that the system with input $T_m(\cdot)$ and output $\omega_m(\cdot)$ is passive. It follows that

$$\mathbf{Re}[h_m(j\omega)] \geq 0 \tag{2.157}$$

for all $\omega \in [-\infty, +\infty]$. From energy arguments we have been able to show that

$$-180^\circ \leq \angle\frac{\theta_m}{T_m}(j\omega) \leq 0^\circ. \tag{2.158}$$

2.13.2 Systems with Several Resonances

Passivity

Consider a motor driving n inertias in a serial connection with springs and dampers. Denote the motor torque by T_m and the angular velocity of the motor shaft by ω_m. The energy in the system is

$$
\begin{aligned}
V(\omega_m, \theta_m, \theta_{Li}) = \tfrac{1}{2}J_m\omega_m^2 &+ \tfrac{1}{2}K_{01}(\theta_m - \theta_{L1})^2 \\
&+\tfrac{1}{2}J_{L1}\omega_{L1}^2 + \tfrac{1}{2}K_{12}(\theta_{L1} - \theta_{L2})^2 + \ldots \\
&+\tfrac{1}{2}J_{L,n-1}\omega_{L,n-1}^2 + \tfrac{1}{2}K_{n-1,n}(\theta_{L,n-1} - \theta_{Ln})^2 \\
&+\tfrac{1}{2}J_{Ln}\omega_{Ln}^2
\end{aligned}
$$

Clearly, $V(\cdot) \geq 0$. Here J_m is the motor inertia, ω_{Li} is the velocity of inertia J_{Li}, while $K_{i-1,i}$ is the spring connecting inertia $i-1$ and i and $D_{i-1,i}$ is the coefficient of the damper in parallel with $K_{i-1,i}$. The index runs over $i = 1, 2, \ldots, n$. The system therefore satisfies the equation

$$\dot{V}(t) = T_m(t)\omega_m(t) - d(t) \tag{2.159}$$

where

$$d(t) = D_{12}(\omega_{L1}(t) - \omega_{L2}(t))^2 + \ldots + D_{n-1,n}(\omega_{L,n-1}(t) - \omega_{Ln}(t))^2 \geq 0 \tag{2.160}$$

represents the power that is dissipated in the dampers, and it follows that the system with input T_m and output ω_m is passive. If the system is linear, then the passivity implies that the transfer function

$$h_m(s) = \frac{\omega_m}{T_m}(s) \tag{2.161}$$

has the phase constraint

$$|\angle h_m(j\omega)| \leq 90^\circ \tag{2.162}$$

for all $\omega \in [-\infty, +\infty]$. It is quite interesting to note that the only information that is used to find this phase constraint on the transfer function is that the system is linear, and that the load is made up from passive mechanical components. It is not even necessary to know the order of the system dynamics, as the result holds for an arbitrary n.

2.13.3 Two Motors Driving an Elastic Load

In this section we will see how passivity considerations can be used as a guideline for how to control two motors that actuate on the same load through elastic interconnections consisting of inertias, springs and dampers as shown in Figure 2.18.

The motors have inertias J_{mi}, angle q_{mi} and motor torque T_{mi} where $i \in \{1, 2\}$. Motor 1 is connected to the inertia J_{L1} with a spring with stiffness K_{11} and a damper D_{11}. Motor 2 is connected to the inertia J_{L2} with a spring with stiffness K_{22} and a damper D_{22}. Inertia J_{Li} has angle q_{Li}. The two inertias are connected with a spring with stiffness K_{12} and a damper D_{12}.

The total energy of the system is

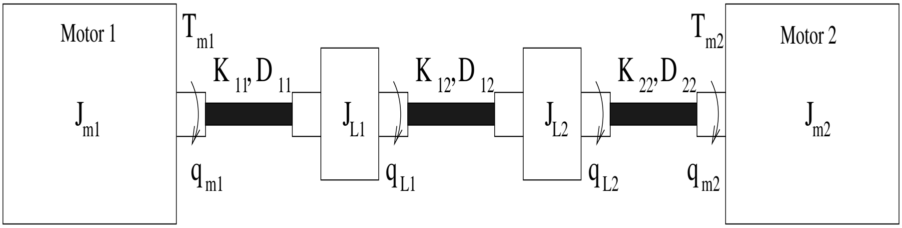

Fig. 2.18. Two motors actuating on one load

$$V(q_{m1}, q_{m2}, q_{Li}) = \frac{1}{2}[J_{m1}q_{m1}^2 + J_{m2}q_{m2}^2 + J_{L1}q_{L1}^2 + J_{L2}q_{L2}^2$$
$$+K_{11}(q_{m1} - q_{L1})^2 + K_{22}(q_{m2} - q_{L2})^2 + K_{12}(q_{L1} - q_{L2})^2]$$

and the time derivative of the energy when the system evolves is

$$\dot{V}(t) = T_{m1}\dot{q}_{m1}(t) + T_{m2}\dot{q}_{m2}(t) - D_{11}(\dot{q}_{m1}(t) - \dot{q}_{L1}(t))^2$$
$$+D_{22}(\dot{q}_{m2}(t) - \dot{q}_{L2}(t))^2 \quad +D_{12}(\dot{q}_{L1}(t) - \dot{q}_{L2}(t))^2$$

It is seen that the system is passive from $(T_{m1}, T_{m2})^T$ to $(\dot{q}_{m1}, \dot{q}_{m2})^T$. The system is multivariable, with controls T_{m1} and T_{m2} and outputs q_{m1} and q_{m2}. A controller can be designed using multivariable control theory, and passivity might be a useful tool in this connection. However, here we will close one control loop at a time to demonstrate that independent control loops can be constructed using passivity arguments. The desired outputs are assumed to be $q_{m1} = q_{m2} = 0$. Consider the PD controller

$$T_{m2} = -K_{p2}q_{m2} - K_{v2}\dot{q}_{m2} \tag{2.163}$$

for motor 2 which is passive from $\dot{q}_{m2}$ to $-T_{m2}$. The mechanical analog of this controller is a spring with stiffness K_{p2} and a damper K_{v2} which is connected between the inertia J_{m2} and a fixed point. The total energy of the system with this mechanical analog is

$$V(q_{m1}, q_{m2}, q_{L1}, q_{L2}) = \frac{1}{2}[J_{m1}q_{m1}^2 + J_{m2}q_{m2}^2 + J_{L1}q_{L1}^2 + J_{L2}q_{L2}^2$$
$$+K_{11}(q_{m1} - q_{L1})^2 + K_{22}(q_{m2} - q_{L2})^2$$
$$+K_{12}(q_{L1} - q_{L2})^2 + K_{p2}q_2^2]$$

and the time derivative is

$$\dot{V}(t) = T_{m1}(t)\dot{q}_{m1}(t) - D_{11}(\dot{q}_{m1}(t) - \dot{q}_{L1}(t))^2 + D_{22}(\dot{q}_{m2}(t) - \dot{q}_{L2}(t))^2$$
$$+D_{12}(\dot{q}_{L1}(t) - \dot{q}_{L2}(t))^2 - K_{v2}\dot{q}_2^2(t)$$

It follows that the system with input T_{m1} and output $\dot{q}_{m1}$ is passive when the PD controller is used to generate the control T_{m2}. The following controller can then be used:

$$T_1(s) = K_{v1}\beta\frac{1+T_i s}{1+\beta T_i s}\dot{q}_1(s) = K_{v1}[1+(\beta-1)\frac{1}{1+\beta T_i s}]sq_1(s) \tag{2.164}$$

This is a PI controller with limited integral action if $\dot{q}_1$ is considered as the output of the system. The resulting closed loop system will be BIBO stable independently from system and controller parameters, although in practice, unmodelled dynamics and motor torque saturation dictate some limitations on the controller parameters. As the system is linear, stability is still ensured even if the phase of the loop transfer function becomes less that $-180°$ for certain frequency ranges. Integral effect from the position can therefore be included for one of the motors, say motor 1. The resulting controller is

$$T_1(s) = K_{p1}\frac{1+T_i s}{T_i s}q_1(s) + K_{v1}sq_1 \tag{2.165}$$

In this case the integral time constant T_i must be selected *e.g.* by Bode diagram techniques so that stability is ensured.

2.14 Strictly Positive Real (SPR) Systems

Consider again the definition of Positive Real transfer function in Definition 2.28. The following is the standard definition of Strictly Positive Real (SPR) transfer functions.

> **Definition 2.42 (Strictly Positive Real).** *A rational transfer function $H(s) \in \mathbb{C}^{m\times m}$ that is not identically zero for all s, is strictly positive real (SPR) if $H(s-\epsilon)$ is PR for some $\epsilon > 0$.*

Let us now consider two simple examples:

Example 2.43. The transfer function of an asymptotically stable first order system is given by

$$H(s) = \frac{1}{s+\lambda} \tag{2.166}$$

where $\lambda > 0$. Replacing s by $\sigma + j\omega$ we get

$$H(s) = \frac{1}{(\sigma+\lambda)+j\omega} = \frac{\sigma+\lambda-j\omega}{(\sigma+\lambda)^2+\omega^2} \tag{2.167}$$

Note that $\forall$ $\mathbf{Re}[s] = \sigma > 0$ we have $\mathbf{Re}[H(s)] \geq 0$. Therefore $H(s)$ is PR. Furthermore $H(s-\epsilon)$ for $\epsilon = \frac{\lambda}{2}$ is also PR and thus $H(s)$ is also SPR.

Example 2.44. Consider now a simple integrator (*i.e.* take $\lambda = 0$ in the previous example)

$$H(s) = \frac{1}{s} = \frac{1}{\sigma + j\omega} = \frac{\sigma - j\omega}{\sigma^2 + \omega^2}. \tag{2.168}$$

It can be seen that $H(s) = \frac{1}{s}$ is PR but not SPR.

In view of Theorem 2.6, one may wonder whether an SPR transfer function is ISP, OSP. See Examples 4.62, 4.64, 4.65.

2.14.1 Frequency Domain Conditions for a Transfer Function to be SPR

The definition of SPR transfer functions given above is in terms of conditions in the s complex plane. Such conditions become relatively difficult to be verified as the order of the system increases. The following theorem establishes conditions in the frequency domain ω for a transfer function to be SPR.

Theorem 2.45 (Strictly Positive Real). *[226] A rational transfer function $h(s)$ is SPR if*

1. *$h(s)$ is analytic in $\mathbf{Re}[s] \geq 0$, i.e. the system is asymptotically stable*
2. *$\mathbf{Re}[h(j\omega)] > 0$, for all $\omega \in (-\infty, \infty)$ and*
3. *a) $\lim_{\omega^2 \to \infty} \omega^2 \mathbf{Re}[h(j\omega)] > 0$ when $r = 1$,*
   b) $\lim_{\omega^2 \to \infty} \mathbf{Re}[h(j\omega)] > 0$, $\lim_{|\omega| \to \infty} \frac{h(j\omega)}{j\omega} > 0$ when $r = -1$,

where r is the relative degree of the system. ∎

Proof: *Necessity:* If $h(s)$ is SPR, then from Definition 2.42, $h(s - \epsilon)$ is PR for some $\epsilon > 0$. Hence, there exists an $\epsilon^* > 0$ such that for each $\epsilon \in [\,0, \epsilon^*)$, $h(s - \epsilon)$ is analytic in Re$[s] < 0$. Therefore, there exists a real rational function $W(s)$ such that [8]

$$h(s - \epsilon) + h(-s + \epsilon) = W(s - \epsilon)W(-s + \epsilon) \tag{2.169}$$

where $W(s)$ is analytic and nonzero for all s in Re $[s] > -\epsilon$. Let $s = \epsilon + j\omega$; then from (2.169) we have

$$2 \text{ Re } [h(j\omega)] = |W(j\omega)|^2 > 0, \qquad \forall \omega \in (-\infty, \infty) \tag{2.170}$$

Now $h(s)$ can be expressed as

$$h(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{2.171}$$

If $m = n - 1$, *i.e.*, $r = 1$, $b_{n-1} \neq 0$, then from (2.171) it follows that $b_{n-1} > 0$ and $a_{n-1} b_{n-1} - b_{n-2} - \epsilon b_{n-1} > 0$ for $h(s - \epsilon)$ to be PR, and

$$\lim_{\omega^2 \to \infty} \omega^2 \text{ Re } [h(j\omega)] = a_{n-1}b_{n-1} - b_{n-2} \geq \epsilon b_{n-1} > 0 \tag{2.172}$$

If $m = n + 1$, *i.e.*, $r = -1$, $b_{n+1} \neq 0$, then

$$\text{Re } [h(j\omega - \epsilon)] = \frac{1}{|a(j\omega - \epsilon)|^2} \left[(b_n - b_{n+1}a_{n-1} - \epsilon b_{n+1})\, \omega^{2n} + \cdots \right] \tag{2.173}$$

Since Re $[h(j\omega - \epsilon)] \geq 0 \; \forall \; \omega \in (-\infty, \infty)$ and

$$\lim_{|\omega| \to \infty} \frac{h(j\omega - \epsilon)}{j\omega} = b_{n-1} \geq 0,$$

then $b_{n+1} > 0$, $b_n - b_{n+1}a_{n-1} \geq \epsilon b_{n+1} > 0$, and therefore 3.b) follows directly.

Sufficiency; Let (A, b, c, d, f) be a minimal state representation of $h(s)$, *i.e.*,

$$h(s) = c(sI - A)^{-1}b + d + fs \tag{2.174}$$

From (2.174) we can write

$$h(s-\epsilon) = c(sI-A)^{-1}b+d+fs+\epsilon \left[c(sI - A - \epsilon I)^{-1}(sI - A)^{-1}b - f \right] \tag{2.175}$$

Hence,

$$\mathbf{Re}\,[h(j\omega - \epsilon)] = \mathbf{Re}\,[h(j\omega)] + \epsilon \mathbf{Re}\,[g(j\omega - \epsilon)] \tag{2.176}$$

where $g(j\omega - \epsilon) = c(j\omega I_n - A - \epsilon I)^{-1}(j\omega I_n - A)^{-1}b - f$. There exists an $\epsilon^* > 0$ such that for all $\epsilon \in [0, \epsilon^*)$ and $\omega \in (-\infty, \infty)$, $(j\omega I_n - A - \epsilon I)^{-1}$ is analytic. Therefore for each $\epsilon \in [0, \epsilon^*)$, $|\mathbf{Re}\;[g(j\omega - \epsilon)]| < k_1 < \infty$ for all $\omega \in (-\infty, \infty)$ and some $k_1 > 0$. If $r = 0$, then Re $[h(j\omega)] > k_2 > 0$ for all ω and some $k_2 > 0$. Therefore

$$\mathbf{Re}\,[h(j\omega - \epsilon)] = \mathbf{Re}\,[h(j\omega)] + \epsilon \mathbf{Re}\,[g(j\omega - \epsilon)] > k_2 - \epsilon k_1 > 0 \tag{2.177}$$

for all $\omega \in (-\infty, \infty)$ and $0 < \epsilon < \min\{\epsilon^*, k_2/k_1\}$. Hence, $h(s - \epsilon)$ is PR and therefore $h(s)$ is SPR.

If $r = 1$, then Re $[h(j\omega)] > k_3 > 0$ for all $|\omega| < \omega_0$ and ω^2 Re $[h(j\omega)] > k_4 > 0$ for all $|\omega| \geq \omega_0$, where ω_0, k_3, k_4 are finite positive constants. Similarly, $\left|\omega^2 \text{ Re } [g(j\omega - \epsilon)]\right| < k_5$ and $|\text{Re } [g(j\omega - \epsilon)]| < k_6$ for all $\omega \in (-\infty, \infty)$ and some finite positive constants k_5, k_6. Therefore, $\mathbf{Re}\;[h(j\omega - \epsilon)] > k_3 - \epsilon k_6$ for all $|\omega| < \omega_0$ and $\omega^2 \mathbf{Re}\;[h(j\omega - \epsilon)] > k_4 - \epsilon k_5$ for all $|\omega| \geq \omega_0$. Then, for $0 < \epsilon < \min\{k_3/k_6, \epsilon^*, k_4/k_5\}$ and $\forall \omega \in (-\infty, \infty)$, $\mathbf{Re}\;[h(j\omega - \epsilon)] > 0$. Hence, $h(s - \epsilon)$ is PR and therefore $h(s)$ is SPR.

If $r = -1$, then $d > 0$ and therefore

$$\mathbf{Re}\,[h(j\omega - \epsilon)] > d - \epsilon k_1 \tag{2.178}$$

Hence, for each ϵ in the interval $[0, \min\{\epsilon^*, d/k_1\})$, $\mathbf{Re}\ [h(j\omega - \epsilon)] > 0$ for all $\omega \in (-\infty, \infty)$. Since

$$\lim_{\omega \to \infty} \frac{h(j\omega)}{j\omega} = f > 0$$

then

$$\lim_{\omega \to \infty} \frac{h(j\omega - \epsilon)}{j\omega} = f > 0$$

and therefore, all the conditions of Definition 2.28 and Theorem 2.38 are satisfied by $h(s-\epsilon)$; hence $h(s-\epsilon)$ is PR, *i.e.*, $h(s)$ is SPR and the sufficiency proof is complete. ∎

Remark 2.46. It should be noted that when $r = 0$, conditions 1 and 2 of the Theorem, or 1 and $\mathbf{Re}[h(j\omega)] > \delta > 0$ for all $\omega \in [-\infty, +\infty]$, are both necessary and sufficient for $h(s)$ to be SPR.

Notice that $H(s)$ in (2.166) satisfies condition 3.a), but $H(s)$ in (2.168) does not. Let us now give a multivariable version of Theorem 2.45, whose proof is given in [256] and is based on [226, 508].

Theorem 2.47. *Let $H(s) \in \mathbb{C}^{m\times m}$ be a proper rational transfer matrix, and suppose that det$(H(s) + H^T(s))$ is not identically zero. Then $H(s)$ is SPR if and only if*

- *$H(s)$ has all its poles with negative real parts*
- *$H(j\omega) + H^T(-j\omega) > 0$ for all $\omega \in \mathbb{R}$*
  and one of the following three conditions is satisfied:
  - $H(\infty) + H^T(\infty) > 0$
  - $H(\infty) + H^T(\infty) = 0$ *and* $\lim_{\omega\to\infty} \omega^2[H(j\omega) + H^T(-j\omega)] > 0$
  - $H(\infty) + H^T(\infty) \geq 0$ *(but not zero nor nonsingular)and there exist positive constants σ and δ such that*

$$\omega^2 \sigma_{\min}[H(j\omega) + H^T(-j\omega)] \geq \sigma, \ \ \forall |\omega| \geq \delta \tag{2.179}$$

∎

2.14.2 Necessary Conditions for $H(s)$ to be PR (SPR)

In general, before checking all the conditions for a specific transfer function to be PR or SPR, it is useful to check first that it satisfies a set of necessary conditions. The following are necessary conditions for a system to be PR (SPR)

- $H(s)$ is (asymptotically) stable.
- The Nyquist plot of $H(j\omega)$ lies entirely in the (closed) right half complex plane.
- The relative degree of $H(s)$ is either $r = 0$ or $r = \pm 1$.
- $H(s)$ is (strictly) minimum-phase, *i.e.* the zeros of $H(s)$ lie in $\mathbf{Re}[s] \leq 0$ $(\mathbf{Re}[s] < 0)$.

Remark 2.48. In view of the above necessary conditions it is clear that unstable systems or nonminimum phase systems are not positive real. Furthermore proper transfer functions can be PR only if their relative degree is 0 or 1. This means for instance that a double integrator, *i.e.* $H(s) = \frac{1}{s^2}$ is not PR. This remark will turn out to be important when dealing with passivity of nonlinear systems. In particular for a robot manipulator we will be able to prove passivity from the torque control input to the velocity of the generalized coordinates but not to the position of the generalized coordinates.

2.14.3 Tests for SPRness

Stating necessary and sufficient conditions for a transfer function to be PR or SPR is a first fundamental step. A second step consists in usable criteria which allow one to determine if a given rational function is SPR or not. Work in this direction may be found in [31,132,146,177,205,341,396,455,504,528,536]. We can for instance quote a result from [455].

Theorem 2.49. *[455] Consider* $H(s) = C(sI_n - A)^{-1}B \in \mathbb{C}$*.* $H(s)$ *is SPR if and only if 1)* $CAB < 0$*, 2)* $CA^{-1}B < 0$*, 3)* A *is stable, 4)* $A(I_n - \frac{ABC}{CAB})A$ *has no eigenvalue on the open negative real axis* $(-\infty, 0)$*. Consider now* $H(s) = C(sI_n - A)^{-1}B + D \in \mathbb{C}$*,* $D > 0$*.* $H(s)$ *is SPR if and only if 1)* A *is stable, 2) the matrix* $(A - \frac{BC}{D})A$ *has no eigenvalue on the closed negative real axis* $(-\infty, +\infty]$*.* ■

Stability means here that all the eigenvalues are in the open left-half of the complex plane $\mathbf{Re}[s] < 0$, and may be called strict stability. An interpretation of SPRness is that (A, B, C, D) with $D \neq 0$ is SPR if and only if the matrix pencil $A^{-1} + \lambda(A - \frac{BC}{D})$ is nonsingular for all $\lambda > 0$ [455].

2.14.4 Interconnection of Positive Real Systems

One of the important properties of positive real systems is that the inverse of a PR system is also PR. In addition the interconnection of PR systems in parallel or in negative feedback (see Figure 2.19) inherit the PR property. More specifically we have the following properties (see [226]):

- $H(s)$ is PR (SPR) $\Leftrightarrow \frac{1}{H(s)}$ is PR (SPR).
- If $H_1(s)$ and $H_2(s)$ are SPR so is $H(s) = \alpha_1 H_1(s) + \alpha_2 H_2(s)$ for $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_1 + \alpha_2 > 0$.

- If $H_1(s)$ and $H_2(s)$ are SPR, so is $H(s) = \frac{H_1(s)}{1+H_1(s)H_2(s)}$.

Remark 2.50. Note that a transfer function $H(s)$ need not be proper to be PR or SPR. For instance, the non-proper transfer function s is PR.

Remark 2.51. Let us recall that if (A, B, C, D) is a realization of the transfer function $H(s) \in \mathbb{C}$, *i.e.* $C(sI_n - A)^{-1}B + D = H(s)$, and if $D \neq 0$, then $(A - \frac{BC}{D}, \frac{B}{D}, -\frac{C}{D}, \frac{1}{D})$ is a realization of a system with transfer function $\frac{1}{H(s)}$ (see for instance [246, p.76]).

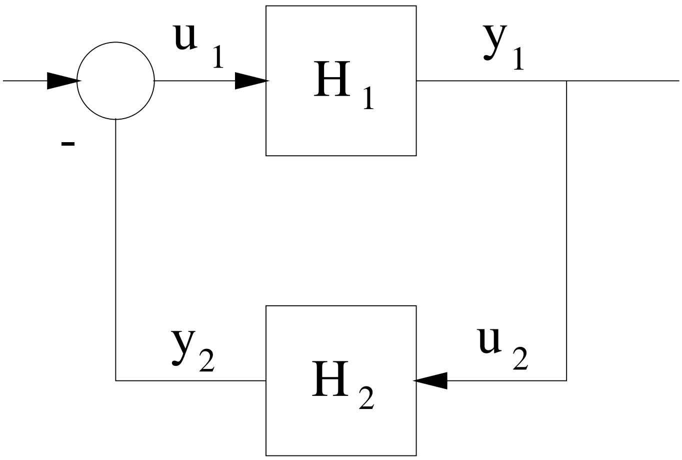

Fig. 2.19. Negative feedback interconnection of H_1 and H_2

2.14.5 Special Cases of Positive Real Systems

We will now introduce two additional definitions of classes of systems. Both of them are PR systems but one of them is weaker than SPR systems and the other is stronger. Weak SPR (WSPR) are important because they allow the extension of the KYP Lemma presented in Chapter 3 for systems other than PR. They are also important because they allow to relax the conditions for stability of the negative feedback interconnection of a PR system and an SPR system. We will actually show that the negative feedback interconnection between a PR system and a WSPR produces an asymptotically stable system. Both properties will be seen later.

Remark 2.52. Consider again an electric circuit composed of an inductor in parallel with a capacitor. Such a circuit will exhibit sustained oscillatory behavior. If we have instead a lossy capacitor in parallel with a lossy inductor, it is clear that the energy stored in the system will be dissipated. However, it is sufficient that at least one of the two is a lossy element (either a lossy capacitor or a lossy inductor) to guarantee that the oscillatory behavior will asymptotically converge to zero. This example motivates the notion of weakly SPR transfer function.

Definition 2.53. (Weakly SPR) *A rational function $H(s) \in \mathbb{C}$ is weakly SPR (WSPR) if*

1. *$H(s)$ is analytic in $\mathbf{Re}[s] \geq 0$.*
2. *$\mathbf{Re}[H(j\omega)] > 0$, for all $\omega \in (-\infty, \infty)$.* ■

In the multivariable case one replaces the second condition by $H(j\omega) + H^T(-j\omega) > 0$ for all $\omega \in I\!R$. It is noteworthy that a transfer function may be WSPR but not be SPR; see an example below. WSPRness may be seen as an intermediate notion between PR and SPR. See Section 5.3 for more analysis on WSPR systems, which shows in particular and in view of Examples 4.62 and 4.64 that WSPR is not SPR.

Definition 2.54. (Strong SPR) *A rational function $H(s) \in \mathbb{C}$ is strongly SPR (SSPR) if*

1. *$H(s)$ is analytic in $\mathbf{Re}[s] \geq 0$.*
2. *$\mathbf{Re}[H(j\omega)] \geq \delta > 0$, for all $\omega \in [-\infty, \infty]$ and some $\delta \in I\!R$.* ■

In the multivariable case the second condition for SSPRness becomes $H(j\omega) + H^T(-j\omega) > 0$ for all $\omega \in I\!R$ and $H(\infty) + H^T(\infty) > 0$, or as $H(j\omega) + H^T(-j\omega) > \delta I_m$ for all $\omega \in [-\infty, \infty]$. From Theorem 2.6, it can be seen that a SSPR transfer function is ISP. If the system has a minimal state space realization (A, B, C, D) then $H(s) + H^T(-s) = C(sI_n - A)^{-1}B - B^T(sI_n + A^T)^{-1}C^T + D + D^T$ so that the second condition implies $D + D^T > 0 \Rightarrow D > 0$. This may also be deduced from the fact that $C(sI_n - A)^{-1}B + D = \sum_{i=1}^{+\infty} CA^{i-1}Bs^{-i} + D$. The next result may be useful to characterize SSPR functions.

Lemma 2.55. *[146] A proper reational matrix $H(s) \in \mathbb{C}^{m\times m}$ is SSPR if and only if its principal minors $H_i(s) \in \mathbb{C}^{i\times i}$ are proper rational SSPR matrices, respectively, for $i = 1, ..., m-1$, and det$(H(j\omega)+H^T(-j\omega)) > 0$ for all $\omega \in I\!R$.*

The next lemma is close to Theorem 2.34.

Lemma 2.56. *Let $G(s) \in \mathbb{C}^{m\times m}$ be a proper rational matrix satisfying det$(I_m + G(s)) \neq 0$ for $\mathbf{Re}[s] \geq 0$. Then the proper rational matrix $H(s) = (I_m + G(s))^{-1}(I_m - G(s)) \in \mathbb{C}^{m\times m}$ is SSPR if and only if $G(s)$ is strictly bounded real.*

Let us now illustrate the various definitions of PR, SPR and WSPR functions on examples.

Example 2.57. Consider again an asymptotically stable first order system

$$H(s) = \frac{1}{s+\lambda}, \quad \text{with} \quad \lambda > 0 \tag{2.180}$$

Let us check the conditions for $H(s)$ to be SPR.

1. $H(s)$ has only poles in $\mathbf{Re}[s] < 0$
2. $H(j\omega)$ is given by

$$H(j\omega) = \frac{1}{\lambda + j\omega} = \frac{\lambda - j\omega}{\lambda^2 + \omega^2} \tag{2.181}$$

Therefore,

$$\mathbf{Re}[H(j\omega)] = \frac{\lambda}{\lambda^2 + \omega^2} > 0 \;\; \forall \omega \in (-\infty, \infty) \tag{2.182}$$

- $\lim_{\omega^2 \to \infty} \omega^2 \mathbf{Re}[H(j\omega)] = \lim_{\omega^2 \to \infty} \frac{\omega^2 \lambda}{\lambda^2+\omega^2} = \lambda > 0$

Therefore $\frac{1}{s+\lambda}$ is SPR. However $\frac{1}{s+\lambda}$ is not SSPR because there does not exist a $\delta > 0$ such that $\mathbf{Re}[H(j\omega)] > \delta$, for all $\omega \in [-\infty, \infty]$ since $\lim_{\omega^2 \to \infty} \frac{\lambda}{\lambda^2+\omega^2} = 0$.

Example 2.58. Similarly it can be proved that $H(s) = \frac{1}{s}$ and $H(s) = \frac{s}{s^2+\omega^2}$ are PR but they are not WSPR. $H(s) = 1$ and $H(s) = \frac{s+a^2}{s+b^2}$ are both SSPR.

The following is an example of a system that is WSPR but is not SPR.

Example 2.59. Consider the second order system

$$H(s) = \frac{s+\alpha+\beta}{(s+\alpha)(s+\beta)} \;;\; \alpha, \;\; \beta > 0 \tag{2.183}$$

Let us verify the conditions for $H(s)$ to be WSPR. $H(j\omega)$ is given by

$$\begin{aligned} H(j\omega) &= \frac{j\omega+\alpha+\beta}{(j\omega+\alpha)(j\omega+\beta)} \\ &= \frac{(j\omega+\alpha+\beta)(\alpha-j\omega)(\beta-j\omega)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} \\ &= \frac{(j\omega+\alpha+\beta)(\alpha\beta-j\omega(\alpha+\beta)-\omega^2)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} \end{aligned} \tag{2.184}$$

Thus

$$\begin{aligned} \mathbf{Re}[H(j\omega)] &= \tfrac{\omega^2(\alpha+\beta)+(\alpha+\beta)(\alpha\beta-\omega^2)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} \\ &= \tfrac{\alpha\beta(\alpha+\beta)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} > 0, \text{ for all } \omega \in (-\infty, \infty) \end{aligned} \tag{2.185}$$

so $H(s)$ is weakly SPR. However $H(s)$ is not SPR since

$$\lim_{\omega^2 \to \infty} \frac{\omega^2 \alpha\beta(\alpha+\beta)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)} = 0 \tag{2.186}$$

Example 2.60. [213] The transfer function $\frac{s+\alpha}{(s+1)(s+2)}$ is

- PR if $0 \leq \alpha \leq 3$
- WSPR if $0 < \alpha \leq 3$
- SPR if $0 < \alpha < 3$

Let us point out that other definitions exist for positive real transfer functions, like the following one:

Definition 2.61. *[430] [γ-**PR**] Let $0 < \gamma < 1$. The transfer function $H(s) \in \mathbb{C}^{m \times m}$ is said to be γ-positive real if it is analytic in $\mathbf{Re}[s] \geq 0$ and satisfies*

$$(\gamma^2 - 1)H^\star(s)H(s) + (\gamma^2 + 1)(H^\star(s) + H(s)) + (\gamma^2 - 1)I_m \geq 0 \tag{2.187}$$

for all $s \in \mathbf{Re}[s] \geq 0$. ■

Then the following holds:

Proposition 2.62. *[430] If a system is $\gamma-$positive real, then it is SSPR. Conversely, if a system is SSPR, then it is $\gamma-$positive real for some $0 < \gamma < 1$.*

■

For single input-single output systems ($m = 1$) the index γ can be used to measure the maximal phase difference of transfer functions. The transfer function $H(s) \in \mathbb{C}$ is $\gamma-$PR if and only if the Nyquist plot of $H(s)$ is in the circle centered at $\frac{1+\gamma^2}{1-\gamma^2}$ and radius $\frac{2\gamma}{1-\gamma^2}$.

Lemma 2.63. *[430] Let $m = 1$. If the system (A, B, C, D) with transfer function $H(s) = C(sI_n - A)^{-1}B + D$ is $\gamma-$PR, then*

$$|arg(H(s))| \leq arctan\left(\frac{2\gamma}{1-\gamma^2}\right) \quad \textit{for all } \mathbf{Re}[s] \geq 0 \tag{2.188}$$

■

Other classes of PR systems exist which may slightly differ from the above ones; see *e.g.* [149,245]. In particular a system is said to be extended SPR if it is SPR and if $H(j\infty)+H^T(-j\infty) > 0$. From the series expansion of a rational transfer matrix one deduces that $H(j\omega) = \sum_{i=1}^{+\infty} CA^{i-1}B(j\omega)^{-i} + D$ which implies that $D + D^T > 0$. The definition of SSPRness in [245, Definition 3] and Definition 2.54 are not the same, as they impose that $H(\infty)+H^T(\infty) \geq 0$ only, with $\lim_{\omega\to\infty} \omega^2[H(j\omega) + H^T(-j\omega)] > 0$ if $H(\infty) + H^T(\infty)$ is singular. The notion of marginally SPR (MSPR) transfer functions is introduced in [245]. MSPR functions satisfy inequality 2 of Definition 2.53, however they are allowed to possess poles on the imaginary axis.

2.15 Applications

2.15.1 SPR and Adaptive Control

The concept of SPR transfer functions is very useful in the design of some type of adaptive control schemes. This will be shown next for the control of an unknown plant in a state space representation and it is due to Parks [394] (see also [240]). Consider a linear time-invariant system in the following state space representation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2.189}$$

with state $x(t) \in I\!R^n$, input $u(t) \in I\!R$ and output $y(t) \in I\!R$. Let us assume that there exists a control input

$$u = -L^T x + r(t) \tag{2.190}$$

where $r(t)$ is a reference input and $L \in I\!R^n$, such that the closed loop system behaves as the reference model

$$\begin{cases} \dot{x}_r(t) = (A - BL^T)x_r(t) + Br(t) \\ \\ y_r(t) = Cx_r(t) \end{cases} \tag{2.191}$$

We also assume that the above reference model has an SPR transfer function. From the Kalman-Yakubovich-Popov Lemma, which will be presented in detail in the next chapter, this means that there exists a matrix $P > 0$, a matrix L', and a positive constant ε such that

$$\begin{cases} A_{cl}^T P + PA_{cl} = -L'L'^T - \varepsilon P \\ \\ PB \qquad\qquad = C^T \end{cases} \tag{2.192}$$

where

$$A_{cl} = A - BL^T$$

Since the system parameters are unknown, let us consider the following adaptive control law:

$$\begin{aligned} u &= -\hat{L}^T x + r(t) \\ &= -L^T x + r(t) - \tilde{L}^T x \end{aligned} \tag{2.193}$$

where $\hat{L}$ is the estimate of L and $\tilde{L}$ is the parametric error

$$\tilde{L}(t) = \hat{L}(t) - L$$

Introducing the above control law into the system (2.189) we obtain

$$\dot{x}(t) = (A - BL^T)x(t) + B(r(t) - \tilde{L}^T(t)x(t)) \tag{2.194}$$

Define the state error $\tilde{x} = x - x_r$ and the output error $e = y - y_r$. From the above we obtain

$$\begin{cases} \frac{d\tilde{x}}{dt}(t) = A_{cl}\tilde{x}(t) - B\tilde{L}^T(t)x(t) \\ e(t) \quad = C\tilde{x}(t) \end{cases} \tag{2.195}$$

Consider the following Lyapunov function candidate

$$V(\tilde{x}, \tilde{L}) = \tilde{x}^T P \tilde{x} + \tilde{L}^T P_L \tilde{L} \tag{2.196}$$

where $P > 0$ and $P_L > 0$. Therefore

$$\dot{V}(\tilde{x}, \tilde{L}) = \tilde{x}^T (A_{cl}^T P + P A_{cl})\tilde{x} - 2\tilde{x}^T P B \tilde{L}^T x + 2\tilde{L}^T P_L \frac{d\tilde{L}}{dt}$$

Choosing the following parameter adaptation law

$$\frac{d\hat{L}}{dt}(t) = P_L^{-1} x(t) e(t) = P_L^{-1} x(t) C \tilde{x}(t)$$

we obtain

$$\dot{V}(\tilde{x}, \tilde{L}) = \tilde{x}^T (A_{cl}^T P + P A_{cl})\tilde{x} - 2\tilde{x}^T (PB - C^T)\tilde{L}^T x$$

Introducing (2.192) in the above we get

$$\dot{V}(\tilde{x}) = -\tilde{x}^T (L' L'^T + \varepsilon P)\tilde{x} \leq 0 \tag{2.197}$$

It follows that $\tilde{x}, x$ and $\tilde{L}$ are bounded. Integrating the above we get

$$\int_0^t \tilde{x}^T(s)(L' L'^T + \varepsilon P)\tilde{x}(s) ds \leq V(\tilde{x}(0), \tilde{L}(0))$$

Thus $\tilde{x} \in \mathcal{L}_2$. From (2.195) it follows that $\frac{d\tilde{x}}{dt}(\cdot)$ is bounded and we conclude that $\tilde{x}(\cdot)$ converges to zero.

2.15.2 Adaptive Output Feedback

In the previous section we presented an adaptive control based on the assumption that there exists a state feedback control law such that the resulting closed-loop system is SPR. In this section we present a similar approach but this time we only require output feedback. In the next section we will present the conditions under which there exists an output feedback that renders the closed loop SPR. The material in this section and the next have been presented in [219]. Consider again the system (2.189) in the MIMO (multiple-input multiple-output) case, *i.e.*, with state $x(t) \in I\!R^n$, input $u(t) \in I\!R^m$ and output $y(t) \in I\!R^p$. Assume that there exists a constant output feedback control law

$$u(t) = -Ky(t) + r(t) \tag{2.198}$$

such that the closed loop system

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) + Br(t) \\ \\ y(t) = Cx(t) \end{cases} \tag{2.199}$$

with

$$\bar{A} = A - BKC$$

is SPR, *i.e.* there exists a matrix $P > 0$, a matrix L', and a positive constant ε such that [3]

$$\begin{cases} \bar{A}^T P + P\bar{A} = -L'L'^T - \varepsilon P \\ \\ PB \qquad\quad = C^T \end{cases} \tag{2.200}$$

Since the plant parameters are unknown, consider the following adaptive controller for $r(t) = 0$:

$$u(t) = -\hat{K}(t)y(t)$$

where $\hat{K}(t)$ is the estimate of K at time t. The closed loop system can be written as

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) - B(\hat{K}(t) - K)y(t) \\ \\ y(t) = Cx(t) \end{cases}$$

Define

$$\tilde{K}(t) = \hat{K}(t) - K$$

[3] Similarly to in the foregoing section, this is a consequence of the Kalman-Yakubovich-Popov Lemma for SPR systems.

and consider the following Lyapunov function candidate

$$V(x, \tilde{K}) = x^T P x + \mathrm{tr}\left(\tilde{K}^T \Gamma^{-1} \tilde{K}\right)$$

where $\Gamma > 0$ is an arbitrary positive definite matrix. The time derivative of $V(\cdot)$ along the system trajectories is given by

$$\dot{V}(x, \tilde{K}) = x^T (\bar{A}^T P + P\bar{A})x - 2x^T PB\tilde{K}y + 2\mathrm{tr}\left(\tilde{K}^T \Gamma^{-1} \frac{d}{dt}\left(\tilde{K}\right)\right)$$

Introducing (2.189) and (2.200) we obtain

$$\dot{V}(x, \tilde{K}) = x^T (\bar{A}^T P + P\bar{A})x - 2\mathrm{tr}\left(\tilde{K}yy^T - \tilde{K}^T \Gamma^{-1} \frac{d}{dt}\left(\tilde{K}\right)\right)$$

Choosing the parameter adaptation law

$$\frac{d}{dt}\left(\hat{K}\right)(t) = \Gamma y(t) y^T(t)$$

and introducing (2.192) we obtain

$$\dot{V}(x) = -x^T (L' L'^T + \varepsilon P)x \leq 0$$

Therefore $V(\cdot)$ is a Lyapunov function and thus $x(\cdot)$ and $\hat{K}(\cdot)$ are both bounded. Integrating the above equation it follows that $x \in \mathcal{L}_2$. Since $\dot{x}(\cdot)$ is also bounded we conclude that $x(t) \to 0$ as $t \to 0$.

Hence the proposed adaptive control law stabilizes the system as long as the assumption of the existence of a constant output feedback that makes the closed-loop transfer matrix SPR is satisfied. The conditions for the existence of such control law are established in the next section. Further work on this topic may be found in [42] who relax the symmetry of the Markov parameter CB.

2.15.3 Design of SPR Systems

The adaptive control scheme presented in the previous section motivates the study of constant output feedback control designs such that the resulting closed-loop is SPR. The positive real synthesis problem is important in its own right and has been investigated by [179, 428, 480, 505]. This problem is quite close to the so-called *passification* or *passivation* by output feedback [153, 156, 280]. Necessary and sufficient conditions have been obtained in [219] for a linear system to become SPR under constant output feedback. Furthermore, they show that if no constant feedback can lead to an SPR closed-loop system, then no dynamic feedback with proper feedback transfer matrix can do it neither. Hence, there exists an output feedback such that the closed-loop system is SPR if and only if there exists a constant output feedback rendering the closed-loop system SPR.

Consider again the system (2.189) in the MIMO case, *i.e.*, with state $x(t) \in I\!R^n$, input $u(t) \in I\!R^m$ and output $y(t) \in I\!R^p$ and the constant output feedback in (2.198). The closed loop is represented in Figure 2.20 where $G(s)$ is the transfer function of the system (2.189). The equation of the closed-loop $T(s)$ of Figure 2.20 is given in (2.199).

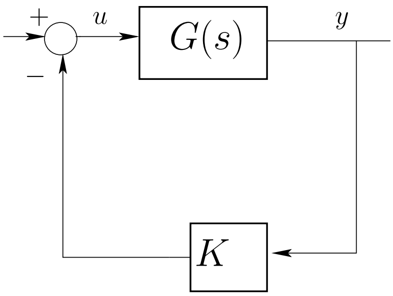

Fig. 2.20. Closed-loop system $T(s)$ using constant output feedback

Theorem 2.64. *[41] Any strictly proper strictly minimum-phase system (A, B, C) with the $m \times m$ transfer function $G(s) = C(sI_n - A)^{-1}B$ and with $CB > 0$ and symmetric, can be made SPR via constant output feedback.* ■

The fact that the zeroes of the system have to satisfy $\mathbf{Re}[s] < 0$ is crucial. Consider $G(s) = \frac{s^2+1}{(s+1)(s+2)(s+5)}$. There does not exist any static output feedback $u = ky + w$ which renders the closed-loop transfer function PR. Indeed if $\omega^2 = \frac{9-k}{8-k}$ then $\mathbf{Re}[T(j\omega)] < 0$ for all $k < 0$. Therefore the strict minimum phase assumption is necessary. Recall that a static state feedback does not change the zeroes of a linear time invariant system. We now state the following result where we assume that B and C are full rank.

Theorem 2.65 (SPR synthesis [219]). *There exists a constant matrix K such that the closed-loop transfer function matrix $T(s)$ in Figure 2.20 is SPR if and only if*

$$B^T C = C^T B > 0$$

and there exists a positive definite matrix X such that

$$C_\perp^T herm\{B_\perp X B_\perp^T A\} C_\perp < 0$$

When the above conditions hold, K is given by

$$K = C^\dagger Z(I - C_\perp(C_\perp^T Z C_\perp)^{-1} C_\perp^T Z) C^{\dagger T} + S$$

where $Z = herm\{PA\}$ *and* $P = C(B^T C)^{-1} C^T + B_\perp X B_\perp^T$, *and* S *is an arbitrary positive definite matrix.* ■

The notation used above is $\text{herm}\{X\} \overset{\Delta}{=} \frac{1}{2}(X + X^*)$, and $X_\perp$ is defined as $X_\perp^T X = 0$ and $X_\perp^T X_\perp = I_n, X \in I\!R^{n \times n}$.

In the single-input single-output case, the necessary condition $B^T C > 0$ implies the relative degree of $G(s)$ is one. It is noteworthy that the above two results apply to systems with no feedthrough term, *i.e.* $D = 0$. An answer is provided in [480, Theorem 4.1], where this time one considers a dynamic output feedback. The system (A, B, C, D) is partitioned as $B = [B_1 \quad B_2]$, $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{pmatrix}$. It is assumed that (A, B_2) is stabilizable and that (A, C_2) is detectable. The closed-loop system is said *internally stable* if the matrix $\begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ & \\ B_K C_2 & A_K \end{pmatrix}$ is stable (has eigenvalues with strictly negative real parts), where (A_K, B_K, C_K, D_K) is the dynamic feedback controller.

Theorem 2.66. *[480] There exists a strictly proper (dynamic) state feedback such that the closed-loop system is internally stable and extended SPR if and only if there exists two matrices* F *and* L *such that*

- $D_{11} + D_{11}^T > 0$
- *The algebraic Riccati inequality*

$$(A + B_2 F)^T P + P(A + B_2 F) + (C_1 + D_{12} F - B_1^T P)^T (D_{11} + D_{11}^T)^{-1}.$$
$$.(C_1 + D_{12} F - B_1^T P) < 0 \tag{2.201}$$

  has a positive definite solution P_f
- *The algebraic Riccati inequality*

$$(A + LC_2)^T G + G(A + LC_2) + (B_1 + LD_{12} - GC_1^T)^T (D_{11} + D_{11}^T)^{-1}.$$
$$.(B_1 + LD_{12} - GC_1^T) < 0 \tag{2.202}$$

  has a positive definite solution G_f,
- *The spectral radius* $\rho(G_f P_f) < 1$. ■

The conditions such that a system can be rendered SPR via static state feedback are relaxed when an observer is used in the control loop. However this creates additional difficulty in the analysis because the closed-loop system loses its controllability. See Section 3.4 for more information. Other works related with the material exposed in this section, may be found in [49, 50, 177, 205, 330, 448, 465, 497, 515, 516]. Despite there being no close relationship with the material of this section, let us mention [19] where model reduction which preserves passivity is considered. Spectral conditions for a single-input single-output system to be SPR, are provided in [455]. The SPRness is also used in identification of LTI systems [12]. Robust stabilisation when a PR uncretainty is considered is studied in [180].

3

Kalman-Yakubovich-Popov Lemma

The Kalman-Yakubovich-Popov Lemma (also called the Yakubovich-Kalman-Popov Lemma) is considered to be one of the cornerstones of Control and Systems Theory due to its applications in absolute stability, hyperstability, dissipativity, passivity, optimal control, adaptive control, stochastic control and filtering. Despite its broad applications the Lemma has been motivated by a very specific problem which is called the *absolute stability Lur'e problem* [321,408]. The first results on the Kalman-Yakubovich-Popov Lemma are due to Yakubovich [518,519]. The proof of Kalman [247] was based on *factorization of polynomials*, which were very popular among electrical engineers. They later became the starting point for new developments. Using general factorization of matrix polynomials, Popov [407,409] obtained the Lemma in the multivariable case. In the following years the Lemma was further extended to the infinite dimensional case (Yakubovich [520], Brusin [87], Likhtarnikov and Yakubovich [300]) and discrete-time case (Szegö and Kalman [483]).

The Kalman-Yakubovich-Popov Lemma (which will be shortly denoted as the KYP Lemma) establishes an equivalence between the conditions in the frequency domain for a system to be positive real, an input-output relationship of the system in the time domain, and conditions on the matrices describing the state-space representation of the system. A proof of this Lemma in the multivariable case is also due to Anderson [11]. This result is very useful in the stability analysis of dynamical systems and is also extensively used in the analysis of adaptive control schemes. We will use this Lemma to prove the Passivity Theorem which ensures the stability of a closed loop system composed of two passive systems connected in negative feedback. Both results are extensively used in the analysis and synthesis of dynamical systems. The reader is referred to the survey [36] for more details on the history of the KYP Lemma.

3.1 The Positive Real Lemma

3.1.1 PR Transfer Functions

Let us consider a multivariable linear time-invariant system described by the following state-space representation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ x(0) = x_0 \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n, u(t), y(t) \in \mathbb{R}^m$ with $n \geq m$. The Positive Real Lemma can be stated as follows [8].

Lemma 3.1 (Positive Real Lemma or KYP Lemma). *Let the system in (3.1) be controllable and observable. The transfer function* $H(s) = C(sI_n - A)^{-1}B + D$, *with* $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$ *is PR with* $H(s) \in \mathbb{R}^{m\times m}$, $s \in \mathbb{C}$, *if and only if there exists matrices* $P = P^T > 0$, $P \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times m}$ *and* $W \in \mathbb{R}^{m\times m}$ *such that:*

$$\begin{aligned} PA + A^T P &= -LL^T \\ PB - C^T &= -LW \\ D + D^T &= W^T W \end{aligned} \tag{3.2}$$

The proof will be given below.

Example 3.2. Let us point out an important fact. It is assumed in Lemma 3.1 that the representation (A, B, C, D) is minimal. Then PRness of the transfer function $C(sI_n - A)^{-1}B + D$ is equivalent to the solvability of the set of equations (3.2) with $P = P^T > 0$. Consider now the following scalar example, where $(A, B, C, D) = (-\alpha, 0, 0, 1)$, with $\alpha > 0$. The transfer function is $H(s) = 0$ that is PR. The set of equations (3.2) takes the form $\begin{pmatrix} -2\alpha p & 0 \\ 0 & 2 \end{pmatrix} \leq 0$, which is satisfied for any $p \geq 0$. Obviously, however, this system is neither controllable nor observable. This example shows that the minimality assumption is not necessary for the set of equations (3.2) to possess a positive definite solution. We shall come back on this topic in Section 3.3.

The first equation above is known as the Lyapunov equation. Note that LL^T is not positive definite but necessarily semi-positive definite as long as $m < n$. The third equation above can be interpreted as the factorization of $D + D^T$. For the case $D = 0$, the above set of equations reduces to the first two equations with $W = 0$. If one comes back to the frequency domain (Definitions 2.28 and 2.29) one sees that the stability is taken care of by the first equation in (3.2) while the other equations rather deal with the positivity. As recalled in the introduction of this chapter, the first published version of the KYP Lemma was by Yakubovich [518, 519] in 1962, with $D = 0$. The set of equations (3.2) can also be written as

$$\begin{bmatrix} -PA - A^T P & C^T - PB \\ C - B^T P & D + D^T \end{bmatrix} = \begin{bmatrix} L \\ W^T \end{bmatrix} \begin{bmatrix} L^T & W \end{bmatrix} \geq 0 \tag{3.3}$$

From (3.2) it follows that $B^T PB - B^T C^T = -B^T LW$. So if $W = 0$ one gets $CB = B^T PB \geq 0$. If B is full column rank then $CB > 0$. Thus the first non-zero Markov parameter of the system is CB, which means that the uniform relative degree of the system is equal to $r = (1, ..., 1)^T \in \mathbb{R}^m$. Before presenting the proof of the KYP Lemma, let us state a number of interesting results, which link the set of equations (3.2), the positive realness, and a new tool that is named a dissipation equality.

Corollary 3.3. *Let the system in (3.1) be controllable and observable, and let* $D = 0$. *Assume that* $C(sI_n - A)^{-1}B$ *is PR. Then*

$$\int_0^t u^T(s)y(s)ds = V(x(t)) - V(x_0) - \frac{1}{2}\int_0^t x^T(s)(A^T P + PA)x(s)d \tag{3.4}$$

for all $t \geq 0$, *with* $V(x) = \frac{1}{2}x^T Px$, P *satisfies the LMI in (3.3), and the equality is computed along state trajectories starting at* $x(0) = x_0$ *and driven by* $u(\cdot)$ *on* $[0, t]$. ■

Proof: Positive realness and minimality imply that (3.2) is satisfied. By simple calculation of the integral $\int_0^t u^T(s)y(s)ds$ and using the KYP Lemma conditions, premultiplying $\dot{x}(t)$ by P, (3.4) follows. ■

The same holds if $D \neq 0$, as the reader may check. We shall see in the next chapter that $V(x)$ is a *storage function* for the system (A, B, C), and that the equality in (3.4) is a *dissipation equality.* One may rewrite it as follows, with an obvious "physical" interpretation:

$$\begin{aligned} \underbrace{V(x(t))}_{\text{energy at time } t} = \underbrace{V(x_0)}_{\text{initial energy}} &+ \underbrace{\frac{1}{2}\int_0^t x^T(s)(A^T P + PA)x(s)ds}_{\text{dissipated energy}} + \\ &+ \underbrace{\int_0^t u^T(s)y(s)ds}_{\text{externally supplied energy}} \end{aligned} \tag{3.5}$$

where we recall that $A^TP+PA \leq 0$. A dynamical system which satisfies (3.5) along its trajectories is named *dissipative*. Notice that the minimality of the triple (A, B, C) is used in Corollary 3.3, which therefore shows that PRness implies the dissipation equality (3.4). However the following is also true.

Corollary 3.4. *Let the triple (A, B, C) be given, where the matrices have appropriate dimensions. Suppose that the KYP Lemma set of equations (3.2) is solvable, i.e. there exists a triple $(P = P^T > 0, L, W)$ that solves (3.2). Then the dissipation equality (3.4) holds along the system's trajectories.* ■

Proof: One has $\dot{x}(t) = Ax(t) + Bu(t) \Leftrightarrow P\dot{x}(t) = PAx(t) + PBu(t) \Rightarrow x^T(t)P\dot{x}(t) = x^T(t)PAx(t) + x^T(t)PBu(t) \Leftrightarrow x^T(t)P\dot{x}(t) - x^T(t)PAx(t) - x^T(t)PBu(t) + u^T(t)y(t) = u^T(t)y(t)$. Integrating between 0 and t we deduce that $\frac{1}{2}x^T(t)Px(t) - \frac{1}{2}x^T(0)Px(0) - \frac{1}{2}\int_0^t x^T(\tau)(PA + A^TP)x(\tau)d\tau + \int_0^t u^T(\tau)(B^TP - C)x(\tau)d\tau = \int_0^t u^T(\tau)y(\tau)d\tau$. From the second equation in (3.2) we get that $\frac{1}{2}x^T(t)Px(t) - \frac{1}{2}x^T(0)Px(0) - \frac{1}{2}\int_0^t x^T(\tau)(PA+A^TP)x(\tau)d\tau = \int_0^t u^T(\tau)y(\tau)d\tau$ which is (3.4). ■

The interest of Corollary 3.4 is that no minimality on (A, B, C) is required [1]. We let the reader treat the case where $D \neq 0$, using Proposition A.63. Corollary 3.3 assumes minimality but shows a stronger result, namely that $H(s) \in PR \Leftrightarrow (3.2) \Rightarrow (3.4)$. The issues linked to minimality and the KYP Lemma are examined in Section 3.3.

One notices from (3.4) that if $x_0 = 0$ then $\int_0^t u^T(s)y(s)ds \geq 0$: this inequality is always true for positive real transfer functions. This is to be linked with Definition 2.1 (the "constant" β is shown to be equal to $-V(x_0)$), and to Theorem 2.2: the function $V(t)$ in Theorem 2.2 actually is a function of the state x and is not an explicit function of time! As the reader may have guessed, it plays the role of a Lyapunov function for the uncontrolled system $\dot{x}(t) = Ax(t)$.

Corollary 3.3 proves that a minimal system satisfying (3.2) satisfies (3.4). It is also of interest to show the converse: suppose that the system (3.1) with $D = 0$ satisfies (3.4) for some positive definite quadratic function $V(x)$. Then does it satisfy the KYP Lemma conditions? The answer is yes. Indeed notice first that the dissipation equality (3.4) is equivalent to its infinitesimal form

$$u^T(t)y(t) = x^T(t)P\dot{x}(t) - \frac{1}{2}x^T(t)(A^TP + PA)x(t) \tag{3.6}$$

since it holds for all $t \geq 0$. Continuing the calculations we get

$$u^T(t)Cx(t) = x^T(t)P(Ax(t) + Bu(t)) - \frac{1}{2}x^T(t)(A^TP + PA)x(t) \tag{3.7}$$

[1] Let $A \in \mathbb{R}^{n\times n}$ be the transition matrix. Minimality of n is equivalent to having (A, B) controllable and (A, C) observable.

so that $u^T(t)Cx(t) = x^T(t)PBu(t)$. Since this equality holds for any $u(\cdot)$ one must have $C^T = PB$. This shows that the second KYP Lemma condition is true. Now suppose that more generally the system satisfies a dissipation equality as

$$\int_0^t u^T(s)y(s)ds = V(x(t)) - V(x_0) - \frac{1}{2}\int_0^t x^T(s)Qx(s)ds \tag{3.8}$$

with $Q \leq 0$ and $V(x) = \frac{1}{2}x^T Px$, $P = P^T > 0$. Then the uncontrolled system is stable in the sense of Lyapunov since $V(x(t)) \leq V(x(0))$ for all $t \geq 0$. Thus $A^T P + PA \leq 0$ from Lyapunov's Theorem. Using once again the infinitesimal version of the dissipation equality we get

$$u^T(t)y(t) = x^T(t)(PA + A^T P)x(t) - \frac{1}{2}x^T(t)Qx(t)$$

This must hold for any admissible input. Rewriting this equality with $u(\cdot) \equiv 0$ we obtain that necessarily $PA+A^T P = -Q = -LL^T$ for some matrix L. Thus we have proved the following.

Corollary 3.5. *Let (3.8) hold along the system's trajectories with* $Q \leq 0$*,* $V(x) = \frac{1}{2}x^T Px$*,* $P = P^T > 0$*. Then the KYP Lemma set of equations (3.2) also hold.* ∎

Remark 3.6. In the case $D \neq 0$, assuming that the dissipation equality (3.8) holds yields after time-derivation

$$u^T(C - B^T P)x + u^T Du - \frac{1}{2}x^T(A^T P + PA)x = -\frac{1}{2}x^T Qx \geq 0 \tag{3.9}$$

since $Q \leq 0$. In a matrix form this leads to

$$(x^T \;\; u^T)\begin{pmatrix} -A^T P - PA & C^T - PB \\ C - B^T P & D + D^T \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} \geq 0 \tag{3.10}$$

Using Proposition A.63, (3.2) follows.

We have seen in the proofs of Theorems 2.6 and 2.21 that Parseval's Theorem allows us to assert that if $H(s)$ is PR then $\int_0^t u^T(\tau)y(\tau)d\tau \geq 0$, where the underlying assumption is that $x(0) = 0$, and conversely (see Corollary 2.35). Obviously the dissipation equality implies $\int_0^t u^T(\tau)y(\tau)d\tau \geq 0$ when $x(0) = 0$.

Therefore concatenating all these results we get the following.

KYP Lemma matrix equality (3.2)

$\Updownarrow$ ((A, B, C, D) minimal)

PR transfer function $\Leftrightarrow \int_0^t u^T(\tau)y(\tau)d\tau \geq 0$ when $x(0) = 0$

$\Updownarrow$

Dissipativity with quadratic storage function

These developments and results somewhat shed new light on the relationships between PR transfers, passivity, dissipation, and the KYP Lemma set of equations. However we have not yet proved the KYP Lemma, *i.e.* the fact that the frequency domain conditions for positive realness, are equivalent to the LMI in (3.2) when (A, B, C, D) is minimal. Several proofs of the KYP Lemma appeared in the book [8].

Proof of the KYP Lemma: The proof that is reproduced now is taken from Anderson's work [11].

Sufficiency: This is the easy part of the proof. Let the set of equations in (3.2) be satisfied. Then

$$
\begin{aligned}
H(s) + H^T(\bar{s}) &= D^T + D + B^T(\bar{s}I_n - A^T)^{-1}C^T + C(sI_n - A)^{-1}B. \\
&.W^TW + B^T\left[(\bar{s}I_n - A^T)^{-1}P + P(sI_n - A)^{-1}\right]B+ \\
&+B^T(\bar{s}I_n - A^T)^{-1}LW + W^TL^T(sI_n - A)^{-1}B \\
&= W^TW + B^T(\bar{s}I_n - A^T)^{-1}\left[P(s+\bar{s}) - PA - A^TP\right](sI_n - A)^{-1}+ \\
&+B^T(\bar{s}I_n - A^T)^{-1}LW + W^TL^T(sI_n - A)^{-1}B \\
&= W^TW + B^T(\bar{s}I_n - A^T)^{-1}LW + W^TL^T(sI_n - A)^{-1}B+ \\
&+B^T(sI_n - A^T)^{-1}LL^T(sI_n - A)^{-1}B+ \\
&+B^T(\bar{s}I_n - A^T)^{-1}P(sI_n - A)^{-1}B(s+\bar{s}) \\
&= \left[W^T + B^T(\bar{s}I_n - A^T)^{-1}L\right]\left[W + L^T(sI_n - A)^{-1}B\right]+ \\
&+B^T(\bar{s}I_n - A^T)^{-1}P(sI_n - A)^{-1}B(s+\bar{s})
\end{aligned}
\tag{3.11}
$$

which is nonnegative definite for all $\mathbf{Re}[s] > 0$.

Necessity: Suppose that $\text{rank}(H(s)+H^T(-s)) = r$ almost everywhere. From the PRness it follows that there exists an $r \times m$ matrix $W(s)$ such that

$$H(s) + H^T(-s) = W_0^T(-s)W_0(s) \tag{3.12}$$

and

- (i) $W_0(\cdot)$ has elements which are analytic in $\mathbf{Re}[s] > 0$, and in $\mathbf{Re}[s] \geq 0$ if $H(s)$ has elements which are analytic in $\mathbf{Re}[s] \geq 0$.
- (ii) $\text{Rank}(W_0(s) = r$ in $\mathbf{Re}[s] > 0$.
- (iii) $W_0(s)$ is unique save for multiplication on the left by an arbitrary orthogonal matrix.

This is a Youla factorization. Suppose that all poles of $H(s)$ are in $\mathbf{Re}[s] < 0$ (the case when poles may be purely imaginary will be treated later). Equivalently all the eigenvalues of A have negative real parts,*i.e.* A is asymptotically stable. From Lemmas A.66 and A.68 (with a slight adaptation to allow for the direct feedthrough term) it follows that there exists matrices L and $W = W_0(\infty)$ such that $W_0(s)$ has a minimal realization (A, B, L, W), with two minimal realizations for $H(s) + H^T(-s) = W_0^T(-s)W_0(s)$ being given by

$$(A_1, B_1, C_1, W^TW) = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} B \\ C^T \end{bmatrix}, \begin{bmatrix} C^T \\ -B \end{bmatrix}, W^TW \right\} \tag{3.13}$$

and

$$(A_3, B_3, C_3, W^TW) = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} B \\ PB + LW \end{bmatrix}, \begin{bmatrix} PB + LW \\ -B \end{bmatrix}, W^TW \right\} \tag{3.14}$$

where P is the unique symmetric positive definite solution of $PA + A^TP = -LL^T$. From Lemma A.69 there exists nonsingular matrices T commuting with $\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$ and such that $T\begin{bmatrix} B \\ C^T \end{bmatrix} = \begin{bmatrix} B \\ PB + LW \end{bmatrix}$ and $(T^{-1})^T\begin{bmatrix} C^T \\ -B \end{bmatrix} = \begin{bmatrix} PB + LW \\ -B \end{bmatrix}$. By Corollaries A.67, A.17 and A.70 there exists T_1 commuting with A such that $T_1B = B$, and $(T_1^{-1})^TC^T = PB + LW$. Now since T_1 commutes with A one has

$$[B, AB, ...] = [T_1B, AT_1B, ...] = [T_1B, T_1AB, ...] = T_1[B, AB, ...] \tag{3.15}$$

The matrix $[B, AB, ...]$ has rank n because of the minimality of the realization. Thus $T_1 = I_n$ and thus $PB + LW = C^T$. The third equation in (3.2) follows by setting $s = \infty$ in $H(s) + H^T(-s) = W_0^T(-s)W_0(s)$.

In a second step let us relax the restriction on the poles of $H(s)$. In this case $H(s) = H_1(s) + H_2(s)$ where $H_1(s)$ has purely imaginary axis poles, and $H_2(s)$ has all its poles in $\mathbf{Re}[s] < 0$, and both $H_1(s)$ and $H_2(s)$ are positive real. Now from Lemma A.71 it follows that there exists $P_1 = P_1^T > 0$ such that $P_1A_1 + A_1^T P_1 = 0$ and $P_1B_1 = C_1^T$, where (A_1, B_1, C_1) is a minimal realization of $H_1(s)$. For $H_2(s)$ we may select a minimal realization (A_2, B_2, C_2, D_2) and using the material just proved above we may write

$$\begin{cases} P_2A_2 + A_2^T P_2 = -L_2L_2^T \\ \\ P_2B_2 = C_2^T - L_2W \\ \\ W^TW = D_2 + D_2^T \end{cases} \tag{3.16}$$

It can be verified that the KYP Lemma set of equations (3.2) is satified by taking $P = P_1+P_2$, $A = A_1+A_2$, $B^T = [B_1^T \;\; B_2^T]$, $C = [C_1 \;\; C_2]$, $L^T = [0 \;\; L_2^T]$. Moreover with (A_1, B_1, C_1) and (A_2, B_2, C_2, D_2) minimal realization sof $H_1(s)$ and $H_2(s)$, (A, B, C, D_2) is a minimal realization of $H(s)$. Indeed the degree of $H(s)$ is the sum of the degrees of $H_1(s)$ and $H_2(s)$ which have no common poles. It just remains to verify that the equations (3.2) hence constructed are valid under any (full rank) coordinate transformation, since they have been established for a particular form $A_1 + A_2$. ■

The KYP Lemma has been derived in the so-called behavioural framework in [162].

3.1.2 A Digression to Optimal Control

We will deal at several places in the book with optimal control and its link with dissipativity. Let us nevertheless point out a first relationship. Provided $D + D^T$ is full-rank (*i.e.* $D + D^T > 0$ in view of (3.2)), the matrix inequality in (3.3) is equivalent to the following algebraic Riccati inequation:

$$-PA - A^TP - (C - B^TP)^T(D + D^T)^{-1}(C - B^TP) \geq 0 \tag{3.17}$$

Equivalence means that the LMI and the Riccati inequality possess the same set of solutions P. The KYP Lemma says that if the transfer function $D+C(sI_n-A)^{-1}B$ is PR and (A, B, C, D) is minimal, then they both possess at least one solution $P = P^T > 0$. Let us recall that the optimal control problem

$$\min_{u \in \mathcal{U}} \mathcal{J}(x_0, u) = \int_0^{+\infty} (x^T(t)Qx(t) + u^T(t)Ru(t))dt \tag{3.18}$$

under the constraints (3.1) and with $R > 0$, $Q \geq 0$, has the solution $u^\star(x) = -R^{-1}B^TPx$ where P is a solution of the Riccati equation $-PA - A^TP + PBR^{-1}B^TP = Q \geq 0$. When the cost function contains cross terms $2x^TSu$ then P is the solution of the Riccati equation $-PA - A^TP - (S - B^TP)R^{-1}(S^T - PB) = Q \geq 0$ and the optimal control is $u^\star(x) = -R^{-1}(S^T - B^TP)x$. The Belmann function for these problems is the quadratic function $V(x) = x^TPx$ and $V(x_0) = \min_{u \in \mathcal{U}} \mathcal{J}(x_0, u)$. If $Q > 0$ then $P > 0$ and $V(x)$ is a Lyapunov function for the closed-loop system $\dot{x}(t) = Ax(t) + Bu^\star(x(t))$, as can be checked by direct calculation of $\dot{V}(x(t))$ along the closed-loop trajectories.

Therefore the Riccati inequality in (3.17) corresponds to the Riccati inequation of an infinite horizon LQ problem whose cost matrix is given by

$$\begin{bmatrix} Q & C^T \\ C & D + D^T \end{bmatrix} \tag{3.19}$$

where $D + D^T = W^TW$ $(= R)$ is the weighting matrix corresponding to u in the cost function, $S = C$ and $Q = LL^T \geq 0$. The equivalence between (3.3) and the Riccati inequality also holds with strict inequality (> 0) in both (3.3) and (3.17). To recapitulate, the positive realness of the controllable and observable LTI system (3.1) is equivalent to the KYP Lemma conditions (3.2), which are equivalent to the linear matrix inequality (3.3), which is equivalent to the Riccati inequality (3.17), whose solution provides the optimal feedback control that corresponds to the optimal control problem in infinite horizon with cost matrix (3.19). All this is relying on the condition $D + D^T > 0$. The controllability assumption on the system (3.1) can be interpreted in the light of the optimal control problem, in the sense that controllability implies the existence of some $u(\cdot)$ such that $\mathcal{J}(x_0, u) < +\infty$.

The proof of the equivalence between the Riccati inequality and the linear matrix inequality follows from Theorem A.61, which is instrumental in characterizing dissipative systems with Riccati and partial differential inequalities. The reader may have a look at Appendix A.5 where several results of matrix algebra are recalled. We may apply Lemma A.62 to the matrix $M = \begin{bmatrix} D + D^T & C - B^TP \\ C^T - PB & -PA - A^TP - LL^T \end{bmatrix}$. Then $\text{rank}(M) = m$ is equivalent to the Riccati equality

$$PA + A^TP + LL^T + (C^T - PB)(D + D^T)^{-1}(C - B^TP) = 0 \tag{3.20}$$

which is (3.17) with $=$ instead of $\geq$. As we shall see further in the book, a Riccati equation for a PR system corresponds in the nonlinear case to a partial differential inequation (Hamilton-Jacobi inequalities), whose solutions serve as Lyapunov functions candidates. The set of solutions is convex and possesses two extremal solutions (which will be called the *available storage* and the *required supply*) which satisfy the algebraic Riccati equation, i.e. (3.17) with equality, see Section 4.4.3, Lemma 4.47 and Proposition 4.48. More details between the KYP Lemma and optimal control will be given in Section 3.8. The case when $D + D^T = 0$ and $D + D^T \geq 0$ will be treated in Section 4.6. Such cases possess some importance. Indeed PR functions may not have a realization with a full rank matrix D. Let us end this subsection by recalling another equivalence: the system (A, B, C, D) with a minimal realization and $D + D^T > 0$ is PR if and only if the Hamiltonian matrix

$$\begin{pmatrix} A - B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -A^T + C^T(D+D^T)^{-1}B^T \end{pmatrix} \tag{3.21}$$

has no pure imaginary eigenvalues. This is a way to characterize SSPR transfer matrices. Indeed notice that

$$H(s) = C(sI_n - A)^{-1}B + D = \sum_{i=1}^{+\infty} CA^{i-1}Bs^{-i} + D$$

so that $H(\infty) = D$. The SSPRness thus implies by Definition 2.54 (2) that $D \geq \delta > 0$ (or $D + D^T \geq \delta I_m > 0$ if $m \geq 2$). It is noteworthy that $D + D^T > 0 \Leftrightarrow D > 0$; however D is not necessarily symmetric.

3.1.3 Duality

The linear matrix inequality (3.3) thus defines a set $\mathcal{P}$ of matrices $P > 0$.

Lemma 3.7 (duality). *Let (A, B, C, D) be such that the set $\mathcal{P}$ is not empty. The inverse $P^{-1} \in \mathcal{P}^{-1}$ of any element of $\mathcal{P}$ is a solution of the dual problem (A^T, C^T, B^T, D).* ■

Remember that the *adjoint* system is defined as $(-A^T, C^T, B^T, D)$.

Proof of Lemma 3.7: Clearly if $P > 0$ then $P^{-1} > 0$. From the following matrix relation

$$\begin{aligned} &\begin{bmatrix} -AP^{-1} - P^{-1}A^T & B - P^{-1}C^T \\ B^T - CP^{-1} & R \end{bmatrix} = \\ &= \begin{bmatrix} -P^{-1} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} -A^TP - PA & C^T - PB \\ C - B^TP & R \end{bmatrix} \begin{bmatrix} -P^{-1} & 0 \\ 0 & I_n \end{bmatrix} \end{aligned} \tag{3.22}$$

one sees that $P^{-1} \in \tilde{\mathcal{P}}$ if $P \in \mathcal{P}$, because the two matrices

$$\begin{bmatrix} -AP^{-1} - P^{-1}A^T & B - P^{-1}C^T \\ & \\ B^T - CP^{-1} & R \end{bmatrix}$$

and

$$\begin{bmatrix} -A^T P - PA & C^T - PB \\ & \\ C - B^T P & R \end{bmatrix}$$

are simultaneously negative definite. The set $\tilde{\mathcal{P}}$ is the set that solves the KYP Lemma linear matrix inequality for the dual system.

■

3.1.4 Positive Real Lemma for SPR Systems

Consider the set of equations in (3.2) and Definition 2.42 of a SPR transfer function. Assume that a realization of the input-output system is given by the quadruple (A, B, C, D), *i.e.* $C(sI_n - A)^{-1}B + D = H(s)$, and (A, B, C, D) is minimal. Then $H(s - \epsilon) = C(sI_n - \epsilon I_n - A)^{-1}B + D$, and a realization of $H(s - \epsilon)$ is given by $(A + \epsilon I_n, B, C, D)$. Saying that $H(s - \epsilon)$ is PR is therefore equivalent to stating that $(A + \epsilon I_n, B, C, D)$ satisfies the KYP Lemma set of equations (3.2), provided $(A + \epsilon I_n, B, C, D)$ is minimal. Therefore (A, B, C, D) is SPR if and only if $(A + \epsilon I_n)^T P + P(A + \epsilon I_n) = -LL^T$ and the last two equations in (3.2) hold, with $P = P^T > 0$. The first equation can be rewritten as $A^T P + PA = -LL^T - 2\epsilon P$. As is well known, this implies that the matrix A is Hurwitz, *i.e.* all its eigenvalues have negative real parts. Indeed consider the Lyapunov function $V(x) = x^T Px$. Then along trajectories of the system $\dot{x}(t) = Ax(t)$ one obtains $\dot{V}(x(t)) = x^T(t)(-LL^T - 2\epsilon P)x(t) \leq -2\epsilon V(x(t))$. Consequently the system is exponentially stable. This in particular shows that SPR transfer functions have poles with negative real parts, and confirms Theorem 2.45.

The Lefschetz-Kalman-Yakubovich Lemma

We now present the Lefschetz-Kalman-Yakubovich (LKY) Lemma which gives necessary and sufficient conditions for a system in state space representation to be SPR.

Lemma 3.8 (Multivariable LKY Lemma). *[485] Consider the system in (3.1), with $m \geq 2$. Assume that the rational transfer matrix $H(s) = C(sI - A)^T B + D$ has poles which lie in $\mathbf{Re}[s] < -\gamma$ where $\gamma > 0$ and (A, B, C, D) is a minimal realization of $H(s)$. Then $H(s - \mu)$ for $\mu > 0$ is PR if and only if a matrix $P = P^T > 0$, and matrices L and W exist such that*

$$\begin{cases} PA + A^T P = -LL^T - 2\mu P \\ PB - C^T = -LW \\ D + D^T = W^T W. \end{cases} \tag{3.23}$$

■

The conditions in (3.23) are more stringent than those in (3.3). Notice that the first line in (3.23) can be rewritten as

$$P(\mu I_n + A) + (A^T + \mu I_n)P = -LL^T \tag{3.24}$$

which allows one to recover (3.3) with A changed to $\mu I_n + A$. The transfer function of the triple $(\mu I_n + A, B, C)$ precisely is $H(s-\mu)$. Thus (3.23) exactly states that $(\mu I_n + A, B, C)$ is PR and satisfies (3.3).

It is assumed in Lemma 3.8 that the system is multivariable, *i.e.* $m \geq 2$. The LKY Lemma for monovariable systems ($m = 1$) is as follows.

Lemma 3.9 (Monovariable LKY Lemma). *[485] Consider the system in (3.1), with $m = 1$. Suppose that A is such that det$(sI_n - A)$ has only zeroes in the open left-half plane. Suppose (A, B) is controllable, and let $\mu > 0$, $L = L^T > 0$ be given. Then a real vector q and a real matrix $P = P^T > 0$ satisfying*

$$\begin{cases} PA + A^T P = -qq^T - \mu L \\ \quad PB - C^T = \sqrt{2D}q \end{cases} \tag{3.25}$$

exist if and only if $H(s)$ is SPR and μ is sufficiently small.

■

Lemma 3.8 is not an extension of Lemma 3.9 because the matrix $L = L^T > 0$ is arbitrary in Lemma 3.9. We now state a result that concerns Definition 2.61.

Lemma 3.10. *[430] Assume that the triple (A, B, C) is controllable and observable. The system whose realization is (A, B, C, D) is $\gamma-$positive real if and only if there exist matrices L and W such that*

$$\begin{cases} PA + A^T P = -(1-\gamma^2)C^T C - L^T L \\ \\ PB = (1+\gamma^2)C^T - (1-\gamma^2)C^T D - L^T W \\ \\ W^T W = (\gamma^2 - 1)I_m + (\gamma^2 - 1)D^T D + (\gamma^2 + 1)(D + D^T) \end{cases} \tag{3.26}$$

■

Time Domain Conditions for Strict Positive Realness

The next result is due to J.T. Wen [508] who established different relationships between conditions in the frequency domain and the time domain for SPR systems.

Lemma 3.11 (KYP Lemma for SPR Systems). *Consider the LTI, minimal (controllable and observable) system (3.1) whose transfer matrix is given by*

$$H(s) = D + C(sI_n - A)^{-1}B \tag{3.27}$$

where the minimum singular value $\sigma_{\min}(B) > 0$. Assume that the system is exponentially stable. Consider the following statements:

1. 1) There exist $P > 0$, P, $L \in I\!R^{n\times n}$, $\mu_{\min}(L) \stackrel{\Delta}{=} \epsilon > 0$, $Q \in I\!R^{m\times n}$, $W \in I\!R^{m\times m}$ that satisfy the Lur'e equations

$$A^T P + PA = -Q^T Q - L \tag{3.28}$$

$$B^T P - C = W^T Q \tag{3.29}$$

$$W^T W = D + D^T \tag{3.30}$$

1′) Same as 1) except L is related to P by

$$L = 2\mu P \tag{3.31}$$

for some $\mu > 0$.
2) There exists $\eta > 0$ such that for all $\omega \in I\!R$

$$H(j\omega) + H^*(j\omega) \geq \eta I_m \tag{3.32}$$

3) For all $\omega \in I\!R$

$$H(j\omega) + H^*(j\omega) > 0 \tag{3.33}$$

4) For all $\omega \in I\!R$

$$H(j\omega) + H^*(j\omega) > 0 \tag{3.34}$$

and

$$\lim_{\omega\to\infty} \omega^2 \left(H(j\omega) + H^*(j\omega)\right) > 0 \tag{3.35}$$

5) The system can be realized as the driving point impedance of a multiport dissipative network.
6) The Lur'e equations with $L = 0$ are satisfied by the internal parameter set $\left(A + \mu I_n, B, C, D\right)$ corresponding to $T(j\omega - \mu)$ for some $\mu > 0$.

7) For all $\omega \in I\!R$, there exists $\mu > 0$ such that

$$H(j\omega - \mu) + H^*(j\omega - \mu) \geq 0 \tag{3.36}$$

8) There exists a positive constant ρ and a constant $\xi(x_0) \in I\!R$, $\xi(0) = 0$, such that for all $t \geq 0$

$$\int_0^t u^T(s)y(s)ds \geq \xi(x_0) + \rho \int_0^t \|u(s)\|^2 ds \tag{3.37}$$

9) There exists a positive constant γ and a constant $\xi(x_0)$, $\xi(0) = 0$, such that for all $t \geq 0$

$$\int_0^t e^{\gamma s} u^T(s)y(s)ds \geq \xi(x_0) \tag{3.38}$$

10) There exists a positive constant α such that the following kernel is positive in $\mathcal{L}_2\left(I\!R_+; I\!R^{m\times m}\right)$:

$$K(t-s) = D\delta(t-s) + Ce^{(A+\alpha I)(t-s)}B.\mathbf{I}(t-s) \tag{3.39}$$

where δ and $\mathbf{I}$ denote the Dirac measure and the step function, respectively.

11) The following kernel is coercive in $\mathcal{L}_2\left([0,T]; I\!R^{m\times m}\right)$, for all T:

$$K(t-s) = D\delta(t-s) + Ce^{A(t-s)}B.\mathbf{I}(t-s) \tag{3.40}$$

These statements are related as follows:

$$(1)\left\{\begin{array}{cccccccccccccc}
\begin{array}{c}\Longleftarrow\\ \Rightarrow\\ (\text{if } D>0)\end{array} & (2) & \Leftrightarrow & (8) & \Leftrightarrow & (11) & & & & & & & & \\
& & & \Downarrow & & & & & & & & & & \\
\begin{array}{c}\Longleftarrow\\ \Longrightarrow\\ (\text{if } D=0)\end{array} & (1') & \Longleftrightarrow & (4) & \Longleftrightarrow & (5) & \Longleftrightarrow & (6) & \Longleftrightarrow & (7) & \Longleftrightarrow & (9) & \Longleftrightarrow & (10) \\
& & & \Downarrow & & & & & & & & & & \\
& & & (3) & & & & & & & & & &
\end{array}\right.$$

Proof:

$(2) \Rightarrow (1)$

Consider the optimization problem of finding $\widehat{u} \in \mathcal{L}_2((-\infty,\infty); I\!R^m)$ to minimize

$$J_f = \int_{-\infty}^{\infty} \left\{-\widehat{x}^*(j\omega)F^T\widehat{x}(j\omega) + 2\widehat{u}^*(j\omega)\widehat{y}(j\omega)\right\} d\omega$$

where the superscript * denotes complex conjugate transposition and $\widehat{x}$, $\widehat{y}$ and $\widehat{u}$ are the Fourier transforms of the x, x and u, respectively. By writing $\widehat{x}$, in terms of the initial condition and the input, the optimization index can be expanded as

$$J_f = \int_{-\infty}^{\infty}\{-((j\omega I_n - A)^{-1}x_0 + (j\omega I_n - A)^{-1}B\widehat{u}(j\omega))^* F^T F((j\omega I_n - A)^{-1}x_0$$
$$+(j\omega I_n - A)^{-1}B\widehat{u}(j\omega) + \widehat{u}^*(j\omega)[(C(j\omega I_n - A)^{-1}B + D)^*$$
$$+(C(j\omega I_n - A)^{-1}B + D)]\widehat{u}(j\omega)$$
$$+2\widehat{u}^*(j\omega)C(j\omega I_n - A)^{-1}x_0\}d\omega$$

Consider the problem as an $\mathcal{L}_2-$optimization. Then

$$J_f = \langle R\widehat{u}, \widehat{u}\rangle + \langle r, \widehat{u}\rangle + k,$$

where the inner products are in the $\mathcal{L}_2$ sense. A unique solutions exists if R is a coercive $\mathbf{L}(\mathcal{L}_2)$ (the space of bounded operators in $\mathcal{L}_2$) operator. Now,

$$R = H^*(j\omega) + H(j\omega) - B^T(-j\omega I_n - A^T)^{-1}F^T F(j\omega I_n - A)^{-1}B$$

By condition (2), if

$$\eta > \left\|F(j\omega I_n - A)^{-1}B\right\|_{H^\infty}^2$$

then the operator R is coercive. By Plancherel's Theorem, J_f can be transformed back to the time domain as

$$J = \int_{-\infty}^{\infty}\left[-x(t)^T F^T F x(t) + 2u^T(t)y(t)\right]dt$$

Since a unique solution of the optimal control problem exists, the necessary conditions from the maximum principle must be satisfied. The Hamiltonian is given by

$$\mathbf{H}(x,u) = -x^T F^T F x + 2u^T(Cx + Du) + \lambda^T(Ax + Bu)$$

where λ is the costate or the Lagrange multiplier. The feedforward D in $u^T Du$ can be regarded as the symmetrized D. Since condition (2) implies $D > 0$, there exists $W > 0$ such that $D + D^T = W^T W$. The optimal u is obtained by minimizing $\mathbf{H}$:

$$u = -\frac{1}{2}W^{-1}W^{-T}\left(2Cx + B^T\lambda\right).$$

The costate equation is governed by

$$\dot{\lambda} = 2F^T F x - 2C^T u - A^T \lambda.$$

It can be shown [88] that λ depends linearly on x. Let $\lambda = -2Px$. Then

$$\left(PA + A^T P + F^T F\right) x = \left(C - B^T P\right)^T u$$

$$= -\left(C - B^T P\right)^T W^{-1} W^{-T} \left(C - B^T P\right) x$$

Since the equality holds for all x, we have

$$\begin{cases} PA + A^T P = -F^T F - Q^T Q \\ C = B^T P - W^T Q \end{cases} \tag{3.41}$$

The first equation implies $P > 0$. By identifying L with $F^T F$ and choosing $F^T F > 0$ and

$$\sigma_{\min}^2(F) < \frac{\eta}{\|(j\omega I - A)^{-1} B\|_{H^\infty}^2}$$

condition (1) is proved.

(1) $\Rightarrow$ (2)

(When $D > 0$)

Given the Lur'e equations, compute the Hermitian part of the transfer function as follows :

$$\begin{array}{l}
H(j\omega) + H^*(j\omega) = D + D^T + C(j\omega I - A)^{-1} B + B^T(-j\omega I_n - A^T)^{-1} C^T \\
= W^T W + \left(B^T P - W^T Q\right) (j\omega I_n - A)^{-1} B \\
+ B^T(-j\omega I_n - A^T)^{-1} \left(PB - Q^T W\right) \\
= W^T W + B^T(-j\omega I_n - A^T)^{-1} [(-j\omega I_n - A^T)^{-1} P \\
+ P(j\omega I_n - A)](j\omega I_n - A)^{-1} B \\
- W^T Q(j\omega I_n - A)^{-1} B - B^T(-j\omega I - A^T)^{-1} Q^T W \\
= W^T W + B^T(-j\omega I_n - A^T)^{-1} \left(Q^T Q + L\right) (j\omega I_n - A)^{-1} B) \\
- W^T Q(j\omega I_n - A)^{-1} B - B^T(-j\omega I_n - A^T)^{-1} Q^T W \\
= \left(W^T - B^T(-j\omega I_n - A^T)^{-1} Q^T\right) \left(W - Q(j\omega I_n - A)^{-1} B\right) \\
+ B^T(-j\omega I_n - A^T)^{-1} L(j\omega I_n - A)^{-1} B \geq 0
\end{array}$$

Assume condition (2) is false. Then there exist $\{u_n\}$, $\|u_n\| = 1$, and $\{\omega_n\}$ such that

$$0 \leq \langle (H(j\omega_n) + H^*(j\omega_n)) u_n, u_n \rangle \leq \frac{1}{n}$$

As $n \to \infty$, if $\omega_n \to \infty$, then

$$\langle (H(j\omega_n) + H^*(j\omega_n)) u_n, u_n \rangle \to \langle D u_n, u_n \rangle \geq \mu_{\min}(D) > 0$$

which is a contradiction since the left-hand side converges to zero. Hence, u_n and ω_n are both bounded sequences and therefore contain convergent subsequences u_{n_k} and ω_{n_k}. Let the limits be u_o and ω_o. Then

$$\langle \left(H\left(j\omega_n\right) + H^*\left(j\omega_n\right)\right) u_n, u_n \rangle = 0.$$

This implies

$$Wu_o - Q\left(j\omega_o I_n - A\right)^{-1} Bu_o = 0$$

$$L^{1/2}\left(j\omega_o I_n - A\right)^{-1} Bu_o = 0$$

Since $L > 0$, the second equality implies

$$\left(j\omega_o I_n - A\right)^{-1} Bu_o = 0$$

Substituting back to the first equality yields $Wu_o = 0$. The positive definiteness of W (by assumption $D > 0$) implies contradiction. Hence, condition (2) is satisfied.

(2) $\Longrightarrow$ (8)

Since (2) $\Longrightarrow$ (1), the Lur'e equation holds. Let

$$V(x) = \frac{1}{2} x^T P x$$

Then

$$\begin{aligned}
\dot{V}(x(t)) &= x(t)PAx(t) + x(t)^T PBu(t) \\
&= -\frac{1}{2}x^T(t)Lx(t) - \frac{1}{2}\left\|Qx(t)^2\right\| + u^T(t)Cx\left(t\right) + u^T(t)W^TQx(t) \\
&= -\frac{1}{2}x^T(t)Lx(t) - \frac{1}{2}\left\|Qx(t)^2\right\| - u^T(t)Du(t) + u^T(t)W^TQx(t) + \\
&\quad +u^T(t)y(t) \\
&\leq -\frac{\epsilon}{2}\left\|x(t)\right\|^2 + u^T(t)y(t) - \frac{1}{2}\left\|Qx(t) - Wu(t)\right\|^2 \\
&\leq -\frac{\epsilon}{2}\left\|x(t)\right\|^2 + u^T(t)y(t)
\end{aligned}$$

By integrating both sides [421] for all $t \geq 0$ we get

$$\int_0^t u^T(s)y(s)ds \geq -V(x_o) \tag{3.42}$$

Since (3.32) remains valid if D is replaced by $D \to \epsilon$ for ϵ sufficiently small, (3.42) holds with y replaced by

$$y_1 = Cx + (D - \epsilon)u$$

Then (3.42) becomes

$$\int_0^t u^T(s)y(s)ds \geq \epsilon \int_0^t \|u(s)\|^2 ds - V(x_o)$$

Identifying $-V(x_o)$ with $\xi(x_o)$ and ϵ with ρ in (3.37), condition (8) follows.

(8) $\Longrightarrow$ (2)

Let $t \to \infty$ in (3.37), then

$$\int_0^t u^T(s)y(s)ds \geq \xi(x_o) + \rho \int_0^\infty \|u(t)\|^2 dt$$

In particular, for $x_o = 0$,

$$\int_0^t u^T(s)y(s)ds \geq \rho \int_0^\infty \|u(t)\|^2 dt$$

By Plancherel's Theorem,

$$\int_{-\infty}^\infty \hat{u}^*(j\omega)\hat{y}(j\omega)d\omega \geq \rho \int_{-\infty}^\infty \|\hat{u}(\omega)\|^2 d\omega,$$

for all $u \in L_2$. Suppose that for each $\eta > 0$, there exists $w \in \mathbb{C}$ and $\omega_o \in \mathbb{R}$ such that

$$w^* H(j\omega)w > \eta \|w\|^2$$

By the continuity of $w^* H(j\omega)w$ in ω, there exists an interval Ω around ω_o of length r such that

$$w^* H(j\omega)w > \eta \|w\|^2$$

for all $\omega \in \Omega$. Let

$$\hat{u}(j\omega) = \begin{cases} w & if\ \omega \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $\hat{u} \in L_2$. Then

$$\int_{-\infty}^\infty \hat{u}^*(j\omega)\hat{y}(j\omega)d\omega = \int_{-\infty}^\infty \hat{u}^*(j\omega)T(j\omega)\hat{u}(j\omega)d\omega < r\eta \|w\|^2,$$

and

$$\rho \int_{-\infty}^\infty \|\hat{u}(\omega)\|^2 d\omega = r\rho \|w\|^2.$$

If $\eta < \rho$, this is a contradiction. Hence, there exists an interval $\eta > 0$ such that (3.32) holds.

(8) $\Longrightarrow$ (11)

Condition (11) follows directly from condition (8).

(11) $\Longrightarrow$ (8)

The implication is obvious if $x_o = 0$. In the proof of (8) $\Longrightarrow$ (2), x_o is taken to be zero. Therefore, for $x = 0$ (11) $\Longrightarrow$ (8) $\Longrightarrow$ (2). It has already been shown that (2) $\Longrightarrow$ (8). Hence, (11) $\Longrightarrow$ (2) $\Longrightarrow$ (8).

(1′) $\Longrightarrow$ (1)

By definition.

(if $D = 0$)

If $D = 0$, then $W = 0$. Rewrite (3.28) as

$$A^T P + PA = -Q^T Q - L + 2\mu P - 2\mu P$$

For μ small enough,

$$Q^T Q + L - 2\mu P \geq 0$$

Hence, there exists Q_1 such that

$$A^T P + PA = -Q_1^T Q_1 - 2\mu P$$

Since (3.29) is independent of Q_1 when $D = 0$, (1′) is proved.

(1′) $\Longrightarrow$ (6)

By straightforward manipulation

(6) $\Longrightarrow$ (7)

Same as in (1) $\Longrightarrow$ (2) except L is replaced $2\mu P$.

(7) $\Longrightarrow$ (6)

Positive Real (or KYP) Lemma

(4) $\Longrightarrow$ (7)

For $\mu > 0$ sufficiently small, $A - \mu I_n$ remains strictly stable. Now, by direct substitution

$$\begin{aligned} H(j\omega - \mu) + H^*(j\omega - \mu) &= D + D^T + C(j\omega I_n - A - \mu I_n)^{-1}B + \\ &\quad + B^T(-j\omega I_n - A^T - \mu I_n)^{-1}C^T \\ &= H(j\omega) + H^*(j\omega) \\ &\quad + \mu\left[C(j\omega I_n - A)^{-1}(j\omega I_n - A - \mu I_n)^{-1}B + \right. \\ &\quad \left. + B^T(-j\omega I_n - A^T - \mu I_n)^{-1}(-j\omega I_n - A^T)^{-1}C^T\right] \end{aligned} \tag{3.43}$$

Therefore for any $w \in \mathbb{C}^m$,

$$\begin{aligned} &2w^* H(j\omega - \mu)w \geq 2w^* H(j\omega)w - \\ &- 2\mu \|C\| \|B\| \left\|(j\omega I_n - A)^{-1}\right\| \left\|(j\omega I_n - A - \mu I_n)^{-1}\right\| \|w\|^2 \end{aligned} \tag{3.44}$$

Since

$$\|(j\omega I_n - A)x\| \geq |(|\omega| - \|A\|) \|x\||$$

it follows [375]

$$\left\|(j\omega I_n - A)^{-1}\right\| \leq \frac{1}{||\omega| - \|A\||}$$

Then

$$2w^* H(j\omega - \mu)w \geq 2w^* H(j\omega)w - \frac{2\mu \|C\| \|B\| \|w\|^2}{||\omega| - \|A\|| \ \ ||\omega| - \|A - \mu I\||}$$

By (3.34), for all $\omega \in \Omega$, Ω is compact in R, there exists $k > 0$, k dependent on Ω, such that

$$2w^* H(j\omega)w \geq k \|w\|^2 \tag{3.45}$$

By (2.7b), for all ω sufficiently large, there exists $g > 0$ such that

$$2w^* H(j\omega)w \geq \frac{g}{\omega^2} \|w\|^2 \tag{3.46}$$

Hence, there exists $\omega_1 \in R$ large enough so that (3.45) and (3.46) hold with some g and k dependent on ω_1. Then, for $|\omega| \leq \omega_1$,

$$\begin{aligned} 2w^* H(j\omega - \mu)w &\geq k \|w\|^2 - \frac{2\mu \|C\| \|B\| \|w\|^2}{||\omega| - \|A\|| \ \ ||\omega| - \|A - \mu I_n\||} \qquad (3.47) \\ &\geq k \|w\|^2 - \mu \left\{ \sup_{|\omega| \leq \omega_1} \frac{2 \|C\| \|B\| \|w\|^2}{||\omega| - \|A\|| \ \ ||\omega| - \|A - \mu I_n\||} \right\} \end{aligned}$$

and for $|\omega| > \omega_1$,

$$\begin{aligned} 2w^* H(j\omega - \mu)w &\geq \frac{g}{\omega^2} \|w\|^2 - \frac{2\mu \|C\| \|B\| \|w\|^2}{||\omega| - \|A\|| \ \ ||\omega| - \|A - \mu I_n\||} \qquad (3.48) \\ &\geq \frac{\|w\|^2}{\omega^2} \cdot \left(g - \mu \left\{ \sup_{|\omega| > \omega_1} \frac{2 \|C\| \|B\| \|w\|^2}{||\omega| - \|A\|| \ \ ||\omega| - \|A - \mu I\||} \right\} \right) \end{aligned}$$

The terms in curly brackets in (3.47) and (3.48) are finite. Hence, there exists μ small enough such that (3.47) and (3.48) are both non-negative, proving condition (7).

(7) $\Longrightarrow$ (4)

From (7) $\Longrightarrow$ (6), the minimal realization (A, B, C, D) associated $T(j\omega)$ satisfies the Lur'e equation with $L = 2\mu P$. Following the same derivation as in (1) $\Longrightarrow$ (2), for all $w \in \mathbb{C}^m$, we have

$$\begin{aligned}
&w^*(H(j\omega-\mu)+H^*(j\omega-\mu))w = \\
&= w^*(W^T+B^T(-j\omega I_n-A^T)^{-1}Q^T)(W+Q(-j\omega I_n-A^T)^{-1}B)w \\
&+2\mu w^*B^T(-j\omega I_n-A^T)^{-1}P(-j\omega I_n-A^T)^{-1}Bw \\
&\geq 2\mu w^*B^T(-j\omega I_n-A^T)^{-1}P(-j\omega I_n-A^T)^{-1}Bw \\
&\geq \tfrac{2\mu\mu_{\min}(P)\sigma_{\min}(B)}{||\omega|-\|A\||^2}\,\|w\|^2
\end{aligned} \tag{3.49}$$

Since P is positive definite and, by assumption, $\sigma_{\min}(B)>0$, $T(j\omega)$ is positive for all $\omega \in I\!R$.

It remains to show (3.35). Multiply both side of the inequality above by ω^2, then

$$\omega^2 w^*(H(j\omega)+H^*(j\omega))w \geq \frac{\omega^2 2\mu\mu_{\min}(P)\sigma^2_{\min}(B)}{||\omega|-\|A\||^2}\,\|w\|^2$$

As $\omega^2 \to \infty$, the lower bound converges to $\omega^2 2\mu\mu_{\min}(P)\sigma^2_{\min}(B)$ which is positive.

(7) $\Longrightarrow$ (5)

If (3.36) is satisfied, $H(j\omega-\mu)$ corresponds to the driving point impedance of a miltiport passive network [8]. Hence, $H(j\omega)$ corresponds to the impedance of the same network, with all C replaced by C in parallel with resistor of conductance μC and L replaced by L in series with a resistor of resistance μL. Since all C, L elements are now lossy, or dissipative, $H(j\omega)$ is the driving point impedance of a dissipative network.

(7) $\Longrightarrow$ (5)

Reversing the above argument, if $H(j\omega)$ is the driving point impedance of a dissipative network, all L and C elements a lossy. Hence, by removing sufficiently small series resistance in L and parallel conductance in C, the network will remain passive. Hence, again by [8], condition (7) is satisfied.

(6) $\Longrightarrow$ (9)

Let

$$V(t,x(t)) = \frac{1}{2}e^{\gamma^t}x^T P x$$

Then

$$\begin{aligned}
&\dot{V}(t,x(t)) = \\
&= \frac{1}{2}e^{\gamma^t}x^T(t)Px(t)+\frac{1}{2}e^{\gamma^t}x^T(t)(PA+A^TP)x(t)+e^{\gamma^t}x^T(t)PBu(t) \\
&\leq \gamma V(t,x(t)) - \frac{\epsilon}{2}\frac{V(t,x(t))}{\|P\|} - e^{\gamma^t}\|Qx(t)-Wu(t)\|^2 + e^{\gamma^t}u^T(t)y(t) \\
&\leq -\left(\frac{\epsilon}{2\|P\|}-\gamma\right)V(t,x(t)) + e^{\gamma^t}u^T(t)y(t)
\end{aligned}$$

Choose $0<\gamma<\epsilon/2\|P\|$. Then by comparison principle, for all $T \geq 0$,

$$\int_0^t e^{\gamma^s} u^T(s)y(s)ds \geq -x_o^T P x_o$$

(9) $\Longrightarrow$ (6)
Define

$$\begin{cases} u_1(t) = e^{(\gamma/2)^t} u(t) \\ y_1(t) = e^{(\gamma/2)^t} y(t) \\ x_1(t) = e^{(\gamma/2)^t} x(t) \end{cases} \tag{3.50}$$

where $\gamma > 0$ is as given in (3.38). then

$$\begin{cases} \dot{x}_1(t) = \left(A + \frac{\gamma}{2} \cdot I\right) x_1(t) + Bu_1(t) \\ \\ y_1(t) = Cx_1(t) + Du_1(t) \end{cases} \tag{3.51}$$

The corresponding transfer function is

$$\begin{aligned} H_1(j\omega) &= D + C\left(j\omega I_n - A - \frac{\gamma}{2} I_n\right)^{-1} B \\ &= H\left(j\omega - \frac{\gamma}{2}\right) \end{aligned}$$

By setting $t = \infty$ and $x_o = 0$ in (3.38),

$$\int_0^t u_1^T(s)y_1(s)ds \geq 0$$

By Plancherel's Theorem,

$$\int_{-\infty}^{\infty} \hat{u}_1^*(j\omega)(T_1(j\omega) + T_1^*(j\omega))\hat{u}_1(j\omega) \geq 0$$

Since this holds true for all $\hat{u}_1(j\omega) \in \mathcal{L}_2$

$$H_1(j\omega) + H_1^*(j\omega) \geq 0$$

Equivalently

$$H\left(j\omega - \frac{\gamma}{2}\right) - H^*\left(j\omega - \frac{\gamma}{2}\right) \geq 0$$

proving (7)
(9) $\Longrightarrow$ (10)
Use the transformation in (3.50), then condition (10) follows directly from condition (9) with $\alpha = \gamma/2$.
(10) $\Longrightarrow$ (9)

If $x_o = 0$, (10) $\Longrightarrow$ (9) is obvious. Since in the proof of (9) $\Longrightarrow$ (6), only the $x_o = 0$ case is considered, it follows, for the $x_o = 0$ case, (10) $\Longrightarrow$ (9) $\Longrightarrow$ (6). It has already been shown that (6) $\Longrightarrow$ (9). Hence, (10) $\Longrightarrow$ (6) $\Longrightarrow$ (9).

(2) $\Longrightarrow$ (4) $\Longrightarrow$ (3)

The implications are obvious. ■

Remark 3.12. Stating $H(j\omega) + H^\star(j\omega) \geq \delta I_n$ for all $\omega \in I\!R = (-\infty, +\infty)$, is equivalent to stating $H(j\omega) + H^\star(j\omega) > 0$ for all $\omega \in I\!R \cup \{-\infty, +\infty\} = [-\infty, +\infty]$. This is different from $H(j\omega) + H^\star(j\omega) > 0$ for all $\omega \in I\!R$ because such a condition does not imply the existence of a $\delta > 0$ such that $H(j\omega) + H^\star(j\omega) \geq \delta I_n$ for all $\omega \in I\!R$.

Example 3.13. If $H(s) = \frac{s}{s+1}$, then $H(j\omega) + H^\star(j\omega) = \frac{2\omega^2}{1+\omega^2}$, so $H(s)$ is not SPR despite $\mathbf{Re}[H(\infty)] = 2$. But $H(0) + H^\star(0) = 0$. If $H(s) = \frac{s+2}{s+1}$, then $H(j\omega) + H^\star(j\omega) = \frac{4+\omega^2}{1+\omega^2} \geq 1$ for all $\omega \in [-\infty, +\infty]$. This tranfer function is SSPR. If $H(s) = \frac{1}{s+1}$, then $H(j\omega) + H^\star(j\omega) = \frac{2}{1+\omega^2} > 0$ for all $\omega \in (-\infty, +\infty)$. Moreover $\lim_{\omega \to +\infty} \frac{2\omega^2}{1+\omega^2} > 0$, so $H(s)$ is SPR.

Further works on the characterization of PR or SPR transfer functions can be found in [10, 49, 50, 132, 177, 205, 339, 340, 396, 448, 455, 479, 504, 530].

3.1.5 Descriptor Variable Systems

The KYP Lemma can be extended to a class of linear systems larger than (3.1). Let us consider the following class of linear time invariant systems

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = Cx(t) + Du(t) \\ \\ x(0^-) = x_0 \end{cases} \tag{3.52}$$

with $A, E \in I\!R^{n \times n}$, $B, C \in I\!R^{n \times m}$, and $D \in I\!R^{m \times m}$. When the matrix E is singular (*i.e.*, its rank is $< n$) then the system in (3.52) is called *singular* or *descriptor system*. Throughout this section we shall assume that rank$(E) < n$ since otherwise we are back to the classical regular case. Descriptor systems arise in various fields of applications, like for instance constrained mechanical systems, or electrical circuits, since Kirschoff's laws directly yield algebraic equality constraints on the state. The next assumption will be supposed to hold throughout the whole of this section.

Assumption 1 *The pair (E, A) is regular, i.e. det$(sE - A)$ is not identically zero, $s \in \mathbb{C}$.*

Let us recall some facts about (3.52). If the pair (E, A) is regular, then there exists two square invertible matrices U and V such that the system can be transformed into its Weierstrass canonical form

$$\begin{cases} \bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t) \\ y(t) = \bar{C}x(t) + Du(t) \\ x(0^-) = x_0 \end{cases} \tag{3.53}$$

with $\bar{A} = UAV = \begin{pmatrix} A_1 & 0 \\ 0 & I_q \end{pmatrix}$, $\bar{E} = UEV = \begin{pmatrix} I_q & 0 \\ 0 & N \end{pmatrix}$, $\bar{B} = UB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $\bar{C} = CV = (C_1 \;\; C_2)$. The $(n-q)\times(n-q)$ matrix N is nilpotent, *i.e.* $N^l = 0$ for some integer $l \geq 1$. Generally speaking, solutions of (3.52) are not functions of time but distributions (i.e. the general solutions may contain Dirac and derivatives of Dirac measures). The system is called *impulse free* if $N = 0$. To better visualize this, let us notice that the transformed system can be written as [106]

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + B_1 u(t) \\ N\dot{x}_2(t) = x_2(t) + B_2 u(t) \end{cases} \tag{3.54}$$

and the solution of (3.52) is $x = x_1 + x_2$. One has

$$\begin{cases} x_1(t) = \exp(tA_1)x_s(0) + \exp(tA_1) \star B_1 u(t) \\ x_2(t) = -\sum_{i=1}^{l-1} \delta_0^{(i-1)} N^i x_2(0^-) - \sum_{i=0}^{l-1} N^i B_2 u^{(i)}(t) \end{cases} \tag{3.55}$$

When $N = 0$ the variable $x_2(\cdot)$ is just equal to $-B_2u(t)$ at all times. Otherwise an initial state jump may occur, and this is the reason why we wrote the left-limit $x(0^-)$ in (3.52). The exponential modes of the regular pair (E, A) are the finite eigenvalues of $sE - A$, $s \in \mathbb{C}$, such that $\det(sE - A) = 0$.

Definition 3.14. *The descriptor system (3.52) is said to be* admissible *if the pair* (E, A) *is regular, impulse-free and has no unstable exponential modes.*

Proposition 3.15. *[348] The descriptor system (3.52) is admissible and SSPR (Strongly SPR) if and only if there exists matrices* $P \in \mathbb{R}^{n\times n}$ *and* $W \in \mathbb{R}^{n\times m}$ *satisfying*

$$\begin{aligned} & E^T P = P^T E \geq 0, \quad E^T W = 0 \\ & \begin{bmatrix} A^T P + P^T A & A^T W + P^T B - C^T \\ (A^T W + P^T B - C^T)^T & W^T B + B^T W - D - D^T \end{bmatrix} < 0 \end{aligned} \tag{3.56}$$

■

When $E = I_n$ then $W = 0$, $P = P^T$ and we are back to the classical KYP Lemma conditions. In the next theorem PRness is understood as in Definition 2.29.

Theorem 3.16. *[157] If the LMI*

$$\begin{aligned} &E^T P = P^T E \geq 0 \\ &\begin{bmatrix} A^T P + P^T A & P^T B - C^T \\ (P^T B - C^T)^T & -D - D^T \end{bmatrix} \leq 0 \end{aligned} \tag{3.57}$$

has a solution $P \in I\!R^{n\times n}$, then the transfer matrix $H(s)$ is PR. Conversely, let $H(s) = \sum_{i=-\infty}^{p} M_i s^i$ be the expansion of $H(s)$ about $s = \infty$, and assume that $D + D^T \geq M_0 + M_0^T$. Let also the realization of $H(s)$ in (3.52) be minimal. Then if $H(s)$ is PR there exists a solution $P \in I\!R^{n\times n}$ to the LMI in (3.57).

■

Minimality means that the dimension n of E and A is as small as possible. The main difference between Proposition 3.15 and Theorem 3.16 is that it is not supposed that the system is impulsive-free in the latter. When the system is impulse-free, one gets $M_0 = H(\infty) = D - C_2 B_2$, and the condition $D + D^T \geq M_0 + M_0^T$ is not satisfied unless $C_2 B_2 + (C_2 B_2)^T \geq 0$.

Proof: Let us prove the sufficient part of Theorem 3.16. Let s with $\mathbf{Re}[s] > 0$ be any point such that s is not a pole of $H(s)$. The matrix $sE - A$ is nonsingular for such a s. From Proposition A.63 it follows that we can write equivalently the LMI in (3.57) as

$$\begin{cases} A^T P + P^T A = -LL^T \\ P^T B - C = -LW \\ D + D^T \geq W^T W \\ E^T P = P^T E \geq 0 \end{cases} \tag{3.58}$$

for some matrices L and W. From the first and last equations of (3.58) it follows that

$$\begin{aligned} (sE - A)^\star P + P^T(sE - A) &= -A^T P - P^T A + \bar{s} E^T P + s P^T E \\ &= LL^T + \mathbf{Re}[s](e^T P + P^T E) - \\ &\quad - j\mathbf{Im}[s](E^T P - P^T E) \\ &= LL^T + 2\mathbf{Re}[s] E^T P \end{aligned} \tag{3.59}$$

Notice that $(sE - A)F(s) = B$ where $F(s) = (sE - A)^{-1}B$. Thus since $H(s) = C(sE - A)^{-1}B + D$ and the second relation in (3.58) one has

$$\begin{aligned} H(s) &= D + C^T F(s) \\ &= D + W^T L^T F(s) + B^T P F(s) \\ &= D + W^T L^T F(s) + F^\star(s)(sE - A)^\star P F(s) \end{aligned} \tag{3.60}$$

Using now (3.60) and (3.59) and the third relation in (3.58) we obtain

$$\begin{aligned} H(s) + H^\star(s) &= D + D^T + W^T L^T F(s) + F^\star L W + \\ &\quad + F^\star(s)[(sE - A)^\star P + P^T(sE - A)] \\ &\geq W^T W + W^T L^T F(s) + F^\star(s) L W + \\ &\quad + F^\star(s)(LL^T + 2\mathbf{Re}[s] E^T P) F(s) \\ &= (W + L^T F(s))^\star (W + L^T F(s)) + 2\mathbf{Re}[s] F^\star(s)(E^T P) F(s) \end{aligned} \tag{3.61}$$

Since $(W + L^T F(s))^\star(W + L^T F(s) \geq 0$ and since $\mathbf{Re}[s] > 0$ and $E^T P \geq 0$, we have $\mathbf{Re}[s] F^\star(s)(E^T P)F(s) \geq 0$. Thus from (3.61) we obtain

$$H(s) + H^\star(s) \geq 0 \tag{3.62}$$

Recall here that s has been assumed to be any complex with $\mathbf{Re}[s] > 0$ and such that it is not a pole of $H(s)$. Now suppose $H(s)$ has a pole s_0 with $\mathbf{Re}[s_0] > 0$. Then there exists a pointed neighborhood of s_0 that is free of any pole of $H(s)$ and thus $H(s)$ satisfies (3.62) in this domain. However this is impossible if s_0 were a pole of $H(s)$. Therefore $H(s)$ does not have any pole in $\mathbf{Re}[s] > 0$, and (3.62) is true for any $s \in \mathbb{C}$ with $\mathbf{Re}[s] > 0$. Thus $H(s)$ is PR.

■

In the proof we used the fact that the pair (E, A) is regular (see Assumption 1) which equivalently means that the matrix $sE - A$ is singular for only finitely many $s \in \mathbb{C}$.

Example 3.17. [541] Consider

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix} \tag{3.63}$$
$$C = [1 \;\; 1 \;\; 1], \quad D = \tfrac{1}{2}$$

where b is a constant. The pair (E, A) is regular, impulse-free and stable. One has

$$H(s) = \frac{1}{s+1} + \frac{1}{s+2} - b + \frac{1}{2} \tag{3.64}$$

and from

$$H(j\omega) + H(-j\omega) = \frac{2}{\omega^2+1} + \frac{4}{\omega^2+4} - 2b + 1 \tag{3.65}$$

it follows that $H(s)$ is SSPR when $b = 0$ and is not SSPR when $b = 1$.

Another example is treated in Example 4.63. Further results on positive realness of descriptor systems and applications to control synthesis, can be found in [157, 260, 348, 541]. The discrete-time case is analyzed in [287, 517].

3.2 Weakly SPR Systems and the KYP Lemma

A dissipative network is composed of resistors, lossy inductors and lossy capacitors (see Example 3.90 for the case of nonsmooth circuits with ideal diodes). Consider the circuit depicted in Figure 3.1 of an ideal capacitor in parallel with a lossy inductor. Even though this circuit is not only composed of dissipative elements, the energy stored in the network always decreases. This suggests that the concept of SPR may be unnecessarily restrictive for some control applications. This motivates the study of weakly SPR systems and its relationship with the Kalman-Yakubovich-Popov Lemma. The transfer function of the depicted circuit is $\frac{Ls+R}{LCs^2+RCs+1}$. It can be checked from Theorem 2.45 that this is not SPR, since $r = 1$ and $\lim_{\omega \to +\infty} \omega^2 \mathbf{Re}[H(j\omega)] = 0$. Lozano and Joshi [310] proposed the following Lemma which establishes equivalent conditions in the frequency and time domain for a system to be weakly SPR (WSPR).

Lemma 3.18. *[310] [Weakly SPR] Consider the minimal (controllable and observable) LTI system (3.1) whose transfer function is given by*

$$H(s) = D + C(sI_n - A)^{-1}B \tag{3.66}$$

Assume that the system is exponentially stable and minimum-phase. Under such conditions the following statements are equivalent:

1. $\exists\ P > 0, P \in I\!R^{n\times n}, W \in I\!R^{m\times m}, L \in I\!R^{n\times m}$

$$\begin{cases} PA + A^T P = -LL^T \\ PB - C^T \ \ = -LW \\ D + D^T \quad\ \ = W^T W \end{cases} \tag{3.67}$$

   and such that the quadruplet (A, B, L, W) is a minimal realization whose transfer function: $\overline{H}(s) = W + L^T(sI_n - A)^{-1}B$ has no zeros in the $j\omega$ axis (*i.e.* rank $\bar{H}(j\omega) = m, \ \forall \ \omega < \infty$).

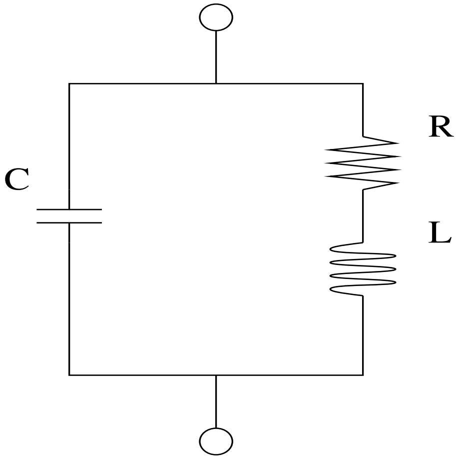

Fig. 3.1. An ideal capacitor in parallel with a lossy inductor.

2. $H(j\omega) + H^*(j\omega) > 0, \;\; \forall \;\; \omega \in I\!R.$
3. The following input-output relationship holds

$$\int_0^t u^t(s)y(s)ds + \beta \geq \int_0^t \bar{y}^T(s)\bar{y}(s)ds, \;\; ,\forall \; t > 0$$

with $\beta = x(0)^T Px(0), P > 0$ and $\bar{y}(s) = \overline{H}(s)u(s)$. ■

Proof: $(\mathbf{1}) \Rightarrow (\mathbf{2})$

Using (3.66) and (3.67) we obtain

$$\begin{aligned}
&H(j\omega) + H^*(j\omega) \\
&= D + D^T + C(j\omega I_n - A)^{-1}B + B^T(-j\omega I_n - A^T)^{-1}C^T \\
&= W^TW + (B^TP + W^TL^T)(j\omega I_n - A)^{-1}B \\
&\quad + B^T(-j\omega I_n - A^T)^{-1}(PB + LW) \\
&= W^TW + B^T(-j\omega I_n - A^T)^{-1}[(-j\omega I_n - A^T)P \\
&\quad + P(j\omega I_n - A)](j\omega I_n - A)^{-1}B + W^TL^T(j\omega I_n - A)^{-1}B \\
&\quad + B^T(-j\omega I_n - A^T)^{-1}LW
\end{aligned}$$

and so

$$
\begin{aligned}
&H(j\omega) + H^*(j\omega) \\
&= W^T W + B^T(-j\omega I_n - A^T)^{-1} L L^T (j\omega I_n - A)^{-1} B \\
&\quad + W^T L^T (j\omega I_n - A)^{-1} B + B^T(-j\omega I_n - A^T)^{-1} L W \\
&= (W + L^T(-j\omega I_n - A)^{-1} B)^T (W + L^T (j\omega I_n - A)^{-1} B)
\end{aligned}
$$

It then follows

$$
H(j\omega) + H^*(j\omega) = \overline{H}^*(j\omega)\overline{H}(j\omega) > 0 \tag{3.68}
$$

Since $\overline{H}(s)$ has no zeros on the $j\omega$-axis, $\overline{H}(j\omega)$ has full rank and, therefore, the right-hand-side of (3.68) is strictly positive.

(**2**) $\Rightarrow$ (**1**)

In view of statement 2, there exists an asymptotically stable transfer function $H(s)$ such that (see [406] or [145])

$$
H(j\omega) + H^*(j\omega) = \overline{H}^*(j\omega)\overline{H}(j\omega) > 0 \tag{3.69}
$$

Without loss of generality let us assume that

$$
\overline{H}(s) = W + J(sI_n - F)^{-1} G \tag{3.70}
$$

with (F, J) observable and the eigenvalues of F satisfying $\lambda_i(F) < 0$ $1 \le i \le n$. Therefore, there exists $\bar{P} > 0$ (see [272]) such that

$$
\bar{P}F + F^T\bar{P} = -JJ^T \tag{3.71}
$$

Using (3.70) and (3.71) we have

$$
\begin{aligned}
\overline{H}^T(-j\omega)\overline{H}(j\omega) &= [W + J(-j\omega I_n - F)^{-1} G]^T \\
&\quad \times [W + J(j\omega I_n - F)^{-1} G] \\
&= W^T W + W^T J (j\omega I_n - F)^{-1} G \\
&\quad + G^T(-j\omega I_n - F^T)^{-1} J^T W + X
\end{aligned} \tag{3.72}
$$

where

$$
\begin{aligned}
X &= G^T(-j\omega I_n - F^T)^{-1} J^T J (j\omega I_n - F)^{-1} G \\
&= -G^T(-j\omega I_n - F^T)^{-1}[\bar{P}(F - j\omega I_n) \\
&\quad + (F^T + j\omega I_n)\bar{P}](j\omega I_n - F)^{-1} G \\
&= G^T(-j\omega I_n - F^T)^{-1}\bar{P}G + G^T\bar{P}(j\omega I_n - F)^{-1} G
\end{aligned} \tag{3.73}
$$

Introducing (3.73) into (3.72) and using (3.69):

$$\begin{aligned}\overline{H}^T(-j\omega)\overline{H}(j\omega) &= W^TW + (W^TJ + G^T\bar{P})(j\omega I_n - F)^{-1}G \\ &\quad + G^T(-j\omega I_n - F^T)^{-1}(J^TW + \bar{P}G) \\ &= H(j\omega) + H^T(-j\omega) \\ &= D + D^T + C(j\omega I_n - A)^{-1}B \\ &\quad + B^T(-j\omega I_n - A^T)^{-1}C^T\end{aligned} \tag{3.74}$$

From (3.74) it follows that $W^TW = D + D^T$. Since the eigenvalues of A and F satisfy $\lambda_i(A) < 0$ and $\lambda_i(F) < 0$, then

$$C(j\omega I_n - A)^{-1}B = (W^TJ + G^T\bar{P})(j\omega I_n - F)^{-1}G \tag{3.75}$$

Therefore the various matrices above can be related through a state space transformation, *i.e.*

$$\begin{cases} TAT^{-1} = F \\ TB = G \\ CT^{-1} = W^TJ + G^T\bar{P} \end{cases} \tag{3.76}$$

Defining $P = T^T\bar{P}T$ and $L^T = JT$ and using (3.71) and (3.76)

$$\begin{aligned}-LL^T &= -T^TJ^TJT \\ &= T^T(\bar{P}F + F^T\bar{P})T \\ &= T^T\bar{P}TT^{-1}FT + T^TF^TT^{-T}T^T\bar{P}T \\ &= PA + A^TP\end{aligned}$$

which is the first equation of (3.67). From (3.76) we get

$$\begin{aligned}C &= W^TJT + G^T\bar{P}T \\ &= W^TL^T + G^TT^{-T}T^T\bar{P}T \\ &= W^TL^T + B^TP\end{aligned} \tag{3.77}$$

which is the second equation of (3.67). $\overline{H}(s)$ was defined by the quadruplet (F, G, J, W) in (3.70) which is equivalent, through a state-space transformation, to the quadruplet $(T^{-1}FT,\ T^{-1}G,\ JT,\ W)$. In view of (3.76) and since $L^T = JT$, $\overline{H}(s)$ can also be represented by the quadruplet (A, B, L^T, W) i.e.

$$\overline{H}(s) = W + L^T(sI_n - A)^{-1}B \tag{3.78}$$

We finally note from (3.69) that $\overline{H}(j\omega)$ has no zeros on the $j\omega$-axis.

(1) $\Rightarrow$ **(3)**

Consider the following positive definite function: $V(x) = \frac{1}{2}x^T Px$. Then using (3.67) we obtain

$$\begin{aligned}
\dot{V}(x) &= \tfrac{1}{2}x^T(PA + A^T P)x + x^T PBu \\
&= -\tfrac{1}{2}x^T LL^T x + u^T B^T Px \\
&= -\tfrac{1}{2}x^T LL^T x + u^T(C - W^T L^T)x \\
&= -\tfrac{1}{2}x^T LL^T x + u^T y - \tfrac{1}{2}u^T(D + D^T)u - u^T W^T L^T x \\
&= -\tfrac{1}{2}x^T LL^T x + u^T y - \tfrac{1}{2}u^T W^T W u - u^T W^T L^T x \\
&= u^T y - \tfrac{1}{2}\left(L^T x + Wu\right)^T \left(L^T x + Wu\right) \\
&= u^T y - \tfrac{1}{2}\bar{y}^T\bar{y}
\end{aligned} \tag{3.79}$$

where $\bar{y}$ is given by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \bar{y}(t) = L^T x(t) + Wu(t) \end{cases} \tag{3.80}$$

Therefore, in view of (3.69)

$$\bar{y}(s) = \overline{H}(s)u(s), \tag{3.81}$$

with $\overline{H}(s) = W + L^T(sI_n - A)^{-1}B$. Integrating (3.79) gives

$$\int_0^t u^T(s)y(s)ds + \beta \geq \frac{1}{2}\int_0^t \bar{y}^T(s)\bar{y}(s)ds \tag{3.82}$$

with $\beta = V(x(0))$.

(3) $\Rightarrow$ **(2)**

Without loss of generality, consider an input u such that $\int_0^t u^T(s)u(s)ds < +\infty$, $\forall\ t \geq 0$. Dividing (3.82) by $\int_0^t u^T(s)u(s)ds$, we obtain

$$\frac{\int_0^t u^T(s)y(s)ds + V(x(0))}{\int_0^t u^T(s)u(s)ds} \geq \frac{\int_0^t \bar{y}^T(s)\bar{y}(s)ds}{\int_0^t u^T(s)u(s)ds} \tag{3.83}$$

This inequality should also hold for $t = \infty$ and $x(0) = 0$, *i.e.*

$$\frac{\int_0^\infty u^T(s)y(s)ds}{\int_0^\infty u^T(s)u(s)ds} \geq \frac{\int_0^\infty \bar{y}^T(s)\bar{y}(s)ds}{\int_0^\infty u^T(s)u(s)ds} \tag{3.84}$$

Since $H(s)$ and $\overline{H}(s)$ are asymptotically stable, $u \in \mathcal{L}_2 \Rightarrow y, \bar{y} \in \mathcal{L}_2$ and we can use Plancherel's Theorem [421], see also Sections 4.1, 4.2 and 4.3 for $\mathcal{L}_p$ functions and their properties. From the above equation we obtain

$$\frac{\int_{-\infty}^{\infty} U^*(j\omega)(H(j\omega)+H^*(j\omega))U(j\omega)d\omega}{\int_{-\infty}^{\infty} U^*(j\omega)U(j\omega)d\omega} \geq \frac{\int_{-\infty}^{\infty} U^*(j\omega)\overline{H}^*(j\omega)\overline{H}(j\omega)U(j\omega)d\omega}{\int_{-\infty}^{\infty} U^*(j\omega)U(j\omega)d\omega}$$

Since $\overline{H}(s)$ has no zeros on the $j\omega$-axis, the right-hand-side of the above equation is strictly positive and so is the left-hand-side for all nonzero $U(j\omega) \in \mathcal{L}_2$, and thus

$$H(j\omega) + H^*(j\omega) > 0, \ \ \forall \ \omega \in (-\infty, \infty)$$

■

3.3 KYP Lemma for Non-minimal Systems

The KYP Lemma as stated above is stated for minimal realizations (A, B, C, D), *i.e.* when there is no pole-zero cancellation in the rational matrix $C(sI_n - A)^{-1}B$. However as Example 3.2 proves, non-minimal realizations may also yield a solvable set of equations (3.2). The KYP Lemma can indeed be stated for stabilizable systems, or more generally for uncontrollable and/or unobservable systems. This is done in [110, 150, 151, 390, 412, 444, 445]. The motivation for such an extension stems from the physics, as it is easy to construct systems (like electrical circuits) which are not controllable or not observable. There are also topics like adaptive control, in which many poles/zeroes cancellation occur, so that controllability of the dynamical systems cannot be assumed. Let us recall a fundamental result. Consider *any* matrices A, B, C, D of appropriate dimensions. Then the KYP Lemma set of equations (3.2) implies that

$$\Pi(j\omega) = C(i\omega I_n - A)^{-1}B - B^T(j\omega + A^T)^{-1}C^T + D + D^T \geq 0 \tag{3.85}$$

for all $\omega \in \mathbb{R}$, where the *spectral density* function $\Pi(\cdot)$ was introduced by Popov, and is named Popov's function, as we already pointed out in Section 2.12, Theorem 2.30 and Proposition 2.31. There we saw that one can characterize a positive operator with the positivity of the associated spectral function. In a word a necessary condition for the solvability of the KYP Lemma set of equations is that the Popov function satisfies (3.85). The spectral function satisfies $\Pi(s) = \Pi(-s)$ with $s \in \mathbb{C}$. In addition, if the pair

(A, B) is controllable, then the inequality (3.85) implies the solvability of the KYP Lemma set of equations, *i.e.* it is sufficient for (3.2) to possess a solution $(P = P^T, L, W)$. It is worth noting that, under *minimality* of (A, B, C, D), that the KYP Lemma set of equations solvability and the positive realness of $H(s) = C(sI_n - A)^{-1}B + D$ are equivalent. Let us notice that $\Pi(j\omega) = H(j\omega) + H^\star(j\omega)$. Let us summarize:

$$
\boxed{
\begin{array}{c}
\text{KYP Lemma equations solvability} \\[2ex]
\Uparrow \text{ (if } (A,B) \text{ controllable)} \qquad \Downarrow \text{ (for all } \omega \in I\!\!R \mid j\omega \text{ is not a pole of } \Pi(s)) \\[2ex]
\Pi(j\omega) \geq 0 \\[2ex]
\Updownarrow \text{ (if } A \text{ is Hurwitz)} \\[2ex]
H(s) = C(sI_n - A)^{-1}B + D \text{ is PR} \\[2ex]
\Downarrow \ (D = 0) \\[2ex]
\text{KYP Lemma equations solvability with } P = P^T > 0
\end{array}
}
$$

The first implication was proved by Kalman [247]. Notice that the second equivalence is stated under no other assumption that all eigenvalues of A have negative real parts. In particular no minimality of (A, B, C, D) is required. The last implication shows that the KYP Lemma solvability is sufficient for PRness of the transfer matrix, without minimality assumption [445] (the proof is led in [445] with $D = 0$). It is important to recall that "KYP Lemma equations solvability" does not mean that P is positive definite, but only the existence of a solution $(P = P^T, L, W)$. When P is searched as a non-negative definite matrix, then we have the following:

$$
\boxed{
\begin{array}{c}
\text{KYP Lemma equations solvability with } P = P^T > 0 \\[2ex]
\Updownarrow \text{ (if } (A,B,C,D) \text{ minimal)} \\[2ex]
C(sI_n - A)^{-1}B \text{ is PR}
\end{array}
}
$$

The original result of Popov, building on earlier works of Kalman and Yakubovich, was as follows:

KYP Lemma equations solvability with $P = P^T \geq 0$
$\Updownarrow$ (if (A, B, C) minimal)
$\Pi(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

One may have a look at Theorem 3.46 where the link between the Popov function positivity and the KYP Lemma set of equations solvability is concerned, and a complete proof is provided. In particular it then becomes clear where the controllability assumption comes into play in this result. However the controllability assumption is not at all necessary for the KYP Lemma set of equations to possess a solution. It is therefore of interest to relax as much as possible this assumption. Perhaps one of the first, if not the first, result relaxing the controllability is due to Meyer [353].

Lemma 3.19 (Meyer-Kalman-Yakubovich Lemma). *Given a scalar $D \geq 0$, vectors B and C, an asymptotically stable matrix A, and a symmetric positive definite matrix L, if*

$$\mathbf{Re}[H(j\omega)] = \mathbf{Re}\left[\frac{D}{2} + C(j\omega I_n - A)^{-1}B\right] > 0 \text{ for all } \omega \in \mathbb{R} \tag{3.86}$$

then there exists a scalar $\epsilon > 0$, a vector q and $P = P^T > 0$ such that

$$\begin{cases} A^T P + PA = -qq^T - \epsilon L \\ PB - C^T = \sqrt{D}q \end{cases} \tag{3.87}$$

■

An application of the MKY Lemma is in Section 8.2.2.

3.3.1 Spectral Factors

The first set of results that we present rely on the factorization of the Popov function, and have been derived by Pandolfi and Ferrante [151, 390]. If $\Pi(s)$ is a rational matrix that is bounded on the imaginary axis and is such that $\Pi(j\omega) \geq 0$, then there exists a matrix $M(s)$ which is bounded in $\mathbf{Re}[s] > 0$ and such that $\Pi(j\omega) = M^T(j\omega)M(j\omega)$. The matrix $M(s)$ of a spectral factorization has as many rows as the normal rank of $\Pi(s)$. The normal rank of a polynomial matrix is defined as the rank of $\Pi(s)$ considered as a rational matrix. If $\Pi(s) \in \mathbb{C}^{m\times m}$, and if $\det(\Pi(s))$ is not the zero function (for instance, if the determinant is equal to $s-1$), $\Pi(s)$ is said to have normal rank m. More generally a polynomial matrix has rank q if q is the largest of the orders of the minors that are not identically zero [246, §6.3.1].

Let us consider an eigenvalue s_0 of A and a Jordan chain of s_0, *i.e.* a finite sequence of vectors satisfying $Av_0 = s_0 v_0$, $Av_i = s_0 v_i + v_{i-1}$, $0 < i \leq r-1$, where r is the length of the Jordan chain. One has

$$e^{At} v_0 = e^{s_0 t} v_0, \;\; e^{At} v_k = e^{s_0 t} \sum_{i=0}^{k} \frac{t^i}{i!} v_{k-i} \tag{3.88}$$

An eigenvalue s_0 may have several Jordan chains, in general in finite number. We suppose these chains have been ordered, and we denote the ith one as $J_{s_0,i}$. The factor $M(s)$ is used together with the Jordan chain $J_{s_0,i} = (v_0, v_1, ..., v_{q-1})$, to construct the following matrix:

$$M_{s_0,i} = \begin{pmatrix} M_0 & 0 & 0 & .... & 0 \\ M_1 & M_0 & 0 & ... & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ M_{r-1} & M_{r-2} & M_{r-3} & ... & M_0 \end{pmatrix} \tag{3.89}$$

One has

$$M_h = \frac{1}{h!} \frac{d^h}{ds^h} M^T(-s_0) = \left[\frac{1}{h!} \frac{d^h}{ds^h} M^T(-s) \right]_{s_0} \tag{3.90}$$

In other words $h!M_h$ is the hth derivative of the function $M^T(-s)$ calculated at $s = s_0$. All the matrices $M_{s_0,i}$ as well as the rational functions $\Pi(s)$ and $M(s)$ are calculable from A, B, C and D. The notation col$[a_0, a_1, ..., a_n]$ is for the column matrix $[a_0 \; a_1 \; ... \; a_n]^T$.

Theorem 3.20. *[390] Let the matrices $M_{s_0,i}$ be constructed from any spectral factor of $\Pi(s)$ and assume that every eigenvalue of A has a negative real part. If the transfer function $H(s)$ is positive real, then there exist matrices L, W and $P = P^T \geq 0$ which solve the KYP Lemma set of equations (3.2), if and only if the following conditions hold for every Jordan chain $J_{s_0,i}$ of the matrix A:*

$$col[C^T v_0, C^T v_1, ..., C^T v_{r-1}] \in \; Im(M_{s_0,i}) \tag{3.91}$$

■

For the proof (that is inspired from [32]) the reader is referred to the paper [390]. It is noteworthy that there is no minimality assumption in Theorem 3.20. However P is only semi-positive definite.

Example 3.21. [390] Let $C = 0$, $B \neq 0$, $D = 0$. Then $\Pi(s) = 0$ and the set of equations $A^T P + PA = -LL^T$, $PB = C^T - LW$ is solvable. One solution is $L = 0$, $P = 0$. This proves that Theorem 3.20 does not guarantee $P > 0$.

A second Theorem relaxes the Hurwitz condition on A.

Theorem 3.22. *[151] Let $A \in I\!R^{n\times n}$, $B \in I\!R^{n\times m}$, $C \in I\!R^{m\times n}$, and $D \in I\!R^{m\times m}$. Assume that $\sigma(A) \cap \sigma(-A^T) = \emptyset$. If the KYP Lemma set of equations (3.2) is solvable, i.e. there exist matrices $P = P^T$, L, W which solve it, then $\Pi(j\omega) \geq 0$ for each ω and the condition (3.91) holds for every Jordan chain $J_{s_0,i}$ of the matrix A. Conversely, let $\Pi(j\omega)$ be nonnegative for each ω and let (3.91) hold for every Jordan chain of A. Then the set of equations (3.2) is solvable. Condition (3.91) does not depend on the specific spectral factor $M(s)$ of $\Pi(s)$.* ■

A matrix A satisfying $\sigma(A) \cap \sigma(-A^T) = \emptyset$ is said unmixed.

Remark 3.23. Until now we have spoken only on controllability, and not of observability. Thus one might think that the unobservable part has no influence neither on (3.85) nor on the solvability of (3.2). Things are more subtle as shown in the next subsection.

3.3.2 Sign-controllability

To start with, let us consider the following system [150]: $A = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, $C = (I \;\; 0)$, $B = \begin{pmatrix} 0 \\ I \end{pmatrix}$, $D = 0$. Then the KYP Lemma set of equations in (3.2) has infinitely many solutions, which can be parametrized as triples

$$\left(\begin{bmatrix} P_1 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} Q_1 \\ 0 \end{bmatrix}, 0 \right)$$

with $P_1 \leq 0$, and $Q_1 Q_1^T = -2P_1$. However the system of equations obtained by eliminating the unobservable subspace associated to (A, C) has no solution, because the second equation for this reduced system takes the form $0 = I - 0$. This example shows that unobservability is not so innocent in the KYP Lemma solvability (which is to be understood here as the existence of a triple $(P = P^T, L, W)$ that solves (3.2)).

The sign-controllability of a pair of matrices is defined in Appendix A.4. Let us assume that (A, B) is sign controllable. Then there exists a feedback $u(t) = Kx(t) + v(t)$ such that the new transition matrix $A + BK$ is unmixed. One can start from a system such that A is unmixed.

Before stating the next Lemma, let us perform a state space transformation. We assume that (A, C) is not observable. The Kalman observability form reads $A = \begin{pmatrix} \bar{A}_1 & 0 \\ \bar{A}_{21} & \bar{A}_2 \end{pmatrix}$, $C = (\bar{C}_1 \;\; 0)$. Let us define

$$\Lambda = [\sigma(\bar{A}_2) \cap \sigma(-\bar{A}_1^T)] \cup [\sigma(\bar{A}_2) \cap \sigma(-\bar{A}_2^T) \cap \sigma(\bar{A}_1)],$$

and select a basis such that

$$\bar{A}_2 = \begin{pmatrix} \tilde{A}_2 & 0 \\ 0 & A_2 \end{pmatrix}$$

with $\sigma(\tilde{A}_2) = \Lambda$, $\sigma(A_2) \cap \Lambda = \emptyset$. Then $\bar{A}_2$ may be partioned conformably as $\bar{A}_2 = \begin{pmatrix} \tilde{A}_{21} \\ \hat{A}_{21} \end{pmatrix}$. Then A and C may be partitioned as

$$A = \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}$$

$$C = (C_1 \;\; 0)$$

with

$$A_1 = \begin{pmatrix} \bar{A}_1 & 0 \\ \tilde{A}_{21} & \tilde{A}_2 \end{pmatrix}$$

$$A_{21} = (\hat{A}_{21} \;\; 0)$$

$$C_1 = (\tilde{C}_1 \;\; 0)$$

One may check that $\sigma(A_2) \cap \sigma(-A_1^T) = \emptyset$. The matrix B can be partitoned conformably with the partitioning of A as $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$. The image space of the matrix $(0 \;\; I)$, where the identity matrix I has the size of A_2, is unobservable for the pair (A, C) and is the largest unobservable subspace such that the corresponding dynamics does not intersect the backwards dynamics of the remainning part, i.e. $\sigma(A_2) \cap \sigma(-A_1^T) = \emptyset$. This space is named the unmixing unobservable subspace. The system (A_1, B_1, C_1, D) obtained from (A, B, C, D) by eliminating the part corresponding to the unmixing unobservable subspace, is called the *mixed+observable* subsystem. When A is unmixed, the mixed+observable subsystem is exactly the observable subsystem. In such a case the unobservable part of the system plays no role in the solvability of the KYP Lemma set of equations (3.2).

Theorem 3.24. *[150] Given a quadruple (A, B, C, D), let A be unmixed and (A_1, B_1, C_1, D) be the matrices associated to the observable subsystem. Then the KYP Lemma set of equations (3.2) possesses solutions $(P = P^T, L, W)$ if and only if the set of equations*

$$\begin{cases} A_1^T P_1 + P_1 A_1 = -L_1 L_1^T \\ P_1 B_1 = C_1^T - L_1 W_1 \\ W_1^T W_1 = D + D^T \end{cases} \tag{3.92}$$

possesses solutions $(P_1 = P_1^T, L_1, W_1)$. ■

Once again we insist on the fact that it is not required here that P nor P_1 be positive definite or even semi positive definite matrices. The result of Theorem 3.24 relies on the unmixity of A. However the following is true, which does not need this assumption.

Theorem 3.25. *[150] The KYP Lemma set of equations (3.2) possesses solutions if and only if (3.92) possesses solutions.* ∎

3.3.3 State Space Decomposition

The result presented in this subsection also relies on a decomposition of the state space into uncontrollable and unobservable subspaces. It was proposed in [444]. Let us start from a system (A, B, C). The Kalman controllability and observability matrices are denoted as K_c and K_o, respectively. The state space of the linear invariant system (A, B, C) is given by the direct sum

$$X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$$

where $sp(K_c) = X_1 \oplus X_2$, $sp(K_c) \cap Ker(K_o) = X_1$, $Ker(K_o) = X_1 \oplus X_3$. The notation $sp(A)$ means the algebraic span of the column vectors of A. Then the following holds.

Theorem 3.26. *[444] Let (A, B, C) be a realization of the rational matrix $H(s)$. Let K be any matrix satisfying*

$$X_1 \oplus X_2 \subseteq sp(K) X_1 \oplus X_2 \oplus X_3$$

Then $H(s)$ is positive real if and only if there exist real matrices $P = P^T \geq 0$ and L such that

$$\begin{cases} K^T(PA + A^T P + LL^T)K = 0 \\ \\ K^T(PB - C^T) = 0 \end{cases} \tag{3.93}$$

If B has full column rank, then $H(s)$ is positive real if and only if there exist real matrices $P = P^T$ and L, with $K^T PK \geq 0$, such that

$$\begin{cases} K^T(PA + A^T P + LL^T)K = 0 \\ \\ PB - C^T = 0 \end{cases} \tag{3.94}$$

∎

3.3.4 A Relaxed KYP Lemma for SPR Functions with Stabilizable Realization

The next result is taken from [110]. Let us consider the system in (3.1) and suppose (A,B,C,D) is a minimal realization, $m \le n$. Suppose that $H(s)+H^T(-s)$ has rank m almost everywhere in the complex plane, *i.e.* it has normal rank m (this avoids redundant inputs and outputs). The following Lemma gives us a general procedure to generate uncontrollable *equivalent* realizations from two minimal realizations of a given transfer matrix $H(s)$. The uncontrollable modes should be similar and the augmented matrices should be related by a change of coordinates as explained next.

Lemma 3.27. *[110] Let (A_i,B_i,C_i,D_i), $i=1,2$ be two minimal realizations of $H(s)$, i.e. $H(s)=C_i(sI_n-A_i)^{-1}B_i+D_i$ for $i=1,2$. Now define the augmented systems*

$$\bar{A}_i=\begin{pmatrix} A_i & 0 \\ \\ 0 & A_{0i}\end{pmatrix}\ \bar{B}_i=\begin{pmatrix} B_i \\ \\ 0\end{pmatrix} \qquad \bar{C}_i=(C_i \ \ C_{0i}) \qquad \bar{D}_i=D_i \tag{3.95}$$

where the dimensions of A_{01} and A_{02} are the same. Moreover there exists a nonsingular matrix T_0 such that $A_{01}=T_0A_{02}T_0^{-1}$ and $C_{01}=C_{02}T_0^{-1}$. Then $(\bar{A}_i,\bar{B}_i,\bar{C}_i,\bar{D}_i)$, $i=1,2$ are two equivalent *realizations.* ■

As a dual result we can generate unobservable augmented realizations of $H(s)$ as established in the following Corollary.

Corollary 3.28. *Let $\Sigma_i\,(A_i,B_i,C_i,D_i)$ for $i=1,2$ be two minimal realizations of $Z(s)$, i.e. $Z(s)=C_i\,(sI-A_i)^{-1}\,B_i+D_i$ for $i=1,2$. Now define the augmented systems:*

$$\overline{A}_i=\begin{pmatrix} A_i & 0 \\ 0 & A_{0i}\end{pmatrix} \quad \overline{B}_i=\begin{pmatrix} B_i \\ B_{i0}\end{pmatrix} \qquad \overline{C}_i=\begin{pmatrix} C_i & 0\end{pmatrix} \qquad \overline{D}_i=D_i \tag{3.96}$$

where the dimensions of A_{01} and A_{02} are the same. Moreover, there exists a nonsingular matrix T_0 such that $A_{01}=T_0A_{02}T_0^{-1}$ and $B_{01}=T_0B_{02}$. Then $\overline{\Sigma}_i\left(\overline{A}_i,\overline{B}_i,\overline{C}_i,\overline{D}_i\right)$ for $i=1,2$ are two equivalent realizations of $H(s)$. ■

Theorem 3.29. *[110] Let $H(s)=\bar{C}(sI_n-\bar{A})^{-1}\bar{B}+\bar{D}$ be an $m\times m$ transfer matrix such that $H(s)+H^T(-s)$ has normal rank m, where $\bar{A}$ is Hurwitz, $(\bar{A},\bar{B})$ is stabilizable, $(\bar{A},\bar{C})$ is observable. Assume that if there are multiple eigenvalues, then all of them are controllable modes or all of them are uncontrollable modes. Then $H(s)$ is SPR if and only if there exist $P=P^T>0$, W, L and a constant $\epsilon>0$ such that*

$$\begin{cases} P\bar{A} + \bar{A}^T P = -L^T L - \epsilon P \\ P\bar{B} = \bar{C}^T - L^T W \\ W^T W = \bar{D} + \bar{D}^T \end{cases} \tag{3.97}$$

■

This theorem is interesting since it states the existence of a positive definite solution to the KYP Lemma set of equations, and not only its solvability with $P = P^T$ or $P = P^T \geq 0$. The assumption that $H(s) + H^T(-s)$ has normal rank m is in order to avoid redundances in inputs and/or outputs. The assumption that the intersection of the set of controllable modes with the set of uncontrollable modes is empty, is used only in the *necessary* part of the proof.

Proof: *Sufficiency:* Let $\mu \in (0, \epsilon/2)$ then from (3.97)

$$P(\overline{A} + \mu I_n) + (\overline{A} + \mu I_n)^T P = -L^T L - (\epsilon - 2\mu) P \tag{3.98}$$

which implies that $(\overline{A} + \mu I_n)$ is Hurwitz and thus $Z(s-\mu)$ is analytic in $\mathbf{Re}[s] \geq 0$. Define now for simplicity

$$\overline{\Phi}(s) \stackrel{\Delta}{=} (sI_n - \overline{A})^{-1}$$

Therefore:

$$\begin{aligned}
&H(s-\mu) + H^T(-s-\mu) = \overline{D} + \overline{D}^T + \overline{C}\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)\overline{C}^T \\
&= W^T W + \left[\overline{B}^T P + W^T L\right]\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)\left[P\overline{B} + L^T W\right] \\
&= W^T W + W^T L\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T W + \\
&+ \overline{B}^T P\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)P\overline{B} \\
&= W^T W + W^T L\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T W + \\
&\overline{B}^T\overline{\Phi}^T(-s-\mu)\left[\overline{\Phi}^{-T}(-s-\mu)P + P\overline{\Phi}^{-1}(s-\mu)\right]\overline{\Phi}(s-\mu)\overline{B} \\
&= W^T W + W^T L\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T W + \overline{B}^T\overline{\Phi}^T(-s-\mu) \\
&\left\{\left[-(s+\mu)I - \overline{A}^T\right]P + P\left[(s-\mu)I - \overline{A}\right]\right\}\overline{\Phi}(s-\mu)\overline{B} \\
&= W^T W + W^T L\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T W + \\
&\overline{B}^T\overline{\Phi}^T(-s-\mu)\left\{-2\mu P - \overline{A}^T P - P\overline{A}\right\}\overline{\Phi}(s-\mu)\overline{B} \\
&= W^T W + W^T L\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T W + \\
&\overline{B}^T\overline{\Phi}^T(-s-\mu)\left\{L^T L + (\epsilon - 2\mu)P\right\}\overline{\Phi}(s-\mu)\overline{B} \\
&= W^T W + W^T L\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T W + \\
&\overline{B}^T\overline{\Phi}^T(-s-\mu)L^T L\overline{\Phi}(s-\mu)\overline{B} + (\epsilon - 2\mu)\overline{B}^T\overline{\Phi}^T(-s-\mu)P\overline{\Phi}(s-\mu)\overline{B} \\
&= \left[W^T + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T\right]\left[W + L\overline{\Phi}(s-\mu)\overline{B}\right] + \\
&(\epsilon - 2\mu)\overline{B}^T\overline{\Phi}^T(-s-\mu)P\overline{\Phi}(s-\mu)\overline{B}
\end{aligned}$$

From the above it follows that $H(j\omega - \mu) + H^T(-j\omega - \mu) \geq 0, \ \forall \omega \in [-\infty, +\infty]$ and $H(s)$ is SPR.

Necessity: Assume that $H(s) \in SPR$. Let $\overline{\Sigma}\left(\overline{A}, \overline{B}, \overline{C}, \overline{D}\right)$ be a stabilizable and observable realization of $H(s)$ and $\Sigma\,(A, B, C, D)$ a minimal realization of $H(s)$. Given that the controllable and uncontrollable modes are different we can consider that the matrix $\overline{A}$ is block diagonal and therefore $H(s)$ can be written as

$$H(s) = \underbrace{\left[C \; C_0 \right]}_{\overline{C}} \underbrace{\begin{bmatrix} sI_n - A & 0 \\ 0 & sI - A_0 \end{bmatrix}^{-1}}_{\left[sI_n - \overline{A}\right]} \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\overline{B}} + \underbrace{D}_{\overline{D}} \tag{3.99}$$

where the eigenvalues of A_0 correspond to the uncontrollable modes. As stated in the preliminaries, the condition $\sigma\,(A) \cap \sigma\,(A_0) = \emptyset$ (where $\sigma\,(T)$ means the spectrum of the square matrix T) means that the pairs (C, A) and (C_0, A_0) are observable if and only if $(\overline{C}, \overline{A}) = \left(\left[C \; C_0 \right], \begin{bmatrix} A & 0 \\ 0 & A_0 \end{bmatrix} \right)$ is observable.

We have to prove that $\overline{\Sigma}\left(\overline{A}, \overline{B}, \overline{C}, \overline{D}\right)$ satisfies the KYP equations (3.97). Note that A, A_0 are both Hurwitz. Indeed A is stable because $\Sigma\,(A, B, C, D)$ is a minimal realization of $H(s)$ which is SPR. A_0 is stable because the system is stabilizable. Thus there exists $\delta > 0$ such that $H(s - \delta) \in PR$ and $H(s - \mu) \in PR$ for all $\mu \in [0, \delta]$. Choose now $\epsilon > 0$ sufficiently small such that $U(s) \stackrel{\Delta}{=} Z(s - \frac{\epsilon}{2}) \in SPR$. Then the following matrices are Hurwitz:

$$\begin{array}{ll} \overline{A}_\epsilon & = \overline{A} + \frac{\epsilon}{2} I \in \mathbf{R}^{(n+n_0)\times(n+n_0)} \\ A_\epsilon & = A + \frac{\epsilon}{2} I \in \mathbf{R}^{n \times n} \\ A_{0\epsilon} & = A_0 + \frac{\epsilon}{2} I \in \mathbf{R}^{n_0 \times n_0}. \end{array} \tag{3.100}$$

Note that $\overline{A}_\epsilon$ is also block diagonal having block elements A_ϵ and $A_{0\epsilon}$ and the eigenvalues of A_ϵ and $A_{0\epsilon}$ are different. Let $\Sigma_\epsilon\,(A_\epsilon, B, C, D)$ be a minimal realization of $U(s)$ and $\overline{\Sigma}_\epsilon\left(\overline{A}_\epsilon, \overline{B}, \overline{C}, \overline{D}\right)$ an observable and stabilizable realization of $U(s)$. Therefore

$$U(s) = C(sI_n - A_\epsilon)^{-1}B + D = \overline{C}(sI_n - \overline{A}_\epsilon)^{-1}\overline{B} + \overline{D} \tag{3.101}$$

Note that the controllability of the pair (A_ϵ, B) follows from the controllability of (A, B). Since $A_{0\epsilon}$ is Hurwitz, it follows that $\left(\overline{A}_\epsilon, \overline{B}\right)$ is stabilizable. From the spectral factorization Lemma for SPR transfer matrices [527], [256, Lemma A.11, pp. 691], or [11], there exists an $m \times m$ stable transfer matrix $V(s)$ such that

$$U(s) + U^T(-s) = V^T(-s)V(s) \tag{3.102}$$

Remark 3.30. Here is used implicitly the assumption that $Z(s)+Z^T(-s)$ has normal rank m, otherwise the matrix $V(s)$ would be of dimensions $(r \times m)$, where r is the normal rank of $Z(s)+Z^T(-s)$.

Let $\Sigma_V(F,G,H,J)$ be a minimal realization of $V(s)$, F is Hurwitz because $V(s)$ is stable; a minimal realization of $V^T(-s)$ is $\Sigma_{V^T}\left(-F^T, H^T, -G^T, J^T\right)$. Now the series connection $V^T(-s)V(s)$ has realization (see [257, p. 15] for the formula of a cascade interconnection)

$$\Sigma_{V^T(-s)V(s)}\left(\begin{bmatrix} F & 0 \\ H^T H & -F^T \end{bmatrix}, \begin{bmatrix} G \\ H^T J \end{bmatrix}, \begin{bmatrix} J^T H & -G^T \end{bmatrix}, \begin{bmatrix} J^T J \end{bmatrix}\right) \tag{3.103}$$

Although we will not require the minimality of $\Sigma_{V^T(-s)V(s)}$ in the sequel, it can be proved to follow from the minimality of $\Sigma_V(F,G,H,J)$, see [11, 256]. Let us now define a nonminimal realization of $V(s)$ obtained from $\Sigma_V(F,G,H,J)$ as follows:

$$\begin{aligned} \overline{F} &= \begin{bmatrix} F & 0 \\ 0 & F_0 \end{bmatrix}, \quad \overline{G} = \begin{bmatrix} G \\ 0 \end{bmatrix} \\ \overline{H} &= \begin{bmatrix} H & H_0 \end{bmatrix}, \quad \overline{J} = J \end{aligned} \tag{3.104}$$

and such that F_0 is similar to $A_{0\epsilon}$ and the pair (H_0, F_0) is observable, *i.e.* $\exists\, T_0$ nonsingular such that

$$F_0 = T_0 A_{0\epsilon} T_0^{-1} \tag{3.105}$$

This constraint will be clarified later on. Since $\sigma(F_0) \cap \sigma(F) = \emptyset$ then the pair

$$(\overline{H}, \overline{F}) = \left(\begin{bmatrix} H & H_0 \end{bmatrix}, \begin{bmatrix} F & 0 \\ 0 & F_0 \end{bmatrix}\right) \tag{3.106}$$

is observable. Thus the nonminimal realization $\overline{\Sigma}_V\left(\overline{F}, \overline{G}, \overline{H}, \overline{J}\right)$ of $V(s)$ is observable and stabilizable. Now a nonminimal realization of $V^T(-s)V(s)$ based on $\overline{\Sigma}_V\left(\overline{F}, \overline{G}, \overline{H}, \overline{J}\right)$

$$\overline{\Sigma}_{V^T(-s)V(s)}\left(\begin{bmatrix} \overline{F} & 0 \\ \overline{H}^T\overline{H} & -\overline{F}^T \end{bmatrix}, \begin{bmatrix} \overline{G} \\ \overline{H}^T\overline{J} \end{bmatrix}, \begin{bmatrix} \overline{J}^T\overline{H} & -\overline{G}^T \end{bmatrix}, \begin{bmatrix} \overline{J}^T\overline{J} \end{bmatrix}\right) \tag{3.107}$$

is (see [257, p. 15])

$$\overline{\Sigma}_{V^T(-s)V(s)} = \left[\begin{array}{cccc|c} F & 0 & 0 & 0 & G \\ 0 & F_0 & 0 & 0 & 0 \\ H^T H & H^T H_0 & -F^T & 0 & H^T J \\ H_0^T H & H_0^T H_0 & 0 & -F_0^T & H_0^T J \\ \hline J^T H & J^T H_0 & -G^T & 0 & J^T J \end{array}\right] \tag{3.108}$$

From the diagonal structure of the above realization, it could be concluded that the eigenvalues of F_0 correspond to uncontrollable modes and the eigenvalues of $(-F_0^T)$ correspond to a unobservable modes. A constructive proof is given below.

Since the pair $(\overline{H}, \overline{F})$ is observable and $\overline{F}$ is stable, there exists a positive definite matrix

$$\overline{K} = \overline{K}^T = \begin{bmatrix} K & r \\ r^T & K_0 \end{bmatrix} > 0 \tag{3.109}$$

solution of the Lyapunov equation

$$\overline{K}\,\overline{F} + \overline{F}^T \overline{K} = -\overline{H}^T \overline{H} \tag{3.110}$$

This explains why we imposed the constraint that (H_0, F_0) should be observable. Otherwise there will not exist a positive definite solution for (3.110).

Define $\overline{T} := \begin{bmatrix} I & 0 \\ \overline{K} & I \end{bmatrix}$; $\overline{T}^{-1} = \begin{bmatrix} I & 0 \\ -\overline{K} & I \end{bmatrix}$ and use it as a change of coordinates for the nonminimal realization $\overline{\Sigma}_{V^T(-s)V(s)}$ above to obtain

$$\overline{\Sigma}_{V^T(-s)V(s)} = \left[\begin{array}{cccc|c} F & 0 & 0 & 0 & G \\ 0 & F_0 & 0 & 0 & 0 \\ 0 & 0 & -F^T & 0 & (J\overline{H} + \overline{G}^T\overline{K})^T \\ 0 & 0 & 0 & -F_0^T & \\ \hline J\overline{H} + \overline{G}^T\overline{K} & & -G^T & 0 & J^TJ \end{array}\right] \tag{3.111}$$

Now it is clear that the eigenvalues of F_0 correspond to uncontrollable modes and the eigenvalues of $(-F_0^T)$ correspond to unobservable modes.

From (3.101) a nonminimal realization of $U(s)$ is $\overline{\Sigma}_\epsilon\left(\overline{A}_\epsilon, \overline{B}, \overline{C}, \overline{D}\right)$. Thus a nonminimal realization for $U^T(-s)$ is $\overline{\Sigma}_\epsilon\left(-\overline{A}_\epsilon^T, \overline{C}^T, -\overline{B}^T, \overline{D}^T\right)$. Using the results in the preliminaries, a nonminimal realization of $U(s) + U^T(-s)$ is

$$\Sigma_{U(s)+U^T(-s)}\left(\begin{bmatrix} \overline{A}_\epsilon & 0 \\ 0 & -\overline{A}_\epsilon^T \end{bmatrix}, \begin{bmatrix} \overline{B} \\ \overline{C}^T \end{bmatrix}, \begin{bmatrix} \overline{C} & -\overline{B}^T \end{bmatrix}, \begin{bmatrix} \overline{D} + \overline{D}^T \end{bmatrix}\right). \tag{3.112}$$

Using (3.102) we conclude that the stable (unstable) parts of the realizations of $U(s) + U^T(-s)$ and $V^T(-s)V(s)$ are identical. Therefore, in view of the block diagonal structure of the system and considering only the stable part we have

$$\begin{aligned} \overline{F} &= \begin{bmatrix} F & 0 \\ 0 & F_0 \end{bmatrix} = R\overline{A}_\epsilon R^{-1} = R\begin{bmatrix} A_\epsilon & 0 \\ 0 & A_{0\epsilon} \end{bmatrix} R^{-1} \\ \overline{G} &= \begin{bmatrix} G \\ 0 \end{bmatrix} = R\overline{B} = R\begin{bmatrix} B \\ 0 \end{bmatrix} \\ J\overline{H} + \overline{G}^T\overline{K} &= \overline{C}R^{-1} = \begin{bmatrix} C & C_0 \end{bmatrix} R^{-1} \\ J^TJ &= \overline{D} + \overline{D}^T \end{aligned} \tag{3.113}$$

The above relationships impose that the uncontrollable parts of the realizations of $U(s)$ and $V(s)$ should be similar. This is why we imposed that F_0 is similar to $A_{0\epsilon}$ in the construction of the nonminimal realization of $V(s)$.

From the Lyapunov equation (3.110) and using $\overline{F} = R\overline{A}_\epsilon R^{-1}$ in (3.113), we get

$$\begin{cases} \overline{KF} + \overline{F}^T\overline{K} & = -\overline{H}^T\overline{H} \\ \overline{K}R\overline{A}_\epsilon R^{-1} + R^{-T}\overline{A}_\epsilon^T R^T\overline{K} & = -\overline{H}^T\overline{H} \\ R^T\overline{K}R\overline{A}_\epsilon + \overline{A}_\epsilon^T R^T\overline{K}R & = -R^T\overline{H}^T\overline{H}R \\ P\overline{A}_\epsilon + \overline{A}_\epsilon^T P & = -L^T L \end{cases} \tag{3.114}$$

where we have used the definitions $P \stackrel{\Delta}{=} R^T\overline{K}R$; $L \stackrel{\Delta}{=} \overline{H}R$. Introducing (3.100) we get the first equation of (3.97). From the second equation of (3.113) we have $\overline{G} = R\overline{B}$. From the third equation in (3.113) and using $W = J$ we get

$$\begin{cases} J\overline{H} + \overline{G}^T\overline{K} & = \overline{C}R^{-1} \\ J^T\overline{H}R + \overline{G}^T R^{-T}R^T\overline{K}R & = \overline{C} \\ W^T L + \overline{B}^T P & = \overline{C} \\ P\overline{B} & = \overline{C}^T - L^T W \end{cases} \tag{3.115}$$

which is the second equation of (3.97). Finally from the last equation of (3.113), we get the last equation of (3.97) because $W = J$. ■

Example 3.31. Consider $H(s) = \frac{s+a}{(s+a)(s+b)}$, for some $a > 0$, $b > 0$, $b \neq a$. Let a nonminimal realization of $H(s)$ be

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{\alpha} \end{pmatrix} u(t) \\ y(t) = [\beta \;\; \alpha]x(t) \end{cases} \tag{3.116}$$

with $\alpha \neq 0$ and $\beta \neq 0$. For all $\epsilon < \min(a, b)$ one has

$$P = \begin{pmatrix} \frac{(a+b-\epsilon)^2\beta^2}{(2b-\epsilon)(2a-\epsilon)} & \alpha\beta \\ \alpha\beta & \alpha^2 \end{pmatrix} > 0$$

for all $a > 0$, $b > 0$, $\alpha \neq 0$, $\beta \neq 0$. The matrix $L = \left[\frac{a+b-\epsilon}{\sqrt{2b-\epsilon}}\beta \;\; \sqrt{2b-\epsilon}\,\alpha\right]$ and P satisfy the KYP Lemma set of equations .

Remark 3.32. Proposition 2.31 states that positivity of an operator is equivalent to the positivity of its Popov's function. There is no mention of stability. This is in accordance with the above versions of the KYP Lemma for which the stability (*i.e.* solvability of the KYP Lemma set of equations with

$P = P^T > 0$) requires more than the mere positivity of the spectral function.

3.4 SPR Problem with Observers

The KYP Lemma for non-controllable systems is especially important for the design of feedback controllers with state observers [111, 241, 242], where the closed-loop system may not be controllable. This may be seen as the extension of the works described in Section 2.15.3 in the case where an observer is added to guarantee that the closed-loop is SPR.

