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# Topics in Percolative and Disordered Systems

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# Preface

The Pan American Advanced Studies Institute (PASI), *Topics in Percolative and Disordered Systems*, took place in January 2012 in Santiago de Chile and Buenos Aires. It brought together mathematicians, physicists and advanced students from Latin America, North America and beyond for an intense 2-week period focused on current research problems in some of the mainstream areas of Probability Theory and Statistical Physics, such as the stochastic Ising model, random walks in random media, the KPZ universality class and interacting particle systems. This volume contains a selection of five peer-reviewed articles that are representative of the topics discussed in the PASI. Two survey articles are presented—one concerns the KPZ universality class (Quastel and Remenik) and the other treats random walks in random media (Drewitz and Ramírez). Other articles present new results about the scaling limit of the stochastic Ising model (Lacoin) and about its coarsening behaviour (Damron, Kogan, Newman and Sidoravicius) and a review of exact computational methods to compute the current of particles through a given site in the asymmetric simple exclusion process (Corwin).

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# Two Ways to Solve ASEP

**Ivan Corwin**

**Abstract** The purpose of this chapter is to describe two approaches to compute exact formulas (which are amenable to asymptotic analysis) for the probability distribution of the current of particles past a given site in the asymmetric simple exclusion process (ASEP) with step initial data. The first approach is via a variant of the coordinate Bethe Ansatz and was developed in work of Tracy and Widom in 2008–2009, while the second approach is via a rigorous version of the replica trick and was developed in work of Borodin, Sasamoto and the author in 2012.

## 1 Introduction

Exact formulas in probabilistic systems are exceedingly important, and when a new one is discovered, it is worth paying attention. This is a lesson that I first learned in relation to the work of Tracy and Widom on the asymmetric simple exclusion process (ASEP) and through my subsequent work on the Kardar–Parisi–Zhang (KPZ) equation. New formulas can enable asymptotic analysis and uncover novel (and universal) limit laws. Comparing new formulas to those already known can help lead to the realization that certain structures or connections exist between disparate areas of study (or at least can suggest such a possibility and provide a guidepost).

The purpose of this chapter is to describe the synthesis of exact formulas for ASEP. There are presently two approaches to compute the current distribution for ASEP on $\mathbb{Z}$ with step initial condition. The first (called here the *coordinate approach*) is due to Tracy and Widom [26–28] in a series of three papers from 2008–2009, while the second (called here the *duality approach*) is due to Borodin, Sasamoto and the author [5] in 2012.

The duality approach is parallel to an approach (also developed in [5]) to study current distribution for another particle system, called $q$-TASEP. Via a limit transition, the duality approach becomes the replica trick for directed polymers. In fact, ASEP and $q$-TASEP should be considered as integrable discrete regularizations of the directed polymer model in which the replica trick (famous for being non-rigorous)

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becomes mathematically rigorous. Underlying the solvability of $q$-TASEP and directed polymers is an integrable structure recently discovered by Borodin and the author [4] called Macdonald processes (which in turn is based on the integrable system surrounding Macdonald symmetric polynomials). It is not presently understood where ASEP could fit into this structure, but the fact that the duality approach applies in parallel for ASEP and $q$-TASEP compels one to look for a higher structure which encompasses both.

## 2 Current Distribution for ASEP

ASEP is an interacting particle system introduced by Spitzer [24] in 1970 (though arising earlier in biology in the work of MacDonald, Gibbs and Pipkin [18] in 1968). Since then, it has become a central object of study in interacting particle systems and non-equilibrium statistical mechanics. Each site of the lattice $\mathbb{Z}$ may be inhabited by at most one particle. Each particle attempts to jump left at rate $q$ and right at rate $p$ ($p + q = 1$), except that jumps which would violate the 'one particle per site rule' are suppressed. We will assume $q > p$, and for later use call $q - p = \gamma$ and $p/q = \tau$ (note that $\gamma > 0$ and $\tau < 1$).

There are two ways of constructing ASEP as a Markov process. The 'occupation process' keeps track of whether each site in $\mathbb{Z}$ is occupied or unoccupied. The state space is $Y = \{0, 1\}^{\mathbb{Z}}$ and for a state $\eta = \{\eta_x\}_{x\in\mathbb{Z}} \in Y$, $\eta_x = 1$ if there is a particle at $x$ and 0 otherwise. This Markov process is denoted $\eta(t)$.

The 'coordinate process' keeps track of the location of each particle. Assume there are only $k$ particles in the system, then the state space $X_k = \{x_1 < \cdots < x_k\} \subset \mathbb{Z}^k$ and for a state $\vec{x} = \{x_1 < \ldots < x_k\} \in X_k$, the value of $x_j$ is the location of particle $j$. We call $X_k$ a Weyl chamber. Because particles cannot hop over each other, the ASEP dynamics preserve particle ordering. This Markov process is denoted $\vec{x}(t)$.

In this chapter, we will be concerned with the 'step' initial condition for ASEP in which every positive integer site is initially occupied and every other site is initially unoccupied. In terms of the occupation process, this corresponds to having $\eta_x(0) = \mathbf{1}_{x>0}$ (here and throughout $\mathbf{1}_E$ is the indicator function for event $E$). Let $N_x(\eta) = \sum_{y\leq x} \eta_y$ and note that $N_0(\eta(t))$ records the number of particles of ASEP which, at time $t$ are to the left of, or at the origin—that is to say, it is the net current of particles to pass the bond 0 and 1 in time $t$.

**Theorem 1** *For ASEP with step initial condition and $q > p$,*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N_0(t/\gamma) - t/4}{2^{-1/3}t^{1/3}} \geq -s\right) = F_{\mathrm{GUE}}(s),$$

*where $F_{\mathrm{GUE}}(s)$ is the GUE Tracy-Widom distribution.*

*Remark 1* The distribution function $F_{\mathrm{GUE}}(s)$ can be defined via a Fredholm determinant as

$$F_{\mathrm{GUE}}(s) = \det(I - K_{\mathrm{Ai}})_{L^2(s,\infty)}$$

where Airy kernel $K_{\mathrm{Ai}}$ acts on $L^2(s,\infty)$ with integral kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt.$$

For $q = 1$ and $p = 0$, result was proved in 1999 by Johansson [13] and for general $q > p$, it was proved by Tracy and Widom [26–28] in 2009, and then reproved via a new formula by Borodin, Sasamoto and the author [5] in 2012. This result confirms that for all $q > p$, ASEP is in the KPZ universality class [15] (see also the review [6]).

In order to prove an asymptotic result (such as above), it is very useful to have a pre-asymptotic (finite $t$) formula to analyze. If the formula does not increase in complexity as $t$ goes to infinity, there is hope to compute its asymptotics. Presently, there are two approaches to computing manageable formulas for the distribution of $N_0(t)$.

## 3 The Coordinate Approach

In [26], Tracy and Widom start by considering the ASEP coordinate process $\vec{x}(t)$ with only $k$ particles. In 1997, Schütz [22] computed the transition probabilities (i.e. Green's function) for ASEP with $k = 2$ particles. The first step in [26] is a generalization to arbitrary $k$. Let $P_{\vec{y}}(\vec{x};t)$ represent the probability that in time $t$, a particle configuration $\vec{y}$ will transition to a second configuration $\vec{x}$. As long as $p \neq 0$, it was proved in [26] that

$$P_{\vec{y}}(\vec{x};t) = \sum_{\sigma\in S_k} \int \cdots \int A_\sigma \prod_{i=1}^{k} \xi_{\sigma(j)}^{x_j - y_{\sigma(j)}-1} e^{\epsilon(\xi_j)t} d\xi_j, \tag{1}$$

where the contour of integration is a circle centered at zero with radius so small as to not contain any poles of $A_\sigma$. Here, $\epsilon(\xi) = p\xi^{-1} + q\xi - 1$ and

$$A_\sigma = \prod \left\{ S_{\alpha\beta} : \{\alpha,\beta\} \text{is an inversion in}\, \sigma \right\}, \qquad S_{\alpha\beta} = -\frac{p + q\xi_\alpha\xi_\beta - \xi_\alpha}{p + q\xi_\alpha\xi_\beta - \xi_\beta}.$$

This result is proved by showing that that $P_{\vec{y}}(\vec{x};t)$ solves the master equation for $k$-particle ASEP

$$\frac{d}{dt}u(\vec{x};t) = ((L^k)^* u)(\vec{x};t), \qquad u(\vec{x};0) = \mathbf{1}_{\vec{x}=\vec{y}}.$$

Here $(L^k)^*$ is the adjoint of the generator of the $k$-particle ASEP coordinate process (this just means that the role of $p$ and $q$ are switched in going between $L^k$ and $(L^k)^*$). For $k = 1$, $L^1$ and $(L^1)^*$ act on function $f : \mathbb{Z} \to \mathbb{R}$ as

$$(L^1 f)(x) = q\,[f(x-1) - f(x)] + p\,[f(x+1) - f(x)],$$
$$\left((L^1)^* f\right)(x) = p\,[f(x-1) - f(x)] + q\,[f(x+1) - f(x)].$$

For $k > 1$, the generator $L^k$ and its adjoint depend on the location of $\vec{x}$ in the Weyl chamber, reflecting the fact that certain particle jumps are not allowed near the boundary of the Weyl chamber.

Quoting a footnote in [26]:

> The idea in Bethe Ansatz (see, e.g. [16, 25, 30]), applied to 1-D $k$-particle quantum mechanical problems, is to represent the wave function as a linear combination of free particle eigenstates and to incorporate the effect of the potential as a set of $k-1$ boundary conditions. The remarkable feature of models amendable to Bethe Ansatz is that the boundary conditions for $k \geq 3$ introduce no more new conditions ... The application of Bethe Ansatz to the evolution equation (master equation) describing ASEP begins with Gwa and Spohn [9] with subsequential developments by Schütz [22].

To see this in practice, assume that one wants to solve

$$\frac{d}{dt} u(\vec{x}; t) = \left((L^k)^* u\right)(\vec{x}; t), \qquad u(\vec{x}; 0) = u_0(\vec{x})$$

for $\vec{x}$ in the Weyl chamber $X_k$.

**Proposition 1** 1 *If* $v : \mathbb{Z}^k \times \mathbb{R}_+ \to \mathbb{R}$ *solves the 'free evolution equation with boundary condition':*

(1) *For all* $\vec{x} \in \mathbb{Z}^k$

$$\frac{d}{dt} v(\vec{x}; t) = \sum_{j=1}^{k} \left([L^1]_j^* v\right)(\vec{x}; t);$$

(2) *For all* $\vec{x} \in \mathbb{Z}^k$ *such that* $x_{j+1} = x_j + 1$ *for some* $1 \leq j \leq k-1$,

$$p v(x_1, \ldots, x_j, x_{j+1} - 1, \ldots, x_k; t) + q v(x_1, \ldots, x_j + 1, x_{j+1}, \ldots, x_k; t)$$
$$- v(\vec{x}; t) = 0;$$

(3) *For all* $\vec{x} \in X_k$, $v(\vec{x}; 0) = u_0(\vec{x})$;

*Then, for all* $t \geq 0$ *and* $\vec{x} \in X_k$, $u(\vec{x}; t) = v(\vec{x}; t)$.

In (1) above, $[L^1]_j^*$ means to apply $(L^1)^*$ in the $x_j$ variable. In fact, some growth conditions must be imposed to ensure that $u$ and $v$ match (see Propositions 4.9 and 4.10 of [5]) but we will not dwell on this presently.

This reformulation of the master equation involves only $k-1$ boundary conditions and is amendable to Bethe Ansatz—hence one is led to postulate Eq. (1). It remains to check the Ansatz (i.e. $P_{\vec{y}}(\vec{x}; t)$ solves the reformulated equation). The $A_\sigma$ is just right to enforce the boundary condition. The only challenge (which requires an involved residue calculation) is to check the initial data, since there are a total of $k!$ integrals.

The transition probabilities for $k$-particle ASEP is only the first step towards Theorem 1. The next step is to integrate out the locations of all but one particle, so as to compute the transition probability for a given particle $x_m$. The formula for the location of the $m^{th}$ particle at time $t$ involves a summation (indexed by certain subsets of $\{1, \dots, k\}$) of contour integrals. These formulae are a result of significant residue calculations and combinatorics.

At this point we are only considering $k$ particles, whereas for the asymptotic problem, we want to consider step initial conditions. This is achieved by taking $y_j = j$ for $1 \le j \le k$ and taking $k$ to infinity. After further manipulations, the $m^{th}$ particle location distribution formula has a clear limit as $k$ goes to infinity. This is the first formula for step initial condition and it is given by an infinite series of contour integrals.

In [27], this infinite series is recognized as equal to a transform of a Fredholm determinant. By the simple relationship between the location of the $m^{th}$ particle of ASEP and $N_0(t)$ (defined earlier), this shows that

$$\mathbb{P}(N_0(t) = m) = \frac{-\tau^m}{2\pi \mathrm{i}} \int \frac{\det(I - \zeta K_1)}{(\zeta;\tau)_{m+1}} d\zeta, \tag{2}$$

where the integral in $\zeta$ is over a contour enclosing $\zeta = q^{-k}$ for $0 \le k \le m-1$ and $(a;\tau)_n = (1-a)(1-\tau a)\cdots\left(1-\tau^{n-1}a\right)$. Here, $\det(I - \zeta K_1)$ is the Fredholm determinant with the kernel of $K : L^2(C_R) \to L^2(C_R)$ given by

$$K_1(\xi, \xi') = q \frac{e^{\epsilon(\xi)t}}{p + q\xi\xi' - \xi},$$

and the contour $C_R$ a sufficiently large circle centered at zero.

There remains, however, a significant challenge to proving Theorem 1 from the above formula. As $m$ increases, the kernel $K_1$ has no clear limit, and the denominator term $(\zeta;\tau)_{m+1}$, behaves widely as $\zeta$ varies on its contour of integration. Much of [28] is devoted to reworking the above formula into one for which asymptotics can be performed. This is done through significant functional analysis. The final formula, from which Theorem 1 is proved by asymptotics is (leaving off the contours of integration),

$$\mathbb{P}(N_0(t) \ge m) = \int \frac{d\mu}{\mu} (\mu;\tau)_\infty \det(I + \mu J), \tag{3}$$

where the kernel of $J$ is given by

$$\begin{aligned} J(\eta, \eta') &= \int \exp\left\{\Psi_{t,m,x}(\zeta) - \Psi_{t,m,x}(\eta')\right\} \frac{f(\mu, \zeta/\eta')}{\eta'(\zeta - \eta)} d\zeta, \\ f(\mu, z) &= \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k \mu} z^k, \\ \Psi_{t,m,x}(\zeta) &= \Lambda_{t,m,x}(\zeta) - \Lambda_{t,m,x}(\xi), \\ \Lambda_{t,m,x}(\zeta) &= -x \log(1-\zeta) + \frac{t\zeta}{1-\zeta} + m \log \zeta. \end{aligned}$$

## 4 The Duality Approach

Duality is a powerful tool in the study of Markov processes. It reveals hidden structures and symmetries of the process, as well as leads to non-trivial systems of ODEs (ordinary differential equation), which expectations of certain observables satisfy. In 1997, Schütz [23] observed that ASEP is self-dual (in a sense which will be made clear below). The fact that duality gives a useful tool for computing the moments of ASEP was first noted by Imamura and Sasamoto [12] in 2011. In 2012, Borodin, Sasamoto and the author [5] used this observation about duality, along with an Ansatz for solving the duality ODEs (which was inspired by the work of Borodin and the author on Macdonald processes [4]) to derive two different formulae for the probability distribution of $N_0(t)$. The first was new and readily amendable to asymptotic analysis necessary to prove Theorem 1, while the second was equivalent to Tracy and Widom's formula (2).

To define the general concept of duality, consider two Markov processes, $\eta(t)$ with state space $Y$ and $\vec{x}(t)$ with state space $X$ (for the moment, we think of these as arbitrary, though after the definition of duality, we will take these as before). Let $\mathbb{E}^{\eta}$ and $\mathbb{E}^{\vec{x}}$ represent the expectation of these two processes (respectively) started from $\eta(0) = \eta$ and $\vec{x}(0) = \vec{x}$. Then, $\eta(t)$ and $\vec{x}(t)$ are dual with respect to a function $H : Y \times X \to \mathbb{R}$, if for all $\eta \in Y$, $\vec{x} \in X$ and $t \geq 0$,

$$\mathbb{E}^{\eta}\left[H(\eta(t), \vec{x})\right] = \mathbb{E}^{\vec{x}}\left[H(\eta, \vec{x}(t))\right].$$

One immediate consequence of duality is that if we define $u_\eta(\vec{x}; t)$ to be the expectations written above, then

$$\frac{d}{dt} u_\eta(\vec{x}; t) = L u_\eta(\vec{x}; t),$$

where $L$ is the generator of $\vec{x}(t)$ and where the initial data is given by $u_\eta(\vec{x}; 0) = H(\eta, \vec{x})$.

Schütz [23] observed that if $\eta(t)$ is the ASEP occupation process and $\vec{x}(t)$ is the $k$-particle ASEP coordinate process with $p$ and $q$ switched from the earlier definition, then these two Markov processes are dual with respect to

$$H(\eta, \vec{x}) = \prod_{j=1}^{k} \tau^{N_{x_j - 1}(\eta)} \eta_{x_j}.$$

The generator of the $p, q$ reversed particle process $\vec{x}(t)$ is equal to $(L^k)^*$, as discussed earlier. Schütz demonstrated this duality in terms of a spin-chain encoding of ASEP by using a commutation relation along with the $U_q[SU(2)]$ symmetry of the chain. A direct proof can also be given in terms of the language of Markov processes [5]. When $p = q$, $\tau = 1$ and this duality reduces to the classical duality of correlation functions for the symmetric simple exclusion process (see [17] Chap. 8, Theorem 1).

As before, we focus on step initial condition, so that $\eta_x = \mathbf{1}_{x \geq 1}$. Duality implies that $u_{\text{step}}(\vec{x}; t) := \mathbb{E}^{\eta}\left[H(\eta(t), \vec{x})\right]$ solves

$$\frac{d}{dt} u_{\text{step}}(\vec{x}; t) = L^k u_{\text{step}}(\vec{x}; t), \qquad u_{\text{step}}(\vec{x}; 0) = \mathbf{1}_{x_1 \geq 1} \prod_{i=1}^{k} \tau^{x_i - 1}. \tag{4}$$

The above system is solved by

$$u_{\text{step}}(\vec{x};t) = \frac{\tau^{k(k-1)/2}}{(2\pi \mathrm{i})^k} \int \cdots \int \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{j=1}^{k} h_{x_j,t}(z_j) dz_j, \tag{5}$$

where

$$h_{x,t}(z) = e^{\epsilon'(z)t} \left(\frac{1+z}{1+z/\tau}\right)^{x-1} \frac{1}{\tau + z}, \qquad \epsilon'(z) = -\frac{z(p-q)^2}{(1+z)(p+qz)},$$

and where the contour of integration for each $z_j$ is a circle around $-\tau$, so small as to not contain $-1$. In order to see this, we use the reformulation of the system (4) in terms of the free evolution equation with boundary condition with ASEP given earlier in Proposition 3.1. Condition (1) is trivially checked since for each $z$, $\frac{d}{dt} h_{x,t}(z) = L^1 h_{x,t}(z)$. Condition (3) is checked via a simple residue calculation. Condition (2) reveals the purpose of the $\frac{z_A - z_B}{z_A - \tau z_B}$ factor. Applying the boundary condition to the integrand above brings out a factor of $z_j - \tau z_j$. This cancels the corresponding term in the denominator and the resulting integral is simultaneous symmetry and antisymmetry in $z_j$ and $z_{j+1}$. Hence, the integral must equal zero, which is the desired boundary condition (2).

The inspiration for this simple solution to the system of ODEs came from analogous formulas which solve free evolution equations with boundary condition for various versions of the delta Bose gas (see Sect. 5 for a brief discussion). For the delta Bose gas and certain integrable discrete regularizations, the formulas arose directly from the structure of Macdonald processes [4]. ASEP does not fit into that structure, but the existence of similar formulas suggests the possibility of a yet higher structure.

A change of variables reveals some similarities to the integrand in (1). Letting

$$\xi_j = \frac{1+z_j}{1+z_j/\tau} \tag{6}$$

we have

$$\frac{z_A - z_B}{z_A - \tau z_B} = q \frac{\xi_A - \xi_B}{p + q\xi_A \xi_B - \xi_B}, \qquad h_{x_j,t}(z_j) dz_j = e^{\epsilon(\xi_j)t} \xi_j^{x_j - 1} \frac{d\xi_j}{\tau - \xi_j}.$$

The system (4) could also be solved via Tracy and Widom's formula (see formula 1 earlier) for the Green's function for $(L^k)^*$ (as suggested in [12]) but the resulting formula would involve the sum of $k!$ $k$-fold contour integrals. Symmetrizing (5) via combinatorial identities, and making the above change of variables, one does recover that formula. The reversal of this procedure is a rather unnatural anti-symmetrization, which explains why (5) was not previously known.

A suitable summation of $H(\eta, \vec{x})$ over $\vec{x}$ gives $\tau^{kN_x(\eta)}$. Using this, and formula (5), [5] proves that for ASEP with step initial condition,

$$\mathbb{E}\left[\tau^{kN_0(t)}\right] = \frac{\tau^{k(k-1)/2}}{(2\pi \mathrm{i})^k} \int \cdots \int \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{j=1}^{k} e^{\epsilon'(z_j)t} \frac{dz_j}{z_j}, \tag{7}$$

where $N_0(t) = N_0(\eta(t))$ and where the contour of integration for $z_j$ includes $0, -\tau$ but not $-1$ or $\tau$ times the contours for $z_{j+1}$ through $z_k$. This is to say, that the contours of integration respect a certain nesting structure.

At this point, the utility of having a single $k$-fold nested contour integral formula for the moments of $\tau^{N_0(t)}$ becomes clear. There are two ways to deform the contours of integration in (7) so that all coincide with each other. The first involves expanding them all to be a circle containing $-\tau$ and 0, but not $-1$. There are many poles encountered in the course of this deformation and the residues can be indexed by a partition. This leads to

$$\mathbb{E}\left[\tau^{kN_0(t)}\right] = k_\tau! \sum_{\substack{\lambda\vdash k\\ \lambda=1^{m_1}2^{m_2}\cdots}} \frac{1}{m_1!m_2!\cdots} \frac{(1-\tau)^k}{(2\pi\iota)^{\ell(\lambda)}} \int\cdots\int \det\left[\frac{-1}{w_i\tau^{\lambda_i}-w_j}\right]_{i,j=1}^{\ell(\lambda)} \times \prod_{j=1}^{\ell(\lambda)} e^{t\sum_{i=0}^{\lambda_j-1}\epsilon'(\tau^i w_j)} dw_j, \tag{8}$$

where $k_\tau! = (\tau;\tau)_k(1-\tau)^{-k}$ is the $\tau$-deformed factorial, and $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ is a partition of $k$ (i.e. $\sum\lambda_i = k$) with $\ell(\lambda)$ nonzero parts, and multiplicity $m_j$ of the value $j$. The structure of these residues is very similar to the string states indexing the eigenfunctions of the attractive delta Bose gas (see Sect. 5).

The final step in the duality approach is to use these moment formulas to recover the distribution of $N_0(t)$. This is done via the $\tau$-deformed Laplace transform Hahn [10] introduced in 1949. The left-hand side of the below equation is the transform of $\tau^{N_0(t)}$ with spectral variable $\zeta$.

$$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_0(t)};\tau)_\infty}\right] = \sum_{k=0}^{\infty} \frac{\zeta^k\mathbb{E}\left[\tau^{kN_0(t)}\right]}{(\tau;\tau)_k}. \tag{9}$$

The right-hand side above comes from the left-hand side by expanding the $\tau$-deformed exponential inside the expectation (using the $\tau$-deformed Binomial theorem) and then interchanging the summation over $k$ with the expectation. This interchange of summation and integration is justified here for $\zeta$ small enough because $|\tau^{kN_0(t)}| \leq 1$ deterministically (in contrast to (15) Sect. 5).

Substituting (8) into the series on the right-hand side of (9), one recognizes a Fredholm determinant. The kernel of the determinant can be rewritten using a Mellin-Barnes integral representation and the result is (leaving off the contours of integration):

$$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_0(t)};\tau)_\infty}\right] = \det\left(I + K_\zeta\right), \tag{10}$$

where the kernel of $K_\zeta$ is

$$K_\zeta(w,w') = \frac{1}{2\pi \mathrm{i}} \int \frac{\pi}{\sin(-\pi s)}(-s)^\zeta \frac{g(w)}{g(\tau^s w)} \frac{ds}{w'-\tau^s w}, \qquad g(w) = e^{\gamma t \frac{\tau}{\tau+w}}.$$

The $\tau$-Laplace transform can easily be inverted to give the distribution of $N_0(t)$ and asymptotics of the above formula are readily performed (see Sect. 9 of [5]) resulting in Theorem 1.

There is a second choice for how to deform the nested contours in (8) to all coincide. The terminal contour of this deformation is a small circle around $-\tau$, and again there are certain poles encountered during the deformation. The combinatorics of the residues here is simpler than in the first case, and one finds the following Fredholm determinant formula,

$$\mathbb{E}\left[\frac{1}{\left(\zeta\tau^{N_0(t)};\tau\right)_\infty}\right] = \frac{\det(I - \zeta K_2)}{(\zeta;\tau)_\infty} \tag{11}$$

where the kernel of $K_2$ is

$$K_2(w, w') = \frac{e^{\epsilon'(w)t}}{\tau w - w'}.$$

Performing the change of variables (6) and inverting this $\tau$-Laplace transform, one recovers Tracy and Widom's formula (2). As in Tracy and Widom's work, this formula is not yet suitable for asymptotics and must be manipulated significantly to get to the form of (3).

## 5 Duality Approach as a Rigorous Replica Trick

Besides the Schütz duality, Borodin, Sasamoto and the author discovered that ASEP is also self-dual with respect to

$$H(\eta, \vec{x}) = \prod_{j=1}^{k} \tau^{N_{x_j}(\eta)},$$

for $k = 1$. This shows that $\mathbb{E}\left[\tau^{N_x(\eta(t))}\right]$ solves the heat equation with generator $L^1$. In fact, this is essentially Gärtner's 1988 observation [8] that $\tau^{N_x(\eta(t))}$ solves a certain discrete multiplicative stochastic heat equation. A multiplicative stochastic heat equation has a Feynman-Kac representation which shows that the solution can be interpreted as a partition function for a directed polymer in a disorder given by the noise of the stochastic heat equation.

In 1997, Bertini and Giacomin [2] showed that under a certain 'weakly asymmetric' scaling, $\tau^{N_x(\eta(t))}$ converges to the solution to the continuum multiplicative stochastic heat equation (SHE) with space–time white noise $\xi(x,t)$:

$$\frac{d}{dt} Z(x,t) = \frac{1}{2}\frac{d^2}{dx^2} Z(x,t) + Z(x,t)\,\xi(x,t).$$

This convergence result did not include when $\eta(0)$ is step initial condition and was extended to that case by Amir, Quastel and the author [1]. The corresponding initial data for the SHE is $Z(x,0) = \delta_{x=0}$ where $\delta$ is the Dirac delta function. The logarithm of the solution to the SHE (formally) solves the KPZ equation,

$$\frac{d}{dt}h(x,t) = \frac{1}{2}\frac{d^2}{dx^2}h(x,t) + \frac{1}{2}\left(\frac{d}{dx}h(x,t)\right)^2 + \xi(x,t). \tag{12}$$

See [6] for more details on the KPZ equation.

Duality of ASEP translates into the fact that the moments of the SHE solve the attractive 1-D imaginary-time delta Bose gas (Lieb–Liniger model with delta interaction) [14]. Define $\bar{Z}(\vec{x};t) = \mathbb{E}[Z(x_1,t)\cdots Z(x_k,t)]$ for $Z$ with $\delta_{x=0}$ initial data. Then $\bar{Z}$ solves the system

$$\frac{d}{dt}\bar{Z}(\vec{x};t) = H_1\bar{Z}(\vec{x};t), \qquad \bar{Z}(\vec{x};0) = \prod_{j=1}^{k}\delta_{x_j=0}, \tag{13}$$

where $H_\kappa$ is the Lieb–Liniger Hamiltonian with delta interaction with strength $\kappa \in \mathbb{R}$:

$$H_\kappa = \frac{1}{2}\sum_{j=1}^{k}\frac{d^2}{dx_j^2} + \kappa\sum_{i<j}\delta_{x_i=x_j}.$$

The Lieb–Liniger model with delta interaction was the second system solved by the Bethe Ansatz (over 30 years after Bethe [3] solved the spin -1/2 isotropic Heisenberg model). This was accomplished by Lieb and Liniger in 1963 for the repulsive system ($\kappa < 0$). A year later, McGuire similarly solved the attractive system ($\kappa > 0$). In their context, solving the system meant writing down eigenfunctions for $H_\kappa$. The structure of the eigenfunctions for the repulsive versus attractive cases is different. In the attractive case, there are extra eigenfunctions which are called 'string states' due to the strings of quasi-momenta with which they are indexed (or physically corresponding to bound states of particle clusters). Completeness of these eigenfunctions was not shown until later [7, 11, 19, 20, 29].

For the purposes of understanding the moments of the SHE, it is not necessary to diagonalize $H_\kappa$, but rather just to solve the system (13) for $\kappa = 1$. Just as with ASEP, this system can be written as a 'free evolution equation with boundary condition'. The free evolution is just according to the $k$ variable Laplacian and the boundary condition is that for all $1 \le j \le k-1$,

$$\left(\frac{d}{dx_j} - \frac{d}{dx_{j+1}} - \kappa\right) v(\vec{x};t)\big|_{x_j\to x_{j+1}} = 0.$$

This system can be solved via an analogous formula to (5): For $x_1 \le \cdots \le x_k$ and $\kappa \in \mathbb{R}$,

$$\bar{Z}(\vec{x};t) = \frac{1}{(2\pi \mathrm{i})^k}\int\cdots\int \prod_{1\le A<B\le k}\frac{z_A - z_B}{z_A - z_B - \kappa}\prod_{j=1}^{k}\exp\left\{\frac{z_j^2}{2}t + x_j z_j\right\} dz_j \tag{14}$$

where the contour of integration for $z_j$ is along $\alpha_j + i\mathbb{R}$ for any $\alpha_1 > \alpha_2 + \kappa > \cdots \alpha_k + (k-1)\kappa$. When $\kappa < 0$, all the $\alpha_j$ can be chosen as 0 and hence the integral occurs on i$\mathbb{R}$, whereas for $\kappa > 0$, the contours must be spaced horizontally. In the $\kappa > 0$ case, the contours can be deformed to i$\mathbb{R}$. The singularities and associated residues encountered have a very similar structure to those seen earlier in (8) in the context to the first ASEP contour deformation. The disparity between residue combinatorics accounts for the difference in the structure of the eigenfunctions and the occurrence of string stated for $\kappa > 0$. In fact, Heckeman and Opdam's 1997 proof of the completeness of the Bethe Ansatz relied on a formula equivalent to (14).

Given expressions for all of the moments of the SHE, one wants to recover the distribution of $Z(x,t)$. Since $Z(x,t)$ is non-negative, its Laplace transform characterizes its distribution. Näively one writes,

$$\mathbb{E}\left[e^{\zeta Z(t,x)}\right] = \sum_{k=0}^{\infty} \frac{\zeta^k \mathbb{E}[Z(t,x)^k]}{k!}. \tag{15}$$

However, the right-hand side is known to make no mathematical sense and the interchange of expectation and summation is totally unjustifiable. The moments of the SHE grow like $e^{ck^3}$ and thus the right-hand side is extremely divergent. One can see that cutting off the summation also fails to remedy the situation in any way.

What should be clear now is that ASEP is an integrable discrete regularization of the SHE (or equivalently the KPZ equation) and the duality approach to solving it is a rigorous version of the replica trick for the SHE. By taking the weakly asymmetric limit of the $\tau$-deformed Laplace transform formulas described above, one finds a Fredholm determinant formula for $\mathbb{E}\left[e^{\zeta Z(t,x)}\right]$. This can be done from either the new formula (10) in [5] or Tracy and Widom's formula (3). It appears that (10) is very amendable to asymptotic analysis.

Using (3), the derivation of the Laplace transform of $Z(t,x)$ involves extremely careful asymptotic analysis which was performed in 2010 rigorously by Amir, Quastel and the author [1] and independently and in parallel (though non-rigorously) by Sasamoto and Spohn [21]. Very soon afterwards, Calabrese, Le Doussal and Rosso, as well as Dotsenko showed how to formally recover this Fredholm determinant from summing the divergent series on the right-hand side of (15). The formal manipulations of divergent series that goes into this can be see as shadows of the rigorous duality approach explained above for ASEP. It can also be seen as a shadow of a parallel duality approach for q-TASEP [4, 5], another integrable discrete regularization of the SHE.

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# Coarsening in 2D Slabs

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**Abstract** We study coarsening; that is, the zero-temperature limit of Glauber dynamics in the standard Ising model on slabs $S_k = \mathbb{Z}^2 \times \{0, \ldots, k-1\}$ of all thicknesses $k \geq 2$ (with free and periodic boundary conditions in the third coordinate). We show that with free boundary conditions, for $k \geq 3$, some sites fixate for large times and some do not, whereas for $k = 2$, all sites fixate. With periodic boundary conditions, for $k \geq 4$, some sites fixate and others do not, while for $k = 2$ and 3, all sites fixate.

## 1 Introduction

Coarsening models have been extensively studied in the Physics literature; see, for example, [5, Chap. 9] and the references therein. These are stochastic Ising models at some low temperature $T_1$ whose initial state is chosen from the equilibrium distribution at a higher temperature $T_2$. The special case of the coarsening model we consider here is the case where $T_1 = 0$ and $T_2 = \infty$ (that is, uniformly random initial state). The states are assignments of $\pm 1$ to the vertices of some graph and the most commonly studied graph is $\mathbb{Z}^d$ (with nearest neighbour edges) or finite box approximations to $\mathbb{Z}^d$ (with, for example, free or periodic boundary conditions).

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For $d = 1$, the dynamics is exactly that of the standard voter model and it is an old result [1] that almost surely every site flips (between $+1$ and $-1$) infinitely often. For $d = 2$, it was shown in [6], that still every site flips infinitely often, but it is an open problem to determine what happens for $d \geq 3$. In [7], it was proposed, based on numerical results in the related issue of "persistence" (sites which do not flip for a long time) from [8] that the flipping results for $d = 1, 2$ might change by dimension 4 or 5. But in fact, the situation is unclear even in dimension 3. In this chapter, in a first attempt to shed some light on the possible difference between $\mathbb{Z}^2$ and $\mathbb{Z}^3$, we study coarsening in slabs of varying thickness $k$ so as to interpolate between the full 2-D and 3-D lattices. To our surprise, there is more interesting structure in this $k$-dependence than we originally suspected.