Theorem 3.33. *[111] Consider a system with stable transfer function $H(s) \in \mathbb{C}^{m\times m}$, and its state space realization*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = Cx(t) \end{cases} \tag{3.117}$$

where (A, B) is stabilizable and (A, C) is observable. Then there exists a gain observer L and an observer

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + LC(x(t) - \hat{x}(t)) \\ \\ z(t) = M\hat{x}(t) \end{cases} \tag{3.118}$$

such that $\sigma(A - LC)$ is in the open left-hand complex plane, and the transfer function between $u(\cdot)$ and the new output $z(t) = M_0 \begin{pmatrix} x \\ \hat{x} - x \end{pmatrix} = M\hat{x}(t)$, with $M = B^T P$, is characterized by a state space realization (A_0, B_0, M_0) that is SPR, where

$$A_0 = \begin{pmatrix} A & 0 \\ 0 & A - LC \end{pmatrix}, \quad B_0 = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

■

The modes associated to the matrix $(A - LC)$ are non-controllable.

3.5 The Feedback KYP Lemma

The feedback KYP Lemma is an extension of the KYP Lemma, when one considers a controller of the form $u(t) = Kx(t)$. This is quite related to the material of Section 2.15.3: which are the conditions under which a system can be made passive (or PR) in closed-loop? Let us consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = Cx(t) \end{cases} \tag{3.119}$$

with the usual dimensions and where all matrices are real.

Definition 3.34. *The system in (3.119) is said to be*

- Minimum phase *if the polynomial* $\det \begin{bmatrix} A - \lambda I_n & B \\ \\ C & 0 \end{bmatrix}$ *is Hurwitz (it has all its zeroes in the open half plane)*
- Strictly minimum phase *if it is minimum phase and the matrix* CB *is nonsingular*
- Hyper minimum phase *if it is minimum phase and the matrix* CB *is positive definite* ■

The next Theorem is close to what is sometimes referred to as Fradkov's Theorem [56].

Theorem 3.35. *[16, 154–156] Let* $rank(B) = m$. *Let* $Q = Q^T \leq 0$. *Then*

- *(A) There exists* $P = P^T > 0$ *and* K *such that* $P(A+BK)+(A+BK)^T P < Q$ *and* $PB = C^T$

  if and only if

- *(B) the system in (3.119) is hyper minimum phase*

  if and only if

- *(C) there exists* $P = P^T > 0$ *and* $\bar{K}$ *such that* $P(A + B\bar{K}C) + (A + B\bar{K}C)^T P < Q$ *and* $PB = C^T$

  if and only if

- *(D) the matrix* CB *is symmetric positive definite and the zero dynamics of the system in (3.119) is asymptotically stable*

Moreover the matrix K *can be chosen as* $K = -\alpha C$ *where* $\alpha > 0$ *is large enough. Assume that in addition* $Ker(C) \subset Ker(Q)$. *Then*

- *(E) There exists* $P = P^T > 0$ *and* K *such that* $A + BK$ *is Hurwitz and* $P(A + BK) + (A + BK)^T P < Q$ *and* $PB = C^T$

if and only if

- *(F) the matrix CB is symmetric positive definite, the pair (A, B) is stabilizable, all the zeroes of the polynomial* $\det \begin{bmatrix} A - \lambda I_n & B \\ C & 0 \end{bmatrix}$ *are in the closed left half plane, and all the pure imaginary eigenvalues of the matrix pencil* $R(\lambda) = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} - \lambda \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ *have only linear elementary divisors $\lambda - j\omega$*

  if and only if

- *(G) the matrix CB is symmetric positive definite, the pair (A, B) is stabilizable and the system (3.119) is weakly minimum phase.* ■

Both matrix equations in (A) and (C) are bilinear matrix inequalities (BMIs). The feedback KYP Lemma extends to systems with a direct feedthrough term $y = Cx + Du$. It is noteworthy that Theorem 3.35 holds for multivariable systems. If $u(t) = Kx(t) + v(t)$, then (A) means that the operator $v \mapsto y$ is SPR. It is known that this control problem is dual to the SPR observer design problem [22]. Related results are in [23]. We recall that a system is said weakly minimum phase if its zero dynamics is Lyapunov stable. The zero dynamics can be explicitly written when the system is written in a special coordinate basis as described in [432–434]. The particular choice for K after item (D) means that the system can be stabilized by output feedback. More work may be found in [153]. The stability analysis of dynamic output feedback systems with a special formulation of the KYP Lemma has been carried out in [241].

3.6 Time-varying Systems

Let us consider the linear system:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \\ x(t_0) = x_0 \end{cases} \tag{3.120}$$

where the functions $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are supposed to be piecewise continuous, and $D(t) \geq \epsilon I_m$, $\epsilon \geq 0$. It is assumed that all (t, x) with $t > t_0$ are reachable from $(t_0, 0)$, and that the system is zero state observable (such controllability and observability conditions may be checked via the controllability and observability grammians, see *e.g.* [467]). It is further assumed that the required supply is continuously differentiable in both t and x, whenever it exists (the required supply is a quantity that will be defined in Definition 4.36. The reader may just want to consider this as a regularity condition on the system (3.120)). The system (3.120) is supposed to be well-posed; see Theorem 3.55, and it defines an operator $\Lambda : u(t) \mapsto y(t)$. The kernel of $\Lambda(\cdot)$ is given by $K(t,r) = C(t)\Phi(t,r)B(r)1(t-r) + B^T(t)\Phi^T(r,t)C^T(tr)1(r-t) + R(t)\delta_{t-r}$, where $1(t) = 0$ if $t < 0$, $1(t) = \frac{1}{2}$ if $t = 0$ and $1(t) = 1$ if $t > 0$, $R(t) = D(t) + D^T(t)$, δ_t is the Dirac measure at t, $\Phi(\cdot,\cdot)$ is the transition matrix of $A(t)$, *i.e.* $\Phi(t,r) = X(t)X^{-1}(r)$ for all t and r, and $\frac{dX}{dt} = A(t)X(t)$. Then $\Lambda(u(t)) = \int_{-\infty}^{t} K(t,r)u(r)dr$.

Lemma 3.36. *The operator $\Lambda(\cdot)$ is nonnegative if and only if there exists an almost everywhere continuously differentiable function $P(\cdot) = P^T(\cdot) \geq 0$ such that on (t_0, t)*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0 \tag{3.121}$$

where

$$\begin{cases} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t) \\ C^T(t) - P(t)B(t) = S(t) \end{cases} \tag{3.122}$$

■

Nonnegativity of $\Lambda(\cdot)$ is understood as in Proposition 2.31.

3.7 Interconnection of PR Systems

We will now study the stability properties of positive real or strictly positive real systems when they are connected in negative feedback. We will consider two PR systems $H_1 : u_1 \to y_1$ and $H_2 : u_2 \to y_2$. H_1 is in the feedforward path and H_2 is in the feedback path(i.e. $u_1 = -y_2$ and $u_2 = y_1$). The stability of the closed loop system is concluded in the following Lemma when H_1 is PR and H_2 is weakly SPR.

Lemma 3.37. *Consider a system $H_1 : u_1 \to y_1$ in negative feedback with a system $H_2 : u_2 \to y_2$ as shown in Figure 3.2, where H_1 is PR and H_2 is WSPR. Under those conditions u_1, u_2, y_1 and y_2 all converge to zero exponentially.*

■

Proof: Let us define the following state-space representation for system $H_1(s)$

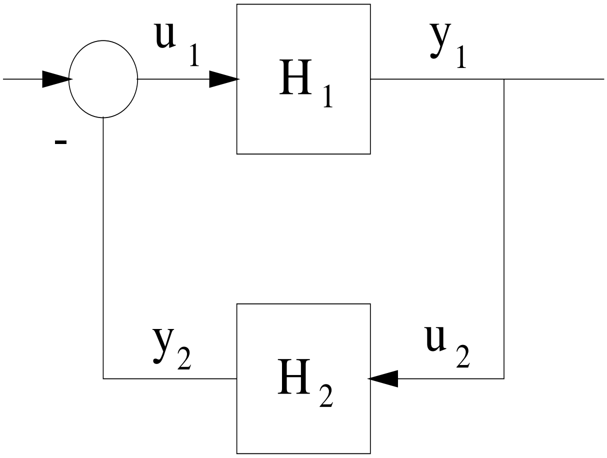

Fig. 3.2. Interconnection of H_1 and H_2

$$\begin{cases} \dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) \\ \\ y_1(t) = C_1 x_1(t) + D_1 u_1(t) \end{cases} \tag{3.123}$$

Since $H_1(s)$ is PR there exists matrices $P > 0, P \in I\!R^{n\times n}, W \in I\!R^{m\times m}, L \in I\!R^{n\times m}$ such that

$$\begin{cases} P_1 A_1 + A_1^T P_1 = -L_1 L_1^T \\ P_1 B_1 - C_1^T \quad = -L_1 W_1 \\ D_1 + D_1^T \qquad = W_1^T W_1 \end{cases} \tag{3.124}$$

Define the following state-space representation for the system $H_2(s)$

$$\begin{cases} \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t) \\ \\ y_2(t) = C_2 x_2(t) + D_2 u_2(t) \end{cases} \tag{3.125}$$

Since $H_2(s)$ is WSPR there exists matrices $P > 0, \in I\!R^{n\times n}, W \in I\!R^{m\times m}, L \in I\!R^{n\times m}$ such that

$$\begin{cases} P_2 A_2 + A_2^T P_2 = -L_2 L_2^T \\ P_2 B_2 - C_2^T \quad = -L_2 W_2 \\ D_2 + D_2^T \qquad = W_2^T W_2 \end{cases} \tag{3.126}$$

and

$$\overline{H}_2(s) = W_2 + L_2^T (sI_n - A_2)^{-1} B_2 \tag{3.127}$$

has no zeros in the $j\omega$-axis. Consider the following positive definite function

$$V_i(x_i) = x_i^T P_i x_i, \quad i = 1, 2.$$

Then using (3.124) and (3.126):

$$\begin{aligned}
\dot{V}_i(x_i) &= (x_i^T A_i^T + u_i^T B_i^T)P_i x_i + x_i^T P_i(A_i x_i + B_i u_i) \\
&= x_i^T(A_i^T P_i + P_i A_i)x_i + 2u_i^T B_i^T P_i x_i \\
&= x_i^T(-L_i L_i^T)x_i + 2u_i^T B_i^T P_i x_i \\
&= -x_i^T L_i L_i^T x_i + 2u_i^T(B_i^T P_i + W_i^T L_i^T)x_i - 2u_i^T W_i^T L_i^T x_i \\
&= -x_i^T L_i L_i^T x_i + 2u_i^T[C_i x_i + D_i u_i] - 2u_i^T D_i u_i - 2u_i^T W_i^T L_i^T x_i \\
&= -x_i^T L_i L_i^T x_i + 2u_i^T y_i - 2u_i^T D_i u_i - 2u_i^T W_i^T L_i^T x_i \\
&\quad -(L_i^T x_i + W_i u_i)^T(L_i^T x_i + W_i u_i) + 2u_i^T y_i
\end{aligned} \tag{3.128}$$

where we have used the fact that

$$2u_i^T D_i u_i = u_i^T(D_i + D_i^T)u_i = u_i^T W_i^T W_i u_i$$

Define $\bar{y}_i = L_i^T x_i + W_i u_i$ and $V(x) = V_1(x_1) + V_2(x_2)$, then

$$\dot{V}(x_1, x_2) = -\bar{y}_1^T \bar{y}_1 - \bar{y}_2^T \bar{y}_2 + 2(u_1^T y_1 + u_2^T y_2)$$

Since $u_1 = -y_2$ and $u_2 = y_1$ it follows that

$$u_1^T y_1 + u_2^T y_2 = -y_2^T y_1 + y_1^T y_2 = 0$$

Therefore

$$\dot{V}(x_1, x_2) = -\bar{y}_1^T \bar{y}_1 - \bar{y}_2^T \bar{y}_2 \leq -\bar{y}_2^T \bar{y}_2,$$

which implies that $V(\cdot)$ is a nondecreasing function and therefore we conclude that $x_i \in \mathcal{L}_\infty$. Integrating the above equation:

$$-V(0) \leq V(t) - V(0) \leq -\int_0^t \bar{y}_2^T(s)\bar{y}(s)_2 ds \tag{3.129}$$

Then

$$\int_0^t \bar{y}_2^T(s)\bar{y}_2(s)ds \leq V(0) \tag{3.130}$$

The feedback interconnection of H_1 and H_2 is a linear system. Since $x_i \in \mathcal{L}_\infty$, the closed loop is at least stable, *i.e.* the closed-loop poles are in the left-half plane or in the jw-axis. This means that u_i, y_i may have an oscillatory behavior. However the equation above means that $\overline{y}_2 \to 0$. By assumption $\bar{H}_2(s)$ has no zeros on the $j\omega$ axis. Since the state is bounded, $u_2(\cdot)$ can not grown unbounded. It follows that $u_2(t) \to 0$ as $t \to +\infty$. This in turn implies that $y_2(t) \to 0$ since H_2 is asymptotically stable. Clearly $u_2(t) \to 0$ and $y_2(t) \to 0$ as $t \to +\infty$. ∎

3.8 Positive Realness and Optimal Control

The material of this section is taken from [513, 514]. As we have already pointed out in Section 3.1.2, strong links exist between dissipativity and optimal control. In this section more details are provided. Close results were also obtained by Yakubovich [349, 520, 523].

3.8.1 General Considerations

Let us start with some general considerations which involve some notions which have not yet been introduced in this book, but will be introduced in the next chapter (actually, the only missing definitions are those of a storage function and a supply rate: the reader may thus skip this part and come back to it after having read Chapter 4). The notions of dissipation inequality and of a storage function have been introduced (without naming them) in (2.3), where the function $V(\cdot)$ is a so-called storage function and is a function of the state $x(\cdot)$ (and is not an explicit function of time). Let us consider the following minimization problem

$$V_f(x_0) \stackrel{\Delta}{=} \min_{u \in \mathcal{L}_{2,e}} \int_0^{+\infty} w(u(s), x(s))ds \tag{3.131}$$

with

$$w(u, x) = u^T R u + 2u^T C x + x^T Q x \tag{3.132}$$

with $R = R^T$, $Q = Q^T$, subject to $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$. It is noteworthy that $V_f(x_0)$ is nothing else but the *value function* of the principle of optimality. The set $\mathcal{L}_{2,e}$ is the extended set of $\mathcal{L}_2$-bounded functions; see Section 4.3.5. If $w(u, x) \geq 0$ for all $x \in I\!R^n$ and all $u \in I\!R^m$ then the value function satisfies

$$V_f(x(0)) \leq V_f(x(t_1)) + \int_0^{t_1} w(u(t), x(t))dt \tag{3.133}$$

for all $t_1 \geq 0$, or, if it is differentiable, the infinitesimal equivalent

$$\frac{\partial V_f}{\partial x}(x)[f(x) + g(x)u] + w(u, x) \geq 0, \ \forall\, x \in I\!R^n, u \in I\!R^m. \tag{3.134}$$

One realizes immediately by rewriting (3.133) as the dissipation inequality

$$-V_f(x(0)) \geq -V_f(x(t_1)) - \int_0^{t_1} w(u(t), x(t))dt \tag{3.135}$$

that $-V_f(\cdot)$ plays the role of a storage function with respect to the supply rate $-w(u, x)$. Let us end this subsection making a small digression on the following well-known fact: why is the optimal function in (3.131) a function of the initial state? To see this intuitively, let us consider the minimization problem

$$\inf_{u \in \mathcal{U}} \int_0^{+\infty} (u^2(t) + x^2(t))dt \tag{3.136}$$

subject to $\dot{x}(t) = u(t)$, $x(0) = x_0$. Let $\mathcal{U}$ consist of smooth functions. Then finiteness of the integral in (3.136) implies that $\lim_{t \to +\infty} x(t) = 0$. Take any constant $a \in \mathbb{R}$. Then

$$\begin{aligned} &\int_0^{+\infty} 2ax(t)u(t)dt = \int_0^{+\infty} 2ax(t)\dot{x}(t)dt = \\ &= \int_0^{+\infty} \frac{d}{dt}[ax^2(t)]dt = [ax^2(t)]_0^{+\infty} = -ax_0^2. \end{aligned} \tag{3.137}$$

So indeed $\inf_{u \in \mathcal{U}} \int_0^{+\infty} (u^2(t) + x^2(t))dt$ is a function of the initial state.

3.8.2 Least Squares Optimal Control

We have already pointed out the relationship which exists between the linear matrix inequality in the KYP Lemma (see Section 3.1.2) and optimal control, through the construction of a Riccati inequality that is equivalent to the linear matrix inequality (LMI) in (3.3). This section is devoted to deepen such relationships. First of all, let us introduce (or re-introduce) the following algebraic tools:

- The linear matrix inequality (LMI)

$$\begin{bmatrix} GA + A^T G + Q & GB + C^T \\ B^T G + C & R \end{bmatrix} \geq 0 \tag{3.138}$$

- The quadratic matrix inequality (QMI) or algebraic Riccati inequality (ARI)

$$GA + A^T G - (GB + C^T)R^{-1}(B^T G + C) + Q \geq 0 \tag{3.139}$$

- The algebraic Riccati equation (ARE)

$$GA + A^T G - (GB + C^T)R^{-1}(B^T G + C) + Q = 0 \tag{3.140}$$

- The frequency-domain inequality (FDI)

$$\begin{aligned} H(\bar{s}, s) = R + C(sI_n - A)^{-1}B + B^T(\bar{s}I_n - A^T)^{-1}C^T + \\ + B^T(\bar{s}I_n - A^T)^{-1}Q(sI_n - A)^{-1}B \geq 0 \end{aligned} \tag{3.141}$$

where $s \in \mathbb{C}$ and $\bar{s}$ is its complex conjugate. Notice that $H(\bar{s}, s)$ can be rewritten as

$$H(\bar{s}, s) = \begin{pmatrix} (\bar{s}I_n - A)^{-1}B \\ I_m \end{pmatrix}^T \begin{pmatrix} Q & C^T \\ C & R \end{pmatrix} \begin{pmatrix} (sI_n - A)^{-1}B \\ I_m \end{pmatrix} \tag{3.142}$$

Remark 3.38. Comparing (3.138) and (3.17) it is expected that $G \leq 0$ in (3.138), and in (3.139) and (3.140) as well.

Let $R = D + D^T$. The function $H(\bar{s}, s)$ in (3.141) is also known as the Popov function $\Pi(s)$, and was formally introduced by Popov in [410] (the first time it has been introduced may even be in [86]). It is worth noting that when $Q = 0$ then $H(\bar{s}, s) = H(s) + H^T(\bar{s})$ where $H(s) = C(sI_n - A)^{-1}B + D$. Thus $H(-j\omega, j\omega) \geq 0$ a the condition for the PRness of $H(s)$. By extension one may also call the function in (3.141) a Popov function [391]. Notice that $H(\bar{s}, s)$ in (3.142) is linked to the system $\dot{x}(t) = Ax(t) + Bu(t)$ as follows. For every $u \in \mathbb{C}^m$ and every $\omega \in I\!R$ such that $j\omega$ is not an eigenvalue of A, we have

$$u^T H(j\omega, -j\omega)u = \begin{pmatrix} x(-j\omega, u) \\ u \end{pmatrix}^T \begin{pmatrix} Q & C^T \\ C & R \end{pmatrix} \begin{pmatrix} x(j\omega, u) \\ u \end{pmatrix} \tag{3.143}$$

where $x(j\omega, u)$ is defined from $j\omega x = Ax + Bu$, *i.e.* $x(j\omega, u) = (j\omega I_n - A)^{-1}Bu$. See for instance Theorem 3.46 for more information on the spectral function and its link with the KYP Lemma set of equations. One sometimes calls any triple of matrices A, B and $\begin{pmatrix} Q & C^T \\ C & R \end{pmatrix}$ a Popov triple.

Remark 3.39. In the scalar case the ARE (3.140) becomes a second order equation $aG^2 + bG + c = 0$ with real coefficients. It is clear that without assumptions on a, b, and c there may be no real solutions. Theorem A.53 in Appendix A.4 states conditions under which an ARE as in (3.140) possesses a real solution.

We will denote the inequality in (3.133) as the DIE (for dissipation inequality), keeping in mind that the real dissipation inequality is in (3.135). Let us introduce the following optimal control problems, with $w(x, u)$ in (3.132).

$$V^+(x_0) \stackrel{\Delta}{=} \min_{u \in \mathcal{L}_{2,e}} \int_0^{+\infty} w(u(s), x(s))ds, \ \lim_{t \to +\infty} x(t) = 0 \tag{3.144}$$

$$V^-(x_0) \stackrel{\Delta}{=} -\min_{u \in \mathcal{L}_{2,e}} \int_0^{+\infty} w(u(s), x(s))ds, \ \lim_{t \to +\infty} x(t) = 0 \tag{3.145}$$

$$V_n(x_0) \overset{\Delta}{=} \min_{u \in \mathcal{L}_{2,e}, t \geq 0} \int_0^t w(u(s), x(s))ds \tag{3.146}$$

These four problems (*i.e.* (3.131), (3.144), (3.145) and (3.146)) are subject to the dynamics $\dot{x}(t) = Ax(t) + Bu(t)$, with initial data $x(0) = x_0$.

Assumption 2 *We assume that the pair* (A, B) *is controllable throughout Section 3.8.2.*

Therefore this assumption will not be repeated. One notes that the four functions in (3.131), (3.144), (3.145) and (3.146) are quadratic functions of the state x_0. Let us summarize few facts:

- $V_n(\cdot) \leq 0$ (take $t = 0$ in (3.146) to deduce that the minimum cannot be positive).
- $V_n(\cdot) \leq V_f(\cdot) \leq V^+(\cdot)$: indeed, if the scalar $\int_0^t w(u(s), x(s))ds$ sweeps a certain domain in $I\!R$ while $t \geq 0$, then the scalar $\int_0^{+\infty} w(u(s), x(s))ds$ must belong to this domain. And similarly if the scalar $\int_0^{+\infty} w(u(s), x(s))ds$ sweeps a certain domain while $u \in \mathcal{L}_{2,e}$, the scalar $\int_0^{+\infty} w(u(s), x(s))ds$ subject to the limit condition must lie inside this domain.
- $V_n(\cdot) < +\infty$,$V_f(\cdot) < +\infty$, $V^+(\cdot) < +\infty$: by controllability the integrand $w(u, x)$ is bounded whatever the final (bounded) state, so the lowerbound is bounded.
- $V^-(\cdot) > -\infty$: note that

  $$-\min_{u \in \mathcal{L}_{2,e}} \int_0^{+\infty} w(u(s), x(s))ds = \max_{u \in \mathcal{L}_{2,e}} -\int_0^{+\infty} w(u(s), x(s))ds.$$

  By controllability one can surely find a control u that drives the system from x_0 to some other state, and such that the scalar $\int_0^{+\infty} w(u(s), x(s))ds$ is bounded. So the supremum surely cannot be $-\infty$.
- $V_f(\cdot)$, $V^+(\cdot)$, and $V_n(\cdot)$ satisfy the DIE (3.133). By direct inspection $V_f(x_0) - V_f(x_1) = \int_{t_0}^{t_1} w(u(s), x(s))ds$ and similarly for the other two functions.
- If for all $x \in I\!R^n$ there exists a $u \in I\!R^m$ such that $w(x, u) \leq 0$ then $V_n(\cdot) = V_f(\cdot)$. A sufficient condition for this is that $R > 0$, or that $Q = 0$.
- If there exists a feedback controller $u(x)$ such that $w(u(x), x) \leq 0$ and such that $\dot{x}(t) = Ax(t) + Bu(x(t))$ has an asymptotically stable fixed point $x = 0$, then $V_n(\cdot) = V_f(\cdot) = V^+(\cdot)$.
- If $w(u, x) = u^T y$ and an output $y = Cx + Du$ is defined, then the optimal control problem corresponds to a problem where the dissipated energy is to be minimized.
- If $w(x, 0) \geq 0$ then the functions $V(\cdot)$ which satisfy the DIE in (3.133) define Lyapunov functions candidate since $-V(\cdot)$ is then non-increasing along the (uncontrolled) system's trajectories, as (3.135) shows.

The second part of the last item is satisfied provided the system is asymptotically stabilizable, which is the case if (A, B) is controllable. The first part may be satisfied if $R = 0$, $Q = 0$, and the matrix $A + BC$ is Hurwitz. The first part of the last-but-one item, is satisfied if $R = 0$, $Q = 0$ (take $u = -Cx$).

Lemma 3.40. *Let $R > 0$. For quadratic functions $V(x) = x^T Gx$, $G = G^T$, the DIE in (3.133) is equivalent to the LMI.* ■

Proof: From (3.134) one obtains

$$2x^T G[Ax + Bu] \geq -w(u, x), \ \forall \, x \in I\!R^n, u \in I\!R^m \tag{3.147}$$

The LMI follows from (3.147). Then the proof is as in Section 3.1.2.

■

Let us now present some Theorems which show how the LMI, the ARI, the ARE and the FDI are related one to each other and to the boundedness properties of the functions $V_f(\cdot)$, $V^+(\cdot)$. The proofs are not provided entirely for the sake of brievity. In what follows, the notation $V(\cdot) > -\infty$ and $V(\cdot) < +\infty$ mean respectively that the function $V : I\!R^n \to I\!R$ is bounded for bounded argument. In other words, given x_0 bounded, $V(x_0)$ is bounded. The controllability of (A, B) is sufficient for the optimum to be bounded [246, p.229].

Theorem 3.41. *The following assertions hold:*

- $V_f(\cdot) > -\infty \Longleftrightarrow$ *there exists a real symmetric solution $G = G^T \leq 0$ to the LMI.*
- $V^+(\cdot) > -\infty \Longleftrightarrow$ *there exists any real symmetric solution $G = G^T$ to the LMI.*
- $V_f(\cdot) > -\infty \Longrightarrow$ *the FDI is satisfied whenever $Re(s) \geq 0$, $s \in \mathbb{C}$.*
- $V^+(\cdot) > -\infty \Longrightarrow$ *the FDI is satisfied along $Re(s) = 0$, $s \in \mathbb{C}$.* ■

Proof: Let us prove the last two items. If there exists a solution $G = G^T$ to the LMI, then

$$\begin{bmatrix} -(I_n\bar{s} - A^T)G - G(I_n s - A) & GB + C^T \\ B^T G + C & R \end{bmatrix} \geq \begin{bmatrix} -2\sigma P & 0 \\ 0 & 0 \end{bmatrix} \tag{3.148}$$

with $s = \sigma + j\omega$, $\sigma \in I\!R$, $\omega \in I\!R$, and $\bar{s} = \sigma - j\omega$. Postmultiplying by

$$\begin{bmatrix} (I_n s - A)^{-1} B \\ I_n \end{bmatrix} \tag{3.149}$$

and premultiplying by $[B^T (I_n\bar{s} - A^T)^{-1} \;\; I_n]$ one obtains

$$H(\bar{s}, s) \geq -2\sigma B^T (I_n\bar{s} - A^T)^{-1} G(I_n s - A)^{-1} B \tag{3.150}$$

From the first item and since $\sigma \geq 0$ one sees that indeed (3.150) implies the FDI (as P is non-positive definite).

■

The following theorems characterize the solutions of the ARE.

Theorem 3.42. *Let $R = R^T > 0$.*

- *The ARE has a real symmetric solution if and only if $H(-j\omega, j\omega) \geq 0$ for all real ω, $j\omega \notin \sigma(A)$. There is then only one such solution denoted as G^+, such that $Re(\lambda(A^+)) \leq 0$, $A^+ = A - BR^{-1}(B^TG^+ + C)$, and only one such solution denoted as G^-, such that $Re(\lambda(A^-)) \geq 0$, $A^- = A - BR^{-1}(B^TG^- + C)$.*
- *Any other real symmetric solution G satisfies $G^- \leq G \leq G^+$.* ■

One recognizes that A^+ and A^- are the closed-loop transition matrices corresponding to a stabilizing optimal feedback in the case of A^+. G^+ is called the stabilizing solution of the ARE. $V^+(\cdot)$ and $V^-(\cdot)$ are in (3.144) and (3.145) respectively. It is noteworthy that if in the first assertion of the Theorem one looks for negative semi-definite solution of the ARE, then the equivalence has to be replaced by "only if". In such a case the positivity of the Popov function is only a necessary condition.

Theorem 3.43. *Assume that $R = R^T > 0$. Then*

- *$V^+(\cdot) > -\infty$ and $V^- < +\infty \iff$ there exists a real symmetric solution to the ARE.*
- *Moreover $V^+(x) = x^TG^+x$ and $V^-(x) = x^TG^-x$.*
- *$V_f(\cdot) > -\infty \iff$ there exists a real symmetric non-positive definite solution to the ARE.*
- *Consequently $V_f(\cdot) > -\infty$ if and only if $G^- \leq 0$. When $G^- < 0$ then $V_f(\cdot) = V^+(\cdot) = x^TG^+x$.*
- *The optimal closed-loop system $\dot{x}(t) = A^+x(t)$ is asymptotically stable if $G^- < 0$ and $G^+ > G^-$, where A^+ is defined in Theorem 3.42.* ■

One can already conclude from the above results that the set of solutions to the KYP Lemma conditions (3.2) possesses a minimum solution $P^- = -G^+$ and a maximum solution $P^+ = -G^-$ when $D + D^T > 0$, and that all the other solutions $P > 0$ of the ARE satisfy $-G^+ \leq P \leq -G^-$. The last two items tell us that if the ARE has a solution $G^- < 0$ then the optimal controller asymptotically stabilizes the system. In this case $\lim_{t \to +\infty} x(t) = 0$ so that indeed $V_f(\cdot) = V^+(\cdot)$.

The function $-V_f(\cdot)$ corresponds to what we shall call the available storage (with respect to the supply rate $w(x, u)$) in Chapter 4. The available storage will be shown to be the minimum solution to the ARE, while the maximum solution will be called the required supply. Also dissipativity will be characterized by the available storage being finite for all $x \in X$ and the required supply being lower-bounded. The material in this section brings some further light on the relationships that exist between optimal control and dissipative systems theory. We had already pointed out a connection in Section 3.1.2. Having in mind that what we call a dissipative linear invariant system is a system which satisfies a dissipation equality as in (3.4), we can rewrite Theorem 3.42 as follows:

Theorem 3.44. *[511] Suppose that the system (A, B, C, D) in (3.1) is controllable and observable and that $D + D^T$ is full rank. Then the ARE $PA + A^T P + (PB - C^T)(D + D^T)^{-1}(B^T P - C) = 0$ has a real symmetric non-negative definite solution if and only if the system in (3.1) is dissipative with respect to the supply rate $u^T y$. If this is the case then there exists one and only one real symmetric solution P^- such that $Re(\lambda(A^-)) \leq 0$, $A^- = A + B(D + D^T)^{-1}(B^T P^- - C)$, and one and only one real symmetric solution P^+ such that $Re(\lambda(A^+)) \geq 0$, $A^+ = A + B(D + D^T)^{-1}(B^T P^+ - C)$. Moreover $0 < P^- \leq P^+$ and every real symmetric solution satisfies $P^- \leq P \leq P^+$. Therefore all real symmetric solutions are positive definite. The inequalities $H(j\omega) + H^T(j\omega) > 0$ for all $\omega \in I\!R$, $Re(\lambda(A^-)) < 0$, $Re(\lambda(A^+)) > 0$, and $P^- < P^+$, hold simultaneously.*

It will be seen later that the matrices P^+ and P^- play a very particular role in the energy properties of a dynamical system (Section 4.4.3, Remark 4.37). Theorem 3.44 will be given a more general form in Theorem 4.58. The matrix P^- is the stabilizing solution of the ARE. Algorithms exist that permit to calculate numericallt the extremal solutions P^- and P^+; see [145, Annexe 5.A] where a Fortran routine is proposed.

Remark 3.45. Let us study the case when $C = 0$ and $Q = 0$, with $R = I_m$ without loss of generality. The ARE then becomes

$$A^T G + GA - GBB^T G = 0 \tag{3.151}$$

and obviously $G = 0$ is a solution. It is the solution that yields the free terminal time optimal control problem of the optimization problem $\int_0^{+\infty} u^T(t)u(t)dt$. If the matrix A is Hurwitz, $G = 0$ is the maximum solution of (3.151). If $-A$ is Hurwitz, $G = 0$ is the minimum solution to the ARE.

Extensions towards the singular case ($R \geq 0$) can be found in [506]; see also Remark 4.94.

3.8.3 The Popov Function and the KYP Lemma LMI

We did not provide most of the proofs of the results of this section, and in particular Theorem 3.42. Let us end this section with a result that links the positivity of the Popov function, and a KYP Lemma LMI, and its complete proof.

Theorem 3.46. *[145] The spectral function*

$$\Pi(s) = [B^T(-sI_n - A^T)^{-1} \;\; I_m] \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} (sI_n - A)^{-1}B \\ I_m \end{pmatrix} \tag{3.152}$$

where the pair (A, B) is controllable, is non-negative if and only if there exists $P = P^T$ such that

$$\begin{pmatrix} Q - A^T P - PA & S - PB \\ S^T - B^T P & R \end{pmatrix} \geq 0$$

■

Before passing to the proof we need some intermediate results.

Lemma 3.47. *[145] Let $\Pi(s)$ be the spectral function in (3.152), which we say is described by the five-tuple (A, B, Q, S, R). Then*

- *i) $\Pi(s)$ is also described by the five-tuple $(A_2, B_2, Q_2, S_2, R_2)$ where*
  - $A_2 = A$
  - $B_2 = B$
  - $Q_2 = Q - A^T P - PA$
  - $S_2 = S - PB$
  - $R_2 = R$

  where $P = P^T$ is any matrix.
- *ii) For $H(s) = I_m - C(sI_n - A + BC)^{-1}B$ where C is any $m \times n$ matrix, the spectral function $H^T(s)\Pi(s)H(s)$ is described by the five-tuple $(A_1, B_1, Q_1, S_1, R_1)$ where*
  - $A_1 = A - BC$
  - $B_1 = B$
  - $Q_1 = Q + C^T QC - SC - C^T S$
  - $S_1 = S - C^T R$
  - $R_1 = R.$

■

Proof: i) Let $\Pi_2(s)$ be the Popov function described by the five-tuple $(A_2, B_2, Q_2, S_2, R_2)$. Then

$$\begin{aligned} \Pi_2(s) - \Pi(s) &= -B^T(sI_n - A^T)^{-1}(A^T P + PA)(sI_n - A)^{-1}B - \\ &\quad -B^T(-sI_n - A^T)^{-1}PB - B^T P(sI_n - A)^{-1}B \\ &= -B^T(-sI_n - A^T)^{-1}[A^T P + PA + P(sI_n - A) + \\ &\quad + (-sI_n - A^T)P](sI_n - A)^{-1}B \\ &= 0. \end{aligned} \tag{3.153}$$

ii) Notice that $(sI_n - A)^{-1}BH(s) = (sI_n - A + BC)^{-1}B$. The Popov function $H^T(s)\Pi(s)H(s)$ can be written as

$$
\begin{aligned}
&H^T(s)\Pi(s)H(s) = \\
&= [B^T(-sI_n - A^T + C^T B^T)^{-1} \;\; H^T(-s)] \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{bmatrix} (sI_n - A + BC)^{-1}B \\ H(s) \end{bmatrix} \\
&= [B^T(-sI_n - A_1^T + C^T B^T)^{-1} \;\; I_m] \begin{pmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{pmatrix} \begin{bmatrix} (sI_n - A_1)^{-1}B \\ I_m \end{bmatrix}
\end{aligned}
\tag{3.154}
$$

which ends the proof. ■

Lemma 3.48. *Let $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{s\times s}$, $C \in \mathbb{R}^{r\times s}$. The solution of the equation $AP + PB = C$ is unique if and only if the set of eigenvalues of A and the set of eigenvalues of $-B$, have no common element.* ■

In the next proof the notation $(A, B, Q, S, R) \overset{(C,P)}{\longrightarrow} (A', B', Q', S', R')$ means that one has applied the two transformations of Lemma 3.47 successively. The two Popov functions which correspond one to each other through such a transformation are simultaneously non-negative.

Proof of Theorem 3.46: Let C be a matrix such that $(A - BC)$ is asymptotically stable. Let J be the unique solution of

$$(A - BC)^T J + J(A - BC) = Q + C^T RC - SC - C^T S^T \tag{3.155}$$

One checks that $(A, B, Q, S, R) \overset{(C,J)}{\longrightarrow} (A - BC, B, 0, H^T, R)$ with $H = S - JB - C^T R$. Under these conditions the positivity of $\Pi(s)$ is equivalent to that of

$$\Pi'(s) = H(sI_n - A + BC)^{-1}B + B^T(-sI_n - A^T + C^T B^T)H^T + R \tag{3.156}$$

i.e. is equivalent to the existence of a matrix $G = G^T > 0$ such that

$$\begin{pmatrix} -(A - BC)^T G - G(A - BC) & H^T - GB \\ H - B^T G & R \end{pmatrix} \tag{3.157}$$

But for $P = G + J$ a direct computation shows that

$$\begin{pmatrix} Q - A^T P - PA & S - PB \\ S^T - B^T P & R \end{pmatrix} =$$

$$= \begin{pmatrix} I_n & C^T \\ 0 & I_n \end{pmatrix} \begin{pmatrix} -(A-BC)^T P - P(A-BC) & H^T - PB \\ H - B^T P & R \end{pmatrix} \begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix} \tag{3.158}$$

which ends the proof. ■

It is noteworthy that the matrix P in Theorem 3.46 is not necessarily positive definite. We will need those results when we deal with hyperstability. We notice that Theorem 3.46 states an equivalence under a controllability assumption of the pair (A, B). But it does not say that it is necessary that (A, B) be controllable for the result to hold; see Section 3.3 for more informations on this point.

Popov's Function and Triples

Remember that given a Popov's function as in (3.142) we call (A, B, Q, C, R) a Popov triple.

Definition 3.49. *Two Popov triples (A, B, Q, C, R) and $(\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{C}, \tilde{R})$ are called* (X, F)-equivalent *if there exist matrices $F \in \mathbb{R}^{m\times n}$ and $X = X^T \in \mathbb{R}^{n\times n}$ such that*

$$\begin{cases} \tilde{A} = A + BF, \quad \tilde{B} = B \\ \tilde{Q} = Q + LF + F^T L^T + F^T RF + \tilde{A}X + XA \\ \tilde{L} = L + F^T R + XB \quad \tilde{R} = R \end{cases} \tag{3.159}$$

One then writes $(A, B, Q, C, R) \sim (\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{C}, \tilde{R})$. Two Popov triples $(A, B, 0, C, R)$ and $(\tilde{A}, \tilde{B}, 0, \tilde{C}, \tilde{R})$ are called dual *if $\tilde{A} = -A^T$, $\tilde{B} = L$, $\tilde{L} = -B$, $\tilde{R} = R$.* ■

From the material which is presented above, it should be clear that a Popov triple can be seen as the representation of a controlled dynamical system $\dot{x}(t) = Ax(t) + Bu(t)$ together with a functional with a quadratic cost as in (3.132). To a Popov triple Σ one can therefore naturally associate a Popov's function (3.142), a Riccati equality, and an extended Hamiltonian pencil

$$\lambda M_\Sigma - N_\Sigma = \lambda \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & B \\ -Q & -A^T & -C^T \\ C & B^T & R \end{bmatrix} \tag{3.160}$$

which shall be denoted as EHP(Σ).

Lemma 3.50. *[225] (a) If $\Sigma = (A, B, Q, C, R) \sim \tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{C}, \tilde{R})$, then $\Pi_\Sigma(s) = S_F^\star(s)\Pi_{\tilde{\Sigma}}(s)S_F(s)$, where $S_F(s) = -F(sI_n - A)^{-1}B + I_m$.*

(b) If $\Sigma = (A, B, 0, C, R)$ and $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, 0, \tilde{C}, \tilde{R})$ are two dual Popov triples, then $\Pi_\Sigma(s) = \Pi_{\tilde{\Sigma}}(s)$. ■

The following holds:

Lemma 3.51. *[224, 225] Let $\Sigma = (A, B, Q, C, R)$ be a Popov triple; the following statements are equivalent:*

- *There exists an invertible block 2×2 matrix V with upper right block zero, such that $R = V^T JV$, where $J = \begin{bmatrix} -I_{m_1} & 0 \\ & \\ 0 & I_{m_2} \end{bmatrix}$, and the Riccati equality $A^T P + PA - (PB + C^T)R^{-1}(B^T P + C) + Q = 0$ has a stabilizing solution P.*
- *Π_Σ has a $J-$spectral factorization $\Pi_\Sigma = G^\star JG$, with G, G^{-1} being rational $m \times m$ matrices with all poles in the left open complex plane.* ■

These tools and results are useful in the H_∞ theory; see [224, Lemma 2, Theorem 3].

3.8.4 A Recapitulating Theorem

Let us state a Theorem proved in [349] and which holds for stabilizable systems (there is consequently also a link with the material of Section 3.3). This theorem summarizes several relationships between the solvability of the KYP Lemma set of equations and the regular optimal control problem, under a stabilizability assumption only.

Theorem 3.52. *Let the pair (A, B) be stabilizable. Then the following assertions are equivalent:*

- *(i) The optimal control problem: (3.131) and (3.132) subject to $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$, is regular, i.e. it has a solution for any $x_0 \in I\!R^n$, and this solution is unique.*
- *(ii) There exists a quadratic Lyapunov function $V(x) = x^* Px$, $P^* = P$, such that the form $\dot{V} + w(u, x) = 2x^* P(Ax + Bu) + w(u, x)$ of the variables $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$ is positive definite.*
- *(iii) The condition $w(u, x) \geq \delta(x^* x + u^* u)$ for any value of $\omega \in I\!R$, $x \in \mathbb{C}^n$, $u \in \mathbb{C}^m$ satisfying $j\omega x = Ax + Bu$, holds for some $\delta > 0$.*
- *(iv) The matrix $R = R^T$ in (3.132) is positive definite and the set of equations $PA + A^* P + Q = kRk^*$, $PB + C^* = -kR$, possesses a solution in the form of real matrices $P = P^T$ and C, such that the controller $u = Cx$ stabilizes the system $\dot{x}(t) = Ax(t) + Bu(t)$.*

- *(v)* $R > 0$ *and* $\det(j\omega J - K) \neq 0$ *for all* $\omega \in \mathbb{R}$, *with* $J = \begin{pmatrix} 0 & -I_n \\ & \\ I_n & 0 \end{pmatrix}$,
  $K = \begin{pmatrix} C^T R^{-1} C - Q & A^T - C R^{-1} B^T \\ & \\ A - B R^{-1} C^T & B R^{-1} B^T \end{pmatrix}$.
- *(vi)* $R > 0$ *and there exist a quadratic form* $V = x^* P x$, $P = P^*$, *and a matrix* $k \in \mathbb{R}^{n \times m}$, *such that* $\dot{V} + w(u, x) = |R^{\frac{1}{2}}(u - k^* x)|^2$ *and the controller* $u = k^* x$ *stabilizes the system* $\dot{x}(t) = Ax(t) + Bu(t)$.
- *(vii) The functional* $V_f(\cdot)$ *in (3.131) is positive definite on the set* $\mathcal{M}(0)$ *of processes* $(x(\cdot), u(\cdot))$ *that satisfy* $\dot{x}(t) = Ax(t) + Bu(t)$ *with* $x(0) = x_0 = 0$, *i.e. there exists* $\delta > 0$ *such that*

$$\int_0^{+\infty} w(u(t), x(t))dt \geq \delta \int_0^{+\infty} (x^T(t)x(t) + u^T(t)u(t))^2 dt$$

  for all $(x(\cdot), u(\cdot)) \in \mathcal{M}(0)$, *where* $\mathcal{M}(x_0)$ *is the set of admissible processes.*

Let at least one of these assertions be valid (which implies that they are all valid). Then there exists a unique pair of matrices (P, k) *which conforms with the requirements of item (iv). In the same way there is a unique pair which complies with the requirements of item (vi) , and the pairs under consideration are the same. Finally any of the items (i) through (vii) implies that for any initial state* $x_0 \in \mathbb{R}^n$ *one has* $V(x_0) = x_0^T P x_0 = \min_{\mathcal{M}(x_0)} V_f(x(\cdot), u(\cdot))$. ■

The set $\mathcal{M}(x_0)$ of admissible processes consists of the set of pairs $(x(\cdot), u(\cdot))$ which satisfy $\dot{x}(t) = Ax(t) + Bu(t)$ with $x(0) = x_0$, with $u \in \mathcal{L}_2$. If (A, B) is controllable then $\mathcal{M}(x_0) \neq \emptyset$ for any $x_0 \in \mathbb{R}^n$.

3.8.5 On the Design of Passive LQG Controllers

The Linear-Quadratic-Gaussian (LQG) controller has attained considerable maturity since its inception in the 1950s and 1960s. It has come to be generally regarded as one of the standard design methods. One attribute of LQG-type compensators is that, although they guarantee closed-loop stability, the compensator itself is not necessarily stable. It would be of interest to characterize the class of LQG-type compensators which are *stable*. Going one step further, if the LQG compensator is restricted to be not only *stable,* but also *passive*, this would define an important subclass. The importance of such compensators is that they would not only be passive, but would also be optimal with respect to an LQG performance criteria. One reason for considering passive compensators is that, when used to control positive real plants, they offer excellent robustness to modeling errors as long as the plant is PR. An important application of passive compensators is vibration suppression in large flexible space structures (LFSS), which are characterized by significant unmodeled dynamics and parameter errors. The linearized elastic-mode dynamics of LFSS [253] with compatible collocated actuators and sensors are PR systems regardless of the unmodeled dynamics or parameter uncertainties can, therefore, be robustly stabilized by an SPR compensator.

The objective of this section is to investigate the conditions under which an LQG-type compensator is SPR, so that one can simultaneously have high performance and robustness to unmodeled dynamics.

Consider a minimal realisation of a PR system expressed by the following state space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v(t) \\ \\ y(t) = \qquad Cx(t) + w(t) \end{cases} \tag{3.161}$$

where $v(\cdot)$ and $w(\cdot)$ are white, zero-mean Gaussian noises. Since the system is PR, we assume, without loss of generality (see Remark 3.54 at the end of this section), that the following equations hold for some matrix $Q_a \geq 0$:

$$A + A^T = -Q_A \leq 0 \tag{3.162}$$

and

$$B = C^T \tag{3.163}$$

The above conditions are equivalent to the Kalman-Yakubovich-Popov Lemma. The LQG compensator for the system (3.161), (3.162) and (3.163) is given by (see [9])

$$u(t) = -u'(t) \tag{3.164}$$

$$\dot{\hat{x}}(t) = \left[A - BR^{-1}B^T P_c - P_f B R_w^{-1} B^T\right] \hat{x}(t) + P_f B R_w^{-1} y(t) \tag{3.165}$$

$$u'(t) = R^{-1}B^T P_c \hat{x}(t) \tag{3.166}$$

where $P_c = P_c^T > 0$ and $P_f = P_f^T > 0$ are the LQ-regulator and the Kalman-Bucy filter Riccati matrices which satisfy the algebraic Riccati equations

$$P_c A + A^T P_c - P_c B R^{-1} B^T P_c + Q = 0 \tag{3.167}$$

$$P_f A^T + A P_f - P_f B R_w^{-1} B^T P_f + Q_V = 0 \tag{3.168}$$

where Q and R are the usual weighting matrices for the state and input, and Q_V and R_W are the covariance matrices of v and w. It is assumed that $Q > 0$ and that the pair $(A, Q_V^{1/2})$ is observable. The main result is stated as follows:

Theorem 3.53. *[312] Consider the PR system in (3.161), (3.162) and (3.163) and the LQG-type controller in (3.164) through (3.168). If* Q, R, Q_v *and* R_w *are such that*

$$Q_v = Q_a + BR^{-1}B^T \tag{3.169}$$

$$R_w = R \tag{3.170}$$

and

$$Q - BR^{-1}B^T \stackrel{\Delta}{=} Q_B > O \tag{3.171}$$

then the controller in (3.165) through (3.166) (described by the transfer function from y *to* u'*) is SPR.*

Proof: Introducing (3.162), (3.169), (3.170) into (3.168), it becomes clear that $\mathrm{P}_f = \mathrm{I}$ is a solution to (3.168). From (3.167) it follows:

$$P_c(A - BR^{-1}B^TP_c - BR^{-1}B^T) + (A - BR^{-1}B^TP_c - BR^{-1}B^T)^TP_c$$

$$= -Q - P_cBR^{-1}B^TP_c - P_cBR^{-1}B^T - BR^{-1}B^TP_c$$

$$= -Q - (P_c + I)BR^{-1}B^T(P_c + I) + BR^{-1}B^T$$

$$= -Q_B - (P_c + I)BR^{-1}B^T(P_c + I) < 0$$

where Q_B is defined in (3.171). In view of (3.163), (3.170) and the above, it follows that the controller in (3.165) and (3.166) is strictly positive real. ■

The above result states that, if the weighting matrices for the regulator and the filters are chosen in a certain manner the resulting LQG-type compensator is SPR. However, it should be noted that this compensator would not be optimal with respect to actual noise covariance matrices. The noise covariance matrices are used herein merely as compensator design parameters and have no statistical meaning. Condition (3.171) is equivalent to introducing an additional term $y^TR^{-1}y$ in the LQ performance index (since $Q = Q_B + CR^{-1}C^T$) and is not particularly restrictive. The resulting feedback configuration is guaranteed to be stable despite unmodeled plant dynamics and parameter inaccuracies, as long as the plant is positive real. One application of such compensators would be for controlling elastic motion of large flexible space structures using collocated actuators and sensors. Further work on passive LQG controllers has been carried out in [99, 160, 165, 179, 237, 238].

Remark 3.54. Consider a positive real system expressed as

$$\begin{cases} \dot{z}(t) = Dz(t) + Fu(t) \\ \\ y(t) = Gz(t) \end{cases} \tag{3.172}$$

Then, there exists matrices $P > 0$ and L such that

$$\begin{cases} PD + D^T P = -LL^T \\ \\ PF = G^T \end{cases} \tag{3.173}$$

Define $x = P^{\frac{1}{2}} z$, where $P^{\frac{1}{2}}$ is a symmetric square root of P [272]. Introducing this definition in (3.172), we obtain a state space representation as the one in (3.161), but with $A = P^{\frac{1}{2}} D P^{-\frac{1}{2}}$, $B = P^{\frac{1}{2}} F$, $C = GP^{-\frac{1}{2}}$. Multiplying the first equation in (3.173) on the left and on the right by $P^{-\frac{1}{2}}$ we obtain (3.162) with $Q_A = P^{-\frac{1}{2}} LL^T P^{-\frac{1}{2}}$. Multiplying (3.173) on the left by $P^{-\frac{1}{2}}$ we obtain (3.163). ■

3.8.6 Summary

Let us recapitulate some of the material in the previous subsections. We consider the two matrix polynomials

$$\begin{aligned} R(P) &= A^T P + PA + (C - B^T P)^T (D + D^T)^{-1} (C - B^T P) \\ S(G) &= AG + GA^T + (B - GC^T)(D^T + D)^{-1} (B - GC^T)^T \end{aligned} \tag{3.174}$$

and the linear invariant system (Σ) : $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$ which is controllable and observable.

Then all the following statements are equivalent one to each other [480]:

- 1) The transfer function of (Σ) is extended SPR.
- 2) There exists a positive definite matrix P such that

$$\begin{bmatrix} A^T P + PA & C^T - PB \\ \\ C - B^T P & -(D + D^T) \end{bmatrix} < 0 \tag{3.175}$$

- 3) $D + D^T > 0$ and the ARI $R(P) < 0$ has a positive definite solution P_i.
- 4) $D + D^T > 0$ and the ARE $R(P) = 0$ has a solution P_e such that $A + (D + D^T) P_E$ has all its eigenvalues with strictly negative real parts.
- 5) There exists a positive definite matrix G such that

$$\begin{bmatrix} AG + GA^T & B - GC^T \\ \\ B^T - CG & -(D^T + D) \end{bmatrix} < 0 \tag{3.176}$$

- 6) $D + D^T > 0$ and the ARI $S(G) < 0$ has a positive definite solution G_i.
- 7) $D + D^T > 0$ and the ARE $S(G) = 0$ has a solution G_e such that $A + (D + D^T)G_e$ has all its eigenvalues with strictly negative real parts.

In addition, assume that any of the above statements 1)–7) holds. Then:

- 8) If the matrix P (resp. P_i) solves the inequality (3.175) (resp. $R(P) < 0$) then its inverse P^{-1} (resp. P_i^{-1}) solves the inequality (3.176) (resp. $S(G) < 0$), and vice-versa.
- 9) The inequalities $0 \le P_e < P_i$ and $0 \le G_e < G_i$ hold.

Lemma 3.7 is used to prove some of the above equivalences. More on Riccati equations can be found in [273, 416]; see also Appendix A.4. Point 2) above and Theorem A.61 show that extended SPR functions and SSPR functions can be tested with the same LMI conditions and are therefore equivalent notions.

Let us recall a fundamental result which is also closely linked to the KYP Lemma solvability under no-controllability assumption of (A, B). Given $A \in I\!R^{n\times n}$, $B \in I\!R^{n\times m}$, $M = M^T \in I\!R^{(n+m)\times(n+m)}$, with $\det(j\omega I_n - A) \neq 0$ for $\omega \in I\!R$ (A does not have imaginary eigenvalues) and (A, B) controllable, the next two statements are equivalent [145, 412]:

- $\begin{pmatrix} (j\omega I_n - A)^{-1}B \\ I_m \end{pmatrix}^\star M \begin{pmatrix} (j\omega I_n - A)^{-1}B \\ I_m \end{pmatrix} \le 0$ for all $\omega \in [-\infty, +\infty]$.
- There exists a matrix $P = P^T \in I\!R^{n\times n}$ such that

$$M + \begin{pmatrix} A^T P + PA & PB \\ B^T P & 0 \end{pmatrix} \le 0 \tag{3.177}$$

When $M = \begin{pmatrix} Q & C^T \\ C & D + D^T \end{pmatrix}$, one recovers the KYP Lemma set of equations. When $Q \ge 0$ then $P \ge 0$ and A is Hurwitz. The corresponding equivalence with strict inequalities holds even if (A, B) is not controllable. This equivalence therefore somewhat generalizes Proposition 2.31. The generalization of this equivalence for a limited range of frequencies $|\omega| \le \bar{\omega}$, has been proposed in [229, 230]. This has important practical consequences.

3.8.7 A Digression on Semidefinite Programming Problems

The above equivalence makes a nice transition to the relationships between semidefinite programming problems (SDP) and the KYP Lemma. Let us consider a SDP of the form

$$\begin{array}{ll}\text{minimize} & q^T x + \sum_{k=1}^{L} \mathbf{Tr}(Q_k P_k) \\ \text{subject to} & \begin{bmatrix} A_k^T P_k + P_k A_k & P_k B_k \\ B_k^T P_k & 0 \end{bmatrix} + \sum_{i=1}^{p} x_i M_{k_i} \geq N_k,\ k = 1, ..., L \end{array} \tag{3.178}$$

where the variables (unknowns) are $x \in I\!R^p$ and $P_k = P_k^T \in I\!R^{n_k \times n_k}$, the problem data are $q \in I\!R^p$, $Q_k = Q_k^T \in I\!R^{n_k \times n_k}$, $A_k \in I\!R^{n_k \times n_k}$, $B_k \in I\!R^{n_k \times m_k}$, $M_{k_i} = M_{k_i}^T \in I\!R^{(n_k+m_k)\times(n_k+m_k)}$, and $N_k = N_k^T \in I\!R^{(n_k+m_k)\times(n_k+m_k)}$. Such a SDP is named a KYP-SDP [498] because of the following. As seen just above the KYP Lemma states a frequency domain inequality of the form

$$\begin{pmatrix} (j\omega I_n - A)^{-1} B \\ I_m \end{pmatrix}^{\star} M \begin{pmatrix} (j\omega I_n - A)^{-1} B \\ I_m \end{pmatrix} \leq 0 \tag{3.179}$$

for all $\omega \in [-\infty, +\infty]$, with M symmetric and A has no imaginary eigenvalue (equivalently the transfer function $C(sI_n - A)^{-1}B + D$ has no poles on the imaginary axis). And (3.179) is equivalent to the LMI in (3.177). The constraints in the KYP-SDP in (3.178) possess the same form as (3.177) where M is replaced by an affine function of the variable x. Let us take $Q_k = 0$, then the KYP-SDP can equivalently be rewritten as

$$\begin{array}{ll}\text{minimize} & q^T x \\ \text{subject to} & \begin{pmatrix} (j\omega I_n - A_k)^{-1} B_k \\ I_m \end{pmatrix}^{\star} (\mathcal{M}_k(x) - N_k) \begin{pmatrix} (j\omega I_n - A_k)^{-1} B_k \\ I_m \end{pmatrix} \geq 0 \\ & k = 1, ..., L \end{array} \tag{3.180}$$

where the optimization variable is x and $\mathcal{M}_k(x) = \sum_{i=1}^{p} x_i M_{k_i}$. Applications of KYP-SDPs are in optimization problems with frequency-domain inequalities, linear systems analysis and design, digital filter design, robust control analysis using integral quadratic constraints, linear quadratic regulators, quadratic Lyapunov functions search, *etc.* More details may be found in [498]. We do not provide more details on this topic since this would bring us too far away from our main interest in this book.

3.9 The Lur'e Problem (Absolute Stability)

3.9.1 Introduction

In this section we study the stability of an important class of control systems. The Lur'e problem has been introduced in [321], it was very popular in the

1950s [2] and can be considered as the first steps towards the synthesis of controllers based on passivity. For a complete account on the Russian school to the Lur'e problem, [524] is mandatory reading. Consider the closed-loop system shown in Figure 3.3. We are interested in obtaining the conditions on the linear system and on the static nonlinearity such that the closed-loop system is stable. This is what is called the Lur'e problem.

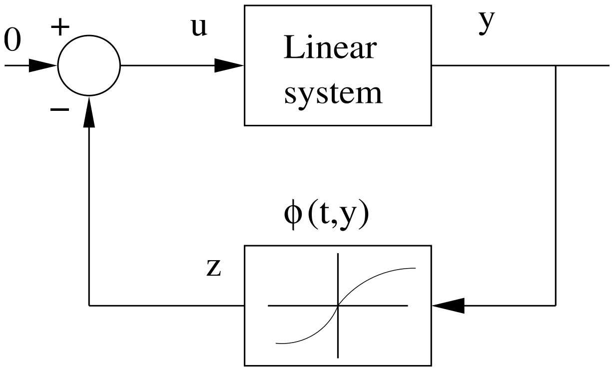

Fig. 3.3. The Lur'e problem

The linear system is given by the following state-space representation:

$$(\Sigma) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = Cx(t) + Du(t), \quad x(0) = x_0 \end{cases} \tag{3.181}$$

with $x(t) \in I\!R^n, u(t), y(t) \in I\!R^m$, $m < n$. The static nonlinearity $\phi : I\!R \times I\!R^m \to I\!R^m$ is possibly time-varying and described by

$$\begin{cases} z(t) = \phi(t, y(t)) \\ u(t) = -z(t) \quad \text{(interconnection relation)} \end{cases} \tag{3.182}$$

The linear system is assumed to be minimal, *i.e.* controllable and observable which means that

$$\text{rank}\left[B \;\; AB \;\; ... \;\; A^{n-1}B\right] = n,$$

[2] Of the 20th century.

and

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

The nonlinearity is assumed to belong to the sector $[a, b]$, *i.e.*:

i) $\phi(t, 0) = 0 \qquad \forall\, t \geq 0$
ii) $[\phi(t, y) - ay]^T [by - \phi(t, y)] \geq 0 \qquad \forall\, t \geq 0, \qquad \forall\, y(t) \in I\!R^m$

In the scalar case ($m = 1$), the static nonlinearity is shown in Figure 3.4.

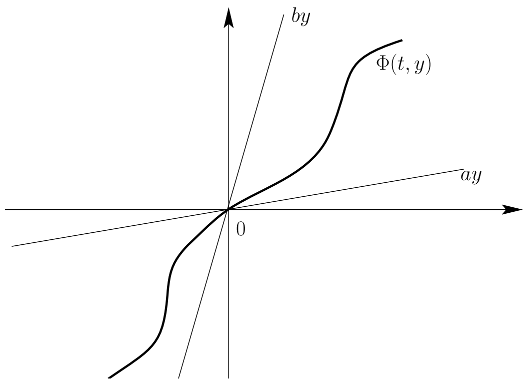

Fig. 3.4. Static nonlinearity for $n = 1$

3.9.2 Well-posedness of ODEs

The function $\phi(\cdot, \cdot)$ must be such that the closed-loop system possesses a unique solution. For an ordinary differential equation $\dot{x}(t) = f(x(t), t)$, the so-called Carathéodory conditions are as follows:

Theorem 3.55. *[107] Let $I = \{(x, t) \mid ||x - x_0|| \leq b, |t - \tau| \leq a, a \in I\!R^+, b \in I\!R^+\}$, and let us assume that $f : I \to I\!R$ satisfies:*

- *(i) $f(x, \cdot)$ is measurable in t for each fixed x*
- *(ii) $f(\cdot, t)$ is continuous in x for each fixed t*
- *(iii) there exists a Lebesgue integrable function $m(\cdot)$ on the interval $|t - \tau| \leq a$ such that $|f(x, t)| \leq m(t)$ for all $(x, t) \in I$*

Then for some $\alpha > 0$ *there exists an absolutely continuous solution* $x(\cdot)$ *on some interval* $|t - \tau| \leq \beta$, $\beta \geq 0$, *satisfying* $x(\tau) = x_0$. ■

One notices that, due to the absolute continuity of the solution $x(\cdot)$, it follows that the equality $\dot{x}(t) = f(x(t), t)$ is satisfied almost everywhere in the Lebesgue measure (*i.e.* for all t in the said interval, except on a set of zero Lebesgue measure). When $f(\cdot, \cdot)$ satisfies $||f(t, x) - f(t, y)|| \leq \psi(|t - \tau|, ||x - y||)$ where $\psi(\cdot, \cdot)$ is continuous and non-negative, then uniqueness of the solution starting at x_0 is guaranteed (and its derivative is unique up to a set of zero Lebesgue measure in the said interval of time). When $f(\cdot, \cdot)$ is a C^r function of both x and t, then local existence and uniqueness of a solution which is also a C^r function of both x and t, is guaranteed [28]. The basic and "classical" well-posedness results for an ordinary differential equation $\dot{x}(t) = f(t, x(t))$ are as follows:

Theorem 3.56 (Local Existence and Uniqueness [96]). *Let* $f(t, x)$ *be continuous in a neighborhood* $\mathbf{N}$ *of* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, *and be locally Lipschitz with Lipschitz constant* k. *Then there exists* $\alpha > 0$ *such that the ODE* $\dot{x}(t) = f(t, x(t))$ *possesses in the interval* $I = [t_0 - \alpha, t_0 + \alpha]$ *one and only one solution* $x : I \to \mathbb{R}^n$ *such that* $x(0) = x_0$. ■

The definition of Lipschitz functions is in Definitions 4.2 and 4.3.

Theorem 3.57 (Global Uniqueness [96]). *Let* $f(t, x)$ *be locally Lipschitz. Let* $I \subset \mathbb{R}$ *be an interval (*I *may be open, closed, unbounded, compact, etc). If* $x_1(\cdot)$ *and* $x_2(\cdot)$ *are two solutions of* $\dot{x}(t) = f(t, x(t))$ *on* I *and if they are equal for some* $t_0 \in I$, *then they are equal on the whole* I. *If in addition* $f(t, x)$ *is continuous in some domain* $U \subset \mathbb{R} \times \mathbb{R}^n$ *and if* $(t_0, x_0) \in U$, *then there exists a maximum interval* $J \ni t_0$ *in which a solution exists, and this solution is unique.* ■

Theorem 3.58 (Continuous Dependence on Initial Data). *Let* $f : W \to \mathbb{R}^n$, $W \subseteq \mathbb{R}^n$ *an open set, be Lipschitz with constant* k. *Let* $x_1(\cdot)$ *and* $x_2(\cdot)$ *be solutions of* $\dot{x}(t) = f(x(t))$ *on the interval* $[t_0, t_1]$. *Then for all* $t \in [t_0, t_1]$, *one has* $||x_1(t) - x_2(t)|| \leq ||x_1(t_0) - x_2(t_0)|| \exp(k(t - t_0))$. ■

The proof of Theorem 3.58 is based on Gronwall's Lemma which is recalled later in the book (Lemma 3.68). It is noteworthy that some of the nonsmooth dynamical systems which are studied in this book do not enjoy the continuity in the initial data property, like Lagrangian systems subject to complementarity conditions (unilateral constraints).

In Section 3.9.4, well-posedness will be extended to multivalued and nonsmooth feedback nonlinearities. Then new tools for studying the well-posedness are required. Concerning the closed-loop system (3.181) and (3.182), one has $f(x(t), t) = Ax(t) - B\phi(t, Cx(t))$ when $D = 0$, and the conditions on $\phi(t, y)$ which assure that the vector field fits within the conditions of Theorems 3.55, 3.56 or 3.57, are easily deduced. It is worth noting that when $D \neq 0$ some care is needed. Indeed one obtains

$$y = Cx - D\phi(t, y), \tag{3.183}$$

and the output mapping makes sense only if Equation (3.183) has a unique solution $y = h(x)$ for all $t \geq 0$ and all $x \in \mathbb{R}^n$. A single-valued mapping $\rho(\cdot)$ is monotone if $\langle x - x', y - y' \rangle \geq 0$ whenever $x = \rho(y)$ and $x' = \rho(y')$. It is strongly monotone if $\langle x - x', y - y' \rangle \geq \alpha ||y - y'||^2$ for some $\alpha > 0$.

Lemma 3.59. *Let $D \geq 0$ and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ be monotone. Then the equation*

$$y = Cx - D\phi(y) \tag{3.184}$$

possesses a unique solution $y = h(x)$ for all $x \in \mathbb{R}^n$. ■

Proof: The proof uses the fact that the generalized equation $0 \in F(x)$ possesses a unique solution provided the mapping $F(\cdot)$ is strongly monotone on $\mathbb{R}^n$ [137, Theorem 2.3.3]. We are thus going to show that the mapping $y \mapsto y + D\phi(y)$ is strongly monotone. Take two couples (x, x') and (y, y') in the graph of this mapping, *i.e.* $x' = x + D\phi(x)$ and $y' = x + D\phi(y)$. Then

$$\begin{aligned}
(x-y)^T(x'-y') &= (x-y)^T(x - y + D\phi(x) - D\phi(y)) \\
&= (x-y)^T(x-y) + (x-y)^T D(\phi(x) - \phi(y)) \\
&\geq (x-y)^T(x-y) + \lambda_{\min}(D)(x-y)^T(\phi(x) - \phi(y)) \\
&\geq (x-y)^T(x-y)
\end{aligned} \tag{3.185}$$

This inequality precisely means that $y \mapsto y + D\phi(y)$ is strongly monotone [137, Definition 2.3.1]. Thus $y \mapsto y + D\phi(y) + \alpha$ for some $\alpha \in \mathbb{R}^m$ is strongly monotone as well. ■

The proof of the above fact applies to generalized equations of the form $0 \in F(x) + N_K(x)$, where $N_K(\cdot)$ is the normal cone to the closed convex set $K \subseteq \mathbb{R}^n$ (we shall come back on convex analysis later in this chapter). It happens that $N_{\mathbb{R}^n}(x) = \{0\}$ for all $x \in \mathbb{R}^n$. But it is worth keeping in mind that the result would still hold by restricting the variable y to some closed convex set. Coming back to the Lur'e problem, one sees that a direct feedthrough of the input in the output is allowed, provided some conditions are respected. Positive real systems with $D > 0$ (which therefore have a uniform vector relative degree $r = (0, ..., 0)^T \in \mathbb{R}^m$), or with $D \geq 0$, meet these conditions.

3.9.3 Aizerman's and Kalman's Conjectures

Lur'e problem in Figure 3.3 can be stated as follows: Find the conditions on (A, B, C, D) such that the equilibrium point $x = 0$ of the closed-loop system is globally asymptotically stable for all nonlinearities $\phi(\cdot)$ in the sector $[a, b]$. Then the system is said to be *absolutely stable*. Another way to formulate it is as follows: suppose the nonlinearity $\phi(\cdot, \cdot)$ belongs to the sector $[0, k]$. The absolute stability problem is to find the value $k^\star \stackrel{\Delta}{=} \inf\{k \geq 0 \mid$ there exists $\phi^\star(\cdot)$ in the sector $[0, k]$ for which the feedback system (3.181) (3.182) is not asymptotically stable$\}$. Equivalently, the feedback system (3.181) (3.182) is asymptotically stable for any nonlinearity in the sector $[0, k^\star]$. In the next sections, we shall first review three celebrated conjectures which happen to be true only in very specific cases. Then we shall see what happens when the feedback nonlinearity $\phi(\cdot, \cdot)$ is no longer a function but a multivalued function. This demands new mathematical tools to be correctly handled, and we shall spend some time on this. Then two celebrated results, the circle criterion and the Popov's criterion, will be presented.

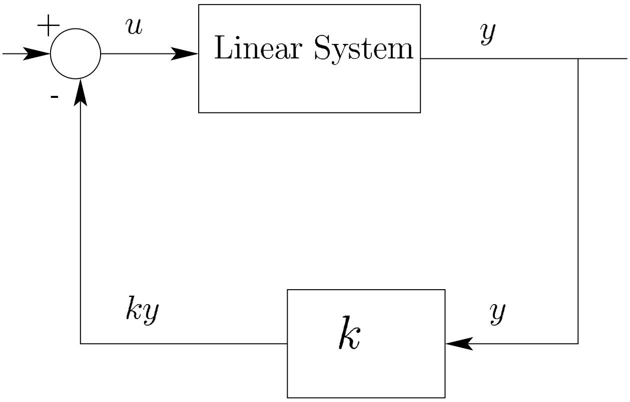

Fig. 3.5. Linear system with a constant output feedback

Conjecture 3.60 (Aizerman's conjecture). If the linear subsystem with $D = 0$ and $m = 1$ in Figure 3.5 is asymptotically stable for all $\phi(y) = ky$, $k \in [a, b]$, then the closed loop system in Figure 3.6 with a time-invariant nonlinearity $\phi(\cdot)$ in the sector $[a, b]$ is also globally asymptotically stable. ■

Aizerman's conjecture states that if the vector field $Ax + b\phi(y)$ is Hurwitz for all linear characteristic functions $\phi(\cdot)$, then the fixed point $x = 0$ should be globally asymptotically stable for any continuously differentiable $\phi(\cdot)$ whose slope remains bounded inside $[a, b]$.

Conjecture 3.61 (Kalman's conjecture). Consider the system in Figure 3.6 with a nonlinearity such that $\phi(t, y) = \phi(y)$ (*i.e.* a time-invariant and continuously differentiable nonlinearity), $m = 1$, $\phi(0) = 0$ and $a \leq \frac{d\phi}{dy}(y) \leq b$. Then the system in (3.181) with $D = 0$ is globally asymptotically stable if it is globally asymptotically stable for all nonlinearities $\phi(y) = ky$, $k \in [a, b]$. ■

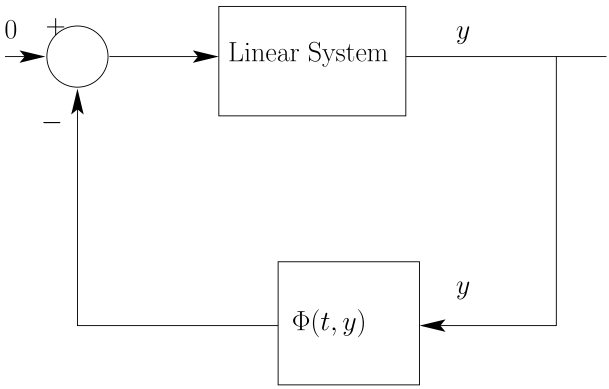

Fig. 3.6. Linear system with a sector nonlinearity in negative feedback

Thus Kalman's conjecture says that if $A - kBC$ is Hurwitz for all $k \in [a, b]$, $x = 0$ should be a globally stable fixed point for (3.181) (3.182) with $\phi(\cdot)$ as described in Conjecture 3.61. However it turns out that both conjectures are false in general. In fact, the absolute stability problem, and consequently Kalman conjecture, may be considered as a particular case of a more general problem known in the Applied Mathematics literature as the Markus-Yamabe conjecture (MYC in short). The MYC can be stated as follows [350]:

Conjecture 3.62 (Markus-Yamabe's conjecture). If a C^1 map $f : I\!R^n \rightarrow I\!R^n$ satisfies $f(0) = 0$ and if its Jacobian matrix $\left.\frac{\partial f}{\partial x}\right|_{x_0}$ is stable **for all** $x_0 \in I\!R^n$, then 0 is a **global attractor** of the system $\dot{x}(t) = f(x(t))$. ■

In other words, the MYC states that if the Jacobian of a system at any point of the state space has eigenvalues with stricty negative real parts, then the fixed point of the system should be globally stable as well. Although this conjecture seems very sound from an intuitive point of view, it is false for $n \geq 3$. Counter examples have been given for instance in [104]. It is however true in dimension 2, *i.e.* $n = 2$. This has been proved in [175]. The proof is highly technical and takes around 40 pages. Since it is, moreover, outside the scope of this monograph dedicated to dissipative systems, it will not be reproduced nor summarized here. This is however one nice example of a result that is apparently quite simple and whose proof is quite complex. The Markus-Yamabe conjecture has been proved to be true for gradient vector fields, *i.e.* systems of the form $\dot{x}(t) = \nabla f(x(t))$ with $f(\cdot)$ of class C^2 [334]. It is clear that the conditions of the Kalman's conjecture with $f(x) = Ax + b\phi(y)$, $\phi(0) = 0$, make it a particular case of the MYC. In short one could say that Kalman's conjecture (as well as Aizerman's conjecture) is a version of MYC for control theory applications. Since, as we shall see in the next subsections, there has been a major interest in developing (sufficient) conditions for Lur'e problem and absolute stability in the Systems and Control community, it is also of significant interest to know the following result:

Theorem 3.63. *[35, 46] Kalman's conjecture is true for dimensions* $n = 1, 2, 3$. *It is false for* $n > 3$. ■

Since it has been shown in [175] that the MYC is true for $n = 1, 2$, it follows immediately that this is also the case for the Kalman's conjecture. Aizerman's conjecture has been shown to be true for $n = 1, 2$ in [163], proving in a different way that Kalman's conjecture holds for $n = 1, 2$. The following holds for the case $n = 3$:

Theorem 3.64 ($n = 3$ [35]). *The system*

$$\begin{cases} \dot{x}(t) = Ax(t) + b\phi(y(t)) \\ \\ y(t) = c^T x(t) \end{cases} \tag{3.186}$$

with $x(t) \in \mathbb{R}^3$, $y(t) \in \mathbb{R}$, $b \in \mathbb{R}^3$, $c \in \mathbb{R}^3$, $\min_y \frac{d\phi}{dy}(y) = 0$, $\max_y \frac{d\phi}{dy}(y) = k \in (0, +\infty)$, $\phi(0) = 0$, *is globally asymptotically stable if the matrices* $A + \frac{d\phi}{dy}(y)c^T \in \mathbb{R}^{n\times n}$ *are Hurwitz for all* $y(t) \in \mathbb{R}$. ■

3.9.4 Multivalued Nonlinearities

It is of interest to extend the Lur'e problem to the case where the static nonlinearity in the feedback loop is not differentiable, or even not a single-valued function (say, a usual function), but is a *multivalued* function. The material in this section is taken from [81]. Before stating the main results, we need to introduce some basic mathematical notions from convex analysis. The reader who wants to learn more on convex analysis and differential inclusions with monotone mappings, is invited to have a look at the textbooks [66, 168, 210, 359].

Basic Facts on Convex and Nonsmooth Analysis

Let $K \subset \mathbb{R}^n$ denote a convex set. Its indicator function is defined as

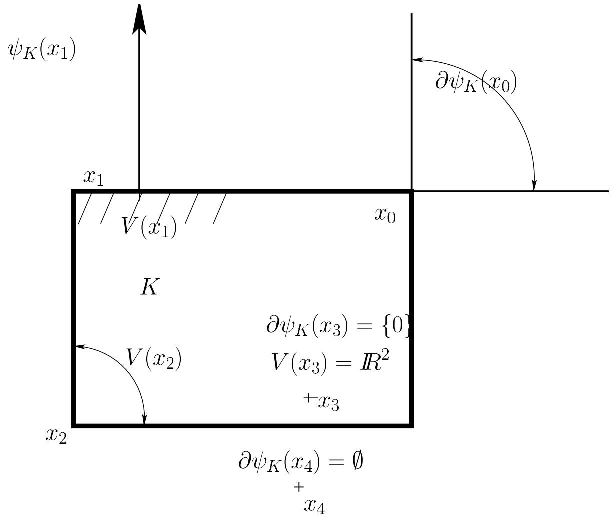

Fig. 3.7. Tangent and normal cones

$$\psi_K(x) = \begin{cases} 0 & \text{if } x \in K \\ \\ +\infty & \text{if } x \notin K \end{cases} \tag{3.187}$$

A convex function $f : I\!R^n \to I\!R$ satisfies $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$ for all $0 < \lambda < 1$, and for all x and y in its (convex) domain of definition. The indicator function $\psi_K(\cdot)$ is convex if and only if K is convex. A convex function is not necessarily differentiable, so that a more general notion of a derivative has to be introduced. The subdifferential of a convex function $f(\cdot)$ at y is denoted as $\partial f(y)$ and is the set of subgradients, *i.e.* vectors $\gamma \in I\!R^n$ satisfying

$$f(x) - f(y) \ge \gamma^T(x - y) \tag{3.188}$$

for all $x \in I\!R^n$. Geometrically, (3.188) means that one can construct a set of affine functions (straight lines) $y \mapsto (x-y)^T\gamma + f(x)$ whose "slope" γ is a subgradient of $f(\cdot)$ at x. The set $\partial f(y)$ may be empty, however if $f(\cdot)$ is convex and $f(y) < +\infty$ then $\partial f(y) \neq \emptyset$ [359]. The simplest example is $f : I\!R \to I\!R^+$, $x \mapsto |x|$. Then

$$\partial f(x) = \begin{cases} -1 & \text{if } x < 0 \\ \\ [-1, 1] & \text{if } x = 0 \\ \\ 1 & \text{if } x > 0 \end{cases} \tag{3.189}$$

One realizes in passing that $\partial|x|$ is the so-called relay characteristic and that $0 \in \partial|0|$: the absolute value function has a minimum at $x = 0$. The subdifferential of the indicator of K (which is convex if K is convex) is given by

$$\partial\psi_K(x) = \begin{cases} \{0\} & \text{if } x \in \text{ Int } (K) \\ \\ N_K(x) & \text{if } x \in \partial K \\ \\ \emptyset & \text{if } x \notin K \end{cases} \tag{3.190}$$

where ∂K is the boundary of K, and

$$N_K(x) = \{z \mid z^T(\zeta - x) \le 0, \forall \zeta \in K\} \tag{3.191}$$

is the outwards normal cone to K at x. Notice that $0 \in N_K(x)$ and that we have drawn the sets $x + N_K(x)$ rather than $N_K(x)$ in Figure 3.7. Also $N_K(x) = \{0\}$ if $x \in \text{Int}(K)$, where $\text{Int}(K) = K \setminus \partial K$. The set in (3.190) is the subdifferential from convex analysis.

Example 3.65. If $K = [a, b]$ then $N_K(a) = I\!R^-$ and $N_K(b) = I\!R^+$.

Remark 3.66. The symbol ∂ is used in three different meanings in this section: boundary of a set, subdifferential and partial derivative. Since this notation is classical we choose not to change it.

Definition 3.67. *Let K be a convex cone. Its* polar *cone (or negative cone) is*

$$K^\star = \{s \in I\!R^n \ \mid \ \langle s, x\rangle \le 0 \textit{ for all } x \in K\} \tag{3.192}$$

■

The inwards tangent cone $T_K(x)$ is the polar cone to $N_K(x)$ and is defined as $T_K(x) = \{z \mid \forall \zeta \in N_K(x), \zeta^T z \le 0\}$. Both the normal and the tangent cones are convex sets. If the set K is defined as $\{x \mid h(x) \ge 0\}$ for some differentiable function $h : I\!R^n \to I\!R^m$, then an alternative definition of the tangent cone at x is [358]

$$T_K(x) = \{v \in I\!R^n \mid v^T \nabla h_i(x) \ge 0, \ \ \forall\, i \in J(x)\} \tag{3.193}$$

with $J(x) = \{i \in \{1, ..., m\} \mid h_i(x) \le 0\}$. One notes that this definition coincides with the first one as long as $x \in K$, and that K needs not be convex to define $T_K(x)$ in (3.193). Some examples are depicted in Figure 3.7; see also [69].

A mapping $\rho(\cdot)$ from X to Y is said to be multivalued if it assigns to each element x of X a subset $\rho(x)$ of Y (which may be empty, contain just one element, or contain several elements). The graph of a mapping $\rho(\cdot)$ is defined as $gph(\rho) = \{(x, y) \ \mid \ y \in \rho(x)\}$. The mappings whose graphs are in Figure 3.8 (c–f) are multivalued. A multivalued mapping $\rho(\cdot)$ is monotone if $(x - x')^T(y - y') \ge 0$ for any couples (x, y) and (x', y') in its graph, *i.e.* $x' \in$

$\rho(y')$ and $x \in \rho(y)$. When $n = 1$ monotone mappings correspond to completely non-decreasing curves. When $\rho(\cdot)$ is single-valued, monotonicity simply means $(\rho(y) - \rho(y'))^T(y - y') \geq 0$ for all y and y'. Let $dom(\rho) = \{x | x \in X, \rho(x) \neq \emptyset\}$ be the domain of $\rho(\cdot)$. Recall that the domain of a (single-valued) function $f(\cdot)$ is $dom(f) = \{x \mid f(x) < +\infty\}$. A monotone mapping $\rho(\cdot)$ is maximal if for any $x \in X$ and any $y \in Y$ such that $\langle y - y_1, x - x_1 \rangle \geq 0$ for any $x_1 \in dom(\rho)$ and any $y_1 \in \rho(x_1)$, then $y \in \rho(x)$. Complete nondecreasing curves in $\mathbb{R}^2$ are the graphs of maximal monotone mappings. Another interpretation is that the graph of a maximal monotone mapping cannot be enlarged without destroying the monotonicity (hence the notion of maximality). Examples of monotone mappings ($n = 1$) are depicted in Figure 3.8. They may represent various physical laws, like dead-zone (a), saturation or elasto-plasticity (b), corner law – unilateral effects, ideal diode characteristic – (c), Coulomb friction (d), MOS transistor ideal characteristic (e), unilateral and adhesive effects (f). Maximal monotone mappings play an important role in the study of infinite dimensional systems. As is illustrated next, they also find nice application in the Lur'e problem. One can see easily that if an operator $H : u \mapsto y$ is monotone, then it is also passive.

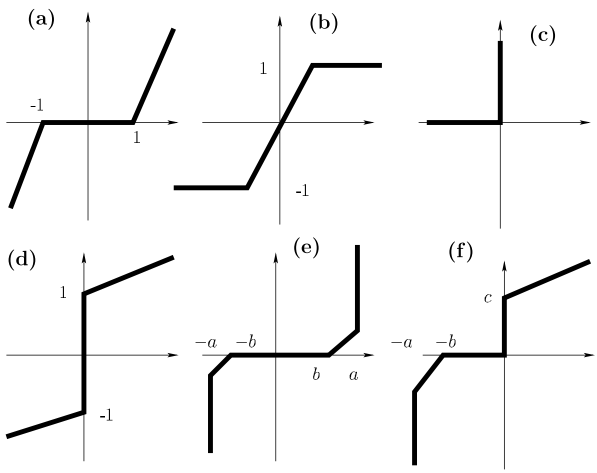

Fig. 3.8. Monotone mappings (one-dimensional case)

We finally end this section by recalling classical tools and definitions which we shall need next:

Lemma 3.68 (Gronwall's Lemma). *Suppose $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function, and $b \geq 0$, $c \geq 0$, are some constants. Then, if $f(t) \leq b + \int_0^t cf(s)ds$ for all $t \geq 0$, one has $f(t) \leq b\exp(ct)$ for all $t \geq 0$.* ■

We recall the definition of an absolutely continuous function.

Definition 3.69. *Let $-\infty < a < b < +\infty$. A function $f : [a,b] \rightarrow \mathbb{R}^n$ is* absolutely continuous *if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ and any family of disjoint intervals (α_1, β_1), (α_2, β_2), ...,(α_n, β_n) in $\mathbb{R}$ satisfying $\sum_{i=1}^n (\beta_i - \alpha_i) < \delta$, one has $\sum_{i=1}^n |f(\beta_i) - f(\alpha_i)| < \epsilon$.* ■

In fact absolutely continuous (AC) functions are usually better known as follows:

Theorem 3.70. *An AC function $f : [a,b] \rightarrow \mathbb{R}$ is almost everywhere differentiable with derivative $\dot{f}(\cdot) \in \mathcal{L}_1$ and $f(x) - f(a) = \int_a^x \dot{f}(t)dt$ for any $a \leq x$.*

■

Theorem 3.70 can also be stated as: there exists a Lebesgue integrable function $g(\cdot)$ such that $f(t) = \int g(\tau)d\tau$ ($d\tau$ being the Lebesgue measure). In a more sophisticated and pedantic language, $df = g(t)dt$ as an equality of measures, which means that $\dot{f}(t) = g(t)$ almost everywhere. A function is Lipschitz continuous if and only if it is absolutely continuous and its derivative $\dot{f}$ is essentially bounded in the sense that there exists a compact set K such that $\dot{f}(t) \in K$ for almost all $t \in [a,b]$. All continuously differentiable (C^1) functions defined on a compact interval of $\mathbb{R}$, are AC. AC functions are of bounded variation (see Definition 6.58) on such an interval and possess a derivative almost everywhere. For functions defined on $\mathbb{R}$ one then may define the notion of local AC, which simply means that the restriction of the function on any bounded interval of $\mathbb{R}$ is AC. Let us now introduce the notion of lower semi-continuity.

Definition 3.71. *Let $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$. The function $f(\cdot)$ is said* lower semi-continuous *(lsc) at $x^\star \in X$ if $\liminf_{x \rightarrow x^\star} f(x) \geq f(x^\star)$.* ■

Obviously a continuous function at $x^\star$ is also lsc at $x^\star$. But the contrary is false (otherwise both properties would make one!). An lsc function can be discontinuous. The sublevel sets are defined as $S_r(f) = \{x \in X \mid f(x) \leq r\}$.

Proposition 3.72. *A function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous on $\mathbb{R}^n$ if and only if the sublevel-sets $S_r(f)$ are closed (possibly empty) for all $r \in \mathbb{R}$.* ■

The subdifferential $\partial\varphi(\cdot)$ of a convex lower semicontinuous function on $\mathbb{R}^n$ is a maximal monotone mapping, and $\partial\varphi(x)$ is a convex closed domain (possibly empty) of $\mathbb{R}^n$. One has for instance $\varphi(x) = \psi_{\mathbb{R}^-}(x)$ in Figure 3.8 (c), $\varphi(x) = |x| + \frac{x^2}{2}$ for (d), $\varphi(x) = \psi_{(-\infty,a]}(x) - \psi_{[-a,+\infty)}(x) +$

$\begin{cases} \frac{a-b}{2}(x-b)^2 \text{ if } |x| \geq b \\ 0 \text{ if } |x| < b \end{cases}$ for (e). If $\varphi(x_1, \cdots, x_m) = \mu_1|x_1| + \cdots + \mu_m|x_m| + \frac{1}{2}x^T x$, then $\partial\varphi(0) = ([-\mu_1, \mu_1], \cdots, [-\mu_m, \mu_m])^T$. Let us now state a classical result of convex analysis, which is a generalization of the chain rule [168].

Proposition 3.73. *Assume that $f : Y \to (-\infty, +\infty]$ is convex and lower semi-continuous. Let $A : X \to Y$ be a linear and continuous operator. Assume that there exists a point $y_0 = Ax_0$ at which $f(\cdot)$ is finite and continuous. Then*

$$\partial(f \circ A)(x) = A^T \partial f(Ax) \tag{3.194}$$

for all $x \in X$. ■

Further generalizations exist, see [415, §10.B]. Let us now state a generalization of the existence and uniqueness results (Theorems 3.55 to 3.57). The next theorem is known as the Hille-Yosida Theorem when the operator $A : x \mapsto Ax$ is linear.

Theorem 3.74 (Existence and uniqueness of solutions of monotone inclusions). *[66, Theorem 3.1] Let A be a maximal monotone operator mapping $\mathbb{R}^n$ into $\mathbb{R}^n$. Then for all $x_0 \in dom(A)$ there exists a unique Lipschitz continuous function $x(\cdot)$ on $[0, +\infty)$ such that*

$$\begin{cases} \dot{x}(t) + Ax(t) \ni 0 \\ \\ x(0) = x_0 \end{cases} \tag{3.195}$$

almost everywhere on $(0, +\infty)$. The function satisfies $x(t) \in dom(A)$ for all $t > 0$, and it possesses a right-derivative for all $t \in [0, +\infty)$. If $x_1(\cdot)$ and $x_2(\cdot)$ are two solutions then $||x_1(t) - x_2(t)|| \leq ||x_1(0) - x_2(0)||$ for all $t \in [0, +\infty)$. In case the operator A is linear then $x(\cdot) \in C^1([0, +\infty), \mathbb{R}^n) \cap C^0([0, +\infty), dom(A))$. Moreover $||x(t)|| \leq ||x_0||$ and $||\dot{x}(t)|| \leq ||Ax(t)|| \leq ||Ax_0||$ for all $t \geq 0$. ■

It is noteworthy that the notion of an operator in Theorem 3.74 goes much further than the mere notion of a linear operator in finite dimension. It encompasses subdifferentials of convex functions, as will be seen next. It also has important applications in infinite-dimensional systems analysis.

Example 3.75. Let $Ax = \begin{cases} +1 & \text{if } x > 0 \\ [0, 1] & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$. Then the solution is

$$x(t) = \begin{cases} (x_0 - t)^+ & \text{if } x_0 \geq 0 \\ x_0 & \text{if } x_0 < 0 \end{cases}$$

The Multivalued Absolute Stability Problem

It is of interest to extend the absolute stability problem with a single-valued feedback nonlinearity, to the case where the operator $\phi : y \mapsto y_L = \phi(y)$ is a maximal monotone operator. The state space equations of the system are given by

$$\begin{cases} \dot{x}(t) \overset{\text{a.e.}}{=} Ax(t) - By_L(t) \\ y(t) = Cx(t) \\ y_L \in \partial\varphi(y), \end{cases} \tag{3.196}$$

where $y(t), y_L(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^n$ and a.e. means almost everywhere in the Lebesgue measure. The fixed points of (3.196) can be characterized with the generalized equation

$$0 \in \{Ax_0\} - B\partial\varphi(Cx_0).$$

One notices that the system in (3.196) is a *differential inclusion*, due to the multivalued right-hand-side. Indeed the subdifferential $\partial\varphi(y)$ is in general multivalued. What is the difference between the differential inclusion in (3.196) and, say, Filippov's systems, which readers from Systems and Control are more familiar with? The main discrepancy between both is that the right-hand-side of (3.196) need not be a compact (bounded) subset of the state space $X \subseteq \mathbb{R}^n$, for all $x \in X$. It can for instance be a normal cone, which is usually not bounded (the normal cone at a of the interval $[a, b]$, $a < b$, is the half line $\mathbb{R}^-$; see Example 3.65). Of course there is a nonzero overlap between the two sets of inclusions: If the feedback loop contains a static nonlinearity as in Figure 3.8 (d), then the inclusion (3.196) can be recast either into the "maximal monotone" formalism, or the "Filippov" formalism. Actually, Filippov's systems are in turn a particular case of what one can name "standard differential inclusions", *i.e.* those inclusions whose right-hand-side is compact, convex, and possesses some linear growth property (see [124] for more details). To summarize, the basic assumptions on the right-hand-sides of both types of inclusions differ so much that their study (mathematics, analysis for control) surely differ a lot as well.

Let us assume that

a) $G(s) = C(sI - A)^{-1}B$, with (A, B, C) a minimal representation, is a SPR transfer matrix. In particular from the KYP Lemma this implies that there exists positive definite matrices $P = P^T$ and $Q = Q^T$ such that $PA + A^T P = -Q$ and $B^T P = C$.
b) B is full column rank, equivalently $\text{Ker}(B) = \{0\}$. Thus $CA^{-1}B + B^T A^{-T} C^T$ is negative definite [3].
c) $\varphi : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is convex lower semi-continuous, so that $\partial\varphi$ is a maximal monotone multivalued mapping.

[3] Indeed $B^T A^{-T} C^T + CA^{-1}B = -B^T A^{-T} Q A^{-1} B < 0$.

Lemma 3.76. *[81] Let assumptions* **a)**–**c)** *hold. If $Cx(0) \in$ dom $\partial\varphi$, then the system in (3.196) has a unique absolutely continuous (AC) solution on $[0, +\infty)$.* ■

Proof: Let R be the square root of P, i.e. $R = R^T > 0$, $RR = P$. Consider the convex lower semi-continuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by $f(z) = \varphi(CR^{-1}z)$. Using **a)** shows that $\mathrm{Ker}(C^T) = \{0\}$ so that $\mathrm{Im}(CR^{-1}) = \mathrm{Im}(C) = \mathbb{R}^m$. From Proposition 3.73 it follows that $\partial f(z) = R^{-1}C^T\partial\varphi(CR^{-1}z)$. Let us prove that the system

$$\begin{cases} \dot{z}(t) \in RAR^{-1}z(t) - \partial f(z(t)) \\ z(0) = Rx(0) \end{cases} \tag{3.197}$$

has a unique AC solution on $[0, +\infty)$. First, to say that $Cx(0) \in$ dom $\partial\varphi$ is to say that $CR^{-1}z(0) \in$ dom $\partial\varphi$, and this just means that $z(0) \in$ dom ∂f. Second, it follows from the KYP Lemma that $RAR^{-1} + (RAR^{-1})^T$ is negative definite. Therefore the multivalued mapping $-RAR^{-1} + \partial f$ is maximal monotone [66, Lemma 2.4]. Consequently the existence and uniqueness result follows from Theorem 3.74.

Now set $x(t) = R^{-1}z(t)$. It is straightforward to check that $x(t)$ is a solution of the system in (3.196). Actually the system in (3.197) is deduced from (3.196) by the change of state vector $z = Rx$. ■

As an example, let us consider dissipative Linear Complementarity Systems (LCS) [83, 94]:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \\ 0 \le y(t) = Cx(t) \perp \lambda \ge 0 \end{cases} \tag{3.198}$$

where (A, B, C) satisfies **a)** and **b)** above, $y(t), \lambda(t) \in \mathbb{R}^m$, and $Cx(0) \ge 0$. The second line in (3.198) is a set of complementarity conditions between y and λ, stating that both these terms have to remain non-negative and orthogonal one to each other. The LCS in (3.198) can be equivalently rewritten as in (3.197) with $\varphi(y) = \psi_{(\mathbb{R}^+)^m}(y)$, noting that

$$0 \le y \perp \lambda \ge 0 \iff -\lambda \in \partial\psi_{(\mathbb{R}^+)^m}(y) \tag{3.199}$$

which is a basic result in convex analysis, where $\psi(\cdot)$ is the indicator function in (3.187). Lemma 3.76is extended in [85] to the case of non-autonomous systems with both locally AC and locally BV inputs, both in the linear and nonlinear cases [4]. The non-autonomous case yields another, more complex, type of differential inclusion named *first order Moreau's sweeping process.*

[4] Linearity refers in this context to the vector fields, not to the system itself that is nonlinear as it is unilaterally constrained.

Remark 3.77. Let us note in passing that Lemma 3.76 applies to nonlinear systems as $\dot{x}(t) = -\sum_{k=0}^{n} x^{2k+1}(t) - y_L(t)$, $y = x$, $y_L \in \partial\varphi(y)$, $x \in \mathbb{R}$. Indeed the dynamics $-y_L \mapsto y$ is strictly dissipative with storage function $V(x) = \frac{x^2}{2}$, so that $P = 1$ and $z = x$.

Let us notice that $y \in \text{dom}\,\partial\varphi$. Finally there exists a Lebesgue integrable function $w(t)$ such that $x(t) = \int w(\tau)d\tau$, where $d\tau$ is the Lebesgue measure. Hence $dx = w(t)dt$ as an equality of measures.

Lemma 3.78. *[81] Let assumptions* **a)**–**c)** *hold, the initial data be such that* $Cx(0) \in$ *dom* $\partial\varphi$*, and assume that the graph of* $\partial\varphi$ *contains* $(0,0)$*. Then:* **i)** $x = 0$ *is the unique solution of the generalized equation* $Ax \in B\partial\varphi(Cx)$ **ii)** *The fixed point* $x = 0$ *of the system in (3.196) is exponentially stable.* ■

Proof: The proof of part **i)** is as follows. First of all notice that $x = 0$ is indeed a fixed point of the dynamics with no control, since $0 \in B\partial\varphi(0)$. Now $Ax \in B\partial\varphi(Cx) \Rightarrow PAx \in PB\partial\varphi(Cx) \Rightarrow x^T PAx = x^T \partial g(x)$, where $g(x) = \varphi(Cx)$ (use Proposition 3.73 to prove this), $g(\cdot)$ is convex as it is the composition of a convex function with a linear mapping, and we used assumption **a)**. The multivalued mapping $\partial g(x)$ is monotone since $g(\cdot)$ is convex. Thus $x^T \partial g(x) \geq 0$ for all $x \in \mathbb{R}^n$. Now there exists $Q = Q^T > 0$ such that $x^T PAx = -\frac{1}{2}x^T Qx < 0$ for all $x \neq 0$. Clearly then x satisfies the generalized equation only if $x = 0$.

Let us now prove part **ii)**. Consider the candidate Lyapunov function $W(x) = \frac{1}{2}x^T Px$. From Lemma 3.76 it follows that the dynamics in (3.196) possesses on $[0, +\infty)$ a solution $x(t)$ which is AC, and whose derivative $\dot{x}(t)$ exists a.e.. The same applies to $W(\cdot)$ which is AC [421, p.189]. Differentiating along the closed-loop trajectories we get

$$\begin{aligned}
\tfrac{d(W\circ x)}{dt}(t) &\overset{\text{a.e.}}{=} x^T(t)Pw(t) \\
&= x^T(t)P(Ax(t) - By_L(t)) = -x^T(t)Qx(t) - x^T(t)PBy_L(t) \\
&= -x^T(t)Qx(t) - x^T(t)C^T y_L(t)
\end{aligned} \tag{3.200}$$

where y_L is any vector that belongs to $\partial\varphi(Cx)$. The equality in the first line means that the density of the measure $d(W \circ x)$ with respect to the Lebesgue measure dt (which exists since $W(x(t))$ is AC) is the function $x^T Pw$. Consequently $\frac{d(W\circ x)}{dt} + x^T Qx \in -x^T C^T \partial\varphi(Cx) = -x^T \partial g(x)$ a.e., where $\frac{d(W\circ x)}{dt}$ is computed along the system's trajectories. Let us consider any $z \in \partial g(x)$. One gets $\frac{d(W\circ x)}{dt} \overset{\text{a.e.}}{=} -x^T Qx - x^T z \leq -x^T Qx$ from the property of monotone multivalued mappings and since $(x, z) = (0, 0)$ belongs to the graph of $\partial g(x)$. The set of time instants at which the inequality $\frac{d(W\circ x)}{dt} \leq -x^T Qx$ is not satisfied is negligible in the Lebesgue measure. It follows that the function of time $W(\cdot)$, which is continuous, is nonincreasing. Thus one has $W(t) - W(0) = \int_0^t (-x^T Qx - x^T z)d\tau \le -\int_0^t x^T Qx d\tau$. Consequently $\frac{1}{2}\lambda_{\min}(P)x^T x \le W(0) - \int_0^t \lambda_{\min}(Q)x^T x d\tau$, where $\lambda_{\min}(\cdot)$ is the smallest eigenvalue. By the Gronwall's Lemma 3.68 one gets that $\frac{1}{2}\lambda_{\min}(P)x^T x \le W(0)\exp\left(-2\frac{\lambda_{\min}(Q)}{\lambda_{\min}(P)}t\right)$ which concludes the proof. ■

It is worth noting that part **i)** of Lemma 3.78 is a particular case of generalized equation $0 \in F(x)$, where $F(\cdot)$ is a maximal monotone operator.

Example 3.79. Let us consider a one degree-of-freedom mechanical system with Coulomb friction

$$m\ddot{q}(t) \overset{\Delta}{=} -\mu \mathrm{sgn}(\dot{q}(t)) + u(t) \tag{3.201}$$

where $q(t)$ is the position of the system, μ is the friction coefficient and the control is given in Laplace transform by $u(s) = H(s)q(s)$. Defining $x_1 = q$ and $x_2 = \dot{q}$ and $u = \alpha q + \beta \dot{q}$ we obtain

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} x(t) - \begin{pmatrix} 0 \\ \frac{\mu}{m} \end{pmatrix} y_L(t) \\ \\ y_L(t) \in \partial|\dot{q}(t)| \\ \\ y(t) = x_2(t) \end{cases} \tag{3.202}$$

The transfer function of the triple (A, B, C) is $G(s) = \frac{\mu}{m}\frac{s}{s^2 - \beta s - \alpha}$, which obviously cannot be SPR but only PR with a suitable choice of $\alpha < 0$ and $\beta < 0$; see Section 2.14. Thus more advanced tools will be needed to study the asymptotic stability of (3.201); see Chapter 7, Section 7.2.5.

Dissipation inequality and storage function:

We consider the same inclusion as in (3.196) but with an input, *i.e.*

$$\begin{cases} \dot{x}(t) \overset{\text{a.e.}}{=} Ax(t) - By_L(t) + Bu(t) \\ \\ y(t) = Cx(t) \\ \\ y_L \in \partial\varphi(y) \end{cases} \tag{3.203}$$

It is then not difficult to calculate that

$$\begin{aligned} \int_0^t u^T(s)y(s)ds &= \int_0^t u^T(s)Cx(s)ds = \int_0^t u^T(s)B^T Px(s)ds \\ &= \int_0^t (\dot{x}(s) - Ax(s) + By_L(s))^T Px(s)ds \\ &\le \tfrac{1}{2}x^T(t)Px(t) - \tfrac{1}{2}x^T(0)Px(0) = W(x(t)) - W(x(0)) \end{aligned} \tag{3.204}$$

Therefore $W(\cdot)$ is a storage function for (3.203) that is smooth despite the system is nonsmooth.We notice that if $Bu(t)$ in (3.203) is replaced by $Eu(t)$ for some matrix E and with both (A,E,C) and (A,B,C) being PR, then the above developments yield that $W(\cdot)$ is a storage function provided the two triples have a set of KYP Lemma equations with the same solution P, so that $B^T P = C$.

Another kind of nonsmooth characteristic that does not fit with the maximal monotone static nonlinearities can be found in [266] where the passivity of an oscillator subject to a Preisach hysteresis is shown. The absolute stability of systems with hysteresis non-linearities is also treated in [393].

3.9.5 Dissipative Evolution Variational Inequalities

Introduction

In this section we introduce a new formalism that is useful in many applications: evolution variational inequalities (in finite dimension). Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear operator. For $(t_0, x_0) \in \mathbb{R} \times K$, we consider the problem $P(t_0, x_0)$: Find a function $t \to x(t)$ $(t \geq t_0)$ with $x \in C^0([t_0, +\infty); \mathbb{R}^n)$, $\frac{dx}{dt} \in \mathcal{L}_{\infty,e}([t_0, +\infty); \mathbb{R}^n)$ and such that:

$$\boxed{\begin{cases} x(t) \in K,\ t \geq t_0 \\ \langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \rangle \geq 0,\ \forall v \in K,\ \text{a.e. } t \geq t_0 \\ x(t_0) = x_0 \end{cases} \qquad (3.205)}$$

Here $\langle ., . \rangle$ denotes the euclidean scalar product in $\mathbb{R}^n$. It follows from standard convex analysis that (3.205) can be rewritten equivalently as the differential inclusion

$$\begin{cases} \frac{dx}{dt}(t) + F(x(t)) \in -N_K(x(t)) \\ x(\cdot) \in K \end{cases} \qquad (3.206)$$

where the definition of the normal cone to a set K is in (3.191). If $K = \{x \mid Cx \geq 0\}$ the reader may use Proposition 3.73 together with (3.187), (3.190) and (3.199) to deduce that (3.206) is the LCS

$$\begin{cases} \frac{dx}{dt}(t) + F(x(t)) = C^T \lambda(t) \\ 0 \leq Cx(t) \perp \lambda(t) \geq 0 \end{cases} \qquad (3.207)$$

Still, another formulation for (3.206) is as follows:

$$\langle \dot{x}(t)+F(x(t),t), v-x(t)\rangle+\varphi(v)-\varphi(x(t)) \geq 0,\ \forall\, v \in I\!R^n,\ \text{a.e. } t \geq 0 \quad (3.208)$$

with $\varphi(x) = \psi_K(x)$ and $x(t) \in dom(\partial\varphi)$, $t \geq 0$, where $dom(\partial\varphi) = \{x \in I\!R^n \mid \partial\varphi \neq \emptyset\}$ is the domain of the multivalued mapping $\partial\varphi$. In general $\varphi(\cdot)$ is a proper convex and lower semi continuous function. One has $dom(\partial\varphi) \subset dom(\varphi) = \{x \in I\!R^n \mid \varphi(x) < +\infty\}$ and $\overline{dom(\partial\varphi)} = \overline{dom(\varphi)}$: the two domains differ only by the boundary. More on the equivalence between various formalisms like the above ones can be found in [84]. The maximal monotone property of operators is at the core of the equivalence. Let us give a well-posedness result, which is one variant of the famous Kato's Theorem [251].

Theorem 3.80. *[167] Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and let $A \in \mathbb{R}^{n\times n}$. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ can be written as*

$$F = F_1 + \Phi'$$

where F_1 is Lipschitz continuous, $\Phi \in C^1(\mathbb{R}^n;\mathbb{R})$ is convex and Φ' denotes its derivative. Let $t_0 \in \mathbb{R}$ and $x_0 \in K$ be given. Then there exists a unique $x \in C^0([t_0,+\infty);\mathbb{R}^n)$ such that

$$\frac{dx}{dt} \in \mathcal{L}_{\infty,e}([t_0,+\infty);\mathbb{R}^n) \quad (3.209)$$

$$x \text{ is right-differentiable on } [t_0,+\infty) \quad (3.210)$$

$$x(t_0) = x_0 \quad (3.211)$$

$$x(t) \in K,\ t \geq t_0 \quad (3.212)$$

$$\langle \frac{dx}{dt}(t) + Ax(t) + F(x(t)), v - x(t)\rangle \geq 0,\ \forall v \in K,\ a.e.\ t \geq t_0 \quad (3.213)$$

■

Suppose that the assumptions of Theorem 3.80 are satisfied and denote by $x(.;t_0,x_0)$ the unique solution of Problem $P(t_0,x_0)$ in (3.205). Suppose now in addition that

$$0 \in K \quad (3.214)$$

and

$$-F(0) \in N_K(0) \quad (3.215)$$

that is

$$\langle F(0), h\rangle \geq 0, \forall\, h \in K$$

Then

$$x(t;t_0,0) = 0,\ t \geq t_0$$

i.e. the trivial solution 0 is the unique solution of problem $P(t_0,0)$.

Lyapunov Stability

Definition 3.81. *The equilibrium point $x = 0$ is said to be stable in the sense of Lyapunov if for every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that for any $x_0 \in K$ with $\|x_0\| \leq \eta$ the solution $x(\cdot; t_0, x_0)$ of problem $P(t_0, x_0)$ satisfies $\|x(t; t_0, x_0)\| \leq \varepsilon$, $\forall t \geq t_0$.*

Definition 3.82. *The equilibrium point $x = 0$ is asymptotically stable if it is stable and there exists $\delta > 0$ such that for any $x_0 \in K$ with $\|x_0\| \leq \delta$ the solution $x(\cdot; t_0, x_0)$ of problem $P(t_0, x_0)$ fulfills $\lim_{t \to +\infty} \|x(t; t_0, x_0)\| = 0$.*

We now give two Theorems inspired from [170] that guarantee that the fixed point of the systems is Lyapunov stable.

Theorem 3.83. *[167] Suppose that the assumptions of Theorem 3.80 together with the condition (3.215) hold. Suppose that there exist $\sigma > 0$ and $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that*

(1)
$$V(x) \geq a(\|x\|),\ x \in K,\ \|x\| \leq \sigma$$
with $a : [0, \sigma] \to \mathbb{R}$ satisfying $a(t) > 0$, $\forall t \in (0, \sigma)$

(2) $V(0) = 0$

(3) $x - V'(x) \in K,\ x \in \partial K,\ \|x\| \leq \sigma$

(4) $\langle Ax + F(x), V'(x) \rangle \geq 0,\ x \in K,\ \|x\| \leq \sigma$

Then the trivial solution of (3.212) and (3.213) is stable. ■

Theorem 3.84. *[167] Suppose that the assumptions of Theorem 3.80 together with the condition (3.215) hold. Suppose that there exist $\lambda > 0$, $\sigma > 0$ and $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that*

(1)
$$V(x) \geq a(\|x\|),\ \text{for all}\ x \in K,\ \|x\| \leq \sigma$$
with $a : [0, \sigma] \to \mathbb{R}$ satisfying $a(t) \geq ct^\tau$, $\forall t \in [0, \sigma]$, for some constants $c > 0$, $\tau > 0$

(2) $V(0) = 0$

(3) $x - V'(x) \in K,\ \text{for all}\ x \in \partial K,\ \|x\| \leq \sigma$

(4) $\langle Ax + F(x), V'(x) \rangle \geq \lambda V(x),\ \text{for all}\ x \in K,\ \|x\| \leq \sigma$

Then the trivial solution of (3.212) and (3.213) is asymptotically stable. ■

Copositive Matrices on a Set

We shall also need the definition of a number of sets of matrices.

Definition 3.85. *[167] The matrix $A \in \mathbb{R}^{n \times n}$ is Lyapunov positive stable on K if there exists a matrix $P \in \mathbb{R}^{n \times n}$ such that*

(1) $\inf_{x\in K\setminus\{0\}} \frac{\langle Px,x\rangle}{\|x\|^2} > 0$

(2) $\langle Ax, [P+P^T]x\rangle \geq 0, \forall x \in K$

(3) $x \in \partial K \Rightarrow [I - [P+P^T]]x \in K$ ■

Definition 3.86. *[167] The matrix $A \in \mathbb{R}^{n\times n}$ is Lyapunov positive strictly-stable on K if there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that*

(1) $\inf_{x\in K\setminus\{0\}} \frac{\langle Px,x\rangle}{\|x\|^2} > 0$

(2) $\inf_{x\in K\setminus\{0\}} \frac{\langle Ax,[P+P^T]x\rangle}{\|x\|^2} > 0$

(3) $x \in \partial K \Rightarrow [I - [P+P^T]]x \in K$ ■

Remark 3.87. Condition (1) of Definitions 3.85 and 3.86 is equivalent to the existence of a constant $c > 0$ such that

$$\langle Px, x\rangle \geq c \parallel x \parallel^2, \ \forall\ x\ \in\ K \tag{3.216}$$

Indeed, set

$$C \stackrel{\Delta}{=} \inf_{x\in K\setminus\{0\}} \frac{\langle Px,x\rangle}{\|x\|^2}$$

If $+\infty > C > 0$ then it is clear that (3.216) holds with $c = C$. If $C = +\infty$ then necessarily $K = \{0\}$ and the relation in (3.216) is trivial. On the other hand, it is clear that if (3.216) holds then $C \geq c > 0$.

Recall that a matrix $P \in \mathbb{R}^{n\times n}$ is said to be copositive on K if

$$\langle Px, x\rangle \geq 0, \ \forall\ x\ \in\ K$$

A matrix $P \in \mathbb{R}^{n\times n}$ is said to be strictly copositive on K if

$$\langle Px, x\rangle > 0, \ \forall\ x\ \in\ K\setminus\{0\}$$

These classes of matrices play an important role in complementarity theory (see *e.g.* [137, 367]). The set of copositive matrices contains that of positive semi definite (PSD) matrices [367, p.174]. Indeed a PSD matrix is necessarily copositive on any set K. However it is easy to construct a matrix that is copositive on a certain set K, but which is not PSD.

Let us here denote by $\mathcal{P}_K$ (resp. $\mathcal{P}_K^+$) the set of copositive (resp. strictly copositive) matrices on K. Let us also denote by $\mathcal{P}_K^{++}$ the set of matrices satisfying condition (1) of Definition 3.85, that is

$$\mathcal{P}_K^{++} = \big\{B \in \mathbb{R}^{n\times n} : \inf_{x\in K\setminus\{0\}} \frac{\langle Bx,x\rangle}{\|x\|^2} > 0\big\}$$

It is clear that

$$\mathcal{P}_K^{++} \subset \mathcal{P}_K^{+} \subset \mathcal{P}_K$$
$$K_1 \subset K_2 \Rightarrow \mathcal{P}_{K_2}^{++} \subset \mathcal{P}_{K_1}^{++}$$

Let us now denote by $\mathcal{L}_K$ the set of Lyapunov positive stable matrices on K and by $\mathcal{L}_K^{++}$ the set of Lyapunov positive strictly-stable matrices on K. We see that

$$\mathcal{L}_K = \big\{A \in \mathbb{R}^{n\times n} : \exists P \in \mathcal{P}_K^{++} \text{ such that } (I - [P + P^T])(\partial K) \subset K$$
$$\text{and } PA + A^T P \in \mathcal{P}_K\big\}$$

and

$$\mathcal{L}_K^{++} = \big\{A \in \mathbb{R}^{n\times n} : \exists P \in \mathcal{P}_K^{++} \text{ such that } (I - [P + P^T])(\partial K) \subset K$$
$$\text{and } PA + A^T P \in \mathcal{P}_K^{++}\big\}$$

Let us note that P needs not be symmetric. In summary, the classical positive definite property of the solutions of the Lyapunov matrix inequality, is replaced by the copositive definite property.

PR Evolution Variational Inequalities

To see how evolution variational inequalities are related to the systems in the foregoing section, let us come back to the system in (3.196):

$$\begin{cases} \dot{x}(t) \overset{\text{a.e.}}{=} Ax(t) - By_L(t) \\ y(t) = Cx(t) \\ y_L \in \partial\varphi(y) \end{cases} \tag{3.217}$$

and let us assume that the convex function $\varphi(y)$ is the indicator of a closed convex set $K \subset I\!R^n$ with $0 \in K$. We therefore rewrite the problem as:

Find $x \in C^0([0,\infty); \mathbb{R}^n)$ such that $\frac{dx}{dt} \in \mathcal{L}_{\infty,e}(0,+\infty;\mathbb{R}^n)$ and

$$\frac{dx}{dt}(t) = Ax(t) - By_L(t), \text{ a.e. } t \geq 0 \tag{3.218}$$

$$y(t) = Cx(t) \tag{3.219}$$

$$y(t) \in K \tag{3.220}$$

$$y_L(t) \in \partial\psi_K(y(t)) \tag{3.221}$$

$$x(0) = x_0 \tag{3.222}$$

Assume there exists a symmetric and invertible matrix $R \in \mathbb{R}^{n\times n}$ such that $R^{-2}C^T = B$. Suppose also that there exists

$$y_0 \stackrel{\Delta}{=} CR^{-1}x_0 \in \text{ Int}(K). \tag{3.223}$$

Then using the change of state vector $z = Rx$ and setting

$$\bar{K} = \{h \in \mathbb{R}^n : CR^{-1}h \in K\} \tag{3.224}$$

we see that problem (3.218) to (3.222) is equivalent to the following one: find $z \in C^0([0,\infty);\mathbb{R}^n)$ such that $\frac{dz}{dt} \in \mathcal{L}_{\infty,e}([0,\infty);\mathbb{R}^n)$ and

$$\langle \frac{dz}{dt}(t) - RAR^{-1}z(t), v - z(t)\rangle \geq 0, \forall v \in \bar{K}, \text{ a.e. } t \geq 0 \tag{3.225}$$

$$z(t) \in \bar{K}, t \geq 0$$

$$z(0) = Rx_0$$

Indeed, it suffices to remark that

$$Cx \in K \Leftrightarrow z \in \bar{K}$$

$$x(0) = x_0 \Leftrightarrow z(0) = Rx_0$$

and

$$\frac{dx}{dt} \in Ax - B\partial\psi_K(Cx) \Leftrightarrow R\frac{dx}{dt} \in RAR^{-1}Rx - RB\partial\psi_K(CR^{-1}Rx)$$

$$\Leftrightarrow \frac{dz}{dt} \in RAR^{-1}z - R^{-1}R^2B\partial\psi_K(CR^{-1}z)$$

$$\Leftrightarrow \frac{dz}{dt} \in RAR^{-1}z - R^{-1}C^T\partial\psi_K(CR^{-1}z)$$

$$\Leftrightarrow \frac{dz}{dt} \in RAR^{-1}z - \partial\psi_{\bar{K}}(z)$$

Indeed, $\psi_{\bar{K}}(z) = (\psi_K \circ CR^{-1})(z)$ and thanks to (3.223) we obtain $\partial\psi_{\bar{K}}(z) = R^{-1}C^T\partial\psi_K(CR^{-1}z)$. We remark also that the set $\bar{K}$ is closed convex with $0 \in \bar{K}$. The variable change $z = Rx$ is exactly the same as the variable change used in Lemma 3.76. The following holds:

Lemma 3.88. *[167] Let $K \subset \mathbb{R}^n$ be a closed convex set containing $x = 0$, and satisfying the condition (3.223). Define $\bar{K}$ as in (3.224). Suppose that there exists a symmetric and invertible matrix $R \in \mathbb{R}^{n\times n}$ such that $R^{-2}C^T = B$.*

i) If $-RAR^{-1} \in \mathcal{L}_{\bar{K}}$ then the trivial equilibrium point of (3.218)–(3.221) is stable.

ii) If $-RAR^{-1} \in \mathcal{L}^{++}_{\bar{K}}$ then then the trivial equilibrium point of (3.218)–(3.221) is asymptotically stable. ■

Example 3.89. **Positive real evolution variational inequalities** Assume that $G(s) = C(sI - A)^{-1}B$, with (A, B, C) a minimal representation, is strictly positive real. From the Kalman-Yakubovitch-Popov Lemma there exist $P = P^T$ positive definite and $Q = Q^T$ positive definite such that $PA + A^T P = -Q$ and $B^T P = C$. Choosing R as the symmetric square root of P, i.e. $R = R^T$, R positive definite and $R^2 = P$, we see that $B^T R^2 = C$ and thus $R^{-2}C^T = B$. Moreover

$$\langle PAx, x\rangle + \langle A^T Px, x\rangle = -\langle Qx, x\rangle, \forall x \in \mathbb{R}^n \tag{3.226}$$

Thus

$$\langle Ax, Px\rangle = -\frac{1}{2}\langle Qx, x\rangle, \forall x \in \mathbb{R}^n \tag{3.227}$$

It results that

$$-\langle RAx, Rx\rangle > 0, \forall x \in \mathbb{R}^n \setminus \{0\} \tag{3.228}$$

Setting $z = Rx$, we see that

$$-\langle RAR^{-1}z, z\rangle > 0, \forall z \in \mathbb{R}^n \setminus \{0\} \tag{3.229}$$

So $-RAR^{-1} \in \mathcal{P}^{++}_{\mathbb{R}^n} \subset \mathcal{P}^{++}_{\bar{K}} \subset \mathcal{L}^{++}_{\bar{K}}$. All the conditions of Lemma 3.88 (part ii)) are satisfied and the trivial solution of (3.218)–(3.221) is asymptotically stable. The results presented in the foregoing section are here recovered. In case $G(s)$ is positive real then Lemma 3.88 (part i)) applies. As shown above (see Lemma 3.78) the equilibrium point is unique in this case.

Example 3.90. **PR electrical circuit** The following example is taken from [82].

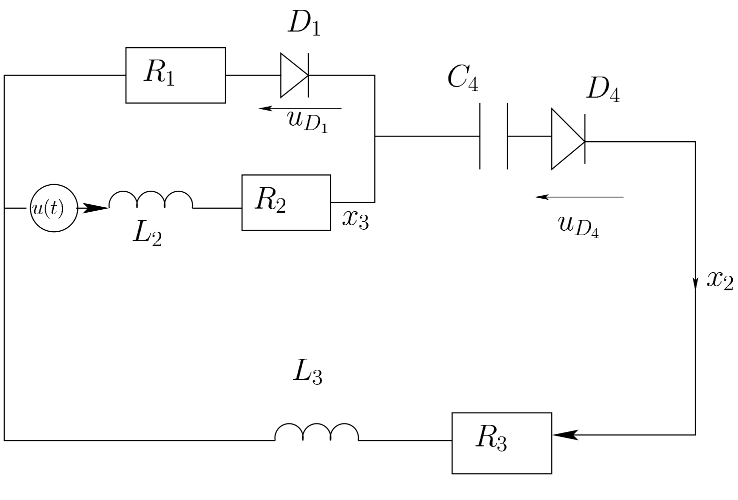

Fig. 3.9. A circuit with ideal diodes

Let us consider the circuit in Figure 3.9 ($R_1, R_2, R_3 \geq 0$, $L_2, L_3 > 0$). One has $0 \leq -u_{D_4} \perp x_2 \geq 0$ and $0 \leq -u_{D_1} \perp -x_3 + x_2 \geq 0$, where u_{D_4} and u_{D_1} are the voltages of the diodes. The dynamical equations are

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\left(\frac{R_1+R_3}{L_3}\right) x_2(t) + \frac{R_1}{L_3} x_3(t) - \frac{1}{L_3 C_4} x_1(t) + \frac{1}{L_3}\lambda_1(t) + \frac{1}{L_3}\lambda_2(t) \\ \dot{x}_3(t) = -\left(\frac{R_1+R_2}{L_2}\right) x_3(t) + \frac{R_1}{L_2} x_2(t) - \frac{1}{L_2}\lambda_1(t) \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} -x_3(t) + x_2(t) \\ x_2(t) \end{pmatrix} \geq 0 \end{cases} \tag{3.230}$$

where $x_1(\cdot)$ is the time integral of the current across the capacitor, $x_2(\cdot)$ is the current across the capacitor, and $x_3(\cdot)$ is the current across the inductor L_2 and resistor R_2, $-\lambda_1$ is the voltage of the diode D_1 and $-\lambda_2$ is the voltage of the diode D_4. The system in (3.230) can be written compactly as the LCS: $\dot{x}(t) = Ax(t) + B\lambda(t)$, $0 \leq \lambda(t) \perp y(t) = Cx(t) \geq 0$, with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{L_3 C_4} & -\frac{R_1+R_3}{L_3} & \frac{R_1}{L_3} \\ 0 & \frac{R_1}{L_2} & -\frac{R_1+R_2}{L_2} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ \frac{1}{L_3} & \frac{1}{L_3} \\ -\frac{1}{L_2} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

The monotonicity (consequently the passivity) of the voltage-current relation $0 \leq u \perp i \geq 0$ at the poles of the diodes is certainly an essential property both for existence and uniqueness of solutions, and for stability. We recall that this relation is a multivalued mapping whose graph is as in Figure 3.8 (c). We set

$$P = \begin{pmatrix} \frac{1}{C_4} & 0 & 0 \\ 0 & L_3 & 0 \\ 0 & 0 & L_2 \end{pmatrix}$$

It is clear that P is symmetric and positive definite. Moreover, we see that $A^T P + PA = -Q$ with

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2(R_1+R_3) & -2R_1 \\ 0 & -2R_1 & 2(R_1+R_2) \end{pmatrix}$$

The matrix $Q \in I\!R^{3\times 3}$ is symmetric and positive semi-definite. Moreover, $PB = C^T$ and the system in (3.230) is positive real, as expected from the physics. We deduce that (3.230) can be rewritten as an evolution variational inequality

$$\begin{cases} \langle \frac{dz}{dt}(t) - RAR^{-1}z(t), v - z(t)\rangle \geq 0, \forall v \in \bar{K}, \text{ a.e. } t \geq 0 \\ \\ z(t) \in \bar{K}, t \geq 0 \end{cases} \tag{3.231}$$

where $z = Rx$, R is a symmetric positive definite square root of P and $\bar{K} = \{h \in \mathbb{R}^n : CR^{-1}h \in K\}$. The change of state matrix R and the new state vector z are easily calculated ($z_1 = \frac{1}{\sqrt{C_4}}x_1, z_2 = \sqrt{L_3}x_2, z_3 = \sqrt{L_2}x_3$). ■

It follows from the above that an extension of the KYP Lemma matrix inequalities to linear evolution variational inequalities is possible at the price of replacing positive definiteness by copositive definiteness of matrices. However what remains unclear is the link with frequency-domain conditions. In other words, we have shown that if the triple (A, B, C) is PR (or SPR), then it satisfies the requirements for the evolution variational inequality in (3.225) to possess a Lyapunov stable equilibrium. Is the converse provable? Certainly the answer is negative, as some examples show that the matrix A can be unstable (with eigenvalues with positive real parts) while $A \in \mathcal{L}_K^{++}$ (thus the corresponding evolution variational inequality has an asymptotically stable fixed point). Extension of the Krasovskii-LaSalle invariance principle to evolution variational inequalities, has been considered in [82]. In Chapter 6, we shall examine second order evolution variational inequalities, which arise in some problems of mechanics with nonsmooth contact laws.

3.10 The Circle Criterion

Let us come back to the Lur'e problem with single-valued nonlinearities in the feedback loop. Consider the observable and controllable system in (3.181). Its transfer function $H(s)$ is

$$H(s) = C(sI_n - A)^{-1}B + D \tag{3.232}$$

Assume that the transfer function $H(s)$ is SPR and is connected in negative feedback with a nonlinearity $\phi(\cdot)$ as illustrated in Figure 3.10. The conditions for stability of such a scheme are stated in the following Theorem.

Theorem 3.91. *Consider the system in Figure 3.10. If $H(s)$ in (3.232) is SPR, the conditions of Lemma 3.59 are satisfied and if $\phi(t, y)$ is in the sector $[0, \infty)$, i.e.:*

i) $\phi(t, 0) = 0 \qquad \forall\, t \geq 0$

ii) $y^T\phi(t, y) \geq 0 \qquad \forall t \geq 0, \qquad \forall\, y \in \mathbb{R}^m$

then the origin is a globally exponentially stable equilibrium point. ■

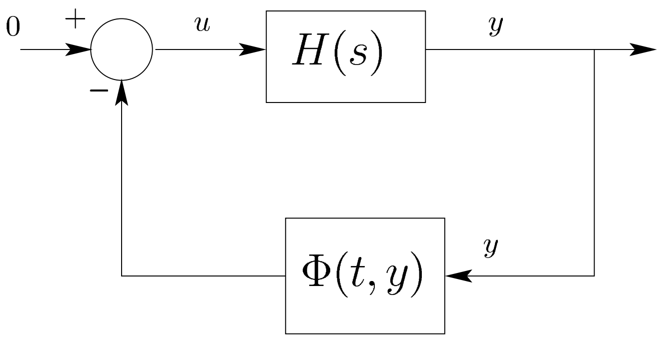

Fig. 3.10. Linear system with a sector nonlinearity in negative feedback

Proof: Since $H(s) = C(sI - A)^{-1}B + D$ is SPR, then there exist $P > 0$, Q and W, $\epsilon > 0$ such that

$$\begin{cases} A^TP + PA = -\epsilon P - Q^TQ \\ B^TP + W^TQ = C \\ W^TW = D + D^T \end{cases} \tag{3.233}$$

Define the Lyapunov function candidate $V(x) = x^TPx$. Then

$$\begin{aligned}\dot{V}(x(t)) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\ &= \left[Ax(t) - B\phi(t, y(t))\right]^T Px(t) + x^T(t)P\left[Ax(t) - B\phi(t, y(t))\right] \\ &= x^T(t)\left(A^TP + PA\right)x(t) - \phi^T(t, y(t))B^TPx(t) - x^T(t)PB\phi(t, y(t))\end{aligned} \tag{3.234}$$

Note that $B^TP = C - W^TQ$. Hence, using the above, (3.181) and the control $u = -\phi(t, y)$, we get

$$\begin{aligned}x^T(t)PB\phi(t, y(t)) &= \phi^T(t, y(t))B^TPx(t) \\ &= \phi^T(t, y(t))Cx(t) - \phi^T(t, y(t))W^TQx(t) \\ &= \phi^T(t, y(t))\left[y(t) - Du(t)\right] - \phi^T(t, y(t))W^TQx(t) \\ &= \phi^T(t, y(t))\left[y(t) + D\phi(t, y(t))\right] - \phi^T(t, y(t))W^TQx(t).\end{aligned}$$

Substituting the above into (3.234) we get

$$\begin{aligned}\dot{V}(x(t)) = &-\epsilon x^T(t)Px(t) - x^T(t)Q^TQx(t) - \phi^T(t)\left(D + D^T\right)\phi(t) \\ &-\phi^T(t)W^TQx(t) - x^T(t)Q^TW\phi(t) - \phi^T(t)y(t) - y^T(t)\phi(t)\end{aligned}$$

Using (3.233) and the fact that $y^T\phi \geq 0$ we have

$$\begin{aligned}\dot{V}(x(t)) &\leq -\epsilon x^T(t)Px(t) - x^T(t)Q^TQx(t) - \phi^T(t,y(t))W^TW\phi(t,y(t)) - \\ &-\phi(t,y(t))^TW^TQx - x^TQ^TW\phi(t,y(t)) \\ &= -\epsilon x^T(t)Px(t) - \left[Qx(t) + W\phi(t,y(t))\right]^T\left[Qx(t) + W\phi(t,y(t))\right] \\ &\leq -\epsilon x^T(t)Px(t)\end{aligned}$$

Define $\bar{z}(t) \overset{\Delta}{=} -\left[Qx(t) + W\phi(t,y(t))\right]^T\left[Qx(t) + W\phi(t,y(t))\right]$ which can also be rewritten as $\dot{V}(x(t)) = -\epsilon V(xt)) + \bar{z}(t)$

Thus

$$\begin{aligned}V(x(t)) &= e^{-\epsilon t}V(0) + \int_0^t e^{-\epsilon(t-\tau)}\bar{z}(\tau)\,d\tau \\ &\leq e^{-\epsilon t}V(0)\end{aligned}$$

Finally the fixed point $x = 0$ is globally exponentially stable. ■

3.10.1 Loop Transformations

The above theorem applies when $\phi(\cdot,\cdot)$ belongs to the sector $[0,\infty)$. In order to use the above result when $\phi(\cdot,\cdot)$ belongs to the sector $[a,b]$ we have to make some loop transformations which are given next.

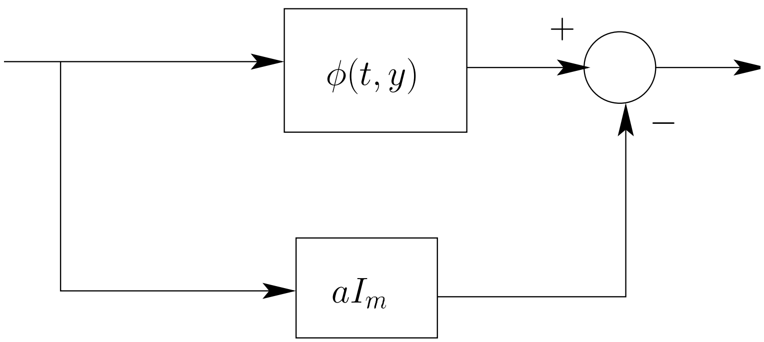

Fig. 3.11. Loop transformations

1) If $\phi(\cdot,\cdot)$ belongs to the sector $[a,b]$ then $\phi_1 \overset{\Delta}{=} \phi(t,y) - a$ belongs to the sector $[0, b-a]$. This is illustrated in Figure 3.11.

2) If $\phi_1(\cdot,\cdot)$ belongs to the sector $[0,c]$ with $c = b - a$ then we can make the transformation indicated in Figure 3.12 where $\bar{y} = \phi_2(t,\bar{u})$ and $\delta > 0$ is arbitrarily small number. Therefore, as is shown next, $\phi_2(\cdot)$ belongs to the sector $[0,\infty)$.

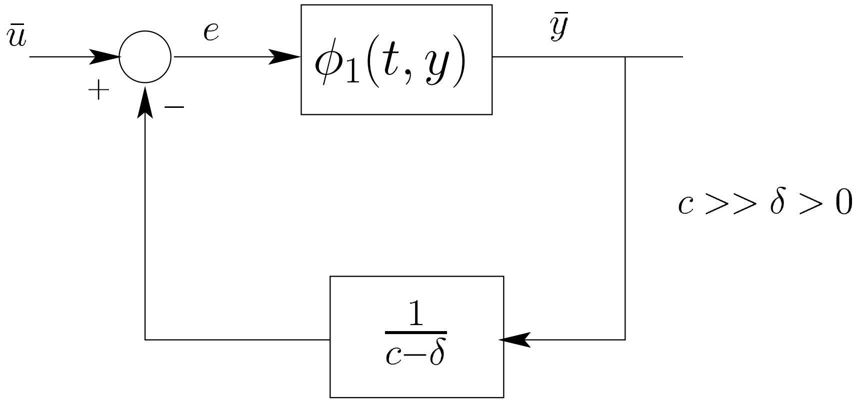

Fig. 3.12. Loop transformations

Note that if $\phi_1 = \bar{c}$, then

$$\bar{y} = \frac{\bar{c}}{1 - \frac{\bar{c}}{c-\delta}} = \frac{\bar{c}\,(c-\delta)}{c - \bar{c} - \delta}\bar{u}$$

Therefore:

1. if $\bar{c} = c$, $\quad \lim_{\delta \to 0} \frac{\bar{y}}{\bar{u}} = \infty$
2. if $\bar{c} = 0$, $\quad \frac{\bar{y}}{\bar{u}} = 0$

Using the two transformations described above, the system in Figure 3.10 can be transformed into the system in Figure 3.13. We then have the following corollary:

Corollary 3.92. *If H_2 in Figure 3.13 is SPR and the nonlinearity $\phi(\cdot,\cdot)$ belongs to the sector $[0,\infty)$ then the closed-loop system is globally exponentially stable.* ■

Note that H_2 is SPR if and only if

$$H_1(j\omega) + H_1^*(j\omega) + \frac{2I}{b-a-\delta} > 0$$

with $H_1(s) = H(s)\left[I + aH(s)\right]^{-1}$ and $\delta << 1$. For $m = 1$ the above result has a graphical interpretation which leads to the **circle criterion**. Suppose

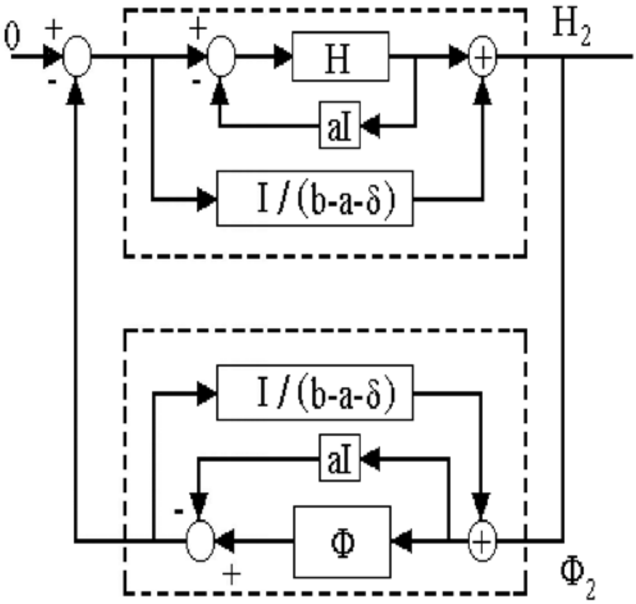

Fig. 3.13. Loop transformations

$z = x + jy$ is a complex number and $a, b \in I\!R$ with $a < b$, $a \neq 0$. Consider the condition

$$\eta = \mathbf{Re}\left\{\frac{z}{1+az} + \frac{1}{b-a}\right\} > 0$$

Now one has

$$\frac{z}{1+az} + \frac{1}{b-a} = \frac{x+jy}{1+a(x+jy)} + \frac{1}{b-a}$$

$$= \frac{x+jy[1+ax-jay]}{(1+ax)^2+y^2a^2} + \frac{1}{b-a}$$

Therefore

$$\eta = \frac{x(1+ax)+ay^2}{(1+ax)^2+y^2a^2} + \frac{1}{b-a} > 0$$

or equivalently

$$\begin{aligned} 0 &< (b-a)\left\{x(1+ax)+ay^2\right\} + (1+ax)^2 + y^2a^2 \\ &= (b-a)\left\{x+ax^2+ay^2\right\} + 1 + 2ax + a^2x^2 + a^2y^2 \\ &= ba\left\{x^2+y^2\right\} + x(b+a) + 1 \end{aligned} \tag{3.235}$$

which implies

$$bay^2 + ba\left(x + \frac{a+b}{2ab}\right)^2 + 1 - \frac{(a+b)^2}{4ab} > 0$$

Note that

$$1 - \frac{(a+b)^2}{4a^2b^2} = \frac{4ab - a^2 - 2ab - b^2}{4ab} = -\frac{(a-b)^2}{4ab}$$

Introducing the above into (3.235) we get

$$bay^2 + ba\left(x + \frac{a+b}{2ab}\right)^2 > \frac{(a-b)^2}{4ab}$$

If $ab > 0$ this can be written as

$$y^2 + ba\left(x + \frac{a+b}{2ab}\right)^2 > \frac{(a-b)^2}{4a^2b^2}$$

or

$$\left|z + \frac{a+b}{2ab}\right| > \frac{|a-b|}{2\,|ab|}$$

If $ab < 0$ then

$$\left|z + \frac{a+b}{2ab}\right| < \frac{|a-b|}{2\,|ab|}$$

Let $D(a,b)$ denote the closed disc in the complex plane centered at $\frac{a+b}{2ab}$ and with radius $\frac{|a-b|}{2|ab|}$. Then

$$\mathbf{Re}\left\{\frac{z}{1+az} + \frac{1}{b-a}\right\} > 0$$

if and only if

$$\left|z + \frac{a+b}{2ab}\right| > \frac{|a-b|}{2\,|ab|}, \qquad ab > 0$$

In other words, the complex number z lies outside the disc $D(a,b)$ in case $ab > 0$ and lies in the interior of the disc $D(a,b)$ in case $ab < 0$. We therefore have the following important result.

Theorem 3.93 (Circle criterion). *Consider again the system for m=1 in Figure 3.13. The closed loop system is globally exponentially stable if:*

(i) $0 < a < b$ *: The plot of* $h(j\omega)$ *lies outside and is bounded away from the disc* $D(a,b)$*. Moreover the plot encircles* $D(a,b)$ *exactly* ν *times in the counter-clockwise direction, where* ν *is the number of eigenvalues of* A *with positive real part.*

(ii) $0 = a < b$ *:* A *is a Hurwitz matrix and*

$$\mathbf{Re}\left\{H(j\omega) + \frac{1}{b}\right\} > 0 \tag{3.236}$$

(iii) $a < 0 < b$ *:* A *is a Hurwitz matrix; the plot of* $h(j\omega)$ *lies in the interior of the disc* $D(a,b)$ *and is bounded away from the circumference of* $D(a,b)$*.*

(iv) $a < b \leq 0$ *: Replace* $h(.)$ *by* $-h(.)$*,* a *by* $-b$*,* b *by* $-a$ *and apply (i) or (ii) as appropriate.*

Remark 3.94. If $b - a \to 0$ the "critical disc" $D(a,b)$ in case (i) shrinks to the "critical point"0 $-1/a$ of the Nyquist criterion. The circle criterion is applicable to time-varying and/or nonlinear systems, whereas the Nyquist criterion is only applicable to linear time invariant systems.

A generalization of the circle criterion for the design of a finite-dimensional controller for unstable infinite-dimensional systems, has been proposed in [509]. The case of an infinite-dimensional linear system, illustrated by an electrical transmission line, is considered in [172].

3.11 The Popov Criterion

Unlike the circle criterion, the Popov criterion [406–408] is applicable only to autonomous single input-single output (SISO) systems:

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ \dot{\xi}(t) = u(t) \\ y(t) = cx(t) + d\xi(t) \\ u(t) = -\phi(y(t)) \end{cases}$$

where $u(t), y(t) \in I\!R$, $\phi : I\!R \to I\!R$ is a time-invariant nonlinearity belonging to the open sector $(0, \infty)$, *i.e.*

$$\phi(0) = 0,\ y\phi(y) > 0,\ \ \forall y \neq 0$$

The linear part can also be written as:

$$\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} b \\ 1 \end{bmatrix} u \\
y(t) &= \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}
\end{aligned} \tag{3.237}$$

Hence the transfer function is

$$h(s) = \frac{d}{s} + c(sI - A)^{-1}b$$

which has a pole at the origin. We can now state the following result:

Theorem 3.95 (Popov's criterion). *Consider the system in (3.237). Assume that*

1. *A is Hurwitz*
2. *(A, b) is controllable*
3. *(c, A) is observable*
4. *$d > 0$*
5. *$\phi(\cdot)$ belongs to the sector $(0, \infty)$*

Then the system is globally asymptotically stable if there exixts $r > 0$ such that $\mathbf{Re}[(1 + j\omega r)h(j\omega)] > 0, \quad \forall \ \omega \in I\!R$.

Remark 3.96. Contrary to Popov's criterion, the circle criterion does not apply to systems with a pole at $s = 0$ and $\phi(\cdot)$ belongs to the sector $(0, \infty)$.

Proof of Popov's criterion: Note that

$$\begin{aligned}
s(sI - A)^{-1} &= (sI - A + A)(sI - A)^{-1} \\
&= I + A(sI - A)^{-1}
\end{aligned}$$

Hence

$$\begin{aligned}
(1 + rs)h(s) &= (1 + rs)\left[\tfrac{d}{s} + c\,(sI - A)^{-1}\,b\right] \\
&= \tfrac{d}{s} + rd + c(sI - A)^{-1}b \\
&\quad + rcb + rcA(sI - A)^{-1}b
\end{aligned}$$

Note that $\frac{d}{j\omega}$ is purely imaginary. From the above and by assumption we have

$$\mathbf{Re}\left[(1+j\omega r)h(j\omega)\right] = \mathbf{Re}\left[r\left(d+cb\right)+c\left(I+rA\right)\left(j\omega-A\right)^{-1}b\right] > 0$$

Define the transfer function

$$g\left(s\right) = r\left(d+cb\right)+c\left(I+rA\right)\left(sI-A\right)^{-1}b$$

i.e. $\{A, b, c\left(I+rA\right), r\left(d+cb\right)\}$ is a minimal realization of $g(s)$. If $\mathbf{Re}\left[g(\omega)\right] > 0$ then there exists $P > 0, q$ and ω and $\epsilon > 0$ such that

$$\begin{cases} A^T P + PA = -\epsilon P - q^T q \\ b^T P + \omega q \quad = c(I+rA) \\ \omega^2 \qquad\qquad = 2r(d+cb) \end{cases}$$

Choose the Lyapunov function candidate

$$V(x,\xi) = x^T Px + d\,\xi^2 + 2r\int_0^y \phi\left(\sigma\right)d\sigma$$

Given that $\phi(\cdot)$ belongs to the sector $[0,\infty)$ it then follows that $\int \phi\left(\sigma\right)d\sigma \geq 0$. Hence $V(x,\xi)$ is positive definite and radially unbounded

$$\begin{aligned}\dot{V}(x,\xi) &= \dot{x}^T\ Px + x^T P\dot{x} + 2d\xi\dot{\xi} + 2r\phi(y)\dot{y} \\ &= (Ax-b\phi)^T Px + x^T P(Ax-b\phi) - \\ &\quad -2d\xi\phi + 2r\phi\left[c\left(Ax-b\phi\right)-d\phi\right]\end{aligned}$$

Note from (3.237) that $d\xi = y - cx$, thus

$$\begin{aligned}\dot{V}(x(t),\xi(t)) &= x^T(t)(A^T P + PA)x(t) - 2\phi(y(t))b^T Px(t) + \\ &\quad +2\phi(y(t))c(I+rA)x(t) - 2r\left(d+cb\right)\phi^2(y(t)) - 2y(t)\phi(y(t)) \\ &= -\epsilon x^T(t)Px(t) - (qx(t)-\omega\phi(y(t)))^2 - \\ &\quad -r\left(d+cb\right)\phi^2(y(t)) - 2y(t)\phi(y(t))\end{aligned}$$

Since $g(j\omega) \to r(d+cb)$ as $\omega \to \infty$ it follows that $r(d+cb) > 0$. Hence

$$\dot{V}(x(t),\xi(t)) \leq -\epsilon x^T(t)Px(t) - 2y(t)\phi(y(t)) \leq 0, \quad \forall\, x \in I\!R^n, \forall\, \epsilon > 0$$

We now show that $\dot{V}(x,\xi) < 0$ if $(x,\xi) \neq (0,0)$. If $x \neq 0$ then $\dot{V}(x,\xi) < 0$ since $P > 0$. If $x = 0$ but $\xi \neq 0$, then $y = d\xi \neq 0$, and $\phi y > 0$ since $\phi(\cdot)$ belongs to the sector $[0,\infty)$. Therefore the system (3.237) is globally asymptotically stable. ■

Corollary 3.97. *Suppose now that $\phi(\cdot)$ belongs to the sector $(0, k)\,, k > 0$. Then the system is globally asymptotically stable if there exists $r > 0$ such that*

$$\mathbf{Re}\left[(1 + j\omega r)h(j\omega)\right] + \frac{1}{k} > 0 \tag{3.238}$$

■

Proof: It follows from the loop transformation in Figure 3.14, where

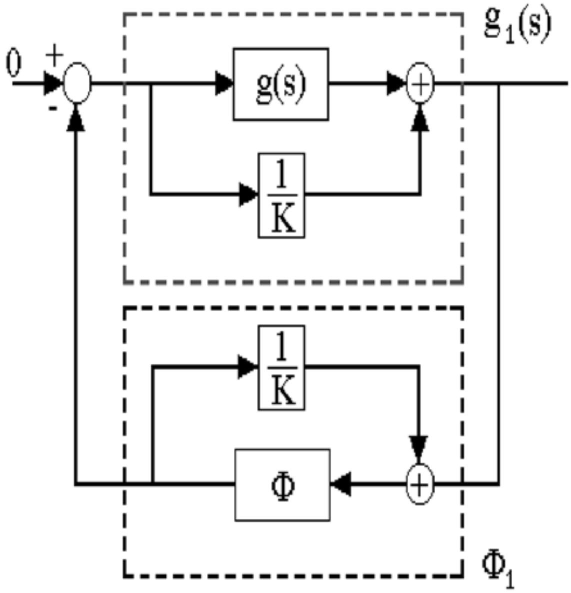

Fig. 3.14. Loop transformations

$$\begin{cases} \phi_1 & = \phi\left[I - \frac{1}{k}\phi\right]^{-1} \\ g_1 & = g(s) + \frac{1}{k} \\ & = (1 + j\omega r)(hj\omega) + \frac{1}{k} \\ \mathbf{Re}(g_1) & = \mathbf{Re}[h(j\omega)] + r\omega\mathbf{Im}[h(j\omega)] + \frac{1}{k} > 0. \end{cases}$$

■

Remark 3.98. The circle and the Popov's criteria owe their great success to the fact that they lend themselves to graphical interpretations as pointed out above for the circle criterion. Consider for instance the inequality in (3.238). Consider the function $M(j\omega) = \mathbf{Re}[h(j\omega)] + j\omega\mathbf{Im}[h(j\omega)]$, $\omega > 0$. Note that $\mathbf{Re}[(1+j\omega r)h(j\omega)] = \mathbf{Re}[h(j\omega)] - r\omega\mathbf{Im}[h(j\omega)] = \mathbf{Re}[M(j\omega)] - r\mathbf{Im}[M(j\omega))]$. Then condition (3.238) means that there must exist a straight line with an arbitrary, fixed slope, passing through the point $\left(-\frac{1}{k}, 0\right)$ in the complex plane, such that the plot of $M(j\omega)$ lies to the right of this line. The slope of this line which is tangent to the plot of $M(j\omega)$ is equal to $\frac{1}{r}$. The line is usually called the *Popov's line*. In the multivariable case the graphical interpretation becomes too complex to remain interesting; see [417].

Further reading: The circle criterion has been introduced in [431, 532, 533] and generalized after. Further results on the absolute stability problem and Popov's criterion, can be found in [56, 102, 136, 166, 181, 182, 196, 200, 212, 217, 218, 220, 221, 258, 265, 293, 335, 336, 369, 382, 395, 456, 487, 503, 538]. These references constitute only a few of all the works that have been published on the topic. The reader is also referred to Section 5.10 on hyperstability. It is also worth reading the European Journal of Control special issue dedicated to V.M. Popov [134]. Generalization of the Popov criterion with Popov multipliers can be found in [48, 190, 244]. An interesting comparative study between the cicle criterion, the Popov criterion, and the small gain Theorem, has been led in [193] on a 4th order spring-mass system with uncertain stiffness. The result in terms of conservativeness is that the Popov criterion design supersedes the circle criterion design and that the small gain design is the most conservative one.

3.12 Discrete-time Systems

3.12.1 The KYP Lemma

In this section we investigate how the KYP Lemma may be extended to discrete-time systems of the following form:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \\ y(k) = Cx(k) + Du(k) \end{cases} \tag{3.239}$$

with $x(k) \in I\!R^n$, $u(k) \in I\!R^m$, $y(k) \in I\!R^m$. The KYP Lemma for systems as (3.239) is due to [211, 483].

Definition 3.99. *A discrete transfer matrix $H(z)$ is positive real if*

- *$H(z)$ has analytic elements in $|z| > 1$, $z \in \mathbb{C}$*
- *$H(z) + H^\star(z) \geq 0$ in $|z| > 1$*

A discrete transfer matrix $H(z)$ is strictly positive real if

- *$H(z)$ has analytic elements in $|z| > 1$*

- $H(e^{j\theta}) + H^\star(e^{j\theta}) > 0$ *for* $\theta \in [0, 2\pi)$

A discrete transfer matrix $H(z)$ *is strongly strictly positive real if it is SPR and* $H(\infty) + G^T(\infty) > 0$. ■

It is noteworthy that the condition $H(z) + H^\star(z) \geq 0$ in $|z| > 1$ implies that $H^T(e^{-j\theta}) + H(e^{j\theta}) \geq 0$ for all real θ such that no element of $H(z)$ has a pole at $z = e^{j\theta}$.

Lemma 3.100. *Let* $H(z) = C(zI_n - A)^{-1}B + D$ *be a square matrix of real rational functions of* z*, with no poles in* $|z| < 1$*. Let* (A, B, C, D) *be a minimal realization of* $H(z)$*. If for* (A, B, C, D) *there exist a real symmetric positive definite matrix* P *and real matrices* L *and* W *such that*

$$\begin{cases} A^TPA - P = -L^TL \\ \\ A^TPB = C^T - LW \\ \\ W^TW = D + D^T - B^TPB \end{cases} \tag{3.240}$$

then the transfer function $H(z)$ *is positive real.* ■

Similarly to their continuous-time counterpart, the KYP Lemma conditions can be written as an LMI, using for instance Proposition A.63. One immediately notices from (3.240) that necessarily $D \neq 0$, otherwise $W^TW = -B^TPB$ (and obviously we assume that $B \neq 0$). If B has full rank m, then D must have full rank m so that $D + D^T > 0$. Therefore a positive real discrete time system with full rank input matrix has a relative degree 0. Consequently in the monovariable case the relative degree is always zero. However it is worth noting that this is true for *passive* systems only, *i.e.* systems which are dissipative with respect to the supply rate $w(u, y) = u^Ty$. If a more general supply rate is used, e.g. $w(u, y) = u^TRu + 2u^TSy + y^TQy$, then the relative degree may not be zero.

When $W = 0$ and $L = 0$ in (3.240) the system is said *lossless*. Then

$$\frac{1}{2}x^T(k+1)Px(k+1) - \frac{1}{2}x^T(k)Px(k) = y^T(k)u(k) \tag{3.241}$$

for all $u(k)$ and $k \geq 0$, which in turn is equivalent to

$$\frac{1}{2}x^T(k+1)Px(k+1) - \frac{1}{2}x^T(0)Px(0) = \sum_{i=0}^{k} y^T(i)u(i) \tag{3.242}$$

for all $x(0)$ and $k \geq 0$. Let us now formulate a KYP Lemma for SPR functions.

Lemma 3.101. *[93, 250] Let* (A, B, C, D) *be a minimal realization of* $H(z)$*. The transfer matrix* $H(z)$ *is SPR if and only if there exist matrices* $P = P^T > 0$*,* L *and* W *such that*

$$\begin{cases} P = A^T PA + L^T L \\ 0 = B^T PA - C + W^T L \\ 0 = D + D^T - B^T PB - W^T W \end{cases} \tag{3.243}$$

is satisfied, the pair (A, L) is observable, and rank$(\hat{H}(z)) = m$ for $z = e^{j\omega}$, $\omega \in I\!R$, where (A, B, L, W) is a minimal realization of $\hat{H}(z)$. ■

Similarly to the continuous time case, PR systems possess stable zeroes. Let us assume that D is full rank. Then the zero dynamics is given by

$$A_0 x(k) = (A - BD^{-1}C)x(k) \tag{3.244}$$

which exactly is the dynamics on the subspace $y(k) = 0$. Then we have the following result:

Proposition 3.102. *[373] Let the system (3.239) be passive. Then the zero dynamics exists and is passive.* ■

Proof: Let us recall that passivity means that the system satisfies

$$V(x(k+1)) - V(x(k)) \le u^T(k)y(k) \tag{3.245}$$

along its trajectories, with $V(x) = \frac{1}{2}x^T Px$ and P is the solution of the KYP Lemma LMI in (3.240). One has $V(A_0 x) - V(x) = x^T Mx$, with $M = (A - BD^{-1}C)^T P(A - BD^{-1}C) - P$. If $M \le 0$ then the zero dynamics is stable. Using the second equality of the KYP Lemma conditions, one obtains

$$\begin{aligned} M = (A^T PA - P) - C^T[D^{-1} + D^{-T}]C + LWD^{-1}C+ \\ +(LWD^{-1}C)^T + C^T D^{-T} B^T PBD^{-1}C \end{aligned} \tag{3.246}$$

Using the equality $C^T D^{-T}(D^T + D)D^{-1}C = C^T[D^{-1} + D^{-T}]C$ and using the third equality of the KYP Lemma conditions(3.240), one gets

$$\begin{aligned} M &= (A^T PA - P) + LWD^{-1}C + (LWD^{-1}C)^T - (D^{-1}C)^T W^T W(D^{-1}C) \\ &= (A^T PA - P) - [L - (D^{-1}C)^T W^T][L - (D^{-1}C)^T W^T]^T + LL^T \end{aligned} \tag{3.247}$$

Injecting the first matrix equality in (3.240) one concludes that $M \le 0$. Therefore PR systems have a stable zero dynamics. ■

Positive real discrete-time transfer functions have proved to be quite useful for identification; see [277, 278, 306]. In particular the so-called Landau's scheme of recursive identification [278] is based on PRness. Further works can be found in [51,90,182,193,249,329,362,373,374,542]. Infinite dimensional discrete time systems and the KYP Lemma extension have been studied in [29]. The time-varying case received attention in [140, 141, 145]. In relation to the relative degree zero property pointed out above, let us state the following result:

Lemma 3.103. *[326] Let $H : I\!R^n \rightarrow I\!R^n$ be a linear operator (possibly time-varying and unstable). Suppose that H is strictly causal, i.e.: if $x(k) = 0$ for all $0 \leq k \leq n-1$ then $H(x(k)) = 0$ for all $0 \leq k \leq n$. Then H is passive if and only if $H = 0$.* ■

Passivity means here that $\sum_{k=0}^{n} x^T(k)H(x(k)) \geq 0$ for all $n \in I\!N$ and all real-valued sequences $\{x(k)\}_{k\geq 0}$. Applications of passivity in discrete-time systems may be found in [112] for the design of repetitive controllers and in [109] for haptic interfaces. The discrete passivity inequality has also been used in the setting of time-discretised differential inclusions where it proves to be a crucial property for the behaviour of the numerical algorithms [2] (see also [338] in the nonlinear framework of Lagrangian systems).

3.12.2 The Tsypkin Criterion

The Tsypkin criterion may be considered as the extension of Popov's and the circle criteria, for discrete time systems. It was introduced in [492–496]. For a discrete-time system of the form

$$x(k+1) = Ax(k) - B\phi(Cx, k) \tag{3.248}$$

Tsypkin proved the absolute stability (i.e. the global asymptotic stability for all $\phi(\cdot,\cdot)$ in the sector $(0,\kappa)$) if the poles of the transfer function $H(z) = C(sI_n - A)^{-1}B$ lie inside the unit disk and

$$\mathbf{Re}[H(z)] + \frac{1}{\kappa} \geq 0 \;\text{ for }\; |z| = 1 \tag{3.249}$$

This is the discrete-time analog of the circle criterion. When $\phi(\cdot)$ is time invariant and monotone, absolute stability holds if there exists a constant $\delta \geq 0$ such that

$$\mathbf{Re}[(1 + \delta(1 - z^{-1}))H(z)] + \frac{1}{\kappa} \geq 0 \;\text{ for all }\; |z| = 1 \tag{3.250}$$

This is the discrete time analog of the Popov's criterion.

We present now the multivariable extension of Tsypkin's result [250]. Let us consider a minimal realization (A, B, C) of the transfer function $H(z)$. The discrete time system with a nonlinearity in the feedback is

$$\begin{cases} x(k+1) = Ax(k) - B\phi(y(k)) \\ \\ y(k) = Cx(k) \end{cases} \tag{3.251}$$

The nonlinearity is described as follows. Let $M = M^T > 0$ be $m \times m$ real matrix. The set $\Phi \ni \phi(\cdot)$ is

$$\begin{aligned} \Phi = \{\phi : I\!R^m \to I\!R^m \ \text{ such that } \ & \phi^T(y)[M^{-1}\phi(y) - y] < 0 \\ & \text{for } \ y \in I\!R^m, y \neq 0, \phi(\cdot) \ \text{ is continuous} \\ & \phi(y) = [\phi_1(y_1), \phi_2(y_2), ..., \phi_m(y_m)]^T, \text{and} \\ & 0 < \tfrac{\phi_i(\sigma) - \phi_i(\hat{\sigma})}{\sigma - \hat{\sigma}}, \sigma \in I\!R, \hat{\sigma} \in I\!R, \sigma \neq \hat{\sigma}, i = 1, ..., m\} \end{aligned} \tag{3.252}$$

When $m = 1$ then we get the usual sector condition $0 < \phi(y)y < My^2$. We also define the matrices

$$A_a = \begin{bmatrix} A & 0_{n\times m} \\ C & 0_m \end{bmatrix}$$

$$B_a = \begin{bmatrix} B \\ 0_m \end{bmatrix}$$

$$C_a = [C \quad - I_m]$$

$$S = [C \ \ 0_m]$$

where O_m denotes the zero $m \times m$ matrix.

Theorem 3.104. *[250] Let (A, B, C) be minimal, N =diag$[N_1, ..., N_m]$ be positive definite, and assume that det$(CA^{-1}B) \neq 0$, and that $(A, C + NC - NCA^{-1})$ is observable. Then*

$$\mathcal{H}(z) = M^{-1} + [I_m + (1 - z^{-1})N]H(z) \tag{3.253}$$

is SPR if and only if there exist matrices $P = P^T > 0$, L and W such that

$$\begin{cases} P = A_a^T P A_a + L^T L \\ \\ 0 = B_a^T P A_a - NC_a - S + W^T L \\ \\ 0 = 2M^{-1} - B_a^T P B_a - W^T W \end{cases} \tag{3.254}$$

Then the following function

$$V(x) = [x^T \;\; y^T]P \begin{bmatrix} x \\ y \end{bmatrix} + 2\sum_{i=1}^{m} \int_0^{y_i} N_i \phi_i(\sigma) d\sigma \tag{3.255}$$

where $y_i = C_i x$, C_i denotes the ith row of C, is a Lyapunov function for the negative feedback interconnection of $H(z)$ and the nonlinearity $\phi(\cdot)$, whose fiexd point is globally asymptotically stable for all $\phi(\cdot) \in \Phi$. ■

Further details on the Tsypkin criterion can be found in [281] and in the special issue [222]. See also [197, 198].

3.12.3 Discretization of PR Systems

In this section we are interested in a problem with a high practical interest: given a PR or SPR continuous time system, is PRness preserved through time discretization? The material is taken from De La Sen [447]. Let us start by recalling some facts and definitions.

Consider the transfer function $H(s) = \frac{N(s)}{M(s)} = H_1(s) + d$, where the relative degree of $H(s)$ is 0, $d \in \mathbb{R}$ and $H_1(s) = \frac{N_1(s)}{M(s)}$. $H_1(s)$ is strictly proper. The system is assumed to be stabilizable and detectable, *i.e.* $N(s) = N_1(s) + dM(s)$ and $M(s)$ may possess common factors in the complex half plane $\mathbf{Re}[s] < 0$. Let (A, B, C, D) be a state representation of $H(s)$. One has $M(s) = \det(sI_n - A)$ and $N(s) = C\mathrm{Adj}(sI_n - A)B + D \det(sI_n - A)$, where $\mathrm{Adj}(\cdot)$ is the adjoint matrix of the square matrix $(\cdot)$. If $M(s)$ and $N(s)$ are coprime then (A, B, C, D) is minimal (controllable and observable) but by assumption if they are not coprime the uncontrollable or unobservable modes are stable.