## 1.1 The Model and Definitions

The slab $S_k,\ k \geq 2$, is the graph with vertex set $\mathbb{Z}^2 \times \{0, 1, \ldots, k-1\}$ and edge set $\mathcal{E}_k = \{\{x, y\} : \|x - y\|_1 = 1\}$. As is usual, we take an initial spin configuration $\sigma(0) = (\sigma_x(0))_{x \in S_k}$ on $\Omega_k = \{-1, 1\}^{S_k}$ distributed using the product measure of $\mu_p,\ p \in [0, 1]$, where

$$\mu_p(\sigma_x(0) = +1) = p = 1 - \mu_p(\sigma_x(0) = -1)\ .$$

The configuration $\sigma(t)$ evolves as $t$ increases according to the zero-temperature limit of Glauber dynamics (the majority rule). To describe this, define the energy (or local cost function) of a site $x$ at time $t$ as

$$e_x(t) = -\sum_{y:\{x,y\} \in \mathcal{E}_k} \sigma_x(t)\sigma_y(t)\ .$$

Note that up to a linear transformation, this is just the number of neighbours $y$ of $x$ such that $\sigma_y(t) \neq \sigma_x(t)$. Each site has an exponential clock with different clocks independent of each other and when a site's clock rings, it makes an update according to the rules

$$\sigma_x(t) = \begin{cases} -\sigma_x(t^-) & \text{if } e_x(t^-) > 0 \\ \pm 1 & \text{with probability } 1/2 \text{ if } e_x(t^-) = 0 \\ \sigma_x(t^-) & \text{if } e_x(t^-) < 0 \end{cases}\ .$$

Write $\mathbb{P}_p$ for the joint distribution of $(\sigma(0), \omega)$, the initial spins and the dynamics realizations.

The main questions we will address involve fixation. We say that the slab $S_k$ fixates for some value of $p$ if

$$\mathbb{P}_p(\text{there exists } T = T(\sigma(0), \omega) < \infty \text{ such that } \sigma_0(t) = \sigma_0(T) \text{for all } t \geq T) = 1\ .$$

We will actually only focus on the case $p = 1/2$, so write $\mathbb{P}$ for $\mathbb{P}_{1/2}$. The setup thus far corresponds to the model with free boundary conditions; in the case of periodic boundary conditions, we consider sites of the form $(x, y, k-1)$ and $(x, y, 0)$ to be neighbours in $S_k$. If $k = 2$, this enforces two edges between $(x, y, 1)$ and $(x, y, 0)$, so that in the computation of energy of a site, that neighbour counts twice.

## 2 Main Results

The first theorem concerns fixation for small $k$. We will prove the case $k = 2$ in the next section; the case $k = 3$ will be treated in a companion chapter [2]. That chapter will also contain a simplified proof of the case $k = 2$, notable for removing the bootstrap percolation comparison used here.

**Theorem 1** *For $k = 2$ with free or periodic boundary conditions, $S_k$ fixates. For $k = 3$ with periodic boundary conditions, $S_k$ fixates.*

The proof of the following theorem is in Sect. 4. The construction used in the proof for $k = 4$ with periodic boundary conditions is considerably more involved and will not be given in this chapter; it will appear in [2].

**Theorem 2** *With $k \geq 4$ and periodic boundary conditions, $S_k$ does not fixate. With $k \geq 3$ and free boundary conditions, $S_k$ does not fixate.*

*Remark 3* The above theorems actually hold with initial state distributed according to the product measure $\mu_p$ for arbitrary $p \in (0, 1)$, not just $p = 1/2$; the proofs are identical. This is in contrast to the situation on the cubic lattice $\mathbb{Z}^d$ for any $d \geq 2$, where fixation is proved to occur for $p$ sufficiently close to 0 or 1 [4].

## 3 Proof of Theorem 1 for $k = 2$

For the free boundary condition case, the theorem follows from the argument in Nanda–Newman–Stein [6]. Specifically, for $v, v' \in S_2$, define $m_t(v', v)$ as the contribution to $e_v(t) - e_v(0)$ due to flips of the spin $\sigma_{v'}$. Write $\pi : S_2 \to \mathbb{Z}^2$ for the projection $\pi(x, y, z) = (x, y)$ and for $v \in S_2$, we use the notation that $\hat{v}$ is the vertex in $S_2$ with $\hat{v} \neq v$ but $\pi(\hat{v}) = \pi(v)$. Then

$$\mathbb{E}\left[e_v(t) - e_v(0)\right] = \sum_{v' \in S_2 : \|v - v'\|_1 \leq 1} \mathbb{E}m_t(v', v) = \mathbb{E}m_t(v, v) + \sum_{v' \in S_2 : \|v - v'\|_1 = 1} \mathbb{E}m_t(v', v) .$$

By symmetry, $\mathbb{E}m_t(v', v) = \mathbb{E}m_t(v, v')$ for all $v'$ so this equals

$$\mathbb{E}m_t(v, v) + \sum_{v' \in S_2 : \|v - v'\|_1 = 1} \mathbb{E}m_t(v, v') .$$

Note that whenever $v$ flips, the sum of the changes of $e_{v'}(t) - e_{v'}(0)$ for all neighbours $v'$ is simply equal to the change of $e_v(t) - e_v(0)$. Therefore,

$$\mathbb{E}\left[e_v(t) - e_v(0)\right] = 2\mathbb{E}m_t(v, v) .$$

Because $v$ has 5 neighbours, $e_v(t)$ decreases by at least 2 each time $\sigma_v$ flips, so $\mathbb{E}m_t(v,v)$ is bounded above by $-2\mathbb{E}N_t(v)$, where $N_t(v)$ is the number of flips of $\sigma_v$ until time $t$. Taking $t$ to infinity and noting that $|e_v(t) - e_v(0)| \leq 10$ for all $t$, we see that almost surely, $\sigma_v$ flips finitely often.

For the periodic case, we will use the following fact several times. If $A \subset \Omega_2$ then we say that $\sigma(t) \in A$ *infinitely often* if the set $\{t : \sigma(t) \in A\}$ is unbounded. To avoid technical issues, we will restrict our attention to $A$'s that are cylinder sets.

**Lemma 1** *If A and B are (cylinder) events in $\Omega_2$ such that*

$$\inf_{\sigma \in A} \mathbb{P}(\sigma(t) \in B \textit{ for some } t \in (0,1] \mid \sigma(0) = \sigma) > 0,$$

*then*

$$\mathbb{P}(\sigma(t) \in A \textit{ infinitely often but } \sigma(t) \in B \text{ finitely often}) = 0.$$

*Proof.* The proof is just an application of the strong Markov property at a sequence of stopping times $(\mathcal{T}_k)$, which could be given by $\mathcal{T}_0 = 0$ and

$$\mathcal{T}_k = \inf\{t \geq \mathcal{T}_{k-1} + 2 : \sigma(t) \in A\}\ .$$

Let us say that a site $v \in S_2$ *fixates* for the realization $(\sigma(0), \omega)$, if there exists $T_v = T_v(\sigma(0), \omega) < \infty$ such that $\sigma_v(t) = \sigma_v(T_v)$ for all $t \geq T_v$. We say that $v$ *fixates from time* $T$ if for all $t \geq T$, $\sigma_v(t) = \sigma_v(T)$.

We now define a process $\tau(t) = (\tau_y(t) : y \in \mathbb{Z}^2)$ from $\sigma(t)$ by declaring $\tau_{\pi(v)}(t) = \sigma_v(t)$ if $\sigma_v(t) = \sigma_{\hat{v}}(t)$. Otherwise, we declare $\tau_{\pi(v)}(t)$ to be grey. In the latter case, we refer to $\pi(v)$ as a $(+/-)$ or $(-/+)$ site if the site of $v, \hat{v}$ with third coordinate 1 is $+1$ or $-1$, respectively. We will use the terms 'flip' and 'fixate' for the configuration $\tau(t)$ as well. Note that with probability one, a site cannot flip from grey to grey; that is, it cannot flip from $(+/-)$ to $(-/+)$ or $(-/+)$ to $(+/-)$. We may say that $\pi(v)$ fixates at $+$; this means that $\pi(v)$ fixates and that its terminal value is $+$. We define $\pi(v)$ fixating at $-$ or at grey (either $(+/-)$ or $(-/+)$) similarly.

**Lemma 2** *With probability one, no site in $\mathbb{Z}^2$ can fixate at grey.*

*Proof.* Let $v \in S_2$ and $A_v \subset \Omega_2$ be the event that $\sigma_v = +1$ but $\sigma_w = -1$ for at least 3 neighbours of $v$ (counting $\hat{v}$ twice). Let $B_v$ be the event that $\sigma_v = -1$ but $\sigma_w = -1$ for at least 3 neighbours of $v$. There is some $c > 0$ such that

$$\mathbb{P}\left(\sigma_v(t) \in B_v \text{ for some } t \in (0,1] \,\middle|\, \sigma(0) = \sigma\right) \geq c \text{ for all } \sigma \in A_v\ .$$

For instance, $v$'s clock may ring before those of all its neighbours and $\sigma_v$ then flips. Using Lemma 1,

$$\mathbb{P}\left(\sigma(t) \in A_v \text{ infinitely often but } \sigma(t) \text{ flips only finitely often}\right) = 0\ .$$

Suppose that for some $y \in \mathbb{Z}^2$ and $t \geq 0$, $\tau_y(t)$ is grey but $\tau_y(T) = \tau_y(t)$ for $T \geq t$. Write $v$ for the site in $S_2$ with $\pi(v) = y$ and third coordinate equal to 1. Assume without loss in generality that $\tau_y$ is $(+/-)$. Note that then $v$ already has at least two unsatisfied neighbours (since we count $\hat{v}$ twice). Therefore, off of the probability zero event above, all neighbours of $v$ with third coordinate equal to 1 must fixate in $\sigma(t)$ at $+1$. Similarly, all neighbours of $\hat{v}$ with third coordinate equal to 0 fixate in $\sigma(t)$ at $-1$. Iterating this argument, with positive probability, the top level of $S_2$ fixates in $\sigma(t)$ at $+1$ and the bottom fixates at $-1$. By ergodicity under spatial translations, this event would have probability 1, but this contradicts symmetry under permuting the top and bottom levels.

**Lemma 3** *With probability one, for all $y \in \mathbb{Z}^2$, if $y-(1,0)$ and $y+(0,1)$ fixate in $\tau(t)$, then so does $y$.*

*Proof.* Suppose that $y-(1,0)$ and $y+(0,1)$ fixate in $\tau(t)$. If they both fixate at $+$ then the argument is not difficult—if $y$ does not fixate, it must be grey (say $(+/-)$) infinitely often. But in this case, we can apply Lemma 1 to the event $A$ that $\sigma_{y-(1,0)}$ and $\sigma_{y+(0,1)}$ are $+1$ with $\tau_y$ equal to $(+/-)$ and $B$ the event that $\sigma_{y-(1,0)}$ and $\sigma_{y+(0,1)}$ are $+1$ with $\tau_y$ equal to $+$. This proves that $\tau_y$ cannot be grey infinitely often without being $+$ infinitely often. But, because two neighbours of $y$ have $\tau$-value fixed at $+$, each $\sigma_v$ with $\pi(v) = y$ has four $+$ neighbours and hence, $\tau_y$ will then remain at $+$ after some time.

Otherwise, $y-(1,0)$ and $y+(0,1)$ fixate at different $\tau$-values, say $+1$ and $-1$ respectively. We will use the following fact: with probability one, each spin $\sigma_v$ can have only finitely many energy-lowering flips. In other words, for each $v \in S_2$ and $t \geq 0$, we can define $F_v(t)$ to be the number of times $s \in (0,t)$, such that $\sigma_v(s^-) \neq \sigma_v(s^+)$ and $e_v(s^+) < e_v(s^-)$. Since the measure $\mathbb{P}$ is invariant under translations, the argument of Newman–Nanda–Stein [6] can be applied to find

$$\lim_{t\to\infty} F_v(t) < \infty \text{ with probability one .}$$

As a consequence of this and Lemma 4, we see that for each $v \in S_2$,

$$\mathbb{P}\left(\sigma_v(t) \text{ disagrees with at least 4 neighbours of } v \text{ in } S_2 \text{ infinitely often}\right) = 0\,. \quad (1)$$

Assume that the $\tau$-value at $y$ does not fixate; then it must be grey (for example $(+/-)$) infinitely often. Note that at each of these times, each $\sigma_v$ spin at a site $v$ with $\pi(v) = y$ disagrees with at least 3 neighbours. From the above remarks, there must be some random time at which these spins no longer disagree with at least 4 neighbours. This implies that at infinitely many of these times, $\tau_y$ must also be identical to $\tau_{y+(1,0)}$ and $\tau_{y-(0,1)}$, so

$$\tau_{y-(1,0)} = +,\ \tau_y = +/-,\ \tau_{y+(0,1)} = - \text{and } \tau_{y+(1,0)} = \tau_{y-(0,1)} = +/-,$$

as in the left diagram of Fig. 1.

We now consider the $\tau$-value of $y+(1,-1)$ at these times $\mathcal{T}$. There must exist one status from the choices $+, -, (+/-)$ and $(-/+)$, such that this spin has this status infinitely often (of the times $\mathcal{T}$). But now, it is elementary (though a bit tedious) to see that in each case, there is a finite sequence of flips that will lead all eight $\sigma_v$'s for $v \in S_2$ with $\pi(v)$ in the set $\{y, y+(1,0), y-(0,1), y+(1,-1)\}$ to have the same

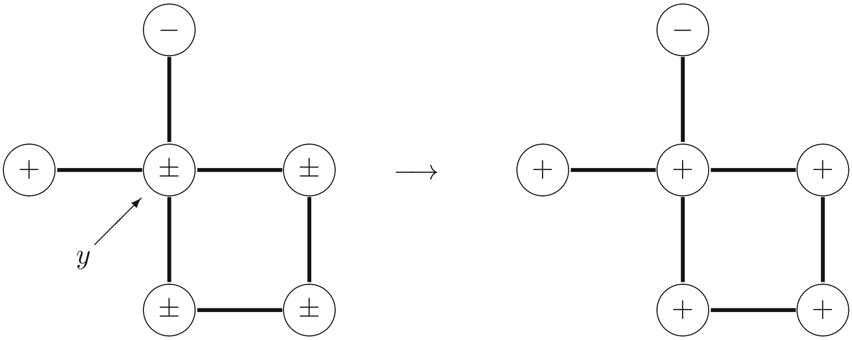


**Fig. 1** The local configuration $\tau$ near $y$ on the left, in which $\tau_y = (+/-)$, $\tau_{y-(1,0)} = +$, $\tau_{y+(0,1)} = -$ and $\tau_y = (+/-)$. The $\tau$-values at $y+(1,0)$ and $y-(0,1)$ are $(+/-)$. In the case depicted, we also have $\tau_{y+(1,-1)} = (+/-)$. One example of a finite sequence of flips that can occur is as follows. $\tau_y$ flips to $+$, $\tau_{y+(1,0)}$ flips to $+$, $\tau_{y-(0,1)}$ flips to $+$ and then $\tau_{y+(1,-1)}$ flips to $+$. This eventually fixates $\tau$-values as on the right

sign—see Fig. 1 for an example. Using Lemma 1 completes the proof, because once they are the same sign, they can never flip again.

To complete the proof, we invoke a comparison to bootstrap percolation, giving a version of van Enter's argument [3] initially due to Straley. For any $\sigma \in \Omega_2$ we identify a configuration $\eta = \eta(\omega) \in \{0,1\}^{\mathbb{Z}^2}$ as follows: We declare $\eta_x = 1$ if all $v$ in the $2 \times 2 \times 2$ block $B_x = 2x + \{0,1\}^3$ have spins of the same sign in $\sigma$. Note that under the coarsening dynamics, all such sites are fixated in $S_2$. For all other sites $x$ we set $\eta_x = 0$. We then run the following discrete time (deterministic) dynamics on $\eta$. We set $\eta(0) = \eta(\omega)$ with $\omega$ distributed by $\mathbb{P}_{1/2}$ and for each $n \in \mathbb{N}$ and $x \in \mathbb{Z}^2$, we set $\eta_x(n) = 1$ if either (a) $\eta_x(n-1) = 1$ or (b) $\eta_{y_1}(n-1) = \eta_{y_2}(n-1) = 1$ for at least two neighbours $y_i$ of $x$ with $\|y_1 - y_2\|_\infty = 1$. Otherwise we set $\eta_x(n) = 0$. This is a modified bootstrap percolation dynamics.

We claim that with probability one, for each $x$, the value $\eta_x(n)$ is 1 for all large $n$. Using Lemma 3, this will prove that all sites in $S_2$ fixate. To show the claim, we briefly summarize the classic argument of [3]. Because the $\eta_x(0)$ variables are independent from site to site, one can show that for some $n$, the probability is positive that all sites in the rectangle $[0,n]^2$ begin with $\eta$-value 1 but that there is no rectangular contour enclosing $[0,n]^2$, all of whose sites begin with $\eta$-value 0. On this event, under our dynamics, such a rectangle will grow to absorb all of space and fix all sites to have $\eta$-value 1. However, by the ergodic theorem, with probability one, some translate of this event will occur and this completes the proof.

## 4 Proof of Theorem 2

We begin by proving the case $k = 3$ with free boundary conditions. The idea is to force a large rectangle on level 3 (that is, with third coordinate equal to 2) to be fixed at $+1$ with a parallel region on level 1 (third coordinate equal to 0) fixed at

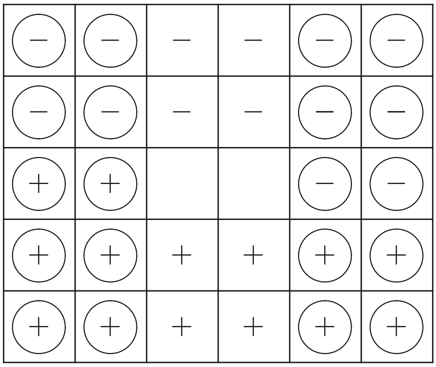

**Fig. 2** Level 1 (in $\mathbb{Z}^2 \times \{1\}$) in the event $A$, for a slab of width 3 with free boundary conditions. The left unmarked box represents the vertex $(0, 0, 1)$. The vertices with circled spins are ones both of whose third coordinate neighbours ("above" and "below") have the same spin. Any configuration in $A$ has the property that spins at vertices above those in the uncircled region are $-1$ and below those are $+1$. The unmarked spins flip infinitely often

$-1$. Spins on the middle level between these regions act like spins in the coarsening model on $\mathbb{Z}^2$.

To stabilize levels 1 and 3, we define for $m, n \in \mathbb{Z}$, the set $P_{m,n} = \{m, m+1\} \times \{n, n+1\} \times \{0, 1, 2\}$ and the "table" of size $n \geq 2$

$$T_n = \Big[\{-n, \ldots, n\}^2 \times \{2\}\Big] \cup P_{-n,-n} \cup P_{-n,n-1} \cup P_{n-1,-n} \cup P_{n-1,n-1} \,.$$

The inverted table of size $n$, $T_n'$, is the reflection of $T_n$ through $\mathbb{Z}^2 \times \{1\}$. Note that if either of these sets are initially monochromatic, then they will be fixed by the coarsening dynamics. Define the event $A \subset \Omega_3$ that,

1. all sites in $P_1 = P_{-2,-2} \cup P_{-2,-1} \cup P_{2,-2}$ have spin $+1$ and all sites in $P_2 = P_{-2,1} \cup P_{2,0} \cup P_{2,1}$ have spin $-1$,
2. all sites in $\{0, 1\} \times \{-2, -1\} \times \{1\}$ have spin $+1$ and all sites in $\{0, 1\} \times \{1, 2\} \times \{1\}$ have spin $-1$,
3. all sites in $T_{10}' \setminus \Big[P_1 \cup P_2\Big]$ have spin $+1$ and
4. all sites in $T_{20} \setminus \Big[T_{10}' \cup P_1 \cup P_2\Big]$ have spin $-1$.

The reader may verify that all sites in $T_{10} \cup T_{20}' \cup P_1 \cup P_2$ are fixated in the event $A$. However, the vertex $(0, 0, 1)$ then has 3 plus neighbours, so by Lemma 1 it must have a plus spin infinitely often. This implies that the vertex $(1, 0, 1)$ has 3 plus neighbours infinitely often and therefore, must have a plus spin infinitely often. By symmetry, the same is true for these vertices and minus spin, meaning they flip infinitely often. As usual, by spatial ergodicity, almost surely some translate of this event occurs and therefore with probability one, not all sites fixate.

The cases $k \geq 3$ with free boundary conditions are handled similarly. We simply add more layers of the construction on top of level 2. To define the event precisely, we set $A$ to be the event defined exactly as in the case of $k = 3$ (above). This event only involves the first three levels $(0 - 2)$ of the slab. Define $A' = \{\sigma\}$ as the event that $\sigma \in A$ and that for all $(x, y, k)$ with $(x, y) \in \{-20, \ldots, 20\}^2$ and $k \geq 3$, we have $\sigma_{(x,y,k)} = \sigma_{(x,y,2)}$. Because in the slab $S_3$, the event $A$ forced all spins for vertices in $\{-20, \ldots, 20\}^2 \times \{2\}$ to be fixed, it is not hard to check that on $A'$, all spins for vertices in $\{-20, \ldots, 20\}^2 \times \{2, \ldots, k - 1\}$ are also fixed. The same argument as before gives that the spins at $(0, 0, 1)$ and $(1, 0, 1)$ do not fixate and consequently the slab does not fixate.

For the case $k \geq 5$ with periodic boundary conditions, we consider again the event $A$ and add duplicate layers of level 2 as before. The only difference is that we also need to add a duplicate layer of the zeroth level at the top (which is the same as level $-1$), in the set $\mathbb{Z}^2 \times \{k - 1\}$. Because we need to duplicate both level 0 and 2, this requires at least 5 layers. The proof is now complete.

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# Selected Topics in Random Walks in Random Environment

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**Abstract** Random walk in random environment (RWRE) is a fundamental model of statistical mechanics, describing the movement of a particle in a highly disordered and inhomogeneous medium as a random walk with random jump probabilities. It has been introduced in a series of papers as a model of DNA chain replication and crystal growth (see Chernov [10] and Temkin [51, 52]), and also as a model of turbulent behavior in fluids through a Lorentz gas description (Sinai 1982 [42]). It is a simple but powerful model for a variety of complex large-scale disordered phenomena arising from fields such as physics, biology, and engineering. While the one-dimensional model is well-understood in the multidimensional setting, fundamental questions about the RWRE model have resisted repeated and persistent attempts to answer them. Two major complications in this context stem from the loss of the Markov property under the averaged measure as well as the fact that in dimensions larger than one, the RWRE is not reversible anymore. In these notes we present a general overview of the model, with an emphasis on the multidimensional setting and a more detailed description of recent progress around ballisticity questions.

We present a review of random walks in random environment. The main focus evolves around several fundamental questions! concerning the existence of invariant probability measures, transience, recurrence, directional transience, and ballisticity. This choice of topics is somewhat biased toward our recent research interests.

The first chapter deals with the question of the existence of an invariant probability measure of the so-called "environmental process"; such a measure is particularly useful if it is absolutely continuous with respect to the law of the environment. The existence and properties of such a measure characterize in some sense the different asymptotic behaviors of the walk, from a general law of large numbers to possibly

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a quenched central limit theorem, and to a variational formula for the rate function in the case of quenched large deviations. After the introduction of basic definitions and concepts, we review the one-dimensional situation, which turns out to be a controlled laboratory of several phenomena which one would expect to encounter in the multidimensional setting. Subsequently, we investigate the latter setting and give some of the corresponding (limited) results which are available in that context.

It is conjectured that for uniformly elliptic and i.i.d. environments, in dimensions $d \geq 2$, directional transience implies ballisticity. The second chapter of these notes reviews this question as well as the progress and understanding which have been achieved toward its resolution. In particular, we introduce the fundamental concept of renewal times. We then proceed to the ballisticity conditions, under which it has been possible to obtain a better understanding of the so-called slowdown phenomena as well as of the ballistic and diffusive behavior in the setting of (uniformly) elliptic environments.

# 1 The Environmental Process and Its Invariant Measures

## *1.1 Definitions*

Throughout these notes, for $x \in \mathbb{R}^d$, we will use the notations $|x|_\infty$, $|x|_1$, and $|x|_2$ for the $L^\infty$, $L^1$, and $L^2$ norms. For a subset $A \subset \mathbb{Z}^d$ we denote by $\partial A$ its external boundary

$$\{x \in \mathbb{Z}^d \backslash A \,:\, \exists y \in A \text{ with } |x - y|_1 = 1\}, \tag{1}$$

and for a subset $B \subset \mathbb{R}^d$ we denote by $\overset{\circ}{B}$ its interior. We write

$$B_p(x_0, r) = \{x \in \mathbb{R}^d : |x - x_0|_p \leq r\} \tag{2}$$

for the closed ball centered in the $x_0$ with radius $r$ in the $p$-norm. In addition, set $B_p(r) := B_p(0, r)$. Furthermore, the set $U := \{e \in \mathbb{Z}^d : |e|_1 = 1\}$ will serve as the set of possible jumps for the random walk to be defined. We will use $C$ to denote constants that can change from one side of an inequality to another, and $c_1, c_2, \ldots$ for constants taking fixed values. Furthermore, if we want to emphasize the dependence of a constant on quantities, e.g., the dimension, we write $C(d)$. We begin with the definition of an environment.

**Definition 1 (Environment)** We define the set

$$\mathcal{P} := \left\{(p(e))_{e \in U} \in [0,1]^U \,:\, \sum_{e \in U} p(e) = 1\right\} \tag{3}$$

of $2d$-vectors $p$ serving as admissible transition probabilities. An *environment* is an element $\omega$ of the *environment space* $\Omega := \mathcal{P}^{\mathbb{Z}^d}$ so that $\omega := (\omega(x))_{x \in \mathbb{Z}^d}$, where $\omega(x) \in \mathcal{P}$. We denote the components of $\omega(x)$ by $\omega(x, e)$.

Let us now define a random walk in a given environment $\omega$.

**Definition 2** **(Random walk in an environment** $\omega$**)** Let $\omega \in \Omega$ be an environment and let $\mathcal{G}$ be the $\sigma$-algebra on $(\mathbb{Z}^d)^{\mathbb{N}}$ defined by the cylinder functions. For $x \in \mathbb{Z}^d$, we define the *random walk in the environment* $\omega$ *starting in* $x$ as the Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$ whose law $P_{x,\omega}$ on $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G})$ is characterized by

$$P_{x,\omega}[X_0 = x] = 1, \quad \text{and}$$

$$P_{x,\omega}[X_{n+1} = y + e \mid X_n = y] = \begin{cases} \omega(y, e), & \text{if } e \in U, \\ 0, & \text{otherwise}, \end{cases}$$

whenever $P_{x,\omega}[X_n = y] > 0$, and 0 otherwise. Furthermore, we denote by

$$p^{(n)}(x, y, \omega) := P_{x,\omega}[X_n = y] \tag{4}$$

the $n$-step transition probability of the random walk in the environment $\omega$.

We will now account for the randomness in the environment. For that purpose, let us endow the environment space $\Omega$ with the product topology and let $\mathbb{P}$ be some probability measure defined on $(\Omega, \mathcal{B}(\Omega))$; here, $\mathcal{B}$ denotes the corresponding Borel $\sigma$-algebra. We call $\mathbb{P}$ the *law of the environment* and for every measurable function $f$ defined on $\Omega$, we denote by $\mathbb{E}[f]$ the corresponding expectation if it exists. It will frequently be useful to assume that

**(IID)** the coordinate maps on the product space $\Omega$ are independent and identically distributed (i.i.d.) under $\mathbb{P}$.

In order to give a relaxation of **(IID)** we introduce the following notation. For each $y \in \mathbb{Z}^d$, let us denote by $t_y$ the translation defined on the environment space $\Omega$ by

$$(t_y\omega)(x, e) := \omega(x + y, e),$$

for every $x \in \mathbb{Z}^d$ and $e \in U$. It will often be useful to assume that

**(ERG)** the family of transformations $(t_x)_{x\in\mathbb{Z}^d}$ is an ergodic family acting on $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$.

In other words, if $A \in \mathcal{B}(\Omega)$ is such that $A = t_x^{-1}(A)$ for every $x \in \mathbb{Z}^d$, then $\mathbb{P}(A) = 0$ or 1. This condition is also called *total ergodicity*. In particular, note that **(IID)** implies **(ERG)**.

For a fixed realization of $\omega$, we now call $P_{x,\omega}$ the *quenched law* of the random walk in random environment (RWRE). Using Dynkin's theorem, it is not hard to show that for each $x \in \mathbb{Z}^d$ and $G \in \mathcal{G}$, the mapping

$$\omega \mapsto P_{x,\omega}[G]$$

is $\mathcal{B}(\Omega)$-measurable. We can therefore define on the space $(\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{B}(\Omega) \otimes \mathcal{G})$ for each $x \in \mathbb{Z}^d$ the semi-direct product $P_{x,\mathbb{P}}$ of the measures $\mathbb{P}$ and $P_{x,\omega}$ by the formula

$$P_{x,\mathbb{P}}[F \times G] := \int_F P_{x,\omega}(G)\, \mathbb{P}(\mathrm{d}\omega). \tag{5}$$

We denote by $P_x$ the marginal law of $P_{x,\mathbb{P}}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$ and call it the *averaged* or *annealed* law of the RWRE. One of the difficulties arising in the study of RWRE is that under the averaged law is generally not Markovian anymore.

We will need the concepts of ellipticity and uniform ellipticity.

**Definition 3 (Ellipticity and uniform ellipticity)** Let $\mathbb{P}$ be a probability measure defined on the space of environments $(\Omega, \mathcal{B}(\Omega))$.

- We say that $\mathbb{P}$ is *elliptic* if
  **(E)**, for every $x \in \mathbb{Z}^d$ we have that

$$\mathbb{P}\,[\min_{e\in U} \omega(x,e) > 0] = 1. \tag{6}$$

- We say that $\mathbb{P}$ is *uniformly elliptic* if

  **(UE)** there exists a constant $\kappa > 0$ such that for every $x \in \mathbb{Z}^d$ we have that

$$\mathbb{P}\,[\min_{e\in U} \omega(x,e) \geq \kappa] = 1. \tag{7}$$

  We will usually call the environment (uniformly) elliptic in such case.

**Remark 1** *This labeling is motivated by operator theory where one has analogous definitions of elliptic and uniformly elliptic differential operators.*

The following auxiliary process will play a significant role in what follows.

**Definition 4 (Environment viewed from the particle)**. Let $(X_n)$ be an RWRE. We define the *environment viewed from the particle* (or also *the environmental process*) as the discrete time process

$$\bar{\omega}_n := t_{X_n}\omega,$$

for $n \geq 0$, with state space $\Omega$.

Apart from taking values in a compact state space, another advantage of the environment viewed from the particle is that even under the averaged measure it is Markovian, as is shown in the next result following Sznitman [6]; however, the cost is that we now deal with an infinite dimensional state space.

**Proposition 1** *Consider an RWRE in an environment with law $\mathbb{P}$. Then, under $P_0$, the process $(\bar{\omega}_n)$ is Markovian with state space $\Omega$, initial law $\mathbb{P}$, and transition kernel*

$$Rf(\omega) := \sum_{e\in U} \omega(0,e) f(t_e\omega), \tag{8}$$

*defined for $f$ bounded measurable on $\Omega$ and initial law $\mathbb{P}$.*

*Proof* Let us first note that for every $x \in \mathbb{Z}^d$, and every bounded measurable function $f$ on $\Omega$,

$$\begin{aligned} E_{x,\omega}[f(\bar{\omega}_1)] = E_{x,\omega}[f(t_{X_1}\omega)] &= \sum_{e\in U} \omega(x,e) f(t_{x+e}\omega) \\ &= \sum_{e\in U} t_x\omega(0,e) f(t_e(t_x\omega)) \\ &= Rf(t_x\omega). \end{aligned} \tag{9}$$

Let $f_i, i = 0, \dots, n+1$ be bounded measurable functions. Note that

$$\begin{aligned} E_{0,\omega}[f_{n+1}(\bar{\omega}_{n+1}(f_n(\bar{\omega}_n)\cdots f_0(\bar{\omega}_0)] &= E_{0,\omega}[f_{n+1}(t_{X_{n+1}}\omega)\cdots f_0(t_{X_0}\omega)] \\ &= E_{0,\omega}[E_{X_n,\omega}(f_{n+1}(t_{X_1}\omega))f_n(\bar{\omega}_n)\cdots f_0(\bar{\omega}_0)] \\ &= E_{0,\omega}[Rf_{n+1}(\bar{\omega}_n)f_n(\bar{\omega}_n)\cdots f_0(\bar{\omega}_0)], \end{aligned}$$

where in the second equality we took advantage of the Markov property of $(X_n)$ under $P_{0,\omega}$, and in the last step we have used (9). Since $Rf_{n+1}(\bar{\omega}_n)$ is $\mathcal{F}_n$-measurable, where $\mathcal{F}_n$ is the natural filtration of $(\bar{\omega}_n)$, it follows from the above that

$$E_{0,\omega}[f_{n+1}(\bar{\omega}_{n+1}) \,|\, \bar{\omega}_0, \dots, \bar{\omega}_n] = Rf_{n+1}(\bar{\omega}_n), \tag{10}$$

which proves the Markov property of the chain $(\bar{\omega}_n)$ under the measure $P_{0,\omega}$. It follows that the transition kernel for the quenched process is given by (8). Integrating $P_{0,\omega}$ with respect to $\mathbb{P}$ we finish the proof.

## *1.2 Invariant Probability Measures of the Environment as Seen from the Random Walk*

We now want to examine the invariant measures of the Markov chain $(\bar{\omega}_n)$. Given an arbitrary probability measure $\mathbb{P}$ on $\Omega$, we define the probability measure $\mathbb{P}R$ through the identity

$$\int Rf\,d\mathbb{P} = \int f\,d(\mathbb{P}R),$$

for every bounded continuous function $f$ on $\Omega$. Whenever $\mathbb{P} = \mathbb{P}R$, we will say that $\mathbb{P}$ is an *invariant probability measure* for the environmental process. We will also need to consider the possibility of having invariant measures which are not necessarily probability measures: similarly to the above, we will say that a measure $\nu$ is invariant for the environmental process if for every bounded continuous function $f$ one has that

$$\int f\,d\nu = \int Rf\,d\nu.$$

It is obvious that any degenerate probability measure which is translation invariant, is an invariant probability measure: this corresponds to any simple random walk. The following lemma is a standard result, but shows that there might be some other ways of constructing more interesting invariant probability measures. Recall that given a sequence of probability measures a limit measure is defined as the limit of any convergent subsequence.

**Lemma 1** *Consider an RWRE and the corresponding environmental process* $(\bar{\omega}_n)$. *Then, if* $\mathbb{P}$ *is any probability measure in* $\Omega$, *there exists at least one limit measure of the Cesàro means*

$$\frac{1}{n+1}\sum_{i=0}^{n} \mathbb{P}R^i. \tag{11}$$

*Furthermore, every limit measure of this Cesàro means is an invariant probability measure for the Markov chain $(\bar{\omega}_n)$.*

*Proof* Let $\mathbb{P}$ be an arbitrary probability measure defined on the space $\Omega$. Denote for each $n \geq 0$ as $\nu_n$ the Cesàro means of (11).

Since the space of probability measures defined on $\Omega$ is compact under the topology of weak convergence, we can extract a weakly convergent subsequence $\nu_{n_k}$, so that the Cesàro means has at least one limit point $\nu$. We claim that $\nu$ is an invariant probability measure. Indeed, it is enough to prove that

$$\int Rf d\nu = \int f d\nu$$

for every bounded continuous function $f$. But since the transition kernel $R$ maps bounded and continuous functions to bounded and continuous functions, we have that

$$\begin{aligned}\int Rf d\nu &= \lim_{k\to\infty} \int Rf d\nu_{n_k} = \lim_{k\to\infty} \frac{1}{n_k+1}\sum_{i=0}^{n_k}\int f d(\nu R^{i+1}) \\ &= \lim_{k\to\infty}\left(\frac{1}{n_k+1}\sum_{i=0}^{n_k}\int f d(\nu R^i) + \frac{1}{n_k+1}\int f d(\nu R^{n_k+1}) - \frac{1}{n_k+1}\int f d\nu\right) \\ &= \lim_{k\to\infty}\int f d\nu_{n_k} = \int f d\nu.\end{aligned}$$

Knowing only the existence of an invariant probability measure turns out not to be very helpful. We will see that what we really need is to find one which is absolutely continuous with respect to the law $\mathbb{P}$ of the environment.