We assume that the system is sampled with a zero-order hold device of sampling period T_s, and we denote $t_k = kT_s$, $x_k = x(t_k)$ and so on. The continuous time system (A, B, C, D) becomes when discretized a discrete time system

$$\begin{cases} x_{k+1} = \Phi x_k + \Gamma u_k \\ \\ y_{k+1} = C x_{k+1} + D u_{k+1} \end{cases} \tag{3.256}$$

for all $k \geq 0$, $k \in \mathbb{N}$, $\Phi = \exp(AT_s)$, $\Gamma = \left(\int_0^{T_s} \exp(A(T_s - \tau)) d\tau\right) B$. The discrete transfer function from $u(z)$ to $y(z)$, $z \in \mathbb{C}$, is given by

$$G(z) = \tfrac{N_d(z)}{M_d(z)} = Z\left(\tfrac{1 - \exp(-T_s s)}{s} H(s)\right) = G_1(z) + D, G_1(z) = \tfrac{N_{1d}(z)}{M_d(z)} \tag{3.257}$$

where $G_1(z)$ has relative degree 1 and real coefficients

$$\begin{cases} N_{1d}(z) = C\mathrm{Adj}(zI_n - \Phi)\Gamma \\ \\ M_d(z) = \det(zI_n - \Phi) = z^n + \sum_{i=1}^n m_i z^{n-i}, \end{cases} \tag{3.258}$$

where $\mathrm{Adj}(zI_n - \Phi) = \sum_{i=0}^{n-1}\left(\sum_{k=0}^{n-1-i} s_k z^{n-k-1}\right)\Phi^i$, n is the dimension of the state vector x, $N_d(z) = N_{1d}(z) + DM(z)$, the degree of the polynomial N_{1d} is $n-1$ and the degree of N_d and M_d is n. It is well-known that the poles of $G(z)$ and of $G_1(z)$ are equal to $\exp(\lambda_A T_s)$ for each eigenvalue λ_A of the matrix A, so that the stability is preserved through discretization. However such is not the same for the zeros of $G_1(z)$ which depend on the zeros and the poles of $H_1(s)$, and on the sampling period T_s. It cannot be guaranteed that these zeros are in $|z| < 1$. It is therefore clear that the preservation of PRness imposes further conditions.

Let us denote H_0 the set of stable transfer functions, possibly critically stable (*i.e.* with pairs of purely imaginary conjugate poles). Let us denote G_1 the set of discrete stable transfer functions, possible critically stable.

Theorem 3.105. *Consider $H_1(s) \in H_0$ with a numerator $N_1(s)$ of degree $n-1$, fulfilling the following conditions:*

- *$H_1(s)$ has a nonempty set of critically stable poles C_h with at most one simple pole at $s = 0$, and any number $\mathcal{N} \geq 0$ of simple critically stable complex conjugate poles $s = \pm js_i$ ($i = 1, 2, ..., \mathcal{N}_0$, $\mathcal{N} = 2\mathcal{N}_0$).*
- *The residuals for all the critically stable poles are real and nonnegative.*

Consider $H(s) = H_1(s) + d$, its discretized transfer function $G(z) = \frac{z-1}{z} Z\left(\frac{H(s)}{s}\right) = G_1(z) + D$, and its transformed transfer function $G_z(w) = G\left(z \stackrel{\Delta}{=} \frac{1+w}{1-w}\right)$. Then the following hold:

- *(i) $G^{-1} \in G_1$ (equivalently $G_z^{-1} \in H_0$) for all sufficiently large absolute values of D, provided that $-\frac{\pi}{2} < Arg(G_z(w)) < \frac{\pi}{2}$ for $w = \frac{e^{T_s s-1}}{e^{T_s s+1}}$ for all $s \in C_h$.*
- *(ii) If (i) holds then there is a constant $\bar{D} > 0$ such that for all $D \geq \bar{D}$, $G(z)$ is (discrete) positive real and $G_z(w)$ is (continuous) positive real.*

■

It is interesting to note that (ii) is directly related to the comment made right after the KYP Lemma 3.100. The homographic transformation $w = \frac{z-1}{z+1}$ transforms the region $|z| \leq 1$ into $\mathbf{Re}[w] \leq 0$, consequently the stability of $G_z(w)$ follows if all its poles are inside $\mathbf{Re}[w] \leq 0$.

4

Dissipative Systems

In this chapter we will further study the concept of dissipative systems which is a very useful tool in the analysis and synthesis of control laws for linear and nonlinear dynamical systems. One of the key properties of a dissipative dynamical system is that the total energy stored in the system decreases with time. Dissipativeness can be considered as an extension of PR systems to the nonlinear case. Some relationships between Positive Real and Passive systems have been established in Chapter 2. There exist several important subclasses of dissipative nonlinear systems with slightly different properties which are important in the analysis. Dissipativity is useful in stabilizing mechanical systems like fully actuated robots manipulators [71], robots with flexible joints [6, 72, 78, 80, 318], underactuated robot manipulators, electric motors, robotic manipulation [25], learning control of manipulators [26,27], fully actuated and underactuated satellites [133], combustion engines [176], power converters [18, 135, 234, 235, 458, 460], neural networks [122, 203, 528, 529], smart actuators [171], piezo-electric structures [269], haptic environments and interfaces [109,128,284,285,289,309,333,422,423,454], particulate processes [131], process and chemical systems [108, 152, 457, 459, 525], missile guidance [283], model helicopters [332], magnetically levitated shafts [355,356], biological and physiological systems [191, 192], flat glass manufacture [526], visual feedback control [252], *etc.* Some of these examples will be presented in the following chapters.

Dissipative systems theory is intimately linked to Lyapunov stability theory. There exists tools from the dissipativity approach that can be used to generate Lyapunov functions. A difference between the two approaches is that the state of the system and the equilibrium point are notions that are required in the Lyapunov approach while the dissipative approach is rather based on input-output behavior of the plant. The input-output properties of a closed loop system can be studied using $\mathcal{L}_p$ stability analysis. The properties of $\mathcal{L}_p$ signals can then be used to analyze the stability of a closed loop control system. $\mathcal{L}_p$ stability analysis has been studied by Desoer and Vidyasagar [125]. A clear presentation of this notions will also be given in this book since they are very useful in the stability analysis of control systems and in particular in the control of robot manipulators. Popov introduced in 1964 the notion of hyperstability which will be defined precisely in Section 5.10 and is in fact quite close to dissipativity. This together with the celebrated Popov's criterion for absolute stability, Popov multipliers [244], the Popov controllability criterion, Popov parameters [246], certainly places V.M. Popov as one of the major contributors in dissipative systems and modern control theories. As quoted from [153]: *V.M. Popov was the first who studied passivity in detail for linear control systems and gave its characterization in terms of frequency domain inequality meaning positive realness of the system.* Dissipativeness of dynamical systems as it is known in the "modern" Systems and Control community has been introduced by Willems [510, 511]. Hill and Moylan [206, 207] carried out an extension of the Kalman-Yakubovich-Popov (KYP) Lemma to the case of nonlinear systems with state space representations that are affine in the input. Byrnes *et al.* [89] further developed the concept of dissipative systems and characterized the class of dissipative systems by obtaining some necessary conditions for a nonlinear system to be dissipative and studied the stabilization of dissipative systems.

Before presenting the definitions of dissipative systems we will study some properties of $\mathcal{L}_p$ signals which will be useful in studying the stability of closed loop control systems.

4.1 Normed Spaces

We will briefly review next the notation and definitions of normed spaces, $\mathcal{L}_p$ norms and properties of $\mathcal{L}_p$ signals. For a more complete presentation the reader is referred to [125] or any monograph on mathematical analysis [419–421]. Let E be a linear space over the field K (typically K is $I\!R$ or the complex field $\mathbb{C}$). The function $\rho(.)$, $\rho : E \to I\!R^+$ is a norm on E if and only if:

1. $x \in E$ and $x \neq 0 \;\Rightarrow \rho(x) > 0,\; \rho(0) = 0$
2. $\rho(\alpha x) = |\alpha|\rho(x), \forall \alpha \in K, \forall x \in E$
3. $\rho(x + y) \leq \rho(x) + \rho(y),\; \forall x, y \in E$ (triangle inequality)

4.2 $\mathcal{L}_p$ Norms

Let $x : I\!R \to I\!R$ be a function, and let $|\cdot|$ denote the absolute value. The most common signal norms are the $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_p$ and $\mathcal{L}_\infty$ norms which are respectively defined as

$$|| \, x \, ||_1 \stackrel{\triangle}{=} \int |x(t)| \, dt$$
$$|| \, x \, ||_2 \stackrel{\triangle}{=} \left(\int |x(t)|^2 dt \right)^{\frac{1}{2}}$$

$$\| x \|_p \stackrel{\triangle}{=} \left(\int |x(t)|^p dt\right)^{\frac{1}{p}} \text{ for } 2 \leq p < +\infty$$

$$\begin{aligned} \| x \|_\infty &\stackrel{\triangle}{=} \operatorname{ess\,sup}_{t \in I\!R} |x(t)| dt \\ &= \inf\{a|\ |x(t)| < a,\ a.e.\} \\ &= \sup_{t>0} |x(t)| \end{aligned}$$

where the integrals have to be understood on $I\!R$, *i.e.* $\int = \int_{I\!R}$ or, if the signals are defined on $I\!R^+$, as $\int_0^{+\infty}$. We say that a function $f(\cdot)$ belongs to $\mathcal{L}_p$ if and only if f is locally Lebesgue integrable (*i.e.* $|\int_a^b f(t)dt| < +\infty$ for any $I\!R \ni b \geq a$) and $\|f\|_p < +\infty$. To recapitulate:

- For $1 \leq p < +\infty$, $\mathcal{L}_p(I) = \{f : I \to I\!R, f(\cdot)$ is Lebesgue measurable and $\left(\int_I |f(t)|^p dt\right)^{\frac{1}{p}} < +\infty\}$.
- $\mathcal{L}_\infty(I) = \{f : I \to I\!R, f(\cdot)$ is Lebesgue measurable, defined and bounded almost everywhere on $I\}$.

Most of the time we shall write $\mathcal{L}_p$ instead of $\mathcal{L}_p(I)$, especially when $I = I\!R^+$. In order to encompass multivariable systems, it is necessary to introduce the norm for vector functions $f : I\!R \to I\!R^n$, where $f_i \in \mathcal{L}_p$ for each $1 \leq i \leq n$ and $||f||_p \stackrel{\Delta}{=} \left[\sum_{i=1}^n ||f_i||_p^2\right]^{\frac{1}{2}}$.

Proposition 4.1. *If $f \in \mathcal{L}_1 \bigcap \mathcal{L}_\infty$ then $f \in \mathcal{L}_p$ for all $1 \leq p \leq +\infty$.* ■

Proof: Since $f \in \mathcal{L}_1$, the set $A \stackrel{\triangle}{=} \{t|\ |f(t)| \geq 1\}$ has finite Lebesgue measure. Therefore, since $f \in \mathcal{L}_\infty$

$$\int_A |f(t)|^p dt < \infty,\ \forall\, p \in [1, +\infty)$$

Define the set $B \stackrel{\triangle}{=} \{t|\ |f(t)| < 1\}$. Then we have

$$\int_B |f(t)|^p dt \leq \int_B |f(t)| dt < \int |f(t)| dt < \infty, \forall\, p \in [1, +\infty)$$

Finally

$$\int |f(t)|^p dt = \int_A |f(t)|^p dt + \int_B |f(t)|^p dt < +\infty$$

■

4.2.1 Relationships Between $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_\infty$ Spaces.

In order to understand the relationship between $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_\infty$ spaces let us consider the following examples that have been introduced in [125]:

- $f_1(t) = 1$
- $f_2(t) = \frac{1}{1+t}$
- $f_3(t) = \frac{1}{1+t}\frac{1+t^{\frac{1}{4}}}{t^{\frac{1}{4}}}$
- $f_4(t) = e^{-t}$
- $f_5(t) = \frac{1}{1+t^2}\frac{1+t^{\frac{1}{4}}}{t^{\frac{1}{4}}}$
- $f_6(t) = \frac{1}{1+t^2}\frac{1+t^{\frac{1}{2}}}{t^{\frac{1}{2}}}$

It can be shown that (see Figure 4.1)

- $f_1 \notin \mathcal{L}_1$, $f_1 \notin \mathcal{L}_2$ and $f_1 \in \mathcal{L}_\infty$
- $f_2 \notin \mathcal{L}_1$, $f_2 \in \mathcal{L}_2$ and $f_2 \in \mathcal{L}_\infty$
- $f_3 \notin \mathcal{L}_1$, $f_3 \in \mathcal{L}_2$ and $f_3 \notin \mathcal{L}_\infty$
- $f_4 \in \mathcal{L}_1$, $f_4 \in \mathcal{L}_2$ and $f_4 \in \mathcal{L}_\infty$
- $f_5 \in \mathcal{L}_1$, $f_5 \in \mathcal{L}_2$ and $f_5 \notin \mathcal{L}_\infty$
- $f_6 \in \mathcal{L}_1$, $f_6 \notin \mathcal{L}_2$ and $f_6 \notin \mathcal{L}_\infty$

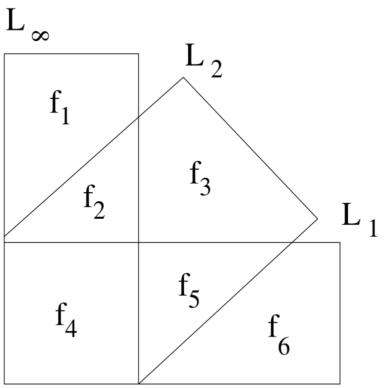

Fig. 4.1. Relationships between L_1, L_2 and L_∞

4.3 Review of Some Properties of $\mathcal{L}_p$ Signals

The following facts are very useful to prove convergence of signals under different conditions.

Fact 1: If $V : I\!R \to I\!R$ is a non-decreasing function (see figure 4.2) and if $V(t) \leq M$ for some $M \in I\!R$ and all $t \in I\!R$, then $V(\cdot)$ converges.
Proof: Since $V(\cdot)$ is non-decreasing, then $V(\cdot)$ can only either increase or remain constant. Assume that $V(\cdot)$ does not converge to a constant limit. Then $V(\cdot)$ has to diverge to infinity since it cannot oscillate. In other words there exists a strictly increasing sequence of time instants t_1, t_2, t_3 ... and a $\delta > 0$ such that $V(t_i) + \delta < V(t_{i+1})$. However this leads to a contradiction since V has upper-bound M. Therefore, the sequence $V(t_i)$ has a limit for any sequence of time instants $\{t_i\}_{i \geq 1}$ so that $V(\cdot)$ converges. ■

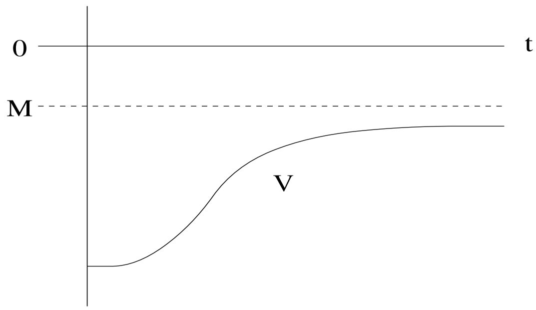

Fig. 4.2. A nondecreasing function $V(\cdot)$.

Examples:

- $\int_0^t |s(\tau)| d\tau < \infty \;\Rightarrow \int_0^t |s(\tau)| d\tau$ converges
- Let $V(\cdot)$ be differentiable. Then $V(\cdot) \geq 0$ and $\dot{V}(\cdot) \leq 0 \;\Longrightarrow\; V(\cdot)$ converges.

Fact 2: If $\int_0^t |f(t')| dt'$ converges then $\int_0^t f(t') dt'$ converges. **Proof:** In view of the assumption we have

$$\infty > \int_0^t |f(t')| dt' = \int_{t|f(t)>0} |f(t')| dt' + \int_{t|f(t)\leq 0} |f(t')| dt'$$

Then both integrals in the right-hand side above converge. We also have

$$\int_0^t f(t') dt' = \int_{t|f(t)>0} |f(t')| dt' - \int_{t|f(t)\leq 0} |f(t')| dt'$$

Then $\int_0^t f(\tau) d\tau$ converges too. ■
Fact 3: $\dot{f} \in \mathcal{L}_1$ implies that f has a limit.
Proof: By assumption we have

$$|f(t) - f(0)| = |\int_0^t \dot{f}(s) ds| \leq \int_0^t |\dot{f}(s)| ds < \infty$$

Using Fact 1 it follows that $\int_0^t |\dot{f}(s)|ds$ converges. This implies that $\int_0^t \dot{f}(s)ds$ converges which in turn implies that $f(\cdot)$ converges too. ■

Fact 4: If $f \in \mathcal{L}_2$ and $\dot{f} \in \mathcal{L}_2$ then $f(t) \to 0$ as $t \to +\infty$ and $f \in \mathcal{L}_\infty$.

Proof: Using the assumptions

$$
\begin{aligned}
|f^2(t) - f^2(0)| &= |\int_0^t \tfrac{d}{ds}[f^2(s)]ds| \\
&\leq \int_0^t |\tfrac{d}{ds}[f^2(s)]|ds \\
&= 2\int_0^t |f(s)\dot{f}(s)|ds \\
&\leq \int_0^t f^2(s)ds + \int_0^t \dot{f}^2(s)ds \\
&< +\infty
\end{aligned}
\tag{4.1}
$$

In view of Fact 3 it follows that $|\frac{d}{dt}[f^2]| \in \mathcal{L}_1$ which implies that $\int_0^t \frac{d}{ds}[f^2(s)]ds$ converges which in turn implies that f^2 converges. But by assumption $\int_0^t f^2(s)ds < \infty$, then f has to converge to zero. Clearly $f \in \mathcal{L}_\infty$. ■

Fact 5: $f \in \mathcal{L}_1$ and $\dot{f} \in \mathcal{L}_1 \Rightarrow f \to 0$.

Proof: Using Fact 3 it follows that $\dot{f} \in \mathcal{L}_1 \Rightarrow f$ has a limit. Since in addition we have $\int_0^t |f(s)|ds < \infty$ then f has to converge to zero. ■

Before presenting further results of $\mathcal{L}_p$ functions, some definitions are in order.

Definition 4.2. *The function $(t,x) \mapsto f(t,x)$ is said to be globally Lipschitz (with respect to x) if there exists a bounded $k \in I\!R^+$ such that*

$$
|f(t,x) - f(t,x')| \leq k|x - x'|, \ \forall\ x,\ x' \in I\!R^n, t \in I\!R^+ \tag{4.2}
$$

■

Definition 4.3. *The function $(t,x) \mapsto f(t,x)$ is said to be locally Lipschitz (with respect to x) if (4.2) holds for all $x \in K$, where $K \subset I\!R^n$ is a compact set. Then k may depend on K.* ■

Example 4.4. Let $f : x \mapsto x^2$. Then $f(\cdot)$ is locally Lipschitz in $[-1,1]$ since $|x^2 - y^2| = |x-y||x+y| \leq 2|x-y|$, for all $x, y \in [-1,1]$.

Definition 4.5. *The function $(t,x) \mapsto f(t,x)$ is said to be Lipschitz with respect to time if there exists a bounded k such that*

$$
|f(t,x) - f(t',x)| \leq k|t - t'|, \ \forall\ \ x \in I\!R^n, t, t' \in I\!R^+
$$

■

Definition 4.6. *The function $f(\cdot)$ is uniformly continuous in a set $\mathcal{A}$ if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$:*

$$|t - t'| < \delta \Rightarrow |f(t) - f(t')| < \epsilon, \ \ \forall \ \ t, t' \ \ \in \mathcal{A}$$

Remark 4.7. Uniform continuity and Lipschitz continuity are two different notions. Any Lipschitz function is uniformly continuous. However the inverse implication is not true. For instance the function $x \mapsto \sqrt{x}$ is uniformly continuous on $[0, 1]$, but it is not Lipschitz on $[0, 1]$. This may be easily checked from the definitions. The criterion in Fact 6 is clearly a sufficient condition only ("very sufficient", one should say!) to assure uniform continuity of a function. Furthermore, uniform continuity has a meaning on a set. Asking whether a function is uniformly continuous at a point is meaningless [420].

Fact 6: $\dot{f} \in \mathcal{L}_\infty \Rightarrow f$ is uniformly continuous.
Proof: $\dot{f} \in \mathcal{L}_\infty$ implies that f is Lipschitz with respect to time t and that $f(\cdot)$ is uniformly continuous.
Fact 7: If $f \in \mathcal{L}_2$ and is Lipschitz with respect to time then $\lim_{t \to +\infty} f(t) = 0$.
Proof: By assumption: $\int_0^t f^2(s)ds < \infty$ and $|f(t) - f(t')| \leq k|t - t'|, \ \forall t, t'$. Assume that

$$|f(t_1)| \geq \epsilon \ \text{ for some } \ \ t_1, \epsilon > 0$$

and

$$|f(t_2)| = 0 \ \text{ for some } \ t_2 \geq t_1$$

then

$$\epsilon \leq |f(t_1) - f(t_2)| \leq k|t_1 - t_2|$$

i.e. $|t_1 - t_2| \geq \frac{\epsilon}{k}$. We are now interested in computing the smallest upper-bound for $\int_{t_1}^{t_2} f^2(t)dt$. We will therefore assume that in the interval of time (t_1, t_2) the function $f(\cdot)$ decreases at maximum rate which is given by k in the equation above. We therefore have (see Figure 4.3):

$$\int_{t1}^{t2} f^2(s)ds \geq \frac{\epsilon^2 \frac{\epsilon}{k}}{2} = \frac{\epsilon^3}{2k}$$

Since $f \in \mathcal{L}_2$, it is clear that the number of times $|f(t)|$ can go from 0 to ϵ is finite on $I\!R$. Since $\epsilon > 0$ is arbitrary, we conclude that $f(t) \to 0$ as $t \to \infty$.■
Fact 8: If $f \in \mathcal{L}_p (1 \leq p \leq \infty)$ and if f is uniformly continuous, then $f(t) \to 0$ as $t \to +\infty$.
Proof: This result can be proved by contradiction following the proof of Fact 7. ■ **Fact 9:** If $f_1 \in \mathcal{L}_2$ and $f_2 \in \mathcal{L}_2$, then $f_1 + f_2 \in \mathcal{L}_2$.
Proof: The result follows from

$$\begin{aligned}\int (f_1(t) + f_2(t))^2 \, dt &= \int (f_1^2(t) + f_2^2(t) + 2 f_1(t) f_2(t))dt \\ &\leq 2 \int (f_1^2(t) + f_2^2(t))dt < +\infty\end{aligned}$$

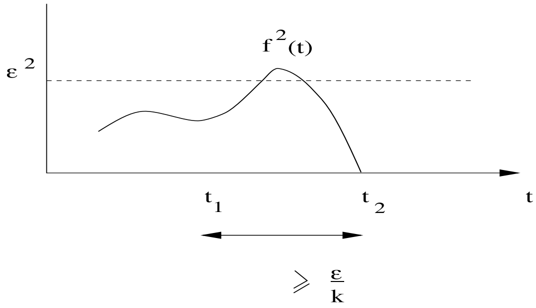

Fig. 4.3. Proof of Fact 7

■

The following Lemma describes the behavior of an asymptotically stable linear system when its input is $\mathcal{L}_2$ bounded.

Lemma 4.8. *Consider the state space representation of a linear system*

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.3}$$

with $u(t) \in I\!R^m, x(t) \in I\!R^n$ *and* A *exponentially stable. If* $u \in \mathcal{L}_2$ *then* $x \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \dot{x} \in \mathcal{L}_2$ *and* $\lim_{t\to+\infty} x(t) = 0$. ■

Remark 4.9. The system above with $u \in \mathcal{L}_2$ does not necessarily have an equilibrium point. Therefore, we cannot use the Lyapunov approach to study the stability of the system.

Proof of Lemma 4.8: Since A is exponentially stable then there exists $P = P^T > 0, Q > 0$ such that

$$PA + A^T P = -Q$$

which is the well known Lyapunov equation. Consider the following positive definite function

$$V(x,t) = x^T P x + k \int_t^\infty u^T(s)u(s)ds$$

where k is a constant to be defined later. $V(\cdot,\cdot)$ is not a Lyapunov function since the system may not have an equilibrium point. Note that since $u \in \mathcal{L}_2$, there exists a constant k' such that

$$\int_0^t u^T(s)u(s)ds + \int_t^\infty u^T(s)u(s)ds = k' < \infty$$

Taking the derivative with respect to time we obtain

$$u^T(t)u(t) + \frac{d}{dt}\left[\int_t^\infty u^T(s)u(s)ds\right] = 0$$

Using the above equations we get

$$\dot{V}(x(t),t) = \dot{x}(t)Px(t) + x^T(t)P\dot{x}(t) - ku^T(t)u(t)$$

$$= (x^T(t)A^T + u^T(t)B^T)Px(t) + x^T(t)P(Ax(t) + Bu(t)) -$$

$$-ku^T(t)u(t)$$

$$= x^T(t)(A^TP + PA)x(t) + 2u^T(t)B^TPx(t) - ku^T(t)u(t)$$

$$= -x^T(t)Qx(t) + 2u^T(t)B^TPx(t) - ku^T(t)u(t) \tag{4.4}$$

Note that

$$\begin{aligned} 2u^TB^TPx &\leq 2|u^TB^TPx| \\ &\leq 2\|u\|\ \|B^TP\|\ \|x\| \\ &\leq 2\|u\|\ \|B^TP\|\left[\tfrac{2}{\lambda_{min}Q}\right]^{\frac{1}{2}}\left[\tfrac{\lambda_{min}Q}{2}\right]^{\frac{1}{2}}\|x\| \\ &\leq \|u\|^2\|B^TP\|^2\tfrac{2}{\lambda_{min}Q} + \tfrac{\lambda_{min}Q}{2}\|x\|^2 \end{aligned} \tag{4.5}$$

where we have used the inequality $2ab \leq a^2 + b^2$, for all $a, b \in I\!R$. Choosing $k = \|B^TP\|^2\frac{2}{\lambda_{min}Q}$ we get

$$\dot{V}(x(t),t) \leq -\frac{\lambda_{min}Q}{2}\|x(t)\|^2$$

Therefore $V(\cdot,\cdot)$ is a non-increasing function and thus $V \in \mathcal{L}_\infty$ which implies that $x \in \mathcal{L}_\infty$. Integrating the above equation we conclude that $x \in \mathcal{L}_2$. From the system equation we conclude that $\dot{x} \in \mathcal{L}_2$ (see also Fact 9). Finally $x, \dot{x} \in \mathcal{L}_2 \Longrightarrow \lim_{t\to+\infty} x(t) = 0$ (see Fact 4). ■

A more general result is stated in the following Theorem which can be found in [125, p.59] where $*$ denotes the convolution product.

Theorem 4.10. *Consider the exponentially stable and strictly proper system*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{4.6}$$

and its transfer function

$$H(s) = C(sI_n - A)^{-1}B$$

If $u \in \mathcal{L}_p$*, then* $y = h * u \in \mathcal{L}_p \cap \mathcal{L}_\infty$*,* $\dot{y} \in \mathcal{L}_p$ *for* $p = 1, 2$ *and* ∞*. For* $p = 1, 2$*, then* $\lim_{t \to +\infty} y(t) = 0$*.*

■

The function $h(\cdot)$ in the Theorem is the inverse Laplace transform of $H(s)$. Theorem 4.10 is a consequence of the Datko-Pazy Theorem [123, 399] formulated in an infinite-dimensional framework.

4.3.1 Example of Applications of the Properties of $\mathcal{L}_p$ Functions in Adaptive Control

Let us first briefly review the Gradient type Parameter Estimation Algorithm, which is widely used in adaptive control and in parameter estimation. Let $y(t) \in I\!R, \phi(t) \in I\!R^n$ be measurable functions [1] which satisfy the following linear relation:

$$y(t) = \theta^T \phi(t)$$

where $\theta(t) \in I\!R^n$ is an unknown constant vector. Define $\hat{y}(t) = \phi(t)^T \hat{\theta}(t)$ and $e(t) = \hat{y}(t) - y(t)$; then

$$e(t) = \tilde{\theta}(t)^T \phi(t) \tag{4.7}$$

where $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$. Note that $\frac{d\tilde{\theta}}{dt} = \frac{d\hat{\theta}}{dt}$. Define the following positive function

$$V(\tilde{\theta}, \phi) = \frac{1}{2} e^2 \tag{4.8}$$

then

$$\dot{V}(\tilde{\theta}, \phi) = \frac{\partial V}{\partial \tilde{\theta}} \frac{d\tilde{\theta}}{dt} + \frac{\partial V}{\partial \phi} \frac{d\phi}{dt} \tag{4.9}$$

Let us choose the following parameter adaptation algorithm:

$$\frac{d\hat{\theta}}{dt}(t) = -\left(\frac{\partial V}{\partial \tilde{\theta}}\right)^T \tag{4.10}$$

Introducing (4.7) and (4.8) into (4.10) gives

$$\frac{d\hat{\theta}}{dt}(t) = -e\left(\frac{\partial e}{\partial \tilde{\theta}}\right)^T = -\phi e$$

The parameter adaptation law (4.10) is motivated by the fact that when $\dot{\phi} = 0$, then introducing (4.10) into (4.9) leads to

---

[1] Here measurable is to be taken in the physical sense, not in the mathematical one. In other words we assume that the process is well-equipped with suitable sensors.

$$\dot{V}(\tilde{\theta}, \phi) = -\left(\frac{\partial V}{\partial \tilde{\theta}}\right)\left(\frac{\partial V}{\partial \tilde{\theta}}\right)^T < 0$$

Let $W(\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^T\tilde{\theta}$, then $\dot{W}(\tilde{\theta}) = \tilde{\theta}^T\dot{\tilde{\theta}} = -\tilde{\theta}^T\phi e$. Integrating we obtain

$$\int_0^t (-\tilde{\theta}^T\phi)edt = W(\tilde{\theta}(t)) - W(\tilde{\theta}(0)) \geq -W(\tilde{\theta}(0))$$

We conclude that the operator $H : e \rightarrow -\tilde{\theta}^T\phi$ is passive.

Example 4.11. **(Adaptive control of a simple nonlinear system)** Let

$$\dot{x}(t) = f(x(t))^T\theta' + bu(t)$$

where $u(t), x(t) \in I\!R$. Define

$$\begin{cases} \theta = \frac{\theta'}{b} \\ \tilde{\theta}(t) = \hat{\theta}(t) - \theta \\ \dot{\hat{\theta}}(t) = f(x(t))x(t) \\ u(t) = -\hat{\theta}^T(t)f(x(t)) - x(t) + v(t) \end{cases}$$

and

$$V(\tilde{\theta}, x) = \frac{b}{2}\tilde{\theta}^T\tilde{\theta} + \frac{1}{2}x^2$$

Then along trajectories of the system we get

$$\begin{aligned} \dot{V}(\tilde{\theta}(t), x(t)) &= b\tilde{\theta}^T(t)\dot{\tilde{\theta}}(t) + x(t)\dot{x}(t) \\ &= b\tilde{\theta}(t)^T f(x(t))x(t) + x(t)\left(f(x(t))^T\theta' + bu(t)\right) \\ &= bx(t)[(\hat{\theta}(t) - \theta)^T f(x(t)) + \theta^T f(x(t)) + u(t)] \\ &= -bx^2(t) + bx(t)v(t) \end{aligned} \tag{4.11}$$

From the last equation it follows that for $v = 0$, $V(\cdot)$ is a non-increasing function and thus $V, x, \tilde{\theta} \in \mathcal{L}_\infty$. Integrating the last equation it follows that $x \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Assume that $f(\cdot)$ has the required property so that $x \in \mathcal{L}_\infty \Rightarrow f(x) \in \mathcal{L}_\infty$. It follows that $u \in \mathcal{L}_\infty$ and also $\dot{x} \in \mathcal{L}_\infty$. $x \in \mathcal{L}_2$ and $\dot{x} \in \mathcal{L}_\infty$ implies $\lim_{t \rightarrow +\infty} x(t) = 0$. Let us note from the last line of (4.11) that the operator $H : v \mapsto x$ is output strictly passive (OSP) as will be defined later.

■

In order to present the Passivity theorem and the Small gain theorem we will require the notion of extended spaces. We will next present a brief introduction to extended spaces. For a more detailed presentation the reader is referred to [125].

4.3.2 Linear Maps

Definition 4.12 (Linear maps). *Let E be a linear space over K ($\mathbb{R}$ or $\mathbb{C}$). Let $\tilde{\mathcal{L}}(E,E)$ be the class of all linear maps from E into E. $\tilde{\mathcal{L}}(E,E)$ is a linear space satisfying the following properties* $\forall\, x \in E$, $\forall\, A, B \in \tilde{\mathcal{L}}(E,E)$, $\forall\, \alpha \in K$:

$$\begin{cases} (A+B)x = Ax + Bx \\ (\alpha A)x \quad\; = \alpha(Ax) \\ (AB)x \quad\; = A(Bx) \end{cases} \tag{4.12}$$

■

4.3.3 Induced Norms

Definition 4.13 (Induced Norms). *Let $|.|$ be a norm on E and $A \in \tilde{\mathcal{L}}(E,E)$. The induced norm of the linear map A is defined as*

$$\begin{aligned} \|A\| &\triangleq \sup_{x \neq 0} \tfrac{|Ax|}{|x|} \\ &= \sup_{|z|=1} |Az| \end{aligned} \tag{4.13}$$

■

4.3.4 Properties of Induced Norms

If $\|A\| < \infty$ and $\|B\| < \infty$ then the following properties hold for all $x \in E, \alpha \in K$

1. $|Ax| \le \|A\||x|$
2. $\|\alpha A\| = |\alpha|\|A\|$
3. $\|A+B\| \le \|A\| + \|B\|$
4. $\|AB\| \le \|A\|\|B\|$

Example 4.14. Let H be a linear map defined on E in terms of an integrable function $h : \mathbb{R} \to \mathbb{R}$

$$H : u \to Hu \triangleq h * u, \;\forall\, u \in \mathcal{L}^{\infty}$$

i.e.

$$(Hu)(t) = \int_0^t h(t-\tau)u(\tau)d\tau, \;\forall\, t \in \mathbb{R}^+$$

Assume that $\|h\|_1 = \int_0^\infty |h(t)|dt < \infty$.

Theorem 4.15. *Under those conditions the following properties hold:*

a) $H : \mathcal{L}^\infty \to \mathcal{L}^\infty$

b) $\|H\|_\infty = \|h\|_1$ *and* $\|h * u\|_\infty \leq \|h\|_1 \|u\|_\infty, \ \forall u \in \mathcal{L}^\infty$

and the right-hand side can be made arbitrarily close to the left-hand side of the inequality by appropriate choice of u. ■

Proof: By definition and from (4.13) we obtain

$$\begin{aligned}
\|H\|_\infty &= \sup_{||u||_\infty = 1} ||Hu||_\infty \\
&= \sup_{\|u\|_\infty = 1} \|h * u\|_\infty \\
&= \sup_{\|u\|_\infty = 1} \sup_{t \geq 0} |(h * u)(t)| \\
&= \sup_{\|u\|_\infty = 1} \left[\sup_{t \geq 0} \left| \int_0^t h(t-\tau) u(\tau) d\tau \right| \right] \\
&\leq \sup_{\|u\|_\infty = 1} \left[\sup_{t \geq 0} \int_0^t |h(t-\tau)| \, |u(\tau)| \, d\tau \right]
\end{aligned}$$

Since $\|u\|_\infty = 1$ we have

$$\begin{aligned}
\|H\|_\infty &\leq \sup_{t \geq 0} \int_0^t |h(t-\tau)| \, d\tau \\
&= \sup_{t \geq 0} \int_0^t |h(t-\tau)| \, d\tau \\
&= \sup_{t \geq 0} \int_0^t |h(t')| \, dt' \\
&\leq \int_0^\infty |h(t')| \, dt' = \|h\|_1
\end{aligned}$$

We can choose $u_t(\tau) = \text{sgn}[h(t-\tau)], t \in I\!\!N$. Thus

$$(h * u_t)(t) = \int_0^t |h(t-\tau)| d\tau \leq \|h * u_t\|_\infty$$

Therefore

$$\begin{aligned}
\int_0^t |h(\tau')| d\tau' &= \int_0^t |h(t-\tau)| d\tau \\
&\leq \|h * u_t\|_\infty \\
&\leq \|H\|_\infty \leq \|h\|_1 \\
&= \int_0^\infty |h(t')| dt'
\end{aligned} \tag{4.14}$$

Letting $t \to \infty$ it follows that $\|H\|_\infty = \|h\|_1$. ■

4.3.5 Extended Spaces

Consider a function $f : \mathbb{R}^+ \to \mathbb{R}$ and let $0 \leq T < +\infty$. Define the truncated function

$$f_T(t) = \begin{cases} f(t) \text{ if } t \leq T \\ \\ 0 \quad\;\; \text{if } t > T \end{cases} \tag{4.15}$$

The function f_T is obtained by truncating $f(\cdot)$ at time T. Let us introduce the following definitions:

$\mathcal{T}$: subset of $\mathbb{R}^+$ (typically, $\mathcal{T} = \mathbb{R}^+$ or $\mathbb{N}$),

$\mathcal{V}$: normed space with norm $\|.\|$ (typically $\mathcal{V} = \mathbb{R}, \mathbb{R}^n, \mathbb{C}, \mathbb{C}^n$),

$\mathcal{F} = \{f \mid f : \mathcal{T} \to \mathcal{V}\}$ the set of all functions mapping $\mathcal{T}$ into $\mathcal{V}$.

The normed linear subspace $\mathcal{L}$ is given by

$$\mathcal{L} \stackrel{\triangle}{=} \{f : \mathcal{T} \to \mathcal{V} \mid \|f\| < \infty\}$$

Associated with $\mathcal{L}$ is the extended space $\mathcal{L}_e$ defined by

$$\mathcal{L}_e \stackrel{\triangle}{=} \{f : \mathcal{T} \to \mathcal{V} \mid \forall T \in \mathcal{T}, \|f_T\| < \infty\}$$

In other words, the sets $\mathcal{L}_{p,e}$ or simply $\mathcal{L}_e$, consist of all Lebesgue measurable functions f such that every truncation of f belongs to the set $\mathcal{L}_p$ (but f may not belong to $\mathcal{L}_p$ itself, so that $\mathcal{L}_p \subset \mathcal{L}_{p,e}$). The following properties hold for all $f \in \mathcal{L}_{p,e}$:

1. The map $t \to \|f_t\|$ is monotonically increasing
2. $\|f_t\| \to \|f\|$ as $t \to +\infty$

Remark 4.16. One sometimes speaks of $\mathcal{L}_{p,loc}$, which means that $\left(\int_I |f(t)|^p dt\right)^{\frac{1}{p}} < +\infty$ for all compact intervals $I \subset \mathbb{R}$. Obviously $\mathcal{L}_{p,loc} = \mathcal{L}_{p,e}$.

We can now introduce the notion of gain of an operator which will be used in the small gain Theorem and the passivity Theorem.

4.3.6 Gain of an Operator

In the next Definition, we consider a general operator with state, input, and output signal spaces. In particular, the input-output mapping is assumed to be causal, invariant under time shifts, and it depends on the initial state x_0.

Definition 4.17. *[206] Consider an operator $H : \mathcal{L}_e \to \mathcal{L}_e$. H is weakly finite-gain stable (WFGS) if there exist a function $\beta(\cdot)$ and a constant k such that*

$$\|(Hu)_T\| \leq k\|u_T\| + \beta(x_0)$$

for all admissible $u(\cdot)$ and all x_0. If $\beta(x_0) = 0$, we call H finite-gain stable (FGS).

In a more rigorous way, the input-output operator H should be denoted as $H(x_0)$ or H_{x_0} as it may depend on x_0. This is a situation completely analogous to that of passive operator as in Definition 2.1, where the constant β may in general depend on the initial state x_0. One may be surprized that the notion of *state* intervenes in a definition that concern purely input-output operators (or systems). Some definitions, indeed, do not mention such a dependence. This is related to the very basic definition of what a system is, and well-posedness. Then the notions of input, output and state can hardly be decoupled, in general.

We call the *gain of H* the number k (or $k(H)$) defined by

$$k(H) = \inf\{\bar{k} \in I\!R^+ / \ \exists \bar{\beta} : \|(Hu)_T\| \leq \bar{k}\|u_T\| + \bar{\beta}, \ \forall u \in \mathcal{L}_e, \forall T \in I\!R^+\}$$

Let us recall the case of linear time invariant systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = Cx(t) + Du(t), \quad x(0) = x_0 \in I\!R^n. \end{cases} \tag{4.16}$$

Theorem 4.18. *Suppose that the matrix A has all its eigenvalues with negative real parts ($\Longleftrightarrow \dot{x}(t) = Ax(t)$ is asymptotically stable). Then the system (4.16) is finite-gain stable where the norm can be any $\mathcal{L}_p$ with $1 \leq p \leq +\infty$. In other words $u \in \mathcal{L}_p \Longrightarrow y \in \mathcal{L}_p$ and $||y||_p \leq k_p||u||_p$ for some $k_p < +\infty$.* ■

A rather complete exposition of input/output stability can be found in [500, Chapter 6].

4.3.7 Small Gain Theorem

This Theorem gives sufficient conditions under which a bounded input produces a bounded output (BIBO).

Theorem 4.19 (Small gain). *Consider $H_1 : \mathcal{L}_e \to \mathcal{L}_e$ and $H_2 : \mathcal{L}_e \to \mathcal{L}_e$. Let $e_1, e_2 \in \mathcal{L}_e$ and define (see Figure 4.4)*

$$\begin{cases} u_1 = e_1 + H_2 e_2 \\ u_2 = e_2 - H_1 e_1 \end{cases} \tag{4.17}$$

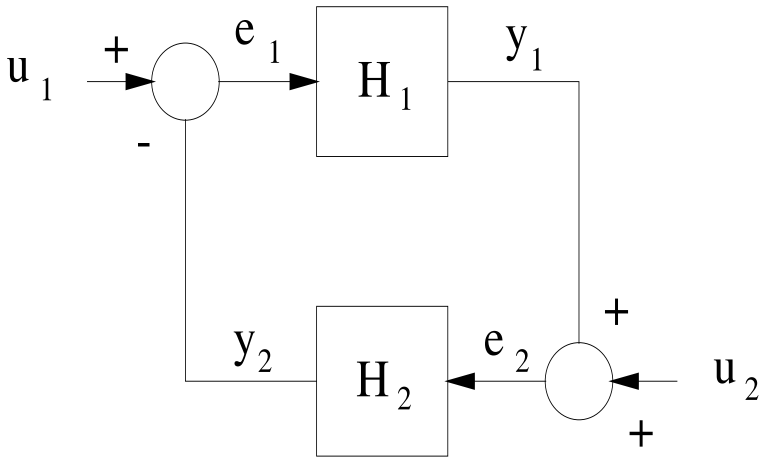

Fig. 4.4. Closed-loop system with two external inputs

Suppose there are constants $\beta_1, \beta_2, \gamma_1, \gamma_2 \geq 0$ such that for all $t \in \mathbb{R}^+$:

$$\begin{cases} \|(H_1 e_1)_T\| \leq \gamma_1 \|e_{1T}\| + \beta_1 \\ \|(H_2 e_2)_T\| \leq \gamma_2 \|e_{2T}\| + \beta_2 \end{cases} \tag{4.18}$$

Under those conditions, if $\gamma_1 \gamma_2 < 1$, then:

i)

$$\begin{cases} \|e_{1t}\| \leq (1-\gamma_1\gamma_2)^{-1}(\|u_{1t}\| + \gamma_2 \|u_{2t}\| + \beta_2 + \gamma_2 \beta_1) \\ \|e_{2t}\| \leq (1-\gamma_1\gamma_2)^{-1}(\|u_{2t}\| + \gamma_1 \|u_{1t}\| + \beta_1 + \gamma_1 \beta_2) \end{cases}$$

ii) If in addition, $\|u_1\|, \|u_2\| < +\infty$, then e_1, e_2, y_1, y_2 have finite norms. ■

Proof: From (4.17) we have

$$\begin{array}{l} e_{1t} = u_{1t} - (H_2 e_2)_t \\ e_{2t} = u_{2t} + (H_1 e_1)_t \end{array} \tag{4.19}$$

Then

$$\|e_{1t}\| \leq \|u_{1t}\| + \|(H_2 e_2)_t\| \leq \|u_{1t}\| + \gamma_2 \|e_{2t}\| + \beta_2$$

$$\|e_{2t}\| \leq \|u_{2t}\| + \|(H_1 e_1)_t\| \leq \|u_{2t}\| + \gamma_1 \|e_{1t}\| + \beta_1$$

Combining these two inequalities we get

$$\|e_{1t}\| \leq \|u_{1t}\| + \beta_2 + \gamma_2(\|u_{2t}\| + \gamma_1 \|e_{1t}\| + \beta_1)$$

$$\|e_{2t}\| \leq \|u_{2t}\| + \beta_1 + \gamma_1(\|u_{1t}\| + \gamma_2 \|e_{2t}\| + \beta_2)$$

Finally

$$\|e_{1t}\| \leq (1-\gamma_1\gamma_2)^{-1}\left[\|u_{1t}\| + \gamma_2\|u_{2t}\| + \beta_2 + \gamma_2\beta_1\right]$$

$$\|e_{2t}\| \leq (1-\gamma_1\gamma_2)^{-1}\left[\|u_{2t}\| + \gamma_1\|u_{1t}\| + \beta_1 + \gamma_1\beta_2\right]$$

The remainder of the proof follows immediately. ■

Clearly to be consistent with Definition 4.17, the constants β_1, β_2, γ_1 and γ_2 may also depend on initial states $x_{1,0}$ and $x_{2,0}$. This obviously does not modify the above calculations. A general notion of dissipativity will be introduced, and some links with the gain theory will be established in Sections 4.4 and 5.1.

4.4 Dissipative Systems

4.4.1 Definitions

We will now review the definitions and properties of dissipative systems. Most of the mathematical foundations on this subject are due to Willems [512], and Hill and Moylan [206, 207]. One difficulty when looking at the literature on the subject, is that there are many different notions of dissipativity which are introduced in many papers published here and there. One of the goals of this chapter is to present all of them in one shot and also the existing relationships between them. Consider a causal nonlinear system $(\Sigma) : u(t) \mapsto y(t);\ u(t) \in \mathcal{L}_{pe}, y(t) \in \mathcal{L}_{pe}$, represented by the following state-space representation affine in the input:

$$(\Sigma)\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) + j(x(t))u(t) \\ x(0) = x_0 \end{cases} \tag{4.20}$$

where $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$, $f(\cdot), g(\cdot), h(\cdot)$ and $j(\cdot)$ possess sufficient regularity so that the system with inputs in $\mathcal{L}_{2,e}$ is well-posed (see Section 3.9.2), and $f(0) = h(0) = 0$. In other words the origin $x = 0$ is a fixed point for the uncontrolled (free) system, and there is no output bias at $x = 0$. The state space is denoted as $X \subseteq \mathbb{R}^n$. Let us call $w(t) = w(u(t), y(t))$ the *supply rate* and be such that for all admissible $u(\cdot)$ and $x(0)$ and for all $t \in \mathbb{R}^+$

$$\int_0^t |w(u(s), y(s))| ds < +\infty \tag{4.21}$$

i.e. we are assuming $w(\cdot)$ to be locally Lebesgue integrable independently of the input and the initial conditions. In an electric circuit $\int_0^t w(s)ds$ can be associated with the energy supplied to the circuit in the interval $(0, t)$, *i.e.* $\int_0^t v(s)i(s)\ ds$ where $v(\cdot)$ is the voltage at the terminals and $i(\cdot)$ the current entering the circuit, see the example in Chapter 1. In the following, what we will often call an admissible input, simply means that the ordinary differential equation in (4.20) possesses a unique differentiable solution. Hence it is sufficient that the vector field $f(x(t)) + g(x(t))u(t)$ satisfies the Carathéodory conditions (see Theorem 3.55): $u(\cdot)$ may be a Lebesgue measurable function of time.

Definition 4.20 (Dissipative System). *The system (Σ) is said to be dissipative if there exists a so-called storage function $V(x) \geq 0$ such that the following dissipation inequality holds:*

$$V(x(t)) \leq V(x(0)) + \int_0^t w(u(s), y(s))ds \tag{4.22}$$

along all possible trajectories of (Σ) starting at $x(0)$, for all $x(0)$, $t \geq 0$ (said differently: for all admissible controllers $u(\cdot)$ that drive the state from $x(0)$ to $x(t)$ on the interval $[0, t]$).

It follows from Lemma 3.1 and Corollary 3.3 that controllable and observable LTI systems with a positive real transfer functions, are dissipative with quadratic storage functions (see also [489] in the context of behavioural approach to linear dynamical systems). Two comments immediately arise from Definition 4.20: first storage functions are defined up to an additive constant; second, if the system is dissipative with respect to supply rates $w_i(u, y)$, $1 \leq i \leq m$, then the system is also dissipative with respect to any supply rate of the form $\sum_{i=1}^m \alpha_i w_i(u, y)$, with $\alpha_i \geq 0$ for all $1 \leq i \leq m$. One notices that the Definition 4.20 (sometimes referred to as Willems' dissipativity) does not require any regularity on the storage functions: it is a very general definition. Actually, storage functions do possess some regularity properties under suitable assumptions, see Section 4.4.5. When one imposes that the storage functions be of class C^r for some integer $r \geq 0$, then one speaks of C^r-dissipativity. A third comment may be done: Willems' definition postulates that dissipativity holds whenever a storage function exists. Some other authors like Hill and Moylan, start from a definition that is closer to Definition 2.1, and then prove the existence of storage functions.

Example 4.21. Let us continue with Example 3.2. The input-output product satisfies $\int_0^t u(t')y(t')dt' = \int_0^t u^2(t')dt' \geq 0$ for any initial data $x(0)$. Now choose $V(x) = \frac{1}{2}x^2$. One has $V(x(t)) \leq V(x(0))$ since solutions strictly decrease. Thus $V(x(t)) - V(x(0)) \leq 0$ and $V(x(t)) - V(x(0)) \leq \int_0^t u(t')y(t')dt'$: the system is dissipative, though neither observable nor controllable (but, it is stable).

It is noterworthy that (4.22) is equivalent to the following: there exists $W(\cdot)$ such that $V(x_1) - V(x_0) \le W(x_1, x_0)$ with

$$W(x_1, x_0) = \inf_{u(\cdot)\in\mathcal{U}} \int_0^t w(u(s), y(s))ds \tag{4.23}$$

along admissible controls which drive the state from x_0 to x_1 on the time interval $[0, t]$. In the following we shall use either 0 or t_0 to denote the initial time for (4.20). Dissipativity is also introduced by Hill and Moylan [207] as follows:

Definition 4.22. *The system (Σ) is dissipative with respect to the supply rate $w(u, y)$ if for all admissible $u(\cdot)$ and all $t_1 \ge t_0$ one has*

$$\int_{t_0}^{t_1} w(u(t), y(t))dt \ge 0 \tag{4.24}$$

with $x(t_0) = 0$ and along trajectories of (Σ). ■

This corresponds to imposing that storage functions satisfy $V(0) = 0$. This is justified by the fact that storage functions will often, if not always, be used as Lyapunov functions for studying the stability of an equilibrium of (Σ) with zero input $u(\cdot)$. In a slightly more general setting, one may assume that the controlled system has a fixed point $x^\star$ (corresponding to some input $u^\star$, and with $f(x^\star)+g(x^\star)u^\star = 0$, $y^\star = h(x^\star)+j(x^\star)u^\star$, and $w(u^\star, y^\star) = 0$), and that $V(x^\star) < +\infty$. Then changing $V(\cdot)$ to $V(\cdot) - V(x^\star)$ one obtains $V(x^\star) = 0$ (we could even have stated this as an assumption in Definition 4.20, as done for instance in [510]). In the sequel of this chapter, we shall encounter some results in which dissipativity is indeed assumed to hold with $V(0) = 0$. Such results were originally settled to produce Lyapunov functions, precisely. Hill and Moylan start from (4.24) and then prove the existence of storage functions, adding properties to the system. The motivation for introducing Definition 4.22 is clear from Corollary 3.3, as it is always satisfied for linear invariant positive real systems with minimal realizations.

Another definition [206] states that the system is *weakly dissipative* with respect to the supply rate $w(u, y)$ if $\int_{t_0}^{t_1} w(u(t), y(t))dt \ge -\beta(x(t_0))$ for some $\beta(\cdot) \ge 0$ with $\beta(0) = 0$ [531] (we shall see later the relationship with Willems' Definition; it is clear at this stage that weak dissipativity implies dissipativity in Definition 4.22, and that Willem's dissipativity implies weak dissipativity provided $V(0) = 0$). Still, another definition is as follows [232]:

Definition 4.23. *The system (Σ) is said dissipative with respect to the supply rate $w(u, y)$ if there exists a locally bounded nonnegative function $V : \mathbb{R}^n \to \mathbb{R}$, such that*

$$V(x) \ge \sup_{t\ge 0, u\in\mathcal{U}} \left\{ V(x(t)) - \int_0^t w(u(s), y(s))ds : \ x(0) = x \right\} \tag{4.25}$$

where the supremum is therefore computed with respect to all trajectories of the controlled system with initial condition x and admissible inputs. ■

This definition requires the local boundedness of storage functions (a real valued function is locally bounded, if $\sup_{x \in K} \mid f(x) \mid \leq C$ for some bounded constant $C > 0$ and any compact set K of its domain). This additional property happens to be important for further characterization of the storage functions as solutions of partial differential inequalities (see Section 4.6). Apart from this additonal property, one sees that if $V(x)(= V(x(0))$ satisfies (4.25), then it satisfies (4.22). Conversely since (4.22) is satisfied for all $t \geq 0$ and for all admissible $u(\cdot)$, if $V(x(0))(= V(x))$ satisfies (4.22) then it satisfies (4.25). Thus under the local boundedness assumption, Willems' original definition and the definition stemming from (4.25), are equivalent. The fact that Definition 4.20 implies Definition 4.22 provided that $V(0) = 0$ is clear. The converse will be investigated in Section 4.5.2.

There is another definition of dissipativity that is sometimes used by Hill and Moylan:

Definition 4.24. *The system* (Σ) *is said to be* cyclo-dissipative *with respect to the supply rate* $w(u, y)$ *if*

$$\int_{t_0}^{t_1} w(u(s), y(s)ds \geq 0 \tag{4.26}$$

for all $t_1 \geq t_0$*, all admissible* $u(\cdot)$*, whenever* $x(t_0) = x(t_1) = 0$. ■

The difference with Definition 4.20 is that the state boundary conditions are forced to be the equilibrium of the uncontrolled system: trajectories start and end at $x = 0$. A cyclo-dissipative system absorbs energy for any cyclic motion passing through the origin. Cyclo-dissipativity and dissipativity are related as follows:

Theorem 4.25. *[209] Suppose that the system* (Σ) *defines a causal input-output operator* $H_{x(0)}$*, and that the supply rate is of the form* $w(u, y) = y^T Qy + 2y^T Su + u^T Ru$*, with* Q *non-positive definite. Suppose further that the system is zero state detectable (i.e.* $u(t) = 0, y(t) = 0 \; \forall \; t \geq 0 \Longrightarrow \lim_{t \to \infty} x(t) = 0$*). Then dissipativity in the sense of Definition 4.22 and cyclo-dissipativity of* (Σ) *are equivalent properties.* ■

The proof of this Theorem relies on the definition of another concept known as ultimate dissipativity, which we do not wish to introduce here for the sake of briefness (roughly, this is dissipativity but only with $t = +\infty$ in (4.22)). The reader is therefore referred to [209] for the proof of Theorem 4.25 (which is a concatenation of Definitions 2, 3, 8 and Theorems 1 and 2 in [209]). Let us recall that an operator $H : u \mapsto y = H(u, t)$ is causal if the following holds: for all admissible inputs $u(\cdot)$ and $v(\cdot)$ with $u(\tau) = v(\tau)$ for all $\tau \leq t$, then $H(u(\cdot), t) = H(v(\cdot), t)$. In other words, the output depends only on the past values of the input, and not on future values. It is noteworthy here that causality may hold for a class of inputs and not for another class.

A local definition of dissipative systems is possible. Roughly, the dissipativity inequality should be satisfied as long as the system's state remains inside a closed domain of the state space [404].

Definition 4.26 (Locally dissipative system). *Let X be the system's state space. Let $\mathcal{U}_e = \{u(\cdot) \mid u_t(\cdot) \in \mathcal{U}$ for all $t \in \mathbb{R}\}$. The dynamical system is locally dissipative with respect to the supply rate $w(u,y)$ in a region $\Omega \subseteq X$ if*

$$\int_0^t w(u(s), y(s))ds \geq 0 \tag{4.27}$$

for all $u \in \mathcal{U}_e$, $t \geq 0$, such that $x(0) = 0$ and $x(s) \in \Omega$ for all $0 \leq s \leq t$. ∎

Still, another notion is known as the *quasi-dissipativity*:

Definition 4.27. *[403] The system (Σ) is said quasi-dissipative with respect to the supply rate $w(u,y)$ if there exists a constant $\alpha \geq 0$ such that it is dissipative with respect to the supply rate $w(u,y) + \alpha$.* ∎

Actually, dissipativity is understood here as weak dissipativity, *i.e.*

$$\int_0^t w(u(s), y(s))ds \geq \beta$$

with $\beta \leq 0$ (see Definition 2.1). The interest of quasi-dissipativity is that a quasi-dissipative system can contain a source of energy with finite power.

To conclude this subsection, we have at our disposal several notions of dissipativity: Willems', Hill and Moylan's, Definition 2.1, weak dissipativity (which is intermediate between Definition 2.1 and Willems'), cyclo-dissipativity, quasi-dissipativity, ultimate dissipativity, local dissipativity, Definition 4.23. There are more (like $J-$dissipativity [397], which is used in specific settings like H_∞ control), exponential dissipativity (see Theorem 5.68), definitions taylored to systems with time-varying parameters [302], and Popov's hyperstability.

Remark 4.28. Some properties are stated as $\int_0^t$ for all $t \geq 0$, and others as $\int_{t_0}^{t_1}$ for all $t_1 \geq t_0$. If trajectories (state) are independent of the initial time but depend only on the elapsed time, clearly both ways of stating dissipativity are equivalent.

4.4.2 The Signification of β

The next result helps to understand the signification of the constant β (apart from the fact that, as we shall see later, one can exhibit examples which prove that the value of $\beta(0)$ when β is a function of the initial state, has a strong influence on the stability of the origin $x = 0$). The supply rate that is considered is the general supply rate $w(u, y) = y^T Qy + 2y^T Su + u^T Ru$, where $Q = Q^T$ and $R = R^T$ (but no other assumptions are made, so that Q and R may be zero). The definition of weak dissipativity is as seen above, but in a local setting, *i.e.* an operator $G : \mathcal{U} \to \mathcal{Y}$ which is denoted as G_{x_0} as it may depend on the initial state. For a region $\Omega \subset I\!R^n$ we denote $G(\Omega)$ the family of operators G_{x_0} for all $x_0 \in \Omega$. Considering such domain Ω may be useful for systems with multiple equilibria, see Example 4.34. Mimicking the definition of weak finite gain (Definition 4.17):

Definition 4.29. *[206] The operator $G(\Omega)$ is said weakly $w(u, y)-$dissipative if there exists a function $\beta : \Omega \to I\!R$ such that*

$$\int_0^t w(u(s), y(s))ds \geq \beta(x_0), \tag{4.28}$$

for all admissible $u(\cdot)$, all $t \geq 0$, and all $x_0 \in \Omega$. If $\beta(x_0) = 0$ in Ω then the operator is called $w(u, y)-$dissipative. ■

This definition has some local taste as it involves possibly several equilibria of the system (the set Ω). Therefore when time comes to settle some stability of these equilibria, it may be that only local stability can be deduced. We also need a reachability definition. The distance of x to Ω is $d(x, \Omega) = \inf_{x_0 \in \Omega} ||x - x_0||$.

Definition 4.30. *[206] A region $X_1 \subset I\!R^n$ is uniformly reachable from $\Omega \subset I\!R^n$ if there exists a class $\mathcal{K}$ function $\alpha(\cdot)$, and for every $x_1 \in X_1$ there exists $x_0 \in \Omega$, a finite $t_1 \geq t_0$ and an admissible $u(\cdot)$ such that the trajectory of the controlled system originating from x_0 at t_0 satisfies $x(t_1) = x_1$ and $\int_0^{t_1} u(s)^T u(s)ds \leq \alpha(d(x_1, \Omega))$.* ■

Uniform reachability means that a state x_1 can be driven from some other state x_0 with an input that is small if the distance between the two states is small. It is local in nature.

Theorem 4.31. *[206] If $G(\Omega)$ is weakly $w(u, y)-$dissipative, and X_1 is uniformly reachable from Ω, then $G(X_1)$ is weakly $w(u, y)-$dissipative.* ■

Proof: Take any $x_1 \in X_1$ and any $t_1 > t_0$, any $x_0 \in \Omega$, any $u(\cdot) \in \mathcal{U}$ such that the controlled trajectory emanating from x_0 at t_0 ends at x_1 at t_1. The value of $u(t)$ for $t > t_1$ is arbitrary. The inequality in (4.28) can be rewritten as

$$\int_0^t w(u(s), y(s))ds \geq \beta_{new}(x_1) \tag{4.29}$$

with $\beta_{new}(x_1) = \beta(x_0) - \int_0^{t_1} w(u(s), G_{x_0}(u(s)))ds$, and we used the fact that the operator is invariant under time shifts. The value $\beta_{new}(x_1)$ depends only on x_1 and not on $u(\cdot)$ because the control that intervenes in the definition of $\beta_{new}(x_1)$ is the specific control which drives x_0 to x_1. Thus G_{x_1} is weakly dissipative. ■

If $\beta(x_0) = 0$ then the system is dissipative with respect to one initial state (in the sense of Definition 4.22 if $x_0 = 0$). But it is weakly dissipative with respect to other reachable initial states. Therefore a way to interpret β is that it allows to take into account non-zero initial states. In Example 4.60 we will see that weak finite-gain stability is not enough to assure that the uncontrolled state space representation of the system has a Lyapunov stable fixed point. It follows from this analysis that defining passivity as $\int_0^t u^T(s)y(s)ds \geq 0$ for any initial state makes little sense if the initial state is not included in the definition (or implicitly assumed to be zero).

The equivalence between Willems' definition and weak dissipativity follows from the following:

Theorem 4.32. *[206] For some $X_1 \subseteq X$, $G(X_1)$ is weakly dissipative if and only if there exists a function $V : X_1 \rightarrow I\!R$, with $V(x) \geq 0$ for all $x \in X$, such that*

$$V(x_1) + \int_{t_0}^{t} w(u(s), y(s))ds \geq V(x_2) \tag{4.30}$$

for all $x_1 \in X_1$, all admissible $u(\cdot) \in \mathcal{U}$, all $t \geq t_0$, with $y(t) = G_{x_1}(u(t))$ and $x(t) = x_2$ is the state starting at x_1 at t_0.

Proof: Let us denote $V(u, y, t_0, t) \stackrel{\Delta}{=} \int_{t_0}^t w(u(s), y(s))ds$. By time-invariance $V(u, y, t_0, t)$ depends only on $t - t_0$ but not on t and t_0 separately. Let $V(x_1) = -\inf_{u(\cdot)\in\mathcal{U}, t\geq t_1} V(u, G_{x_1}u, t_1, t)$. Because of $t \geq t_1$, t may be chosen as t_1 and consequently $V(x_1) \geq 0$. For any $t_2 \geq t_1$ and $t \geq t_2$ one has $V(x_1) \geq -V(u, G_{x_1}u, t_1, t_2) - V(u, G_{x_2}u, t_2, t)$, where $x(t_2) = x_2$ is the state which starts at x_1 at time t_1 and with the control u on $[t_1, t_2]$. Since this inequality holds for all u, it is true in particular that $V(x_1) \geq -V(u, G_{x_1}u, t_1, t_2) - \inf_{u(\cdot)\in\mathcal{U}, t\geq t_2} V(u, G_{x_2}u, t_2, t)$ from which (4.30) follows. The inequality (4.28) implies that $V(x_1) \leq -\beta(x_1)$ so that $0 \leq V(x) < +\infty$ for all $x \in X_1$. Now starting from (4.30) one sees that $V(x_1) + \int_{t_0}^t w(u(s), y(s))ds \geq 0$ which shows that the system is $w(u, y)-$dissipative. ■

Moreover:

Theorem 4.33. *[206] Assume that $X_1 \subseteq X$ is uniformly reachable from $\Omega \subseteq X$. Then $G(\Omega)$ is $w(u, y)-$dissipative if and only if there exists a function*

$V : X_1 \rightarrow I\!R$ satisfying the conditions of Theorem 4.32 and $V(x) = 0$ for all $x \in \Omega$. ■

Proof: If $G(\Omega)$ is $w(u,y)-$dissipative and X_1 is reachable from Ω, then Theorem 4.31 shows that G_{x_1} is $w(u,y)-$dissipative. Following the same steps as in the proof of Theorem 4.32, the only thing that remains to be shown is that $V(x) = 0$ for all $x \in \Omega$. The bounds $0 \leq V(x) \leq \beta(x)$ for all $x \in X_1$ and Definition 4.29 imply it. The converse is a direct consequence of (4.30). ■

Thus summarizing Theorems 4.25, 4.32 and 4.33:

cyclo-dissipativity
Definition 4.24

$\Updownarrow$ (if ZSD and $Q \leq 0$)

Hill and Moylan's dissipativity
Definition 4.22

$\Uparrow$ (if reachability)

weak $w(u,y)-$dissipativity [$w(u,y)-$dissipativity + reachability]
Definition 4.29

$\Updownarrow$

Willems' dissipativity [Willems' dissipativity $+V(0) = 0$]
Definition 4.20

(if local boundedness of the storage function)
$\Updownarrow$

Definition 4.23

The link between $w(u,y)-$dissipativity and dissipativity in Definition 4.22 can also be established from Theorem 4.33. The equivalence between weak $w(u,y)-$dissipativity and the other two, supposes that the required dynamical system that is under study is as (4.20), so in particular $0 \in \Omega$.

Example 4.34. [206] To illustrate Definition 4.29 consider the following system:

$$\begin{cases} \dot{x}(t) = -\alpha \sin(x(t)) + 2\gamma u(t) \\ y(t) = -\alpha \sin(x(t)) + \gamma u(t) \\ x(0) = x_0 \end{cases} \tag{4.31}$$

with $\alpha > 0$. Then $V(x) = \alpha(1-\cos(x))$, $V(x_0) = 0$ for all $x_0 = \pm 2n\pi$, $n \in I\!N$. Thus $\Omega = \{x_0 \mid x_0 = \pm 2n\pi\}$. This system is finite-gain stable, and the equilibria are (locally) asymptotically stable.

4.4.3 Storage Functions (Available, Required Supply)

Having in mind this preliminary material, the next natural question is, given a system, how can we find $V(x)$? This question is closely related to the problem of finding a suitable Lyapunov function in the Lyapunov second method. As will be seen next, a storage function can be found by computing the maximum amount of energy that can be extracted from the system.

Definition 4.35 (Available Storage). *The available storage $V_a(\cdot)$ of the system (Σ) is given by*

$$0 \le V_a(x) = \sup_{x=x(0),u(\cdot),t\ge 0} -\left\{\int_0^t w(u(s),y(s))ds\right\} \tag{4.32}$$

where $V_a(x)$ is the maximum amount of energy which can be extracted from the system with initial state $x = x(0)$.

The supremum is taken over all admissible $u(\cdot)$, all $t \ge 0$, all signals with initial value $x(0) = x$, and the terminal boundary condition $x(t)$ is left free. It is clear that $0 \le V_a(x)$ (just take $t = 0$ to notice that the supremum cannot be negative). When the final state is not free but constrained to $x(t) = 0$ (the equilibrium of the uncontrolled system), then one speaks of the *virtual* available storage, denoted as $V_a^\star(\cdot)$ [209]. Another function plays an important role in dissipative systems, called the required supply. We recall that the state space of a system is said reachable from the state $x^\star$ if, given any x and t there exist a time $t_0 \le t$ and an admissible controller $u(\cdot)$ such that the state can be driven from $x(t_0) = x^\star$ to $x(t) = x$. The state space X is connected provided every state is reachable from every other state.

Definition 4.36 (Required Supply). *The required supply $V_r(\cdot)$ of the system (Σ) is given by*

$$V_r(x) = \inf_{u(\cdot),t\ge 0} \left\{\int_{-t}^0 w(u(s),y(s))ds\right\} \tag{4.33}$$

with $x(-t) = x^\star$, $x(0) = x$, and it is assumed that the system is reachable from $x^\star$. The function $V_r(x)$ is the required amount of energy to be injected in the system to go from $x(-t)$ to $x(0)$.

The infimum is taken over all trajectories starting from $x^\star$ at t and ending at $x(0) = x$ at time 0, and all $t \geq 0$ (or, said differently, over all admissible controllers $u(\cdot)$ which drive the system from $x^\star$ to x on the interval $[-t, 0]$). If the system is not reachable from $x^\star$, one may define $V_r(x) = +\infty$.

Remark 4.37. The optimal "extraction" control policy which allows one to obtain the available storage in case of an LTI system as in (3.1) is given by $u = (D+D^T)^{-1}(B^T P^- - C)x$, and the optimal "supply" control policy which allows one to obtain the required supply is given by $u = (D+D^T)^{-1}(B^T P^+ - C)x$, where P^+ and P^- are as in Theorem 3.44.

Remark 4.38. Contrary to the available storage, the required supply is not necessarily positive, see however Lemma 4.45. When the system is reversible, the required supply and the available storage coincide [512]. It is interesting to define accurately what is meant by *reversibility* of a dynamical system. This is done thanks to the definition of a third energy function, the *cycle energy*:

$$\boxed{V_c(x) = \inf_{u(\cdot), t_0 \leq t_1, x(t_0)=0} \int_{t_0}^{t_1} u(t)^T y(t) dt \qquad (4.34)}$$

where the infimum is taken over all admissible $u(\cdot)$ which drive the system from $x(t_0) = 0$ to x. The cycle energy is thus the minimum energy it takes to cycle a system between the equilibrium $x = 0$ and a given state x. One has $V_a(\cdot) + V_c(\cdot) = V_r(\cdot)$ (assuming that the system is reachable so that the required supply is not identically $+\infty$). Then the following is in order:

Definition 4.39 (Reversibility). *Let a dynamical system be passive in the sense of Definition 2.1 with $\beta = 0$, and let its state space representation be reachable. The system is* irreversible *if $V_c(x) = 0$ only if $x = 0$. It is said* uniformly irreversible *if there exists a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that for all $x \in I\!R^n$: $V_c(x) \geq \alpha(||x||)$. The system is said to be* reversible *if $V_c(x) = 0$ for all $x \in I\!R^n$,* i.e. *if $V_a(\cdot) = V_r(\cdot)$.* ■

The following is taken from [209].

Example 4.40. Let us consider the one-dimensional system

$$\begin{cases} \dot{x}(t) = -x(t) + u(t) \\ y(t) = x(t) + \frac{1}{2}u(t) \\ x(0) = x_0. \end{cases} \qquad (4.35)$$

This system is dissipative with respect to the supply rate $w(u, y) = uy$. Indeed $\int_0^t u(s)y(s)ds = \int_0^t [(\dot{x}(s) + x(s))x(s) + \frac{1}{2}u^2(s)]ds = \left[\frac{x^2(s)}{2}\right]_0^t + \int_0^t (x^2(s) +$

$\frac{1}{2}u^2(s))ds \geq -\frac{x^2(0)}{2}$. Then $V_a(x) = \frac{2-\sqrt{3}}{2}x^2$ and $V_r(x) = \frac{2+\sqrt{3}}{2}x^2$. Indeed the available storage and required supply are the extrema solutions of the Riccati equation $A^T P + AP + (PB - C^T)(D + D^T)^{-1}(B^T P - C) = 0$, which is in this case $p^2 - 4p + 1$. Moreover the available storage and the virtual available storage (where the terminal state is forced to be $x = 0$) are the same. One sees that $V(x) = \frac{1}{2}x^2$ is a storage function.

The following results link the boundedness of the functions introduced in Definitions 4.35 and 4.36 to the dissipativeness of the system. As an example, consider again an electrical circuit. If there is an ideal battery in the circuit, the energy that can be extracted is not finite. Such a circuit is not dissipative. The following results are due to Willems [510, 511].

Theorem 4.41. *[510, 511] The available storage $V_a(\cdot)$ in* (4.32)*, is finite for all $x \in X$ if and only if (Σ) in (4.20) is dissipative in the sense of Definition 4.20. Moreover, $0 \leq V_a(x) \leq V(x)$ for all $x \in X$ for dissipative systems and $V_a(\cdot)$ is itself a possible storage function.*

Proof:

($\Rightarrow$) In order to show that $V_a(x) < \infty \Rightarrow$ the system (Σ) in (4.20) is dissipative, it suffices to show that the available storage V_a in (4.32) is a storage function i.e. it satisfies the dissipation inequality

$$V_a(x(t)) \leq V_a(x(0)) + \int_0^t w(t)dt$$

But this is certainly the case because the available storage $V_a(x(t))$ at time t is not larger than the available storage $V_a(x(0))$ at time 0 plus the energy introduced into the system in the interval $[0, t]$.

($\Leftarrow$) Let us now prove that if the system (Σ) is dissipative then $V_a(x) < \infty$. If (Σ) is dissipative then there exists $V(x) \geq 0$ such that

$$V(x(0)) + \int_0^t w(t)dt \geq V(x(t)) \geq 0$$

From the above and (4.32) it follows that

$$V(x(0)) \geq \sup_{x=x(0), t\geq 0, u} \left\{ -\int_0^t w(t)dt \right\} = V_a(x)$$

Since the initial storage function $V(x(0))$ is finite it follows that $V_a(x) < +\infty$. The last part of the Theorem follows from the definitions of $V_a(\cdot)$ and $V(\cdot)$ (see (4.25)). ■

Therefore dissipativeness can be tested by attempting to compute $V_a(x)$: if it is locally bounded, it is a strorage function and the system is dissipative with respect to the supply rate $w(u,y)$. This is a *variational approach.* Compare (4.32) with (4.25). It clearly appears why, among all possible storage functions satisfying (4.25), the available storage is the "smallest" one. Testing the dissipativity of the system (Σ) is by Theorem 4.41 equivalent to testing whether or not $\inf_{u\in\mathcal{U}} \int_0^{+\infty} w(u(t),y(t))dt$ under the constraints $\dot{x}(t) = f(x(t)) + g(x(t))u(t)$, $x(0) = x_0$, is finite for all $x_0 \in \mathbb{R}^n$. As we saw in Section 3.8.2 the value of this infimum yields the negative of the available storage function. Similar results can be derived from the cyclo-dissipativity:

Lemma 4.42. *[209] Let the system (Σ) be cyclo-dissipative. Then*

- *(i) $V_r(x(0)) < +\infty$ for any reachable state $x(0)$ and with $x(-t) = 0$*
- *(ii) $V_a^\star(x(0)) > -\infty$ for any controllable state $x(0)$*
- *(iii) $V_a^\star(0) = V_r(0) = 0$ if $x(-t) = 0$*
- *(iv) $V_r(x) \geq V_a^\star(x)$ for any state $x \in X$* ■

Controllability means in this context that there exists an admissible $u(\cdot)$ that drives the state trajectory towards $x = 0$ at a time $t \geq 0$. **Proof:** (i) and (ii) are a direct consequence of reachability and controllability, and the fact that $w(u(s),y(s))$ is integrable. Now let $x(0)$ be both reachable and controllable. Let us choose a state trajectory which passes through the points $x(-t) = x(t) = 0$, and with $x(0) = x$. Then $\int_{-t}^{0} w(u(s),y(s))ds + \int_0^t w(u(s),y(s))ds \geq 0$, from the definition of cyclo-dissipativity. From the definitions of $V_a^\star(\cdot)$ (paragraph below Definition 4.35) and $V_r(\cdot)$, (iv) follows using that $\int_{-t}^{0} w(u(s),y(s))ds \geq -\int_0^t w(u(s),y(s))ds$. (iv) remains true even in the case of uncontrollability and unreachability, as in such a case $V_r(x(0)) = +\infty$ and $V_a(x(0)) = -\infty$. ■

Similarly to the above results concerning the available storage, we have the following:

Theorem 4.43. *[510, 511] The system (Σ) in (4.20) is dissipative in the sense of Definition 4.20 if and only if the required supply satisfies $V_r(x) \geq -K > -\infty$ for all $x \in X$ and some $K \in \mathbb{R}$. Moreover, $0 \leq V_a(x) \leq V(x) \leq V_r(x)$ for all $x \in X$ for dissipative systems.*

Before presenting the next Lemma, let us introduce a notion of reachability.

Definition 4.44 (Locally $w-$uniformly reachable). *[209] The system (Σ) is said to be locally $w-$uniformly reachable at the state $x^\star$ if there exists a neighborhood Ω of $x^\star$ and a class $\mathcal{K}$ function $\rho(\cdot)$ such that for each $x \in \Omega$*

there exist $t \geq 0$ and an admissible $u(\cdot)$ driving the system from $x^\star$ to x on the interval $[0, t)$ and

$$|\int_0^t w(u(s), y(s))ds| \leq \rho(||x - x^\star||) \tag{4.36}$$

The system is said to be locally uniformly $w-$reachable in Ω if it is locally uniformly $w-$reachable at all states $x^\star \in \Omega$. ■

A way to characterize such a property is indicated later; see Proposition 4.76. The following provides informations on whether or not the required supply may serve as a storage function. It is extracted from [401, Theorem 2].

Lemma 4.45. *Let the system (Σ) be dissipative in the sense of Definition 2.1 with respect to the supply rate $w(u, y)$, and locally $w-$uniformly reachable at $x^\star$. Let $V(\cdot)$ be a storage function. Then the function $V_r(\cdot) + V(x(0))$ is a continuous storage function.* ■

One sees that if the storage function satisfies $V(0) = 0$ and if $x(0) = 0$ then the required supply is a storage function. The value $V(x(0))$ plays the role of the bias $-\beta$ in Definition 2.1. When $V(0) = 0$ the system has zero bias at the equilibrium $x = 0$. In fact a variant of Theorem 4.41 can be stated as follows, where dissipativity is checked through $V_a(\cdot)$ provided the system (Σ) is reachable from some state $x^\star$.

Lemma 4.46. *[442] Assume that the state space of (Σ) is reachable from $x^\star \in X$. Then (Σ) is dissipative in the sense of Definition 4.20 if and only if $V_a(x^\star) < +\infty$.* ■

The conditions of Theorem 4.41 are less stringent since reachability is not assumed. However in practice, systems of interest are often reachable so that Lemma 4.46 is important.

Notice that given two storage functions $V_1(\cdot)$ and $V_2(\cdot)$ for the same supply rate, it is not difficult to see from the dissipation inequality that for any constant $\lambda \in [0, 1]$ then $\lambda V_1(\cdot) + (1 - \lambda)V_2(\cdot)$ is still a storage function. More formally:

Lemma 4.47. *The set of all possible storage functions of a dissipative system is convex. In particular $\lambda V_a(\cdot) + (1 - \lambda)V_r(\cdot)$ is a storage function provided the required supply is itself a storage function.*

Proof: Let $V_1(\cdot)$ and $V_2(\cdot)$ be two storage functions. Let $0 \leq \lambda \leq 1$ be a constant. Then it is an easy computation to check that $\lambda V_1(\cdot) + (1 - \lambda)V_2(\cdot)$ is also a storage function. Since the available storage and the required supply are storage functions, the last part follows. ■

The available storage and the required supply can be characterized as follows:

Proposition 4.48. *Consider the system* (Σ) *in (4.20). Assume that it is zero state observable* ($u(t) = 0$ *and* $y(t) = 0$ *for all* $t \geq 0$ *imply that* $x(t) = 0$ *for all* $t \geq 0$), *with a reachable state space* X, *and that it is dissipative with respect to* $w(u, y) = 2u^T y$. *Let* $j(x) + j^T(x)$ *have full rank for all* $x \in X$. *Then* $V_a(\cdot)$ *and* $V_r(\cdot)$ *are solutions of the partial differential equality:*

$$\begin{aligned}&\nabla V^T(x) f(x) +\\ &+ \left(h^T(x) - \tfrac{1}{2}\nabla V^T(x) g(x)\right)\left(j(x) + j^T(x)\right)^{-1}\left(h(x) - \tfrac{1}{2} g^T(x)\nabla V(x)\right) = 0\end{aligned} \tag{4.37}$$

Before presenting the proof we need an intermediate result:

Lemma 4.49. *Let a function* $V(\cdot)$ *be differentiable. Let* $W(x) = -\nabla V^T(x) f(x)$ *and* $S(x) = h^T(x) - \frac{1}{2}\nabla V^T(x) g(x)$. *Then along any trajectory of* (Σ) *in (4.20) and for all* t_1 *and* t_0 *with* $t_1 \geq t_0$, *one has*

$$\begin{aligned}&\int_{t_0}^{t_1} 2u(t)^T y(t) dt =\\ &= [V(x(t))]_{t_0}^{t_1} + \int_{t_0}^{t_1} [1 \;\; u^T(t)] \begin{pmatrix} W(x(t)) & S(x(t)) \\ S^T(x(t)) & j(x(t)) + j^T(x(t)) \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} dt\end{aligned} \tag{4.38}$$

■

Proof: The proof is led by calculating the integral of the right-hand-side of (4.38). ■

Proof of Proposition 4.48: Let us rewrite the available storage as

$$V_a(x) = - \inf_{x = x(0), u(\cdot), t \geq 0} - \left\{ \int_0^t w(u(s), y(s)) ds \right\} \tag{4.39}$$

Using Lemma 4.49 one gets

$$V_a(x) =$$

$$= - \inf_{x=x(0),u(\cdot),t\geq 0} \left\{ [V_a(x(t))]_0^t + \int_0^t [1 \;\; u^T(t)]\mathcal{D}(x(t)) \begin{pmatrix} 1 \\ u \end{pmatrix} dt \right\} \tag{4.40}$$

$$= V_a(x) - \inf_{x=x(0),u(\cdot),t\geq 0} \left\{ V_a(x(t)) + \int_0^t [1 \;\; u^T(t)]\mathcal{D}(x(t)) \begin{pmatrix} 1 \\ u \end{pmatrix} dt \right\}$$

where we used that $x(0) = x$ and $\mathcal{D}(x) = \begin{pmatrix} W_a(x(t)) & S_a(x(t)) \\ S_a^T(x(t)) & j(x(t)) + j^T(x(t)) \end{pmatrix}$.

Therefore

$$0 = - \inf_{x=x(0),u(\cdot),t\geq 0} \left\{ V_a(x(t)) + \int_0^t [1 \;\; u^T(t)]\mathcal{D}(x(t)) \begin{pmatrix} 1 \\ u \end{pmatrix} dt \right\} \tag{4.41}$$

If the infimum exists and since $j(x(t))+j^T(x(t))$ is supposed to be full rank, it follows that its Schur complement $W_a(x)-S_a(x)(j(x)+j^T(x))^{-1}S_a^T(x) = 0$ (see Lemma A.62), which exactly means that $V_a(\cdot)$ satisfies the partial differential inequality (4.37). A similar proof may be made for the required supply. ■

In the linear time invariant case, and provided the system is observable and controllable, then $V_a(x) = x^T P_a x$ and $V_r(x) = x^T P_r x$ satisfy the above partial differential equality, which means that P_a and P_r are the extremal solutions of the Riccati equation $A^T P + PA + (PB - C^T)(D + D^T)^{-1}(B^T P - C) = 0$. Have a look at Theorems 3.42, 3.43 and 3.44, and Theorem 4.43. One especially deduces that the set of solutions $P = P^T > 0$ of the KYP Lemma set of equations in (3.2) has a maximum P_r and a minimum P_a elements, and that all other solutions satisfy $0 < P_a \leq P \leq P_r$. What is called G^+ in Theorem 3.43 and is equal to $-P_a$ and what is called G^- is equal to $-P_r$ (it is worth recalling that minimality of (A, B, C, D) is required in the KYP Lemma solvability with positive definite symmetric solutions, and that the relaxation of the minimality requires some care; see Section 3.3). Similarly P^- and P^+ in Theorem 3.44 are equal to P_a and P_r respectively.

The following is a consequence of Theorem 2.2 and relates to a notion introduced at the beginning of this book for input-output systems, to the notion of dissipativity introduced for state space systems.

Theorem 4.50 (Passive systems). *Suppose that the system (Σ) in (4.20) is dissipative with supply rate $w(u, y) = u^T y$ and storage function $V(\cdot)$ with $V(0) = 0$, i.e. for all $t \geq 0$:*

$$V(x(t)) \leq V(x(0)) + \int_0^t u^T(s)y(s)ds \tag{4.42}$$

Then the system is passive. ■

Passivity is defined in Definition 2.1. Let us recall that a positive real (PR) system is passive; see Corollary 2.35.

Definition 4.51 (Strictly state passive systems). *A system* (Σ) *in (4.20) is said to be* strictly state passive *if it dissipative with supply rate* $w = u^T y$ *and the storage function* $V(\cdot)$ *with* $V(0) = 0$*, and there exists a positive definite function* $\mathcal{S}(x)$ *such that for all* $t \geq 0$*:*

$$V(x(t)) \leq V(x(0)) + \int_0^t u^T(s)y(s)ds - \int_0^t \mathcal{S}(x(t))dt \tag{4.43}$$

If the equality holds in the above and $\mathcal{S}(x) \equiv 0$*, then the system is said to be* lossless . ■

Some authors [228] also introduce a notion of *weak strict passivity* that is more general than the strict state passivity: the function $\mathcal{S}(x)$ is replaced by a dissipation function $\mathcal{D}(x,u) \geq 0$, $\mathcal{D}(0,0) = 0$. One gets a notion that is close to (4.55). The notion of weak strict passivity is meant to generalize WSPR functions to nonlinear systems.

Theorem 4.52. *[510] Suppose that the system* (Σ) *in (4.20) is lossless with a minimum value at* $x = x^\star$ *such that* $V(x^\star) = 0$*. If the state space is reachable from* $x^\star$ *and controllable to* $x^\star$*, then* $V_a(\cdot) = V_r(\cdot)$ *and thus the storage function is unique and given by* $V(x) = \int_{t_1}^0 w(u(t), y(t))dt$ *with any* $t_1 \leq 0$ *and* $u \in \mathcal{U}$ *such that the state trajectory starting at* $x^\star$ *at* t_1 *is driven by* $u(\cdot)$ *to* $x = 0$ *at* $t = 0$*. Equivalently* $V(x) = -\int_0^{t_1} w(u(t), y(t))dt$ *with any* $t_1 \geq 0$ *and* $u \in \mathcal{U}$ *such that the state trajectory starting at* x *at* $t = 0$ *is driven by* $u(\cdot)$ *to* $x^\star$ *at* t_1*.* ■

Remark 4.53. If the system (Σ) in (4.20) is dissipative with supply rate $w = u^T y$ and the storage function $V(\cdot)$ satisfies $V(0) = 0$ with $V(\cdot)$ positive definite, then the system and its zero dynamics are Lyapunov stable. This can be seen from the dissipativity inequality (4.22) by taking u or y equal to zero.

Example 4.54 (passivity $\subset$ *dissipativity).* Consider $H(s) = \frac{1-s}{1+s}$. From Theorem 4.18 this system has a finte $\mathcal{L}_p$-gain for all $1 \leq p \leq +\infty$ and it is dissipative with respect to all supply rates $w(u,y) = \gamma|u|^p - \delta|y|^p$, $1 \leq p \leq +\infty$. However $H(s) \notin PR$ and it is not passive, *i.e.* it is not dissipative with respect to the supply rate $w(u,y) = uy$.

A general supply rate has been introduced by [207] which is useful to distinguish different types of strictly passive systems and will be useful in the Passivity Theorems presented in the next section. Let us reformulate some notions introduced in Definition 2.1 in terms of supply rate, where we recall that $\beta \leq 0$.

Definition 4.55 (General Supply Rate). *Let us consider a dissipative system, with supply rate*

$$w(u, y) = y^T Q y + u^T R u + 2y^T S u \tag{4.44}$$

with $Q = Q^T$, $R = R^T$. If $Q = 0$, $R = -\varepsilon I_m$, $\varepsilon > 0$, $S = \frac{1}{2} I_m$, the system is said to be input strictly passive (ISP), i.e.

$$\int_0^t y^T(s)u(s)ds \geq \beta + \epsilon \int_0^t u^T(s)u(s)ds$$

If $R = 0$, $Q = -\delta I_m$, $\delta > 0$, $S = \frac{1}{2} I_m$, the system is said to be output strictly passive (OSP), i.e.

$$\int_0^t y^T(s)u(s)ds \geq \beta + \delta \int_0^t y^T(s)y(s)ds$$

If $Q = -\delta I_m$, $\delta > 0$, $R = -\varepsilon I_m$, $\varepsilon > 0$, $S = \frac{1}{2} I_m$, the system is said to be very-strictly passive (VSP), i.e.

$$\int_0^t y^T(s)u(s)ds + \beta \geq \delta \int_0^t y^T(s)y(s)ds + \epsilon \int_0^t u^T(s)u(s)ds$$

Note that Definitions 4.51 and 4.55 do not imply in general the *asymptotic* stability of the considered system. For instance $\frac{s+a^2}{s}$ is ISP as stated in Definition 4.55; see also Theorem 2.6. Though this will be examined at several places of this book, let us explain at once the relationship between the finite-gain property of an operator as in Definition 4.17, and dissipativity with respect to a general supply rate. Assume that the system (Σ) is dissipative with respect to the general supply rate, *i.e.*

$$V(x(t)) - V(x(0)) \leq \int_0^t [y^T(s)Qy(s) + u^T(s)Ru(s) + 2y^T(s)Sy(s)]ds \tag{4.45}$$

for some storage function $V(\cdot)$. Let $S = 0$. Then it follows that

$$-\int_0^t y^T(s)Qy(s)ds \leq \int_0^t u^T(s)Ru(s)ds + V(x(0)) \tag{4.46}$$

Let $Q = -\delta I_m$ and $R = \epsilon I_m$, $\delta > 0$, $\epsilon > 0$. Then we get

$$\int_0^t y^T(s)Qy(s)ds \leq \frac{\epsilon}{\delta} \int_0^t u^T(s)Ru(s)ds + V(x(0)) \tag{4.47}$$

so that the operator $u \mapsto y$ has a finite $\mathcal{L}_2$-gain with a bias equal to $V(x(0))$. Dissipativity with supply rates $w(u,y) = -\delta y^T y + \epsilon u^T u$ will be commonly met, and is sometimes called the H_∞-behaviour supply rate of the system. Therefore dissipativity with $Q = -\delta I_m$ and $R = \epsilon I_m$ and $S = 0$ implies finite-gain stability. What about the converse? The following is true:

Theorem 4.56. *[206] The system is dissipative with respect to the general supply rate in (4.44) with zero bias ($\beta = 0$) and with $Q < 0$, if and only if it is finite-gain stable.*

We note that the constant k in Definition 4.17 may be zero, so that no condition on the matrix R is required in this Theorem. The $\Longrightarrow$ implication has been shown just above. The $\Longleftarrow$ implication holds because of zero bias. Then it can be shown that $0 \leq \int_0^t [y^T(s)Qy(s) + u^T(s)Ru(s) + 2y^T(s)Sy(s)]ds$. Dissipativity is here understood in the sense of Hill and Moylan in Definition 4.22.

Remark 4.57. A dynamical system may be dissipative with respect to several supply rates, and with different storage functions corresponding to those supply rates. Consider for instance a linear time invariant system that is asymptotically stable: it may be SPR (thus passive) and it has a finite gain and is thus dissipative with respect to a H_∞ supply rate.

Let us make an aside on linear invariant systems. A more general version of Theorem 3.44 is as follows. We consider a general supply rate with $Q \leq 0$ and $\bar{R} = R + SD + D^T S + D^T QD > 0$. We denote $\bar{S} = S + D^T Q$. Then

Theorem 4.58. *[531] Consider the system (A, B, C, D) with A asymptotically stable. Suppose that*

$$-\int_0^t w(u(s), y(s))ds \leq -\frac{\epsilon}{2}\int_0^t u^T(s)u(s)ds + \beta(x_0) \tag{4.48}$$

where $\beta(\cdot) \geq 0$ and $\beta(0) = 0$. Then

- *There exists a solution $P \geq 0$ to the ARE*

$$A^T P + PA + (PB - C^T \bar{S}^T)\bar{R}^T(B^T P - \bar{S}C) - C^T QC = 0 \tag{4.49}$$

such that $A^\star = A + B\bar{R}^{-1}(B^T P - \bar{S}C)$ is asymptotically stable, and

- *there exists a solution $\bar{P} > 0$ to the ARI*

$$A^T \bar{P} + \bar{P}A + (\bar{P}B - C^T \bar{S}^T)\bar{R}^T(B^T \bar{P} - \bar{S}C) - C^T QC < 0 \tag{4.50}$$

Conversely, suppose that there exists a solution $P \geq 0$ *to the ARE (4.49) such that the matrix* $A^{\star} = A + B\bar{R}^{-1}(B^T P - \bar{S}C)$ *is asymptotically stable. Then the matrix* A *is asymptotically stable and the system* (A, B, C, D) *satisfies (4.48) with the above supply rate.* ∎

We shall see in Section 4.5 and Chapter 5 that this can be generalized to a class of nonlinear systems.

4.4.4 Examples

Example 4.59. At several places we have insisted on the essential role played by the constant β in Definition 2.1. Let us illustrate here how it may influence the Lyapunov stability of dissipative systems. For instance let us consider the following example, brought to our attention by David Hill, where the open-loop system is unstable:

$$\begin{cases} \dot{x}(t) = x(t) + u(t) \\ y(t) = -\frac{\alpha x(t)}{1+x^4(t)} \\ x(0) = x_0 \end{cases} \tag{4.51}$$

with $\alpha > 0$. Let us note that

$$\begin{aligned} \int_{t_0}^{t_1} u(t)y(t)dt &= -\int_{t_0}^{t_1} (\dot{x}(t) - x(t))\frac{\alpha x(t)}{1+x^4(t)} dt \\ &\geq -\frac{\alpha}{2}[\arctan(x^2(t_1)) - \arctan(x^2(t_0))] \end{aligned} \tag{4.52}$$

Thus the system is passive with respect to the storage function $V(x) = \frac{\alpha}{2}(\frac{\pi}{2} - \arctan(x^2))$ and $V(x) > 0$ for all finite $x \in I\!R^n$. Hence the system is dissipative despite the fact that the open-loop is unstable. Note however that $-V(0) = \beta(0) < 0$ and that the system loses its observability at $x = \infty$. We shall come back later on conditions that assure the stability of dissipative systems. ∎

Example 4.60. [206] The system is now given by

$$\begin{cases} \dot{x}(t) = x(t) + \frac{\alpha x(t)}{1+x^4(t)} + 2\gamma u(t) \\ y(t) = -\frac{\alpha x(t)}{1+x^4(t)} + \gamma u(t) \\ x(0) = x_0 \end{cases} \tag{4.53}$$

with $\alpha > 0$. Then we get that

$$\int_0^t (\gamma^2 u^T(s)u(s) - y^T(s)y(s))ds \geq V(x(t)) - V(x_0) \tag{4.54}$$

with the same $V(x)$ as in the previous example. Thus the system is weakly finite-gain stable, but the unique equilibrium of the uncontrolled system $x = 0$ is Lyapunov unstable. We notice that the system in (4.53) is not passive. Therefore weak finite-gain stability is not sufficient to guarantee the Lyapunov stability. ■

In view of the above generalizations of the dissipativity and supply rate, a dissipation equality that is more general than the one in Definition 4.51 can be introduced with a so-called *dissipation function* $\mathcal{D}(x, u, t) \geq 0$ for all $x \in X$, admissible u, and $t \geq 0$, such that along trajectories of the system (Σ) one gets

$$V(x(t), t) = V(x(0), 0) + \int_0^t w(u(s), y(s))ds + \mathcal{D}(x(0), u, t) \tag{4.55}$$

Example 4.61. Let us continue with Example 4.40. Let us consider the storage functions $V(x) = \frac{1}{2}Cx^2$, with $2-\sqrt{3} \leq C \leq 2+\sqrt{3}$. It is easily computed that the dissipation function is $\mathcal{D}(x, u, t) = \int_0^t [C(x(s) - \gamma_c u(s))^2 + R_c u^2(s)]ds$, with $\gamma_c = \frac{C-1}{2C}$ and $R_c = \frac{1}{2} - C\gamma_c^2$. The choice for this notation stems from the electrical circuit interpretation where C is a capacitor and R_c is a resistance. It is worth noting that for each value of the coefficient C, then there is a different physical realization (different resistors, capacitors), but the state equations (4.35) remain the same. Comparing with Definition 4.51, one has $\mathcal{S}(x) = x^2$ when $C = 1$. Comparing with the ISP Definition 4.55 one has $\epsilon = R_c$, provided $R_c > 0$. An interesting interpretation is in term of phase lag. Let us choose the two outputs as $y_1 = \sqrt{R_c}u$ and $y_2 = \sqrt{C}(x - \gamma_c u)$. Then the transfer function between $y_2(s)$ and $u(s)$ (the Laplace transforms of both signals) is equal to $\sqrt{C}\frac{1-\gamma_c-\gamma_c s}{1+s}$. As C varies from $2-\sqrt{3}$ to $2+\sqrt{3}$, γ_c varies monotonically from $-\frac{1}{2}(\sqrt{3}+1)$ to $\frac{1}{2}(\sqrt{3}-1)$. Thus the phase lag of $y_2(s)$ with respect to $u(s)$ increases monotonically with C. Let us now study the variation of the dissipation function $\mathcal{D}(x, u, t)$ with C. For small C the low-dissipation trajectories are those for which $||x||$ is decreasing. For large C, the low-dissipation trajectories are those for which $||x||$ is increasing. There are two extreme cases, as expected: when $C = 2-\sqrt{3}$, then $V(x) = V_a(x)$ and it is possible to drive the state to the origin with an arbitrarily small amount of dissipation. In other words, the stored energy can be extracted from the system. Doing the converse (driving the state from the origin to some other state) produces a large amount of dissipation. The other extreme is for $C = 2+\sqrt{3}$, then $V(x) = V_r(x)$. In this case any state is reachable from the origin with an arbitrarily small amount of dissipation. The converse (returning the state to the origin) however dissipates significantly. This illustrates that for small C

the dissipation seems to be concentrated at the beginning of a trajectory which leaves the origin $x = 0$ and returns back to the origin, and that the opposite behaviour occurs when C is large. This simple example therefore allows one to exhibit the relationship between phase lag and dissipation delay.

Example 4.62. If a system (A, B, C, D) is SPR and the vector relative degree $r = (1 \; ... \; 1)^T \in I\!R^m$ (*i.e.* $D = 0$), then the system is OSP. Indeed from the KYP Lemma 3.11, defining $V(x) = x^T P x$ one obtains $\dot{V}(x(t)) = -x^T(t)(QQ^T + L)x(t) + 2y^T(t)u(t)$ along the system's solutions. Integrating and taking into account that $L = 2\mu P$ is full rank, the result follows. It is noteworthy that the converse is not true. Any transfer function of the form $\frac{s+\alpha}{s^2+as+b}$, $b > 0$, $0 < a < 2\sqrt{b}$ is SPR if and only if $0 < \alpha < a$. However $\frac{s}{s^2+s+1}$ is not SPR (obvious!) but it defines an OSP system. One realization is given by $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = -x_1(t) - x_2(t) + u(t)$, $y(t) = x_2(t)$. One checks that $\int_0^t u(s)y(s)ds \geq -\frac{1}{2}(x_1^2(0) + x_2^2(0)) + \int_0^t y^2(s)ds$. Thus SPRness is only sufficient for OSPness, but it is not necessary.

Example 4.63. Consider the non-proper system $y(t) = \dot{u}(t) + au(t)$, $a > 0$, with relative degree $r = -1$. This system is SSPR and ISP since $\mathbf{Re}\,[j\omega + a] = a$ and

$$\int_0^t u(s)y(s)ds = \frac{u^2(t)}{2} + a\int_0^t u^2(s)ds$$

This plant belongs to the descriptor-variable systems (see Section 3.1.5), with state space representation:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ 0 = -x_1(t) + u(t) \\ y(t) = x_2(t) + au(t) \end{cases}$$

This can be rewritten as

$$\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ u(t) \end{pmatrix} \\ y(t) = (0 \; 1)x(t) + au(t) \end{cases} \tag{4.56}$$

This system is regular since $\det(sE - A) = 1$. The conditions of Proposition 3.15 and of Theorem 3.16 can be checked on this example. PRness can be checked with $P = 0$, while SSPRness amounts to finding $p_{21} > 0$, $p_{11} \neq p_{22}$, and w_{21} such that $\alpha w_{21} + \beta < 0$, with $\alpha = -(p_{11} - p_{22})^2 - (p_{11} - p_{22})(p_{22} - 1)$, $\beta = (p_{11} - p_{22})^2 a + p_{21}(p_{11} - p_{22})(p_{22} - 1) + p_{21}(p_{22} - 1)^2$.

Example 4.64. If a system (A, B, C, D) is SPR and if the matrix

$$\bar{Q} \triangleq \begin{pmatrix} Q + LL^T & LW \\ \\ W^T L^T & D + D^T \end{pmatrix}$$

is positive definite with $Q = -A^T P - PA$, then the system is VSP. This can be proved by using again $V(x) = x^T Px$. Let us denote $\bar{x} = \begin{pmatrix} x \\ u \end{pmatrix}$. Differentiating and using the KYP Lemma 3.11, one gets $\dot{V}(x(t)) = -\bar{x}^T(t)\bar{Q}\bar{x}(t) + 2y^T(t)u(t)$. One deduces that

$$\int_{t_0}^{t_1} u^T(t)y(t)dt \geq -V(x(t_0)) + \delta \int_{t_0}^{t_1} u^T(s)u(s)ds + \alpha \int_{t_0}^{t_1} y^T(s)y(s)ds$$

for some $\delta > 0$ and $\alpha > 0$ small enough [2]. Note that the condition $\bar{Q} > 0$ implies that the vector relative degree of (A, B, C, D) is equal to $(0 \, ... \, 0)^T$, which implies that the matrix $D \neq 0$. Indeed $D + D^T = W^T W$ and $W = 0$ implies that $\bar{Q}$ does not have full rank. In the monovariable case $m = 1$, then $r = 0$. In the multivariable case, $\bar{Q} > 0$ implies that W has full rank m. Indeed we can rewrite $\bar{Q} > 0$ as $x^T(Q + LL^T)x + u^T W^T Wu - 2x^T LWu > 0$. If W has rank $p < m$, then we can find a $u \neq 0$ such that $Wu = 0$. Therefore for the couple $x = 0$ and such a u, one has $\bar{x}^T \bar{Q}\bar{x} = 0$ which contradicts $\bar{Q} > 0$. We deduce that $r = (0 \, ... \, 0)^T \in I\!R^m$. *VSP linear invariant systems possess a uniform relative degree 0.*

Example 4.65. If a system (A, B, C, D) is SPR, then it is strictly passive with $\mathcal{S}(x) = x^T Qx$. This can be proved using the KYP Lemma.

Example 4.66. Consider the system $H(s) = \frac{1}{s+a}, a > 0$. We will now prove that the system is $H(s)$ is OSP. The system is described by

$$\dot{y}(t) = -ay(t) + u(t)$$

Let us consider the positive definite function $V(y) = \frac{1}{2}y^2$. Then

$$\dot{V}(y(t)) = y(t)\dot{y}(t) = -ay^2(t) + u(t)y(t)$$

Integrating we obtain

$$-V(y(0)) \leq V(y(t)) - V(y(0)) = -a \int_0^t y^2(s)ds + \int_0^t u(s)y(s)ds$$

$$\Longrightarrow \int_0^t u(s)y(s)ds + V(0) \geq a \int_0^t y^2(s)ds$$

Thus the system is OSP. Taking $a = 0$, we can see that the system whose transfer function is $\frac{1}{s}$, defines a passive system (the transfer function being PR).

[2] Once again we see that the system has zero bias provided $x(t_0) = 0$. But in general $\beta(x(t_0)) \neq 0$.

Remark 4.67. As we saw in Section 2.9 for linear systems, there exists a relationship between passive systems and $\mathcal{L}_2-$gain [125]. Let $\Sigma : u \to y$ be a passive system as in Definition 2.1. Define the input-output transformation $u = \gamma w + z$, $y = \gamma w - z$, (compare with (2.85)) then

$$\beta \leq \int_0^t u^T(s)y(s)ds = \int_0^t (\gamma^2 w^T(s)w(s) - z^T(s)z(s))ds$$

which is equivalent to

$$\int_0^t z^T(s)z(s)ds \leq \int_0^t \gamma^2 w^T(s)w(s)ds - \beta$$

which means that the system $\Sigma' : w \mapsto z$ has a finite $\mathcal{L}_2-$gain.

Example 4.68 ($\mathcal{L}_2$-gain). Let us consider the system $\dot{x}(t) = -x(t) + u(t)$, $y(t) = x(t)$. This system is dissipative with respect to the H_∞ supply rate $w(u,y) = \gamma^2 u^2 - y^2$ if and only if there exists a storage function $V(x)$ such that $\int_0^t (\gamma^2 u^2(\tau) - y^2(\tau))d\tau \geq V(x(t)) - V(x(0))$. Equivalently the infinitesimal dissipation inequality holds, *i.e.* $\gamma^2 u^2(t) - y^2(t) - \dot{V}(x(t))(-x(t) + u(t)) \geq 0$. Consider $V(x) = px^2$. The infinitesimal dissipation inequality then becomes $\gamma^2 u^2(t) - x^2(t) - 2px(t)(-x(t) + u(t)) \geq 0$. In a matrix form this is equivalent to having the matrix $\begin{pmatrix} 2p-1 & -p \\ -p & \gamma^2 \end{pmatrix} \geq 0$. This holds if and only if

$$\gamma^2(2p-1) - p^2 \geq 0 \tag{4.57}$$

This polynomial in p has a real solution if and only if $\gamma^2 \geq 1$. This polynomial is a Riccati inequality whose solvability is equivalent to $\gamma^2 \geq 1$. The system has an $\mathcal{L}_2$ gain equal to 1, and the condition that $\gamma^2 \geq 1$ agrees with this. Indeed the fact that the system is dissipative with respect to the above H_∞ supply rate implies that the H_∞-norm of its transfer function is $\leq \gamma$ (this is known as the Bounded Real Lemma; see Section 5.9).

This example together with Example 4.64 illustrates that the same system can be dissipative with respect to several supply rates, and with different storage functions.

Proposition 4.69. *Consider the system represented in Figure 4.5, where $\phi(\cdot)$ is a static nonlinearity, $q \geq 0$ and $\sigma\phi(\sigma) \geq 0$ for all $\sigma \in I\!R$. Then $H : u \mapsto y$ is passive.* ■

Proof: Let us adopt the classical notation $\langle u|y\rangle_t \stackrel{\Delta}{=} \int_0^t u(s)y(s)ds$. Then

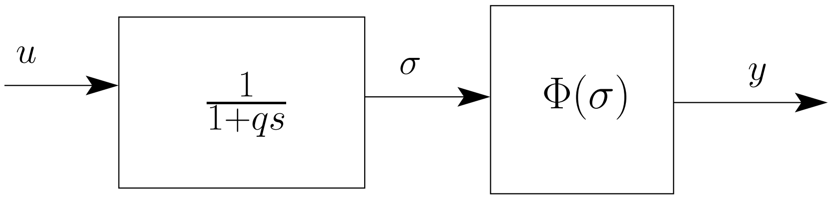

Fig. 4.5. A linear system and a static nonlinearity in cascade

$$\begin{aligned}\langle y \mid u\rangle_t &= \langle \phi(\sigma)|u\rangle_t \\ &= \langle \phi(\sigma)|q\dot{\sigma}+\sigma\rangle_t \\ &= q\int_0^t \phi(\sigma(s))\dot{\sigma}(s)ds + \int_0^t \sigma(s)\phi[\sigma(s)]ds \\ &= q\int_{\sigma(0)}^{\sigma(t)} \phi(\sigma)d\sigma + \int_0^t \sigma(s)\phi(\sigma(s))ds \\ &\geq q\int_0^{\sigma(t)} \phi(\sigma)d\sigma - q\int_0^{\sigma(0)} \phi[\sigma]d\sigma\end{aligned} \tag{4.58}$$

where we have used the fact that $\sigma(t)\phi(\sigma(t)) \geq 0$ for all $t \geq 0$. Note that $V(\sigma) = \int_0^\sigma \phi(\xi)d\xi \geq 0$ and is therefore qualified as a storage function, $\sigma(\cdot)$ being the state of this system.

Proposition 4.70. *If a system is output-strictly passive, then it is also weakly finite gain stable, i.e. OSP $\Rightarrow$ WFGS.* ■

Proof: The following upperbound can be computed:

$$\begin{aligned}\delta\int_0^t y^2(s)ds &\leq \beta + \int_0^t u(s)y(s)ds \\ &\leq \beta + \int_0^t u(s)y(s)ds + \tfrac{1}{2}\int_0^t(\sqrt{\lambda}u(s) - \tfrac{y(s)}{\sqrt{\lambda}})^2dt \\ &= \beta + \tfrac{\lambda}{2}\int_0^t u^2(s)ds + \tfrac{1}{2\lambda}\int_0^t y^2(s)ds\end{aligned} \tag{4.59}$$

Choosing $\lambda = \frac{1}{\delta}$ one gets

$$\frac{\delta}{2}\int_0^t y^2(s)ds \leq \beta + \frac{1}{2\delta}\int_0^t u^2(s)dt$$

which ends the proof. ■

Several results are given in [512] which concern the Lyapunov stability of systems which are finite-gain stable. They are not presented in this section since they rather belong to the kind of results presented in Section 5.1.

Example 4.71. Let us consider two linear systems in parallel, *i.e.*

$$\begin{cases} y_1(t) = k_1 u(t) \\ \dot{y}_2(t) = -a y_2(t) + k_2 u(t) \\ y(t) \ \ = y_1(t) + y_2(t) \end{cases} \tag{4.60}$$

where $a > 0$. Thus, for some constants β and k_3

$$\begin{aligned} \textstyle\int_0^t u(s)y(s)dt &= \textstyle\int_0^t u(s)y_1(s)ds + \int_0^t u(s)y_2(s)ds \\ &\geq k_1 \textstyle\int_0^t u^2(s)ds + \beta + k_3 \int_0^2 y_2^2(s)ds \\ &\geq \tfrac{k_1}{2} \textstyle\int_0^t u^2(s)ds + \beta + k' \int_0^t (y_1^2(s) + y_2^2(s))ds \\ &\geq \tfrac{k_1}{2} \textstyle\int_0^t u^2(s)ds + \beta + \tfrac{k'}{2} \int_0^t (y_1(s) + y_2(s))^2 ds \end{aligned} \tag{4.61}$$

where $k' \leq k_3$ and $k' \leq \frac{1}{2k_1}$. So the system $(\Sigma) : u \mapsto y$ is VSP.

4.4.5 Regularity of the Storage Functions

Until now we have not said a lot on the properties of the storage functions: are they differentiable (in x)? Continuous? Discontinuous? We now state some results which guarantee some regularity of storage functions. As we already pointed out, storage functions are potential Lyapunov functions candidate. It is well-known that Lyapunov functions need not be smooth, neither differentiable.

Continuous Storage Functions

Probably the first result in this direction is the following Lemma, for which we first need a preliminary definition.

Definition 4.72. *[209] A function $V : X \rightarrow I\!R$ is called a* virtual storage function *if it satisfies $V(0) = 0$ and*

$$V(x_0) + \int_{t_0}^{t_1} w(u(s)y(s)ds \geq V(x_1) \tag{4.62}$$

for all $t_1 \geq t_0$ and all admissible $u(\cdot)$, with $x(t_0) = x_0$ and $x(t_1) = x_1$. ■

Clearly if in addition one imposes that $V(x) \geq 0$ then one gets storage functions.

Lemma 4.73. *[209] Let the system (Σ) be locally w-uniformly reachable in the sense of Definition 4.44. Then any virtual storage function which exists for all $x \in X$ is continuous.* ■

Proof: Consider an arbitrary state $x_0 \in X$, and let a virtual storage function be $V(\cdot)$. Then for any x_1 in a neighborhood Ω of x_0, it follows from (4.62) that

$$V(x_0) + \int_{t_0}^{t_1} w(u(s), y(s)ds \geq V(x_1) \tag{4.63}$$

where the time t_1 corresponds to t in (4.36) and the controller $u(\cdot)$ is the one in Definition 4.44 (in other words, replace $[0, t]$ in (4.36) by $[t_0, t_1]$). From (4.36) and (4.63) and considering transitions in each direction between x_0 and x_1, one deduces that $\mid V(x_1) - V(x_0) \mid \leq \rho(\| x_1 - x_0 \|)$. Since x_1 is arbitrary in Ω and since $\rho(\cdot)$ is continuous, it follows that $V(\cdot)$ is continuous at x_0.

■

The next result concerns storage functions. Strong controllability means local $w-$uniform reachability in the sense of Definition 4.44, plus reachability, plus controllability. We recall that a system is controllable if every state $x \in X$ is controllable, *i.e.* given $x(t_0)$, there exists $t_1 \geq t_0$ and an admissible $u(\cdot)$ on $[t_0, t_1]$ such that the solution of the controlled system satisfies $x(t_1) = 0$ (sometimes this is named controllability to zero). Reachability is defined before Definition 4.36. Dissipativity in the next Theorem, is to be understood in Hill and Moylan's way; see (4.24).

Theorem 4.74. *[209] Let us assume that the system (Σ) in (4.20) is strongly controllable. Then the system is cyclo-dissipative (resp. dissipative in the sense of Definition 4.22) if and only if there exists a continuous function $V : X \to I\!R$ satisfying $V(0) = 0$ (resp. $V(0) = 0$ and $V(x) \geq 0$ for all $x \in X$) and $V(x(t)) \leq w(u(t), y(t))$ for almost all $t \geq 0$ along the system's trajectories.* ■

A relaxed version of Theorem 4.74 is as follows:

Theorem 4.75. *[401] Let the system $\dot{x}(t) = f(x(t), u(t))$ be dissipative in the sense of Definition 2.1 with supply rate $w(x, u)$, and locally $w-$uniformly reachable at the state $x^\star$. Assume that for every fixed u, the function $f(\cdot, u)$ is continuously differentiable, and that both $f(x, u)$ and $\frac{\partial f}{\partial x}(x, u)$ are continuous in x and u. Then the set $\mathcal{R}(x^\star)$ of states reachable form $x^\star$ is an open and connected set of X, and there exists a continuous function $V : \mathcal{R}(x^\star) \to I\!R^+$ such that for every $x_0 \in \mathcal{R}(x^\star)$ and every admissible $u(\cdot)$*

$$V(x(t)) - V(x_0) \leq \int_0^t w(x(s), u(s))ds \tag{4.64}$$

along the solution of the controlled system with $x(0) = x_0$. An example of such a function is $V_r(x) + \beta$, where β is a suitable constant and $V_r(x)$ is the required supply as in Definition 4.36. ■

We have already stated the last part of the Theorem in Lemma 4.45. The proof of Theorem 4.75 relies on an extended version of the continuous dependence of solutions with respect to initial conditions, and we omit it here. Let us now state a result that is more constructive, in the sense that it relies on verifiable properties of the system. Before this, we need the following intermediate Proposition.

Proposition 4.76. *[401] If the linearization of the vector field $f(x) + g(x)u$ around $x = 0$, given by $\dot{z}(t) = Az(t) + Bu(t)$ with $A = \frac{\partial f}{\partial x}(0)$ and $B = \frac{\partial g}{\partial x}(0)$, is controllable, then the system (Σ) in (4.20) is locally $w-$uniformly reachable at $x = 0$.* ■

Of course, controllability of the tangent linearization is here equivalent to having the Kalman matrix of rank n. This sufficient condition for local $w-$uniform reachability is easy to check, and one sees in passing that all time-invariant linear systems which are controllable, also are local $w-$uniformly reachable. Then the following is true, where dissipativity is understood in Hill and Moylan's sense; see (4.24):

Corollary 4.77. *[401] Let the system (Σ) be dissipative and suppose its tangent linearization at $x = 0$ is controllable. Then there exists a continuous storage function defined on the reachable set $\mathcal{R}(x^\star)$.* ■

Refinements and generalizations can be found in [402]. In Section 4.5 generalizations of the Kalman-Yakubovich-Popov Lemma will be stated which hold under the restriction that the storage functions (see then as the solutions of partial differential inequalities) are continuously differentiable (of class C^1 on the state space X). It is easy to exhibit systems for which no C^1 storage function exists. This will pose a difficulty in the extension of the KYP Lemma, which relies on some sort of infinitesimal version of the dissipation inequality. Indeed the PDIs will have then to be interpreted in a more general sense. More will be said in section 4.6. Results on dissipative systems depending on time-varying parameters, with continuous storage functions may be found in [302].

Differentiable Storage Functions

Let us end this section on regularity with a result that shows that in the one-dimensional case, the existence of locally Lipschitz storage functions implies the existence of continuous storage functions whose restriction to $\mathbb{R}^n \setminus \{x = 0\}$ is continuously differentiable. Such a set of functions is denoted as C_0^1. We specialize here to systems which are dissipative with respect to the supply rate $w(u, y) = \gamma^2 u^T u - y^T y$. This is a particular choice of the general supply rate in (4.44). In the differentiable case, the dissipation inequality in its infinitesimal form is

$$\nabla V^T(x(t))[f(x(t)+g(x(t))u] \leq \gamma^2 u^T(t)u(t) - y^T(t)y(t) \tag{4.65}$$

Let us define the following generalized derivative of the (non-differentiable) function $V(\cdot)$ at x

$$\partial V(x) = \lim_{h\to 0}\inf \frac{1}{\mid h \mid}[V(x+h) - V(x) - \zeta^T h] \tag{4.66}$$

where $\zeta \in \mathbb{R}^n$. When $\partial V(x) \geq 0$, one calls ζ a *viscosity subgradient* of $V(\cdot)$ at x. The set of all such vectors ζ, *i.e.* $D^- V(x)$, is possibly empty, but can also be non-single-valued (in other words: multivalued!). The viscosity subgradient is also sometimes called a *regular subgradient* [415, Equation 8(4)]. In case the function $V(\cdot)$ is proper convex, then the viscosity subgradient is the same as the subgradient from convex analysis defined in (3.188) [415, Proposition 8.12], and if $V(\cdot)$ is differentiable it is the same as the usual Euclidean gradient. An introduction to viscosity solutions is given in Section A.3 in the Appendix. With this machinery in mind, one may naturally rewrite (4.65) as

$$\zeta^T[f(x(t)+g(x(t))u] \leq \gamma^2 u^T(t)u(t) - y^T(t)y(t),\ \forall\, \zeta \in \partial V(x) \tag{4.67}$$

for all $x \in X \setminus \{0\}$ and all admissible $u(\cdot)$ (see Proposition A.52 in the Appendix). If the function $V(\cdot)$ is differentiable, then (4.67) becomes the usual infinitesimal dissipation inequality $\nabla V^T(x)[f(x(t)+g(x(t))u] \leq \gamma^2 u^T(t)u(t) - y^T(t)y(t)$. As we saw in Section 3.9.4, it is customary in nonsmooth and convex analysis, to replace the usual gradient by a set of subgradients. The set of all continuous functions $V : \mathbb{R}^n \to \mathbb{R}^+$ that satisfy (4.67) is denoted as $\mathcal{W}(\Sigma, \gamma^2)$. The set of all functions in $\mathcal{W}(\Sigma, \gamma^2)$ which are proper (radially unbounded) and positive definite, is denoted as $\mathcal{W}_\infty(\Sigma, \gamma^2)$.

Theorem 4.78. *[418] Let $n = m = 1$ in (4.74) and assume that the vector fields $f(x)$ and $g(x)$ are locally Lipschitz. Assume that for some $\gamma > 0$ there exists a locally Lipschitz $V \in \mathcal{W}_\infty(\Sigma, \gamma^2)$. Then $\mathcal{W}_\infty(\Sigma, \gamma^2) \cap C_0^1 \neq \emptyset$.* ■

The proof is rather long and technical so we omit it here. This result means that for scalar systems, there is no gap between locally Lipschitz and C_0^1 cases. When $n \geq 2$ the result is no longer true as the following examples prove [418].

Example 4.79. [418] Consider the system (Σ_1) with $n = m = 2$:

$$\begin{cases} \dot{x}_1(t) = \mid x_1(t) \mid (-x_1(t) + \mid x_2(t) \mid + u_1(t)) \\ \\ \dot{x}_2(t) = x_2(t)(-x_1(t) - \mid x_2(t) \mid + u_2(t)) \end{cases} \tag{4.68}$$

Let us define $V_1(x) = 2 \mid x_1 \mid + 2 \mid x_2 \mid$, which is a proper, positive definite and globally Lipschitz function. Moreover $V_1 \in \mathcal{W}_\infty(\Sigma_1, 1)$. However it is not C_0^1 and any function that is C_0^1 and which belongs to $\mathcal{W}(\Sigma_1, 1)$, is neither positive definite nor proper [418, Proposition 2.2].

Example 4.80. [418] Consider the system (Σ_2) with $n = m = 2$:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + x_2(t) + u_1(t) \\ \\ \dot{x}_2(t) = 3x_2^{\frac{4}{3}}(t)(-x_1(t) - x_2(t) + u_2(t)) \end{cases} \tag{4.69}$$

Let us consider $V_2(x_1, x_2) = x_1^2 + x_2^{\frac{2}{3}}$. This function is proper, positive definite, and continuous. Moreover $V_2 \in \mathcal{W}_\infty(\Sigma_2, 1)$. However any locally Lipschitz function in $\mathcal{W}(\Sigma_2, 1)$ is neither positive definite nor proper.

Things are however not so dramatic as the next Theorem shows:

Theorem 4.81. *[418] For any system (Σ) with locally Lipschitz vector fields $f(x)$ and $g(x)$,*

$$\inf\{\gamma \mid \mathcal{W}_\infty(\Sigma, \gamma^2) \neq \emptyset\} = \inf\{\gamma \mid \mathcal{W}_\infty(\Sigma, \gamma^2) \cap C_0^1 \neq \emptyset\} \tag{4.70}$$

■

In other words, Theorem 4.81 says that, given a γ, if one is able to exhibit at least one function in $\mathcal{W}_\infty(\Sigma, \gamma^2)$, then increasing slightly γ allows one to get the existence of a function that is both in $\mathcal{W}_\infty(\Sigma, \gamma^2)$ and in C_0^1. This is a sort of regularization of the storage function of a system that is dissipative with respect to the supply rate $w(u, y) = \gamma^2 u^T u - y^T y$.

Remark 4.82. The results hold for systems which are affine in the input, as in (4.20). For more general systems they may not remain true.

Example 4.83. Let us lead some calculations for the system and the Lyapunov function of Example 4.79. We get

$$\begin{array}{c} \partial V_1(x) = \begin{pmatrix} 2 \text{ or } -2 \text{ or } [-2, 2] \\ \\ 2 \text{ or } -2 \text{ or } [-2, 2] \end{pmatrix} \\ \\ \qquad\qquad \uparrow \qquad\quad \uparrow \qquad\quad \uparrow \\ \\ \qquad\qquad x_i > 0 \quad x_i < 0 \quad x_i = 0 \end{array} \tag{4.71}$$

Thus the left hand side of (4.67) is

$$\begin{cases} \zeta_1 |x_1| (-x_1 + |x_2| + u_1) \\ \\ \zeta_2 x_2 (-x_1 - |x_2| + u_2) \end{cases} \tag{4.72}$$

Thus we may write the first line, taking (4.71) into account, as

$$\begin{cases} 2(-x_1^2 + x_1|x_2| + x_1u_1) & \text{if } x_1 > 0 \\ 2(x_1^2 - x_1|x_2| - x_1u_1) & \text{if } x_1 < 0 \\ [-2|x_1|(-x_1 + |x_2| + u_1); 2|x_1|(-x_1 + |x_2| + u_1)] = \{0\} & \text{if } x_1 = 0 \end{cases} \tag{4.73}$$

and similarly for the second line. It happens that $V(\cdot)$ is not differentiable at $x = 0$, and that $f(0)+g(0)u = 0$. Let $y_1 = x_1$, $y_2 = x_2$. Consider the case $x_1 > 0$, $x_2 > 0$. We obtain $-2y^T y+2y^T u \leq -2y^T y+y^T y+u^T u = -y+y^T y+u^T u$. For $x_2 > 0$ and $x_1 = 0$ we obtain $-2y_2^2+2y_2u_2 \leq -y+y^T y+u^T u = -y_2^2+u^T u$.

4.5 Nonlinear KYP Lemma

4.5.1 A Particular Case

The KYP Lemma for linear systems can be extended for nonlinear systems having state-space representations affine in the input. In this section we will consider the case when the plant output y is not a function of the input u. A more general case will be studied in the next section. Consider the following nonlinear system

$$(\Sigma) \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) \\ x(0) = x_0 \end{cases} \tag{4.74}$$

where $x(t) \in I\!R^n$, $u(t) \in I\!R^m$, $y(t) \in I\!R^m$, $f : I\!R^n \to I\!R^n$ with $f(0) = 0$, $h(0) = 0$, $g : I\!R^n \to I\!R^{n\times m}$, $h : I\!R^n \to I\!R^m$, are smooth functions of x. We then have the following result.

Lemma 4.84 (KYP Lemma for nonlinear systems). *Consider the nonlinear system (4.74). The following statements are equivalent.*

(1) There exists a $\mathcal{C}^1$ storage function $V(x) \geq 0$, $V(0) = 0$ and a function $\mathcal{S}(x) \geq 0$ such that for all $t \geq 0$:

$$V(x(t)) - V(x(0)) = \int_0^t y^T(s)u(s)ds - \int_0^t \mathcal{S}(x(s))ds \tag{4.75}$$

The system is strictly passive for $\mathcal{S}(x) > 0$, passive for $\mathcal{S}(x) \geq 0$ and lossless for $\mathcal{S}(x) = 0$.

(2) There exists a $\mathcal{C}^1$ non-negative function $V : X \to I\!R$ with $V(0) = 0$, such that

$$\begin{cases} L_f V(x) = -\mathcal{S}(x) \\ L_g V(x) = h^T(x) \end{cases} \tag{4.76}$$

where $L_gV(x) = \frac{\partial V(x)}{\partial x} g(x)$. ■

Remark 4.85. Note that if $V(x)$ is a positive definite function (i.e. $V(x) > 0$), then the system $\dot{x}(t) = f(x(t))$ has a stable equilibrium point at $x = 0$. If in addition $\mathcal{S}(x) > 0$ then $x = 0$ is an asymptotically stable equilibrium point.

Proof of Lemma 4.84:

- (1)⇒ (2). By assumption we have

$$V(x(t)) - V(x(0)) = \int_0^t y^T(s)u(s)ds - \int_0^t \mathcal{S}(x(s))ds \tag{4.77}$$

Taking the derivative with respect to t and using (4.74)

$$\begin{aligned} \tfrac{d(V\circ x)}{dt}(t) &= \tfrac{\partial V(x)}{\partial x}\dot{x}(t) \\ &= \tfrac{\partial V(x)}{\partial x}(f(x(t)) + g(x(t))u(t)) \\ &\triangleq L_fV(x(t)) + L_gV(x(t))u(t) \\ &= y^T(t)u(t) - \mathcal{S}(x(t)) \ \ \text{(see (4.74))} \end{aligned} \tag{4.78}$$

Taking the partial derivative with respect to u, we get $L_fV(x) = -\mathcal{S}(x)$ and therefore $L_gV(x) = h^T(x)$.
- (2)⇒(1). From (4.74) and (4.76) we obtain

$$\frac{d(V\circ x)}{dt}(t) = L_fV(x(t)) + L_gV(x(t))u(t) = -\mathcal{S}(x(t)) + h^T(x(t))u(t)$$

Integrating the above we obtain (4.74). ■

Remark 4.86. From these developments, the dissipativity equality in (4.75) is equivalent to its infinitesimal version $\dot{V} = L_fV + L_gVu = h^T(x)u(t) - \mathcal{S}(x) = \langle u, y\rangle - \mathcal{S}(x)$. Obviously this holds under the assumption that $V(\cdot)$ is sufficiently regular (differentiable). No differentiability is required in the general Willems' Definition of dissipativity, however. Some authors [228] systematically define dissipativity with C^1 storage functions satisfying $\alpha(||x||) \leq V(x) \leq \beta(||x||)$ for some class-$\mathcal{K}_\infty$ functions, and infinitesimal dissipation equalities or inequalities. Such a definition of dissipativity is therefore much more stringent than the basic definitions of Section 4.4.

4.5.2 Nonlinear KYP Lemma in the General Case

We will now consider the more general case in which the system is described by the following state-space representation affine in the input:

$$(\Sigma)\begin{cases}\dot{x}(t) = f(x(t)) + g(x(t))u(t)\\ \\ y(t) = h(x(t)) + j(x(t))u(t),\end{cases} \tag{4.79}$$

where $x(t) \in I\!R^n$, $u(t) \in I\!R^m$, $y(t) \in I\!R^m$, and $f : I\!R^n \rightarrow I\!R^n$, $g : I\!R^n \rightarrow I\!R^{n\times m}$, $h : I\!R^n \rightarrow I\!R^m$, $j : I\!R^n \rightarrow I\!R^{m\times m}$, are smooth functions of x with $f(0) = 0, h(0) = 0$. What follows may be seen as settling the material of Definiton 2.1, Theorem 2.2 and Corollary 2.3 in the context of dissipative systems.

Assumption 3 *The state space of the system at (4.79) is reachable from the origin. More precisely given any x_1 and t_1, there exists $t_0 \leq t_1$ and an admissible control $u(\cdot)$ such that the state can be driven from $x(t_0) = 0$ to $x(t_1) = x_1$.*

Assumption 4 *The available storage $V_a(\cdot)$, when it exists, is a differentiable function of x.*

These two assumptions are assumed to hold throughout this section. Consider the general supply rate:

$$\begin{aligned} w(u, y) &= y^T Q y + 2y^T S u + u^T R u \\ &= \left[y^T u^T\right] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \end{aligned} \tag{4.80}$$

with $Q = Q^T, R = R^T$. We then have the following Theorem which is due to Hill and Moylan [207], and concerns the dissipativity as in Definition 4.22.

Lemma 4.87 (NL KYP Lemma: general case). *The nonlinear system* (4.79) *is dissipative in the sense of Definition 4.22 with respect to the supply rate* $w(u,y)$ *in* (4.80) *if and only if there exists functions* $V : \mathbb{R}^n \to \mathbb{R}$, $L : \mathbb{R}^n \to \mathbb{R}^q$, $W : \mathbb{R}^n \to \mathbb{R}^{q\times m}$ *(for some integer* q*), with* $V(\cdot)$ *differentiable, such that:*

$$\begin{aligned} V(x) &\geq 0 \\ V(0) &= 0 \\ \nabla V^T(x) f(x) &= h^T(x) Q h(x) - L^T(x) L(x) \\ \tfrac{1}{2} g^T(x) \nabla V(x) &= \hat{S}^T(x) h(x) - W^T(x) L(x) \\ \hat{R}(x) &= W^T(x) W(x) \end{aligned} \tag{4.81}$$

where

$$\begin{cases} \hat{S}(x) \overset{\Delta}{=} Qj(x) + S \\ \hat{R}(x) \overset{\Delta}{=} R + j^T(x) S + S^T j(x) + j^T(x) Q j(x) \end{cases} \tag{4.82}$$

Proof:
Sufficiency. From (4.80), (4.79), (4.81) and (4.82) we obtain

$$\begin{aligned} w(u,y) &= y^T Q y + 2y^T S u + u^T R u \\ &= (h(x) + j(x)u)^T Q (h(x) + j(x)u) + 2(h(x) + j(x)u)^T S u + u^T R u \\ &= h^T(x) Q h(x) + 2u^T j^T(x) Q h(x) + u^T j^T(x) Q j(x) u + u^T R u + \\ &\quad + 2u^T j^T(x) S u + 2h^T(x) S u \\ &= h^T(x) Q h(x) + 2u^T j^T(x) Q h(x) + u^T \hat{R}(x) u + 2h^T(x) S u \end{aligned} \tag{4.83}$$

so that

$$\begin{aligned}
w(u,y) &= \nabla V^T(x)f(x) + L^T(x)L(x) + u^T\hat{R}(x)u + 2u^T[S^T + j^T(x)Q]h(x) \\
&= \nabla V^T(x)f(x) + L^T(x)L(x) + u^T\hat{R}(x)u + 2u^T\hat{S}^T(x)h(x) \\
&= \nabla V^T(x)f(x) + L^T(x)L(x) + u^TW^T(x)W(x)u + u^Tg^T(x)\nabla V(x)+ \\
&\quad +2u^TW^T(x)L(x) \\
&= \nabla V^T(x)\dot{x} + (L(x)+W(x)u)^T(L(x)+W(x)u) \\
&\geq \nabla V^T(x)\dot{x} = \dot{V}(x)
\end{aligned} \tag{4.84}$$

Integrating the above we get

$$\int_0^t w(s)ds \geq V(x(t)) - V(x(0)) \tag{4.85}$$

Necessity. We will show that the available storage function $V_a(x)$ is a solution to the set of equations (4.81) for some $L(\cdot)$ and $W(\cdot)$. Since the system is reachable from the origin, there exists $u(.)$ defined on $[t_{-1}, 0]$ such that $x(t_{-1}) = 0$ and $x(0) = x_0$. Since the system (4.79) is dissipative it satisfies (4.24), then there exists $V(x) \geq 0, V(0) = 0$ such that:

$$\begin{aligned}
\textstyle\int_{t_{-1}}^t w(s)ds &= \textstyle\int_{t_{-1}}^0 w(t)dt + \int_0^t w(s)ds \\
&\geq V(x(t)) - V(x(t_{-1})) \\
&\geq 0
\end{aligned}$$

Remember that $\int_{t_{-1}}^t w(s)ds$ is the energy introduced into the system. From the above we have

$$\int_0^t w(s)ds \geq -\int_{t_{-1}}^0 w(t)dt$$

The right-hand side of the above depends only on x_0. Hence, there exists a bounded function $C(\cdot) \in \mathbb{R}$ such that

$$\int_0^t w(s)ds \geq C(x_0) > -\infty$$

Therefore the available storage is bounded:

$$0 \leq V_a(x) = \sup_{x=x(0),t_1\geq 0,u} \left\{-\int_0^t w(s)ds\right\} < +\infty.$$

Dissipativeness in the sense of Definition 4.22 implies that $V_a(0) = 0$ and the available storage $V_a(x)$ is itself a storage function, *i.e.*

$$V_a(x(t)) - V_a(x(0)) \le \int_0^t w(s)ds \ \ \forall \ \ t \ge 0$$

or

$$0 \le \int_0^t (w(s) - \frac{dV_a}{dt}(s))ds \ \ \forall \ \ t \ge 0$$

Since the above inequality holds for all $t \ge 0$, taking the derivative in the above it follows that

$$0 \le w(u,y) - \frac{d(V_a \circ x)}{dt} \stackrel{\Delta}{=} d(x,u)$$

Introducing (4.79)

$$\begin{aligned} d(x,u) &= w(u,y) - \tfrac{d(V_a \circ x)}{dt} \\ &= w[u, h(x) + j(x)u] - \tfrac{\partial V_a}{\partial x}(x)\left[f(x) + g(x)u\right] \\ &\ge 0 \end{aligned} \tag{4.86}$$

Since $d(x,u) \ge 0$ and since $w(u,y) = y^T Q y + 2y^T S u + u^T R u$, it follows that $d(x,u)$ is quadratic in u and may be factored as

$$d(x,u) = \left[L(x) + W(x)u\right]^T \left[L(x) + W(x)u\right]$$

for some $L(x) \in I\!\!R^q, W(x) \in I\!\!R^{q \times m}$ and some integer q. Therefore from the two previous equations and the system (4.79) and the Definitions in (4.82) we obtain

$$\begin{aligned} d(x,u) &= -\tfrac{\partial V_a}{\partial x}(x)\left[f(x) + g(x)u\right] + (h(x) + j(x)u)^T Q (h(x) + j(x)u) + \\ &\quad + 2(h(x) + j(x)u)^T S u + u^T R u \\ &= -\nabla V_a^T(x) f(x) - \nabla V_a^T(x) g(x) u + h^T(x) Q h(x) + \\ &\quad + 2h^T(x)\left[Qj(x) + S\right]u + u^T\left[R + j^T(x)S + S^T j(x) + j^T(x)Qj(x)\right]u \\ &= -\nabla V_a^T(x) f(x) - \nabla V_a^T(x) g(x) u + h^T(x) Q h(x) + \\ &\quad + 2h^T(x)\hat{S}(x)u + u^T \hat{R}(x) u \\ &= L^T(x)L(x) + 2L^T(x)W(x)u + u^T W^T(x) W(x) u \end{aligned} \tag{4.87}$$

which holds for all x, u. Equating coefficients of like powers of u we get:

$$\begin{aligned}\nabla V_a^T(x)f(x) &= h^T(x)Qh(x) - L^T(x)L(x)\\ \tfrac{1}{2}g^T(x)\nabla V_a(x) &= \hat{S}^T(x)h(x) - W^T(x)L(x)\\ \hat{R}(x) &= W^T(x)W(x)\end{aligned} \tag{4.88}$$

which concludes the proof. ∎

If cyclo-dissipativity is used instead of dissipativity, then the first two conditions on the storage function $V(\cdot)$ can be replaced by the single condition that $V(0) = 0$ [209]. Consequently, Lemma 4.87 proves that:

Hill-Moylan's dissipativity + reachability from $x = 0$ + C^1 available storage
$\Updownarrow$
Willems' dissipativity with one C^1 storage function $V(\cdot)$ with $V(0) = 0$.

Actually the Lemma proves the $\Rightarrow$ sense, and the $\Leftarrow$ sense is obvious. Using the sufficiency part of the proof of the above Theorem we have the following Corollary, which holds under Assumptions 3 and 4:

Corollary 4.88. *[207] If the system (4.79) is dissipative with respect to the supply rate $w(u, y)$ in (4.80), then there exists $V(x) \geq 0, V(0) = 0$ and some $L : X \to I\!R^q$, $W : X \to I\!R^{q\times m}$ such that*

$$\frac{d(V \circ x)}{dt} = -\left[L(x) + W(x)u\right]^T \left[L(x) + W(x)u\right] + w(u, y).$$

∎

Under the conditions of Corollary 4.88, the dissipation function in (4.55) is equal to $\mathcal{D}(x(0), u, t) = \int_0^t \left[L(x(s)) + W(x(s))u(s)\right]^T \left[L(x(s)) + W(x(s))u(s)\right] ds$. What about generalizations of the KYP Lemma when storage functions may not be differentiable (even possibly discontinuous)? The extension passes through the fact that the conditions (4.81) and (4.82) can be rewritten as a partial differential inequality which is a generalization of a Riccati inequation (exactly as in Section 3.1.2 for the linear time invariant case). Then relax the notion of solution to this PDI to admit continuous (or discontinuous) storage functions. See Section 4.6.

Remark 4.89. The Lemma 4.84 is a special case of Lemma 4.87 for

$$Q = 0,\ R = 0,\ S = \frac{1}{2}I,\ j = 0$$

In that case (4.81) reduces to

$$\begin{cases} \nabla V^T(x)f(x) = -L^T(x)L(x) = -S(x) \\ \\ g^T(x)\nabla V(x) = h(x) \end{cases} \tag{4.89}$$

Remark 4.90. If $j(x) \equiv 0$, then the system in (4.79) cannot be ISP (that corresponds to having $R = -\epsilon I$ in (4.80) for some $\epsilon > 0$). Indeed if (4.79) is dissipative with respect to (4.80) we obtain along the system's trajectories:

$$\begin{aligned} \tfrac{d(V\circ x)}{dt}(t) &= w(u(t), y(t)) \\ &= h^T(x(t))Qh(x(t)) - L(x(t))L^T(x(t)) + 2h^T(x(t))\hat{S}(x(t))u(t) \\ &-L^T(x(t))W(x(t))u(t) \\ &= (y(t) - j(x(t))u(t))^T Q(y(t) - j(x(t))u(t)) - L(x(t))L^T(x(t)) \\ &+2(y(t) - j(x(t))u(t))^T[Qj(x(t)) + S]u(t) - L^T(x(t))W(x(t)) \\ &= y^T(t)Qy(t) - 2y^T(x(t))Qj(x(t))u(t) + u^T(t)j^T(x(t))Qj(x(t))u(t) \\ &-L(x(t))L^T(x(t)) \\ &+2y^T(t)Qj(x(t))u(t) + 2y^T(t)Su(t) - 2u^T(t)j^T(x(t))Qj(x(t))u(t) \\ &-2u^T(t)j^T(x(t))Su(t) \\ &= y^T(t)Qy(t) + 2y^T(t)Su(t) - \epsilon u^T(t)u(t) \end{aligned} \tag{4.90}$$

If $j(x) = 0$ we get $-L(x)L^T(x) = -\epsilon u^T u$ which obviously cannot be satisfied with x and u considered as independent variables (except if both sides are constant and identical). This result is consistent with the linear case (a PR or SPR function has to have relative degree 0 to be ISP).

4.5.3 Time-varying Systems

All the results presented until now deal with time-invariant systems. This is partly due to the fact that dissipativity is a tool that is used to study and design stable closed-loop systems, and the Krasovskii-LaSalle invariance principle is at the core of stability proofs (this will be seen in Chapter 5). As far as only dissipativity is in question, one can say that most of the tools we have presented in the foregoing sections, extend to the case:

$$(\Sigma_t)\begin{cases} \dot{x}(t) = f(x(t,t)) + g(x(t),t)u(t) \\ \\ y(t) = h(x(t),t) + j(x(t),t)u(t) \end{cases} \tag{4.91}$$

where the well-posedness conditions are assumed to be fulfilled (see section 3.9.2). The available storage and required supply are now defined as

$$V_a(t_0, x) = \sup_{x=x(t_0),u(\cdot),t_1\geq t_0} -\int_{t_0}^{t_1} w(u(t), y(t))dt \tag{4.92}$$

and

$$V_r(t_0, x) = \inf_{u(\cdot),t\leq t_0} \int_{t}^{t_0} w(u(t), y(t))dt \tag{4.93}$$

Then one has:

Lemma 4.91. *Let Assumptions 3 and 4 hold for (4.91). Suppose moreover that the required supply $V_r(t, x)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}$. The system (4.91) is dissipative in the sense of Definition 2.1 with $\beta = 0$ if and only if there exists a continuous almost everywhere differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $V(t, x) \geq 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $V(t, 0) = 0$ for all $t \in \mathbb{R}$, and such that*

$$\begin{pmatrix} -\nabla V^T(x)f(x) - \frac{\partial V}{\partial t} & h^T(x) - \frac{1}{2}\nabla V^T(x)g(x,t) \\ h(x) - \frac{1}{2}g^T(x,t)\nabla V(x) & j(x,t) + j^T(x,t) \end{pmatrix} \geq 0 \tag{4.94}$$

■

4.5.4 Nonlinear-in-the-input Systems

So far only nonlinear systems which are linear in the input have been considered in this book. It seems that there is no KYP Lemma extension for systems of the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \\ x(0) = x_0 \end{cases} \tag{4.95}$$

with $f(0, 0) = 0$ and $h(0, 0) = 0$. It is assumed that $f(\cdot, \cdot)$ and $h(\cdot, \cdot)$ are smooth functions (infinitely differentiable).

Proposition 4.92. *[303] Let $\Omega = \{x \in \mathbb{R}^n \mid \frac{\partial V}{\partial x} f(x, 0) = 0\}$. Necessary conditions for the system in (4.95) to be passive with a C^2 storage function $V(\cdot)$ are that*

- *(a)* $\frac{\partial V}{\partial x} f(x, 0) \leq 0$
- *(b)* $\frac{\partial V}{\partial x} g(x, 0) = h^T(x, 0)$ *for all* $x \in \Omega$

- *(c)* $\sum_{i=1}^{n} \frac{\partial^2 f_i}{\partial u^2}(x,0).\frac{\partial V}{\partial x_i} \leq \frac{\partial h}{\partial u}^T(x,0) + \frac{\partial h}{\partial u}(x,0)$ *for all* $x \in \Omega$

where $f_i(x,u)$ *is the ith component of the vector function* $f(x,u)$. ■

Proof: [303] Consider an auxiliary function $F : I\!R^n \times I\!R^m \rightarrow I\!R$ defined as $F(x,u) = \frac{\partial V}{\partial x} f(x,u) - h^T(x,u)u$. Since the system in (4.79) is passive, it is clear that $F(x,u) \leq 0$ for all $u \in I\!R^m$. Therefore (a) follows by setting $u = 0$. For all $x \in \Omega$, one has $F(x,0) = \frac{\partial V}{\partial x} f(x,0) = 0$. Thus $F(x,u) \leq F(x,0) = 0$ for all $x \in \Omega$ and for all $u \in I\!R^m$. In other words $F(x,u)$ attains its maximum at $u = 0$ on the set Ω. Let us now define $g_0(x) = \frac{\partial f}{\partial u}(x,0)$. We obtain for all $x \in \Omega$

$$\begin{cases} 0 = \frac{\partial F}{\partial u}(x,0) & = \frac{\partial V}{\partial x}\frac{\partial f}{\partial u}(x,0) - h^T(x,0) \\ 0 \geq \frac{\partial^2 F}{\partial u^2}(x,0) = & \frac{\partial((\partial V/\partial x)(\partial f/\partial u))}{\partial u}|_{u=0} \quad - \left(\frac{\partial h}{\partial u}(x,0) + \frac{\partial h}{\partial u}^T(x,0)\right) \\ & = \sum_{i=1}^{n} \frac{\partial^2 f_i}{\partial u^2}(x,0).\frac{\partial V}{\partial x_i} \quad - \left(\frac{\partial h}{\partial u}(x,0) + \frac{\partial h}{\partial u}^T(x,0)\right) \end{cases} \tag{4.96}$$

from which (b) and (c) follow. ■

4.6 Dissipative Systems and Partial Differential Inequalities

As we have seen in Section 4.4.5, storage functions are continuous under some reasonable controllability assumptions. However it is a much stronger assumption to suppose that they are differentiable, or of class C^1. The versions of the KYP Lemma that have been presented above, rely on the property that $V(\cdot)$ is C^1. Here we show how to relax this property, by considering the infinitesimal version of the dissipation inequality: this is a partial differential inequality which represents the extension of the KYP Lemma to the case of continuous, non-differentiable storage functions.

4.6.1 The linear invariant case

First of all and before going on with the nonlinear affine-in-the-input case, let us investigate a novel path to reach the conclusions of Section 3.1.2. We consider the linear time-invariant system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Dx(t) \end{cases} \tag{4.97}$$

Let us define the Hamiltonian function

$$\boxed{H(x,p) = \sup_{u \in \mathbb{R}^m} [p^T(Ax + Bu) - w(u,y)] \qquad (4.98)}$$

where the supply rate is chosen as $w(u,y) = u^T y$. By rearranging terms one gets

$$H(x,p) = p^T Ax + \sup_{u \in \mathbb{R}^m} [(p^T B - x^T C^T)u - u^T Du] \tag{4.99}$$

$D > 0$

Let us assume that $D > 0$ ($\Longleftrightarrow D + D^T > 0$), so that the maximizing u is given by

$$u^\star = (D + D^T)^{-1}(B^T p - Cx) \tag{4.100}$$

and the matrix $D + D^T$ arises from the derivation of $u^T Du$. Injecting $u^\star$ into $H(x,p)$ and rewriting $u^T Du$ as $\frac{1}{2}u^T(D + D^T)u$, one obtains

$$H(x,p) = p^T Ax + \frac{1}{2}(B^T p - Cx)^T(D + D^T)^{-1}(B^T p - Cx) \tag{4.101}$$

Let us now consider the quadratic function $V(x) = \frac{1}{2}x^T Px$, $P = P^T$, and $H(x,P) \stackrel{\Delta}{=} H(x, \frac{\partial V}{\partial x})$. We obtain

$$H(x,P) = x^T PAx + \frac{1}{2}(B^T Px - Cx)^T(D + D^T)^{-1}(B^T Px - Cx) \tag{4.102}$$

Now imposing that $H(x,P) \leq 0$ for all $x \in \mathbb{R}^n$ and using $x^T PAx = \frac{1}{2}x^T(A^T P + PA)x$ we get

$$A^T P + PA + (PB - C^T)(D + D^T)^{-1}(B^T P - C) \leq 0, \tag{4.103}$$

which is the Riccati inequality in (3.17). We have therefore shown that under the condition $D > 0$ the inequality $H(x, \frac{\partial V}{\partial x}) \leq 0$ is equivalent to the Riccati inequality in (4.103), thus to the matrix inequality in (3.3).

$D = 0$

Let us now investigate what happens when $D = 0$. Following the same reasoning one finds that the maximizing input does not exist (the function to maximize is $(p^T B - x^T C^T)u$) so that it is necessary for the supremum to have a meaning (to be different from $+\infty$) that $p^T B - x^T C^T = 0$ for all $x \in \mathbb{R}^n$. Choosing the same storage function as above it follows that $H(x, \frac{\partial V}{\partial x}) \leq 0$ yields $PA + A^T P \leq 0$ and $PB = C^T$.

$D \geq 0$

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function not identically $+\infty$, minorized by an affine function. Then the *conjugate* function of $f(\cdot)$ is defined by [210, Definition 1.1.1]

$$f^\star(z) \stackrel{\Delta}{=} \sup_{u \in \mathrm{dom}(f)} [z^T u - f(u)] \tag{4.104}$$

Doing the analogy with (4.98) one finds $f(u) = u^T D u$, $z = B^T p - Cx$, and $H(z)$ is the sum of the conjugate of $f(u)$ and $p^T A x$. It is a basic result from convex analysis that if $D + D^T > 0$ then

$$f^\star(z) = z^T (D + D^T)^{-1} z, \tag{4.105}$$

from which one straightforwardly recovers the previous results and the Riccati inequality. We also saw what happens when $D = 0$. Let us now investigate the case $D + D^T \geq 0$. We get [210, Example 1.1.4]:

$$f^\star(z) = \begin{cases} +\infty & \text{if } z \notin \mathrm{Im}(D + D^T) \\ z^T (D + D^T)^\dagger z & \text{if } z \in \mathrm{Im}(D + D^T) \end{cases} \tag{4.106}$$

where $(D + D^T)^\dagger$ is the Moore-Penrose pseudo-inverse of $(D + D^T)$. Replacing z by its value we obtain

$$H(x, p) = p^T A x +$$

$$+ \begin{cases} +\infty & \text{if } B^T p - Cx \notin \mathrm{Im}(D + D^T) \\ (B^T p - Cx)^T (D + D^T)^\dagger (B^T p - Cx) & \text{if } B^T p - Cx \in \mathrm{Im}(D + D^T) \end{cases} \tag{4.107}$$

Setting $p = \frac{\partial V}{\partial x}$ and $V = \frac{1}{2} x^T P x$ with $P = P^T$ it follows from $H(x, p) \leq 0$ for all $x \in \mathbb{R}^n$, that P is the solution of a degenerate Riccati inequality (DRI):

$$\begin{cases} \textbf{(i)} \ \ \mathrm{Im}(B^T P - C) \subseteq \mathrm{Im}(D + D^T) \\ \textbf{(ii)} \ \ PA + A^T P + (B^T P - C)^T (D + D^T)^\dagger (B^T P - C) \leq 0 \end{cases} \tag{4.108}$$

Is (4.108) equivalent to the KYP Lemma conditions? The following can be proved:

- (3.2) $\Longrightarrow$ (4.108) (**i**),
- The conditions in (3.2) are equivalent to

$$\begin{cases} \textbf{(i)} \;\; LL^T - LW(W^TW)^{\dagger}W^TL \geq 0 \\ \\ \textbf{(ii)} \;\; LW[I_m - W^TW(W^TW)^{\dagger}] = 0 \end{cases} \tag{4.109}$$

whose proof can be deduced almost directly from Lemma A.65 noticing that $W^TW \geq 0$.

Notice that (4.109) **(ii)** is equivalently rewritten as

$$PB - C^T = PB - C^T(D + D^T)(D + D^T)^{\dagger} \tag{4.110}$$

It follows from (4.110) and standard matrix algebra [272, p.78,p.433] that $\mathrm{Im}(B^TP - C)$ $=\mathrm{Im}[(D + D^T)^{\dagger}(D + D^T)(B^TP - C)] \subseteq \mathrm{Im}[(D + D^T)^{\dagger}(D + D^T)] \subseteq \mathrm{Im}((D + D^T)^{\dagger}) = \mathrm{Im}(D + D^T)$. Thus (4.110) $\Longleftrightarrow$ (4.109) **(ii)** $\Longleftrightarrow$ (4.108) **(i)**. Now obviously (4.109) **(i)** is nothing else but (4.108) **(ii)**. We therefore conclude that the conditions of the KYP Lemma in (3.2) are equivalent to the degenerate Riccati inequality (4.108).

To summarize:

(ARI) in (4.103) [$\Longleftrightarrow$ KYP conditions (3.2)]
$\Uparrow \;\; (D > 0)$
Hamiltonian function in (4.98)
$\overset{D=0}{\Longrightarrow}$ LMI in (3.2) with $W = 0$
$\Downarrow \;\; (D \geq 0)$
DRI in (4.108) or RORE in (A.40)

It is worth noting that there is no minimality assumption in (4.97).

Remark 4.93. In the degenerate case $D+D^T \geq 0$ with $\mathrm{rank}(D+D^T) = r < m$, there exists an orthogonal transformation $\Gamma = [\Gamma_1 \;\; \Gamma_2]$ such that

$$\begin{pmatrix} \Gamma_1^T \\ \\ \Gamma_2^T \end{pmatrix} (D + D^T)[\Gamma_1 \;\; \Gamma_2] = \begin{pmatrix} R_1 & 0 \\ \\ 0 & 0 \end{pmatrix} \tag{4.111}$$

with $R_1 > 0$. When $H(s)$ is PR the transfer function $\Gamma^T H(s)\Gamma = \Gamma^T C(sI_n - A)^{-1}B\Gamma + \Gamma^T D\Gamma$ is PR [506].

Remark 4.94 (Singular optimal control). As we saw in Section 3.1.2 and Section 3.8, the link between passivity (the KYP Lemma) and optimal control exists when $R = D + D^T > 0$. The optimal control problem is then regular. There must exist a link between the KYP Lemma conditions with $D + D^T \geq 0$ and singular optimal control problems. We consider the optimal control with cost function $w(u, x) = u^T y = u^T(Cx + Du) = \frac{1}{2}u^T Ru + x^T Cu$.

Let $\mathrm{rank}(D+D^T)=r<m$, and $s=m-r$ be the dimension of the singular control. Let $n \leq s$ and partition B and C as $B=[B_1\ B_2]$ and $C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, with $B_1 \in I\!R^{n\times r}$, $B_2 \in I\!R^{n\times s}$, $C_1 \in I\!R^{r\times n}$, $C_2 \in I\!R^{s\times n}$. Then (A,B,C,D) is PR if and only if $D+D^T \geq 0$ and there exists $P=CB(BB)>0$ satisfying $PB=C^T$ and

$$\begin{pmatrix} -PA-A^TP & -PB_1+C_1^T \\ \\ -B_1^TP+C_1 & R_1 \end{pmatrix} \geq 0 \tag{4.112}$$

The proof can be found in [506]. It is based on the fact that when $D+D^T$ is not full rank, then (3.3) can be rewritten as $-PB_2+C_2^T=0$ and (4.112).

Remark 4.95. In [213] an algorithm is proposed which allows one to construct a reduced Riccati equation for the case $D+D^T \geq 0$. The authors start from the KYP Lemma LMI for the WSPR case (then indeed D is not full rank otherwise the transfer would be SSPR). We recall this algorithm and this important result on a degenerate Riccati equation in Appendix A.4.

4.6.2 The Nonlinear Case $y=h(x)$

We consider in this section the system (Σ) in (4.74). Let us first state the following Theorem, which shows what kind of partial differential inequality, the storage functions of dissipative systems (*i.e.* systems satisfying (4.25)) are solutions of. Let us define the Hamiltonian function

$$H(x,p)=p^Tf(x)+\sup_{u(\cdot)\in\mathbf{U}}[p^Tg(x)u-w(u,y)] \tag{4.113}$$

Also let $V_\star(x)=\lim_{z\to x}\inf V(z)$ be the lower semi-continuous envelope of $V(\cdot)$. A locally bounded function $V : X \to I\!R$ is a weak or a viscosity solution to the partial differential inequality $H(x,\nabla V)\leq 0$ for all $x\in X$, if for every C^1 function $\phi : X \to I\!R$ and every local minimum $x_0 \in I\!R^n$ of $V_\star-\phi$, one has $H(x_0,\frac{\partial}{\partial x}\phi(x_0))\leq 0$. The PDI $H(x,\nabla V)\leq 0$ for all $x\in X$ is also called a *Hamilton-Jacobi inequality.* The set $\mathbf{U}$ plays an important role in the study of the HJI, and also for practical reasons (for instance, if u is to be considered as a disturbance, then it may be assumed to take values in some compact set, but not in the whole of $I\!R^m$). Let us present the following theorem, whose proof is inspired by [304]. Only those readers familiar with partial differential inequalities and viscosity solutions should read it. The others can safely skip the proof. The next Theorem concerns the system in (4.74), where $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are supposed to be continuously differentiable, with $f(0)=0$, $h(0)=0$ (thus $x=0$ is a fixed point of the uncontrolled system), and $\frac{\partial f}{\partial x}$, $\frac{\partial g}{\partial x}$ and $\frac{\partial h}{\partial x}$ are globally bounded.

Theorem 4.96. *[232]* **(i)** *If the system (Σ) in (4.74) is dissipative in the sense of Definition 4.23 with storage function $V(\cdot)$, then $V(\cdot)$ satisfies the partial differential inequality*

$$H(x, \nabla V(x)) = \nabla V^T(x) f(x) + \sup_{u(\cdot)\in \mathbf{U}} [\nabla V^T(x) g(x) u - w(u, y)] \leq 0 \; in \; \mathbb{R}^n \tag{4.114}$$

(ii) *Conversely, if a nonnegative locally bounded function $V(\cdot)$ satisfies (4.114), then (Σ) is dissipative and $V_\star(x)$ is a lower semi-continuous storage function.*

■

The suprema in (4.113) and (4.114) are computed over all admissible $u(\cdot)$. It is noteworthy that the PDI in (4.114) is to be understood in a weak sense ($V(\cdot)$ is a viscosity solution), which means that $V(\cdot)$ needs not be continuously differentiable to be a solution. The derivative is understood as the viscosity derivative, see (4.66) and Appendix A.3.

In short, Theorem 4.96 says that a dissipative system as (Σ) in (4.74) possesses a storage function that is at least lower semi-continuous.

Proof of Theorem 4.96:

(i) Let $\phi(\cdot) \in C^1(\mathbb{R}^n)$ and suppose that $V_\star - \phi$ attains a local minimum at the point $x_0 \in \mathbb{R}^n$. Let us consider a constant input u ($u(t) = u$ for all $t \geq 0$), and let $x(t)$ be the corresponding trajectory with initial condition $x(0) = x_0$. For sufficiently small $t \geq 0$ we get

$$V_\star(x_0) - V_\star(x(t)) \leq \phi(x_0) - \phi(x(t)) \tag{4.115}$$

since $V_\star - \phi$ attains a local minimum at the point $x_0 \in \mathbb{R}^n$. Since the system (Σ) is dissipative in the sense of Definition 4.23 with storage function $V(\cdot)$, and since $V_\star(\cdot)$ satisfies the dissipation inequality each time its associated storage $V(\cdot)$ does, it follows that

$$V_\star(x_0) - V_\star(x(t)) \geq - \int_0^t w(u, y(s)) ds \tag{4.116}$$

Combining (4.115) and (4.116) one obtains

$$\frac{\phi(x(t)) - \phi(x_0}{t} - \frac{1}{t}\int_0^t w(u, y(s) ds \leq 0 \tag{4.117}$$

By letting $t \to 0$, $t > 0$, one gets

$$\nabla \phi^T(x_0) + \nabla \phi^T(x_0) g(x_0) u - w(u, h(x_0)) \leq 0 \tag{4.118}$$

Since this inequality holds for all u, it follows that

$$H(x_0, \nabla \phi(x_0)) = \nabla \phi^T(x_0) f(x_0) + \sup_{u\in \mathbf{U}} [\nabla \phi^T(x_0) g(x_0) u - w(u, h(x_0))] \leq 0 \tag{4.119}$$

holds for all $u \in \mathbf{U}$. We have therefore proved that V is a viscosity solution of (4.114).

(ii) Let us define $U_R = \{u \in \mathbf{U} \mid \| u \| \leq R\}$, $R > 0$. Let $\mathcal{U}_R$ denote the set of controllers with values in U_R. Since $V_\star(\cdot)$ is lower semi continuous, there exists a sequence $\{\Psi_i\}_{i=1}^{\infty}$ of locally bounded functions such that $\Psi_i \leq V_\star$ and $\Psi_i \to V_\star$ as $i \to +\infty$, $\Psi_i \geq V_\star$. Let $\tau > 0$ and define

$$Z_R^i(x,s) = \sup_{u \in \mathcal{U}_R} \left\{ \Psi_i(x(\tau)) - \int_s^\tau w(u(r)y(r))dr \;\mid\; x(s) = x \right\} \tag{4.120}$$

Then $Z_R^i(\cdot)$ is continuous and is the unique solution of

$$\begin{cases} \frac{\partial Z_R^i}{\partial t} + (\nabla Z_R^i)^T(x,s)f(x) + \sup_{u \in U_R}[(\nabla Z_R^i)^T(x,s)g(x)u - w(u,y)] = 0 \\ \text{in } I\!R^n \times (0,\tau) \\ Z_R^i(x,\tau) = \psi_i(x) \text{ in } I\!R^n \end{cases} \tag{4.121}$$

Compare (4.120) and (4.121) with (4.25) and (4.113) respectively. By definition of a so-called viscosity supersolution, it follows that precisely $V_\star(\cdot)$ is a viscosity supersolution of this partial differential equality (roughly, because $V_\star(\cdot)$ upperbounds $\Psi_i(\cdot)$ and is a viscosity solution of (4.114)). By the comparison Theorem it follows for all integer $i \geq 1$ that

$$V_\star(x) \geq Z_R^i(x,s) \;\forall\; (x,s) \in I\!R^n \times [0,\tau] \tag{4.122}$$

Setting $s = 0$ yields

$$V_\star(x) \geq \sup_{u \in \mathcal{U}_R} \left\{ \Psi_i(x(\tau)) - \int_0^\tau w(u(r),y(r))dr \;\mid\; x(0) = x \right\} \tag{4.123}$$

Letting $i \to +\infty$ we obtain

$$V_\star(x) \geq \sup_{u \in \mathcal{U}_R} \left\{ V_\star(x(\tau)) - \int_0^\tau w(u(r),y(r))dr \;\mid\; x(0) = x \right\} \tag{4.124}$$

Letting $R \to +\infty$

$$V_\star(x) \geq \sup_{u \in \mathcal{U}} \left\{ V_\star(x(\tau)) - \int_0^\tau w(u(r),y(r))dr \;\mid\; x(0) = x \right\} \tag{4.125}$$

where we recall that $\mathcal{U}$ is just the set of admissible inputs, *i.e.* locally square Lebesgue integrable functions of time (locally $\mathcal{L}_2$) such that (4.21) is satisfied.

This last inequality holds for all $\tau \geq 0$, so that (4.25) holds. Consequently (Σ) is dissipative and $V_\star(\cdot)$ is a storage function.

■

When specializing to passive systems then the following holds:

Corollary 4.97. *[232] The system (Σ) in (4.74) is passive if and only if there exists a locally bounded non-negative function $V(\cdot)$ such that $V(0) = 0$ and*

$$\nabla V^T(x)f(x) + \sup_{u(\cdot)\in\mathbf{U}} [\nabla V^T(x)g(x)u - u^T y] \leq 0 \text{ in } \mathbb{R}^n \tag{4.126}$$

In case $\mathbf{U} = \mathbb{R}^m$ then (4.126) reads

$$\begin{cases} \nabla V^T(x)f(x) \leq 0 \\ \\ \nabla V^T(x)g(x) = h(x) \end{cases} \tag{4.127}$$

for all $x \in \mathbb{R}^n$. ■

In (4.127), solutions are supposed to be weak, *i.e.*: if $\Xi(\cdot) \in C^1(\mathbb{R}^n)$ and $V_\star - \Xi$ attains a local minimum at $x_0 \in \mathbb{R}^n$, then

$$\begin{cases} \nabla \Xi^T(x_0)f(x_0) \leq 0 \\ \\ \nabla \Xi^T(x_0)g(x_0) = h(x_0). \end{cases} \tag{4.128}$$

One sees that the set of conditions in (4.128) is nothing else but (4.76) expressed in a weak (or viscosity) sense.

4.6.3 The Nonlinear Case $y = h(x) + j(x)u$

We now consider systems as in (4.79), and the supply rate is $w(u, y) = \gamma^2 u^T u - y^T y$ ($Q = -I_m$, $R = \gamma^2 I_m$, $S = 0$ in Definition 4.55). The dissipation inequality then reads

$$V(x(t)) - V(x(0)) \leq \int_0^t [\gamma^2 u^T(s)u(s) - y^T(s)y(s)]ds \tag{4.129}$$

If one supposes that $V(0) = 0$ and $x(0) = 0$ then it follows from (4.129) that

$$0 \leq V(x(t)) \leq \int_0^t [\gamma^2 u^T(s)u(s) - y^T(s)y(s)]ds \tag{4.130}$$

from which one deduces that

$$\int_0^t y^T(s)y(s)ds \leq \gamma^2 \int_0^t u^T(s)u(s)ds \tag{4.131}$$

which simply means that the system defines an input-output operator H_x which has a finite $\mathcal{L}_2-$gain at most γ (see Definition 4.17), and $H_{x=0}$ has zero bias. An argument of local $w-$uniform reachability assures that storage functions are continuous. Let us assume that $V(\cdot)$ is a smooth storage function. Then the dissipation inequality (4.129) is equivalent to its infinitesimal form

$$\nabla V^T(x)[f(x)+g(x)u]+(h(x)+j(x)u)^T(h(x)+j(x)u)-\gamma^2u^Tu \leq 0. \quad (4.132)$$

Since the dissipation inequality is required to hold for a certain set $\mathbf{U}$ of admissible inputs, the infinitesimal form (4.132) is a Hamilton-Jacobi inequality $H(x, \nabla V(x)) \leq 0$, with Hamiltonian function

$$H(x,p) = \sup_{u\in\mathbf{U}}[p^T(f(x)+g(x)u)+(h(x)+j(x)u)^T(h(x)+j(x)u)-\gamma^2u^Tu] \quad (4.133)$$

If in addition the term $\Delta(x) = \gamma^2 I_m - j(x)^T j(x) > 0$ for all $x \in X$, then the Hamiltonian can be written in a explicit way as

$$\begin{aligned} H(x,p) = p^T[f(x)+g(x)\Delta^{-1}(x)j(x)^T h(x)] + \tfrac{1}{4}p^T g(x)\Delta^{-1}(x)g(x)^T p + \\ + h(x)^T[I_m + j(x)\Delta^{-1}(x)j(x)^T]h(x) \end{aligned} \quad (4.134)$$

Let us note once again that if $u(\cdot)$ is considered as a disturbance, and not a control input, then it makes perfect sense to consider the set $\mathbf{U}$ in which the disturbance is supposed to live. This is also the case if the admissible inputs are bounded because of physical saturations. Those developments are then at the core of the H_∞ theory for nonlinear systems [442]. Similarly to the above, the obstacle in studying such PDIs is that storage functions may not be differentiable: in general they are only continuous. How does this machinery extends to such a case? Once again weak (or viscosity) solutions are the key.

Theorem 4.98. *[33] Suppose that $V : X \to \mathbb{R}^+$ is continuous. Then $V(\cdot)$ is a storage function for the system (Σ) in (4.79) if and only if it is a viscosity solution of the Hamilton-Jacobi inequality $H(x, \nabla V(x)) \leq 0$ for all $x \in X$, with $H(\cdot,\cdot)$ given in (4.133).* ■

Under some conditions, the available storage and required supply are proved to be the viscosity solutions of Hamilton-Jacobi equalities, thereby extending (4.37).

Assumption 5 *Given $x_0 \in \mathbb{R}^n$ and $t_1 < t_2$ with $t_2 - t_1$ sufficiently small, there exists a bounded set $B_{x_0} \subset \mathbb{R}^m$ such that*

$$\sup_{\substack{u \in \mathcal{L}_2([t_1,t_2]) \\ u(t) \in B_{x_0}}} \left\{ V_a(x(t_2)) - V_a(x_0) - \int_{t_1}^{t_2} (\gamma^2 u^T(t)u(t) - y^T(t)y(t))dt \right\} = 0 \quad (4.135)$$

where $x(t)$ and $y(t)$ correspond to the solution initialized at x_0 and controlled by $u(\cdot)$ on $[t_1, t]$.

Assumption 6 *Given $x_0 \in I\!R^n$ and $t_0 < t_1$ with $t_1 - t_0$ sufficiently small, there exists a bounded set $B_{x_0} \subset I\!R^m$ such that*

$$\sup_{\substack{u \in \mathcal{L}_2([t_0,t_1]) \\ u(t) \in B_{x_0}}} \left\{ V_r(x_0) - V_a(x(t_0)) - \int_{t_0}^{t_1} (\gamma^2 u^T(t)u(t) - y^T(t)y(t))dt \right\} = 0 \tag{4.136}$$

where $x(t)$ and $y(t)$ correspond to the solution initialized at x_0 and controlled by $u(\cdot)$ on $[t_1, t]$.

Theorem 4.99. *[33] Assume that the system in (4.79) has finite-gain at most γ and is uniformly controllable, so that $V_a(\cdot)$ and $V_r(\cdot)$ are both well-defined continuous storage functions. Then*

- *$V_a(\cdot)$ is a viscosity solution of $-H(x, \nabla V(x)) = 0$ if Assumption 6 is satisfied.*
- *$V_r(\cdot)$ is a viscosity solution of $H(x, \nabla V(x)) = 0$ if Assumption 5 is satisfied.*

■

Remark 4.100. • Storage functions that satisfy (4.81) can also be shown to be the solutions of the following partial differential inequation:

$$\nabla V^T(x)f(x) + (h^T(x) - \frac{1}{2}\nabla V^T(x)g(x))\hat{R}^{-1}(x)(h(x) - \frac{1}{2}g^T(x)\nabla V(x)) \leq 0 \tag{4.137}$$

when $\hat{R} = j(x) + j^T(x)$ is full-rank, $R = 0$, $Q = 0$, $S = \frac{1}{2}I$. The proof is exactly the same as in the linear time invariant case (Section 3.1.2). The available storage and the required supply satisfy this formula (that is similar to a Riccati equation) as an equality (Proposition 4.48).

- In the linear invariant case, the equivalent to Hamilton-Jacobi inequalities are Riccati equations, see Section 3.1.2. This also shows the link with optimal control. Hamilton-Jacobi equalities also arise in the problem of inverse optimal control, see section 4.6.5.
- In the time varying case (4.91), the PDI in (4.137) becomes

$$\begin{aligned} &\tfrac{\partial V}{\partial t}(x,t) + \nabla V^T(x,t)f(x,t) + \\ &+(h^T(x,t) - \tfrac{1}{2}\nabla V^T(x,t)g(x,t))\hat{R}^{-1}(x,t)(h(x,t) - \tfrac{1}{2}g^T(x,t)\nabla V(x,t)) \leq 0 \end{aligned} \tag{4.138}$$

■

In order to illustrate the above developments let us present an example, taken from [116].

Example 4.101. Consider the following system

$$\begin{cases} \dot{x}_1(t) = x_1(t)[(r^2(t)-1)(r^2(t)-4) + r(t)(r^2(t)-4)u(t)] - x_2(t) \\ \dot{x}_2(t) = x_2(t)[(r^2(t)-1)(r^2(t)-4) + r(t)(r^2(t)-4)u(t)] + x_1(t) \\ y(t) = r^2(t) - 1, \quad r = \sqrt{x_1^2 + x_2^2} \end{cases} \tag{4.139}$$

In polar coordinates one gets

$$\begin{cases} \dot{r}(t) = r(t)(r^2(t)-1)(r^2(t)-4) + r(t)(r^2(t)-4)u(t) \\ \dot{\theta}(t) = 1 \mod [2\pi] \\ y(t) = r^2(t) - 1 \end{cases} \tag{4.140}$$

The set $\mathbf{S} = \{x \in I\!R^2 \mid r = 1\}$ is invariant under the uncontrolled dynamics ($u = 0$), and is asymptotically stable. The open set $\mathbf{R} = \{x \in I\!R^2 \mid 0 < r < 2\}$ is the largest basin of attraction of $\mathbf{S}$ (still with $u = 0$). Moreover all points in $\mathbf{R}$ are reached from $\mathbf{S}$ in finite time by suitable control. Invariance of $\mathbf{S}$ is easy to check as $f(x) = x_1^2 + x_2^2 - 1$ is a first integral of the uncontrolled system. The objective is to prove that the system in (4.139) is dissipative with respect to the supply rate $w(u, y) = \gamma^2 u^T u - y^T y$, for all $\gamma \geq 1$. Let us look for a storage function of the form $V(r^2)$. Thus $\frac{\partial V}{\partial x}(x) = (2x_1 \;\; 2x_2)\frac{dV}{d(r^2)}$. The pre- Hamiltonian function $PH(\cdot)$ (that is the function to be supremized in (4.113)) is equal to

$$PH(r, u) = 2\frac{dV}{d(r^2)}r^2[(r^2-1)(r^2-4) + r(r^2-4)u] - \gamma^2 u^T u + (r^2-1)^2 \tag{4.141}$$

and the maximizing controller is

$$u = \frac{1}{\gamma^2} r^2 (r^2-4)^2 \frac{dV}{d(r^2)} \tag{4.142}$$

So the Hamilton-Jacobi inequality in (4.114) reads on $0 < r < 2$:

$$\begin{aligned} H(r, \nabla V(r)) = \left[r^2(r^2-4)\tfrac{dV}{d(r^2)} + (r^2-1)\right]^2 - \\ - \left(1 - \tfrac{1}{\gamma^2}\right) r^4 (r^2-4)^2 \left(\tfrac{dV}{d(r^2)}\right)^2 \leq 0 \end{aligned} \tag{4.143}$$

Obviously this PDI has a solution if and only if $\gamma \geq 1$. By inspection one sees that any solution to the ordinary differential equation $r^2(r^2-4)\frac{dV}{d(r^2)} + (r^2-1) = 0$ with minimal set condition $V(1) = 0$ solves this HJI. One such solution is given by

$$V(r) = -\frac{1}{4}\ln(r^2) - \frac{3}{4}\ln(4-r^2) + \frac{3}{4}\ln(3) \tag{4.144}$$

This $V(r)$ is locally bounded on the set $\mathbf{R}$, $V(r) \geq 0$, it is radially unbounded for all $x \to \partial\mathbf{R}$ (all states approaching the boundary of $\mathbf{R}$, in particular the origin), and $V(r) = 0$ on the circle $\mathbf{S}$. Therefore the system in (4.139) is dissipative with respect to supply rates $w(u, y) = \gamma^2 u^T u - y^T y$, for all $\gamma \geq 1$. The exhibited storage function is differentiable. One can check by calculation that $\dot{V}(r) = -\frac{1}{r}(r^2-1)^2 \leq 0$ along trajectories of the uncontrolled system and for all $x \in \mathbf{R}$. One has $\dot{V}(r) = 0$ for all $x \in \mathbf{S}$. ■

Let us summarize the developments in this section and the foregoing ones, on the characterization of dissipative systems.

$H(x, \nabla V(x)) \leq 0$ with Hamiltonian function in (4.113) or (4.133) or (4.134)
$\Downarrow$
PDI in (4.114) or (4.126) or in Theorem 4.98, general lsc storage functions (viscosity solutions)
$\Downarrow$
PDI in (4.137) or (4.76), C^1 storage functions
$\Updownarrow$
nonlinear KYP Lemma 4.84 or 4.87 with C^1 storage functions
$\Downarrow$
Riccati inequality (3.17) for LTI systems
$\Updownarrow$
KYP Lemma for LTI systems
$\Updownarrow$
PR transfer functions

where the "implications" just mean that the problems are decreasing in mathematical complexity.

4.6.4 Recapitulation

Let us take advantage of the presentation of this section, to recapitulate some tools that have been introduced throughout the foregoing: Riccati inequalities, Hamiltonian function, Popov's functions, and Hermitian forms. A Hermitian form has the general expression

$$\mathcal{H}(x,y) = [x^T \;\; y^T]\Sigma \begin{bmatrix} x \\ \\ y \end{bmatrix} \tag{4.145}$$

with $x \in I\!R^n$, $y \in I\!R^n$, $\Sigma = \begin{bmatrix} Q & Y^T \\ \\ Y & R \end{bmatrix}$, $Q \in I\!R^{n\times n}$, $Y \in I\!R^{n\times n}$, $R \in I\!R^{n\times n}$, $Q = Q^T$, $R = R^T$. Let $y = Px$ for some $P = P^T \in I\!R^{n\times n}$. Then

$$\mathcal{H}(x, Px) = 0 \;\text{ for all }\; x \in I\!R^n$$

if and only if

$$Q + PY + Y^T P + PRP = 0 \quad (P = P^T).$$

The proof is done by calculating explicitly $\mathcal{H}(x, Px)$. The analogy with (4.102) and (4.103) is straightforward (with equalities instead of inequalities). A solution to the ARE is stabilizing if the ODE $\dot{x}(t) = \frac{d\mathcal{H}}{dy}|_{y=Px} = 2(Y + RP)x(t)$ is globally asymptotically stable. The results of Theorems 3.42, 3.43, 3.44 and 4.58 allow us to assert that stabilizing solutions exist in important cases.

Linking this with the spectral (or Popov's) function $\Pi(s)$ in Theorems 2.30 and 3.46, or (3.141) (3.142), we see that taking $x = (j\omega I_n - A)^{-1}B$ and $y = I_m$ in (4.145) (with appropriate dimensions of the matrices $Y \in I\!R^{m\times n}$ and $R \in I\!R^{m\times m}$) yields that $\Pi(j\omega)$ is a rational Hermitian matrix valued function defined on the imaginary axis. The positivity of $\Pi(j\omega)$ is equivalent to the passivity of the system with realization (A, B, Y), which in turn can be characterized by a LMI (the KYP Lemma set of equations) which in turn is equivalent to an ARI: the loop is closed!

4.6.5 Inverse Optimal Control

A particular optimal control problem is to find the control input $u(\cdot)$ that minimizes the integral action $\int_0^\infty [q(x(t)) + u^T(t)u(t)]dt$ under the dynamics in (4.74), where $q(x)$ is continuously differentiable and positive definite. From standard dynamic programming arguments it is known that the optimal input is $u^\star(x) = -\frac{1}{2}g^T(x)\frac{\partial V^\star}{\partial x}^T(x)$, where $V^\star(\cdot)$ is the solution of the partial differential equation, called a Hamilton-Jacobi-Bellman equation:

$$\frac{\partial V^\star}{\partial x}(x)f(x) - \frac{1}{4}\left(\frac{\partial V^\star}{\partial x}(x)g(x)g^T(x)\frac{\partial V^\star}{\partial x}^T(x)\right) + q(x) = 0 \tag{4.146}$$

Moreover $V^\star(x(t)) = \inf_{u(\cdot)} \int_t^\infty [q(x(\tau)) + u^T(\tau)u(\tau)]d\tau$, $V^\star(0) = 0$. One recognizes that $u^\star(x)$ is nothing else but a static feedback of the passive output of the system (4.74) with storage function $V^\star(\cdot)$. Applying some of the results in this section and in Section 5.4 one may additionally study the stability of the closed-loop system with the optimal input (see in particular Theorem 5.24). Let us consider the linear time-invariant case with quadratic cost $q(x) = x^T Qx$. Then one looks for storage functions of the form $V(x) = x^T Px$. The Hamilton-Jacobi-Bellman equation in (4.146) then becomes the Riccati equation

$$PA + A^T P - PBB^T P + Q = 0 \tag{4.147}$$

The optimal controller is classically given by $u^\star(x) = -B^T Px$ (recall that $\nabla V(x) = \frac{\partial V}{\partial x}^T(x) = 2Px$). It is worth comparing (4.147) with (3.17) (take $D + D^T = I_m$, $C = 0$, and the cost is $PA + A^T P + Q$). See also (3.138).

Let us now describe the so-called *inverse optimal control problem* [363,365]. We are given the system

$$\dot{x}(t) = f(x(t)) + Bu(t), \quad x(0) = x_0 \tag{4.148}$$

where $f(\cdot)$ is smooth, $f(0) = 0$, and B is a constant matrix. We are also given a performance index

$$V = \lim_{t\to+\infty}\left[\eta(x(t)) + \int_0^t (L^T(x(s))L(x(s)) + u^T(s)u(s))ds\right] \tag{4.149}$$

with $\eta(x) \geq 0$ for all $x \in X$, $\eta(0) = 0$, $L(0) = 0$, and a feedback controller

$$u^\star(x) = -k(x). \tag{4.150}$$

Let us assume that $u^\star(x)$ is optimal with respect to the performance index (4.149), and let us denote the minimum value of V as $\phi(x_0)$. In general, there is not a unique $L(x)$ and $\eta(x)$ for which the same controller is optimal. In other words there may exist many different $L(x)$, to which correspond different $\phi(x)$, for which the same controller is optimal. The inverse optimal control problem is as follows: given the system (4.148) and the controller (4.150), a pair $(\phi(\cdot), L(\cdot))$ is a solution of the inverse optimal control problem if the performance index (4.149) is minimized by (4.150), with minimum value $\phi(x_0)$. In other words, the inverse approach consists of designing a stabilizing feedback control law, and then to show that it is optimal with respect to a meaningful and well defined cost functional.

Lemma 4.102. *[365] Suppose that the system in (4.148) and the controller in (4.150) are given. Then a pair $(\phi(\cdot), L(\cdot))$ is a solution of the inverse optimal control problem if and only if $\phi(x)$ and $L(x)$ satisfy the equations*

$$\begin{cases} \nabla\phi^T(x)[f(x) - \frac{1}{2}Bk(x)] = -L^T(x)L(x) \\ \\ \frac{1}{2}B^T\nabla\phi(x) = k(x) \\ \\ \phi(0) = 0 \\ \\ \phi(x) \geq 0 \textit{ for all } x \in X \qquad \blacksquare \end{cases} \tag{4.151}$$

The following should not be surprising to the reader who has followed the previous developments.

Lemma 4.103. *[365] A necessary and sufficient condition for the existence of a solution to the inverse optimal control problem, is that the system*

$$\begin{cases} \dot{x}(t) = f(x(t)) - \frac{1}{2}Bk(x(t)) + Bu \\ \\ y(t) = k(x(t)) \end{cases} \tag{4.152}$$

be passive. If this is the case, then there exists two solutions $(\phi_a(\cdot), L_a(\cdot))$ and $(\phi_r(\cdot), L_r(\cdot))$ of (4.151) such that all other solutions satisfy $\phi_a(x) \leq \phi(x) \leq \phi_r(x)$ for all $x \in X$. ■

Indeed the equations in Lemma 4.102 are nothing else but the KYP Lemma conditions for the system (4.152). The interpretation of $\phi_a(x)$ and $\phi_r(x)$ as the available storage and required supply, respectively, is obvious as well. One recovers the HJB equation (4.146) replacing $g(x)$ by B and $q(x)$ by $L^T(x)L(x)$.

Remark 4.104. The inverse optimal control problem was first solved by Kalman [248] in the case of linear systems with linear state feedback. Other works can be found in [142].

Let us end this section with a result that completes the above ones. We consider the system

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ \\ y(t) = h(x(t)) + j(x(t))u(t) \\ \\ x(0) = x_0 \end{cases} \tag{4.153}$$

where all the mappings are continuously differentiable and $f(0) = 0$, $h(0) = 0$. Let us define the set of stabilizing controllers:

$\mathcal{S}(x_0) = \{u(\cdot) \mid u \in \mathcal{U} \text{ and solution of (4.153) satisfies } x(t) \to 0 \text{ as } t \to +\infty\}$

We also consider a nonlinear nonquadratic performance criterion

$$J(x_0, u(\cdot)) = \int_0^{\infty} [L(x(t)) + u^T(t)Ru(t)]dt \tag{4.154}$$

with $L : \mathbb{R}^n \to \mathbb{R}^+$, $0 < R \in \mathbb{R}^{m\times m}$.

Theorem 4.105. *[363, 502] Consider the system in (4.153) with the performance index in (4.154). Let us assume that there exists a continuously differentiable and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$, satisfying*

$$L(x) + \nabla^T V(x) f(x) - \frac{1}{4}\nabla^T V(x) g(x) R^{-1} g^T(x) \nabla V(x) = 0 \tag{4.155}$$

Moreover let $h(x) = L(x)$ and suppose that the new system in (4.153) is zero-state observable. Then the origin $x = 0$ of the closed-loop system

$$\dot{x}(t) = f(x(t)) - g(x(t))\phi(x(t)), \quad x(0) = x_0, \quad t \geq 0 \tag{4.156}$$

is globallly asymptotically stable with the feedback control input

$$u(x) = -\phi(x) = -\frac{1}{2}R^{-1}g^T(x)\nabla V(x) \tag{4.157}$$

The action in (4.154) is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}^n \tag{4.158}$$

and we have $J(x_0, \phi(x(\cdot))) = V(x_0)$, $x_0 \in \mathbb{R}^n$ ■

The extension of Theorem 4.105 towards the output feedback case is given in [99, Theorem 6.2]. The equation in (4.155) is a Hamilton-Jacobi-Bellman equation. Consider the Hamiltonian function

$$H(x, p, u) = L(x) + u^T Ru + p^T(f(x) + g(x)u) \tag{4.159}$$

One may calculate that the HJB equation in (4.155) is in fact

$$\min_{u\in\mathcal{U}} H(x, u, \nabla V(x)) = 0,$$

using the strict convexity of the integrand in (4.154) (since $R > 0$), so that the minimizing input is $u(x) = -\frac{1}{2}R^{-1}g^T(x)p$. Various application examples may be found in [502], like the stabilization of the controlled Lorenz equations, the stabilization of the angular velocity with two actuators, and with one actuator.

4.7 Nonlinear Discrete-time Systems

The material of this section is taken mainly from [90]. The following class of systems is considered:

$$\begin{cases} x(k+1) = f(x(k)) + g(x(k))u(k) \\ \\ y(k) = h(x(k)) + j(x(k))u(k) \end{cases} \tag{4.160}$$

where $x(k) \in I\!R^n$, $u(k) \in I\!R^m$, $y(k) \in I\!R^m$, and the functions $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ and $j(\cdot)$ are smooth mappings. It is assumed that $f(0) = 0$ and $h(0) = 0$.

Definition 4.106. *The dynamical system in (4.160) is said dissipative with respect to the supply rate $w(u, y)$ if there exists a nonnegative function $V : I\!R^n \to I\!R$ with $V(0) = 0$ called a storage function, such that for alll $u \in I\!R^m$ and all $k \in I\!N$ one has*

$$V(x(k+1)) - V(x(k)) \leq w(u(k), y(k)), \tag{4.161}$$

or equivalently

$$V(x(k+1)) - V(x(0)) \leq \sum_{i=0}^{k} w(u(i), y(i)) \tag{4.162}$$

for all k, $u(k)$ and $x(0)$. The inequality (4.162) is called the dissipation inequality in the discrete-time setting. ■

Similarly to the continuous-time case we have

Definition 4.107. *The dynamical system in (4.160) is said passive if it is dissipative with respect to the supply rate $w(u, y) = u^T y$. It is said strictly passive if $V(x(k+1)) - V(x(k)) < u^T(k)y(k)$ for all $u(k)$ unless $x(k)$ is identically zero. Equivalently the system is strictly passive if there exists a positive definite function $\mathcal{S} : I\!R^n \to I\!R$ such that $V(x(k+1)) - V(x(k)) \leq u^T(k)y(k) - \mathcal{S}(x(k))$ for all $u(k)$ and all k. It is said lossless if $V(x(k+1)) - V(x(k)) = u^T(k)y(k)$ for all $u(k)$ and all k, equivalently $V(x(k+1)) - V(x(0)) = \sum_{i=0}^{k} u^T(i)y(i)$ for all $u(k)$ and all k.* ■

It is of interest to present the extension of the KYP Lemma for such nonlinear discrete-time systems, that is the nonlinear counterpart to Lemma 3.100.

Lemma 4.108 (KYP Lemma). *[90] The system (4.160) is lossless with a C^2 storage function if and only if*

$$\begin{cases} V(f(x)) = V(x) \\ \frac{\partial V}{\partial z}(z)|_{z=f(x)} g(x) = h^T(x) \\ g^T(x) \frac{\partial^2 V}{\partial z^2}(z)|_{z=f(x)} g(x) = j^T(x) + j(x) \\ V(f(x)) + g(x)u) \ \ \textit{is quadratic in} \ \ u \end{cases} \tag{4.163}$$

■

Proof: *Necessity:* If the system is lossless there exists a nonnegative storage function $V(x)$ such that

$$\begin{aligned} V(f(x(k) + g(x(k))u(k)) - V(x(k)) = h^T(x(k))u(k) + \\ + \tfrac{1}{2}u^T(k)[j(x(k)) + j^T(x(k))]u(k) \end{aligned} \tag{4.164}$$

for all $u(k) \in \mathbb{R}^m$ and all $k \in \mathbb{N}$. Setting $u(k) = 0$ one gets the first equality in (4.108). Now one may calculate that (from now on we drop the k argument in the functions)

$$\frac{\partial V(f(x) + g(x)u)}{\partial u} = \frac{\partial V}{\partial z}|_{z=f(x)+g(x)u} = h^T(x) + u^T[j^T(x) + j(x)] \tag{4.165}$$

and

$$\begin{aligned} \tfrac{\partial^2 V(f(x)+g(x)u)}{\partial u^2} &= g^T(x) \tfrac{\partial^2 V}{\partial z^2}|_{z=f(x)+g(x)u} g(x) \\ &= j(x) + j^T(x) \end{aligned} \tag{4.166}$$

Equations (4.165) and (4.166) imply the second and third equations in (4.108). The last condition in (4.108) follows easily from (4.164).

Sufficiency: Suppose that the last condition in (4.108) is satisfied. One deduces that

$$V(f(x)) + g(x)u) = A(x) + B(x)u + u^T C(x)u \tag{4.167}$$

for all $u \in \mathbb{R}^m$ and some functions $A(x)$, $B(x)$, $C(x)$. From the Taylor expansion of $V(f(x)) + g(x)u)$ at $u = 0$ we obtain

$$\begin{cases} A(x) = V(f(x)) \\ B(x) = \frac{\partial V(f(x)+g(x)u)}{\partial u}|_{u=0} = \frac{\partial V}{\partial z}|_{z=f(x)} g(x) \\ C(x) = \frac{\partial^2 V(f(x)+g(x)u)}{\partial u^2}|_{u=0} = \frac{1}{2} g^T(x) \frac{\partial^2 V}{\partial z^2}|_{z=f(x)} g(x) \end{cases} \tag{4.168}$$

From the first three equations of (4.108) it follows that

$$V(f(x) + g(x)u) - V(x) = y^T u \tag{4.169}$$

for all $u \in \mathbb{R}^m$, which concludes the proof. ■

Further results on nonlinear dissipative discrete-time systems may be found in [185, 371, 373, 374].

4.8 PR tangent system and dissipativity

The topic of this section is the following: consider a nonlinear system with sufficiently regular vector field, and its tangent linearization about some point $(x^\star, u^\star)$. Suppose that the tangent linearization is positive real, or strictly positive real. Then, is the nonlinear system locally dissipative? Or the converse?

Let us consider the following nonlinear system:

$$(\Sigma)\quad \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ \\ y(t) = h(x(t)) \\ \\ x(0) = x_0 \end{cases} \tag{4.170}$$

where $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are continuously differentiable functions of x, $f(0) = 0$, $h(0) = 0$. Let us denote $A = \frac{\partial f}{\partial x}(0)$, $B = \frac{\partial g(x)u}{\partial u}(x = 0, u = 0) = g(0)$, $C = \frac{\partial h}{\partial x}(0)$. The tangent linearization of the system in (4.170) is the linear time-invariant system

$$(\Sigma_t)\quad \begin{cases} \dot{z}(t) = Az(t) + Bu(t) \\ \\ \zeta(t) = Cz(t) \\ \\ z(0) = x_0 \end{cases} \tag{4.171}$$

The problem is as follows: under which conditions are the following equivalences true?

$$(\Sigma_t) \in \mathrm{PR} \overset{?}{\Longleftrightarrow} (\Sigma) \text{ is locally passive}$$

$$(\Sigma_t) \in \mathrm{SPR} \overset{?}{\Longleftrightarrow} (\Sigma) \text{ is locally strictly dissipative}$$

It also has to be said whether dissipativity is understood in Willems' sense (existence of a storage function), or in Hill and Moylan's sense. Clearly one will also be interested in knowing whether or not the quadratic storage functions for (Σ_t) are local storage functions for (Σ). Important tools to study the above two equivalences, will be the local stability, the local controllability, and the local observability properties of (Σ) when (A, B) is controllable, (A, C) is observable, and A has only eigenvalues with nonpositive real parts. For instance local $w-$uniform reachability of (Σ) (Definition 4.44) is implied by the controllability of (Σ_t) (Proposition 4.76). One can thus already state that if A has eigenvalues with negative real parts, and if (A, B) is controllable and (A, C) is observable, then (Σ) has properties that make it a good candidate for local dissipativity with positive definite storage functions and a Lyapunov asymptoticaly stable fixed point of $\dot{x}(t) = f(x(t))$ (see Lemmas 5.18 and 5.20 in the next chapter).

Example 4.109. Let us consider the scalar system

$$(\Sigma) \quad \begin{cases} \dot{x}(t) = \frac{1}{2}x^2(t) + (x(t)+1)u(t) \\ \\ y(t) = x(t) \\ \\ x(0) = x_0 \end{cases} \tag{4.172}$$

Then its tangent linearization around $x = 0$ is

$$(\Sigma_t) \quad \begin{cases} \dot{z}(t) = u(t) \\ \\ \zeta(t) = z(t) \\ \\ z(0) = x_0 \end{cases} \tag{4.173}$$

The tangent system (Σ_t) is an integrator $H(s) = \frac{1}{s}$. It is PR, though the uncontrolled (Σ) is unstable (it may even have finite escape times).

Example 4.110. Let us consider the scalar system

$$(\Sigma) \quad \begin{cases} \dot{x}(t) = x^2(t) - x(t) + (x^3(t) + x(t) + 1)u(t) \\ \\ y(t) = x^2(t) + x(t) \\ \\ x(0) = x_0 \end{cases} \tag{4.174}$$

Then the tangent linearization around $z = 0$ is

$$(\Sigma_t) \quad \begin{cases} \dot{z}(t) = -z(t) + u(t) \\ \\ \zeta(t) = z(t) \\ \\ z(0) = x_0 \end{cases} \tag{4.175}$$

The tangent system has transfer function $H(s) = \frac{1}{s+1} \in$ SPR. The uncontrolled (Σ) is locally stable (take $V(x) = \frac{x^2}{2}$). However (Σ) in (4.174) is not dissipative with this storage function and the supply rate uy since $y \neq g^T(x)\frac{\partial V}{\partial x}(x)$. Consider now

$$(\Sigma) \quad \begin{cases} \dot{x}(t) = x^2(t) - x(t) + u(t) \\ y(t) = x(t) \\ x(0) = x_0 \end{cases} \tag{4.176}$$

whose tangent linearization is in (4.175). This system is locally stable with Lyapunov function $V(x) = \frac{x^2}{2}$, and $y = g^T(x)\frac{\partial V}{\partial x}(x)$. Easy computation yields that $\int_0^t u(s)y(s)ds \geq V(x(t)) - V(x(0))$ for $x \in (-1,1)$. Hence $V(x)$ is a storage function for (4.176), which is locally dissipative in $(-1,1) \ni x$.

■

Let us present a result which states under which conditions the tangent linearization of a dissipative system, is a SPR system. Consider the system

$$(\Sigma) \quad \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) + j(x(t))u(t) \\ x(0) = x_0 \end{cases} \tag{4.177}$$

with the dimensions for signals used throughout this book, $f(0) = 0$ and $h(0) = 0$. The notion of dissipativity that is used is that of exponential dissipativity,*i.e.* dissipativity with respect to $\exp(\epsilon t)w(u(t), y(t))$ for some $\epsilon > 0$.

Assumption 7 *There exists a function $\kappa : \mathbb{R}^m \to \mathbb{R}^m$, $\kappa(0) = 0$, such that $w(\kappa(y), y) < 0$, $y \neq 0$.*

Assumption 8 *The available storage function $V_a(\cdot)$ is of class C^3.*

Assumption 9 *The system is completely reachable if for all $x_0 \in \mathbb{R}^n$ there exists a finite $t_0 \leq 0$, and an admissible input defined on $[t_0, 0]$ which can drive the state $x(\cdot)$ from the origin $x(t_0) = 0$ to $x(0) = x_0$.*

Theorem 4.111. *[99] Let $Q = Q^T \in \mathbb{R}^{m\times m}$, $S = S^T \in \mathbb{R}^{m\times m}$, $R = R^T \in \mathbb{R}^{=\times m}$, and assume that Assumptions 7, 8 and 9 hold, and that the system in (4.177) is exponentially dissipative with respect to the general supply rate $w(u,y) = y^TQy + 2y^TSu + u^TRu$. Then there exists matrices $P \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{p\times n}$, $W \in \mathbb{R}^{p\times m}$, $P = P^T \geq 0$, and a scalar $\epsilon > 0$ such that*

$$\begin{cases} A^TP + PA + \epsilon P - C^TQC + L^TL = 0 \\ PB - C^T(QD + S) + L^TW = 0 \\ R + S^TD + D^TS + D^TQD - W^TW = 0 \end{cases} \tag{4.178}$$

with $A = \frac{\partial f}{\partial x}(0)$, $B = g(0)$, $C = \frac{\partial h}{\partial x}(0)$, $D = j(0)$. *If in addition the pair* (A, C) *is observable then* $P > 0$. ■

A similar result was proved in [187]. Theorem 4.111 proves that under some conditions a dissipative system possesses a positive real tangent linearization. What about the converse,*i.e.* if the tangent linearization is positive real, is the system (locally) dissipative? The following brings an answer.

Theorem 4.112. *[442] Consider the system in (4.177) and suppose that* $j(0) = 0$. *Suppose that the tangent linearization is dissipative with respect to the supply rate* $w(u, y) = y^T Qy + 2y^T Su + u^T Ru$, *with* $R > 0$, *and* $w(0, y) \leq 0$ *for all* y. *Suppose that the matrix*

$$\begin{pmatrix} A - BR^{-1}SC & BR^{-1}B^T \\ \\ C^T QC & -(A - BR^{-1}SC)^T \end{pmatrix} \tag{4.179}$$

has no purely imaginary eigenvalues, and that A *is asymptotically stable. Then there exists a neighborhood* $\mathbf{N} \subset I\!R^n$ *of* $x = 0$ *and* $V : \mathbf{N} \to I\!R$ *with* $v(0) = 0$, $\frac{\partial V}{\partial x}(0) = 0$ *such that* $\frac{\partial V}{\partial x}[f(x) + g(x)u] \leq w(u, h(x) + j(x)u)$ *for all* $x \in \mathbf{N}$ *and all* $u \in \mathbf{U} \subset I\!R^m$, $V(x) \geq 0$ *for all* $x \in \mathbf{N}$. *Consequently the system in (4.177) is locally dissipative in* $\mathbf{N}$ *with respect to* $w(u, y)$. ■

One remarks that the matrix (4.179) corresponds to the transition matrix of the Hamiltonian system of the first order necessary condition of the Pontryagin principle for the Bolza problem, with a cost function equal to $u^T Ru + x^T C^T QCx$, under the constraint $\dot{x}(t) = (A - BR^{-1}SC)x(t) + Bu(t)$. The two above examples do not fit within the framework of Theorem 4.112 as the dissipativity of the tangent linearizations holds with respect to the supply rate $w(u, y) = u^T y$, and thus $R = 0$.

4.9 Infinite-dimensional Systems

4.9.1 An Extension of the KYP Lemma

The first extensions of the KYP Lemma to the infinite-dimensional case have been achieved by Yakubovich *et al.* [300, 301, 520, 521]. Let us briefly report an extension of the KYP Lemma. We consider a system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = Cx(t) + Du(t), \quad x(0) = x_0 \in X \end{cases} \tag{4.180}$$

where X is a real Hilbert space. The operator $A : dom(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $U(t)$. The operators $B : I\!R^m \to X$,

$C : X \rightarrow \mathbb{R}^m$, $D : \mathbb{R}^m \rightarrow \mathbb{R}^m$, are assumed to be bounded [3]. The solution of (4.180) is

$$x(t) = U(t)x_0 + \int_0^t U(t-s)Bu(s)ds \tag{4.181}$$

Definition 4.113. *The operator $H : \mathcal{L}_{2,e} \rightarrow \mathcal{L}_{2,e}$ is said (γ, ξ)-passive if*

$$\int_0^t e^{\gamma s}(Hu)^T(s)u(s)ds \geq \xi \int_0^t e^{\gamma s}||u(s)||^2 ds \tag{4.182}$$

for all $u \in \mathcal{L}_{2,e}$. ∎

We have the following:

Lemma 4.114. *[509] Let $H : \mathcal{L}_{2,e} \rightarrow \mathcal{L}_{2,e}$ be defined by $y = H(u)$ and (4.180). Suppose that the C_0-semigroup associated with H satisfies $||U(t)|| \leq Me^{-\sigma t}$ for some $M \geq 1$ and $\sigma > 0$. Then for $\gamma < 2\sigma$, $\xi < \sigma_{\min}(D)$, H is $(\gamma, \xi)-$passive if and only if for each $\xi_0 < \xi$, there exist bounded linear operators $0 < P = P^T : X \rightarrow X$, $L \gg 0 : X \rightarrow X$, $Q : X \rightarrow \mathbb{R}^m$, and a matrix $W \in \mathbb{R}^{m \times m}$, such that*

$$\begin{cases} (A^*P + PA + 2\gamma P + L + Q^*Q)x = 0 \;\; \textit{for all} \;\; x \in dom(A) \\ \\ B^*P = C - W^*Q \\ \\ W^*W = D + D^* - 2\xi_0 I_m \end{cases} \tag{4.183}$$

∎

$dom(A)$ is the domain of the operator A. A semigroup that satisfies the condition of the lemma is said exponentially stable. The notation $L(\cdot) \gg 0$ means that $L(\cdot)$ is a positive operator that is bounded invertible (or coercive).

4.9.2 The Wave Equation

This section presents an example of an infinite-dimensional system which is dissipative: the wave equation. Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary Γ. Let us denote $Q = \Omega \times (0, +\infty)$ and $\Sigma = \Gamma \times (0, +\infty)$. The problem is to find a function $u(x,t) : \bar{\Omega} \times [0, +\infty) \rightarrow \mathbb{R}$ such that

[3] An operator may here be much more general than a linear operator represented by a constant matrix $A \in \mathbb{R}^{m \times n}$: $x \mapsto Ax \in \mathbb{R}^m$. For instance the Laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, or the D'Alembertian $\frac{\partial^2}{\partial t^2} - \Delta$ are operators.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{on } Q \\ u = 0 & \text{on } \Sigma \\ u(x,0) = u_0(x) & \text{on } \Omega \\ \frac{\partial u}{\partial t}(x,0) = v_0(x) & \text{on } \Omega \end{cases} \tag{4.184}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x^2}$ is the Laplacian with respect to state variables, $u_0(\cdot)$ and $v_0(\cdot)$ are data. The system in (4.184) is called the *wave equation*: this is an hyperbolic equation. When $n = 1$ and $\Omega = (0,1)$, (4.184) models the small vibrations of a free rope. For each $t \geq 0$, the graph of the function $x \in \Omega \mapsto u(x,t)$ coincides with the rope configuration at time t. When $n = 2$, it models the small vibrations of an elastic shell. From a general point of view, (4.184) models wave propagation in an elastic homogeneous medium $\Omega \subset \mathbb{R}^n$. The second condition in (4.184) is the Dirichlet boundary condition. It means that the rope is fixed on the boundary Γ. The third and fourth conditions in (4.184) are the Cauchy initial data for the system (initial position and initial velocity). It is assumed that the boundary data and Ω satisfy some regularity conditions, so that the solution of (4.184) exists and is unique as a $C^2(\mathbb{R}^+)$ and $\mathcal{L}_2(\Omega)$ function (we do not present here the rigorous definition of the functional spaces which are needed to correctly define the solution, because this would bring us much too far in such a brief presentation). The interesting part for us is:

Lemma 4.115. *Along the solutions of (4.184) one has*

$$\left\| \frac{\partial u}{\partial x}(t) \right\|_{2,\Omega}^2 + ||\nabla u(t)||_{2,\Omega}^2 = ||v_0||_{2,\Omega}^2 + ||\nabla u_0||_{2,\Omega}^2 \tag{4.185}$$

for all $t \geq 0$. ■

One has is $\left\|\frac{\partial u}{\partial x}(t)\right\|_{2,\Omega}^2 = \int_\Omega \left\|\frac{\partial u}{\partial x}(t)\right\| dx$ and $||\nabla u(t)||_{2,\Omega}^2 = \int_\Omega \left\|\frac{\partial u}{\partial x_i}(x,t)\right\|^2 dx$. The equality in (4.185) means that the system is lossless (energy is conserved). Notice that the wave equation may be rewritten as a first order system

$$\begin{cases} \frac{\partial u}{\partial t} - v = 0 & \text{on } Q \\ \frac{\partial v}{\partial t} - \Delta u = 0 & \text{on } Q \end{cases} \tag{4.186}$$

If $X = \begin{pmatrix} u \\ v \end{pmatrix}$ then (4.186) becomes $\frac{dX}{dt} + AX = 0$ with $A = \begin{pmatrix} 0_n & -I_n \\ -\Delta & 0_n \end{pmatrix} X$. It happens that the operator $A + I_{2n}$ is maximal monotone. We retrieve here this notion that we used also in the case of finite-dimensional nonsmooth systems in Section 3.9.4.

4.9.3 The Heat Equation

The notation is kept form the foregoing subsection. The heat equation is given as

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 & \text{on } Q \\ u = 0 & \text{on } \Sigma \\ u(x,0) = u_0(x) & \text{on } \Omega \end{cases} \tag{4.187}$$

The variable u may be the temperature in the domain Ω. Under the assumption that $u_0 \in \mathcal{L}_2(\Omega)$, there exists a unique solution $u(x,t)$ for (4.187) in $C^1(I\!R^+)$ which is itself $\mathcal{L}_2(\Omega)$. Moreover:

Lemma 4.116. *Along the solutions of (4.187) one has*

$$\int_\Omega ||u(x,t)||^2 dx + \int_0^t \|\nabla u(t)\|_{2,\Omega}^2 \, dt = \frac{1}{2}||u_0||_{2,\Omega}^2 \tag{4.188}$$

for all $t \geq 0$*, where* $\|\nabla u(t)\|_{2,\Omega}^2 = \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i}(x,t) \right\|^2 dx$. ■

The operator $A : u \mapsto -\Delta u$ is maximal monotone. The equality in (4.188) means that the temperature decreases on Q at a fixed position x.

Let us mention more work on infinite dimensional systems that may be found in [30,39,40,57–59,117–120,172,196,224,307,391,507,509]. The case of a parabolic equation describing the temperature control problem for a homogeneous rod of unit length is provided in [57, §4].

4.10 Further Results

Nonnegative systems: the theory of dissipative systems and the KYP Lemma have also been applied to nonnegative systems [191,192]. Nonnegative dynamical systems are derived from mass and energy balance considerations that involve states whose values are nonnegative. For instance in ecological models, the quantity of fishes in a lake cannot be negative (if the mathematical model allows for such negative values then surely it is not a good model). A matrix $A \in I\!R^{n\times m}$ is nonnegative if $A_{ij} \geq 0$ for all $1 \leq i \leq n$ and all $1 \leq j \leq m$. It is positive if the strict inequality > 0 holds. A matrix $A \in I\!R^{n\times n}$ is called essentially nonnegative (positive) if $-A$ is a **Z**-matrix, i.e. if $A_{ij} \geq 0$ (> 0) for all $1 \leq i \leq n$ and all $1 \leq j \leq n$ with $i \neq j$. A matrix $A \in I\!R^{n\times n}$ is essentially nonnegative if and only if $\exp(At)$ is nonnegative for all $t \geq 0$. A sufficient condition for the solutions of the system $\dot{x}(t) = Ax(t)$, $x(0) = x_0 \geq 0$, $t \geq 0$, to satisfy $x(t) \geq 0$ for all $t \geq 0$, is that A be essentially nonnegative.

Let us now consider a system whose realization is the quadruple (A, B, C, D), with $A \in \mathbb{R}^{n\times n}$ being essentially nonnegative, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{l\times m}$ being nonnegative matrices. Suppose also that the inputs are restricted to nonnegative values, *i.e.* $u(t) \geq 0$ for all $t \geq 0$. Then the system is nonnegative in the sense that $x(t) \geq 0$ and $y(t) \geq 0$ for all $t \geq 0$ [191, Lemma 2.2].

Theorem 4.117 (KYP Lemma for nonnegative systems). *[191] Let $q \in \mathbb{R}^l$ and $r \in \mathbb{R}^m$. Consider the nonnegative dynamical system with realization (A, B, C, D) where A is essentially nonnegative, B, C and D are nonnegative. Then the system is exponentially dissipative with respect to the supply rate $w(u, y) = q^T y + r^T u$ if and only if there exist nonnegative vectors $p \in \mathbb{R}^n$, $l \in \mathbb{R}^n$, and $w \in \mathbb{R}^m$, and a scalar $\epsilon \geq 0$ such that*

$$\begin{cases} A^T p + \epsilon p - C^T q + l = 0 \\ \\ B^T p - D^T q - r + w = 0 \end{cases} \tag{4.189}$$

■

Clearly when $\epsilon = 0$ the system is simply dissipative and no longer exponentially dissipative. This result extends to positive nonlinear systems.

The word *dissipative* is sometimes used in a different context in the theory of dynamical systems, see *e.g.* [98].

5

Stability of Dissipative Systems

In this chapter, various results concerning the stability of dissipative systems are presented. First, the input/output properties of several feedback interconnections of passive systems are reviewed. Then the conditions under which storage functions are Lyapunov functions are given in detail. The chapter ends with an introduction to H_∞ theory for nonlinear systems that is related to a specific dissipativity property, and with a section on Popov's hyperstability.

5.1 Passivity Theorems

In this section we will study the stability of the interconnection in negative feedback of different types of passive systems. We will first study closed-loop interconnections with one external input (one-channel results) and then interconnections with two external inputs (two-channel results). The implicit assumption in the passivity theorems is that the problem is well-posed, *i.e.* that all the signals belong to $\mathcal{L}_{2e}$.

Remark 5.1. Different versions of passivity theorems can be obtained depending on the properties of the subsystems in the interconnections. We will only consider here the most classical versions.

5.1.1 One-channel Results

Theorem 5.2 (Passivity (one-channel) [207]). *Assume that both H_1, H_2 are pseudo-VSP,* i.e.

$$\int_0^t y_1^T(s)u_1(s)ds + \beta_1 \geq \delta_1 \int_0^t y_1^T(s)y_1(s)ds + \epsilon_1 \int_0^t u_1^T(s)u_1(s)ds$$

$$\int_0^t y_2^T(s)u_2(s)ds + \beta_2 \geq \delta_2 \int_0^t y_2^T(s)y_2(s)ds + \epsilon_2 \int_0^t u_2^T(s)u_2(s)ds$$

with

$$\delta_1 + \epsilon_1 > 0,\ \delta_2 + \epsilon_2 > 0$$

The feedback closed-loop system (see Figure 5.1) is finite gain stable if

$$\delta_2 \geq 0, \epsilon_1 \geq 0, \epsilon_2 + \delta_1 > 0,$$

where ϵ_2 or δ_1 may be negative. ■

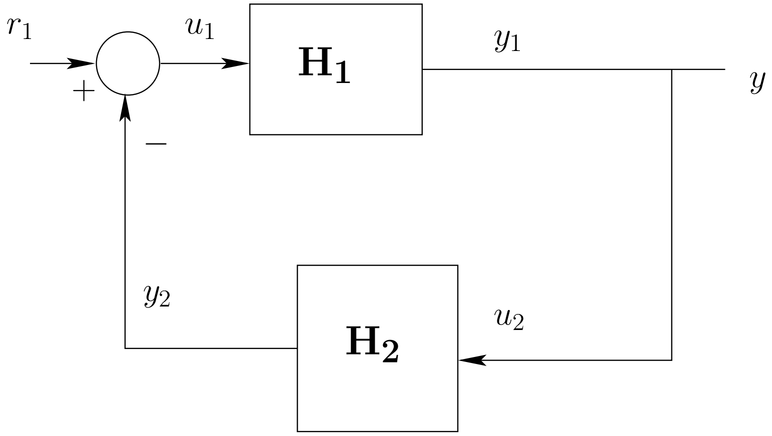

Fig. 5.1. Closed-loop system with one external input

Corollary 5.3. *The feedback system in Figure 5.1 is $\mathcal{L}_2-$finite-gain stable if*

1. H_1 is passive and H_2 is ISP *i.e.* $\epsilon_1 \geq 0, \epsilon_2 > 0, \delta_1 \geq 0, \delta_2 \geq 0$
2. H_1 is OSP and H_2 is passive *i.e.* $\epsilon_1 \geq 0, \epsilon_2 \geq 0, \delta_1 > 0, \delta_2 \geq 0$

■

Proof: Let $\langle r|y\rangle_t = \int_0^t r(s)y(s)ds$. Then

$$\begin{aligned}
\langle r|y\rangle_t &= \langle u_1 + y_2|y\rangle_t \\
&= \langle u_1|y_1\rangle_t + \langle y_2|u_2\rangle_t \\
&\geq \beta_1 + \epsilon_1\|u_1\|^2 + \delta_1\|y_1\|_t^2 + \beta_2 + \epsilon_2\|u_2\|^2 + \delta_2\|y_2\|_t^2 \\
&\geq \beta_1 + \beta_2 + (\delta_1 + \epsilon_2)\|y\|_t^2,
\end{aligned} \tag{5.1}$$

where $\|y\|_t^2 = \langle y|y\rangle_t$. Using the Schwartz' inequality we have

$$\langle r|y\rangle_t = \int_0^t r(s)y(s)ds \le \left[\int_0^t r^2(s)ds\right]^{\frac{1}{2}} \left[\int_0^t y^2(s)ds\right]^{\frac{1}{2}} = \|r\|_t\|y\|_t$$

Then

$$\|r\|_t\|y\|_t \ge \langle r|y\rangle_t \ge \beta_1 + \beta_2 + (\delta_1 + \epsilon_2)\|y\|_t^2$$

For any $\lambda \in I\!R$ the following holds

$$\begin{aligned} \tfrac{1}{2\lambda}\|r\|_t^2 + \tfrac{\lambda}{2}\|y\|_t^2 &= \tfrac{1}{2}\left(\tfrac{1}{\sqrt{\lambda}}\|r\|_t - \sqrt{\lambda}\|y\|_t\right)^2 + \|r\|_t\|y\|_t \\ &\ge \beta_1 + \beta_2 + (\delta_1 + \epsilon_2)\|y\|_t^2 \end{aligned} \tag{5.2}$$

Choosing $\lambda = \delta_1 + \epsilon_2$ we get

$$\frac{\|r\|_t^2}{2(\delta_1 + \epsilon_2)} \ge \beta_1 + \beta_2 + \frac{(\delta_1 + \epsilon_2)}{2}\|y\|_t^2$$

which concludes the proof. ■

5.1.2 Two-channel Results

Consider now the system depicted in Figure 5.2 where r_1, r_2 can represent disturbances, initial condition responses or commands. Assume well-posedness.

Theorem 5.4. (Passivity (two-channel) [500]) *Assume H_1, H_2 are pseudo VSP. The feedback system is $\mathcal{L}_2-$finite-gain stable if*

$$\epsilon_1 + \delta_2 > 0$$

$$\epsilon_2 + \delta_1 > 0$$

where ϵ_i, δ_i may be negative. ■

Corollary 5.5. *The feedback system is $\mathcal{L}_2-$finite-gain stable if*

1. *H_1, H_2 are ISP ($\epsilon_1 > 0, \epsilon_2 > 0, \delta_1 = \delta_2 = 0$)*
2. *H_1, H_2 are OSP ($\delta_1 > 0, \delta_2 > 0, \epsilon_1 = \epsilon_2 = 0$)*
3. *H_1 is VSP, H_2 is passive ($\epsilon_1 > 0, \delta_1 > 0, \delta_2 = \epsilon_2 = 0$)*
4. *H_1 is passive, H_2 is VSP ($\epsilon_2 > 0, \delta_2 > 0, \delta_1 = \epsilon_1 = 0$)* ■

Proof:

$$\begin{aligned} \langle u_1|y_1\rangle_t + \langle y_2|u_2\rangle_t &= \langle r_1 - y_2|y_1\rangle_t + \langle y_2|y_1 + r_2\rangle_t \\ &= \langle r_1|y_1\rangle_t + \langle y_2|r_2\rangle_t \\ &\ge \beta_1 + \epsilon_1\|u_1\|^2 + \delta_1\|y_1\|_t^2 + \beta_2 \\ &+\epsilon_2\|u_2\|_t^2 + \delta_2\|y_2\|_t^2 \end{aligned} \tag{5.3}$$

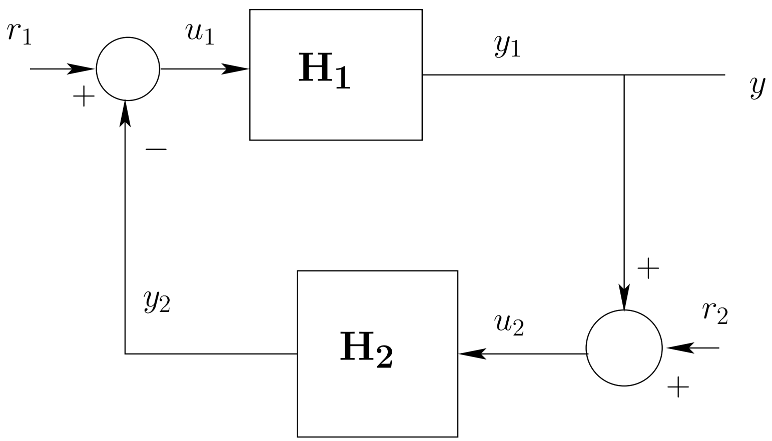

Fig. 5.2. Closed-loop system with two external inputs

Note that

$$\begin{aligned} \|u_1\|_t^2 &= \int_0^t u_1^T(s)u_1(s)ds \\ &= \int_0^t (r_1(s) - y_2(s))^T(r_1(s) - y_2(s))ds \\ &\geq -2\langle r_1|y_2\rangle_t + \|y_2\|_t^2 \end{aligned} \tag{5.4}$$

and similarly

$$\|u_2\|_t^2 \geq 2\langle r_2|y_1\rangle_t + \|y_1\|_t^2$$

Then

$$\begin{aligned} \langle r_1|y_1\rangle_t + \langle y_2|r_2\rangle_t &+ 2\epsilon_1\langle r_1|y_2\rangle_t - 2\epsilon_2\langle r_2|y_1\rangle_t \geq \\ &\beta_1 + \beta_2 + (\epsilon_1 + \delta_2)\|y_2\|_t^2 + (\epsilon_2 + \delta_1)\|y_1\|_t^2 \end{aligned} \tag{5.5}$$

Note that for any $\lambda \in \mathbb{R}$, for $i = 1, 2$ we have

$$\begin{aligned} \langle r_i|y_i\rangle_t &\leq \|y_i\|_t\|r_i\|_t + \tfrac{1}{2}\left(\tfrac{1}{\sqrt{\lambda_i}}\|r_i\|_t - \sqrt{\lambda_i}\|y_i\|_t\right)^2 \\ &\leq \tfrac{1}{2\lambda_i}\|r_i\|_t^2 + \tfrac{2\lambda_i}{2}\|y_i\|_t^2 \end{aligned} \tag{5.6}$$

We choose $\lambda_1 = \frac{\epsilon_2 + \lambda_1}{2}$ and $\lambda_2 = \frac{\epsilon_1 + \epsilon_2}{2}$:

- If $\epsilon_1 = 0$ then $2\epsilon_1\langle r_1|y_2\rangle_t \leq 0$

- If $\epsilon_1 > 0$ then for any $\lambda_1' \in I\!R$

$$2\epsilon_1 \langle r_1 | y_2 \rangle_t \leq \frac{\epsilon_1}{\lambda_1'} \|r_1\|_t^2 + \epsilon_1 \lambda_1' \|y_2\|_t^2$$

Let us choose $\lambda_1' = \frac{\lambda_1''}{\epsilon_1}$ and $\lambda_1'' = \frac{\epsilon_1 + \delta_2}{4}$. Therefore

$$\begin{array}{c} \beta_1 + \beta_2 + \frac{(\epsilon_1+\delta_2)}{4} \|y_2\|_t^2 + \frac{(\epsilon_2+\delta_1)}{4} \|y_1\|_t^2 \\ \leq \|r_1\|_t^2 \left(\frac{1}{\epsilon_2+\delta_1} + \frac{4\epsilon_1^2}{\epsilon_1+\delta_2}\right) + \|r_2\|_t^2 \left(\frac{1}{\epsilon_1+\delta_2} + \frac{4\epsilon_1^2}{\epsilon_1+\delta_2}\right) \end{array} \tag{5.7}$$

which concludes the proof. ■

Boundedness of the closed-loop signals can be ensured if H_1 and H_2 have finite gain as can be seen from the following Lemma, which is no longer a purely input/output result but involves the state of the system.

Lemma 5.6. *Consider again the negative feedback interconnection of H_1 and H_2 as in Figure 5.2. Assume that the operators H_1 and H_2 are pseudo VSP* i.e.

$$\int_0^t u_i^T(s) y_i(s) ds = V_i(x_i) - V_i(x_i(0)) + \epsilon_i \int_0^t u_i^T(s) u_i(s) ds + \delta_i \int_0^t y_i^T(s) y_i(s) ds$$

with $V_1(\cdot)$ and $V_2(\cdot)$ positive definite functions. Then the origin is an asymptotically stable equilibrium point if:

$$\epsilon_1 + \delta_2 > 0$$

and

$$\epsilon_2 + \delta_1 > 0$$

and both H_1 and H_2 are zero-state observable (i.e. $u_i \equiv 0, y_i \equiv 0 \Rightarrow x_i = 0$). ■

Proof: Consider the positive definite function which is the sum of the two storage functions for H_1 and H_2, *i.e.*:

$$V(x) = V_1(x_1) + V_2(x_2)$$

Then using the dissipativity inequalities in their infinitesimal form we get along the trajectories of the system

$$\begin{array}{rl} \dot{V}(x(t)) = & \sum_{i=1}^{2} \left[u_i^T(t) y_i(t) - \epsilon_i u_i^T(t) u_i(t) - \delta_i y_i^T(t) y_i(t) \right] \\ = & -(\epsilon_1 + \delta_2) u_1^T(t) u_1(t) - (\epsilon_2 + \delta_1) y_1^T(t) y_1(t) \end{array} \tag{5.8}$$

The result follows from the Krasovskii-LaSalle Theorem and the assumption guaranteeing that $y_i \equiv 0, u_i \equiv 0 \Rightarrow x_i = 0$. If in addition $V_1(\cdot)$ and $V_2(\cdot)$ are radially unbounded, then one gets global stability. ■

Roughly speaking, the foregoing lemma says that the feedback interconnection of two dissipative systems is asymptotically stable provided an observability property holds. Let us now state a result which uses the quasi-dissipativity property as defined in Definition 4.27. Each subsystem H_1 and H_2 of the interconnection is supposed to be dissipative with respect to a general supply rate of the form $w_i(u_i, y_i) = y_i^T Q_i y_i + 2y_i^T S_i u_i + u_i^T R_i u_i$, with $Q_i^T = Q_i$ and $R_i^T = R_i$. Before stating the next Proposition, we need a preliminary definition:

Definition 5.7. *A system $\dot{x}(t) = f(x(t), u(t))$, $y(t) = h(x(t))$ has uniform finite power gain $\gamma \geq 0$ if it is quasi-dissipative with supply rate $w(u, y) = \gamma^2 u^T u - y^T y$.* ■

The following holds:

Proposition 5.8. *[403] Suppose that the systems H_1 and H_2 are quasi-dissipative with respect to supply rates $w_1(u_1, y_1)$ and $w_2(u_2, y_2)$, respectively. Suppose there exists $\rho > 0$ such that the matrix*

$$Q_\rho = \begin{pmatrix} Q_1 + \rho R_2 & -S_1 + \rho S_2^T \\ -S_1^T + \rho S_2 & R_1 + \rho Q_2 \end{pmatrix} \tag{5.9}$$

is negative definite. Then the feedback system in Figure 5.2 has uniform finite power gain. ■

Proof: taking into account the interconnections $u_1 = r_1 - y_2$ and $u_2 = r_2 + y_1$, it follows that

$$\begin{aligned} & w_1(u_1, y_1) + \rho w_2(u_2, y_2) = \\ & = \left[y_1^T \; y_2^T \right] Q_\rho \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \left[y_1^T \; y_2^T \right] S_\rho \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \\ & + \left[r_1^T \; r_2^T \right] R_\rho \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \end{aligned} \tag{5.10}$$

for some matrices S_ρ and R_ρ. Since $Q_\rho < 0$ it follows that there exists $\mu > 0$ and $\eta > 0$ such that

$$-\eta(y_1^T y_1 + y_2^T y_2) + \mu(r_1^T r_1 + r_2^T r_2) \geq w_1(u_1, y_1) + \rho w_2(u_2, y_2) \tag{5.11}$$

Integrating from $t = 0$ to $t = \tau \geq 0$ and using the fact that H_1 and H_2 are quasi-dissipative with constants $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, we obtain

$$\int_0^\tau [-\eta(y_1^T(t)y_1(t) + y_2^T(t)y_2(t)) + \mu(r_1^T(t)r_1(t) + r_2^T(t)r_2(t))]dt + \\ +(\alpha_1 + \rho\alpha_2)\tau + \beta_1 + \rho\beta_2 \geq 0 \tag{5.12}$$

where $\beta_1 \geq 0$ and $\beta_2 \geq 0$ are the bias for H_1 and H_2. ■

This proof is really an input/output system stability result as it does not mention the state. Let us mention a result in [466] that contains a version of the passivity Theorem, using the so-called *secant condition* for the stability of polynomials of the form $p(s) = (s + a_1)(s + a_2)...(s + a_n) + b_1 b_2 ... b_n$, with all $a_i > 0$ and all $b_i > 0$. This $p(s)$ is the characteristic polynomial of the matrix $A = \begin{pmatrix} -a_1 & 0 & .... & 0 & -b_1 \\ b_2 & -a_2 & .... & 0 & 0 \\ \dots & \dots & & & \dots \\ & & & & \\ 0 & 0 & .... & b_n & -a_n \end{pmatrix}$. The secant condition states that A is Hurwitz provided that $\frac{b_1 \dots b_n}{a_1 \dots a_n} < \left(\sec\frac{\pi}{n}\right)^n = \frac{1}{(\cos(\frac{\pi}{n}))^n}$.

5.1.3 Lossless and WSPR Blocks Interconnection

It is known that the feedback interconnection of a PR and a SPR blocks yields an asymptotically stable system; see Lemma 3.37. In the case of nonlinear systems, and using a pure input/output definition of passivity (as in Definition 2.1 where β is not assumed to depend on the initial state value) the passivity Theorem provides $\mathcal{L}_2$-stability results for the interconnection of a passive block with an ISP, OSP or a VSP block (see *e.g.* [500]). Lyapunov stability can be obtained when the blocks are passive in the sense of Willems (*i.e.* the state intervenes in the definition). The goal of the following lemma is to present stability results with slightly relaxed requirements on the feedback block. More precisely, we will deal with the interconnection of lossless blocks with WSPR blocks. The results presented in this section relax the conditions of the passivity Theorem as was conjectured in [310].

We now consider the negative feedback interconnection of a lossless (possibly nonlinear) system with a linear WSPR system and prove the stability of the closed-loop system.

Lemma 5.9. *Assume that H_1 in Figure 5.2 is zero-state observable and lossless with a radially unbounded positive definite storage function $V_1(x_1)$, whereas H_2 is WSPR. Then the feedback interconnection of H_1 and H_2 is Lyapunov globally asymptotically stable.* ■

Proof: Consider $V(x_1, x_2) = x_2^T P_2 x_2 + 2V_1(x_1)$, where $V_1(\cdot)$ is a radially unbounded positive definite storage function for H_1. In view of the assumptions and of the KYP Lemma, there exists matrices P_2, L_2, W_2 such that Equations (3.2) are satisfied for H_2. Then

$$\begin{aligned}\dot{V}(x_1,x_2) &= -x_2^T L_2^T L_2 x_2 + 2x_2^T P_2 B_2 u_2 + 2u_1^T y_1 \\ &= -x_2 L_2^T L_2 x_2 + 2u_2^T (C_2 - W_2^T L_2^T) x_2 + 2u_1^T y_1 \\ &= -x_2^T L_2^T L_2 x_2 - 2u_2^T (W_2^T L_2^T x_2 + D_2 u_2) \\ &= -x_2^T L_2^T L_2 x_2 - 2u_2^T W_2^T L_2^T x_2 - u_2^T (D_2 + D_2^T) u_2 \\ &= -(u_2^T W_2^T + x_2^T L_2^T)(W_2 u_2 + L_2 x_2) \\ &= -\bar{y}_2^T \bar{y}_2 \end{aligned} \tag{5.13}$$

The above ensures that $x^T = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix} = 0$ is a stable equilibrium point, which implies that the state x is bounded. Moreover the transfer function

$$\bar{H}_2(s) = W_2 + L_2(sI - A_2)^{-1} B_2$$

has no zeros on the imaginary axis (see Lemma 3.18). Note that $\bar{Y}_2(s) = \bar{H}_2(s) U_2(s)$. Therefore, when $\bar{y}_2(t) \equiv 0$, $u_2(t)$ can only either exponentially diverge or exponentially converge to zero. However, if $u_2(t)$ diverges, it follows from $\bar{y}_2(t) = W_2 u_2 + L_2 x_2 \equiv 0$ that x_2 should also diverge which is a contradiction. It then follows that u_2 should converge to zero. Note that for $u_2 = 0$ the H_2 system reduces to $\dot{x}_2 = A_2 x_2$ with A_2 Hurwitz. Therefore if $\bar{y}_2(t) \equiv 0$, then $x_2 \to 0$. On the other hand $u_2 = y_1$ and so we also have $y_1 \to 0$. In view of the zero-state observability of $H_{1,}$ we conclude that $x_1 \to 0$.

Hence, from the Krasovskii-La Salle invariance set Theorem, the largest invariant set S inside the set $\bar{y}_2 \equiv 0$ is reduced to $x = 0$ plus all the trajectories such that x tends to the origin. Therefore, the origin $x = 0$ is asymptotically stable. Moreover, when $V_1(x_1)$ is radially unbounded any trajectory is bounded, and the equilibrium is globally asymptotically stable. ■

Another proof can be found in [213]. It makes use of the material in Appendix A.4.1 which possesses its own interest.

5.1.4 Large-scale Systems

Large-scale systems consist of an interconnection of N subsystems H_i, which are all dissipative. It is assumed here that the subsystems are dissipative in the sense of Definition 4.22 and with respect to a general supply rate $w_i(u_i, y_i) = y_i^T Q_i y_i + 2y_i^T S_i u_i + u_i^T R_i u_i$. The interconnection relationship is

$$u_i = u_{e,i} - \sum_{j=1}^{N} H_{ij} y_j \tag{5.14}$$

where u_i is the input of subsystem H_i, y_i is its ouput, $u_{e,i}$ is an external input, and all the H_{ij} are constant matrices. Grouping the inputs, outputs and external inputs as N-vectors u, y and u_e respectively, one may rewrite (5.14) as

$$u = u_e - Hy \tag{5.15}$$

where $H \in I\!R^{N\times N}$. Let us define Q =diag(Q_i), S =diag(S_i) and R =diag(R_i), and the matrix

$$\hat{Q} = SH + H^T S^T - H^T RH - Q \tag{5.16}$$

Theorem 5.10. *[364] The overall system with input $u_e(\cdot)$ and output $y(\cdot)$ and the interconnection in (5.15) is $\mathcal{L}_2$−finite-gain stable if $\hat{Q} > 0$ in (5.16).* ■

Proof: For each subsystem H_i we have by assumption

$$\int_{t_0}^{t_1} w_i(u_i(t), y_i(t))dt \geq 0 \tag{5.17}$$

for all $t_1 \geq t_0$. By summation over all i one obtains

$$\int_{t_0}^{t_1} w(u(t), y(t))dt \geq 0 \tag{5.18}$$

Using (5.15) and (5.16) one obtains

$$\int_{t_0}^{t_1} [y^T(t)\hat{Q}y(t) - 2y^T(t)\hat{Q}^{\frac{1}{2}}\hat{S}u_e(t)]dt \leq \int_{t_0}^{t_1} u_e^T(t)Ru_e(t)dt \tag{5.19}$$

with $\hat{S} = \hat{Q}^{-\frac{1}{2}}(S - H^T R)$. Let $\alpha > 0$ be a finite real such that $R + \hat{S}^T\hat{S} \leq \alpha^2 I_N$. Clearly one can always find such a scalar. Then one finds after some manipulation

$$\int_{t_0}^{t_1} [\hat{Q}^{\frac{1}{2}}y(t) - \hat{S}u_e(t)]^T[\hat{Q}^{\frac{1}{2}}y(t) - \hat{S}u_e(t)]dt \leq \alpha^2 \int_{t_0}^{t_1} u_e^T(t)u_e(t)dt, \tag{5.20}$$

so that

$$\int_{t_0}^{t_1} y^T(t)y(t)dt \leq k^2 \int_{t_0}^{t_1} u_e^T(t)u_e(t)dt \tag{5.21}$$

with $k = ||\hat{Q}^{-\frac{1}{2}}||(\alpha + ||\hat{S}||)$. ■

Let us recall that we assumed at the beginning of this section that all signals belong to the extended space $\mathcal{L}_{2,e}$ (more rigorously: the inputs are in $\mathcal{L}_{2,e}$ and we assume that the systems are well-posed in the sense that the outputs also belong to $\mathcal{L}_{2,e}$). Under such an assumption, one sees that stating (5.21) for all $t_1 \geq t_0 \geq 0$ is equivalent to stating $||y||_{2,t} \leq k||u_e||_{2,t}$ for all

$t \geq 0$, where $||\cdot||_{2,e}$ is the extended $\mathcal{L}_2$ norm. One notes that Theorem 5.10 is constructive in the sense that the interconnections $H_{i,j}$ may be chosen or designed so that the Riccati inequality $\hat{Q} > 0$ in (5.16) is satisfied. The literature on large-scale systems stability is abundant, and an early reference to be read for more informations and results is [499]. Let us end this subsection with a result which will allow us to make a link between the interconnection strucuture, and so-called $\mathbf{M}-$matrices.

Theorem 5.11. *[364] Let the subsystem H_i have a $\mathcal{L}_2-$finite-gain γ_i and suppose that all subsystems are single input single output (SISO). Let $\Gamma =$diag(γ_i), and $A = \Gamma H$. Then if there exists a diagonal positive definite matrix P such that*

$$P - A^T P A > 0 \tag{5.22}$$

the interconnected system is $\mathcal{L}_2-$finite-gain stable. ■

A sufficient condition for the existence of a matrix P as in the theorem is that the matrix B made of the entries $b_{ii} = 1 - |a_{ii}|$, $b_{ij} = -|a_{ij}|$ for $i \neq j$, has all its leading principal minors positive. Such a matrix is called an $\mathbf{M}-$matrix.

Further works on large-scale systems may be found in [184, 185].

5.2 Positive Definiteness of Storage Functions

In this section we will study the relationship between dissipativeness and stability of dynamical systems. Let us first recall that in the case of linear systems, the plant is required to be asymptotically stable to be WSPR, SPR or SSPR. For a PR system it is required that its poles be in the left-half plane and the poles in the $j\omega-$axis be simple and have non-negative associated residues. Consider a dissipative system as in Definition 4.20. It can be seen that if $u = 0$ or $y = 0$, then $V(x(t)) \leq V(x(0))$. If in addition the storage function is positive definite, then we can conclude that the system $\dot{x}(t) = f(x(t))$ has a Lyapunov stable fixed point $x = 0$, and the system's zero dynamics is stable. Furthermore, if the system is strictly passive (*i.e.* $\mathcal{S}(x) > 0$ in (4.51)) then the system $\dot{x}(t) = f(x(t))$, and the system's zero dynamics are both asymptotically stable (see Theorem 4.10).

Let us now consider passive systems as given by Definition 2.1. The two following Lemmae will be used to establish the conditions under which a passive system is asymptotically stable.

Definition 5.12 (locally ZSD). *A nonlinear system (4.79) is locally zero-state detectable (ZSD)[locally Zero state observable (ZSO)] if there exists a neighborhood $\mathbf{N}$ of 0 such that for all $x(t) \in \mathbf{N}$*

$$u(t) = 0, h(x(t)) = 0, \;\; \forall\, t \geq 0 \;\Rightarrow \lim_{t \to +\infty} x(t) \to 0 \;\; [x(t) = 0 \textit{ for all } t \geq 0]$$

If $\mathbf{N} = I\!R^n$ *the system is ZSD [ZSO].* ■

Lemma 5.13. *[207] Consider a dissipative system with a general supply rate $w(u, y)$, and let Assumptions 3 and 4 of Section 4.5.2 hold. Assume that:*

1. *The system is zero-state observable*
2. *For any $y \neq 0$ there exists some u such that $w(u, y) < 0$*

Then all the solutions to the NL-KYP set of equations (4.81) are positive definite.

Proof: We have already seen that the available storage

$$V_a(x) = \sup_{x=x(0),t\geq 0,u} \left\{ -\int_0^t w(s)ds \right\}$$

is a (minimum) solution of the KYP-NL set of equations (4.81), see the necessity part of the proof of Lemma 4.87 and Theorem 4.41. Recall that $0 \leq V_a(x) \leq V(x)$. If we choose u such that $w(u, y) \leq 0$ on $[t_0, \infty)$, with strict inequality on a subset of positive measure, then $V_a(x) > 0$, $\forall y \neq 0$. Note from the equation above that the available storage $V_a(x)$ does not depend on $u(t)$ for $t \in [t_0, \infty)$. When $y = 0$ we can choose $u = 0$ and therefore $x = 0$ in view of the zero-state observability assumption. We conclude that $V_a(x)$ is positive definite and that $V(x)$ is also positive definite (see Definition A.9). ■

Lemma 5.14. *Under the same conditions of the previous lemma, the free system $\dot{x} = f(x)$ is (Lyapunov) stable if $Q \leq 0$ and asymptotically stable if $Q < 0$, where Q is the weighting matrix in the general supply rate (4.80).* ■

Proof: From Corollary 4.88 and Lemma 4.87 there exists $V(x) > 0$ for all $x \neq 0$, $V(0) = 0$, such that (using (4.81) and (4.82))

$$\begin{aligned}
\tfrac{d(V\circ x)}{dt}(t) &= -\left[L(x(t)) + W(x(t))u(t)\right]^T \left[L(x(t)) + W(x(t))u(t)\right] + \\
&+y^T(t)Qy(t) + 2y^T(t)Su(t) + u^T(t)Ru(t) \\
&= -L^T(x(t))L(x(t)) - 2L^T(x(t))W(x(t))u(t) - u^T(t)W^T(x(t))\times \\
&\times W(x(t))u(t) + (h(x(t)) + j(x(t))u)^T Q(h(x(t)) + j(x(t))u(t)) + \\
&+2(h(x(t)) + j(x(t))u(t))^T Su(t) + u^{T(t)}Ru(t)
\end{aligned} \tag{5.23}$$

so that

$$
\begin{aligned}
\frac{d(V\circ x)}{dt}(t) &= -L^T(x(t))L(x(t)) - u^T(t)W^T(x(t))W(x(t))u(t)+\\
&+u^T(t)[R + j^T(x(t))Qj(x(t)) + j^T(x(t))S + S^T j(x(t))]u(t)+\\
&+2[-L^T(x(t))W(x(t)) + h^T(x(t))(Qj(x(t)) + S)]u(t)+\\
&+h^T(x(t))Qh(x(t))\\
&= -L^T(x(t))L(x(t)) - u^T(t)\hat{R}(x(t))u(t) + u^T(t)\hat{R}(x(t))u(t)+\\
&+2[-L^T W(x(t)) + h^T(x(t))\hat{S}(x(t))]u(t) + h^T(x(t))Qh(x(t))\\
&= -L^T(x(t))L(x(t)) + \nabla V^T(x(t))g(x(t))u(t) + h^T(x(t))Qh(x(t))
\end{aligned}
\tag{5.24}
$$

For the free system $\dot{x}(t) = f(x(t))$ we have

$$
\frac{d(V\circ x)}{dt}(t) = -L^T(x(t))L(x(t)) + h^T(x(t))Qh(x(t)) \leq h^T(x(t))Qh(x(t)) \leq 0
$$

If $Q < 0$ then $\frac{d(V\circ x)}{dt}(t) \leq 0$ which implies stability of the system. If $Q \leq 0$ we use Krasovskii-LaSalle Invariance Principle. The invariant set is given by $\Omega : \{\xi | h(\xi) = y = 0\}$ and therefore $x(\cdot)$ converges to the set Ω. In view of the zero-state observability we conclude that $x(t) \to 0$ asymptotically. One sees that under the conditions of Lemma 5.13 and with $Q < 0$, then necessarily $x = 0$ is an isolated fixed point of $\dot{x}(t) = f(x(t))$. ■

Example 5.15. Let us come back to Example 4.59. The system in (4.51) is not zero state detectable, since $u \equiv 0$ and $y \equiv 0$ do not imply $x \to 0$ as $t \to +\infty$. And the uncontrolled (or free) system is exponentially unstable ($\dot{x}(t) = x(t)$). This shows the necessity of the ZSD condition.

Corollary 5.16. *[207] Consider a dissipative system with a general supply rate $w(u, y)$. Assume that:*

1. *The system is zero-state observable(i.e. $u(t) \equiv 0$ and $y(t) \equiv 0 \Rightarrow x(t) = 0$)*
2. *For any $y \neq 0$ there exists some u such that $w(u, y) < 0$*

Then passive systems (i.e. $Q = R = 0, S = I$) and input strictly passive systems (ISP) (i.e. $Q = 0, 2S = I, R = -\epsilon$) are stable, while output passive systems (OSP) (i.e. $Q = -\delta, 2S = I, R = 0$) and very strictly passive systems (VSP) (i.e. $Q = -\delta, 2S = I, R = -\epsilon$) are asymptotically stable. ■

Before stating the next lemmas let us introduce another notion of zero state detectability.

Definition 5.17. *A dynamical system is said to be locally zero state detectable in a region Ω_z if for any $x_0 \in \Omega_z$, $x_0 \neq 0$, such that the solution $x(t) \in \Omega$, for all $0 \leq t \leq \tau$ for some $\tau > 0$, with $u(\cdot) = 0$, there exists a continuous function $\alpha : I\!R \to I\!R^+$, $\alpha(0) = 0$, $\alpha(w) > 0$ for all $w \neq 0$, such that*

$$\int_0^t y^T(t')y(t')dt' \geq \alpha(||x_0||) \tag{5.25}$$

for some $t < +\infty$ such that $0 \leq t \leq \tau$. If in addition for any sequence $\{w_n\} \in \Omega$, one has $\alpha(w_n) \to +\infty$ as $||w_n|| \to +\infty$, the system is said to be locally uniformly zero state detectable in Ω_z with respect to Ω. ■

Clearly a system that is ZSD according to this definition is also ZSD in the sense of Definition 5.12. Sometimes a system that satisfies the first part of Definition 5.17 is called uniformly observable. The local versions of Lemmas 5.13 and 5.14 are as follows:

Lemma 5.18. *[404] Let the dynamical system in (4.79) be*

- *Locally dissipative with respect to a general supply rate (4.80) in a region $\Omega \subset I\!R^n$,*
- *Locally $w-$uniformly reachable in a region Ω_c with respect to Ω*
- *Locally uniformly zero state detectable in Ω_z with respect to Ω*

Suppose that $\Omega_z \cap \Omega_c \neq \emptyset$. Then the dynamical system has all its storage functions $V : \Omega_z \cap \Omega_c \to I\!R$ continuous, $V(0) = 0$, and $V(x) > 0$ for all $x \in \Omega_z \cap \Omega_c$. Moreover for any sequence $\{x_n\} \in \Omega_z \cap \Omega_c$, $V(x_n) \to +\infty$ as $||x_n|| \to +\infty$. ■

We will also say that a system is said to be locally reachable with respect to Ω in a region $\Omega_r \subseteq \Omega$, if every state $x_1 \in \Omega_r$ is locally reachable with respect to Ω from the origin $x = 0$ and for all $t_0 \in I\!R$, with an input that keeps the state trajectory inside Ω.

Definition 5.19. *A system is said locally connected with respect to Ω in a region $\Omega_{con} \subseteq \Omega$, if any $x_1 \in \Omega_{con}$ is locally reachable with respect to Ω from any $x_0 \in \Omega_{con}$, and for all $t_0 \in I\!R$.* ■

Now we are ready to state the main result which concerns the local stability deduced from local dissipativity.

Lemma 5.20. *[404] Let the dynamical system in (4.79) be*

- *Locally dissipative with respect to a general supply rate (4.80) in a region $\Omega \subset I\!R^n$*
- *Locally $w-$uniformly reachable in a region Ω_c with respect to Ω*
- *Locally uniformly zero state detectable in Ω_z with respect to Ω*
- *Locally connected in a region Ω_{con} with respect to Ω*

- *Locally Lipschitz continuous in Ω*

and be such that there exists a feedback controller $u^\star(x)$ such that $w(u^\star, y) < 0$ for all $y \neq 0$, $u^\star(0) = 0$ and $u^\star(\cdot)$ drives the system from $x_0 \in \Omega$ to $x_1 \in \Omega$ while keeping the trajectory inside Ω. Suppose that the region $\Omega_c \cap \Omega_z \cap \Omega_{con}$ contains an open neighborhood of $x = 0$. Then if $Q < 0$ the origin $x = 0$ is asymptotically stable. ■

The above conditions imply that all the defined regions contain $x = 0$. We now state a result which is based on the notion of weak $w(u, y)-$dissipativity (Definition 4.29) and is interesting as it applies to systems with multiple equilibria, and makes no assumption on the differentiability of the storage functions. This theorem is linked to Theorems 4.31, 4.32 and 4.33. $d(x, \Omega) = \inf_{y \in \Omega} ||x - y||$ denotes the distance from x to Ω.

Theorem 5.21. *[206] Suppose that $G(\Omega)$ is $w(u, y)-$dissipative for some $Q < 0$. Let $X_1 = \{x \mid d(x, \Omega) \leq d_1\}$ for some $d_1 > 0$, be uniformly reachable from Ω and zero state observable with respect to Ω. Then there exists some $d_2 > 0$ (dependent on d_1) such that, with input $u(\cdot) \equiv 0$, all state trajectories starting in $X_2 = \{x \mid d(x, \Omega) \leq d_2\}$ remain in X_1, and asymptotically approach Ω.* ■

As an illustration one may consider Example 4.34. Let us now introduce the following definition:

Definition 5.22 (Proper function). *A function $V : x \rightarrow I\!R$ is said to be proper if for each $a > 0$, the set $V^{-1}[0, a] = \{x : 0 \leq V(x) \leq a\}$ is compact (closed [1] and bounded).* ■

A variant of Lemma 5.14 is as follows:

Lemma 5.23. *[442] Let $V(\cdot) \geq 0$ be a solution of (4.76), with $\mathcal{S}(x) = \epsilon h^T(x)h(x)$, $\epsilon > 0$, $V(0) = 0$ and $V(x) > 0$, $x \neq 0$, and suppose that the system in (4.74) is zero-state detectable. Then $x = 0$ is a locally asymptotically stable equilibrium of $\dot{x}(t) = f(x(t))$. If additionally $V(\cdot)$ is proper then $x = 0$ is globally asymptotically stable.* ■

5.3 WSPR Does not Imply OSP

In this subsection we prove that if a system is WSPR (Weakly Strictly Positive Real), it does not necessarily imply that the system is OSP (Output Strictly Passive). The proof is established by presenting a counterexample. The passivity Theorems concern interconnections of two blocks, where the feedback block must be either ISP, OSP or VSP. The interest of the results in Section

[1] A set is closed if it contains its limit points.

5.1.3 is that the conditions on the feedback block are relaxed to WSPR. We prove now that the following transfer function (which is WSPR; see Example 2.59)

$$H(s) = \frac{s+a+b}{(s+a)(s+b)} \tag{5.26}$$

is not OSP. This proves that in general WSPR $\not\Rightarrow$ OSP. A minimal state space representation (A, B, C) for $H(s)$ is given by $A = \begin{pmatrix} 0 & 1 \\ -ab & -a-b \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = (1, 0)$. Let us choose $a = 1$, $b = 2$, $x(0) = 0$, $u = \sin(\omega t)$. Then

$$y(t) = \int_0^t [2\exp(\tau - t) - \exp(2\tau - 2t)]\sin(\omega\tau)d\tau \tag{5.27}$$

It can be shown that

$$y(t) = f_1(\omega)\cos(\omega t) + f_2(\omega)\sin(\omega t) \tag{5.28}$$

with $f_1(\omega) = -\frac{\omega^3 - 7\omega}{(1+\omega^2)(4\omega^2)}$, and $f_2(\omega) = \frac{6}{(1+\omega^2)(4\omega^2)}$. It can also be proved that

$$\int_0^t u(\tau)y(\tau)d\tau = -\frac{f_1(\omega)}{4\omega}[\cos(2\omega t) - 1] + \frac{f_2(\omega)}{2}[t - \frac{\sin(2\omega t)}{2\omega}] \tag{5.29}$$

and that

$$\begin{aligned} \textstyle\int_0^t y^2(\tau)d\tau = f_1^2(\omega)\left[\frac{t}{2} + \frac{\sin(2\omega t)}{4\omega}\right] + f_2^2(\omega)\left[\frac{t}{2} - \frac{\sin(2\omega t)}{\omega}\right] \\ -f_1(\omega)f_2(\omega)\left[\frac{\cos(2\omega t)}{2\omega} - 1\right] \end{aligned} \tag{5.30}$$

Let us choose $t_n = \frac{2n\pi}{\omega}$ for some integer $n > 0$. When $\omega \to +\infty$, then $\int_0^{t_n} u(\tau)y(\tau)d\tau = \frac{f_2(\omega)2n\pi}{4\omega}$, whereas

$$\int_0^{t_n} y^2(\tau)d\tau = \frac{2n\pi(f_1^2(\omega) + f_2^2(\omega))}{4\omega} + f_1(\omega)f_2(\omega)\left(1 - \frac{1}{2\omega}\right)$$

It follows that $\int_0^{t_n} u(\tau)y(\tau)d\tau \underset{\omega \to \infty}{\simeq} \frac{\alpha}{\omega^5}$ while $\int_0^{t_n} y^2(\tau)d\tau \underset{\omega \to \infty}{\simeq} \frac{\gamma}{\omega^3}$ for some positive real α and γ. Therefore we have found an input $u(t) = \sin(\omega t)$ and a time t such that the inequality $\int_0^t u(\tau)y(\tau)d\tau \geq \delta \int_0^t y^2(\tau)d\tau$ cannot be satisfied for any $\delta > 0$, as $\omega \to +\infty$.

5.4 Stabilization by Output Feedback

5.4.1 Autonomous Systems

Consider a causal nonlinear system $(\Sigma) : u(t) \to y(t);\ u(t) \in \mathcal{L}_{pe}, y(t) \in \mathcal{L}_{pe}$ represented by the following state-space representation affine in the input:

$$(\Sigma) \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ \\ y(t) = h(x(t)) + j(x(t))u(t) \end{cases} \tag{5.31}$$

where $x(t) \in I\!R^n$, $u(t), y(t) \in I\!R^m$, $f(\cdot), g(\cdot), h(\cdot)$, and $j(\cdot)$ are smooth functions of x and $f(0) = h(0) = 0$. We can now state the following result:

Theorem 5.24 (Global asymptotic stabilization [89]). *Suppose (*5.31*) is passive and locally ZSD. Let $\phi(y)$ be any smooth function such that $\phi(0) = 0$ and $y^T\phi(y) > 0$, $\forall y \neq 0$. Assume that the storage function $V(x) > 0$ is proper. Then, the control law $u = -\phi(y)$ asymptotically stabilizes the equilibrium point $x = 0$. If in addition (*5.31*) is ZSD then $x = 0$ is globally asymptotically stable.* ■

Proof: By assumption, $V(x) > 0$ for all $x \neq 0$. Replacing $u = -\phi(y)$ in (4.42) we obtain

$$V(x(t)) - V(x(0)) \leq -\int_0^t y^T(s)\phi(y(s))ds \leq 0$$

It follows that $V(x(t)) \leq V(x(0)) < \infty$, which implies that $\|x(t)\| < \infty$ for all $t \geq 0$, and thus $\|y(t)\| < \infty$. Therefore $V(x(\cdot))$ is non-increasing and thus converges. In the limit, the left hand side of the inequality is 0 , i.e. $\int_0^t y^T(s)\phi(y(s))ds \to 0$ as $t \to \infty$. Thus $y(t) \to 0$ as $t \to +\infty$ and u also converges to 0. Since the system is locally ZSD, then $x(t) \to 0$ as $t \to +\infty$. If in addition the system is globally ZSD, then $x = 0$ is globally asymptotically stable. ■

Lemma 5.25. *Suppose the system (*5.31*) is passive and zero state observable, with feedback control law $u = -\phi(y)$, $\phi(0) = 0$. Then the storage function of the closed-loop system is positive definite, i.e. $V(x) > 0$, for all $x \neq 0$.* ■

Proof: Recall that the available storage satisfies $0 \leq V_a(x) \leq V(x)$ and

$$\begin{aligned} V_a(x) &= \sup_{x=x(0), t\geq 0, u} \left\{ -\int_0^t y^T(s)u(s)ds \right\} \\ &= \sup_{x=x(0), t\geq 0, u} \left\{ \int_0^t y^T(s)\phi(y(s))ds \right\} \end{aligned} \tag{5.32}$$

If $V_a(x) = 0$, then necessarily $y(t) = 0$. In view of zero state observability, $y = 0 \Rightarrow x = 0$. Thus $V_a(x)$ vanishes only at $x = 0$ and so does $V(x)$. ■

The asymptotic stabilization by output feedback of nonlinear systems nonlinear in the input as in (4.95) continues to hold [303, Theorem 4.4]. Further results on the asymptotic stabilization by output feedback may be found in [430, Theorem 6].

5.4.2 Time-varying Nonlinear Systems

In this section we consider systems of the form

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t) \\ \\ y(t) = h(t, x(t)) \end{cases} \tag{5.33}$$

where $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are continuous functions $I\!R^+ \times I\!R^n \to I\!R^n$, $f(t,0) = 0$ and $h(t,0) = 0$ for all $t \geq 0$. It is further supposed that $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $h(\cdot;\cdot)$ are uniformly bounded functions. Since the system is not autonomous, it is no longer possible to apply the arguments based on the Krasovskii-LaSalle invariance principle. An extension is proposed in [286] which we summarize here. Before stating the main result, some definitions are needed.

Definition 5.26. *[286] Let $g : I\!R^+ \times X \to I\!R^m$ be a continuous function. An unbounded sequence $\gamma = \{t_n\}$ in $I\!R^+$ is said to be an* admissible sequence *associated with $g(\cdot)$ if there exists a continuous function $g_\gamma : I\!R^+ \times X \to I\!R^m$ such that the associated sequence $\{g_n \mid (t,x) \mapsto g(t+t_n, x)\}$ converges uniformly to $g_\gamma(\cdot)$ on every compact subset of $I\!R^+ \times X$. The function $g_\gamma(\cdot)$ is uniquely determined and called the* limiting function *of $g(\cdot)$ associated with γ.*

Definition 5.27. *[286] Let $g : I\!R^+ \times X \to I\!R^m$ be a continuous function. It is said to be an* asymptotically almost periodic *(AAP) function if, for any unbounded sequence $\{t_n\}$ in $I\!R^+$ there exists a subsequence γ of $\{t_n\}$ so that γ is an admissible sequence associated with $g(\cdot)$.* ■

The set of all admissible sequences associated with an AAP function $g(\cdot)$ is denoted as $\Gamma(g)$. As an example, any continuous function $g : X \to I\!R^m$, $x \mapsto g(x)$, has all its limiting functions equal to itself. A function $g : I\!R^+ \times I\!R^p \to I\!R^m$ that is continuous and such that $g(\cdot, x)$ is periodic for each fixed x, has limiting functions which can be written as time-shifting functions $g_{t_0} : (t,x) \mapsto g(t+t_0, x)$ of $g(\cdot,\cdot)$ for some $t_0 > 0$.

Lemma 5.28. *[286] Suppose that $g : I\!R^+ \times X \to I\!R^m$ is uniformly continuous and bounded on $I\!R^+ \times \kappa$ for every compact $\kappa \subset X$. Then $g(\cdot,\cdot)$ is an AAP function.* ■

Let $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ be AAP functions. To the system in (5.33) one associates its reduced limiting system

$$\begin{cases} \dot{z}(t) = f_\gamma(t, z(t)) \\ \\ \zeta(t) = h_\gamma(t, z(t)) \end{cases} \tag{5.34}$$

The following assumption is made, which is a simplified zero state detectability hypothesis:

Assumption 10 *For any admissible sequence $\gamma \in \Gamma(f) \cap \Gamma(h)$ and any bounded solution $z : I\!R^+ \to X$ of the reduced limiting system in (5.34) satisfying the equation $h_\gamma(t, z(t)) = 0$ for all $t \geq 0$, it holds that either the origin is a $\omega-$limit point of $z(\cdot)$ or $z(t_0) = 0$ for some $t_0 \geq 0$.* ■

Let us now recall the KYP property for time-varying systems (this is (4.76) with the explicit dependence on time):

Assumption 11 *There exists a continuously differentiable, positive definite and proper storage function $V : I\!R^n \to I\!R^+$ such that:*

$$\begin{cases} \frac{\partial V}{\partial x}(x) f(t,x) \leq 0 \;\; \forall\, t \geq 0,\; \forall\, x \in I\!R^n \\ \\ h(t,x) = \left[\frac{\partial V}{\partial x}(x) g(t,x)\right]^T, \;\; \forall\, t \geq 0,\; \forall\, x \in I\!R^n \end{cases} \tag{5.35}$$

■

We are now ready for the following proposition:

Proposition 5.29. *[286] Consider a system of the form (5.33), with the output feedback law $u = -ky$, $k > 0$. Let Assumption 11 hold, and Assumption 10 hold with the output function $\tilde{h}(t,x) = \left[\frac{\partial V}{\partial x}(x)[f(t,x), g(t,x)]\right]^T$. Let $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ be both AAP functions. Then the origin of the closed-loop system is uniformly globally asymptotically stable. Conversely, the uniform global asymptotic stability implies Assumption 10 when $f(\cdot,\cdot)$ and $\tilde{h}(\cdot,\cdot)$ are locally Lipschitz continuous, uniformly in t.* ■

There is in fact a strong link between AAP functions and the condition of persistency of excitation of a bounded matrix-valued function $\psi : I\!R^+ \to I\!R^{p\times q}$, which states that $\int_t^{t+\epsilon} \psi(s)\psi^T(s) ds \geq \alpha I_p$ for some $\epsilon > 0$, some $\alpha > 0$ and all $t \geq 0$. The persistency of excitation is a well-known condition which guarantees the convergence of parameters in adaptive control of linear time invariant systems, and is consequently a tool which allows to prove the asymptotic convergence towards the equilibrium. When $h : (t,x) \to \psi^T(t)x$ is an AAP function, the persistency of excitation can be interpreted as a nonzero property of limiting functions.

5.4.3 Evolution Variational Inequalities

Let us come back on the evolution variational inequalities as in Section 3.9.5. We consider the linear case, that is a system similar to the system in (3.218)–(3.222) and its transformed form (3.225). We however consider now the controlled case, *i.e.*:

$$\langle \frac{dz}{dt}(t) - RAR^{-1}z(t) - RFu(t), v - z(t)\rangle \geq 0, \forall v \in \bar{K}_u, \text{ a.e. } t \geq 0 \tag{5.36}$$

with an output $y = Cx + Du = CR^{-1}z + Du$, $\bar{K}_u = \{h \in I\!R^n \mid (CR^{-1}h + Du \in K\}$. Remember that $R^2B = C^T$. The "input" matrix B is hidden in this formulation, but we recall that the variational inequality (5.36) is equivalent to the inclusion

$$\begin{cases} \dot{x}(t) - Ax(t) - Fu \in BN_K(y(t)) \\ y(t) = Cx(t) + Du \\ y(t) \in K, \ \forall\, t \geq 0 \end{cases} \tag{5.37}$$

via the state transformation $z = Rx$. We consider a static state feedback $u = Gx$. We are therefore back to the case of an output with no feedthrough term $y = (C + DG)x \in K$ and $\bar{K} = \{h \in I\!R^n \mid CR^{-1}h + DFGh \in K\}$. The closed-loop system thus becomes

$$\langle \frac{dz}{dt}(t) - R(A + FG)R^{-1}z(t), v - z(t)\rangle \geq 0, \forall v \in \bar{K}, \text{ a.e. } t \geq 0 \tag{5.38}$$

There are two steps in the stabilization design:

- To render the triple $(A + FG, B, C + DG)$ positive real
- To study the asymptotic stability

One notes that we could also consider an output feedback $u = Gy$, in which case the first step would be to test the PRness of the triple $(A+FGC, B, C+DGC)$. If $F = B$ and $D = 0$ the first step can be solved using Theorem 2.64 and involves conditions on the triple (A, B, C).

Lemma 5.30. *[82] Let $(A+FGC, B, C+DGC)$ be positive real. If ker$[R(A+FGC)R^{-1} + R^{-1}(A + FGC)^TR] \cap \bar{K} = \{0\}$, then the trivial solution of the system*

$$\begin{cases} \langle \frac{dz}{dt}(t) - R(A + FGC)R^{-1}z(t), v - z(t)\rangle \geq 0, \forall v \in \bar{K}, \text{ a.e. } t \geq 0 \\ z(t) \in \bar{K}, t \geq 0 \\ z(0) = Rx_0, \ R^2B = C^T \\ \bar{K} = \{h \in I\!R^n \mid CR^{-1}h \in K\} \\ y(t) \in K \ \ \forall\, t \geq 0 \end{cases} \tag{5.39}$$

is asymptotically stable. ■

One sees that the output feedback stabilization problem for evolution variational inequalities of the form (5.36) is consequently more complex than the usual problem of rendering a system SPR by static output feedback, as it involves two "input" matrices: F which is the controller matrix, B which characterizes the convex set $\bar{K}$ in which the state $z(\cdot)$ lives.

5.5 Equivalence to a Passive System

Byrnes, Isidori and Willems [89] have found sufficient conditions for a nonlinear system to be feedback equivalent to a passive system with a positive definite storage function. See Chapter A for a short review on differential geometry tools for nonlinear systems. Consider a nonlinear system described by

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_0 \\ \\ y(t) = h(x(t)) \end{cases} \tag{5.40}$$

Definition 5.31. *The system (5.40) is feedback equivalent to a passive system if there exists a feedback $u(x,t) = \alpha(x) + \beta(x)v(t)$ such that the closed-loop system $(f(x) + g(x)\alpha(x), g(x)\beta(x), h(x))$ is passive.* ■

This is an extension to the nonlinear case of what is reported in Sections 3.5 and 2.15.3 ([2]). This is often referred to as the problem of passification of nonlinear systems [400]. The system has relative degree $\{1, \ldots, 1\}$ at $x = 0$ if $L_g h(0) = \frac{\partial h(x)}{\partial x} g(x)|_{x=0}$ is a non singular $(m \times m)$ matrix. If in addition the vector field $g_1(x), \ldots, g_m(x)$ is involutive then the system can be written in the normal form

$$\begin{cases} \dot{z}(t) = q(z(t), y(t)) \\ \\ \dot{y}(t) = b(z(t), y(t)) + a(z(t), y(t))u(t) \end{cases} \tag{5.41}$$

where

$$\begin{cases} b(z, y) = L_f h(x) \\ \\ a(z, y) = L_g h(x) \end{cases} \tag{5.42}$$

The normal form is globally defined if and only if

H1: $L_g h(x)$ is non singular for all x
H2: The columns of $g(x)[L_g h(x)]^{-1}$ form a complete vector field
H3: The vector field formed by the columns of $g(x)[L_g h(x)]^{-1}$ commutes

The zero dynamics describes the internal dynamics of the system when $y \equiv 0$ and is characterized by

$$\dot{z}(t) = q(z(t), 0)$$

Define the manifold $Z^* = \{x : h(x) = 0\}$ and

$$\tilde{f}(x) = f(x) + g(x)u^*(x) \tag{5.43}$$

[2] The problem of rendering the quadruple (A, B, C, D) passive by pole shifting is to find $\alpha \in \mathbb{R}$ such that $(A + \alpha I_n, B, C, D)$ is PR.

with

$$u^*(x) = -[L_g h(x)]^{-1} L_f h(x) \tag{5.44}$$

Let $f^*(x)$ be the restriction to Z^* of $\tilde{f}(x)$. Then the zero dynamics is also described by

$$\dot{x}(t) = f^*(x(t)) \text{ for all } x \in Z^* \tag{5.45}$$

Definition 5.32. *Assume that the matrix $L_g h(0)$ is nonsingular. Then the system (5.40) is said to be*

1. *Minimum phase if $z = 0$ is an asymptotically stable equilibrium of (5.45),*
2. *Weakly minimum phase if $\exists$ $W^*(z) \in C^r, r \geq 2$, with $W^*(z)$ positive definite, proper and such that $L_{f^*} W^*(z) \leq 0$ locally around $z = 0$* ■

These definitions become global if they hold for all z and H1-H3 above hold.

Definition 5.33. *x^0 is a regular point of (5.40) if rank$\{L_g h(0)\}$ is constant in a neighborhood of x^0.* ■

Recall that a necessary condition for a strictly proper transfer to be PR is to have relative degree equal to 1. The next theorem extends this fact for multivariable nonlinear systems. We will assume in the sequel that rank $g(0) =$ rank $dh(0) = m$.

Theorem 5.34. *[89] Assume that the system (5.40) is passive with a C^2 positive definite storage function $V(x)$. Suppose $x = 0$ is a regular point. Then $L_g h(0)$ is nonsingular and the system has a relative degree $\{1, \ldots, 1\}$ at $x = 0$.* ■

Proof: If $L_g h(0)$ is singular, there exists $u(x) \neq 0$ for x in the neighborhood $\mathbf{N}(0)$ of $x = 0$ such that

$$L_g h(x) u(x) = 0$$

Since rank$\{dh(x)\} = m$, for all $x \in N(0)$, we have

$$\gamma(x) = g(x) u(x) \neq 0$$

for all $x \in N(0)$. Given that the system (5.40) is passive it follows that $L_g V(x) = h^T(x)$ so that

$$L_\gamma^2 V(x) = L_\gamma [L_g V(x) u(x)] = L_\gamma [u^T(x) h(x)]$$

where

$$L_\gamma[u^T h] = \frac{\partial(u^T h)}{\partial x}\gamma$$

and

$$\begin{aligned}
\tfrac{\partial(u^T h)}{\partial x}(x) &= \left[\tfrac{\partial(u^T h)}{\partial x_1}, \ldots, \tfrac{\partial(u^T h)}{\partial x_n}\right] \\
&= [\tfrac{\partial u^T}{\partial x_1}h(x) + u^T(x)\tfrac{\partial h}{\partial x_1}; \ldots; \tfrac{\partial u^T}{\partial x_n}h(x) + u^T(x)\tfrac{\partial h}{\partial x_n}] \\
&= h^T(x)[\tfrac{\partial u}{\partial x_1}, \ldots, \tfrac{\partial u}{\partial x_2}] + u^T(x)[\tfrac{\partial h}{\partial x_1}, \ldots, \tfrac{\partial h}{\partial x_n}] \\
&= h^T(x)\tfrac{\partial u}{\partial x} + u^T(x)\tfrac{\partial h}{\partial x}
\end{aligned} \tag{5.46}$$

Then

$$\begin{aligned}
L_\gamma^2 V(x) &= L_\gamma[u^T(x)h(x)] \\
&= h^T(x)L_\gamma u(x) + u^T(x)L\gamma h(x) \\
&= (L_\gamma u(x))^T h(x) + u^T(x)L_\gamma h(x) \\
&= v^T(x)h(x)
\end{aligned} \tag{5.47}$$

with

$$v^T(x) = (L_\gamma u(x))^T + u^T(x)L_\gamma$$

Let $\phi_t^\gamma(x_{t_0})$ denote the flow of the vector field $\gamma(\cdot)$, i.e. the solution of $\dot{\xi}(t) = \gamma(\xi(t))$ for $\xi_0 = x(t_0)$. Define $f(t) = V(\phi_t^\gamma(0))$. Using Taylor's Theorem for $n = 2$ we have

$$f(t) = f(0) + f^{(1)}(0)t + f^{(2)}(s)\frac{1}{2}t^2$$

where $0 \leq s \leq t$. Note that

$$\begin{cases}
f(t) \quad = V(\phi_t^\gamma(0)) \\
f^{(1)}(t) = \frac{\partial V(\phi_t^\gamma(0))}{\partial \xi}\dot{\xi} = \frac{\partial V(\phi_t^\gamma(0))}{\partial \xi}\gamma(\xi(t)) = L_\gamma V(\phi_t^\gamma(0)) \\
f^{(2)}(t) = \frac{\partial f^{(1)}(t)}{\partial \xi}\dot{\xi} = L_\gamma f^{(1)}(t) = L_\gamma^2 V(\phi_t^\gamma(0))
\end{cases} \tag{5.48}$$

Therefore

$$V(\phi_t^\gamma(0)) = V(0) + L_\gamma V(0)t + L_\gamma^2 V(\phi_s^\gamma(0))\frac{1}{2}t^2$$

Given that $V(0) = 0$ we have

$$\begin{aligned} L_\gamma V(0) &= \frac{\partial V(x)}{\partial x} g(x)u(x)|_{x=0} \\ &= L_g V(x) u(x)|_{x=0} \\ &= h^T(0)u(0) = 0 \end{aligned} \tag{5.49}$$

Thus

$$V(\phi_t^\gamma(0)) = v^T(\phi_s^\gamma(0))h(\phi_s^\gamma(0))\frac{1}{2}t^2$$

Recall that $\frac{\partial h(x)}{\partial x} g(x)u(x) = 0$, for all x and in particular we have $\frac{\partial h(\xi)}{\partial \xi} g(\xi)u(\xi) = 0$ which implies that $\frac{\partial h(\xi)}{\partial \xi}\dot{\xi} = 0 \Rightarrow \dot{h}(\xi) = 0 \Rightarrow h(\xi) = \alpha$ where $\alpha \in I\!R$ is a constant. Thus $h(\phi_t^\gamma(0)) = h(0) = 0$ and then $V(\phi_t^\gamma(0)) = 0 \Rightarrow \phi_t^\gamma(0) = 0 \Rightarrow \gamma(0)) = 0$ which is a contradiction. Therefore $L_g h(0)$ must be nonsingular. ■

Recall that a necessary condition for a strictly proper transfer to be PR is to be minimum phase. The next theorem extends this fact to general nonlinear systems. A function $V : I\!R^n \to I\!R^+$ is non degenerate in a neighborhood of $x = 0$ if its Hessian matrix $\frac{\partial^2 V}{\partial x^2}(x)$ has full rank n in this neighborhood.

Theorem 5.35. *[89] Assume that the system (5.40) is passive with a C^2 positive definite storage function $V(\cdot)$. Suppose that either $x = 0$ is a regular point or that $V(\cdot)$ is non degenerate. Then the system zero-dynamics locally exists at $x = 0$ and the system is weakly minimum phase.* ■

Proof: In view of Theorem 5.34 the system has relative degree $\{1 \dots 1\}$ at $x = 0$ and therefore its zero-dynamics locally exists at $x = 0$. Define the positive definite function $W^* = V|_{Z^*}$ with $Z^* = \{x : h(x) = 0\}$. Since the system is passive we have $0 \geq L_f V(x)$ and $L_g V(x) = h^T(x)$. Define $f^*(x) = f(x) + g(x)u^*(x)$ and $u^*(x) = -[L_g h(x)]^{-1} L_f h(x)$. Thus:

$$\begin{aligned} 0 &\geq L_f V(x) \\ &= L_{f^*} V(x) - L_g V(x) u^*(x) \\ &= L_{f^*} V(x) - h^T(x) u^*(x) \\ &= L_{f^*} V(x) \end{aligned} \tag{5.50}$$

along any trajectory of the zero dynamics ($h(x) = 0$). ■

The two theorems above show essentially that any passive system with a positive definite storage function, under mild regularity assumptions, necessarily has relative degree $\{1 \dots 1\}$ at $x = 0$ and is weakly minimum phase. These two conditions are shown to be sufficient for a system to be feedback equivalent to a passive system as stated in the following theorem.

Theorem 5.36. *[89] Suppose $x = 0$ is a regular point. Then the system (5.40) is locally feedback equivalent to a passive system with C^2 storage function $V(\cdot)$ which is positive definite, if and only if (5.40) has relative degree $\{1 \dots 1\}$ at $x = 0$ and is weakly minimum phase.*

This has been extended to the relative degree zero case in [435]. Specifically one considers systems of the classical form

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_0 \\ \\ y(t) = h(x(t)) + j(x(t))u(t) \end{cases} \tag{5.51}$$

with $x(t) \in I\!R^n$, $u(t) \in I\!R^m$, $y(t) \in I\!R^m$, $f(\cdot)$ and $g(\cdot)$ are smooth vector fields, $f(0) = 0$, $h(0) = 0$, rank$[g(0)] = m$. The notion of invertibilty will play a role in the next result, and is therefore introduced now.

Definition 5.37. *The system in (5.51) is invertible at $x = 0$ with relative order 1 if*

- *(i) The matrix $j(x)$ has constant rank $m - p$ in a neighborhood $\mathbf{N}$ of $x = 0$*
- *(ii) If $D(x)$ is a $p \times m$ matrix of smooth functions such that, for all $x \in \mathbf{N}$: rank$[D(x)] = p$ and $D(x)j(x) = 0$, then the $(m+p) \times m$ matrix*

$$H(x) = \begin{pmatrix} j(x) \\ \\ L_g[D(x)h(x)] \end{pmatrix}$$

has constant rank m for all $x \in \mathbf{N}$.

If this property holds for $\mathbf{N} = I\!R^n$ then the system is said uniformly invertible with relative order 1. ■

The following links invertibilty with passivity.

Proposition 5.38. *[435] Consider the system in (5.51), let $\mathbf{N}$ be a neighborhood of $x = 0$ and assume that*

- *(i) $j(x)$ has constant rank $m - p$ for all $x \in \mathbf{N}$.*

Let $D(x)$ be a $p \times m$ matrix the rows of which are linearly independent for all $x \in \mathbf{N}$. Let

$$H(x) = \begin{pmatrix} j(x) \\ \\ L_g[D(x)h(x)] \end{pmatrix}$$

and assume that

- *(ii) $H(x)$ has constant rank for all $x \in \mathbf{N}$.*

Suppose that the system is passive with a C^2 positive definite storage function $V(\cdot)$. Then there is a neighborhood $\hat{\mathbf{N}} \subseteq \mathbf{N}$ such that the system is invertible with relative order 1 for all $x \in \hat{\mathbf{N}}$. ■

We then have the following proposition on feedback equivalence to a passive system

Proposition 5.39. *[435] Consider a system as in (5.51) and assume that it satisfies the regularity hypotheses of Proposition 5.38. Then there exists a regular static state feedback which locally transforms the system into a passive system having a C^2 positive definite storage function, if and only if the system is invertible with relative order 1 and is weakly minimum phase.* ■

The notion of weak minimum phase for (5.51) is similar to that for systems as in (5.40), except that the input $u^*(x)$ is changed, since the output is changed. The zero-dynamics input is calculated as the unique solution of

$$H(x)u^*(x) + \begin{pmatrix} j(x) \\ \\ L_f[D(x)h(x)] \end{pmatrix} = 0$$

and is such that the vector field $f^*(x) = f(x) + g(x)u^*(x)$ is tangent to the submanifold $Z^* = \{x \in \mathbf{N} \mid D(x)h(x) = 0\}$. The proof of Proposition 5.39 relies on the cross-term cancellation procedure and a two-term Lyapunov function, so that the results of Section 7.3.3 may be applied to interpret the obtained closed-loop as the negative feedback interconnection of two dissipative blocks. Further works on feedback equivalence to a passive system can be found in [56, 99, 130, 156, 161, 236, 303, 414]. The adaptive feedback passivity problem has been analyzed in [453].

Remark 5.40. Most of the results on feedback equivalence to a passive system, relative degree, zero dynamics, are extended to nonlinear discrete-time systems in [372, 374].

5.6 Cascaded Systems

Cascaded systems are important systems that appear in many different practical cases. We will state here some results concerning this type of systems which will be used later in the book. Consider a cascaded system of the following form

$$\begin{cases} \dot{\zeta}(t) = f_0(\zeta(t)) + f_1(\zeta(t), y(t))y(t) \\ \\ \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ \\ y(t) = h(x(t)) \end{cases} \tag{5.52}$$

The first equation above is called the *driven system* while the second and third equations are called the *driving system.*

Theorem 5.41. *[89] Consider the cascaded system (5.52). Suppose that the driven system is globally asymptotically stable and the driven system is (strictly) passive with a C^r , $r \geq 2$, storage function $V(\cdot)$ which is positive definite. The system (5.52) is feedback equivalent to a (strictly) passive system with a C^r storage function which is positive definite.* ■

The cascaded system in (5.52) can also be globally stabilized using a smooth control law as is stated in the following Theorem for which we need the following definitions. Concerning the *driving system* in (5.52) we define the associate distribution [227, 381]

$$\mathcal{D} = \text{span}\{ad_f^k g_i : \; 0 \leq k \leq n-1, \; 1 \leq i \leq m\} \tag{5.53}$$

and the following set

$$\mathcal{S} = \{x \in X : \; L_f^m L_\tau \; V(x) = 0, \text{ for all } \; \tau \in \mathcal{D}, \;\; 0 \leq m < r\} \tag{5.54}$$

Theorem 5.42. *[89] Consider the cascaded system (5.52). Suppose that the driven system is globally asymptotically stable and the driven system is passive with a C^r , $r \geq 1$, storage function $V(\cdot)$ which is positive definite and proper. Suppose that $\mathcal{S} = \{0\}$. Then the system (5.52) is globally asymptotically stabilizable by the smooth feedback*

$$u^T(\zeta, x) = -L_{f_1(\zeta, h(x))} U(\zeta) \tag{5.55}$$

where $U(\cdot)$ is a Lyapunov function for the driven system part of the cascaded system (5.52). ■

Some additional comments on the choice of u in (5.55) are given in Subsection 7.3.3, where the role of cross-terms cancellation is highlighted. Further work on the stabilization of cascaded systems using dissipativity may be found in [101].

5.7 Input-to-State Stability (ISS) and Dissipativity

Close links between passive systems and Lyapunov stability have been shown to exist in the foregoing sections. This section demonstrates that more can be said. E.D. Sontag has introduced the following notion of input-to-state stability (ISS): given a system

$$\dot{x}(t) = f(x(t), u(t)), \;\; x(0) = x_0 \tag{5.56}$$

where $f(\cdot, \cdot)$ is locally Lipschitz, $f(0,0) = 0$, and $\mathcal{U}$ is a set of measurable locally essentially bounded functions from $I\!R^+$ into $I\!R^m$, one studies the input-to-state mapping $(x_0, u(\cdot)) \mapsto x(\cdot)$ and its stability (a notion that will be defined next). The material in this section is to be considered as a brief introduction to the field of ISS and is taken from [468, 469]. We shall be especially interested by the relationships with dissipativity, as the reader may expect.

The problem is the following: assume that the equilibrium $x = 0$ of the free system $\dot{x}(t) = f(x(t), 0)$ is globally asymptotically stable. The question is to determine if this property implies, or is equivalent to: $[\lim_{t\to+\infty} u(t) \to 0 \Longrightarrow \lim_{t\to+\infty} x(t) \to 0]$, or to: $[u(\cdot)$ bounded $\Longrightarrow x(\cdot)$ bounded $]$. Equivalence is known to be true for linear time invariant systems. This is not the case for nonlinear systems, as the following example proves: $\dot{x}(t) = -x(t) + (x^2(t) + 1)u(t)$ with $u(t) = (2t+2)^{-\frac{1}{2}}$. The trajectory which starts at $x_0 = \sqrt{2}$ is given by $x(t) = (2t+2)^{\frac{1}{2}}$ which is unbounded, though $\lim_{t\to+\infty} u(t) \to 0$ and $\dot{x}(t) = -x(t)$ is globally asymptotically stable.

Definition 5.43. *The system (5.56) is ISS if:*

- *For each x_0 there is a unique solution in $C^0(I\!R^+, I\!R^n)$*
- *The map $I\!R^n \times \mathcal{U} \to C^0(I\!R^+, I\!R^n)$, $(x_0, u) \mapsto x(\cdot)$ is continuous at $(0, 0)$*
- *There exists a nonlinear asymptotic gain $\gamma(\cdot)$ of class $\mathcal{K}$ so that*

$$\lim \sup_{t\to+\infty} ||x(t, x_0, u)|| \leq \gamma(||\, u\, ||_\infty)$$

uniformly on x_0 in any compact set and all $u \in \mathcal{U}$.

Continuity in the second item means that for any sequence $\{x_{0,n}\}$ such that $\lim_{n\to+\infty} x_{0,n} = x_0$ and any sequence $\{u_n\}$ such that $\lim_{n\to+\infty} u_n = u$, then the solution $x(t; x_{0,n}, u_n) \to x(t; x_0, u)$ as $n \to +\infty$. Then the following holds:

Theorem 5.44. *[468, 469] The system (5.56) is ISS if and only if there exists a class $\mathcal{KL}$-function $\beta(\cdot, \cdot)$, and two functions $\gamma_0(\cdot)$, $\gamma_1(\cdot)$ of class $\mathcal{K}$ such that*

$$||x(t, x_0, u)|| \leq \beta(||x_0||, t) + \gamma_0\left(\int_0^t e^{s-t}\gamma_1(||u(s)||)ds\right) \tag{5.57}$$

for all $t \geq 0$. Equivalently, the system is ISS if

$$||x(t, x_0, u)|| \leq \beta(||x_0||, t) + \gamma(||u||_\infty) \tag{5.58}$$

for some class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and some class $\mathcal{K}$ function $\gamma(\cdot)$. ■

Let us now define an ISS-Lyapunov function.

Definition 5.45. *A differentiable storage function $V(\cdot)$ is an ISS-Lyapunov function if there exists two functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ of class $\mathcal{K}_\infty$ such that*

$$\nabla V^T(x)f(x,u) \leq \alpha_1(||u||) - \alpha_2(||x||) \tag{5.59}$$

for all $x \in I\!R^n$, $u \in I\!R^m$. Equivalently, a storage function with the property that there exist two class-$\mathcal{K}$ functions $\alpha(\cdot)$ and $\chi(\cdot)$ such that

$$||x|| \geq \chi(||u||) \Longrightarrow \nabla V^T(x)f(x,u) \leq -\alpha(||x||) \tag{5.60}$$

holds for all $x \in I\!R^n$ and all $u \in I\!R^m$, is an ISS-Lyapunov function. ■

One notices that (5.59) means that along trajectories of the system $\frac{dV \circ x}{dt}(t) \leq \alpha_1(||u(t)||) - \alpha_2(||x(t)||)$. One also immediately realizes that (5.59) is a dissipation inequality (in its infinitesimal form, so that indeed some differentiability of the storage function is required). Integrating on any interval $[t_0, t_1]$ we get that along the system's trajectories

$$V(x(t_1)) - V(x(t_0)) \leq \int_0^t w(u(s), x(s))ds, \tag{5.61}$$

where the supply rate $w(u,x) = \alpha_1(||u||) - \alpha_2(||x||)$. The dissipation inequality (5.61) might be written even if $V(\cdot)$ is not differentiable, using the notion of viscosity solutions. However, as far as ISS is concerned, the following holds:

Theorem 5.46. *[469] The system in (5.56) is ISS if and only if it admits a smooth ISS-Lyapunov function.* ■

This strong result shows that ISS is more stringent that dissipativity. We recall that smooth means infinitely differentiable.

Example 5.47. [468] Consider $\dot{x}(t) = -x^3(t) + x^2(t)u_1(t) - x(t)u_2(t) + u_1(t)u_2(t)$. When u_1 and u_2 are zero, the origin $x = 0$ is globally asymptotically stable. This can be easily checked with the Lyapunov function $V(x) = \frac{x^2}{2}$. One also has $\nabla V^T(x)(-x^3 + x^2u_1 - xu_2 + u_1u_2) \leq -\frac{2}{9}x^4$, provided that $3|u_1| \leq |x|$ and $3|u_2| \leq x^2$. This is the case if $||u|| \leq \nu(||x||)$, with $\nu(r) = \min(\frac{r}{3}, \frac{r^2}{3})$. Thus $V(x) = \frac{x^2}{2}$ is an ISS-Lyapunov function with $\alpha(r) = \frac{2}{9}r^4$ and $\chi = \nu^{-1}$ in (5.60).

Let us now introduce a slightly different property known as the *integral ISS* (in short iISS):

Definition 5.48. *The system in (5.56) is iISS provided that there exist functions $\alpha(\cdot)$ and $\gamma(\cdot)$ of class $\mathcal{K}_\infty$, and a function $\beta(\cdot,\cdot)$ of class $\mathcal{KL}$ such that*

$$\alpha(||x(t)||) \leq \beta(||x_0||, t) + \int_0^t \gamma(||u(s)||)ds \tag{5.62}$$

holds along solutions of the system. ■

An iISS-Lyapunov function is defined as follows:

Definition 5.49. *A smooth storage function $V(\cdot)$ is an iISS-Lyapunov function for the system in (5.56) if there is a function $\gamma(\cdot)$ of class $\mathcal{K}_\infty$ and a positive definite function $\alpha : I\!R^+ \to I\!R^+$, such that*

$$\dot{V}(x(t), u(t)) \leq -\alpha(||x(t)||) + \gamma(||u||) \tag{5.63}$$

for all $x(t) \in I\!R^n$ and all $u(t) \in I\!R^m$. ■

Notice that $\dot{V}(x(t), u(t)) = \frac{\partial V}{\partial x}(f(x(t), u(t))$. The fact that (5.63) is a dissipation inequality (in its infinitesimal form) with supply rate $w(x, u) = -\alpha(||x(t)||) + \gamma(||u||)$ is obvious. Since every class $\mathcal{K}_\infty$ function is also positive definite, an ISS-Lyapunov function is also an iISS-Lyapunov function. But the converse is not true. Similarly to Theorem 5.46 one has the following:

Theorem 5.50. *[469] The system in (5.56) is iISS if and only if it admits a smooth iISS-Lyapunov function.* ■

Example 5.51. Let us present an example of a scalar system that is not ISS but is iISS. Consider

$$\dot{x}(t) = -\tan^{-1}(x(t)) + u(t) \tag{5.64}$$

This system is not ISS because the input $u(t) \equiv \frac{\pi}{2}$ gives unbounded trajectories. But it is iISS. Indeed choose $V(x) = x \tan^{-1}(x)$. Then

$$\dot{V}(x(t), u(t)) \leq -\left(\tan^{-1}(|x(t)|\right)^2 + 2|u(t)| \tag{5.65}$$

and consequently this storage function $V(\cdot)$ is an iISS-Lyapunov function.

More can be said about iISS stability, as the following shows:

Theorem 5.52. *The system in (5.56) is iISS if and only if the uncontrolled system $\dot{x}(t) = f(x(t), 0)$ has a globally asymptotically stable fixed point $x = 0$ and there is a smooth storage function $V(\cdot)$ such that for some function $\sigma(\cdot)$ of class $\mathcal{K}_\infty$*

$$\dot{V}(x(t), u(t)) \leq \sigma(||u(t)||) \tag{5.66}$$

for all $x(t) \in I\!R^n$ and all $u(t) \in I\!R^m$. ■

Let us now state a result on ISS in which zero-state detectability (Definition 5.12) intervenes:

Theorem 5.53. *[469] A system is iISS if and only if there exists a continuous output function $y = h(x)$, $h(0) = 0$, which provides zero-state detectability and dissipativity in the following sense: there exists a storage function $V(\cdot)$ and a function $\sigma(\cdot)$ of class $\mathcal{K}_\infty$ and a positive definite function $\alpha(\cdot)$ so that*

$$\dot{V}(x(t), u(t)) \leq \sigma(||u(t)||) - \alpha(h(x(t))) \tag{5.67}$$

for all $x(t) \in I\!R^n$ and all $u \in I\!R^m$. ■

The next results may be seen as a mixture of results between the stability of feedback interconnections as in Figure 5.2, the ISS property, and quasi-dissipative systems. Two definitions are needed before stating the results.

Definition 5.54. *A dynamical system* $\dot{x}(t) = f(x(t), u(t))$, $y(t) = h(x(t))$, *with* $f(\cdot, \cdot)$ *and* $h(\cdot)$ *locally Lipschitz functions, is* strongly finite time detectable *if there exists a time* $t > 0$ *and a function* $\kappa(\cdot)$ *of class* $\mathcal{K}_\infty$ *such that for any* $x_0 \in I\!R^n$ *and for any* $u \in \mathcal{U}$ *the following holds:*

$$\int_0^t (u^T(s)u(s) + y^T(s)y(s))ds \geq \kappa(||x_0||) \tag{5.68}$$

■

This property is to be compared to the zero state detectability in Definition 5.17. Roughly, a system that is strongly finite time detectable and starts with a large initial state, must have either a large input or a large output, or both. A system that is ZSD in the sense of Definition 5.17 must have a large output when the initial state is large.

Definition 5.12 $\Longleftarrow$ Definition 5.17 $\Longrightarrow$ Definition 5.54 $\Longrightarrow$ Definition 5.12

Definition 5.55. *[403] The system* $\dot{x}(t) = f(x(t), u(t))$, $y(t) = h(x(t))$ *is input-to-state ultimately bounded (ISUB), or has input-to-state ultimately bounded trajectories, if for given* $a \geq 0$ *and* $r \geq 0$, *one has*

$$\begin{array}{c} ||x(0)|| \leq r \;\; \textit{and} \;\; \sup_{t\geq 0} ||u(t)|| \leq a \\ \Downarrow \\ \textit{there exist}\; C_{a,r} \geq r \;\; \textit{such that} \;\; \sup_{t\geq 0} ||x(t)|| \leq C_{a,r} \\ \textit{and} \\ \textit{there exist}\; D \geq 0 \; \textit{(independent of } r\textit{) and } t_r \geq 0 \; \textit{(independent of } a\textit{)} \\ \textit{such that} \;\; \sup_{t \geq t_r} ||x(t)|| \leq D \end{array} \tag{5.69}$$

■

This definition is closely related to the ISS with respect to a compact invariant set. However, ISUB implies only boundedness, not stability, and is therefore a weaker property. The next proposition is an intermediate result which we give without proof.

Proposition 5.56. *[403] Suppose that the system* $\dot{x}(t) = f(x(t), u(t))$, $y(t) = h(x(t))$ *has uniform finite power gain, with a locallly bounded radially unbounded storage function, and is strongly finite time detectable. Then it is ISUB.* ■

The definition of a finite power gain is in Definition 5.7. Then we have the following.

Theorem 5.57. *[403] Suppose that each of the subsystem H_1 and H_2 in Figure 5.2 has the dynamics $\dot{x}_i(t) = f_i(x_i(t), u_i(t))$, $y_i(t) = h_i(x_i(t))$, $i = 1, 2$, and is*

- *Quasi-dissipative with general supply rate $w_i(u_i, y_i)$, with a locally bounded radially unbounded storage function*
- *Strongly finite time detectable*

Suppose that there exists $\rho > 0$ such that the matrix Q_ρ in (5.9) is negative definite. Then the feedback system is ISUB. ■

Proof: From Proposition 5.8 one sees that the feedback system has uniform finite power gain. Suppose that $V_1(\cdot)$ and $V_2(\cdot)$ are locallly bounded radially unbounded storage functions for H_1 and H_2 respectively. Then $V_1(\cdot)+\rho V_2(\cdot)$ is a locally bounded radially unbounded storage function of the feedback system. Let us now show that the feedback system is strongly finite-time detectable. We have

$$\begin{aligned} &\int_0^{t_1} [r_1^T(s)r_1(s) + y_2^T(s)y_2(s) + y_1^T(s)y_1(s)]ds \\ &\geq \int_0^{t_1} [u_1^T(s)u_1(s) + y_1^T(s)y_1(s)]ds \\ &\geq \kappa_1(||x_1(0)||) \end{aligned} \tag{5.70}$$

and

$$\begin{aligned} &\int_0^{t_2} [r_2^T(s)r_2(s) + y_2^T(s)y_2(s) + y_1^T(s)y_1(s)]ds \\ &\geq \int_0^{t_1} [u_2^T(s)u_2(s) + y_2^T(s)y_2(s)]ds \\ &\geq \kappa_2(||x_2(0)||) \end{aligned} \tag{5.71}$$

for some $t_1 > 0$, $t_2 > 0$, $\kappa_1(\cdot)$ and $\kappa_2(\cdot) \in \mathcal{K}_\infty$. Combining (5.70) and (5.71) we obtain

$$\begin{aligned} &\int_0^{t^\star} [r_1^T(s)r_1(s) + r_2^T(s)r_2(s) + y_2^T(s)y_2(s) + y_1^T(s)y_1(s)]ds \\ &\geq \tfrac{1}{2}[\kappa_1(||x_1(0)||) + \kappa_2(||x_2(0)||)] \\ &\geq \kappa_\star(\max\{||x_1(0)||, ||x_2(0)||\}) \end{aligned} \tag{5.72}$$

where $t^\star = \max(t_1, t_2)$, and $\kappa_\star(\cdot) = \frac{1}{2}\min\{\kappa_1(\cdot), \kappa_1(\cdot)\} \in \mathcal{K}_\infty$. Using Proposition 5.56, the result follows. ■

The literature on ISS stability is abundant and our objective in this section was just to mention the connections with dissipativity. The interested reader should have a look at [469] and the bibliography therein to realize the richness of this field.

5.8 Passivity of Linear Delay Systems

The above developments focus on particular classes of smooth finite dimensional dynamical systems. Let us investigate another type of systems that does not fit within these classes, namely time-delayed systems. Stability and control of linear systems with delayed state are problems of recurring interest since the existence of a delay in a system representation may induce instability, oscillations or bad performances for the closed-loop scheme. In this section we shall consider the *passivity* problem of a linear system described by differential equations with delayed state. The interconnection schemes with passive systems will be also treated. The proposed approach is based on an appropriate Lyapunov-Krasovskii functional construction. The material presented in this section follows the analysis given in [377]; see also [291, 380]. The corresponding results may include or not delay information and are expressed in terms of solutions of some algebraic Riccati equations. The results presented here can be extended to the multiple delays case by an appropriate choice of the Lyapunov functional.

5.8.1 Systems with State Delay

Consider the following system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t-\tau) + Bu(t) \\ \\ y(t) = Cx(t) \end{cases} \tag{5.73}$$

where $x(t) \in I\!R^n, y(t) \in I\!R^p, u(t) \in I\!R^p$ are the state, the output and the input of the system and τ denotes the delay. The matrices $A \in I\!R^{n\times n}$, $A_1 \in I\!R^{n\times n}$ and $B \in I\!R^{n\times m}$ are constant. Time-delay systems may be seen as infinite-dimensional systems. In particular the initial data for (5.73) is a function $\phi : [-\tau, 0] \to I\!R^n$ that is continuous in the uniform convergence topology (*i.e.* $||\phi|| = \sup_{-\tau \le t \le t} ||\phi(t)||$). The initial condition is then denoted as $x(t_0 + \theta) = \phi(\theta)$ for all $\theta \in [-\tau, 0]$. There exists a unique continuous solution [201, Theorem 2.1] which depends continuously on the initial data $(x(0), \phi)$ in the following sense: the solution of (5.73) is denoted as

$$x_t(\theta) = \begin{cases} x(t+\theta) \text{ if } \; t+\theta \ge 0 \\ \\ \phi(t+\theta) \text{ if } \; -\tau \le t+\theta \le 0 \end{cases} \tag{5.74}$$

with $\theta \in [-\tau, 0]$. Let $\{\phi_n(\cdot)\}_{n\geq 0}$ be a sequence of functions that converge uniformly towards $\phi(\cdot)$. Then $x_n(0) \to x(0)$, and $x_{t,n}(\cdot)$ converges uniformly towards $x_t(\cdot)$. The transfer function of the system in (5.73) is given by $G(\lambda) = C(\lambda - A - A_1 e^{-\tau})^{-1}B$, with $\lambda \in \rho(A + A_1 e^{-\tau}) \in \mathbb{C}$ where $\rho(M) = \{\lambda \in \mathbb{C} \mid (\lambda I_n - M)$ is full rank$\}$ for $M \in \mathbb{R}^{n\times n}$ [178].

The main result of this section can be stated as follows:

Lemma 5.58. *If there exists positive definite matrices $P > 0$ and $S > 0$ and a scalar $\gamma \geq 0$ such that*

$$\begin{cases} \Gamma \stackrel{\Delta}{=} A^T P + PA + PA_1 S^{-1} A_1^T P + S < \gamma C^T C \\ \\ C = B^T P \end{cases} \tag{5.75}$$

then the system (5.73) satisfies the following inequality:

$$\int_0^t u^T(s)y(s)ds \geq \frac{1}{2}\left[V(x(t),t) - V(x(0),0)\right] - \frac{1}{2}\gamma \int_0^t y^T(s)y(s)ds \tag{5.76}$$

where

$$V(x(t),t) = x(t)^T P x(t) + \int_{t-\tau}^t x(s)^T S x(s) ds \tag{5.77}$$

■

Remark 5.59. Note that the system (5.73) is passive only if $\gamma = 0$. Roughly speaking for $\gamma > 0$ we may say system (5.73) is less than output strictly passive. This gives us an extra degree of freedom for choosing P and S in (5.75) since inequality in (5.75) becomes more restrictive for $\gamma = 0$. We can expect to be able to stabilize the system (5.73) using an appropriate passive controller as will be seen in the next section. Note that for $\gamma < 0$ the system is output strictly passive but this imposes stronger restrictions on the system (see (5.75)).

Proof: From (5.73) and the above conditions we have

$$
\begin{aligned}
2\int_0^t u^T(s)y(s)ds &= 2\int_0^t u^T(s)Cx(s)ds \\
&= 2\int_0^t u^T(s)B^T Px(s)ds \\
&= \int_0^t u^T(s)B^T Px(s)ds + \int_0^t x(s)^T PBu(s)ds \\
&= \int_0^t \Big\{ \left(\tfrac{dx}{ds} - Ax(s) - A_1x(s-\tau)\right)^T Px(s) + \\
&\quad + x^T(s)P\left(\tfrac{dx}{ds} - Ax(s) - A_1x(s-\tau)\right)\Big\}\, ds \\
&= \int_0^t \Big\{ \tfrac{d(x^T(s)Px(s))}{ds} - x(s)^T(A^TP + PA)x(s) - \\
&\quad - x(s-\tau)^T A_1^T Px(s) - x(s)^T PA_1x(s-\tau)\Big\}\, ds \\
&= \int_0^t \Big\{ \tfrac{dV(s)}{ds} - x(s)^T \mathit{\Gamma} x(s) + I(x(s), x(s-\tau))\Big\}\, ds
\end{aligned}
\tag{5.78}
$$

where $\mathit{\Gamma}$ is given by (5.75) and

$$
\begin{aligned}
I(x(t), x(t-\tau)) &= \left[S^{-1}A_1^T Px(t) - x(t-\tau)\right]^T S \\
&\quad \times \left[S^{-1}A_1^T Px(t) - x(t-\tau)\right]
\end{aligned}
$$

Note that $V(x,t)$ is a positive definite function and $I(x(t), x(t-\tau)) \geq 0$ for all the trajectories of the system. Thus from (5.76) and (5.78) it follows that:

$$
\begin{aligned}
\int_0^t u^T(s)y(s)ds &\geq \tfrac{1}{2}\left[V(x(t),t) - V(x(0),0)\right] - \tfrac{1}{2}\int_0^t x^T(s)\mathit{\Gamma} x(s)ds \\
&\geq \tfrac{1}{2}\left[V(x(t),t) - V(x(0),0)\right] - \tfrac{1}{2}\gamma\int_0^t x^T(s)C^TCx(s)ds \\
&\geq -\tfrac{1}{2}V(x(0),0) - \tfrac{1}{2}\gamma\int_0^t y^T(s)y(s)ds \quad \forall\, t > 0
\end{aligned}
\tag{5.79}
$$

Therefore if $\gamma = 0$ then the system is passive. ■

5.8.2 Interconnection of Passive Systems

Let us consider the block interconnection depicted in Figure 5.1 where H_1 represents the system (5.73) and H_2 is the controller which is input strictly passive as defined above, i.e. for some $\varepsilon > 0$

$$
\int_0^t u_2^T(s)y_2(s)ds \geq -\beta_2^2 + \varepsilon\int_0^t u_2^T(s)u_2(s)ds \tag{5.80}
$$

for some $\beta \in I\!R$ and for all $t \geq 0$. H_2 can be a finite dimensional linear system for example. For the sake of simplicity we will consider H_2 to be an asymptotically stable linear system. We will show next that the controller satisfying the above property will stabilize the system (5.73). From Lemma 5.58, the interconnection scheme and (5.80) we have

$$\begin{cases} u_1 = u \\ y_1 = y \\ u_2 = y_1 \\ y_2 = -u_1. \end{cases} \tag{5.81}$$

Therefore from (5.76) and (5.80) we have

$$\begin{aligned} 0 &= \int_0^t u_1^T(s)y_1(s)ds + \int_0^t u_2^T(s)y_2(s)ds \\ &\geq -\tfrac{1}{2}V(x(0),0) - \tfrac{1}{2}\gamma \int_0^t y_1^T(s)y_1(s)ds - \beta_2^2 + \varepsilon \int_0^t u_2^T(s)u_2(s)ds \\ &\geq -\beta^2 + (\varepsilon - \tfrac{1}{2}\gamma)\int_0^t y_1^T(s)y_1(s)ds \end{aligned}$$

where $\beta^2 = \frac{1}{2}V(x(0),0) + \beta_2^2$. If $\varepsilon - \frac{1}{2}\gamma > 0$ then y_1 is $\mathcal{L}_2$. Since H_2 is an asymptotically stable linear system with an $\mathcal{L}_2$ input, it follows that the corresponding output y_2 is also $\mathcal{L}_2$. Given that the closed-loop system is composed of two linear systems, the signals in the closed-loop cannot have peaks. Therefore all the signals converge to zero which means the stability of the closed loop system.

5.8.3 Extension to a System with Distributed State Delay

Let us consider the following class of distributional convolution systems:

$$\begin{cases} \dot{x}(t) = \mathcal{A} * x(t) + Bu(t) \\ \\ y(t) = Cx(t) \end{cases} \tag{5.82}$$

where $\mathcal{A}$ denotes a distribution of order 0 on some compact support $[-\tau, 0]$. Let us choose

$$\mathcal{A} = A\delta(\theta) + A_1\delta(\theta - \tau_1) + A_2(\theta) \tag{5.83}$$

where $\delta(\theta)$ represents the Dirac delta functional and $A_2(\theta)$ is a piece-wise continuous function. Due to the term $A_2(\theta)$ the system has a distributed delay. For the sake of simplicity we shall consider $A_2(\theta)$ constant. The system (5.82) becomes

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-\tau_1) + \int_{-\tau}^0 A_2x(t+\theta)d\theta + Bu(t) \\ \\ y(t) = Cx(t) \end{cases} \tag{5.84}$$

Some details on the well-posedness of such state delay systems are provided in Section A.6 in the Appendix.

Lemma 5.60. *If there exists positive definite matrices* $P > 0$, $S_1 > 0$ *and* $S_2 > 0$ *and a scalar* $\gamma \geq 0$ *such that*

$$\begin{cases} \Gamma(\tau) \overset{\Delta}{=} A^T P + PA + PA_1 S_1^{-1} A_1^T P + S_1 + \tau(PA_2 S_2^{-1} A_2^T P + S_2) \\ \qquad < \gamma C^T C \\ C \quad = B^T P \end{cases} \tag{5.85}$$

then the system (5.84) verifies the following inequality:

$$\int_0^t u(s)^T y(s) ds \geq \frac{1}{2} \left[V(t) - V(0) \right] - \frac{1}{2} \gamma \int_0^t y(s)^T y(s) ds \tag{5.86}$$

where

$$\begin{aligned} V(x(t), t) = x(t)^T P x(t) + \int_{t-\tau_1}^t x(s)^T S_1 x(s) ds + \\ + \int_{-\tau}^0 \left(\int_{t+\theta}^t x(s)^T S_2 x(s) ds \right) d\theta. \end{aligned} \tag{5.87}$$

■

Proof: We shall use the same steps as in the proof of Lemma 5.58. Thus from (5.84) and the above conditions we have

$$\begin{aligned} 2\int_0^t u^T(s) y(s) ds = 2\int_0^t u^T(s) C x(s) ds = 2\int_0^t u^T(s) B^T P x(s) ds \\ = \int_0^t u^T(s) B^T P x(s) ds + \int_0^t x^T(s) P B u(s) ds \\ = \int_0^t \left\{ \frac{dx}{ds} - A x(s) - A_1 x(s - \tau_1) - \right. \\ \left. - \int_{-\tau}^0 A_2 x(s+\theta) d\theta \right\}^T P x(s) ds + \\ + \int_0^t x(s)^T P \left\{ \frac{dx}{ds} - A x(s) - A_1 x(s - \tau_1) - \right. \\ \left. - \int_{-\tau}^0 A_2 x(s+\theta) d\theta \right\} ds \end{aligned} \tag{5.88}$$

We also have

$$
\begin{aligned}
2\int_0^t u^T(s)y(s)ds &= \int_0^t \left\{ \frac{d(x(s)^T Px(s))}{ds} - x(s)^T(A^TP + PA)x(s) - \right.\\
&\quad \left. - x(s-\tau_1)^T A_1^T Px(s) - x^{T(s)} PA_1 x(s-\tau_1) \right\} ds\\
&\quad - \int_0^t \left\{ x^T(s)P\int_{-\tau}^0 A_2 x(s+\theta)d\theta + \right.\\
&\quad \left. + \left[\int_{-\tau}^0 x^T(s+\theta)A_2^T d\theta\right] Px(s) \right\} ds\\
&= \int_0^t \left\{ \frac{dV(s)}{ds} - x(s)^T \Gamma(\tau)x(s) + I_1(x(s), x(s-\tau_1)) + \right.\\
&\quad \left. + I_2(x(s), x(s+\theta)) \right\} ds
\end{aligned}
\tag{5.89}
$$

where $\Gamma(\tau)$ is given by (5.85) and

$$
\begin{aligned}
I_1(x(t), x(t-\tau_1)) &= \left[S_1^{-1}A_1^T Px(t) - x(t-\tau_1)\right]^T S_1 \times\\
&\quad \times \left[S_1^{-1}A_1^T Px(t) - x(t-\tau_1)\right]
\end{aligned}
\tag{5.90}
$$

$$
\begin{aligned}
I_2(x(t), x(t+\theta)) &= \int_{-\tau}^0 \left[S_2^{-1}A_2^T Px(t) - x(t+\theta)\right]^T S_2 \times\\
&\quad \times \left[S_2^{-1}A_2^T Px(t) - x(t+\theta)\right] d\theta
\end{aligned}
\tag{5.91}
$$

Note that $V(t)$ is a positive definite function and $I_1(x(t), x(t-\tau_1)) \geq 0$ and $I_2(x(t), x(t+\theta)) \geq 0$ for all the trajectories of the system. Thus from (5.85) and (5.87) it follows tha

$$
\begin{aligned}
\int_0^t u^T(s)y(s)ds &\geq \frac{1}{2}\left[V(x(t),t) - V(x(0),0)\right] - \frac{1}{2}\int_0^t x^T(s)\Gamma(\tau)x(s)ds\\
&\geq \frac{1}{2}\left[V(x(t),t) - V(x(0),0)\right] - \frac{1}{2}\gamma\int_0^t x^T(s)C^TCx(s)ds\\
&\geq -\frac{1}{2}V(x(0),0) - \frac{1}{2}\gamma\int_0^t y^T(s)y(s)ds \quad \forall t > 0
\end{aligned}
\tag{5.92}
$$

Therefore if $\gamma = 0$ then the system is passive. ■

Remark 5.61. The presence of a distributed delay term in the system (5.84) imposes extra constraints in the solution of inequality (5.85). Note that for $\tau = 0$ we recover the previous case having only a point state delay. Extensions of the result presented in this section can be found in [413]. Other work may be found in [15,100,158,192,328,331]. The passification of time-delay systems with an observer-based dynamic output feedback is considered in [173]. Results for systems with delay both in the state and the input may be found in [452]. The stability and L_2-gain of a class of switching systems with delay with time-continuous solutions have been analysed in [481].

Remark 5.62. Note also that given that the system (5.84) satisfies the inequality (5.92), it can be stabilized by an input strictly passive system as described in the previous section. Furthermore due to the form of the Riccati equation the upper bound for the (sufficient) distributed delay τ (seen as a parameter) may be improved by feedback interconnection for the same Lyapunov-based construction. Such result does not contradict the theory since the derived condition is only sufficient, and not necessary and sufficient.

5.8.4 Absolute Stability

Let us end this section on time-delay systems by noting that the absolute stability problem for systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau) + Dw(t) \\ y(t) = Mx(t) + Nx(t-\tau) \\ w(t) = -\phi(t, y(t)) \end{cases} \tag{5.93}$$

has been studied in [202], with $x(\theta) = \phi(\theta)$ for all $\theta \in [-\tau, 0]$, $\tau > 0$ is the constant delay and $\phi : I\!R^+ \times I\!R^m \to I\!R^m$ is a static, piecewise continuous in t and globally Lipschitz continuous in y nonlinearity. This nonlinearity satisfies the sector condition $[\phi(t, y(t)) - K_1 y(t)]^T[\phi(t, y(t)) - K_2 y(t)] \leq 0$ where K_1 and K_2 are constant matrices of appropriate dimensions and $K = K_1 - K_2$ is symmetric positive definite. One says that the nonlinearity belongs to the sector $[K_1, K_2]$. The following result holds:

Proposition 5.63. *[202] For a given scalar $\tau > 0$, the system (5.93) is globally uniformly asymptotically stable for any nonlinear connection in the sector $[0, K]$ if there exists a scalar $\epsilon \geq 0$, real matrices $P > 0$, $Q > 0$, $R > 0$ such that*

$$\begin{pmatrix} A^TP + PA + Q - R & PB + R & PD - \epsilon M^T K^T & \tau A^T R \\ (PB + R)^T & -Q - R & -\epsilon N^T K^T & \tau B^T R \\ (PD - \epsilon M^T K^T)^T & (-\epsilon N^T K^T)^T & -2\epsilon I_m & \tau D^T R \\ (\tau A^T R)^T & (\tau B^T R)^T & (\tau D^T R)^T & -R \end{pmatrix} < 0 \tag{5.94}$$

■

Other works on absolute stability of time-delay systems can be found in [53, 159, 204, 295, 298, 299, 411, 464, 537].

5.9 Nonlinear H_∞ Control

In this section we first briefly recall basic results on H_∞ control of linear time invariant systems, then a brief review of the nonlinear case is done. We finish with an extension of the finite power gain notion. It has already been seen in the case of linear time invariant systems that there exists a close relationship between bounded realness and positive realness; see *e.g.* Theorem 2.23. Here we investigate similar properties starting from the so-called Bounded Real Lemma.

5.9.1 Introduction

Let us recall that the input/output mapping $u \mapsto y = H(u)$ of a linear time invariant system (A, B, C) with stable transfer function $H(s)$ has the H_∞ norm equal to

$$||H||_\infty = \sup_{u(s)\in H_2} \frac{||y(s)||_2}{||u(s)||_2} = \sup_{\omega\in I\!R} \sigma_{\max}\left(H(j\omega)\right) = \sup_{u(t)\neq 0} \frac{||y(t)||_2}{||u(t)||_2} \tag{5.95}$$

where H_2 is the Hardy space of functions $\mathbb{C} \to \mathbb{C}^n$ analytic in $\mathrm{Re}(s) > 0$, $||f(s)||_2 = \sqrt{\frac{1}{2}\int_{-\infty}^{+\infty} ||f(j\omega)||^2 d\omega} = ||f||_2 = \sqrt{\int_0^{+\infty} ||f(t)||^2 dt} < +\infty$, by Parseval's equality, provided $f \in \mathcal{L}_2(I\!R^+)$. Thus the H_∞ norm exactly corresponds to the $\mathcal{L}_2$-gain of the said operator, and its nonlinear extension corresponds to having

$$\int_0^t y^T(\tau)y(\tau)d\tau \leq \gamma^2 \int_0^t u^T(\tau)u(\tau)d\tau \tag{5.96}$$

for all $t \geq 0$. Let us recall the following, known as the Bounded Real Lemma:

Lemma 5.64 (Bounded Real Lemma). *Consider the system $\dot{x}(t) = Ax(t)+Bu(t)$, $y(t) = Cx(t)$. Let (A, B) be controllable and (A, C) be observable. The following statements are equivalent:*

- $||H||_\infty \leq 1$
- *The Riccati equation $A^T P + PA + PBB^T P + C^T C = 0$, has a solution $P > 0$* ■

The Strict Bounded Real Lemma is as follows:

Lemma 5.65 (Strict Bounded Real Lemma). *Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$. The following statements are equivalent:*

- *A is asymptotically stable and $||H||_\infty < 1$*

- *There exists a matrix* $\bar{P} > 0$ *such that* $A^T\bar{P} + \bar{P}A + \bar{P}BB^T\bar{P} + C^TC < 0$
- *The Riccati equation* $A^TP + PA + PBB^TP + C^TC = 0$ *has a solution* $P \geq 0$ *with* $A + BB^TP$ *asymptotically stable* ■

The Strict Bounded Real Lemma therefore makes no controllability nor observability assumptions. In order to make the link with the bounded realness of rational functions as introduced in Definition 2.24, let us recall that a transfer function $H(s) \in \mathbb{C}^{m\times m}$ is bounded real if and only if all the elements of $H(s)$ are analytic in $\mathrm{Re}(s) \geq 0$ and $||H||_\infty \leq \gamma$, or equivalently $\gamma^2 I_m - H^\star(j\omega)H(j\omega) \geq 0$ for all $\mathrm{Re}(s) \geq 0$, $\gamma > 0$. We have only replaced the upperbound 1 in Definition 2.24 by γ. The transfer function $H(s)$ is said to be *strictly bounded real* if there exists $\epsilon > 0$ such that $H(s-\epsilon)$ is bounded real. It is *strongly bounded real* if it is bounded real and $\gamma^2 I_m - DD^T > 0$, where $D = G(\infty)$.

The extension of the above lemmas to the relative degree 0 case where a direct feedthrough matrix $D \neq 0$ exists is as follows:

Lemma 5.66. *The transfer matrix* $H(s) = C(sI_n - A)^{-1}B + D$ *of the system* (A, B, C, D) *is stable and has an* H_∞*-norm* $||H||_\infty < \gamma$ *if and only if there exists a matrix* $P = P^T > 0$ *such that*

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I_m & D^T \\ C & D & -\gamma I_m \end{bmatrix} < 0 \tag{5.97}$$

■

From Theorem A.61 and the LMI (5.97) one is able to recover a Riccati inequality. We first notice that the inverse of the negative definite matrix $\tilde{D} = \begin{pmatrix} -\gamma I_m & D^T \\ D & -\gamma I_m \end{pmatrix}$ is equal to

$$\tilde{D}^{-1} = \begin{pmatrix} (-\gamma I_m + \frac{1}{\gamma}D^TD)^{-1} & D^T(DD^T - \gamma^2 I_m)^{-1} \\ D(D^TD - \gamma^2 I_m)^{-1} & (\frac{1}{\gamma}DD^T - \gamma I_m)^{-1} \end{pmatrix}.$$

Notice that in particular one has $-\gamma^2 I_m + D^TD < 0$ ($\Leftrightarrow \gamma^2 I_m - D^TD > 0$) and $-\gamma^2 I_m + DD^T < 0$, still using Theorem A.61 applied to $\tilde{D} < 0$. This indeed secures that the terms in $\tilde{D}^{-1}$ are defined. One then calculates the following identities:

$$\begin{cases} D^T(DD^T - \gamma^2 I_m)^{-1} = (D^T D - \gamma^2 I_m)^{-1} D^T \\ (-\gamma^2 I_m + DD^T)^{-1} D = D(D^T D - \gamma^2 I_m)^{-1} \\ \gamma^2 (DD^T - \gamma^2 I_m)^{-1} = D(D^T D - \gamma^2 I_m)^{-1} - I_m \end{cases} \tag{5.98}$$

From Theorem A.61 we can rewrite the LMI as

$$A^T P + PA - (PB \;\; C^T)\tilde{D}^{-1} \begin{pmatrix} B^T P \\ C \end{pmatrix} < 0,$$

$P = P^T > 0$. After some manipulations and using the above identities one gets

$$A^T P + PA - (PB + C^T D)(D^T D - \gamma^2 I_m)^{-1}(B^T P + D^T C) + C^T C < 0$$

This Riccati inequality tells us that the system is dissipative with respect to the H_∞ supply rate $w(u, y) = \gamma^2 u^T u - y^T y$. This can be checked using for instance the KYP Lemma 4.87 with the right choice of the matrices Q, R and S. Using Theorem A.61 one can further deduce that the Riccati inequality is equivalent to the LMI: find $P = P^T > 0$ such that

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma I_m \end{bmatrix} < 0 \tag{5.99}$$

The equivalence between the LMI in (5.97) and the LMI in (5.99) can be shown using once again Theorem A.61, considering this time the Schur complement of the matrix $-\gamma I_m$ in (5.97). We therefore have shown the equivalence between two LMIs and one Riccati inequality which all express the Bounded Real Lemma. We once again stress the fundamental role played by Theorem A.61. The main result of this part is summarized as follows.

Let $\gamma^2 I_m - D^T D > 0$. The existence of a positive definite solution $P = P^T$ to the ARI

$$A^T P + PA + (B^T P + D^T C)^T (\gamma^2 I_m - D^T D)^{-1}(B^T P + D^T C) + C^T C < 0$$

implies that the system (A, B, C, D) is strictly dissipative with respect to the supply rate $\gamma^2 u^T u - y^T y$, which in turn implies that $||H||_\infty < \gamma$.

Letting $D \to 0$ and $\gamma = 1$ one recovers the Riccati equation for (A, B, C) in Lemma 5.65. The following results hold also true and somewhat extend the above:

Theorem 5.67. *[99] Let (A, B, C, D) be a minimal realization of the transfer function $H(s) \in \mathbb{C}^{m\times m}$, with input $y(\cdot)$ and output $y(\cdot)$. Then the following statements are equivalent:*

- *$H(s)$ is strictly bounded real*
- *$H(s)$ is exponentially finite gain, i.e.*

$$\int_0^t \exp(\epsilon\tau)y^T(\tau)y(\tau)d\tau \le \gamma^2 \int_0^t \exp(\epsilon\tau)u^T(\tau)u(\tau)d\tau$$

  for all $t \ge 0$ and some $\epsilon > 0$
- *There exists matrices $P = P^T > 0$, $L \in \mathbb{R}^{n\times p}$, $W \in \mathbb{R}^{p\times m}$ and a scalar $\epsilon > 0$ such that*

$$\begin{cases} A^TP + PA + \epsilon P + C^TC + LL^T = 0 \\ PB + C^TD + LW = 0 \\ \gamma^2 I_m - D^TD - W^TW = 0 \end{cases} \tag{5.100}$$

Furthermore $H(s)$ is strongly bounded real if and only if there exists $P = P^T > 0$ and $R = R^T > 0$ such that

$$A^TP + PA + (B^TP + D^TC)^T(\gamma^2 I_m - D^TD)^{-1}(B^TP + D^TC) + R = 0 \tag{5.101}$$

■

From Proposition A.63 the set of equations in (5.100) is equivalent to the LMI

$$\begin{bmatrix} A^TP + PA + \epsilon P + C^TC & PB + C^TD \\ D^TC + B^TP & D^TD - \gamma^2 I_m \end{bmatrix} \le 0$$

Similarly the next theorem holds that concerns positive realness.

Theorem 5.68. *[99] Let (A, B, C, D) be a minimal realization of the transfer function $H(s) \in \mathbb{C}^{m\times m}$, with input $y(\cdot)$ and output $y(\cdot)$. Then the following statements are equivalent:*

- *$H(s)$ is strictly positive real*
- *$H(s)$ is exponentially passive, i.e. $\int_0^t \exp(\epsilon\tau)u^T(\tau)y(\tau)d\tau \ge 0$, for all $t \ge 0$*
- *The conditions of the Lefschetz-Kalman-Yakubovich Lemma 3.8 in (3.23) are satisfied*

Furthermore $H(s)$ is SSPR if and only if there exists $P = P^T > 0$ and $R = R^T > 0$ such that

$$A^T P + PA + (B^T P - C)^T (D^T + D)^{-1}(B^T P - C) + R = 0 \tag{5.102}$$

■

The Riccati equations in Theorems 5.67 and 5.68 can be deduced from Lemma A.62. Notice that the Riccati equations are not identical from one theorem to the other, since the considered supply rates differ: the first one concerns the H_∞ supply rate, while the second one concerns the passivity supply rate. The exponential dissipativity can also be expressed via the existence of a storage function and the dissipation inequality is then $\exp(\epsilon t)V(x(t)) - V(x(0)) \leq \int_0^t \exp(\epsilon\tau)u^T(\tau)y(\tau)d\tau$, for all $t \geq 0$. If $V(\cdot)$ is continuously differentiable, then the infinitesimal form of the dissipation inequality is $\dot{V}(x(t)) + \epsilon V(x(t)) \leq u^T(t)y(t)$ for all $t \geq 0$. Another definition of exponential dissipativity has been introduced in [156], which is strict passivity (Definition 4.51) with the storage functions that satisfy $\alpha_1||x||^2 \leq V(x) \leq \alpha_2||x||^2$ and $\alpha_3||x||^2 \leq S(x)$ for some $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$. Such a definition was motivated by a result of Krasovskii [263]. If a system is exponentially dissipative in this sense then the uncontrolled system is exponentially stable. The definition in Theorem 5.68 is more general since the exponential dissipativity implies the strict dissipativity: in case the storage function satisfies $\alpha_1||x||^2 \leq V(x) \leq \alpha_2||x||^2$ then the second condition is also satisfied with $S(x) = V(x)$. The exponential finite gain property has been used in [481,539]to study the stability of switched systems with delay and time-continuous solutions.

Notice that the material presented in Section 3.8.3 finds application in the H_∞ problem, via the so-called *four-block Nehari problem*. This may be seen as an extension of the Bounded Real Lemma; see [224, Lemma 2, Theorem 3]. Further results on H_∞ control in the so-called behavorial framework, may be found in [491].

Remark 5.69 (Finite $\mathcal{L}_p$-gain). A system has a finite $\mathcal{L}_p$-gain if it is dissipative with respect to a supply rate of the form

$$w(u, y) = \gamma||u||^p - \delta||y||^p \tag{5.103}$$

for some $\gamma > 0$, $\delta > 0$. It is noteworthy that such supply rates satisfy the condition 2 in Lemma 5.13 in a strong sense since $w(0, y) < 0$ for all $y \neq 0$.