**Example 1** *Let us consider the case $d = 1$. Assume* **(E)** *to be fulfilled and define*

$$\rho(x,\omega) := \frac{\omega(x,-1)}{\omega(x,1)} \quad \text{and} \quad \rho(\omega) := \rho(0,\omega). \tag{12}$$

*If $\mathbb{E}[\rho] < 1$ and the environment $(\omega(x))_{x\in\mathbb{Z}}$ is i.i.d. under the law $\mathbb{P}$, we will prove in this lecture that*

$$\nu(d\omega) := f(\omega)\mathbb{P}(d\omega),$$

*where*

$$\begin{aligned}f(\omega) := C\,(1+\rho(0,\omega))\,(1+\rho(1,\omega)+\rho(1,\omega)\rho(2,\omega) \\ +\rho(1,\omega)\rho(2,\omega)\rho(3,\omega)+\cdots) < \infty,\end{aligned}$$

*for some constant $C > 0$, is an invariant probability measure for the process $(\bar{\omega}_n)$, cf. also Theorem 3 below.*

## *1.3 Transience and Recurrence in the One-Dimensional Model*

The focus of this section will be on one-dimensional RWRE under the assumption **(E)** and ergodicity properties of the law $\mathbb{P}$ of the environment. In this context, we will derive explicit necessary and sufficient conditions in terms of the environment for the walk being transient or recurrent. It turns out that in this case the model is reversible in the following sense: for $\mathbb{P}$-a.a. environments $\omega$ it is possible to find a measure defined on $\mathbb{Z}$ for which the random walk $(X_n)$ in environment $\omega$ is reversible. This observation partly explains the fact that many explicit computations can be performed, and even explicit conditions characterizing particular behaviors of the walk can be found.

The following lemma of Kesten [23] will prove useful.

**Lemma 2** *Given any stationary sequence of random variables* $(Y_n)_{n\geq 0}$ *with law* $P$ *such that* $P[\lim_{n\to\infty}\sum_{k=0}^{n} Y_n = \infty] = 1$ *one has* $P\left[\liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} Y_k > 0\right] = 1$.

In what follows, we will say that a function $f$ is *Lebesgue integrable in the extended sense* if its Lebesgue integral exists, possibly taking the values $\infty$ or $-\infty$.

**Theorem 1** *Consider an RWRE in dimension* $d = 1$ *in an environment with law* $\mathbb{P}$ *such that* **(E)** *holds. Assume* **(ERG)** *and that* $\mathbb{E}[\log\rho]$ *is Lebesgue integrable in the extended sense. Then the following are satisfied.*

1. *If* $\mathbb{E}[\log\rho] < 0$ *then the random walk is* $P_0$*-a.s. transient to the right, i.e.,*

$$\lim_{n\to\infty} X_n = \infty, \qquad P_0 - a.s.$$

2. *If* $\mathbb{E}[\log\rho] > 0$ *then the random walk is* $P_0$*-a.s. transient to the left, i.e.,*

$$\lim_{n\to\infty} X_n = -\infty, \qquad P_0 - a.s.$$

3. *If* $\mathbb{E}[\log\rho] = 0$ *then the random walk is* $P_0$*-a.s. recurrent and*

$$\limsup_{n\to\infty} X_n = \infty \qquad \text{and} \qquad \liminf_{n\to\infty} X_n = -\infty \qquad P_0 - a.s.$$

The above theorem was first proved within the context of branching processes in i.i.d. random environments by Smith and Wilkinson in 1969 [43] (see also [25, Remark 8] and the references therein). In 1975 it was proved by Solomon [44] for i.i.d. environments and afterwards extended to ergodic environments by Alili [1]. Here, we present a proof based on the method of Lyapunov functions (see [11] and [18]). The so-called Sinai's regime corresponds to the recurrent case under the additional assumption that $0 < \mathbb{E}[(\log\rho)^2] < \infty$. In [41], Sinai proved that under these conditions the position of the walk at time $n$ is typically of order $(\log n)^2$ under $P_0$. We will see in Sect. 1.4, that the dichotomy expressed by Theorem 1 is an expression of the different possibilities concerning the existence of an invariant measure (not necessarily a probability measure) for the environmental process which is absolutely continuous with respect to $\mathbb{P}$: (2) and (1) occur when there exists such a measure; (3) occurs when such a measure does not exist.

*Proof* We want to find a martingale defined in terms of the environment which discriminates between transience and recurrence through the use of the martingale convergence theorem. Let us furthermore try to find such a martingale of the form $f(X_n)$, where

$$f(x) = \sum_{j=0}^{x-1} \Delta_j,$$

for $x \geq 0$, and for some sequence $(\Delta_j)$ which will be chosen appropriately. In fact, using this convergence we will deduce the desired asymptotics from the properties of the limit of that martingale. Now note that with $q(x) := \omega(x,1)$ and $p(x) := \omega(x,-1)$,

$$E_{x,\omega}[f(X_{n+1}) - f(X_n) \,|\, X_n = y] = \begin{cases} -p(y)\Delta_{y-1} + q(y)\Delta_y, & \text{if} \quad y \geq 2, \\ p(1)\Delta_{-1} + q(1)\Delta_1, & \text{if} \quad y = 1, \\ p(y)\Delta_{y-2} - q(y)\Delta_{y-1}, & \text{if} \quad y \leq 0. \end{cases}$$

But if $f(X_n)$ is a martingale, the left-hand side of this display must vanish and we should have that

$$\Delta_1 = -\rho_1 \Delta_{-1},$$

and that

$$\Delta_y = \rho_y \Delta_{y-1} \qquad \text{for} \quad y \geq 2,$$
$$\Delta_{y-2} = \rho_y^{-1} \Delta_{y-1} \qquad \text{for} \quad y \leq 0,$$

where we have used the shorthand notation $\rho_y := \rho(y,\omega)$. Choosing $\Delta_0 = -1$, $\Delta_1 = -\rho_1$ and $\Delta_{-1} = 1$ we deduce that

$$f(x) = \begin{cases} -\sum_{0\leq j\leq x-1} \prod_{i=1}^{j} \rho_i, & \text{if} \quad x \geq 0, \\ \sum_{x\leq j\leq -1} \prod_{i=j+1}^{0} \rho_i^{-1}, & \text{if} \quad x < 0, \end{cases}$$

serves our purposes, where $\prod_{i=1}^{0} \rho_i := 1$. Hence, $f$ is harmonic with respect to the generator of the quenched RWRE and $f(X_n)$, is an $\mathcal{G}_n$-martingale under the probability measure $P_{0,\omega}$, where $\mathcal{G}_n$ is the natural $\sigma$-algebra generated by the random walk. Now, by the ergodic theorem, we have $\mathbb{P}$-a.s. that

$$\prod_{i=1}^{x} \rho_i = \exp\left\{x(\mathbb{E}[\,\log\rho] + o(1))\right\},$$

as $x \to \infty$, while when $x \to -\infty$, one has

$$\prod_{i=x+1}^{-1} \rho_i = \exp\left\{x(\mathbb{E}[\,\log\rho] + o(1))\right\}.$$

We now see that in the case $\mathbb{E}[\,\log\rho] < 0$, there is a constant $C > 0$ such that $\mathbb{P}$-a.s.

$$\lim_{x\to\infty} f(x) = -C, \tag{13}$$

and

$$\lim_{x \to -\infty} f(x) = \infty.$$

It follows that $\mathbb{P}$-a.s.

$$E_{0,\omega}[f(X_n)_-] = \sum_{x=1}^{\infty} f(x) P_{0,\omega}[X_n = x] < \infty.$$

By the martingale convergence theorem $P_0$-a.s.

$$\lim_{n \to \infty} f(X_n) \quad \text{exists.} \tag{14}$$

Now, by ellipticity, it is easy to see that $P_0$-a.s., only the following three possibilities can occur:

1. $\limsup_{n \to \infty} X_n = \infty$ and $\liminf_{n \to \infty} X_n = -\infty$.
2. $\lim_{n \to \infty} X_n = \infty$.
3. $\lim_{n \to \infty} X_n = -\infty$.

By (14) and (13) we conclude that necessarily case (2) mentioned above occurs. By a similar analysis we see that if $\mathbb{E}[\log \rho] > 0$, case (3) occurs. Let us now consider the case

$$\mathbb{E}[\log \rho] = 0.$$

If $\rho$ was almost surely constant and hence equal to 1, the above setting would be reduced to simple random walk, for which the corresponding result is canonical knowledge. Therefore, without loss of generality, we can assume that $\mathbb{E}[(\log \rho)^2] > 0$, and equally that $\mathbb{P}[\log \rho > 0] > 0$. Then, by Lemma 2 and the ergodicity of $\mathbb{P}$, we can conclude that $\mathbb{P}$-a.s.,

$$\limsup_{x \to \infty} \sum_{i=1}^{x} \log \rho_i > -\infty.$$

It follows that $\mathbb{P}$-a.s. one has that

$$\lim_{x \to \infty} f(x) = -\infty,$$

and similarly that

$$\lim_{x \to -\infty} f(x) = \infty.$$

If we define for $A > 0$ the stopping times $T_A := \inf\{k \geq 0 : X_k \geq A\}$ and $S_A := \inf\{k \geq 0 : X_k \leq -A\}$, we see that $f(X_{n \wedge T_A})$ and $f(X_{n \wedge S_A})$ are martingales such that $E_{0,\omega}[f(X_{n \wedge T_A})_+] < \infty$ and $E_{0,\omega}[f(X_{n \wedge S_A})_-] < \infty$, respectively. Hence, by the martingale convergence theorem we conclude that the limits

$$\lim_{n \to \infty} f(X_{n \wedge T_A}), \qquad \lim_{n \to \infty} f(X_{n \wedge S_A}),$$

exist. The only possibility is that $\mathbb{P}$-a.s. we have that $P_{0,\omega}$-a.s., $X_n$ eventually hits both $A$ and $-A$. Since $A$ was chosen arbitrarily, this proves part 3 of the theorem.

### 1.4 Computation of an Absolutely Continuous Invariant Measure in Dimension $d = 1$

In 1999, Alili [1] proved a one-dimensional result which establishes the existence of an invariant measure for the environment process as seen from the random walk with respect to the initial law of the environment. The proof we present here, is due to Conze and Guivarc'h [13] (see also [34]). We will say that **(B+)** is satisfied if

$$\mathbb{E}\left[(1+\rho_0)\sum_{j=0}^{\infty}\prod_{k=0}^{j-1}\rho_{k+1}\right] < \infty,$$

while we will say that **(B−)** is satisfied if

$$\mathbb{E}\left[(1+\rho_0^{-1})\sum_{j=0}^{\infty}\prod_{k=-1}^{-j}\rho_k^{-1}\right] < \infty.$$

Note that in the i.i.d. case **(B+)** reduces to $\mathbb{E}[\rho_0] < 1$ while **(B−)** to $\mathbb{E}[\rho_0^{-1}] < 1$.

**Theorem 2** *(Alili) Consider a RWRE with law $\mathbb{P}$ fulfilling* **(E)** *and* **(ERG)** *in dimension $d = 1$. Then the following holds.*

1. *Assume that $\mathbb{E}[\log\rho] = 0$. If $\mathbb{E}[(\log\rho)^2] > 0$, then there are no invariant measures which are absolutely continuous with respect to $\mathbb{P}$. If $\mathbb{E}[(\log\rho)^2] = 0$, $\mathbb{P}$ is the unique invariant measure of the environmental process absolutely continuous with respect to $\mathbb{P}$ (up to multiplicative constants).*
2. *If $\mathbb{E}[\log\rho] > 0$ but* **(B+)** *is not satisfied, or if $\mathbb{E}[\log\rho] < 0$ but* **(B−)** *is not satisfied, the environment viewed from the random walk has a unique invariant measure $\nu$ (up to multiplicative constants) which is absolutely continuous with respect to $\mathbb{P}$, but which is not a probability measure.*
3. *If* **(B+)** *is satisfied, there exists a unique invariant probability measure $\nu$ which is absolutely continuous with respect to $\mathbb{P}$. Furthermore,*

$$\frac{d\nu}{d\mathbb{P}} = C(1+\rho_0)\sum_{j=0}^{\infty}\prod_{k=0}^{j-1}\rho_{k+1},$$

   *for some constant $C > 0$.*
4. *If* **(B−)** *is satisfied, there exists a unique invariant probability measure $\nu$ which is absolutely continuous with respect to $\mathbb{P}$. Furthermore*

$$\frac{d\nu}{d\mathbb{P}} = C(1+\rho_0^{-1})\sum_{j=0}^{\infty}\prod_{k=-1}^{-j}\rho_k^{-1},$$

   *for some constant $C > 0$.*

We will see soon how this result exhibits a relationship between the existence of an absolutely continuous invariant probability measure and the ballisticity of the random walk: in dimension $d = 1$, the existence of an absolute continuous invariant probability measure is equivalent to ballisticity. We will give more details about this soon.

*Sketch of the proof* We will start proving part 2. Note that if $\nu$ is an invariant measure, we have that for every bounded measurable function $f$

$$\int (q(0,\omega)f(t_1\omega) + p(0,\omega)f(t_{-1}\omega))\ \nu(d\omega) = \int f(\omega)\,\nu(d\omega).$$

Now if $\nu$ is absolutely continuous with respect to $\mathbb{P}$ with density $\phi$, the above equation is equivalent to

$$q(0,t_{-1}\omega)\phi(t_{-1}\omega) + p(0,t_1\omega)\phi(t_1\omega) = \phi(\omega)$$

holding for $\nu$-a.a. $\omega$. We then have that

$$h \circ t_1^2 - \left(\frac{1}{1-q}h\right) \circ t_1 + \rho^{-1}h = 0,$$

where $h := p\phi$ and where we have written $p = p(0,\omega)$ and $q = q(0,\omega)$. If we now define

$$\widetilde{h} := h \circ t_1 - \rho^{-1}h,$$

we conclude that for every $x \in \mathbb{Z}$,

$$\widetilde{h} \circ t_x - \widetilde{h} = 0.$$

But since $\mathbb{P}$ is ergodic with respect to $(t_x)_{x\in\mathbb{Z}}$, we conclude that $\widetilde{h}$ is $\mathbb{P}$-a.s. equal to a constant $C$. Assume that $C = 0$. Then $\widetilde{h} = 0$ is equivalent to

$$h(t_1\omega) = \rho^{-1}(\omega)h(\omega).$$

We claim that the only solution in this case is $h = 0$. Indeed, using induction on $n$ we have that

$$h(t_n\omega) = h(\omega)\prod_{j=0}^{n-1}\rho^{-1}(t_j\omega).$$

If $\mathbb{E}[\log\rho] > 0$, by the ergodic theorem this would imply that a.s.

$$\lim_{n\to\infty} h(t_n\omega) = 0.$$

Now integrating with respect to $\mathbb{P}$, using its stationarity and the fact that $h(t_n\omega)$, $n \in \mathbb{N}$, are uniformly integrable, we conclude that

$$\int h(\omega)\mathbb{P}(d\omega) = \lim_{n\to\infty}\int h(t_n\omega)\mathbb{P}(d\omega) = 0,$$

so that

$$h = 0.$$

Using a similar argument one arrives at the same conclusion when $\mathbb{E}[\log \rho] < 0$.

So let us assume that $C \neq 0$. In this case we have that

$$h = (\rho^{-1} h) \circ t_{-1} + C. \tag{15}$$

Now choose a constant $h_0$ and define recursively

$$h_{n+1} := (\rho^{-1} h_n) \circ t_1^{-1} + C. \tag{16}$$

If we can prove that $h_n$ converges $\mathbb{P}$-a.s. as $n \to \infty$, then the limit should be a solution to (15). Now from (16) we can deduce

$$h_n(\omega) = C \sum_{j=0}^{n-1} \prod_{k=0}^{j-1} \rho^{-1}(t_{k+1}^{-1}\omega) + \left( \prod_{k=1}^{n} \rho^{-1}(t_k^{-1}\omega) \right) h_0.$$

Taking the limit when $n \to \infty$, we conclude the case in which $\mathbb{E}[\log \rho] > 0$, in combination with the ergodic theorem, that $h_n$ converges $\mathbb{P}$-a.s. to

$$h(\omega) = c \sum_{j=0}^{\infty} \prod_{k=0}^{j-1} \rho^{-1}(t_{k+1}^{-1}\omega).$$

Thus,

$$\phi(\omega) = (1 + \rho^{-1}(\omega)) \sum_{j=0}^{\infty} \prod_{k=-1}^{-j} \rho^{-1}(t_k \omega).$$

This proves part 2 of the proposition. To prove part 3, note that Jensen's inequality and **(B-)** imply that $\mathbb{E}[\log \rho] < \infty$. Therefore, the measure with density $\phi$ already defined can be normalized to define a probability measure. Similarly, one can prove part 4. The proof of part 1 in the case $\mathbb{E}[(\log \rho)^2] > 0$ is analogous to the proof of the recurrent case of Theorem 1. The case $\mathbb{E}[(\log \rho)^2] = 0$ is trivial, since in this case we would be in the situation of simple random walk.

## *1.5 Absolutely Continuous Invariant Measures and Some Implications*

The existence of an invariant probability measure which is absolutely continuous with respect to the initial distribution of the environment will turn out to be crucial in the study of the model. We recall that the environmental process has been defined in Definition 4, which considered as a trajectory has state space $\Gamma := \Omega^{\mathbb{N}}$. Furthermore, define the law $P_\omega$ defined on its Borel $\sigma$-algebra $\mathcal{B}(\Gamma)$ through the identity

$$P_\omega[A] := P_{0,\omega}[(\bar{\omega}_n) \in A], \tag{17}$$

for any Borel subset $A$ of $\Gamma$ endowed with the product topology. Furthermore, for any probability measure $\nu$ defined in $\Omega$, we define

$$P_\nu := \int P_\omega \nu(d\omega). \tag{18}$$

We will denote by $\theta : \Gamma \to \Gamma$ the canonical shift on $\Gamma$ defined by

$$\theta(\omega_0, \omega_1, \dots) := (\omega_1, \omega_2, \dots). \tag{19}$$

The following result of Theorem 3 was proved by Kozlov in [27]. For its proof, we will follow Sznitman in [6].

**Theorem 3** *(Kozlov) Consider a RWRE in an environment with law $\mathbb{P}$ fulfilling* **(E)** *and* **(ERG)**. *Assume that there exists an invariant probability measure $\nu$ for the environment seen from the random walk which is absolutely continuous with respect to $\mathbb{P}$. Then the following are satisfied:*

1. *$\nu$ is equivalent to $\mathbb{P}$.*
2. *The environment as seen from the random walk with initial law $\nu$ is ergodic.*
3. *$\nu$ is the unique invariant probability measure for the environment as seen from the particle which is absolutely continuous with respect to $\mathbb{P}$.*
4. *The Cesàro means*

$$\frac{1}{n+1}\sum_{i=0}^{n} \mathbb{P}R^i$$

*converges weakly to $\nu$.*

*Proof of part 1* Let $f$ be the Radon-Nikodym derivative of $\nu$ with respect to $\mathbb{P}$ and consider the event $E := \{f = 0\}$. In order to prove the desired result it will be sufficient to show $\mathbb{P}[E] = 0$.

Since $\nu$ is invariant, we have that

$$\int f \cdot (R1_E)\, d\mathbb{P} = (\nu R)[E] = \nu[E] = \int_{\{f=0\}} d\mathbb{P} = 0.$$

It follows that $\mathbb{P}$-a.s. on the event $E^c = \{f > 0\}$ one has that $R1_E = 0$. Therefore, using the fact that $R1_E \leq 1$, one has that for every $e \in U$,

$$1_E(\omega) \geq R1_E(\omega) = \sum_{e' \in U} \omega(0, e') 1_E(t_{e'}\omega) \geq \omega(0, e) 1_E(t_e \omega), \qquad \mathbb{P} - a.a.\, \omega.$$

From the ellipticity assumption and the fact that $1_E(\omega)$ and $1_E(t_e\omega)$ for $e \in U$ only take the values 0 or 1 we have that for such $e$,

$$1_E(\omega) \geq 1_E(t_e \omega), \qquad \mathbb{P} - a.s.$$

Now using the fact that $\mathbb{P}[E] = \mathbb{P}[t_e^{-1}E]$ we conclude that for each $e \in U$ one has

$$1_E = 1_{t_e^{-1}E}, \qquad \mathbb{P} - a.a.\, \omega.$$

Thus, we iteratively obtain that for each $x \in \mathbb{Z}^d$,

$$1_E = 1_{t_x^{-1}(E)}, \qquad \mathbb{P} - a.s.$$

It follows that the event

$$\widetilde{E} := \bigcap_{x \in \mathbb{Z}^d} t_x^{-1}(E),$$

is invariant under the action of the family $(t_y)_{y \in \mathbb{Z}^d}$ and that it differs form the event $E$ on an event of $\mathbb{P}$-probability 0. Since, $\mathbb{P}$ is ergodic with respect to the family $(t_y)_{y \in \mathbb{Z}^d}$ we conclude that

$$\mathbb{P}[E] = \mathbb{P}[\widetilde{E}] \in \{0, 1\}. \tag{20}$$

But since $\int_{E^c} f \, d\mathbb{P} = \int f \, d\mathbb{P} = 1$ we know that $\mathbb{P}[E^c] > 0$, which in combination with $\mathbb{P}[\Omega] = 1$ and (20) implies $\mathbb{P}[E] = 0$. Hence, $\mathbb{P}$ is equivalent to $\nu$.

*Proof of part 2* We will prove that if $A \in \mathcal{B}(\Gamma)$ is invariant so that $\theta^{-1}(A) = A$ then $P_\nu[A]$ (cf. (18) and (19)) is equal to 0 or 1. For $\omega \in \Omega$ define

$$\phi(\omega) := P_\omega[A].$$

We claim that

$$(\phi(\bar{\omega}_n))_{n \geq 0}$$

is a $P_\nu$-martingale with the canonical filtration on $\Gamma$. In fact, note that since $A$ is invariant, we have that $1_A = 1_A \circ \theta_n$ and hence,

$$E_\nu[1_A \,|\, \bar{\omega}_0, \ldots, \bar{\omega}_n] = E_\nu[1_A \circ \theta_n \,|\, \bar{\omega}_0, \ldots, \bar{\omega}_n] = P_{\bar{\omega}_n}[A] = \phi(\bar{\omega}_n),$$
$$P_\nu - a.a. \, (\bar{\omega}_n). \tag{21}$$

It follows from (1.5) and the martingale convergence theorem that

$$\lim_{n \to \infty} \phi(\bar{\omega}_n) = 1_A((\bar{\omega}_n)_{n \in \mathbb{N}}), \qquad P_\nu - a.a. \, (\bar{\omega}_n) \tag{22}$$

Let us now prove that there is a set $B \in \mathcal{B}(\Omega)$ such that $\nu$-a.s.

$$\phi = 1_B. \tag{23}$$

In fact, assume that (23) is not satisfied. Then there is an interval $[a, b] \subset (0, 1)$ with $a < b$ such that

$$\nu[\phi \in [a, b]] > 0. \tag{24}$$

Also, by the ergodic theorem we have that $P_\nu$-a.s.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\phi^{-1}([a,b])}(\bar{\omega}_k) = \Psi := E_\nu[1_{\phi^{-1}([a,b])}(\bar{\omega}_0) \,|\, \mathcal{I}],$$

where $\mathcal{I} := \{A \in \Gamma : \theta^{-1}(A) = A\}$ is the $\sigma$-field of invariant events. Now, by (24),

$$E_\nu[\Psi] = P_\nu[\phi(\omega_0) \in [a, b]] = \nu[\phi \in [a, b]] > 0.$$

But this contradicts (22). Hence, (23) holds. Let us now prove that $\nu$-a.s.

$$R1_B = 1_B. \tag{25}$$

Indeed, we have that $P_\nu$-a.s. it is true that

$$1_B(\omega_0) = E_\nu[1_B(\omega_1) \,|\, \omega_0] = R1_B(\omega_0).$$

Since $P_\nu[A] = \nu[B]$, it is then enough to prove that

$$\nu[B] \in \{0, 1\}. \tag{26}$$

Now, $\mathbb{P}$-a.s. we have that

$$1_B(\omega) = R1_B(\omega) = \sum_{|e|_1=1} \omega(0, e) 1_B(t_e\omega).$$

Using ellipticity, this implies that $\mathbb{P}[B] \in \{0, 1\}$, which again by part *1* of this theorem implies (26).

*Proof of parts 3 and 4* Let $g$ be a bounded measurable function on $\Omega$. Let $\nu$ be any invariant probability measure for the transition kernel $R$ that is absolutely continuous with respect to $\mathbb{P}$. By part 2 and the ergodic theorem we have that $P_\nu$-a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\omega_k) = \int g\, d\nu.$$

Now, by part $(i)$ of this theorem, the above convergence also occurs $P_{\mathbb{P}}$-a.s. Hence, we have that

$$\lim_{n\to\infty} E_0 \left[ \frac{1}{n} \sum_{k=0}^{n-1} g(\omega_k) \right] = \int g\, d\nu.$$

This proves the uniqueness of $\nu$ and part $(iv)$. □

An important generalization of Kozlov's theorem was obtained by Rassoul-Agha in [32]. There, he shows that under the assumption that the random walk is directionally transient, the environment satisfies a certain mixing and uniform ellipticity condition, and if there exists an invariant probability measure which is absolutely continuous with respect to the initial law $\mathbb{P}$ in certain half-spaces, a conclusion analogous to Kozlov's theorem holds.

In [30], Lenci generalizes Kozlov's theorem to environments which are not necessarily elliptic. Lenci admits the possibility that the environment is ergodic with respect to some subgroup $\Gamma$ strictly smaller than $\mathbb{Z}^d$, which is a stronger condition than total ergodicity, and which enables him to relax the ellipticity condition. Furthermore, in Bolthausen–Sznitman [7], an example of a RWRE which does not satisfy the ellipticity condition **(E)** and for which there are no invariant probability measures for the environmental process which are absolutely continuous with respect to the initial law of the environment is presented (see also [32]).

## 1.6 The Law of Large Numbers, Directional Transience and Ballisticity

For the purposes of applying Kozlov's theorem, it would be important to understand how to reconstruct the random walk from the canonical environmental process. Now, let us note that if we denote by $\Omega_{per}$ the periodic environments so that

$$\Omega_{per} := \{\omega \in \Omega : \omega = t_x\omega \text{ for some } x \in \mathbb{Z}^d, x \neq 0\},$$

whenever $\omega \in \Omega \backslash \Omega_{per}$ and $\omega'$ is a translation of $\omega$, this translation is uniquely defined. This observation would enable us to express the increments of the random walk as a function of the environmental process whenever the initial condition is not periodic. Assuming that the initial law $\mathbb{P}$ of the environment is ergodic, and noting that the set of periodic environments is invariant under translations, we can see that $\mathbb{P}[\Omega_{per}]$ equals either 0 or 1. Nevertheless, assuming **(ERG)**, may happen that $\mathbb{P}[\Omega_{per}] = 1$, a situation where a priori we cannot perform this reconstruction (and which is impossible if we assume even **(IID)**). We will therefore prove directly the ergodicity of the increments of the random walk.

Our first application of Kozlov's theorem will relate the so-called transient regime with the ballistic one.

**Definition 5** **(Transience in a given direction)** For $l \in \mathbb{S}^{d-1}$ define the event

$$A_l := \{\lim_{n\to\infty} X_n \cdot l = \infty\} \tag{27}$$

of directional transience in direction $l$. We will call an RWRE *transient in direction* $l$ if $P_0[A_l] = 1$.

**Definition 6** **(Ballisticity in a given direction)** Let $l \in \mathbb{S}^d$. We say that an RWRE is ballistic in direction $l$, if $P_0$-a.s.

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0. \tag{28}$$

We will see in Chap. 3, that the limit on the left-hand side of (28) always exists, and is even known to be deterministic in dimensions $d = 2$.

Let us now consider for each $x \in \mathbb{Z}^d$ the *local drift* at site $x$ is defined as

$$d(x,\omega) := \sum_{e\in U} \omega(x,e)e = E_{x,\omega}[X_1 - X_0].$$

We then have the following corollary to Kozlov's theorem.

**Corollary 1** *Consider an RWRE in an environment with law* $\mathbb{P}$ *fulfilling* **(E)** *and* **(ERG)**. *Furthermore, assume that there exists an invariant probability measure for the environment seen from the particle, denoted by* $\nu$, *which is absolutely continuous with respect to* $\mathbb{P}$. *Then a law of large number is satisfied so that* $P_{0,\mathbb{P}}$*-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = \int d(0,\omega)\nu(d\omega) =: v.$$

*Furthermore, if the walk is transient in a given direction* $l$, *it is necessarily ballistic in that direction so that* $v \cdot l \neq 0$.

*Proof* We will follow Sabot [37]. Define for $n \geq 1$,

$$\Delta X_n := X_n - X_{n-1}.$$

This is a process with state space $\mathcal{U} := U^{\mathbb{N}}$. In a slight abuse of notation to (19), we define the canonical shift $\theta : \mathcal{U} \to \mathcal{U}$ via

$$\theta(\Delta X_1, \Delta X_2, \dots) := (\Delta X_2, \Delta X_3, \dots). \tag{29}$$

Note that the process $(\Delta X_n)_{n \geq 1}$ is stationary under the law $P_{0,\nu}$. We will show that in fact the transformation $\theta$ is ergodic with respect to the space $(\mathcal{U}, \mathcal{B}(\mathcal{U}), P_{0,\nu})$, where $\mathcal{B}(\mathcal{U})$ is the Borel $\sigma$-field of $\mathcal{U}$. Let $A \in \mathcal{B}(\mathcal{U})$ be invariant so that $\theta^{-1}(A) = A$ and define

$$\psi(x, \omega) := P_{x,\omega}[(\Delta X_n) \in A].$$

We claim that

$$(\psi(X_n, \omega))_{n \geq 0}$$

is a martingale with respect to the canonical filtration on $\mathcal{U}$ generated by $(X_n)$. Indeed,

$$P_{0,\omega}[(\Delta X_m) \in A \mid X_0, \dots, X_n] = P_{X_n,\omega}[(\Delta X_m) \in A] = \psi(X_n, \omega).$$

Therefore, taking the limit when $n \to \infty$, and for any $\omega$, the martingale convergence theorem yields that

$$\lim_{n\to\infty} \psi(0, \bar{\omega}_n) = \lim_{n\to\infty} \psi(X_n, \omega) = 1_A((\Delta X_n)) \qquad P_{0,\omega} - a.s. \tag{30}$$

We now have by the ergodic and Kozlov's theorems that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \psi(0, \bar{\omega}_n) = \int \psi(0, \omega)\nu(d\omega) \qquad P_{0,\nu} - a.s.$$

The limit (30) now implies that

$$P_{0,\nu}\,[(\Delta X_n) \in A] \in \{0, 1\},$$

which gives us the claimed ergodicity. We thus have that

$$\lim_{n\to\infty} \frac{X_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Delta X_k = \int d(0, \omega)\nu(d\omega) \qquad P_{0,\nu} - a.s.$$

By Kozlov's theorem, we can conclude that the above convergence occurs $P_{0,\mathbb{P}}$-a.s. The second claim of the corollary is immediate from Lemma 2 above.

Rassoul-Agha in [32], obtains a version of Corollary 1 where transience is replaced by the so-called Kalikow's condition [22], a stronger mixing assumption than ergodicity is required, but it is necessary only to assume the existence of an invariant probability measure which is absolutely continuous with respect to the initial law only on appropriate half-spaces.

On the other hand, combining Corollary 1 with Theorem 2, we can now easily derive the following result for one-dimensional case, originally proved by Solomon [44] for the i.i.d. case and later extended by Alili [1] to the ergodic case.

**Theorem 4** *Consider a RWRE in dimension $d = 1$ in an environment with law $\mathbb{P}$ fulfilling* **(E)** *and* **(ERG)**. *Then, there exists a deterministic $v \in \mathbb{R}^d$ such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \qquad P_0 - a.s.$$

*Furthermore,*

1. *If* **(B+)** *is satisfied, then*

$$v = \mathbb{E}\big[(1-\rho_0)\sum_{j=0}^{\infty}\prod_{k=0}^{j-1}\rho_{k+1}\big].$$

2. *If* **(B−)** *is satisfied, then*

$$v = \mathbb{E}\big[(1-\rho_0^{-1})\sum_{j=0}^{\infty}\prod_{k=-1}^{-j}\rho_{-k}^{-1}\big].$$

3. *If neither* **(B+)** *nor* **(B−)** *are satisfied, then*

$$v = 0.$$

Since, case (3) mentioned above shows that $X_n/n$ converges to 0, one immediately is led to the question of the typical order of $X_n$ in this case. The answer to this problem (and further interesting insight) has been obtained by Kesten, Kozlov, and Spitzer [24]: In fact, there is a direct connection between the exponent $\kappa \in (0,1)$ characterized by

$$\mathbb{E}[\rho^\kappa] = 1,$$

and the typical order of $X_n$ in this case, which is $n^\kappa$. We refer the reader to [24] for further details.

In addition, from the above discussion we see that in dimension $d = 1$, if the family of integer shifts is ergodic with respect to the law $\mathbb{P}$ of the environment, the walk being transient to the right or left does not ensure the existence of an invariant probability measure for the environmental process which is absolutely continuous with respect to $\mathbb{P}$. Let us give two examples which show that this situation could also occur for dimensions $d \geq 2$.