The paper [305] concerns the standard H_∞ problem and relationships between LMI, ARE, ARI, and is worth reading.

5.9.2 Closed-loop Synthesis: Static State Feedback

Let us make an aside on the problem of designing a feedback $u(t) = v(t) + Kx(t)$ applied to the linear time invariant system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ x(0) = x_0 \end{cases} \tag{5.104}$$

so that the closed-loop system is dissipative with respect to the supply rate $w(v, y) = u^T Ru - y^T Jy$. Such systems, when they possess a storage function $x^T Px$, are named $(R, P, J)-$dissipative [478]. The feedback gain K has to be chosen in such a way that the closed-loop system $(A + BK, B, C + DK, D)$ is $(R, P, J)-$dissipative. This gives rise to the following set of matrix equations:

$$\begin{cases} A^T P + PA + C^T JC = K^T RK \\ PB + C^T JD = -K^T R \\ D^T JD = R \end{cases} \tag{5.105}$$

A suitable choice of the matrices P, R and J allows one to obtain several standard one-block or two-block results, to which Riccati equalities correspond. This is summarized as follows, where the dimensions are not given explicitly but follow from the context. The notation $y = \begin{bmatrix} y \\ u \end{bmatrix}$ means that the signal y is split into two subsignals, one still called the output y, the other one being the input u. The following ingredients (LMI and Riccati equalities) have already been seen in this book, under slightly different forms. This is once again the opportunity to realize how the supply rate modifications influence the type of problem one is solving.

- Let $y = \begin{bmatrix} y \\ u \end{bmatrix}$, $C = \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$, $J = \begin{bmatrix} I_m & 0 \\ 0 & R \end{bmatrix}$. The matrix R in J and R in (5.105) are the same matrix. With this choice of input and matrices one obtains from (5.105) the standard LQR Riccati equation. Indeed one gets

$$\begin{cases} A^T P + PA + C^T C = K^T RK \\ B^T P = -RK \end{cases} \tag{5.106}$$

with $R \geq 0$, $J \geq 0$, $P \geq 0$. If $R > 0$ then one can eliminate K to get the Riccati equation

$$A^T P + PA + C^T C - PBR^{-1} B^T P = 0 \tag{5.107}$$

- Let $y = \begin{bmatrix} y \\ u \end{bmatrix}$, $C = \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} D \\ I_m \end{bmatrix}$, $J = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}$. This time one gets the normalized coprime factorization problem, still with $J \geq 0$, $P \geq 0$, $R \geq 0$. From (5.105) it follows that

$$\begin{cases} A^T P + PA + C^T C = K^T RK \\ PB + C^T D = -K^T R \\ D^T D + I_m = R \end{cases} \tag{5.108}$$

If $R > 0$ then both R and K can be eliminated and we obtain the normalized coprime factorization Riccati equation

$$A^T P+PA+C^T C-(PB+C^T D)(I_m+D^T D)^{-1}(B^T P+D^T C) = 0 \tag{5.109}$$

- Let $y = \begin{bmatrix} y \\ u \end{bmatrix}$, $C = \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} D \\ I_m \end{bmatrix}$, $J = \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_m \end{bmatrix}$. We obtain the Bounded Real Lemma, and (5.105) becomes

$$\begin{cases} A^T P + PA + C^T C = K^T RK \\ C^T D + PB = -K^T R \\ R = D^T D - \gamma^2 I_m \end{cases} \tag{5.110}$$

If γ is such that $R < 0$ and $P \geq 0$, one can eliminate R and K from the above and obtain the Bounded Real Lemma Riccati equality

$$A^T P+PA+C^T C+(PB+C^T D)(\gamma^2 I-D^T D)^{-1}(B^T P+D^T C) = 0 \tag{5.111}$$

- Let $y = \begin{bmatrix} y \\ u \end{bmatrix}$, $C = \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} D \\ I_m \end{bmatrix}$, $J = -\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$. We obtain the Positive Real Lemma, and (5.105) becomes the set of equations of the KYP Lemma, i.e.

$$\begin{cases} A^T P + PA = K^T RK \\ C^T - PB = K^T R \\ R = -(D + D^T) \end{cases} \tag{5.112}$$

One has $R \leq 0$ and it is required that $P \geq 0$. If the matrix $D + D^T$ is invertible, then one can eliminate both R andf K to obtain the Positive Real Lemma Riccati equation

$$A^T P + PA + C^T C + (PB - C)(D^T D)^{-1}(B^T P - C) = 0 \tag{5.113}$$

- Let $y = \begin{bmatrix} y \\ u \end{bmatrix}$, $C = \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} D_{11} & D_{12} \\ 0 & I_m \end{bmatrix}$, $J = \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_m \end{bmatrix}$, $u = \begin{bmatrix} w \\ u \end{bmatrix}$, $B = [B_1 \quad B_2]$. With such a choice we obtain the H_∞ full information problem. In this problem $P \geq 0$. If $D_{12} = 0$ then (5.105) becomes:

$$\begin{cases} A^T P + PA + C^T C = [K_1^T \;\; K_2^T] R \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \\ \\ R \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = - \left(\begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} P + \begin{bmatrix} D_{11}^T \\ 0 \end{bmatrix} C \right) \\ \\ R = \begin{bmatrix} D_{11}^T D_{11} & 0 \\ \\ 0 & -\gamma^2 I_m \end{bmatrix} \end{cases} \tag{5.114}$$

A system that is dissipative with respect to this choice of the supply rate is called $J-$dissipative. For more details on the $J-$dissipative approach and its application in $H_\infty-$control, one is referred to [397].

5.9.3 Closed-loop Synthesis: PR Dynamic Feedback

The problem that is of interest here, and which is in a sense of the same nature as the problem treated in Section 3.8.5, is about the design of a robust controller that is also PR. Let us consider the dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ \\ z(t) = C_1 x(t) + D_{12} u(t) \\ \\ y(t) = C_2 x(t) + D_{21} w(t) \end{cases} \tag{5.115}$$

The signal $u(\cdot)$ is the controller, $w(\cdot)$ is a disturbance. Let us denote $H_{ij}(s) = C_i(sI_n - A)^{-1}B_j + D_{ij}$, $s \in \mathbb{C}$. Since $D_{11} = 0$ and $D_{22} = 0$, the transfer matrices $H_{11}(s)$ and $H_{22}(s)$ are strictly proper. In a compact notation one has

$$\begin{pmatrix} z(s) \\ y(s) \end{pmatrix} = H(s) \begin{pmatrix} w(s) \\ u(s) \end{pmatrix} \tag{5.116}$$

with $H(s) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (sI_n - A)^{-1}(B_1 \quad B_2) + \begin{pmatrix} 0 & D_{12} \\ D_{21} & 0 \end{pmatrix}$. The objective of the control task is to construct a positive real controller with transfer matrix $K(s)$ such that

$$||T_{zw}(s)||_\infty = ||H_{11}(s) + H_{12}(s)K(s)(I_m - H_{22}(s)K(s))^{-1}H_{21}(s)||_\infty < \gamma \tag{5.117}$$

for some $\gamma > 0$. Some assumptions are in order:

Assumption 12 • *(i)* $D_{11} = D_{22} = 0$.
- *(ii) The pair* (A, B_1) *is stabilizable, the pair* (A, C_1) *is detectable.*
- *(iii) The pair* (A, B_2) *is stabilizable, the pair* (A, C_2) *is detectable.*
- *(iv)* $D_{12}^T(C_1 \;\; D_{12}) = (0 \;\; R)$ *with* R *invertible.*
- *(v)* $\begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} D_{21}^T = \begin{pmatrix} 0 \\ N \end{pmatrix}$ *with* N *invertible.* ■

Assumptions (ii) and (iii) will guarantee that some Riccati equations in (5.120) and (5.121) possess a solution, respectively. Assumptions (iv) and (v) concern the exogeneous signal $w(\cdot)$ and how it enters the transfer $H(s)$: $w(\cdot)$ includes both plant disturbances and sensor noise, which are orthogonal, and the sensor noise weighting matrix is nonsingular. Assumption (iv) means that $C_1 x$ and $D_{12}u$ are orthogonal so that the penalty on $z = C_1 x + D_{12} u$ includes a nonsingular penalty on the control u.

Let us disregard for the moment that the controller be PR. We obtain the so-called central controller

$$K(s) = -F_c(sI_n - A_c)^{-1} Z_c L_c \tag{5.118}$$

where the various vectors and matrices satisfy

$$\begin{array}{ll} (i) & A_c = A + \gamma^{-2} B_1 B_1^T P_c + B_2 F_c + Z_c L_c C_2 \\ (ii) & F_c = -R^{-1} B_2^T P_c \\ (iii) & L_c = -Y_c C_2^T N^{-1} \\ (iv) & Z_c = (I_m - \gamma^{-2} Y_c P_c)^{-1} \end{array} \tag{5.119}$$

with $P_c = P_c^T \geq 0$, $Y_c = Y_c^T \geq 0$, $\rho(Y_c P_c) < \gamma^2$, and these matrices are solutions of the Riccati equations

$$A^T P_c + P_c A + P_c[\gamma^{-2} B_1 B_1^T - B_2 R^{-1} B_2^T] P_c + C_1^T C_1 = 0 \tag{5.120}$$

and

$$A^T Y_c + Y_c A + Y_c[\gamma^{-2} C_1 C_1^T - C_2 N^{-1} C_2^T] Y_c + B_1^T B_1 = 0. \tag{5.121}$$

The next step is to guarantee that the controller is PR. To this end an additional assumption is made:

Assumption 13 • *The triple* (A, B_2, C_2) *satisfies the assumptions of Theorem 3.29.*
- *The transfer matrix* $H_{22}(s)$ *is PR, equivalently there exists* $P = P^T > 0$ *and* $Q = Q^T \geq 0$ *such that* $A^T P + PA + Q = 0$ *and* $B_2^T P = C_2$.

Proposition 5.70. *[238] Let* $B_1 B_1^T = P^{-1} Q P^{-1} - \gamma^{-2} P^{-1} C_1^T C_1 P^{-1} + C_2 N^{-1} C_2^T$*, and* $N = R$*. Then the controller transfer matrix* $-K(s)$ *given in (5.119) through (5.121) is PR if*

$$
\begin{aligned}
Q_r = {} & C_1^T C_1 - (Y_c^{-1} - \gamma^{-2} P_c Y_c P_c) B_2 R^{-1} B_2^T (Y_c^{-1} - \gamma^{-2} P_c Y_c P_c) - \\
& - \gamma^{-2} A^T P_c P_c A - \gamma^{-2} Y_c P_c P_c Y_c - \\
& - \gamma^2 P_c (I_n - \gamma^{-2} Y_c P_c) B_1 B_1^T (I_n - \gamma^{-2} P_c Y_c) P_c + \\
& + (Z_c^{-T} P_c + Y_c^{-1})^T B_2 R^{-1} B_2^T (Z_c^{-T} P_c + Y_c^{-1}) + \\
& + (A P_c + Y_c P_c)^T (A P_c + Y_c P_c) + \gamma^{-6} P_c Y_c P_c B_1 B_1^T P_c Y_c P_c
\end{aligned}
\tag{5.122}
$$

is positive definite. ■

Proof: The proof consists of showing that with the above choices of B_1 and of the matrix $Q_r > 0$, then there exists $P_r = P_r^T > 0$ and $Q_c = Q_c^T \geq 0$ such that

$$A_c^T P_r + P_r A_c + Q_c = 0 \tag{5.123}$$

and

$$C_2 Y_c Z_c^T P_r = B_2^T P_c \tag{5.124}$$

where in fact $Q_c = Q_r$. The fact that $C_2 = B_2^T P$ implies that

$$B_2^T P Y_c (I_n - \gamma^{-2} P_c Y_c)^{-1} P_r = B_2^T P_c \tag{5.125}$$

A solution to this equation is given by

$$P Y_c Z_c^T P_r = P_c \tag{5.126}$$

Now let us consider

$$
\begin{cases}
(i) \;\; Y_c = P^{-1} \\
(ii) \; P_r = Z_c^{-T} P_c
\end{cases}
\tag{5.127}
$$

We can remark that

$$
\begin{aligned}
P_r = Z_c^{-T} P_c &= (I_n - \gamma^{-2} P_c Y_c) P_c \\
&= P_c - \gamma^{-2} P_c Y_c P_c = P_r^T
\end{aligned}
\tag{5.128}
$$

and that $Y_c = P^{-1}$ is a solution of equation (5.121), *i.e.*

$$AP^{-1} + P^{-1}A^T + P^{-1}(\gamma^{-2}C_1^T C_1 - C_2^T N^{-1} C_2)P^{-1} + B_1 B_1^T = 0 \quad (5.129)$$

Indeed let us pre- and postmultiply (5.129) with P. This gives

$$PA + A^T P + \gamma^{-2}C_1^T C_1 - C_2^T N^{-1} C_2 + PB_1 B_1^T P = 0 \quad (5.130)$$

The choice made for $B_1 B_1^T$ reduces (5.130) to the KYP Lemma Equation $A^T P + PA + Q = 0$. This shows that $Y_c = P^{-1}$ is a solution of equation (5.121). Now inserting (5.119)(i), (5.122) and (5.127)(ii) into (5.123) reduces this equation to (5.120). This proves that the above choices for A_c, P_r, Q_r guarantee that (5.123) is true with $Q_c = Q_r$. In other words we have shown that with the choices for the matrices A_c, P_r and Q_r the KYP Lemma first Equation (5.123) is satisfied as it reduces to the KYP Lemma equation $A^T P + PA + Q = 0$ which is supposed to be satisfied. The second equation is also satified because $B_2^T P = C_2$ is supposed to hold. ■

Let us end these two subsections by mentioning the work in [20, 21] in which the H_∞ problem is solved with a nonsmooth quadratic optimization problem, making use of the same tools from nonsmooth analysis that we saw in various places of this book (subderivatives, subgradients). The Bounded Real Lemma has been extended to a class of nonlinear time-delayed systems in [15]; see also [378, 379] for details on the H_∞ control of delayed systems. Other, related results, may be found in [430] using the $\gamma-$PRness property (see Definition 2.61). A discrete-time version of the Bounder Real Lemma is presented in [470].

5.9.4 Nonlinear H_∞

A nonlinear version of the Bounded Real Lemma is obtained from (4.81) (4.82) setting $Q = -I_m$, $S = 0$, $R = \gamma^2 I_m$. One obtains

$$\begin{cases} \hat{S}(x) = -j(x) \\ \hat{R}(x) = \gamma^2 I_m - j^T(x)j(x) = W^T(x)W(x) \\ \frac{1}{2}g^T(x)\nabla V(x) = -j^T(x)j(x) - W^T(x)L(x) \\ \nabla V^T(x)f(x) = -h^T(x)h(x) - L^T(x)L(x) \\ V(x) \geq 0, \quad V(0) = 0 \end{cases} \quad (5.131)$$

which we can rewrite as the LMI

$$\begin{pmatrix} \nabla V^T(x)f(x) + h^T(x)h(x) & \left[\frac{1}{2}g^T(x)\nabla V(x) + j^T(x)j(x)\right]^T \\ \frac{1}{2}g^T(x)\nabla V(x) + j^T(x)j(x) & -\gamma^2 I_m + j^T(x)j(x) \end{pmatrix} =$$
$$= - \begin{bmatrix} L^T(x) \\ W^T(x) \end{bmatrix} [L(x) \quad W(x)] \leq 0 \tag{5.132}$$

From (5.132) one easily gets a Hamilton-Jacobi inequality that is a generalization of the above ones, using Proposition A.63.

Let us now pass to the main subject of this subsection. Given a plant of the form

$$\begin{cases} \dot{x}(t) = A(x(t)) + B_1(x(t))w(t) + B_2(x(t))u(t) \\ z(t) = C_1(x(t)) + D_{12}(x(t))u(t) \\ y(t) = C_2(x(t)) + D_{21}(x(t))w(t) \\ x(0) = x_0 \end{cases} \tag{5.133}$$

with $A(0) = 0$, $C_1(0) = 0$, $C_2(0) = 0$, $C_2(\cdot)$ and $D_{21}(\cdot)$ are continuously differentiable, the nonlinear H_∞ control problem is to construct a state feedback

$$\begin{cases} \dot{\zeta}(t) = a(\zeta(t)) + b(\zeta(t))y(t) \\ u(t) = c(\zeta(t)) \end{cases} \tag{5.134}$$

with continuously differentiable $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $a(0) = 0$, $c(0) = 0$, $\dim(\zeta(t)) = l$, such that there exists a storage function $V : I\!R^n \times I\!R^l \to I\!R^+$ such that

$$V(x(t_1), \zeta(t_1)) - V(x(t_0), \zeta(t_0)) \leq \int_{t_0}^{t_1} \{\gamma^2 w^T(t)w(t) - z^T(t)z(t)\}dt \tag{5.135}$$

for any $t_1 \geq t_0$, along the closed-loop trajectories. The controller $u(\cdot)$ may be static, *i.e.* $u = u(x)$. One may also formulate (5.135) as

$$\int_{t_0}^{t_1} z^T(t)z(t)dt \leq \gamma^2 \int_{t_0}^{t_1} w^T(t)w(t)dt + \beta(x(t_0)) \tag{5.136}$$

for some non negative function $\beta(\cdot)$ with $\beta(0) = 0$.

Theorem 5.71. *Let $B_1(\cdot)$ and $B_2(\cdot)$ be bounded, all data in (5.133) have bounded first derivatives, $D_{12}^T D_{12} = I_m$, $D_{21}D_{21}^T = I_q$, D_{21} and D_{12} be constant. Consider the state feedback $u(x)$. If the closed-loop system satisfies (5.136) there exists a storage function $V(x) \geq 0$, $V(0) = 0$, such that the Hamilton-Jacobi equality*

$$\nabla V(x)(A(x) - B_2(x)C_1(x)) - \tfrac{1}{2}\nabla V(x)[B_2(x)B^T(x) - \gamma^2 B_1(x)B_1^T(x)]\nabla V^T(x)$$

$$+\tfrac{1}{2}C_1^T(I - D_{12}D_{12}^T)C_1(x) = 0 \tag{5.137}$$

is satisfied, where the function $V(\cdot)$ may be continuous but not differentiable so that the PDE (5.137) has to be interpreted in the viscosity sense. Conversely, if (5.137) has a smooth solution $V(x) > 0$ for $x \neq 0$, $V(0) = 0$, then the state feedback controller $u(x) = -(D_{12}^T C_1(x) + B_2^T(x)\nabla V^T(x))$ makes the closed-loop system satisfy (5.136). The stability of the closed-loop system is guaranteed provided that the system

$$\begin{cases} \dot{x}(t) = A(x(t)) + B_2(x(t))u(x(t)) + B_1(x(t))w(t) \\ \\ z(t) = C_1(x(t)) + D_{12}(x(t))u(x(t)) \end{cases} \tag{5.138}$$

is zero state detectable. ■

Much more material can be found in [33, 129, 288, 397] and the books [257,442]. Extensions of the strict Bounded Real Lemma 5.65 to the nonlinear affine in the input case, where storage functions are allowed to be lower semi continuous only, has been proposed in [233].

5.9.5 More on Finite-power-gain Systems

We have already introduced the notion of finite power gain in Definition 5.7. Here we refine it a little bit, which gives rise to the characterization of a new quantity (the power bias) with a partial differential equality involving a storage function. The material is taken from [129]. In particular an example will show that storage functions are not always differentiable and that tools based on viscosity solutions may be needed. We consider systems of the form

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ \\ y(t) = h(x(t)) + j(x(t))u(t) \end{cases} \tag{5.139}$$

with the usual dimensions of vectors, and all vector fields are continuously differentiable on $\mathbb{R}^n$. It is further assumed that $||g(x)||_\infty < +\infty$, $||j(x)||_\infty < +\infty$, and that $\frac{\partial f}{\partial x}(x)$, $\frac{\partial g}{\partial x}(x)$, $\frac{\partial h}{\partial x}(x)$, $\frac{\partial j}{\partial x}(x)$ are (globally) bounded.

Definition 5.72. *The system (5.139) has finite power gain $\leq \gamma$ if there exists finite non-negative functions $\lambda : \mathbb{R}^n \to \mathbb{R}$ (the power bias) and $\beta : \mathbb{R}^n \to \mathbb{R}$ (the energy bias) such that*

$$\int_0^t y^T(s)y(s)ds \leq \gamma^2 \int_0^t u^T(s)u(s)ds + \lambda(x)t + \beta(x) \tag{5.140}$$

for all admissible $u(\cdot)$ (here $u \in \mathcal{L}_{2,e}$), all $t \geq 0$ and all $x \in \mathbb{R}^n$. ■

The presence of the term $\lambda(x)t$ may be explained as follows: defining the norm

$$||y||_{fp} = \sqrt{\lim \sup_{t \to +\infty} \left\{ \frac{1}{t} \int_0^t y^T(s) y(s) ds \right\}} \tag{5.141}$$

and dividing both sides of (5.140) by t and letting $t \to +\infty$ one obtains

$$||y||_{fp} \leq \gamma^2 ||u||_{fp} + \lambda(x) \tag{5.142}$$

It is noteworthy that (5.140) implies (5.142) but not the contrary. Moreover the link between (5.140) and dissipativity is not difficult to make, whereas it is not clear with (5.142). Since (5.142) is obtained in the limit as $t \to +\infty$, possibly the concept of ultimate dissipativity could be suitable. This is why finite power gain is defined as in Definition 5.72.

Proposition 5.73. *[129] Any system with finite power gain $\leq \gamma$ and zero power bias has an $\mathcal{L}_2-$gain $\leq \gamma$. Conversely, any system with $\mathcal{L}_2-$gain $\leq \gamma$ has a finite power gain with zero power bias.* ■

From (5.140) let us define the quantity

$$\phi(t,x) = \sup_{u \in \mathcal{L}_{2,e}} \left\{ \int_0^t (y^T(s) y(s) - \gamma^2 u^T(s) u(s)) ds \;\; | \;\; x(0) = x \right\} \tag{5.143}$$

This represents the energy that can be extracted from the system on $[0,t]$. It is non decreasing in t and one has for all $t \geq 0$ and all $x \in I\!R^n$:

$$\phi(t,x) \leq \lambda(x)t + \beta(x) \tag{5.144}$$

Definition 5.74. *The available power $\lambda_a(x)$ is the most average power that can be extracted from the system over an infinite time when initialized at x, i.e.*

$$\lambda_a(x) = \lim \sup_{t \to +\infty} \left\{ \frac{\phi(t,x)}{t} \right\} \tag{5.145}$$

■

Proposition 5.75. *[129] Suppose that the system has finite power gain $\leq \gamma$ with power bias and energy pair (λ, β). Then the available power is finite, with $\lambda_a(x) \leq \lambda(x)$ for all $x \in I\!R^n$.* ■

One realizes that the framework of finite power gain systems tends to generalize that of dissipative systems.

Example 5.76. [129] Consider the scalar system $\dot{x}(t) = ax(t) + bu(t)$, $y(t) = c(x(t))$, where $c(\cdot)$ is a saturation

$$c(x) = \begin{cases} -c\epsilon \text{ if } x < -\epsilon \\ \\ cx \quad \text{if } |x| \leq \epsilon \\ \\ c\epsilon \quad \text{if } x > \epsilon \end{cases} \tag{5.146}$$

For this system one has

$$\lambda_a = \begin{cases} \frac{a^2\epsilon^2}{b^2}\left(\frac{b^2c^2}{a^2} - \gamma^2\right) \text{ if } \gamma < |\frac{bc}{a}| \\ \\ 0 \qquad\qquad\qquad\quad \text{if } \gamma \geq |\frac{bc}{a}| \end{cases} \tag{5.147}$$

The power gain $\gamma^\star = \inf\{\gamma \geq 0 \mid (5.140) \text{ holds}\}$ thus depends on the power bias:

$$\gamma^\star = \begin{cases} |\frac{b}{a\epsilon}|\sqrt{c^2\epsilon^2 - \lambda} \text{ if } \lambda \in [0, c^2\epsilon^2) \\ \\ 0 \qquad\qquad\qquad \text{if } \lambda \in [c^2\epsilon^2, +\infty) \end{cases} \tag{5.148}$$

We are now going to characterize the property of finite power gain through a partial differential equation, similarly to what has been developed in Section 4.6.

Theorem 5.77. *[129] Let the system in (5.139) satisfy*

$$j^T(x)j(x) - \gamma^2 I_m < 0 \tag{5.149}$$

Suppose that the system has finite power gain $\leq \gamma$. Then there exists a finite viscosity solution pair (λ, V) of the partial differential inequality

$$H(x, \nabla V(x)) \leq \lambda \tag{5.150}$$

where $H(x,p) = \max_{v \in I\!R^m} H(x,p,v)$ and

$$H(x,p,v) = p^T(f(x)+g(x)v)+(h(x)+j(x)v)^T(h(x)+j(x)v)-\gamma^2 v^T v \tag{5.151}$$

Conversely, if there is a viscosity solution pair (λ, V) to the partial differential inequality in (5.150), then the system has finite power gain $\leq \gamma$. If $V(\cdot)$ is continuously differentiable, the worst case disturbance is given by $v^\star = argmax_{v \in I\!R^m} H(x, \nabla V(x), v)$. ■

The following may be useful for calculations:

Theorem 5.78. *[129] Suppose there exists a Lipschitz continuous solution pair (λ, V) of the partial differential equality*

$$H(x, \nabla V(x)) = \lambda \tag{5.152}$$

Then the power bias λ is minimal, i.e. $\lambda_a = \lambda$ and is consequently unique. ∎

Example 5.79. Let us continue with the above example. The system is scalar, so that the partial differential equality (5.152) reduces to a quadratic in $\nabla V(x)$. One may compute that for $\gamma \geq |\frac{bc}{a}|$

$$V(x) = \begin{cases} -\frac{\gamma^2 a x^2}{b^2}(1 - \sqrt{1-\mu^2}) & \text{if } |x| < \varepsilon \\ -\frac{\gamma^2 a x^2}{b^2} + \frac{\gamma^2 a |x|}{b^2}\sqrt{x^2 - \mu^2\epsilon^2} - \frac{\gamma^2 a \epsilon^2}{b^2}\log\left(\frac{|x| + \sqrt{x^2 - \mu^2\varepsilon^2}}{\epsilon + \epsilon\sqrt{1-\mu^2}}\right) & \text{if } |x| \geq \epsilon \end{cases} \tag{5.153}$$

where $\mu = |\frac{bc}{\gamma a}|$, and for $\gamma < |\frac{bc}{a}|$:

$$V(x) = \begin{cases} -\frac{\gamma^2 a x^2}{b^2} - \frac{\gamma^2 a}{b^2}\sqrt{\mu^2 - 1}\left(|x|\sqrt{\epsilon^2 - x^2} + \epsilon^2 \arcsin\left(\frac{|x|}{\epsilon}\right)\right) & \text{if } |x| < \epsilon \\ -\frac{\gamma^2 a x^2}{b^2} + \frac{\gamma^2 a |x|}{b^2}\sqrt{x^2 - \epsilon^2} - \frac{\gamma^2 a \epsilon^2}{b^2}\log\left(\frac{|x| + \sqrt{x^2 - \epsilon^2}}{\epsilon}\right) - \\ -\frac{\gamma^2 a \epsilon^2 \pi}{2b^2}\sqrt{\mu^2 - 1} & \text{if } |x| \geq \epsilon \end{cases} \tag{5.154}$$

It is expected from these expressions that the function $V(x)$ may not be differentiable everywhere, so that viscosity solutions have to be considered.

Let us end with a generalized version of the small gain theorem.

Theorem 5.80. *[129] Consider a feedback interconnection as in Figure 5.2. Suppose that the subsystems H_1 and H_2 are both causal and with finite power gain $\leq \gamma_1$ and γ_2, respectively, and power bias λ_1 and λ_2, respectively. If $\gamma_1\gamma_2 < 1$ then for all inputs $||r_1||_{fp} < +\infty$ and $||r_2||_{fp} < +\infty$, the closed-loop interconnection is stable in the sense that $||u_1||_{fp} < +\infty$, $||u_2||_{fp} < +\infty$, $||y_1||_{fp} < +\infty$, $||y_2||_{fp} < +\infty$, where the norm $||\cdot||_{fp}$ is defined in (5.141).* ∎

5.10 Popov's Hyperstability

The notion of hyperstable system has been introduced by Popov in 1964 [405, 409]. It grew out of the concept of absolute stability which was reviewed in Section 3.9. Let us consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{5.155}$$

and the quadratic functional

$$\eta(0,t) = \left[x^T(s)Ax(s)\right]_0^t + \int_0^t [x^T(s)\; u^T(s)] \begin{pmatrix} Q & S \\ \\ S^T & R \end{pmatrix} \begin{bmatrix} x(s) \\ \\ u(s) \end{bmatrix} ds \tag{5.156}$$

for all $t \geq 0$. It is assumed that (A, B) is controllable.

Definition 5.81. *The pair (5.155) (5.156) is* hyperstable *if for any constant* $\gamma \geq 0$, $\delta \geq 0$, *and for every input* $u(\cdot)$ *such that*

$$\eta(0,t) \leq \gamma^2 + \delta \sup_{0 \leq \tau \leq t} ||x(\tau)||, \quad \forall\, t \geq 0 \tag{5.157}$$

there exists a $k \in I\!R^+$ *such that*

$$||x(t)|| \leq k(\gamma + \delta + ||x(0)||), \quad \forall\, t \geq 0 \tag{5.158}$$

Moreover if $\lim_{t \to +\infty} ||x(t)|| = 0$ *the pair (5.155) and (5.156) is asymptotically hyperstable.* ■

Definition 5.82. *The pair (5.155) and (5.156) has the* minimal stability *property if for any initial condition* $x(0)$ *there exists a control input* $u_m(\cdot)$ *such that the trajectory of (5.155) satisfies*

- $\lim_{t \to +\infty} ||x(t)|| = 0$
- $\eta(0,t) \leq 0$, *for all* $t \geq 0$ ■

The following theorem is taken from [145], and generalizes the results in [13, 223, 276, 277].

Theorem 5.83. *[145] Suppose that the pair (5.155) and (5.156) has the minimal stability property. Then the pair (5.155) and (5.156) is*

- *Hyperstable if and only if the spectral function*

$$\Pi(s) = [B^T(-sI_n - A^T)^{-1} \;\; I_m] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI_n - A)^{-1}B \\ \\ I_m \end{bmatrix} \tag{5.159}$$

  is nonnegative
- *Asymptotically hyperstable if this spectral function is nonnegative and* $\Pi(j\omega) > 0$ *for all* $\omega \in I\!R$ ■

It is worth recalling here Proposition 2.31, Theorem 3.46, as well as the equivalence at the end of Section 3.8.6 between the spectral function positivity and the KYP Lemma set of equations solvability.

Proof: Let us prove the first item of the Theorem.

Hyperstability implies positivity:

Let us consider the Hermitian matrix

$$\Sigma(s) = [B^T(\bar{s}I_n - A^T)^{-1} \;\; I_m] \begin{bmatrix} Q & S \\ \\ S^T & R + (s+\bar{s})A \end{bmatrix} \begin{bmatrix} (sI_n - A)^{-1}B \\ \\ I_m \end{bmatrix} \tag{5.160}$$

and let us prove that $\Sigma \geq 0$ for all $\mathbf{Re}[s] > 0$ is implied by the hyperstability. Indeed suppose that for some s_0 with $\mathbf{Re}[s_0] > 0$, $\Sigma(s_0) < 0$. Then there exists a nonzero vector $u_0 \in \mathbb{C}^m$ such that $u_0^\star \Sigma(s_0) u_0 \leq 0$. For the input $u(t) = u_0 \exp(s_0 t)$ with the initial data $x(0) = (s_0 I_n - A)^{-1} B u_0$, one has $x(t) = (s_0 I_n - A)^{-1} B u_0 \exp(s_0 t)$. Clearly $||x(t)||$ is increasing with the same rate as $\exp(\mathbf{Re}[s_0]t)$, and it cannot satisfy an inequality as (5.158). On the other hand the constraint (5.157) is satisfied since for all $t \geq 0$ one has $\eta(0,t) = u_0^\star \Sigma(s_0) u_0 \int_0^t \exp(r\mathbf{Re}[s_0]\tau) d\tau \leq 0$. Consequently $\Sigma(s)$ is Hermitian positive for all s with $\mathbf{Re}[s] > 0$. By continuity one concludes that $\Pi(j\omega) = \Sigma(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

Positivity implies hyperstability:

Take any symmetric matrix G and notice that the functional in (5.156) can be rewritten as

$$\eta(0,t) = [x^T(A+G)x]_0^t + \int_0^t [x^T \; u^T] \begin{bmatrix} Q - A^TG - GA & S - GB \\ \\ S^T - B^TG & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} d\tau \tag{5.161}$$

If one considers the matrix $G = P_r$ that is the maximal solution of the KYP Lemma set of equations (see *e.g.* the arguments after Proposition 4.48), then

$$\eta(0,t) = [x^T(A+P_r)x]_0^t + \int_0^t ||\lambda^\star x(\tau) + \nu^\star u(\tau)|| d\tau \tag{5.162}$$

for some $\lambda^\star$ and $\nu^\star$. Let $u_m(\cdot)$ be an input which renders $\eta(0,t) \leq 0$, introduced via the minimal stability assumption. If $x_m(\cdot)$ is the corresponding state trajectory with initial condition $x_m(0) = x_0$, then $x_0^T(A+P_r)x_0 \geq x^T(t)(A+P_r)x(t)$ for all $t \geq 0$, which implies, since $\lim_{t\to+\infty} x(t) = 0$

for $u(\cdot) = u_m(\cdot)$), that $x_0^T(A + P_r)x_0 \geq 0$ for all x_0. Thus the matrix $A + P_r$ is semi positive definite. Suppose that there exists x_0 such that $x_0^T(A + P_r)x_0 = 0$. The condition that $\eta(0,t) \leq 0$ for the input $u_m(\cdot)$ implies that $\lambda^\star x(\tau) + \nu^\star u_m(\tau) = 0$. In other words the state trajectory $x_m(\cdot)$ of the dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = \lambda^\star x(t) + \nu^\star u_m(t) \end{cases} \tag{5.163}$$

with initial state $x_m(0) = x_0$ and the input $u_m(\cdot)$, results in an identically zero output $y(\cdot)$. The inverse system of the system in (5.163), which is given by

$$\begin{cases} \dot{x}(t) = (A - B(\nu^\star)^{-1}\lambda^\star)x(t) + B(\nu^\star)^{-1}y(t) \\ \\ u(t) = -(\nu^\star)^{-1}\lambda^\star x(t) + (\nu^\star)^{-1}y(t) \end{cases} \tag{5.164}$$

has an unstable transfer function. It is deduced that one has $\lim_{t\to+\infty} ||x_m(t)|| \neq 0$ when applying an identically zero input $y(\cdot)$ to (5.164). The assumption is contradicted. Thus $A + P_r$ is positive definite. There exists two scalars $\alpha > 0$ and $\beta > 0$ such that

$$0 < \alpha^2||x||^2 \leq x^T(A + P_r)x \leq \beta^2||x||^2 \tag{5.165}$$

If the input $u(\cdot)$ satisfies the constraint (5.157), one has

$$\alpha^2||x(t)||^2 \leq \delta \sup_{0\leq\tau\leq t} ||x(\tau)|| + \beta^2||x(0)||^2 + \gamma^2 \tag{5.166}$$

and

$$\alpha^2||x(t)||^2 \leq \delta \sup_{0\leq\tau\leq t} ||x(\tau)|| + (\beta||x(0)|| + \gamma)^2 \tag{5.167}$$

from which it follows that

$$||x(t)|| \leq \frac{\gamma + \delta + \beta||x(0)||}{\alpha} \leq \sup\left(\frac{1}{\alpha}, \frac{\beta}{\alpha}\right)[\gamma + \delta + ||x(0)||] \tag{5.168}$$

and the proof is done. ■

Further work on hyperstability may be found in [97,354,446,449–451]. The name "hyperstability" is used in a different context than Popov's one in other fields of science; see *e.g.* [262].

6

Dissipative Physical Systems

In this chapter we shall present a class of dissipative systems which correspond to models of physical systems and hence embed in their structure the conservation of energy (first principle of thermodynamics) and the interaction with their environment through pairs of conjugated variables with respect to the power. First, we shall recall three different definitions of systems obtained by an energy based modeling: controlled Lagrangian, input-output Hamiltonian systems and port controlled Hamiltonian systems. We shall illustrate and compare these definitions on some very simple examples. Second we shall treat a class of systems which gave rise to numerous stabilizing control using passivity theory and corresponds to models of robotic manipulators. In each worked case we show how the main functions associated to a dissipative system (the available storage, the required supply, storage functions) can be computed analytically and related to the energy of the physical system.

6.1 Lagrangian Control Systems

Lagrangian systems arise from variational calculus and gave a first general analytical definition of physical dynamical systems in analytical mechanics [1, 271, 294]. They also allow to describe the dynamics of various engineering systems as electromechanical systems or electrical circuits. They also gave rise to intensive work in control in order to derive different control laws by taking into account the structure of the system's dynamics derived from energy based modeling [437, 439]. In this section we shall present the definition of controlled Lagrangian systems and particular attention will be given to the expression of the interaction of a system with its environment.

6.1.1 Definition and Properties

In this section we shall briefly recall the definition of Lagrangian systems with external forces on $\mathbb{R}^n$ and the definition of Lagrangian control systems derived from it.

Definition 6.1 (Lagrangian systems with external forces). *Consider a* configuration manifold $Q = \mathbb{R}^n$ *whose points are denoted by* $q \in \mathbb{R}^n$ *and are called* generalized coordinates. *Denote by* $TQ = \mathbb{R}^{2n}$ *its tangent space and its elements by* $(q, \dot{q}) \in \mathbb{R}^{2n}$ *where* $\dot{q}$ *is called* generalized velocity. *A* Lagrangian system with external forces *on the configuration space* $Q = \mathbb{R}^n$ *is defined by a real function* $L(q, \dot{q})$, *from the tangent space* TQ *to* $\mathbb{R}$ *called* Lagrangian function *and the* Lagrangian equations*:*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial L}{\partial q}(q, \dot{q}) = F \tag{6.1}$$

where $F(t) \in \mathbb{R}^n$ *is the vector of* generalized forces *acting on the system and* $\frac{\partial F(x)}{\partial x}$ *denotes the gradient of the function* $F(x)$ *with respect to* x. ■

Remark 6.2. In this definition the configuration space is the real vector space $\mathbb{R}^n$ to which we shall restrict ourselves hereafter, but in general one may consider a differentiable manifold as configuration space [294]. Considering real vector spaces as configuration spaces corresponds actually to considering a *local* definition of a Lagrangian system.

If the vector of external forces $F(\cdot)$ is the vector of control inputs, then the Lagrangian control system is fully actuated. Such models arise for instance for fully actuated kinematic chains [366] .

Example 6.3 (Harmonic oscillator with external force). Let us consider the very simple example of the linear mass-spring system consisting in a mass attached to a fixed frame through a spring and subject to a force F. The coordinate q of the system is the position of the mass with respect to the fixed frame and the Lagrangian function is given by $L(q, \dot{q}) = K(\dot{q}) - U(q)$ where $K(\dot{q}) = \frac{1}{2}m\dot{q}^2$ is the kinetic co-energy of the mass and $U(q) = \frac{1}{2}kq^2$ is the potential energy of the spring. Then the Lagrangian system with external force is

$$m\ddot{q}(t) + kq(t) = F(t) \tag{6.2}$$

■

Lagrangian systems with external forces satisfy, by construction, a power balance equation that leads to some passivity property.

Lemma 6.4 (Lossless Lagrangian systems with external forces). *A Lagrangian system with external forces (6.1) satisfies the following* power balance equation*:*

$$F^T \dot{q} = \frac{dH}{dt} \tag{6.3}$$

where the real function $H(\cdot)$ *is obtained by the* Legendre transformation *of the Lagrangian function* $L(q, \dot{q})$ *with respect to the generalized velocity* $\dot{q}$ *and is defined by*

$$H(q, p) = \dot{q}^T p - L(q, \dot{q}) \tag{6.4}$$

where p *is the vector of* generalized momenta:

$$p(q, \dot{q}) = \left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) \tag{6.5}$$

and the Lagrangian function is assumed to be hyperregular *[294] in such a way that the map from the generalized velocities* $\dot{q}$ *to the generalized momenta* p *is bijective. If moreover the function* $H(\cdot)$ *is bounded from below, then the Lagrangian system with external forces is lossless with respect to the supply rate:* $F^T \dot{q}$ *with storage function* $H(\cdot)$*.* ■

Proof: let us first compute the power balance equation by computing $F^T \dot{q}$ using the Lagrangian equation (6.1) and the definition of the generalized momentum (6.5). We get

$$\begin{aligned} \dot{q}^T F &= \dot{q}^T \left[\tfrac{d}{dt} \left(\tfrac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) - \tfrac{\partial L}{\partial q}(q, \dot{q}) \right] \\ &= \dot{q}^T \tfrac{d}{dt} p - \dot{q}^T \tfrac{\partial L}{\partial q} \\ &= \tfrac{d}{dt} \left(\dot{q}^T p \right) - \ddot{q}^T p + \ddot{q}^T \tfrac{\partial L}{\partial \dot{q}} - \tfrac{d}{dt} L(q, \dot{q}) \\ &= \tfrac{d}{dt} \left(\dot{q}^T p - L(q, \dot{q}) \right) \\ &= \tfrac{dH}{dt} \end{aligned} \tag{6.6}$$

Then, using as outputs the generalized velocities and assuming that the function $H(\cdot)$ is bounded from below, the Lagrangian system with external forces is passive and lossless with storage function $H(\cdot)$. ■

Remark 6.5. The name *power balance equation* for (6.3) comes from the fact that for physical systems, the supply rate is the power ingoing the system due to the external force F and that the function $H(\cdot)$ is equal to the total energy of the system. ■

Example 6.6. Consider again Example 6.3 of the harmonic oscillator. In this case the supply rate is the mechanical power ingoing the system and the storage function is $H(p, q) = K(p) + U(q)$ and is the total energy of the system, *i.e.* the sum of the elastic potential and kinetic energy.

Actually the definition of Lagrangian systems with external forces may be too restrictive, as, for instance, the external forces F may not correspond to actual inputs. For example they may be linear functions of the inputs u:

$$F = J^T(q)u \tag{6.7}$$

where $J(q)$ is a $p \times n$ matrix depending on the generalized coordinates q. This is the case when for instance the dynamics of a robot is described in generalized coordinates for which the generalized velocities are not colocated to the actuators' forces and torques. Then the matrix $J(q)$ is the Jacobian of the geometric relations between the actuators' displacement and the generalized coordinates [366]. This system remains lossless with storage function $H(q,p)$ defined in (6.4) by choosing the outputs: $y = J(q)\dot{q}$.

In order to cope with such situations, a more general definition of Lagrangian systems with external controls is given and consists in considering that the input is directly modifying the Lagrangian function [437, 439].

Definition 6.7 (Lagrangian control system). *Consider a* configuration space $Q = I\!R^n$ *and its tangent space* $TQ = I\!R^{2n}$*, an input vector space* $\mathcal{U} = I\!R^p$*. A* Lagrangian control systems *is defined by a real function* $L(q, \dot{q}, u)$ *from* $TQ \times \mathcal{U}$ *to* $I\!R$*, and the equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}, u)\right) - \frac{\partial L}{\partial q}(q, \dot{q}, u) = 0 \tag{6.8}$$

■

This definition includes the Lagrangian systems with external forces (6.1) by choosing the Lagrangian function to be

$$L_1(q, \dot{q}, F) = L(q, \dot{q}) + q^T F \tag{6.9}$$

It includes as well the case when the the external forces are given by (6.7) as a linear function of the inputs where the matrix $J(q)$ is the Jacobian of some geometric function $C(q)$ from $I\!R^n$ to $I\!R^p$:

$$J(q) = \frac{\partial C}{\partial q}(q) \tag{6.10}$$

Then the Lagrangian function is given by

$$L_1(q, \dot{q}, F) = L(q, \dot{q}) + C(q)^T u \tag{6.11}$$

However it also encompasses Lagrangian systems where the inputs do not appear as forces as may be seen on the following example.

Example 6.8. Consider the harmonic oscillator, but assume now that the spring is no longer attached to a fixed basis but to a moving basis with its position u considered as an input. Let us choose as coordinate q, the position of the mass with respect to the fixed frame. The displacement of the spring then becomes $q - u$ and the potential energy becomes: $U(q, u) = \frac{1}{2}k(q - u)^2$ and the Lagrangian becomes

$$L(q, \dot{q}, u) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}k(q - u)^2 \tag{6.12}$$

The Lagrangian control systems then becomes

$$m\ddot{q}(t) + kq(t) = ku(t) \tag{6.13}$$

Lagrangian control systems also allow one to consider more inputs that the number of generalized velocities as may be seen on the next example.

Example 6.9. Consider again the harmonic oscillator and assume that the basis of the spring is moving with controlled position u_1 and that there is a force u_2 exerted on the mass. Consider a gain as generalized coordinate, the position $q \in I\!R$ of the mass with respect to an inertial frame. Then considering the Lagrangian function

$$L(q, \dot{q}, u) = \frac{1}{2}m(\dot{q})^2 - \frac{1}{2}k(q - u_1)^2 + qu_2 \tag{6.14}$$

one obtains the Lagrangian control system

$$m\ddot{q}(t) + k(q(t) - u_1(t)) - u_2(t) = 0 \tag{6.15}$$

This system has two inputs and one generalized coordinate. ■

Lagrangian control systems were derived first to treat mechanical control systems, as robots for instance, but they may also be derived for other types of systems like electrical circuits or electromechanical systems [243]. However for such systems the definition of the configuration space is no more based on some geometric configuration like for mechanical systems. The choice of the configuration variables is based on the definition of some potential functions associated with the different energies involved in the physical system. In particular for electrical circuits, the definition of Lagrangian systems describing their dynamical behaviour has led to numerous different definitions [47, 103]. Furthermore the Lagrangian formulation is in competition with two other formulations: the Brayton-Moser formulation and the Hamiltonian formulation which will be treated in the next section. Therefore we shall not present the different formulations of the dynamics of LC-circuits, but only present one of them as an example.

Example 6.10 (An LC circuit of order 3). Consider the LC circuit depicted in Figure 6.1.

We shall follow the procedure proposed by Chua and McPherson [103], in order to establish a Lagrangian formulation of its dynamical behaviour. **The first step** consists in defining the space of *generalized velocities*. One

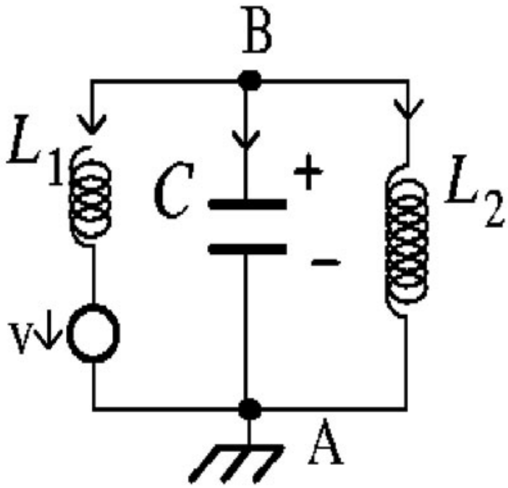

Fig. 6.1. LC circuit

considers a maximal tree in the circuit graph (called spanning tree) that is a maximal set of edges without loops, and that furthermore contains a maximal number of capacitors. The generalized velocities are then defined as the vector of voltages of the capacitors in the tree and currents in the inductors in the co-tree. Denoting the edges by the element which they connect, the circuit may be partitioned into the spanning tree: $\Gamma = \Gamma_1 \cup \Gamma_2 = \{C\} \cup \{S_u\}$ and its cotree: $\Lambda = \Lambda_1 \cup \Lambda_2 = \{L_1\} \cup \{L_2\}$. Hence one may choose as vector of generalized velocities the voltages of the capacitors in the tree Γ_1 and the currents of the inductors in the cotree Λ_2:

$$\dot{q} = \begin{pmatrix} v_C \\ i_{L_2} \end{pmatrix} \tag{6.16}$$

where v_C denotes the voltage at the port of the capacitor and i_{L_2} denotes the current in the inductor labeled L_2. The vector of generalized coordinates is hence obtained by integration of the vector of generalized velocities:

$$q = \begin{pmatrix} \phi_C \\ Q_{L_2} \end{pmatrix} \tag{6.17}$$

Note that this definition of the variables is somewhat unnatural as it amounts to associate flux-type variables with capacitors and charge-like variables with inductors (see the discussions in [344, 482]). **The second step** consists in the definition of the Lagrangian function which describes both the electro-magnetic energy of the circuit and the Kirchhoff's laws. The Lagrangian function is constructed as the sum of four terms:

$$L(q, \dot{q}, u) = \hat{\mathcal{E}}(\dot{q}) - \mathcal{E}(q) + \mathcal{C}(q, \dot{q}) + \mathcal{I}(q, u) \tag{6.18}$$

The function $\hat{\mathcal{E}}(\dot{q})$ is the sum of the electric coenergy of the capacitors in the tree Γ_1 and the magnetic coenergy of the inductors in the cotree Λ_2 which is, in this example, in the case of linear elements:

$$\hat{\mathcal{E}}(\dot{q}) = \frac{1}{2} C {v_C}^2 + \frac{1}{2} L_2 {i_{L_2}}^2 = \frac{1}{2} C \dot{q}_1^2 + \frac{1}{2} L_2 \dot{q}_2^2 \tag{6.19}$$

The function $\mathcal{E}(q)$ is the sum of the magnetic energy of the inductors in the cotree Λ_1 and the electric energy of the capacitors in the tree Γ_2 which is

$$\mathcal{E}(q) = \frac{1}{2L_1} {\phi_{L_1}}^2 = \frac{1}{2L_1} (q_1 + q_{10})^2 \tag{6.20}$$

where the relation between the flux ϕ_{L_1} of the inductor L_1 was obtained by integrating the Kirchhoff's mesh law on the mesh consisting of the capacitor C and the inductor L_1 yielding $\phi_{L_1} = (q_1 + q_{10})$ and q_{10} denotes some real constant which may be chosen to be null. The function $\mathcal{C}(q, \dot{q})$ accounts for the coupling between the capacitors in the tree Γ_1 and inductors in the cotree Λ_2 depending on the topological interconnection between them and is

$$\mathcal{C}(q, \dot{q}) = i_{L_2} \phi_C = \dot{q}_2 q_1 \tag{6.21}$$

The function $\mathcal{I}(q, u)$ is an interaction potential function describing the action of the source element and is:

$$\mathcal{I}(q, u) = q_{L_2} u = q_2 u \tag{6.22}$$

The Lagrangian control system is then :

$$C \ddot{q}_1(t) - \dot{q}_2(t) + \frac{1}{L_1} (q_1(t) + q_{10}) = 0 \tag{6.23}$$

$$L_2 \ddot{q}_2(t) + \dot{q}_1(t) - u(t) = 0 \tag{6.24}$$

Note that this system is of order 4 (it has 2 generalized coordinates) which does not correspond to the order of the electrical circuit which, by topological inspection, would be 3; indeed one may choose a maximal tree containing the capacitor and having a cotree containing the 2 inductors. We shall come back to this remark and expand it in the sequel when we shall treat the same example as a port controlled Hamiltonian system. ■

This example illustrates that, although the derivation of Lagrangian system is based on the determination of some energy functions and other physical properties of the system, its structure may not agree with the physical insight. Indeed the Lagrangian control systems are defined on the state space TQ, the tangent space to the configuration space. This state space has a very special structure; it is endowed with a symplectic form which is used to give an intrinsic definition of Lagrangian systems [294]. A very simple property of this state space is that its dimension is even (there are as many generalized coordinates as generalized velocities). Already this property may be in contradiction with the physical structure of the system.

Lagrangian control systems, in the same way as the Lagrangian systems with external forces, satisfy, by construction, a power balance equation and losslessness passivity property [68].

Lemma 6.11 (Lossless Lagrangian control systems). *A Lagrangian control system, (Definition 6.7), satisfies the following* power balance equation

$$u^T z = \frac{dE}{dt} \tag{6.25}$$

where

$$z_i = -\sum_{i=1}^{n} \frac{\partial^2 H}{\partial q_j \partial u_i} \frac{\partial H}{\partial p_j} + \sum_{i=1}^{n} \frac{\partial^2 H}{\partial p_j \partial u_i} \frac{\partial H}{\partial q_j} \tag{6.26}$$

and the real function E is obtained by the Legendre transformation *of the Lagrangian function $L(q, \dot{q})$ with respect to the generalized velocity $\dot{q}$ and the inputs and is defined by:*

$$E(q, p, u) = H(q, p, u) - u^T \frac{\partial H}{\partial u} \tag{6.27}$$

with

$$H(q, p, u) = \dot{q}^T p - L(q, \dot{q}, u) \tag{6.28}$$

where p is the vector of generalized momenta

$$p(q, \dot{q}, u) = \left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) \tag{6.29}$$

and the Lagrangian function is assumed to be hyperregular *[294] in such a way that the map from the generalized velocities $\dot{q}$ to the generalized momenta p is bijective for any u.*

If moreover the Hamiltonian (6.28) is affine in the inputs (hence the function E is independent of the inputs), the controlled Lagrangian system will be called affine Lagrangian control system*. And assuming that $E(q, p)$ is bounded from below, then the Lagrangian system with external forces is lossless with respect to the supply rate $u^T z$ with storage function $E(q, p)$.* ■

As we have seen above, the affine Lagrangian control systems are lossless with respect to the storage function $E(q, p)$ which in physical systems may be chosen to be equal to the internal energy of the system. However, in numerous systems, dissipation has to be included. For instance for robotic manipulator, the dissipation will be due to the friction at the joints and in the actuators. This may be done by modifying the definition of Lagrangian control systems and including dissipating forces as follows:

Definition 6.12 (Lagrangian control system with dissipation). *Consider a* configuration space $Q = \mathbb{R}^n$ *and its tangent space* $TQ = \mathbb{R}^{2n}$, *an input vector space* $\mathcal{U} = \mathbb{R}^p$. *A* Lagrangian control systems with dissipation *is defined by a Lagrangian function* $L(q, \dot{q}, u)$ *from* $TQ \times \mathcal{U}$ *to* $\mathbb{R}$, *a function* $R(\dot{q})$ *from* TQ *to* $\mathbb{R}$, *called* Rayleigh dissipation function *and which satisfies*

$$\dot{q}^T \frac{\partial R}{\partial \dot{q}}(\dot{q}) \geq 0 \tag{6.30}$$

and the equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}, u)\right) - \frac{\partial L}{\partial q}(q, \dot{q}, u) + \frac{\partial R}{\partial \dot{q}} = u \tag{6.31}$$

■

Example 6.13. Consider the example of the vertical motion of a magnetically levitated ball. There are three types of energy involved: the magnetic energy, the kinetic energy of the ball and its potential energy. The vector of generalized coordinates may be chosen as a vector in $\mathbb{R}^2$ where q_1 denotes a primitive of the current in the inductor (according to the procedure described in Example 6.10); $q_2 = z$ is the altitude of the sphere. The Lagrangian function may then be chosen as the sum of three terms:

$$L(q, \dot{q}, u) = \hat{\mathcal{E}}_m\left(q, \dot{q}\right) + \hat{\mathcal{E}}_k\left(\dot{q}\right) - \mathcal{U}\left(q\right) + \mathcal{I}\left(q, u\right) \tag{6.32}$$

The function $\hat{\mathcal{E}}_m\left(q, \dot{q}\right)$ is the magnetic coenergy of the inductor and depends on the currents in the coil as well on the altitude of the sphere:

$$\hat{\mathcal{E}}_m\left(q, \dot{q}\right) = \frac{1}{2} L\left(q_2\right) \dot{q}_1^2 \tag{6.33}$$

where

$$L\left(q_2\right) = L_0 + \frac{k}{q_2 - z_0} \tag{6.34}$$

The function $\hat{\mathcal{E}}_k\left(\dot{q}\right)$ is the kinetic coenergy of the ball

$$\hat{\mathcal{E}}_k\left(\dot{q}\right) = \frac{1}{2} m \dot{q}_2^2 \tag{6.35}$$

The function $\mathcal{U}\left(q\right)$ denotes the potential energy due to the gravity

$$\mathcal{U}\left(q\right) = g q_2 \tag{6.36}$$

The interaction potential is

$$\mathcal{I}\left(q, u\right) = q_1 u \tag{6.37}$$

In order to take into account the dissipation represented by the resistor R, one also define the following Rayleigh potential function:

$$\mathcal{R}(\dot{q}) = \frac{1}{2} R \dot{q}_1^2 \tag{6.38}$$

This leads to the following Lagrangian control system with dissipation:

$$L(q_2(t))\,\ddot{q}_1(t) + \frac{\partial L}{\partial q_2}(q_2(t))\,\dot{q}_2(t)\dot{q}_1(t) + R\dot{q}_1(t) - u(t) = 0 \tag{6.39}$$

$$m\ddot{q}_2(t) - \frac{1}{2}\frac{\partial L}{\partial q_2}(q_2(t))\,\dot{q}_1^2(t) + g = 0 \tag{6.40}$$

6.1.2 Simple Mechanical Systems

An important subclass of Lagrangian contol systems is given by the so-called simple mechanical systems where the Lagrangian function takes a particular form.

Definition 6.14 (Simple mechanical systems with external forces). *The Lagrangian system for a simple mechanical system is a Lagrangian system with external forces according to Definition 6.1 with Lagrangian function:*

$$L(q, \dot{q}) = T(q, \dot{q}) - U(q) \tag{6.41}$$

where $U(q)$ is a real function from the configuration space Q on $\mathbb{R}$ and is called potential energy *and $T(q, \dot{q})$ is a real function from TQ on $\mathbb{R}$, called* kinetic energy *and is defined by*

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} \tag{6.42}$$

where the matrix $M(q) \in \mathbb{R}^{n\times n}$ is symmetric positive definite and is called the inertia matrix. ■

Considering the special form of the Lagrangian function, the Lagrangian equations (6.1) may be written in some special form which is particularly useful for deriving stabilizing controllers as will be presented in the subsequent chapters.

Lemma 6.15 (Lagrangian equations for simple mechanical systems). *The Lagrangian equations (6.1) for a simple mechanical system may be written*

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) = F(t) \tag{6.43}$$

where $g(q) = \frac{dU}{dq}(q) \in \mathbb{R}^n$,

$$C(q, \dot{q}) = \sum_{k=1}^{n} \Gamma_{ijk}\dot{q}_k \tag{6.44}$$

and Γ_{ijk} are called the Christoffel's symbols *associated with the inertia matrix $M(q)$ and are defined by*

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i}\right) \tag{6.45}$$

■

A property of Christoffel's symbols which is easily derived but is of great importance for the derivation of stabilizing control laws, is given below. What is denoted as $\dot{M}(q)$ is the time derivative of the time function $M(q(t))$.

Lemma 6.16. *The Christoffel's symbols (6.45) satisfy the following property: the matrix $\dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric. Equivalently $\dot{M}(q) = C(q,\dot{q}) + C^T(q,\dot{q})$.* ■

The notation $\dot{M}(q)$ is, without any dependence on time in $M(q)$, a little meaningless. Its meaning is that $\dot{M}(q) = \left(\frac{\partial M_{ij}}{\partial q}\dot{q}\right)_{i,j}$, so that $\dot{M}(q(t)) = \frac{d}{dt}M(q(t))$. Let us note that an immediate consequence is that $\dot{M}(q) = C(q,\dot{q}) + C^T(q,\dot{q})$.

Remark 6.17. A consequence of the Lemma is that:

$$\dot{q}^T\left(\dot{M}(q) - 2C(q,\dot{q})\right)\dot{q} = 0 \tag{6.46}$$

and hence reflects that the generalized inertial forces $\left[\dot{M}(q) - 2C(q,\dot{q})\right]\dot{q}$ do not work. This may be seen as follows:

$$\begin{aligned}
\tau^T\dot{q} = \tfrac{dH}{dt}(q,p) &= \dot{q}^T M(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + g(q) \\
&= \dot{q}^T\left[-C(q,\dot{q})\dot{q} - g(q) + \tau\right] + \tfrac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + g(q) \\
&= \dot{q}^T\tau + \tfrac{1}{2}\dot{q}^T\left[\dot{M}(q) - 2C(q,\dot{q})\right]\dot{q}
\end{aligned} \tag{6.47}$$

from which (6.46) follows. Such forces are sometimes called *gyroscopic* [351]. It is noteworthy that (6.46) does not mean that the matrix $\dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric. Skew-symmetry is true only for the particular definition of the matrix $C(q,\dot{q})$ using Christoffel's symbols.

Remark 6.18. The definition of a positive definite symmetric inertia matrix for simple mechanical systems, may be expressed in some coordinate independent way by using so-called *Riemannian manifolds* [1]. In [442, Chapter 4] the properties of the Christoffell's symbols, that shall be used in the sequel for the synthesis of stabilizing controllers, may also be related to properties of Riemannian manifolds.

A class of systems which typically may be represented in this formulation is the dynamics of multibody systems, for which systematic derivation procedures were obtained (see [366] and the references herein).

6.2 Hamiltonian Control Systems

There is an alternative to the Lagrangian formulation of the dynamics of physical controlled systems that is the Hamiltonian formalism. This formalism has been derived from the Lagrangian one in the end of the nineteenth century and has now become the fundamental structure of the mathematical description of physical systems [1, 294]. In particular it allowed one to deal with symmetry and reduction and also to describe the extension of classical mechanics to quantum mechanics.

6.2.1 Input-output Hamiltonian Systems

Lagrangian systems may be transformed to standard Hamiltonian systems by using the Legendre transformation [1, 294].

Lemma 6.19 (Legendre transformation of a Lagrangian system). *Consider a Lagrangian system with external forces and define the vector of* generalized momenta*:*

$$p(q,\dot{q}) = \left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) \in I\!R^n \tag{6.48}$$

Assume that the map from generalized velocities to generalized momenta is invertible, and consider the Legendre transformation with respect to $\dot{q}$ *of the Lagrangian function, called* Hamiltonian function*:*

$$H_0(q,p) = \dot{q}^T p - L(q,\dot{q}) \tag{6.49}$$

Then the Lagrangian system with external forces is equivalent to the following standard Hamiltonian system*:*

$$\begin{aligned} \dot{q}(t) &= \tfrac{\partial H_0}{\partial p}(q(t),p(t)) \\ \dot{p}(t) &= -\tfrac{\partial H_0}{\partial q}(q(t),p(t)) + F(t) \end{aligned} \tag{6.50}$$

■

There is an alternative way of writing these equations as follows (we drop the arguments):

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J_s \begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \frac{\partial H_0}{\partial p} \end{pmatrix} + \begin{pmatrix} 0_n \\ I_n \end{pmatrix} F \tag{6.51}$$

where J_s is the following matrix, called *symplectic matrix*:

$$J_s = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \tag{6.52}$$

This symplectic matrix is the local representation, in canonical coordinates, of the symplectic Poisson tensor field which defines the geometric structure of the state space of standard Hamiltonian systems (the interested reader may find an precise exposition to symplectic geometry in [294].)

In the same way as a Lagrangian system with external forces may be expressed as a control Lagrangian system (for which the inputs are an argument of the Lagrangian function), the standard Hamiltonian system with external forces (6.51) may be expressed as Hamiltonian system where the Hamiltonian function depends on the inputs $H(q,p,u) = H_0(q,p) - q^T F$ which yields

$$\begin{pmatrix} \dot{q} \\ \\ \dot{p} \end{pmatrix} = J_s \left(\begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \\ \frac{\partial H_0}{\partial p} \end{pmatrix} + \begin{pmatrix} -I_n \\ \\ 0_n \end{pmatrix} F \right) = J_s \begin{pmatrix} \frac{\partial H_0}{\partial q} \\ \\ \frac{\partial H_0}{\partial p} \end{pmatrix} + \begin{pmatrix} 0_n \\ \\ I_n \end{pmatrix} F \tag{6.53}$$

As the simplest example let us consider the harmonic oscillator with an external force.

Example 6.20 (Harmonic oscillator with external force). First let us recall that in its Lagrangian representation (see Example 6.3), the state space is given by the position of the mass (with respect to the fixed frame) and its velocity. Its Lagrangian is $L(q, \dot{q}, F) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 + q^T F$. Hence the (generalized) momentum is $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$. The Hamiltonian function, obtained through the Legendre transformation is $H(q,p,F) = H_0(q,p) - q^T F$ where the Hamiltonian function $H_0(q,p)$ represents the total internal energy $H_0(q,p) = K(p) + U(q)$, the sum of the kinetic energy $K(p) = \frac{1}{2}\frac{p^2}{m}$ and the potential energy $U(q)$. The Hamiltonian system becomes

$$\begin{pmatrix} \dot{q}(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} kq(t) \\ \frac{p(t)}{m} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(t) \tag{6.54}$$

■

Hamiltonian systems with external forces may be generalized to so-called *input-output Hamiltonian systems* [68] for which the Hamiltonian function depends on the inputs. In the sequel we shall restrict ourselves to systems for which the Hamiltonian function depends linearly on the inputs, which actually constitute the basis of the major part of the work dedicated to the system theoretic analysis and the control of Hamiltonian systems [68, 381, 439].

Definition 6.21 (Input-output Hamiltonian systems). *An* input-output Hamiltonian system *on* $\mathbb{R}^{2n}$ *is defined by a Hamiltonian function*

$$H(x) = H_0(x) - \sum_{i=1}^{m} H_i(x)\, u_i \tag{6.55}$$

composed of the sum of the internal Hamiltonian $H_0(x)$ *and a linear combination of* m interaction Hamiltonian functions $H_i(x)$ *and the dynamic equations*

$$\begin{cases} \dot{x} = J_s dH_0(x) + \sum_{i=1}^m J_s dH_i(x) u_i \\ \tilde{y}_i = H_i(x),\ i = 1, .., m \end{cases} \tag{6.56}$$

denoting the state by $x^T = (q^T, p^T) \in I\!R^{2n}$ *and the gradient of a function* H *by* $dH = \frac{dH}{dx} \in I\!R^{2n}$. ■

One may note that an input-output Hamiltonian system (6.56) is a nonlinear system affine in the inputs in the sense of [227, 381]. It is composed of a Hamiltonian drift vector field $J_s dH_0(q, p)$ and the input vector fields $J_s dH_i(q, p)$ are also Hamiltonian and generated by the interaction Hamiltonian functions.

The outputs are the Hamiltonian interaction functions and are called *natural outputs* [68]. We may already note here that these outputs, although called "natural", are not the outputs conjugated to the inputs for which the system is passive as will be shown in the sequel.

Example 6.22. Consider again Example 6.9. The state space is given by the displacement of the spring and its velocity. Its Lagrangian is

$$L(q, \dot{q}, F) = \frac{1}{2} m(\dot{q} + u_1)^2 - \frac{1}{2} k q^2 + q u_2 \tag{6.57}$$

Hence the generalized momentum is: $p = \frac{\partial L}{\partial \dot{q}} = m(\dot{q} + u_1)$ The Hamiltonian function, obtained through the Legendre transformation with respect to $\dot{q}$ is

$$H(q, p, u_1, u_2) = \dot{q}^T p - L(q, \dot{q}, u_1, u_2) = H_0(q, p) - p u_1 - q u_2 \tag{6.58}$$

where the Hamiltonian function $H_0(q, p) = \frac{1}{2} \frac{p^2}{m} + \frac{1}{2} k q^2$ represents, as in the preceding example, the sum of the kinetic and the elastic potential energy. The interaction potentials are the momentum of the mass $H_1(q, p) = p$, for the input u_1 which represents the controlled velocity of the basis and the the displacement of the spring $H_2(q, p) = q$ for the input u_2 which is the external force exerted on the mass. The dynamics is now described by the following input-output Hamiltonian system:

$$\begin{pmatrix} \dot{q}(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} kq(t) \\ \frac{p(t)}{m} \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u_1(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2(t) \tag{6.59}$$

■

Note that the definition of the generalized momentum p corresponds to a generalized state space transformation involving the input u_1. Consequently in the Hamiltonian formulation (6.59) the derivative of the input no longer appears, contrary to the Lagrangian dynamics in (6.15). Moreover, like affine Lagrangian control systems, input-output Hamiltonian systems satisfy a power balance equation, however considering, instead of the natural outputs $\tilde{y}_i$ (6.56), their derivatives.

Lemma 6.23 (Lossless input-output Hamiltonian systems). *An input-output Hamiltonian system (according to Definition 6.21), satisfies the following* power balance equation:

$$u^T \dot{\tilde{y}} = \frac{dH_0}{dt} \tag{6.60}$$

If, moreover, the Hamiltonian function $H_0(x)$ is bounded from below, then the input-output Hamiltonian system is lossless with respect to the supply rate: $u^T \dot{\tilde{y}}$ with storage function $H_0(q,p)$. ■

Let us comment on this power balance equation using the example of the harmonic oscillator with moving frame and continue Example 6.22.

Example 6.24. The natural outputs are then the momentum of the system: $\tilde{y}_1 = H_1(q,p) = p$ which is conjugated to the input u_1 (the velocity of the basis of the spring) and the displacement of the spring $\tilde{y}_2 = H_2(q,p) = q$ which is conjugated to the input u_2 (the external force exerted on the mass). The passive outputs defining the supply rate are then

$$\dot{\tilde{y}}_1 = \dot{p} = -kq + u_2 \tag{6.61}$$

and

$$\dot{\tilde{y}}_2 = \dot{q} = \frac{p}{m} - u_1 \tag{6.62}$$

Computing the supply rate, the terms in the inputs cancel each other and one obtains

$$\dot{\tilde{y}}_1 u_1 + \dot{\tilde{y}}_2 u_2 = kqu_1 + u_2 \frac{p}{m} \tag{6.63}$$

This is precisely the sum of the mechanical power supplied to the mechanical system by the source of displacement at the basis of the spring and the source of force at the mass. This indeed is equal to the variation of the total energy of the mechanical system. However it may be noticed that the natural outputs as well as their derivatives are *not* the variables which one uses in order to define the interconnection of this system with some other mechanical system: the force at the basis of the spring which should be used to write a force balance equation at that point and the velocity of the mass m which should be used in order to write the kinematic interconnection of the mass (their dual variables are the input variables). In general, input- output Hamiltonian systems (or their Lagrangian counterpart) are not well suited for expressing their interconnection.

Example 6.25. Consider the LC circuit of order 3 in Example 6.10. In the Lagrangian formulation, the generalized velocities were $\dot{q}_1 = V_C$ the voltage of the capacitor, $\dot{q}_2 = i_{L_2}$ the current of the inductor L_2 and the generalized coordinates were some primitives denoted by $q_1 = \phi_C$ and $q_2 = Q_{L_2}$. The Lagrangian function was given by $L(q,\dot{q},u) = \hat{\mathcal{E}}(\dot{q}) - \mathcal{E}(q) + \mathcal{C}(q,\dot{q}) + \mathcal{I}(q,u)$ where $\hat{\mathcal{E}}(\dot{q})$ is the sum of the electric coenergy of the capacitor and of the inductor L_2, $\mathcal{E}(q)$ is the magnetic energy of the inductor L_1, $\mathcal{C}(q,\dot{q})$ is a coupling function between the capacitor and the inductor L_2 and $\mathcal{I}(q,u)$ is the interaction potential function.

Let us now define the generalized momenta. The first momentum variable is

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \frac{\partial \hat{\mathcal{E}}}{\partial \dot{q}_1} + \frac{\partial \hat{\mathcal{C}}}{\partial \dot{q}_1} = \frac{\partial \hat{\mathcal{E}}}{\partial \dot{q}_1} = C\dot{q}_1 = Q_C \tag{6.64}$$

and is the *electrical charge of the capacitor*, *i.e.* its energy variable. The second momentum variable is

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = \frac{\partial \hat{\mathcal{E}}}{\partial \dot{q}_2} + \frac{\partial \hat{\mathcal{C}}}{\partial \dot{q}_2} = L_2\dot{q}_2 + q_1 = \phi_{L_2} + \phi_C \tag{6.65}$$

and is *the sum* of the *the total magnetic flux of the inductor* L_2 (its energy variable) and of the fictitious flux at the capacitor ϕ_C. The Hamiltonian function is obtained as the Legendre transformation of $L(q,\dot{q},u)$ with respect to $\dot{q}$:

$$H(q,p,u) = \dot{q}_1 p_1 + \dot{q}_2 p_2 - L(q,\dot{q},u) = H_0(q,p) - H_i(q)u \tag{6.66}$$

where $H_i = q_2$ and H_0 is

$$H_0(q,p) = \frac{1}{2L_1}q_1^2 + \frac{1}{2C}p_1^2 + \frac{1}{2L_2}(p_2 - q_1)^2 \tag{6.67}$$

Note that the function $H_0(p,q)$ is the total electromagnetic energy of the circuit as the state variables are equal to the energy variables of the capacitors and inductors. Indeed using Kirchhoff's law on the mesh containing the inductor L_1 and the capacitor C, up to a constant $q_1 = \phi_C = \phi_{L_1}$ is the magnetic flux in the inductor, by definition of the momenta $p_1 = Q_C$ is the charge of the capacitor and $p_2 - q_1 = \phi_{L_2}$ is the magnetic flux of the inductor L_2. This input-output Hamiltonian system again has order 4 (and not the order of the circuit). But one may note that the Hamiltonian function H_0 does not depend on q_2. Hence it has a symmetry and the drift dynamics may be reduced to a third order system (the order of the circuit) and in a second step to a second order system [294]. However the interaction Hamiltonian depends on the symmetry variable q_2, so the *controlled* system may not be reduced to a lower order input-output Hamiltonian system. The power balance equation (6.60) becomes $\frac{dH_0}{dt} = u\dot{q}_2 = i_{L_2}u$ which is exactly the power delivered by the source as the current i_{L_2} is also the current flowing in the voltage source. ■

The preceding input-output Hamiltonian systems may be extended by considering more general structure matrices than the symplectic structure matrix J_s which appear in the reduction of Hamiltonian systems with symmetries [294]. Indeed one may consider so-called *Poisson structure matrices* that are matrices $J(x)$ depending on $x(t) \in I\!R^{2n}$, skew-symmetric and satisfying the *Jacobi identities*:

$$\sum_{k,l=1}^{n}\left(J_{lj}\frac{\partial J_{ik}}{\partial x_l}(x)+J_{li}(x)\frac{\partial J_{kj}}{\partial x_l}(x)+J_{lk}\frac{\partial J_{ji}}{\partial x_l}(x)\right)=0 \tag{6.68}$$

Remark 6.26. These structure matrices are the local definition of Poisson brackets defining the geometrical structure of the state-space [1,294] of Hamiltonian systems defined on differentiable manifold endowed with a Poisson bracket. Such systems appear for instance in the Hamiltonian formulation of a rigid body spinning around its center of mass (the Euler-Poinsot problem) [294].

Remark 6.27. Poisson structure matrices may be related to symplectic structure matrices as follows. Note first that, by its skew-symmetry, the rank of the structure matrix of a Poisson bracket at any point is even, say $2n$ (then one says also that the Poisson bracket has the rank $2n$). Suppose moreover that the structure matrix has constant rank $2n$ in a neighborhood of a point $x_0 \in M$. Then the Jacobi identities (6.68) ensure the existence of *canonical coordinates* $(q,p,r)=(q_1,..,q_n,p_1,..,p_n,r_1,..,r_l)$ where $(2n+l)=m$, such that the $m\times m$ structure matrix $J(q,p,r)$ is given as follows:

$$J(q,p,r)=\begin{pmatrix} 0_n & I_n & 0_{n\times l}\\ -I_n & 0_n & 0_{n\times l}\\ 0_{l\times n} & 0_{l\times n} & 0_{l\times l}\end{pmatrix} \tag{6.69}$$

One may hence see a symplectic matrix appear associated with the first $2n$ coordinates. The remaining coordinates correspond to so-called distinguished functions or Casimir functions which define an important class of dynamical invariants of the Hamiltonian system [294]. ■

With such structure matrices, the input-output Hamiltonian systems may be generalized to Poisson control systems as follows [381].

Definition 6.28 (Poisson control systems). *A* Poisson control system *on $\mathbb{R}^n$ is defined by a Poisson structure matrix $J(x)$, a Hamiltonian function $H(x)=H_0(x)-\sum_{i=1}^{m}H_i(x)\,u_i$ composed of the sum of the internal Hamiltonian $H_0(x)$ and a linear combination of m* interaction Hamiltonian functions *$H_i(x)$ and the dynamic equations:*

$$\dot{x}=J(x)dH_0(x)-\sum_{i=1}^{m}J(x)dH_i(x)\,u_i \tag{6.70}$$

■

6.2.2 Port Controlled Hamiltonian Systems

As the examples of the LC circuit and of the levitated ball have shown, although the input-output Hamiltonian systems represent the dynamics of physical systems in a way that the conservation of energy is embedded in the model, they fail to represent accurately some other of their structural properties. Therefore another type of Hamiltonian systems, called *port controlled Hamiltonian systems* was introduced which allow to represent both the energy conservation as well as some other structural properties of physical systems, mainly related to their internal interconnection structure [342, 442].

Definition 6.29 (Port controlled Hamiltonian system). *A* port controlled Hamiltonian system *on $\mathbb{R}^n$ is defined by a skew-symmetric structure matrix $J(x)$, a real-valued Hamiltonian function $H_0(x)$, m input vector fields $g_i(x)$ and the dynamic equations*

$$\begin{cases} \dot{x} = J(x)dH_0(x) + \sum_{i=1}^{m} g_i(x)U_i \\ \\ y_i = g_i^T(x)dH_0(x) \end{cases} \tag{6.71}$$

■

One may note that port controlled Hamiltonian system, as the input output Hamiltonian systems, are affine with respect to the inputs [227, 381].

Remark 6.30. The system-theoretic properties of port controlled Hamiltonian systems were investigated in particular concerning the external equivalence, but as this subject goes beyond the scope of this book, the reader is referred to [441] [442, Chapter 4].

The systems (6.71) have been called *port controlled Hamiltonian system* in allusion to the network concept of the interaction through ports [342,441,442]. In this case the Hamiltonian function corresponds to the internal energy of the system, the structure matrix corresponds to the interconnection structure associated with the energy flows in the system [343–345] and the interaction with the environment of the network is defined through pairs of port variables [342, 441]. Moreover, the underlying modeling formalism is a network formalism which provides a practical frame to construct models of physical systems and roots on a firmly established tradition in engineering [62] which found its achievement in the bond graph formalism [63, 342, 398].

Port controlled Hamiltonian systems differ from input-output Hamiltonian systems in three ways which we shall illustrate below on some examples. First, the structure matrix $J(x)$ does not have to satisfy the Jacobi identities (6.68); such structure matrices indeed arise in the reduction of simple mechanical systems with non-holonomic constraints [440]. Second the input vector fields are no more necessarily Hamiltonian, that is they may not derive from an interaction potential function. Third, the definition of the output is changed. The most simple examples of port controlled Hamiltonian system consist in elementary energy storing systems, corresponding for instance to a linear spring or a capacitor.

Example 6.31 (Elementary energy storing systems). Consider the following first order port controlled Hamiltonian system:

$$\begin{cases} \dot{x}(t) = u(t) \\ \\ y = \frac{dH_0}{dx}(x) \end{cases} \tag{6.72}$$

where $x(t) \in I\!R^n$ is the state variable, $H_0(x)$ is the Hamiltonian function and the structure matrix is equal to 0. In the scalar case, this system represents the *integrator* which is obtained by choosing the Hamiltonian function to be: $H_0 = \frac{1}{2}\, x^2$. This system represents also a *linear spring* , where the state variable $x(\cdot)$ is the displacement of the spring and the energy function is the elastic potential energy of the spring (for instance $H(x) = \frac{1}{2}\, k\, q^2$ where k is the stiffness of the spring). In the same way (6.72) represents a *capacitor* with x being the charge and H_0 the electrical energy stored in the capacitor, or an *inductance* where x is the total magnetic flux and H_0 is the magnetic energy stored in the inductance.

In $I\!R^3$ such a system represents the point mass in the three-dimensional Euclidean space with mass m where the state variable $x(t) \in I\!R^3$ is the momentum vector, the input $u \in I\!R^3$ is the vector of forces applied on the mass, the output vector $y(t) \in I\!R^3$ is the velocity vector and the Hamiltonian function is the kinetic energy $H_0(x) = \frac{1}{2m} x^T x$.

It may be noted that such elementary systems may take more involved forms when the state variable belongs to some manifold different from $I\!R^n$, as it is the case for instance for spatial springs which deform according to rigid body displacements [143, 144, 308, 345]. ■

Like affine Lagrangian control systems and input-output Hamiltonian systems, port controlled Hamiltonian systems satisfy a power balance equation and under some assumption on the Hamiltonian function are lossless.

Lemma 6.32 (Losslessness of port controlled Hamiltonian systems). *A port controlled Hamiltonian system (according to Definition 6.29), satisfies the following* power balance equation:

$$u^T y = \frac{dH_0}{dt} \tag{6.73}$$

If moreover the Hamiltonian function $H_0(x)$ is bounded from below, then the port controlled Hamiltonian system is lossless with respect to the supply rate $u^T y$ with storage function $H_0(x)$. ■

Again in the case when the Hamiltonian function is the energy, the balance equation corresponds to a power balance expressing the conservation of energy. Let us now consider a slightly more involved example, the LC circuit of order 3 treated here above, in order to comment on the structure of port controlled Hamiltonian sytems as well as to compare it to the structure of input output and Poisson control systems.

Example 6.33 (LC circuit of order 3). Consider again the circuit of Example 6.10. According to the partition of the interconnection graph into the spanning tree: $\Gamma = \{C\} \cup \{S_u\}$ and its cotree: $\Lambda = \{L_1\} \cup \{L_2\}$, one may write Kirchhoff's mesh law for the meshes defined by the edges in Λ and the node law corresponding to the edges in Γ as follows:

$$\begin{pmatrix} i_C \\ v_{L_1} \\ v_{L_2} \\ -i_S \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_C \\ i_{L_1} \\ i_{L_2} \\ v_S \end{pmatrix} \tag{6.74}$$

Now, taking as state variables the energy variables of the capacitor (the charge Q_C, the total magnetic fluxes ϕ_{L_1} and ϕ_{L_2} in the two inductors) one identifies immediately the first three components of the left hand side in (6.74) as the time derivative of the state vector $x = (Q_C, \phi_{L_1}, \phi_{L_2})^T$. Denoting by $H_C(Q_C)$, $H_{L_1}(\phi_{L_1})$ and $H_{L_2}(\phi_{L_2})$ the electric and magnetic energies stored in the elements, one may identify the coenergy variables as follows: $v_C = \frac{\partial H_C}{\partial Q_C}$, $i_{L_1} = \frac{\partial H_{L_1}}{\partial \phi_{L_1}}$ and $i_{L_2} = \frac{\partial H_{L_2}}{\partial \phi_{L_2}}$. Hence the first three components of the vector on the right hand side of Equation (6.74) may be interpreted as the components of the gradient of the total electromagnetic energy of the LC circuit $H_0(x) = H_C(Q_C) + H_{L_1}(\phi_{L_1}) + H_{L_2}(\phi_{L_2})$. Hence the dynamics of the LC circuit may be written as the following port controlled Hamiltonian system:

$$\begin{cases} \dot{x}(t) = JdH_0(x(t)) + gu(t) \\ \\ y = g^T dH_0(x) \end{cases} \tag{6.75}$$

where the structure matrix J and the input vector g are part of the matrix describing Kirchhoff's laws in (6.74) (*i.e.* part of the fundamental loop matrix associated with the tree Γ):

$$J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.76}$$

The input is $u = v_S$ and the output is the current with generator sign convention $y = -i_S$. In this example the power balance equation (6.73) is simply interpreted as the time derivative of the total electromagnetic energy being the power supplied by the source. Actually this formulation is completely general to LC circuits and it may be found in [344] as well as the comparison with the formulation in terms of Lagrangian or input-output Hamiltonian systems [47, 344].

The port controlled Hamiltonian formulation of the dynamics of the LC circuit may be compared with the input-output formulation derived in the Example 6.25. First, one may notice that in the port controlled Hamiltonian formulation, the information on the topology of the circuit and the information about the elements (*i.e.* the energy) is represented in two *different* objects: the structure matrix and the input vector on the one side and the Hamiltonian function on the other side. In the input-output Hamiltonian formulation this information is captured solely in the Hamiltonian function (with interaction potential), in the same way as in the Lagrangian formulation in Example 6.10. Second, the port controlled Hamiltonian system is defined with respect to a non-symplectic structure matrix and its order coincides with the order of the circuit, whereas the input-output system is given (by definition) with respect to a symplectic (even order) structure matrix of order larger than the order of the circuit. Third, the definition of the state variables in the port controlled system corresponds simply to the energy variables of the different elements of the circuit whereas in the input-output Hamiltonian system, they are defined for the total circuit and for instance the flux of capacitor L_2 does not appear as one of them. Finally, although the two structure matrices of the port controlled and the input output Hamiltonian systems may be related by projection of the dynamics using the symmetry in q_2 of the input output Hamiltonian system, the *controlled* systems remain distinct. Indeed, consider the input vector g; it is clear that it is not in the image of the structure matrix J. Hence there exist no interaction potential function which generates this vector and the port controlled Hamiltonian formulation *cannot* be formulated as an input output Hamiltonian system or Poisson control system. ■

In order to illustrate a case where the energy function defines some interdomain coupling, let us consider the example of the iron ball in magnetic levitation. This example may be seen as the one-dimensional case of general electromechanical coupling arising in electrical motors or actuated multibody systems.

Example 6.34. Consider again the example of the vertical motion of a magnetically levitated ball as treated in Example 6.13. Following a bond graph modeling approach, one defines the state space as being the variables defining the energy of the system. Here the state vector is then $x = (\phi, z, p_b)^T$ where ϕ is the magnetic flux in the coil, z is the altitude of the sphere and p_b is the kinetic momentum of the ball. The total energy of the system is composed of three terms: $H_0(x) = H_{mg}(\phi, z) + \mathcal{U}(z) + H_{kin}(p_b)$ where $H_{mg}(\phi, z)$ denotes the magnetic energy of the coil and is

$$H_{mg}(\phi, z) = \frac{1}{2}\frac{1}{L(z)}\phi^2 \tag{6.77}$$

where $L(z)$ is given in (6.34), $\mathcal{U}(z) = gz$ is the gravitational potential energy and $H_{kin}(p_b) = \frac{1}{2m}p^2$ is the kinetic energy of the ball. Hence the gradient of the energy function H_0 is the vector of the coenergy variables: $\frac{\partial H_0}{\partial x} = (v_L, f, v_b)$ where v_L is the voltage at the coil:

$$v_L = \frac{\partial H_{mg}}{\partial \phi} = \frac{\phi}{L(z)} \tag{6.78}$$

The sum of the gravity force and the electromagnetic force is given by $f = g - f_{mg}$:

$$f_{mg} = \frac{1}{2} \frac{\phi^2}{L^2(z)} \frac{\partial L}{\partial z}(z) \tag{6.79}$$

and $v_b = \frac{p_b}{m}$ is the velocity of the ball. Then from Kirchhoff's laws and the kinematic and static relations in the system, it follows that the dynamics may be expressed as a port controlled Hamiltonian system (6.71) where the structure matrix is constant:

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{6.80}$$

and the input vector is constant:

$$g = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{6.81}$$

Note that the structure matrix is already in canonical form. In order to take into account the dissipation represented by the resistor R, one also defines the following dissipating force $v_R = -Ri_R = -Ri_L$ which may be expressed in a Hamiltonian-like format as a *Hamiltonian-system with dissipation* [121].

Let us compare now the port controlled Hamiltonian formulation with the Lagrangian or input output Hamiltonian formulation. Therefore recall first the input output Hamiltonian system obtained by the Legendre transformation of the Lagrangian system of Example 6.13. The vector of the momenta is

$$p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) = \begin{pmatrix} \phi \\ p_b \end{pmatrix} \tag{6.82}$$

and the Hamiltonian function obtained by Legendre transformation of the Lagrangian function, defined in Example 6.13, is

$$H(q, p) = H_0(x) - q_1 u \tag{6.83}$$

Hence the state space of the input-output representation is the state space of the port controlled system augmented with the variable q_1 (the primitive if the current in the inductor). Hence the order of the input output Hamiltonian system is 4 and larger than 3, the *natural* order of the system (a second order mechanical system coupled with a first order electrical circuit), which is precisely the order of the port controlled Hamiltonian system. Moreover the state variable "in excess" is q_1 and is precisely the symmetry variable of the internal Hamiltonian function $H_0(x)$ in $H(q, p)$. In an analogous way as in the LC circuit example above, this symmetry variable defines the interaction Hamiltonian, hence the *controlled* input-output Hamiltonian system may not be reduced. And again one may notice that the input vector g does not belong to the image of the structure matrix J, hence cannot be generated by any interaction potential function. ■

Now we shall compare the definitions of the outputs for input-output Hamiltonian or Poisson control systems and port controlled Hamiltonian systems. Consider the port controlled system (6.71) and assume that the input vector fields are Hamiltonian, *i.e.* there exists interaction Hamiltonian functions such that $g_i(x) = J(x)dH_i(x)$. The port conjugated outputs are then $y_i = dH_0^T(x)g_i(x) = dH_0^T(x)J(x)dH_i(x)$. The natural outputs are $\tilde{y}_i = H_i(x)$. Using the drift dynamics in (6.71), their derivatives are computed as

$$\dot{\tilde{y}}_i = dH_i^T(x)\dot{x} = y_i + \sum_{j=1, j\neq i}^{m} u_j dH_i^T(x)J(x)dH_j(x) \tag{6.84}$$

Hence the passive outputs of both systems differ, in general, by some skew symmetric terms in the inputs. This is related to the two versions of the Kalman-Yakubovich-Popov Lemma where the output includes or not a skew symmetric feedthrough term.

Example 6.35 (Mass-spring system with moving basis). Consider again the mass-spring system with moving basis and its input-output model treated in Examples 6.22 and 6.24. The input vector fields are Hamiltonian, hence we may compare the definition of the passive outputs in the input-output Hamiltonian formalism and in the port controlled Hamiltonian formalism. The derivatives of the natural outputs derived in Example 6.24 are $\dot{\tilde{y}}_2 = \dot{q} = \frac{p}{m} - u_1$ and $\dot{\tilde{y}}_1 u_1 + \dot{\tilde{y}}_2 u_2 = u_1(kq) + u_2\frac{p}{m}$. The port conjugated outputs are $y_1 = (-1, 0)\begin{pmatrix} kq \\ \frac{p}{m} \end{pmatrix} = -kq$ and $y_2 = (0, 1)\begin{pmatrix} kq \\ \frac{p}{m} \end{pmatrix} = \frac{p}{m}$. These outputs, contrary to the natural outputs and their derivatives, are precisely the interconnection variables needed to write the kinematic and static relation for interconnecting this mass-spring system to some other mechanical systems. ∎

The mass-spring example shows how the different definitions of the pairs of input-output variables for input-output and port controlled Hamiltonian systems, although both defining a supply rate for the energy function as storage function, are fundamentally different with respect to the interconnection of the system with its environment. One may step further and investigate the *interconnection* of Hamiltonian and Lagrangian systems which preserve their structure. It was shown that the port controlled Hamiltonian systems may be interconnected in a structure preserving way by so-called *power continuous interconnections* [121, 346]. Therefore a generalization of port controlled Hamiltonian systems to *implicit* port controlled Hamiltonian systems (encompassing constrained systems) was used in [121,346,441,442]. However this topic is beyond the scope of this section and we shall only discuss the interconnection of Lagrangian and Hamiltonian systems on the example of the ball in magnetic levitation.

Example 6.36 (Levitated ball as the interconnection of two subsystems). We have seen that the dynamics of the levitated ball may be formulated as a third order port controlled Hamiltonian system where the coupling between the potential and kinetic energy is expressed in the structure matrix (the symplectic coupling) and the coupling through the electromagnetic energy in the Hamiltonian function. However it also allows one to express this system as the coupling, through a passivity preserving interconnection, of two port controlled Hamiltonian systems. Therefore one may conceptually split the physical properties of the iron ball into purely electric and purely mechanical ones. Then the electromechanical energy transduction is represented by a second order port controlled Hamiltonian system:

$$\begin{pmatrix} \dot{\phi}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_{mg}}{\partial \phi}(\phi(t), z(t)) \\ \frac{\partial H_{mg}}{\partial z}(\phi(t), z(t)) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(t) \quad (6.85)$$

with output equations

$$i_S = (1, 0) \begin{pmatrix} \frac{\partial H_{mg}}{\partial \phi} \\ \frac{\partial H_{mg}}{\partial z} \end{pmatrix} \quad (6.86)$$

$$(6.87)$$

$$y_1 = f_{mg} = (0, 1) \begin{pmatrix} \frac{\partial H_{mg}}{\partial \phi} \\ \frac{\partial H_{mg}}{\partial z} \end{pmatrix} \quad (6.88)$$

The second subsystem simply represents the dynamics of a ball in vertical translation submitted to the action of an external force u_2:

$$\begin{pmatrix} \dot{q}(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_2}{\partial q}(q(t), p(t)) \\ \frac{\partial H_2}{\partial p}(q(t), p(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2(t) \quad (6.89)$$

where the Hamiltonian H_2 is the sum of the kinetic and the potential energy of the ball: $H_2(q, p) = \frac{1}{2m} p^2 + gq$ and the conjugated output is the velocity of the ball:

$$y_2 = (0, 1) \begin{pmatrix} \frac{\partial H_2}{\partial q} \\ \frac{\partial H_2}{\partial p} \end{pmatrix} \quad (6.90)$$

Consider the interconnection defined by:

$$u_1 = y_2 \quad (6.91)$$

$$u_2 = -y_1 \quad (6.92)$$

It is clear that this interconnection satisfies a power balance: $u_1 y_1 + u_2 y_2 = 0$. Hence it may be proved [121, 346, 442] that the interconnection of the two port controlled Hamiltonian systems leads to a port controlled Hamiltonian system (actually much more general interconnection relations may be considered, involving also constraints). In this example a simple elimination of the variables involved in the interconnection leads to the port controlled Hamiltonian system with Hamiltonian function $H_{tot} = H_{mg} + H_2$ and structure matrix

$$J_{tot} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \tag{6.93}$$

Considering lines 2 and 3 of the structure matrix, one deduces that the variations of z and q satisfy

$$\dot{z} - \dot{q} = 0 \tag{6.94}$$

This is precisely a Casimir function, *i.e.* a dynamical invariant of any Hamiltonian system defined with respect to the structure matrix J_{tot}. Hence it is possible to identify (up to an arbitrary constant) the two positions z and q, thus to reduce this system to the three-dimensional port controlled Hamiltonian system presented here above. It is clear that this splitting is not possible using the input output Hamiltonian system or Poisson control systems as the subsystem 1 in (6.85) has a non-symplectic (null) structure matrix and the input vector hence are not Hamiltonian (else they would be null too).

As a conclusion to this section we shall present an extension of lossless port control Hamiltonian systems to dissipative system, called *port controlled Hamiltonian systems with dissipation* introduced in [121]. The main difference is that the skew-symmetry of the structure matrix J is no more required, hence the structure matrix is in general an addition of a skew-symmetric matrix and a symetric positive matrix.

Definition 6.37 (Port Controlled Hamiltonian system with dissipation). *A* port controlled Hamiltonian system *on $I\!R^n$ is defined by a skew-symmetric structure matrix $J(x)$, a symmetric positive matrix $R(x)$, a real-valued Hamiltonian function $H_0(x)$, m input vector fields $g_i(x)$ and the dynamic equations*

$$\begin{cases} \dot{x} = (J(x), R(x))\, \frac{\partial H_0}{\partial x}(x) + \sum_{i=1}^m g_i(x) U_i \\ y_i = g_i^T(x) \frac{\partial H_0}{\partial x}(x) \end{cases} \tag{6.95}$$

■

Of course such a system is no more lossless, but it still satisfies a power balance equation and under some assumption on the Hamiltonian system, a passivity property.

Lemma 6.38 (Dissipativity of Port Controlled Hamiltonian systems). *A port controlled Hamiltonian system with dissipation (according to Definition 6.37) satisfies the following* power balance equation:

$$u^T y = \frac{dH_0}{dt} + \frac{\partial H_0}{\partial x}^T (x) R(x) \frac{\partial H_0}{\partial x}(x). \tag{6.96}$$

If, moreover, the Hamiltonian function $H_0(x)$ is bounded from below, then the port controlled Hamiltonian system with dissipation is dissipative with respect to the supply rate $u^T y$ with storage function $H_0(x)$. ■

As an example recall the levitated ball as the interconnection of two subsystems.

Example 6.39. Consider first the magnetic part. Considering the losses in the coil amounts to add to the skew symmetric structure matrix defined in (6.85) the symmetric positive matrix:

$$R = \begin{pmatrix} -R & 0 \\ 0 & 0 \end{pmatrix} \tag{6.97}$$

Then the total system also becomes a port controlled Hamiltonian system with a symmetric matrix $R_{tot} = \mathrm{diag}(-R \;\; 0_3)$. ■

6.3 Rigid Joint–Rigid Link Manipulators

In this section and in the next ones we shall recall the simple models corresponding to electromechanical systems, which motivated numerous results on passivity-based control. We shall recall and derive their passivity properties, and we illustrate some concepts introduced in the previous sections and chapters. Actually the results in the next sections of the present chapter will serve as a basis for introducing the control problem in Chapter 7. Our aim now is to show how one can use the passivity properties of the analyzed processes, to construct globally stable control laws. We shall insist on the calculation of storage functions, and it will be shown at some places (see for instance Section 7.3) that this can be quite useful to derive Lyapunov functions for closed-loop systems.

The dynamics of the mechanism constituting the mechanical part of a robotic manipulator is given by a simple mechanical system according to Definition 6.14 and Lemma 6.15:

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) = \tau(t) \tag{6.98}$$

From Lemma 6.11, it follows that they are lossless systems with respect to the supply rate $\tau^T \dot{q}$ with storage function $E(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + V(q)$ and $g(q) = \frac{\partial V}{\partial q}$ is the gradient of the gravitation potential energy $V(q)$.

6.3.1 The Available Storage

We have seen that storage functions play an important role in the dissipativity theory. In particular the dissipativity of a system can be characterized by the available storage $V_a(q,\dot{q})$ and the required supply $V_r(q,\dot{q})$ functions. Let us focus now on the calculation of the available storage function (see Definition 4.35), which represents the maximum internal energy contained in the system that can be extracted from it. More formally recall that we have

$$\begin{aligned} V_a(q_0,\dot{q}_0) &= -\inf_{\tau:(0,q_0,\dot{q}_0)\rightarrow} \int_0^t \tau^T(s)\dot{q}(s)ds \\ &= \sup_{\tau:(0,q_0,\dot{q}_0)\rightarrow} -\int_0^t \tau^T(s)\dot{q}(s)ds \end{aligned} \tag{6.99}$$

The notation $\inf_{\tau:(0,q_0,\dot{q}_0)\rightarrow}$ means that one performs the infinimization over all trajectories of the system on intervals $[0,t]$, $t \geq 0$, starting from the extended state $(0,q_0,\dot{q}_0)$, with $(q_0,\dot{q}_0) = (q(0),\dot{q}(0))$, with admissible inputs (at least the closed-loop system must be shown to be well-posed). In other words the infinimization is done over all trajectories $\phi(t;0,q_0,\dot{q}_0,\tau)$, $t \geq 0$. From (6.99) one obtains

$$\begin{aligned} V_a(q_0,\dot{q}_0) &= \sup_{\tau:(0,q_0,\dot{q}_0)\rightarrow} -\left\{\left[\frac{1}{2}\dot{q}^T M(q)\dot{q}\right]_0^t + U_g(q(t)) - U_g(q(0))\right\} \\ &= \tfrac{1}{2}\dot{q}(0)^T M(q(0))\dot{q}(0) + U_g(q(0)) \\ &= E(q_0,\dot{q}_0) \end{aligned} \tag{6.100}$$

where we have to assume that $U_g(q) \geq -K > -\infty$ for some $K < +\infty$, so that we may assume that the potential energy has been normalized to secure that $U_g(q) \geq 0$ for all $q \in I\!R^n$. It is not surprizing that the available storage is just the total initial mechanical energy of the system (but we shall see in a moment that for certain systems this is not so evident).

Remark 6.40. We might have deduced that the system is dissipative since $V_a(q,\dot{q}) < +\infty$ for any bounded state; see Theorem 4.41. On the other hand, $V_a(q,\dot{q})$ must be bounded since we already know that the system is dissipative with respect to the chosen supply rate.

Remark 6.41. In Section 6.1 we saw that the addition of Rayleigh dissipation enforces the dissipativity property of the system. Let us recalculate the available storage of a rigid joint-rigid link manipulator when the dynamics is given by

$$M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + g(q(t)) + \frac{\partial R}{\partial \dot{q}}(t) = \tau(t) \tag{6.101}$$

One has:

$$\begin{aligned}
V_a(q_0,\dot{q}_0) &= \sup_{\tau:(0,q_0,\dot{q}_0)\to} -\int_0^t \tau^T \dot{q} ds \\
&= \sup_{\tau:(0,q_0,\dot{q}_0)\to} \left\{ -\left[\frac{1}{2}\dot{q}^T M(q)\dot{q}\right]_0^t - [U_g(q)]_0^t - \int_0^t \dot{q}^T \frac{\partial R}{\partial \dot{q}} ds \right\} \\
&= \tfrac{1}{2}\dot{q}(0)^T M(q(0))\dot{q}(0) + U_g(q(0)) \\
&= E(q_0,\dot{q}_0)
\end{aligned} \tag{6.102}$$

since $\dot{q}^T \frac{\partial R}{\partial \dot{q}} \geq \delta \dot{q}^T \dot{q}$ for some $\delta > 0$. One therefore concludes that the dissipation does not modify the available storage function, which is a logical feature from the intuitive physical point of view (the dissipation and the storage are defined independently).

6.3.2 The Required Supply

Let us now compute the required supply $V_r(q,\dot{q})$ as in Definition 4.36, with the same assumption on $U_g(q)$. Recall that it is given in a variational form by:

$$V_r(q_0,\dot{q}_0) = \inf_{\tau:(-t,q_t,\dot{q}_t)\to(0,q_0,\dot{q}_0)} \int_{-t}^0 \tau^T(s)\dot{q}(s)ds \tag{6.103}$$

where $(q_t,\dot{q}_t) = (q(-t),\dot{q}(-t))$, $(q_0,\dot{q}_0) = (q(0),\dot{q}(0))$, $t \geq 0$. Thus this time the minimization process is taken over all trajectories of the system, joining the extended states $(-t,q_t,\dot{q}_t)$ and $(0,q_0,\dot{q}_0)$ (i.e. $(q_0,\dot{q}_0) = \phi(0;-t,q_t,\dot{q}_t,\tau)$). For the rigid manipulator case one finds

$$\begin{aligned}
V_r(q_0,\dot{q}_0) &= \inf_{\tau:(-t,q_t,\dot{q}_t)\to(0,q_0,\dot{q}_0)} [E(q_0,\dot{q}_0) - E(q(-t),\dot{q}(-t))] \\
&= E(0) - E(-t)
\end{aligned} \tag{6.104}$$

Note that $V_r(\cdot)$ hence defined is not necessarily positive. However if we compute it from $(-t,q_t,\dot{q}_t) = (-t,0,0)$ then indeed $V_r(\cdot) \geq 0$ is a storage function. Here one trivially finds that $V_r(q_0,\dot{q}_0) = E(q_0,\dot{q}_0)$ $(= V_a(q_0,\dot{q}_0))$.

Remark 6.42. The system is reachable from any state $(q_0,\dot{q}_0)$ (actually, this system is globally controllable). Similarly to the available storage function property, the system is dissipative with respect to a supply rate if and only if the required supply $V_r \geq -K$ for some $K > -\infty$; see Theorem 4.41. Here we can take $K = E(-t)$.

6.4 Flexible Joint–Rigid Link Manipulators

In this section we consider another class of systems which corresponds to models of manipulators whose joints are no longer assumed to be perfectly rigid, but can be fairly modelled by a linear elasticity. Their simplified dynamics can be written as

$$\begin{cases} M(q_1(t))\ddot{q}_1(t) + C(q_1(t), \dot{q}_1(t))\dot{q}_1(t) + g(q_1(t)) = K(q_2(t) - q_1(t)) \\ \\ J\ddot{q}_2(t) = K(q_1(t) - q_2(t)) + u(t) \end{cases} \tag{6.105}$$

where $q_1(t) \in I\!R^n$ is the vector of rigid links angles, $q_2(t) \in I\!R^n$ is the vector of motor shaft angles, $K \in I\!R^{n\times n}$ is the joint stiffness matrix and $J \in I\!R^{n\times n}$ is the motor shaft inertia matrix (both assumed here to be constant and diagonal). It is a simple mechanical system in Lagrangian form (6.43), we can say that $M(q) = \begin{pmatrix} M(q_1) & 0 \\ 0 & J \end{pmatrix}$, $C(q,\dot{q}) = \begin{pmatrix} C(q_1,\dot{q}_1) & 0 \\ 0 & 0 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 \\ u \end{pmatrix}$, $g(q) = \begin{pmatrix} g(q_1) \\ 0 \end{pmatrix} + \begin{pmatrix} K(q_2 - q_1) \\ K(q_1 - q_2) \end{pmatrix}$. Actually the potential energy is given by the sum of the gravity and the elasticity terms, $U_g(q_1)$ and $U_e(q_1,q_2) = \frac{1}{2}(q_2 - q_1)^T K(q_2 - q_1)$ respectively. The dynamics of flexible joint-rigid link manipulators can be seen as the interconnection of the simple mechanical system representing the dynamics of the rigid joint-rigid link manipulators with a set of linear Lagrangian systems with external forces representing the inertial dynamics of the rotor, interconnected by the rotational spring representing the compliance of the joints. It may be seen as the power continuous interconnection of the corresponding three port controlled Hamiltonian systems in a way completely similar to the example of the levitated ball (Example 6.36). We shall not detail the procedure here but summarize it on Figure 6.2. As a result it follows that the system is passive, lossless with respect to the supply rate $u^T\dot{q}_2$ with storage function being the sum of the kinetic energies and potential energies of the different elements. We shall see in Section 6.6 that including actuator dynamics produces similar interconnected systems, but with quite different interconnection terms. These terms will be shown to play a crucial role in the stabilizability properties of the overall system.

Remark 6.43. The model in (6.105) was proposed by Spong [471] and is based on the assumption that the rotation of the motor shafts due to the link angular motion does not play any role in the kinetic energy of the system, compared to the kinetic energy of the rigid links. In other words the angular part of the kinetic energy of each motor shaft rotor is considered to be due to its own rotation only. This is why the inertia matrix is diagonal. This assumption seems satisfied in practice for most of the manipulators. It is also satisfied (mathematically speaking) for those manipulators whose actuators are all mounted at the base, known as parallel-drive manipulators (the Capri robot presented

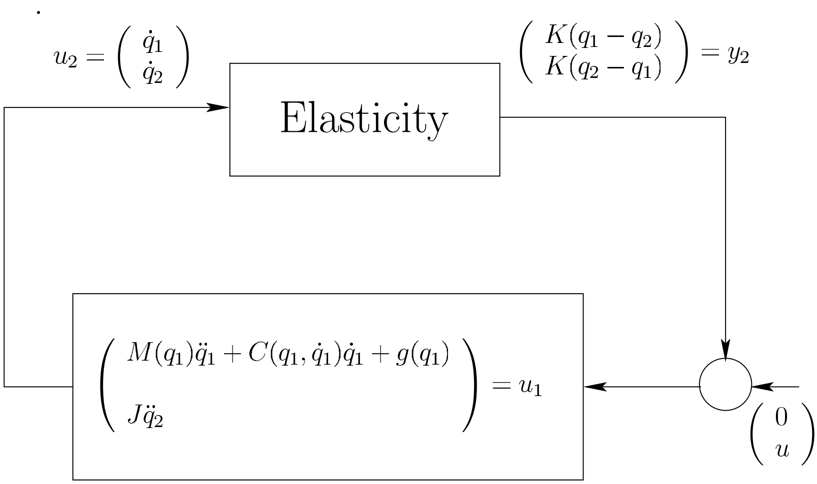

Fig. 6.2. Flexible joint–rigid link: interconnection as two passive blocks

in chapter 9 is a parallel-drive manipulator). If this assumption is not satisfied [488], the inertia matrix takes the form $M(q) = \begin{pmatrix} M(q_1) & M_{12}(q_1) \\ M_{12}^T(q_1) & J \end{pmatrix}$. ■

The particular feature of the model in (6.105) is that it is static feedback linearizable and possesses a triangular structure [325] that will be very useful when we deal with control.

Let us now prove in some other way that the system is passive (*i.e.* dissipative with respect to the supply rate $\tau^T \dot{q} = u^T \dot{q}_2$). We get for all $t \geq 0$:

$$\begin{aligned} \int_0^t u^T(s)\dot{q}_2(s)ds &= \int_0^t \left[J\ddot{q}_2(s) + K(q_2(s) - q_1(s))\right]^T \dot{q}_2(s)ds \pm \\ &\pm \int_0^t (q_2(s) - q_{1(s)})^T K\dot{q}_1(s)ds \\ &= \left[\tfrac{1}{2}\dot{q}_2^T J\dot{q}_2\right]_0^t + \left[\tfrac{1}{2}(q_2 - q_1)^T K(q_2 - q_1)\right]_0^t + \\ &+ \int_0^t (q_2(s) - q_1(s))^T K\dot{q}_1(s)ds \end{aligned} \tag{6.106}$$

The last integral term can be rewritten as

$$\int_0^t (q_2 - q_1)^T K\dot{q}_1 ds = \int_0^t \dot{q}_1^T \left[M(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + g(q_1)\right] ds \tag{6.107}$$

Looking at the rigid joint–rigid link case, one sees that

$$\int_0^t (q_2 - q_1)^T K\dot{q}_1 ds = \left[\frac{1}{2}\dot{q}_1^T M(q_1)\dot{q}_1 + U_g(q_1)\right]_0^t \tag{6.108}$$

Therefore grouping (6.106) and (6.108) one obtains

$$\begin{aligned}\int_0^t u^T \dot{q}_2 ds \geq & -\tfrac{1}{2}\dot{q}_2(0)^T J \dot{q}_2(0) \\ & -\tfrac{1}{2}\dot{q}_1(0)^T M(q_1(0)\dot{q}_1(0) \\ & -\tfrac{1}{2}[q_2(0) - q_1(0)]^T K[q_2(0) - q_1(0)] - U_g(q_1(0))\end{aligned} \tag{6.109}$$

The result is therefore true whenever $U_g(q_1)$ is bounded from below.

Remark 6.44. One could have thought of another decomposition of the system as depicted in Figure 6.3. In this case the total system is broken down into two Lagrangian control systems with input being the free end of the springs with respect to each submodel. The subsystem with generalized coordinate q_1 (i.e. representing the dynamics of the multibody system of the robot) is analogous to the harmonic oscillator of Example 6.12 and with input q_2. The dynamics of the rotors (with generalized coordinates q_2) is again analogous to an additional external force u. But the interconnection of these two subsystems is defined by : $u_1 = q_2$ and $u_2 = q_1$ involving the *generalized coordinates* which are not passive outputs of the subsystems.

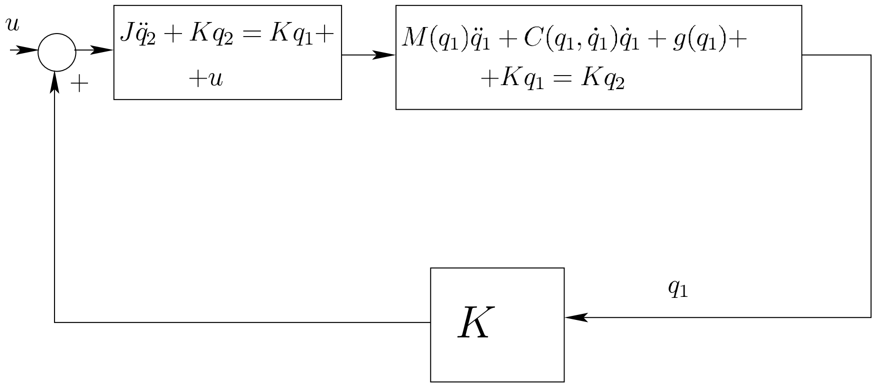

Fig. 6.3. Flexible joint–rigid link manipulator

Remark 6.45. Let us point out that manipulators with prismatic joints cannot be passive, except if those joints are horizontal. Hence all those results on open-loop dissipativity hold for revolute joint manipulators only. This will not at all preclude the application of passivity tools for any sort of joints when we deal with feedback control —for instance it suffices to compensate for gravity to avoid this problem—.

6.4.1 The Available Storage

Mimicking the rigid joint-rigid link case, one finds that

$$
\begin{aligned}
V_a(q,\dot{q}) = E(q,\dot{q}) &= \tfrac{1}{2}\dot{q}_1^T M(q_1)\dot{q}_1 + \tfrac{1}{2}\dot{q}_2^T J\dot{q}_2 \\
&\quad +\tfrac{1}{2}[q_1 - q_2]^T K[q_1 - q_2] + U_g(q_1)
\end{aligned}
\tag{6.110}
$$

6.4.2 The Required Supply

From Subsection 6.3.2 one finds that the energy required from an external source to transfer the system from the extended state

$$(-t, q_1(-t), q_2(-t), \dot{q}_1(-t), \dot{q}_2(-t)) = (-t, q_{1t}, q_{2t}, \dot{q}_{1t}, \dot{q}_{2t})$$

to

$$(0, q_1(0), q_2(0), \dot{q}_1(0), \dot{q}_2(0)) = (0, q_{10}, q_{20}, \dot{q}_{10}, \dot{q}_{20}),$$

is given by

$$
\begin{aligned}
V_r(q_1(0), q_2(0), \dot{q}_1(0), \dot{q}_2(0)) &= E(q_1(0), q_2(0), \dot{q}_1(0), \dot{q}_2(0)) - \\
&\quad - E(q_1(-t), q_2(-t), \dot{q}_1(-t), \dot{q}_2(-t))
\end{aligned}
\tag{6.111}
$$

The KYP Lemma Conditions

Recall from the Positive Real (or Kalman-Yacubovich-Popov) Lemma 4.84 that a system of the form

$$
\begin{cases}
\dot{x} = f(x) + g(x)u \\
y = h(x)
\end{cases}
\tag{6.112}
$$

is passive (dissipative with respect to the supply rate $u^T y$) if and only if there exists at least one function $V(t,x) \geq 0$ such the following conditions are satisfied:

$$
\begin{cases}
h^T(x) = \frac{\partial V}{\partial x}(x) g(x) \\
\frac{\partial V}{\partial x}(x) f(x) \geq 0
\end{cases}
\tag{6.113}
$$

The **if** part of this Lemma tells us that an unforced system that is Lyapunov stable with Lyapunov function $V(\cdot)$ is passive when the output has the particular form in (6.113). The **only if** part tells us that given an output function, then passivity holds only if the searched $V(\cdot)$ does exist.

Now let us assume that the potential function $U_g(q_1)$ is finite for all $q \in \mathcal{C}$. Then it follows that the available storage calculated in (6.110) is a storage function, hence it satisfies the conditions in (6.113) when $y = JJ^{-1}\dot{q}_2 = \dot{q}_2$ and

u is defined in (6.105). More explicitly the function $E(q, \dot{q})$ in (6.110) satisfies the partial differential equations (in (6.105) one has $g^T(x) = (0, 0, 0, J^{-1})$)

$$\begin{cases} \frac{\partial E}{\partial \dot{q}_2}^T J^{-1} = \dot{q}_2^T \\ \\ \frac{\partial E}{\partial q_1}^T \dot{q}_1 + \frac{\partial E}{\partial \dot{q}_1}^T M(q_1)^{-1} \left[-C(q_1, \dot{q}_1)\dot{q}_1 - g(q_1) + K(q_2 - q_1)\right] + \\ \\ + \frac{\partial E}{\partial q_2}^T \dot{q}_2 + \frac{\partial E}{\partial \dot{q}_2} J^{-1} \left[K(q_1 - q_2)\right] = 0 \end{cases} \tag{6.114}$$

6.5 A Bouncing System

We may conclude from the preceding examples that in general, for mechanical systems, the total mechanical energy is a storage function. However the calculation of the available storage may not always be so straightforward as the following example shows. Let us consider a one degree-of-freedom system composed of a mass striking a compliant obstacle modelled as a spring-dashpot system. The dynamical equations for contact and non-contact phase are given by

$$m\ddot{q}(t) = \tau(t) + \begin{cases} -f\dot{q}(t) - kq(t) & \text{if } q(t) > 0 \\ \\ 0 & \text{if } q(t) \le 0 \end{cases} \tag{6.115}$$

It is noteworthy that the system in (6.115) is nonlinear since the switching condition depends on the state. Moreover existence of a solution with q continuously differentiable is proved in [392] when τ is a Lipschitz continuous function of time, q and $\dot{q}$. The control objective is to stabilize the system at rest in contact with the obstacle. To this aim let us choose the input

$$\tau = -\lambda_2 \dot{q} - \lambda_1 (q - q_d) + v \tag{6.116}$$

with $q_d > 0$ constant, $\lambda_1 > 0$, $\lambda_2 > 0$ and v is an auxiliary signal. The input in (6.116) is a PD controller but can also be interpreted as an input transformation. Let us now consider the equivalent closed-loop system with input v and output $\dot{q}$, and supply rate $w = v\dot{q}$. The available storage function is given by

$$V_a(x_0, \dot{x}_0) = \sup_{\tau:(0, q_0, \dot{q}_0) \to} - \int_{t_0}^{t} v(s)\dot{q}(s)ds \tag{6.117}$$

Due to the system's dynamics in (6.115) we have to consider two cases:

- **Case** $q_0 \le 0$: Let us denote $\Omega_{2i} = [t_{2i}, t_{2i+1}]$ the time intervals such that $q(t) \le 0$, and $\Omega_{2i+1} = [t_{2i+1}, t_{2i+2}]$ the intervals such that $q(t) > 0$, $i \in \mathbb{N}$. From (6.116) and (6.115) one has

$$V_a(q_0,\dot q_0) =$$

$$= \sup_{\tau:(0,q_0,\dot q_0)\rightarrow} -\sum_{i\geq 0}\left\{\int_{\Omega_{2i}}(m\ddot q(s)+\lambda_2\dot q(s)+\lambda_1 q(s)-\lambda_1 q_d)\dot q(s)ds\right\}$$

$$-\sum_{i\geq 0}\left\{\int_{\Omega_{2i+1}}(m\ddot q(s)+\lambda_2\dot q(s)+(\lambda_1+k)q(s)-\lambda_1 q_d)\dot q(s)ds\right\}$$

$$= \sup_{\tau:(0,q_0,\dot q_0)\rightarrow} \sum_{i\geq 0}\left\{-\left[m\frac{\dot q^2}{2}\right]_{t_{2i}}^{t_{2i+1}} - \left[\frac{\lambda_1}{2}(x-x_d)^2\right]_{t_{2i}}^{t_{2i+1}} - \lambda_2\int_{\Omega_{2i}}\dot q^2(t)dt\right\}$$

$$+\sum_{i\geq 0}\left\{-\left[m\frac{\dot q^2}{2}-\frac{\lambda_1+k}{2}\left(q-\frac{\lambda_1 q_d}{\lambda_1+k}\right)^2\right]_{t_{2i+1}}^{t_{2i+2}} - (\lambda_2+f)\int_{\Omega_{2i+1}}\dot q^2(t)dt\right\} \tag{6.118}$$

In order to maximize the terms between brackets it is necessary that the integrals $-\int_{\Omega_i}\dot q^2(t)dt$ be zero and that $\dot q(t_{2i+1}) = 0$. In view of the system's controllability, there exists an impulsive input v that fulfils these requirements [246] (let us recall that this impulsive input is applied while the system evolves in a free-motion phase, hence has linear dynamics). In order to maximize the second term $-\left[\frac{\lambda_1}{2}(q-q_d)^2\right]_{t_0}^{t_1}$ it is also necessary that $q(t_1) = 0$. Using similar arguments, it follows that $\dot q(t_{2i+2}) = 0$ and that $q(t_2) = \frac{\lambda_1 q_d}{\lambda_1+k}$. This reasoning can be iterated to obtain the optimal path which is $(q_0,\dot q_0)\rightarrow(0,0)\rightarrow(\frac{\lambda_1 q_d}{\lambda_1+k},0)$ where all the transitions are instantaneous. This leads us to the following available storage function:

$$V_a(q_0,\dot q_0) = m\frac{\dot q_0^2}{2}+\frac{\lambda_1 q_0^2}{2}-\lambda_1 q_d q_0+\frac{\lambda_1^2 q_d^2}{2(\lambda_1+k)} \tag{6.119}$$

- **Case** $q_0 > 0$: Using a similar reasoning one obtains

$$V_a(q_0,\dot q_0) = m\frac{\dot q_0^2}{2}+\frac{(\lambda_1+k)}{2}\left(q_0-\frac{\lambda_1 q_d}{\lambda_1+k}\right)^2 \tag{6.120}$$

Notice that the two functions in (6.119) and (6.120) are not equal. Their concatenation yields a positive definite function of $(\tilde q,\dot q) = (0,0)$ with $\tilde q = q - \frac{\lambda_1 q_d}{\lambda_1+k}$, that is continuous at $q = 0$, but not differentiable (this is in accordance with [33, Proposition]).

Remark 6.46. Let us now consider the following systems

$$m\ddot q(t)+\lambda_2\dot q(t)+\lambda_1(q(t)-q_d) = v(t) \tag{6.121}$$

and

$$m\ddot{q}(t) + (\lambda_2 + f)\dot{q}(t) + \lambda_1(q(t) - q_d) + kq(t) = v(t) \tag{6.122}$$

that represent the persistent free motion and the persistent contact motion dynamics respectively. The available storage function for the system in (6.121) is given by (see Remark 6.41)

$$V_a(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}\lambda_1(q - q_d)^2 \tag{6.123}$$

whereas it is given for the system in (6.122) by

$$V_a(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}\lambda_1(q - q_d)^2 + \frac{1}{2}kq^2 \tag{6.124}$$

It is clear that the functions in (6.119) and (6.123), (6.120) and (6.124), are respectively not equal. Notice that this does not preclude that the concatenation of the functions in (6.123) and (6.124) yield a storage function for the system (in which case it must be larger than the concatenation of the functions in (6.119) and (6.120) for all $(q, \dot{q})$). In fact an easy inspection shows that the functions in (6.123) and (6.124) are obtained by adding $\frac{1}{2}\frac{\lambda_1 k q_d^2}{\lambda_1 + k}$ to those in (6.119) and (6.120) respectively. Thus their concatenation indeed yields a storage function for the system in (6.115) with input (6.116). ■

An open issue is to study the conditions under which the available storage function of the piecewise continuous system

$$\begin{cases} \dot{x}(t) = f_i(x(t), u(t)) \text{ if } C_i x(t) \geq 0 \\ \\ \dot{x}(t) = g_i(x(t), u(t)) \text{ if } C_i x(t) < 0 \\ \\ i \in \{1, \cdots, m\} \end{cases} \tag{6.125}$$

can be deduced as a concatenation of the available storages of the independent systems $\dot{x} = f_i(x, u)$ and $\dot{x} = g_i(x, u)$. More generally one should study the dissipativity of switching systems. An important contribution can be found in [540], where a suitable definition of dissipativity is proposed that involves several supply rates and storage functions [1]. Other contributions are in [534, 535].

6.6 Including Actuator Dynamics

6.6.1 Armature-controlled DC Motors

In all the foregoing examples it has been assumed that the control is directly provided by the generalized torque τ. In reality the actuators possess their

[1] It is noteworthy that the class of systems considered in [534, 535, 540] does not encompass the nonsmooth systems which are examined elsewhere in this book, like unbounded differential inclusions, variational inequalities. They are *different* types of dynamical systems.

own dynamics, and the torque is just the output of a dynamical system. In practice the effect of neglecting those dynamics may deteriorate the closed-loop performance [79]. In other words, the dynamics in (6.43) are replaced by a more accurate armature-controlled DC motor model as:

$$\begin{cases} M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) = \tau = K_t I(t) \\ \\ RI(t) + L\frac{dI}{dt}(t) + K_t \dot{q}(t) = u(t) \end{cases} \tag{6.126}$$

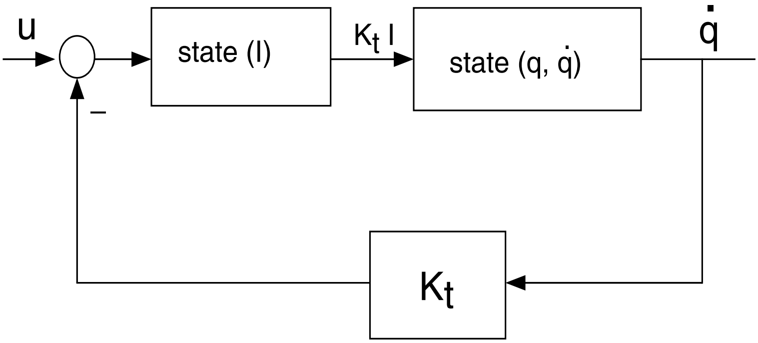

Fig. 6.4. Manipulator + armature-controlled DC motor

where R, L, K_t are diagonal constant matrices with strictly positive entries, $R \in \mathbb{R}^{n\times n}$ is a matrix whose jth entry is the resistance of the jth motor armature circuit, $L \in \mathbb{R}^{n\times n}$ has entries which represent the inductances of the armature, $K_t \in \mathbb{R}^{n\times n}$ represents the torque constants of each motor, $u \in \mathbb{R}^n$ is the vector of armature voltage, $I \in \mathbb{R}^n$ is the vector of armature currents. For the sake of simplicity we have assumed that all the gear ratios that might relate the various velocities are equal to one. Moreover the inertia matrix $M(q)$ is the sum of the manipulator and the motorschaft inertias. The new control input is therefore u, see Figure 6.4. For the moment we are interested in deriving the passivity properties of this augmented model. We shall see further that the (disconnected) dynamics of the motor are strictly output passive with respect to the supply rate $u^T I$.

Remark 6.47. One may consider this system as the interconnection of two subsystems as in Figure 6.5. One notes at once a strong similarity between the model in (6.126) and the example of the magnetic ball in Example 6.34. The difference is that there is no coupling through the energy function (no state variable in common) but that the simple mechanical system, representing the dynamics of the mechanical part, is non-linear. The interconnection structure

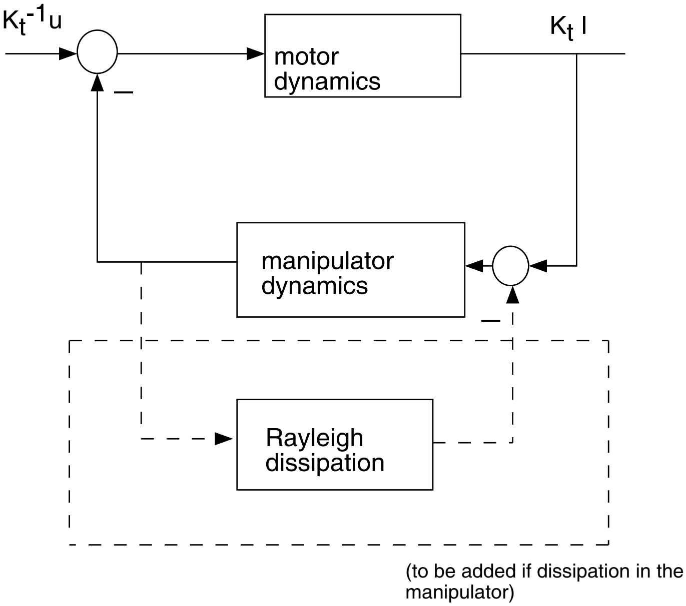

Fig. 6.5. Negative feedback interconnection in two dissipative blocks

is best seen on the formulation using port controlled Hamiltonian systems as follows and illustrated in Figure 6.5. The Legendre transformation of the simple mechanical system leads to the definition of the momentum vector $p = \frac{\partial L}{\partial \dot{q}} = M(q)\dot{q}$, the Hamiltonian function $H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + U(q)$ and the following port controlled Hamiltonian system:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} + \begin{pmatrix} 0_n \\ I_n \end{pmatrix} \tau \tag{6.127}$$

$$\tag{6.128}$$

$$y_{mech} = (0_n, I_n) \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \dot{q} \tag{6.129}$$

where the input τ represents the electromechanical forces. The dynamics of the motors is described by the following port controlled Hamiltonian system with dissipation with state variable being the total magnetic flux $\phi = LI$ and the magnetic energy being $H_{mg} = \frac{1}{2L}\phi^2$:

$$\dot{\phi} = -R\frac{\partial H_{mg}}{\partial \phi} + u + u_{mg} \tag{6.130}$$

$$y_{mg} = \frac{\partial H_{mg}}{\partial \phi} = I \tag{6.131}$$

where u_{mg} represents the electromotive forces. Note that the structure matrix consists only of a negative definite part, thus it is purely an energy dissipating system. The interconnection between the two subsystems is defined by the following power continuous interconnection:

$$\tau = K_t y_{mg} \tag{6.132}$$

$$u_{mg} = -K_t y_{mech} \tag{6.133}$$

A simple elimination leads to the following port controlled Hamiltonian system with dissipation

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{\phi} \end{pmatrix} = \left[\begin{pmatrix} 0_n & I_n & 0_n \\ -I_n & 0_n & K_t \\ 0_n & -K_t & 0_n \end{pmatrix} + \begin{pmatrix} 0_{2n} & 0_{2n\times n} \\ 0_{n\times 2n} & -R \end{pmatrix} \right] \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H_{mg}}{\partial \phi} \end{pmatrix} \tag{6.134}$$

$$+ \begin{pmatrix} 0_n \\ 0_n \\ I_n \end{pmatrix} u$$

$$y = (0_n, 0_n, I_n) \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H_{mg}}{\partial \phi} \end{pmatrix} = I \tag{6.135}$$

From this formulation of the system as interconnected port controlled Hamiltonian with dissipation, the interconnected system is seen to be passive with supply rate $u^T I$ and storage function $H(q,p) + H_{mg}(\phi)$.

Passivity with Respect to the Supply Rate $u^T I$

Let us calculate directly the value of $\langle u, I\rangle_t$, where the choice of this supply rate is motivated by an (electrical) energy expression:

$$\begin{aligned} \langle u, I\rangle_t &= \int_0^t I^T \left[RI + L\frac{dI}{dt} + K_v \dot{q}\right] \\ &= \int_0^t I(s)^T RI(s)ds + \tfrac{1}{2}\left[I(s)^T LI(s)\right]_0^t \\ &+\tfrac{1}{2}\left[\dot{q}(s)^T M(q(s))\dot{q}(s)\right]_0^t + \left[U_g(q(s))\right]_0^t \\ &\geq \int_0^t I(s)^T RI(s)ds - \tfrac{1}{2}I(0)^T LI(0) \\ &-\tfrac{1}{2}\dot{q}(0)^T M(q(0))\dot{q}(0) - U_g(q(0)) \end{aligned} \tag{6.136}$$

where we used the fact that $R > 0$, $L > 0$. One sees that the system in (6.126) is even strictly output passive when the output is $y = K_t I$. Indeed $I^T RI \geq \lambda_{\min}(R) y^T y$ where $\lambda_{\min}(R)$ denotes the minimum eigenvalue of R.

Available Storage and Required Supply

Using the same supply rate as in Subsection 6.6.1, one gets

$$\begin{aligned} V_a(q, \dot{q}, I) &= \tfrac{1}{2} I^T L I + \tfrac{1}{2} \dot{q}^T M(q) \dot{q} + U_g(q) \\ &= V_r(q, \dot{q}, I) \end{aligned} \tag{6.137}$$

Necessity and Sufficiency for the Supply Rate to be $u^T I$

The supply rate $u^T I$ has been chosen according to the definition of conjugated port variables of port controlled Hamiltonian systems. In the sequel, we shall prove that no other form on the port variables may be chosen to define a supply rate for another storage function. Therefore let us introduce a more general supply rate of the form $u^T A^T B I$ for some constant matrices A and B of suitable dimensions. Our goal is to show that if the system is dissipative with respect to this new supply rate, then necessarily (and sufficiently) $A = \frac{1}{\alpha} U^{-1} K_t^{-1}$ and $B = \alpha K_t U$, where $\alpha \neq 0$ and U is a full-rank symmetric matrix. Let us compute the available storage associated to this supply rate, *i.e.*

$$\begin{aligned} V_a(q_0, \dot{q}_0, I_0) &= \sup_{u_2:(0, q_0, \dot{q}_0, I_0)} -\int_0^t u^T(s) A^T B I(s) ds \\ &= \sup_{u_2:(0, q_0, \dot{q}_0, I_0)} -\left\{ \frac{1}{2} [I^T L A^T B I]_0^t + \int_0^t I^T R A^T B I ds \right. \\ &\quad \left. + \int_0^t \dot{q}^T K_t A^T B K_t^{-1} \left[M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + g(q) \right] ds \right\} \end{aligned} \tag{6.138}$$

It follows that the necessary conditions for $V_a(q, \dot{q}, I)$ to be bounded are that $LA^T B \geq 0$ and $RA^T B \geq 0$. Moreover the last integral concerns the dissipativity of the rigid joint-rigid link manipulator dynamics. We know storage functions for this dynamics, from which it follows that an output of the form $K_t^{-1} B^T A K_t \dot{q}$ does not satisfy the (necessary) Kalman-Yakubovic-Popov property, except if $K_t^{-1} B^T A K_t = I_n$. One concludes that the only supply rate with respect to which the system is dissipative must satisfy

$$\begin{cases} K_t^{-1} B^T A K_t = I_n \\ LA^T B \geq 0 \\ RA^T B \geq 0 \end{cases} \tag{6.139}$$

Hence $A = \frac{1}{\alpha}U^{-1}K_t^{-1}$ and $B = \alpha K_t U$ for some $\alpha \neq 0$ and some full-rank matrix $U = U^T$.

6.6.2 Field-controlled DC Motors

Now consider the model of rigid joint-rigid link manipulators actuated by field-controlled DC motors:

$$\begin{cases} L_1 \frac{dI_1}{dt} + R_1 I_1 = u_1 \\ L_2 \frac{dI_2}{dt} + R_2 I_2 + K_t(I_1)\dot{q} = u_2 \\ M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + K_v\dot{q} = \tau = K_t(I_1)I_2 \end{cases} \tag{6.140}$$

where I_1, I_2 are the vectors of currents in the coils of the n motors actuating the manipulator, L_1 and L_2 denote their inductances, R_1 and R_2 are the resistors representing the losses in the coils. The matrix $K_t(I_1)$ represent the electromechanical coupling and is defined by a constant diagonal matrix K_t as follows:

$$K_t(I_1) = \mathrm{diag}(k_{t1}I_{11}, \cdots, k_{tn}I_{1n}) = K_t I_1 \tag{6.141}$$

with $k_{ti} > 0$. The last equation is the Lagrangian control system representing the dynamics of the manipulator with n degrees of freedom defined in (6.98) where the diagonal matrix K_v is positive definite and represents the mechanical losses in the manipulator.

In order to reveal the passive structure of the system, we shall again, like in the preceding case, assemble it as the interconnection of two passive port controlled Hamiltonian systems. Therefore let us split this system in two parts: the magnetic part and the mechanical part and interconnect them through a power continuous interconnection. The first port controlled Hamiltonian system with dissipation represents the magnetic energy storage and the electromechanical energy transduction. The state variables are the total magnetic fluxes in the coils $\phi = (\phi_1, \phi_2)^T$ defining the magnetic energy $H_{mg} = \frac{1}{2}\left(\frac{1}{L_1}\phi^2 + \frac{1}{L_1}\phi^2\right)$ and becomes

$$\begin{aligned} \dot{\phi} = &\begin{pmatrix} -R_1 & 0_n \\ 0_n & -R_2 \end{pmatrix} \begin{pmatrix} \frac{\partial Hmg}{\partial \phi_1} \\ \frac{\partial Hmg}{\partial \phi_2} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2 \\ &+ \begin{pmatrix} 0 \\ K_t \frac{\phi_1}{L_1} \end{pmatrix} u_{mg} \end{aligned} \tag{6.142}$$

with the conjugated outputs associated to the voltages u_1 and u_2:

$$y_1 = (1,0) \begin{pmatrix} \frac{\partial Hmg}{\partial \phi_1} \\ \frac{\partial Hmg}{\partial \phi_2} \end{pmatrix} = I_1 \text{and} \;\; y_2 = (0,1) \begin{pmatrix} \frac{\partial Hmg}{\partial \phi_1} \\ \frac{\partial Hmg}{\partial \phi_2} \end{pmatrix} = I_2 \tag{6.143}$$

and the output conjuguated to the electromotive force u_{mg} is

$$y_{mg} = \left(0, K_t \frac{\phi_1}{L_1}\right) \begin{pmatrix} \frac{\partial Hmg}{\partial \phi_1} \\ \\ \frac{\partial Hmg}{\partial \phi_2} \end{pmatrix} \tag{6.144}$$

where the two conjugated port variables u_{mg} and y_{mg} define the interconnection with the mechanical system. The second port controlled Hamiltonian system with dissipation represents the dynamics of the manipulator and was presented above:

$$\begin{pmatrix} \dot{q} \\ \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0_n & I_n \\ \\ -I_n & -K_v \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \\ \frac{\partial H}{\partial p} \end{pmatrix} + \begin{pmatrix} 0_n \\ I_n \end{pmatrix} u_{mech} \tag{6.145}$$

$$y_{mech} = (0_n, I_n) \begin{pmatrix} \frac{\partial H}{\partial q} \\ \\ \frac{\partial H}{\partial p} \end{pmatrix} = \dot{q} \tag{6.146}$$

where one notes that the dissipation defined by the matrix K_t was included in the structure matrix. The interconnection of the two subsystems is defined as an elementary negative feedback interconnection:

$$u_{mech} = y_{mg} \tag{6.147}$$

$$u_{mg} = -y_{mech} \tag{6.148}$$

Again a simple elimination of the interconnection variables leads to the port controlled Hamiltonian system with dissipation, with Hamiltonian being the sum of the Hamiltonian of the subsystems: $H_{tot}(\phi, q, p) = H_{mg}(\phi) + H(q, p)$ and structure matrice with skew-symmetric part

$$J_{tot} = \begin{pmatrix} 0_n & 0_n & 0_n \\ \\ 0_n & 0_n & -K_t \frac{\phi_1}{L_1} \\ \\ 0_n & K_t \frac{\phi_1}{L_1} & 0_n \end{pmatrix} \tag{6.149}$$

and symmetric positive structure matrix:

$$R_{tot} = \mathrm{diag}(-R_1, -R_2, -K_v) \tag{6.150}$$

Hence the complete system is passive with respect to the supply rate of the remaining port variables: $u_1 y_1 + y_2 u_2$ and with storage function being the total energy H_{tot}.

Passivity of the Manipulator Plus Field-controlled DC Motor

Motivated by the preceding physical analysis of the field-controlled DC motor, using the integral formulation of the passivity, let us prove the dissipativity with respect to the supply rate $u_1^T I_1 + u_2^T I_2$:

$$\begin{aligned}
\langle u_1, I_1\rangle_t + \langle u_2, I_2\rangle_t &\geq -\tfrac{1}{2} I_1(0)^T L_1 I_1(0) + \textstyle\int_0^t I_1^T(s) R_1 I_1(s) ds \\
&\quad -\tfrac{1}{2} I_2(0)^T L_2 I_2(0) + \textstyle\int_0^t I_2^T(s) R_2 I_2(s) ds + \\
&\quad + \textstyle\int_0^t \dot{q}^T(s) K_t(I_1(s)) I_2(s) ds \\
&\geq -\tfrac{1}{2} I_1(0)^T L_1 I_1(0) - \tfrac{1}{2} I_2(0)^T L_2 I_2(0) \\
&\quad -\tfrac{1}{2}\dot{q}(0)^T M(q(0))\dot{q}(0) - U_g(q(0))
\end{aligned} \tag{6.151}$$

which proves the statement.

Remark 6.48 (Passivity of the motors alone). The dynamics of a field-controlled DC motor is given by

$$\begin{cases}
L_1 \frac{dI_1}{dt}(t) + R_1 I_1(t) = u_1(t) \\
L_2 \frac{dI_2}{dt}(t) + R_2 I_2(t) + K_v(I_1(t))\dot{q}(t) = u_2(t) \\
J\ddot{q}(t) = K_t(I_1(t)) I_2(t) - K_{vt}\dot{q}(t)
\end{cases} \tag{6.152}$$

where $J \in \mathbb{R}^{n\times n}$ is the rotor inertia matrix. It follows that the (disconnected) actuator is passive with respect to the supply rate $u_1^T I_1 + u_2^T I_2$. Actually we could have started by showing the passivity of the system in (6.152) and then proceeded to showing the dissipativity properties of the overall system in (6.140) using a procedure analog to the interconnection of subsystems. Similar conclusions hold for the armature-controlled DC motor whose dynamics is given by

$$\begin{cases}
J\ddot{q}(t) = K_t I(t) \\
RI(t) + L\frac{dI}{dt}(t) + K_t\dot{q}(t) = u(t)
\end{cases} \tag{6.153}$$

and which is dissipative with respect to $u^T I$. This dynamics is even output strictly passive (the output is $y = I$ or $y = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$) due to the resistance.

The Available Storage

The available storage function of the system in (6.140) with respect to the supply rate $u_1^T I_1 + u_2^T I_2$ is found to be, after some calculations:

$$V_a(I_1, I_2, q, \dot{q}) = \frac{1}{2} I_1^T L_1 I_1 + \frac{1}{2} I_2^T L_2 I_2 + \frac{1}{2}\dot{q}^T M(q)\dot{q} + U_g(q) \qquad (6.154)$$

This is a storage function and a Lyapunov function of the unforced system in (6.140).

Remark 6.49. Storage functions for the disconnected DC motors are given by $V_{adc}(I, q, \dot{q}) = \frac{1}{2}\dot{q}^T J\dot{q} + \frac{1}{2} I^T L I$ and $V_{fdc}(I_1, I_2, q, \dot{q}) = \frac{1}{2}\dot{q}^T J\dot{q} + \frac{1}{2} I_1^T L_1 I_1 + \frac{1}{2} I_2^T L_2 I_2$. Notice that they are not positive definite functions of the state $(q, \dot{q}, I)$ but they are positive definite functions of the partial state $(\dot{q}, I)$. Hence the fixed point $(\dot{q}, I) = (0, 0)$ (or $(\dot{q}, I_1, I_2) = (0, 0, 0)$) is asymptotically stable. ■

Notice that the actuator dynamics in (6.152) with input (u_1, u_2) and output (I_1, I_2) (which are the signals from which the supply rate is calculated, hence the storage functions) is zero-state detectable: $((u_1, u_2) \equiv (0, 0)$ and $I_1 = I_2 = 0) \Longrightarrow \dot{q} = 0$ (but nothing can be concluded on q), and is strictly output passive. From Lemma 5.13 one may conclude at once that any function satisfying the Kalman-Yacubovich-Popov conditions is indeed positive definite.

Remark 6.50. The model of field-controlled DC motors in (6.152) is similar to that of induction motors, that may be given in some reference frame by (here we show the model for one motor whereas in (6.152) the dynamics represent a system composed of n motors):

$$\begin{cases} L\dot{z}(t) + C(z(t), u_3(t))z(t) + Rq(t) = E\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + d(t) \\ \\ y(t) = L_{sr}(I_2(t)I_3(t) - I_1(t)I_4(t)) \end{cases} \qquad (6.155)$$

where $z^T = [I_1, I_2, I_3, I_4, \dot{q}] \in I\!R^5$, $u^T = [u_1, u_2, u_3] \in I\!R^3$, $d^T = [0, 0, 0, 0, d_5]$, $L = \mathrm{diag}(L_e, vJ) \in I\!R^{5\times 5}$, $C(z, u_3) = \begin{bmatrix} C_e(u_3, \dot{q}) & -c(\dot{q}) \\ & \\ c^T(\dot{q}) & 0 \end{bmatrix} \in I\!R^{5\times 5}$, $E = \begin{bmatrix} I_2 \\ 0_{3\times 2} \end{bmatrix} \in I\!R^{5\times 2}$, $R = \mathrm{diag}(R_e, vb) \in I\!R^{5\times 5}$. $L_e \in I\!R^{4\times 4}$ is a matrix of inductance, $v \in I\!R$ is the number of pole pairs, $J \in I\!R$ is the rotor inertia, $R_e \in I\!R^{4\times 4}$ is the matrix of resistance, $b \in I\!R$ is the coefficient of motor damping, u_1 and u_2 are stator voltages, u_3 is the primary frequency, I_1 and I_2 are stator currents, I_3 and I_4 are rotor currents, $\dot{q}$ is the rotor angular velocity, $d_5 = -vy_l$ where y_L is the load torque. Finally $y(t) \in I\!R$ is the generated torque, where $L_{sr} \in I\!R$ is the mutual inductance.

It can be shown that this model shares properties with the Euler-Lagrange dynamics. In particular [385] the matrix $C(z, u_3)$ satisfies the skew-symmetry requirement for a certain choice of its definition (which is not unique), and $z^T C(z, u_3)z = 0$ (similarly to workless forces). Also this system is strictly passive with respect to the supply rate $I_1 u_1 + I_2 u_2$, with storage function $H(z) = \frac{1}{2} z^T L z$ and function $S(z) = z^T R z$ (see Definition 4.51).

6.7 Passive Environment

In this section we shall briefly treat systems which may be considered as models of manipulators in contact with their environment through their end-effector or some other body (for instance in assembly tasks or in cooperation with other robots). These systems are part of a more general class of constrained dynamical systems or implicit dynamical systems which constitute still an open problem for their simulation and control. More precisely we shall consider simple mechanical systems which are subject to two types of constraints. First, we shall consider ideal, *i.e.* workless, constraints on the generalized coordinates or velocities which again may be split into integrable constraints which may be expressed on the generalized coordinates and non-holonomic constraints which may solely be expressed in terms of the generalized velocities. Second we shall consider the case when the environment itself is a simple mechanical system and hence consider two simple mechanical systems related by some constraints on their generalized coordinates.

6.7.1 Systems with Holonomic Constraints

Let us consider first a robotic manipulator whose motion is constrained by some m bilateral kinematic constraints, for instance following a smooth surface while keeping in contact. Its model may be expressed as a simple mechanical system (6.43) of order $2n$ with $m < n$ *kinematic constraints* of order zero, and defined by some real function ϕ from the space of generalized coordinates $\mathbb{R}^n$ in $\mathbb{R}^m$:

$$\phi(q) = 0 \tag{6.156}$$

Let us assume moreover that the Jacobian $J(q) = \frac{\partial \phi}{\partial q}$ is of rank m everywhere and the kinematic constraints (6.156) define a smooth submanifold Q_c of $\mathbb{R}^n$. Then by differentiating the constraints (6.156) one obtains kinematic constraints of order 1, defined on the velocities:

$$J(q)\dot{q} = 0 \tag{6.157}$$

The two sets of constraints (6.156) and (6.157) define now a submanifold S on the state space $T\mathbb{R}^n = \mathbb{R}^{2n}$ of the simple mechanical system (6.43):

$$S = \left\{ (q, \dot{q}) \in \mathbb{R}^{2n} : \phi(q) = 0, J(q)\dot{q} = 0 \right\} \tag{6.158}$$

The dynamics of the constrained simple mechanical system is then described by the following system:

$$\begin{cases} M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) = \tau(t) + J^T(q(t))\lambda(t) \\ \\ J(q(t))\dot{q}(t) = 0 \end{cases} \tag{6.159}$$

where $\lambda \in I\!R^m$ is the m dimensional vector of the Lagrangian multipliers associated with the constraint (6.156). They define the reaction forces $F_r = J^T(q)\lambda$ associated with the constraint which enforce the simple mechanical system to remain on the constraint submanifold S defined in (6.158).

Remark 6.51. Note that the constrained system (6.159) may be viewed as a port controlled Hamiltonian system with conjugated port variables λ and $y = J(q)\dot{q}$ interconnected to a power continous constraint relation defined by $y = 0$ and $\lambda \in I\!R^m$. It may then be shown that this defines an *implicit* port controlled Hamiltonian system [121,346]. More general definition of kinematic constraints were considered in [345, 347].

Remark 6.52. Constrained dynamical systems are the subject of numerous works which are impossible to present here in any detail, and we refer the interested reader to [337] for a brief historical presentation and presentation of related Hamiltonian and Lagrangian formulation as well as to [438] for a Hamiltonian formulation in some more system theoretic setting.

Remark 6.53. Note that the kinematic constraint of order zero (6.156) is not included in the definition of the dynamics (6.159). Indeed it is not relevant to it, in the sense that this dynamics is valid for any constraint $\phi(q) = c$ where c is a constant vector and may be fixed to zero by the appropriate initial conditions. ■

One may reduce the constrained system to a simple mechanical system of order $2(n-m)$ by using an adapted set of coordinates as proposed by McClamroch and Wang [324]. Using the Theorem of implicit functions, one may find, locally, a function ρ from $I\!R^{n-m}$ to $I\!R^m$ such that

$$\phi(\rho(q_2), q_2) = 0 \tag{6.160}$$

Then define the change of coordinates:

$$z = \tilde{\mathcal{Q}}(q) = \begin{pmatrix} q_1 - \rho(q_2) \\ q_2 \end{pmatrix} \tag{6.161}$$

Its inverse is then simply

$$q = \mathcal{Q}(z) = \begin{pmatrix} z_1 - \rho(z_2) \\ z_2 \end{pmatrix} \tag{6.162}$$

In the new coordinates (6.161), the constrained simple mechanical system becomes

$$\begin{cases} \tilde{M}(z(t))\ddot{z}(t) + \tilde{C}(z(t), \dot{z}(t))\dot{z}(t) + \tilde{g}(z(t)) = \frac{\partial \mathcal{Q}}{\partial \tilde{q}}^T(t)\tau(t) + \begin{pmatrix} I_m \\ 0_{n-m} \end{pmatrix} \lambda(t) \\ \dot{z}_1(t) = \begin{pmatrix} I_m & 0_{n-m} \end{pmatrix} \dot{z}(t) = 0 \end{cases} \tag{6.163}$$

where the inertia matrix is defined by

$$\tilde{M}(z) = \frac{\partial \mathcal{Q}}{\partial \tilde{q}}^T(\mathcal{Q}(z))M(\mathcal{Q}(\tilde{q}))\frac{\partial \mathcal{Q}}{\partial \tilde{q}}(\mathcal{Q}(z)) \tag{6.164}$$

and $\tilde{g}(z)$ is the gradient of the potential function $\tilde{U}(\mathcal{Q}(z))$. The kinematic constraint is now expressed in a canonical form in (6.163) or in its integral form $z_1 = 0$. The equations in (6.163) may be interpreted as follows: the second equation corresponds to the motion along the tangential direction to the constraints. It is not affected by the interaction force since the constraints are assumed to be frictionless. It is exactly the reduced-order dynamics that one obtains after having eliminated m coordinates, so that the $n-m$ remaining coordinates z_2 are independent. Therefore the first equation must be considered as an algebraic relationship that provides the value of the Lagrange multiplier as a function of the system's state and external forces.

Taking into account the canonical expression of the kinematic constraints, the constrained system may then be reduced to the simple mechanical system of order $2(n-m)$ with generalized coordinates z_2, and inertia matrix (defining the kinetic energy) being the submatrix $\tilde{M}_r(z_2)$ obtained by extracting the last $n-m$ columns and rows from $\tilde{M}(z)$ and setting $z_1 = 0$. The input term is obtained by taking into account the expression of $\mathcal{Q}$ and computing its Jacobian:

$$\frac{\partial \mathcal{Q}^T}{\partial z} = \begin{pmatrix} I_m & 0_{m\times(n-m)} \\ -\frac{\partial \rho}{\partial q_2}(\mathcal{Q}(z) & I_{n-m} \end{pmatrix} \tag{6.165}$$

The reduced dynamics is then a simple mechanical system with inertia matrix $\tilde{M}_r(z)$ and is expressed by

$$\tilde{M}_r(z(t))\ddot{z}(t) + \tilde{C}_r(z(t), \dot{z}(t))\dot{z}(t) + \tilde{g}_r(z(t)) = \left(-\frac{\partial \rho}{\partial q_2}(z_2(t)), I_{n-m}\right)\tau(t) \tag{6.166}$$

The port conjuguated output to τ is then

$$y_r(t) = \begin{pmatrix} -\frac{\partial \rho}{\partial q_2}(q_2(t)) \\ I_{n-m} \end{pmatrix} \dot{z}_2(t) \tag{6.167}$$

Hence the restricted system is passive and lossless with respect to the supply rate $\tau^T y_r$ and storage function being the sum of the kinetic and potential energy of the constrained system.

Remark 6.54. We have considered the case of simple mechanical systems subject to holonomic kinematic constraints, that means kinematic constraints of order 1 in (6.157), that fulfil some integrability conditions which guarantee the existence of kinematic constraints of order 0 (6.156). If this is not the case, the constraints are said to be *non-holonomic*. This means that the system may no more be reduced to a lower order simple mechanical system. As we shall not treat them in the sequel, we do not give a detailed presentation and give a sketch of the results indicating only some references. These systems may still be reduced by choosing an adapted set of velocities (in the case of a Lagrangian formulation) or momenta in the case of a Hamiltonian formulation) and then projecting the dynamics along a subspace of velocities or momenta [95, 337, 440]. This dynamics cannot be expressed as a controlled Lagrangian systems, however it has been proved that it may still be expressed as a port controlled Hamiltonian system for which the structure matrix does not satisfy the Jacobi identities (6.68) [264, 440].

6.7.2 Compliant Environment

The General Dynamics

The general dynamical equations of a rigid joint-rigid link manipulator in permanent contact with an obstacle (that is also a Euler-Lagrange system and can be for instance another –uncontrolled– kinematic chain) are given by

$$\begin{cases} M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + g(q(t)) = \tau(t) + F_q(t) \\ M_e(x(t))\ddot{x}(t) + C_e(x(t),\dot{x}(t))\dot{x}(t) + \frac{dR_e}{d\dot{x}}(t) + g_e(x(t)) + K_e x(t) = F_x(t) \end{cases} \tag{6.168}$$

where $q(t) \in I\!R^n$, $x(t) \in I\!R^m$, $m < n$, $F_q(t) \in I\!R^n$ and $F_x(t) \in I\!R^m$ represent the generalized interaction force in coordinates q and in coordinates x respectively. In other words, if $x = \phi(q)$ for some function $\phi(\cdot)$, then $\dot{x} = \frac{d\phi}{dx}^T(q)\dot{q} = J(q)\dot{q}$, and $F_q = J^T(q)F_x$. If we view the system in (6.168) as a whole, then the interaction force becomes an internal force. The virtual work principle (for the moment let us assume that all contacts are frictionless) tells us that for any virtual displacements δq and δx, one has $\delta x^T F_x = -\delta q^T F_q$. This can also be seen as a form of the principle of mutual actions. Let us further assume that $\text{rank}(\phi) = m$ and that $K_e > 0$. Let us note that the relation $x = \phi(q)$ relates the generalized displacements of the controlled subsystem to those of the uncontrolled one, *i.e.* to the deflexion of the environment. With this in mind, one can define following McClamroch a nonlinear transformation $q = \mathcal{Q}(z)$, $z = \mathcal{Q}^{-1}(q) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} K_e\phi(q_1,q_2) \\ q_2 \end{bmatrix}$, $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \Omega(z_1,z_2) \\ z_2 \end{bmatrix}$, $\dot{q} = T(z)\dot{z}$, with $T(z) = \begin{bmatrix} \frac{\partial \Omega}{\partial z_1}^T & \frac{\partial \Omega}{\partial z_2}^T \\ 0 & I_{n-m} \end{bmatrix}$, where $z_1(t) \in I\!R^m$, $z_2(t) \in I\!R^{n-m}$, and $\phi(\Omega(z_1,z_2),z_2) = z_1$ for all z in the configuration space. Notice that from the rank assumption on $\phi(q)$ and due to the procedure to split z into z_1 and z_2

(using the implicit function Theorem), the Jacobian $T(z)$ is full-rank. Moreover $z_2 = q_2$ where q_2 are the $n-m$ last components of q. In new coordinates z one has $z_1 = x$ and

$$\begin{cases} \bar{M}(z(t))\ddot{z}(t) + \bar{C}(z(t),\dot{z}(t))\dot{z}(t) + \bar{g}(z(t)) = \bar{\tau}(t) + \begin{pmatrix} \lambda_{z_1}(t) \\ 0 \end{pmatrix} \\ M_e(z_1(t))\ddot{z}_1(t) + C_e(z_1(t),\dot{z}_1(t))\dot{z}_1(t) + \frac{dR_e}{dz_1}(t) + g_e(z_1(t)) + \\ +K_e z_1(t) = -\lambda_{z_1}(t) \end{cases} \tag{6.169}$$

where $\lambda_{z_1}(t) \in I\!R^m$, $\bar{M}(z) = T(z)^T M(q) T(z)$, and $\bar{\tau} = T^T(z)\tau$. In a sense this coordinate change splits the generalized coordinates into "normal" direction z_1 and "tangential" direction z_2, similarly as in Subsection 6.7.1. The virtual work principle tells us that $\delta z^T F_z = -\delta z_1 \lambda_{z_1}$ for all virtual displacement δz, hence the form of F_z in (6.169) where the principle of mutual actions clearly appears. The original system may appear as having $n+m$ degrees of freedom. However since the two subsystems are assumed to be bilaterally coupled, the number of degrees of freedom is n. This is clear once the coordinate change in (6.169) has been applied. The system in (6.168) once again has a cascade form where the interconnection between both subsystems is the contact interaction force.

Remark 6.55. An equivalent representation as two passive blocks is shown in Figure 6.6. As an exercise one may consider the calculation of the storage functions associated to each block.

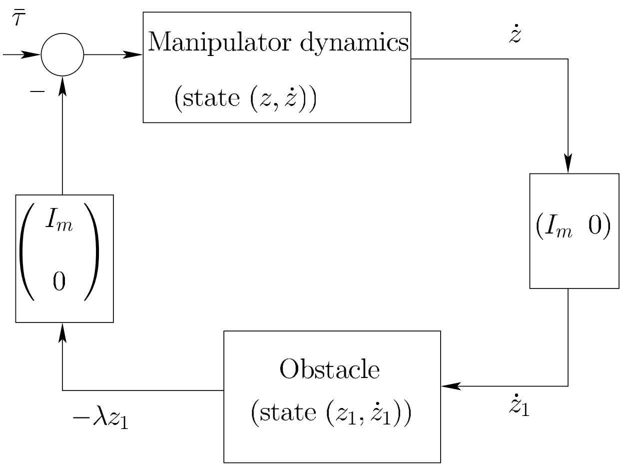

Fig. 6.6. Manipulator in bilateral contact with a dynamical passive obstacle

Dissipativity Properties

Let us assume that the potential energy terms $U_g(z)$ and $U_{g_e}(z_1)$ are bounded from below. This assumption is clearly justified by the foregoing developments on passivity properties of Euler-Lagrange systems. Now it is an evident choice that the suitable supply rate is given by $(\bar{\tau}^T + F_z^T)\dot{z} - \lambda_{z_1}^T \dot{z}_1$. Notice that although one might be tempted to reduce this expression to $\bar{\tau}^T \dot{z}$ since $F_z^T \dot{z} = \lambda_{z_1}^T \dot{z}_1$, it is important to keep it since they do represent the outputs and inputs of different subsystems: one refers to the controlled system while the other refers to the uncontrolled obstacle. Let us calculate the available storage of the total system in (6.169):

$$
\begin{aligned}
V_a(z,\dot{z}) &= \sup_{\bar{\tau}:(0,z(0),\dot{z}(0))\rightarrow} - \int_0^t \left\{(\bar{\tau}^T + F_z^T)\dot{z} - \lambda_{z_1}^T \dot{z}_1\right\} ds \\
&= \tfrac{1}{2}\dot{z}^T(0)M(z(0))\dot{z}(0) + \tfrac{1}{2}\dot{z}_1^T(0)M_e(z_1(0))z_1(0) \\
&+ \tfrac{1}{2}z_1^T(0)K_e z_1(0) + U_g(z(0)) + U_{g_e}(z_1(0))
\end{aligned}
\tag{6.170}
$$

Hence the system is dissipative since $V_a(\cdot)$ is bounded for bounded state. Since we introduced some Rayleigh dissipation in the environment dynamics, the system has some strict passivity property.

6.8 Nonsmooth Lagrangian Systems

The material in this section may be seen as the continuation of what we exposed in Chapter 3, Sections 3.9.4 and 3.9.5. The notation is the same.

6.8.1 Systems with C^0 Solutions

Let us introduce a class of nonsmooth Lagrangian systems, which are mechanical systems subject to some nonsmooth friction forces. Let $\Phi : I\!R^l \rightarrow I\!R \cup \{+\infty\}$ be a convex proper and lower semicontinuous function. Let $M = M^T > 0 \in I\!R^{n\times n}$, $C \in I\!R^{n\times n}$, $K \in \ RR^{n\times n}$, $H_1 \in I\!R^{n\times l}$, $H_2 \in I\!R^{l\times n}$ be constant matrices. For $(t_0, q_0, \dot{q}_0) \in I\!R \times I\!R^n \times I\!R^n$, with $H_2\dot{q}_0 \in D(\partial\Phi)$, we consider the problem [3]: Find a function $q : t \mapsto q(t)$ $(t \geq t_0)$ with $q \in C^1([t_0, +\infty); I\!R^m)$ such that

- (i) $\ddot{q}(\cdot) \in \mathcal{L}_{\infty,e}([t_0, +\infty); I\!R^m)$
- (ii) $\dot{q}(\cdot)$ is right-differentiable on $[t_0, +\infty)$
- (iii) $q(t_0) = q_0$, $\dot{q}(t_0) = \dot{q}_0$
- $$H_2\dot{q}(t) \in dom(\partial\Phi) \tag{6.171}$$

  for all $t \geq t_0$

- $$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \in -H_1 \partial \Phi(H_2 \dot{q}(t)) \tag{6.172}$$

  a.e. on $[t_0, +\infty)$.

We recall that $dom(\partial \Phi)$ denotes the domain of the subdifferential of the convex function $\Phi(\cdot)$. The term $H_1 \partial \Phi(H_2 \cdot)$ is supposed to model the unilaterality of the contact induced by friction forces (for instance the Coulomb friction model). Unilaterality is not at the position level as it is in the next section, but at the velocity level. This is important because it means that the solutions are much more "gentle". Notice that if the system is considered as a first order differential system, then as the section title indicates solutions $(q(\cdot), \dot{q}(\cdot))$ are time continuous.

Theorem 6.56. *[3] Suppose that*

- *(a) There exists a matrix $R = R^T \in \mathbb{R}^{n \times n}$, nonsingular, such that*

$$R^{-2} H_2^T = M^{-1} H_1 \tag{6.173}$$

- *(b) There exists $y_0 = H_2 R^{-1}$ $(x_0 \in \mathbb{R}^n)$ at which $\Phi(\cdot)$ is finite and continuous.*

Let $t_0 \in \mathbb{R}$, $q_0, \dot{q}_0 \in \mathbb{R}^n$ with $H_2 \dot{q}_0 \in dom(\partial \Phi)$. Then there exists a unique $q \in C^1([t_0, +\infty); \mathbb{R}^m)$ satisfying conditions (i) (ii) (iii) and (6.171) and (6.172).

■

We do not go into the details of the proof of this well-posedness result. Let us just mention that thanks to the existence of the matrix R one can perform a variable change $z = Rq$ which allows one to rewrite the system as a first-order system

$$\begin{cases} \dot{x}(t) + Ax(t) \in -\partial \phi(x) \\ \\ x(t_0) = x_0 \end{cases} \tag{6.174}$$

with $A = \begin{pmatrix} 0_{n \times n} & -I_n \\ \\ RM^{-1}KR^{-1} & RM^{-1}CR^{-1} \end{pmatrix}$, $x = \begin{pmatrix} z \\ \dot{z} \end{pmatrix}$. The function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous and is defined as $\phi(x) = \chi(\dot{z})$, with $\chi(\dot{z}) = (\Phi \circ H_2 R^{-1})(\dot{z})$. The well-posedness of the system in (6.174) can be shown relying on a Theorem quite similar to Theorem 3.80, with a slight difference as the variational inequality that concerns (6.174) is of the form

$$\langle \dot{x}(t) + Ax(t), v - x(t) \rangle + \phi(v) - \phi(x(t)) \geq 0, \ \ \forall\, v \in \mathbb{R}^n, \ \text{ a.e. in } [t_0, +\infty) \tag{6.175}$$

This reduces to (3.213) if one chooses $\phi(\cdot)$ as the indicator function of a convex set K, and is in turn equivalent to an unbounded differential inclusion. Indeed one has

$$
\begin{array}{c}
\langle Mu + q, v - u \rangle + \phi(v) - \phi(u) \geq 0, \;\; \forall \;\; v \in I\!R^n \\
\Updownarrow \\
Mu + q \in -\partial\phi(u)
\end{array}
\tag{6.176}
$$

for any proper, convex, lower semicontinuous function with closed domain, $M \in I\!R^{n\times n}$, $q \in I\!R^n$. The stability analysis of these mechanical systems will be led in Section 7.2.5.

6.8.2 Systems with BV Solutions

We deal in this section with mechanical Lagrangian systems, subject to frictionless unilateral constraints on the position and impacts. This material is necessary to study the stability issues as will be done in Section 7.2.4. There are some tools from convex analysis which have already been introduced in Section 3.9.4, and which are useful in this setting as well. More precisely, let us consider the following class of unilaterally constrained mechanical systems

$$
\begin{cases}
M(q(t))\ddot{q}(t) + F(q(t),\dot{q}(t)) = \nabla h(q(t))\lambda(t) \\
q(0) = q_0, \dot{q}(0^-) = \dot{q}_0 \\
0 \leq h(q(t)) \perp \lambda(t) \geq 0 \\
\dot{q}(t_k^+) = -e\dot{q}(t_k^-) + (1+e)\mathrm{prox}_{M(q(t_k))}[\dot{q}(t_k^-), V(q(t_k))]
\end{cases}
\tag{6.177}
$$

In (6.177) $M(q) = M^T(q) > 0$ is the $n \times n$ inertia matrix, $F(q,\dot{q}) = C(q,\dot{q})\dot{q} + \frac{\partial U}{\partial q}(q)$ where $C(q,\dot{q})\dot{q}$ denotes centripetal and Coriolis generalized forces, whereas $U(q)$ is a smooth potential energy from which conservative forces derive, and $h(\cdot) : I\!R^n \mapsto I\!R^m$. We assume that $h(q_0) \geq 0$. The set $V(q)$ is the tangent cone to the set $\Phi = \{q \in I\!R^n \;\mid\; h(q) \geq 0\}$; see (3.193): $V(q) = T_\Phi(q)$. The impact times are generically denoted as t_k, the left-limit $\dot{q}(t_k^-) \in -V(q(t_k))$ whereas the right-limit $\dot{q}(t_k^+) \in V(q(t_k))$. The third line in (6.177) is a collision mapping that relates pre- and post-impact generalized velocities, and $e \in [0,1]$ is a restitution coefficient [327]. The notation $\mathrm{prox}_{M(q)}$ means the proximation in the kinetic metric, *i.e.* the metric defined as $x^T M(q) y$ for $x, y \in I\!R^n$: the vector $\frac{\dot{q}(t_k^+)+e\dot{q}(t_k^-)}{1+e}$ is the closest vector to the pre-impact velocity, inside $V(q(t_k))$ (it can therefore be computed through a quadratic programme) [358]. In particular the impact law in (6.177) implies that the kinetic energy loss at time t_k satisfies (see [327], [69, p.199, p.489], [34])

$$T_L(t_k) = -\frac{1}{2}\frac{1-e}{1+e}\left(\dot{q}(t_k^+) - \dot{q}(t_k^-)\right)^T M(q(t_k))\left(\dot{q}(t_k^+) - \dot{q}(t_k^-)\right) \leq 0 \quad (6.178)$$

Remark 6.57. The formulation of the unilateral constraints in (6.177) does not encompass all closed domains $\Phi = \{q \mid h(q) \geq 0\}$, as simple non-convex cases with so-called reentrant corners prove [77]. It can be used to describe admissible domains Φ which are defined either by a single constraint (i.e. $m = 1$), or with $m < +\infty$ where convexity holds at nondifferentiable points of the boundary $\partial\Phi$ (such sets are called *regular* [105]). It is easy to imagine physical examples that do not fit within this framework, *e.g.* a ladder.

Let us note that the tangent cone $V(q(t))$ is assumed to have its origin at $q(t)$ so that $0 \in V(q(t))$ to allow for post-impact velocities tangential to the admissible set boundary $\partial\Phi$. The second line in (6.177) is a set of complementarity conditions between $h(q)$ and λ, stating that both these terms have to remain non-negative and orthogonal one to each other. Before passing to the well-posedness results for (6.177), let us define a function of bounded variation.

Definition 6.58. *Let $f : [a,b] \to I\!R$ be a function, and let the* total variation *of $f(\cdot)$ be defined as*

$$\mathcal{V}(x) = \sup \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})|, \quad (a \leq x \leq b) \quad (6.179)$$

where the supremum is taken along all integers N, and all possible choices of the sequence $\{t_i\}$ such that

$$a = t_0 < t_1 < .... < t_N = x$$

The function $f(\cdot)$ is said of bounded variation *on $[a,b]$ if $\mathcal{V}(b) < +\infty$.* ■

One should not confuse BV functions with piecewise continuous functions. We say that a function $f : I \to J$ is piecewise continuous if there exists a constant $\delta > 0$ and a finite partition of I into intervals (a_i, b_i), with $I = \cup_i [a_i, b_i]$, and $b_i - a_i \geq \delta$ for all i, and $f(\cdot)$ is continuous on each (a_i, b_i) with left limit at a_i and right-limit at b_i. Thus piecewise continuous functions are a different class of functions. There are well-known examples of continuous functions which are not BV, like $f : x \mapsto x \sin(\frac{1}{x})$ defined on $[0,1]$. Clearly $f(0) = 0$ but the infinite oscillations of $f(\cdot)$ as x approaches 0 hamper the bounded variation. In addition, piecewise continuity precludes finite accumulations of discontinuities. BV functions are such that given any t there exists $\sigma > 0$ such that the function is continuous on $(t, t+\sigma)$. But this σ may not be uniform with respect to t. Definition 6.58 holds whatever the function $f(\cdot)$, even if $f(\cdot)$

is not AC. One may consult [357] for more informations on BV functions. One speaks of local bounded variation (LBV) functions when $f : \mathbb{R} \to \mathbb{R}$ and $f(\cdot)$ is BV on all compact intervals $[a, b]$. LBV functions possess very interesting properties, some of which are recalled below.

Assumption 14 *The gradients* $\nabla h_i(q) = \frac{\partial h}{\partial q}^T(q)$ *are not zero at the contact configurations* $h_i(q) = 0$*, and the vectors* ∇h_i*,* $1 \le i \le m$*, are independent. Furthermore the functions* $h(\cdot)$*,* $F(q, \dot{q})$*,* $M(q)$ *and the system's configuration manifold are real analytic, and* $||F(q, \dot{q})||_q \le d(q, q(0)) + ||\dot{q}||_q$*, where* $d(\cdot, \cdot)$ *is the Riemannian distance and* $||\cdot||_q$ *is the norm induced by the kinetic metric.*

Then the following results hold, which are essentially a compilation of Proposition 32, Theorems 8 and 10, and Corollary 9 of [34]:

i) Solutions of (6.177) exist on $[0, +\infty)$ such that $q(\cdot)$ is absolutely continuous (AC), whereas $\dot{q}(\cdot)$ is right-continuous of local bounded variation (RCLBV). In particular the left and right-limits of these functions exist everywhere.
ii) The function $q(\cdot)$ cannot be supposed to be everywhere differentiable. One has $q(t) = q(0) + \int_0^t v(s)ds$ for some function $v(\cdot) \overset{\text{a.e.}}{=} \dot{q}(\cdot)$. Moreover $\dot{q}(t^+) = v(t^+)$ and $\dot{q}(t^-) = v(t^-)$ [270].
iii) Solutions are unique (however in general they do not depend continuously on the initial conditions).
iv) The acceleration $\ddot{q}$ is a *measure* dv, which is the sum of two measures: an atomic measure $d\mu_a$, and a Lebesgue integrable function which we denote $\ddot{q}(\cdot)$, *i.e.* $dv = d\mu_a + \ddot{q}(t)dt$. The atoms correspond to the impact times [358]. See Remark 6.59 for some comments on this decomposition.
v) The set of impact times is countable. In many applications one has $d\mu_a = \sum_{k \ge 0} [\dot{q}(t_k^+) - \dot{q}(t_k^-)]\delta_{t_k}$, where δ_t is the Dirac measure and the sequence $\{t_k\}_{k \ge 0}$ can be ordered, i.e. $t_{k+1} > t_k$. However phenomena like accumulations of left-accumulations of impacts may exist (at least bounded variation does not preclude them). In any case the ordering may not be possible. This is a sort of complex Zeno behaviour ([2]). In the case of elastic impacts ($e = 1$) it follows from [34, Prop.4.11] that $t_{k+1} - t_k \ge \delta > 0$ for some $\delta > 0$. Hence solutions are piecewise continuous in this case.
vi) Any quadratic function $W(\cdot)$ of $\dot{q}$ is itself RCLBV, hence its derivative is a measure dW [358]. Consequently $dW \le 0$ has a meaning and implies that the function $W(\cdot)$ does not increase [127, p.101].

These results enable one to lead a stability analysis safely. Let us now introduce a new formulation of the dynamics in (3.205), which can be written as the following Measure Differential Inclusion (MDI): [358]

[2] *I.e.* all phenomena involving an infinity of events in a finite time interval, and which occur in various types of hybrid systems like Filippov's inclusions, *etc.*

$$-M(q(t))dv - F(q(t), v(t^+))dt \in \partial\psi_{V(q(t))}(w(t)) \subseteq \partial\psi_{\Phi}(q(t)) \tag{6.180}$$

where $w(t) = \frac{v(t^+)+ev(t^-)}{1+e} \in \partial V(q(t))$ from (6.177). If $e = 0$ then $w(t) = v(t^+)$, if $e = 1$ then $w(t) = \frac{v(t^+)+v(t^-)}{2}$. Moreover when $v(\cdot)$ is continuous then $w(t) = v(t)$. The term MDI has been coined by Moreau, and (6.180) may also be called Moreau's second order sweeping process. The inclusion in the right-hand-side of (6.180) is proved as follows: for convenience let us rewrite the following definitions for a convex set Φ:

$$N_{\Phi}(q) = \{z \mid z^T\xi \leq 0, \forall \xi \in V(q)\} \tag{6.181}$$

which precisely means that the normal cone is the polar cone of the tangent cone (see Definition 3.67), and

$$\partial\psi_{V(q)}(w) = \{z \mid z^T(\eta - \dot{q}) \leq 0, \forall \eta \in V(q)\} \tag{6.182}$$

Since $V(q)$ as it is defined in (3.193) (replace K by Φ) is a cone and since $\dot{q} \in V(q)$, one can choose $\eta = \xi + \dot{q}$ with $\xi \in V(q)$ as a particular value for η. Thus if $z \in \partial\psi_{V(q)}(w)$ one gets $z^T\eta \leq z^T\dot{q}(t^+)$ and introducing $\eta = \xi + \dot{q}$, one gets $z^T\xi \leq 0$ so that $z \in N_{\Phi}(q)$. Therefore Moreau's inclusion in (6.180) is proved.

Let us note that the cones are to be understood as being attached to the same origin in the inclusion. Moreover some natural identifications between spaces (the dual $T^*_{\dot{q}}T_q\mathcal{Q}$ at $\dot{q}$ of the tangent space $T_q\mathcal{Q}$ at q to the configuration space $\mathcal{Q}$, and the cotangent space $T^*_q\mathcal{Q}$) have been made, thanks to the linear structure of these spaces in which the cones $\partial\psi_{V(q)}(\cdot)$ and $N_{\Phi}(q)$ are defined. This allows one to give a meaning to the inclusion in (6.180). This is just a generalization of the well-known identification between the space of velocities and that of forces acting on a particle in a three-dimensional space, which both are identified with $I\!R^3$. More details are in [34] and [358].

What happens at impact times? It is well-known in Mechanics that the dynamics become algebraic at an impact time [69]. Such is the case for the measure differential inclusion in (6.180). Let x and z be two vectors of a linear Euclidean space E, V be a closed convex cone of E, and N be the polar cone to V. Then from Moreau's Lemma of the two cones [210, p.51] [69, lemma D1], one has $(x - z) \in \partial\psi_V(x) \iff x = \text{prox}[V, z] \iff z - x = \text{prox}[N, z]$. Times t_k are atoms of the measure dv in (6.180). *Via* a suitable base change, the kinetic metric at an impact time can be considered as a Euclidean metric since $q(\cdot)$ is continuous at t_k, and in particular all the identifications between various dual spaces can be done. One gets from (6.180): $-M(q(t_k))[\dot{q}(t_k^+) - \dot{q}(t_k^-)] \in \partial\psi_{V(q(t_k))}(w(t_k^+)) \iff \dot{q}(t_k^+) + e\dot{q}(t_k^-) = \text{prox}_{M(q(t_k))}[V(q(t_k)), (1 + e)\dot{q}(t_k^-)] \iff \dot{q}(t_k^+) + e\dot{q}(t_k^-) = (1 + e)\text{prox}_{M(q(t_k))}[V(q(t_k)), \dot{q}(t_k^-)$, where the second equivalence is proved in [358].

When $\dot{q}(t)$ is discontinuous, (6.180) implies that Moreau's collision rule in (6.177) is satisfied. The term $\psi_{V(q(t))}(w(t))$ can be interpreted as a velocity potential. The MDI in (6.180), whose left-hand-side is a measure and whose right-hand-side is a cone, has the following meaning [327, 357]: there exists a positive measure $d\mu$ such that both dt and dv possess densities with respect to $d\mu$, denoted respectively as $\frac{dt}{d\mu}(\cdot)$ and $\frac{dv}{d\mu}(\cdot)$. One also has $\frac{dt}{d\mu}(t) = \lim_{\epsilon \to 0, \epsilon > 0} \frac{dt([t,t+\epsilon])}{d\mu([t,t+\epsilon])}$ [360], [357, p.9], which shows the link with the usual notion of a derivative. The choice of $d\mu$ is not unique because the right-hand-side is a cone [358]. However by the Lebesgue-Radon-Nikodym Theorem [419], the densities $\frac{dt}{d\mu}(\cdot)$ and $\frac{dv}{d\mu}(\cdot)$ are unique functions for a given $d\mu$. To shed some light on this, let us consider for instance $d\mu = dt + \Sigma_{k \geq 0} \delta_{t_k}$, which corresponds to applications where the system is subject to impacts at times t_k and otherwise evolves freely. Then $\frac{dt}{d\mu}(t_k) = 0$ (the Lebesgue measure dt and the Dirac measure δ_t are mutually singular) whereas $\frac{dv}{d\mu}(t_k) = v(t_k^+) - v(t_k^-)$ (t_k is an atom of the measure dv). When $t \neq t_k$ then $\frac{dt}{d\mu}(t) = 1$ and $\frac{dv}{d\mu}(t) = \dot{v}(t)$.

Therefore the meaning of (6.180) is that there exists a positive measure $d\mu$ with respect to which both dt and dv possess densities, and

$$-M(q(t))\frac{dv}{d\mu}(t) - F(q(t), v(t^+))\frac{dt}{d\mu}(t) \in \partial\psi_{V(q(t))}(w(t)) \subseteq \partial\psi_{\Phi}(q(t)) \quad (6.183)$$

holds $d\mu-$almost everywhere. In a sense, densities replace derivatives, for measures. When dealing with measure differential equations or inclusions, it is then natural to manipulate densities instead of derivatives. In general one can choose $d\mu = |dv| + dt$ [357, p.90], where $|dv|$ is the absolute value of dv, or $d\mu = ||v(t)||dt + d\mu_a$, or $d\mu = dt + d\mu_a$. It is fundamental to recall at this stage, that the solution of (6.183) does not depend on this choice. For instance, if $d\mu = ||v(t)||dt + d\mu_a$ then for all $t \neq t_k$, $\frac{dt}{d\mu}(t) = \frac{1}{||v(t)||}$ and $\frac{dv}{d\mu}(t) = \frac{\ddot{q}(t)}{||v(t)||}$. Whereas if $d\mu = dt + d\mu_a$ then for all $t \neq t_k$, $\frac{dt}{d\mu}(t) = 1$ and $\frac{dv}{d\mu}(t) = \ddot{q}(t)$.

Remark 6.59. The above mathematical framework is more than just a mathematical fuss. Indeed as noted in [358], introducing the velocity into the right-hand-side of the dynamics as done in (6.180), not only allows one to get a compact formulation of the nonsmooth dynamics (see Figure 6.7 in this respect), but it also paves the way towards the consideration of friction in the model. In turn it is clear that introducing friction, is likely to complicate the dynamics. Especially the above framework paves the way towards more complex cases where the measure dv may contain a third term $d\mu_{na}$ which is a nonatomic measure singular with respect to the Lebesgue measure dt (assumption 14 implies that $d\mu_{na} = 0$ [34]). In summary the dynamics in (6.183) is rich enough to encompass complex behaviours involving solutions which may be far from merely piecewise continuous. This is a consequence of replacing functions by the more general notion of measure, at the price of a more involved model. In fact using measures allows one to encompass somewhat complex

Zeno behaviours occurring in unilaterally constrained mechanical systems in a rigorous manner.

Dissipative Systems Interpretation

Let us end this section with a dissipativity interpretation of Moreau's inclusion in (6.180). The dynamics in (6.180) has the interpretation as in Figure 6.7, where $\xi \in \partial\psi_{V(q(t))}(w(t))$. Since $\partial\psi_{V(q(t))}(w(t)) \subseteq N_{\Phi}(q) = V^{\star}(q)$ (the cone polar to $V(q)$), the feedback loop in Figure 6.7 contains the cone complementarity problem (or complementarity problem over a cone [113, p.31])

$$N_{\Phi}(q) \supseteq \partial\psi_{V(q(t))}(w(t)) \ni \xi \perp w(t) \in V(q) \tag{6.184}$$

When $m = 1$ and $q \in \partial\Phi$, one has $V(q) = I\!R^{+}$ and $N_{\Phi}(q) = I\!R^{-}$ in a suitable frame attached to q, and the graph of the multivalued mapping is the so-called corner law. In general this is an example of an m-dimensional monotone multivalued mapping $w(t) \mapsto \xi$. It is noteworthy that the feedback loop in Figure 6.7 contains both the complementarity conditions and the collision mapping in (6.177). A quite similar structure can be found for the dynamics in (3.198).

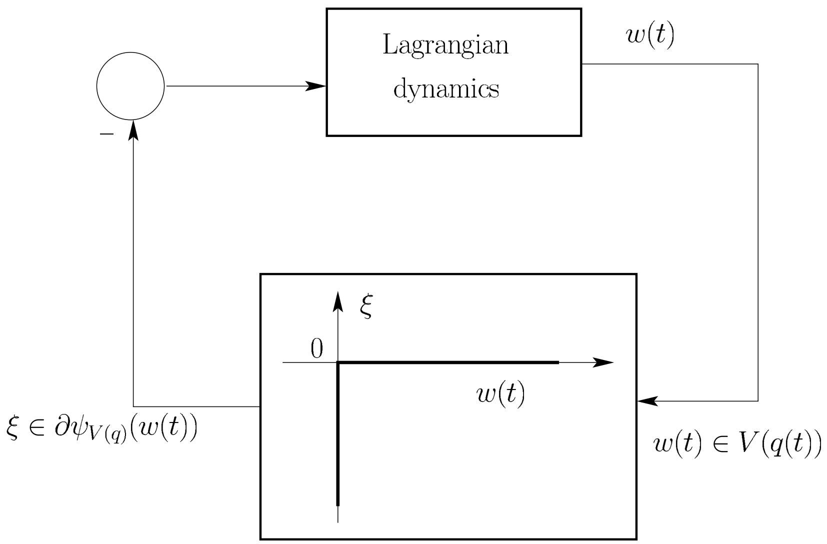

Fig. 6.7. Unilaterally constrained Lagrangian system

This interpretation of the dynamics motivates us to search for a dissipation equality applying to (6.177), with a supply rate handling both continuous and discontinuous motions. This will be done in Section 7.2.4 when the manipulations leading the stability analysis have been presented.

7

Passivity-based Control

This chapter is devoted to investigating how the dissipativity properties of the various systems examined in the foregoing chapter can be used to design stable and robust feedback controllers. We start with a classical result of mechanics, which actually is the basis of Lyapunov stability and Lyapunov functions theory. The interest of this result is that its proof hinges on important stability analysis tools, and allows one to make a clear connection between Lyapunov stability and dissipativity theory. The next section is a brief survey on passivity-based control methods, a topic that has been the object of numerous publications.

7.1 Brief Historical Survey

The fundamental works on dissipative systems and positive real transfer function which are exposed in the foregoing chapters have been mainly motivated by the stability and stabilization of electrical networks. It is only at the beginning of the 1980s that work on mechanical systems and the use of dissipativity in their control started to appear, with the seminal paper by Takegaki and Arimoto [484]. Roughly speaking, two classes of feedback controllers have emerged:

- **Passivity-based controllers:** the control input is such that the closed-loop system can be interpreted as the negative interconnection of two dissipative subsystems. The Lyapunov function of the total system is close to the process total energy, in the sense that it is the sum of a quadratic function $\frac{1}{2}\zeta^T M(q)\zeta$ for some ζ depending on time, generalized positions q and velocities $\dot{q}$, and a term looking like a potential energy. Sometimes additional terms come into play, like in adaptive control where the on-line estimation algorithm provides supplementary state variables. Such algorithms have been motivated originally by trajectory tracking and adaptive

motion control of fully actuated robot manipulators. The machinery behind this is dissipative systems and Lyapunov stability theory. This chapter will describe some of these schemes in great detail, consequently we do not insist on passivity-based controllers in this short introduction.
- **Controlled Lagrangian (or Hamiltonian):** the objective is not only to get a two-block dissipative interconnection, but also to preserve a Lagrangian (or a Hamiltonian) structure in closed-loop. In other words, the closed-loop system is itself a Lagrangian (or a Hamiltonian) systems with a Lagrangian (or Hamiltonian) function, and its dynamics can be derived from a variational principle such as Hamilton's principle. In essence, one introduces a feedback that changes the kinetic energy tensor $M(q)$. Differential geometry machinery is the underlying tool. The same applies to port-Hamiltonian systems which we saw in Chapter 6. Regulation tasks for various kind of systems (mechanical, electromechanical, underactuated) have been the original motivations of such schemes. The method is described in Section 7.9.

Related terms are potential energy shaping, energy shaping, damping injection or assigment, energy balancing. The very starting point for all those methods, is the Lagrange-Dirichlet (or Lejeune-Dirichlet) Theorem which is described in Section 7.2. It is difficult to make a classification of the numerous schemes that have been developed along the above two main lines. Indeed this would imply to highlight the discrepancies between:

- Trajectory tracking *vs* regulation
- Full actuation *vs* underactuation
- Fixed parameters *vs* adaptive control
- Static feedback *vs* dynamic feedback
- Smooth systems *vs* nonsmooth systems
- Constrained systems *vs* unconstrained systems
- Rigid systems *vs* flexible systems
- *etc.*

We will therefore rather present the contributions in a chronological order, as they appeared in the literature. As said above, the starting point may be situated in 1981 with [484]. The challenge then in the Systems and Control and the Robotics communities was about nonlinear control of fully actuated manipulators for trajectory tracking purpose, and especially the design of a scheme allowing for parameter adaptive control. The first robot adaptive control algorithms were based on tangent linearization techniques [216]. Then two classes of schemes emerged: those requiring an inversion of the dynamics and acceleration measurement or inversion of the inertia matrix $M(q)$ [7,114, 115,477], and those avoiding such drawbacks [215,255,352,425–427,461–463]. Despite the fact that they were not originally designed with dissipativity in mind, the schemes of the second class were all proved to belong to passivity-based schemes in [71] (the schemes in [425,461] were proved to be hyperstable in [275], while the term passivity-based was introduced in [384]). Then many schemes have been designed, which more or less are extensions of the previous ones but adapted to constrained systems, systems in contact with a flexible environment, *etc.*

The next step, as advocated in [384], was to solve the trajectory tracking problem in the adaptive control context, for flexible joint robots. This was done in [70, 72, 316, 318], using what has been called afterwards *backstepping*, together with a specific parametrization to guarantee the linearity in the unknown parameters, and a differentiable parameter projection. The adaptive control of flexible joint manipulators is a non-trivial problem combining these three ingredients. See [78] for further comparisons between this scheme and schemes designed with the backstepping approach, in the fixed parameters case. Almost at the same time the regulation problem with passivity-based control of induction motors was considered in [385, 386], using a key idea of [316, 318]. The control of induction motors then was a subject of excitation for several years.

Later came controlled Lagrangian and Hamiltonian methods as developed by Bloch, Leonard, Mardsen [54, 55] and in [52, 387], to cite a few.

7.2 The Lagrange-Dirichlet Theorem

In this section we present a stability result that was first stated by Lagrange in 1788 and subsequently proved rigorously by Dirichlet. It provides sufficient conditions for a conservative mechanical system to possess a Lyapunov stable fixed point. The case of Rayleigh dissipation is also presented. The developments base on the dissipativity results of Chapter 4.

7.2.1 Lyapunov Stability

Let us consider the Euler-Lagrange dynamics in (6.1) or that in (6.43). Let us further make the following:

Assumption 15 *The potential energy* $U(q)$ *is such that* **i)** $\frac{dU}{dq}(q) = 0 \Leftrightarrow q = q_0$ *and* **ii)** $\frac{d^2U}{dq^2}(q_0) > 0$. ■

In other words $U(q)$ is locally convex around $q = q_0$ and q_0 is a critical point of the potential energy. Hence the point $(q_0, 0)$ is a fixed point of the dynamics in (6.1). Then it follows that the considered system with input τ, output $\dot{q}$ and state $(q - q_0, \dot{q})$, is zero-state observable (see Definition 5.12). Indeed if $\tau \equiv 0$ and $\dot{q} \equiv 0$, it follows from (6.43) that $g(q) = \frac{dU}{dq} = 0$, hence $q = q_0$. The following is then true:

Theorem 7.1 (Lagrange-Dirichlet). *Let Assumption 15 hold. Then the fixed point* $(q, \dot{q}) = (q_0, 0)$ *of the unforced system in (6.1) is Lyapunov stable.* ■

Proof: First of all notice that the local (strict) convexity of $U(q)$ around q_0 precludes the existence of other q arbitrarily close to q_0 and such that $\frac{dU}{dq}(q_1) = 0$. This means that the point $(q_0, 0)$ is a strict local minimum for the total energy $E(q, \dot{q})$. We have seen that $E(q, \dot{q})$ is a storage function provided that $U(q)$ remains positive. Now it suffices to define a new potential energy as $U(q) - U(q_0)$ to fulfil this requirement, and at the same time to guarantee that the new $E(q, \dot{q})$ satisfies $E(0, 0) = 0$, and is a positive definite function (locally at least) of $(0, 0)$. Since this is a storage function, we deduce from the dissipation inequality (which is actually here an equality) that for all $\tau \equiv 0$ one gets

$$E(0) = E(t) - \int_0^t \tau^T(s)\dot{q}(s)ds = E(t) \tag{7.1}$$

Therefore the fixed point of the unforced system is locally Lyapunov stable. Actually we have just proved that the system evolves on a constant energy level (what we already knew) and that the special form of the potential energy implies that the state remains close enough to the fixed point when initialized close enough to it. Notice that (7.1) is of the type (4.75) with $\mathcal{S}(x) = 0$: the system is lossless. All in all, we did not make an extraordinary progress. Before going ahead with asymptotic stability, let us give an illustration of Theorem 7.1.

Example 7.2. Let us consider the dynamics of planar two-link revolute joint manipulator with generalized coordinates the link angles (q_1, q_2) (this notation is not to be confused with that employed for the flexible joint-rigid link manipulators). We do not need here to develop the whole stuff. Only the potential energy is of interest to us. It is given by

$$U(q) = a_1 \sin(q_1) + a_2 \sin(q_1 + q_2) \tag{7.2}$$

where $a_1 > 0$ and $a_2 > 0$ are constant depending on masses, dimensions and gravity. It is easy to see that $\frac{dU}{dq} = \begin{pmatrix} a_1 \cos(q_1) + a_2 \cos(q_1 + q_2) \\ a_2 \cos(q_1 + q_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ implies that $q_1 + q_2 = (2n + 1)\frac{\pi}{2}$ and $q_1 = (2m + 1)\frac{\pi}{2}$ for $n, m \in I\!N$. In particular $q_1 = -\frac{\pi}{2}$ and $q_2 = 0$ (*i.e.* $n = m = -1$) is a point that satisfies the requirements of assumption 15. One computes that at this point $\frac{d^2U}{dq^2} = \begin{pmatrix} a_1 + a_2 & a_2 \\ a_2 & a_2 \end{pmatrix}$ that is positive definite since it is symmetric and its determinant is $a_1 a_2 > 0$. Intuitively one notices that global stability is not possible for this example since the unforced system possesses a second fixed point when $q_1 = \frac{\pi}{2}$, $q_2 = 0$, which is not stable.

7.2.2 Asymptotic Lyapunov Stability

Let us now consider the dynamics in (6.31). The following is true:

Lemma 7.3. *Suppose that Assumption 15 holds. The unforced Euler-Lagrange dynamics with Rayleigh dissipation satisfying* $\dot{q}^t \frac{\partial R}{\partial \dot{q}} \geq \delta \dot{q}^T \dot{q}$ *for some* $\delta > 0$, *possesses a fixed point* $(q, \dot{q}) = (q_0, 0)$ *that is locally asymptotically stable.* ∎

Proof: It is not difficult to prove that the dynamics in Definition 6.12 defines an OSP system (with the velocity $\dot{q}$ as the output signal). Therefore the system now defines as well an output strictly passive operator $\tau \mapsto \dot{q}$. We could directly compute the derivative of $E(q, \dot{q})$ along the trajectories of the unforced system to attain our target. Let us however use passivity. We know (see Remark 4.86) that the dissipation inequality is equivalent to its infinitesimal form, *i.e.*

$$\frac{dV}{dx}^T f(x, \tau) = \tau^T x_2 - x_2^T \frac{\partial R}{\partial x_2} \tag{7.3}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$, $f(x, u)$ denotes the system vector field in state space notations, and $V(x)$ is any storage function. Let us take $V(x) = E(q, \dot{q})$. We deduce that

$$\dot{E} = \frac{dE}{dx}^T f(x, 0) = -\delta \dot{q}^T \dot{q} \tag{7.4}$$

The only invariant set inside the set $\{(q, \dot{q}) : \dot{q} \equiv 0\}$ is the fixed point $(q_0, 0)$. Resorting to Krasovskii-La Salle Invariance Theorem one deduces that the trajectories converge asymptotically to this point, provided that the initial conditions are chosen in a sufficiently small neighborhood of it. Notice that we could have used Corollary 5.16 to prove the asymptotic stability.

Remark 7.4.
- **Convexity:** Convex properties at the core of stability in mechanics: in statics the equilibrium positions of a solid body lying on a horizontal plane, submitted to gravity, are characterized by the condition that the vertical line that passes by its center of mass crosses the convex hull of the contact points of support. In dynamics Assumption 15 shows that without a convexity property (maybe local), the stability of the fixed point is generically impossible to obtain.
- The Lagrange-Dirichlet Theorem also applies to constrained Euler-Lagrange systems as in (6.163). If Rayleigh dissipation is added and if the potential energy satisfies the required asumptions, then the $(z_2, \dot{z}_2)$ dynamics are asymptotically stable. Thus $\ddot{z}_2(t)$ tends towards zero as well so that $\lambda_{z_1}(t) = \bar{g}_1(z_2(t))$ as $t \to +\infty$.
- It is clear that if Assumption 15 is strengthened to having a potential energy $U(q)$ that is globally convex, then its minimum point is globally Lyapunov stable.
- Other results generalizing the Lagrange-Dirichlet Theorem for systems with cyclic coordinates (*i.e.* coordinates such that $\frac{\partial T}{\partial q_i}(q) = 0$) were given by Routh and Lyapunov; see [351].

Remark 7.5. It is a general result that output strict passivity together with zero-state detectability yields under certain conditions asymptotic stability; see Corollary 5.16. One basic idea for feedback control may then be to find a control law that renders the closed-loop system strictly output passive with respect to some supply rate, and such that the closed-loop operator is zero-state detectable with respect to the considered output.

Example 7.6. Let us come back on the example in section 6.5. As we noted the concatenation of the two functions in (6.119) and (6.120) yields a positive definite function of $(\tilde{q}, \dot{q}) = (0, 0)$ with $\tilde{q} = q - \frac{\lambda_1 q_d}{\lambda_1 + k}$, that is continuous at $q = 0$. The only invariant set for the system in (6.115) with the input in (6.116) is $(q, \dot{q}) = (\frac{\lambda_1 q_d}{\lambda_1 + k}, 0)$. Using the Krasovskii-La Salle invariance Theorem one concludes that the point $\tilde{q} = 0$, $\dot{q} = 0$ is globally asymptotically uniformly Lyapunov stable.

7.2.3 Invertibility of the Lagrange-Dirichlet Theorem

One question that comes to one's mind is that, since the strong assumption on which the Lagrange-Dirichlet Theorem relies is the existence of a minimum point for the potential energy, what happens if $U(q)$ does not possess a minimum point? Is the equilibrium point of the dynamics unstable in this case? Lyapunov and Chetaev stated the following:

Theorem 7.7. *(a) If at a position of isolated equlibrium $(q, \dot{q}) = (q_0, 0)$ the potential energy does not have a minimum, and, neglecting higher order terms, it can be expressed as a second order polynomial, then the equilibirum is unstable. (b) If at a position of isolated equilibrium $(q, \dot{q}) = (q_0, 0)$ the potential energy has a maximum with respect to the variables of smallest order that occur in the expansion of this function, then the equilibrium is unstable. (c) If at a position of isolated equilibrium $(q, \dot{q}) = (q_0, 0)$ the potential energy, which is an analytical function, has no minimum, then this fixed point is unstable.* ■

Since $U(q) = U(q_0) + \frac{dU}{dq}(q_0)(q - q_0) + \frac{1}{2}(q - q_0)^T \frac{d^2U}{dq^2}(q - q_0) + \mathrm{o}[(q - q_0)^T(q - q_0)]$, and since q_0 is a critical point of $U(q)$, the first item tells us that the Hessian matrix $\frac{d^2U}{dq^2}$ is not positive definite, otherwise the potential energy would be convex and hence the fixed point would be a minimum. Without going into the details of the proof since we are interested in dissipative systems, not unstable systems, let us note that the trick consisting of redefining the potential energy as $U(q) - U(q_0)$ in order to get a positive storage function no longer works. Moreover, assume there is only one fixed point for the dynamical equations. It is clear at least in the one degree-of-freedom case that if $\frac{d^2U}{dq^2}(q_0) < 0$ then $U(q) \to -\infty$ for some q. Hence the available storage function that contains a term equal to $\sup_{\tau:(0,q(0),\dot{q}(0))\to} [U(q(t))]_0^t$ cannot be bounded, assuming that the state space is reachable. Thus the system cannot be dissipative, see Theorem 4.41.

7.2.4 The Lagrange-Dirichlet Theorem for Nonsmooth Lagrangian Systems (BV Solutions)

Let us consider the class of Lagrangian systems as in Section 6.8.2, *i.e.* fully actuated Lagrangian systems with complementarity conditions and impacts. The constraints are supposed to be frictionless. First notice that since $F(q,0) = \frac{\partial U}{\partial q}$ and $0 \in V(q)$, fixed points of (6.180) satisfy the generalized equation $0 \in \partial\psi_{\Phi}(q^\star) + \frac{\partial U}{\partial q}(q^\star)$ which in particular implies $q^\star \in \Phi$. Conditions under which such a generalized equation possess at least one solution, and numerical algorithms to compute one solution, exist [169]. In the following we shall assume for simplicity that the solutions are isolated, or even more: that it is unique.

Lemma 7.8. *Consider a mechanical system as in (6.177). Assume that the potential function $U(q)$ is radially unbounded. Then if $\psi_{\Phi}(q) + U(q)$ has a strict global minimum at q^*, the equilibrium point $(q^*, 0)$ is globally Lyapunov stable.* ■

Let us note that Φ needs not be convex in general (for instance the equilibrium may exist in $\mathrm{Int}(\Phi)$, or it may belong to $\partial\Phi$ but be forced by the continuous dynamics; see Figure 7.1 for planar examples with both convex and non-convex Φ; it is obvious that in the depicted non-convex case all points $(q^*, 0)$ with $q^* \in \partial\Phi$ are fixed points of the dynamics).

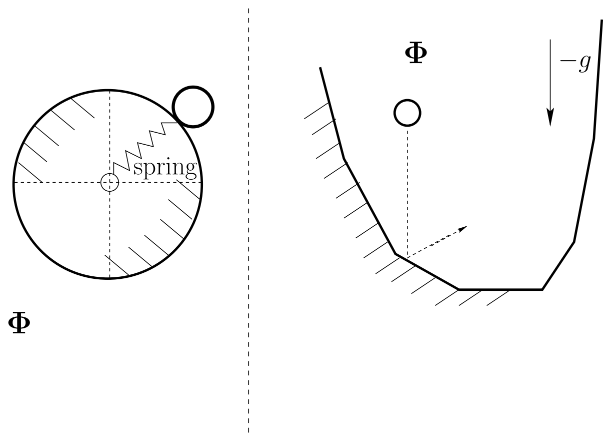

Fig. 7.1. Convex and nonconvex admissible sets

Proof: The proof may be led as follows. Let us consider the nonsmooth Lyapunov candidate function

$$W(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \psi_{\Phi}(q) + U(q) - U(q^*) \tag{7.5}$$

Since the potential $\psi_\Phi(q) + U(q)$ has a strict global minimum at q^* equal to $U(q^*)$ and is radially unbounded, this function $W(\cdot)$ is positive definite on the whole state space and is radially unbounded. Also $W(q, \dot q) \le \beta(||q||, ||\dot q||)$ for some class $\mathcal{K}$ function $\beta(\cdot)$ is satisfied on Φ ($\ni q(t)$ for all $t \ge 0$). The potential function $\psi_\Phi(q) + U(q)$ is continuous on Φ. Thus $W(q, \dot q)$ in (7.5) satisfies the requirements of a Lyapunov function candidate on Φ, despite the indicator function has a discontinuity on $\partial\Phi$ (but is continuous on the closed set Φ; see (3.187)). Moreover since (6.180) secures that $q(t) \in \Phi$ for all $t \ge 0$, it follows that $\psi_\Phi(q(t)) = 0$ for all $t \ge 0$. In view of this one can safely discard the indicator function in the subsequent stability analysis. Let us examine the variation of $W(q, \dot q)$ along trajectories of (6.183). In view of the above discussion, one can characterize the measure dW by its density with respect to $d\mu$ and the function W decreases if its density $\frac{dW}{d\mu}(t) \le 0$ for all $t \ge 0$. We recall Moreau's rule for differentiation of quadratic functions of RCLVB functions [357, pp.8-9]: let $u(\cdot)$ be RCLBV, then $d(u^2) = (u^+ + u^-)du$ where u^+ and u^- are the right-limit and left-limit functions of $u(\cdot)$. Let us now compute the density of the measure dW with respect to $d\mu$:

$$\begin{aligned} \frac{dW}{d\mu}(t) &= \frac{1}{2}\left[\dot q(t^+) + \dot q(t^-)\right]^T M(q(t)) \frac{dv}{d\mu}(t) + \frac{\partial U}{\partial q}\frac{dq}{d\mu}(t) \\ &+ \frac{1}{2}\frac{\partial}{\partial q}\left(\dot q(t^+)^T M(q(t)) \dot q(t^+)\right)\frac{dq}{d\mu}(t) \end{aligned} \tag{7.6}$$

where $dq = v(t)dt$ since the function $v(\cdot)$ is Lebesgue integrable. Let us now choose $d\mu = dt + d\mu_a$. Since $\frac{dt}{d\mu}(t_k) = 0$ and $\frac{dq}{d\mu}(t_k) = 0$ whereas $\frac{dv}{d\mu}(t_k) = v(t_k^+) - v(t_k^-) = \dot q(t_k^+) - \dot q(t_k^-)$, it follows from (7.6) that at impact times one gets

$$\frac{dW}{d\mu}(t_k) = \frac{1}{2}\left[\dot q(t_k^+) + \dot q(t_k^-)\right]^T M(q(t))\left[\dot q(t_k^+) - \dot q(t_k^-)\right] = T_L(t_k) \le 0 \tag{7.7}$$

where $T_L(t_k)$ is in (6.178). Let the matrix function $\dot M(q, \dot q)$ be defined by $\dot M(q(t), \dot q(t)) = \frac{d}{dt}M(q(t))$. Let us use the expression of $F(q, \dot q)$ given after (6.177), and let us assume that Christoffel's symbols of the first kind are used to express the vector $C(q, \dot q)\dot q = \dot M(q, \dot q) - \frac{1}{2}\left[\frac{\partial}{\partial q}\left(\dot q^T M(q(t))\dot q\right)\right]^T$. Then the matrix $\dot M(q, \dot q) - 2C(q, \dot q)$ is skew-symmetric; see Lemma 6.16. Now if $t \ne t_k$, one gets $\frac{dv}{d\mu}(t) = \dot v(t) = \ddot q(t)$ and $\frac{dt}{d\mu}(t) = 1$ [357, p.76] and one can calculate from (7.6), using the dynamics and the skew-symmetry property (see Lemma 6.16):

$$\begin{aligned} \frac{dW}{d\mu} = \frac{dW}{dt} &= -\dot q^T C(q, \dot q)\dot q + \frac{1}{2}\dot q^T \dot M(q, \dot q)\dot q - \dot q^T z_1 \\ &= -\dot q^T z_1 \end{aligned} \tag{7.8}$$

where $z_1 \in -\partial\psi_{V(q(t))}(w(t))$ and $W(\cdot)$ is defined in (7.5). To simplify the notation we have dropped arguments in (7.8), however $\dot q$ is to be understood as $\dot{q}(t) = \dot{q}(t^+)$ since $t \neq t_k$. Now since for all $t \geq 0$ one has $\dot{q}(t^+) \in V(q)$ [358] which is polar to $\partial\psi_\Phi(q(t))$, and from Moreau's inclusion in (6.180), it follows that $z_1^T\dot{q}(t^+) \geq 0$. Therefore the *measure* dW is non-positive. Consequently the *function* $W(\cdot)$ is non-increasing [127, p.101]. We finally notice that the velocity jump mapping in (6.177) is a projection and is therefore Lipschtiz continuous as a mapping $\dot{q}(t_k^-) \mapsto \dot{q}(t_k^+)$, for fixed $q(t_k)$. In particular it is continuous at $(q^\star, 0)$, so that a small pre-impact velocity gives a small post-impact velocity. All the conditions for Lyapunov stability of $(q^\star, 0)$ are fulfilled and Lemma 7.8 is proved. ■

> The main feature of the proof is that one works with *densities* (which are functions of time) and not with the measures themselves, in order to characterize the variations of the Lyapunov function.

Remark 7.9. • The above result holds also locally thanks to the continuity property of the impact mapping in (6.177).

- The inclusion of the indicator function $\psi_\Phi(q(t))$ in the Lyapunov function not only guarantees its positive definiteness (which anyway is assured along solutions of (6.183) which remain in Φ), but it also allows one to consider cases where the smooth potential has a minimum that is outside Φ. Saying "$\psi_\Phi(q) + U(q)$ has a strict minimum at q^*" is the same as saying "$U(q)$ has a strict minimum at q^* inside Φ". Since the indicator function has originally been introduced by Moreau as a potential associated to unilateral constraints, it finds here its natural use. In fact we could have kept the indicator function in the stability analysis. This would just add a null term $\dot{q}(t^+)^T z_2 \frac{dt}{d\mu}(t)$ in the right-hand-side of (7.6), with $z_2 \in \partial\psi_\Phi(q(t))$.
- As alluded to above, taking $e = 1$ in (6.177) ensures that there is no accumulation of impacts, thus the sequence of impact times $\{t_k\}_{k\geq 0}$ can be ordered, $d\mu_a = \sum_{k\geq 0} \delta_{t_k}$, and velocities are piecewise continuous. Then a much simpler formulation can be adopted by separating continuous motion phases occurring on intervals (t_k, t_{k+1}) from impact times. The system is therefore non-Zeno for $e = 1$ and if Assumption 14 holds.
- One doesn't need to make further assumptions on the measure $d\mu_a$ to conclude, and one sees that this conclusion is obtained directly applying general differentiation rules of RCLBV functions. The dynamics might even contain dense sets of velocity discontinuities, (7.6) and (7.7) would continue to hold. This shows that using the MDI formalism in (6.180) or (6.183) places the stability analysis in a much more general perspective than, say, restricting $\dot{q}(\cdot)$ to be piecewise continuous.
- Other work on energy-based control of a class of nonsmooth systems may be found in [188, 195].

A Dissipation Inequality

Let us now derive a dissipation inequality for the dynamical system (6.177). To that end let us take advantage of the compact formalism (6.183). We consider a Lebesgue measurable input $\tau(\cdot)$ so that (6.183) becomes

$$-M(q(t))\frac{dv}{d\mu}(t) - F(q(t), v(t^+))\frac{dt}{d\mu}(t) - \tau(t)\frac{dt}{d\mu} \in \partial\psi_{V(q(t))}(w(t)) \tag{7.9}$$

Following (6.184) let ξ denote a measure that belongs to the normal cone to the tangent cone $\partial\psi_{V(q(t))}(w(t))$, and let us denote $\frac{dR}{d\mu}(\cdot)$ its density with respect to μ. The system in (7.9) is dissipative with respect to the generalized supply rate

$$\langle \frac{1}{2}(v(t^+) + v(t^-)), \tau(t)\frac{dt}{d\mu} + \frac{dR}{d\mu}(t)\rangle \tag{7.10}$$

Noting that $\xi = \nabla h(q)\lambda$ for some measure λ we obtain

$$\langle \frac{1}{2}(v(t^+) + v(t^-)), \tau(t)\frac{dt}{d\mu} + \nabla h(q)\frac{d\lambda}{d\mu}(t)\rangle \tag{7.11}$$

where we recall that $v(\cdot)$ satisfies the properties in item ii) in Section 6.8.2 and that outside impacts (*i.e.* outside atoms of the measure dR) one has $\frac{dt}{d\mu} = 0$ because the Lebesgue measure has no atom. It is noteworthy that (7.11) is a generalization of the Thomson-Tait's Formula of Mechanics [69, §4.2.12], which expresses the work performed by the contact forces during an impact. The supply rate in (7.11) may be split into two parts: a function part and a measure part. The function part describes what happens outside impacts, and one has $\frac{1}{2}(v(t^+) + v(t^-) = v(t) = \dot{q}(t)$. The measure part describes what happens at impacts t_k. Then one gets

$$\begin{aligned}\langle (v(t_k^+) + v(t_k^-)), \nabla h(q)\tfrac{d\lambda}{d\mu}(t_k)\rangle &= \langle (v(t_k^+) + v(t_k^-)), M(q(t_k)(v(t_k^+) - v(t_k^-))\rangle \\ &= v^T(t_k^+)M(q(t_k))v(t_k^+) - v^T(t_k^-)M(q(t_k))v(t_k^-) = 2T_L(t_k) \le 0\end{aligned} \tag{7.12}$$

where we used the fact that the dynamics at an impact time is algebraic: $M(q(t_k))(v(t_k^+) - v(t_k^-)) = \nabla h(q)\frac{d\lambda}{d\mu}(t_k)$ with a suitable choice of the basis measure μ. The storage function of the system is nothing else but its total energy. It may be viewed as the usual smooth energy $\frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q)$, or as the *unilateral energy* $\frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q) + \psi_\Phi(q)$, which is nonsmooth on $\mathbb{R}^n \times \mathbb{R}^n$. It is worth remarking, however, that the nonsmoothness of the storage function is not a consequence of the impacts, but of the complementarity condition $0 \le h(q) \perp \lambda \ge 0$.

Further Reading and Discussion

The foregoing developments concern a specific class of nonsmooth dynamical systems involving state jumps and measures. Other classes of systems with impulsive terms exist, which can be written as

$$\begin{cases} \dot{x}(t) = F(x(t), t) & \text{if } t \neq t_k \\ x(t^+) - x(t^-) = S(x(t^-)) & \text{if } t = t_k \\ x(0^-) = x_0 \end{cases} \tag{7.13}$$

where some assumptions are made on the set of times t_k, see for instance [183,184,186,189]. Such assumptions always make the set of state jump times, a very particular case of the set of discontinuities of a LBV function. It is noteworthy that the systems in (7.13) and in (6.177) are different dynamical systems. Most importantly the complementarity conditions are absent from (7.13). Another class of impulsive systems is that of measure differential equations (MDE), or impulsive ODEs. Let us consider one example:

$$\dot{x}(t) = \sin\left(x(t) + \frac{5\pi}{4}\right) + \cos\left(x(t) + \frac{3\pi}{4}\right)\dot{u}(t), \quad x(0^-) = x_0, \quad x(t) \in \mathbb{R} \tag{7.14}$$

where $u(\cdot)$ is of bounded variation. Applying [65, Theorem 2.1], this MDE has a unique global generalized solution. Consider now

$$\begin{cases} \dot{x}(t) = \sin\left(x(t) + \frac{5\pi}{4}\right) + \cos\left(x(t) + \frac{3\pi}{4}\right)\lambda(t), \quad x(0^-) = x_0, \quad x(t) \in \mathbb{R} \\ 0 \leq x(t) \perp \lambda(t) \geq 0 \end{cases} \tag{7.15}$$

Suppose that $x_0 = 0$. Then if $\lambda(0) = 0$ one gets $\dot{x}(0) = \sin(\frac{5\pi}{4}) < 0$. It is necessary that there exists a $\lambda(0) > 0$ such that $\dot{x}(0) \geq 0$. However since $\cos(\frac{3\pi}{4}) < 0$, this is not possible and necessarily $\dot{x}(0) < 0$. If $x_0 < 0$, then an initial jump must occur and $x(0^+) \geq 0$. If $x(0^+) = 0$ the previous analysis applies. One sees that defining generalized solutions as in [65, Definition 2.1] is not sufficient. Therefore the complementarity system in (7.15) is not well-posed, despite its resemblance with the MDE in (7.14). One notices that the class of nonsmooth Euler-Lagrange systems considered for instance in [194, 195] and in (6.177) are, in the same way, different classes of nonsmooth dynamical systems (the discrepancy being the same as the one between (7.13) and (7.15)). In other words, the considered models are not the same, since the models in [194, 195] do not incorporate the complementarity conditions.

7.2.5 The Lagrange-Dirichlet Theorem for Nonsmooth Lagrangian Systems (C^0 Solutions)

Let us now pass to the stability analysis of the systems presented in Section 6.8.1. The set of stationary solutions of (6.171) and (6.172) is given by

$$\mathcal{W} = \{\bar{q} \in I\!R^m \mid K\bar{q} \in -H_1 \partial\Phi(0)\} \tag{7.16}$$

Definition 7.10. *A stationary solution* $\bar{q} \in \mathcal{W}$ *is* stable *provided that for any* $\epsilon > 0$ *there exists* $\eta(\epsilon) > 0$ *such that for any* $q_0 \in I\!R^n$, $\dot{q}_0 \in I\!R^n$, $H_2\dot{q}_0 \in D(\partial\Phi)$, *with* $\sqrt{||q_0 - \bar{q}||^2 + ||\dot{q}_0||^2} \leq \eta$, *the solution* $q(\cdot, t_0, q_0, \dot{q}_0)$ *of the problem (i) (ii) (iii) (6.171) (6.172) satisfies*

$$\sqrt{||q(t,t_0,q_0,\dot{q}_0) - \bar{q}||^2 + ||\dot{q}(t,t_0,q_0,\dot{q}_0)||^2} \leq \epsilon, \quad \forall\, t \geq t_0 \tag{7.17}$$

■

We then have the following theorems which we give without proofs.

Theorem 7.11. *[3] Let the assumptions of Theorem 6.56 hold, and* $0 \in D(\partial\Phi)$. *Suppose in addition that*

- $RM^{-1}CR^{-1} \geq 0$
- $RM^{-1}KR^{-1} > 0$ *and is symmetric*

Then the set $\mathcal{W} \neq \emptyset$ *and any stationary solution* $\bar{q} \in \mathcal{W}$ *of (6.171) and (6.172) is stable.* ■

A variant is as follows:

Theorem 7.12. *[3] Let the assumptions of Theorem 6.56 hold, and* $0 \in D(\partial\Phi)$. *Let* $\bar{q} \in \mathcal{W}$ *be a stationary solution of (6.171) and (6.172). Suppose that*

- $\langle RM^{-1}CR^{-1}z + RM^{-1}K\bar{q}, z\rangle + \Phi(H_2R^{-1}z) - \Phi(0) \geq 0, \;\; z \in I\!R^n$
- $RM^{-1}KR^{-1} > 0$ *and is symmetric*

Then $\bar{q}$ *is stable.* ■

The next theorem concerns the attractivity of the stationary solutions, and may be seen as an extension of the LaSalle invariance principle. Let $d[s,\mathcal{M}] = \inf_{m\in\mathcal{M}} ||s - m||$ be the distance from a point $s \in I\!R^n$ to a set $\mathcal{M} \subset I\!R^n$.

Theorem 7.13. *[3] Let the assumptions of Theorem 6.56 hold, and* $0 \in D(\partial\Phi)$. *Suppose that:*

- $RM^{-1}KR^{-1} > 0$ *and is symmetric*
- $\langle RM^{-1}CR^{-1}z + RM^{-1}K\bar{q}, z\rangle + \Phi(H_2R^{-1}z) - \Phi(0) > 0, \;\; z \in I\!R^n \setminus \{0\}$

- *$D(\partial\Phi)$ is closed*

  Then for any $q_0 \in \mathbb{R}^n$, $\dot{q}_0 \in \mathbb{R}^n$, $H_2\dot{q}_0 \in D(\partial\Phi)$, the orbit

$$\Omega(q_0,\dot{q}_0) = \{(q(\tau,t_0,q_0,\dot{q}_0),\dot{q}(\tau,t_0,q_0,\dot{q}_0)) \mid \tau \geq t_0\} \tag{7.18}$$

is bounded and

$$\lim_{\tau\to+\infty} d[q(\tau,t_0,q_0,\dot{q}_0),\mathcal{W}] = 0, \quad \lim_{\tau\to+\infty} \dot{q}(\tau,t_0,q_0,\dot{q}_0) = 0 \tag{7.19}$$

■

The proof is led with the help of the quadratic function $V(x) = \frac{1}{2}(q-\bar{q})^T R^2 M^{-1}K(q-\bar{q}) + \frac{1}{2}\dot{q}^T R^2\dot{q}$. Notice that $(q-\bar{q})^T R^2 M^{-1}K(q-\bar{q}) = (q-\bar{q})^T R(RM^{-1}KR^{-1})R(q-\bar{q})$. More on the attractivity properties of similar evolution problems can be found in [4].

Example 7.14. We are given the dynamics

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) \in -\partial\Phi(\dot{q}(t)) \tag{7.20}$$

of a one-degree-of-freedom system acted upon by a spring with stiffness $k > 0$ and with viscous friction $c > 0$. Coulomb's friction is obtained by setting $\Phi(z) = \mu|z|$. Then $\mathcal{W} = \left[-\frac{\mu}{k},\frac{\mu}{k}\right]$, and $\lim_{\tau\to+\infty} d[q(\tau,t_0,q_0,\dot{q}_0),\mathcal{W}] = 0$, $\lim_{\tau\to+\infty}\dot{q}(\tau,t_0,q_0,\dot{q}_0) = 0$. The mass stops somewhere within $\mathcal{W}$, as expected. Actually one may even expect convergence in finite time. Finite-time convergence properties for a class of differential inclusions have been shown in [5, 91].

7.2.6 Conclusion

These theorems generalize the Lagrange-Dirichlet Theorem for a class of nonsmooth systems. It is worth recalling that the subdifferential of a proper convex lower semicontinuous mapping, defines a maximal monotone mapping (see Section 3.9.4 where some basic results of convex analysis are recalled). The system in (6.171) and (6.172) can consequently be seen as the feedback interconnection of a Lagrangian system and a monotone mapping. Both subsystems can be described as follows:

$$\begin{cases} M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = u_1, \quad y_1(t) = \dot{q}(t) \\ \\ u_2(t) = \dot{q}(t), \quad y_2(t) \in H_1\partial\Phi(H_2u_1(t)), \quad y_2(t) = -u_1(t) \end{cases} \tag{7.21}$$

More precisely the variable change defined in (6.173) allows one to rewrite the dynamics (6.172) as

$$\ddot{z}(t) + RM^{-1}CR^{1}\dot{z}(t) + RM^{-1}KR^{-1}z(t) \in -\partial\chi(\dot{z}(t)) \tag{7.22}$$

with $\chi(w) = (\Phi \circ H_2R^{-1})(w)$ for all $w \in I\!R^n$, and

$$\partial\chi(w) = R^{-1}H_2^T\partial\Phi(H_2R^{-1}w)$$

for all $w \in I\!R^n$. It is clear that $\chi(\cdot)$ is proper convex lower semicontinuous so that its subdifferential defines a maximal monotone mapping. Let the assumptions of Theorem 7.11 be fulfilled. The feedback interconnection is described as

$$\begin{cases} \ddot{z}(t) + RM^{-1}CR^{-1}\dot{z}(t) + RM^{-1}KR^{-1}z(t) = u_1(t), \;\; y_1(t) = \dot{z}(t) \\ \\ u_2(t) = \dot{z}(t), \;\; y_2(t) \in \partial\chi(\dot{z}(t)), \;\; y_2(t) = -u_1(t) \end{cases} \tag{7.23}$$

and both subsystems are passive. This interpretation together with the one at the end of section 6.8.2 allow us to conclude that "maximal monotone" differential inclusions permit to nicely recast such nonsmooth systems into a sound and established framework which extends the usual passivity theorems.

7.3 Rigid Joint–Rigid Link Systems: State Feedback

In this subsection we shall present various feedback controllers that assure some stability properties for the rigid joint-rigid link model in (6.98). We start with the regulation problem and then generalize to the tracking case. In each case we emphasize how the dissipativity properties of the closed-loop systems constitute the basis of the stability properties.

7.3.1 PD Control

Let us consider the following input:

$$\tau = -\lambda_1\dot{q} - \lambda_2\tilde{q} \tag{7.24}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are the constant feedback gains (for simplicity we consider them as being scalars instead of positive definite $n \times n$ matrices, this is not very important for what follows), $\tilde{q} = q - q_d$, $q_d \in I\!R^n$ is a constant desired position. The closed-loop system is given by

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) + \lambda_1\dot{q}(t) + \lambda_2\tilde{q}(t) = 0 \tag{7.25}$$

Two paths are possible: we can search for the available storage function of the closed-loop system in (7.25) which is likely to provide us with a Lyapunov function, or we can try to interpret this dynamics as the negative interconnection of two passive blocks and then use the passivity theorem (more exactly one of its numerous versions) to conclude on stability. To fix the ideas we develop both paths in detail.

The Closed-loop Available Storage

First of all notice that in order to calculate an available storage we need a supply rate, consequently we need an input (that will be just an auxiliary signal with no significance). Let us therefore just add a term u in the right-hand-side of (7.25) instead of zero. In other words we proceed as we did for the example in Section 6.5: we make an input transformation and the new system is controllable. Let us compute the available storage along the trajectories of this new input-output system, assuming that $U(q)$ is bounded from below, *i.e.* $U(q) \geq U_{\min} > -\infty$ for all $q \in Q$:

$$
\begin{aligned}
V_a(q_0, \dot{q}_0) &= \sup_{u:(0,q_0,\dot{q}_0)\rightarrow} -\int_0^t u^T(s)\dot{q}(s)ds \\
&= \sup_{u:(0,q_0,\dot{q}_0)\rightarrow} -\int_0^t \dot{q}^T(s)\left\{M(q(s))\ddot{q}(s) + C(q(s),\dot{q}(s))\dot{q}(s) + g(q(s))\right. \\
&\quad \left. +\lambda_1\dot{q}(s) + \lambda_2\tilde{q}(s)\right\} ds \\
&= \sup_{u:(0,q_0,\dot{q}_0)\rightarrow} \left\{-\left[\frac{1}{2}\dot{q}^T(s)M(q(s))\dot{q}(s)\right]_0^t - [U(q(t))]_0^t - \right. \\
&\quad \left. -\left[\tfrac{1}{2}\lambda_2\tilde{q}^T(s)\tilde{q}(s)\right]_0^t - \lambda_1\int_0^t \dot{q}^T(s)\dot{q}(s)ds\right\} \\
&= \tfrac{1}{2}\dot{q}(0)^T M(q(0))\dot{q}(0) + U(q(0)) + \tfrac{1}{2}\lambda_2\tilde{q}(0)^T\tilde{q}(0)
\end{aligned}
\tag{7.26}
$$

where we used the fact that $\dot{q}^T[\dot{M}(q,\dot{q}) - 2C(q,\dot{q})]\dot{q} = 0$ for all $q \in Q$ and all $\dot{q} \in T_qQ$; see Lemma 6.16 [1]. Let us now make a little stop: we want to show some stability property for the unforced system in (7.25), so what is the fixed point of this system? Letting $\dot{q} \equiv 0$ in (7.25) one finds

$$
g(q) + \lambda_2\tilde{q} = 0. \tag{7.27}
$$

Let us state the following:

Assumption 16 *The equations in (7.27) possess a finite number of isolated roots $q = q_i$. Moreover the q_is are strict local minima of $U(q)$.* ■

Then we have the following:

Lemma 7.15. *Assume that Assumption 16 is true. The rigid joint-rigid link manipulator dynamics in (6.98) with PD controller in (7.24) has locally asymptotically stable fixed points $(q,\dot{q}) = (q_i, 0)$.* ■

[1] Actually this equality is is always true, even if the matrix $\dot{M}(q,\dot{q}) - 2C(q,\dot{q})$ is not skew-symmetric.

Proof: From the second part of Assumption 16 it follows that the available storage V_a in (7.26) is a storage function for the closed-loop system with input u (fictitious) and output $\dot{q}$. Next this also allows us to state that $V_{pd}(q - q_i, \dot{q}) \stackrel{\Delta}{=} V_a(q, \dot{q}) - U(q_i)$, is a Lyapunov function for the unforced system in (7.25): indeed this is a storage function and the conditions of Lemma 5.13 are satisfied. Now let us calculate the derivative of this function along the trajectories of (7.25):

$$\begin{aligned} \dot{V}_{pd}(q(t) - q_i, \dot{q}(t)) &= -\lambda_1 \dot{q}^T(t)\dot{q}(t) + \dot{q}^T(t)\left[-g(q(t)) + \frac{dU}{dq}(t)\right] \\ &= -\lambda_1 \dot{q}^T(t)\dot{q}(t) \end{aligned} \tag{7.28}$$

One therefore just has to apply the Krasovskii-La Salle Lemma to deduce that the fixed points $(q_i, 0)$ are locally asymptotically Lyapunov stable. Lyapunov second method guarantees that the basin of attraction B_{r_i} of each fixed point has a strictly positive measure. ■

Remark 7.16 (Potential energy shaping). One remarks that asymptotic stability has been obtained in part because the PD control injects some strict output passivity inside the closed-loop system. This may be seen as a forced damping. On the other hand the position feedback may be interpreted as a modification of the potential energy so as to shape it adequately for control purposes. It seems that this technique was first advocated by Takegaki and Arimoto in [484].

Remark 7.17. The PD control alone cannot compensate for gravity. Hence the system will converge to a configuration that is not the desired one. Clearly increasing λ_2 reduces the steady-state error. But increasing gains is not always desirable in practice, due to measurement noise in the sensors.

Equivalent Closed-loop Interconnections

Since the closed-loop system possesses several equilibrium points, the underlying passivity properties of the complete closed-loop system must be local in nature, *i.e.* they hold whenever the state remains inside the balls B_{r_i} [404]. It is however possible that each block of the interconnection, when considered separately, possesses global dissipativity properties. But the interconnection does not.

A first interconnection:

Looking at (7.25) one is tempted to interpret those dynamics as the interconnection of two subsystems with respective inputs u_1, u_2 and outputs y_1 and y_2, with $y_1 = u_2$ and $y_2 = -u_1$, and

$$\begin{cases} u_1 = -\lambda_1 \dot{q} - \lambda_2 \tilde{q} \\ \\ y_1 = \dot{q} \end{cases} \tag{7.29}$$

Evidently this is motivated by the fact that the rigid joint-rigid link manipulator dynamics in (7.25) defines a passive operator between u_1 and y_1, with state vector $\begin{pmatrix} \tilde{q} \\ \dot{q} \end{pmatrix}$ and dynamics

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) = u_1(t) \tag{7.30}$$

Let us write this second subsystem in state space form as

$$\begin{cases} \dot{z}_1 = u_2 \\ \\ y_2 = \lambda_2 z_1 + \lambda_1 u_2 \end{cases} \tag{7.31}$$

with $z_1(0) = q(0) - q_d$. Its transfer matrix is given by $H_{pd}(s) = \frac{\lambda_2 + \lambda_1 s}{s} I_n$ where I_n is the $n \times n$ identity matrix. Thus $H_{pd}(s)$ is PR; see Definition 2.28. From Theorem 5.2 and Corollary 5.3 it follows that $\dot{q} \in \mathcal{L}_2(\mathbb{R}^+)$. Notice that this is a consequence of the fact that $H_{pd}(s)$ defines an input strictly passive operator; see Theorem 2.6 2). We cannot say much more if we do not pick up the storage functions of each subsystem. Now the second subsystem has dynamics such that the associated operator $u_2 \mapsto y_2$ is input strictly passive (hence necessarily of relative degree zero) and with storage function $\frac{\lambda_2}{2} z_1^T z_1$. From the fact that $\dot{z}_1 = \dot{q}$ and due to the choice of the initial data, one has for all $t \geq 0$: $z_1(t) = \tilde{q}(t)$. It is easy to see then that the first subsystem (the rigid joint-rigid links dynamics) has a storage function equal to $\frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q) - U(q_i)$. The sum of both storage functions yields the desired Lyapunov function for the whole system. The interconnection is depicted in Figure 7.2.

Remark 7.18. Looking at the dynamics of both subsystems it seems that the total system order has been augmented. But the interconnection equation $y_1 = z_1$ may be rewriten as $\dot{z}_1 = \dot{q}$. This defines a dynamical invariant $z_1 - q = q_0$, where $q_0 \in \mathbb{R}^n$ is fixed by the initial condition $z_1(0) = q(0) - q_d$. Hence the system (7.30) and (7.31) may be reduced to the subspace $z_1 - q = -q_d$ and one recovers a system of dimension $2n$ (in other words the space $(q, \dot{q}, z_1)$ is foliated by invariant manifolds $z_1 - q = -q_d$).

Remark 7.19. In connection with the remarks at the beginning of this subsubsection, let us note that the fixed points of the first unforced (*i.e.* $u_1 \equiv 0$) subsystem are given by $\{(q, \dot{q}) : g(q) = 0, \dot{q} = 0\}$, while those of the unforced second subsystem are given by $\{z_1 : \dot{z}_1 = 0 \Rightarrow \tilde{q} = \tilde{q}(0)\}$. Thus the first subsystem has Lyapunov stable fixed points which correspond to its static equilibrium, while the fixed point of the second subsystem corresponds to the desired static position q_d. The fixed points of the interconnected blocks are

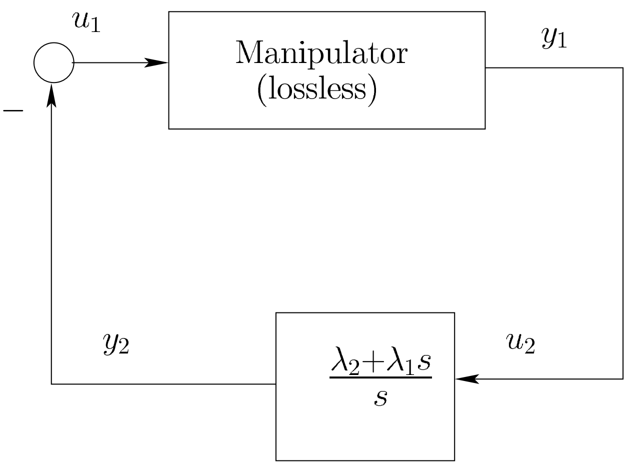

Fig. 7.2. The first equivalent representation

given by the roots of (7.27). If one looks at the system from a pure input-output point of view, such a fixed points problem does not appear. However if one looks at it from a dissipativity point of view, which necessarily implies that the input-output properties are related to the state space properties, then it becomes a necessary step.

Remark 7.20. $H_{pd}(s)$ provides us with an example of a passive system that is ISP but obviously not asymptotically stable, only stable (see Corollary 5.16).

A second interconnection:

A second possible interpretation of the closed-loop system in (7.25) is made of the interconnection of the two blocks:

$$\begin{cases} u_1 = -\lambda_2 \tilde{q},\ y_1 = \dot{q} \\ \\ u_2 = y_1, \qquad y_2 = \lambda_2 \tilde{q} \end{cases} \tag{7.32}$$

The first subsystem then has the dynamics

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) + \lambda_1 \dot{q}(t) = u_1(t) \tag{7.33}$$

from which one recognizes an output strictly passive system, while the second one has the dynamics

$$\begin{cases} \dot{z}_1 = u_2 \\ \\ y_2 = \lambda_2 z_1 \end{cases} \tag{7.34}$$

with $z_1(0) = q(0) - q_d$. One can check that it is a passive lossless system since $\langle u_2, y_2 \rangle_t = \int_0^t \lambda_2 \tilde{q}^T(s)\dot{q}(s)ds = \frac{\lambda_2}{2}[\tilde{q}^T\tilde{q}(t) - \tilde{q}^T\tilde{q}(0)]$, with storage function $\frac{\lambda_2}{2}\tilde{q}^T\tilde{q}$. Therefore applying the passivity Theorem (see theorem 5.2 and Corollary 5.3), one still concludes that $\dot{q} \in \mathcal{L}_2(I\!R^+)$. We however may go a little further with this decomposition. Indeed consider the system with input $u = u_1 + y_2$ and output $y = y_1$. This defines an output strictly passive operator $u \mapsto y$. Setting $u \equiv y \equiv 0$ one obtains that $(q - q_i, \dot{q}) = (0, 0)$. Hence this closed-loop system is zero-state observable. Since the storage function (the sum of both storage functions) we have exhibited is positive definite with respect to this error equation fixed point, and since it is proper, it follows that the equilibrium point of the unforced system (*i.e.* $u \equiv 0$) is globally asymptotically stable. This second interconnection is depicted in Figure 7.3.

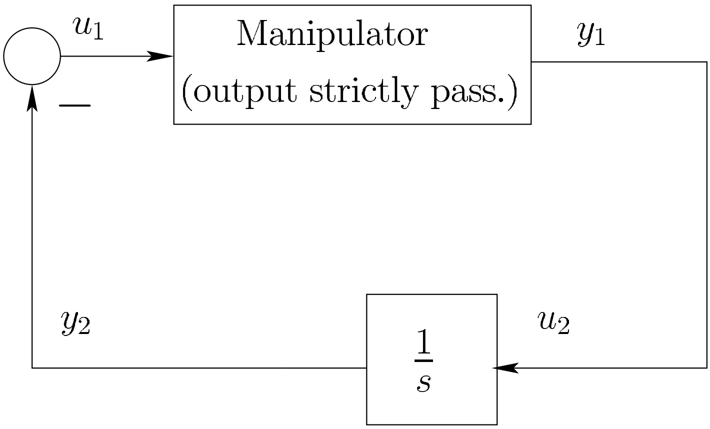

Fig. 7.3. The second equivalent representation

In conclusion it is not very important whether we associate the strict passivity property to one block or the other. What is important is that we can *systematically* associate to these dissipative subsystems some Lyapunov functions that are *systematically* deduced from their passivity property. This is a fundamental property of dissipative systems that one can calculate Lyapunov functions for them. It has even been originally the main motivation for studying passivity, at least in the field of control and stabilization of dynamic systems.

7.3.2 PID Control

The PID control is also a feedback controller that is widely used in practice. Let us investigate whether we can redo the above analysis for the PD controller. If we proceed in the same manner, we decompose the closed-loop dynamics

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) + \lambda_1 \dot{q}(t) + \lambda_2 \tilde{q}(t) + \lambda_3 \int_0^t \tilde{q}(s)ds = 0 \tag{7.35}$$

into two subsystems, one of which corresponds to the rigid joint-rigid link dynamics, and the other one to the PID controller itself. The input and output signals of this interconnection are this time chosen to be

$$\begin{cases} u_1 = -\lambda_1 \dot{q} - \lambda_2 \tilde{q} - \lambda_3 \int_0^t \tilde{q}(s)ds = -y_2 \\ \\ y_1 = \dot{q} = u_2 \end{cases} \tag{7.36}$$

The dynamics of the PID block is given by (compare with (7.31)):

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \\ \dot{z}_2(t) = u_2(t) \\ \\ y_2 = \lambda_1 u_2 + \lambda_2 z_2 + \lambda_3 z_1 \end{cases} \tag{7.37}$$

The transfer matrix of this linear operator is given by (compare with (2.64) and (2.65))

$$H_{pid}(s) = \frac{\lambda_1 s^2 + \lambda_2 s + \lambda_3}{s^2} I_n \tag{7.38}$$

Thus it has a a double pole with zero real part and it cannot be a PR transfer matrix, see Theorem 2.38. This can also be checked by calculating $\langle u_2, y_2 \rangle_t$ that contains a term $\int_0^t u_2(s)z_1(s)ds$ which cannot be lower bounded.

If one chooses $u_2' = \tilde{q}$ then the PID block transfer matrix becomes

$$H_{pid}(s) = \frac{\lambda_1 s^2 + \lambda_2 s + \lambda_3}{s} I_n \tag{7.39}$$

which this time is a PR transfer function for a suitable choice of the gains, and one can check that

$$\langle u_2', y_2 \rangle_t = \frac{\lambda_1}{2} [\tilde{q}^T(s)\tilde{q}(s)]_0^t + \lambda_2 \int_0^t \tilde{q}^T(s)\tilde{q}(s)ds + \frac{\lambda_3}{2} \left[\int_0^t \tilde{q}^T(s)\tilde{q}(s)ds \right]_0^t \tag{7.40}$$

which shows that the system is even input strict passive (but the transfer function is not SPR otherwise this system would be strictly passive (in the state space sense); see Example 4.65, which it is not from inspection of (7.40)). However this change of input is suitable for the PID block, but not for the rigid joint-rigid link block, that we know is not passive with respect to the supply rate $u_1^T \tilde{q}$ because of the relative degree of this output. As a consequence the dynamics in (7.35) cannot be analyzed through the passivity theorem.

Remark 7.21. Notice however that the system in (7.35) can be shown to be locally Lyapunov stable [17] with a Lyapunov function $V(z)$, where $z(t) = \begin{pmatrix} \int_0^t \tilde{q}(s)ds \\ \tilde{q}(t) \\ \dot{q}(t)) \end{pmatrix}$. Let us add a fictitious input τ in the right-hand-side of (7.35) instead of zero. From the KYP lemma we know that there exists an output y (another fictitious signal) such that this closed-loop is passive with respect to the supply rate $\tau^T y$. One has $y = (0,0,1)\frac{\partial V}{\partial z}$.

7.3.3 More about Lyapunov Functions and the Passivity Theorem

Before going on with controllers that assure tracking of arbitrary (smooth enough) desired trajectories, let us investigate in more detail the relationships between Lyapunov stable systems and the passivity theorem (which has numerous versions, but is always based on the study of the interconnection of two dissipative blocks). From the study we made about the closed-loop dynamics of the PD and PID controllers, it follows that if one has been able to transform a system (should it be open or closed-loop) as in Figure 3.2 and such that both blocks are dissipative, then the sum of the respective storage functions of each block is a suitable Lyapunov function candidate. Now one might like to know whether a Lyapunov stable system possesses some dissipativity properties. More precisely, we would like to know whether a system that possesses a Lyapunov function, can be interpreted as the interconnection of two dissipative subsystems. Let us state the following [74, 317]:

Lemma 7.22. *Let $\mathcal{L}$ denote a set of Lyapunov stable systems with equilibrium point $(x_1, x_2) = (0, 0)$, where (x_1, x_2) generically denotes the state of systems in $\mathcal{L}$. Suppose the Lyapunov function $V(x_1, x_2, t)$ satisfies*

1.
$$V(x_1, x_2, t) = V_1(x_1, t) + V_2(x_2, t) \tag{7.41}$$
where $V_1(\cdot)$, $V_2(\cdot)$ are positive definite radially unbounded functions
2.
$$\dot{V}(x_1, x_2, t) \leq -\gamma_1\beta_1(\| x_1 \|) - \gamma_2\beta_2(\| x_2 \|) \tag{7.42}$$
along trajectories of systems in $\mathcal{L}$, where $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are class $\mathcal{K}$ functions, and $\gamma_1 \geq 0$, $\gamma_2 \geq 0$.

Suppose there exist functions $F_1(\cdot)$ and $F_2(\cdot)$ such that for all x_1, x_2 and $t \geq t_0$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}^T F_1(x_1, t) \leq -\gamma_1\beta_1(\| x_1 \|) \tag{7.43}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}^T F_2(x_1, t) \leq -\gamma_2\beta_1(\| x_2 \|) \tag{7.44}$$

and $F_i(0,t) = 0$, dim x_i'=dim x_i for $i = 1,2$, for all $t \geq t_0$. Then there exists a set $\mathcal{P}$ of Lyapunov stable systems, with the same Lyapunov function $V(x_1', x_2', t)$, that can be represented as the feedback interconnection of two (strictly) passive subsystems with states x_1' and x_2' respectively. These systems are defined as follows:

$$\begin{cases} \dot{x}_1'(t) = F_1(x_1'(t),t) + G_1(x_1'(t), x_2'(t), t)u_1 \\ \\ y_1 = G_1^T(x_1'(t), x_2'(t), t)\frac{\partial V_1}{\partial x_1'}(x_1', t) \end{cases} \tag{7.45}$$

$$\begin{cases} \dot{x}_2'(t) = F_2(x_2'(t),t) + G_2(x_1'(t), x_2'(t), t)y_1 \\ \\ y_2 = G_2^T(x_1'(t), x_2'(t), t)\frac{\partial V_2}{\partial x_2'}(x_2', t) = -u_1 \end{cases} \tag{7.46}$$

where $G_1(\cdot)$ and $G_2(\cdot)$ are arbitrary smooth nonzero functions [2], which can be shown to define the inputs and the outputs of the interconnected systems. ■

The proof of Lemma 7.22 is straightforward from the KYP property of the outputs of passive systems. Note that Lemma 7.22 does not imply any relationship between the system in $\mathcal{L}$ and the system in $\mathcal{P}$ other than the fact that they both have the same Lyapunov function structure. That is why we used different notations for their states (x_1', x_2') and (x_1, x_2). We are now interested in establishing sufficient conditions allowing us to transform a system in $\mathcal{L}$ into a system in $\mathcal{P}$ having the particular form given in (7.45) and (7.46). These conditions are discussed next. Suppose (Σ_L) has the following form (notice that this is a *closed-loop* form):

$$\begin{cases} \dot{x}_1(t) = F_1(x_1(t),t) + G_1(x_1(t), x_2(t), t)u_1 \\ \\ y_1 = h_1(x_1(t), t) = u_2 \end{cases} \tag{7.47}$$

$$\begin{cases} \dot{x}_2(t) = F_2(x_2(t),t) + G_2(x_1(t), x_2(t), t)u_2 \\ \\ y_2 = h_2(x_2(t), t) = u_1 \end{cases} \tag{7.48}$$

From (7.42) we thus have

$$\begin{aligned} \dot{V}(x_1, x_2, t) = & \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}^T F_1(x_1, t) + \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2}^T F_2(x_2, t) \\ & + \frac{\partial V_1}{\partial x_1}^T G_1(x_1, x_2, t) h_2(x_2, t) + \frac{\partial V_2}{\partial x_2}^T G_2(x_1, x_2, t) \\ & \times h_1(x_1, t) \\ & \leq -\gamma_1 \beta_1(\| x_1 \|) - \gamma_2 \beta_2(\| x_2 \|) \end{aligned} \tag{7.49}$$

[2] We assume that the considered systems have 0 as a unique equilibrium point.

with inequalities (7.43) and (7.44) satisfied for both systems in (7.47) and (7.48). Now let us rewrite (Σ_L) in (7.47) (7.48) as follows (we drop the arguments for convenience; $u_1 = h_2(x_2)$, $u_2 = h_1(x_1)$):

$$\begin{cases} \dot{x}_1 = (F_1 + G_1 u_1 - \bar{g}_1 \bar{u}_1) + \bar{g}_1 \bar{u}_1 \\ \bar{y}_1 = \bar{g}_1^T \frac{\partial V_1}{\partial x_1} = \bar{u}_2 \end{cases} \tag{7.50}$$

$$\begin{cases} \dot{x}_2 = (F_2 + G_2 u_2 - \bar{g}_2 \bar{u}_2) + \bar{g}_2 \bar{u}_2 \\ \bar{y}_2 = \bar{g}_2^T \frac{\partial V_2}{\partial x_2} = -\bar{u}_1 \end{cases} \tag{7.51}$$

Notice that $(\tilde{\Sigma}_L)$ in (7.50) and (7.51) and (Σ_L) in (7.47) and (7.48) strictly represent the same system. We have simply changed the definition of the inputs and of the outputs of both subsystems in (7.47) and (7.48). Then the following Lemma is true:

Lemma 7.23. *Consider the closed-loop Lyapunov stable system (Σ_L) in (7.47) and (7.48), satisfying (7.49), with F_1 and F_2 satisfying (7.43) and (7.44). A sufficient condition for (Σ_L) to be able to be transformed into a system in $\mathcal{P}$ is that the following two inequalities are satisfied:*

1.

$$\frac{\partial V_1}{\partial x_1}^T \left(G_1 h_2 + \bar{g}_1 \bar{g}_2^T \frac{\partial V_2}{\partial x_2} \right) \leq 0 \tag{7.52}$$

2.

$$\frac{\partial V_2}{\partial x_2}^T \left(G_2 h_1 - \bar{g}_2 \bar{g}_1^T \frac{\partial V_1}{\partial x_1} \right) \leq 0 \tag{7.53}$$

for some non-zero, smooth matrices $\bar{g}_1$, $\bar{g}_2$ of appropriate dimensions, and with

$$\begin{aligned} &F_1(0,t) + G_1(0,x_2,t)u_1(0,x_2,t) - \bar{g}_1(0,x_2,t)\bar{u}_1(0,x_2,t) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall\, x_2, \forall\, t \geq 0 \\ &F_2(0,t) + g(x_1,0,t)u_2(x_1,0,t) - \bar{g}_2(x_1,0,t)\bar{u}_2(x_1,0,t) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall\, x_1, \forall\, t \geq 0 \end{aligned} \tag{7.54}$$

■

Notice that these conditions are sufficient only for transforming the system in $\mathcal{P}$; see Remark 7.29.

Proof: The proof of Lemma 7.23 is straightforward. Inequalities (7.52) and (7.53) simply guarantee that $\frac{\partial V_i}{\partial x_i}(f_i + g_i u_i - \bar{g}_i \bar{u}_i) \leq -\gamma_i \beta_i(\| x_i \|)$, and (7.54) guarantees that $\dot{x}_i = f_i + g_i u_i - \bar{g}_i \bar{u}_i$ has $x_i = 0$ as equilibrium point. Thus $(\tilde{\Sigma}_L)$ is in $\mathcal{P}$. ■

Example 7.24. Consider the following system:

$$\begin{cases} \dot{x}_1(t) = F_1(x_1(t),t) + G_1(x_1(t),x_2(t),t)u_1 \\ \\ y_1(t) = \frac{\partial V_1}{\partial x_1}(x_1,t) = u_2 \end{cases} \tag{7.55}$$

$$\begin{cases} \dot{x}_2(t) = F_2(x_2(t),t) - G_1^T(x_1(t),x_2(t),t)u_2 \\ \\ y_2(t) = \frac{\partial V_2}{\partial x_2}(x_2,t) = u_1 \end{cases} \tag{7.56}$$

with $\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i}^T f_i \leq -\gamma_i\beta_i(\| x_i \|)$, $\gamma_i \geq 0$, $f_i(0,t) = 0$ for all $t \geq t_0$, and V satisfies (7.41) and (7.42). Then we get along trajectories of (7.55) and (7.56): $\dot{V} = \dot{V}_1 + \dot{V}_2 \leq -\gamma_1\beta_1(\| x_1 \|) - \gamma_2\beta_2(\| x_2 \|)$. However the subsystems in (7.55) and (7.56) are not passive, as they do not verify the KYP property. The conditions (7.52) and (7.53) reduce to

$$\frac{\partial V_1}{\partial x_1}^T G_1 \frac{\partial V_2}{\partial x_2} + \frac{\partial V_1}{\partial x_1}^T \bar{g}_1\bar{g}_2^T \frac{\partial V_2}{\partial x_2} = 0 \tag{7.57}$$

as in this case $\frac{\partial V_1}{\partial x_1}^T G_1 h_2 = -\frac{\partial V_2}{\partial x_2}^T G_2 h_1$. Now choose $\bar{g}_1 = -G_1$, $\bar{g}_2 = 1$, $\bar{u}_1 = -\frac{\partial V_2}{\partial x_2}$, $\bar{u}_2 = -G_1^T \frac{\partial V_1}{\partial x_1}$: (7.57) is verified. ∎

In conclusion, the system in (7.55) and (7.56) is not convenient because its outputs and inputs have not been properly chosen. By changing the definitions of the inputs and outputs of the subsystems in (7.55) and (7.56), leaving the closed-loop system unchanged, we transform the system such that it belongs to $\mathcal{P}$. In most of the cases, the functions g_i, h_i and f_i are such that the only possibility for the equivalent systems in (7.50) and (7.51) to be Lyapunov stable with Lyapunov functions $V_1(\cdot)$ and $V_2(\cdot)$ respectively is that $\bar{g}_i\bar{u}_i \equiv g_i u_i$, *i.e.* we only have to rearrange the inputs and the outputs to prove passivity. From Lemma 7.23 we can deduce the following result:

Corollary 7.25. *Consider the system in (7.47) and (7.48). Assume (7.49) is satisfied, and that* $\frac{\partial V_1}{\partial x_1}^T G_1 h_2 = -\frac{\partial V_2}{\partial x_2}^T G_2 h_1$ *(let us denote this equality as the Cross Terms Cancellation Equality CTCE). Then* **i)** *If one of the subsystems in (7.47) or (7.48) is passive, the system in (7.47) and (7.48) can be transformed into a system that belongs to* $\mathcal{P}$. **ii)** *If the system in (7.47) and (7.48) is autonomous, it belongs to* $\mathcal{P}$. ∎

Proof: Using the CTCE, one sees that inequalities in (7.52) and (7.53) reduce either to:

$$\frac{\partial V_1}{\partial x_1}^T G_1 h_2 + \frac{\partial V_1}{\partial x_1}^T \bar{g}_1\bar{g}_2^T \frac{\partial V_2}{\partial x_2} = 0 \tag{7.58}$$

or to

$$-\frac{\partial V_2}{\partial x_2}^T G_2 h_1 + \frac{\partial V_2}{\partial x_2}^T \bar{g}_2\bar{g}_1^T \frac{\partial V_1}{\partial x_1} = 0 \tag{7.59}$$

Suppose that the system in (7.48) is passive. Then $h_2 = G_2^T \frac{\partial V_2}{\partial x_2}$, thus it suffices to choose $\bar{g}_2 = G_2$, $\bar{g}_1 = -G_1$. If the system in (7.47) is passive, then $h_1 = G_1^T \frac{\partial V_1}{\partial x_1}$, and we can take $\bar{g}_2 = G_2$, $\bar{g}_1 = G_1$. The second part of the corollary follows from the fact that one has for all x_1 and x_2:

$$\frac{\partial V_1}{\partial x_1}^T G_1(x_1) h_2(x_2) = -\frac{\partial V_2}{\partial x_2}^T G_2(x_2) h_1(x_1) \tag{7.60}$$

Then (7.47) (7.48) can be transformed into a system that belongs to $\mathcal{P}$. Necessarily $h_2(x_2) = G_2^T \frac{\partial V_2}{\partial x_2}$ and $h_1(x_1) = -G_1^T \frac{\partial V_1}{\partial x_1}$, or $h_2(x_2) = -G_2^T \frac{\partial V_2}{\partial x_2}$ and $h_1(x_1 = G_1^T \frac{\partial V_1}{\partial x_1}$, which correspond to solutions of (7.58) or (7.59) respectively. ■

In the case of linear time invariant systems, one gets $G_1 C_2 P_2^{-1} + \bar{G}_1 \bar{G}_2^T = 0$ or $-G_2 C_1 P_1^{-1} + \bar{G}_2 \bar{G}_1^T = 0$ instead of (7.58) and (7.59) respectively. Supposing either $C_2 = G_2^T P_2$ or $C_1 = G_1^T P_1$ the result follows and the passive interconnection is found.

Example 7.26. Throughout this chapter and Chapter 8 we shall see several applications of Lemmas 7.22 and 7.23. In particular it happens that the cancellation of cross terms in Lyapunov functions derivatives has been widely used for stabilization and almost systematically yields an interpretation *via* the passivity theorem. To illustrate those results let us reconsider the PD controller closed-loop dynamics in (7.25). Let us start from the knowledge of the Lyapunov function deduced from the storage function in (7.26). Letting $x_1 = (q, \dot{q})$ be the state of the rigid joint-rigid link dynamics and $x_2 = z_1$ be the state of the second subsystem in (7.31), one sees that the sum of the storage functions associated to each of these blocks forms a Lyapunov function that satisfies the conditions of Lemma 7.22. Moreover the conditions of Corollary 7.25 are satisfied as well, in particular the CTCE. Indeed from (7.32) we get (but the same could be done with the interconnection in (7.29))

$$\begin{cases} \frac{\partial V_1}{\partial x_1}^T G_1(x_1) h_2(x_2) = \left(g^T(q), \dot{q}^T M(q)\right) \begin{pmatrix} 0 \\ M^{-1}(q) \end{pmatrix} (-\lambda_2 \tilde{q}) \\ \qquad\qquad\qquad\qquad = -\lambda_2 \dot{q}^T \tilde{q} \\ \frac{\partial V_2}{\partial x_2}^T G_2(x_2) h_1(x_1) = \lambda_2 \tilde{q}^T \dot{q} \end{cases} \tag{7.61}$$

Hence the dynamics in (7.25) can indeed be interpreted as the negative feedback interconnection of two dissipative blocks. As another example, consider Theorem 5.42: notice that choosing the controller u of the driving system as $u^T = -\left(L_{f_1} U(\zeta)\right)$ exactly corresponds to a CTCE. Hence the closed-loop system thereby constructed can be analyzed through the passivity theorem. This is the mechanism used in [317].

Such closed-loop interpretations of Lyapunov stable systems are not fundamental from a stability point of view, since the system is already known to be stable. However they have been widely used in the Systems and Control literature since they provide an elegant manner to analyze the closed-loop system. Moreover they may provide the designer with ideas linked to the properties of interconnections of passive systems. We shall illustrate again the application of Lemmas 7.22 and 7.23 and Corollary 7.25 in the sequel; see in particular Sections 7.4, 7.6 and Chapter 8.

7.3.4 Extensions of the PD Controller for the Tracking Case

The tracking problem for the model in (6.98) can be easily solved using a linearizing feedback that renders the closed-loop system equivalent to a double integrator. Then all the classical machinery for linear systems can be applied. However we are not interested here in following this path. We would rather like to see how the PD control may be extended to the tracking case, *i.e.* how we can preserve and use the system dissipativity to derive a globally stable controller guaranteeing tracking of any sufficiently differentiable desired trajectory.

A First Extension of the PD Controller: the Paden and Panja Scheme

The first idea is a direct extension of the PD structure, applying the control [389]:

$$\tau = M(q(t))\ddot{q}_d(t) + C(q(t),\dot{q}(t))\dot{q}_d(t) + g(q(t)) - \lambda_1\dot{\tilde{q}}(t) - \lambda_2\tilde{q}(t) \tag{7.62}$$

with $q_d(\cdot) \in C^2(I\!R^+)$. Setting q_d constant one retrieves a PD controller with gravity compensation. The closed-loop system is given by:

$$M(q(t))\ddot{\tilde{q}}(t) + C(q(t),\dot{q}(t))\dot{\tilde{q}}(t) + \lambda_1\dot{\tilde{q}}(t) + \lambda_2\tilde{q}(t) = 0 \tag{7.63}$$

This closed-loop dynamics resembles the one in (7.25). This motivates us to study its stability properties by splitting it into two subsystems as

$$\begin{cases} M(q(t))\ddot{\tilde{q}}(t) + C(q(t),\dot{q}(t))\dot{\tilde{q}}(t) = u_1(t) = -y_2(t) \\ \\ y_1(t) = \dot{\tilde{q}}(t) = u_2(t) \end{cases} \tag{7.64}$$

and

$$\begin{cases} \dot{z}_1(t) = u_2(t) \\ \\ y_2(t) = \lambda_1 u_2(t) + \lambda_2 z_1(t) \\ \\ z_1(0) = q(0) - q_d(0) \end{cases} \tag{7.65}$$

Let us make the following Assumption (see Lemma 6.16):

Assumption 17 *The matrix $C(q,\dot{q})$ is written in such a way that $\dot{M}(q,\dot{q}) - 2C(q,\dot{q})$ is skew-symmetric.* ■

Then one computes that

$$\begin{aligned}\langle u_1, y_1\rangle_t &= \int_0^t \dot{\tilde{q}}^T(\tau)\left[M(q(\tau))\ddot{\tilde{q}}(\tau) + C(q(\tau),\dot{q}(\tau))\dot{\tilde{q}}(\tau)\right]d\tau \\ &= \frac{1}{2}\left[\dot{\tilde{q}}^T(\tau)M(q(\tau))\dot{\tilde{q}}(\tau)\right]_0^t \\ &\geq -\frac{1}{2}\dot{\tilde{q}}(0)^T M(q(0))\dot{\tilde{q}}(0)\end{aligned} \tag{7.66}$$

and that

$$\begin{aligned}\langle u_2, y_2\rangle_t &= \lambda_1 \int_0^t \dot{\tilde{q}}^T(\tau)\dot{\tilde{q}}(\tau)d\tau + \frac{1}{2}\left[\tilde{q}(s)^T\tilde{q}(s)\right]_0^t \\ &\geq -\frac{1}{2}\tilde{q}(0)^T\tilde{q}(0)\end{aligned} \tag{7.67}$$

Notice that the second block is input strictly passive. Similarly to the PD controller analysis, one concludes that the dynamics in (7.63) can indeed be transformed into the interconnection of two passive blocks. We could also have deduced from Lemma 7.23 that such an interconnection exists, checking that $V(\tilde{q},\dot{\tilde{q}}) = \frac{1}{2}\dot{\tilde{q}}M(q)\dot{\tilde{q}} + \frac{1}{2}\lambda_2\tilde{q}^T\tilde{q}$ is a Lyapunov function for this system, whose derivative along the trajectories of (7.63) is semi-negative definite (*i.e.* $\gamma_1 = 0$ in Lemma 7.22) (we let the reader do the calculations by him/herself). However one cannot apply the Krasovskii-La Salle Theorem to this system because it is not autonomous (the inertia and Coriolis matrices depend explicitly on time when the state is considered to be $(\tilde{q},\dot{\tilde{q}})$). One has to resort to Matrosov's Theorem to prove the asymptotic stability (see Theorem A.35 and Lemma A.36 in the Appendix) [389]. Equivalent representations (that are to be compared to the ones constructed for the PD control in Subsection 7.3.1) are depicted in Figures 7.4 and 7.5.

The Slotine and Li Controller (Passivity Interpretation)

The above scheme has the advantage of being quite simple. However its extension to the adaptive case (when the inertia parameters are supposed to be unknown, one needs to introduce some on-line adaptation) is really not straightforward. One big challenge in the Robotics and Systems and Control fields during the 1980s was to propose a feedback controller that guarantees tracking and which extends also to an adaptive version (which will be presented in Section 8.1.1). Let us consider the following input [425, 461] [3]:

$$\tau(q(t),\dot{q}(t),t) = M(q(t))\ddot{q}_r(t) + C(q(t),\dot{q}(t))\dot{q}_r(t) + g(q(t)) - \lambda_1 s(t) \tag{7.68}$$

[3] It seems that what is now widely known as the Slotine and Li scheme, was also designed in [425] at the same time so that the Slotine and Li scheme could be named the Slotine-Li-Sadegh-Horowitz scheme.

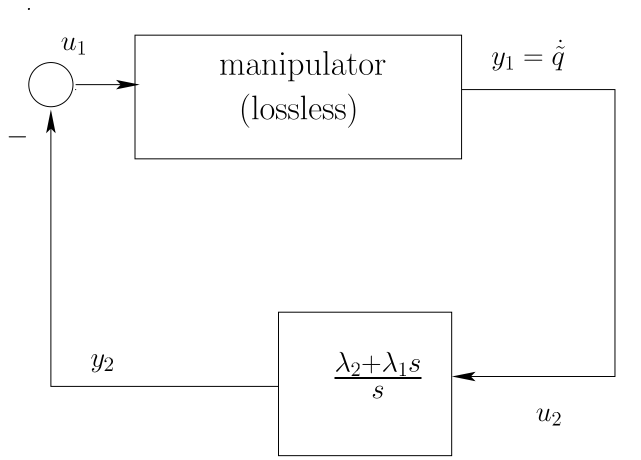

Fig. 7.4. First interconnection: lossless manipulator dynamics

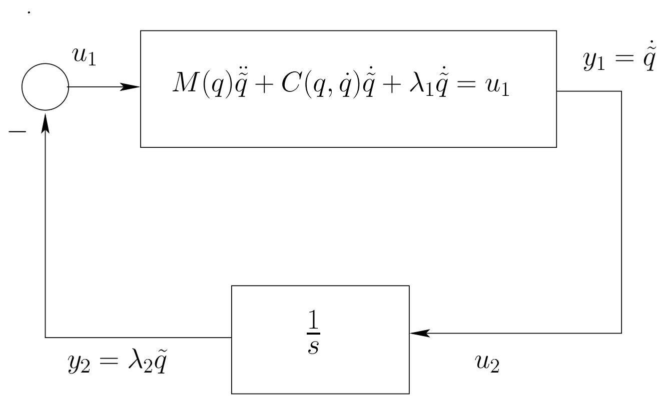

Fig. 7.5. Second interconnection: OSP manipulator dynamics

where $\dot{q}_r(t) = \dot{q}_d(t) - \lambda\tilde{q}(t)$, $s(t) = \dot{q}(t) - \dot{q}_r(t) = \dot{\tilde{q}}(t) + \lambda\tilde{q}(t)$, and we recall that $q_d(\cdot)$ is supposed to be in $C^2(I\!R^+)$. Introducing (7.68) into (6.98) one obtains

$$M(q(t))\dot{s}(t) + C(q(t),\dot{q}(t))s(t) + \lambda_1 s(t) = 0 \tag{7.69}$$

Notice that contrary to the scheme in (7.62), setting q_d constant in (7.68) does not yield the PD controller. However the controller in (7.68) can be seen as a PD action ($\lambda_1 s$) with additional nonlinear terms whose role is to assure some tracking properties. Before going on let us note that the whole closed-loop dynamics is not in (7.69) since this is an nth order system with state s, whereas the whole system is $2n$-th order. To complete it one needs to add to (7.69):

$$\dot{\tilde{q}}(t) = -\lambda\tilde{q}(t) + s(t) \tag{7.70}$$

Therefore the complete closed-loop dynamical system is given by

$$\boxed{\begin{array}{l} M(q(t))\dot{s}(t) + C(q(t),\dot{q}(t))s(t) + \lambda_1 s(t) = 0 \\[2mm] \dot{\tilde{q}}(t) = -\lambda\tilde{q}(t) + s(t) \\[2mm] \tilde{q}(0) = \tilde{q}_0, \dot{\tilde{q}}(0) = \dot{\tilde{q}}_0 \end{array}}$$

It should be clear from now all the foregoing developments that the subsystem in (7.69) defines a passive operator between $u_1 = -\lambda_1 s = -y_2$ and $y_1 = s = u_2$, with storage function $V_1(s,t) = \frac{1}{2}s^T M(q)s$ (which is a Lyapunov function for this subsystem which is zero-state observable). This is strongly based on Assumption 17. The equivalent feedback interconnection of the closed-loop is shown in Figure 7.6.

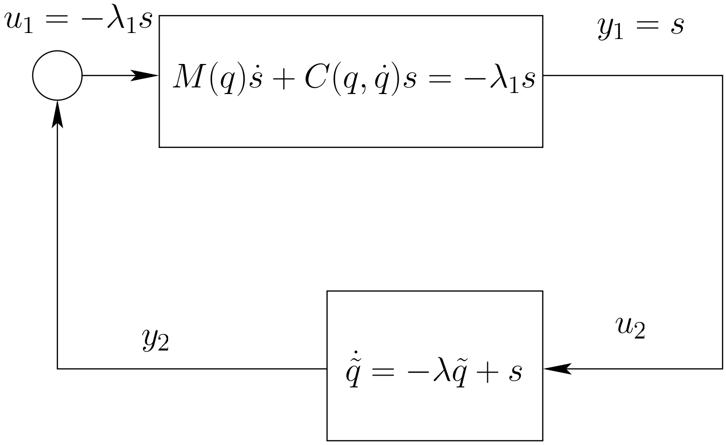

Fig. 7.6. Closed-loop equivalent representation

Remark 7.27. The subsystem in (7.69) can be at once proved to define an asymptotically stable system since one can view it as the interconnection of a passive mapping $u \mapsto y = \dot{q}$ with zero-state detectable dynamics $M(q)\dot{s} + C(q,\dot{q})s = u$, with a static output feedback $u = -\lambda_1 y$. Hence Theorem 5.24 applies and one concludes that $s(t) \to 0$ as $t \to +\infty$. ■

The second subsystem obtained from (7.70) can be rewritten as

$$\begin{cases} \dot{z}_1(t) = -\lambda z_1(t) + u_2(t) \\ \\ y_2 = \lambda_1 u_2 \end{cases} \tag{7.71}$$

It therefore has a relative degree $r_2 = 0$, and the state is not observable from the output y_2. However it is zero-state detectable since $\{y_2 = u_2 = 0\} \Rightarrow \lim_{t\to+\infty} z_1(t) = 0$. We also notice that this system is very strictly passive since

$$\begin{aligned} \langle u_2, y_2 \rangle_t &= \lambda_1 \int_0^t u_2^T(s) u_2(s) ds \\ &= \tfrac{1}{\lambda_1} \int_0^t y_2^T(s) y_2(s) ds \\ &= \tfrac{\lambda_1}{2} \int_0^t u_2^T(s) u_2(s) ds + \tfrac{1}{2\lambda_1} \int_0^t y_2^T(s) y_2(s) ds \end{aligned} \tag{7.72}$$

Let us compute storage functions for this system. Let us recall from (4.137) that for systems of the form $\dot{x} = f(x,t) + g(x,t)u, y = h(x,t) + j(x,t)u$ with $j(x,t) + j^T(x,t) = R$ full-rank, the storage functions are solutions of the partial differential inequality (that reduces to a Riccati inequation in the linear case)

$$\frac{\partial V}{\partial x}^T f(x,t) + \frac{\partial V}{\partial t} + \left(h^T - \frac{1}{2}\frac{\partial V}{\partial x}^T g\right) R^{-1} \left(h - \frac{1}{2} g \frac{\partial V}{\partial x}\right) \leq 0 \tag{7.73}$$

and that the available storage $V_a(\cdot)$ and the required supply $V_r(\cdot)$ (with $x(-t) = 0$) satisfy (7.73) as an equality. Thus the storage functions $V(z_1)$ for the system in (7.71) are solutions of

$$-\lambda \frac{dV}{dz_1}^T z_1 + \frac{1}{4\lambda_1} \frac{dV}{dz_1}^T \frac{dV}{dz_1} \leq 0 \tag{7.74}$$

If we set the equality it follows that the two solutions satisfy

$$\begin{cases} \frac{dV}{dz_1}(t) = 0 \\ \\ \frac{dV}{dz_1}(t) = 4\lambda\lambda_1 z_1(t) \end{cases} \tag{7.75}$$

for all $t \geq 0$, from which one deduces that $V_a(z_1) = 0$ and $V_r(z_1) = 2\lambda\lambda_1 z_1^T z_1$, whereas any other storage function satisfies $0 = V_a(z_1) \leq V(z_1) \leq V_r(z_1)$.

Remark 7.28. Let us retrieve the available storage and the required supply from their variational formulations (notice that the system in (7.71) is controllable so that the required supply can be defined):

$$V_a(z_1(0)) = \sup_{u_2:(0,z_1(0))\rightarrow} -\int_0^t \lambda_1 u_2^T u_2 ds = 0 \tag{7.76}$$

which means that the best strategy to recover energy from this system through the output y_2 is to leave it at rest (so as to recover nothing, actually!), and

$$\begin{aligned} V_r(z_1(0)) &= \inf_{u_2:(-t,0)\rightarrow(0,z_1(0))} \int_{-t}^0 u_2^T y_2 ds \\ &= \inf_{u_2:(-t,0)\rightarrow(0,z_1(0))} \lambda_1 \int_{-t}^0 \left\{(\dot{z}_1^T + \lambda z_1^T)(\dot{z}_1 + \lambda z_1)\right\} ds \\ &= \lambda_1 \lambda z_1^T(0) z_1(0) \end{aligned} \tag{7.77}$$

where the last step is performed by simple integration of the cross term and dropping the other two terms which are always positive, for any control strategy. ■

We conclude that a suitable Lyapunov function for the closed-loop system in (7.69) (7.70) is given by the sum

$$V(s,\tilde{q},t) = \frac{1}{2} s^T M(\tilde{q},t) s + 2\lambda\lambda_1 \tilde{q}^T \tilde{q} \tag{7.78}$$

It is noteworthy that we have really deduced a Lyapunov function from the knowledge of some passivity properties of the equivalent interconnection form of the closed-loop system. Historically, the closed-loop system in (7.69) and (7.70) has been studied first using the storage function of the first subsystem in (7.69) only, and then using additional arguments to prove the asymptotic convergence of the whole state towards zero [461]. It is only afterwards that the Lyapunov function for the whole closed-loop system has been proposed [472]. We have shown here that it is possible to construct it directly from passivity arguments. It must therefore be concluded on this example that the dissipativity properties allow one to directly find out the right Lyapunov function for this system.

Remark 7.29. Lemmas 7.22 and 7.23 can in general be used if one starts from the knowledge of the Lyapunov function. However the cross-term-cancellation-equality (CTCE) is not satisfied since

$$\begin{cases} \frac{\partial V_1}{\partial x_1}^T G_1(x_1) h_2(x_2) = s^T M(q) M^{-1}(q) \lambda_1 s = \lambda_1 s^T s \\ \\ \frac{\partial V_2}{\partial x_2}^T G_2(x_2) h_1(x_1) = -\lambda\lambda_1 \tilde{q}^T s \end{cases} \tag{7.79}$$

This comes from the fact that this time one has to add $\frac{\partial V_1}{\partial x_1}^T G_1(x_1)h_2(x_2) + \frac{\partial V_2}{\partial x_2}^T G_2(x_2)h_1(x_1) = -\lambda_1 s^T s + \lambda\lambda_1 \tilde{q}^T s$ to $\frac{\partial V_2}{\partial x_2}^T F_2(x_2) = -2\lambda^2\lambda_1\tilde{q}^T\tilde{q}$ in order to get the inequality in (7.49). One may also check that the inequalities in (7.52) and (7.53) can hardly be satisfied by any $\bar{g}_1$ and $\bar{g}_2$. Actually the conditions stated in Lemma 7.23 and Corollary 7.25 are sufficient only. For instance from (7.49) one can change the inequalities in (7.52) and (7.53) to incorporate the terms $\frac{\partial V_1}{\partial x_1}^T F_1(x_1,t)$ and $\frac{\partial V_2}{\partial x_2}^T F_2(x_2,t)$ in the conditions required for the matrices $\bar{g}_1$ and $\bar{g}_2$. Actually Lemmae 7.22 and 7.23 will be useful when we deal with adaptive control; see Chapter 8, in which case the CTCE is generally satisfied.

The Slotine and Li Controller (Stability Analysis)

There are two ways to prove the stability for the closed-loop system in (7.69) and (7.70). The first proof is based on the positive function $V(s,\tilde{q},t) = \frac{1}{2}s^T M(q)s$ (which we denoted as $V_1(s,t)$ above), where one notices that $q(t) = \tilde{q}(t) + q_d(t)$. Hence the explicit time-dependency in $V(s,\tilde{q},t)$. This proof makes use of Lemma 4.8. This proof does not show Lyapunov stability but merely shows the boundedness of all signals as well as the asymptotic convergence of the tracking error and its derivative towards zero. The second prof is based on the Lyapunov function (candidate) in (7.78). Lyapunov stability of the error (closed-loop) system equilibrium point is then concluded.

First stability proof:

Let us consider

$$V(s,\tilde{q},t) = \frac{1}{2}s^T M(q)s, \tag{7.80}$$

and let us calculate its derivative along the solutions of (7.69):

$$\begin{aligned}
\dot{V}(s,\tilde{q},t) &= s^T(t)M(q(t))\dot{s}(t) + \tfrac{1}{2}s^T(t)\dot{M}(q(t),\dot{q}(t))s(t)\\
&= s^T(t)(-C(q(t),\dot{q}(t)) - \lambda_1 s(t)) + \tfrac{1}{2}s^T(t)\dot{M}(q(t),\dot{q}(t))s(t)\\
&= -\lambda_1 s^T(t)s(t) + s^T(t)[-C(q(t),\dot{q}(t)) + \tfrac{1}{2}\dot{M}(q(t),\dot{q}(t))]s(t)\\
&= -\lambda_1 s^T(t)s(t) \le 0
\end{aligned} \tag{7.81}$$

where the last equality is obtained thanks to the skew-symmetry property (Lemma 6.16). Let us now integrate both sides of (7.81):

$$V(s(t),q(t)) - V(s(0),q(0)) \le -\int_0^t s^T(\tau)s(\tau)d\tau \tag{7.82}$$

which implies that

$$\int_0^t s^T(\tau)s(\tau)d\tau \leq V(s(0), q(0)) \tag{7.83}$$

since $V(\cdot,\cdot) \geq 0$. Therefore $s(\cdot)$ is in $\mathcal{L}_2$. Let us now consider the system in (7.70). This is an asymptotically stable system whose state is $\tilde{q}(\cdot)$ and whose input is $s(\cdot)$. Applying Lemma 4.8 we deduce that $\tilde{q} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\dot{\tilde{q}} \in \mathcal{L}_2$, and $\lim_{t\to+\infty} \tilde{q}(t) = 0$. Furthermore since $V(s(t), \tilde{q}(t), t) \leq V(s(0), \tilde{q}(0), 0)$, it follows that for bounded initial data, $||s(t)|| < +\infty$, *i.e.* $s \in \mathcal{L}_\infty$. Therefore $\dot{\tilde{q}} \in \mathcal{L}_\infty$ as well, and from Fact 6 (Section 4.1) the function $\tilde{q}(\cdot)$ is uniformly continuous. Using (7.69) it follows that $\dot{s} \in \mathcal{L}_\infty$, so using Fact 6 and then Fact 8 we conclude that $s(t) \to 0$ as $t \to +\infty$. Thus $\dot{\tilde{q}}(t) \to 0$ as $t \to +\infty$. All the closed-loop signals are bounded and the tracking error $(\tilde{q}, \dot{\tilde{q}})$ converges globally asymptotically to zero. However we have not proved the Lyapunov stability of the equilibrium point of the closed-loop error system (7.69) and (7.70).

Lyapunov stability proof:

Let us now consider the positive definite function in (7.78). Computing its derivative along the closed-loop system (7.69) and (7.70) trajectories yields

$$\dot{V}(\tilde{q}(t), \dot{\tilde{q}}(t)) = -\lambda_1 \dot{\tilde{q}}^T(t)\dot{\tilde{q}}^T(t) - \lambda^2\lambda_1 \tilde{q}^T(t)\tilde{q}(t) \leq 0 \tag{7.84}$$

from which the global asymptotic Lyapunov stability of the fixed point $(\tilde{q}, \dot{\tilde{q}}) = (0, 0)$ follows. The skew-symmetry property is used once again to compute the derivative. It was further shown in [472] that when the system has only revolute joints then the stability is uniform. This comes from the fact that in such a case, the inertia matrix $M(q)$ contains only bounded (smooth) functions like $\cos(\cdot)$ and $\sin(\cdot)$ and is thus bounded, consequently the Lyapunov function is also upperbounded by some class $\mathcal{K}$ function. It is interesting to see how the technology influences the stability.

In both stability proofs, one can conclude about exponential convergence. Indeed for the first proof one has $\dot{V}(s, \tilde{q}, t) \leq -\lambda_1 s^T(t)s(t) \leq -\frac{\lambda_1}{\lambda_{\min} M(q)} V(s, \tilde{q}, t)$. Therefore $s(\cdot)$ converges to zero exponentially fast, and so do $\tilde{q}(\cdot)$ and $\dot{\tilde{q}}(\cdot)$. The interest of the above proof is that when we deal with the adaptive case, then exponential stability will be lost, and the stability proof is then identical to the above one.

7.3.5 Other Types of State Feedback Controllers

The use of the property in Assumption 17 is not mandatory. Let us describe now a control scheme proposed in [239], that can be classified in the set of passivity-based control schemes, as will become clear after the analysis. Let us consider the following control input:

$$\begin{aligned}\tau = &-\tfrac{1}{2}\dot{M}(q(t),\dot{q}(t))[\dot{\tilde{q}}(t) + \lambda\tilde{q}(t)] + M(q(t))[\ddot{q}_r(t) - \lambda\dot{\tilde{q}}(t)] + \\ &+ C(q(t),\dot{q}(t))\dot{q}(t) + g(q(t)) - \left(\lambda_d + \tfrac{\lambda}{\lambda_1}\right)\dot{\tilde{q}}(t) - \lambda\lambda_d\tilde{q}(t)\end{aligned} \tag{7.85}$$

Introducing (7.85) into the dynamics (6.98) one obtains:

$$M(q(t))\dot{s}(t) + \frac{1}{2}\dot{M}(q(t),\dot{q}(t))s(t) + \left(\lambda_d + \frac{\lambda}{\lambda_1}\right)\dot{\tilde{q}}(t) + \lambda\lambda_d\tilde{q}(t) = 0 \tag{7.86}$$

which we can rewrite equivalently as

$$\begin{aligned}&M(q(t))\dot{s}(t) + C(q(t),\dot{q}(t))s(t) + \left(\lambda_d + \tfrac{\lambda_1^2}{2}\right)\dot{\tilde{q}}(t) + \tfrac{\lambda_d\lambda_2}{\lambda_1}\tilde{q}(t) = \\ &= -\tfrac{1}{2}\dot{M}(q(t),\dot{q}(t))s(t) + C(q(t),\dot{q}(t))s(t)\end{aligned} \tag{7.87}$$

These two representations of the same closed-loop system are now analyzed from a "passivity theorem" point of view. Let us consider the following negative feedback interconnection:

$$\begin{cases} u_1 = -y_2 = -\tfrac{1}{2}\dot{M}(q,\dot{q})s + C(q,\dot{q})s \\ \\ u_2 = y_1 = s \end{cases} \tag{7.88}$$

where the first subsystem has dynamics $M(q(t))\dot{s}(t) + C(q(t),\dot{q}(t))s(t) + \left(\lambda_d + \frac{\lambda}{\lambda_1}\right)\dot{\tilde{q}}(t) + \lambda_d\lambda\tilde{q}(t) = u_1(t)$ while the second one is a static operator between $u_2 = s$ and y_2 given by $u_2(t) = \frac{1}{2}\dot{M}(q(t),\dot{q}(t))s(t) - C(q(t),\dot{q}(t))s(t)$. It is easily checked that if Assumption 17 is satisfied then

$$\langle u_2, y_2\rangle_t = \frac{1}{2}\int_0^t s^T(\tau)[\dot{M}(q(\tau),\dot{q}(\tau)) - 2C(q(\tau),\dot{q}(\tau))]s(\tau)d\tau = 0 \tag{7.89}$$

and that the available storage of the second block is the zero function as well. Concerning the first subsystem one has

$$\begin{aligned}\langle u_1, y_1\rangle_t &= \textstyle\int_0^t s^T(\tau)\left[M(q(\tau))\dot{s} + C(q(\tau),\dot{q}(\tau))s(\tau) + \left(\lambda_d + \frac{\lambda}{\lambda_1}\right)\dot{\tilde{q}}(\tau) + \lambda_d\lambda\tilde{q}(\tau)\right]d\tau \\ &= \tfrac{1}{2}[s^{T(\tau)}M(q(\tau))s(\tau)]_0^t + \tfrac{1}{2}\left(2\lambda\lambda_d + \tfrac{\lambda^2}{\lambda_1}\right)[\tilde{q}^T(\tau)\tilde{q}(\tau)]_0^t \\ &+ \textstyle\int_0^t \left\{\left(\lambda_d + \frac{\lambda}{\lambda_1}\right)\dot{\tilde{q}}^T(\tau)\dot{\tilde{q}}(\tau) + \lambda^2\lambda_d\tilde{q}^T(\tau)\tilde{q}(\tau)\right\}d\tau \\ &\geq -\tfrac{1}{2}s(0)^T M(q(0))s(0) - \tfrac{1}{2}\left(2\lambda\lambda_d + \tfrac{\lambda^2}{\lambda_1}\right)\tilde{q}(0)^T\tilde{q}(0)\end{aligned} \tag{7.90}$$

which proves that it is passive with respect to the supply rate $u_1^T y_1$. It can also be calculated that the available storage function of this subsystem is given by:

$$\begin{aligned} V_a(\tilde{q}(0), s(0)) = \sup_{u_1:[\tilde{q}(0),s(0)]\to} & -\int_0^t s^T(\tau)\left\{M(q(\tau))s(\tau) + C(q(\tau),\dot{q}(\tau))s(\tau)\right. \\ & \left. + \left(\lambda_d + \tfrac{\lambda}{\lambda_1}\right)\dot{\tilde{q}}(\tau) + \lambda\lambda_d\tilde{q}(\tau)\right\} d\tau \\ & = \tfrac{1}{2}s(0)^T M(q(0))s(0) + \left(\lambda\lambda_d + \tfrac{\lambda^2}{2\lambda_1}\right)\tilde{q}^T(0)\tilde{q}(0) \end{aligned} \tag{7.91}$$

Since this subsystem is zero-state detectable ($u_1 \equiv s \equiv 0 \Rightarrow \tilde{q} \to 0$ as $t \to +\infty$) one concludes that the available storage in (7.91) is actually a Lyapunov function for the corresponding unforced system, whose fixed point $(\tilde{q}, s) = (0,0)$ (or $(\tilde{q}, \dot{\tilde{q}}) = (0,0)$) is asymptotically stable. This also holds for the complete closed-loop system since the second block has storage functions equal to zero and the dynamics in (7.86) is zero-state detectable when one considers the input to be u in the left-hand-side of (7.86) and $y = y_1 = s$ (set $u \equiv 0$ and $s \equiv 0$ and it follows from (7.86) that $\tilde{q} \to 0$ exponentially). Actually, the derivative of $V_a(\tilde{q}, s)$ in (7.91) along trajectories of the first subsystem is given by:

$$\dot{V}_a(\tilde{q}(t), s(t)) = -\left(\lambda_d + \frac{\lambda}{\lambda_1}\right)\dot{\tilde{q}}^T(t)\dot{\tilde{q}}(t) - \lambda^2\lambda_d\tilde{q}^T(t)\tilde{q}(t) \le 0 \tag{7.92}$$

It is noteworthy that the result in (7.92) can be obtained without using the skew-symmetry property in assumption 17 at all. But skew-symmetry was used to prove the dissipativity of each block in (7.88).

Remark 7.30. Originally the closed-loop system in (7.86) has been proven to be Lyapunov stable using the Lyapunov function

$$V(\dot{\tilde{q}}, \tilde{q}) = \frac{1}{2}\dot{\tilde{q}}^T M(q)\dot{\tilde{q}} + \dot{\tilde{q}}^T M(q)\tilde{q} + \frac{1}{2}\tilde{q}^T[\lambda^2 M(q) + \lambda_1 I_n]\tilde{q} \tag{7.93}$$

which can be rearranged as

$$V(s, \tilde{q}) = \frac{1}{2}s^T M(q)s + \frac{1}{2}\lambda_1\tilde{q}^T\tilde{q} \tag{7.94}$$

The derivative of $V(\cdot)$ in (7.93) or (7.94) along closed-loop trajectories is given by:

$$\dot{V}(\tilde{q}(t), \dot{\tilde{q}}(t)) = -\dot{\tilde{q}}^T(t)\left(\lambda_d + \frac{\lambda_1}{\lambda}\right)\dot{\tilde{q}}(t) - 2\lambda_d\lambda\dot{\tilde{q}}^T(t)\tilde{q}(t) - \lambda^2\lambda_d\tilde{q}^T(t)\tilde{q}(t) \tag{7.95}$$

Notice that $V_a(\cdot)$ in (7.91) and $V(\cdot)$ in (7.94) are not equal one to each other. One concludes that the passivity analysis of the closed-loop permits to discover a (simpler) Lyapunov function.

Remark 7.31. The foregoing stability analysis does not use the CTCE of Lemma 7.23. One concludes that the schemes that are not based on the skew-symmetry property in Assumption 17 do not lend themselves very well to an analysis through the passivity Theorem. We may however consider the controller in (7.85) to be passivity-based since it does not attempt at linearizing the system, similarly to the Slotine and Li scheme.

7.4 Rigid Joint–Rigid Link: Position Feedback

Usually most manipulators are equipped with position and velocity sensors, and controlled point-to-point with a PD. The tracking case requires more, as we saw. However the controllers structure becomes more complicated, hence less robust. It is of some interest to try to extend the separation principle for linear systems (a stable observer can be connected to a stabilizing controller without destroying the closed-loop stability), towards some classes of nonlinear systems. The rigid joint-rigid link manipulator case seems to constitute a good candidate, due to its nice properties. At the same time such systems are nonlinear enough, so that the extension is not trivial. In the continuity of what has been done in the preceding sections, we shall investigate how the dissipativity properties of the Slotine and Li and of the Paden and Panja schemes can be used to derive (locally) stable controllers not using velocity feedback.

In the following we shall start by the regulation case (see Section 7.4.1), and then analyze the tracking of trajectories (see Sections 7.4.2 and 7.4.3).

7.4.1 P + Observer Control

In this subsection we present the extension of the PD controller when the velocity is not available as done in [44, 488]. Basically the structure of output (position) feedback controllers is that of the original input where the velocity $\dot{q}$ is replaced by some estimated value. Let us consider the dynamics in (6.98) with the controller:

$$\begin{cases} \tau = g(q_d) - \lambda_1 \tilde{q} - \frac{1}{\lambda_2}(\tilde{q} - z) \\ \\ \dot{z} = \lambda_3(\tilde{q} - z) \end{cases} \tag{7.96}$$

so that the closed-loop dynamics is given by

$$\begin{cases} M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + g(q(t)) - g(q_d) + \frac{1}{\lambda_2}(\tilde{q}(t) - z(t)) = -\lambda_1 \tilde{q}(t) \\ \dot{z}(t) - \dot{q}(t) = \lambda_3(\tilde{q}(t) - z(t)) - \dot{q}(t) \end{cases} \tag{7.97}$$

Let us now make a direct application of Corollary 7.25. Let us first rewrite (7.97) in a state-space form, with $x_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} \tilde{q} \\ \dot{q} \end{pmatrix}$ and $x_2 = \tilde{q} - z$. We obtain

$$\begin{cases} \dot{x}_{11}(t) &= x_{12}(t) \\ \dot{x}_{12}(t) &= -M^{-1}(x_{11}(t) + q_d)[C(x_{11}(t) + q_d, x_{12}(t))x_{12}(t) + g(x_{11}(t) + q_d) \\ & \quad -g(q_d) + \lambda_1 x_{11}(t)] + M^{-1}(x_{11}(t) + q_d)h_2(x_2(t)) \\ \dot{x}_2(t) &= -\lambda_3 x_2(t) + h_1(x_1(t)) \\ h_2(x_2) &= -\frac{1}{\lambda_2} x_2 \\ h_1(x_1) &= x_{12} \end{cases} \tag{7.98}$$

where $h_1(\cdot)$ and $h_2(\cdot)$ are as in (7.47) and (7.48). The closed-loop scheme can be shown to be globally asymptotically Lyapunov stable with the Lyapunov function $V(x_{11}, x_{12}, x_2) = V_1(x_{11}, x_{12}) + V_2(x_2)$ defined as

$$\begin{aligned} V_1(x_{11}, x_{12}) = \lambda_2 \Big[& \tfrac{1}{2} x_{12}^T M(x_{11} + q_d)x_{12} + \tfrac{\lambda_1}{2} x_{11}^T x_{11} + U_g(x_{11} + q_d) \\ & -U_g(q_d) - x_{11}^T g(q_d) \Big] \end{aligned} \tag{7.99}$$

and

$$V_2(x_2) = \frac{1}{2} x_2^T x_2 \tag{7.100}$$

It can be shown that $V_1(\cdot)$ is positive definite and has a global minimum at $(x_{11}, x_{12}) = (0, 0)$ provided $\lambda_1 \geq \gamma$ where γ is a Lipschitz constant for $g(\cdot)$. Differentiating $V(\cdot)$ along the trajectories of (7.97) or equivalently (7.98) one finds

$$\dot{V}(x_2) = -\lambda_3 x_2^T x_2 \tag{7.101}$$

where the CTCE is satisfied since $\frac{\partial V_1}{\partial x_1}^T G_1 h_2 = -x_{12}^T x_2 = -\frac{\partial V_2}{\partial x_2}^T G_2 h_1$. Since the system is autonomous, Corollary 7.25 **ii)** applies. Now it is easy to see that the second subsystem with state vector x_2, input $u_2 = h_1(x_1)$ and ouput $y_2 = -h_2(x_2)$ is passive:

$$\begin{aligned}\langle u_2, y_2\rangle_t &= \int_0^t \tfrac{1}{\lambda_2} x_2^T(s) u_2(s) ds \\ &= \int_0^t \tfrac{1}{\lambda_2} x_2^T(s)(\dot{x}_2(s) + \lambda_3 x_2(s)) ds \\ &= \tfrac{1}{2\lambda_2}[x_2^T(s) x_2(s)]_0^t + \tfrac{\lambda_3}{\lambda_2}\int_0^t x_2^T(s) x_2(s) ds\end{aligned} \tag{7.102}$$

and one recognizes a storage function $S_2(x_2)$ equal to $\frac{1}{\lambda_2}V_2$ with V_2 in (7.100). Notice that the second subsystem (with state x_2) is strictly passive in the sense of Lemma 4.84, but it is also output strictly passive. The other subsystem is defined with input $u_1 = -y_2 = h_2(x_2)$ and output $y_1 = u_2 = h_1(x_1)$ and is passive as one can check:

$$\begin{aligned}&\langle u_1, y_1\rangle_t = \langle x_{12}, h_2\rangle_t = \\ &= \int_0^t x_{12}^T(s)[M(x_{11}(s) + q_d)\dot{x}_{12}(s) + C(x_{11}(s) + q_d, x_{12}(s))x_{12}(s) \\ &\quad + g(x_{11}(s) + q_d)x_{12}(s) - g(q_d)x_{12}(s) + \lambda_1 x_{11}(s) x_{12}(s)] ds \\ &= S_1(t) - S_1(0),\end{aligned} \tag{7.103}$$

where we used $\dot{x}_{11}(t) = x_{12}(t)$ in the calculation.

Remark 7.32.
- In connection with Remark 7.18, let us note that this time the closed-loop scheme has an order strictly larger than the open-loop one.
- One has $V(x_1, x_2) = V_1(x_1) + V_2(x_2) = \lambda_2 S_1(x_1) + \lambda_2 S_2(x_2)$. This is due to the particular choice of $h_1(x_1)$ and $h_2(x_2)$.
- The output strict passivity plus zero state detectability properties of the second block is important because it is precisely these properties that allow one to use the Krasovskii-La Salle Theorem to prove the asymptotic stability.

7.4.2 The Paden and Panja + Observer Controller

The material that follows is mainly taken from [45]. In fact it is to be expected that the separation principle does not extend completely to the nonlinear systems we deal with. Indeed the presented schemes assure local stability only (more exactly they assure semi-global stability, *i.e.* the region of attraction of the closed-loop fixed point can be arbitrarily increased by increasing some feedback gains). In what follows we shall not develop the whole stability proofs. We shall just focus on the passivity interpretation of the obtained closed-loop system, and in particular on the local stability that results from the fact that the storage function satisfies the dissipation inequality locally only.

The foregoing subsection was devoted to an extension of PD controllers and concerns global regulation around a fixed position only. It is of interest to consider the tracking case which is, as one expects, much more involved due to the non-autonomy of the closed-loop scheme. Let us consider the following fixed parameter scheme (compare with the expression in (7.62)):

$$\textbf{Controller} \begin{cases} \tau = M(q)\ddot{q}_d + C(q,\dot{q}_0)\dot{q}_d + g(q) - \lambda_1(\dot{q}_0 - \dot{q}_r) \\ \dot{q}_r(t) = \dot{q}_d(t) - \lambda_2 e(t) \\ \dot{q}_0(t) = \dot{\hat{q}}(t) - \lambda_3 \tilde{q}(t) \end{cases} \tag{7.104}$$

$$\textbf{Observer} \begin{cases} \dot{\hat{q}}(t) = z(t) + \lambda_4 \tilde{q}(t) = z(t) + (\lambda_6 + \lambda_3)\tilde{q}(t) \\ \dot{z}(t) = \ddot{q}_d(t) + \lambda_5 \tilde{q}(t) = \ddot{q}_d(t) + \lambda_6\lambda_3\tilde{q}(t) \end{cases}$$

where $e = q - q_d(t)$ is the tracking error, $\tilde{q} = q - \hat{q}$ is the estimation error, $\lambda_i > 0$ for all $i = 1, \cdots, 6$. Let us denote $s_1 = \dot{q} - \dot{q}_r = \dot{e} + \lambda_2 e$ and $s_2 = \dot{q} - \dot{q}_0 = \dot{\tilde{q}} + \lambda_3 \tilde{q}$, so that $(\dot{q}_0 - \dot{q}_r) = s_1 - s_2$. Introducing (7.104) into (6.98) and using some properties of the matrix $C(q,\dot{q})$ (like the fact that $C(q,y)x = C(q,x)y$ and $C(q, z + \alpha x)y = C(q,z)y + \alpha C(q,x)y$ for all $x, y \in I\!R^n$ and $\alpha \in I\!R$) one gets the following closed-loop error equation:

$$\begin{cases} M(q(t))\ddot{e}(t) + C(q(t),\dot{q}(t))s_1(t) + \lambda_1 s_1(t) = \lambda_1 s_2(t) + C(q(t),\dot{q}(t))\lambda_2 e(t) - \\ - C(q(t), s_2(t))\dot{q}_d(t) \\ \dot{e}(t) = -\lambda_2 e(t) + s_1(t) \\ M(q(t))\dot{s}_2(t) + C(q(t),\dot{q}(t))s_2(t) + [\lambda_6 M(q(t)) - \lambda_1 I_n]s_2(t) = -\lambda_1 s_1(t) + \\ + C(q(t), s_2(t) - \dot{q}(t))\dot{e}(t) \\ \dot{\tilde{q}}(t) = -\lambda_3 \tilde{q}(t) + s_2(t) \end{cases} \tag{7.105}$$

Define $K_1(q,e) = \lambda_2^2[2\frac{\lambda_1}{\lambda_2} - M(q)]$ and $K_2(q,\tilde{q}) = 2\lambda_3\lambda_1$. It can be shown using the positive definite function

$$\begin{aligned} V(e, s_1, \tilde{q}, s_2) = \tfrac{1}{2}s_1^T M(q)s_1 + \tfrac{1}{2}e^T K_1(q,e)e + \tfrac{1}{2}s_2^T M(q)s_2 \\ + \tfrac{1}{2}\tilde{q}^T K_2(q,\tilde{q})\tilde{q} \end{aligned} \tag{7.106}$$

that for a suitable choice of the initial data within a ball B_r whose radius r is directly related to the control gains, the closed-loop fixed point $(e, s_1, \tilde{q}, s_2) =$

$(0, 0, 0, 0)$ is (locally) exponentially stable. As pointed out above r can actually be varied by varying λ_6 or λ_1, making the scheme semi-global. An intuitive decomposition of the closed-loop system in (7.105) is as follows, noting that $M(q)\ddot{e} = M(q)\dot{s}_1 - \lambda_2 M(q)e$:

$$\begin{cases} \bar{M}(q)\dot{s} + \bar{C}(q,\dot{q})s = u_1,\ \dot{\tilde{q}} = -\lambda_2\tilde{q} + s_1,\ \dot{e} = -\lambda_3 e + s_2 \\ y_1 = s,\quad u_2 = y_1, \qquad\qquad y_2 = -T(q,\dot{q},s) = -u_1 \end{cases} \tag{7.107}$$

where

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \tag{7.108}$$

$$T(q,\dot{q},s) = -\begin{bmatrix} \lambda_1 s_2 + \lambda_2 C(q,\dot{q})\dot{e} - C(q,\dot{q}_d)s_2 + \lambda_2 M(q)\dot{e} \\ -\lambda_1 s_1 + C(q, s_2 - \dot{q})\dot{e} \end{bmatrix} \tag{7.109}$$

$$\bar{M}(q) = \mathrm{diag}[M(q), M(q)] \tag{7.110}$$

$$\bar{C}(q,\dot{q}) = \mathrm{diag}[C(q,\dot{q}), C(q,\dot{q})] \tag{7.111}$$

The first subsystem is clearly passive with respect to the supply rate $u_1^T y_1$. The second subsystem is a memoryless operator $u_2 \mapsto -T(q,\dot{q},u_2)$. If it can be shown that locally $-u_2^T T(q,\dot{q},u_2) \geq -\delta u_2^T u_2$, then the system with input $u = u_1 + y_2$ and output $y = y_1$ is output strictly passive. Indeed

$$\begin{aligned} \langle u, y\rangle_t &= \langle u_1 + y_2, y\rangle_t = \langle u_1, y_1\rangle_t + \langle y_2, u_2\rangle_t \\ &\geq -\tfrac{1}{2}s(0)^T\bar{M}(q(0))s(0) + \delta\int_0^t u_2^T(s)u_2(s)ds \end{aligned} \tag{7.112}$$

for some $\delta > 0$. In other words the function in (7.106) satisfies the dissipation inequality along the closed-loop trajectories: $\frac{dV}{dx}^T[f(x) + g(x)u] \leq u^T h(x) - \delta h^T(x)h(x)$ for all u and x locally only, where $x^T = (e^T, s_1^T, \tilde{q}^T, s_2^T)$ and $y = h(x)$. Then under suitable zero-state detectability properties, any storage function which is positive definite with respect to the closed-loop fixed point is a strict (local) Lyapunov function. Notice that the total closed-loop system is zero-state detectable since $y_1 = s \equiv 0$ and $u \equiv 0$ implies that $y_2 \equiv 0$, hence $u_1 \equiv 0$ and $e \to 0$ and $\tilde{q} \to 0$ as $t \to +\infty$.

7.4.3 The Slotine and Li + Observer Controller

Let us consider the following fixed parameter scheme:

$$
\textbf{Controller} \begin{cases} \tau = M(q)\ddot{q}_r + C(q,\dot{q}_0)\dot{q}_r + g(q) - \lambda_1(\dot{q}_0 - \dot{q}_r) - \lambda_2 e \\ \dot{q}_r(t) = \dot{q}_d(t) - \lambda(\hat{q}(t) - q_d(t)) \\ \dot{q}_0(t) = \dot{\hat{q}}(t) - \lambda(q(t) - \hat{q}(t)) \end{cases}
$$

$$
\textbf{Observer} \begin{cases} \dot{\hat{q}}(t) = z(t) + \lambda_3(q(t) - \hat{q}(t)) \\ \dot{z}(t) = \ddot{q}_r(t) + \lambda_4(q(t) - \hat{q}(t)) + \lambda_2 M^{-1}(q(t))[q_d(t) - \hat{q}(t)] \end{cases} \tag{7.113}
$$

Introducing (7.113) into (6.98) one obtains the closed-loop error equation

$$
\begin{cases} M(q(t))\dot{s}_1(t) + C(q(t),\dot{q}(t))s_1(t) + \lambda_1 s_1(t) + \lambda_2 e(t) = \\ = \lambda_1 s_2(t) - C(q(t), s_2(t))\dot{q}_r(t) \\ \dot{e}(t) = -\lambda(e(t) - \tilde{q}(t)) + s_1(t), \quad \dot{\tilde{q}}(t) = -\lambda\tilde{q}(t) + s_2(t) \\ M(q(t))\dot{s}_2(t) + C(q(t),\dot{q}(t))s_2(t) + (\lambda_6 M(q(t)) - \lambda_1 I_n)s_2(t) + \lambda_2\tilde{q}(t) = \\ \qquad = -\lambda_2 s_1(t) + C(q(t), s_1(t))[s_2(t) - \dot{q}(t)] \end{cases} \tag{7.114}
$$

with $\lambda_3 = \lambda_6 + \lambda$, $\lambda_4 = \lambda_6\lambda$. Again a natural decomposition of the closed-loop scheme is similarly done as in the previous case, *i.e.*

$$
\begin{cases} \bar{M}(q(t))\dot{s}(t) + \bar{C}(q(t),\dot{q(t)})s = u_1(t),\ \dot{e}(t) = -\lambda(e(t) - \tilde{q}(t)) + s_1(t), \dot{\tilde{q}}(t) \\ \qquad\qquad = -\lambda\tilde{q}(t) + s_2(t) \\ y_1(t) = s(t), u_2(t) = y_1(t), \qquad y_2(t) = -T(q(t),\dot{q}(t),s(t)) = -u_1(t) \end{cases} \tag{7.115}
$$

where this time

$$
T(q,\dot{q},s) = \begin{bmatrix} \lambda_1 s_1 - [\lambda_1 + C(q, s_1 - \dot{q})]s_2 \\ \lambda_1 - C(q, s_2 - \dot{q})s_1 + [\lambda_6 M(q) - \lambda_1 I_n]s_2 \end{bmatrix} \tag{7.116}
$$

It can be shown that locally $T(q,\dot{q},s) > 0$ so that $\langle u_2, y_2\rangle_t \geq \delta\int_0^t u_2^T(s)u_2(s)ds$ for some $\delta > 0$. The same conclusions as above follow about semi-global asymptotic Lyapunov stability of the closed-loop fixed point.

7.5 Flexible Joint–Rigid Link: State Feedback

7.5.1 Passivity-based Controller: The Lozano and Brogliato Scheme

In Section 6.4 we saw how the dissipativity properties derived for the rigid joint-rigid link manipulator case extend to the flexible joint-rigid link case, and we presented what we called *passivity-based* schemes. Considering the Lyapunov function in (7.78) let us try the following [70, 72, 316, 318]:

$$\begin{aligned} V(\tilde{q}_1, \tilde{q}_2, s_1, s_2) = &\tfrac{1}{2} s_1^T M(q_1) s_1 + \tfrac{1}{2} s_2^T J s_2 + \lambda\lambda_1 \tilde{q}_1^T \tilde{q}_1 + \lambda\lambda_1 \tilde{q}_2 \tilde{q}_2 \\ &+ \tfrac{1}{2} (\tilde{q}_1 - \tilde{q}_2)^T K (\tilde{q}_1 - \tilde{q}_2) \end{aligned} \tag{7.117}$$

The various signals have the same definition as in the rigid case. One sees that similarly to (7.78) this positive definite function mimics the total energy function of the open-loop unforced system. In order to make it a Lyapunov function for the closed-loop system, one can classically compute its derivative along the trajectories of (6.105) and try to find out a u that makes its derivative negative definite. Since we already have analyzed the rigid joint-rigid link case, we can intuitively guess that one goal is to get a closed-loop system of the form

$$\begin{cases} M(q_1(t))\dot{s}_1(t) + C(q_1(t), \dot{q}_1(t)) s_1(t) + \lambda_1 s_1(t) = f_1(s_1(t), s_2(t), \tilde{q}_1(t), \tilde{q}_2(t)) \\ \\ J\dot{s}_2(t) + \lambda_1 s_2(t) = f_2(s_1(t), s_2(t), \tilde{q}_1(t), \tilde{q}_2(t)) \end{cases} \tag{7.118}$$

For the moment we do not fix the functions $f_1(\cdot)$ and $f_2(\cdot)$. Since the Lyapunov function candidate preserves the form of the system's total energy, it is also to be strongly expected that the potential energy terms appear in the closed-loop dynamics. Moreover we desire that the closed-loop system consists of two passive blocks in negative feedback. Obviously $V(\cdot)$ in (7.117) contains the ingredients for Lemmas 7.22 and 7.23 to apply. The first block may be chosen with state vector $x_1 = \begin{pmatrix} \tilde{q}_1 \\ s_1 \\ \tilde{q}_2 \\ s_2 \end{pmatrix}$. We know it is passive with respect to the supply rate $u_1^T y_1$ with input $u_1 = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ and output $y_2 = \begin{pmatrix} K(\tilde{q}_1 - \tilde{q}_2) \\ -K(\tilde{q}_1 - \tilde{q}_2) \end{pmatrix}$. One storage function for this subsystem is

$$V_1(x_1, t) = \frac{1}{2} s_1^T M(q_1) s_1 + \frac{1}{2} s_2^T J s_2 + \lambda\lambda_1 \tilde{q}_1^T \tilde{q}_1 + \lambda\lambda_1 \tilde{q}_2 \tilde{q}_2 \tag{7.119}$$

However notice that we have not fixed the input and output of this subsystem, since we leave for the moment $f_1(\cdot)$ and $f_2(\cdot)$ free. Now the second subsystem must have a storage function equal to:

$$V_2(x_2,t) = \frac{1}{2}\left(\tilde{q}_1 - \tilde{q}_2\right)^T K \left(\tilde{q}_1 - \tilde{q}_2\right) \tag{7.120}$$

and we know it is passive with respect to the supply rate $u_2^T y_2$, with an input $u_2 = y_1$ and an output $y_2 = -u_1$, and from (7.120) with a state vector $x_1 = K\left(\tilde{q}_1 - \tilde{q}_2\right)$. Its dynamics is consequently given by

$$\dot{x}_2 = -\lambda x_2 + K(s_2 - s_1). \tag{7.121}$$

In order for Lemmas 7.22 and 7.23 to apply we also require the CTCE to be satisfied, *i.e.* $\frac{\partial V_1}{\partial x_1}^T G_1 h_2 = -\frac{\partial V_2}{\partial x_2}^T G_2 h_1$, where we get from (7.118)

$$s_1^T f_1 + s_2^T f_2 = -\left(\tilde{q}_2 - \tilde{q}_1\right)^T K(s_2 - s_1) \tag{7.122}$$

from which one deduces that $f_2(s_1, s_2, \tilde{q}_1, \tilde{q}_2) = K(\tilde{q}_1 - \tilde{q}_2)$ and $f_1(s_1, s_2, \tilde{q}_1, \tilde{q}_2) = K(\tilde{q}_2 - \tilde{q}_1)$. Thus since we have fixed the input and output of the second subsystem so as to make it a passive block, we can deduce from Lemma 7.23 that the closed-loop system that consists of the feedback interconnection of the dynamics in (7.118) and (7.121) can be analyzed through the passivity theorem.

Notice however that we have not yet checked whether a state feedback exists that assures this closed-loop form. This is what we develop now. Let us consider the following controller:

$$\begin{cases} u = J\ddot{q}_{2r} + K(q_{2d} - q_{1d}) - \lambda_1 s_2 \\ \\ q_{2d} = K^{-1} u_r + q_{1d} \end{cases} \tag{7.123}$$

where $\dot{q}_{2r} = \dot{q}_{2d} - \lambda \tilde{q}_2$ and u_r is given by the rigid joint-rigid link controller in (7.68), *i.e.*

$$u_r = M(q_1)\ddot{q}_r + C(q_1, \dot{q}_1)\dot{q}_r + g(q_1) - \lambda_1 s_1 \tag{7.124}$$

It is noteworthy that the controller is thus formed of two controllers similar to the one in (7.68): one for the first “rigid link” subsystem and the other for the motorschaft dynamics. The particular form of the interconnection between them makes it possible to pass from the first dynamics to the second one easily. It should be noted that the form in (7.123) and (7.124) depends on the state $(\tilde{q}_1, s_1, \tilde{q}_2, s_2)$ only, and not on any acceleration nor jerk terms.

To recapitulate, the closed-loop error dynamics is given by

$$\boxed{\begin{array}{l} M(q_1(t))\dot{s}_1(t) + C(q_1(t), \dot{q}_1(t))s_1(t) + \lambda_1 s_1(t) = K(\tilde{q}_2(t) - \tilde{q}_1(t)) \\ J\dot{s}_2(t) + \lambda_1 s_2(t) = K(\tilde{q}_1(t) - \tilde{q}_2(t)) \\ \dot{\tilde{q}}_1(t) = -\lambda\tilde{q}_1(t) + s_1(t) \\ \dot{\tilde{q}}_2(t) = -\lambda\tilde{q}_2(t) + s_2(t) \end{array} \quad (7.125)}$$

It is possible to replace the potential energy terms in (7.117) by

$$\left(\int_0^t [s_1 - s_2]d\tau\right)^T K \left(\int_0^t [s_1 - s_2]d\tau\right) \tag{7.126}$$

This does not modify significantly the structure of the scheme, apart from the fact that this introduces a dynamic state feedback term in the control loop. Actually as shown in [72] the static state feedback scheme has the advantage over the dynamic one of not constraining the initial conditions on the open-loop state vector and on $q_{1d}(0)$, $\dot{q}_{1d}(0)$ and $\ddot{q}_{1d}(0)$. The stability of the scheme with the integral terms as in (7.126) may be shown using the function

$$V(s_1, s_2, z) = \frac{1}{2}s_1^T M(q_1)s_1 + \frac{1}{2}s_2^T J s_2 + \frac{1}{2}z^T K z \tag{7.127}$$

with

$$\begin{cases} q_{2d} = q_{1d} - \lambda x + K^{-1}(-s_1 + M(q_1)\ddot{q}_{1r} + C(q_1, \dot{q}_1)\dot{q}_{1r} + g(q_1)) \\ \dot{q}_{1r}(t) = \dot{q}_{1d}(t) - \lambda\tilde{q}_1(t) \\ \dot{x}(t) = \tilde{q}_1(t) - \tilde{q}_2(t) \\ z(t) = \lambda x(t) + (\tilde{q}_1(t) - \tilde{q}_{2(t)}) \quad (\dot{z}(t) = s_1(t) - s_2(t)) \\ u = -s_2 - J[-\ddot{q}_{2d} + \lambda\dot{\tilde{q}}_2] - K[q_{1d} - q_{2d} - \lambda x] \end{cases}$$

Then one gets along closed-loop trajectories $\dot{V}(s_1, s_2, z) = -s_1^T s_1 - s_2^T s_2$. See [72] for more details.

Remark 7.33. A strong property of the controller in (7.123) and (7.124) in closed-loop with the dynamics in (6.105), with the Lyapunov function in (7.117), is that they converge towards the closed-loop system in (7.69) and (7.70) when $K \to +\infty$ (all the entries diverge). Indeed one notices that

$K(q_{2d} - q_{1d}) = u_r$ for all K and that $q_{2d} \to q_{1d}$ as $K \to \infty$. Noting that all the closed-loop signals remain uniformly bounded for any K and introducing these results into u in (7.123) one sees that $u = J\ddot{q}_r + u_r - \lambda_1 s_1$ which is exactly the controller in (7.68) applied to the system in (6.105), letting $q_1 \equiv q_2$ and adding both subsystems. We therefore have constructed a real family of controllers that share some fundamental features of the plant dynamics.

A Recursive Algorithm Construction

A close look at the above developments, shows that the control scheme in (7.123) and (7.124) is based on a two-step procedure:

- The control of the first equation in (6.105) using q_{2d} as a fictitious input. Since q_{2d} is not the input, this results in an error term $K(\tilde{q}_2 - \tilde{q}_1)$.
- A specific transformation of the second equation in (6.105) that makes the control input u explicitly appear. The controller is then designed in such a way that the closed-loop dynamics possesses a Lyapunov function as in (7.117).

> This is typically an instance of what has been called afterwards the *backstepping* design method and *passivity-based controllers*. It is the first time these two techniques have been applied simultaneously for tracking control of Lagrangian systems.

Stability Proof

The stability proof for the fixed parameters Lozano and Brogliato scheme, very much mimics that of the Slotine and Li scheme. One may for instance choose as a quadratic function

$$V(\tilde{q}_1, \tilde{q}_2, s_1, s_2) = \frac{1}{2} s_1^T M(q_1) s_1 + \frac{1}{2} s_2^T J s_2 + \frac{1}{2} \left(\tilde{q}_1 - \tilde{q}_2\right)^T K \left(\tilde{q}_1 - \tilde{q}_2\right) \quad (7.128)$$

instead of the Lyapunov function candidate in (7.117). The function in (7.128) is the counterpart for flexible joint systems, of the function in (7.80). Let us compute the derivative of (7.128) along the trajectories of the error system (7.125):

$$
\begin{aligned}
\dot V(\tilde q_1(t), \tilde q_2(t), s_1(t), s_2(t)) &= s_1^T(t)M(q_1(t))\dot s_1(t) + s_2^T(t)J\dot s_2(t)+ \\
&+\tfrac{1}{2}s_1^T(t)\dot M(q_1(t))s_1(t) \\
&+ (\tilde q_1(t) - \tilde q_2(t))^T K \left(\dot{\tilde q}_1(t) - \dot{\tilde q}_2(t)\right) \\
&= s_1^T(t)[\tfrac{1}{2}\dot M(q_1(t)) - C(q_1(t), \dot q_1(t))s_1(t) - \lambda_1 s_1(t) \\
&+K(\tilde q_2(t) - \tilde q_1(t))]+ \\
&s_2^T[-\lambda_1 s_2(t) + K(\tilde q_1(t) - \tilde q_2(t))] + (\tilde q_1(t) - \tilde q_2(t))^T \\
&K\left(-\lambda_1 \tilde q_1(t) + s_1(t) + \lambda_1 \tilde q_2(t) - s_2(t)\right) \\
&= -\lambda_1 s_1^T(t)s_1(t) - \lambda_1 s_2^T(t)s_2(t)- \\
&-\lambda_1(\tilde q_1(t) - \tilde q_2(t))^T K(\tilde q_1(t) - \tilde q_2(t)) \le 0
\end{aligned}
\tag{7.129}
$$

It follows from (7.129) that all closed-loop signals are bounded on $[0, +\infty)$, and that $s_1 \in \mathcal{L}_2$, $s_2 \in \mathcal{L}_2$. Using similar arguments as for the first stability proof of the Slotine and Li controller in Section 7.3.4, one concludes that $\tilde q_1(t)$, $\tilde q_2(t)$, $\dot{\tilde q}_1(t)$ and $\dot{\tilde q}_2(t)$ all tend towards zero as $t \to +\infty$. One may again also conclude on the exponential convergence of these functions towards zero noticing that $\dot V(\tilde q_1, \tilde q_2, s_1, s_2) \le \beta V(\tilde q_1, \tilde q_2, s_1, s_2)$ for some $\beta > 0$.

It is also possible to lead a stability analysis using the Lyapunov function candidate in (7.117). We reiterate that the quadratic function in (7.128) cannot be named a Lyapunov function candidate for the closed-loop system (7.125), since it is not a radially unbounded nor positive definite function of the state $(\tilde q_1, \tilde q_2, \dot{\tilde q}_1, \dot{\tilde q}_2)$.

7.5.2 Other Globally Tracking Feedback Controllers

A Recursive Method for Control Design

As pointed out one may also view the passivity-based controller in (7.123) as the result of a procedure that consists of stabilizing first the rigid part of the dynamics, using the signal $q_{2d}(t)$ as a fictitious intermediate input, and then looking at the rest of the dynamics. However instead of looking at the rest as a whole and considering it as a passive second order subsystem, one may treat it step by step: this is the core of a popular method known under the name of *backstepping.* Let us develop it now for the flexible joint-rigid link manipulators.

- **Step 1:** Any type of globally stabilizing controller can be used. Let us still use u_r in (7.124), *i.e.* let us set

$$q_{2d} = K^{-1}u_r + q_1 \tag{7.130}$$

so that we get

$$M(q_1(t))\dot{s}_1(t) + C(q_1(t), \dot{q}_1(t))s_1(t) + \lambda_1 s_1(t) = K\tilde{q}_2(t) \tag{7.131}$$

The system in (7.131) with $\tilde{q}_2 \equiv 0$ thus defines a globally uniformly asymptotically stable system with Lyapunov function $V_1(\tilde{q}_1, s_1) = \frac{1}{2}s_1^T M(q_1)s_1 + \lambda\lambda_1\tilde{q}_1^T\tilde{q}_1$. The interconnection term is therefore quite simple (as long as the stiffness matrix is known!). Let us take its derivative to obtain

$$\dot{\tilde{q}}_2(t) = \dot{q}_2(t) - \dot{q}_{2d}(t) = \dot{q}_2(t) + f_1(q_1(t), \dot{q}_1(t)q_2(t)) \tag{7.132}$$

where $f_1(\cdot)$ can be computed using the dynamics (actually $\dot{q}_{2d}$ is a function of the acceleration $\ddot{q}_1$ which can be expressed in terms of q_1, $\dot{q}_1$ and q_2 by simply inverting the first dynamical equation in (6.105)).

- **Step 2:** Now if $\dot{q}_2$ was the input we would set $\dot{q}_2 = -f_1(q_1, \dot{q}_1 q_2) - \lambda_2\tilde{q}_2 - Ks_1$ so that the function $V_2 = V_1 + \frac{1}{2}\tilde{q}_2^T\tilde{q}_2$ has a negative definite derivative along the partial closed-loop system in (7.131) and

$$\dot{\tilde{q}}_2(t) = -\lambda_2\tilde{q}_2(t) - Ks_1(t) \tag{7.133}$$

However $\dot{q}_2$ is not an input, so that we shall rather define a new error signal as $e_2 = \dot{q}_2 - e_{2d}$, with $e_{2d} = -f_1(q_1, \dot{q}_1 q_2) - \lambda_2\tilde{q}_2 - Ks_1$. One obtains

$$\begin{aligned}\dot{e}_2(t) &= \ddot{q}_2(t) - \dot{e}_{2d}(t) = \ddot{q}_2(t) + f_2(q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t)) \\ &= J^{-1}[K(q_1(t) - q_2(t)) + u(t)] + f_2(q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t))\end{aligned} \tag{7.134}$$

- **Step 3:** Since the real control input appears in (7.134) this is the last step. Let us choose

$$u = K(q_2 - q_1) + J[-f_2(q_1, \dot{q}_1, q_2, \dot{q}_2) - e_2 - \tilde{q}_2] \tag{7.135}$$

so that we get:

$$\dot{e}_2(t) = -\lambda_3 e_2(t) - \tilde{q}_2(t) \tag{7.136}$$

where the term $-\tilde{q}_2$ has been chosen to satisfy the CTCE (see Lemma 7.23) when the function V_2 is augmented to

$$V_3(\tilde{q}_1, s_1, \tilde{q}_2, e_2) = V_2 + \frac{1}{2}e_2^T e_2 \tag{7.137}$$

Then along the closed-loop trajectories of the system in (7.131) (7.118) (7.136) one gets

$$\begin{aligned}\dot{V}_3(\tilde{q}_1(t), s_1(t), \tilde{q}_2(t), e_2(t)) = -\lambda_1\dot{\tilde{q}}_1^T(t)\dot{\tilde{q}}_1(t) - \lambda^2\lambda_1\tilde{q}_1^T(t)\tilde{q}_1(t) - \\ -\tilde{q}_2^T(t)\tilde{q}_{2(t)} - e_2^T(t)e_2(t)\end{aligned} \tag{7.138}$$

which shows that this closed-loop system is globally uniformly exponentially stable.

It is noteworthy that e_2 is not the time derivative of q_2. Therefore the backstepping method hinges upon a state variable transformation which actually depends on the system dynamics in the preceding steps.

Remark 7.34.
- The control law in (7.135) can be computed from the definition of q_{2d} in (7.130) and $\dot{q}_{2d}$ as well as $\ddot{q}_{2d}$ are to be calculated using the dynamics to express the acceleration $\ddot{q}_1$ and the jerk $q_1^{(3)}$ as functions of positions and velocities only (take the first dynamical equation in (6.105) and invert it to get the acceleration. Differentiate it again and introduce the expression obtained for the acceleration to express the jerk). Clearly u is a complicated nonlinear function of the state, but it is a static state feedback. This apparent complexity is shared by all the nonlinear controllers described in Section 7.5. Notice however that it is only a matter of additions and multiplications, nothing else!
- We noticed in Remark 7.33 that the passivity-based controller tends towards the Slotine and Li input when the joint stiffness tends to infinity. This is no longer the case with the backstepping controller derived here. Even more, after some manipulations, it can be shown [78] that the controller in (7.135) can be equivalently rewritten as

$$\begin{cases} u = J[\ddot{q}_{2d} - (\lambda_2 + \lambda_3)\dot{\tilde{q}}_2 - (1 + \lambda_2\lambda_3)\tilde{q}_2 - K(\dot{s}_1 + s_1)] \\ \\ q_{2d} = K^{-1}u_r + q_1 \end{cases} \tag{7.139}$$

where it immediately appears that the term $K(\dot{s}_1 + s_1)$ is not bounded as K grows without bound. Here comes into play the "flexibility" of the backstepping method: let us modify the function V_2 above to $V_2 = V_1 + \frac{1}{2}\tilde{q}_2^T K \tilde{q}_2$. Then in step 2 it is sufficient to choose $\dot{q}_2 = -f_1(q_1, \dot{q}_1 q_2) - \lambda_2\tilde{q}_2 - s_1$, so that the final input becomes

$$\begin{cases} u = J[\ddot{q}_{2d} - (\lambda_2 + \lambda_3)\dot{\tilde{q}}_2 - (1 + \lambda_2\lambda_3)\tilde{q}_2 - (\dot{s}_1 + s_1)] \\ \\ q_{2d} = K^{-1}u_r + q_1 \end{cases} \tag{7.140}$$

Such a modification may appear at first sight quite innocent, easy to do, and very slight: it is not! The experimental results presented in Chapter 9 demonstrate it. Actually the term $K(\dot{s}_1 + s_1)$ introduces a high-gain in the loop that may have disastrous effects. This may be seen through simulations, see [78]. It is noteworthy that even with quite flexible systems (some of the reported experiments were led with a system whose stiffness is $k = 3.5$ Nm/rad) this term makes the control law in (7.135) behave less satisfactorily than the one in (7.140). More details can be found in Chapter 9.
- This recursive design method applies to all systems that possess a triangular structure [325]. See [67] for a survey of backstepping methods for flexible joint manipulators.

- Compare (7.139) and (7.140) to (7.123). Although these controllers have the same degree of complexity and can be considered as similar, they have significant discrepencies as explained above. For instance in (7.123) one has $K(q_{2d} - q_{1d}) = u_r$ while in (7.139) and (7.140), $K(q_{2d} - q_{1d}) = u_r + \tilde{q}_1$.