**Example 2** *Let $d = 2$. Consider a random walk in an environment $(\omega(x))_{x\in\mathbb{Z}^2}$ of the form $\omega(x) := (\omega(x,e))_{e\in U}$ with a law $\mathbb{P}$ such that $\mathbb{P}[\omega(x,e) = 1/4] = 1$ for $e = e_2$ and $e = -e_2$ and $\omega(x,e_1) = q(x)$ while $\omega(x,-e_1) = p(x) = \frac{1}{2} - q(x)$, with $\mathbb{E}[\log(p(x)/q(x))] < 0$ and $\mathbb{E}[p(x)/q(x)] = 1$. Assume also that for every $x \in \mathbb{Z}^2$, $(\omega(x+ne_1))_{n\in\mathbb{Z}}$ are i.i.d. under $\mathbb{P}$ while*

$$\mathbb{P}[\omega(x+e_2) = \omega(x)] = 1$$

*In other words, the environment is constant in the direction $e_2$, but it is i.i.d. in the direction $e_1$, see Fig. 1 also. It is easy to check that the shifts $(\theta_x)_{x\in\mathbb{Z}^d}$ form an ergodic family with respect to $\mathbb{P}$. Also, the walk is transient in direction $e_1$, but not ballistic in that direction and there are no invariant probability measures for the environmental process which are absolutely continuous with respect to $\mathbb{P}$ (cf. Corollary 1).*

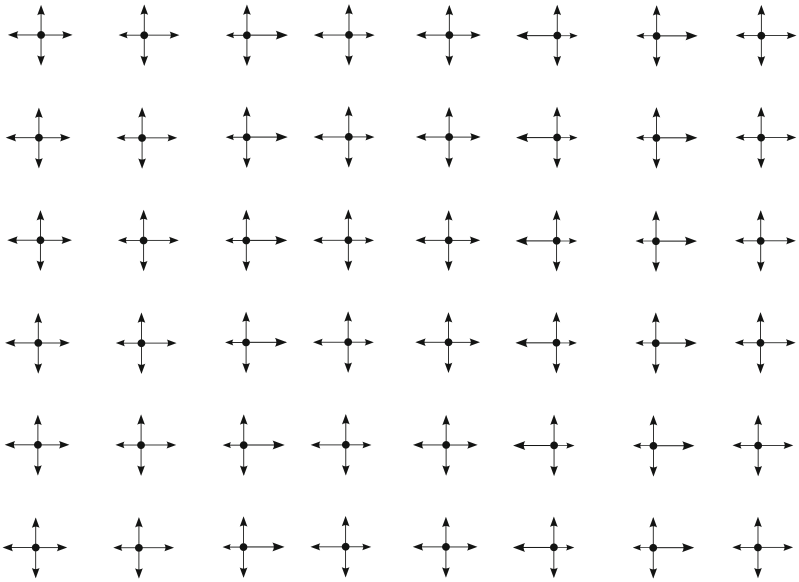

**Fig. 1** A sketch of an environment which is i.i.d. in direction $e_1$ and constant in direction $e_2$

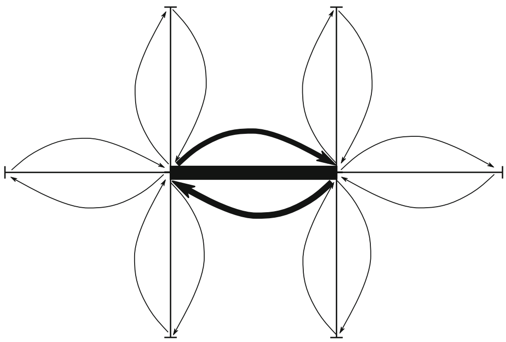

**Fig. 2** A trap produced by an elliptic environment

**Example 3** *Let $\epsilon > 0$. Furthermore, take $\phi$ to be any random variable taking values on the interval $(0, 1/4)$ and such that the expected value of $\phi^{-1/2}$ is infinite, while for every $\epsilon > 0$, the expected value of $\phi^{-(1/2-\epsilon)}$ is finite. Let Z be a Bernoulli random variable of parameter $1/2$. We now define $\omega_1(0, e_1) = 2\phi$, $\omega_1(0, -e_1) = \phi$, $\omega_1(0, -e_2) = \phi$ and $\omega_1(0, e_2) = 1 - 4\phi$ and $\omega_2(0, e_1) = 2\phi$, $\omega_2(0, -e_1) = \phi$, $\omega_2(0, e_2) = \phi$ and $\omega_2(0, -e_2) = 1 - 4\phi$. We then let the environment at site 0 be given by the random variable $\omega(0, \cdot) := Z\omega_1(0, \cdot) + (1 - Z(0))\omega_2(0, \cdot)$, and extend this to an i.i.d. environment on $\mathbb{Z}^d$. This environment has the property that traps can appear, where the random walk gets caught in an edge, as shown in Fig. 2. Furthermore, as we will show, it is not difficult to check that the random walk in* *this random environment is transient in direction $e_1$ but not ballistic. Hence, due to Corollary 1 there exists no invariant probability measure for the environment seen from the particle, which in addition is absolutely continuous with respect to $\mathbb{P}$.*

These are two examples of walks which are transient in a given direction but not ballistic, and for which there is no invariant probability measure for the environmental process absolutely continuous with respect to the initial law $\mathbb{P}$ of the environment. It is natural, hence, to raise the following questions:

**Open Question 1** *Assume given an RWRE fulfilling* **(ERG)** *and* **(E)**. *Furthermore, assume the RWRE is transient in a given direction. Is the existence of an invariant probability measure for the environmental process which is absolutely continuous with respect to $\mathbb{P}$ equivalent to ballisticity in the given direction?*

**Open Question 2** *Let $d \geq 2$. Assume given a RWRE for which* **(UE)** *and* **(IID)** *are fulfilled, and which is transient in direction $l \in \mathbb{S}^{d-1}$. Is the RWRE necessarily ballistic in direction $l$?*

As it is discussed above, Example 1 shows that if the hypothesis **(UE)** is replaced by **(E)** in the Open Question 2, then its answer is negative. The following proposition gives an indication of how much ellipticity should be required.

**Proposition 2** *Consider a random walk in an i.i.d. environment. Assume that*

$$\max_{e \in U} \mathbb{E}\left[\frac{1}{1-\omega(0,e)\omega(0,-e)}\right] = \infty. \tag{31}$$

*Then the walk is not ballistic in any direction.*

*Proof* Fix $e \in U$ and define the first exit time of the random walk from the edge between 0 and $e$ as

$$T_{\{0,e\}} := \min\{n \geq 0 : \mathrm{X_n} \notin \{0,e\}\}.$$

We then have for every $k \geq 0$, using the notation $\omega_1 := \omega(0,e)$ and $\omega_2 := \omega(0,-e)$ that

$$P_{0,\omega}[T_{\{0,e\}} > 2k] = (\omega_1\omega_2)^k$$

and

$$\sum_{k=0}^{\infty} P_{0,\omega}[T_{\{0,e\}} > 2k] = \frac{1}{1-\omega_1\omega_2}. \tag{32}$$

Using (31), this implies that

$$E_0[T_{\{0,e\}}] = \infty.$$

We can now show using the strong Markov property under the quenched measure and the i.i.d. nature of the environment, that for each natural $m > 0$, the time $T_m := \min\{n \geq 0 : X_n \cdot l > m\}$ can be bounded from below by the sum of a sequence of random variables $\widetilde{T}_1, \ldots, \widetilde{T}_m$ which under the averaged measure are i.i.d. and distributed as $T_{\{0,e\}}$. This proves that $P_0$-a.s. $T_m/m \to \infty$ which implies that the random walk is not ballistic in direction $l$.

Based now on Proposition 1 we have the following extended version of the Open Question 1.

**Open Question 3** *Let $d \geq 2$. Is it the case that every random walk fulfilling* **(E)** *and* **(IID)***, and satisfying*

$$\max_{e \in U} \mathbb{E}\left[\frac{1}{1-\omega(0,e)\omega(0,-e)}\right] < \infty,$$

*and which is transient in direction $l \in \mathbb{S}^{d-1}$, is ballistic in direction $l$?*

For the case of an environment fulfilling **(IID)** and having a Dirichlet law, the above question was answered positively by Sabot [37] in dimensions $d \geq 3$ (see also the work of Campos and Ramírez [8]).

## 1.7 Transience, Recurrence and a Quenched Invariance Principle

Similarly to the case of simple random walk, one of the most basic questions for an RWRE is a classification in terms of transience and recurrence. As simple as this question is to pose, it is still far from being completely understood. In fact, a natural question is the following one.

**Open Question 4** *Is it the case that in dimensions $d \geq 3$, an RWRE fulfilling* **(E)** *and* **(IID)** *is transient?*

This question has been answered only in the case of the so-called Dirichlet environment (see [36]) and essentially also for balanced environments (see [29]). It is intimately related to the quenched central limit theorem. In this section, we will discuss how Kozlov's theorem can be used for balanced random walks to derive such a theorem, from which eventually transience in direction $d \geq 3$ can be deduced.

Consider the subset of the set of environments

$$\Omega_0 := \{\omega \in \Omega : \omega(x,e) = \omega(x,-e) \text{ for all } x \in \mathbb{Z}^d, e \in U\}.$$

We will say that the law $\mathbb{P}$ of the environment of an RWRE is *balanced* if

$$\mathbb{P}[\Omega_0] = 1,$$

where in particular we use that $\Omega_0$ is a measurable subset of $\Omega$. The following result was proved by Lawler in [29].

**Theorem 5** *Consider a random walk with an environment which has a law $\mathbb{P}$ fulfilling* **(UE)** *as well as* **(ERG)***, and which is balanced. Then there exists an invariant measure for the environmental process which is absolutely continuous with respect to $\mathbb{P}$.*

The above result is one of the few instances in which it has been possible to construct an absolutely continuous invariant measure for the environmental process in dimensions $d \geq 2$ (for non-nestling random walks at low disorder; Bolthausen and Sznitman also make such a construction in [7]; for random environment with Dirichlet law Sabot characterizes the cases when this happens in [37]). As a corollary, Lawler can prove the following.

**Corollary 2** *Under the conditions of Theorem 5 for $\mathbb{P}$-a.e. $\omega$, under $P_{0,\omega}$, the sequence $X_{[n\cdot]}/\sqrt{n}$ converges in law on the Skorokhod space $D([0,\infty);\mathbb{R}^d)$ to a non-degenerate Brownian motion with a diagonal and deterministic covariance matrix $A := \{a_{i,j}\}$, $a_{i,j} = a_i\delta_{i,j}$.*

*Proof* Let us first explain how to prove the convergence of the finite-dimensional ditributions. Note that for every $\theta \in \mathbb{R}^d$ sufficiently close to 0 and using $\mathbb{P}[\Omega_0] = 1$ we have that

$$e^{iX_n\cdot\theta - \sum_{k=0}^{n-1}\ln\left(2\sum_{j=1}^{d}\cos(e_j\cdot\theta)\omega(X_k,e_j)\right)}$$

is a martingale in $n$ with respect to the law $P_{0,\omega}$. Therefore, rescaling $\theta$ by $\theta/\sqrt{n}$ we see that for all $n$ large enough,

$$E_{0,\omega}\left[e^{i\frac{X_n}{\sqrt{n}}\cdot\theta - \sum_{k=0}^{n-1}\ln\left(2\sum_{j=1}^{d}\cos\left(\frac{\theta_j}{\sqrt{n}}\right)\omega(X_k,e_j)\right)}\right] = 1.$$

Hence, it is enough to prove that there exist constants $\{a_i : 1 \le i \le d\}$ such that $P_0$-a.s. one has that

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\ln\left(2\sum_{j=1}^{d}\cos\left(\frac{\theta_j}{\sqrt{n}}\right)\omega(X_k,e_j)\right) = -\sum_{j=1}^{d}\frac{a_j}{2}\theta_j^2. \tag{33}$$

Now, by Taylor's theorem,

$$\cos(x) = 1 - \frac{x^2}{2!} + h_1(x)x^2,$$

where $\lim_{x\to 0} h_1(x) = 0$. Hence,

$$\cos\left(\frac{\theta_j}{\sqrt{n}}\right) = 1 - \frac{\theta_j^2}{2n} + \frac{\theta_j^2}{n}h_1\left(\frac{\theta_j}{\sqrt{n}}\right),$$

and for each $k \ge 0$,

$$2\sum_{j=1}^{d}\cos\left(\frac{\theta_j}{\sqrt{n}}\right)\omega(X_k,e_j) = 1 - \sum_{j=1}^{d}\frac{\theta_j^2}{n}\omega(X_k,e_j) + 2\sum_{j=1}^{d}\frac{\theta_j^2}{n}h_1\left(\frac{\theta_j}{\sqrt{n}}\right)\omega(X_k,e_j). \tag{34}$$

A second application of Taylor's theorem gives that

$$\ln(1-x) = -x + h_2(x)x,$$

where $\lim_{x\to 0} h_2(x) = 0$. Thus, using (34) we have that,

$$\ln\left(2\sum_{j=1}^{d}\cos\left(\frac{\theta_j}{\sqrt{n}}\right)\omega(X_k,e_j)\right)$$

$$= -\sum_{j=1}^{d} \frac{\theta_j^2}{n} \bar{\omega}_k(0, e_j) + 2 \sum_{j=1}^{d} \frac{\theta_j^2}{n} h_1\left(\frac{\theta_j}{\sqrt{n}}\right) \bar{\omega}_k(0, e_j)$$

$$+ \left( \sum_{j=1}^{d} \frac{\theta_j^2}{n} \bar{\omega}_k(0, e_j) - 2 \sum_{j=1}^{d} \frac{\theta_j^2}{n} h_1\left(\frac{\theta_j}{\sqrt{n}}\right) \bar{\omega}_k(0, e_j) \right) h_2,$$

where

$$h_2 = h_2 \left( \sum_{j=1}^{d} \frac{\theta_j^2}{n} \bar{\omega}_k(0, e_j) - 2 \sum_{j=1}^{d} \frac{\theta_j^2}{n} h_1\left(\frac{\theta_j}{\sqrt{n}}\right) \bar{\omega}_k(0, e_j) \right),$$

and where we recall that the environmental process $(\bar{\omega}_n)$ has been introduced in Definition 4. It then follows that if we are able to prove that for each $1 \le j \le d$, $P_0$-a.s. one has that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \bar{\omega}_k(0, e_j) = \frac{a_j}{2}, \tag{35}$$

then we have proven (33). To prove (35), by Kozlov's theorem, it is enough to use Theorem 5 which ensures the existence of a measure $\nu$ which is an invariant measure for the process $(\bar{\omega}_n)$ and which is absolutely continuous with respect to $\mathbb{P}$. The prove the convergence to Brownian motion we can use the martingale convergence theorem ([48]).

We will now explain the main ideas in the proof of Theorem 5, the details of which can be found, for example, in Sznitman [6]. We will construct an invariant measure by approximating it with invariant measures with respect to the environmental processes on finite spaces. Configurations of the environment on these finite spaces will then correspond to periodic configurations on the full space. The point is to do this in such a way that the density of these invariant measures with respect to periodized versions of the measure $\mathbb{P}$, has an $L_p$ norm for some $p > 1$, which is uniformly bounded in the size of the boxes.

We introduce for $x \in \mathbb{Z}^d$ the equivalence classes

$$\hat{x} := x + (2N+1)\mathbb{Z}^d \in \mathbb{Z}^d/((2N+1)\mathbb{Z}^d).$$

In addition we define for $\omega \in \Omega_0$ the corresponding periodized version $\omega_N$ of $\omega$ so that $\omega_N(y) = \omega(x)$ for $y \in \mathbb{Z}^d$ and $x \in B_\infty(N)$ such that $\hat{y} = \hat{x}$,. and set

$$\Omega_N := \{\omega_N : \omega \in \Omega_0\}.$$

It is straightforward to see that the random walk in the environment $\omega_N$ has an invariant measure of the form

$$m_N := \frac{1}{(2N+1)^d} \sum_{x\in B_\infty(N)} \Phi_N(x)\delta_{\hat{x}},$$

for some function $\Phi_N$ on $B_\infty(N)$ such that $\sum_{x\in B_\infty(N)} \Phi_N = (2\,N+1)^d$.

Now define a probability measure on $\Omega_N$ by

$$\nu_N := \frac{1}{(2N+1)^d} \sum_{x\in B_\infty(N)} \Phi_N(x)\delta_{t_x\omega_N}.$$

Now introduce the sequence of measures

$$\mathbb{P}_N := \frac{1}{(2N+1)^d} \sum_{x\in B_\infty(N)} \delta_{t_x\omega_N}.$$

By the multidimensional ergodic theorem (see [17, Theorem VIII.6.9]), we have that

$$\lim_{N\to\infty} \mathbb{P}_N = \mathbb{P} \qquad \mathbb{P}-a.s.$$

Also, one can see that $\nu_N$ is absolutely continuous with respect to $\mathbb{P}_N$,

$$d\nu_N =: f_N\, d\mathbb{P}_N,$$

with

$$\int f_N^{\frac{d}{d-1}}\, d\mathbb{P}_N \le \frac{1}{(2N+1)^d} \sum_{x\in B_\infty(N)} \Phi_N(x)^{\frac{d}{d-1}}.$$

Hence, for every bounded measurable function $g$ on $\Omega$ we have that

$$\left|\int g\, d\nu_N\right| \le \left(\int |g|^d\, d\mathbb{P}_N\right)^{\frac{1}{d}} \left(\int f_N^{\frac{d}{d-1}}\, d\mathbb{P}_N\right)^{\frac{d-1}{d}} \le \|g\|_{L^d(\mathbb{P}_N)} \|\Phi_N\|_{L^{\frac{d}{d-1}}}.$$

where we write $L^d$ for the corresponding space with respect to the normalized counting measure on $B_\infty(N)$. Now, assume that there is a constant $C$ such that for every $N$,

$$\|\Phi_N\|_{L^{\frac{d}{d-1}}} \le C. \tag{36}$$

Using the compactness of $\Omega$ and Prohorov's theorem, we can extract a subsequence $\nu_{N_k}$ of $\nu_N$ which converges weakly to some limit $\nu$ as $k \to \infty$. Then we would obtain that

$$\left|\int g\, d\nu\right| \le C\|g\|_{L^d(\mathbb{P})},$$

which would prove that $\nu$ is absolutely continuous with respect to $\mathbb{P}$. Note also that Kozlov's theorem (Theorem 3) ensures that $\nu$ is deterministic. Let us now prove (36). For that purpose, suppose that for every function $h \in L^d(\mathbb{P}_N)$,

$$\sup_{x\in B_\infty(N),\omega_N} \left| E_{x,\omega_N}\left[\sum_{k=0}^{\infty} \left(1-\frac{1}{N^2}\right)^k h(X_k)\right]\right| \le CN^2\|h\|_{L^d(\mathbb{P}_N)}. \tag{37}$$

We claim that (37) implies (36). Indeed,

$$\begin{aligned}\|\Phi_N\|_{L^{\frac{d}{d-1}}} &= \sup_{h:\|h\|_{L^d}\le 1}(\Phi_N,h)\\ &= \sup_{h:\|h\|_{L^d}\le 1}\frac{1}{N^2}\sum_{k=0}^{\infty}\left(1-\frac{1}{N^2}\right)^k\frac{1}{(2N+1)^d}\sum_{x\in B_\infty(N)}\Phi_N(x)h(x)\\ &= \sup_{h:\|h\|_{L^d}\le 1}\sum_{k=0}^{\infty}\frac{1}{N^2}\left(1-\frac{1}{N^2}\right)^k\frac{1}{(2N+1)^d}\sum_{x\in B_\infty(N)}\Phi_N(x)E_{x,\omega_N}[h(X_k)],\end{aligned}$$

which would yield (36). We now claim that (37) is a consequence of the inequality

$$\|Q_\omega f\|_\infty \le CN^2\left(\frac{1}{(2\,N+1)^d}\sum_{x\in B_\infty(N)}|f(x)|^d\right)^{\frac{1}{d}}, \tag{38}$$

where

$$Q_\omega f(x) := E_{x,\omega_N}\left[\sum_{k=0}^{S_N-1} f(X_k)\right]$$

and

$$S_N := \inf\{n\ge 0: |X_n|_\infty \ge N\}.$$

To prove (37) assuming (38), define $\tau_0 := 0$ and

$$\tau_1 := \tau = \inf\{n\ge 0: |X_n - X_0|_\infty \ge N\},$$

as well as recursively for $k\ge 1$, $\tau_{k+1} := \tau\circ\theta_{\tau_k} + \tau_k$. Then, for each $\rho\in[0,1)$ we have that

$$\begin{aligned}E_{x,\omega_N}\left[\sum_{k=0}^{\infty}\rho^k f(X_k)\right] &= E_{x,\omega_N}\left[\sum_{m=0}^{\infty}\sum_{\tau_m\le k<\tau_{m+1}}\rho^k f(X_k)\right]\\ &\le \sum_{m=0}^{\infty}\sup_{x\in\mathbb{Z}^d}E_{x,\omega_N}[\rho^\tau]^m \sup_{x\in\mathbb{Z}^d}\left|(Q_{t_x\omega}(t_x f))(0)\right|\\ &\le CN^2\frac{1}{|B_\infty(N)|^{1/d}}\|f\|_{L^d}\frac{1}{1-\sup_x E_{x,\omega_N}[\rho^\tau]}.\end{aligned}$$

Now, for every $K>0$ we have $E_{x,\omega_N}[\rho^\tau]\le P_{x,\omega_N}[\tau\le K]+\rho^K P_{x,\omega_N}[\tau>K]$. But since the random walk $(X_n)_{n\ge 0}$ is a martingale, by Doob's martingale inequality we have that for every $\lambda>0$,

$$\lambda N P_{0,t_x\omega}\Big[\sup_{0\le k\le K}|X_k^i|\ge\lambda N\Big]\le C'K^{1/2},$$

for some constant $C'>0$. Choosing $K=CN^2$ for an appropriate constant $C$, we have that for an appropriate choice of $\lambda$,

$$P_{x,\omega}[\tau\le K]\le\sum_{i=1}^{d}P_{0,t_x\omega}\left[\sup_{0\le k\le K}|X_k|_\infty\ge\lambda N\right]\le C'\frac{1}{\lambda}C^{1/2}N\le\frac{1}{2}.$$

To finish the proof, it remains to establish (38). As explained in Sznitman [6], one can follow the methods developed by Kuo and Trudinger [28] to obtain pointwise estimates for linear elliptic difference equations with random coefficients. One uses the fact that $u = Q_\omega f$ is a solution of the equation

$$(L_\omega u)(x) = -f(x), \qquad \text{for } x \in B_\infty(N),$$
$$u(x) = 0, \qquad \text{for } x \in \partial B_\infty(N);$$

where

$$(L_\omega g)(x) = \sum_{e \in U} \omega(x,e)(g(x+e) - g(x)) \tag{39}$$

and the so-called *normal mapping* (see [28]) defined for $x \in B_\infty(N)$ as

$$\chi_u(x) := \{p \in \mathbb{R}^d : u(z) \leq u(x) + p \cdot (z - x), \text{ for } z \in B_\infty(N) \cup \partial B_\infty(N)\}.$$

To conclude that

$$\omega_d \frac{(\max u)^{\mathrm{d}}}{(2N)^d} = |B_2(\max u/(2\mathrm{N}))| \leq \sum_{\mathrm{x} \in B_\infty(N)} |\chi_u(x)| \leq \sum_{x \in B_\infty(N)} \frac{f(x)^d}{\kappa^d},$$

where $\omega_d$ is the volume of a sphere unit radius, which proves (38).

Theorem 5 and Corollary 2 have recently been been extended by Guo and Zeitouni in [21] to the elliptic case. Further progress has been made by Berger and Deuschel in [4]. They introduce the following concept which is considerably weaker than ellipticity.

**Definition 7 (Genuinely $d$-dimensional environment)** We say that an environment $\omega \in \Omega$ is a *genuinely $d$-dimensional environment* if for every $e \in U$ there exists a $y \in \mathbb{Z}^d$ such that $\omega(y, e) > 0$. We say that the law $\mathbb{P}$ of an environment is *genuinely $d$-dimensional* if environments are genuinely $d$-dimensional under $\mathbb{P}$ with probability one.

**Theorem 6** *([4]) Consider a RWRE in an i.i.d., balanced and genuinely d-dimensional environment. Then the quenched invariance principle holds with a deterministic non-degenerate diagonal covariance matrix.*

In [56], Zeitouni proves as a corollary of Lawler's quenched central limit theorem for balanced random walks the following result.

**Theorem 7** ([56, Theorem 3.3.22]) Under the conditions of Theorem 5, the random walk is transient in dimensions $d \geq 3$ and recurrent in dimension $d = 2$.

## *1.8 One-Dimensional Quenched Large Deviations*

The following result was first derived by Greven and den Hollander [20] to the case of an i.i.d. environment and then extended by Comets, Gantert, and Zeitouni [12] for ergodic environments.

**Theorem 8** *(Greven-den Hollander, Comets–Gantert–Zeitouni) Consider an RWRE in dimension $d = 1$. Assume that $\mathbb{E}[\log \rho] \leq 0$ and that the environment fulfills* **(E)** *and is totally ergodic. Then, there exists a deterministic rate function $I : \mathbb{R} \to [0, \infty]$ such that*

1. *For every open set $G \subset \mathbb{R}$ we have that*

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{0,\omega}\left[\frac{X_n}{n} \in G\right] \geq - \inf_{x\in G} I(x) \qquad \mathbb{P} - a.s.$$

2. *For every closed set $C \subset \mathbb{R}$ we have that*

$$\limsup_{n\to\infty} \frac{1}{n} \log P_{0,\omega}\left[\frac{X_n}{n} \in C\right] \leq - \inf_{x\in C} I(x) \qquad \mathbb{P} - a.s.$$

*Furthermore, $I$ is continuous and convex, and it is finite exactly on $[-1, 1]$.*

The strategy used by Comets, Gantert and Zeitouni in [12] to prove Theorem 8 is based on obtaining a recursion relation for the moment generating function $\phi(\lambda) := E_{0,\omega}[e^{\lambda T_1}]$, where for $k \geq 1$, $T_k := \inf\{n \geq 0 : X_n = k\}$, which leads to a continuous fraction expansion of it. This leads to a large deviation principle for $T_k/k$ with rate function given by the expression

$$I(t) = \sup_{\lambda\in\mathbb{R}} (\lambda t - E_0[\phi(\lambda)]).$$

As is often the case, the expression for the rate function is much more explicit in $d = 1$ than in higher dimensions (cf. also Sect. 1.10 for the latter). In addition to the above, in [12] the following is also shown.

**Theorem 9** *Consider an RWRE satisfying the hypotheses of Theorem 8. Assume that the support of the law of $\omega(0, 1)$ intersects both $\left(0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right)$. Then the rate function $I$ of Theorem 8 satisfies the following properties:*

1. *For $x \in (0, 1]$ we have that $I(-x) = I(x) - x\mathbb{E}[\log \rho]$.*
2. *$I(x) = 0$ if and only if $x \in [0, v]$, with $v$ denoting the limiting velocity $\lim_{n\to\infty} X_n$ (see also (74) below).*

Part 1 of Theorem 8 shows that the slope of the rate function to the left of the origin does not vanish. A similar phenomenon is expected to happen for every transient random walk fulfilling **(IID)** and **(UE)** in dimensions $d \geq 2$. This behavior is expected to be connected to the resolution of a conjecture about the equivalence of two particular ballisticity conditions (see (77) below), which will be discussed in Chap. 2.

## *1.9 Multidimensional Quenched Large Deviations*

In [54] Varadhan presented a short proof of the quenched large deviation principle for the RWRE in general ergodic environments. His method is based on the use of the superadditive ergodic theorem.

Note that by the Markov property for each environment $\omega$ the $n$-step transition probability of the random walk (see (4)) satisfies for each natural numbers $n$ and $m$ and $x, y \in \mathbb{Z}^d$ the inequality

$$p^{(n+m)}(0, x + y) \geq p^{(n)}(0, x)p^{(m)}(x, x + y). \tag{40}$$

We would like to take logarithms on both sides to obtain a superadditive quantity and then apply the subadditive ergodic theorem. Nevertheless, there are two types of degeneracy that complicate this operation:

1. $p^{(0)}(x, y, \omega) = 0$ for $x \neq y$;
2. $p^{(n)}(x, y, \omega) = 0$ whenever $n$ and $|x - y|_1$ do not have the same parity.

To avoid them Varadhan introduced the following *smoothed transition probabilities*, defined for each $c > 0$ and $\omega, x, y$ and non-negative real $t$,

$$q_c(x, y, t) := \sup_{m \geq 0}\{p^{(m)}(x, y, \omega)e^{-c|m-t|}\}.$$

This regularization method is related to homogenization methods already developed within the context of the stochastic Hamilton–Jacobi equation (see for example, [26], [34]).

**Theorem 10** *(Varadhan) Consider an RWRE fulfilling* **(UE)** *and* **(ERG)**. *Then, there exists a convex rate function* $I : \mathbb{R} \to [0, \infty]$ *such that*

1. *For every open set* $G \subset \mathbb{R}^d$ *we have that*

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{0,\omega}\left[\frac{X_n}{n} \in G\right] \geq -\inf_{x\in G} I(x) \qquad \mathbb{P} - a.s.$$

2. *For every closed set* $C \subset \mathbb{R}^d$ *we have that*

$$\limsup_{n\to\infty} \frac{1}{n} \log P_{0,\omega}\left[\frac{X_n}{n} \in C\right] \leq -\inf_{x\in C} I(x) \qquad \mathbb{P} - a.s.$$

*Furthermore,* $I$ *is continuous in* $\overset{\circ}{B}_1(1)$, *lower-semicontinuous in* $B_1(1)$ and $I(x) = \infty$ for $x \notin B_1(1)$.

We will present here the proof of Theorem 10 given by Campos, Drewitz, Rassoul-Agaha, Ramírez and Seppäläinen in [9] and which is valid also for time-dependent random environments satisfying certain ergodicity conditions — we refer the reader to [9] for further details on the time dependent setting.

The idea is to avoid the degeneracy issues discussed related to point (2) above, by considering the random walk at even and odd times separately.

Let us begin modifying our random walk model, admitting the possibility that the walk does not move after one step, so that the set of jumps after one step is now $U' := U \cup \{0\}$ and

$$\omega(0,0) \geq \kappa.$$

We will call this random walk, the *random walk in random environment with holding times*. We will denote by $P^h_{x,\omega}$ its quenched law starting from $x$ and by

$$p_h^{(n)}(x, y, \omega) := P^h_{x,\omega}[X_n = y]$$

its $n$-step transition probabilities. For $x \in \mathbb{R}^d$, we will define

$$[x] := ([x_1], \ldots, [x_d]) \in \mathbb{Z}^d.$$

Let us define for $n \geq 0$, $R_n$ as the set of sites that the random walk can visit with positive probability at time $n$. Thus, $R_0 := \{0\}$, $R_1 := U'$ while for $n \geq 1$,

$$R_{n+1} := \{y \in \mathbb{Z}^d : y = x + e \text{ for some } x \in R_n \text{ and } e \in U'\} = R_{n+1} + (R_n + U).$$

It is easy to check that $B_1(1)$ equals the set of limit points of the sequence of sets $R_n/n$. Furthermore,

$$R_n = (nB_1(1)) \cap \mathbb{Z}^d \tag{41}$$

(see also Lemma 3.1 in [9]). We will now prove the following.

**Proposition 3** *Consider a random walk in random environment with holding times, and which fulfills* **(UE)** *and* **(ERG)** *to hold. Then, for each $x \in \mathbb{Q}^d$ we have that $\mathbb{P}$-a.s. the limit*

$$I(x) := -\lim_{n\to\infty} \frac{1}{n} \log p_h^{(n)}(0, [nx]) \tag{42}$$

*exists, is convex and deterministic. Furthermore, $I(x) < \infty$ for $x \in \mathbb{Q}^d \cap \mathring{B}_1(1)$.*

*Proof* Note that from we can check that if $x \notin B_1(1)$, for every $n \geq 1$ one has that $nx \notin nB_1(1)$ so that $nx \notin R_n$, and thus $p_h^{(n)}(0, [nx]) = 0$. This proves that $I(x) = \infty$ if $x \notin B_1(1)$.

Let us now consider an $x \in \mathbb{Q}^d \cap \mathring{B}_1(1)$. Note that there exists a $k \in \mathbb{N}$ and a $y \in \mathbb{Z}^d \cap k\mathring{B}_1(1)$ such that $x = k^{-1}y$; in addition, $y \in R_k$.

We will now introduce an auxiliary function $\widetilde{I}$ and then show that it in fact equals the expression given for $I$ in (42). Indeed, by the convexity of $B_1(1)$, the subadditive ergodic theorem [31] and (40), we have that

$$\widetilde{I}(k^{-1}y) := -\lim_{m\to\infty} \frac{1}{mk} \log p_h^{(mk)}(0, my)$$

exists $\mathbb{P}$-a.s. Furthermore, this definition is independent of the representation of $x$. Indeed, if $x = k^{-1}y_1 = l^{-1}y_2$ for some $k, l \in \mathbb{N}$, $y_1 \in \mathbb{Z}^d \cap k\mathring{B}_1(1)$ and $y_2 \in \mathbb{Z}^d \cap l\mathring{B}_1(1)$, we have that

$$\widetilde{I}(k^{-1}y_1) = -\lim_{n\to\infty}\frac{1}{nlk}\log p_h^{(nlk)}(0,nly_1) = -\lim_{n\to\infty}\frac{1}{nlk}\log p_h^{(nlk)}(0,nky_2)$$
$$= \widetilde{I}(l^{-1}y_2).$$

We will next prove that $\widetilde{I}$ is deterministic on $\mathbb{Q}^d \cap \mathbb{Z}^d$. Let $x \in \mathbb{Q}^d \cap \overset{\circ}{B}_1(1)$. There exists a $k \in \mathbb{N}$ and a $y \in \mathbb{Z}^d \cap k\overset{\circ}{B}_1(1)$ such that $x = k^{-1}y$. Now it is enough to prove that for each $z \in U$ one has that

$$\widetilde{I}(x,\omega) \le \widetilde{I}(x,t_z\omega) = -\lim_{m\to\infty}\frac{1}{mk}\log p_h^{(mk)}(z,my+z).$$

But for each $n \in \mathbb{N}$, we have that

$$-\frac{1}{mnk}\log p_h^{(mnk)}(0,mny) \le -\frac{1}{mnk}\log p_h^{(mnk)}(0,z) - \frac{1}{mnk}\log p_h^{(mnk)}(z,mny).$$

By uniform ellipticity, the first term in the right-hand side of the above inequality tends to 0 as $m \to \infty$. Therefore,

$$\widetilde{I}(x,\omega) = -\lim_{m\to\infty}\frac{1}{mnk}\log p_h^{(mnk)}(0,mny) \le -\liminf_{m\to\infty}\frac{1}{mnk}\log p_h^{(mnk)}(z,mny).$$

On the other hand,

$$-\frac{1}{mnk}\log p_h^{(mnk)}(z,mny)$$
$$\le -\frac{1}{mnk}\log p_h^{((m-1)nk)}(z,(m-1)ny+z) - \frac{1}{mnk}\log p_h^{(nk-1)}((m-1)ny+z,mny).$$

Now, since $z \in U$, one can check that $p_h^{(nk-1)}((m-1)ny+z,mny) \ge \kappa^{nk-1}$, so that the last term of the above inequality tends to 0 when $m \to \infty$. We can then conclude that $\widetilde{I}(x,\omega) \le \widetilde{I}(x,t_z\omega)$.