A Passivity Theorem Interpretation

As we pointed out the procedure relies on the CTCE at each step. Since the first subsystem in (7.131) is output strictly passive with respect to the supply rate $u_1^T y_1$ with $u_1 = K\tilde{q}_2$ and $y_1 = s_1$, we are tempted to apply the result of Lemmas 7.22 and 7.23 to interpret the closed-loop scheme in (7.131), (7.118) and (7.136) as an interconnection of passive blocks. From the developments concerning the rigid joint-rigid link case we know that the first subsystem can be seen as the interconnection of two passive blocks in (7.69) and (7.71). However, now the first subsystem is passive when the input is changed to $u_1 = K\tilde{q}_2 - \lambda_1 s_1$. We shall therefore define four subsystems as follows:

$$
\begin{array}{l}
(H1)\left\{\begin{array}{l}
(H11): M(q_1(t))\dot{s}_1(t) + C(q_1(t),\dot{q}_1(t))s_1(t) = K\tilde{q}_2(t) - \lambda_1 s_1(t) \\[2ex]
\qquad\qquad u_{11}(t) = K\tilde{q}_2(t) - \lambda_1 s_1(t), y_{11}(t) = s_1(t), \text{state } s_1 \\[2ex]
(H12): \dot{\tilde{q}}_1(t) = -\lambda_1 \tilde{q}_1(t) + s_1(t) \\[2ex]
\qquad\qquad u_{12}(t) = s_1(t), y_{12}(t) = \lambda_1 s_1(t), \text{state } \tilde{q}_1
\end{array}\right. \\[8ex]
(H2)\left\{\begin{array}{l}
(H21): \dot{\tilde{q}}_2(t) = -\lambda_2 \tilde{q}_2(t) + e_2(t) - K s_1(t) \\[2ex]
\qquad\qquad u_{21}(t) = e_2(t) - K s_1(t), y_{21}(t) = \tilde{q}_2(t), \text{state } \tilde{q}_2 \\[2ex]
(H22): \dot{e}_2(t) = -\lambda_3 e_2(t) - \tilde{q}_2(t) \\[2ex]
\qquad\qquad u_{22(t)} = \tilde{q}_2(t), y_{22}(t) = -e_2(t), \text{state } e_2
\end{array}\right.
\end{array}
\tag{7.141}
$$

Then the closed-loop system can be viewed as the negative feedback interconnection of the block $(H1)$ with $u_1 = u_{11} + y_{12} = K\tilde{q}_2$, $y_1 = y_{11}$, with the block $(H2)$ with input $-K^{-1}u_2 = s_1 = y_1$ and output $-Ky_2 = -K\tilde{q}_2 = -u_1$. This is depicted in Figure 7.7.

Remark 7.35. The backstepping procedure also yields a closed-loop system that can be analyzed through the passivity Theorem. However the major difference with the passivity-based method is that the block $(H2)$ is not related to any physical relevant energetical term. In a sense this is similar to what one would get by linearizing the rigid joint–rigid link dynamics, applying a new linear feedback so as to impose some second order linear dynamics which may define an "artificial" passive system.

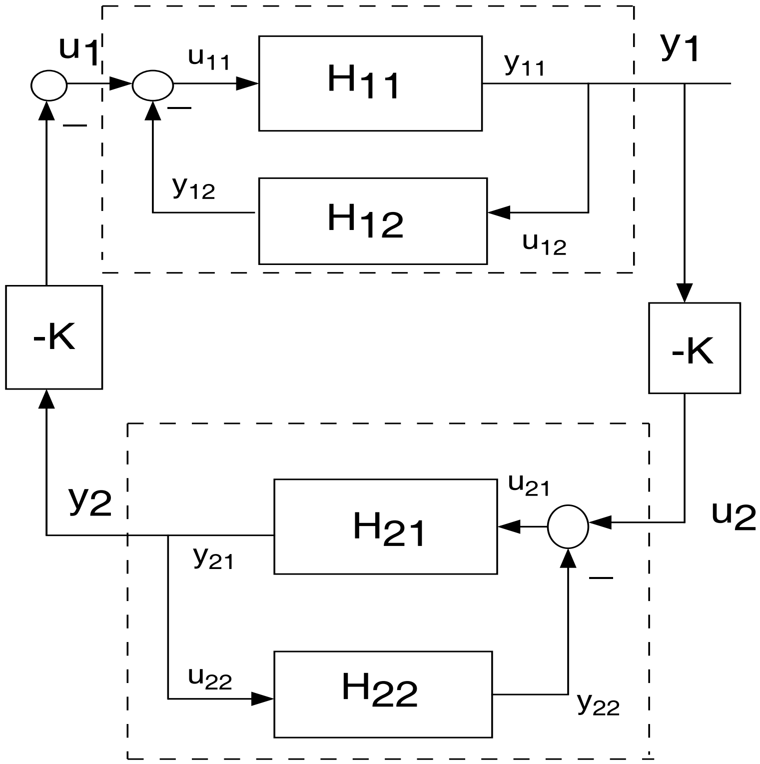

Fig. 7.7. Flexible joint-rigid link (equivalent interpretation)

7.6 Flexible Joint–Rigid Link: Output Feedback

7.6.1 PD Control

We have seen in Section 7.3.1 that a PD controller stabilizes globally and asymptotically rigid joint-rigid link manipulators. It is a combination of passivity and detectability properties that makes such a result hold: the former is a guide for the choice of a Lyapunov function, while the latter allows the Krasovskii-La Salle invariance principle to apply. More precisely, the output strict passivity property is crucial, because output strict passivity together with zero-state detectability of a system, imply its asymptotic stability in the sense of Lyapunov (see Corollary 5.16). Let us consider the dynamics in (6.105) and the following controller:

$$u = -\lambda_1 \dot{q}_2 - \lambda_2 (q_2 - q_d) \tag{7.142}$$

with q_d a constant signal, so that the closed-loop system is given by

$$\begin{cases} M(q_1(t))\ddot{q}_1(t) + C(q_1(t), \dot{q}_1(t))\dot{q}_1(t) + g(q_1(t)) = K(q_2(t) - q_1(t)) \\ \\ J\ddot{q}_2(t) + \lambda_1\dot{q}_2(t) + \lambda_2(q_2(t) - q_d) = K(q_1(t) - q_2(t)) \end{cases} \tag{7.143}$$

Let us proceed as for the rigid joint-rigid link case, i.e. let us first "guess" a Lyapunov function candidate from the available storage function, and then show how the application of the passivity Theorem applies equally well.

The Closed-loop Available Storage

Similarly as for the rigid joint-rigid link case, one may guess that a PD controller alone will not enable one to stabilize *any* fixed point. The closed-loop fixed point is given by

$$\begin{cases} g(q_1) = K(q_2 - q_1) \\ \\ \lambda_2(q_2 - q_d) = K(q_1 - q_2) \end{cases} \tag{7.144}$$

and we may assume for simplicity that this set of nonlinear equations (which are not in general algebraic but transcendental) possesses a unique root $(q_1, q_2) = (q_{10}, q_{20})$. We aim at showing the stability of this point. To compute the available storage of the closed-loop system in (7.143) we consider a fictitious input u in the second dynamical equation, while the output is taken as $\dot{\tilde{q}}_2$. Then we obtain the following:

$$\begin{aligned} V_a(\tilde{q}_1, \dot{q}_1, \tilde{q}_2, \dot{q}_2) &= \sup_{u:(0,\tilde{q}_1(0),\dot{q}_1(0),\tilde{q}_2(0),\dot{q}_2(0))\to} -\int_0^t \dot{q}_2^T(s)u(s)ds \\ &= \sup_{u:(0,\tilde{q}_1(0),\dot{q}_1(0),\tilde{q}_2(0),\dot{q}_2(0))\to} -\int_0^t u^T[J\ddot{q}_2 + K(q_2 - q_1) + \lambda_1\dot{q}_2 + \lambda_2\tilde{q}_2]ds \\ &= \tfrac{1}{2}\dot{q}_1(0)^T M(q_1(0))\dot{q}_1(0) + U_g(q_1(0)) + \tfrac{1}{2}\dot{q}_2(0)^T J\dot{q}_2(0) + \\ &\quad + \tfrac{1}{2}(q_2(0) - q_1(0))^T K(q_2 - q_1) + \tfrac{1}{2}\lambda_2\tilde{q}_2^T(0)\tilde{q}_2(0) \end{aligned} \tag{7.145}$$

where $\tilde{q}_i = q_i - q_{i0}$, $i = 1, 2$. Now the supply rate satisfies $w(0, \dot{q}_2) \leq 0$ for all $\dot{q}_2$, and obviously $(\tilde{q}_1, \dot{q}_1, \tilde{q}_2, \dot{q}_2) = (0, 0, 0, 0)$ is a strict (global) minimum of V_a in (7.145) provided $U_g(q_1)$ has a strict minimum at q_{10}. Notice that $\tilde{q}_2 = 0 \Rightarrow (q_1 - q_2) = 0 \Rightarrow g(q_1) = 0$ so that $q_1 = q_{10}$ is a critical point for $U_g(q_1)$ (that we might assume to be strictly globally convex, but this is only sufficient). Hence from Lemmae 5.13 and 4.8 one deduces that the closed-loop system in (7.143) is Lyapunov stable. To show asymptotic stability, one has to resort to the Krasovskii-La Salle invariance principle.

Closed-loop Feedback Interconnections

Motivated by the rigid joint-rigid link case let us look for an equivalent feedback interconnection such that the overall system is strictly output passive and zero-state detectable. To this end let us consider the following two blocks:

$$\begin{cases} u_1 = K(q_1 - q_2),\ y_1 = \dot{q}_2 \\ \\ u_2 = y_1, \qquad\qquad y_2 = -u_1 \end{cases} \tag{7.146}$$

where the first block has the dynamics $J\ddot{q}_2(t) + \lambda_1\dot{q}_2(t) + \lambda_2(q_2(t) - q_d) = K(q_1(t) - q_2(t))$, while the second one has the dynamics $M(q_1(t))\ddot{q}_1(t) + C(q_1(t), \dot{q}_1(t))\dot{q}_1(t) + g(q_1(t)) = K(q_2(t) - q_1(t))$. It is easy to calculate the following:

$$\begin{array}{l} \langle u_1, y_1\rangle_t \geq -\frac{1}{2}\dot{q}_2(0)^T J\dot{q}_2(0) - \lambda_2(q_2(0) - q_d)^T(q_2(0) - q_d) \\ \\ \qquad\qquad +\lambda_1 \int_0^t \dot{q}_2^T(s)\dot{q}_2(s)ds \\ \\ \langle u_2, y_2\rangle_t \geq -\frac{1}{2}[q_1(0) - q_2(0)]^T K[q_1(0) - q_2(0)] \\ \\ \qquad\qquad -\frac{1}{2}\dot{q}_1(0)^T M(q_1(0))\dot{q}_1(0) - U_g(q_1(0)) \end{array} \tag{7.147}$$

from which one deduces that the first block is output strictly passive (actually if we added Rayleigh dissipation in the first dynamics, the second block would not be output strictly passive with the proposed decomposition). Each block possesses its own storage functions which are Lyapunov functions for them. The concatenation of these two Lyapunov functions forms the available storage in (7.145). Let us now consider the overall system with input $u = u_1 + y_2$ and output $y = y_1$. Setting $u \equiv y \equiv 0$ implies $\tilde{q}_2 \equiv 0$ and $\dot{q}_1 \to 0$, $\tilde{q}_1 \to 0$ asymptotically. The system is zero-state detectable. Hence by Lemmae 5.13 and 4.8 its fixed point is globally asymptotically Lyapunov stable.

Remark 7.36 (Collocation). The *collocation* of the sensors and the actuators is an important feature for closed-loop stability. It is clear here that if the PD control is changed to

$$u(t) = -\lambda_1\dot{\tilde{q}}_1(t) - \lambda_2\tilde{q}_1(t) \tag{7.148}$$

then the above analysis no longer holds. It can even be shown that there are some gains for which the closed-loop system is unstable [473]. One choice for the location of the sensors may be guided by the passivity property between their output and the actuators torque (in case the actuator dynamics is neglected).

7.6.2 Motor Position Feedback

A position feedback controller similar to the one in Section 7.4.1 can be derived for flexible joint-rigid link manipulators [43]. It may be seen as a PD controller with the velocity feedback replaced by an observer feedback. It is given by

$$\begin{cases} u(t) = g(q_d) - \lambda_1 \tilde{q}_2(t) - \frac{1}{\lambda_2}(\tilde{q}_2(t) - z(t)) \\ \\ \dot{z}(t) = \lambda_3(\tilde{q}_2(t) - z(t)) \end{cases} \tag{7.149}$$

with $\tilde{q}_2 = q_2 - q_d + K^{-1}g(q_d)$, and q_d is the desired position for q_1. The analysis is quite close to the one done for the rigid joint-rigid link case. Due to the autonomy of the closed-loop (q_d is constant) Corollary 7.25 is likely to apply. The stability proof bases on the following global Lyapunov function:

$$\begin{aligned} V(\tilde{q}_1, \dot{q}_1\ \tilde{q}_2, \dot{q}_2) = \lambda_2 \big(&\tfrac{1}{2}\dot{q}_1^T M(q_1)\dot{q}_1 + \tfrac{1}{2}\dot{q}_2^T J \dot{q}_2 + \tfrac{1}{2}\tilde{q}_1^T K \tilde{q}_1 \\ &+ \tfrac{1}{2}\tilde{q}_2^T (K + \lambda_1 I_n)\tilde{q}_2 \big) - 2\lambda_2 \tilde{q}_1^T K \tilde{q}_2 + \tfrac{1}{2}(\tilde{q}_2 - z)^T(\tilde{q}_2 - z) \end{aligned} \tag{7.150}$$

Compare with $V(\cdot) = V_1(\cdot) + V_2(\cdot)$ in (7.99) and (7.100): the structure of $V(\cdot)$ in (7.150) is quite similar. It is a positive definite function provided $K + \frac{dg(q)}{dq}(q_d) > 0$ and $\lambda_1 I_n + K - K\left(K + \frac{dg(q)}{dq}(q_d)\right)^{-1} > 0$, for all q_d. This implies that K and λ_1 are sufficiently large. The decomposition into two subsystems as in (7.98) can be performed, choosing $x_2 = \tilde{q}_2 - z$ and $x_1^T = (\tilde{q}_1^T, \dot{q}_1^T, \tilde{q}_2^T, \dot{q}_2^T) = (x_{11}^T, x_{12}^T, x_{13}^T, x_{14}^T)$. The closed-loop scheme is given by

$$\begin{cases} \dot{x}_{11}(t) = x_{12}(t) \\ \\ \dot{x}_{12}(t) = -M(x_{11}(t) + q_d)[C(x_{11}(t) + q_d, x_{12}(t))x_{12}(t) + K(x_{11}(t) - x_{12}(t)) \\ \\ \quad +g(x_{11}(t) + q_d) - g(q_d)] \\ \\ \dot{x}_{13}(t) = x_{14}(t) \\ \\ \dot{x}_{14}(t) = J^{-1}[K(x_{11}(t) - x_{13}(t)) - g(q_d) - \lambda_1 x_{13}(t) - \frac{1}{\lambda_2}x_2(t)] \\ \\ \dot{x}_{2(t)} = -\lambda_3 x_{2(t)} + x_{14}(t) \end{cases} \tag{7.151}$$

Define $h_2(x_2) = \frac{1}{\lambda_2}x_2$ and $h_1(x_1) = x_{14}$. It follows that the CTCE is satisfied since $\frac{\partial V_1}{\partial x_1}^T G_1 h_2 = -x_{14}^T x_2 = -\frac{\partial V_2}{\partial x_2}^T G_2 h_1$. Indeed one may calculate that $G_1^T = (0, 0, 0, J^{-1}) \in I\!R^{n \times 4n}$ whereas $G_2 = I_n$. Hence once again Corollary 7.25 applies and the closed-loop system can be interpreted *via* the passivity theorem.

Remark 7.37. • Battilotti et al [44] have presented a result that allows one to recast the dynamic position feedback controllers presented in this subsection and in Section 7.4 into the same general framework. It is based on passifiability and detectability properties. The interpretation of the P + observer schemes in Subsections 7.4.1 and 7.6.2 *via* Corollary 7.25 is however original.

- It is also possible to derive a globally stable P + observer controller using only the measurement of q_1 [44]. Its structure is however more complex than the above one. This is easily understandable since in this case the actuators and sensors are non-collocated. Energy shaping is used in [254] to globally stabilize flexible joint–rigid link manipulators. PD control for flexible joint–rigid link manipulators with disturbances and actuatir dynamics is analysed in [315].

7.7 Including Actuator Dynamics

7.7.1 Armature-controlled DC Motors

We have seen in Section 6.6 that the available storage of the interconnection between the rigid joint-rigid link manipulator model and the armature-controlled DC motor is given by

$$V_a(q, \dot{q}, I) = \frac{1}{2} I^T L I + \frac{1}{2} \dot{q}^T M(q) \dot{q} + U_g(q) \tag{7.152}$$

Motivated by the method employed for the design of stable controllers for rigid joint-rigid link and flexible joint-rigid link manipulators, let us consider the following positive definite function:

$$V(\tilde{q}, s, \tilde{I}) = \frac{1}{2} \tilde{I}^T L \tilde{I} + \frac{1}{2} s^T M(q) s + +2\lambda\lambda_1 \tilde{q}^T \tilde{q} \tag{7.153}$$

where $s = \dot{\tilde{q}} + \lambda \tilde{q}$. Let us consider the dynamics in (6.126) which we recall here for convenience:

$$\begin{cases} M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) = \tau(t) = K_t I(t) \\ \\ RI(t) + L\frac{dI}{dt}(t) + K_t \dot{q}dt(t) = u(t) \end{cases} \tag{7.154}$$

Let us set

$$I_d = K_t^{-1}[M(q)\ddot{q}_r + C(q, \dot{q})\dot{q}_r + g(q) - \lambda_1 s] \tag{7.155}$$

where $s = \dot{q} - \dot{q}_r$, so that the manipulator dynamics in (6.126) becomes

$$M(q(t))\dot{s}(t) + C(q(t), \dot{q}(t))s(t) + \lambda_1 s(t) = K_t \tilde{I}(t) \tag{7.156}$$

where $\tilde{I} = I - I_d$. Then it is easy to see that the control input

$$u = RI - k_v \dot{q} + L^{-1} \dot{I}_d - L^{-1} K_t s - \tilde{I} \tag{7.157}$$

(which is a state feedback) leads to

$$\dot{\tilde{I}}(t) = -\tilde{I}(t) + L^{-1}K_t s(t) \tag{7.158}$$

Taking the derivative of $V(\tilde{q}, s, \tilde{I})$ in (7.153) along closed-loop trajectories in (7.156) and (7.158) one gets:

$$\dot{V}(\tilde{q}(t), s(t), \tilde{I}(t)) = -\tilde{I}^T(t)L\tilde{I}(t) - \lambda_1\dot{\tilde{q}}^T(t)\dot{\tilde{q}}(t) - \lambda^2\lambda_1\tilde{q}^T(t)\tilde{q}(t) \tag{7.159}$$

showing that the closed-loop fixed point $(\tilde{q}, s, \tilde{I}) = (0, 0, 0)$ is globally asymptotically uniformly stable in the sense of Lyapunov.

Remark 7.38 (Regulation of cascade systems). Consider the system in (7.154) with Rayleigh dissipation in the manipulator dynamics. Let us write the second subsystem in (7.154) as

$$\dot{I}(t) = -L^{-1}RI(t) - L^{-1}K_t\dot{q}(t) + L^{-1}u(t) \tag{7.160}$$

Let $L^{-1}u = L^{-1}K_v\dot{q} + \mathbf{u}$ so that we obtain the cascade system

$$\begin{cases} M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) + \frac{dR}{d\dot{q}}(t) = K_t y(t) \\ \\ \dot{I}(t) = -L^{-1}RI(t) + u(t) \\ \\ y(t) = I(t) \end{cases} \tag{7.161}$$

The terms corresponding to (5.52) can be easily identified by inspection. One sees that the conditions of Theorem 5.42 are satisfied (provided the potential energy $U(q)$ satisfies the requirements of Assumption 15), so that this (partially) closed-loop system is feedback equivalent to a strictly passive system. In other words there exists a feedback input $\mathbf{u} = \alpha(I, q, \dot{q}) + \mathbf{v}$ such that there exists a positive definite function $V(I, q, \dot{q})$ of the fixed point $(I, q, \dot{q}) = (0, 0, 0)$ and a positive definite function $S(I, q, \dot{q})$ such that

$$V(t) - V(0) = \int_0^t \mathbf{v}^T(s)y(s)ds - \int_0^t S(I(s), q(s), \dot{q}(s))ds \tag{7.162}$$

Thus the unforced system (*i.e.* take $\mathbf{v} = 0$) has a globally asymptotically stable fixed point (in the sense of Lyapunov).

A similar analysis for the field-controlled DC motor case can be led. The dissipativity properties of the driven and the driving subsystems allow the designer to construct a globally stabilizing feedback law.

Remark 7.39 (Nested passive structure). The computation of $\dot{V}(\cdot)$ relies on a CTCE as required in Lemma 7.23 and Corollary 7.25. Thus if we had started from the *a priori* knowledge of the function $V(\cdot)$ we could have deduced that the closed-loop system can be analyzed as the negative feedback interconnection of two passive blocks, one with input $u_1 = K_t\tilde{I}$ and output $y_1 = s$ and dynamics in (7.156), the second one with dynamics in (7.158) and $u_2 = y_1$, $y_2 = -u_1$. Recall from Section 7.3.4 that the first subsystem can be in turn decomposed as a negative feedback interconnection of two passive blocks given in (7.69) and (7.70): the overall system therefore possesses a structure of nested negative feedback interconnections of passive systems.

7.7.2 Field-controlled DC Motors

Let us recall the model of rigid joint-rigid link manipulators in cascade with a field-controlled DC motor:

$$\begin{cases} L_1 \frac{dI_1}{dt}(t) + R_1 I_1(t) = u_1(t) \\ L_2 \frac{dI_2}{dt}(t) + R_2 I_2(t) + K_t(I_1(t))\dot{q}(t) = u_2(t) \\ M(q(t))\ddot{q}(t) + C(q(t), \dot{q(t)})\dot{q}(t) + g(q(t)) + K_{vt}\dot{q}(t) = \tau = K_t(I_1(t))I_2(t) \end{cases} \tag{7.163}$$

The regulation problem around the constant fixed points $(q, \dot{q}, I_1, I_2) = (q_0, 0, I_{1d}, 0)$ or $(q_0, 0, 0, I_{2d})$ is solvable, where q_0 is as in assumption 15. Indeed the subsystem can be seen as a cascade system as in (5.52) that satisfies the requirements of Theorem 5.42. Hence it is feedback equivalent to a strictly passive system (in the sense of Theorem 5.36), whose unforced version is Lyapunov globally asymptotically stable. One remarks that the tracking control problem is quite similar to that of the flexible joint-rigid link manipulators with torque input. However this time the matrix that premultiplies I_2 is no longer constant invertible. Actually $K_t(I_1)$ may pass through singular values each time $I_{1i} = 0$ for some $i \in \{1, \cdots, n\}$. The extension of the regulation case is therefore not trivial. Nevertheless if the goal is to track a reference trajectory for $(q, \dot{q})$ only, then one may keep I_1 constant such that $K_t(I_1)$ remains full-rank, through a suitable u_{21}, so that the armature-controlled DC motor case is recovered.

Remark 7.40. All the preceeding developments apply to flexible joint-rigid link manipulators. Notice also that induction motors have the same complexity as field-controlled DC motors for control since the generated torque for each motor is given by $\tau = L_{sr}(I_2 I_3 - I_1 I_4)$; see Remark 6.50 for details.

7.8 Constrained Mechanical Systems

In real robotic tasks, the manipulators seldom evolve in a space free of obstacles. A general task may be thought as involving free-motion as well as constrained motion phases. In this section we shall focus on the case when the system is assumed to be in a permanent contact with some environment.

In other words the constraint between the controlled system and the obstacle is supposed to be bilateral. In all the sequel we assume that the potential energy of the controlled system $U_g(z)$ and of the passive environment $U_{g_e}(z_1)$ each have a unique strict minimum, and to simplify further that they are positive (they have been chosen so).

7.8.1 Regulation with a Position PD Controller

Before going on with particular environment dynamics, let us analyze the regulation problem for the system in (6.169). To this end let us define the PD control

$$\bar{\tau} = -\lambda_2 \tilde{z} - \lambda_1 \dot{z} \tag{7.164}$$

where $\tilde{z} = z(t) - z_d$, z_d a constant signal. Since we have assumed that the constraints are bilateral, we do not have to restrict z_d to a particular domain of the state space (*i.e.* we do not care about the sign of the interaction force). Let us "invent" a Lyapunov function candidate by mimicking the available storage in (6.170), i.e.

$$\begin{aligned} V(\tilde{z}, \dot{z}, z_1) &= \tfrac{1}{2}\dot{z}^T M(z)\dot{z} + \tfrac{1}{2}\dot{z}_1 M_e(z_1)\dot{z}_1 \\ &\quad + \tfrac{1}{2}\lambda_2 \tilde{z}^T \tilde{z} + U_g(z) + U_{g_e}(z_1) + \tfrac{1}{2} z_1^T K_e z_1 \end{aligned} \tag{7.165}$$

Instead of computing the derivative of this function along the closed-loop system (6.169) and (7.164), let us decompose the overall system into two blocks. The first block contains the controlled subsystem dynamics, and has input $u_1 = F_z = \begin{pmatrix} \lambda_z \\ 0 \end{pmatrix}$, output $y_1 = \dot{z}$. The second block has the dynamics of the environment, output $u_2 = -\lambda_z$ and input $u_2 = \dot{z}$. These two subsystems are passive since

$$\begin{aligned} \langle u_1, y_1 \rangle_t &= \int_0^t \dot{z}^T \left[\bar{M}(z)\ddot{z} + \bar{C}(z,\dot{z})\dot{z} + \bar{g}(z) + \lambda_2 \tilde{z} + \lambda_2 \dot{z}\right] ds \\ &\geq -\tfrac{1}{2} z(0)^T \bar{M}(z(0)) z(0) - U_g(z(0)) - \tfrac{1}{2}\lambda_2 z(0)^T z(0) \end{aligned} \tag{7.166}$$

and

$$\begin{aligned} &\langle u_2, y_2 \rangle_t = \\ &= \int_0^t \dot{z}_1^T \left[M_e(z_1)\ddot{z}_1 + C_e(z_1, \dot{z}_1)\dot{z}_1 + \tfrac{dR_e}{d\dot{z}_1} + K_e z_1 + g_e(z_1)\right] ds \\ &\geq -\tfrac{1}{2}\dot{z}_1(0)^T M_e(z_1(0))\dot{z}_1(0) - \tfrac{1}{2} z_1(0)^T K_e z_1(0) - U_{g_e}(z_1(0)) \end{aligned} \tag{7.167}$$

Now the inputs and outputs have been properly chosen so that the two subsystems are already in the required form for the application of the passivity theorem. Notice that they are both controllable and zero-state detectable from the chosen inputs and outputs. Therefore the storage functions that appear in the right-hand-sides of (7.166) and (7.167) are Lyapunov functions (see Lemmae 5.13 and 4.8) and their concatenation is the Lyapunov function candidate in (7.165) which is a Lyapunov function. The asymptotic stability of the closed-loop system fixed point can be shown using the Krasovskii-La Salle Theorem, similarly to the case of rigid joint-rigid link manipulators controlled by a PD feedback. Notice that similarly to (7.27) the fixed points are given as solutions of the following equation (obtained by summing the dynamics of the two subsystems)

$$\begin{pmatrix} K_e z_1 + \bar{g}_1(z) + g_e(z_1) + \lambda_2 \tilde{z}_1 \\ \lambda_2 \tilde{z}_2 + \bar{g}_2(z) \end{pmatrix} = \begin{pmatrix} 0_{m\times 1} \\ 0_{(n-m)\times 1} \end{pmatrix} \tag{7.168}$$

We may assume that this equation has only one root $z = z_i$ so that the fixed point $(z, \dot{z}) = (z_i, 0)$ is globally asymptotically stable.

Remark 7.41. It is noteworthy that this interpretation works well because the interconnection between the two subsystems satisfies Newton's principle of mutual actions. The open-loop system is therefore "ready" for a decomposition through the passivity theorem.

Remark 7.42. Let us note that there is no measurement of the environment state in (7.164). The coordinate change presented in Section 6.7.2 just allows one to express the generalized coordinates for the controlled subsystem in a frame that coincides with a "natural" frame associated to the obstacle. It is clear however that the transformation relies on the exact knowledge of the obstacle geometry.

The next step, that consists of designing a passivity-based nonlinear controller guaranteeing some tracking properties in closed-loop, has been performed in [320]. It has been extended in [368] when the geometry of the obstacle surface is unknown (it depends on some unknown parameters) and has to be identified (then an adaptive version is needed). Further works using closed-loop passivity may be found in [296, 297].

7.8.2 Holonomic Constraints

Let us now analyze the case when $M_e(z_1)\ddot{z}_1 = 0$ and the contact stiffness K_e and damping $R_e(\dot{z}_1)$ tend to infinity, in which case the controlled subsystem is subject to a bilateral holonomic constraint $\phi(q) = 0$ [4]. In the transformed coordinates (z_1, z_2) the dynamics is given in (6.163); see Subsection 6.7.1. We saw that the open-loop properties of the unforced system transport from the

[4] Actually the way these coefficients tend to infinity is important to pass from the compliant case to the rigid body limit. This is analyzed for instance in [323] through a singular perturbation approach.

free-motion to the reduced constrained motion systems. Similarly, it is clear that any feedback controller that applies to the dynamics in (6.98) applies equally well to the reduced order dynamics $(z_2, \dot{z}_2)$ in (6.163). The real problem now (which has important practical consequences) is to design a controller such that the contact force tracks some desired signal. Let us investigate the extension of the Slotine and Li scheme in this framework. The controller in (7.68) is slightly transformed into

$$\begin{cases} \bar{\tau}_1 = \bar{M}_{12}\ddot{z}_{2r} + \bar{C}_{12}(z_2, \dot{z}_2)\dot{z}_{2r} + \bar{g}_1 - \lambda_2\lambda_{zd} \\ \\ \bar{\tau}_2 = \bar{M}_{22}\ddot{z}_{2r} + \bar{C}_{22}(z_2, \dot{z}_2)\dot{z}_{2r} + \bar{g}_2 - \lambda_2 s_2 \end{cases} \tag{7.169}$$

where all the terms keep the same definition as for (7.68). λ_d is some desired value for the contact force λ_{z_1}. The closed-loop system is therefore given by

$$\begin{cases} \bar{M}_{12}(z_2(t))\dot{s}_2(t) + \bar{C}(z_2(t), \dot{z}_2(t))s_2(t) = \lambda_2(\lambda_{z_1}(t) - \lambda_d) \\ \\ \bar{M}_{22}(z_2(t))\dot{s}_2(t) + C(z_2(t), \dot{z}_2(t))s_2(t) + \lambda_1 s_2(t) = 0 \\ \\ \dot{\tilde{z}}_2(t) = -\lambda\tilde{z}_2(t) + s_2(t) \end{cases} \tag{7.170}$$

The dissipativity properties of the free-motion closed-loop system are similar to those of (7.69) and (7.70). Notice that due to the asymptotic stability properties of the fixed point $(\tilde{z}_2, s_2)$ one gets $\lambda_{z_1}(t) \to \lambda_d(t)$ as $t \to +\infty$.

7.8.3 Nonsmooth Lagrangian Systems

In practice one often has to face *unilateral* or *inequality* constraints where (6.162) is replaced by $\phi(q) \geq 0$, which models the fact that contact may be lost or established with obstacles. Let us just point out that this yields to *nonsmooth* systems containing impact rules (or state reinitializations) and so-called *complementarity* relationships between λ_{z_1} and z_1, of the form

$$\lambda_{z_1} \geq 0,\ z_1 \geq 0,\ \lambda_{z_1}^T z_1 = 0 \tag{7.171}$$

The inclusion of such complementarity conditions into the dynamics, yields a Lagrangian complementarity system as (6.177) or a measure differential inclusion as in (6.180) . See Section 6.8.2 for more developments on nonsmooth systems. The trajectory tracking problem for such systems has been studied in [60, 75, 76]. Specific stability notions are developed that take into account the subtleties of this problem. For instance the times of first impacts when one wants to stabilize the system on a surface $z_1 = 0$ is ususally unknown, as well as the time of detachment from this surface. It is shown in [60] that the Slotine and Li controller is a suitable basic nonlinear controller to achieve the stability requirements developed therein, because of its exponential convergence property, and also because the quadratic Lyapunov function (7.80) is close to the kinetic energy of the open-loop system (consequently it should possess nice properties at impacts, following the kinetic energy variation in (6.178)). A “switching” or “hybrid” Slotine and Li controller is designed in [60].

More details on nonsmooth mechanical systems dynamics and control can be found in [69, 75, 76, 164].

7.9 Controlled Lagrangians

Until now we have focussed in this chapter on passivity-based controllers, designed for trajectory tracking and adaptive control. Let us briefly introduce the method of controlled Lagrangians. As said in the introduction of the chapter, the objective is to shape both the kinetic and potential energies, with a suitable feedback. Let us describe the method in the simplest case, *i.e.* a fully actuated Lagrangian system

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q) = \tau \tag{7.172}$$

The objective is to design τ in such a way that the closed-loop system becomes

$$M_c(q(t))\ddot{q}(t) + C_c(q(t), \dot{q}(t))\dot{q}(t) + g_c(q) = 0 \tag{7.173}$$

where $M_c(q)$ is a desired kinetic tensor, and $g_c(q) = \nabla U_c(q)$ where $U_c(q)$ is a desired potential energy. Let us propose

$$\tau = M(q)M_c^{-1}(q)[-C_c(q, \dot{q})\dot{q} - g_c(q)] + C(q, \dot{q})\dot{q} + g(q) \tag{7.174}$$

Injecting (7.174) into (7.172) one obtains

$$M(q(t))\ddot{q}(t) = M(q(t))M_c^{-1}(q(t))[-C_c(q(t), \dot{q}(t))\dot{q}(t) - g_c(q(t))] \tag{7.175}$$

Since $M(q)$ is full rank one can rewrite (7.175) as (7.173). The fully actuated case is therefore quite trivial, and the methods owns its interest to the underactuated case. Let us therefore consider

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q) = G(q(t)\tau \tag{7.176}$$

for some $n \times m$ matrice $G(q)$ with $\mathrm{rank}(G(q)) = m$ for all $q \in I\!R^n$. There exists a matrix $G^{\perp}(q)$ such that $G^{\perp}(q)G(q) = 0$ for all q. Also $\mathrm{Im}(G^{\perp}(q)) + \mathrm{Im}(G^T(q)) = I\!R^{2n}$, and both subspaces are orthogonal. It is thus equivalent to rewrite (7.176) as

$$\begin{cases} G^{\perp}(q)\{M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + g(q)\} = 0 \\ \\ G^T(q)\{M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + g(q)\} = G^T(q)G(q)\tau \end{cases} \tag{7.177}$$

where one notices that $G^T(q)G(q)$ is a $m \times m$ invertible matrix. Obviously the same operation may be applied to the objective system, *i.e.*

$$\begin{cases} G^{\perp}(q)\{M_c(q(t))\ddot{q}(t) + C_c(q(t),\dot{q}(t))\dot{q}(t) + g_c(q)\} = 0 \\ \\ G^T(q)\{M_c(q(t))\ddot{q}(t) + C_c(q(t),\dot{q}(t))\dot{q}(t) + g_c(q)\} = 0 \end{cases} \tag{7.178}$$

One says that the two systems (7.177) and (7.178) *match* if they possess the same solutions for any initial data $(q(0),\dot{q}(0))$. It is easy to see that by choosing

$$\tau = (G^T(q)G(q))^{-1}G^T(q)\left\{M(q)M_c^{-1}(q)[-C_c(q,\dot{q})\dot{q} - g_c(q)] + C(q,\dot{q})\dot{q} + g(q)\right\} \tag{7.179}$$

one obtains

$$G^T(q)\{M_c(q(t))\ddot{q}(t) + C_c(q(t),\dot{q}(t))\dot{q}(t) + g_c(q)\} = 0. \tag{7.180}$$

It then remains to examine what happens with the rest of the closed-loop dynamics. Matching between (7.173) and (7.176) occurs if and only if $G^{\perp}(q)\{M_c(q(t))\ddot{q}(t) + C_c(q(t),\dot{q}(t))\dot{q}(t) + g_c(q)\} = 0$ holds along the solutions of the closed-loop system (7.176) and (7.179). In other words matching occurs if and only if

$$\begin{aligned} M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + g(q) - G(q(t)\tau = \\ = M_c(q(t))\ddot{q}(t) + C_c(q(t),\dot{q}(t))\dot{q}(t) + g_c(q) \end{aligned} \tag{7.181}$$

Note that if there is matching then we can also express the acceleration as

$$\begin{aligned} \ddot{q} = M^{-1}(q)G(q)\tau - M^{-1}(q)[C(q,\dot{q})\dot{q} + g(q)] - \\ -M_c^{-1}(q)[C_c(q,\dot{q})\dot{q} + g_c(q)] \end{aligned} \tag{7.182}$$

so that

$$G(q)\tau = -M(q)M_c(q)[C_c(q,\dot{q})\dot{q} + g_c(q)] + C(q,\dot{q})\dot{q} + g(q) \tag{7.183}$$

and premultiplying by $G^{\perp}(q)$ one gets

$$G^{\perp}(q)\left\{-M(q)M_c(q)[C_c(q,\dot{q})\dot{q} + g_c(q)] + C(q,\dot{q})\dot{q} + g(q)\right\} = 0 \tag{7.184}$$

Consequently matching between (7.173) and (7.176) occurs if and only if (7.184) holds and τ is as in (7.179).

Remark 7.43. All these developments may be led within a differential geometry context [55]. This does not help in understanding the underlying simplicity of the method (on the contrary it may obscure it). However it highlights the fact that the equality in (7.184) is in fact a partial differential equation for $M_c(q)$ and $U_c(q)$. Consequently the controlled Lagrangian method boils down to solving a PDE.

8

Adaptive Control

This chapter is dedicated to present so-called *direct adaptive* controllers applied to mechanical and to linear invariant systems. We have already studied some applications of dissipativity theory in the stability of adaptive schemes in Chapters 1, 2, 3 and 4. Direct adaptation means that one has been able to rewrite the fixed parameter input u in a form that is linear with respect to some unknown parameters, usually written as a vector $\theta \in \mathbb{R}^p$, *i.e.* $u = \phi(x,t)\theta$, where $\phi(x,t)$ is a known matrix (called the regressor) function of measurable [1] terms. The parameters θ_i, $i \in \{1, \cdots, p\}$, are generally nonlinear combinations of the physical parameters (for instance in the case of mechanical systems, they will be nonlinear combinations of moments of inertia, masses). When the parameters are unknown, one cannot use them in the input. Therefore one replaces θ in u by an estimate, that we shall denote $\hat{\theta}$ in the sequel. In other words, $u = \phi(x,t)\theta$ is replaced by $u = \phi(x,t)\hat{\theta}$ at the input of the system, and $\hat{\theta}$ is estimated on-line with a suitable identification algorithm. As a consequence, one easily imagines that the closed-loop system stability analysis will become more complex. However through the passivity theorem (or the application of Lemma 7.23) the complexity reduces to adding a passive block to the closed-loop system that corresponds to the estimation algorithm dynamics. The rest of the chapter is composed of several examples that show how this analysis mechanism work. It is always assumed that the parameter vector is constant: the case of time-varying parameters, although closer to the reality, is not treated here due to the difficulties in deriving stable adaptive controllers in this case. This is a topic in itself in adaptive control theory and is clearly outside the scope of this book.

[1] In the technological sense, not in the mathematical one.

8.1 Lagrangian Systems

8.1.1 Rigid Joint–Rigid Link Manipulators

In this subsection we first examine the case of a PD controller with an adaptive gravity compensation. Indeed it has been proved in Section 7.3.1 that gravity hampers asymptotic stability of the desired fixed point, since the closed-loop system possesses an equilibrium that is different from the desired one. Then we pass to the case of tracking control of n degree-of-freedom manipulators.

PD + Adaptive Gravity Compensation

A First Simple Extension

Let us consider the following controller + estimation algorithm:

$$\begin{cases} \tau(t) = -\lambda_1 \dot{q}(t) - \lambda_2 \tilde{q}(t) + Y_g(q(t))\hat{\theta}_g(t) \\ \\ \dot{\hat{\theta}}_g(t) = \lambda_3 Y_g^T \dot{q}(t) \end{cases} \tag{8.1}$$

where we suppose that the gravity generalized torque $g(q) = Y_g(q)\theta_g$ for some known matrix $Y_g(q) \in I\!R^{n\times p}$ and unknown vector θ_g, and $\tilde{\theta}_g = \theta_g - \hat{\theta}_g$. The estimation algorithm is of the gradient type, and we know from Subsection 4.3.1 that such an estimation law defines a passive operator $\dot{q} \mapsto \tilde{\theta}_g^T Y_g(q)$, with storage function $V_2(\tilde{\theta}_g) = \frac{1}{2}\tilde{\theta}^T\tilde{\theta}$. This strongly suggests one should decompose the closed-loop system obtained by introducing (8.1) into (6.98) into two blocks as follows:

$$\begin{cases} M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + \lambda_1\dot{q}(t) + \lambda_2\tilde{q}(t) = -Y_g(q(t))\tilde{\theta}(t) \\ \\ \dot{\tilde{\theta}}_g(t) = \lambda_3 Y_g^T(t)\dot{q}(t) \end{cases} \tag{8.2}$$

Obviously the first block with the rigid joint-rigid link dynamics and input $u_1 = -Y_g(q)\tilde{\theta}(= -y_2)$ and output $y_1 = \dot{q}(= u_2)$ defines an output strictly passive operator with storage function $V_1(\tilde{q},\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{\lambda_2}{2}\tilde{q}^T\tilde{q}$; see Subsection 7.3.1. One is tempted to conclude about the asymptotic stability with a Lyapunov function $V(\tilde{q},\dot{q},\tilde{\theta}) = V_1(\tilde{q},\dot{q}) + V_2(\tilde{\theta}_g)$. However notice that the overall system with input $u = u_1 + y_2$ and output $y = y_1$, although output strictly passive, is not zero-state detectable. Indeed $u \equiv y \equiv 0$ implies $\lambda_2\tilde{q} = Y_g(q)\tilde{\theta}_g$ and $\dot{\tilde{\theta}}_g = 0$, nothing more. Hence very little has been gained by adding an estimation of the gravity, despite the passivity theorem applies well.

How to Get Asymptotic Stability?

The lack of zero-state detectability of the system in (8.2) is an obstacle to the asymptotic stability of the closed-loop scheme. The problem is therefore to keep the negative feedback interconnection structure of the two highlighted blocks, while introducing some detectability property in the loop. However the whole state is now $(q, \dot{q}, \tilde{\theta}_g)$ and it is known in identification and adaptive control theory that the estimated parameters converge to the real ones (*i.e.* $\tilde{\theta}_g(t) \to 0$) only if some persistent excitation conditions are fullfilled. Those conditions are related to the spectrum of the signals entering the regressor matrix $Y_g(q)$. Such a result is hopeless here since we are dealing with regulation. Hence the best one may expect to obtain is convergence of $(\tilde{q}, \dot{q})$ towards zero. We may however hope that there exists a feedback adaptive controller that can be analyzed through the passivity theorem and such that the underlying storage function can be used as a Lyapunov function with Krasovskii-La Salle Theorem to prove asymptotic convergence. Let us consider the estimation algorithm proposed in [488]:

$$\dot{\tilde{\theta}}_g(t) = \lambda_3 Y_g^T(t)\left(\lambda_4 \dot{q}(t) + \frac{2\tilde{q}(t)}{1 + 2\tilde{q}^T(t)\tilde{q}(t)}\right) \tag{8.3}$$

Note that this is still a gradient update law. It defines a passive operator $\left(\lambda_4\dot{q} + \frac{2\tilde{q}}{1+2\tilde{q}^T\tilde{q}}\right) \mapsto Y_g(q)\tilde{\theta}_g$, not $\dot{q} \mapsto Y_g(q)\tilde{\theta}_g$. We therefore have to look at the dissipativity properties of the subsystem with dynamics $M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + \lambda_1\dot{q}(t) + \lambda_2\tilde{q}(t) = u_1(t)$, $y_1(t) = \left(\lambda_4\dot{q}(t) + \frac{2\tilde{q}(t)}{1+2\tilde{q}^T(t)\tilde{q}(t)}\right)$: this is new compared to what we have seen until now in this book. Let us analyze it in detail:

$$\begin{array}{l}
\langle u_1, y_1\rangle_t = \\[2ex]
= \int_0^t \left(\lambda_4\dot{q} + \frac{2\tilde{q}}{1+2\tilde{q}^T\tilde{q}}\right)^T [M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \lambda_1\dot{q} + \lambda_2\tilde{q}]ds \\[2ex]
= q\int_0^t \left\{\lambda_4\dot{q}^T(\lambda_2\tilde{q} + \lambda_1\dot{q}) + \frac{d}{ds}\left(\frac{\lambda_4}{2}\dot{q}^T M(q)\dot{q} + \frac{2\dot{q}^T M(q)\tilde{q}}{1+2\tilde{q}^T\tilde{q}}\right)\right\} ds+ \\[2ex]
+ \int_0^t \left\{-\frac{2\dot{q}^T M(q)\dot{q} + 2\dot{q}^T C(q,\dot{q})\tilde{q}}{1+2\tilde{q}^T\tilde{q}} + \frac{8\dot{q}^T M(q)\tilde{q}\dot{q}^T\tilde{q}}{1+2\tilde{q}^T\tilde{q}} 2\frac{\tilde{q}}{1+2\tilde{q}^T\tilde{q}}(\lambda_2\tilde{q} + \lambda_1\dot{q})\right\} ds \\[2ex]
\geq \frac{\lambda_4\lambda_2}{2}\left[\tilde{q}^T\tilde{q}\right]_0^t + \left[\frac{\lambda_4}{2}\dot{q}^T M(q)\dot{q} + \frac{2\dot{q}^T M(q)\tilde{q}}{1+2\tilde{q}^T\tilde{q}}\right]_0^t + \lambda_4\lambda_1\int_0^t \dot{q}^T\dot{q}ds+ \\[2ex]
+ \int_0^t \left\{2\lambda_2\frac{\tilde{q}^T\tilde{q}}{1+2\tilde{q}^T\tilde{q}} - \left(4\lambda_M + \frac{k_c}{\sqrt{2}}\right)\dot{q}^T\dot{q} - 2\frac{\lambda_1||\dot{q}||.||\tilde{q}||}{1+2\tilde{q}^T\tilde{q}}\right\} ds \\[2ex]
\geq -\frac{\lambda_4\lambda_2}{2}\tilde{q}(0)^T\tilde{q}(0) - \frac{\lambda_4}{2}\dot{q}(0)^T M(q(0))\dot{q}(0) + \frac{2\dot{q}(0)^T M(q(0))\tilde{q}(0)}{1+2\tilde{q}(0)^T\tilde{q}(0)} + \\[2ex]
+\lambda_4\lambda_1\int_0^t \dot{q}^T(s)\dot{q}(s)ds
\end{array} \tag{8.4}$$

where we have used the fact that due to the skew-symmetry of $\dot{M}(q)-2C(q,\dot{q})$ we have $\dot{M}(q) = C(q,\dot{q}) + C^T(q,\dot{q})$, and where

$$\lambda_4 > \max\left\{\frac{1}{\lambda_1}\left(\frac{\lambda_1^2}{2\lambda_2} + 4\lambda_M + \frac{k_c}{\sqrt{2}}\right), \frac{2\lambda_M}{\sqrt{\lambda_m \lambda_2}}\right\}$$

with $\lambda_m I_n \leq M(q) \leq \lambda_M I_n$, $||C(q,\dot{q})|| \leq k_c||\dot{q}||$ for any compatible matrix and vector norms. Under these gain conditions, one sees from (8.4) that the first subsystem is passive with respect to the supply rate $u_1^T y_1$, and a storage function is given by

$$V_1(\tilde{q},\dot{q}) = \frac{\lambda_4\lambda_2}{2}\tilde{q}^T\tilde{q} + \frac{\lambda_4}{2}\dot{q}^T M(q)\dot{q} + \frac{2\dot{q}^T M(q)\tilde{q}}{1+2\tilde{q}^T\tilde{q}} \tag{8.5}$$

The first subsystem even possesses some strict passivity property; see (8.4). Finally a complete storage function is provided by the sum $V(\tilde{q},\dot{q},\tilde{\theta}_g) = V_1(\tilde{q},\dot{q}) + V_2(\tilde{\theta}_g)$, and it can be shown that its derivative is semi-negative definite and that the largest invariant set contained in the set $\dot{V} \equiv 0$ is contained in the set $(q,\dot{q}) = (0,0)$ which ends the proof.

Remark 8.1. The storage function associated to the first subsystem is quite original. It looks like the available storage of the closed-loop system when a PD controller is applied, but the added term comes from "nowhere"! Our analysis has been done easily because we knew beforehand that such a storage function was a good one. The intuition behind it is not evident. It was first discovered in [261] and then used in [488].

The Adaptive Slotine and Li Controller

Let us now pass to the controller presented in Subsection 7.3.4 in (7.68). It turns out that this scheme yields a much more simple stability analysis than the PD with adaptive gravity compensation: this is due to the fact that as pointed out earlier, it uses the inertia matrix explicitly even for regulation.

Gradient Estimation Law

Consider the following controller:

$$\begin{cases} \tau(t) = \hat{M}(q(t))\ddot{q}_r(t) + \hat{C}(q(t),\dot{q}(t))\dot{q}_r(t) + \hat{g}(q(t)) - \lambda_1 s(t) \\ \qquad = Y(q(t),\dot{q}(t),t)\hat{\theta}(t) \\ \qquad = M(q(t))\ddot{q}_r(t) + C(q(t),\dot{q}(t))\dot{q}_r(t) + g(q(t)) - \lambda_1 s(t) - Y(q(t),q\dot{(t)},t)\tilde{\theta} \\ \dot{\hat{\theta}}(t) = \lambda_2 Y^T(t)(q(t),\dot{q}(t),t)s(t), \quad \lambda_2 > 0 \end{cases} \tag{8.6}$$

where we used the fact that the fixed parameter controller can be rewritten under the required linear form $Y(q, \dot{q}, t)\hat{\theta}$, where θ is a vector of unknown inertia parameters. Actually one has $M(q(t))\ddot{q}(t)+C(q(t),\dot{q}(t))\dot{q}(t)+g(q(t)) = Y(q(t),\dot{q}(t),\ddot{q}(t))\theta$. The closed-loop system is therefore given by

$$\boxed{\begin{cases} M(q(t))\dot{s}(t) + C(q(t),\dot{q}(t))s(t) + \lambda_1 s(t) = Y(q(t),\dot{q}(t),t)\tilde{\theta}(t) \\ \dot{\tilde{q}}(t) = -\lambda\tilde{q}(t) + s(t) \\ \dot{\tilde{\theta}}(t) = \lambda_2 Y^T(t)(q(t),\dot{q}(t),t)s(t) \end{cases} \qquad (8.7)}$$

The interpretation through the passivity theorem is obvious: the update law in (8.6) is a gradient that defines a passive operator $s \mapsto Y(q, \dot{q}, t)\tilde{\theta}$ and the first subsystem has state $(\tilde{q}, s)$. From the developments in Subsection 7.3.4 one therefore sees that the adaptive version of the Slotine and Li controller just yields a closed-loop system that is identical to the one in (7.69) (7.70) with an additional passive block interconnected to the two previous ones in (7.69) and (7.71); see Figure 8.1 and compare with Figure 7.6. The storage function follows immediately. Similarly to the PD with adaptive compensation scheme, one cannot expect to get asymptotic stability of the whole state because of the parameter estimates that generally do not converge towards the real ones. Let us consider the quadratic function

$$V(s,t) = \frac{1}{2}s^T M(q)s + \frac{1}{2}\tilde{\theta}^T\tilde{\theta} \tag{8.8}$$

Computing its derivative along the closed-loop trajectories and u sing the same arguments as for the first stability proof of the fixed parameters Slotine and Li controller in Section 7.3.4, one easily concludes on the global convergence of all signals but $\tilde{\theta}(t)$ to zero as $t \to +\infty$, and on the boundedness of all signals on $[0, +\infty)$.

Remark 8.2. Historically the passivity interpretation of the Slotine and Li scheme has been deduced from Lemma 7.23, see [71, 74], where most of the adaptive schemes (including *e.g.* [426]) designed for rigid manipulators have been analyzed through the passivity theorem. Indeed this is based on a CTCE as defined in Lemma 7.23. Actually the first subsystem in (8.7) with state s has relative degree one between its input $u_1 = Y(q, q, t)\tilde{\theta}$ and its output $y_1 = s$. As we shall remark when we have presented the adaptive control of linear invariant systems with relative degree one, the CTCE is ubiquitous in direct adaptive control. The extension of the Slotine and Li scheme to the case of force-position control when the system is in permanent contact with a flexible

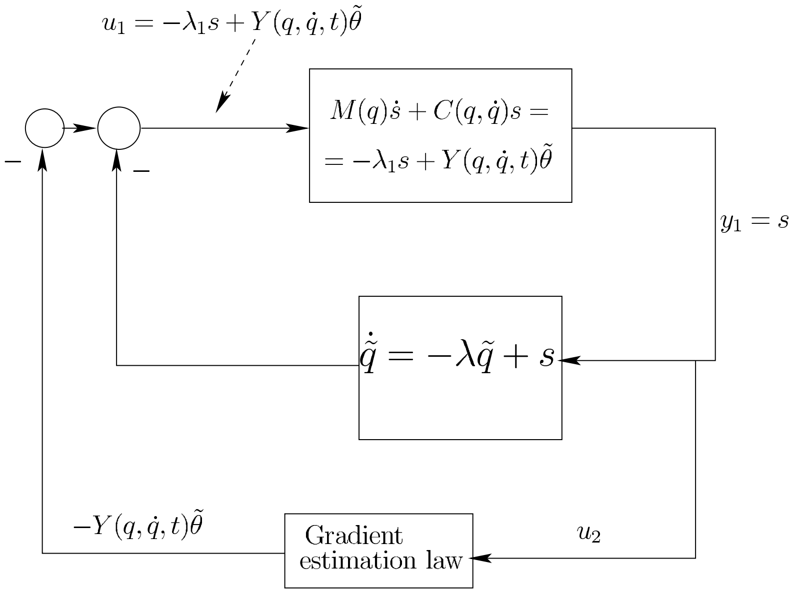

Fig. 8.1. Closed-loop equivalent representation (adaptive case)

stiff environment, has been done in [320] (see also [368] for an extension of the scheme in [320]).

Least-squares Estimation Law

Until now we have presented only gradient-type update laws. It is clear that the estimation block can be replaced by any system that has $\tilde{\theta}$ inside the state and is passive with respect to the same supply rate. The classical recursive least-squares estimation algorithm does not satisfy such requirements. However it can be "passified" as explained now. First of all let us recall the form of the classical least-squares algorithm:

$$\begin{cases} \dot{\hat{\theta}}_{ls}(t) = P(t)Y^T(q(t),\dot{q}(t),t)s(t) \\ \\ \dot{P}(t) = -PY(q(t),\dot{q}(t),t)Y^T(q(t),\dot{q}(t),t)P(t), \quad P(0) > 0 \end{cases} \tag{8.9}$$

The required passivity property is between s and $-Y(q,\dot{q},t)\tilde{\theta}$ (recall we have defined $\tilde{\theta} = \theta - \hat{\theta}$). Let us compute the available storage of this system:

$$
\begin{aligned}
V_a(\tilde{\theta}, P) &= \sup_{s:(0,\tilde{\theta}(0),P(0))\to} -\int_0^t s^T Y \tilde{\theta} ds \\
&= \sup_{s:(0,\tilde{\theta}(0),P(0))\to} -\frac{1}{2}\left[\tilde{\theta}^T P^{-1}\tilde{\theta}\right]_0^t + \frac{1}{2}\int_0^t \tilde{\theta}^T \dot{P}^{-1}\tilde{\theta} ds \\
&= \sup_{s:(0,\tilde{\theta}(0),P(0))\to} -\frac{1}{2}\left[\tilde{\theta}^T P^{-1}\tilde{\theta}\right]_0^t + \frac{1}{2}\int_0^t \tilde{\theta}^T Y Y^T \tilde{\theta} ds
\end{aligned}
\tag{8.10}
$$

where we used the fact that $\dot{P}^{-1} = YY^T$. One remarks that the available storage in (8.10) is not "far" from being bounded: it would suffice that $Y^T\tilde{\theta}$ be L^2-bounded. However it seems difficult to prove this. Consequently let us propose the following modified least-squares estimation algorithm [2]:

$$
\begin{cases}
\hat{\theta}(t) = \hat{\theta}_{ls}(t) + \mathcal{S}(t) \\
\dot{P}(t) = \alpha(t)\left[-P(t)\left(\frac{Y^T(t)Y(t)}{1+\mathrm{tr}(Y^T(t)Y(t))} + \lambda R\right)P(t) + \lambda P(t)\right] \\
\alpha(t) = \frac{s^T(t)Y(t)Y^T(t)s(t)}{(1+s^T(t)s(t))(1+\mathrm{tr}(Y^T(t)Y(t)))} \\
A = \frac{Y^T(t)Y(t)}{1+\mathrm{tr}(Y^{T(t)}Y(t))} + \lambda R \\
\mathcal{S}(t) = \frac{Y^T(t)}{1+\mathrm{tr}(Y^T(t)Y(t))}\frac{s(t)}{1+s^T(t)s(t)}\left(\hat{\theta}_{ls}^T(t)A\hat{\theta}_{ls}(t) + M(1+\lambda\lambda_{\max}(R))\right) \\
\lambda \geq 0, \quad R > 0 \\
\lambda_{\min}(R) I_n \leq P^{-1}(0) \leq \left(\lambda_{\max}(R) + \frac{1}{\lambda}\right) I_n \\
M \geq \theta^T\theta
\end{cases}
\tag{8.11}
$$

Then the following is true [73, 319]:

Lemma 8.3. *(a)* $\lambda_{\min}(R) \leq \lambda_i(P^{-1}) \leq \lambda_{\max}(R) + \frac{1}{\lambda}$, *where* $\lambda_i(P^{-1})$ *denotes the eigenvalues of* P^{-1}. *(b)* $\int_0^t -s^T Y\tilde{\theta} ds = \frac{1}{2}\left[\tilde{\theta}_{ls}^T P^{-1}\tilde{\theta}_{ls}\right]_0^t - \frac{1}{2}\int_0^t \tilde{\theta}_{ls}^T \dot{P}^{-1}\tilde{\theta}_{ls} d\tau - \int_0^t s^T Y \mathcal{S} d\tau$, *where* $\tilde{\theta}_{ls} = \theta - \hat{\theta}_{ls}$, *where* $\hat{\theta}_{ls}$ *is the classical least-squares estimate* $\dot{\hat{\theta}}_{ls} = PY^T s$. *(c)* $-\frac{1}{2}\int_0^t \tilde{\theta}_{ls}^T \dot{P}^{-1}\tilde{\theta}_{ls} d\tau - \int_0^t s^T Y \mathcal{S} d\tau \geq 0$. ■

[2] Let us note that the denomination "least-squares" somewhat loses its original meaning here, since it is not clear that the proposed scheme minimizes any quadratic criterion. However the name least-squares is kept for obvious reasons.

It follows that the mapping $s \mapsto -Y\tilde{\theta}$ is passive with storage function $\frac{1}{2}\tilde{\theta}_{ls}^T P^{-1}\tilde{\theta}_{ls}$. The proof of Lemma 8.3 is not given here for the sake of briefness and also because despite its originality, it has not been proved that such passified least-square yields better closed-loop performance than the simple gradient update law (for instance in terms of parameter convergence speed and of robustness). It is therefore to be seen more like a theoretical exercise (find out how to passify the classical least-squares) rather than something motivated by applications. The interest for us here is to illustrate the modularity provided by passivity-based controllers. As we shall see further, it applies equally well to adaptive control of relative degree one and two linear invariant systems.

8.1.2 Flexible Joint–Rigid Link Manipulators: The Adaptive Lozano and Brogliato Algorithm

In this section we provide the complete exposition of the adaptive version of the scheme of Section 7.5.1, which is the only adaptive scheme proposed in the literature solving both the linearity-in-the-paramaters and the a priori knowledge of the stiffness matrix K issues, and at the same time guaranteeing the global convergence of the tracking errors, the boundedness of all the closed-loop signals, with only positions and velocity measurements (no acceleration feedback). It has been published in [72, 318]. This scheme uses ingredients from [461] and from [426] in the stability proof.

The starting point for the stability analysis of the adaptive version is the quadratic function

$$\boxed{\begin{aligned} V(s_1, s_2, \tilde{q}_1, \tilde{\theta}, \tilde{q}_2) = \tfrac{1}{2}s_1^T M(q_1)s_1 + \tfrac{1}{2}\det(M(q_1))s_2^T J s_2 + \\ +\tfrac{1}{2}(\tilde{q}_1 - \tilde{q}_2)^T K(\tilde{q}_1 - \tilde{q}_2) + +\tfrac{1}{2}\sigma_p \tilde{q}_1^T \tilde{q}_1 + \tfrac{1}{2}\tilde{\theta}^T\tilde{\theta} \end{aligned} \quad (8.12)}$$

where $\sigma_p > 0$ is a feedback gain and $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$ is the parameter error vector. We do not define what θ is at this stage, because this vector of unknown parameters will be constructed in proportion as the stability proof progresses. Actually it will be proved in Lemma 8.6 below that the nonadaptive control law may be written as

$$\theta_5^T h(q_1)u + Y_6(q_1, \dot{q}_1, q_2, \dot{q}_2)\theta_6 = 0 \tag{8.13}$$

where $\theta_5^T h(q_1)$ =det$(M(q_1)$. Thus a nice property that will be used is that $M(q_1) > 0$ so that $\det(M(q_1)) > 0$: the controller hence defined is not singular. This is used when the parameter vector θ_5 is replaced by its estimate $\hat{\theta}_5(t)$, by defining a suitable projection algorithm. Another issue is that of the *a priori* knowledge of the diagonal matrix K =diag(k_{ii}) which has to be replaced by an estimate $\hat{K}$ in the controller. Since the fictitious input q_{2d} is defined with K^{-1}, its adaptive counterpart will be defined with $\hat{K}^{-1}(t)$, so that $\hat{K}(t)$ has to be nonsingular. Moreover the signal q_{2d} has to be twice differentiable. This implies that $\hat{K}(t)$ will have in addition to be twice differentiable as well. The two parameter projection algorithms are given as follows. We define $\theta_K = (k_{11}, k_{22}, ..., k_{nn})^T$, and we assume that a lower bound αI_n on $M(q_1)$ is known.

The Parameter-adaptation Algorithms

It is possible to define a subspace spanned by $h(q_1)$ as $S = \{v \mid v = h(q_1) \text{ for some } q_1\}$, and a set $\Lambda = \{v \mid v^T h \geq \alpha^n \text{ for all } h \in S\}$. The set Λ is convex, and $\theta_5 \in \Lambda$. The first parameter adaptation law is as follows:

$$\dot{\hat{\theta}}_5(t) = \begin{cases} h(q_1(t))u^T(t)s_2(t) & \text{if } \hat{\theta}_5(t) \in \text{Int}(\Lambda) \\ P_r[h(q_1(t))u^T(t)s_2(t)] & \text{if } \hat{\theta}_5(t) \in \partial(\Lambda) \\ & \text{and } [h(q_1(t))u^T(t)s_2(t)]^T\hat{\theta}_5^{\perp} > 0 \end{cases} \tag{8.14}$$

where $P_r[\cdot]$ denotes the orthogonal projection onto Λ, $\partial(\Lambda)$ is the boundary of Λ, and $\hat{\theta}_5^{\perp}$ is the vector normal to $\partial(\Lambda)$ at $\hat{\theta}_5(t)$, and

$$\dot{\hat{\theta}}_6(t) = Y_6^T(q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t))s_2(t) \tag{8.15}$$

The gradient update laws in (8.14) and (8.15) will then be used to define the adaptive controller as

$$\hat{\theta}_5^T(t)h(q_1(t))u(t) + Y_6(q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t))\hat{\theta}_6(t) = 0 \tag{8.16}$$

The second projection algorithm is as follows, and concerns the estimate of the stiffness matrix K:

$$\dot{\hat{\theta}}_k^i(t) = \begin{cases} x^i(t) & \text{if } \hat{\theta}_k^i(t) \geq \delta_k \\ x^i(t) & \text{if } \hat{\theta}_k^i(t) \geq \frac{\delta_k}{2} \text{ and } x^i(t) \geq 0 \\ \left[f(\hat{\theta}_k^i(t))\right]^{-x^i(t)} x^i(t) & \text{if } \delta_k \geq \hat{\theta}_k^i(t) \geq \frac{\delta_k}{2} \text{ and } x^i(t) \leq 0 \end{cases} \tag{8.17}$$

where $x^i(t) = Y_{2d}^i(q_1(t), \dot{q}_1(t), q_{1d}(t), q_{2d}(t))$ and $0 < \delta_k \leq \min \theta_k^i$. The row vector $Y_{2d}(\cdot)$ is defined as

$$\begin{cases} s_1^T K[q_{2d} - q_{1d}] = \theta_k^T \mathrm{diag}(s_1^i)[q_{2d} - q_{1d}] = Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d})\theta_k \\ \\ Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d}) = [q_{2d} - q_{1d}]^T \mathrm{diag}(s_1^i) \end{cases} \tag{8.18}$$

The function $f(\cdot)$ has to be chosen as a smooth function $0 \le f(\hat{\theta}_k^i) \le 1$ with $f(\frac{\delta_k}{2}) = 0$ and $f(\delta_k) = 1$. This implies that the parameter projection in (8.17) is twice differentiable and that $\hat{\theta}_k^i(t) \ge \frac{\delta_k}{2}$ for all $t \ge 0$ and all $1 \le i \le n$ [3].

■

The rational behind the choice for the various functions appearing in these update laws, will be clarified. We now introduce a Lemma that will be useful in constructing a function $q_{2d}(\cdot)$ whose second derivative $\ddot{q}_{2d}(\cdot)$ depends only on position and velocity.

Lemma 8.4. *[72] One has $M(q_1(t))s_1(t) = Y_{4f}(t)\theta_4$, where $\dot{Y}_{4f}(t) + Y_{4f}(t) = Y_4(q_1(t), \dot{q}_1(t), q_2(t))$ for some $Y_4(q_1(t), \dot{q}_1(t), q_2(t))$.* ■

Proof: Let us filter the first dynamical equation in (6.105) as

$$\frac{1}{1+s}\left[M(q_1(t))\ddot{q}_1(t) + C(q_1(t), \dot{q}_1(t))\dot{q}_1(t) + g(q_1(t)) - K(q_2(t) - q_1(t))\right] = 0 \tag{8.19}$$

where we implicitly mean that $\frac{1}{1+s}[f(t)]$ is the Laplace transform of $f(t)$. Now we have (we drop the time argument for simplicity)

$$\begin{aligned} &\tfrac{1}{1+s}[M(q_1)\ddot{q}_1] = M(q_1)\dot{q}_1 - M(q_1(0))\dot{q}_1(0) - \\ &-\tfrac{1}{1+s}[M(q_1)\dot{q}_1 - M(q_1(0)\dot{q}_1(0)] - \tfrac{1}{1+s}[\dot{M}(q_1, \dot{q}_1)\dot{q}_1] \end{aligned} \tag{8.20}$$

which follows from $M(q_1)\ddot{q}_1 = \frac{d}{dt}(M(q_1)\dot{q}_1) - \dot{M}(q_1, \dot{q}_1)\dot{q}_1$. Now

$$\frac{1}{1+s}[M(q_1)\ddot{q}_1] = \int_0^t \exp(-t+\tau)M(q_1(\tau))d\tau \tag{8.21}$$

Then using integration by parts one gets

$$\begin{aligned} &\tfrac{1}{1+s}[M(q_1)\ddot{q}_1] = \\ &= \exp(-t)\Bigg[\Big[\exp(\tau)\Big(M(q_1) - M(q_1(0)\dot{q}_1(0) - \textstyle\int_0^\tau \dot{M}(q_1(y), \dot{q}_1(y))dy\Big)\Big]_0^t \\ &- \textstyle\int_0^t \exp(\tau)\Big(M(q_1(\tau))\dot{q}_1(\tau) - M(q_1(0)\dot{q}_1(0) - \int_0^\tau \dot{M}(q_1(y), \dot{q}_1(y))dy\Big)d\tau\Bigg] \end{aligned} \tag{8.22}$$

[3] Another type of C^n projections is presented in [92], whose motivation is quite in the spirit of this one, see e.g. [92, §III].

which finally yields

$$
\begin{aligned}
&\tfrac{1}{1+s}[M(q_1)\ddot{q}_1] = \\
&= M(q_1)\dot{q}_1 - M(q_1(0))\dot{q}_1(0) - \textstyle\int_0^t \dot{M}(q_1(y),\dot{q}_1(y))dy - \\
&- \textstyle\int_0^t \exp(-t+\tau)\left(M(q_1)\dot{q}_1 - M(q_1(0))\dot{q}_1(0) - \int_0^\tau \dot{M}(q_1(y),\dot{q}_1(y))dy\right)d\tau
\end{aligned}
\tag{8.23}
$$

Still integrating by parts we get

$$
\begin{aligned}
&\textstyle\int_0^t \exp(-t+\tau)\dot{M}(q_1(\tau),\dot{q}_1(\tau))\dot{q}_1(\tau)d\tau = \\
&= \textstyle\int_0^t \dot{M}(q_1(\tau),\dot{q}_1(\tau))\dot{q}_1(\tau)d\tau - \int_0^t \exp(-t+\tau)\left(\int_0^\tau \dot{M}(q_1(y),\dot{q}_1(y))\dot{q}_1(y)dy\right)d\tau
\end{aligned}
\tag{8.24}
$$

from which we can deduce (8.20) combining (8.23) and (8.24). Now using (8.19) and (8.20) we obtain

$$
\begin{aligned}
M(q_1)\dot{q}_1 &= M(q_1(0))\dot{q}_1(0) + \tfrac{1}{s+1}\left[M(q_1)\dot{q}_1 - M(q_1(0))\dot{q}_1(0)\right] + \\
&+ \tfrac{1}{s+1}[\dot{M}(q_1,\dot{q}_1)\dot{q}_1] - \tfrac{1}{s+1}[C(q_1\dot{q}_1)\dot{q}_1 + g(q_1) + Kq_1] + \\
&+ \tfrac{1}{s+1}[Kq_2]
\end{aligned}
\tag{8.25}
$$

The terms between brackets can be written as $Y_i(q_1,\dot{q}_1)\theta_i$ for some constant vector θ_i. Therefore $\frac{1}{s+1}[Y_i(q_1,\dot{q}_1)\theta_i] = \frac{1}{s+1}[Y_i(q_1,\dot{q}_1)]\theta_i \overset{\Delta}{=} Y_{if}(t)\theta_i$ with $\dot{Y}_{if}(t) + Y_{if}(t) = [Y_i(q_1(t),\dot{q}_1(t))$. It follows that (8.20) can be written as $M(q_1)s_1 = Y_{4f}(t)\theta_4$ with $\dot{Y}_{4f}(t) + Y_{4f}(t) = [Y_4(q_1(t),\dot{q}_1(t),q_2(t))$. ■

Let us now proceed with the stability proof, which we start by differentiating the function (8.12) along the system's trajectories. The controller $u(\cdot)$ will then be constructed step by step within the proof. Afterwards we shall recapitulate and present compact forms of the input and the closed-loop system. We obtain

$$
\begin{aligned}
\dot{V}(s_1,s_2,\tilde{q}_1,\tilde{\theta},\tilde{q}_2) &= s_1^T[M(q_1)\dot{s}_1 + C(q_1,\dot{q}_1)]s_1 + \det(M(q_1))s_2^T J\dot{s}_2 + \\
&+ (\tilde{q}_1 - \tilde{q}_2)^T K(\dot{\tilde{q}}_1 - \dot{\tilde{q}}_2) + \sigma_p \tilde{q}_1^T \dot{\tilde{q}}_1 + \tilde{\theta}^T \dot{\tilde{\theta}} + \\
&+ \tfrac{1}{2}\tfrac{d}{dt}[\det(M(q_1))]s_2^T J s_2
\end{aligned}
\tag{8.26}
$$

Notice that

$$(\tilde{q}_1-\tilde{q}_2)^T K(\dot{\tilde{q}}_1-\dot{\tilde{q}}_2) = (\tilde{q}_1-\tilde{q}_2)^T K(-\lambda\tilde{q}_1+s_1+\lambda\tilde{q}_2-s_2)$$
$$= -(\tilde{q}_1-\tilde{q}_2)^T K(\tilde{q}_1-\tilde{q}_2)+(s_1-s_2)^T K(\tilde{q}_1-\tilde{q}_2) \tag{8.27}$$

Introducing this in (8.26) we obtain

$$\dot{V}(s_1,s_2,\tilde{q}_1,\tilde{\theta},\tilde{q}_2) \le s_1^T\left[M(q_1)\dot{s}_1+C(q_1,\dot{q}_1)s_1+K(\tilde{q}_1-\tilde{q}_2)\right]+$$
$$+\, s_2^T\left[\det(M(q_1))J\dot{s}_2+\tfrac{1}{2}\tfrac{d}{dt}[\det(M(q_1))Js_2-K(\tilde{q}_1-\tilde{q}_2)]\right]+\sigma_p\tilde{q}_1^T\dot{\tilde{q}}_1+\tilde{\theta}^T\dot{\tilde{\theta}} \tag{8.28}$$

where the skew-symmetry property of Lemma 6.16 has been used to introduce the term $C(q_1,\dot{q}_1)s_1$. Let us manipulate the first term between brackets in the right-hand-side of (8.28):

$$\begin{aligned} T_1 &= s_1^T\left[M(q_1)\dot{s}_1+C(q_1,\dot{q}_1)s_1+K(\tilde{q}_1-\tilde{q}_2)\right] \\ &= s_1^T\left\{M(q_1)(\ddot{q}_1-\ddot{q}_{1d}+\lambda\dot{\tilde{q}}_1)+C(q_1,\dot{q}_1)(\dot{\tilde{q}}_1+\lambda\tilde{q}_1)+K(q_{2d}-q_{1d})\right\} \\ &= s_1^T[\Delta_1+\Delta_2+K(q_{2d}-q_{1d})] \end{aligned} \tag{8.29}$$

where we define

$$\begin{aligned} \Delta_1 &= (M(q_{1d})-M(q_1)\ddot{q}_{1d}+(C(q_{1d},\dot{q}_{1d})-C(q_1,\dot{q}_1))\dot{q}_{1d}+g(q_{1d}) \\ &\quad -g(q_1)+\lambda(M(q_1)\dot{\tilde{q}}_1+C(q_1,\dot{q}_1)\tilde{q}_1 \end{aligned} \tag{8.30}$$

$$\Delta_2 = -M(q_{1d})\ddot{q}_{1d}-C(q_{1d},\dot{q}_{1d})\dot{q}_{1d}-g(q_{1d}) \tag{8.31}$$

We now need a technical result from [426].

Lemma 8.5. *[426, Lemma 1] The following inequality holds:*

$$\begin{aligned} & s_1^T[M(q_{1d})\ddot{q}_{1d}+C(q_{1d},\dot{q}_{1d})\dot{q}_{1d}+g(q_{1d})-M(q_1)(\ddot{q}_{1d}-\lambda\dot{\tilde{q}}_1)- \\ & -C(q_1,\dot{q}_1)(\dot{q}_{1d}-\lambda\tilde{q}_1)-g(q_1)]s_1 \le \\ & \le s^T(\lambda M(q_1)+b_1I_n)s+s^T(-\lambda^2M(q_1)+b_2I_n)\tilde{q}_1+b_3(s^Ts\,\|\tilde{q}_1\|+ \\ & +\lambda\|s\|\,\tilde{q}_1^T\tilde{q}_1) \end{aligned} \tag{8.32}$$

for some positive bounded functions $b_1(\cdot)$, $b_2(\cdot)$, $b_3(\cdot)$ of $q_{1d}(\cdot)$, $\dot{q}_{1d}(\cdot)$, and $\ddot{q}_{1d}(\cdot)$. ■

This allows us to upperbound the term $s_1^T \Delta_1$ as follows:

$$s_1^T \Delta_1 \leq s_1^T(\lambda M(q_1) + b_1 I_n) + s_1^T(-\lambda^2 M(q_1) + b_2 I_n)\tilde{q}_1 + \\ + b_3(s_1^T s_1 \; ||\tilde{q}_1|| + \lambda ||s_1|| \; \tilde{q}_1^T \tilde{q}_1) \tag{8.33}$$

Now notice that

$$\begin{aligned} s_1^T s_1 \; ||\tilde{q}_1|| + \lambda ||s_1|| \; \tilde{q}_1^T \tilde{q}_1 &= \tfrac{s_1^T s_1}{4} + \tfrac{\lambda \tilde{q}_1^T \tilde{q}_1}{4} - s_1^T s_1 \left(\tfrac{1}{2} - \tilde{q}_1^T \tilde{q}_1\right)^2 - \\ &\quad -\lambda \tilde{q}_1^T \tilde{q}_1 \left(\tfrac{1}{2} - s_1^T s_1\right)^2 + (1+\lambda) s_1^T s_1 \; \tilde{q}_1^T \tilde{q}_1 \\ &\leq \tfrac{s_1^T s_1}{4} + \tfrac{\lambda \tilde{q}_1^T \tilde{q}_1}{4} + (1+\lambda) s_1^T s_1 \; \tilde{q}_1^T \tilde{q}_1 \end{aligned} \tag{8.34}$$

Introducing (8.34) into (8.33) we get

$$s_1^T \Delta_1 \leq a_1 s_1^T s_1 + a_2 \tilde{q}_1^T \tilde{q}_1 + a_3 s_1^T s_1 \; \tilde{q}_1^T \tilde{q}_1 \tag{8.35}$$

where $a_1(\cdot)$, $a_2(\cdot)$ and $a_3(\cdot)$ are positive bounded functions of q_{1d}, $\dot{q}_{1d}$, $\ddot{q}_{1d}$, and of the dynamic model parameters. Now from (8.31) and the fact that the various terms of the dynamical model are linear in the parameters, we can write

$$\Delta_2 = Y_d(q_{1d}, \dot{q}_{1d}, \ddot{q}_{1d})\theta_1 \tag{8.36}$$

where the matrix $Y_d(q_{1d}, \dot{q}_{1d}, \ddot{q}_{1d})$ is of appropriate dimensions and θ_1 is a vector of constant parameters. Now since K is a diagonal matrix we can write

$$s_1^T K(q_{2d} - q_{1d}) = \theta_k^T \mathrm{diag}(s_1^i)(q_{2d} - q_{1d}) \tag{8.37}$$

with $\theta_k = (k_{11}, k_{22}, ..., k_{nn})^T$. From (8.37) we have

$$s_1^T K(q_{2d} - q_{1d}) = Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d})\theta_k \tag{8.38}$$

where

$$Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d}) = (q_{2d} - q_{1d})^T \mathrm{diag}(s_1^i) \tag{8.39}$$

(we recall that s_1^i denotes the ith element of the vector $s_1 \in I\!R^n$). Now injecting (8.38) into (8.29) we obtain

$$T_1 = s_1^T(\Delta_1 + \Delta_2) - Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d})\tilde{\theta}_k + Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d})\hat{\theta}_k \pm \\ \pm(\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1) s_1^T M(q_1) s_1 \tag{8.40}$$

where $\tilde{\theta}_k(t) = \hat{\theta}_k(t) - \theta_k$, $\sigma_v > 0$, $\sigma_n > 0$. The last term in (8.40) will be used to compensate the term $s_1^T \Delta_1$. Now from Lemma 8.4 we have $M(q_1(t))s_1(t) = Y_{4f}(t)\theta_4$. Introducing this into (8.40) we obtain

$$
\begin{aligned}
T_1 = s_1^T(\Delta_1 + \Delta_2) - Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d})\tilde{\theta}_k + Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d})\hat{\theta}_k + \\
+(\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1)s_1^T Y_{4f}(t)\theta_4 - (\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1)s_1^T M(q_1)s_1
\end{aligned}
\tag{8.41}
$$

Provided $\hat{k}_{ii} > 0$ for all $1 \leq i \leq n$, we can safely define the function $q_{2d}(\cdot)$ as follows:

$$
\hat{K}(q_{2d} - q_{1d}) = -(\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1)Y_{4f}(t)\hat{\theta}_4 - Y_d(q_{1d}, \dot{q}_{1d}, \ddot{q}_{1d})\hat{\theta}_1 - \sigma_p \tilde{q}_1 \tag{8.42}
$$

where $\hat{K}$ =diag$(\hat{k}_{ii})$ and $\hat{\theta}_k = (\hat{k}_{11}, \hat{k}_{22}, ..., \hat{k}_{nn})^T$. Introducing (8.42) into (8.39) we obtain

$$
\begin{aligned}
Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d})\hat{\theta}_k &= \hat{\theta}_k \text{diag}(s_1^i)(q_{2d} - q_{1d}) = s_1^T \hat{K}(q_{2d} - q_{1d}) \\
&= -(\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1)s_1^T Y_{4f}(t)\hat{\theta}_4 - s_1^T Y_d(q_{1d}, \dot{q}_{1d}, \ddot{q}_{1d})\hat{\theta}_1 - \\
&\quad -\sigma_p s_1^T \tilde{q}_1
\end{aligned}
\tag{8.43}
$$

where $\sigma_p > 0$. Introducing (8.43) and (8.36) into (8.41) we obtain

$$
\begin{aligned}
T_1 = s_1^T \Delta_1 - s_1^T Y_d \tilde{\theta}_1 - Y_{2d}\tilde{\theta}_k - (\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1)(s_1^T Y_{4f}\tilde{\theta}_4 + s_1^T M(q_1)s_1) - \\
-\sigma_p s_1^T \tilde{q}_1
\end{aligned}
\tag{8.44}
$$

Furthermore from (8.35) we have that

$$
\begin{aligned}
s_1^T\ -(\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1)s_1^T M(q_1)s_1 - \lambda \sigma_p \tilde{q}_1^T \tilde{q}_1 \leq \\
\leq -s_1^T s_1(\lambda_{\min}(M(q_1))\ \sigma_v - a_1) - \tilde{q}_1^T \tilde{q}_1(\lambda \sigma_p - a_2) - \\
-s_1^T s_1\ \tilde{q}_1^T \tilde{q}_1(\lambda_{\min}(M(q_1))\ \sigma_n - a_3)
\end{aligned}
\tag{8.45}
$$

If σ_v, σ_p, σ_n are chosen large enough so that

$$
\begin{cases}
\lambda_{\min}(M(q_1))\ \sigma_v - a_1 \geq \delta_0 > 0 \\
\lambda \sigma_p - a_2 \geq \delta_1 > 0 \\
(\lambda_{\min}(M(q_1))\ \sigma_n - a_3 \geq 0
\end{cases}
\tag{8.46}
$$

we obtain

$$
\begin{aligned}
T_1 \leq & -\delta_0 s_1^T s_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 - \sigma_p \dot{\tilde{q}}_1^T \tilde{q}_1 - s_1^T Y_d(q_{1d}, \dot{q}_{1d}, \ddot{q}_{1d}) \tilde{\theta}_1 - \\
& -Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d}) \tilde{\theta}_k - (\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1) s_1^T Y_{4f}(t) \tilde{\theta}_4
\end{aligned}
\tag{8.47}
$$

Combining (8.28), (8.29) and (8.47) we obtain

$$
\begin{aligned}
& \dot{V}(s_1, s_2, \tilde{q}_1, \tilde{\theta}, \tilde{q}_2) \leq -\delta_0 s_1^T s_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 - s_1^T Y_d(q_{1d}, \dot{q}_{1d}, \ddot{q}_{1d}) \tilde{\theta}_1 - \\
& - Y_{2d}(q_1, \dot{q}_1, q_{1d}, q_{2d}) \tilde{\theta}_k - (\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1) s_1^T Y_{4f}(t) \tilde{\theta}_4 + \tilde{\theta}^T \dot{\hat{\theta}} + s_2^T T_2
\end{aligned}
\tag{8.48}
$$

with

$$
T_2 = \det(M(q_1)) \dot{s}_2 + \frac{J}{2} \frac{d}{dt} [\det(M(q_1))] - K(\tilde{q}_1 - \tilde{q}_2) \tag{8.49}
$$

Let us define

$$
\theta = [\theta_k^T \; \theta_1^T \; \theta_4^T \; \theta_5^T \; \theta_6^T]^T \tag{8.50}
$$

where the precise definition of θ_5 and θ_6 will be given later. Let us introduce the parameter update laws:

$$
\begin{cases}
\dot{\hat{\theta}}_1(t) = Y_d^T(q_{1d}(t), \dot{q}_{1d}(t), \ddot{q}_{1d}(t)) s_1(t) \\
\\
\dot{\hat{\theta}}_4(t) = (\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1) Y_{4f}(t) s_1(t)
\end{cases}
\tag{8.51}
$$

where we recall that $M(q_1)s_1 = Y_{4f}(t)\theta_4$ with

$$
\dot{Y}_{4f}(t) + Y_{4f}(t) = Y_4(q_1(t), \dot{q}_1(t), q_2(t))
$$

from Lemma 8.4. Now let us introduce (8.50), (8.51) and (8.17) into (8.48), in order to obtain

$$
\dot{V}(s_1, s_2, \tilde{q}_1, \tilde{\theta}, \tilde{q}_2) \leq -\delta_0 s_1^T s_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 + \tilde{\theta}_5^T \dot{\hat{\theta}}_5 + \tilde{\theta}_6^T \dot{\hat{\theta}}_6 + s_2^T T_2 \tag{8.52}
$$

where the equality

$$
(\dot{\hat{\theta}}_k - Y_{2d})^T \tilde{\theta}_k = \sum_i (\dot{\hat{\theta}}_k^i - Y_{2d}^i)(\hat{\theta}_k^i - \theta_k^i) \tag{8.53}
$$

has been used. The expression for the controller is obtained from the following lemma:

Lemma 8.6. *[318] The term T_2 in (8.49) can be expressed as*

$$T_2 = \theta_5^T h(q_1)u + Y_6(q_1, \dot{q}_1, q_2, \dot{q}_2)\theta_6 \tag{8.54}$$

with $det(M(q_1)) = \theta_5^T h(q_1) > \alpha^n$ for some $\alpha > 0$ and all $q_1 \in I\!R^n$. The vectors θ_5 and θ_6 are unknown parameters and $h(q_1)$ and $Y_6(q_1, \dot{q}_1, q_2, \dot{q}_2)$ are known functions. ■

Proof: From (6.105) and (8.49) we can deduce that

$$\begin{aligned} T_2 &= \det(M(q_1))[u + K(q_1 - q_2)] + J\det(M(q_1))(-\ddot{q}_{2d} + \lambda\dot{\tilde{q}}_2) \\ &+ \tfrac{J}{2}\tfrac{d}{dt}\det(M(q_1))s_2 - K(\tilde{q}_2 - \tilde{q}_1) \end{aligned} \tag{8.55}$$

Since $M(q_1) > 0$ then $\det(M(q_1)) > 0$ and the linearity-in-the-parameters property of the dynamical equations allows one to write $\det(M(q_1)) = \theta_5^T h(q_1)$. Considering the second order time-derivative of (8.42), it can be proved that $\det(M(q_1))\ddot{q}_{2d}$ is a linear-in-the-parameters function of positions and velocities (notice that the way Y_{4f} is defined plays a crucial role here) and of the acceleration $\ddot{q}_1$. Similarly $\dot{q}_{2d}$ is a measurable signal (*i.e.* a function of positions and velocities); see (8.39), Lemma 8.4, (8.42) and (8.17). However notice that $\det(M(q_1))\ddot{q}_1$ is a function of q_1, $\dot{q}_1$, and q_2. Thus $\ddot{q}_{2d}$ is a function of positions and velocities only. We conclude that T_2 can indeed be written in a compact form as in (8.54). ■

In view of Lemma 8.6 we obtain

$$\begin{aligned} \dot{V}(s_1, s_2, \tilde{q}_1, \tilde{\theta}, \tilde{q}_2) &\le -\delta_0 s_1^T s_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 + \tilde{\theta}_5^T \dot{\hat{\theta}}_5 + \tilde{\theta}_6^T \dot{\hat{\theta}}_6 + s_2^T \theta_5^T h(q_1)u + \\ &+ s_2^T Y_6(q_1, \dot{q}_1, q_2, \dot{q}_2)\theta_6 \\ &= -\delta_0 s_1^T s_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 + \tilde{\theta}_5^T \dot{\hat{\theta}}_5 - s_2^T h^T(q_1)\tilde{\theta}_5^T u + s_2^T h(q_1)\hat{\theta}_5^T u \\ &+ s_2^T Y_6(q_1, \dot{q}_1, q_2, \dot{q}_2)\theta_6 \end{aligned} \tag{8.56}$$

Introducing the parameters adaptation laws in (8.14) and (8.15) and the adaptive control law in (8.16), into (8.56), we get

$$\dot{V}(s_1, s_2, \tilde{q}_1, \tilde{\theta}, \tilde{q}_2) \le -\delta_0 s_1^T s_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 + \tilde{\theta}_5^T[\dot{\hat{\theta}}_5 - h(q_1)u^T s_2]. \tag{8.57}$$

The term $h(q_1)u^T s_2$ can be broken down as

$$h(q_1)u^T s_2 = P_r(h(q_1)u^T s_2) + P_r^{\perp}(h(q_1)u^T s_2), \tag{8.58}$$

where we recall that $P_r(z)$ denotes the orthogonal projection on the hyperplane tangent to $\partial(\Lambda)$ at z and $P_r^{\perp}(z)$ is the component of z that is perpendicular to this hyperplane at z. Then using (8.15) we obtain

$$\tilde{\theta}_5^T[\dot{\hat{\theta}}_5 - h(q_1)u^T s_2] = \begin{cases} 0 & \text{if } \hat{\theta}_5 \in \operatorname{Int}(\Lambda) \\ -\tilde{\theta}_5^T(h(q_1)u^T s_2) \leq 0 & \text{if } \hat{\theta}_5 \in \partial(\Lambda) \\ & \text{and } (h(q_1)u^T s_2)^T \hat{\theta}_5^{\perp} > 0 \end{cases} \tag{8.59}$$

Consequently we finally obtain

$$\dot{V}(s_1, s_2, \tilde{q}_1, \tilde{\theta}, \tilde{q}_2) \leq -\delta_0 s_1^T s_1 - \delta_1 \tilde{q}_1^T \tilde{q}_1 \tag{8.60}$$

It immediately follows from (8.12), (8.61), Lemma 4.8 and Theorem 4.10 that $\tilde{\theta}(\cdot)$, $s_1(\cdot)$, $s_2(\cdot)$, $\tilde{q}_1(\cdot)$, $\dot{\tilde{q}}_1(\cdot)$, $\dot{\tilde{q}}_2(\cdot)$ and $\tilde{q}_2(\cdot)$ are bounded functions of time on $[0, +\infty)$ [4]. Moreover $s_1 \in \mathcal{L}_2$. Using the same reasonning as in the proof of the fixed parameters Slotine and Li or Lozano and Brogliato schemes, we deduce that $\tilde{q}_1(t) \to 0$ as $t \to +\infty$. It is deduced from (8.25) that the term $\frac{1}{s+1}[q_2]$ is bounded, so that q_{2d} is bounded also, and consequently both $q_2(\cdot)$ and $\ddot{q}_1(\cdot)$ are bounded. The boundedness of $\dot{q}_2(\cdot)$ follows from differentiating (8.42), which proves that $\dot{q}_{2d}(\cdot)$ is bounded. Thus $\dot{q}_2(\cdot)$ is bounded. The boundedness of the controller u can be inferred from (8.16). One deduces that $\ddot{q}_2(\cdot)$ is bounded on $[0, +\infty)$.

Recapitulation

The closed-loop system that results from the controller defined in (8.16), (8.14), (8.15), (8.51) and (8.17) does not have a form as simple and intuitive as the closed-loop system of the Slotine and Li adaptive controller, or of the closed-loop system of the Lozano and Brogliato fixed parameters controller. This seems however to be an intrinsic property of the adaptive scheme for (6.105), because one needs to invert the first dynamical equation to avoid the acceleration $\ddot{q}_1(t)$ measurement. Consequently the matrix $M^{-1}(q_1)$ necessarily appears in the fixed parameters scheme, and it is a nonlinear-in-the-parameters function. The adaptation for the matrix K may be avoided in practice if one is able to estimate it accurately enough. But *the linearity-in-the-parameters issue is unavoidable and intrinsic to such controlled dynamics.*

After a certain number of manipulations based on the above developments we may write the closed-loop dynamics as follows:

[4] It is clear that the desired trajectory $q_{1d}(t)$ and its first and second derivatives, are chosen as bounded functions of time. Any other choice would be silly.

$$
\boxed{
\begin{array}{l}
M(q_1(t))\dot{s}_1(t) + C(q_1(t), \dot{q}_1(t))s_1(t) = K(\tilde{q}_2 - \tilde{q}_1) + \tilde{K}(q_{1d}(t) - q_{2d}(t)) \\
-\,(\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1) Y_{4f}(t)\tilde{\theta}_4(t) - Y_d(q_{1d}(t), \dot{q}_{1d}(t), \ddot{q}_{1d}(t))\tilde{\theta}_1(t) - \sigma_p \tilde{q}_1 \\
-(\sigma_v + \sigma_n \tilde{q}_1^T \tilde{q}_1) M(q_1(t))s_1(t) + \Delta W(q_1(t), \dot{q}_1(t), q_{1d}(t), \dot{q}_{1d}(t), \ddot{q}_{1d}(t)) \\
\\
\text{with } \Delta W(q_1(t), \dot{q}_1(t), q_{1d}(t), \dot{q}_{1d}(t), \ddot{q}_{1d}(t)) = M(q_1(t))[\ddot{q}_{1d}(t) - \lambda \dot{\tilde{q}}_1(t)] + \\
+C(q_1(t), \dot{q}_1(t))[\dot{q}_{1d}(t) - \lambda \tilde{q}_1(t)] - g(q_1(t)) + Y_d(q_{1d}(t), \dot{q}_{1d}(t), \ddot{q}_{1d}(t))\theta_1 \\
\\
\tilde{\theta}_5^T(t) h(q_1(t)) u(t) + Y_6(q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t))\tilde{\theta}_6(t) = 0 \\
\text{Updates laws in (8.14), (8.15), (8.17) and (8.51)} \\
\dot{\tilde{q}}_i(t) = -\lambda \tilde{q}_i(t) + s_i(t), \quad i = 1, 2
\end{array}
}
\tag{8.61}
$$

where we recall that $Y_d(q_{1d}(t), \dot{q}_{1d}(t), \ddot{q}_{1d}(t))\theta_1 = -M(q_{1d})\ddot{q}_{1d} - C(q_{1d}, \dot{q}_{1d})\dot{q}_{1d} - g(q_{1d})$; see (8.31). It is worth comparing (8.61) with (7.125) to measure the gap between adaptive control and fixed-parameter control, and comparing (8.61) with (8.7) to measure the gap between the flexible-joint case and the rigid-joint case.

Remark 8.7. As we saw in Section 7.5.1, the fixed parameters Lozano and Brogliato scheme is a passivity-based controller using a backstepping design method. The adaptive scheme is a highly non-trivial extension, where the linearity-in-the-parameters and the unknown stiffness matrix issues imply the use of very specific update laws, and hampers the direct application of backstepping methods designed elsewhere for some classes of nonlinear systems.

8.1.3 Flexible Joint–Rigid Link Manipulators: The Backstepping Algorithm

Let us now investigate how the backstepping approach may be used to solve the adaptive control problem for flexible joint manipulators. We will assume that K is a known matrix. We have to solve two main problems in order to extend the fixed parameter scheme presented in Subsection 7.5.2 towards an adaptive version:

1) The input u in must be LP (Linear in some set of Parameters).

2) The signals $\tilde{q}_2$ and e_2 have to be available on line because they will be used in the update laws.

To solve 1), we can use the idea introduced in [318] which consists of adding the determinant of the inertia matrix $\det(M(q_1))$ in the Lyapunov function $V_1(\cdot)$ (see the foregoing subsection on the adaptive passivity-based scheme). As we explained the nonlinearity in the unknown parameters comes from the terms containing the inverse of the inertia matrix $M^{-1}(q_1)$. Premultiplying by $\det(M(q_1))$ allows us to retrieve LP terms, as $\det(M(q_1))M^{-1}(q_1)$ is indeed LP (the price to pay is an overparametrization of the controller). Moreover 2) implies that q_{2d} (see (7.130)) and e_{2d} (see after (7.133)) are available on line, and thus do not depend on unknown parameters. We can proceed as follows:

- **Step 1:** The right-hand-side of (6.105) can be written as $Y_1(q_1, \dot{q}_1, t)\theta_1^*$. Thus we choose q_{2d} in (7.130) as

$$Kq_{2d} = Y_1(q_1, \dot{q}_1, t)\hat{\theta}_1 \tag{8.62}$$

where $\hat{\theta}_1$ stands for an estimate of θ_1^*. Thus

$$\tilde{q}_2 = q_2 - K^{-1}Y_1(q_1, \dot{q}_1, t)\hat{\theta}_1 \tag{8.63}$$

Adding $\pm Y_1(\cdot)\theta_1^*$ to the right-hand-side of the first equation in (6.105) and differentiating (8.63), one obtains:

$$\begin{cases} M(q_1(t))\dot{s}_1(t) + C(q_1(t), \dot{q}_1(t))s_1(t) + \lambda_1 s_1(t) = K\tilde{q}_2(t) - Y_1(t)\theta_1^* \\ \\ \dot{\tilde{q}}_2(t) = \dot{q}_2(t) - K^{-1}\frac{d}{dt}(Y_1(t)\theta_1^*) \end{cases} \tag{8.64}$$

- **Step 2:** Now consider e_{2d} defined after (7.133). The first two terms are available but the third term is a function of unknown parameters and it is not LP (it contains M^{-1}). Assume now that V_2 is replaced by

$$V_{2a} = V_r(\tilde{q}_1, s_1, t) + \frac{1}{2}\tilde{\theta}_1^T\tilde{\theta}_1 + \frac{1}{2}\det(M(q_1))\tilde{q}_2^T\tilde{q}_2 \tag{8.65}$$

Setting $\dot{q}_2 = e_{2d} + e_2$, *i.e.* $\dot{\tilde{q}}_2 = e_{2d} + e_2 - K^{-1}\frac{d}{dt}(Y_1\hat{\theta}_1)$, we get along trajectories of (8.64):

$$\begin{aligned} \dot{V}_{2a} \leq & -\lambda_1\dot{\tilde{q}}_1^T\dot{\tilde{q}}_1 - \lambda^2\lambda_1\tilde{q}_1^T\tilde{q}_1 - s_1^TY_1\tilde{\theta}_1 + \dot{\hat{\theta}}_1^T\tilde{\theta}_1 + \tilde{q}_2^TKs_1 + \\ & +\tilde{q}_2^T\det(M(q_1))e_2 + \tilde{q}_2^T\det(M(q_1))[e_{2d} - \dot{q}_{2d}] + \\ & +\tilde{q}_2^T\frac{d}{dt}\{\tfrac{(det(M(q_1))}{2}\}\tilde{q}_2 \end{aligned} \tag{8.66}$$

Let us denote $\det(M) = Y_2(q_1)\theta_2^*$, and choose

$$Y_2\hat{\theta}_2 e_{2d} = -Y_3(q_1, \dot{q}_1, q_2, t)\hat{\theta}_3 - \tilde{q}_2 \tag{8.67}$$

where

$$Y_3(q_1, \dot{q}_1, q_2, t)\theta_3^* = \frac{d}{dt}\left\{\frac{\det(M(q_1))}{2}\right\}\tilde{q}_2 - \det(M(q_1))\dot{q}_{2d} + Ks_1 \tag{8.68}$$

Choose also

$$\dot{\hat{\theta}}_1(t) = Y_1^T(q_1(t), \dot{q}_1(t), t)s_1(t) \tag{8.69}$$

Thus we obtain

$$\dot{V}_{2a} \leq -\lambda_1\dot{\tilde{q}}_1^T\dot{\tilde{q}}_1 - \lambda^2\lambda_1\tilde{q}_1^T\tilde{q}_1 + \tilde{q}_2^T\det(M(q_1))e_2 + \tilde{q}_2^T[Y_2\theta_2^{\star T}e_{2d} + Y_3\theta_3^{\star T}] \tag{8.70}$$

(we drop the arguments for convenience). Introducing $\pm\tilde{q}_2^T Y_2\hat{\theta}_2 e_{2d}$ we obtain

$$\begin{aligned} \dot{V}_{2a} \leq & -\lambda_1\dot{\tilde{q}}_1^T\dot{\tilde{q}}_1 - \lambda^2\lambda_1\tilde{q}_1^T\tilde{q}_1 + \tilde{q}_2^T\det(M(q_1))e_2 - \\ & -\tilde{q}_2^T e_{2d}Y_2\tilde{\theta}_2 + \tilde{q}_2^T Y_3\tilde{\theta}_3 - \tilde{q}_2^T\tilde{q}_2 \end{aligned} \tag{8.71}$$

Define $V_{3a} = V_{2a} + \frac{1}{2}\tilde{\theta}_2^T\tilde{\theta}_2 + \frac{1}{2}\tilde{\theta}_3^T\tilde{\theta}_3$ and set

$$\dot{\hat{\theta}}_3(t) = -Y_3^T(q_1(t), \dot{q}_1(t), q_2(t), t)\tilde{q}_2(t) \tag{8.72}$$

$$\dot{\hat{\theta}}_2(t) = -Y_2^T(q_1(t))e_{2d}^T(t)\tilde{q}_2(t) \tag{8.73}$$

We therefore obtain

$$\dot{V}_{3a} \leq -\lambda_1\dot{\tilde{q}}_1^T\dot{\tilde{q}}_1 - \lambda^2\lambda_1\tilde{q}_1^T\tilde{q}_1 + \tilde{q}_2^T\det(M(q_1))e_2 - \tilde{q}_2^T\tilde{q}_2 \tag{8.74}$$

Remark 8.8. In order to avoid any singularity in the control input, the update law in (8.73) has to be modified using a projection algorithm, assuming that θ_2^* belongs to a known convex domain. We refer the reader to the foregoing subsection for details about how this domain may be calculated, and the stability analysis related to the projection. For the sake of clarity of this presentation, we do not introduce this modification here, although we know it is necessary for the implementaton of the algorithm.

- **Step 3:** At this stage our goal is partially reached, as we have defined signals $\tilde{q}_2$ and e_2 available on line. Now consider the function

$$V_{4a} = V_{3a} + \frac{1}{2}\det(M(q_1))e_2^T e_2 \tag{8.75}$$

We obtain

$$\begin{aligned} \dot{V}_{4a} \leq & -\lambda_1\dot{\tilde{q}}_1^T\dot{\tilde{q}}_1 - \lambda^2\lambda_1\tilde{q}_1^T\tilde{q}_1 + \tilde{q}_2^T\det(M(q_1))e_2 - \\ & -\tilde{q}_2^T\tilde{q}_2 + e_2^T[v - \dot{e}_{2d}] + e_2^T\frac{d}{dt}\left\{\frac{\det(M(q))}{2}\right\}e_2 \end{aligned} \tag{8.76}$$

Notice that

$$-\det(M(q_1))\dot{e}_{2d} + \frac{d}{dt}\left\{\frac{\det(M(q_1))}{2}\right\} e_2 = Y_4(q_1, \dot{q}_1, q_2, \dot{q}_2)\theta_4^* \tag{8.77}$$

for some Y_4 and θ_4^* matrices of suitable dimensions. Let us denote this time $\det(M) = Y_2(q_1)\theta_5^*$ (this is strictly equal to $Y_2(q_1)\theta_2^*$ defined above, but we choose a different notation because the estimate of θ_5^* is going to be chosen differently). Let us choose $v = -\tilde{q}_2 + w$ and

$$Y_2\hat{\theta}_5 w = -Y_4\hat{\theta}_4 - e_2 \tag{8.78}$$

We obtain

$$\dot{V}_{4a} \leq -\lambda_1 \dot{\tilde{q}}_1^T \dot{\tilde{q}}_1 - \lambda^2\lambda_1 \tilde{q}_1^T \tilde{q}_1 - \tilde{q}_2^T \tilde{q}_2 - e_2^T w Y_2 \tilde{\theta}_5 + e_2^T Y_4 \tilde{\theta}_4 - e_2^T e_2 \tag{8.79}$$

Finally we choose as a Lyapunov function for the whole closed-loop system

$$V = V_{4a} + \frac{1}{2}\tilde{\theta}_4^T\tilde{\theta}_4 + \frac{1}{2}\tilde{\theta}_5^T\tilde{\theta}_5 \tag{8.80}$$

and the following update laws:

$$\dot{\hat{\theta}}_4(t) = -Y_4^T(q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t))e_2(t) \tag{8.81}$$

$$\dot{\hat{\theta}}_5(t) = -Y_2^T(q_1(t))w^T(t)e_2(t) \tag{8.82}$$

(a projection algorithm has to be applied to $\hat{\theta}_5$; see Remark 8.8 above). We obtain

$$\dot{V} \leq -\lambda_1 \dot{\tilde{q}}_1^T \dot{\tilde{q}}_1 - \lambda^2\lambda_1 \tilde{q}_1^T \tilde{q}_1 - \tilde{q}_2^T \tilde{q}_2 - e_2^T e_2. \tag{8.83}$$

We therefore conclude that $\hat{\theta} \in \mathcal{L}_\infty$, $\tilde{q}_2$, e_2, $\tilde{q}_1, s_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $q_2 \in \mathcal{L}_\infty$, (see (8.63)) , $\dot{q}_2 \in \mathcal{L}_\infty$. Finally from the definition of s_1 and Theorem 4.10 we conclude that $\tilde{q}_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\dot{\tilde{q}}_1 \in \mathcal{L}_2$ and $\tilde{q}_1 \to 0$ as $t \to +\infty$.

To conclude this subsection, one may say that the backstepping procedure does not bring much more than the passivity-based one to the adaptive control problem for flexible joint Lagrangian systems. The fact that the fictitious input q_{2d} is premultiplied by an unknown term K, creates a difficulty that has been solved in [318] but has never been tackled in the "backstepping" literature. The linearity-in-the-parameters problem solution also is an original one, motivated by the physics of the process, and whose solution also was proposed in [318] and nowehere else, to the best of the authors' knowledge.

8.2 Linear Invariant Systems

The problem of adaptive control of linear invariant systems has been a very active field of research since the beginning of the 1960s. Two paths have been followed: the indirect approach which consists of estimating the process parameters, and using those estimated values into the control input, and the direct approach that we described in the introduction of this chapter. The direct approach has many attractive features, among them the nice passivity properties of the closed-loop system, which actually is a direct consequence of Lemma 7.23. This is what we develop now.

8.2.1 A Scalar Example

Before passing to more general classes of systems, let us reconsider the following first order system similar to the one presented in Subsection 1.4:

$$\dot{x}(t) = a^* x(t) + b^* u(t) \tag{8.84}$$

where $x(t) \in I\!R$, a^* and b^* are constant parameters, and $u(t) \in I\!R$ is the input signal. The control objective is to make the state $x(\cdot)$ track some desired signal $x_d(\cdot) \in I\!R$ defined as follows:

$$\dot{x}_d(t) = -x_d(t) + r(t) \tag{8.85}$$

where $r(\cdot)$ is some time function. Let us assume first that a^* and b^* are known to the designer and define the tracking error as $e = x - x_d$. Then it is easy to see that the input

$$u = \frac{1}{b^*}(r - (a^* + 1)x) \tag{8.86}$$

forces the closed-loop to behave like $\dot{e}(t) = -e(t)$ so that $e(t) \to 0$ as $t \to +\infty$. Let us assume now that a^* and b^* are unknown to the designer, but that it is known that $b^* > 0$. Let us rewrite the input in (8.86) as $u = \theta^{*T}\phi$, where $\theta^{*T} = (-\frac{a^*+1}{b^*}, \frac{1}{b^*})$ and $\phi^T = (x, r)$ are the vector of unknown parameters and the regressor, respectively. Clearly it is possible to rewrite the error dynamics as

$$\dot{e}(t) = -e(t) + b^* \left(-\theta^{*T}(t)\phi(t) + u(t)\right) \tag{8.87}$$

Since the parameters are unknown, let us choose (following the so-called *certainty equivalence principle*, which is not a principle but mainly a heuristic method) the control as

$$u(t) = \hat{\theta}^T(t)\phi(t) \tag{8.88}$$

where $\hat{\theta}^T = (\hat{\theta}_1, \hat{\theta}_2)$ is a vector of control parameters to be estimated online. Notice that we intentionally do not impose any structure on these parameters, since they are not meant to represent the plant parameters, but the control parameters: this is what is called *direct* adaptive control. An *indirect* adaptive scheme would aim at estimating the plant parameters and then introducing these estimates in the control input: this is not the case in what we shall describe in this part of the book. Introducing (8.88) into (8.87) we obtain

$$\dot{e}(t) = -e(t) + b^* \tilde{\theta}^T(t)\phi(t) \tag{8.89}$$

where $\tilde{\theta} = \hat{\theta} - \theta^*$. The reader may have a look now at (8.2) and (8.7) to guess what will follow. The dynamics in (8.89) may be rewritten as $[e](s) = \frac{1}{1+s} b^* [\tilde{\theta}^T \phi](s)$, where $[\cdot](s)$ denotes the Laplace transform and $s \in \mathbb{C}$. Consequently a gradient estimation algorithm should suffice to enable one to analyze the closed-loop scheme with the passivity theorem, since $\frac{b^*}{1+s}$ is SPR. Let us choose

$$\dot{\hat{\theta}}(t) = -\phi(t)e(t) \tag{8.90}$$

As shown in Subsection 4.3.1, this defines a passive operator $e \mapsto -\tilde{\theta}^T \phi$. The rest of the stability analysis follows as usual (except that since we deal here with a time-varying system, one has to resort to Barbalat's Lemma to prove the asymptotic convergence of $e(\cdot)$ towards 0. The zero state detectability property plus Krasovskii-La Salle invariance Lemma do not suffice so that the various results exposed in Section 5.1 cannot be directly applied).

Remark 8.9.
- The system in (8.85) is called a model of reference, and this adaptive technique approach is called the Model Reference Adaptive Control MRAC, a term coined by Landau [274].
- One can easily deduce the storage functions associated to each subsystem and form a Lyapunov candidate function for the overall closed-loop scheme.
- One may also proceed with a Lyapunov function analysis, and then retrieve the passivity interpretation using the results in Subsection 7.3.3.
- We have supposed that $b^* > 0$. Clearly we could have supposed $b^* < 0$. However when the sign of b^* is not known, then the design becomes much more involved. A solution consists of an indirect adaptive scheme with a modified estimation algorithm [311]. The above passivity design is lost in such schemes.

8.2.2 Systems with Relative Degree $r = 1$

Let us consider the following controllable and observable system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \\ y(t) = C^T x(t) \end{cases} \tag{8.91}$$

with $u(t) \in I\!R$, $y(t) \in I\!R$, $x(t) \in I\!R^n$, whose transfer function is given by

$$H(s) = k\frac{B(s)}{A(s)} = C^T(sI_n - A)^{-1}B \tag{8.92}$$

where s is the Laplace variable. The constant k is the high-frequency gain of the system, and we assume in the following that

- $k > 0$
- $A(s)$ and $B(s)$ are monic polynomials, and $B(s)$ is Hurwitz (the system has strictly stable zero dynamics), with known order $m = n - 1$

The problem is basically that of cancelling the dynamics of the process with a suitable dynamic output feedback in order to get a closed-loop system whose dynamics matches that of a given *reference model* with input $r(t)$ and output $y_m(t)$. The reference model transfer function is given by

$$H_m(s) = k_m \frac{B_m(s)}{A_m(s)} \tag{8.93}$$

where $H_m(s)$ is chosen as a SPR transfer function.

The control problem is that of output tracking, *i.e.* one desires to find out a differentiator-free dynamic output feedback such that all closed-loop signals remain bounded, and such that $\lim_{t\to+\infty} |y(t) - y_m(t)| = 0$. It is clear that one chooses $r(t)$ bounded so that $y_m(t)$ is. Due to the fact that the parameters of the polynomials $A(s)$ and $B(s)$ as well as k are unknown, the exact cancellation procedure cannot be achieved. Actually the problem can be seen as follows: in the ideal case when the process parameters are known, one is able to find out a dynamic output controller of the following form

$$\begin{cases} u(t) = \theta^T \phi((t)r, \omega_1^T(t), y(t), \omega_2^T(t)) & \\ \dot{\omega}_1(t) = \Lambda\omega_1(t) + bu(t), & \dot{\omega}_2(t) = \Lambda\omega_2(t) + by(t) \\ \phi^T = [r, \omega_1^T, y, \omega_2^T], & \theta^T = [k_c, \theta_1, \theta_0, \theta_2] \end{cases} \tag{8.94}$$

with $\omega_1(t)$, θ_1, θ_2 and $\omega_2(t) \in I\!R^{n-1}$, $\theta_0 \in I\!R$, and (Λ, b) is controllable. One sees immediately that u in (8.94) is a dynamic output feedback controller with a feedforward term. The set of gains $[k, \theta_1, \theta_0, \theta_2]$ can be properly chosen such that the closed-loop transfer function is

$$H_0(s) = \frac{k_c k B(s)\lambda(s)}{(\lambda(s) - C(s))A(s) - kB(s)D(s)} = H_m(s) \tag{8.95}$$

where the transfer function of the feedforward term is given by $\frac{\lambda(s)}{\lambda(s)-C(s)}$ while that of the feedback term is given by $\frac{D(s)}{\lambda(s)}$. $C(s)$ has order $n-2$ and $D(s)$ has order $n-1$. Notice that $\lambda(s)$ is just the characteristic polynomial of the matrix Λ, *i.e.* $\lambda(s) = (sI_{n-1} - \Lambda)^{-1}$ and is therefore Hurwitz. We do not develop further the model matching equations here (see *e.g.* [370] or [436] for details). Let us just denote the set of "ideal" controller parameters such that (8.95) holds as θ^*. In general those gains will be combinations of the process parameters. Let us now write down the state space equations of the whole system. Notice that we have

$$\dot{z}(t) \stackrel{\Delta}{=} \begin{bmatrix} \dot{x}(t) \\ \\ \dot{\omega}_1(t) \\ \\ \dot{\omega}_2(t) \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ \\ 0 & \Lambda & 0 \\ \\ bC^T x(t) & 0 & \Lambda \end{bmatrix} z(t) + \begin{bmatrix} B \\ \\ b \\ \\ 0 \end{bmatrix} u(t) \tag{8.96}$$

from which one deduces using (8.94) that

$$\dot{z}(t) = \begin{bmatrix} A + B\theta_0^{*T} C^T & B\theta_1^{*T} & B\theta_2^{*T} \\ \\ b\theta_0^{*} C^T & \Lambda + b\theta_1^{*T} & b\theta_2^{*T} \\ \\ bC^T & 0 & \Lambda \end{bmatrix} z(t) + \begin{bmatrix} Bk^* \\ \\ bk^* \\ \\ 0 \end{bmatrix} r(t) \tag{8.97}$$

Now since the process parameters are unknown, so is θ^*. The controller in (8.94) is thus replaced by its estimated counterpart, *i.e.* $u = \hat{\theta}\phi$. This gives rise to exactly the same closed-loop structure as in (8.97), except that θ^* is replaced by $\hat{\theta}$. Notice that the system in (8.97) is not controllable nor observable, but it is stabilizable and detectable. Also its transfer function is exactly equal to $H_0(s)$ when the input is $r(t)$ and the output is y. This is therefore a SPR transfer function.

Now we have seen in the manipulator adaptive control case that the classical way to proceed is to add and substract $\theta^{*T}\phi$ in the right-hand-side of (8.97) in order to get (see (8.96) and (8.97)) a system of the form

$$\dot{z}(t) = A_m z(t) + B_m \tilde{\theta}^T(t)\phi(t) + B_m k^* r(t) \tag{8.98}$$

where A_m is given in the right-hand-side of (8.97) while B_m is in the right-hand-side of (8.96) (actually in (8.97) the input matrix is given by $B_m k^*$). We are now ready to set the last step of the analysis: to this end notice that we can define the same type of dynamical structure for the reference model as the one that has been developed for the process. One can define filters of the input $r(t)$ and of the output $y_m(t)$ similarly to the ones in (8.94). Let us denote their state as $\omega_{1m}(\cdot)$ and $\omega_{2m}(\cdot)$, whereas the total reference model state will be denoted as $z_m(\cdot)$. In other words one is able to write

$$\dot{z}_m(t) = A_m z_m(t) + B_m k^* r(t) \tag{8.99}$$

Defining $e(t) = z(t) - z_m(t)$ and introducing (8.99) into (8.98) one gets the following error equation:

$$\dot{e}(t) = A_m e(t) + B_m \tilde{\theta}^T(t)\phi(t) \tag{8.100}$$

This needs to be compared with (8.7) and (8.2). Let us define the signal $e_1 = C_m^T e = C^T(x - x_m)$: clearly the transfer function $C_m^T(sI_{3n-2} - A_m)^{-1}B_m$

is equal to $H_m(s)$ which is SPR by definition. Hence the subsystem in (8.100) is strictly passive with input $\tilde{\theta}^T \phi$ and output e_1 (in the sense of Lemma 4.84) and is also output strictly passive since it has relative degree $r = 1$ (see Example 4.62). A gradient estimation algorithm of the form

$$\dot{\tilde{\theta}}(t) = -\lambda_1 \phi(t) e_1(t) \tag{8.101}$$

where $\lambda_1 > 0$, is passive with respect to the supply rate $u_2 y_2$ with $y_2 = -\tilde{\theta}^T \phi$ and $u_2 = e_1$. Due to the stabilizability properties of the first block in (8.100), it follows from the Meyer-Kalman-Yakubovich Lemma that the overall system is asymptotically stable. Indeed there exists a storage function $V_1(e) = e^T P e$ associated to the first block, and such that $V(e, \tilde{\theta}) = V_1(x) + \frac{1}{2}\tilde{\theta}^T \tilde{\theta}$ is a Lyapunov function for the system in (8.100) and (8.101), *i.e.* one gets $\dot{V} = -e^T q q^T e \leq 0$. Notice that in general the closed-loop system is not autonomous, hence the Krasovskii-La Salle Theorem does not apply. One has to resort to Barbalat's Lemma (see the Appendix) to prove the asymptotic convergence of the tracking error e towards 0. Notice also that the form of $\dot{V}$ follows from a CTCE so that Lemma 7.23 directly applies.

8.2.3 Systems with Relative Degree $r = 2$

Let us now concentrate on the case when the plant in (8.91) and (8.92) has relative degree two. Let us pass over the algebraic developments that allow one to show that there is a controller such that when the process parameters are known, then the closed-loop system has the same transfer function as the model reference. Such a controller is a dynamic output feedback of the form $u = \theta^{*T} \phi$. It is clear that one can repeat exactly the above relative degree one procedure to get a system as in (8.100) and (8.101). However this time $H_m(s)$ cannot be chosen as a SPR transfer function, since it has relative degree two! Thus the interconnection interpretation through the passivity theorem no longer works. The basic idea is to modify the input u so that the transfer function between the estimator output and the first block output e_1 is no longer $H_m(s)$ but $(s+a)H_m(s)$ for some $a > 0$ such that $(s+a)H_m(s)$ is SPR. To this end let us define a filtered regressor $\bar{\phi} = \frac{1}{s+a}[\phi]$, i.e. $\dot{\bar{\phi}} + a\bar{\phi} = \phi$. Since we aim at obtaining a closed-loop system such that $e_1 = H_m(s)(s+a)\tilde{\theta}^T \bar{\phi}$, let us look for an input that realizes this goal:

$$
\begin{aligned}
e_1 &= H_m(s)(s+a)\tilde{\theta}^T\bar{\phi} \\
&= H_m(s)[\dot{\tilde{\theta}}^T\bar{\phi} + \tilde{\theta}^T\dot{\bar{\phi}} + a\tilde{\theta}^T\bar{\phi}] \\
&= H_m(s)[\dot{\tilde{\theta}}^T\bar{\phi} + \tilde{\theta}^T(\phi - a\bar{\phi}) + a\tilde{\theta}^T\bar{\phi}] \\
&= H_m(s)[\dot{\tilde{\theta}}^T\bar{\phi} + \tilde{\theta}^T\phi] \\
&= H_m(s)[u - \theta^{*T}\phi]
\end{aligned}
\tag{8.102}
$$

It follows that a controller of the form

$$
u(t) = \dot{\hat{\theta}}^T(t)\bar{\phi}(t) + \hat{\theta}^T(t)\phi(t) \tag{8.103}
$$

will be suitable. Indeed one can proceed as for the relative degree one case, *i.e.* add and substract $\theta^{*T}\phi$ to u in order to get $\dot{z}(t) = A_m z(t) + B_m(\dot{\tilde{\theta}}^T(t)\bar{\phi}(t) + \tilde{\theta}^T(t)\phi(t))$ such that the transfer function between $\tilde{\theta}^T\bar{\phi}$ and e_1 is $H_m(s)(s+a)$. Then the update law can be logically chosen as

$$
\dot{\tilde{\theta}}(t) = -\lambda_1\bar{\phi}(t)e_1(t) \tag{8.104}
$$

(compare with (8.101)), and the rest of the proof follows.

8.2.4 Systems with Relative Degree $r \geq 3$

The controller in (8.103) is implementable without differentiation of the plant output y because the derivative $\dot{\hat{\theta}}$ is available. The extension of the underlying idea towards the case $r \geq 3$ would imply it is possible to have at one's disposal an estimation algorithm that provides the higher order derivatives of the estimates: this is not the case of a simple gradient update law. The relative degree problem has been for a long time a major obstacle in direct adaptive control theory. The next two paragraphs briefly present two solutions: the first one uses the backstepping method that we already used in Subsection 7.5.2 to derive a globally stable tracking controller for the flexible joint-rigid link manipulators. It was presented in [267]. The second method is due to Morse [361]. It can be considered as an extension of the controllers in Subsections 8.2.2 and 8.2.3. It is based on the design of update laws which provide $\hat{\theta}$ as well as its derivatives up to the order $r-1$. In the following we shall restrict ourselves to the presentation of the closed-loop error equations: the whole developments would take us too far.

The Backstepping Approach

Given a plant defined as in (8.91), $r = n - m$, it is possible to design $u(t)$ such that the closed-loop system becomes

$$\begin{cases} \dot{z}(t) = A(z(t), t, \Gamma)z(t) + b(z(t), t, \Gamma)(\omega^T(t)\tilde{\theta}(t) + \varepsilon_2) \\ \dot{\tilde{\theta}}(t) = -\Gamma\omega b^T(t)(z(t), t, \Gamma)z(t) \\ \dot{\varepsilon}(t) = A_0\varepsilon(t) \\ \dot{\tilde{\eta}}(t) = A_0\tilde{\eta}(t) + e_n z_1(t) \\ \dot{\tilde{\zeta}}(t) = A_b\tilde{\zeta}(t) + \bar{b}z_1(t) \end{cases} \tag{8.105}$$

where $\tilde{\theta}(t) \in I\!R^{(m+n)\times 1}$, $\omega(t) \in I\!R^{(m+n)\times 1}$, $\bar{b} \in I\!R^{m\times 1}$, $b \in I\!R^{r\times 1}$, $z(t) \in I\!R^{r\times 1}$, $e_n \in I\!R^{n\times 1}$ and is the $n-$th coordinate vector in $I\!R^n$, , $\tilde{\eta}(t) \in I\!R^{n\times 1}$, $\tilde{\zeta}(t) \in I\!R^{m\times 1}$. $z_1(t)$ is the first component of $z(t)$ and $z_1(t) = y(t) - y_r(t)$ is the tracking error, $y_r(t)$ is the reference signal; all other terms come from filtered values of the input $u(t)$ and the output $y(t)$. A_b and A_0 are stable matrices. What is important in the context of our study is that the closed-loop system in (8.105) can be shown to be stable using the function

$$V(z, \varepsilon, \tilde{\theta}, \tilde{\eta}, \tilde{\zeta}) = V_z(z) + V_\varepsilon(\varepsilon) + V_{\tilde{\theta}}(\tilde{\theta}) + V_{\tilde{\eta}}(\tilde{\eta}) + V_{\tilde{\zeta}}(\tilde{\zeta}) \tag{8.106}$$

whose time derivative along trajectories of (8.105) is

$$\dot{V} \le -\sum_{i=1}^{n^*} \lambda_i z_i^2 - \lambda_\varepsilon \varepsilon^T \varepsilon - \lambda_{\tilde{\eta}} \tilde{\eta}^T \tilde{\eta} - \lambda_{\tilde{\zeta}} \tilde{\zeta}^T \tilde{\zeta} \tag{8.107}$$

with $V_z(\cdot)$,$V_\varepsilon(\cdot)$,$V_{\tilde{\theta}}(\cdot)$,$V_{\tilde{\eta}}(\cdot)$,$V_{\tilde{\zeta}}(\cdot)$ positive definite functions, $\lambda_i > 0$, $1 \le i \le$, $\lambda_\varepsilon > 0$, $\lambda_{\tilde{\eta}} > 0$, $\lambda_{\tilde{\zeta}} > 0$.

Now let us have a look at the equations in (8.105): note that we can rewrite the closed-loop system similarly as in (7.47) and (7.48) as follows ($\bar{e}_1$ is the first component vector in $I\!R^r$):

$$\begin{pmatrix} \dot{z}(t) \\ \dot{\tilde{\eta}}(t) \\ \dot{\tilde{\zeta}}(t) \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ e_n\bar{e}_1^T & A_0 & 0 \\ \bar{b}\bar{e}_1^T & 0 & A_b \end{pmatrix} \begin{pmatrix} z(t) \\ \tilde{\eta}(t) \\ \tilde{\zeta}(t) \end{pmatrix} + \begin{pmatrix} b\omega^T\tilde{\theta}(t) \\ 0_n \\ 0_m \end{pmatrix} + \begin{pmatrix} b\varepsilon_2(t) \\ 0_n \\ 0_m \end{pmatrix} \tag{8.108}$$

$$\dot{\tilde{\theta}}(t) = -\Gamma\omega(t)b^T z(t). \tag{8.109}$$

We can thus directly conclude from Lemma 7.23 that the closed-loop system can be transformed into a system in $\mathcal{P}$ [5]. With the notations of the preceding section, we get $V_1 = V_{\tilde{\theta}} = \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}$, $V_2 = V_z + V_{\tilde{\eta}} + V_{\tilde{\zeta}}$, $y_2 = -u_1 = z$, $y_1 = u_2 = b\omega^T\tilde{\theta}$,

[5] ε_2 can be seen as a $\mathcal{L}_2$-bounded disturbance and is therefore not important in our study.

$g_1{=}\Gamma\omega b^T$ $(\frac{\partial V_{\tilde\theta}}{\partial\tilde\theta}={\Gamma}^{-1}\tilde\theta)$, $g_2{=}\begin{pmatrix} I_r \\ 0_{n\times r} \\ 0_{m\times r}\end{pmatrix}$, $(\frac{\partial V_2}{\partial x_2}{=}\begin{pmatrix} \frac{\partial V_z}{\partial z} \\ \frac{\partial V_{\tilde\eta}}{\partial\tilde\eta} \\ \frac{\partial V_{\tilde\zeta}}{\partial\tilde\zeta}\end{pmatrix})$. The CTCE is verified as $\frac{\partial V_1}{\partial x_1}^T g_1 u_1{=}-\frac{\partial V_2}{\partial x_2}^T g_2 u_2{=}-z^T b\omega^T\tilde\theta$.

Morse's High Order Tuners

Similarly to the preceding case, we only present here the closed-loop equations without entering into the details on how the different terms are obtained. The interested reader can consult the original paper [361,383] for a comprehensive study of high order tuners. The closed-loop equations are the following:

$$\dot e(t) = -\lambda e(t) + q_0\tilde\theta^T(t)\omega(t) + q_0\sum_{i=1}^m \omega_i(t)\bar c z_i(t) + \varepsilon \tag{8.110}$$

$$\dot z_i(t) = \bar A z_i(t)(1+\mu\omega_i^2(t)) - \mathrm{sign}(q_0)\bar A^{-1}\bar b\omega_i(t)e(t), i\in\mathbf{m} \tag{8.111}$$

$$k_i(t) - h_i(t) = \bar c z_i(t), i\in\mathbf{m} \tag{8.112}$$

$$\dot{\tilde\theta}(t) = -\mathrm{sign}(q_0)\omega(t)e(t) \tag{8.113}$$

where $\mathbf{m} = \{1,...,m\}$, e is the scalar tracking error, $\lambda > 0$, q_0 is the high frequency gain of the open-loop system, $|q_0| \le \bar q$, $k\in I\!R^m$ is the vector of estimated parameters to be tuned, $h(\cdot)$ is an internal signal of the high order tuner, $\tilde\theta = h - q_P$, $q_P \in I\!R^m$ is a vector of unknown parameters, $(\bar c, \bar A, \bar b)$ is the minimal realization of a stable transfer function, $\omega\in I\!R^m$ is a regressor, and ε is an exponentially decaying term due to non-zero initial conditions. k_i and h_i denote the ith component of k and h respectively, whereas μ is a constant satisfying $\mu > \frac{2m\bar q\|\bar c^T\|.\|P\bar A^{-1}\bar b\|}{\lambda}$. It is proved in [361] that the system in (8.110) through (8.113) is stable using the function

$$V(e,\tilde\theta,z) = e^2 + |q_0|\tilde\theta^T\tilde\theta + \delta\sum_{i=1}^m z_i^T P z_i \tag{8.114}$$

where $\bar A^T P + P\bar A = -I_m$, $\delta = \frac{\bar q\|\bar c^T\|}{\|P\bar A^{-1}\bar b\|}$. The time derivative of $V(\cdot)$ along trajectories of (8.110) through (8.113) is given by

$$\dot V(e,\tilde\theta,z) \le -\lambda^\star e^2 + \frac{1}{\lambda^\star}\varepsilon^2 \tag{8.115}$$

with $\lambda^\star = \lambda - \frac{2m\bar q\|\bar c^T\|.\|P\bar A^{-1}\bar b\|}{\mu}$. Now let us rewrite the system in (8.110) through (8.113) as follows:

$$\begin{pmatrix} \dot{e}(t) \\ \dot{z}_1(t) \\ \dot{z}_2(t) \\ \ldots \\ \dot{z}_m(t) \end{pmatrix} =$$

$$= \begin{pmatrix} -\lambda & q_0\omega_1\bar{c} & \ldots & \ldots\ \ldots & q_0\omega_m\bar{c} \\ -sgn(q_0)\bar{A}^{-1}\bar{b}\omega_1 & \bar{A}(1+\mu\omega_1^2) & 0 & \ldots\ \ldots & 0 \\ -sgn(q_0)\bar{A}^{-1}\bar{b}\omega_2 & 0 & \bar{A}(1+\mu\omega_2^2) & 0\ \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots\ \ldots & \ldots \\ -sgn(q_0)\bar{A}^{-1}\bar{b}\omega_m & 0 & \ldots & \ldots\ 0 & \bar{A}(1+\mu\omega_m^2) \end{pmatrix} \times \tag{8.116}$$

$$\times \begin{pmatrix} e(t) \\ z_1(t) \\ z_2(t) \\ \ldots \\ z_m(t) \end{pmatrix} + \begin{pmatrix} q_0\tilde{\theta}^T(t)\omega(t) \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix} + \begin{pmatrix} \varepsilon(t) \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

$$\dot{\tilde{\theta}}(t) = -\mathrm{sgn}(q_0)\omega(t)e(t) \tag{8.117}$$

We conclude from Corollary 4 that the system in (8.116) (8.117) belongs to $\mathcal{P}$, with $V_1 = |q_0|\tilde{\theta}^T\tilde{\theta}$, $V_2 = e^2 + \delta\sum_{i=1}^m z_i^T P z_i$, $g_1 = -q_0\omega$, $g_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$, $u_1 = -y_2 = -e$, $u_2 = y_1 = q_0\omega^T\tilde{\theta}$. (We can neglect ε in the analysis or consider it as a $\mathcal{L}_2$-bounded disturbance).

Comparing Euqtaions (8.108) and (8.109) and Equations (8.116) and (8.117) we conclude that the closed-loop error equations in both cases are very much similar. However, this similarity is limited to the closed-loop system stability analysis. First, the basic philosophies of each scheme are very different: Roughly speaking, the *high order tuners* philosophy aims at rendering the operator between the tracking error and the estimates strictly passive (using a control input that is the extension of "classical" certainty equivalent control laws), while preserving stability of the overall system with an appropriate update law. On the contrary, the *backstepping* method is based on the use of a very simple "classical" update law (a passive gradient), and the difficulty is to design a control input (quite different in essence from the certainty equivalent control laws) which guarantees stability. Second, notice that $\tilde{\theta}$ in (8.109) truly represents the unknown parameters estimates, while $\tilde{\theta}$ in (8.117) is the difference between the vector of unknown plant parameters and a signal $h(\cdot)$ internal to the high order update law (the control input being computed with the estimates k and their derivatives up to the plant relative degree minus one). Third, the tracking error in the *backstepping* scheme is part of a r-dimensional differential equation (see the first equation in (8.105)), while it is the solution of a first order equation in the *high order tuner* method (see (8.110)).

In [383], it is proved that the high order tuner that leads to the error equations in (8.110) through (8.113) defines a passive operator between the tracking error e and $(k - q_P)^T \omega$, and that this leads to nice properties of the closed-loop system, such as guaranteed speed of convergence of the tracking error towards zero. In [268], it has been shown that the backstepping method also possesses interesting transient performances. Such results tend to prove that the schemes that base on passivity properties possess nice closed-loop properties. Other types of adaptive controllers using passivity have been studied in [388].

9

Experimental Results

In this chapter we present experimental results on three experimental mechanical systems. They illustrate the applicability of the methodologies exposed in the foregoing chapters. The first set of experiments concerns flexible-joint manipulators, whose dynamics and control have been thoroughly explained. The second focuses on an underactuated system, the inverted pendulum, which does not fall into the classes of mechanical systems presented so far. The reader is referred to the introduction of Chapter 4 where a list of applications of passivity to control design is given.

9.1 Flexible Joint Manipulators

9.1.1 Introduction

The state feedback control problem of flexible joint manipulators has constituted an interesting challenge in the Systems and Control and in the Robotics scientific communities. It was motivated by practical problems encountered for instance in industrial robots equipped with harmonic drives, that may decrease the tracking performances, or even sometimes destabilize the closed-loop system. Moreover as we pointed out in the previous chapter, it represented at the end of the 1980s (twentieth century) a pure academic problem, due to the particular structure of the model. From a historical point of view, the main directions that have been followed to solve the tracking and adaptive control problems have been: singular perturbation techniques (the stability results then require a high enough stiffness value at the joints so that the stability theoretical results make sense in practice) [475, 476], and nonlinear global tracking controllers derived from design tools such as the backstepping or the passivity-based techniques. We have described these last two families of schemes in the previous chapter; see Sections 7.5 and 7.5.2. In this section we aim at illustrating on two laboratory processes how these schemes work in practice and whether they bring significant performance improvement with respect to PD and the Slotine and Li controllers (which can both be cast into the passivity-based schemes, but do not *a priori* incorporate flexibility effects in their design). What follows is taken from [79,80]. More generally the goal of this section is to present experimental results for passivity-based controllers with increasing complexity, starting from the PD input. Let us stress that the reliability of the presented experimental works is increased by the fact that theoretical and numerical investigations predicted reasonably well the obtained behaviours of the real closed-loop plants; see [78]. The experimental results that follow should not be considered as a definitive answer to the question: "What is the best controller?". Indeed the answer to such a question may be very difficult, possibly impossible to give. Our goal is only to show that the concepts that were presented in the previous chapters may provide good results in practice.

9.1.2 Controller Design

In this work the model as introduced in [471] is used; see (6.105). As we saw in Section 6.4 this model possesses nice passivity properties as well as a triangular structure that make it quite attractive for control design; see Sections 7.5, 7.5.2 and 7.6.1. Only fixed parameter controllers are considered here. As shown in [78] (see (7.123) and (7.140)), the three nonlinear controllers for flexible joint manipulators which are tested can be written shortly as follows:

Controller 1

$$\begin{cases} u = J[\ddot{q}_{2d} - 2\dot{\tilde{q}}_2 - 2\tilde{q}_2 - K(\dot{s}_1 + s_1)] + K(q_2 - q_1) \\ q_{2d} = K^{-1}u_R + q_1 \end{cases} \tag{9.1}$$

Controller 2

$$\begin{cases} u = J[\ddot{q}_{2d} - 2\dot{\tilde{q}}_2 - 2\tilde{q}_2 - (\dot{s}_1 + s_1)] + K(q_2 - q_1) \\ q_{2d} = K^{-1}u_R + q_1 \end{cases} \tag{9.2}$$

Controller 3

$$\begin{cases} u = J\ddot{q}_{2r} + K(q_{2d} - q_{1d}) - B_2 s_2 \\ q_{2d} = K^{-1}u_R + q_{1d} \end{cases} \tag{9.3}$$

where $u_R = M(q_1)\ddot{q}_{1r} + C(q_1, \dot{q}_1)\dot{q}_{1r} + g(q_1) - \lambda_1 s_1$ is as in (7.124). The signals $\dot{q}_{1r} = \dot{q}_{1d} - \lambda\tilde{q}_1$, $s_1 = \dot{\tilde{q}}_1 + \lambda\tilde{q}_1$ are the classical signals used in the design of this controller (the same definitions apply with subscript 2). Let us reiterate that the expressions in (9.1), (9.2) and (9.3) are equivalent closed-loop representations. In particular no acceleration measurement is needed for the implementation, despite the fact that $\dot{s}_1$ may appear in the equivalent form of u.

As pointed out in Remark 7.33, the last controller is in fact an improved version (in the sense that it is a static state feedback) of the dynamic state feedback proposed in [72, 318], that can be written as

$$\begin{cases} u = \quad J\ddot{q}_{2r} - K[q_{1d} - q_{2d} - \int_0^t (\lambda_1 \tilde{q}_1 - \lambda_2 \tilde{q}_2) d\tau] - \lambda_2 s_2 \\ q_{2d} = p[pI + \lambda_2]^{-1} \left\{ K^{-1} u_R + q_{1d} - \int_0^t (\lambda_1 \tilde{q}_1 - \lambda_2 q_2) d\tau \right\} \end{cases} \tag{9.4}$$

with $p \in \mathbb{C}$. This controller has not been considered in the experiments, because it is logically expected not to provide better results than its simplified counterpart: it is more complex, but based on the same idea. Controllers 1 and 2 are designed following a backstepping approach. The two backstepping controllers differ from the fact that in Controller 2, the joint stiffness K no longer appears before $\dot{s}_1 + s_1$ in the right-hand-side of the u-equation. This modification is expected to decrease significantly the input magnitude when K is large. This will indeed be confirmed experimentally.

In [78] these controllers have been commented and discussed from several points of views. Most importantly it was shown that when the joint stiffness grows unbounded (*i.e.* the rigid manipulator model is retrieved), then the only controller that converges to the rigid Slotine and Li control law is the passivity-based Controller 3 in (9.3). In this sense, it can be concluded that Controller 3 is the extension of the rigid case to the flexible joint case, which cannot be stated for the other two control laws. We believe that this elegant physical property plays a major role in the closed-loop behaviour of the plant. As shown in Section 7.5.2 the backstepping schemes presented here *do* possess some closed-loop passivity properties. However they are related to transformed coordinates, as the reader may see in Section 7.5.2. On the contrary, the passivity-based schemes possess this property in the original generalized coordinates $\tilde{q}$: consequently they are closer to the physical system than the other schemes. This is to be considered as an intuitive explanation of the good experimental results obtained with passivity-based schemes (PD, Slotine and Li, and Controller 3).

9.1.3 The Experimental Devices

This subsection is devoted to present the two experimental devices in detail: a planar two degree-of-freedom (dof) manipulator, and a planar system of two pulleys with one actuator. They are shown in photographs 9.34 and 9.35 respectively. We shall concentrate on two points: the mechanical structure and the real time computer connected to the process. Actually we focus essentially in this description on the first plant, that was a two dof planar manipulator of the Laboratoire d'Automatique de Grenoble, France, named Capri. The second process is much more simple and is depicted in Figure 9.1. It can be considered as an equivalent one dof flexible joint manipulator. Its dynamics is linear. Its physical parameters are given by: $I_1 = 0.0085$ kg.m^2, $I_2 = 0.0078$ kg.m^2, $K = 3.4$ Nm/rad.

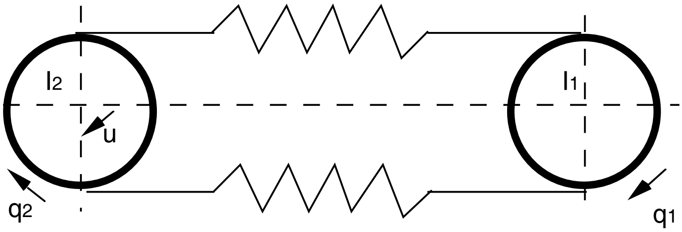

Fig. 9.1. A one dof flexible joint manipulator.

Mechanical Structure of the Capri Robot

The Capri robot is a planar mechanism constituted by two links, of respective lengths 0.16 and 0.27 m, connected by two hubs. The first link is an aluminium AU4G, U-frame to improve stiffness, with respect to the forearm which can be designed less rigid. The second link has a more peculiar structure because it supports the applied forces: It is designed as a pipe of diameter 0.05 m, and it is equipped with force piezo-electric sensors. The force magnitude, point of application and orientation can be measured and calculated. The sides of the forearm with Kistler quartz load washers can measure extension and compression forces, and the half-spherical extremity possesses a Kistler three components force transducer (only two of them are used) from which it is possible to calculate the magnitude and the orientation of the applied force. In this work these force measurement devices are not needed, since we are concerned by motion control only.

The robot arm is actuated by two DC motors located at the underside of the basement table (therefore the Capri robot is a parallel-drive manipulator for which the model in (6.105) is the "exact" one; see Remark 6.43). They are coupled to the links by reducers (gears and notched belts), each of them with ratio 1/50. The first motor (Infranor MX 10) delivers a continuous torque of 30 N.cm and a peak torque of 220 N.cm for a total weight of 0.85 kg. The second motor (Movinor MR 08) provides a continuous torque of 19 N.cm and a peak torque of 200 N.cm, for a weight of 0.65 kg. The drive arrangement is such that the weight is not boarded on the links, to increase speed. Both motors are equipped with a 500 pulses/turn incremental encoder and a DC tachometer making joint position q_2 and velocity $\dot{q}_2$ available for feedback. The position q_1 is measured by a potentiometer mounted on the last link. In the experiments the velocity $\dot{q}_1$ has been obtained by differentiating the position signal (a filtering action has been incorporated by calculating the derivative from one measurement every four only, *i.e.* every four sampling times).

The effective working area of the robot arm is bounded by sensors: an inductive sensor prevents the first arm from doing more than one turn, *i.e.* $q_{11} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (see Figure 9.2 for the definition of the angles). Two microswitches prevents the second arm from overlapping on the first one. They both inhibit the inverters (Infranor MSM 1207) controlling the DC motors.

Remark 9.1. The Capri robot has been modeled as a parallel-drive rigid-link robot, with the second joint elastic. It is clear that such a model is only a crude approximation of the real device. Some approximations may be quite justified, like the rigidity of the first joint and of the links. Some others are much more inaccurate.

i) The belt that couples the second actuator and the second joint is modeled as a spring with constant stiffness, which means that only the first mode of its dynamic response is considered.
ii) There is some clearance in the mechanical transmission (especially at the joints, due to the belts and the pulleys), and a serious amount of dry friction.
iii) The frequency inverters that deliver the current to the motors possess a nonsymmetric dead zone. Therefore, different amounts of current are necessary to start motion in one direction or the other.
iv) The value of $\dot{q}_1$ used in the algorithm and obtained by differentiating a potentiometer signal is noisy, despite a filtering action.
v) The inertial parameters have been calculated by simply measuring and weighting the mechanical elements of the arms. The second joint stiffness has been measured statically off-line. It has been found to be 50 Nm/rad. This value has been used in the experiments without any further identification procedure.
vi) Some saturation on the actuators currents has been imposed by software, for obvious safety reasons. Since nothing *a priori* guarantees stability when the inputs are saturated, the feedback gains have to be chosen so that the control input remains inside these limits.

Some of these approximations stem from the process to be controlled, and cannot be avoided (points **i**, **ii**, **iii**): this would imply modifying the mechanical structure. The measurement noise effects in **iv** could perhaps be avoided via the use of observers or of position dynamic feedbacks. However on one hand the robustness improvement is not guaranteed and would deserve a deep analytical study. On the other hand the structure of the obtained schemes would be significantly modified (compare for instance the schemes in Sections 7.3.4 and 7.4 respectively). A much more simple solution consists of replacing the potentiometer by an optical encoder. The saturation in **vi** is necessary to protect the motors, and has been chosen in accordance with the manufacturer recommendations and our own experience on their natural "robustness". The crude identification procedure in **v** has been judged sufficient, because the aim of the work was not to make a controller perform as well as possible in view of an industrial application, but rather to compare several controllers and to show that nonlinear control schemes behave well. In view of this the most important fact is that they be all tested with the same (acceptable) parameters values, *i.e.* if one controller proves to behave correctly with these set of parameters, do the others behave as well or not? Another problem is that of the choice of the control parameters, *i.e.* feedback gains. We will come back on this important point later.

Real-time Computer

A real-time computer was connected to both processes in the workshop of the Laboratoire d'Automatique de Grenoble. It consisted of a set of DSpace boards and a host PC. The PC is a HP Vectra running at 66 MHz with 8 Mo of RAM and a hard disk of 240 Mo. The DSpace system is made of:

- A DS 1002 floating-point processor board built around the Texas Instruments TMS/320C30 digital signal processor. This processor allows 32 bits floating point computation at 33 MFlops. A static memory of 128 K words of 32 bits is available on this board. A 2 K words dual-port RAM is used simultaneously by the host PC and the DSP.
- A DS 2002 multi-channel ADC board with 2 A/D 16 bits resolution converters (5 μs conversion time) and a 16 channel multiplexer for each converter.
- A DS 2001 D/A converter board comprising 5 parallel analog output channels with 12 bits DAC (3 μs conversion time)
- A DS 3001 incremental encoder board with 5 parallel input channels. A 4-fold pulse multiplication, a digital noise filter and a 24 bits width counter are used for each channel.
- A DS 4001 digitak I/O and timer board with 32 digital I/O lines configurable as inputs or outputs in groups of 8 lines.

All these boards are attached together by the 32 bits PHS-Bus at a 16 MB/sec transfer speed. They are located in a separate rack connected to the host PC by a cable between two adaptation boards.

The PC is used for developments and supervision of the application. Several softwares are available for the DSpace system:

- SED30 and MON30 are used to configure the hardware.
- C30 is the Texas Instruments Compiler for the TMS320C30.
- TRACE30W is a graphical real-time software which permits to display the selected variables of the application.

The application itself was made of two parts: The control algorithm runing on the DSP, sampled at 1 ms in our case, and the dialogue interface running on the PC which allows the operator to supervise the execution of the control through the dual port memory. To guarantee repeatability of the experiments, there was an initialization procedure that was to be activated each time the origins have been lost, or at the beginning of the experiments.

9.1.4 Experimental Results

In this section we present the experimental results obtained by implementing the three controllers described above on each plant. A PD controller as in (7.142), and the scheme in (7.68) have also been implemented, as if the manipulator had both joints rigid (i.e. one replaces q in (7.68) by q_2). This allows to dissociate clearly the effects of the nonlinearities (the reference trajectories have been chosen fast enough so that Coriolis and centrifugal effects are effective), from the effects of the flexibility (once the "rigid" controllers are implemented, one can see how the "flexible" ones improve the closed-loop behaviour, if they do). In the case of the linear system in Figure 9.1, the scheme in (7.68) reduces to a PD control.

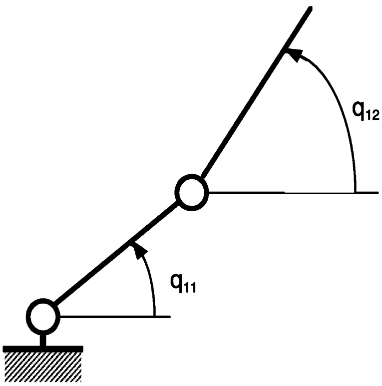

Fig. 9.2. Joint angles on the Capri robot

In order to perform the experiments, three different desired trajectories have been implemented for the Capri robot (see Figure 9.2 for the definition of the angles, due to the fact that the Capri robot is a parallel-drive manipulator):

- Desired trajectory 1: $q_{1d} = \begin{pmatrix} q_{11d} \\ q_{12d} \end{pmatrix} = \begin{pmatrix} 0.8\sin(ft) \\ -0.8\sin(ft) \end{pmatrix}$
- Desired trajectory 2: $q_{1d} = \begin{pmatrix} 0.4\sin(2ft) \\ 0.8\sin(ft) \end{pmatrix}$

- Desired trajectory 3: $q_{1d} = \begin{pmatrix} \frac{b^5}{(s+b)^5}[g(t)] \\ \\ -\frac{b^5}{(s+b)^5}[g(t)] \end{pmatrix}$

with $f = \omega(1 - \exp(-at))^4$, $a = 14$, $\omega = 9$ rad/s, $g(t)$ is a square function with magnitude 0.8 rad, period 5 s, and $b = 30$. The variable s is the Laplace transform variable. The choice for f allows one to smooth sufficiently the desired orbit to be tracked, as required by the theoretical developments. The other parameters values have been chosen so that the nonlinearities and the flexibilities effects are significant. Concerning the system in Figure 9.1, two desired trajectories have been chosen: $q_{1d} = sin(\omega t)$ and $q_{1d} = \frac{b^5}{(s+b)^5}[g(t)]$. The parameters ω and b have been varied as indicated in the figures captions. These time functions, which are sufficiently different to one another, have been chosen to permit to conclude about the capability of adaptation of the controllers to a modification of the desired motion. This is believed to constitute an important property in applications, since it dispenses the user from retuning the control gains between two different tasks. As a matter of fact, the following criteria have been retained to evaluate the performance of the controllers:

- The tracking error during the steady-state regime is an important parameter for performance evaluation. The quadratic errors sums $e_i = \int_{10}^{20} \tilde{q}_i^2(t)dt$ for each joint ($i = 1, 2$ for the Capri robot and $i = 3$ for the pulleys) and the maximum tracking error (pulleys) have been computed on-line.
- The shape and magnitude of the input signal.
- The capabilities of the various control schemes to provide an acceptable performance for any of the above desired motions, without having to retune the feedback gains.

The transient behaviour has not been included in this list. This will be explained from the observation of the experimental results. Let us emphasize that the presented results therefore concern two quite different plants (one nonlinear with high stiffness, the other one linear and with high flexibility), and with significantly different motions. They are consequently expected to provide an objective view of the capabilities of the various controllers.

Remark 9.2 (Feedback gains tuning method). Two methods have been employed to tune the gains. From a general point of view, one has to confess that one of the main drawbacks of nonlinear controllers such as backstepping and passivity-based ones, is that Lyapunov-like analysis does not provide the designer or the user with any acceptable way to tune the gains. The fact that increasing the gains accelerates the convergence of the Lyapunov function towards zero, is a nice theoretical result, that happens to be somewhat limited in practice.

Concerning the Capri robot, experiments were started with the first link fixed with respect to the base, *i.e.* with only the second link to be controlled.

The gains of the PD input were chosen from the second-order approximation obtained by assuming an infinite joint stiffness. From the fact that the Slotine and Li scheme in (7.68) mainly consists of a PD action plus a nonlinear part, these values have been used as a basis for the tuning of the gains λ and λ_1 in (7.68). The full-order system is linear of order 4 (a one degree-of-freedom flexible joint manipulator). The gains were tuned by essentially placing the closed-loop poles according to simple criteria like an optimal response time, nonoscillatory modes. In all cases, the desired trajectory 1 was used to determine a first set of gains. This provided a basis to choose the gains for the complete robot. Experiments were started with trajectory 1, and the gains were modified in real-time (essentially by increasing them in a heuristic manner) until the performance observed through the TRACE30W could no more be improved. Then trajectories 2 and 3 were tested, and the gains modified again if needed.

It has to be stressed that even in the linear case (like for the pulley-system), tuning the gains of such nonlinear controls is not evident. Indeed the gains appear quite nonlinearly in the state feedback, and their influence on the closed-loop dynamics is not obvious. For instance it is difficult to find a region in the gain space of the passivity-based controller in (9.3), such that the gains can be modified and at the same time the poles remain real.

In view of these limitations and of the lack of a systematic manner to calculate optimal feedback gains, advantage has been taken in [79] of the pulley-system linearity. Since this system is linear, the controllers in (9.1), (9.2) and (9.3) reduce to linear feedbacks of the form $u = Gx + h(t)$, where $h(t)$ accounts for the tracking terms. De Larminat [279] has proposed a systematic (and more or less heuristic) method to calculate the matrix G for LQ controllers. Actually one should notice that despite the fact that the nonlinear backstepping and passivity-based controllers have a linear structure when applied to a linear system, their gains appear in a very nonlinear way in the state feedback matrix G. As an example, the term multiplying q_1 for the scheme in (9.3) is equal to $-(\lambda\lambda_2 + k)\frac{\lambda_1\lambda}{k} + (\lambda_2 + I_2\lambda)\frac{\lambda I_1+\lambda_1}{I_1} + I_2\frac{\lambda_1\lambda}{I_1}$ (the gains λ_1 and λ_2 can be introduced in (7.123) and (7.124) respectively instead of using only one gain in both expressions, so that the passivity-based controller has three gains). The tuning method proposed in [279] that applies to LQ controllers allows one to choose the weighting matrices of the quadratic form to be minimized, in accordance with the desired closed-loop bandwidth (or cut-off frequency $\omega_c(CL)$). The advantages of this method are that the user focuses on one closed-loop parameter only to tune the gains, which is quite appreciable in practice. Therefore one gets an "optimal" state feedback matrix G_{LQ}, with a controller $u = G_{\mathrm{LQ}}x$ in the case of regulation. Since the various controllers used in the experiments yield some state feedback matrices G_{PD}, G_{BACK1}, G_{BACK2} and G_{MES} respectively, which are (highly) nonlinear functions of the gains as shown above, we choose to calculate their gains so that the norms $||G_{\mathrm{LQ}} - G_{\mathrm{CONT}}||$ are minimum. This amounts to solving a nonlinear set of equations $f(Z) = 0$ where Z is the vector of gains. This is in general a hard task, since we do not know *a priori* any root (otherwise the job would be done!). This has been done numerically by constructing a grid in the gain space of each scheme and minimizing the above norm with a standard optimization routine. The experimental results prove that the method may work well, despite possible improvements (especially in the numerical way to solve $f(Z) = 0$). Its extension towards the nonlinear case remains an open problem. ■

The quadratic error sums e_1, e_2 are reported in Tables 9.1 and 9.2. The error e_3 is in Table 9.3. The maximum tracking errors $|q_1 - q_d|_{\max}$ for the pulley-system are reported in Table 9.4. All the results for the pulley-system in Tables 9.3 and 9.4 concern the desired motion $q_{1d} = sin(\omega t)$. In each case the presented figures represent an average of several experiments. Concerning trajectories 2 and 3 in Tables 9.1 and 9.2, the results outside brackets have been obtained after having retuned the feedback gains. The ones in brackets have been obtained using the same gains as for trajectory 1. When they are not modified, it means that we have not been able to improve the results. A cross x indicates that no feedback gains have been found to stabilize the system.

The next results that concern the Capri robot are reported in Figures 9.3–9.21 and 9.33. The tracking errors $\tilde{q}_{11}$, $\tilde{q}_{12}$ and the inputs (currents) I_{c1} and I_{c2} at each motor, are depicted in Figures 9.3–9.13. Figures 9.14–9.21 contain results concerning the transient behaviour when the second link position tracking errors are initially of 0.4 rad. The inputs I_{c1} and I_{c2} are the calculated ones, not the true input of the actuators (they coincide as long as there is no saturation, *i.e.* $I_{c1} \leq 2$ A and $I_{c2} \leq 2$ A). The results concerning the pulley-system are in Figures 9.22–9.32. The signals $q_d(t)$ and $q_1(t)$ are shown in the upper boxes, and the torque input u is depicted in the lower boxes.

The following comments can be made:

Adaptation to the Desired Motion

The gains of the PD controller that correspond to the tests on the Capri robot, reported in Tables 9.1 and 9.2, are given in Table 9.5. They show that significant changes have been necessary from one desired motion to the next. One sees that the PD gains have had to be modified drastically to maintain a reasonable performance level. On the contrary it is observable from Tables 9.1 and 9.2 that even without any gain modification, the other controllers still perform well in general. In any case the modifications have seldom exceeded 50 % and concerned very few gains [80]. Since this is also true for the Slotine and Li controller, we conclude that the insensitivity of the performance with respect to desired motion changes is essentially due to the compensation of the nonlinearities.

The Slotine and Li controller seems to provide the most invariant performance with respect to the desired motion. This is especially apparent for

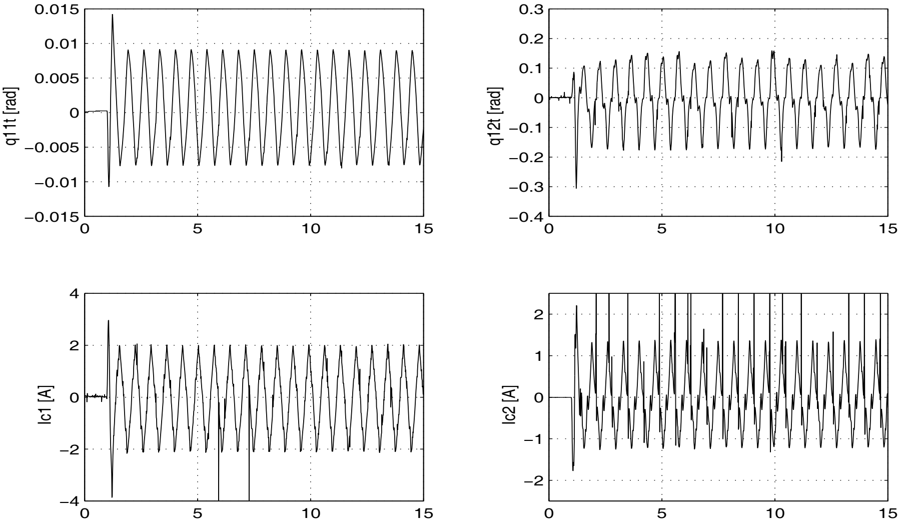

Fig. 9.3. PD controller, desired trajectory 1

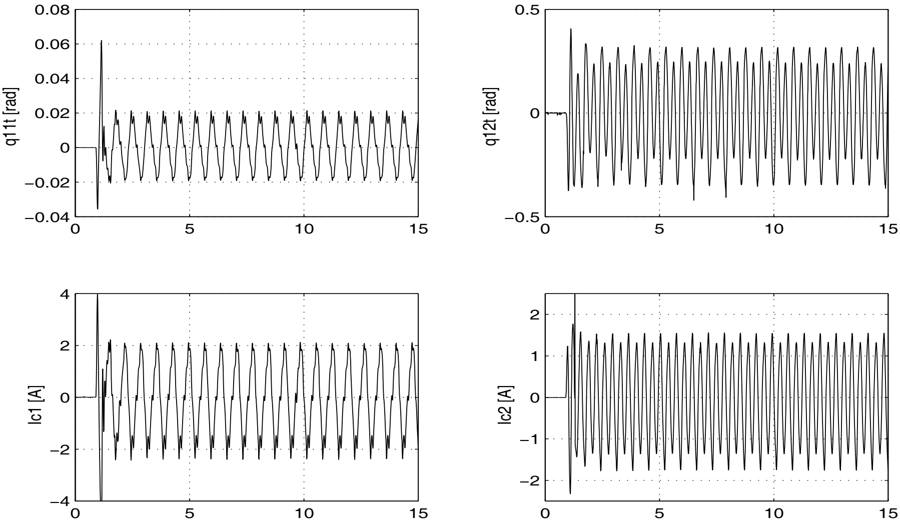

Fig. 9.4. PD controller, desired trajectory 2

trajectory 2 on the Capri experiments. In this case it provides the best error e_2, even after having retuned the gains for Controllers 2 and 3. This may be explained by the fact that the input in (7.68) is much smoother than the others (see Figure 9.7). This in turn may be a consequence of its simplicity, and from the fact that it does not use the noisy potentiometer signal.

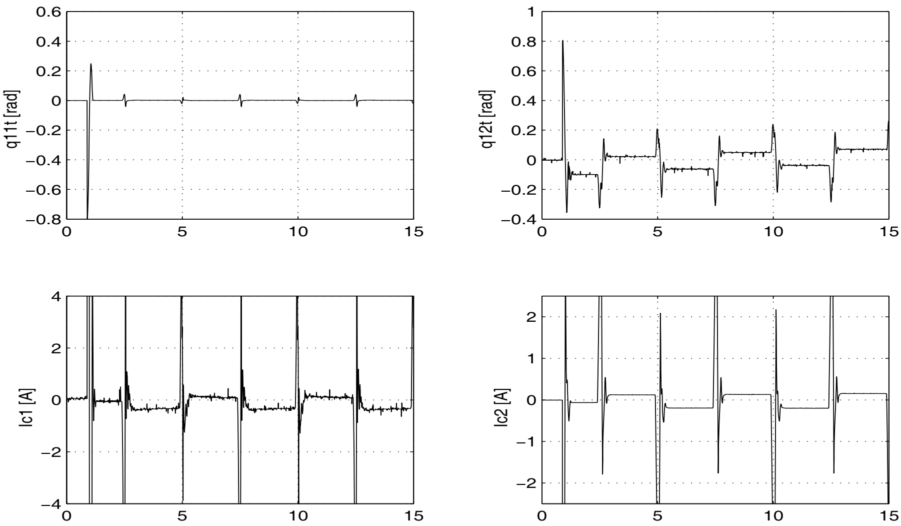

Fig. 9.5. PD controller, desired trajectory 3

Backstepping Controllers

For the Capri experiments, it has not been possible to find feedback gains that stabilize controller 1. On the contrary this has been possible for the pulley-system, see Figures 9.30, 9.23 and 9.26. This confirms the fact that the modification of the intermediate Lyapunov function (see (7.139) and (7.140)) may play a significant role in practice, and that the term $K(s_1 + \dot{s}_1)$ is a high-gain in the loop if K is large.