We will now prove that $I$ is well defined in $\mathbb{Q}^d \cap \overset{\circ}{B}_1(1)$ and that it equals $\widetilde{I}$ there. Let $x \in \mathbb{Q}^d \cap \overset{\circ}{B}_1(1)$. Furthermore, choose $k$ such that $kx \in \mathbb{Z}^d$ and given $n \in \mathbb{N}$ define

$$m := \left[\frac{n}{k}\right].$$

Necessarily, we can find a sequence $z_1, \ldots, z_{n-mk} \in U$ such that

$$[nx] = mkx + z_1 + \cdots + z_{n-mk}.$$

Hence, by superadditivity and uniform ellipticity we have that

$$-\frac{1}{n}\log p_h^{(n)}(0,[nx]) \le -\frac{1}{n}\log p_h^{(mk)}(0,mkx) - \frac{1}{n}\log\kappa^{n-mk}.$$

Therefore

$$-\limsup_{n\to\infty}\frac{1}{n}\log p_h^{(n)}(0,[nx])\le \widetilde{I}(x).$$

Using a similar argument we can establish that

$$-\liminf_{n\to\infty}\frac{1}{n}\log p_h^{(n)}(0,[nx])\ge \widetilde{I}(x).$$

We want now to extend Proposition 3 to $x\in\mathbb{R}^d$. To do this, we will need to establish a lemma which in some sense shows that the quantity $-\log p^{(n)}(0,[nx])$ is continuous as a function of $x$. For each $x\in\mathbb{Z}^d$ we define $s(x)$ as the minimum number $n$ of steps required for the random walk to move from 0 to $x$, so that

$$s(x):=\min\{n\ge 0 : x\in R_n\}.$$

We will now define a norm in $\mathbb{R}^d$ as follows. For each $y\in\partial B_1(1)$ we set $\|y\|:=1$. Then, for each $x\in\mathbb{R}^d$ of the form $x=ay$ for some $a\ge 0$, we define $\|x\|:=a$. Since $B_1(1)$ is convex, symmetric (in the sense that $x\in B_1(1)$ implies that $-x\in B_1(1)$), this implies that this defines a norm. It is easy to check that for every $x\in\mathbb{Z}^d$,

$$\|x\|\le s(x)\le\|x\|+1. \tag{43}$$

**Lemma 3** *Let $z\in B_1(1)$ and $x\in\overset{\circ}{B}_1(1)$.*

1. *For each natural n there exists an $n_2$ such that*

$$n\le n_2\le n+\frac{4d+1}{1-\|x\|}+n\frac{\|x-z\|}{1-\|x\|}+1, \tag{44}$$

*and such that*

$$-\log p_h^{(n_2)}(0,[n_2x])\le -\log p_h^{(n)}(0,[nz])-\log\kappa^{n_2-n}.$$

2. *Similarly, whenever $\|x-z\|<1-\|x\|$, there exists an $n_0$ such that for each natural $n\ge n_0$ there exists an $n_1$ such that*

$$n-\frac{4d+1}{1-\|x\|}-n\frac{\|x-z\|}{1-\|x\|}-1\le n_1\le n \tag{45}$$

*and such that*

$$-\log p_h^{(n)}(0,[nz])\le -\log p_h^{(n_1)}(0,[n_1x])-\log\kappa^{n-n_1}.$$

*Proof* To prove part 1 of the lemma, it is enough to show that there exists an $n_2\ge n$ satisfying (44) and such that

$$s([n_2x]-[nz])\le n_2-n. \tag{46}$$

But by (43) and the fact that $\|x - [x]\| \le d$ we see that

$$s([n_2 x] - [nz]) \le \|[n_2 x] - [nz]\| + 1 \le \|[n_2 x] - [nx]\| + \|[nx] - [nz]\| + 1$$
$$\le \|(n_2 - n)x\| + \|n(x - z)\| + 4d + 1 = (n_2 - n)\|x\| + n\|x - z\| + 4d + 1.$$

This shows that (46) is satisfied whenever

$$n_2 \ge n + \frac{4d + 1}{1 - \|x\|} + n\frac{\|x - z\|}{1 - \|x\|}.$$

To prove part 2 of the lemma, note that it is enough to show that there exists an $n_1 \le n$ satisfying (45) and

$$s([nz] - [n_1 x]) \le n - n_1.$$

But,

$$s([nz] - [n_1 x]) \le n\|z - x\| + (n - n_1)\|x\| + 4d + 1$$

which is equivalent to

$$n_1 \le n - \frac{4d + 1}{1 - \|x\|} - n\frac{\|z - x\|}{1 - \|x\|}.$$

We are now in a position to extend Proposition 3 to the following.

**Proposition 4** *Consider a random walk in random environment with holding times, where the law $\mathbb{P}$ of the environment is totally ergodic. Then, for each $x \in \mathbb{R}^d$ we have that $\mathbb{P}$-a.s. the limit*

$$I(x) := -\lim_{n\to\infty} \frac{1}{n} \log p_h^{(n)}(0, [nx])$$

*exists, is convex and deterministic. Furthermore, $I(x) < \infty$ if and only if $x \in B_1(1)$.*

*Proof* Let $z \in \mathbb{R}^d \cap \overset{\circ}{B}_1(1)$. Choose a point $x$ with rational coordinates such that $\|z - x\| < 1 - \|x\|$ and $\frac{1}{1-\|x\|} \le 2\frac{1}{1-\|z\|}$. By Lemma 3, for each $n \ge n_0$ we can find $n_1$ and $n_2$ satisfying (44) and (45) and such that

$$-\frac{n_2}{n}\frac{1}{n_2} \log p_h^{(n_2)}(0, [n_2 x]) \le -\frac{1}{n} \log p_h^{(n)}(0, [nz]) + b\left(\frac{n_2}{n} - 1\right)$$

and

$$-\frac{1}{n} \log p_h^{(n)}(0, [nz]) \le -\frac{n_1}{n}\frac{1}{n_1} \log p_h^{(n_1)}(0, [n_1 x]) + b\left(1 - \frac{n_1}{n}\right),$$

where $b := -\log\kappa$. From inequalities (44) and (45) of Lemma 3 and by Proposition 3 we can then conclude that

$$I(x) \le -\liminf_{n\to\infty} \frac{1}{n} \log p_h^{(n)}(0, [nz]) + C(z)b\|x - z\|$$

and

$$-\liminf_{n\to\infty} \frac{1}{n} \log p_h^{(n)}(0, [nz]) \le I(x) + C(z)b\|x - z\|,$$

where $C(z) := 2\frac{1}{1-\|z\|}$. Letting $x \to z$ we conclude that $I$ is well defined on $\mathbb{R}^d \cap \overset{\circ}{B}_1(1)$.

We are now in a position to extend the function $I$ of Proposition 4 from $\overset{\circ}{B}_1(1)$ to $B_1(1)$ as

$$I(x) := \begin{cases} I(x), & \text{if} \quad x \in \overset{\circ}{B}_1(1), \\ \liminf_{\overset{\circ}{B}_1(1) \ni y \to x} I(y), & \text{if} \quad x \in \partial B_1(1). \end{cases}$$

We will show that this is in fact the rate function of Theorem 10, but of an RWRE with holding times. Let us first show that $I$ satisfies the requirements of Theorem 10. By uniform ellipticity, it is clear that $I(x) \leq |\log \kappa|$ whenever $x \in B_1(1)$. Also, the proof of Proposition 4 shows that $I$ is continuous in $\overset{\circ}{B}_1(1)$. Furthermore, it is obvious that $I$ is convex and lower-semicontinuous in $B_1(1)$.

Now, note that if $G$ is an open subset of $\mathbb{R}^d$ and $x \in G$, the sequence $[nx]$ is in $nG \cap \mathbb{Z}^d$ and

$$P^h_{0,\omega}\left[\frac{X_n}{n} \in G\right] \geq P^h_{0,\omega}[X_n = [nx]].$$

In combination with Proposition 4 we therefore conclude that

$$\liminf_{n\to\infty} \frac{1}{n} \log P^h_{0,\omega}\left[\frac{X_n}{n} \in G\right] \geq -\inf_{x\in G} I(x).$$

Let us now consider a compact set $C \subset \overset{\circ}{B}_1(1)$. We then have that

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log P^h_{0,\omega}\left[\frac{X_n}{n} \in C\right] &\leq \limsup_{n\to\infty} \sup_{x\in C} \frac{1}{n} \log p_h^{(n)}(0, [nx]) \\ &= \inf_n \sup_{x\in C} \sup_{m\geq n} \frac{1}{m} \log p_h^{(n)}(0, [mx]).\end{aligned}$$

Now, through a contradiction argument and an application of Lemma 3, one can prove that

$$\sup_{x\in C} \sup_{m\geq n} \frac{1}{m} \log p_h^{(n)}(0, [mx]) \leq -\inf_{x\in C} I(x).$$

This shows that

$$\limsup_{n\to\infty} \frac{1}{n} \log P^h_{0,\omega}\left[\frac{X_n}{n} \in C\right] \leq -\inf_{x\in C} I(x). \tag{47}$$

Standard arguments using uniform ellipticity enable us now to extend (47) from compact sets to closed sets.

One can now derive Theorem 10 for the plain RWRE from the RWRE with holding times as follows. Define the *even lattice* as $\mathbb{Z}^d_{even} := \{x \in \mathbb{Z}^d : |x|_1 \text{ is even}\}$. Using the fact that since $\mathbb{Z}^d_{even}$ is a free Abelian group it is isomorphic to $\mathbb{Z}^d$, we can apply Proposition 4 for the RWRE with holding times to deduce an analogous result for the random walk $Y_n := X_{2n}$ at even times. On the other hand, using the equality

$$P_{0,\omega}\left[\frac{X_{2n+1}}{2n+1} \in A\right] = \sum_{i=1}^{2d} \omega(0, e_i) P_{e_i,\omega}\left[\frac{X_{2n}}{2n} \in A\right],$$

and the asymptotic behavior previously proved at even times, in combination with the assumption of uniform ellipticity, we can deduce the large deviation principle of Theorem 10.

## 1.10 Rosenbluth's Variational Formula for the Multidimensional Quenched Rate Function

The drawback of Theorem 10 is that it gives very little information about the rate function of the quenched large deviations of the random walk. A partial remedy to this was obtained by Rosenbluth [35] in his Ph.D. thesis in 2006, where he derived a variational expression for the rate function. To state Rosenbluth's result, it is more natural to define the RWRE in an abstract setting, where we first define the dynamics of the environmental process. In analogy to the set of admissible transition kernels $\mathcal{P}$ defined in 3, we denote by $\mathcal{Q}$ the set of measurable functions $q : \Omega \times U \mapsto [0, 1]$ such that $\sum_{e\in U} f(\omega, e) = 1$ for all $\omega \in \Omega$. Define the function $p \in \mathcal{Q}$ via $p(\omega, e) := \omega(0, e)$, corresponding to the transition probabilities of the canonical RWRE. Let us call $\mathcal{D}$ the set of measurable functions $\phi : \Omega \to [0, \infty)$ such that $\int \phi d\mathbb{P} = 1$.

**Theorem 11** *Assume that* **(ERG)** *is fulfilled and that there is an* $\alpha > 0$ such that

$$\max_{e\in U} \int |\ln p(\omega, e)|^{d+\alpha} \mathbb{P}(d\omega) < \infty.$$

*Then the RWRE satisfies a large deviation principle with rate function*

$$I(x) := \sup_{\lambda\in\mathbb{R}^d} \{\lambda \cdot x - \Lambda(\lambda)\},$$

*where*

$$\Lambda(\lambda) := \sup_{q\in\mathcal{Q}} \sup_{\phi\in\mathcal{D}} \inf_h \sum_{e\in U} \int \left(\lambda \cdot e - \ln \frac{q(\omega, e)}{p(\omega, e)} + h(\omega) - h(T_e\omega)\right) q(\omega, e)\phi(\omega)\mathbb{P}(d\omega).$$

**Remark 2** *The integrability assumption in the above theorem is fulfilled if* **(UE)** *holds true, for example.*

Note that using canonical LDP machinery, one can show that it is enough to prove that

$$\lim_{n\to\infty} \log E_{P_\omega}[e^{\lambda\cdot X_n}] = \Lambda(\lambda).$$

We will just give an idea of the proof of the above theorem deriving the lower bound in the above limit. In analogy to the definition of $P_\omega$ in (17), given $q \in \mathcal{Q}$, we denote by $Q_\omega$ the law of the corresponding Markov chain $(\bar{\omega}_n)_{n\geq 0}$ starting from $\omega$. We then have

$$E_{P_\omega}\left[e^{\lambda\cdot X_n}\right] = E_{Q_\omega}\left[e^{\lambda\cdot X_n} \frac{dP_\omega}{dQ_\omega}\right]$$
$$= E_{Q_\omega}\left[\exp\left\{\lambda \cdot X_n - \sum_{k=0}^{n-1} \ln \frac{q(t_{X_k}\omega, X_{k+1} - X_k)}{p(t_{X_k}\omega, X_{k+1} - X_k)}\right\}\right].$$

By Jensen's inequality it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \ln E_{P_\omega}\left[e^{\lambda\cdot X_n}\right] \geq \lim_{n\to\infty} E_{Q_\omega}\left[\frac{1}{n}\lambda \cdot X_n - \frac{1}{n}\sum_{k=0}^{n-1}\sum_{e\in U} \ln \frac{q(t_{X_k}\omega, e)}{p(t_{X_k}\omega, e)}\right]. \quad (48)$$

Now note that the expectation of the second term of (48) can be written as

$$E_{Q_\omega}\left[\frac{1}{n}\sum_{k=0}^{n-1}\ln\frac{q(t_{X_k}\omega,e)}{p(t_{X_k}\omega,e)}\right]=\frac{1}{n}\sum_{k=0}^{n-1}\sum_{e\in U}\ln\frac{q(t_{X_k}\omega,e)}{p(t_{X_k}\omega,e)}q(t_{X_k}\omega,e).$$

Let us now assume that the chain $(\bar{\omega}_n)_{n\geq 0}$ under $Q_\omega$ has an invariant measure $\nu$ which is absolutely continuous with respect to $\mathbb{P}$. Let us call $\phi$ the Radon–Nikodym derivative of $\nu$ with respect to $\mathbb{P}$. By Kozlov's theorem (Theorem 3), we know that the measure $\nu$ is such that $Q_\nu := \int Q_\omega \nu(d\omega)$ is ergodic (with respect to the time shifts). It follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sum_{e\in U}\ln\frac{q(t_{X_k}\omega,e)}{p(t_{X_k}\omega,e)}q(t_{X_k}\omega,e)$$
$$=\int\sum_{e\in U}\ln\frac{q(\omega,e)}{p(\omega,e)}q(\omega,e)\phi(\omega)\mathbb{P}(d\omega)\qquad Q_\nu-a.a.\,\omega,$$

and hence that

$$\lim_{n\to\infty}E_{Q_\omega}\left[\frac{1}{n}\sum_{k=0}^{n-1}\sum_{e\in U}\ln\frac{q(t_{X_k}\omega,e)}{p(t_{X_k}\omega,e)}q(t_{X_k}\omega,e)\right]$$
$$=\int\sum_{e\in U}\ln\frac{q(\omega,e)}{p(\omega,e)}q(\omega,e)\phi(\omega)\mathbb{P}(d\omega)\qquad \mathbb{P}-a.a.\,\omega.$$

On the other hand, by the law of large numbers, we can see that the behavior of the first term on the right-hand side of (48) is characterized by

$$\lim_{n\to\infty}E_{Q_\omega}\left[\frac{1}{n}\lambda\cdot X_n\right]=\int\sum_{e\in U}\lambda\cdot e q(\omega,e)\phi(\omega)\mathbb{P}(d\omega)\qquad \mathbb{P}-a.a.\,\omega.$$

It follows that if we call $\mathcal{Q}_0$ the set of transition probabilities $q$ for which there is an invariant measure $\nu_q$ which is absolutely continuous with respect to $\mathbb{P}$ (and which is unique, by part 3 of Kozlov's theorem), with $\phi_q=\frac{d\nu_q}{d\mathbb{P}}$ we have by (48) that

$$\Lambda(\lambda)\geq\sup_{q\in\mathcal{Q}_0}\sum_{e\in U}\int\left((\lambda,e)-\ln\frac{q(\omega,e)}{p(\omega,e)}\right)q(\omega,e)\phi_q(\omega)\mathbb{P}(d\omega).$$

Now note that for $\phi\in\mathcal{D}$, the following are equivalent

$$\phi=\phi_q$$

and

$$\inf_h\int\sum_{e\in U}\left(h(\omega)-h(T_e\omega)\right)q(\omega,e)\phi(\omega)\mathbb{P}(d\omega)=0.$$

Similarly,

$$\phi \neq \phi_q$$

and

$$\inf_h \int \sum_{e\in U} (h(\omega) - h(T_e\omega))\, q(\omega, e)\phi(\omega)\mathbb{P}(d\omega) = -\infty.$$

Therefore, we conclude that

$$\sup_{q\in\mathcal{Q}_0} \sum_{e\in U} \int \left(\lambda \cdot e - \ln \frac{q(\omega, e)}{p(\omega, e)}\right) q(\omega, e)\phi_q(\omega)\mathbb{P}(d\omega)$$
$$= \sup_{q\in\mathcal{Q},\phi\in\mathcal{D}} \inf_h \sum_{e\in U} \int \left(\lambda \cdot e - \ln \frac{q(\omega, e)}{p(\omega, e)} + h(\omega) - h(T_e\omega)\right) q(\omega, e)\phi_q(\omega)\mathbb{P}(d\omega),$$

which finishes the sketch of the proof for the lower bound.

A level 2 large deviation principle version of Rosenbluth's variational formula was derived by Yilmaz in [55]. Subsequently, a level 3 version was derived by Rassoul-Agha and Seppäläinen in [33].

## 2 Ballistic Behavior and Trapping in Higher Dimensions

In Chap. 1 we have already considered some situations in which one has been able to obtain information not only on transience and ballisticity, but also on the diffusive behavior of RWRE as well as its large deviations; in these situations, this supplied us with a rather precise understanding of the asymptotic behavior. The content of this chapter is a more general analysis of RWRE in terms of the coarser scales of (directional) transience and ballistic behavior.

### *2.1 Directional Transience*

As we have seen in Chap. 1, the question of whether under appropriate conditions an RWRE in dimension $d \geq 3$ is transient, remains essentially unsolved. More is known, however, about "transience in a given direction" which has been introduced in Definition 5, and we will see how this concept plays a role in the investigation of ballistic behavior of RWRE also. In fact, some quite challenging questions concerning RWRE are related to that notion, too, as we will see in this chapter.

In the following, we will tacitly use for $x \in \mathbb{Z}^d$ the equivalence of the conditions

$$\text{“}P_x[A_l] = 1\text{”}, \tag{49}$$

and

$$\text{“for } \mathbb{P}\text{-almost all } \omega \text{ one has } P_{x,\omega}[A_l] = 1\text{”}.$$

Note that this equivalence is a direct consequence of the definition of the averaged measure below (5).

The following result has essentially been proven by Kalikow [22] and has been refined in [49, 60].

**Lemma 4** *Consider an RWRE satisfying* **(E)** and **(IID)**. Then for every $l \in \mathbb{S}^{d-1}$ we have that

$$P_0[A_l \cup A_{-l}] \in \{0, 1\}.$$

Of course, the above zero-one law seems incomplete and one would like to have a zero-one law for the event $A_l$ already. Intriguingly, however, it is still not known if such a statement holds in full generality.

**Open Question 5** *Consider a RWRE satisfying the assumptions* **(E)** *and* **(IID)**. *Is it true that for every $l \in \mathbb{S}^{d-1}$ one has*

$$P_0[A_l] \in \{0, 1\}? \tag{50}$$

As we have seen in Theorem 1, statement (50) holds true for $d = 1$. In dimension two, it has been proven to hold true by Zerner and Merkl [60]. In fact, it is also shown in that source, that if one assumes the environment to be stationary and ergodic with respect to lattice translations only, it can indeed happen that $P_0[A_l] \notin \{0, 1\}$.

Apart from leading to interesting problems on its own, the events $A_l$ also play a key role in the next section in order to define a renewal structure for RWRE.

## *2.2 Renewal Structure*

In order to prove some of the main asymptotic results for RWRE in the directionally transient regime, we will define a renewal structure which will help us to decompose the RWRE in terms of finite i.i.d. (apart from its initial part; see Corollary 3) trajectories. The first use of this renewal structure in the context of RWRE is due to Kesten, Kozlov, and Spitzer [24] in the one-dimensional case, and it has then been generalized to the higher-dimensional case by Sznitman and Zerner [49]. It can be introduced as follows: given a direction $l \in \mathbb{S}^{d-1}$, it is the first time that the random walk reaches a new maximum level in direction $l$ and such that after this time it never goes below this maximum in direction $l$. Thus, an easy way to define the renewal time $\tau_1$ is via

$$\tau_1 := \min\Big\{n \geq 1 : \max_{0 \leq m \leq n-1} X_m \cdot l < X_n \cdot l \leq \inf_{m \geq n} X_m \cdot l\Big\}. \tag{51}$$

Another way to put it is that $\tau_1$ is the first time that the last exit time from a half space of the form $\{x \cdot l < r\}$, some $r \in \mathbb{R}$, coincides with the first entrance time into its complement.

In order to introduce notation which is used in the computations below, we give another definition of $\tau_1$ in terms of a sequence of stopping times; it is slightly more involved. Consider

$$H_u^l := \inf\{n \geq 1 : X_n \cdot l > u\}$$

for $u \in \mathbb{R}$ as well as

$$D := \inf\{n \geq 0 : X_n \cdot l < X_0 \cdot l\}$$

which are stopping times with respect to the canonical filtration. Furthermore, set

$$S_0 := 0, \qquad R_0 := X_0 \cdot l.$$

In a slight abuse of notation and similarly to (29), we will now use $\theta$ to denote the canonical shift on $(\mathbb{Z}^d)^{\mathbb{N}}$, i.e.,

$$\theta : (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots),$$

and for $n \geq 1$, we define $\theta_n$ to be the $n$-fold composition of $\theta$. Using this notation, for $k \geq 1$ we now introduce the stopping times

$$S_k := H_{R_{k-1}}^l, \qquad D_k := \begin{cases} D \circ \theta_{S_k} + S_k, & \text{if } S_k < \infty, \\ \infty, & \text{otherwise}, \end{cases},$$
$$R_k := \sup\{X_m \cdot l : 0 \leq m \leq D_k\}. \tag{52}$$

We then define

$$K := \inf\{k \geq 0 : S_k < \infty, D_k = \infty\}, \tag{53}$$

and the first *renewal time*,

$$\tau_1 := S_K.$$

Note that $\tau_1$ is not a stopping time with respect to the canonical filtration anymore, since in order to determine whether $\{S_K = m\}$ occurs one has to “see into the future” of $(X_n)$ after time $m$. One can then recursively define the sequence of regeneration times $(\tau_k)_{k \in \mathbb{N}}$ via

$$\tau_{k+1} = \tau_1 \circ \theta_{\tau_k} + \tau_k, \quad k \geq 1,$$

and set $\tau_0 = 0$. See Fig. 3 for an illustration of the above renewal structure.

**Remark 3**

- *Note here that, although not emphasized explicitly in the notation, the definition of the sequence $(\tau_n)$ depends on the choice of the direction $l$; if the very choice of $l$ matters, it will usually be clear from the context.*
- *If working with directions $l$ having rational coordinates, Definition 3 works fine. However, for general directions $l \in \mathbb{S}^{d-1}$, one might under some circumstances run into slightly more technical argumentations — e.g., for guaranteeing that each time a renewal time occurs, the walker has gained some height bounded away*

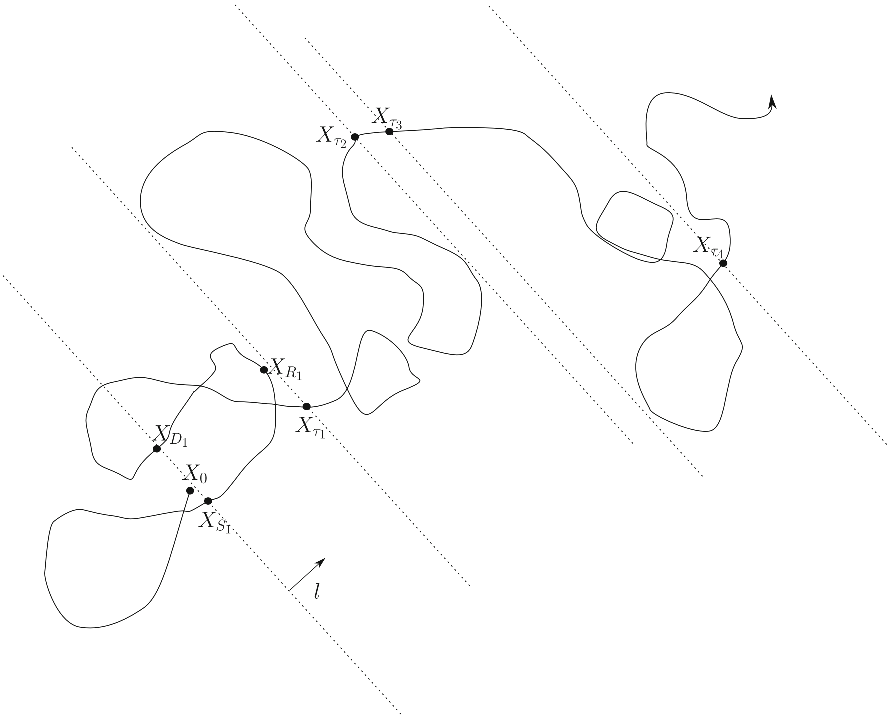


**Fig. 3** Sketch of the renewal structure

*from* 0 *in direction l (see for example [45, (1.63)]); however, these complications do not pose any serious problems.*
*Note, on one hand, that one way to avoid this kind of technicalities is to replace* $H^l_{R_{k-1}}$ *in 52 by* $H^l_{R_{k-1}+a}$ *for some* $a > 0$, *as is done for example in [49]. On the other hand, however, formulas such as in Lemma 6 would result to be more complicated, and therefore we stick to the definition given above.*

The following lemma illustrates the role of the events $A_l$ from (27) in the definition of the renewal structure described above.

**Lemma 5** *Assume* **(E)** *and* **(IID)** *to hold. Let furthermore* $l \in \mathbb{S}^{d-1}$ *and assume that*

$$P_0[A_l] > 0. \tag{54}$$

*Then the following are satisfied:*

1.
$$P_0[D = \infty] > 0;$$

2.
$$P_0[A_l \Delta \{K < \infty\}] = 0. \tag{55}$$

In words, Lemma 5, part 1 states that if the walk has a positive probability of finally escaping to infinity in direction $l$, then it must have a positive probability of doing so "at once", i.e., without entering the half-space $\{x \in \mathbb{Z}^d : x \cdot l < 0\}$. Part 2 then ensures that on $A_l$, the above renewal structure is a.s. well-defined.

*Proof* Let us first prove part 1. Let 54 be fulfilled and assume that

$$P_0[D = \infty] = 0, \qquad \text{i.e.,} \qquad P_0[D < \infty] = 1.$$

From the invariance of $\mathbb{P}$ under spatial translations, it follows that for all $x \in \mathbb{Z}^d$ we have

$$P_x[D < \infty] = 1.$$

Using 49, we deduce that for $\mathbb{P}$-almost all $\omega$ we would get that for all $x \in \mathbb{Z}^d$,

$$P_{x,\omega}[D < \infty] = 1.$$

Therefore, iteratively applying the strong Markov property at the return times of the walk to the half-space $\{x \in \mathbb{Z}^d : x \cdot l \le 0\}$, we obtain that $P_0$-a.s.,

$$\liminf_{n\to\infty} X_n \cdot l \le 0,$$

which is a contradiction to (54).

We now prove part $(ii)$. Recalling the definition of $K$ from (53), we note that

$$\{K < \infty\} \subset A_{-l}^c.$$

In combination with the zero-one law of Lemma 4, we therefore infer that

$$P_0[\{K < \infty\} \backslash A_l] = 0. \tag{56}$$

On the other hand, observe that for $k \ge 1$,

$$\begin{aligned}
P_0[R_k < \infty] &= P_0[S_k < \infty, R_k < \infty] = \mathbb{E}[E_{0,\omega}[S_k < \infty, P_{X_{S_k},\omega}[D < \infty]]] \\
&= \sum_{x\in\mathbb{Z}^d} \mathbb{E}[P_{0,\omega}[S_k < \infty, X_{S_k} = x] P_{x,\omega}[D < \infty]] \\
&= \sum_{x\in\mathbb{Z}^d} P_0[S_k < \infty, X_{S_k} = x] P_0[D < \infty] \\
&= P_0[S_k < \infty] P_0[D < \infty] \le P_0[S_{k-1} < \infty, R_{k-1} < \infty] P_0[D < \infty],
\end{aligned}$$

where to obtain the penultimate equality we used assumption **(IID)** in combination with the fact that $P_{0,\omega}[S_k < \infty, X_{S_k} = x]$ and $P_{x,\omega}[D < \infty]$ are measurable with respect to a disjoint set of coordinates in $\Omega$.

It follows that

$$P_0[R_k < \infty] \le P_0[D < \infty]^k.$$

Using part 1 of this lemma, this again implies $P_0[K < \infty \mid A_l] = 1$, which again yields

$$P_0[A_l \backslash \{K < \infty\}] = 0$$

and hence in combination with (56) finishes the proof. □

The next result is contained in [49, Proposition 1.4]

**Proposition 5** *Denote*

$$\mathcal{G}_1 := \sigma\big(\tau_1, (X_k)_{0\le k\le \tau_1}, (\omega(y,\cdot))_{\{y\,:\,y\cdot l < X_{\tau_1}\cdot l\}}\big).$$

*Then the joint distribution of*

$$\big((X_n - X_{\tau_1})_{n\ge\tau_1}, (\omega(y,\cdot))_{y\cdot l\ge X_{\tau_1}\cdot l}\big)$$

*under* $P_0[\,\cdot \mid A_l, \mathcal{G}_1]$ *equals the joint distribution of*

$$\big((X_n)_{n\ge 0}, (\omega(y,\cdot))_{y\cdot l\ge 0}\big)$$

*under* $P_0[\,\cdot \mid D = \infty]$.

In particular, one can infer inductively that on $A_l$, the sequence of renewal times $(\tau_n)$ is well-defined.

As a corollary of a slight generalization of the above result, Sznitman and Zerner [49] obtain the following.

**Corollary** 3 *Under* $P_0[\,\cdot \mid A_l]$, *the variables* $(X_{\tau_k} - X_{\tau_{k-1}}, \tau_k)_{k\ge 1}$ *are an independent family. Furthermore,* $(X_{\tau_k} - X_{\tau_{k-1}}, \tau_k)_{k\ge 2}$, *under* $P_0[\,\cdot \mid A_l]$ *are identically distributed as* $(X_{\tau_1} - X_0, \tau_1)$ *under* $P_0[\,\cdot \mid A_l]$.

On an intuitive level, the idea behind the proof of Corollary 3 is that the environments that the walk sees between different renewal times are i.i.d., which can then be transferred to the behavior of the walk itself.

## *2.3 A General Law of Large Numbers*

Recall that we have already seen a law of large numbers in Corollary 1; however, the assumptions for that result included the existence of an invariant measure $\nu$ for the environmental process such that $\nu$ was absolutely continuous with respect to $\mathbb{P}$. We have seen that in some special cases (cf. e.g. Theorem 5), one can ensure the existence of such a measure $\nu$. On the other hand, however, not much is known about when such $\nu$ exists, and it would be desirable to have a law of large numbers that holds without this assumption.

The following theorem is such a result and constitutes a slight refinement of the directional laws of large numbers by Zerner [58, Theorem 1] and Zeitouni [56, Theorem 3.2.2].

**Theorem 12** Assume **(IID)** and **(E)** to hold. Then in dimensions $d \ge 2$, there exists a direction $\nu \in \mathbb{S}^{d-1}$, and $v_1, v_2 \in [0,1]$ (all deterministic) such that $P_0$-a.s.

$$\lim_{n\to\infty} \frac{X_n}{n} = v_1 \nu 1_{A_\nu} - v_2 \nu 1_{A_{-\nu}}. \tag{57}$$

**Remark 4** *Let us remark here that on the level of the law of large numbers (in contrast to the central limit theorem or large deviation results), the average result directly implies the* $\mathbb{P}$*-a.s. quenched result due to (49).*

Since the conjectured zero-one law of Open Question 5 is still eluding its complete resolution, the right-hand side of (57) might be a non-degenerate random variable. In dimensions larger or equal to five, Berger [2] has shown that at least one of the velocities $v_1$ and $v_2$ must vanish. In dimension two, the zero-one law of Zerner and Merkl [60] mentioned after Open Question 5 leads to the following corollary of Theorem 12.

**Corollary 4** *Assume* **(IID)** *and* **(UE)** *to hold. Then in dimension* $d = 2$*, there exists a direction* $\nu \in \mathbb{S}^{d-1}$*, and* $v_1 \in [0, 1]$ *(all deterministic) such that* $P_0$*-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = v_1 \nu.$$

To prove Theorem 12, we need the following lemma.

**Lemma 6** *Assume* **(IID)** *and* **(E)** *to be fulfilled. Then for* $l = (l_1, \dots, l_d) \in \mathbb{Z}^d$ *such that* $\gcd(l_1, \dots, l_d) = 1$*, one has*

$$E_0[X_{\tau_1} \cdot l \mid D = \infty] = \frac{1}{P_0[D = \infty \mid A_l] \lim_{i\to\infty} P_0[H^l_{i-1} < \infty, X_{H^l_i} \cdot l = i]} < \infty. \tag{58}$$

*(Note that in a slight abuse of notation we use* $l \in \mathbb{Z}^d$ *instead of* $l \in \mathbb{S}^{d-1}$ *here.)*

In the case $l = (1, 0, \dots, 0)$, the proof of Lemma 6 can be found in [50, Lemma 3.2.5] and is based on an argument by Zerner. See [14, Lemma 2.5] for how (in the context of a different renewal structure) the generalization to $l$ as in Lemma 6 works and how to obtain the finiteness of (58).

*Proof of Theorem 12* The proof is split into several pieces.

1. We start with proving the following version of a directional law of large numbers, which can be found in [50, Theorem 3.2.2]. It states that for $l \in \mathbb{S}^{d-1}$ with

$$P_0[A_l \cup A_{-l}] = 1 \tag{59}$$

there exist $v_l, v_{-l} \in [0, 1]$ such that $P_0$-a.s.

$$\lim_{n\to\infty} \frac{X_n \cdot l}{n} = v_l 1_{A_l} - v_{-l} 1_{A_{-l}}. \tag{60}$$

We will prove this result here for $l \in \mathbb{Z}^d$, which is slightly easier notation wise. Without loss of generality, assume that $P_0[A_l] > 0$. Then, by the standard law of large numbers in combination with Corollary 3, $P_0[\,\cdot \mid A_l]$-a.s. we have that

$$\lim_{k\to\infty} \frac{\tau_k}{k} = E_0[\tau_1 \mid D = \infty],$$

and

$$\lim_{k\to\infty} \frac{X_{\tau_k} \cdot l}{k} = E_0[X_{\tau_1} \cdot l \mid D = \infty].$$

From this we conclude that $P_0[\,\cdot\mid A_l]$-a.s.

$$\lim_{k\to\infty}\frac{X_{\tau_k}\cdot l}{\tau_k}=\frac{E_0[X_{\tau_1}\cdot l\mid D=\infty]}{E_0[\tau_1\mid D=\infty]}=:v_l, \tag{61}$$

which due to Lemma 6 is a finite quantity. Using the fact that the $\tau_k$ and $X_{\tau_k}\cdot l$ are increasing in $k$, one obtains the sandwiching

$$\frac{X_{\tau_k}\cdot l}{\tau_{k+1}}\le\frac{X_n\cdot l}{n}\le\frac{X_{\tau_{k+1}}\cdot l}{\tau_k}$$

for $\tau_k\le n<\tau_{k+1}$. In combination with (61) we infer that

$$\lim_{n\to\infty}\frac{X_n\cdot l}{n}=v_l$$

$P_0[\,\cdot\mid A_l]$-a.s. By exchanging $l$ for $-l$ in the above, in combination with (59) we therefore obtain (60).

2. Next, we will use [58, Theorem 1] which states that assuming **(IID)**, **(E)** and $P_0[A_e\cup A_{-e}]=0$, one has for any $e\in U$, that

$$\lim_{n\to\infty}\frac{X_n\cdot e}{n}=0,\qquad P_0-a.s. \tag{62}$$

On a very coarse heuristic level, the proof of that result is as follows by contradiction: Let

$$P_0[A_l\cup A_{-l}]=0, \tag{63}$$

and assume that

$$\limsup_{n\to\infty}\frac{X_n\cdot l}{n}>0$$

with positive probability. Then, if one partitions $\mathbb{Z}^d$ into slabs orthogonal to $l$ which are of positive finite thickness, there exists a constant $C$ such that with positive probability, the walk visits each of a positive fraction of the slabs for at most $C$ time steps. One can next deduce that, denoting the first entrance position of the walk in such a slab by $x$, there exists a positive number $r$ and a vector $z$ such that with positive probability, the walk visits the slab for the last time at its $r$-th visit to $x+z$. From this one is then able to deduce that one must have $P_0[A_l]>0$, a contradiction to (63). We refer the reader to [58] for more details.

An inspection of the proof in [58] yields that by slightly modifying it, one obtains (62) for $e$ replaced by arbitrary $l\in\mathbb{S}^{d-1}$. In combination with the result of (60), and due to the zero-one law of Lemma 4, we may therefore omit assumption (59) and still obtain that (60) holds true.

3. Using (60), we obtain that $\lim_{n\to\infty}X_n/n$ exists $P_0$-a.s. and, also $P_0$-a.s, takes values in a set of cardinality at most $2^d$. One can then take advantage of similar arguments as Goergen on page 1112 of [19] in order to show that $P_0$-a.s. $\lim_{n\to\infty}X_n/n$ takes values in a set of two elements which are collinear, which

finishes the proof. Indeed, assume there were $v_1, v_2$ not collinear such that $P_0[\lim_{n\to\infty} X_n/n = v_i] > 0$ for $i = 1, 2$. Then for any $l$ such that

$$l \cdot v_1, l \cdot v_2 > 0 \tag{64}$$

one obtains by (60) and the fact that

$$\big\{\lim_{n\to\infty} X_n/n = v_1\big\} \cup \big\{\lim_{n\to\infty} X_n/n = v_2\big\} \subset A_l,$$

that

$$l \cdot v_1 = v_l = l \cdot v_2. \tag{65}$$

Since the set of vectors $l$ fulfilling (64) is open, we can let $l$ vary along a set of basis vectors fulfilling (64) and hence conclude that (65) holds for a set of vectors $l$ which form a basis. This implies $v_1 = v_2$, a contradiction to the assumption that $v_1$ and $v_2$ were collinear. This yields Theorem 12.

**Remark 5** *It is useful to observe from part 1 of the proof of Lemma 6 that*

$$v_l \neq 0 \text{ if and only if } E_0[\tau_1 \mid D = \infty] < \infty. \tag{66}$$

*This condition is in general hard to check—it will be one of the principal goals of the remaining part of these notes to investigate conditions that ensure $v_l \neq 0$.*

## 2.4 Ballisticity

We have seen in Theorem 12 that a version of a law of large numbers is valid. This, however, did not tell us anything practical about the fundamental question of whether $v_1$ and $v_2$ are equal to or different from 0 (except for the one-dimensional setting of Theorem 4, Remark 5, and the result of Berger [2] alluded to above). Here, we will address this question and for this purpose recall the concept of ballisticity in a given direction (see Definition 6).

**Remark 6** *If a RWRE is ballistic in a direction $l$ according to Definition 6, then one can deduce that $P_0$-a.s., the limit*

$$\lim_{n\to\infty} \frac{X_n \cdot l}{n} \quad \text{exists, is positive, and is } P_0\text{-a.s. constant.} \tag{67}$$

*Indeed, if (28) is fulfilled, then $P_0[A_l] = 1$ and hence the renewal structure as introduced in Sect. 2.2 is $P_0$-a.s. well-defined (cf. Lemma 5). Similarly to the proof of Theorem 12 one obtains that $P_0$-a.s.,*

$$\lim_{k\to\infty} \frac{X_{\tau_k} \cdot l}{\tau_k} = \frac{E_0[X_{\tau_1} \cdot l \mid D = \infty]}{E_0[\tau_1 \mid D = \infty]} \tag{68}$$

*exists; using (28) we then infer that the expression in (68) must be positive, which implies (67).*

If one wants to investigate the occurrence of ballistic behavior in higher dimensions, it is obvious that one cannot expect as simple conditions as in the one-dimensional case (cf. Theorem 4) As a partial remedy, Sznitman [47] has introduced conditions which in some sense can be considered a higher-dimensional analog to the conditions given in Theorem 4 for dimension one. These conditions have turned out to be useful in a plethora of different contexts of RWRE.

**Definition 8 (Conditions $(T)_\gamma$, $(T')$ and $(T)$).** Assume $l \in \mathbb{S}^{d-1}$ and $\gamma \in (0,1]$. We say that condition $(T)_\gamma|l$ is satisfied if there exists a neighborhood $V_l$ of $l$ such that for every $l' \in V_l$ one has that

$$\limsup_{L\to\infty} \frac{1}{L^\gamma} \log P_0[H_{bL}^{-l'} < H_L^{l'}] < 0. \tag{69}$$

We say that condition $(T)|l$ is satisfied if condition $(T)_1|l$ holds. Finally, we say that condition $(T')|l$ is satisfied if for every $\gamma \in (0,1)$, condition $(T)_\gamma|l$ is satisfied. Also, if the precise value of $l$ is irrelevant, then we often write $(T)_\gamma$ instead of $(T)_\gamma|l$, and analogously for the remaining conditions.

Intuitively, if the walk escapes in direction $l'$ and is "well-behaved", then the probability in (69) corresponds to that of a rare event and, due to the independence structure of the environment, should decay reasonably fast.

**Example 4** *Zerner and Sznitman [57, 45] have introduced a classification of RWREs in terms of the support of the law of the random variable*

$$d(0,\omega) = \sum_{e\in U} \omega(0,e)\cdot e; \tag{70}$$

*The random variable $d(0,\cdot)$ is the local drift at the origin. Denote by $C \subset D$ (cf. 2) the convex hull of the support of the law of $d(0,\omega)$. An RWRE is called*

1. non-nestling *if*
$$0 \notin C;$$
2. marginally nestling *if*
$$0 \in \partial C;$$
3. plain nestling *if*
$$0 \in \overset{\circ}{C}.$$

*In terms of investigating their ballistic behavior, the non-nestling and marginally nestling RWREs are easier to handle than the nestling ones. This is due to the fact that their behavior "dominates" that of i.i.d. variables with positive expectation. We leave it to the reader to prove that non-nestling RWRE satisfy condition $(T)$.*

For future purposes it will be helpful to also consider the corresponding polynomial analogues.

**Definition 9 (Conditions** $(\mathcal{P}^*)_M, (\mathcal{P}^*)_0$**).** Assume $M > 0$ and $l \in \mathbb{S}^{d-1}$ to be given. We say that condition $(\mathcal{P}^*)_M|l$ (sometimes referred to as $(\mathcal{P}^*)_M$ or $(\mathcal{P}^*)$ also) is fulfilled, if there exists a neighborhood $V_l$ of $l$ such that for all $l' \in V_l$ and for all $b > 0$ we have

$$\lim_{L\to\infty} L^M P_0[H_{bL}^{-l'} < H_L^{l'}] = 0. \tag{71}$$

In addition, we define $(\mathcal{P}^*)_0$ to hold if for all $l'$ in a neighborhood of $l$ and for all $b > 0$ we have

$$\lim_{L\to\infty} P_0[H_{bL}^{-l'} < H_L^{l'}] = 0. \tag{72}$$

**Remark 7**

- *In the following we will give some fundamental results that were mostly proven under the assumption of condition* $(T')$*. However, in anticipation of Theorem 8 below, we will instead formulate them assuming* $(\mathcal{P})_M$ *for* $M > 15d + 5$ *only.*
- *Also, note that due to Theorem 8 it is actually sufficient to assume* $(\mathcal{P})_M$ *(see Definition 11) instead of* $(\mathcal{P}^*)_M$*, both for* $M > 15d + 5$*, in what follows. This condition is a priori weaker and has the advantage that it can be checked on finite boxes already. However, since it is more complicated to state and needs notation introduced only later on, we will not give its exact definition here yet.*

There is an alternative formulation for the conditions $(T)_\gamma$, which instead of considering slab exit estimates involves transience and the (stretched) exponential integrability of the renewal radii.

**Theorem 13** *(47, Cor. 1.5]) Assume* **(IID)** *and* **(UE)** *to hold, and let furthermore* $d \geq 1$ *and* $\gamma \in (0, 1]$*. Then the following are equivalent.*

1. *Condition* $(T)_\gamma|l$ *is satisfied.*
2. *One has* $P_0[A_l] = 1$ *(note that this ensures that* $\tau_1$ *is well-defined) and there exists a constant* $C > 0$ *such that*

$$E_0\Big[\exp\big\{C^{-1} \max_{0\leq i\leq\tau_1} |X_i|_1^\gamma\big\}\Big] < \infty. \tag{73}$$

Note that the first part of the condition (2) in Theorem 13 in combination with the law of large numbers of Theorem 12 already supplies us with the fact that $P_0$-a.s., $\lim_n X_n/n$ converges to a deterministic vector. Therefore, due to Theorem 14 below, the second part of condition (2) in Theorem 13 can be seen as guaranteeing that this deterministic limit is different from 0. Note, however, that an affirmative answer to the Open Question 2 would imply that the transience assumption $P_0[A_l]$ is already sufficient and the integrability condition of (73) is not needed for having a non-zero limiting velocity, i.e., ballisticity.

These stretched exponential integrability assumptions on the renewal radii have been used by Sznitman (see [47]) to deduce the following: In dimensions larger than or equal to two, $(T')$ implies a law of large numbers with non-zero limiting velocity as well as an invariance principle for the RWRE, so that diffusively rescaled it converges to Brownian motion under the averaged measure.

**Theorem 14** *([47, Theorem 3.3]) Assume* **(IID)** *and* **(UE)** *to hold. Furthermore, assume $d \geq 2$ and let $(\mathcal{P})_M|l$ is fulfilled for some $l \in \mathbb{S}^{d-1}$ and $M > 15d + 5$. Then:*

1. *The RWRE is ballistic, i.e., one has $P_0$-a.s. that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v \neq 0, \tag{74}$$

   *where $v$ is deterministic.*
2. *Under $P_0$ and with*

$$B_t^n := \frac{1}{\sqrt{n}}(X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v), \quad t \geq 0,$$

   *the sequence of processes $((B_t^n)_{t\geq 0})_{n\in\mathbb{N}}$ converges in law on the Skorokhod space $D([0,\infty);\mathbb{R}^d)$ to Brownian motion with non-degenerate covariance matrix as $n \to \infty$.*

**Remark 8** *Recently, there has also been initiated the investigation of ballisticity and related topics for the situation where* **(IID)** *holds, but the condition* **(UE)** *has been replaced by the weaker* **(E)**. *In this context, in order to obtain results comparable to the ones above, one then has to make assumptions on the decay of the random variables $\omega(0,e)$ at $0$. These assumptions can be used to apply large deviations estimates in order to obtain that with high $\mathbb{P}$-probability, for sufficiently long paths, the probability of following them is comparable at least to a situation where one has uniform ellipticity; see Sect. 2.12 as well as Campos and Ramírez [8] for further details.*

**Open Question 6** *Theorem 14 states that the conditions $(\mathcal{P})_M$ for $M > 15d + 5$ do imply a ballistic behavior. Vice versa, one can ask if (74) already implies the validity of $(\mathcal{P})_M$ for $M > 15d + 5$. This question is intimately linked to the slope of the large deviation principle rate function in the origin.*

As observed in Remark 5, in order to guarantee a positive limiting velocity, and therefore to prove Theorem 14 1, it is enough to show the integrability of $\tau_1$ with respect to $P[\,\cdot \mid D = \infty]$. On the other hand, in order to deduce Theorem 14 (2), an essential part of the proof is to establish the square integrability of $\tau_1$ with respect to $P[\,\cdot \mid D = \infty]$ (see also Theorem 4.1 in [45]). Both of these integrability conditions are a direct consequence of the following recent result of Berger.

**Theorem 15** *([3, Proposition 2.2]) Let* **(UE)**, **(IID)**, *and $(\mathcal{P})_M|l$ be fulfilled for some $l \in \mathbb{S}^{d-1}$ and $M > 15d + 5$. Then, for $d \geq 4$ and every $\alpha < d$ one has that*

$$P_0[\tau_1 \geq u] \leq \exp\{-(\log u)^\alpha\}$$

*for all $u$ large enough.*

In the plain nestling case, this asymptotics is very close to being optimal as can be seen by the use of so-called naïve traps (see proof of [45, Theorem 2.7] for a more restricted version of these traps and [48] also). These correspond to balls within which the local drift points in the direction of the origin, see Fig. 4 as well. Using such traps one gets the following.

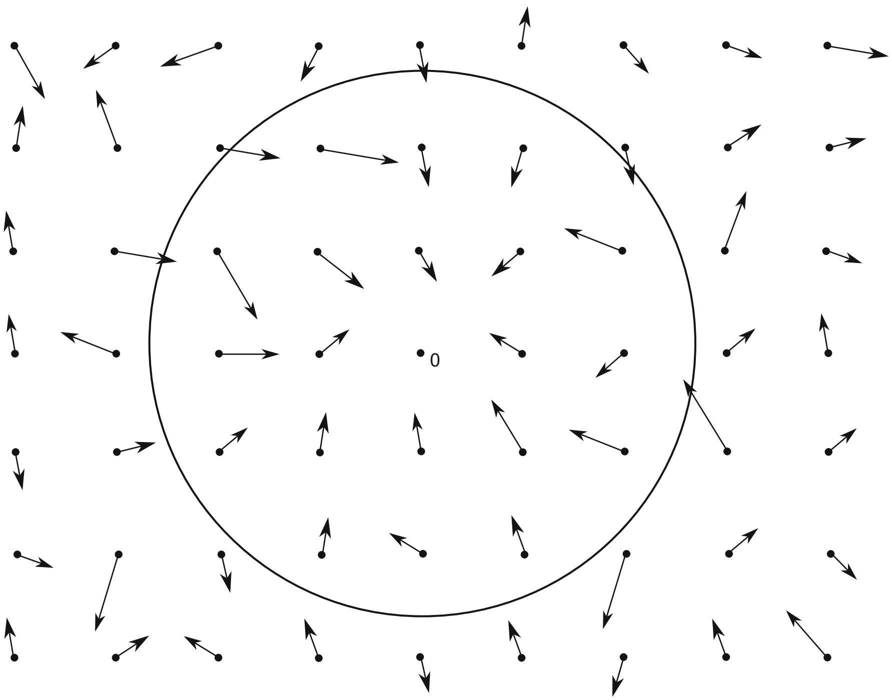


**Fig. 4** A realization of a naive trap: The local drifts within the ball point toward the origin whereas the local drifts outside the ball are arbitrary

**Theorem 16** *([45, Theorem 2.7] [48]) Assume* **(UE)**, **(IID)**, *and* $(\mathcal{P})_M|l$ *to be fulfilled for some* $l \in \mathbb{S}^{d-1}$ *and* $M > 15d + 5$. *Then, for* $d \geq 2$ *there exists a constant* $C$ *such that one has*

$$P_0[\tau_1 \geq u] \geq \exp\{-C(\log u)^d\}$$

*for all* $u$ *large enough.*

As a corollary to Theorem 15, Berger obtained the following large deviations upper bound, essentially matching Sznitman's lower bound for the nestling case in [45, Sect. 5]. In this result, we write $v$ for the $P_0$-a.s. non-zero limit of $X_n/n$, cf. Theorem 12 1.

**Theorem 17** *([3]) Let* **(UE)** *and* **(IID)** *be fulfilled. Assume furthermore that* $d \geq 4$ *and that* $(\mathcal{P})_M|l$ *is fulfilled for some* $l \in \mathbb{S}^{d-1}$ *and* $M > 15d + 5$. *Then for* $\alpha \in (0, d)$, $y \in \{tv : t \in [0, 1)\}$, *and* $\varepsilon \in (0, |y - v|_2)$, *one has*

$$P_0[|X_n/n - y|_2 < \varepsilon] < \exp\{-(\log n)^\alpha\}$$

*for all* $n$ *large enough.*

**Remark 9** *In the results of [3], one has the standing assumption that $d \geq 4$. This assumption (in combination with $(T')$) is used in order to deduce that, on $\mathbb{P}$-average, two independent random walks in the same environment do not meet too often. It is plausible that a refinement of the methods in [3] might still yield corresponding results in $d = 3$; however, it seems that for the case $d = 2$ one essentially needs some further new ideas.*

## 2.5 How to Check $(T')$ on Finite Boxes

The conditions $(T)_\gamma$ in any of the formulations of Theorem 13, as well as the condition $(\mathcal{P}^*)$, are asymptotic in nature and therefore generally not easy to check. In this context, the effective criterion introduced by Sznitman [47] proves to be a helpful tool for checking these conditions on finite boxes already. It can be seen as an analog to the ballisticity conditions of Solomon (cf. Theorem 4) in higher dimensions.[1]

In order to introduce this criterion, for positive numbers $L$, $L'$ and $\widetilde{L}$ as well as a space rotation $R$ around the origin we define the

$$\textit{box specification } \mathcal{B}(R, L, L', \widetilde{L}) \text{ as the box} B := \Big\{ x \in \mathbb{Z}^d \, : \, x \in R((-L, L') \times (-\widetilde{L}, \widetilde{L})^{d-1}) \Big\}.$$

Recalling the notation of (1), we introduce

$$\rho_{\mathcal{B}}(\omega) := \frac{P_{0,\omega}[H_{\partial B} \neq H_{\partial_+ B}]}{P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}]},$$

where for a subset $A \subset \mathbb{Z}^d$, we use the notation

$$H_A := \inf\{n \geq 0 \, : \, X_n \in A\},$$

as well as

$$\partial_+ B := \Big\{ x \in \partial B : R(e_1) \cdot x \geq L', |R(e_j) \cdot x|_2 < \widetilde{L} \; \forall j \in \{2, \ldots, d\} \Big\}.$$

We will sometimes write $\rho$ instead of $\rho_{\mathcal{B}}$ if the box we refer to is clear from the context.

**Definition 10** Given $l \in \mathbb{S}^{d-1}$, the *effective criterion with respect to* $l$ is satisfied if for some $L > c_1$ and $\widetilde{L} \in [3\sqrt{d}, L^3)$, we have that

$$\inf_{\mathcal{B},a} \left\{ c_2 \Big( \ln \frac{1}{\kappa} \Big)^{3(d-1)} \widetilde{L}^{d-1} L^{3(d-1)+1} \mathbb{E}[\rho_{\mathcal{B}}^a] \right\} < 1. \tag{75}$$

[1] Note that, while the condition $(\mathcal{P})_M$ of Definition 11 also is effective in the sense that it can be checked on finite boxes, the proof that it implies $(T')$ takes advantage of the effective criterion (cf. Definition 11 and Theorem 20)—we therefore do introduce this criterion here.

Here, when taking the infimum, $a$ runs over $[0, 1]$ while $\mathcal{B}$ runs over the

$$\text{box specifications } \mathcal{B}(R, L-2, L+2, \widetilde{L}) \text{ with } R, \text{ a rotation around the origin such that } R(e_1) = l. \tag{76}$$

Furthermore, $c_1$ and $c_2$ are dimension dependent constants.

The effective criterion is of significant importance due to the combination of the facts that it can be checked on finite boxes (in comparison to $(T')$ which is asymptotic in nature) and that it is equivalent to $(T')$, cf. Theorem 19 below.

**Theorem 18** ([47]) *Let* **(IID)** *and* **(UE)** *be fulfilled. Then for each* $l \in \mathbb{S}^{d-1}$ *the following conditions are equivalent.*

1. *The effective criterion with respect to* $l$ *is satisfied.*
2. $(T')|l$ *is satisfied.*

In the proof of Theorem 18, the estimate (75) serves as a seed estimate for an involved multi-scale renormalization scheme. We refer to the original source for the lengthy proof of this fundamental result, and to p. 239 ff. of [48] for a reasonably detailed proof sketch.

## *2.6 Interrelation of Stretched Exponential Ballisticity Conditions*

While a priori $(T)_\gamma$ is a weaker condition than smaller $\gamma$ is, Sznitman [47] showed that for each $\gamma \in (0.5, 1)$, the conditions $(T)_\gamma$ and $(T')$ are equivalent. This equivalence has been extended by Drewitz and Ramírez [15] to some dimension dependant interval $(\gamma_d, 1)$, with $\gamma_d \in (0.366, 0.388)$, for all $d \geq 2$. Furthermore, it has been conjectured (see p. 227 in [48]) that

$$\text{the conditions } (T)_\gamma|l \text{ are equivalent for all } \gamma \in (0, 1]. \tag{77}$$

**Theorem 19** ([16, 5]) *Assume* $d \geq 2$, **(UE)** *and* **(IID)** *to hold. Then, for* $l \in \mathbb{S}^{d-1}$, *the conditions* $(T)_\gamma|l$, $\gamma \in (0, 1)$, *are all equivalent.*

**Open Question 7** *It is still not known if* $(T')$ *is actually equivalent to condition* $(T)$; *however, in some sense there is not missing "too much" in some sense (see [47, Proposition 2.3]).*

According to Theorem 19, in order to check $(T')$, it is sufficient to check $(T)_\gamma$ for any $\gamma$ small enough but positive. As alluded before already, we will see in the next section that it is sufficient to establish the polynomial conditions $(\mathcal{P}^*)_M$ or $(\mathcal{P})_M$ for $M$ large enough.

## *2.7 The Condition* $(\mathcal{P})_M$

The main result of this section will be that of [5], namely that for $M$ large enough, $(\mathcal{P})_M$ already implies the conditions $(T)_\gamma$ and hence all its consequences such as ballistic behavior and an invariance principle.

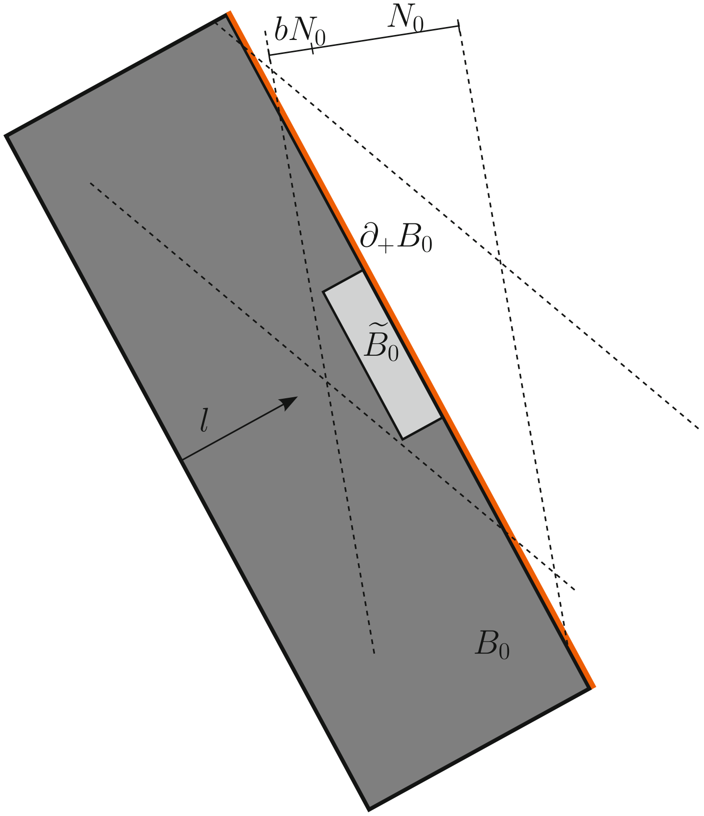


**Fig. 5** A box $B_0$ and the middle frontal part $\widetilde{B}_0$; the dashed lines illustrate the slabs from the definition of $(\mathcal{P}^*)_M|l$, shifted by some $x \in \widetilde{B}_0$; it is visually apparent here how condition $(\mathcal{P}^*)$ implies condition $(\mathcal{P})$

We will be guided by the presentation in [5], however, we will omit a significant share of the more technical parts of the proof and try to give a less rigorous and more intuitive description instead.

The main result of this section is the following.

**Theorem 20** ([5]) *Assume $d \geq 2$,* **(IID)** *and* **(UE)** *to be fulfilled. Let $l \in \mathbb{S}^{d-1}$ and assume that $(\mathcal{P}^*)_M|l$ or $(\mathcal{P})_M|l$ holds for some $M > 15d + 5$. Then $(T')|l$ holds.*

**Remark 10** *The condition $M > 15d + 5$ looks quite arbitrary, and is indeed not the weakest condition possible. However, since with the methods we used it does not seem possible to significantly weaken this condition, we refrain from trying to do so.*

We are going to introduce some of the notations needed for the proof of Theorem 20 as well as give two propositions that play a fundamental role in the proof.

Let

$$c_3 = \exp\left\{100 + 4d(\ln\kappa)^2\right\}, \tag{78}$$

let $N_0 \geq c_3$ be an even integer, and set $N_{-1} := 2N_0/3$. Using the notation

$$\pi_l : \mathbb{R}^d \ni x \mapsto (x \cdot l)\, l \in \mathbb{R}^d \tag{79}$$

to denote the orthogonal projection on the space $\{\lambda l : \lambda \in \mathbb{R}\}$, we introduce the box

$$B := \left\{ y \in \mathbb{Z}^d : -\frac{N_0}{2} < (y - x) \cdot l < N_0, |\pi_{l^\perp}(y - x)|_\infty < 25N_0^3 \right\}, \tag{80}$$

as well as their frontal parts

$$\widetilde{B} := \left\{ y \in \mathbb{Z}^d : N_0 - N_{-1} \leq (y - x) \cdot l < N_0, |\pi_{l^\perp}(y - x)|_\infty < N_0^3 \right\}. \tag{81}$$

In addition, we define

$$\partial_+ B := \{ y \in \partial B : (y - x) \cdot l \geq N_0 \}. \tag{82}$$

To simplify notation, throughout we will denote a typical box of scale $k$ by $B_k$, and its middle frontal part by $\widetilde{B}_k$.

**Definition 11** Let $l \in \mathbb{S}^{d-1}$ and $M > 0$. We say that $(\mathcal{P})_M | l$ is fulfilled if

$$\sup_{x \in \widetilde{B}_0} P_x[H_{\partial B_0} \neq H_{\partial_+ B_0}] < N_0^{-M} \tag{83}$$

holds for some $N_0 \geq c_3$.

## 2.8 An Intermediate Condition Between $(\mathcal{P})_M$ and $(T)_\gamma$

We need a little further notation for stating this result in particular. To start with, for a given generic $l = l_1 \in \mathbb{S}^{d-1}$, we choose $l_2, \ldots, l_d$ arbitrarily in such a way that $l_1, \ldots, l_d$ forms an orthonormal basis of $\mathbb{R}^d$.

For $L > 0$, define

$$\mathcal{D}_L^l := \left\{ x \in \mathbb{Z}^d : -L \leq x \cdot l \leq 10L,\ |x \cdot l_k| \leq \frac{L^3 \ln \ln L}{\ln L}\ \forall k \in \{2, \ldots, d\} \right\}$$

as well as its *frontal boundary part*

$$\partial_+ \mathcal{D}_L^l := \left\{ x \in \partial \mathcal{D}_L^l : \pi_l(x) \cdot l > 10L,\ |x \cdot l_k| \leq \frac{L^3 \ln \ln L}{\ln L}\ \forall k \in \{2, \ldots, d\} \right\}.$$

In the following we will refer to the condition that

$$\text{for } l' \in \mathbb{S}^{d-1} \text{one has} P_0\big[H_{\partial \mathcal{D}_L^l} < H_{\partial_+ \mathcal{D}_L^l}\big] \leq \exp\left\{ -L^{\frac{(1+o(1))\ln 2}{\ln \ln L}} \right\}, \tag{84}$$

as $L \to \infty$.

**Definition 12** If (84) holds for all $l'$ in a neighborhood of $l \in \mathbb{S}^{d-1}$, then we say that condition $(T)_{\gamma_L}|l$ is fulfilled.

Since $\gamma_L$ tends to 0 as $L$ tends to infinity, one observes that the condition $(T)_{\gamma_L}$ is weaker than $(T)_\gamma$ for any $\gamma > 0$.

On the other hand, while the condition $(T)_{\gamma_L}$ is a priori stronger than all of the polynomial conditions $(\mathcal{P}^*)_M$, $M > 0$, it can be shown that it is a consequence of $(\mathcal{P}^*)_M$ once $M$ is chosen large enough. This is the content of Proposition 6 below.

## 2.9 Strategy of the Proof of Theorem 20

Using Theorem 18, we observe that in order to prove Theorem 21, it is sufficient to establish the effective criterion departing from $(\mathcal{P})_M|l$ with $M$ large enough. On a heuristic level, we will do so via two renormalization schemes:

1. The first one starts with assuming condition $(\mathcal{P})_M|l$ for some $M$ large enough and derives the intermediate condition $(T)_{\gamma_L}$ introduced in Definition 12.

   **Proposition 6** (Sharpened averaged exit estimates) *Assume* **(IID)** *and* **(UE)** *to be fulfilled. Let* $M > 15d + 5$, $l \in \mathbb{S}^{d-1}$, *and assume that condition* $(\mathcal{P})_M|l$ *is satisfied. Then* $(T)_{\gamma_L}|l$ *holds.*
   We will not give the technically involved proof of this result and refer to the original source [5] instead.
2. The second renormalization step supplies us with the following large deviations result.

   **Proposition 7** (Weak atypical quenched exit estimates, [5]) *Let* $d \geq 2$ *and assume* **(IID)** *and* **(UE)** *to be fulfilled and let* $(T)_{\gamma_L}|l$ *hold. Then for* $\epsilon(L) := \frac{1}{(\ln \ln L)^2}$, *and any function* $\beta : (0, \infty) \to (0, \infty)$, *one has that*

   $$\mathbb{P}\big[P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}] \leq \frac{1}{2} \exp\{-c_1 L^{\beta(L)}\}\big] \leq 5^d \frac{e}{\lceil L^{\beta(L)-\epsilon(L)}/5^d \rceil !}, \qquad (85)$$

   *where* $B$ *is a box specification as in* 76 *with* $\widetilde{L} = L^3 - 1$, *and*

   $$c_4 := -2d \ln \kappa > 1. \qquad (86)$$

   This result is much less technical to prove, but nevertheless we refer to [5] for its proof in order not to lose the principal thread of these notes.

We do, however, mention that in dimensions $d \geq 4$, Proposition 7 can be strengthened significantly as follows:

**Theorem 21** (Atypical quenched exit estimates, [16]) *Let* $d \geq 4$, *and assume* **(IID)**, **(UE)**, *and* $(T)_\gamma|l$ *to hold for some* $\gamma \in (0,1)$, $l \in \mathbb{S}^{d-1}$. *Fix* $c > 0$ *and* $\beta \in (0,1)$. *Then there exists a constant* $C > 0$ *such that for all* $\alpha \in (0, \beta d)$,

$$\limsup_{L\to\infty} L^{-\alpha} \log \mathbb{P}\Big[P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}] \leq e^{-cL^\beta}\Big] < 0,$$

*where $B$ is a box specification as in (76) with $\widetilde{L} = CL$.*

The proof of this result is significantly more involved than that of Proposition 7. Note that this theorem is very close to being optimal in the sense that its conclusion will not hold in general for $\alpha > \beta d$. In fact, for plain nestling RWRE, this can be shown by the use of naïve traps introduced above.

For the purpose of proving Theorem 20, however, Proposition 7 is sufficient.

## 2.10 Proof of Theorem 20 assuming Propositions 6 and 7

In this section we demonstrate how Propositions 6 and 7 can be employed in order to establish the effective criterion. We will do so by rewriting $\mathbb{E}[\rho_{\mathcal{B}}^a]$ of (75) as a sum of terms typically of the form

$$\mathcal{E}_j := \mathbb{E}\left[\rho_{\mathcal{B}}^a, \frac{1}{2}\exp\{-c_4 L^{\beta_{j+1}}\} < P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}] \le \frac{1}{2}\exp\{-c_4 L^{\beta_j}\}\right] \quad (87)$$

with $\beta_{j+1} > \beta_j$.

Generally, the lower bound on $P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}]$ in (87) yields a control on the integrand $\rho_{\mathcal{B}}^a$ from above, while the upper bound enforces an atypical behavior which will be exploited using Proposition 7. The interplay of the upper bound of the integrand thus obtained with the estimate from Proposition 7 will then determine the asymptotics we obtain for $\mathcal{E}_j$ (cf. also Lemma 8 below).

Our proof of Theorem 20 goes along establishing the effective criterion. We do so by a subtle decomposition of the expectation occurring in (75) into several summands, and in the following we will give some basic lemmas that will prove useful in estimating each of these summands.

For that purpose, we define the quantities

$$\beta_1(L) := \frac{\gamma_L}{2} = \frac{\ln 2}{2\ln\ln L}, \quad (88)$$

$$a := L^{-\gamma_L/3}, \quad (89)$$

and write $\rho$ for $\rho_{\mathcal{B}}$ with some arbitrary box specification of (76) with $\widetilde{L} = L^3 - 1$. We split $\mathbb{E}[\rho^a]$ according to

$$E[\rho^a] = \mathcal{E}_0 + \sum_{j=1}^{n-1} \mathcal{E}_j + \mathcal{E}_n, \quad (90)$$

where

$$n := n(L) := \left\lceil \frac{4(1-\gamma_L/2)}{\gamma_L} \right\rceil + 1,$$

$$\mathcal{E}_0 := \mathbb{E}\left[\rho^a, P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}] > \frac{1}{2}\exp\{-c_4 L^{\beta_1}\}\right],$$

$$\mathcal{E}_j := \mathbb{E}\left[\rho^a, \frac{1}{2}\exp\{-c_4 L^{\beta_{j+1}}\} < P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}] \leq \frac{1}{2}\exp\{-c_4 L^{\beta_j}\}\right]$$

for $j \in \{1, \dots, n-1\}$, and

$$\mathcal{E}_n := \mathbb{E}\left[\rho^a, P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}] \leq \frac{1}{2}\exp\{-c_4 L^{\beta_n}\}\right],$$

with parameters

$$\beta_j(L) := \beta_1(L) + (j-1)\frac{\gamma_L}{4}, \tag{91}$$

for $2 \leq j \leq n(L)$; for the sake of brevity we may sometimes omit the dependence on $L$ of the parameters if that does not cause any confusion. Furthermore, in order to verify equality (90), note that due to the uniform ellipticity assumption **(UE)** and the choice of $c_4$ (cf. (86)), one has for $\mathbb{P}$-a.a. $\omega$ that

$$P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}] > e^{-c_4 L},$$

as well as that

$$\beta_n > 1.$$

To bound $\mathcal{E}_0$ we employ the following lemma.

**Lemma 7** *Let $(T)_{\gamma_L}$ be fulfilled. Then*

$$\mathcal{E}_0 \leq \exp\{c_4 L^{\gamma_L/6} - L^{(1+o(1))\gamma_L/2}\},$$

*as $L \to \infty$.*

*Proof* Jensen's inequality yields

$$\mathcal{E}_0 \leq 2\exp\{c_4 L^{\beta_1 - \gamma_L/3}\} P_0[H_{\partial B} \neq H_{\partial_+ B}]^a.$$

Using (88) in combination with $(T)_{\gamma_L}$ we obtain the desired result.

To deal with the middle summand in the right-hand side of (90), we use the following lemma.

**Lemma 8** *Let* **(IID)***and* **(UE)** *be fulfilled and assume $(T)_{\gamma_L}|l$ to hold. Then for all $L$ large enough we have uniformly in $j \in \{1, \dots, n-1\}$ that*

$$\mathcal{E}_j \leq 2 \cdot 5^d \exp\{c_4 L^{\beta_{j+1} - \gamma_L/3}\} \frac{e}{\lceil L^{\beta_j - \epsilon(L)}/5^d \rceil !}.$$

*proof* Using Markov's inequality, for $j \in \{1, \dots, n-1\}$ we obtain the estimate

$$\mathcal{E}_j \leq 2\exp\{c_4 L^{\beta_{j+1} - \gamma_L/3}\} \mathbb{P}\left[P_{0,\omega}[H_{\partial B} = H_{\partial_+ B}] \leq \frac{1}{2}\exp\{-c_4 L^{\beta_j}\}\right]. \tag{92}$$

Thus, due to Proposition 4, the probability on the right-hand side of (92) can be estimated from above by

$$5^d \frac{e}{\lceil L^{\beta_j - \epsilon(L)}/5^d \rceil !}.$$

When it comes to the term $\mathcal{E}_n$ in (90) we note that it vanishes because of the choice of $c_4$.