Compensation of Nonlinearities

Although the PD algorithm provides a stable closed-loop behaviour in all cases (for the Capri experiments and at the price of very large gain modifications as we pointed out above), its performance is poor for trajectories 1 and 2. The behaviour is much better for trajectory 3. This can be explained since this is almost a regulation task. The improvements obtained with the Slotine and Li scheme show that the Coriolis and centrifugal terms may play an important role depending on the desired motion.

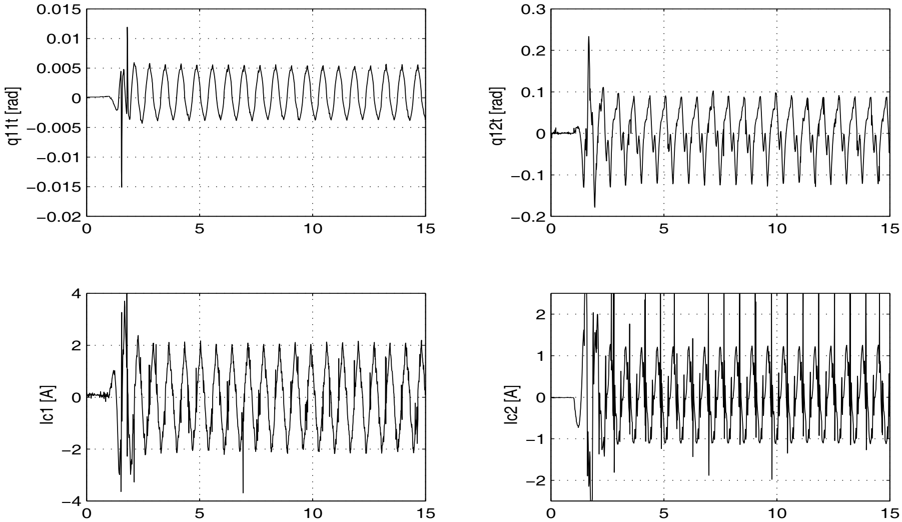

Fig. 9.6. SLI controller, desired trajectory 1

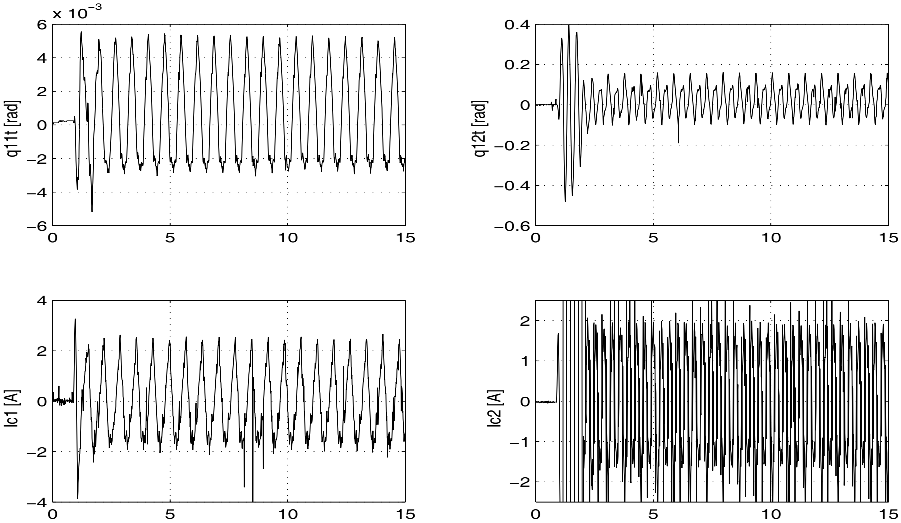

Fig. 9.7. SLI controller, desired trajectory 2

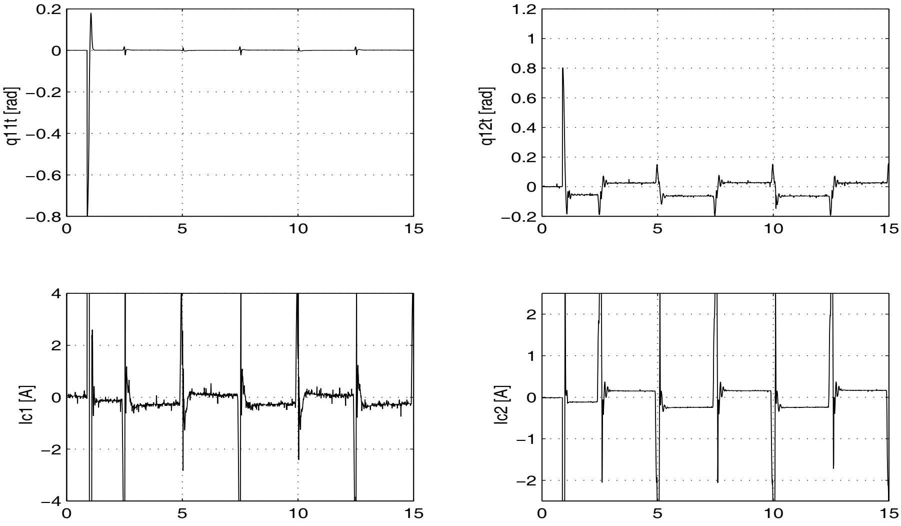

Fig. 9.8. SLI controller, desired trajectory 3

Compensation of Flexibilities

The PD and the Slotine and Li controls behave well for the Capri robot because the joint stiffness is large. The results obtained for the pulley-system show that the behaviour deteriorates a lot if K is small; see Tables 9.3 and 9.4.

Controller Complexity

The rather complex structure of the nonlinear Controllers 1, 2 and 3 is not an obstacle to their implementation with the available real-time computer described above. In particular recall that the acceleraton and jerk are estimated by inverting the dynamics (see Section (7.5)). Such terms have a complicated structure and depend on the system's physical parameters in a nonlinear way. Some experiments have shown that the sampling period (1 ms) could have been decreased to 0.5 ms.

Torque Input

The major problem that prevents certain controllers from behaving correctly is the input magnitude and shape. This has been noted above. The performance of Controllers 2 and 3 may be less good than that of the Slotine and Li algorithm, mainly because of the chattering in the input, inducing vibrations in the mechanical structure. Chattering is particularly present during the

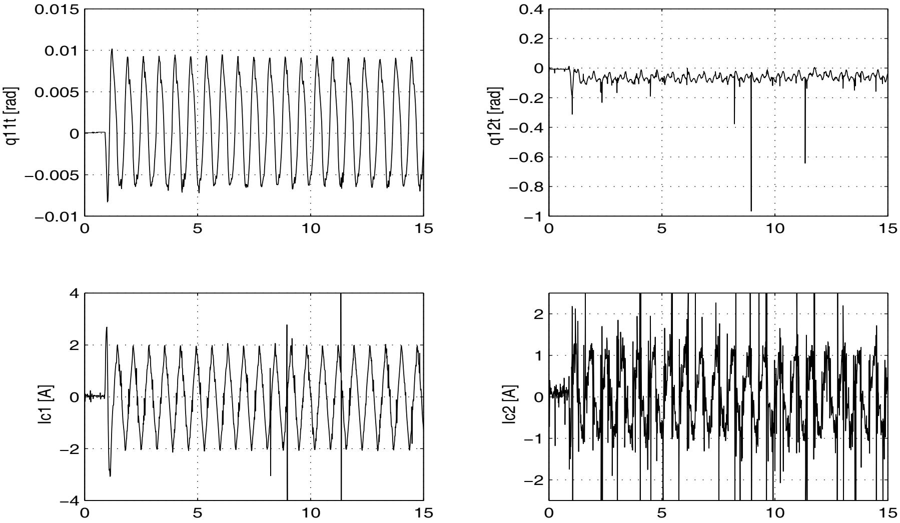

Fig. 9.9. Controller 2, desired trajectory 1

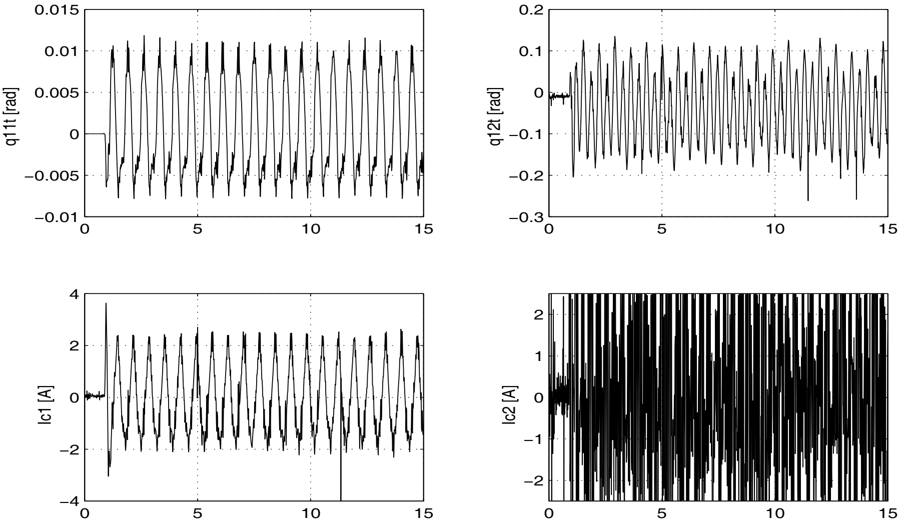

Fig. 9.10. Controller 2, desired trajectory 2

regulation phases in I_{c2} for trajectory 3 and Controllers 2 and 3; see Figures 9.11 and 9.13. On the contrary Figures 9.5 and 9.8 show smooth inputs. It may be expected from Figures 9.18–9.21 that a less noisy velocity $\dot{q}_1$ obtained from a better position measurement would bring the shape of I_{c2} close to the input in figures 9.16 and 9.17. Indeed they differ only in terms of chatter. One concludes that an optical encoder to measure q_1 would be a better solution.

Backstepping *vs* Passivity-based Controls

It is noteworthy that Controllers 2 and 3 possess quite similar closed-loop behaviours; see Figures 9.31 and 9.32, 9.24 and 9.25, 9.27 and 9.28 for the pulley-system, 9.9 and 9.33, 9.10 and 9.12, 9.11 and 9.13 for the Capri robot (although I_{c2} chatters slightly less for Controller 3, see Figures 9.9 and 9.33, and 9.11 and 9.13). The advantage of passivity-based methods is that the controllers are obtained in one shot, whereas the backstepping approach *a priori* leads to various algorithms. This can be an advantage (more degrees of freedom), but also a drawback as Controller 1 behaviour proves. Notice on figures 9.29, 9.30, 9.31 and 9.32 that Controllers 2 and 3 allow one to damp the oscillations much better than Controller 1 and the PD (it is possible that the PD gains could have been tuned in a better way for these experiments; see however the paragraph below on gain tuning for the pulley-system).

Transient Behaviour

The transient behaviour for the tracking error $\tilde{q}_{12}$ can be improved slightly when the flexibilities are taken into account in the controller design. This can be seen by comparing figures 9.6 and 9.7 with figures 9.9 and 9.10, 9.33 and 9.12. The tracking error tends to oscillate more for the Slotine and Li scheme than for the others. Notice that these results have been obtained with initial tracking errors close to zero. However the results in Figures 9.14–9.21 prove that the controllers respond quite well to initial state deviation. The transient duration is around 0.5 s for all the controllers. The tracking errors have a similar shape once the transient has vanished. The only significant difference is in the initial input I_{c2}. The torque is initially much higher for nonzero initial conditions.

Feedback Gains Tuning

The method described in remark 9.2 for tuning the gains in the case of the pulley-system provides good preliminary results. The gains that have been used in all the experiments for the pulley-system have not been modified during the tests on the real device to tentatively improve the results. They have been kept constant. This tends to prove that such a method is quite promising since it relies on the choice of a single parameter (the closed loop bandwidth, chosen as $\omega_c(CL) = 11$ rad/s in the experiments) and is therefore quite attractive for potential users.

The actuators and current drivers neglected dynamics may have a significant influence on the closed-loop behaviour. A close look at Tables 9.3 and 9.4 shows the existence of a resonance phenomenon in the closed-loop. This can be confirmed numerically by replacing u with $u_f = \frac{u}{1+\tau s}$ which allows one to suspect that this actuator neglected dynamics may play a crucial role in the loop. It might be then argued that developing velocity observers for such systems may not be so important, whereas some neglected dynamics, whose influence has received less attention in the literature, have a significant effect.

Remark 9.3. The peaks in the input I_{c2} for trajectory 1 are due to the saturation of the DC tachometers when the trajectory is at its maximum speed. When the saturation stops, the velocity signal delivered by the tachometers has a short noisy transient that results in such peaks in the input. However this has not had any significant influence on the performance, since such peaks are naturally filtered by the actuators (let us recall that the *calculated* inputs are depicted).

9.1.5 Conclusions

In this section we have presented experimental results that concern the application of passivity-based (PD, Slotine and Li, the controller in Subsection 7.5.1) and backstepping controllers, to two quite different laboratory plants which serve as flexible joint-rigid link manipulators. The major conclusion is that passivity-based controllers provide generally very good results. In particular the PD and Slotine and Li algorithms show quite good robustness and provide a high level of performance when the flexibility remains small enough. Tracking with high flexibility implies the choice of controllers which are designed from a model that incorporates the joint compliance. These experimental results illustrate nicely the developments of the foregoing chapter: one goes from the PD scheme to the one in Subsection 7.5.1 by adding more complexity, but always through the addition of new dissipative modules to the controller, and consequently to the closed-loop system. These three schemes can really be considered to belong to the same "family", namely passivity-based controllers. It is therefore not surprizing that their closed-loop behaviour when applied to real plants reproduces this "dissipative modularity": the PD works well when nonlinearities and flexibilities remain small enough, the Slotine and Li algorithm improves the robustness with respect to nonlinearities, and the scheme in Subsection 7.5.1 provides a significant advantage over the other two only if these two dynamical effects are large enough. Finally it is noteworthy that all controllers present a good robustness with respect to the uncertainties listed in Subsection 9.1.3.

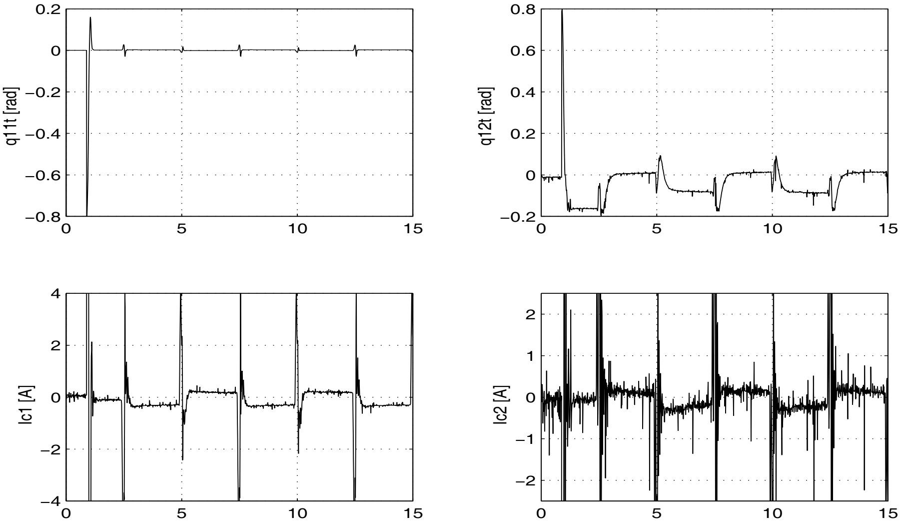

Fig. 9.11. Controller 2, desired trajectory 3

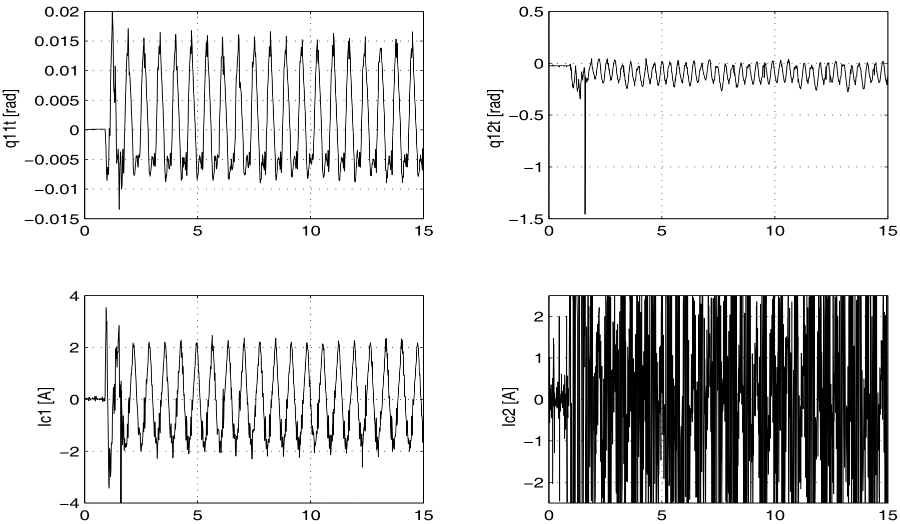

Fig. 9.12. Controller 3, desired trajectory 2

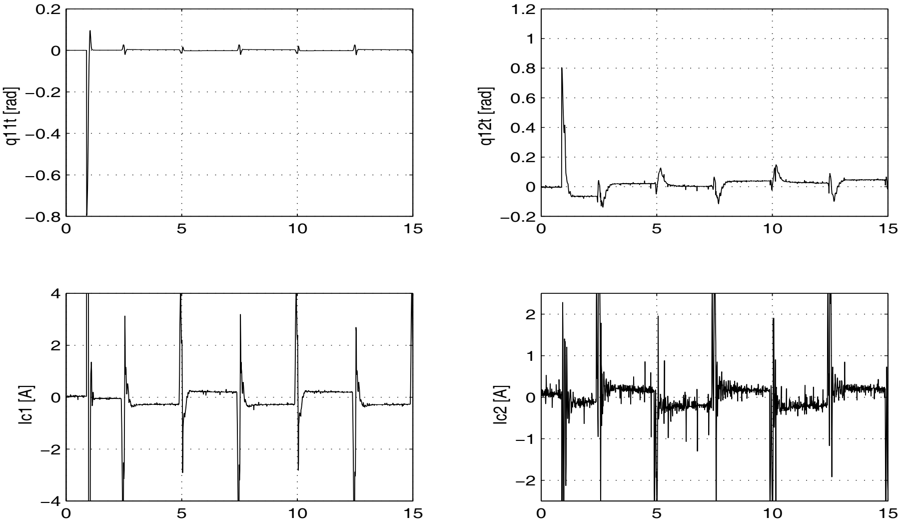

Fig. 9.13. Controller 3, desired trajectory 3

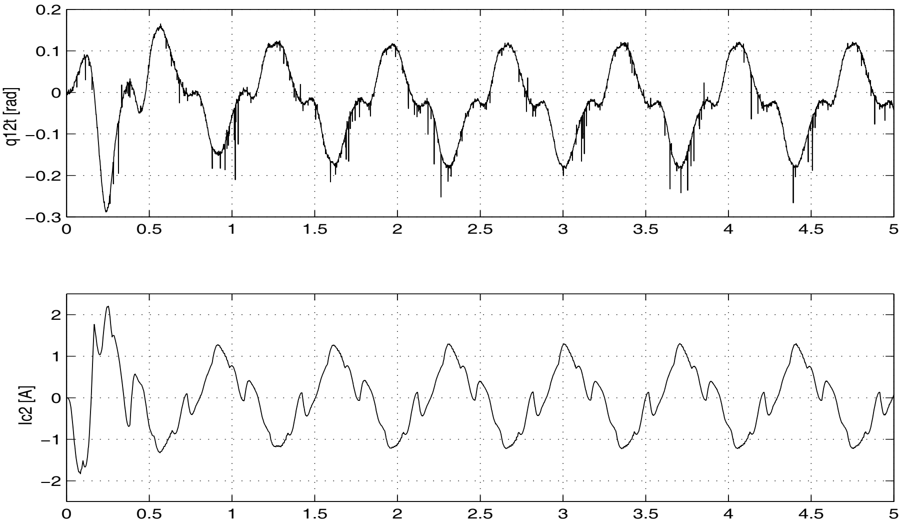

Fig. 9.14. PD controller, desired trajectory 1, zero initial conditions

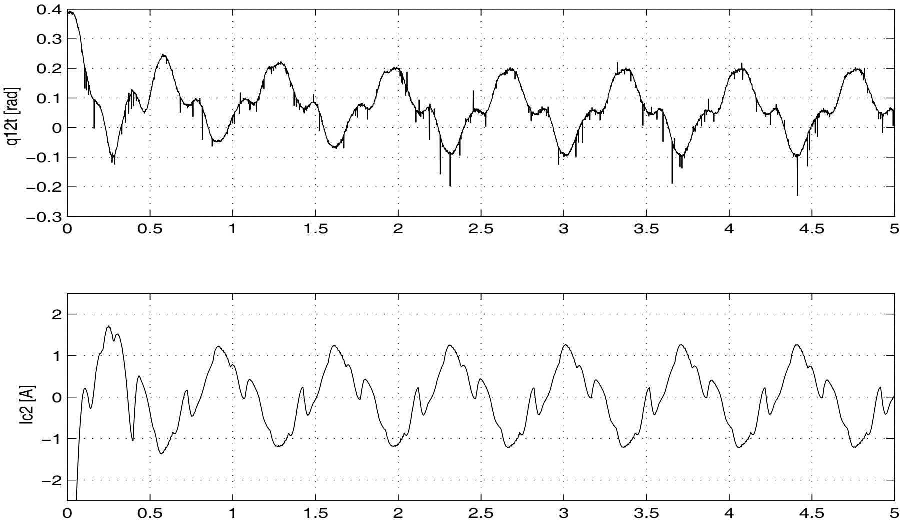

Fig. 9.15. PD controller, desired trajectory 1, nonzero initial conditions

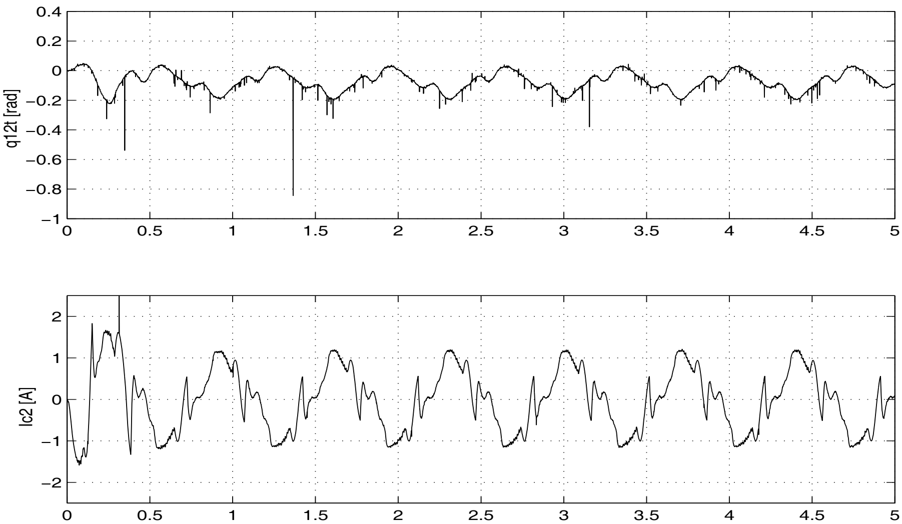

Fig. 9.16. SLI controller, desired trajectory 1, zero initial conditions

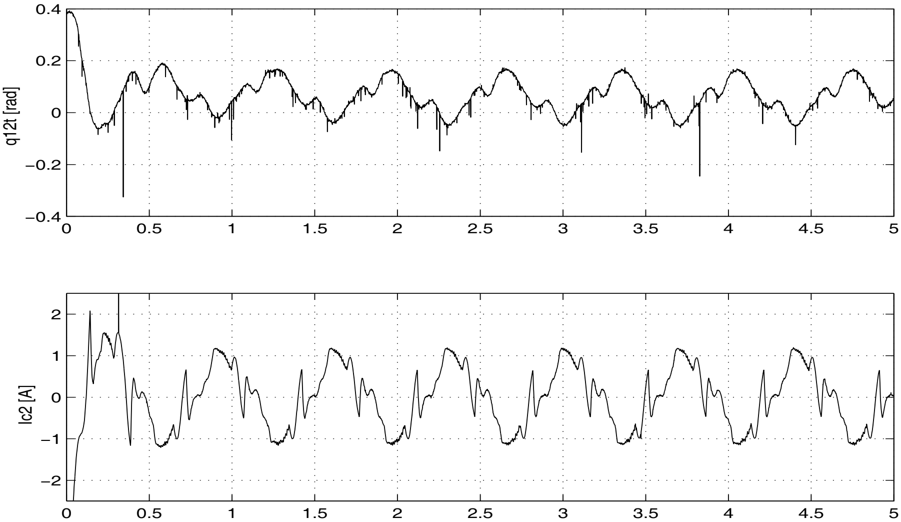

Fig. 9.17. SLI controller, desired trajectory 1, nonzero initial conditions

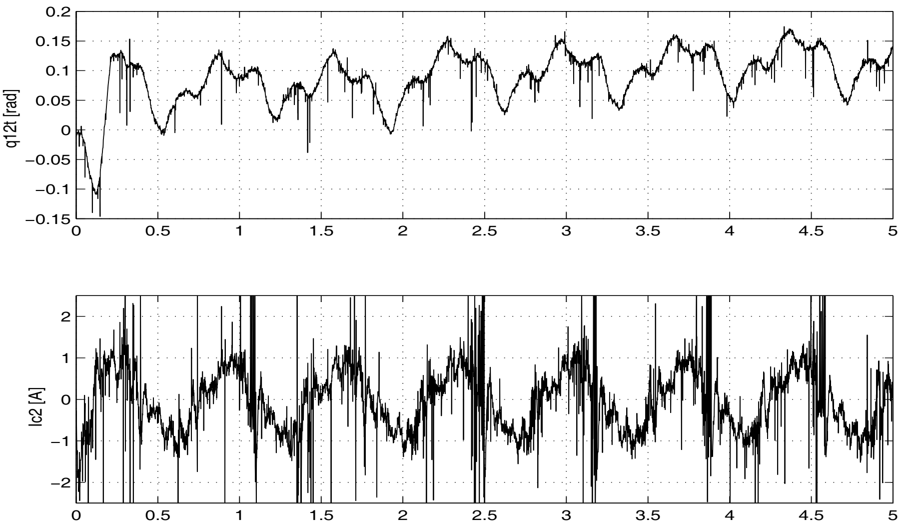

Fig. 9.18. Controller 2, desired trajectory 1, zero initial conditions

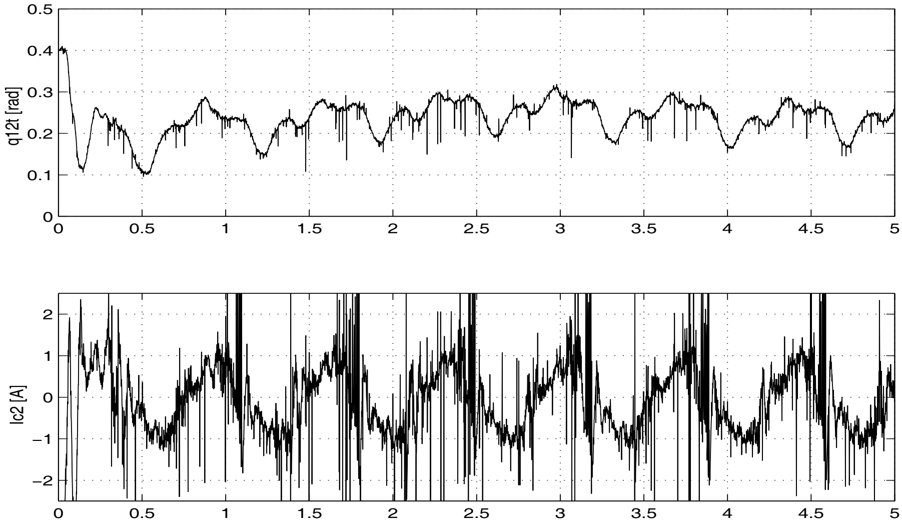

Fig. 9.19. Controller 2, desired trajectory 1, nonzero initial conditions

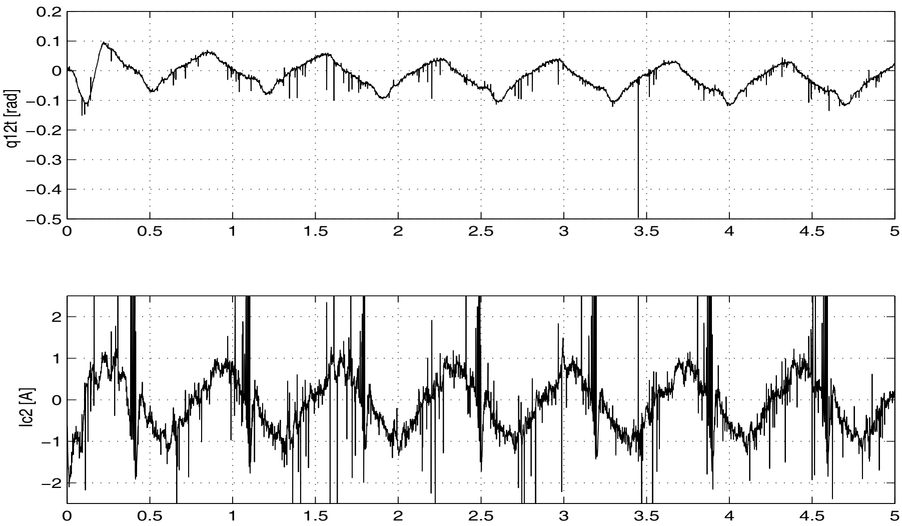

Fig. 9.20. Controller 3, desired trajectory 1, zero initial conditions

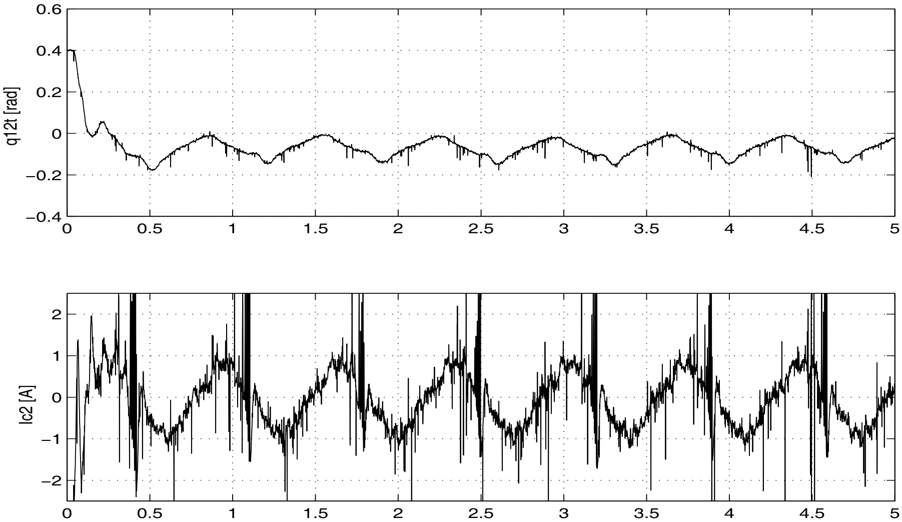

Fig. 9.21. Controller 3, desired trajectory 1, nonzero initial conditions

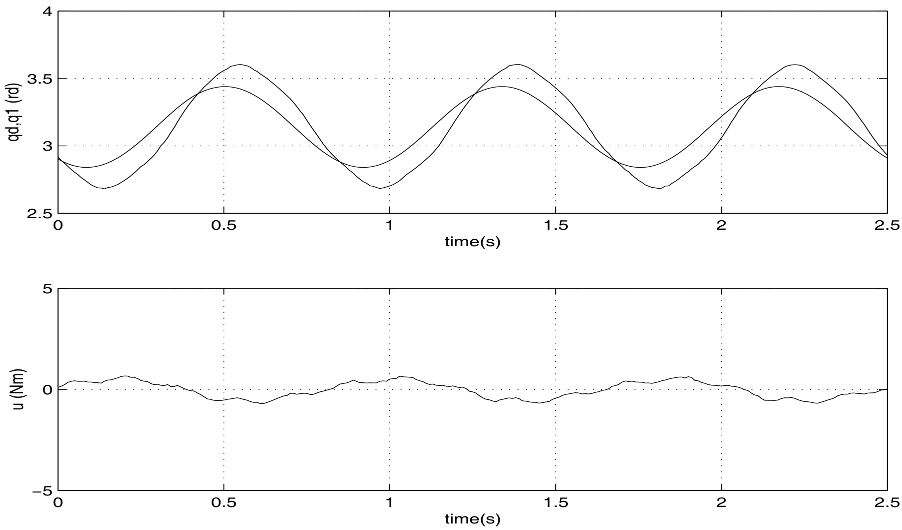

Fig. 9.22. PD controller, $\omega = 7.5$ rad/s.

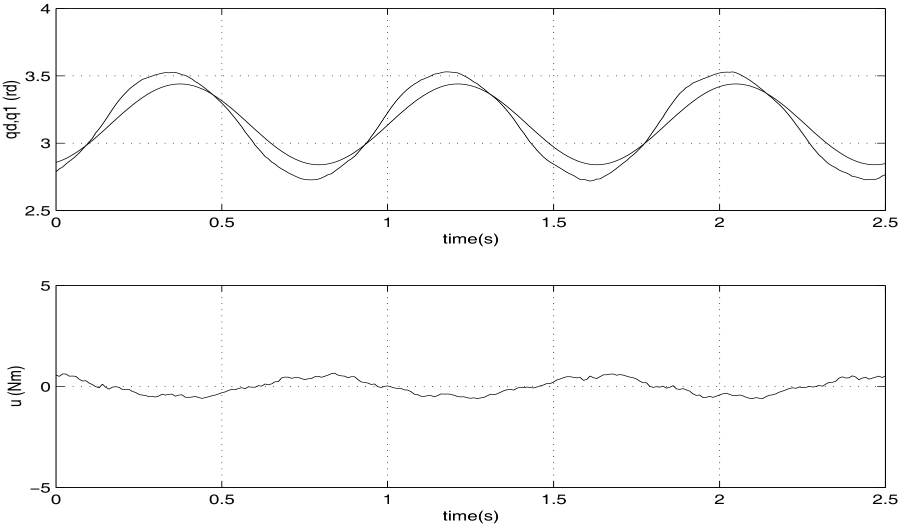

Fig. 9.23. Controller 1, $\omega = 7.5$ rad/s

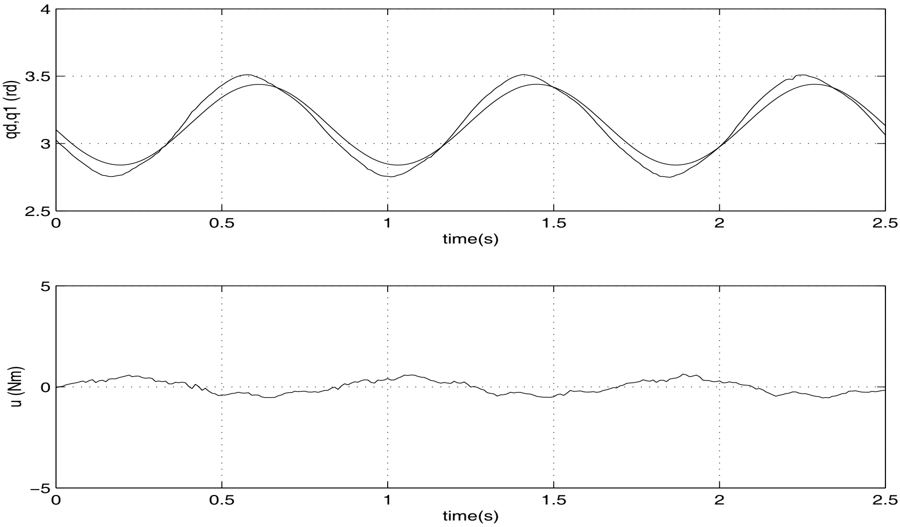

Fig. 9.24. Controller 2, $\omega = 7.5$ rad/s

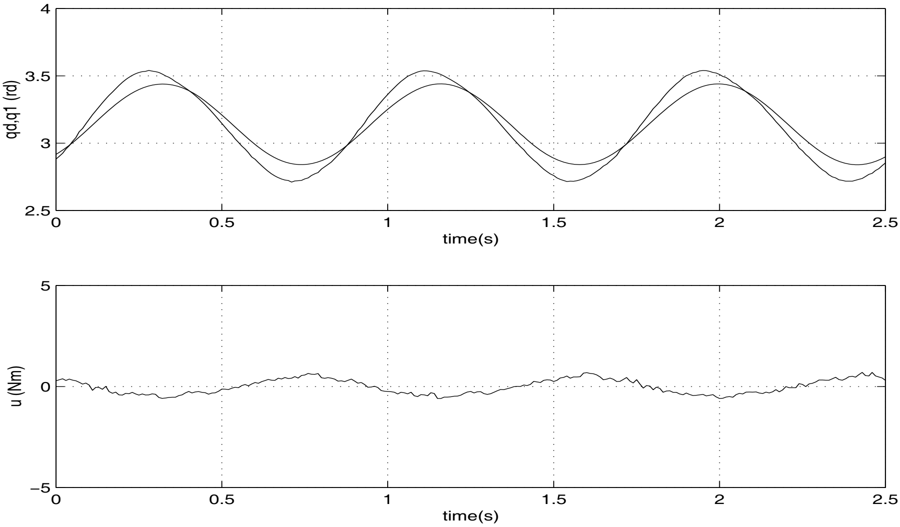

Fig. 9.25. Controller 3, $\omega = 7.5$ rad/s

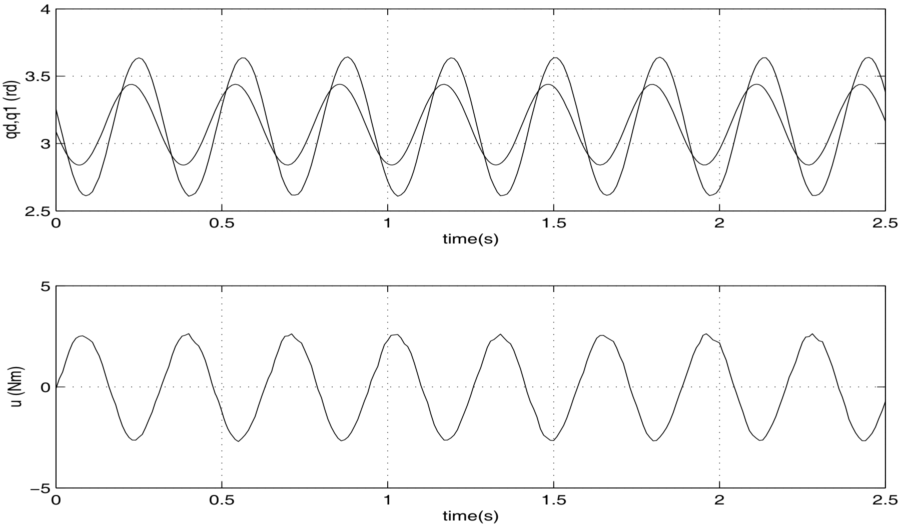

Fig. 9.26. Controller 1, $\omega = 20$ rad/s

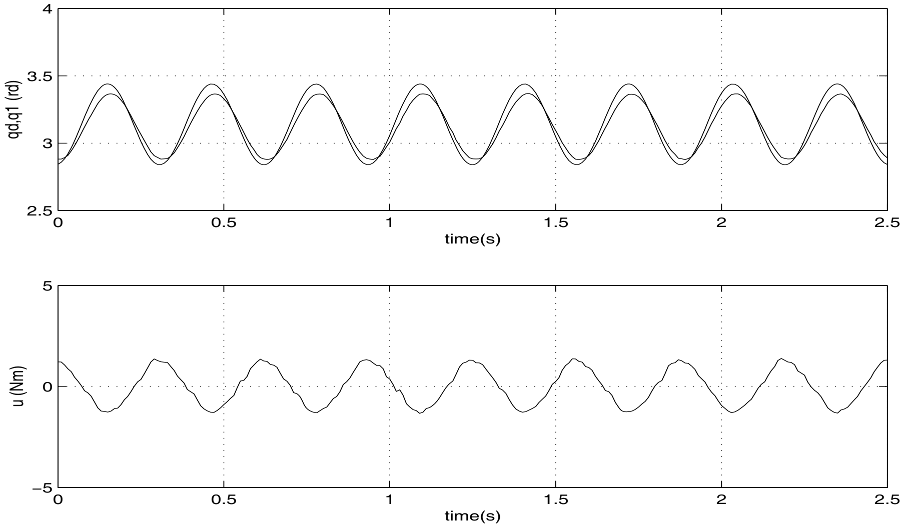

Fig. 9.27. Controller 2, $\omega = 20$ rad/s

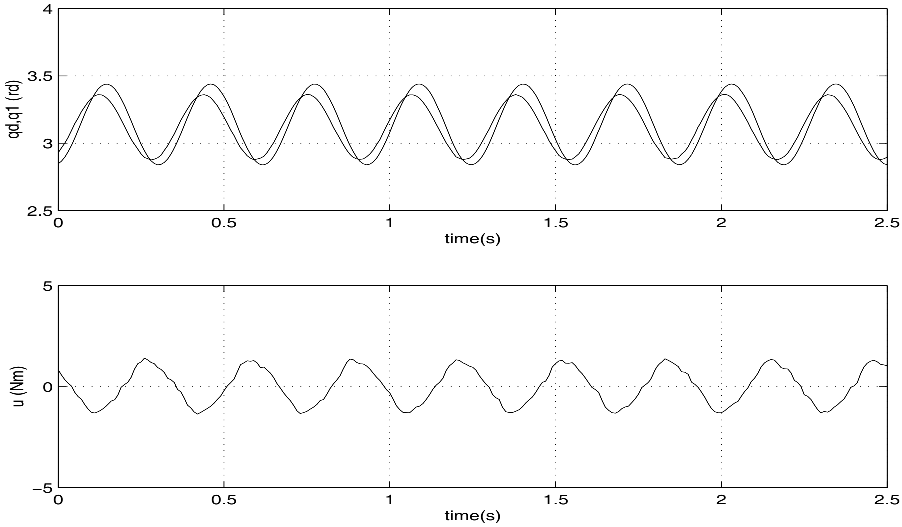

Fig. 9.28. Controller 3, $\omega = 20$ rad/s

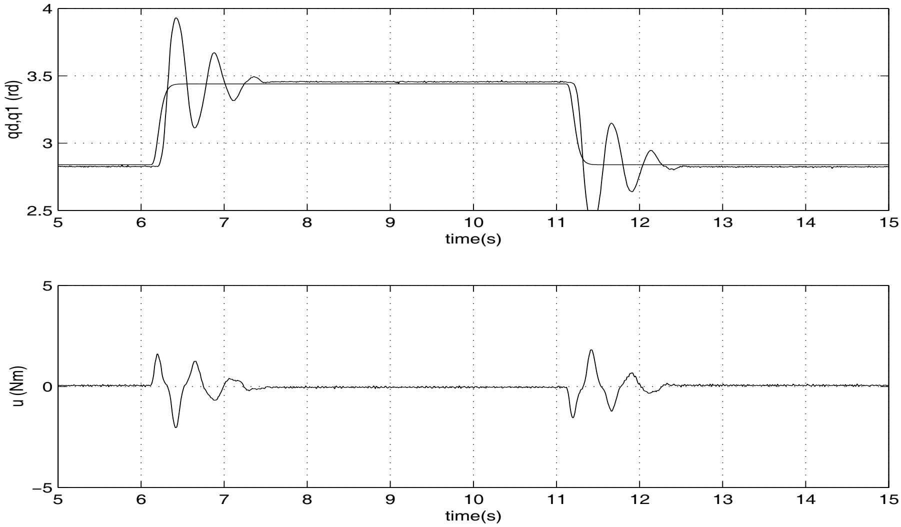

Fig. 9.29. PD controller, $b = 40$

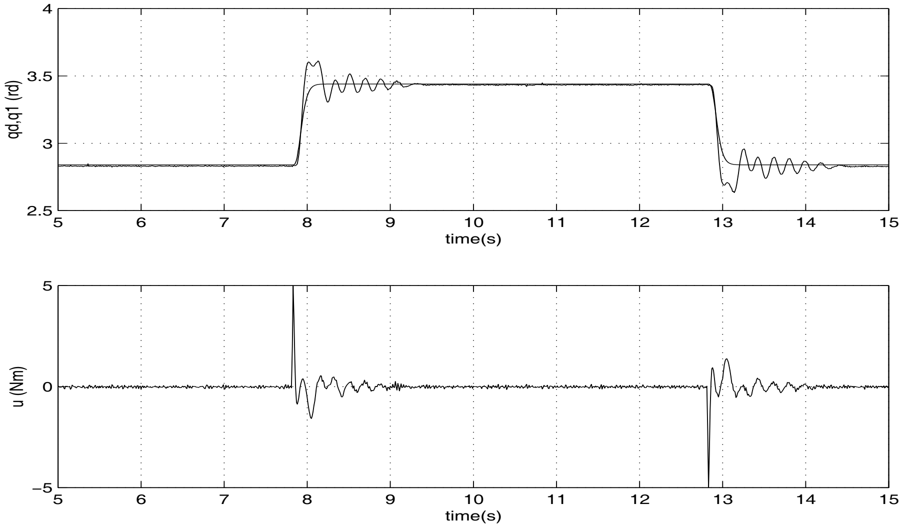

Fig. 9.30. Controller 1, $b = 40$

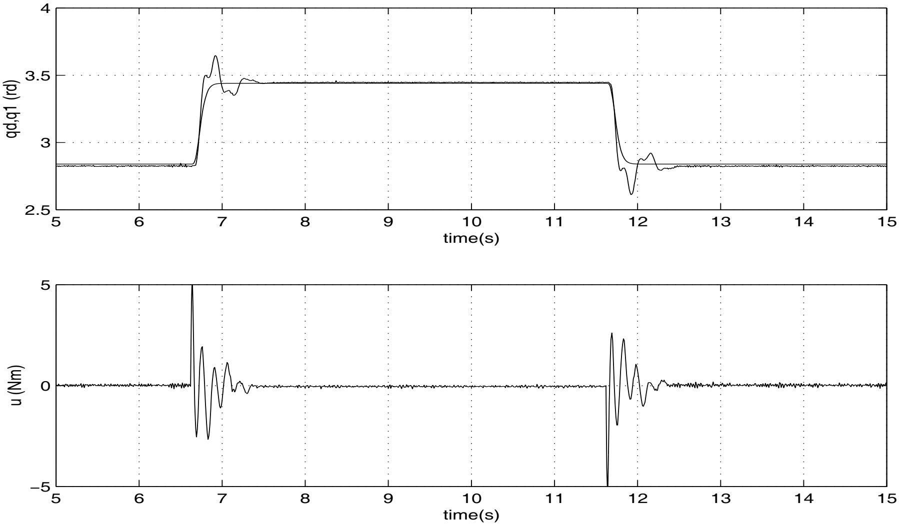

Fig. 9.31. Controller 2, $b = 40$

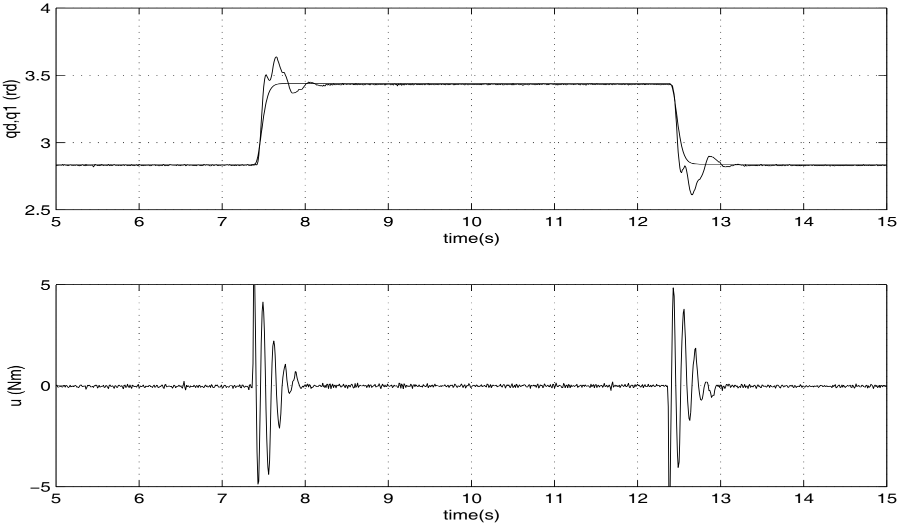

Fig. 9.32. Controller 3, $b = 40$

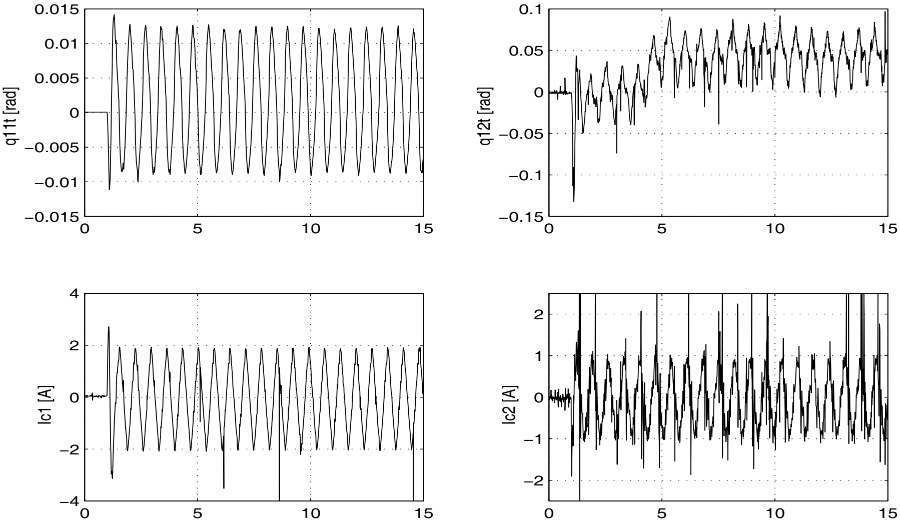

Fig. 9.33. Controller 3, desired trajectory 1

Table 9.1. Quadratic error sums e_1 and e_2 (Capri robot)

Controller	e_1 (traj. 1)	e_2 (traj. 1)	e_1 (traj. 2)	e_2 (traj. 2)
PD	0.346	84.5	1.4 (1.6)	360 (1000)
SLI	0.11	37.9	0.02 (0.034)	40 (51)
Controller 1	x	x	x	x
Controller 2	0.34	12	0.3	75 (173)
Controller 3	0.64	9	0.224 (0.6)	70 (150)

Table 9.2. Quadratic error sums e_1 and e_2 (Capri robot)

Controller	e_1 (traj. 3)	e_2 (traj. 3)
PD	0.3 (0.3)	50 (50)
SLI	0.055 (0.055)	30 (30)
Controller 1	x	x
Controller 2	0.135 (0.135)	30 (30
Controller 3	0.19 (0.19)	15 (15)

Table 9.3. Quadratic error sum e_3 (pulley system)

ω (rad/s)	PD	Control. 1	Control. 2	Control. 3
2.5	0.70	0.21	0.25	0.33
5	3.54	2.57	1.54	2.78
7.5	20.86	8.53	4.17	7.92
10	x	20.60	13.00	19.03
12.5	x	48.07	35.15	36.05
15	x	63.44	53.33	31.03
20	x	37.70	2.97	8.58

Table 9.4. Maximum tracking error (pulley system)

ω (rad/s)	PD	Controller 1	Controller 2	Controller 3
2.5	0.0630	0.0293	0.0374	0.0386
5	0.0943	0.1138	0.0840	0.0983
7.5	0.1946	0.1501	0.1040	0.1472
10	x	0.2428	0.1823	0.2150
12.5	x	0.4138	0.2965	0.2910
15	x	0.4494	0.3418	0.2581
20	x	0.2842	0.0842	0.1364

Table 9.5. Feedback gains (Capri robot)

PD Controller	traj. 1	traj. 2	traj. 3
λ_{21}	1500	650	1500
λ_{22}	250	10	250
λ_{11}	30	4	30
λ_{12}	5	3.5	5

9.2 Stabilization of the Inverted Pendulum

9.2.1 Introduction

The inverted pendulum is a very popular experiment used for educational purposes in modern control theory. It is basically a pole which has a pivot on a cart that can be moved horizontally. The pole moves freely around the cart and the control objective is to bring the pole to the upper unstable equilibrium position by moving the cart on the horizontal plane. Since the angular acceleration of the pole cannot be controlled directly, the inverted pendulum is an underactuated mechanical system. Therefore, the techniques developed for fully-actuated mechanical robot manipulators cannot be used to control the inverted pendulum.

The cart and pole system is also known because the standard nonlinear control techniques are ineffective to control it. Indeed the relative degree of

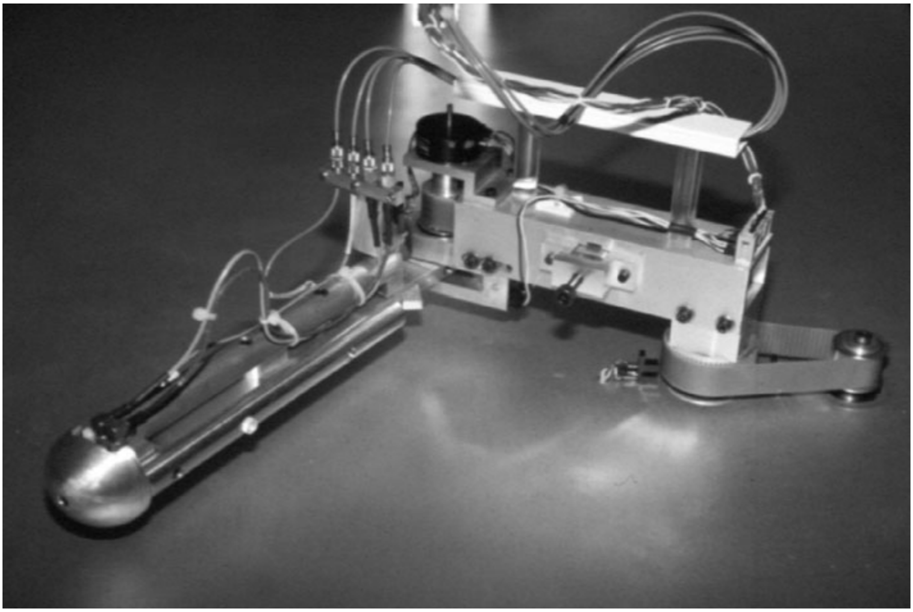

Fig. 9.34. The Capri robot of the Laboratoire d'Automatique de Grenoble

the system is not constant (when the output is chosen to be the swinging energy of the pendulum), the system is not input-output linearizable. Jakubczyk and Respondek [231] have shown that the inverted pendulum is not feedback linearizable. An additional difficulty comes from the fact that when the pendulum swings past the horizontal the controllability distribution does not have a constant rank.

9.2.2 System's Dynamics

Consider the cart and pendulum system as shown in Figure 9.36. We will consider the standard assumptions, *i.e.* massless rod, point masses, no flexibilities and no friction. M is the mass of the cart, m the mass of the pendulum, concentrated in the bob, θ the angle that the pendulum makes with the vertical and l the length of the rod. The equations of motion can be obtained either by applying Newton's second law or by the Euler-Lagrange formulation.

The system can be written as

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + g(q(t)) = \tau(t) \tag{9.5}$$

where:

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix}, \quad M(q) = \begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix} \tag{9.6}$$

Fig. 9.35. The pulley system of the Laboratoire d'Automatique de Grenoble

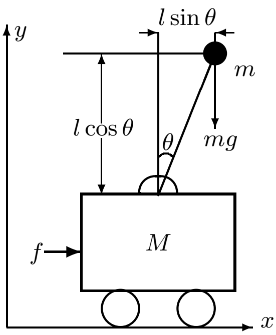

Fig. 9.36. The cart pendulum system

$$C(q, \dot{q}) = \begin{bmatrix} 0 & -ml \sin\theta\dot{\theta} \\ 0 & 0 \end{bmatrix} \tag{9.7}$$

$$g(q) = \begin{bmatrix} 0 \\ -mgl \sin\theta \end{bmatrix} \quad \text{and} \quad \tau = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{9.8}$$

Note that $M(q)$ is symmetric and

$$\begin{aligned} \det(M(q)) &= (M + m)ml^2 - m^2l^2\cos^2\theta \\ &= Mml^2 + m^2l^2\sin^2\theta > 0 \end{aligned} \tag{9.9}$$

Therefore, $M(q)$ is positive definite for all q. From (9.6) and (9.7) it follows that

$$\dot{M}(q, \dot{q}) - 2C(q, \dot{q}) = \begin{bmatrix} 0 & ml \sin\theta\dot{\theta} \\ -ml \sin\theta\dot{\theta} & 0 \end{bmatrix} \tag{9.10}$$

which is a skew-symmetric matrix (see Lemma 6.16). The potential energy of the pendulum can be defined as $U(\theta) = mgl(\cos\theta - 1)$. Note that $U(\theta)$ is related to $g(q)$ as follows:

$$g(q) = \frac{\partial U}{\partial q} = \begin{bmatrix} 0 \\ -mgl \sin\theta \end{bmatrix} \tag{9.11}$$

Passivity of the Inverted Pendulum

The total energy of the cart and pole system is given by

$$\begin{aligned} E(q, \dot{q}) &= K(q, \dot{q}) + U(q) \\ &= \tfrac{1}{2}\dot{q}^T M(q)\dot{q} + mgl(\cos\theta - 1) \end{aligned} \tag{9.12}$$

Therefore from (9.5), (9.6), (9.7), (9.8), (9.10) and (9.11) we obtain:

$$\begin{aligned} \tfrac{d}{dt}E(q(t), \dot{q}(t)) &= \dot{q}^T(t)M(q(t))\ddot{q}(t) + \tfrac{1}{2}\dot{q}^T(t)\dot{M}(q(t))\dot{q}(t) + \dot{q}^T(t)g(q(t)) \\ &= \dot{q}^T(t)(-C(q(t), \dot{q}(t))\dot{q}(t) - g(q(t)) + \tau(t) + \tfrac{1}{2}\dot{M}(q(t))\dot{q}(t)) + \dot{q}^T(t)g(q(t)) \\ &= \dot{q}^T(t)\tau(t) = \dot{x}(t)f(t) \end{aligned} \tag{9.13}$$

Integrating both sides of the above equation we obtain

$$\begin{aligned} \textstyle\int_0^t \dot{x}(t')f(t')dt' &= E(t) - E(0) \\ &\geq -2mgl - E(0) \end{aligned} \tag{9.14}$$

Therefore, the system having f as input and $\dot{x}$ as output is passive. Note that for $f = 0$ and $\theta \in [0, 2\pi[$ the system (9.5) has a subset of two equilibrium points. $(x, \dot{x}, \theta, \dot{\theta}) = (*, 0, 0, 0)$ is an unstable equilibrium point and $(x, \dot{x}, \theta, \dot{\theta}) = (*, 0, \pi, 0)$ is a stable equilibrium point. The total energy $E(q, \dot{q})$ is equal to 0 for the unstable equilibrium point and to $-2mgl$ for the stable equilibrium point. The control objective is to stabilize the system around its unstable equilibrium point, *i.e.* to bring the pendulum to its upper position and the cart displacement to zero simultaneously.

9.2.3 Stabilizing Control Law

Let us first note that in view of (9.12) and (9.6), if $\dot{x} = 0$ and $E(q, \dot{q}) = 0$ then

$$\frac{1}{2} ml^2 \dot{\theta}^2 = mgl(1 - \cos\theta) \tag{9.15}$$

The above equation defines a very particular trajectory which corresponds to a homoclinic orbit. Note that $\dot{\theta} = 0$ only when $\theta = 0$. This means that the pendulum angular position moves clockwise or counter-clockwise until it reaches the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$. Thus our objective can be reached if the system can be brought to the orbit (9.15) for $\dot{x} = 0$, $x = 0$ and $E = 0$. Bringing the system to this homoclinic orbit solves the problem of "swinging up" the pendulum. In order to balance the pendulum at the upper equilibrium position the control must eventually be switched to a controller which guarantees (local) asymptotic stability of this equilibrium [474]. By guaranteeing convergence to the above homoclinic orbit, we guarantee that the trajectory will enter the basin of attraction of any (local) balancing controller. We do not consider in this book the design of the balancing controller.

The passivity property of the system suggests us to use the total energy $E(q, \dot{q})$ in (9.12) in the controller design. Since we wish to bring to zero $x, \dot{x}$ and E we propose the following Lyapunov function candidate:

$$V(q, \dot{q}) = \frac{k_E}{2} E^2(q, \dot{q}) + \frac{k_v}{2} \dot{x}^2 + \frac{k_x}{2} x^2 \tag{9.16}$$

where k_E, k_v and k_x are strictly positive constants. Note that $V(q, \dot{q})$ is a positive semi-definite function. Differentiating $V(q, \dot{q})$ and using (9.13) we obtain

$$\begin{aligned} \dot{V}(q, \dot{q}) &= k_E E \dot{E} + k_v \dot{x} \ddot{x} + k_x x \dot{x} \\ &= k_E E \dot{x} f + k_v \dot{x} \ddot{x} + k_x x \dot{x} \\ &= \dot{x}(k_E E f + k_v \ddot{x} + k_x x) \end{aligned} \tag{9.17}$$

Let us now compute $\ddot{x}$ from (9.5). The inverse of $M(q)$ can be obtained from (9.6), (9.7) and (9.9) and is given by:

$$M^{-1} = \frac{1}{\det(M)} \begin{bmatrix} ml^2 & -ml\cos\theta \\ -ml\cos\theta & M+m \end{bmatrix} \tag{9.18}$$

with $\det(M) = ml^2(M + m\sin^2\theta)$. Therefore we have

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = [det(M(q))]^{-1} \left(\begin{bmatrix} 0 & m^2l^3\dot{\theta}\sin\theta \\ 0 & -m^2l^2\dot{\theta}\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \right.$$

$$\left. + \begin{bmatrix} -m^2l^2g\sin\theta\cos\theta \\ (M+m)mgl\sin\theta \end{bmatrix} + \begin{bmatrix} ml^2f \\ -mlf\cos\theta \end{bmatrix} \right)$$

Thus $\ddot{x}(\cdot)$ can be written as

$$\ddot{x}(t) = \frac{1}{M + m\sin^2\theta(t)} \left[m\sin\theta(t)(l\dot{\theta}^2(t) - g\cos\theta(t)) + f(t) \right] \tag{9.19}$$

Introducing the above in (9.17) one has

$$\dot{V}(q,\dot{q}) = \dot{x}\left[f\left(k_E E + \frac{k_v}{M+m\sin^2\theta} \right) + \frac{k_v m\sin\theta(l\dot{\theta}^2 - g\cos\theta)}{M+m\sin^2\theta} + k_x x \right] \tag{9.20}$$

For simplicity and without loss of generality we will consider $M = m = l = 1$, thus

$$\dot{V}(q,\dot{q}) = \dot{x}\left[f\left(k_E E + \frac{k_v}{1+\sin^2\theta} \right) + \frac{k_v\sin\theta(\dot{\theta}^2 - g\cos\theta)}{1+\sin^2\theta} + k_x x \right] \tag{9.21}$$

We propose a control law such that

$$f\left(k_E E + \frac{k_v}{1+\sin^2\theta} \right) + \frac{k_v\sin\theta(\dot{\theta}^2 - g\cos\theta)}{1+\sin^2\theta} + k_x x = -k_{dx}\dot{x} \tag{9.22}$$

which will lead to

$$\dot{V}(q,\dot{q}) = -k_{dx}\dot{x}^2 \tag{9.23}$$

Note that other functions $f(\dot{x})$ such that $\dot{x}f(\dot{x}) > 0$ are also possible. The control law in (9.22) will have no singularities provided that

$$\left(k_E E + \frac{k_v}{1+\sin^2\theta} \right) \neq 0 \tag{9.24}$$

The above condition will be satisfied if for some $\epsilon > 0$

$$|E| \leq \frac{\frac{k_v}{k_E} - \epsilon}{2} < \frac{\frac{k_v}{k_E}}{\left(1+\sin^2\theta\right)} \tag{9.25}$$

Note that when using the control law (9.22), the pendulum can get stuck at the (lower) stable equilibrium point, $(x, \dot{x}, \theta, \dot{\theta}) = (0, 0, \pi, 0)$. In order to avoid this singular point, which occurs when $E = -2mgl$ (see (9.12)), we require $|E| < 2mgl$ *i.e.* $|E| < 2g$ (for $m = 1$, $l = 1$). Taking also (9.25) into account, we require

$$|E| < c = \min\left(2g, \frac{\frac{k_v}{k_E} - \epsilon}{2}\right) \tag{9.26}$$

Since $V(\cdot)$ is a non-increasing function (see (9.23)), (9.26) will hold if the initial conditions are such that

$$V(0) < \frac{c^2}{2} \tag{9.27}$$

The above defines the region of attraction as will be shown in the next section.

Domain of Attraction

The condition (9.27) imposes bounds on the initial energy of the system. Note that the potential energy $U = mgl(\cos\theta - 1)$ lies between $-2g$ and 0, for $m = l = 1$. This means that the initial kinetic energy should belong to $[0, c+2g)$. Note also that the initial position of the cart $x(0)$ is arbitrary since we can always choose an appropiate value for k_x in $V(\cdot)$ in (9.16). If $x(0)$ is large we should choose k_x small. The convergence rate of the algorithm may however decrease when k_x is small. Note that when the initial kinetic energy $K(q(0), \dot{q}(0))$ is zero, the initial angular position $\theta(0)$ should belong to $(-\pi, \pi)$. This means that the only forbidden point is $\theta(0) = \pi$. When the initial kinetic energy $K(q(0), \dot{q}(0))$ is different from zero, *i.e.* $K(q(0), \dot{q}(0))$ belongs to $(0, c + 2g)$ (see (9.26) and (9.27)), then there are less restrictions on the initial angular position $\theta(0)$. In particular, $\theta(0)$ can even be pointing downwards, *i.e.* $\theta = \pi$ provided that $K(q(0), \dot{q}(0))$ is not zero. Despite the fact that our controller is local, its basin of attraction is far from being small. The simulation example and the real-time experiments will show this feature. For future use we will rewrite the control law f from (9.22) as

$$f = \frac{k_v \sin\theta\left(g\cos\theta - \dot{\theta}^2\right) - \left(1 + \sin^2\theta\right)(k_x x + k_{dx}\dot{x})}{k_v + \left(1 + \sin^2\theta\right) k_E E} \tag{9.28}$$

The stability analysis can be obtained by using the Krasovskii-LaSalle's invariance Theorem. The stability properties are summarized in the following lemma.

Lemma 9.4. *Consider the inverted pendulum system (9.5) and the controller in (9.28) with strictly positive constants k_E, k_v, k_x and k_{dx}. Provided that the* *state initial conditions satisfy the inequalities at Equations (9.26) and (9.27), then the solution of the closed-loop system converges to the invariant set* M *given by the homoclinic orbit (9.15) with* $(x, \dot{x}) = (0, 0)$*. Note that* $f(\cdot)$ *does not necessarily converge to zero.* ■

Proof: The proof can be found in [314]. ■

9.2.4 Simulation Results

In order to observe the performance of the proposed control law based on an energy approach of the system, we have performed simulations on MATLAB© using Simulink©.

We have considered the real system parameters $\bar{M} = M + m = 1.2$, $ml^2 = 0.0097$ and $ml = 0.04$, and $g = 9.804\ ms^{-2}$ of the inverted pendulum at the University of Illinois at Urbana-Champaign. Recall that the control law requires initial conditions such that (9.27) is satisfied. We have chosen the gains $k_E = 1$, $k_v = 1$, $k_x = 10^{-2}$ and $k_{dx} = 1$. These gains have been chosen to increase the convergence rate in order to switch to a linear stabilizing controller in a reasonable time. The algorithm brings the inverted pendulum close to the homoclinic orbit but the inverted pendulum will remain swinging while getting closer and closer to the origin. Once the system is close enough to the origin, *i.e.* $(|x| \leq 0.1, |\dot{x}| \leq 0.2, |\theta| \leq 0.3, |\dot{\theta}| \leq 0.3)$, we switch to the linear LQR controller $f = -K[x \quad \dot{x} \quad \theta \quad \dot{\theta}]^T$ where $K = [44 \quad 23 \quad 74 \quad 11]$. Figure 9.37 shows the results for an initial position:

$$\begin{cases} x = 0.1,\ \dot{x} = 0 \\ \\ \theta = \frac{2\pi}{3} \quad \dot{\theta} = 0 \end{cases} \tag{9.29}$$

Simulations showed that the nonlinear control law brings the system to the homoclinic orbit (see the phase plot in figure 9.37). Switching to the linear controller occurs at time $t = 120\ s$. Note that before the switching the energy E goes to zero and that the Lyapunov function $V(\cdot)$ is decreasing and converges to zero.

9.2.5 Experimental Results

We have performed experiments on the inverted pendulum setting at the University of Illinois at Urbana-Champaign. The parameters of the model used for the controller design and the linear controller gains K are the same as in the previous section. For this experiment we have chosen the gains $k_E = 1$, $k_v = 1.15$, $k_x = 20$ and $k_{dx} = 0.001$. Figure 9.38 shows the results for an initial position:

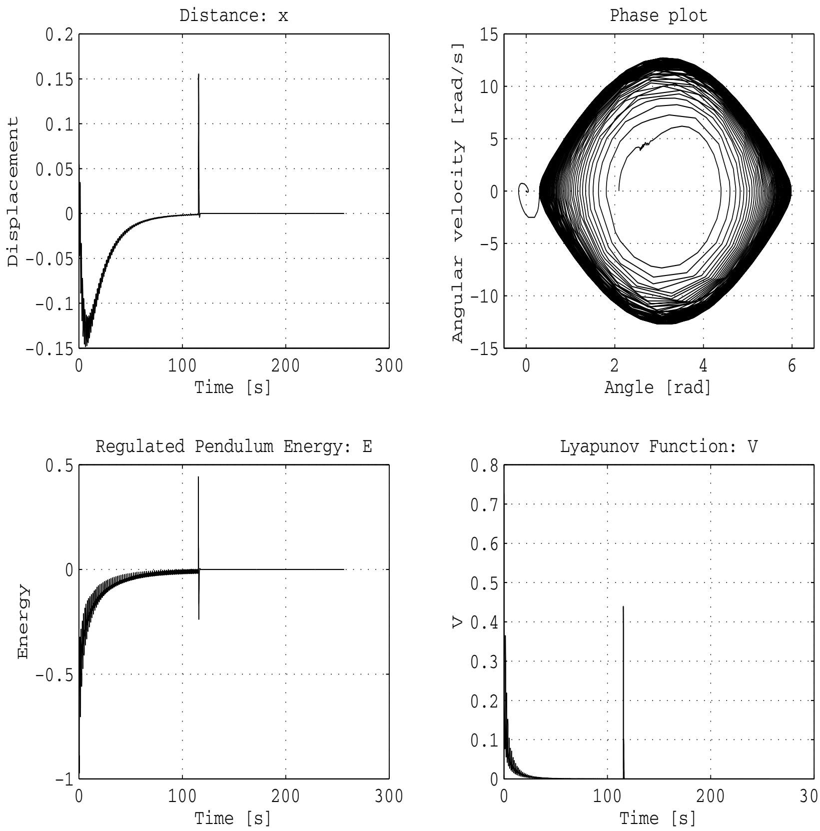

Fig. 9.37. Simulation results for the inverted pendulum

$$\begin{cases} x = 0, & \dot{x} = 0 \\ \theta = \pi + 0.1, & \dot{\theta} = 0.1 \end{cases} \tag{9.30}$$

Real-time experiments showed that the nonlinear control law brings the system to the homoclinic orbit (see the phase plot in Figure 9.38). Switching to the linear controller occurs at time $t = 27\ s$. Note that the control input lies in an acceptable range. Note that in both simulation and experimental results, the initial conditions lie slightly outside the domain of attraction. This proves that the domain of attraction in (9.26) and (9.27) is conservative.

9.3 Conclusions

In the first part of this chapter dedicated to experimental validations of passivity-based control schemes, we have presented a set of experiments on

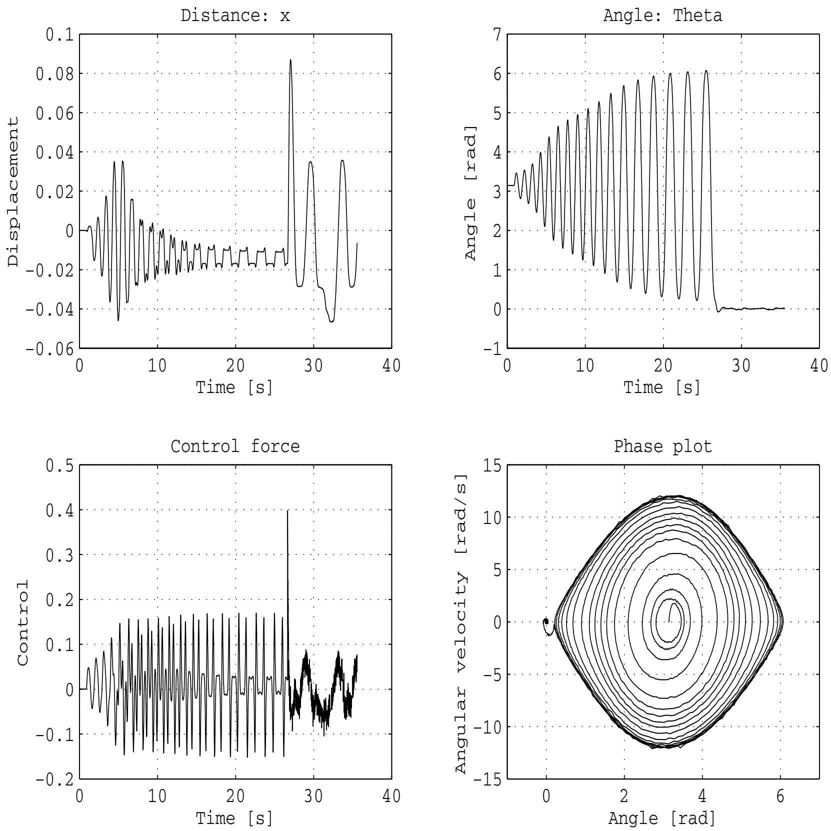

Fig. 9.38. Experimental results for the inverted pendulum

two types of manipulators with flexible joints and rigid links: the first setup is nonlinear, with low flexibility. The second setup is linear but with high flexibility. Various passivity-based controllers, with increasing complexity, have been tested on the two devices. The results are quite encouraging and show that this design concept yields very nice results for robust tracking control. Then we have presented a control strategy for the inverted pendulum that brings the pendulum to a homoclinic orbit, while the cart displacement converges to zero. Therefore the state will enter the basin of attraction of any locally convergent controller. The control strategy is based on the total energy of the system, using its passivity properties. A Lyapunov function is obtained using the total energy of the system. The convergence analysis is carried out using the Krasovskii-LaSalle's invariance principle. The system nonlinearities have not been compensated which has enabled us to exploit the physical properties of the system in the stability analysis. The proposed control strategy is proved to be applicable to a wider class of underactuated mechanical systems (see [138, 139]).

As recalled in the introduction of Chapter 4, there are many other fields of applications to which the passivity-based approach applies and provides good results. Experimental results with passivity-based controllers have been presented in many other papers which are impossible to describe comprehensively in this chapter.

A

Background Material

In this Appendix we present the background for the main tools used throughout the book; namely, Lyapunov stability, differential geometry for nonlinear systems, Riccati equations, viscosity solutions of PDEs, some useful matrix algebra results, and some results that are used in the proof of the KYP Lemma.

A.1 Lyapunov Stability

Let us consider a nonlinear system represented as

$$\dot{x}(t) = f(x(t), t), \quad x(0) = x_0 \tag{A.1}$$

where $f(\cdot)$ is a nonlinear vector function, and $x(t) \in I\!R^n$ is the state vector. We suppose that the system is well-posed, i.e. a unique solution exists globally (see Section 3.9.2 for details on existence, uniqueness and continuous dependence on parameters). We may for instance assume that the conditions of Theorem 3.55 are satisfied. We refer the reader to Theorems 3.83 and 3.84 for extensions of Lyapunov stability to more general systems like evolution variational inequalities. In this Appendix we focus on ODEs.

A.1.1 Autonomous systems

The nonlinear system (A.1) is said to be *autonomous* (or *time-invariant*) if $f(\cdot)$ does not depend explicitly on time, *i.e.*,

$$\dot{x}(t) = f(x(t)) \tag{A.2}$$

Otherwise the system is called *non-autonomous* (or *time-varying*). In this section, we briefly review the *Lyapunov theory* results for autonomous systems while non-autonomous systems will be reviewed in the next section. Lyapunov theory is the fundamental tool for stability analysis of dynamic systems. The basic stability concepts are summarized in the following definitions.

Definition A.1 (Equilibrium). *A state $x^\star$ is an* equilibrium *point of (A.2) if $f(x^\star) = 0$.*

Definition A.2 (Stability). *The equilibrium point $x = 0$ is said to be* stable *if, for any $\rho > 0$, there exists $r > 0$ such that if $\|x(0)\| < r$, then $\|x(t)\| < \rho$ $\forall t \geq 0$. Otherwise the equilibrium point is unstable.*

Definition A.3 (Asymptotic stability). *An equilibrium point $x = 0$ is* asymptotically stable *if it is stable, and if in addition there exists some $r > 0$ such that $\|x(0)\| < r$ implies that $x(t) \to 0$ as $t \to \infty$.*

Definition A.4 (Marginal stability). *An equilibrium point that is Lyapunov stable but not asymptotically stable is called* marginally stable.

Definition A.5 (Exponential stability). *An equilibrium point is* exponentially stable *if there exist two strictly positive numbers α and λ independent of time and initial conditions such that*

$$\|x(t)\| \leq \alpha \|x(0)\| \exp(-\lambda t), \qquad \forall t > 0 \tag{A.3}$$

in some ball around the origin. ■

The above definitions correspond to *local* properties of the system around the equilibrium point. The above stability concepts become *global* when their corresponding conditions are satisfied for *any initial state.*

Lyapunov Linearization Method

Assume that $f(x)$ in (A.2) is continuously differentiable and that $x = 0$ is an equilibrium point. Then, using Taylor expansion, the system dynamics can be written as

$$\dot{x}(t) = \left.\frac{\partial f}{\partial x}\right|_{x=0} x(t) + o(x) \tag{A.4}$$

where o stands for higher-order terms in x. Linearization of the original nonlinear system at the equilibrium point is given by

$$\dot{x}(t) = Ax(t) \tag{A.5}$$

where A denotes the Jacobian matrix of f with respect to x at $x = 0$, i.e.,

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

A linear time-invariant system of the form (A.5) is (asymptotically) stable if A is a (strictly) stable matrix, *i.e.*, if all the eigenvalues of A have (negative) nonpositive real parts. The stability of linear time-invariant systems can be determined according to the following theorem.

Theorem A.6. *The equilibrium state $x = 0$ of the the system (A.5) is asymptotically stable if and only if, given any matrix $Q > 0$, the solution P to the Lyapunov equation*

$$A^T P + PA = -Q \tag{A.6}$$

is positive definite. If Q is only positive semi-definite ($Q \geq 0$), then only stability is concluded. ■

The following theorem somewhat clarifies some points:

Theorem A.7. *[500] Given a matrix $A \in I\!R^{n\times n}$, the following statements are equivalent:*

- *A is a Hurwitz matrix.*
- *There exists* some *positive definite matris $Q \in I\!R^{n\times n}$ such that $A^T P + PA = -Q$ has a corresponding unique solution for P, and this P is positive definite.*
- *For* every *positive definite matrix $Q \in I\!R^{n\times n}$, $A^T P + PA = -Q$ has a unique solution for P, and this solution is positive definite.* ■

The term "corresponding unique solution" means the matrix

$$P = \int_0^\infty \exp(A^T t) Q \exp(At) dt$$

Local stability of the original nonlinear system can be inferred from stability of the linearized system as stated in the following theorem.

Theorem A.8. *If the linearized system is strictly stable (unstable), then the equilibrium point of the nonlinear system is locally asymptotically stable (unstable).* ■

The above theorem does not allow us to conclude anything when the linearized system is marginally stable. Then one has to rely on more sophisticated tools like the invariant manifold theory [256].

Lyapunov's Direct Method

Let us consider the following definitions.

Definition A.9 ((Semi-)definiteness). *A scalar continuous function $V : I\!R^+ \to I\!R^n$ is said to be locally* positive (semi-)definite *if $V(0) = 0$ and $V(x) > 0$ ($V(x) \geq 0$) for $x \neq 0$. Similarly, $V(\cdot)$ is said to be* negative (semi-)definite *if $-V(\cdot)$ is positive (semi-)definite.* ■

Another definition of positive definiteness can be given:

Definition A.10. *A function $V : I\!R^+ \to I\!R^n$ is said to be locally positive definite if it is continuous, $V(0) = 0$, and there exists a constant $r > 0$ and a function $\alpha(\cdot)$ of class $\mathcal{K}$ such that*

$$\alpha(||x||) \leq V(x) \tag{A.7}$$

for all $||x|| \leq r$. ■

It happens that both characterizations are equivalent [500, Lemma 5.2.6]. In fact if $V(0) = 0$ and $V(x) > 0$ when $x \neq 0$, one can always find a class-$\mathcal{K}$ function which locally lowerbounds $V(\cdot)$ in a neighborhood of $x = 0$.

Definition A.11 (Lyapunov function). *$V(x)$ is called a* Lyapunov function *for the system (A.2) if, in a ball B containing the origin, $V(x)$ is positive definite and has continuous partial derivatives, and if its time derivative along the solutions of (A.2) is negative semi-definite, i.e., $\dot{V}(x) = (\partial V/\partial x)f(x) \leq 0$.* ■

The following Theorems can be used for local and global analysis of stability, respectively. Assume that $f(0) = 0$ and that $x^* = 0$ is an isolated fixed point of (A.2).

Theorem A.12. *[Local stability] The equilibrium point $x^* = 0$ of the system (A.2) is (asymptotically) stable in a ball B if there exists a scalar function $V(x)$ with continuous derivatives such that $V(x)$ is positive definite and $\dot{V}(x)$ is negative semi-definite (negative definite) in the ball B.* ■

Theorem A.13. *[Global stability] The equilibrium point of system (A.2) is globally asymptotically stable if there exists a scalar function $V(x)$ with continuous first order derivatives such that $V(x)$ is positive definite, $\dot{V}(x)$ is negative definite and $V(x)$ is radially unbounded, i.e., $V(x) \to \infty$ as $\|x\| \to \infty$.* ■

Clearly the global asymptotic stability implies that 0 is the unique fixed point of (A.2) in the whole state space $I\!R^n$.

Krasovskii-LaSalle's Invariant Set Theorem

Krasovskii-LaSalle's results extend the stability analysis of the previous Theorems when $\dot{V}(\cdot)$ is only negative semi-definite. They are stated as follows.

Definition A.14 (Invariant set). *A set S is an* invariant set *for a dynamic system if every trajectory starting in S remains in S.* ■

Invariant sets include equilibrium points, limit cycles, as well as any trajectory of an autonomous system.

Theorem A.15. *[Krasovskii-LaSalle] Consider the system (A.2) with $f(\cdot)$ continuous, and let $V(x)$ be a scalar function with continuous first partial derivatives. Consider a region Γ defined by $V(x) < \gamma$ for some $\gamma > 0$. Assume that the region Γ is bounded and $\dot{V}(x) \leq 0 \ \forall x \in \Gamma$. Let Ω be the set of all points in Γ where $\dot{V}(x) = 0$, and M be the largest invariant set in Ω. Then, every solution $x(t)$ originating in Γ tends to M as $t \to \infty$. On the other hand, if $\dot{V}(x) \leq 0 \ \forall x$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then all solutions globally asymptotically converge to M as $t \to \infty$.* ■

Some crucial properties for the invariance principle to hold, are that state trajectories are continuous with respect to initial data, and that the $\omega-$limit sets are compact invariant sets. Not all the systems examined in this book possess those properties (for instance the nonsmooth Lagrangian systems of Section 6.8.2 do not necessarily enjoy the continuity-in-the-initial-data property). Another formulation of this result is as follows [351].

Theorem A.16. *Under the same assumptions of Theorem A.15, let K be the set of points not containing whole trajectories of the system for $\leq t \leq \infty$. Then if $\dot{V}(x) \leq 0$ outside of K and $\dot{V}(x) = 0$ inside K, the system is asymptotically stable.* ■

Notice in particular that $\{x = 0\} \notin K$. K can be a surface, a line, etc. In Theorem A.6, notice that if $Q = C^T C$ with (A, C) being an observable pair, then asymptotic stability is obtained again. More formally:

Corollary A.17. *If $C \in I\!R^{m \times n}$ and the pair (A, C) is observable, then the matrix A is asymptotically stable if and only if there exists a matrix $P = P^T > 0$ that is the unique solution of $A^T P + PA + C^T C = 0$.* ■

The proof of this corollary is based on the quadratic function $V(x) = x^T P x$, whose derivative is computed along the solutions of $\dot{x}(t) = Ax(t)$. Then use the Krasovskii-LaSalle Theorem to conclude on the asymptotic stability, using that the Kalman observability matrix is full-rank.

A.1.2 Non-autonomous Systems

In this section we consider non-autonomous nonlinear systems represented by (A.1). The stability concepts are characterized by the following definitions.

Definition A.18 (Equilibrium). *A state $x^\star$ is an* equilibrium *point of (A.1) if $f(x^\star, t) = 0 \ \forall t \geq t_0$.* ■

Definition A.19 (Stability). *The equilibrium point $x = 0$ is* stable *at $t = t_0$ if for any $\rho > 0$ there exists an $r(\rho, t_0) > 0$ such that $\|x(t_0)\| < r \Rightarrow \|x(t)\| < \rho$, $\forall \ t \geq t_0$. Otherwise the equilibrium point $x = 0$ is unstable.* ■

Definition A.20 (Asymptotic stability). *The equilibrium point* $x = 0$ *is* asymptotically stable *at* $t = t_0$ *if it is stable and if it exists* $r(t_0) > 0$ *such that* $\|x(t_0)\| < r(t_0) \Rightarrow x(t) \to 0$ *as* $t \to \infty$. ■

Definition A.21 (Exponential stability). *The equilibrium point* $x = 0$ *is* exponentially stable *if there exist two positive numbers* α *and* λ *such that* $\|x(t)\| \leq \alpha\|x(t_0)\| \exp(-\lambda(t - t_0))\ \forall t \geq t_0$, *for* $x(t_0)$ *sufficiently small.* ■

Definition A.22 (Global asymptotic stability). *The equilibrium point* $x = 0$ *is* globally asymptotically stable *if it is stable and* $x(t) \to 0$ *as* $t \to \infty$ $\forall\, x(t_0) \in I\!R^n$. ■

The stability properties are called *uniform* when they hold independently of the initial time t_0 as in the following definitions.

Definition A.23 (Uniform stability). *The equilibrium point* $x = 0$ *is* uniformly stable *if it is stable with* $r = r(\rho)$ *that can be chosen independently of* t_0. ■

Definition A.24 (Uniform asymptotic stability). *The equilibrium point* $x = 0$ *is* uniformly asymptotically stable *if it is uniformly stable and there exists a ball of attraction* B, *independent of* t_0, *such that* $x(t_0) \in B \Rightarrow x(t) \to 0$ *as* $t \to \infty$. ■

Lyapunov's Linearization Method

Using Taylor expansion, the system (A.1) can be rewritten as

$$\dot{x}(t) = A(t)x(t) + o(x,t) \tag{A.8}$$

where

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{x=0}(t)$$

A linear approximation of (A.1) is given by

$$\dot{x}(t) = A(t)x(t) \tag{A.9}$$

The result of Theorem A.6 can be extended to linear time-varying systems of the form (A.9) as follows.

Theorem A.25. *A necessary and sufficient condition for the uniform asymptotic stability of the origin of the system (A.9) is that a matrix* $P(t)$ *exists such that*

$$V(t,x) = x^T P(t)x > 0$$

and

$$\dot{V}(t,x(t)) = x^T(t)(A^T P(t) + P(t)A + \dot{P}(t))x(t) \leq k(t)V(t,x(t))$$

where $\lim_{t\to\infty} \int_{t_0}^{t} k(\tau)d\tau = -\infty$ *uniformly with respect to* t_0. ■

We can now state the following result.

Theorem A.26. *If the linearized system (A.9) is uniformly asymptotically stable, then the equilibrium point $x^\star = 0$ of the original system (A.1) is also uniformly asymptotically stable.* ■

Lyapunov's Direct Method

We present now the Lyapunov stability theorems for non-autonomous systems. The following Definitions are required.

Definition A.27. *[Function of class $\mathcal{K}$] A continuous* function $\kappa : [0, k) \to \mathbb{R}^+$ *is said to be* of class $\mathcal{K}$ *if*

(i) $\kappa(0) = 0$
(ii)$\kappa(\chi) > 0 \quad \forall \chi > 0$
(iii)$\kappa(\cdot)$ *is nondecreasing*

Statements (ii) and (iii) can also be replaced with

(ii')κ *is strictly increasing*

so that the inverse function $\kappa^{-1}(\cdot)$ *is defined. The function is said to be of class* $\mathcal{K}_\infty$ *if* $k = \infty$ *and* $\kappa(\chi) \to \infty$ *as* $\chi \to \infty$. ■

Definition A.28. *A class $\mathcal{KL}-$function is a function* $\kappa : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\kappa(\cdot, t)$ *is of class* $\mathcal{K}_\infty$ *for each* t *and* $\lim_{t \to +\infty, t \geq 0} \kappa(r, t) = 0$. ■

Based on the definition of function of class $\mathcal{K}$, a modified definition of exponential stability can be given.

Definition A.29 ($\mathcal{K}$-exponential stability). *The equilibrium point* $x = 0$ *is* K-exponentially stable *if there exist a function* $\kappa(\cdot)$ *of class* $\mathcal{K}$ *and a positive number* λ *such that* $\|x(t)\| \leq \kappa(\|x(t_0)\|) \exp(-\lambda(t - t_0))$ $\forall t \geq t_0$, *for* $x(t_0)$ *sufficiently small.* ■

Definition A.30 (Positive definite function). *A* function $V(x, t)$ *is said to be locally (globally)* positive definite *if and only if there exists a function* $\alpha(\cdot)$ *of class* $\mathcal{K}$ *such that* $V(0, t) = 0$ *and* $V(x, t) \geq \alpha(\|x\|)$ $\forall t \geq 0$ *and* $\forall x$ *in a ball* B. ■

Definition A.31 (Decrescent function). *A* function $V(x, t)$ *is locally (globally)* decrescent *if and only if there exists a function* $\beta(\cdot)$ *of class* $\mathcal{K}$ *such that* $V(0, t) = 0$ *and* $V(x, t) \leq \beta(\|x\|)$, $\forall t > 0$ *and* $\forall x$ *in a ball* B. ■

The main Lyapunov stability theorem can now be stated as follows.

Theorem A.32. *Assume that $V(x,t)$ has continuous first derivatives around the equilibrium point $x^\star = 0$. Consider the following conditions on $V(\cdot)$ and $\dot{V}(\cdot)$ where $\alpha(\cdot)$, $\beta(\cdot)$ and $\gamma(\cdot)$ denote functions of class $\mathcal{K}$,and let B_r be the closed ball with radius $r > 0$ and center $x^\star = 0$:*

$$
\begin{aligned}
&(i)\ V(x,t) \geq \alpha(\|x\|) > 0,\ \forall\, x \in B_r,\ \forall\, t \geq t_0 \\
&(ii)\ \dot{V}(x,t) \leq 0 \\
&(iii)\ V(x,t) \leq \beta(\|x\|),\ \forall\, x \in B_r,\ \forall\, t \geq t_0 \\
&(iv)\ \dot{V}(x,t) \leq -\gamma(\|x\|) < 0,\ \forall\, x \in B_r,\ \forall\, t \geq t_0 \\
&(v)\ \lim_{x\to\infty} \alpha(\|x\|) = \infty.
\end{aligned}
\tag{A.10}
$$

Then the equilibrium point $x^\star = 0$ is:

- *Stable if conditions (i) and (ii) hold*
- *Uniformly stable if conditions (i)–(iii) hold*
- *Uniformly asymptotically stable if conditions (i)–(iv) hold*
- *Globally uniformly asymptotically stable if conditions (i)–(iv) hold globally, i.e. $B_r = I\!R^n$ and (v) holds* ■

Barbalat's Lemma

Krasovskii-LaSalle's results are only applicable to autonomous systems. On the other hand, Barbalat's Lemma can be used to obtain stability results when the Lyapunov function derivative is negative semi-definite.

Lemma A.33. *[Barbalat] If the differentiable function $f(\cdot)$ has a finite limit as $t \to \infty$, and if $\dot{f}(\cdot)$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$.* ■

This lemma can be applied for studying stability of non-autonomous systems with Lyapunov Theorem, as stated by the following result.

Lemma A.34. *If a scalar function $V(x,t)$ is lower bounded and $\dot{V}(x,t)$ is negative semi-definite, then $\dot{V}(x,t) \to 0$ as $t \to \infty$ if $\dot{V}(x,t)$ is uniformly continuous in time.* ■

Matrosov's Theorem

Theorem A.35 (Matrosov's Theorem). *Let $\Omega \subset I\!R^n$ be an open connected domain containing the origin $x = 0$. If there exists two continuously differentiable functions $V : [t_0, +\infty) \times \Omega \to I\!R$ and $W : [t_0, +\infty) \times \Omega \to I\!R$, a continuous function $V^\star : \Omega \to I\!R$, three functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$ of class $\mathcal{K}$, such that for every $(x,t) \in [t_0, +\infty) \times \Omega$ one has*

- $\alpha(||x||) \leq V(t,x) \leq \beta(||x||)$
- $\dot{V}(t,x) \leq V^{\star}(x) \leq 0$
- $|W(t,x)|$ *is bounded*
- $\max(d(x,E)), |\dot{W}(t,x)|) \geq \gamma(||x||)$*, where* $E = \{x \in \Omega \mid V^{\star}(x) = 0\}$
- $||f(t,x)||$ *is bounded*

Choosing $a > 0$ *such that the closed ball* $\bar{B}_a \subset \Omega$*, define for all* $t \in [t_0, +\infty)$*:* $V_{t,a}^{-1} = \{x \in \Omega \mid V(t,x) \leq \alpha(a)\}$*. Then*

- *For all* $x_0 \in V_{t_0,a}^{-1}$*,* $x(t)$ *tends to zero asymptotically uniformly in* t_0*,* x_0
- *The origin is uniformly asymptotically stable in the sense of Lyapunov*

■

The following may help in checking the Theorem's conditions.

Lemma A.36. *[389] The fourth condition in Matrosov's Theorem is satisfied if:*

- $\dot{W}(x,t)$ *is continuous in both arguments and depends on time in the following way:* $\dot{W}(x,t) = g(x,\beta(t))$ *where* $g(\cdot)$ *is continuous in both arguments,* $\beta(\cdot)$ *is continuous and its image lies in a bounded set* K*.*
- *There exists a class* $\mathcal{K}$ *function* $\kappa(\cdot)$ *such that* $|\dot{W}(x,t)| \geq \kappa(||x||)$ *for all* $x \in E$ *and alll* $t \geq t_0$*.* ■

A.2 Differential Geometry Theory

Consider a nonlinear affine single-input/single-output system of the form

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ \\ y(t) = h(x(t)) \end{cases} \tag{A.11}$$

where $h : I\!\!R^n \to I\!\!R$ and $f, g : I\!\!R^n \to I\!\!R^n$ are smooth functions. For ease of presentation we assume that the system (A.11) has an equilibrium at $x^* = 0$.

Definition A.37 (Lie derivative). *The* Lie derivative *of* h *with respect to* f *is the scalar*

$$L_f h = \frac{\partial h}{\partial x} f$$

and the higher derivatives satisfy the recursion

$$L_f^i h = L_f(L_f^{i-1} h)$$

with $L_f^0 h = h$*.* ■

Definition A.38 (Lie bracket). *The* Lie bracket *of f and g is the vector*

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g,$$

and the recursive operation is established by

$$ad_f^i g = [f, ad_f^{i-1} g]$$

■

Some properties of Lie brackets are:

$$[\alpha_1 f_1 + \alpha_2 f_2, g] = \alpha_1 [f_1, g] + \alpha_2 [f_2, g]$$

$$[f, g] = -[g, f]$$

and the Jacobi identity

$$L_{ad_g} h = L_f(L_g h) - L_g(L_f h)$$

To define nonlinear changes of coordinates we need the following concept.

Definition A.39 (Diffeomorphism). *A function $\phi(x) : I\!R^n \to I\!R^n$ is said to be a* diffeomorphism *in a region $\Omega \in I\!R^n$ if it is smooth, and $\phi^{-1}(x)$ exists and is also smooth.* ■

A sufficient condition for a smooth function $\phi(x)$ to be a diffeomorphism in a neighbourhood of the origin is that the Jacobian $\partial\phi/\partial x$ be nonsingular at zero. The conditions for feedback linearizability of a nonlinear system are strongly related with the following theorem.

Theorem A.40. *[Frobenius] Consider a set of linearly independent vectors $\{f_1(x), \ldots, f_m(x)\}$ with $f_i(x) : I\!R^n \to I\!R^n$. Then, the following statements are equivalent:*

(i) (Complete integrability) *there exist $n-m$ scalar functions $h_i(x) : I\!R^n \to I\!R$ such that*

$$L_{f_j} h_i = 0 \qquad 1 \leq i \quad j \leq n-m$$

where $\partial h_i/\partial x$ are linearly independent

(ii) (Involutivity) *there exist scalar functions $\alpha_{ijk}(x) : I\!R^n \to I\!R$ such that*

$$[f_i, f_j] = \sum_{k=1}^{m} \alpha_{ijk}(x) f_k(x)$$

■

A.2.1 Normal Form

In this section we present the normal form of a nonlinear system which has been instrumental for the development of the feedback linearizing technique. For this, it is convenient to define the notion of relative degree of a nonlinear system.

Definition A.41 (Relative degree). *The single input-single output system (A.11) has* relative degree r *at* $x = 0$ *if*

(i) $L_g L_f^k h(x) = 0$, *for all* x *in a neighbourhood of the origin and for all* $k < r - 1$
(ii) $L_g L_f^{r-1} h(x) \neq 0$ ∎

It is worth noticing that in the case of linear systems, *e.g.*, $f(x) = Ax$, $g(x) = Bx$, $h(x) = Cx$, the integer r is characterized by the conditions $CA^k B = 0\ \forall k < r - 1$ and $CA^{r-1}B \neq 0$. It is well known that these are exactly the conditions that define the relative degree of a linear system. Another interesting interpretation of the relative degree is that r is exactly the number of times we have to differentiate the output to obtain the input explicitly appearing. Let us now asumme that u and y both have dimension m in (A.11).

Definition A.42 (Vector relative degree). *The multi input-multi output system (A.11) has* vector relative degree $[r_1, r_2, ..., r_m]^T \in I\!R^m$ *at* $x = 0$ *if*

(i) $L_{g_j} L_f^k h(x) = 0$, $\forall x$ *in a neighborhood of the origin and* $\forall k < r_j - 1$
(ii) The matrix $\begin{pmatrix} L_{g_1} L_f^{r_1-1} h_1 & .. & L_{g_m} L_f^{r_1-1} h_1 \\ . & .. & . \\ . & .. & . \\ L_{g_1} L_f^{r_m-1} h_m & .. & L_{g_m} L_f^{r_m-1} h_m \end{pmatrix}$ *is nonsingular in a neighborhood of the origin* ∎

Definition A.43 (Uniform vector relative degree). *Let* $u(t) \in I\!R^m$ *and* $y(t) \in I\!R^m$ *in (A.11). The system is said to have a uniform relative degree* r *if* $r_i = r$ *for all* $1 \leq i \leq m$ *in the previous definition.* ∎

We note that this definition is different from the definition of the uniform relative degree in [227, p.427] where uniformity refers to the fact that the system (single input-single output) has a (scalar) relative degree r at each $x(t) \in I\!R^n$. Here we rather employ uniformity in the sense that the vector relative degree has equal elements. In the linear invariant multivariable case, such a property has favourable consequences as recalled a few lines below.

The functions $L_f^i h$ for $i = 0, 1, \ldots, r - 1$ have a special significance as demonstrated in the following theorem.

Theorem A.44. *[Normal form] If the single input-single output system (A.11) has relative degree $r \leq n$, then it is possible to find $n-r$ functions $\phi_{r+1}(x), \ldots, \phi_n(x)$ so that*

$$\phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{pmatrix} \tag{A.12}$$

is a diffeomorphism $z = \phi(x)$ that transforms the system into the following normal form

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \vdots \\ \dot{z}_{r-1} = z_r \\ \dot{z}_r = b(z) + a(z)u \\ \dot{z}_{r+1} = q_{r+1}(z) \\ \vdots \\ \dot{z}_n = q_n(z) \end{cases} \tag{A.13}$$

Moreover, $a(z) \neq 0$ in a neighborhood of $z_0 = \phi(0)$. ■

A similar canonical form can be derived for the multivariable case, however more it is more involved [227]. In the case of a linear time invariant system (A, B, C), a similar canonical state space realization has been shown to exist in [432], provided $CA^iB = 0$ for all $i = 0, 1, ..., r-2$, and the matrix $CA^{r-1}B$ is nonsingular. This Sannuti's canonical form is quite interesting as the zero dynamics taks the form $\dot{\xi}(t) = A_0\xi(t) + B_0 z_1(t)$: it involves only the output z_1 of the system. The conditions on the Markov parameters are sufficient conditions for the invertibility of the system [429]. Other such canonical state space representations have been derived by Sannuti and co-workers [424, 433, 434], which are usually not mentioned in textbooks.

A.2.2 Feedback Linearization

From the above theorem we see that the state feedback control law

$$u = \frac{1}{a(z)}(-b(z) + v) \tag{A.14}$$

yields a closed-loop system consisting of a chain of r integrators and an $(n-r)$-dimensional autonomous system. In the particular case of $r = n$ we *fully linearize* the system. The first set of conditions for the triple $\{f(x), g(x), h(x)\}$ to have relative degree n is given by the partial differential equation

$$\frac{\partial h}{\partial x}\left(g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)\right) = 0$$

The Frobenius Theorem shows that the existence of solutions to this equation is equivalent to the involutivity of $\{g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)\}$. It can be shown that the second condition, *i.e.* $L_g L_f^{n-1} h(x) \neq 0$ is ensured by the linear independence of $\{g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)\}$.

The preceding discussion is summarized by the following key Theorem.

Theorem A.45. *For the system (A.11) there exists an output function $h(x)$ such that the triple $\{f(x), g(x), h(x)\}$ has relative degree n at $x = 0$ if and only if:*

(i) The matrix $\{g(0), ad_f g(0), \ldots, ad_f^{n-1} g(0)\}$ is full rank
(ii) The set $\{g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)\}$ is involutive around the origin

■

The importance of the preceding theorem can hardly be overestimated. It gives (*a priori* verifiable) necessary and sufficient conditions for full linearization of a nonlinear affine system. However, it should be pointed out that this control design approach requires on one hand the solution of a set of partial differential equations. On the other hand, it is intrinsically nonrobust since it relies on exact cancellation of nonlinearities. In the linear case this is tantamount to pole-zero cancellation.

A.2.3 Stabilization of Feedback Linearizable Systems

If the relative degree of the system $r < n$ then, under the action of the feedback linearizing controller (A.14), there remains an $(n-r)$-dimensional subsystem. The importance of this subsystem is underscored in the proposition below.

Theorem A.46. *Consider the system (A.11) assumed to have relative degree r. Further, assume that the trivial equilibrium of the following $(n-r)$-dimensional dynamical system is* locally *asymptotically stable:*

$$\begin{cases} \dot{z}_{r+1} = q_{r+1}(0, \ldots, 0, z_{r+1}, \ldots, z_n) \\ \vdots \\ \dot{z}_n = q_n(0, \ldots, 0, z_{r+1}, \ldots, z_n) \end{cases}$$

where $q_{r+1}, \ldots, q_n$ are given by the normal form. Under these conditions, the control law (A.14) yields a locally *asymptotically stable closed-loop system.* ■

The $(n-r)$-dimensional system (A.15) is known as the *zero dynamics*. It represents the dynamics of the unobservable part of the system when the input is set equal to zero and the output is constrained to be identically zero. It is worth highlighting the qualifier *local* in the above theorem; in other words, it can be shown that the conditions above are not enough to ensure *global* asymptotic stability.

A.2.4 Further Reading

The original Lyapunov Theorem is contained in [322], while stability of nonlinear dynamic systems is widely covered in [282,292]. The proofs of the theorems concerning Lyapunov stability theorem can be found in [199, 256, 500]. An extensive presentation of differential geometry methods can be found in [227] and the references therein. For the extension to the multivariable case and further details we refer the reader again to [227, 381].

A.3 Viscosity Solutions

This section intends to briefly describe what viscosity solutions of first order nonlinear partial differential equations of the form

$$F(x, V(x), \nabla V(x)) = 0 \tag{A.15}$$

are, where $x \in I\!R^n$, $V : I\!R^n \to I\!R$, ∇ is the differential operator (the Euclidean gradient), and $F : I\!R^n \times I\!R \times I\!R^n \to I\!R$ is continuous. A function $V(\cdot)$ is differentiable at x and with derivative ζ if

$$\lim_{z \to x} \frac{V(z) - V(x) - \zeta^T(z-x)}{|z-x|} = 0 \tag{A.16}$$

and this equality can equivalently be stated with the two inequalities supposed to hold simulatenaously

$$\limsup_{z \to x} \frac{V(z) - V(x) - \zeta^T(z-x)}{|z-x|} \leq 0 \tag{A.17}$$

(in other words ζ satisfies (A.17) if and only if the plane $z \mapsto V(z) + \zeta^T(z-x)$ is tangent from above to the graph of $V(\cdot)$ at x), and

$$\liminf_{z \to x} \frac{V(z) - V(x) - \zeta^T(z-x)}{|z-x|} \geq 0, \tag{A.18}$$

(in other words ζ satisfies (A.18) if and only if the plane $z \mapsto V(z) + \zeta^T(z-x)$ is tangent from below to the graph of $V(\cdot)$ at x). The *superdifferential* of $V(\cdot)$ at x is then defined as the *set*

$$D^+V(x) = \{\zeta \in I\!R^n \ | \ (A.17)\text{holds}\}$$

and the subdifferential of $V(\cdot)$ at x is then defined as the *set*

$$D^-V(x) = \{\zeta \in I\!R^n \ | \ (A.18)\text{holds}\}$$

It is noteworthy that such sets may be empty, see the examples below. Sometimes these sets are named one-sided differentials. The function $V(\cdot)$ is said to be a *viscosity subsolution* of the partial differential equation (A.15) if for each $x \in I\!R^n$ one has

$$F(x, V(x), \zeta) \le 0$$

for all $\zeta \in D^+V(x)$. The function $V(\cdot)$ is said to be a *viscosity supersolution* of the partial differential equation (A.15) if for aeach $x \in I\!R^n$ one has

$$F(x, V(x), \zeta) \ge 0$$

for all $\zeta \in D^-V(x)$. The function $V(\cdot)$ is said to be a *viscosity solution* of the partial differential equation (A.15) if it is both a viscosity subsolution and a viscosity supersolution of this partial differential equation. As we already pointed out in section 4.4.5, in case of proper [1] convex functions the viscosity subdifferential (or subgradient) and the convex analysis subgradient, are the same [415, Proposition 8.12]. We now consider two illustrating examples taken from [64].

Example A.47. Consider the function

$$V(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \in [0,1] \\ 1 & \text{if } x > 1 \end{cases} \tag{A.19}$$

Then $D^+V(0) = \emptyset$, $D^-V(0) = [0, +\infty)$, $D^+V(x) = D^-V(x) = \{\frac{1}{2}\sqrt{x}\}$ if $x \in (0,1)$, $D^+V(1) = [0, \frac{1}{2}]$, $D^-V(1) = \emptyset$.

Example A.48. Consider $F(x, V(x), \nabla V(x)) = 1 - |\frac{\partial V}{\partial x}|$. Then $V : I\!R \to I\!R^+$ $x \mapsto |x|$ is a viscosity solution of $1 - |\frac{\partial V}{\partial x}| = 0$. Indeed $V(\cdot)$ is differentiable at all $x \neq 0$ and one has $D^+V(0) = \emptyset$, and $D^-V(0) = [-1, 1]$. $V(\cdot)$ is indeed a supersolution since $1 - |\zeta| \ge 0$ for all $\zeta \in D^-u(0) = [-1, 1]$.

Example A.49. The same function $V(x) = |x|$ is not a viscosity solution of $-1 + |\frac{\partial V}{\partial x}| = 0$. At $x = 0$ and choosing $\zeta = 0$ one obtains $-1 + |0| = -1 < 0$ so the function is not a supersolution, though it is a viscosity subsolution.

[1] Proper in this context means that $V(x) < +\infty$ for at least one $x \in I\!R^n$, and $V(x) > -\infty$ for all $x \in I\!R^n$.

It is a fact that if $V(\cdot)$ is convex and not differentiable at x then $D^+V(x) = \emptyset$. The following Lemma says a bit more.

Lemma A.50. *Let $V(\cdot)$ be continuous on some interval $I \ni x$. Then:*

- *If $V(\cdot)$ is differentiable at x: $D^+V(x) = D^-V(x) = \{\nabla V(x)\}$*
- *If the sets $D^+V(x)$ and $D^-V(x)$ are both nonempty, then $V(\cdot)$ is differentiable at x and the first item holds*
- *The sets of points where a one-sided differential exists:*

$$I^+ = \left\{x \in I \ \mid\ D^+V(x) \neq \emptyset\right\}$$

and

$$I^- = \left\{x \in I \ \mid\ D^-V(x) \neq \emptyset\right\}$$

are both nonempty. Both I^+ and I^- are dense in I. ■

The second item says that if a function is not differentiable at x then necessarily one of the two sets must be empty. This confirms the above examples. The third item says that the points x where the continuous function $V(\cdot)$ admits a superdifferential and a subdifferential, exist in I and even are numerous in I: they form dense subsets of I (take any point $y \in I$ and any neighborhood of y: there is an x in such a neighborhood at which $V(\cdot)$ has a one-sided differential). There is another way to define a viscosity solution.

Lemma A.51. *Let $V(\cdot)$ be continuous on some interval I. Then*

- *$\zeta \in D^+V(x)$ if and only if there exists a function $\varphi \in C^1(I)$ such that $\nabla\varphi(x) = \zeta$ and $V - \varphi$ has a local maximum at x.*
- *$\zeta \in D^-V(x)$ if and only if there exists a function $\varphi \in C^1(I)$ such that $\nabla\varphi(x) = \zeta$ and $V - \varphi$ has a local minimum at x.* ■

From the first item it becomes clear why a convex function that is not differentiable at x has $D^+V(x) = \emptyset$. Then a continuous function $V(\cdot)$ is a viscosity subsolution of $F(x, V(x), \nabla V(x)) = 0$ if for every C^1 function $\varphi(\cdot)$ such that $V - \varphi$ has a local maximum at x one has $F(x, V(x), \nabla\varphi(x)) \leq 0$. It is a viscosity supersolution of $F(x, V(x), \nabla V(x)) = 0$ if for every C^1 function $\varphi(\cdot)$ such that $V - \varphi$ has a local minimum at x one has $F(x, V(x), \nabla\varphi(x)) \geq 0$.

The following result is interesting:

Proposition A.52. *[418] Given a system $\dot{x}(t) = f(x(t), u(t))$ whose solution on $[t_0, t_1]$ is an absolutely continuous function such that $\dot{x}(t) = f(x(t), u(t))$ for almost all $t \in [t_0, t_1]$, a supply rate $w(x, u)$ such that $w(0, u) \geq 0$, and a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, then:*

$$V(x(t_1)) - V(x(t_0) \leq \int_{t_0}^{t_1} w(x(t), u(t))dt$$

holds for every solution $[t_0, t_1] \to \mathbb{R}^n$

$$\Updownarrow$$

$$\zeta^T f(x, u) \leq w(x, u) \ \ \textit{for every } x \in \mathbb{R}^n,\ u \in \mathcal{U},\ \textit{and } \zeta \in D^- V(x) \tag{A.20}$$

In other words one may write the infinitesimal version of the dissipation inequality when the storage function is not differentiable, by replacing its gradient by a viscosity subgradient.

A.4 Algebraic Riccati Equations

The topic of studying and solving Riccati equations is a wide topic and we do not pretend to cover it in this small appendix. The results we present only aim at showing that under some conditions which are different from the conditions stated in the foregoing chapters, existence of solutions to algebraic Riccati equations can be guaranteed. Let us consider the following algebraic Riccati equation:

$$PDP + PA + A^T P - C = 0, \tag{A.21}$$

where $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{n \times n}$. P is the unknown matrix. Before going on we need a number of definitions. A subspace $\Omega \subset \mathbb{R}^{2n}$ is called $N-$neutral if $x^T N y = 0$ for all $x, y \in \Omega$ (Ω may be $\mathrm{Ker}(N)$, or $\mathrm{Ker}(N^T)$). The neutrality index $\gamma(M, N)$ of a pair of matrices (M, N) is the maximal dimension of a real $M-$invariant $N-$neutral subspace in $\mathbb{R}^{2n}$. A pair of matrices (A, D) is sign controllable if every $\lambda_0 \in \mathbb{R}$ at least one of the subspaces $\mathrm{Ker}(\lambda_0 I_n - A)^n$ and $\mathrm{Ker}(-\lambda_0 I_n - A)^n$ is contained in $\mathrm{Range}[D, AD, ..., A^{n-1}D]$, and for every $\lambda + j\mu \in \mathbb{C}$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, $\mu \neq 0$, at least one of the two subspaces $\mathrm{Ker}[(\lambda^2 + \mu^2)I_n \pm 2\lambda A + A^2]^n$ is contained in $\mathrm{Range}[D, AD, ..., A^{n-1}D]$. Another way to characterize the sign-controllability of the pair (A, D) is: for any $\lambda \in \mathbb{C}$, at least one of the two matrices $(\lambda I_n - A \ \ D)$ and $(-\bar{\lambda} I_n - A \ \ D)$ is full-rank [273]. Sign-controllability of (A, D) implies that there exists a matrix K such that $F = A + DK$ is *unmixed,i.e.* $\sigma(F) \cap \sigma(-F^T) = \emptyset$.

We now define the two matrices in $I\!R^{2n \times 2n}$

$$M = \begin{bmatrix} A & D \\ & \\ C & -A^T \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & I_n \\ & \\ -I_n & 0 \end{bmatrix}$$

Theorem A.53. *[416] Let $D \geq 0$ and (A, D) be sign controllable. Suppose that the matrix M is invertible. Then the following statements are equivalent:*

- *The ARE (A.21) has a real solution.*
- *The ARE (A.21) has a real solution P for which rank$(P - P^T) \leq 2(n - \gamma(M, H))$.*
- *The matrix M has a real $n-$dimensional invariant subspace.*
- *Either n is even, or n is odd and M has a real eigenvalue.*

If $\gamma(M, N) = n$ there exists a real symmetric solution. ■

We recall that an $M-$invariant subspace Ω is a subspace such that for all $v \in \Omega$, $Mv \in \Omega$. Comparing (A.21) to (3.140) one sees that A in (A.21) is replaced by $A + BR^{-1}C$ in (3.140), whereas $BR^{-1}B^T$ in (3.140) plays the role of D in (A.21), and $C^T R^{-1} C + Q$ in (3.140) plays the role of $-C$ in (A.21). Theorems 3.42, 3.43 and 3.44 state stronger results than Theorem A.53 since the negative definiteness (resp. positivite definiteness) of the solution is crucial in the framework of optimal control (resp. dissipative systems). On the other hand the conditions of Theorems 3.42, 3.43 and 3.44 look much simpler than those of Theorem A.53.

Let us now consider the following Riccati inequality:

$$PA + A^T P - PBB^T P + Q > 0 \tag{A.22}$$

Lemma A.54. *[259] Suppose that the pair (A, B) is stabilizable. The following three statements are equivalent:*

- *There exists a symmetric matrix P solving (A.22).*
- *There exists a symmetric matrix P^- such that*

$$P^- A + A^T P^- - P^- BB^T P^- + Q = 0, \ \ \sigma(A - BB^T P^-) \subset \mathbb{C}^-.$$

- *The Hamiltonian matrix $H = \begin{pmatrix} A & -BB^T \\ & \\ -Q & -AT \end{pmatrix}$ has no eigenvalues on the imaginary axis.*

Suppose that one of these conditions hold. Then any solution P of (A.22) satisfies $P < P^-$. ■

The notation $\sigma(A) \in \mathbb{C}^-$ means that all the eigenvalues of A have negative real parts. In case the pair (A, B) is not stabilizable, things are more complex and one has first to perform a decomposition of A and B before proposing a test; see [443].

Further study on Riccati equations, their solvability and their link with the KYP Lemma set of equations solvability, may be found in [37, 38]. A special type of Riccati equations that correspond to the KYP Lemma set of equations for descriptor systems may be found in [501]. See also [290] for upper bounds estimation of solutions to AREs. The problem of the existence of a real symmetric negative semi-definite solution to AREs is a tricky problem [490].

A.4.1 Reduced Riccati Equation for WSPR Systems

The following is taken from Hodaka *et al.* [213]. As we have explained the KYP Lemma set of equations form a LMI which is in turn equivalent to a Riccati equation. When the transfer function $H(s) \in \mathbb{C}^{m\times m}$ is SSPR, then $D + D^T > 0$ and this Riccati equation has a positive definite symmetric solution P. The point now is: what happens when $H(s)$ is not SSPR and when $D \neq 0$? The algorithm that is proposed next, allows one to characterize WSPRness, SPRness and PRness in terms of Riccati equations and the inherent integration of the system. The developments are rather lengthy and need some preliminary results.

Definition A.55. *[429] The transfer function $H(s) = C(sI_n - A)^{-1}B + D \in \mathbb{C}^{m\times m}$ is invertible if there exists a proper transfer function $\hat{H}(s)$ and a nonnegative integer l such that*

$$\hat{H}(s)H(s) = \frac{1}{s^l} I_m \tag{A.23}$$

The least integer l satisfying (A.23) is called the inherent integration of $H(s)$. ■

A $m \times m$ transfer function is invertible if and only if it has rank m over the field of proper transfer functions.

Lemma A.56. *[429] The inherent integration of $H(s)$ is k if and only if for $l = 0, 1, 2, ..., k-1$,*

$$rank(Q_l[A, B, L, W]) - rank(Q_{l-1}[A, B, L, W]) < m \tag{A.24}$$

and $rank(Q_k[A, B, L, W]) - rank(Q_{k-1}[A, B, L, W]) = m$, where $Q_0(A, B, L, W) \stackrel{\Delta}{=} W$, for $l = 1, 2, ...$

$$Q_l(A,B,L,W) \triangleq \begin{pmatrix} W & 0 & 0 & .... & 0 \\ LB & W & 0 & ... & 0 \\ LAB & LB & W & ... & 0 \\ ... & & & & \\ LA^{l-1}B & LA^{l-2}B & LA^{l-3}B & ... & W \end{pmatrix}$$

and rank$(Q_{-1}[A,B,L,W]) \triangleq 0.$ ■

We now proceed with the main developments. Let us assume that $H(s)$ is PR and that $\text{rank}(H(s)+H^T(-s)) = m$. Let $H_0(s) \triangleq H(s)$, and $\Sigma_0 = (A_0, B_0, C_0, D_0)$ is a minimal realization of H_0 with $A_0 \in \mathbb{R}^{n_0 \times n_0}$, $B_0 \in \mathbb{R}^{n_0 \times m}$, $C_0 \in \mathbb{R}^{m \times n_0}$, $D_0 \in \mathbb{R}^{m \times m}$, $n_0 \geq 1$. The rest of the sequence $\{\Sigma_i\}_{i\geq 0}$ is constructed as follows: $\Sigma_i = (A_i, B_i, C_i, D_i)$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m}$, $C_i \in \mathbb{R}^{m \times n_i}$, $D_i \in \mathbb{R}^{m \times m}$, $H_i(s) = C_i(sI_n - A_i)^{-1}B_i + D_i$, and we define $\Phi_i(s) = H_i(s) + H_i^T(-s)$. Let us suppose that $H_i(s)$ is PR. The next three conditions are denoted as Π_i:

- (**a**) (A_i, B_i, C_i, D_i) is minimal.
- (**b**) The KYP Lemma set of equations in (3.2) is satisfied with (A_i, B_i, C_i, D_i), $P_i = P_i^T > 0$ and L_i, W_i, and we denote $R_i = W_i^T W_i = D_i + D_i^T$.
- (**c**) $\text{rank}(\Phi_i(s)) = m$.

Taking $G_i(s) = W_i + L_i^T(sI_{n_i} - A_i)^{-1}B_i$ gives $\Phi_i(s) = G_i^T(-s)G_i(s)$, so that (**c**) can be replaced by $\text{rank}(G_i(s)) = m$ or

$$\det \begin{pmatrix} A_i - sI_{n_i} & B_i \\ L_i^T & W_i \end{pmatrix} \not\equiv 0 \tag{A.25}$$

Let r_i =rank(R_i). If $r_i = m$ then $R_i > 0$ and the algorithm is terminated. In such a case the transfer function $H(s)$ is SSPR. If $r_i < m$, the algorithm proceeds as follows. Since $R_i = R_i^T \geq 0$, there exists an orthogonal matrix $S_i \in \mathbb{R}^{m \times m}$ such that

$$S_i^T R_i S_i = \begin{pmatrix} \bar{R}_i & 0 \\ 0 & 0 \end{pmatrix} \tag{A.26}$$

where $\bar{R}_i \in \mathbb{R}^{r_i \times r_i}$ is positive definite. Partition S_i such that $S_i = [S_{i1} \;\; S_{i2}]$ with $S_{i1} \in \mathbb{R}^{m \times r_i}$, $S_{i2} \in \mathbb{R}^{m \times (m-r_i)}$. Using the nonsingular matrix S_i, let us introduce the matrices

$$[B_{i1} \quad B_{i2}] = [B_i S_{i1} \quad B_i S_{i2}] \tag{A.27}$$

$$[W_{i1} \quad W_{i2}] = [W_i S_{i1} \quad W_i S_{i2}] \tag{A.28}$$

$$\begin{pmatrix} C_{i1} \\ \\ C_{i2} \end{pmatrix} = \begin{pmatrix} S_{i1}^T C_i \\ \\ S_{i2}^T C_i \end{pmatrix} \tag{A.29}$$

$$\begin{pmatrix} D_{i1} & D_{i2} \\ \\ D_{i3} & D_{i4} \end{pmatrix} = \begin{pmatrix} S_{i1}^T D_i S_{i1} & S_{i1}^T D_i S_{i2} \\ \\ S_{i2}^T D_i S_{i1} & S_{i2}^T D_i S_{i2} \end{pmatrix} \tag{A.30}$$

Lemma A.57. *[213] If the property Π_i is satisfied, then:*

- $W_{12} = 0$ *and* $P_i B_{i2} = C_{i2}^T$.
- B_{i2} *has full column rank,* C_{i2} *has full row rank and* $n_i \geq m - r_i$. ∎

Proof: Let us pre and post-multiply the KYP Lemma set of equations of property Π_i (**b**) above, by $\mathrm{diag}(I_{n_i}, S_i^T)$ and $\mathrm{diag}(I_{n_i}, S_i)$, respectively. The first condition follows. Now postmultiplying the matrix in (A.25) by $\mathrm{diag}(I_{n_i}, S_i)$ gives

$$\det \begin{pmatrix} A_i - sI_{n_i} & B_{i1} & B_{i2} \\ \\ L_i^T & W_{i1} & 0 \end{pmatrix} \not\equiv 0 \tag{A.31}$$

This secures that B_{i2} has full column rank and consequently $n_i \geq m - r_i$. Finally it follows from $P_i B_{i2} = C_{i2}^T$ and $Pi > 0$ that C_{i2} has full row rank. ∎

Let $E_i = C_{i2} B_{i2} = B_{i2}^T P_i B_{i2} = E_i^T > 0$ by Lemma A.57. A square root $\Delta_i = E_i^{\frac{1}{2}} = \Delta_i^T > 0$ exists. Moreover there exist matrices $N_i \in \mathbb{R}^{n_{i+1} \times n_i}$ and $M_i \in \mathbb{R}^{n_i \times n_{i+1}}$ such that

$$N_i B_{i2} = 0, \quad C_{i2} M_i = 0, \quad N_i M_i = I_{n_{i+1}}, \tag{A.32}$$

where $n_{i+1} \stackrel{\Delta}{=} n_i - m + r_i$. Let us define a nonsingular matrix $T_i \in \mathbb{R}^{n_i \times n_i}$ as

$$T_i = \begin{pmatrix} N_i \\ \\ \Delta_i^{-1} C_{i2} \end{pmatrix}, \quad T_i^{-1} = [M_i \quad B_{i2} \Delta_i^{-1}] \tag{A.33}$$

Using the full-rank matrices T_i and S_i we can consider transformations of input, output and state variables as $u_i = S_i \tilde{u}_i$, $y_i S_i \tilde{y}_i$, $\tilde{x}_i = T_i x_i$, where

$$\tilde{u}_i = \begin{pmatrix} S_{i1}^T u_i \\ \\ S_{i2}^T u_i \end{pmatrix} \stackrel{\Delta}{=} \begin{pmatrix} u_{i1} \\ \\ u_{i2} \end{pmatrix} \tag{A.34}$$

$$\tilde{y}_i = \begin{pmatrix} S_{i1}^T y_i \\ \\ S_{i2}^T y_i \end{pmatrix} \stackrel{\Delta}{=} \begin{pmatrix} y_{i1} \\ \\ y_{i2} \end{pmatrix} \tag{A.35}$$

$$\tilde{x}_i = \begin{pmatrix} N_i x_i \\ \\ \Delta_i^{-1} C_{i2} x_i \end{pmatrix} \stackrel{\Delta}{=} \begin{pmatrix} x_{i+1} \\ \\ \bar{x}_i \end{pmatrix} \tag{A.36}$$

Let us now introduce matrices with subscripts $(i+1)$ defined by

$$A_{i+1} = N_i A_i M_i, \;\; B_{i+1} = [N_i A_i B_{i2} \Delta_i^{-1} \;\; N_i B_{i1}]$$

$$C_{i+1} = \begin{pmatrix} -\Delta_i^{-1} C_{i2} A_i M_i \\ \\ C_{i1} M_i \end{pmatrix}$$

$$D_{i+1} = \begin{pmatrix} -\Delta_i^{-1} C_{i2} A_i B_{i2} \Delta_i^{-1} & -\Delta_i^{-1} C_{i2} B_{i1} \\ \\ C_{i1} B_{i2} \Delta_i^{-1} & D_{i1} \end{pmatrix} \tag{A.37}$$

$$P_{i+1} = M_i^T P_i M_i, \;\; L_{i+1} = M_i^T L_i$$

$$W_{i+1} = [L_i^T B_{i2} \Delta_i^{-1} \;\; W_{i1}], \;\; R_{i+1} = D_{i+1} + D_{i+1}^T$$

and the $i+1$-th system is defined as $\Sigma_{i+1} = (A_{i+1}, B_{i+1}, C_{i+1}, D_{i+1})$ with $u_{i+1} = (\bar{x}_i^T \;\; u_{i1}^T)$.

Lemma A.58. *[213] If the property Π_i holds for the system Σ_i, then Π_{i+1} holds for the system Σ_{i+1}.* ■

Proof: Define the matrices $\tilde{A}_i = T_i A_i T_i^{-1}$, $\tilde{B}_i = T_i B_i S_i$, $\tilde{C}_i = S_i^T C_i T_i^T$, $\tilde{D}_i = S_i^T D_i S_i$, $\tilde{R}_i = S_i^T R i S_i$, $\tilde{P}_i = T_i^{-T} P_i T_i^{-1}$, $\tilde{L}_i^T = L_i^T T_i^{-1}$, $\tilde{W}_i = W_i S_i$. (**a**): the controllability of (A_i, B_i) is equivalent to that of $(\tilde{A}_i, \tilde{B}_i)$. Since Δ_i is full-rank it follows that (A_{i+1}, B_{i+1}) is controllable. In a similar way one can show that (A_{i+1}, C_{i+1}) is observable. (**b**): since M_i has full column rank and $P_i = P_i^T > 0$, it follows that $P_{i+1} = P_{i+1}^T > 0$. Pre and postmultiplying the KYP Lemma set of equations for Σ_i by diag$(T_i^{-T} \;\; S_i^T)$ and diag$(T_i^{-1} \;\; S_i)$ respectively, we obtain the KYP Lemma set of equations for Σ_{i+1}. (**c**): pre and postmultiplying the matrix in (A.25) by diag$(T_i \;\; I_m)$ and diag$(T_i^{-1} \;\; S_i)$ respectively, and since δ_i is full-rank, it follows that rank$(\Phi_{i+1}(s)) = m$. ■

One sees that the algorithm preserves the PRness, in the sense that if Σ_i is PR then Σ_{i+1} is PR. However there is no guarantee yet that the algorithm terminates.

Lemma A.59. *[213] Assume that the property Π_i holds for the transfer function $H_i(s)$. Consider the transfer functions $G_i(s) = W_i + L_i^T(sI_{n_i} - A_i)^{-1} B_i$ and $G_{i+1}(s)$ as defined in the algorithm. Let k_i and k_{i+1} be the inherent integrations of $G_i(s)$ and $G_{i+1}(s)$ respectively. Then $k_{i+1} = k_i - 1$.* ■

Proof: the equations

$$\operatorname{rank}(Q_l[A_i, B_i, L_i, W_i]) = r_i + \operatorname{rank}(Q_{l-1}[A_{i+1}, B_{i+1}, L_{i+1}, W_{i+1}]) \quad \text{(A.38)}$$

for $l = 0, 1, 2, \ldots$ secure that

$$\operatorname{rank}(Q_{l-1}[A_{i+1}, B_{i+1}, L_{i+1}, W_{i+1}]) - \operatorname{rank}(Q_{l-2}[A_{i+1}, B_{i+1}, L_{i+1}, W_{i+1}])$$

$$= \operatorname{rank}(Q_l[A_i, B_i, L_i, W_i]) - \operatorname{rank}(Q_{l-1}[A_i, B_i, L_i, W_i])$$

for $l = 0, 1, 2, \ldots$. Then it follows from Lemma A.56 that $k_{i+1} = k_i - 1$. Therefore it is sufficient to prove (A.38). From the definitions one has

$$W_{i+1} = [L_i B_{i2} \Delta_i^{-1} \quad W_{i1}]$$

and

$$L_{i+1}^T A_{i+1}^l B_{i+1} = L_i^T M_i (N_i A_i M_i)^l N_i [A_i B_{i2} \Delta_i^{-1} \quad B_{i1}]$$

for $l = 0, 1, 2, \ldots$. By noting these identities and $M_i N_i = I_{n_i} - B_{i2} E_i^{-1} C_{i2}$, we can verify the equation

$$\operatorname{rank}(Q_{l-1}[A_{i+1}, B_{i+1}, L_{i+1}, W_{i+1}]) =$$

$$= \operatorname{rank} \begin{pmatrix} L_i B_{i2} & W_{i1} & 0 & 0 & \ldots & 0 & 0 \\ L_i A_i B_{i2} & L_i B_{i1} & L_i B_{i2} & W_{i1} & \ldots & 0 & 0 \\ \ldots & & & & & \ldots & \\ L_i A_i^{l-1} B_{i2} & L_i A_i^{l-2} B_{i1} & L_i A_i^{l-2} B_{i2} & L_i A_i^{l-3} B_{i2} & \ldots & L_i B_{i2} & W_{i1} \end{pmatrix}$$

We may also obtain

$$\operatorname{rank}(Q_l[A_i, B_i, L_i, W_i]) =$$

$$= \operatorname{rank} \begin{pmatrix} W_{i1} & 0 & 0 & 0 & \ldots & 0 & 0 \\ L_i B_{i1} & L_i B_{i2} & W_{i1} & 0 & \ldots & 0 & 0 \\ \ldots & & & & & \ldots & \\ L_i A_i^{l-1} B_{i1} & L_i A_i^{l-1} B_{i2} & L_i A_i^{l-2} B_{i1} & L_i A_i^{l-2} B_{i2} & \ldots & W_{i1} & 0 \end{pmatrix}$$

Therefore we obtain (A.38). ■

The main result is coming now.

Theorem A.60. *[213] Suppose that the realization (A, B, C, D) is minimal and that $H(s) = C(sI_n - A)^{-1}B + D \in \mathbb{C}^{m\times m}$. If $H(s)$ is PR and rank$(H(s) + H^T(-s)) = m$, then there exists an integer $k \geq 0$ such that the inherent integration of $\Phi(s) = H(s) + H^T(-s)$ is $2k$ and the transfer function $H_k(s)$, which is constructed from the algorithm applied to $H_0(s) = H(s)$ and $(A_0, B_0, C_0, D_0) = (A, B, C, D)$, satisfies $R_k = H_k(\infty) + H_k^T(\infty) > 0$, and any solution P of the KYP Lemma set of equations for $H(s)$ is given by*

$$P = \mathcal{N}_k^T \mathcal{P}_k \mathcal{N}_k \in \mathbb{R}^{n_0\times n_0} \tag{A.39}$$

(notice that $n_0 = n$), where

$$\mathcal{P}_k = diag(P_k, E_{k-1}^{-1}, ..., E_1^{-1}, E_0^{-1})$$

with $\mathcal{P}_k$ a $(n_k + km + \sum_{i=0}^{k-1} r_i) \times (n_k + km + \sum_{i=0}^{k-1} r_i)$ matrix, and

$$\mathcal{N}_k = \begin{pmatrix} N_{k-1}N_{k-2}...N_1N_0 \\ C_{k-1,2}N_{k-2}...N_1N_0 \\ .... \\ C_{12}N_0 \\ C_{02} \end{pmatrix}$$

is a $(n_k + km + \sum_{i=0}^{k-1} r_i) \times n_0$ matrix. The matrix P_k is any symmetric positive definite solution of the Riccati equation:

$$P_kA_k + A_k^T + (C_k^T - P_kA_k)R_k^{-1}(C_k - B_k^TP_k) = 0 \tag{A.40}$$

If $H(s) = C(sI_n - A)^{-1}B + D$ is WSPR then (A.40) has a unique positive definite symmetric stabilizing solution P_k. ■

For the ease of reading let us recall that $N_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $C_{k,2} \in \mathbb{R}^{(m-r_k)\times n_k}$, $E_k \in \mathbb{R}^{(m-r_k)\times(m-r_k)}$. **Proof:** From the KYP Lemma it follows that $\Phi(s) = G(s)G^T(-s)$ holds, where $G(s) = W + L^T(sI_n - A)^{-1}B$. Also since $G(s)$ is invertible there exists a nonnegative integer k such that the inherent integration of $G(s)$ is k. Then it is easy to see that the inherent integration of $\Phi(s)$ is $2k$. Now since (A_0, B_0, C_0, D_0) has the property Π_0 and since the inherent integration of $G(s)$ is k, Lemmas A.58 and A.59 show that we can apply the algorithm to the system Σ_0. Then the system $\Sigma_k = (A_k, B_k, C_k, D_k)$ has the property Π_k and the inherent integration of $G_k(s)$ is zero. The last condition is equivalent to $\det(W_k) \neq 0$, or $R_k = G_k(\infty) + G_k^T(\infty) > 0$. From the property Π_k (**b**) we have $L_k^T = W_k^{-T}(C_k - B_k^TP_k)$ and (A.40) in the case when $n_k > 0$. By examination of $\tilde{P}_i = \begin{pmatrix} P_{i+1} & 0 \\ 0 & I_{n_i - n_{i+1}} \end{pmatrix}$ and $\tilde{P}_i = T_i^{-T}P_iT_i^{-1}$, we get

$$
\begin{aligned}
P_i &= T_i^T \begin{pmatrix} P_{i+1} & 0 \\ 0 & I_{n_i-n_{i+1}} \end{pmatrix} T_i \\
&= [N_i^T \quad C_{i2}^T] \begin{pmatrix} P_{i+1} & 0 \\ 0 & E_i^{-1} \end{pmatrix} \begin{pmatrix} N_i \\ C_{i2} \end{pmatrix}
\end{aligned}
\tag{A.41}
$$

Repeating the equation above, one can obtain (A.39). When $n_k = 0$ it follows from $n_k = n_{k-1} - m + r_{k-1}$ that $n_{k-1} = m - r_{k-1}$. This means that $B_{k-1,2}$ is square and from Lemma A.57 it is nonsingular. This yields

$$P_{k-1} = C_{k-1,2}^T B_{k-1,2}^{-1} = C_{k-1,2} E_{k-1}^{-1} C_{k-1,2}$$

Combining this and (A.41) gives (A.39) with $n_k = 0$. Furthermore if $H(s)$ is WSPR then $\det \begin{pmatrix} A_k - j\omega I_{n_k} & B_k \\ L_k^T & W_k \end{pmatrix} \neq 0$, for all $\omega \in I\!R$, and (A_k, L_k^T) is observable. This means that $H_k(s)$ is SSPR and that the Riccati equation (A.40) has a unique stabilizing solution (see for instance Theorem 3.44). ■

A.5 Some Useful Matrix Algebra

In this section some matrix algebra results are provided, some of which are instrumental in the PR and dissipative systems characterization.

A.5.1 Results Useful for the KYP Lemma LMI

Theorem A.61. *Let $G \in I\!R^{n\times n}$, $g \in I\!R^{n\times m}$, $\Gamma \in I\!R^{m\times m}$ be arbitrary matrices and vector, respectively. Then*

$$
\begin{aligned}
\begin{bmatrix} G & g \\ g^T & \Gamma \end{bmatrix} > 0 &\Longleftrightarrow G > 0 \text{ and } \Gamma - g^T G^{-1} g > 0 \\
&\Longleftrightarrow \Gamma > 0 \text{ and } G - g\Gamma^{-1} g^T > 0 \\
&\Longleftrightarrow \rho(g^T G^{-1} g \Gamma^{-1}) < 1
\end{aligned}
\tag{A.42}
$$

Since proving that $\begin{bmatrix} G & g \\ g^T & \Gamma \end{bmatrix} > 0$ is equivalent to proving that $\begin{bmatrix} \Gamma & g^T \\ g & G \end{bmatrix} > 0$, the equivalence between (3.3) and (3.17) follows from Theorem A.61, identifying Γ with $-PA - A^T P$ and G with $D + D^T$. The matrix $\Gamma - g^T G^{-1} g$ is the so-called Schur complement of G in $\begin{bmatrix} G & g \\ g^T & \Gamma \end{bmatrix}$. Another useful result is the following:

Lemma A.62. *[272] Let $G \in \mathbb{R}^{m\times m}$ be an invertible matrix and Γ be square. Then* rank$\begin{bmatrix} G & g \\ g^T & \Gamma \end{bmatrix} = m$ *if and only if* $\Gamma = g^T G^{-1} g$. ■

Still, another result related to the above is the following:

Proposition A.63. *[157, 260] Let* $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$ *be a real symmetric matrix. Then* $M \geq 0$ *if and only if there exists real matrices* L *and* W *such that* $M_{11} = LL^T$, $M_{12} = LW$, $M_{22} \geq W^T W$. *Moreover* $M > 0$ *if and only if* L *is full rank and* $M_{22} > W^T W$. ■

Proof: Let us prove the first part with ≥ 0. The "if" sense is easy to prove. The "only if" is as follows: Assume $M \geq 0$. Let S be any real square matrix such that $M = S^T S$, i.e. S is a square root of M. Let $S = QR$ be the QR factorization of S with an orthonormal matrix Q and an upper triangular matrix R. Then $M = R^T R$ is a Cholesky factorization of M. Let us partition the matrix R as $R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \triangleq \begin{bmatrix} L^T & W \\ 0 & R_{22} \end{bmatrix}$. From $M = R^T R$ it follows that $M_{11} = LL^T$, $M_{12} = LW$, $M_{22} = W^T W + R_{22}^T R_{22} \geq W^T W$. Therefore L and W satisfy the conditions of the proposition. ■

This proposition allows us to rewrite (3.2) as an inequality. Another result that may be useful for the degenerate case of systems where $D \geq 0$ is the following one.

Lemma A.64. *Let* Q, S, R *be real matrices with* $R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$, $R_1 > 0$. *Then*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} Q & S_1 & S_2 \\ S_1^T & R_1 & 0 \\ S_2^T & 0 & 0 \end{bmatrix} \geq 0 \tag{A.43}$$

if and only if

$$\begin{bmatrix} Q & S_1 \\ S_1^T & R_1 \end{bmatrix} \geq 0 \tag{A.44}$$

and

$$\begin{cases} Q \geq 0 \\ S_2 = 0 \end{cases}, \tag{A.45}$$

where S_1 *and* S_2 *are of appropriate dimensions.* ■

One sees that applying Theorem A.61 the reduced order LMI can be rewritten as the Riccati inequality $Q-S_1^T R_1 S_1 \geq 0$. This is the reduced order Riccati inequality satisfied by a PR system with a feedthrough term $D \geq 0$.

The following is taken from [61] and also concerns the degenerate case when $D \geq 0$, where $A^\dagger$ is the Moore-Penrose pseudo inverse of the matrix A.

Lemma A.65. *Suppose that Q and R are symmetric. Then* $\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \geq 0$ *if and only if $R \geq 0$, $Q - SR^\dagger S^T \geq 0$, $S(I - RR^\dagger) = 0$.* ■

A.5.2 Inverse of Matrices

The following can be found in classical books on matrix algebra or linear systems [214, 246, 272]. Let $A \in I\!R^{m \times m}$ and $C \in I\!R^{n \times n}$ be nonsingular matrices. Then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1} \tag{A.46}$$

so that

$$(I + C(sI - A)^{-1}B)^{-1} = I - C(sI - A + BC)^{-1}B, \tag{A.47}$$

where I has the appropriate dimension. Let now A and B be square nonsingular matrices. Then

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{pmatrix} \tag{A.48}$$

and

$$\begin{pmatrix} A & D \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}DB^{-1} \\ 0 & B^{-1} \end{pmatrix} \tag{A.49}$$

Let A be square nonsingular. Then

$$\begin{pmatrix} A & D \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + EG^{-1}F & -EG^{-1} \\ -G^{-1}F & G^{-1} \end{pmatrix} \tag{A.50}$$

where $G = B - CA^{-1}D$, $E = A^{-1}D$, $F = CA^{-1}$. The matrix G is the Schur complement of A.

A.5.3 Jordan Chain

Let T denote a linear transformation acting on an $n-$dimensional linear space $\mathcal{S}$. A sequence $\{v_0, v_1, ..., v_{r-1}\}$ is called a Jordan chain of length r associated with the eigenvalue λ, if

$$\begin{cases} T(v_0) = \lambda v_0 \\ T(v_1) = \lambda v_1 + v_0 \\ . \\ . \\ . \\ T(v_{r-1}) = \lambda v_{r-1} + v_{r-2} \end{cases} \tag{A.51}$$

The vector v_{r-1} is a generalized eigenvector of T of order r. Equivalently, the vector $(T - \lambda I)^{r-1}(v_{r-1})$ is an eigenvector of T. Equivalently, $(T - \lambda I)^k(v_{r-1}) = 0$ for $k \geq r$. The length of any Jordan chain of T is finite, and the members of a Jordan chain are linearly independent [272, §6.3].

A.5.4 Auxiliary Lemmas for the KYP Lemma Proof

The following results are used in Anderson's proof of the KYP Lemma 3.1.

Lemma A.66. *[11] Let (A, B, C) be a minimal realization for $H(s)$. Suppose that all the poles of $H(s)$ lie in $\mathbf{Re}[s] < 0$. With $H(s)$ and $W_0(s)$ related as in (3.12). Suppose that $W_0(s)$ has a minimal realization (F, G, L). Then the matrices A and F are similar.* ■

Proof: Since (A, B, C) is a realization for $H(s)$, a direct calculation shows that

$$(A_1, B_1, C_1) = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} B \\ C^T \end{bmatrix}, \begin{bmatrix} C^T \\ -B \end{bmatrix} \right\}$$

is a realization of $H(s) + H^T(-s)$. Since $H(s)$ and $H^T(s)$ cannot have a pole in common (the poles of $H(s)$ are in $\mathbf{Re}[s] < 0$ and those of $H^T(-s)$ are in $\mathbf{Re}[s] > 0$) then the degree of $H(s) + H^T(-s)$ is equal to twice the degree of $H(s)$. Thus the triple (A_1, B_1, C_1) is minimal. By direct calculation one finds that

$$\begin{aligned} W_0^T(-s)W_0(s) &= G^T(-sI_n - A^T)^{-1}LL^T(sI_n - A)^{-1}G \\ &= C_2(sI_n - A_2)^{-1}B_2 \end{aligned} \tag{A.52}$$

with

$$(A_2, B_2, C_2) = \left\{ \begin{bmatrix} F & 0 \\ LL^T & -F^T \end{bmatrix}, \begin{bmatrix} G \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -G \end{bmatrix} \right\}$$

Using items (i) and (ii) below (3.12), it can then be shown that the degree of $W_0^T(-s)W_0(s)$ is twice the degree of $W_0(s)$ and therefore the triple (A_2, B_2, C_2) is minimal. Let $P = P^T > 0$ be the unique positive definite solution of $F^T P + PF = -LL^T$. The existence of such a P follows from item (i) below (3.12) and the minimality of (F, G, L). Then one may apply Lemma A.69 below, choosing $\begin{bmatrix} I_n & 0 \\ P & I_n \end{bmatrix}$ to obtain the following alternative realization of $W_0^T(-s)W_0(s)$

$$(A_3, B_3, C_3) = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F^T \end{bmatrix}, \begin{bmatrix} G \\ PG \end{bmatrix}, \begin{bmatrix} PG \\ -G \end{bmatrix} \right\}$$

Since (A_1, B_1, C_1) and (A_3, B_3, C_3) are minimal realizations of the same transfer matrix, and since A has eigenvalues with strictly negative real part, so has F from item (i) below (3.12). The result follows from Lemma A.69.

■

Corollary A.67. *Let $H(s)$ have a minimal realization (A, B, C) and let $H(s)$ and $W_0(s)$ be related as in (3.12). Then there exists matrices K, L such that $W_0(s)$ has a minimal realization (A, K, L). Furthermore, two minimal realizations of $H(s) + H^T(-s) = W_0^T(-s)W_0(s)$ are given by*

$$(A_1, B_1, C_1) = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} B \\ C^T \end{bmatrix}, \begin{bmatrix} C^T \\ -B \end{bmatrix} \right\}$$

and

$$(A_3, B_3, C_3) = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} K \\ PK \end{bmatrix}, \begin{bmatrix} PK \\ -K \end{bmatrix} \right\}$$

where P is uniquely defined by $PA + A^T P = -LL^T$. ■

Lemma A.68. *[11] Let $H(s)$ have a minimal realization (A, B, C) and let $H(s)$ and $W_0(s)$ be related as in (3.12). Then there exists a matrix $\hat{L}$ such that $(A, B, \hat{L})$ is a minimal realization for $W_0(s)$.* ■

Lemma A.69. *Let (A_1, B_1, C_1) and (A_2, B_2, C_2) be two minimal realizations of the rational matrix $H(s)$. Then there exists a nonsingular matrix T such that $A_2 = TA_1T^{-1}$, $B_2 = TB_1$, $C_2 = (T^T)^{-1}C_1$. Conversely if (A_1, B_1, C_1) is minimal and T is nonsingular, then this triple (A_2, B_2, C_2) is minimal.* ■

Corollary A.70. *The only matrices which commute with* $\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$ *are of the form* $\begin{bmatrix} T_1 & 0 \\ 0 & T_1 \end{bmatrix}$, *where* T_1 *and* T_2^T *commute with* A. ■

The next lemma is a specific version of the KYP Lemma that is needed in its proof.

Lemma A.71. *Let* $H(s)$ *be PR and have only purely imaginary poles, with* $H(\infty) = 0$. *Let* (A, B, C) *be a minimal realization of* $H(s)$. *Then there exists* $P = P^T > 0$ *such that*

$$\begin{cases} PA + A^T P = 0 \\ \\ PB = C^T \end{cases} \tag{A.53}$$

■

Proof: The procedure consists in finding a minimal realization (A, B, C) for which the matrix P has an obvious form. Then use the fact that if P satisfies the set of equations (A.53) then $P^\star = (T^T)^{-1} P T^{-1}$ satisfies the corresponding set of equations for the minimal realization $(TAT^{-1}, TB, (T^{-1})^T C)$. Thus if one exhibits a symmetric positive definite P for a particular minimal realization, a symmetric positive definite P will exist for all minimal realizations.

It is possible to write $H(s)$ as $H(s) = \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2}$ where the ω_i are all different and the matrices A_i and B_i satisfy certain requirements [376]. Let us realize each term $(A_i s + B_i)(s^2 + \omega_i^2)^{-1}$ separately with a minimal realization (F_i, G_i, H_i). Select a P_i such that (A.53) is satisfied, so as to obtain a minimal reaization (F, G, H) and a P satisfying (A.53) with $F = \dot{+}_i F_i$, $G^T = [G_1^T \; G_2^T \ldots.]$, $H^T = [H_1^T \; H_2^T \ldots.]$ and $P = \dot{+}_i P_i$, where $\dot{+}$ is the direct sum of the matrices [272, p.145-146]. As a consequence we can consider the realization of the simpler transfer function

$$H(s) = \frac{As + B}{s^2 + \omega_0^2} \tag{A.54}$$

If the degree of $H(s)$ in (A.54) is equal to $2k$, then there exists k complex vectors v_i such that $\bar{v}_i^T v_i = 1$, $v_i^T v_i =: mu_i$, $0 < \mu_i \leq 1$, $\mu_i \in I\!R$, and $H(s) = \sum_{i=1}^k \left[\frac{v_i \bar{v}_i^T}{s - j\omega_0} + \frac{\bar{v}_i v_i^T}{s + j\omega_0} \right]$ [376]. Direct sum techniques allow one to restrict considerations to obtaining a minimal realization for the degree 2, *i.e.*

$$H(s) = \frac{v\bar{v}^T}{s - j\omega_0} + \frac{\bar{v}v^T}{s + j\omega_0} \tag{A.55}$$

Now define $y_1 = \frac{v + \bar{v}}{\sqrt{2}}$ and $y_2 = \frac{v - \bar{v}}{\sqrt{2}}$, and check that

$$H(s) = [y_1 \; y_2] \frac{1}{s^2 + \omega_0^2} \begin{bmatrix} s & \omega_0 \\ -\omega_0 & s \end{bmatrix} \begin{bmatrix} y_1^T \\ y_2^T \end{bmatrix}$$

and then

$$(F, G, H, P) = \left\{ \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}, \begin{bmatrix} y_1^T \\ y_2^T \end{bmatrix}, \begin{bmatrix} y_1^T \\ y_2^T \end{bmatrix}, I_n \right\}$$

defines a minimal realization of (A.55) with the set of equations (A.53) satisfied. ■

A.6 Well-posedness Results for State Delay Systems

In this appendix we provide an existence and uniqueness of solutions for systems as in (5.82) or (5.84). Let us consider the state delay control equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Lx_t + Bu(t), \quad t \geq 0 \\ \\ x(0) = x^0 \\ \\ x_0(\cdot) = \phi(\cdot) \\ \\ y(t) = Cx(t) \end{cases} \tag{A.56}$$

where $A \in I\!R^{n\times n}$, $L : C([-\tau, 0], I\!R^n) \to I\!R^n$ and $B : I\!R^m \to I\!R^n$ are bounded linear operators. Here for a function $z : [-\tau, \infty) \to I\!R^n$, the history of the function $z(\cdot)$ is the function $z_t : [-\tau, 0] \to I\!R^n$ defined by $z_t(\theta) = z(t + \theta)$ for $t \geq 0$ and $\theta \in [-\tau, 0]$. It is assumed further that $u \in \mathcal{L}_{p,e}$.

Definition A.72 (Mild solution). *For the initial condition $x^0 \in I\!R^n$ and $\phi \in C([-\tau, 0], I\!R^n)$, a mild solution of the system (A.56) is the function defined by*

$$\begin{cases} x(t) = e^{tA}x^0 + \int_0^t e^{(t-s)A}\left[Lx_s + Bu(s)\right] ds, \quad t \geq 0 \\ \\ x(t + \theta) = \phi(\theta), \quad -\tau \leq \theta \leq 0 \end{cases} \tag{A.57}$$

■

By using a straightforward argument from fixed point theory, one can see that the system (A.56) possesses a unique mild solution given as in Definition A.72. An example of delay operator is given by

$$Lf = A_1 f(-\tau_1) + \int_{-\tau}^{0} A_2(\theta) f(\theta) d\theta \tag{A.58}$$

where $A_1 \in I\!R^{n\times n}$, $A_2(\theta) \in C([-\tau, 0], I\!R^{n\times n})$, $\tau_1 \geq 0$. More generally, let $\mu : [-\tau, 0] \to \mathcal{L}(I\!R^{n\times n})$ be a function of bounded variation. We define the delay operator by

$$Lf = \int_{-\tau}^{0} d\mu(\theta) f(\theta) \tag{A.59}$$

Now if we set $\mu = A_1 1_{[-\tau,0]}(\cdot) + A_2(\cdot)$ then we obtain the delay operator defined by (A.58). Here $1_{[-\tau,0]}(\cdot)$ is the indicator function of the interval $[-\tau, 0]$ (not the same indicator as the one of convex analysis used elsewhere in this book), *i.e.* the function that takes values 1 in $[-\tau, 0]$ and 0 outside.

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Index

Addendum-Erratum to *Dissipative Systems Analysis and Control*, 2nd Edition

(the first figure indicates the page number)

- Notation section: $\lambda(A)$: an eigenvalue of $A \in \mathbb{R}^{n\times n}$.
- Notation section: $\sigma(A)$: the set of eigenvalues of $A \in \mathbb{R}^{n\times n}$ (*i.e.* the spectrum of A).
- 17, 18, 28: line 2 after (2.22), and also in (2.24), (2.29), (2.30), line 3 in the equation above (2.75): change $u_T(t)$ to $u_t(\tau)$, and $u_T(j\omega)$ to $u_t(j\omega)$.
- 17: line 8, the integration is done over $\tau \in [0, t]$
- 59, lemma 2.55: rational
- 81, Lemma 3.11: $\mu_{\min}(A)$ denotes the minimum eigenvalue of the symmetrized matrix $\frac{1}{2}(A + A^T)$.
- 81: $\sigma_{\min}(B) > 0$ is equivalent to $\mathrm{Ker}(B) = \{0\}$, and to $m < n$ and B of full rank m.
- 81, last line: it is $H(j\omega - \mu)$, not $T(j\omega - \mu)$
- 89 line 6: it is $H(j\omega)$, not $T(j\omega)$
- 88, line -3: it is $H(j\omega)$, not $T(j\omega)$
- 101: in the framed set of implications/equivalences, the last implication has to be reversed and in the last line this is $P = P^T \geq 0$ (hence the last condition is really sufficient for PRness as indicated in the paragraph in between the two framed sets.
- 119: in (3.134) it is $\frac{\partial V_f}{\partial x}(x)[Ax + Bu] + w(x, u) \geq 0$
- 121, line 5 after remark 3.38: Thus $H(-j\omega, j\omega)$ is the condition....
- 123, left hand side of (3.148): P is G
- 124, line 2: P is G
- 125: line 5 after Theorem 3.44: numerically
- Section 3.9 (Lur'e problem and absolute stability): it's also worth reading the survey by R. Shorten et al, "Stability Criteria for Switched and Hybrid Systems", SIAM review, vol.49, no 4, pp.545-592, 2007, and in particular section 6 therein. Their Theorem 6.2 is close to Theorem 2.49 page 57 in the book.
- 147, example 3.75: it is $Ax = 0$ if $x < 0$. The notation x^+ means $\max(0, x)$.
- 149, the dissipativity of linear complementarity systems is throughly investigated in Camlibel, Iannelli, Vasca, "Passivity and complementarity", Mathematical Programming A, 2013, DOI: 10.1007/s10107-013-0678-4 . Closely related results concerning Lur'e set-valued systems are in Brogliato and Goeleven, "Existence, uniqueness of solutions and stability of nonsmooth multivalued Lur'e dynamical systems", Journal of Convex Analysis, vol.20, no 3, 2013, and "Well-posedness, stability and invariance results for a class of multivalued Lur'e dynamical systems", Nonlinear Analysis: Theory, Methods and Applications, vol.74, pp.195-212, 2011.
- 167, in Remark 3.96 this is $[0, \infty)$ (see Theorem 3.91)

- 175, preservation of dissipativity after time-discretization has recently been tackled in S. Greenhalg, V. Acary, B. Brogliato, "Preservation of the dissipativity properties of a class of nonsmooth dynamical systems with the (θ, γ)-algorithm", Numerische Mathematik. See several references that concern this issue in this article. The issue that is tackled concerns the preservation of the strorage function, the supply rate, and the dissipation function, and therefore yields much more stringent conditions than the mere preservation of dissipativity (with possible different supply rate, dissipation function after discretization).

- 170, about the Popov's line. In the book by Aiserman and Gantmacher from 1965 it is pointed outy that Popov criterion also holds in the case of a negative slope $\frac{1}{r}$.

- 192, in (4.18) all T are t

- 197: second line of definition 4.26: $u(t) \in \mathcal{U}$.

- 197-198: the role of the additive constants β which basically accounts for initial stored energy, has also been investigated in D.H. Hill: 'Dissipative nonlinear systems: basic properties and stability analysis", Proc. of IEEE CDC, pp.3259-3264, 16-18 December 1992.

- 204, Theorem 4.43: It is supposed that there exists a storage function $V(\cdot)$ such that $V(x^\star) = 0$, so that $V_r(x^\star) = 0$. From the definition of the required supply in Definition 4.36, this also assumes that the system is reachable from $x^\star$. This is Theorem 2 in [510].

- 202: more on reversibility and its relationship with reciprocity, for LTI systems, is in [511,sections 8 and 9]. In particular Theorem 8 of [51] provides a way to check reversibility.

- 230 and 116: Lemma 3.36 page 116, and Lemma 4.91 page 230, are taken from the book referenced 145 by Faurre, Clerget and Germain. In fact Lemma 3.36 is presented in the book by Faurre et al as a corollary of Lemma 4.91.

- 230, Lemma 4.91: in (4.94) this is $V(t, x)$.

- p.247, line 10: alll should be all.

- 248 and 249, replace all (4.108) by (4.163)

- 251, Theorem 4.111, line 2: $I\!R^{m\times m}$

- Section 5.1: Most of the results presented in this section have been stated in two articles by D.J. Hill and P. Moylan: "Stability results for nonlinear feedback systems", Automatica, vol.13, pp.377-382, 1977, and "General instability results for interconnected systems", SIAM J. Control and Optimization, vol.21, no 2, pp.256-279, 1983. The errata for these papers include the following. In the SIAM paper, the first condition that appears in equation (12), Theorem 7, is redundant. In the Automatica article, condition (iii) of Theorem 4 can be removed, while condition (ii) should be stated with $T\bar{A}(\cdot)$ a nonnegative real valued function.

- 267: from Lemma 5.13 and Corollary 3.4 it may be deduced that the solutions $P = P^T$ of the KYP Lemma LMI, are > 0 if (C, A) is observable.

- 298: in Theorem 5.68, this is "...with input $u(\cdot)$..."

- 305: line before section 5.9.4: Bounded

- 306: Theorem 5.71 is taken from M.R. James et al, SIAM J. Control Optimization, vol.43, no 5, pp.1535-1582, 2005.

- 366: generally speaking BV functions do not satisfy that for any t, there exists a σ such the function is continuous on $[t, t+\sigma)$. This is because at t a BV function may possess an accumulation of jumps on the left, and an accumulation of jumps on the right. But in mechanical systems with frictionless unilateral constraints and piecewise analytic data, accumulations of impacts on the right do not exist (result of Ballard in 2000). So in this particular case the generalized velocity has this property.
- 368: line -8, it is $-\partial\psi_V(x)$.
- 382: ξ in the line after (7.10) refers to the ξ in (6.184)
- section 3.9.4: Notice that for a maximal monotone operator $F : I\!R^n \to I\!R^n$, $x \mapsto y = F(x)$, with $F = \partial f$ where $f(\cdot)$ is convex, proper, lower semi continuous, then the "input-output" product satisfies $x^T y = f(x) + f^*(y)$, where $f^*(\cdot)$ is the dual function of $f(\cdot)$.
- 409: CTCE means Cross Terms Cancellation Equality
- 431 line 2 section 7.8.3: (6.162) is (6.156), and λ_{z_1} is as in (6.169)
- 133, line -2 before (3.176): P_e
- 426, line 6: actuator dynamics
- 431, line 1, (7.169): λ_{z_d} is λ_d
- 536, line 11 in the proof of Lemma A.71: realization
- 564, reference [470]: 1992