*Proof of Theorem 20* It follows from Lemmas 7 and 8, the choice of parameters in (88), (89) and (91), and the fact that $\mathcal{E}_n$ vanishes, that for $L$ large enough, (90) can be bounded from above by

$$\exp\left\{c_4 L^{\gamma_L/6} - L^{(1+o(1))\gamma_L/2}\right\}$$
$$+2 \cdot 5^d n(L) \max_{1 \le j \le n(L)-1} \left( \exp\left\{c_4 L^{\beta_{j+1} - \gamma_L/3}\right\} \frac{e}{\lceil L^{\beta_j - \epsilon(L)}/5^d \rceil !} \right).$$

Thus, we see that for our choice of parameters, (90) tends to zero faster than any polynomial in $L$. Hence, due to (75), the effective criterion holds and Theorem 18 then yields the desired result.

## 2.11 Relation Between Directional Transience and Slab Exit Estimates

The aim of this subsection is to show how the condition of directional transience relates to slab exit estimates such as condition $(\mathcal{P}^*)_M$.

**Lemma 9** *Let $l \in \mathbb{S}^{d-1}$, and suppose that $(\mathcal{P}^*)_M | l$ is satisfied.*

1. *There exists a constant $C$ such that*
$$P_0\left[T^{-l}_{-L} \circ \theta_{T^l_{2L}} \le T^l_{4L} \circ \theta_{T^l_{2L}}\right] \le C L^{d-1-M} \tag{93}$$
*for all $L \in \mathbb{N}$.*
2. *If $M > d$, then $P_0$-a.s. the random walk $X$ is transient in direction $l$, i.e. $P_0[A_l] = 1$.*

*Proof*

1. The general idea of this proof is taken from a stretched exponential analog [47, Theorem 2.11]. Note that
$$\left\{T^l_L < T^{-l}_L\right\} \subset \left\{\limsup_{n\to\infty} X_n \cdot l \ge L\right\}.$$
Due to condition $(\mathcal{P}^*)_M | l$, the probability with respect to $P_0$ of the left-hand side tends to 1 as $L \to \infty$, which implies
$$P_0\left[\limsup_{n\to\infty} X_n \cdot l = \infty\right] = 1. \tag{94}$$

Now choose $l_1, \ldots, l_d \in \mathbb{S}^{d-1} \cap V_l$ (with $V_l$ denoting the neighborhood associated to $l$ in the definition of $(\mathcal{P}^*)_M|l$, see Definition 9) to be linearly independent. If furthermore $l_1, \ldots, l_d$ are chosen sufficiently close to $l$, setting $l_0 := l$ there exists $\delta > 0$ such that for

$$\Delta_L := \{x \in \mathbb{Z}^d : -\delta L \le x \cdot l_j \le L \ \forall 0 \le j \le d\} \tag{95}$$

and

$$\partial_+ \Delta_L := \big\{x \in \partial \Delta_L : \max_{0 \le j \le d} x \cdot l_j > L \text{ and } \min_{0 \le j \le d} x \cdot l_j \ge -\delta L\big\},$$

we have

$$2\delta L \le \min\{x \cdot l : x \in \partial_+ \Delta_\mathrm{L}\}. \tag{96}$$

Now due to 94, we infer that $T_L^l$ is finite $P_0$-a.s. and hence $\theta_{T_L^l}$ is well-defined for all $L > 0$. Thus, we get using the strong Markov property at time $T_{2\delta L}^l$ (applied to the quenched walk) in combination with the translation invariance of $\mathbb{P}$ and (96), that

$$\begin{aligned}
&P_0\big[T_{-\delta L}^{-l} \circ \theta_{T_{2\delta L}^l} \le T_{4\delta L}^l \circ \theta_{T_{2\delta L}^l}\big] \\
&\le P_0\big[T_{\partial \Delta_L} < T_{2\delta L}^l\big] + P_0\big[T_{-\delta L}^{-l} \circ \theta_{T_{2\delta L}^l} \le T_{4\delta L}^l \circ \theta_{T_{2\delta L}^l}, T_{\partial \Delta_L} \ge T_{2\delta L}^l\big] \\
&\le \sum_{j=0}^d P_0\big[T_{\delta L}^{-l_j} < T_L^{l_j}\big] + C L^{d-1} P_0\big[T_{\delta L}^{-l} \le T_{2\delta L}^l\big].
\end{aligned} \tag{97}$$

To obtain the last line we used the fact that, since $l_1, \ldots, l_d$ form a basis, $|\{x \in \Delta_L : x \cdot l \in (2\delta L, 2\delta L + 1]\}| \le C L^{d-1}$ holds. Since, furthermore $(\mathcal{P}^*)_M|l$ is fulfilled we can estimate (97) from above by $C L^{d-1-M}$ which proves the first assertion of the lemma.

2. Using this result in combination with the assumption that $M > d$, Borel–Cantelli's lemma yields that $P_0$-a.s., for eventually all $L \in \mathbb{N}$,

$$T_{4L}^l \circ \theta_{T_{2L}^l} < T_{-L}^{-l} \circ \theta_{T_{2L}^l}.$$

This implies that $P_0[\lim_{n\to\infty} X_n \cdot l = \infty] = 1$. □

We have the following corollary on the relation between transience and the conditions $(\mathcal{P}^*)_M$.

**Corollary 5** *The implications*

$$(\mathcal{P}^*)_M|l \text{ for some } M > d \Longrightarrow P_0[A_{l'}] = 1\ \forall l' \text{ in a neighborhood } V_l \text{ of } l \Longrightarrow (\mathcal{P}^*)_0|l \tag{98}$$

*hold true.*

*Proof* T he first implication is a direct consequence of Lemma 9. To obtain the second implication note that if $P_0[A_{l'}] = 1$ for all $l' \in V_l$, then we have

$$P_0[H_{bL}^{-l'} < H_L^{l'}] \le P_0[A_{l'}, H_{bL}^{-l'} < \infty] \to 0, \quad \text{as } L \to \infty,$$

where we used that $P_0[\,\cdot \mid A_{l'}]$-a.s. one has $\inf_{n\in\mathbb{N}} X_n \cdot l' \in (-\infty, 0]$.

**Remark 11** *The above corollary immediately leads to two questions:*

1. *Which is the minimal $M$ for which the first implication holds?*
2. *Can $(\mathcal{P}^*)_0$ on the right-hand side of the implications be replaced by $(\mathcal{P}^*)_M$ for some $M > 0$, and if so, what is the maximal $M$?*

*These questions are intimately connected to Open Question 2.*

## *2.12 Ellipticity Conditions for Ballistic Behavior*

We have seen in Chap. 2 that there can exist elliptic random walks which are transient in a given direction but which are not ballistic. On the other hand, Proposition 2.17 of this chapter shows that at least some condition on the moments of the jump probabilities of the random environment should be asked if we expect to extend the results of this chapter.

**Definition 13** Consider an RW in an environment $\mathbb{P}$. We say that $\mathbb{P}$ satisfies the ellipticity condition $(E)_\beta$ if there exist positive parameters $\{\beta_e : e \in U\}$ such that

$$2\sum_{e\in U}\beta_e - \sup_{e'}(\beta_{e'} + \beta_{-e'}) > \beta$$

and

$$\mathbb{E}\left[e^{\sum_e \beta_e \log \frac{1}{\omega(0,e)}}\right] < \infty.$$

If in addition there exists a $\bar{\beta}$ such that $\beta_e = \bar{\beta}$ for $e$ such that $e \cdot \hat{v} \geq 0$ (recall that $\hat{v}$ was the asymptotic direction) while $\beta_e \leq \bar{\beta}$ for $e$ such that $e \cdot \hat{v} < 0$, we say that condition $(E)_\beta$ is satisfied directionally. Furthermore, whenever there exists an $\alpha > 0$ such that

$$\sup_e \mathbb{E}\left[\frac{1}{\omega(0,e)^\alpha}\right] < \infty$$

we say that the law $\mathbb{P}$ of the environment satisfies condition $(E')_\alpha$.

We have the following extension of Theorem 20 proved in [8].

**Theorem 22** **(Campos–Ramírez)** *Consider a random walk in an i.i.d. environment which satisfies condition $(E')_\alpha$ for some $\alpha > 0$. Then, if $(\mathcal{P}^*)_M|l$ is satisfied for some $M \geq 15d + 5$, $(T')|l$ is satisfied.*

Furthermore, we have then the following consequence of Theorem 22 proved in [8].

**Theorem 23** **(Campos-Ramírez)** *Consider a random walk in an i.i.d. environment which satisfies condition $(E)_1$ directionally. Then, if $(\mathcal{P}^*)_M|l$ is satisfied for $M \geq 15d + 5$, the walk is ballistic.*

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# The Scaling Limit for Zero-Temperature Planar Ising Droplets: With and Without Magnetic Fields

**Hubert Lacoin**

**Abstract** We consider the continuous time, zero-temperature heat-bath dynamics for the nearest-neighbor Ising model on $\mathbb{Z}^2$ with positive magnetic field. For a system of size $L \in \mathbb{N}$, we start with initial condition $\sigma$ such that $\sigma_x = -1$ if $x \in [-L, L]^2$ and $\sigma_x = +1$ and investigate the scaling limit of the set of – spins when both time and space are rescaled by $L$. We compare the obtained result and its proof with the case of zero-magnetic fields, for which a scaling result was proved by Lacoin et al. (J Eur Math Soc, in press). In that case, the time-scaling is diffusive and the scaling limit is given by anisotropic motion by curvature.

## 1 Introduction

The Ising model is one of the simplest models proposed by statistical mechanics to investigate ferromagnetic properties of metals. It is based on the following simplification of reality: we assume that a ferromagnetic material is composed of microscopic magnets that live on a lattice and can assume only two orientations, up (or $+$) and down (or $-$) called spins; the way the orientations of the micromagnets are determined follows two rules: neighboring magnets like to have the same orientation, and all spins like to align with the external magnetic field if there is one.

The Ising model is a probabilistic model (i.e., a probability law on the set of possible spin orientation) following these rules and is defined using a Boltzmann–Gibbs formalism, which is detailed in the next section. It describes the equilibrium state of a ferromagnet.

The stochastic Ising model, also called Glauber dynamics or heat-bath dynamics for Ising model, is a Markov chain on the set of spin configurations that describes the evolution of a magnet out of equilibrium (e.g., after a brutal change of temperature or of external magnetic fields). This is the object we study in this paper. Our aim is to understand the time needed and the pattern used for a magnetic material to reach its equilibrium state after a change of external conditions. In order to bring a more complete answer to these questions, we consider them in a simplified, but still

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nontrivial setup that is the zero-temperature limit. This brings heuristic intuition for what should happen in the whole low temperature regime.

## 1.1 The Glauber Dynamics for the Ising Model

Set

$$\mathbb{Z}^* := \mathbb{Z} + \frac{1}{2} := \{x + (1/2) \mid x \in \mathbb{Z}\}.$$

Consider the square

$$\Lambda = \Lambda_L := [-L, L]^2 \cap (\mathbb{Z}^*)^2.$$

We define the set of spin configurations on $\Lambda_L$ to be $\{-1, 1\}^{\Lambda_L}$. A generic spin configuration is denoted by $\sigma = (\sigma_x)_{x \in \Lambda_L}$, and $\sigma_x \in \{-1, 1\}$ is called the spin at site $x$. Define the external boundary of $\Lambda$ as

$$\partial\Lambda := \{y \in \mathbb{Z}^d \setminus \Lambda \mid \exists x \in \Lambda, x \sim y\}. \tag{1}$$

The Ising model at inverse temperature $\beta$, with external magnetic fields $h$ and boundary condition $\eta \in \{-1, 1\}^{\partial\Lambda_L}$, is a measure on the set of spin configurations defined by

$$\mu_L^{\beta,h,\eta}(\sigma) := \frac{1}{Z_\Lambda^{h.\beta}} \exp\left(\beta \sum_{\substack{\{x,y\} \subset (\Lambda \cup \partial\Lambda) \\ x \sim y}} \sigma_x \sigma_y + h \sum_{x \in \Lambda} \sigma_x\right), \tag{2}$$

where

$$Z_\Lambda^{h,\beta} := \sum_{\sigma \in \{-1,1\}^\Lambda} \exp\left(\beta \sum_{\substack{\{x,y\} \subset (\Lambda \cup \partial\Lambda) \\ x \sim y}} \sigma_x \sigma_y + h \sum_{x \in \Lambda} \sigma_x\right), \tag{3}$$

and the convention is taken that $\sigma_x := \eta_x$ when $x \notin \Lambda$. Note that the first term in the exponential makes the spins of neighboring sites more likely to agree, whereas the second term underlines that configuration with spins aligned with the magnetic fields are favored.

The heat-bath dynamic for the Ising model (at inverse temperature $\beta$, with external magnetic fields $h$ and boundary condition $\eta$) is a Markov chain on the set of spin configurations $\{-1, 1\}^\Lambda$. We denote the trajectory of the Markov chain by $\sigma(t) = (\sigma_x(t))_{x \in \Lambda}$. One starts from a given configuration $\sigma_0$ and the rules for the evolution are the following:

(i) Sites $x \in \Lambda$ are equipped with independent, rate one, Poisson processes: $(\tau_n^x)_{n \geq 0}$ where $\tau_0^x = 0$, i.e., the increments $(\tau_{n,x} - \tau_{n-1,x})_{n \geq 0, x \in \mathbb{Z}^d}$ are IID exponential variables.

(ii) The spin at site $x$ may change its value only when the clock at $x$ rings, i.e., at time $\tau_n^x$, $n \geq 1$, and its value stays constant on the intervals $[\tau_n^x, \tau_{n+1}^x)$, $n \geq 0$.

(iii) When a clock rings, the spin at $x$ is updated according to the following law independently of the past history of the process:
$-\sigma_x(t_0) = +$ with probability

$$\frac{\exp\left(\beta \sum_{y\sim x} \sigma_y(\tau_{n,x}^-) + h\right)}{2\cosh\left(\beta \sum_{y\sim x} \sigma_y(\tau_{n,x}^-) + h\right)}, \tag{4}$$

$-\sigma_x(t_0) = -$ with probability

$$\frac{\exp\left(-\beta \sum_{y\sim x} \sigma_y(\tau_{n,x}^-) + h\right)}{2\cosh\left(\beta \sum_{y\sim x} \sigma_y(\tau_{n,x}^-) + h\right)}, \tag{5}$$

When $x$ has a neighbor $y$ in $\partial\Lambda$, the value $\sigma_y$ appearing in the Eqs. 4 and 5 is fixed (by convention) to be equal to $\eta_y$ for all time, where $\eta$ is the boundary condition. The update of the spin corresponds to sampling a spin configuration according to the measure $\mu_\Lambda^{\beta,h,\eta}(\cdot\,|\,\sigma_y = \sigma_y(\tau_{n,x}^-), \forall y \neq x)$.

As a consequence of this last remark, $\mu_\Lambda^{\beta,h,\eta}$ is the unique invariant measure for the dynamics, so that the law of $\sigma(t)$ converges to the equilibrium measure $\mu_\Lambda^{\beta,h,\eta}$.

The main questions in the study of dynamics are how much time is needed to reach equilibrium and what is the pattern used by the chain to reach it when we consider the dynamics on a very large domain $\Lambda$. The answer to this question should of course depend on the temperature $\beta$, the magnetic fields $h$, the boundary condition $\eta$, and the initial condition $\sigma_0$.

In what follows, we denote often by $+$ resp. $-$ the spin configuration or boundary condition where all spins are $+$ resp. $-$.

## 1.2 Conjecture and Known Results in the Case $\beta \in (0,\infty)$, $h = 0$

Let us review shortly what is known and conjectured for these kind of dynamics when $\beta \in (0,\infty)$ and $h=0$. The property of the dynamics depends crucially on the equilibrium property of the system. Recall that the two-dimensional Ising model undertakes a phase transition at $\beta_c = \log(1+\sqrt{2})/2$ (see the seminal work of Onsager [23]), which has the following form:

- When $\beta < \beta_c$, the correlation between two spins decays exponentially fast with the distance, and for this reason what happens in the center of $\Lambda_L$ becomes independent of the boundary condition when $L$ tends to infinity. This is called the high temperature phase.
- When $\beta > \beta_c$, on the contrary, long-range correlations are present between spins, and the boundary condition plays a crucial role. In particular, the measure $\mu_L^{\beta,h,+}$ and $\mu_L^{\beta,h,-}$ corresponding to $+$ and $-$ boundary conditio n are very different. This is called the low temperature phase.

In the high temperature phase, the rapid decay of correlation between distant sites makes the evolution of the system in two distant zones of the box $\Lambda_L$ almost independent. Schematically, the box of $\Lambda_L$ can be separated in $O(L^2)$ zones of finite size that come to equilibrium independently. This requires a time of order $\log L$. This prediction has been made rigorous by Lubetzky and Sly, who proved that in that case, the mixing time (i.e., the time to reach equilibrium) for the dynamic in $\Lambda_L$ is equal to $\lambda_\infty \log L(1 + o(1))$, where $\lambda_\infty$ is the relaxation time for the infinite volume dynamics (see [19]). For a formal definition of mixing time and relaxation time and an introduction to the modern theory of Markov chain, we refer to [17].

In the low temperature phase, the behavior of the dynamics depends on the boundary condition, and for the sake of simplicity, we restrict to the case of + boundary condition. In that case, the equilibrium state is biased towards +, and even spins in the center have a larger probability to be + than −. In a sense, one can say that, at equilibrium, the center of the box "knows" what the boundary condition is. Thus, if one starts, e.g., from full − initial condition ($\sigma_x(0) = -1$ for all $x \in \Lambda$), information must travel from the boundary to the center of the box in order to reach equilibrium. For this reason, the mixing time is much longer than in the high temperature phase.

In [18], Lifshitz described a conjectural pattern used by the system with + boundary condition to reach equilibrium that can be described as follows: starting from − initial condition, the system should rapidly reach a state of local equilibrium that looks like the equilibrium measure with − boundary condition (we call this the − phase); then on the time-scale $L^2$, something looking like the true equilibrium measure with + boundary condition, the + phase, should start to appear in the neighborhood of the cubes boundary. The interface between the + and the − phase should move on the diffusive time-scale $L^2$, having a drift in time proportional to its local curvature. As a consequence, the system should reach equilibrium when the bubble formed by the − phase disappears macroscopically, i.e., in a time $O(L^2)$.

For finite $\beta > \beta_c$, this conjecture is far from being on rigorous mathematical ground, but Lifshitz ideas have been used to get bounds on the mixing time. The best to date being by Lubetzky et al. [22] saying that the system reaches equilibrium in a time $L^{\log L}$, still far from the conjecture $L^2$. This gap between the Lifshitz conjecture and the rigorous mathematical result has been one of the incentives to study the simpler zero-temperature version of the model.

Dynamics with different boundary condition or nonzero magnetic fields at low temperature also exhibit interesting behavior like low-temperature-induced metastability (see, e.g., [27]), which we choose not to expose here. Note also that results are available at the critical temperature $\beta_c$ [20], where the equilibrium state of the system is somehow harder to describe.

## 2 Zero-Temperature Dynamics

Given $h \geq 0$, we look at the limiting dynamics when $\beta$ tends to infinity. We call this the zero-temperature limit (recall that $\beta$ is the inverse of the temperature). In what follows, we will consider only + boundary condition: $\eta_x = +1, \forall x \in \partial\Lambda$.

With this setup, on a finite box with + boundary condition, the limit of the Ising measure $\lim_{\beta\to\infty}\mu_L^{\beta,h,+}$ is just the Dirac measure on the full + configuration: $\sigma_x = +1$, $\forall x \in \Lambda_L$. The limiting dynamics is a nondegenerate stochastic process that can be described as follows: the value the spin at $x$ is updated with rate one as before; when a spin is updated, it takes the value of the majority of its neighbors if it is well defined and take value $\pm 1$ with probability $e^{\pm h}/(2\cosh(h))$ if it has the same number of + and − neighbors. The process can also be defined when $h = \infty$; it corresponds to the case where both $\beta$ and $h$ tend to infinity with $h \ll \beta$.

The question concerning the pattern used to reach the equilibrium then takes the following form: starting from a finite domain of $\mathbb{Z}^2$ filled with − spins what is the time needed to reach the whole + configuration and what is the pattern used to reach it.

More precisely: We consider a compact, simply connected subset $\mathcal{D} \subset [-1,1]^2$ whose boundary is a closed smooth curve. Given $L \in \mathbb{N}$, we consider the Markov chain described above with initial condition

$$\sigma_x(0) = \begin{cases} -1 & \text{if } x \in (\mathbb{Z}^*)^2 \cap L\mathcal{D}, \\ +1 & \text{otherwise.} \end{cases} \tag{6}$$

In order to see a set of "−" spins as a subset of $\mathbb{R}^2$, each vertex $x \in (\mathbb{Z}^*)^2$ may be identified with the closed square of side-length one centered at $x$,

$$\mathcal{C}_x := x + [-1/2, 1/2]^2. \tag{7}$$

One defines

$$\mathcal{A}_L(t) := \bigcup_{\{x:\sigma_x(t)=-1\}} \mathcal{C}_x, \tag{8}$$

which is the "− droplet" at time $t$ for the dynamics. Note that the boundary of $\mathcal{A}_L(t)$ is a union of edges of $\mathbb{Z}^2$ (and this is the reason why we defined the Ising model on $(\mathbb{Z}^*)^2$ rather than on $\mathbb{Z}^2$).

We investigate the scaling limit of $\mathcal{A}_L(t)$ and the time needed for it to vanish. The nature of the scaling limit depends really much on the value of the external magnetic fields $h$:

- If $h > 0$, the interface between + and − is always pushed towards the minus region, at linear speed, and thus macroscopic motion is visible on time-scale $L$.
- If $h = 0$, there is no sign that is favored, and the interface will be pushed in the direction of its curvature to reduce its length. This is visible only on the diffusive time scale.

The main aims of this paper are: to review in detail the recent proof (with coauthors [15]) that the scaling limit of $\mathcal{A}_L(t)$ is given by an anisotropic curve shortening flow, and to give a description of the scaling limit in the case $h > 0$ (for the sake of simplicity we will limit our proof to $h = \infty$).

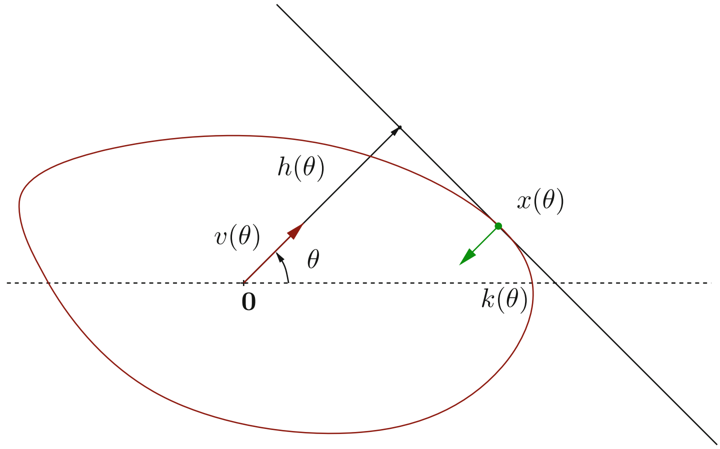


**Fig. 1** A graphic description of the support function $h$. Given $\theta$, consider the point $x(\theta)$ of $\gamma$ that maximizes $x \cdot v(\theta)$ (it is unique if the curve is strictly convex). Then $h(\theta) = x(\theta) \cdot v(\theta)$, and $k(\theta)$ is the norm of the curvature vector of $\gamma$ (*green vector*) at $x(\theta)$. If the tangent to $\gamma$ at $x$ exists, it is normal to $v(\theta)$ and $|h(\theta)|$ is the distance between the tangent and the origin

## 2.1 The Case of Zero Magnetic Fields

Let us focus first on the case of zero-magnetic fields, and make more precise the Lifshitz conjecture [18] concerning low temperature dynamics in this case.

On heuristic grounds, Lifshitz predicted that the boundary of $\mathcal{A}_L(t)$ should follow an anisotropic curve-shortening motion: after rescaling space by $L$ and time by $L^2$ and letting $L$ tend to infinity, the motion of the interface between $\mathcal{A}_L(t)$ and its complement (i.e., between $+$ and $-$ spins) should be deterministic and the local drift of the interface should be proportional to the curvature. An anisotropic factor should appear in front of the curvature to reflect anisotropy of the lattice. Spohn [28] made this conjecture more precise and brought some elements for its proof: Let $\gamma(t, L)$ denote the boundary of the (random) set $(1/L)\mathcal{A}_L(L^2 t)$. Then, for $L \to \infty$, the flow of curve $(\gamma(t, L))_{t \geq 0}$ should converge to a deterministic flow $(\gamma(t))_{t\geq 0}$ and the motion of the limiting curve should be such that the normal velocity vector at a point $x \in \gamma(t)$ is given by the curvature at $x$ multiplied an anisotropic factor $a(\theta_x)$, where $\theta_x$ is the angle made by the outward directed normal to $\gamma(t)$ at $x$ together with the horizontal axis (see Fig. 1). The velocity vector points in the direction of concavity. The function $a(\,\cdot\,)$ has the explicit expression

$$a(\theta) := \frac{1}{2(|\cos(\theta)| + |\sin(\theta)|)^2}. \tag{9}$$

The area enclosed by the curve decreases with a constant speed: $-\int_0^{2\pi} a(\theta)\mathrm{d}\theta$. In particular, the curve $\gamma(t)$ shrinks to a point in a finite time

$$t_0 = \frac{Area(\mathcal{D})}{\int_0^{2\pi} a(\theta)\mathrm{d}\theta} = \frac{Area(\mathcal{D})}{2}.$$

Note that the function $a(\cdot)$ is symmetric around 0 and is periodic with period $\pi/2$. This is inherited from the discrete symmetries of the lattice $(\mathbb{Z}^*)^2$.

In a recent work [15] with coauthors, this conjecture was brought on rigorous ground in the case when the initial condition $\mathcal{D}$ is convex. Let us first give a rigorous definition of motion by curvature and a result concerning its existence: given a convex, smooth, closed curve $\gamma = \partial\mathcal{D}$ in $\mathbb{R}^2$, we parameterize it following a standard convention of convex geometry (cf. e.g., [8] and Fig. 1). For $\theta \in [0, 2\pi]$, let $v(\theta)$ be the unit vector forming an anticlockwise angle $\theta$ with the horizontal axis and let

$$h(\theta) = \sup\{x \cdot v(\theta), x \in \gamma\}, \tag{10}$$

where $\gamma$ denotes the usual scalar product in $\mathbb{R}^2$. The function $\theta \mapsto h(\theta)$ (called "the support function") uniquely determines $\gamma$:

$$\mathcal{D} = \bigcap_{0\le\theta\le 2\pi} \{x \in \mathbb{R}^2 : x \cdot v(\theta) \le h(\theta)\}. \tag{11}$$

With this parameterization, the anisotropic curve shortening evolution reads

$$\begin{cases} \partial_t h(\theta, t) = -a(\theta)k(\theta, t), \\ h(\theta, 0) = h(\theta), \end{cases} \tag{12}$$

where, for a convex curve $\gamma$, $k(\theta) \ge 0$ is the curvature at the point $x(\theta) \in \gamma$ where the outward normal vector makes an angle $\theta$ with the horizontal axis and the time derivative is taken at constant $\theta$ (see [[8], Lemma 2.1] for a proof that Eq. 12 is equivalent to the standard definition of anisotropic motion by curvature).

The existence of a solution to is not straightforward. It was proved under the assumption that $a$ is $C^2$ in [8]. The function $a$ given by Eq. 9 is only Lipshitz due to singularity at $\theta = i\pi/2$, $i = 1, \ldots, 4$, but a proof of existence and uniqueness of the motion was given in [15] for that particular case.

**Theorem 1** ([15], Theorem 2.1). *Let $\mathcal{D} \subset [-1, 1]^2$ be strictly convex and assume that its boundary $\gamma = \partial\mathcal{D}$ is a curve whose curvature $[0, 2\pi] \ni \theta \mapsto k(\theta)$ defines a positive, $2\pi$-periodic, Lipschitz function.*

*Then there exists a unique flow of convex curves $(\gamma(t))_t$ with curvature defined everywhere, such that $\gamma(0) = \gamma$ and that the corresponding support function $h(\theta, t)$ solves Eq. 12 for $t \ge 0$ and satisfies the correct initial condition $h(\theta, 0) = h(\theta)$.*

*The curve $\gamma(t)$ shrinks to a point $x_f \in \mathbb{R}^2$ at time $t_f = Area(\mathcal{D})/2$.*

*For $t < t_f$, $\gamma(t)$ is a smooth curve in the following sense: its curvature function $k(\cdot, t)$ is Lipschitz and bounded away from 0 and infinity on any compact subset of $(0, t_f)$.*

Now let $\mathcal{D}(t)$ denote the flow of a convex shape whose support function is the solution of Eq. 12, with $\mathcal{D}(0) = \mathcal{D}$. Let $B(x, r)$ denote the closed Euclidean ball of radius $r > 0$ and center $x \in \mathbb{R}^2$. For a closed set $\mathcal{C} \subset \mathbb{R}^2$, define the inner and outer $\delta$-neighborhood of $\mathcal{C}$ to be

$$\mathcal{C}^{(\delta)} := \bigcup_{x \in \mathcal{C}} B(x, \delta),$$
$$\mathcal{C}^{(-\delta)} := \left( \bigcup_{x \in \mathcal{C}^c} B(x, \delta) \right)^c. \tag{13}$$

We say that an event, or rather, that a sequence of events $(A_n)_{n \geq 0}$ holds *with high probability* or w.h.p. if the probability of $A_n$ tends to one. Then $\mathcal{D}(t)$ is the scaling limit of $\mathcal{A}_L(t)$ in the following sense.

**Theorem 2** ([15], Theorem 2.2). *Consider the dynamics starting with initial condition given by Eq. 6, where $\mathcal{D}$ is a convex shape satisfying the assumption of Theorem 1. For any $\delta > 0$, one has w.h.p.*

$$\mathcal{D}^{(-\delta)}(t) \subset \frac{1}{L} \mathcal{A}_L(L^2 t) \subset \mathcal{D}^{(\delta)}(t) \quad \textit{for every} \ \ 0 \leq t \leq t_f + \delta \tag{14}$$

$$\mathcal{A}_L(L^2 t) = \emptyset \quad \textit{for every} \ \ t > t_f + \delta. \tag{15}$$

*In particular, one has the following convergence in probability:*

$$\lim_{L \to \infty} \frac{\tau_+}{L^2 Area(\mathcal{D})} = \frac{1}{2}. \tag{16}$$

We can mention a previous related result in the literature: in [4], the authors consider simplified dynamics that do not allow the interface to break in several components. For these dynamics, they present a result similar to Eq. 16 without any statement concerning the limiting shape.

In [3], the drift for the interface at the initial time is studied, and the authors prove that it is proportional to the curvature multiplied by an anisotropy function that is different from $a(\theta)$ of Eq. 9. This difference is explained by the fact that the initial condition considered by the authors, Eq. 6, is very far from being a local equilibrium for the interface dynamics.

**Remark 1** The result presented here concerns only the case where $\mathcal{D}$ is a convex shape. For many reasons, starting with nonconvex initial droplets makes the problem more difficult both on the probabilistic and analytical sides, and this is far from being just a technical point. The main point is that whereas the curve shortening flow we consider is monotone for the the inclusion, this becomes false if the initial condition is nonconvex. Since the first version of this paper has been written, in a collaborative effort with Simenhaus and Toninelli, we have established the existence of the anisotropic motion and the scaling limit of planar Ising droplets starting from an arbitrary shape using Theorem 2 as a building brick and using ideas coming from the work of Grayson concerning isotropic curve shorting flow [10].

We can now compare this result with what happens in the case of positive magnetic fields.

### 2.2 Zero-Temperature Dynamics with Positive Magnetic Fields ($h = \infty$)

Consider the dynamics where sites are still updated with rate 1, but with the following rule of update: when a site is updated, its spin flips to + if it has two or more + neighbors and to – if it has three or more – neighbors. Note that in this case, the update rule is completely deterministic and that the only source of randomness in the process is the one of the Poisson clocks.

In that case, the right time-scale to describe the evolution of $\mathcal{A}_L$ is not $L^2$ but $L$ and the interface is always going towards the – side regardless of the curvature. The main new result of this paper is identifying the scaling limit in this case.

Heuristically, the intensity of the drift of the interface can be deduced from mathematical work on totally asymmetric simple exclusion process (TASEP), more precisely from results giving the scaling limit of the height function (see Sect. 3.2 for detailed explanations). The drift at a point where the interface makes an angle $\theta$ with the horizontal axis is equal to

$$b(\theta) := \frac{|\sin(2\theta)|(|\cos\theta| + |\sin\theta|)}{1 + |\sin 2\theta|}. \tag{17}$$

Let us give a rigorous definition of this shape evolution in the case of convex initial condition. The shape remains convex at all times and one can describe the evolution of the interface in terms of the support function (recall Eq. 10) as follows:

$$\begin{cases} \partial_t h(\theta, t) = -b(\theta), \\ h(\theta, 0) = h(\theta). \end{cases} \tag{18}$$

The problem is that the Eq. 18 is not well-posed. However, there is a notion of a weak solution to Eq. 18 that has a rather simple description, and this is the one that will be of interest to us: given an initial condition $\mathcal{D}$, define

$$\mathcal{D}(t) := \bigcap_{\theta \in [0, 2\pi]} \{x \in \mathbb{R}^2 : x \cdot v(\theta) \le h(\theta) - b(\theta)t\}. \tag{19}$$

It could be shown that the support function of $(\mathcal{D}(t))_{t \ge 0}$ is the unique viscosity solution of Eq. 18 (see [6] for an introduction to this concept) but we will not pain ourselves with such considerations, and rather consider Eq. 19 as a definition.

Our main result is

**Theorem 3** *Given an arbitrary convex $\mathcal{D}$, we consider the dynamics with positive magnetic field defined above, with initial condition given by Eq. 6. Then the renormalized domain of – spins $(\frac{1}{L}\mathcal{A}_L(Lt))_{t \ge 0}$ converges to $(\mathcal{D}(t))_{t \ge 0}$ defined in Eq. 19 in probability, in the topology of time-uniform convergence for the Hausdorf metric. This is to say, for any $\delta > 0$ with probability tending to one, for all $t \ge 0$*

$$\mathcal{D}^{(-\delta)}(t) \subset \frac{1}{L}\mathcal{A}_L(Lt) \subset \mathcal{D}^{(\delta)}(t). \tag{20}$$

For simplicity, we choose to expose the proof in the case where the initial condition is the square $[-1,1]^2$. Starting from a general convex domain is not conceptually more difficult, but it involves notational complication. We sketch the modification needed for the general setup in Sect. 6.

Let us describe $\mathcal{D}(t)$ when $\mathcal{D} = [-1,1]^2$: define the function $g$ on $\mathbb{R} \times \mathbb{R}_+$ by

$$g(x,t) := \begin{cases} \frac{x^2+t^2}{2t} \text{ if } |x| \le t, \\ |x| \text{ if } |x| \ge t, \end{cases} \tag{21}$$

let $\mathcal{D}_1(t)$ denote the epigraph of $g(\cdot,t) - 2$ in the base $(0, \mathbf{f}_1, \mathbf{f}_2)$ where $\mathbf{f}_1 := \frac{\mathbf{e}_1 - \mathbf{e}_2}{2}$, $\mathbf{f}_2 := \frac{\mathbf{e}_1 + \mathbf{e}_2}{2}$, i.e.,

$$\mathcal{D}_1(t) := \{x\mathbf{f}_1 + y\mathbf{f}_2 \mid x \in \mathbb{R}, y \ge g(x,t) - 2\}, \tag{22}$$

and $\mathcal{D}_i$, $i = 2, \ldots, 4$, denote its image by rotation of an angle $(i-1)\pi/2$. Then one has

$$\mathcal{D}(t) := \bigcap_{i=1}^{4} \mathcal{D}_i(t). \tag{23}$$

It is a convex compact set and it is the solution of Eq. 19 when $h(\theta) = \sqrt{2}/(|\cos(\theta + \pi/4)| + |\sin(\theta + \pi/4)|)$ is the support function of $[-1,1]^2$.

**Remark 2** The proof of Theorem 3 also adapts quite easily to the case of positive magnetic fields $h \in (0, \infty)$. In that case, the scaling limits remain the same but time has to be rescaled by a factor $\cosh(h)/\sinh(h)$. The nonconvex case is more delicate as the limiting shape does not have a nice description, and it might split into several connected components. All of this makes the analysis more complicated on a technical point of view but we believe that an analogous result could be proved in that case with some efforts if the initial condition is sufficiently nice, e.g., with smooth boundary.

**Remark 3** Biasing the updates toward + with a magnetic field has a variety of effects on the scaling limit. First, the time scale at which things move on a macroscopic scale drops from $L^2$ to $L$. Furthermore, the limiting shape becomes a lot less regular: Whereas the solution of anisotropic motion by curvature is relatively smooth (the curvature is a Lipshitz function), the solution of Eq. 19 is quite irregular, and it can present some angle. Also whereas flat parts disappear instantaneously with motion by curvature, they can stay for a while for the evolution given by Eq. 19. On a microscopic level also, the fluctuation of the interfaces should belong to different universality classes.

### 2.3 Interpolating Between $h = 0$ and $h > 0$: the Weak Magnetic Field Limit

Remark 2 says that the scaling limit of set of – spins is somehow independent of the intensity of the magnetic fields apart from an armless scaling factor. We want to discuss shortly here a way to obtain a nontrivial intermediate regime (sometimes referred to as a crossover regime) between the cases $h > 0$ and $h = 0$. We give a brief description of a conjecture concerning that case, based on heuristic consideration.

The intermediate regime should take place when $h = h_L \sim \alpha L^{-1}$. When $h \gg L^{-1}$, Theorem 3 should hold, whereas Theorem 2 should be valid when $h \ll L^{-1}$. In this intermediate regime, the scaling should be $L^2$ and the limiting equation for the support function

$$\begin{cases} \partial_t h(\theta,t) = -a(\theta)k(\theta,t) - \alpha b(\theta) \\ h(\theta,0) = h(\theta). \end{cases} \tag{24}$$

The existence and uniqueness of the solution to the above equation is a challenging issue.

Note that this type of scaling has already been studied in the context particle system in the case where the interface between + and – is the graph of a function (see Sect. 3.2) and is often referred to as weakly asymmetric exclusion process (WASEP).

It has been proved in [5, 9, 12] that in a certain sense, the height function of the particle system converges to the solution of

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{\alpha}{2}(1 - (\partial_x u)^2), \tag{25}$$

which is as can be checked by the reader, equivalent to Eq. 24 in this case. These result could be used as a starting point to get the scaling limit starting from a compact convex shape filled with –. Again, the major difficulty seems to be on the analytical side, as on the probabilistic side most of the tools used in the proof of Theorem 2 could also be applied.

### 2.4 Higher Dimensions

Of course, the Ising model and dynamics are also defined in higher dimensions $d \geq 3$, for which one also has a phase transition, and they have also raised a lot of interest (in particular the case $d = 3$ for obvious reasons). Let us briefly talk about what should be true in that case with a focus on low temperature with + boundary condition (for high temperature, we can notice that the result of [20] holds in every dimension).

The Lifshitz conjecture for low-temperature dynamics should hold in fact in any dimension. However, rigorous results are even more difficult to obtain for the model with zero-magnetic fields: for finite $\beta$, the best known bound for the mixing time at low temperature with + boundary condition is super-exponential (it is equal to $O(\exp(L^{d-2}(\log L)^2))$ [29]), which is really far from the conjectured $O(L^2)$.

Concerning the zero-temperature case, results are much more precise: in [2], it has been proved that for $d = 3$, the time necessary for the last – spin to disappear starting from a full cube with + boundary condition is of order $L^2(\log L)^{O(1)}$, a similar upper bound has been derived for arbitrary dimension $d$ in [13] (with no matching lower bound). However, it seems that with the actual tools, we are very far from being able to prove a shape theorem similar to Theorem 2 when $d = 3$, or even to get rid of the $\tau$. A first step would be to derive on a heuristic level the anisotropy function (the equivalent of $a(\theta)$ from Eq. 9), and it seems highly nontrivial.

The anisotropy function should have a lot of singularity when $d \geq 3$. For instance, consider the three-dimensional dynamics on $(\mathbb{Z}^*)^3$ starting from initial condition

$$\begin{cases} \sigma(x) = -1 & \text{if } x \in (\mathbb{Z}^*)^3 \cap (\mathbb{R} \times B(0, L)) \\ \sigma(x) = +1 & \text{else ,} \end{cases} \tag{26}$$

where $B(0, L)$ denote the two-dimensional Euclidean ball. If $L \geq 4$, then this initial configuration is stable, and the system stays forever in that state: every – spin has a strict majority of agreeing neighbors and the same for + spin. However, the mean curvature is well defined and positive at every point of the interface of the cylinder.

For the case of zero-temperature with positive $h$, it can be shown with simple arguments that a cube full of – spins of diameter $L$ needs a time of order $L$ to disappear. Getting the exact asymptotic and a shape theorem is a much more challenging problem.

### *2.5 Organization of the Paper*

In Sect. 3, we present some of the main tools of the proof. They are: general monotonicity property of the dynamics, correspondence with particle systems, and scaling limits for interface dynamics.

In Sect. 4, we sketch in detail the proof of Theorem 2 from [15]

In Sect. 5, we prove our main result, that is Theorem 3, and underline the similarities and difference between this proof and the one of the zero-magnetic fields case.

## 3 Interface Dynamics, Correspondence with Particle Systems and Other Technical Tools

### *3.1 Graphical Construction and Monotonicity*

We present here a construction of the dynamics (called sometimes the *graphical construction*) that yields nice monotonicity properties. We consider a family of independent Poisson clock processes $(\tau^x)_{x\in\bar{\mathbb{Z}}^2}$: to each site $x \in (\mathbb{Z}^*)^2$, one associates an independent random sequence of times $(\tau_n^x)_{n\geq 0}$, that are such that $\tau_0^x = 0$ and $(\tau_{n+1}^x - \tau_n^x)_{n\geq 0}$ are IID exponential variables with mean one. One also defines random variables $(U_{n,x})_{n\geq 0}, x \in (\mathbb{Z}^*)^2$ that are IID Bernoulli variables, that assume value $\pm 1$ with probability $e^{\pm h}/(2\cosh(h))$.

Then, given an initial configuration $\xi \in \{-1,1\}^{(\mathbb{Z}^*)^2}$, one constructs the dynamics $\sigma^\xi(t)$ starting from $\sigma^\xi(0) = \xi$ as follows:

- $(\sigma_x(t))_{t\geq 0}$ is constant on the intervals of the type $[\tau_n^x, \tau_{n+1}^x)$.
- $\sigma_x(\tau_n^x)$ is chosen to be equal to $\pm 1$ if a strict majority of the neighbors of $x$ satisfies $\sigma_y(\tau_n^x) = \pm 1$, and $U_{n,x}$ otherwise (this definition makes sense as, almost surely, two neighbors will not update at the same time).

This construction gives a simple way to define simultaneously the dynamics for all initial conditions and boundary condition, using the same variable $U$ and $\tau$ (we denote by $P$ the associated probability). Moreover, the coupling of dynamics thus obtained preserves the natural order on $\{-1,+1\}^{(\mathbb{Z}^*)^2}$, given by

$$\xi \geq \xi' \Leftrightarrow \xi_x \geq \xi'_x \quad \text{for every } x \in (\mathbb{Z}^*)^2 \tag{27}$$

(this order is just the opposite of the inclusion order for the set of "–" spins, which is, therefore, also preserved). Indeed, if $\xi \geq \xi'$ and $\sigma^\xi$ resp. $\sigma^{\xi'}$ denote the dynamics with initial condition $\sigma$ resp. $\sigma'$ using the same $\tau$ and $U$, with the above construction, one has $P$-a.s.

$$\forall t > 0 \qquad \sigma^\xi(t) \geq \sigma^{\xi'}(t). \tag{28}$$

It also yields monotonicity with respect to boundary condition and other nice properties.

### *3.2 The Height Function of the Simple Exclusion Process*

We deal in this subsection and the next ones, with special initial conditions and boundary conditions, for which the interface between + and – at all times is given by the graph of a function in the coordinate frame $(\mathbf{f}_1, \mathbf{f}_2)$ (recall the definition above Eq. 22). These cases are easier to treat for two reasons: firstly the rescaled interface motion can be written a functional PDE, which is easier to deal with than a flow of curves; secondly, there are bijective correspondences with particle systems that facilitate the probabilistic treatment of these interface problems (and also gives extra motivation to study them).

Results concerning interface models are one of the principal building bricks for the proof of Theorems 2 and 3. We describe the correspondence in this subsection and state convergence results in the symmetric and the asymmetric cases in Sects. 3.3 resp. 3.4.

We say that a set $A \subset (\mathbb{Z}^*)^2$ or $A \subset \Lambda_L$ is *increasing* if and only if

$$\forall x \in A, (y \geq x \Rightarrow y \in A), \tag{29}$$

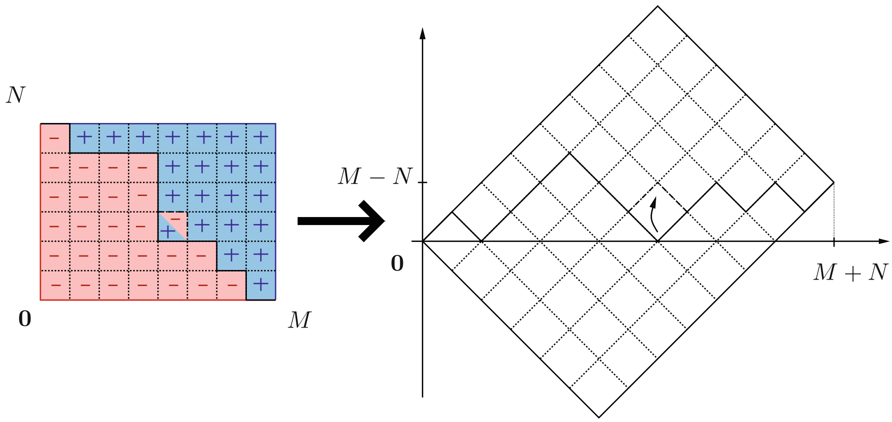


**Fig. 2** One-to-one correspondence between the dynamics in a $N \times M$ rectangle with mixed boundary conditions and corner-flip dynamics on paths. This correspondence is, of course, also valid in the case of a $2L \times 2L$ square. An example of possible flip update is displayed

where the order in $(\mathbb{Z}^*)^2$ is the usual partial order $(y_1, y_2) \geq (x_1, x_2)$ where $y_1 \geq x_1$ and $y_2 \geq x_2$.

We define the interface between $+$ and – for a configuration $\sigma$ to be the topological boundary of

$$\mathcal{A}^-(\sigma) := \bigcup_{\{x:\sigma_x=-1\}} \mathcal{C}_x, \tag{30}$$

which also is equal to the boundary of $\mathcal{A}^+(\sigma)$ with an analogous definition.

If the set of $+$ spins is an increasing set for $\sigma(0)$, then it remains an increasing set for $\sigma(t)$ for all $t \geq 0$ (the only site whose spin can flip has two $+$ neighbors and two $-$ neighbors, and the reader can check that flipping one of them does not break the property). Also, when the set of $+$ spin is increasing, the interface between $+$ and $-$ is the graph of a function $\eta : \mathbb{R} \to \mathbb{R}$ in the frame $(0, \mathbf{f}_1, \mathbf{f}_2)$. The restriction of $\eta$ to $\mathbb{Z}$ is integer-valued and satisfies $|\eta(x+1) - \eta(x)| = 1$ for all $x \in \mathbb{Z}$ and it is linear on every interval of the type $[x, x+1]$, $x \in \mathbb{Z}$. Hence, in a small abuse of notation we can also consider $\eta$ as a function on $\mathbb{Z}$, and see the dynamics as a Markov chain on the state-space

$$\Omega^0 := \{\eta \in \mathbb{Z}^{\mathbb{Z}} \,||\eta(x+1) - \eta(x)| = 1, \forall x \in \mathbb{Z}\}. \tag{31}$$

In the case where the dynamic is restricted to $\Lambda_L$, we choose the boundary condition to be $+$ on the upper and right sides and $-$ on the two opposite sides and the interface function $\eta$ defined on $[-2L, 2L]$ (see Fig. 2).

The dynamic of the interface can be described as follows: sites $x \in \mathbb{Z}$ (or $x \in [-2L+1, 2L-1]$) are equipped with independent Poisson clocks with rate one; when a clock rings at $x$ the path $\eta$ is replaced by $\eta^x$ with probability $e^h/(2\cosh(h))$ and $\eta_x$ with probability $e^h/(2\cosh(h))$, where $\eta^x$ (and $\eta_x$) are respectively the maximum and minimum path in $\Omega$ that coincides with $\eta$ on $\mathbb{Z}\setminus\{x\}$. The paths $\eta_x$ and $\eta^x$ differ if and only if $\eta$ has a local extremum at $x$, i.e., if $\eta(x+1) = \eta(x-1)$).

This description is only formal in the full line case since the set of update time is dense, but is a rigorous definition of the generator of the chain (for a general proof of existence of a chain with such a generator, see [[19], Chap. 1]).

Finally, this interface dynamic can be mapped onto a one-dimensional particle system. For any $x \in \mathbb{Z}$ set

$$\xi(x) = (\eta(x) - \eta(x+1) + 1)/2$$

and say that a particle lies on the site $x$ when $\xi(x) = 1$. With an alternative view, the dynamic can be described as follows: each particle jumps to the right with rate $e^h/(2\cosh(h))$, and to the left with rate $e^{-h}/(2\cosh(h))$, with jumps being canceled if the aimed site is already occupied by another particle. In the case of $\eta$ defined on a segment, we define the particle in the same manner, with the constraint that the extremities of the particle system are "closed," so that the particle cannot run through them. This system is called the simple exclusion process.

When $h = 0$, the particles perform symmetric motions and one speaks about symmetric simple exclusion process (SSEP). When $h \in (0, \infty]$, the particles are biased towards the right and one talks about ASEP (asymmetric). When $h = \infty$ particles can only jump to the right and the system is called TASEP (totally asymmetric).

For all these particle systems, the only infinite volume equilibrium measures are the one under which $\xi(x)$ are IID Bernoulli variables.

## *3.3 Scaling Limit for the Asymmetric Simple Exclusion Process*

This correspondence with one-dimensional particle systems has given to mathematical physicists some additional motivation to study these interface motions and a variety of results for the scaling limit of the interface have been obtained. Let us cite the first one, due to Rost [25] that concerns the case where the initial condition is – in a quadrant and + in the three others, in the totally asymmetric case (recall the definition of the function $g$ in Eq. 21).

**Theorem 4** ([25], Theorem 1). *Consider the stochastic Ising model on $(\mathbb{Z}^*)^2$ at temperature zero with $h = \infty$ and initial condition $\sigma_0 - in\ \mathbb{R}^2_+$ and + elsewhere. Let $\eta(x,t)$ denote the function whose interface in $(0, \mathbf{f}_1, \mathbf{f}_2)$ is the interface between + and –for $\sigma(t)$ $(\eta(x,0) = |x|)$. Then one has for all $\varepsilon > 0$ and $T < \infty$*

$$\lim_{L\to\infty} \mathbb{P}\left[\max_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |\frac{1}{L}\eta(Lx, Lt) - g(x,t)| > \varepsilon\right] = 0. \tag{32}$$

The above result has been developed in order to be able to treat all kinds of initial conditions. Let $u_0$ be a 1-Lipshitz function and let $\sigma_0^L$ be a sequence of initial configuration for which the set of – is an increasing set and the initial interface function

$\eta_0^L$ satisfies:

$$\lim_{L\to\infty}\sup_{x\in\mathbb{R}}\left|\frac{1}{L}\eta_0^L(Lx)-u_0(x)\right|=0. \tag{33}$$

Then define $u(x,t)$ as being the unique *viscosity solution* of

$$\begin{cases}\partial_t u &= \frac{1}{2}(1-(\partial_x u)^2),\\ u(\cdot,0) &= u_0,\end{cases} \tag{34}$$

which can be showed (for instance we refer to in [[26], Eq. (2.8)] and the discussion following it for a more general case) to be equal to (recall Eq. 21),

$$u(x,t) := \inf_y\{u_0(y)+g(x-y,t)\}. \tag{35}$$

We use Eq. 35 as the definition of $u$. It turns out that $u$ is the description of the scaling limit of the interface when the initial condition satisfies Eq. 33. Before stating the theorem, let us explain briefly on heuristic grounds why it is so.

We use the particle system description given at the end of the previous section. With this setup, the height variation $\eta(x,t)-\eta(x,0)$ is equal to twice the number of particle jumping from $x-1$ to $x$ in the time interval $[0,t]$.

We consider the simplified case where the initial profile is linear with slope $s$ and the particle system is in an equilibrium configuration. i.e., $(\xi_x)_{x\in\mathbb{Z}}$ are IID Bernoulli variables of parameter $\rho := (1-s)/2$. In that case, the jump rate of particle from $x-1$ to $x$ is constant and equal to

$$\mathrm{P}[\eta(x-1,t)=1;\eta(x,t)=0]=\rho(1-\rho)=\frac{1}{4}(1-s^2).$$

Assuming that there is some kind of ergodicity in the system, a law of large number should hold and for any fixed $x$ a.s.

$$\eta(t,x)-\eta(0,x)=\frac{1}{2}(1-s^2)t(1+o(1)). \tag{36}$$

The reason why the argument also works when the original density of particle is nonuniform is that locally, the system relaxes quickly to equilibrium so that the field $\eta(x,t)_{x\in\mathbb{Z}}$ looks locally like IID Bernoulli after a small time.

The following theorem was proved in [24] for ASEP in arbitrary dimension. The reader can also refer to [26] for a generalization to the $K$-exclusion process.

**Theorem 5** *Consider the stochastic Ising model at temperature zero with $h=\infty$ and initial condition $\sigma_0^L$ as above (and $\eta^L$ the corresponding interface). Then, the rescaled interface converges in law to $u$ in the following sense: For every $\varepsilon$, for every positive $T$ and $K$*

$$\lim_{L\to\infty}\mathrm{P}\left[\max_{\substack{|x|\le K\\ t\le T}}|\frac{1}{L}\eta^L(Lx,Lt)-u(x,t)|\ge\varepsilon\right]=0. \tag{37}$$

These two results remain valid for the ASEP, but time has to be rescaled by a factor $\gamma(h)=\coth(h)$.

### 3.4 Scaling Limit of the Symmetric Simple Exclusion Process

The case where $h = 0$ corresponds to the case where the particles perform symmetric jumps. In that case, the speed of the particle is zero, and one has to rescale time by $L^2$ to get a nontrivial scaling limit.

It is now a classical result that, in any dimension, the weak limit of the density profile of particle for the SSEP is given by the heat equation (see, e.g., [[11], Chap. 4]). In [15], we proved an analogous strong limit result for the profile $\eta$, in the case of dynamics restricted to a box: consider the zero-temperature dynamics with zero magnetic fields on $\Lambda_L$ with $+$ boundary condition on the upper and right sides and $-$ on the two others. Let $t \mapsto \eta(\cdot, t)$ defined on $[-2L, 2L]$ denote the function whose graph in $(0, \mathbf{f}_1, \mathbf{f}_2)$ is the interface between $+$ and $-$ for the zero-temperature dynamics.

Given a 1-Lipschitz function $v_0 : [-2, 2] \mapsto \mathbb{R}$ with $v_0(\pm 2) = 0$, assume that one starts the dynamics in $\Lambda_L$ with a sequence of initial condition $\sigma_0^L$ for which the interface $\eta$ satisfies

$$\lim_{L\to\infty} \sup_{x\in[-2L,2L]} \left| \frac{1}{L}\eta(x,0) - v_0(Lx) \right| = 0. \tag{38}$$

Let $v$ be the solution of

$$\begin{cases} \partial_t v & = \frac{1}{2}\partial_x^2 v, \\ v(0,t) & = v_0, \forall x \in [-2,2], \\ v(\pm 2, t) & = 0, \forall t \geq 0. \end{cases} \tag{39}$$

**Theorem 6** ([15], Theorem 3.2). *Consider the dynamics on $\Lambda_L$ with the above mentioned initial condition, then for all $T \geq 0$ and $\varepsilon > 0$, w.h.p.*

$$\sup_{t\in[0,T],x\in[-2,2]} \left| \frac{1}{L}\eta(Lx, L^2 t) - v(x,t) \right| \leq \varepsilon \tag{40}$$

*where $v$ is the solution of Eq. 39.*

Let us give shortly a heuristic explanation for Theorem 6. The first thing is to show that the expected value of $(\eta_x)_{x\in[-L,L]}$ satisfies approximately the heat equation.

First notice that the expected drift of $\eta_x$ at time zero depends only on the value of $\eta_x$ and $\eta_{x\pm1}$. If $\eta_x$ is a local maximum, it will jump down by two with rate $1/2$, whereas if it is a local minimum it will jump up by two with the same rate. Else $\eta_x$ has drift zero so that

$$\partial_t \mathbb{E}[\eta_x(t)]|_{t=0} = \begin{cases} 1 & \text{if } \eta_{x\pm1}(0) = \eta_{x-1}(0), \\ -1 & \text{if } \eta_{x\pm1}(0) = \eta_{x+1}(0), \\ 0 & \text{else.} \end{cases} \tag{41}$$

The reader can check that the r. h.s. of the above equation is equal to

$$\frac{1}{2}(\eta_{x+1}(0) + \eta_{x-1}(0) - 2\eta_x(0)) =: \frac{1}{2}\Delta_{\mathrm{d}}\eta_x,$$

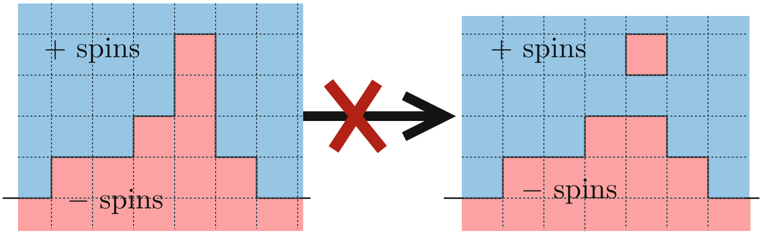


**Fig. 3** An example of spin update that splits the interface into two disconnected components. The interface dynamics presented in this section does not allow this kind of move

where $\Delta_{\mathrm{d}}$ denotes the discrete Laplacian. Hence, using the Markov property, one obtains that $(\mathbb{E}[\eta_x])_{x\in[-L,L]}$ satisfies

$$\partial_t \mathbb{E}[\eta_x(t)] = \frac{1}{2}\Delta_{\mathrm{d}}\mathbb{E}[\eta_x(t)]. \tag{42}$$

Theorem 6 is obtained then by showing that:

- The solution of the discrete heat-equation converges to the solution of the continuous one in the scaling limit (this is classic).
- That $\eta_x(t)$ concentrates around its mean for large values of $L$. This is more delicate. We proved it by proving concentration for all the Fourier coefficient in a base of eigenfunctions of $\Delta_{\mathrm{d}}$.

Projections and trigonometry allows a heuristic derivation of Eq. 12 from Theorem 6 and Eq. 39. Indeed, it is quite reasonable to think that for the original dynamics with shrinking – domain, the local drift of the interface is the same than for these dynamics with modified boundary condition. However, there is a crucial ingredient missing to try to perform a proof.

Indeed Theorem 6 does not say anything about the drift of the interface around the "poles" of $\mathcal{A}_L(t)$, i.e., around the points for which one of the coordinates is extremal. This problem was treated using another correspondence with particle system, first in [28] where a complete sketch of proof was given in a special setup (periodic boundary condition). This study was pushed further in [15] where all the technical details were handled to get a result that could be used to prove Theorem 2. We present this correspondence in the next section.

## *3.5 Dynamics Near the "Poles": Zero Range Process and Scaling Limit*

Near the "poles," the dynamics cannot be easily reduced to an interface dynamics: indeed (see Fig. 3) there are some possibilities for the set of − to break into several connected components. However, one can introduce auxiliary dynamics that cancel the transitions that make $\mathcal{A}_L(t)$ disconnected.

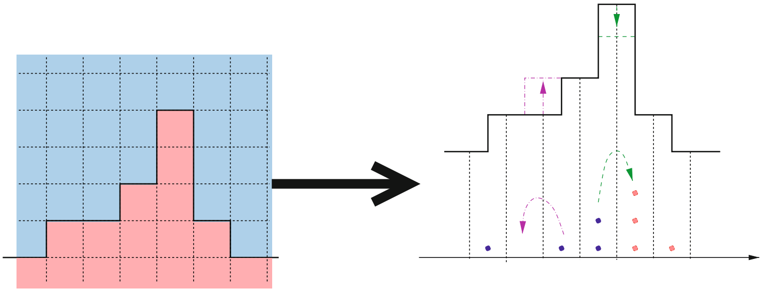

**Fig. 4** Correspondence between interface dynamics and zero-range process. Arrows represent possible motions for the interface and their representation in terms of particle moves. When an A particle jumps on a B particle (*green arrow*), they annihilate

More precisely, we consider $\Lambda_L$ with boundary condition – in the upper half plane and – one the lower half plane, starting with an initial condition such that the interface between $+$ and $-$ is the graph of a function $[-L, L] \to \mathbb{R}$ (plus some vertical lines), and run modified Ising dynamics that discard update if the interface after the update is not a single connected curve. With these dynamics, the interface remains the graph of a function for all time (if one neglects the vertical lines). We call $\eta(x,t)$ the corresponding function (as there is no confusion possible with $\eta$ from the other section): by convention, we choose it to be defined on $[-L, L] \cap \mathbb{Z}^*$ as it is piecewise constant.

In this case also, we can describe the evolution of the gradient as a particle system. For $x \in \{-L, \ldots, L\}$ we set

$$\xi_x(t) := \eta_{x+1} - \eta_{x-1}$$

to be the discrete gradient of $\eta$. We say that each site in $\{-L, \ldots, L\}$ carries $|\xi_x(t)|$ particles. These particles are said to be of type $A$ if $\xi$ is positive and of type $B$ if $\xi$ is negative.

Under the modified Ising dynamics depicted above, the rules for the motion of the particles are the following:

- If there are $k$ particles on a site, they jump left or right with rate $1/(2k)$.
- If a particle of type $A$ meets a particle of type $B$ they annihilate.

This kind of particle system where the jump rate depends on the number of particles on one site is called zero-range process (see Fig. 4 for a scheme of the correspondence of interface dynamics with the particle system), and has been extensively studied in the literature (see, e.g., [1] where the invariant measures of this process are studied).

This correspondence with particle system was underlined in [28], and a partial proof of the scaling limit of the interface motion was given there: the scaling limit of $\xi$ should be the solution of the equation

$$\partial_t w = \frac{\partial_x^2 w}{2(1 + |\partial_x w|^2)}. \tag{43}$$

Although we were not able to complete this proof fully (in particular we miss a statement concerning existence and regularity of the solution in Eq. 43), we proved a partial statement that was sufficient for the purpose of the proof of Theorem 2. Our statement is that, on the macroscopic scale, $\eta(x,t)$ stays close to a deterministic discrete evolution that can be thought of as a discretization of Eq. 43 [[15], Theorem 7]. We record informally here the result for quotes in the rest of the paper.

**Theorem 7** ([15], Theorem 3.4 and Corollary 3.5). *In some weak sense, Eq. 43 describes the evolution of the rescaled interface on the diffusive time-scale.*

## 4 Zero Magnetic Fields: A Detailed Sketch of Proof for Theorem 2

We expose in this section the ideas behind the proof of Theorem 2. The first important step is to reduce the proof to an infinitesimal statement. Using continuity properties of the conjectured scaling limit, we can show that in order to control the evolution of the – domain for all time, it is sufficient to control the motion with a first order precision during a small time $\varepsilon$.

Once this is done, it suffices to iterate the statement as many times as needed (order $\varepsilon^{-1}$) to control the evolution for arbitrary positive time. We do not develop this point further.

Let us state directly the two infinitesimal statements we want to prove: the first one concerns continuity of the interface motion (which has to be used to get continuity of the motion).

**Proposition 1** *Let $\mathcal{D}$ be convex and with a Lipschitz curvature function. For every $\alpha > 0$, w.h.p. (recall definition 13)*

$$\mathcal{A}_L(L^2 t) \subset L\mathcal{D}^{(\alpha)} \quad \text{for every} \quad t \geq 0. \tag{44}$$

*Moreover, for every $\alpha > 0$ there exists $\varepsilon_1(\alpha, k_{\max}) > 0$ such that w.h.p.*

$$\mathcal{A}_L(L^2 t) \supset L\mathcal{D}^{(-\alpha)} \text{ for every } t \in [0, \varepsilon_1]. \tag{45}$$

the second one is the control at first order of the evolution,

**Proposition 2** *For all $\delta > 0$, there exists $\varepsilon_0(\delta, k_{\min}, k_{\max}) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, w.h.p.,*

$$\mathcal{A}_L(L^2 \varepsilon) \subset L\mathcal{D}(\varepsilon(1-\delta)), \tag{46}$$

*and*

$$\mathcal{A}_L(L^2 \varepsilon) \supset L\mathcal{D}(\varepsilon(1+\delta)), \tag{47}$$

*where $\mathcal{D}(t)$ is the solution of Eq. 12.*

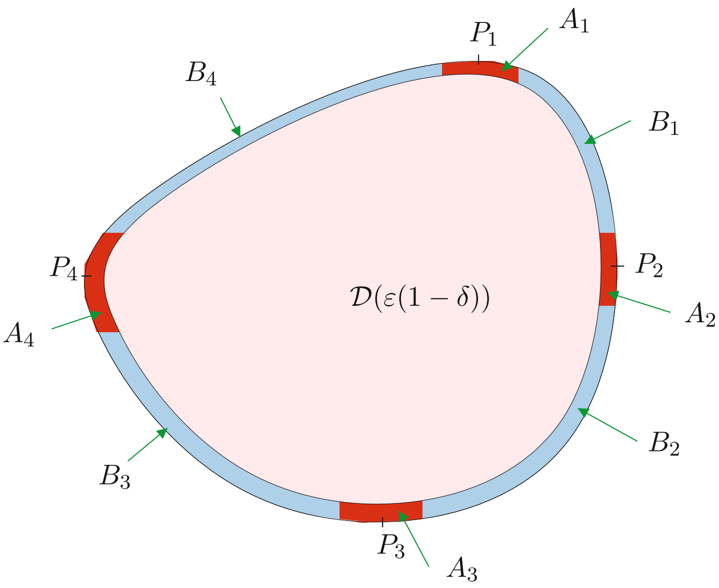


**Fig. 5** Decomposition of $\mathcal{D} \setminus \mathcal{D}(\varepsilon(1-\delta))$ in eight zones: four small ones around the poles $P_i$ that we call $A_i$ and four larger ones away from the poles, called $B_i$

The main work that remains is to prove the two inclusion bounds 46 and 47, as Proposition 1 is more of a technical detail. We detail now the sketch for the proof of the upper inclusion 46. The other bound is proved similarly, but for technical reasons, it is a bit harder to expose its proof.

*Sketch of the proof of Eq. 46.* The idea of the proof is to use the monotonicity properties of the dynamics in order to control it. What we have to show is that after a time $\varepsilon$, all the spins in $\mathcal{D} \setminus \mathcal{D}(\varepsilon(1-\delta))$ (on the rescaled picture) that were initially – have turned + w.h.p. after a time $\varepsilon$.

To do so, we divide $\mathcal{D} \setminus \mathcal{D}(\varepsilon(1-\delta))$ in eight regions and in each of these region, we will try to compare our dynamics with some interface dynamics. We consider four regions around the poles named $(A_i)_{i=1}^4$ and four others named $(B_i)_{i=1}^4$ (see Fig. 5), and one wants to show that each one of them is filled with + after a time $\varepsilon$. In what follows, we will choose the $A_i$ to be very small zones around the poles whereas the $B_i$ will cover a proportion of the boundary of $\mathcal{D}$ close to 1.

Due to rotational symmetries of the problem, the inclusion 46 reduces to proving that w.h.p.

$$\begin{aligned} &\forall x \in B_1, \sigma_x(\varepsilon L^2) = +1, \\ &\forall x \in A_1, \sigma_x(\varepsilon L^2) = +1. \end{aligned} \tag{48}$$

The idea behind this division is that initially, $\gamma$ restricted to the region $A_1$ (resp. $B_1$) is the graph of a function in the coordinate frame $(\mathbf{e}_1, \mathbf{e}_2)$ for $A_1$ and in the frame,

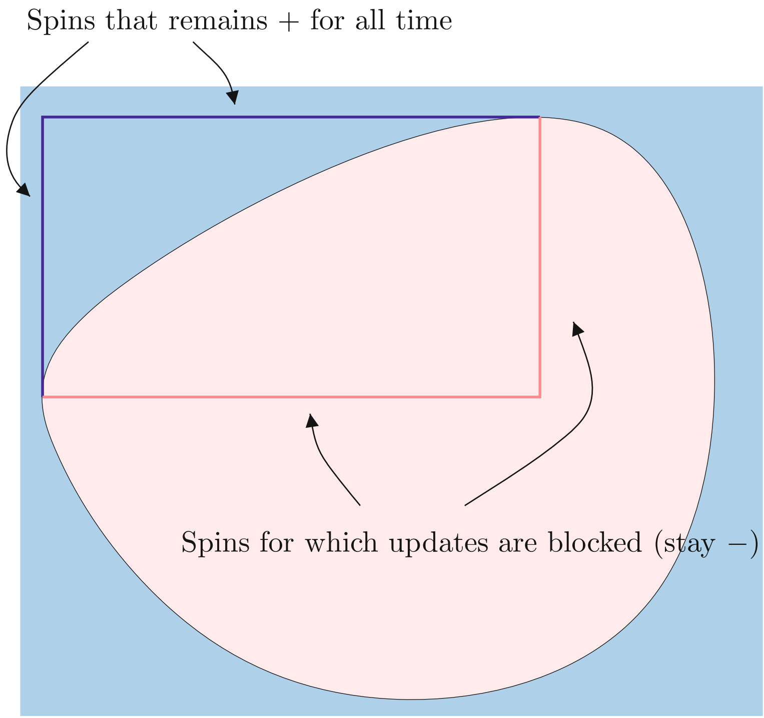


**Fig. 6** In order to control the dynamics on a zone of type $B$, we look at the dynamic restricted to a "quadrant" of the original shape. In order to cut dependence from what happens outside the rectangle, we block the update on a set of spins that have to stay $-$ at all time (*dark red*), and we can do a monotone coupling of these dynamics with the original one. On the two opposite sides of the rectangle (*dark blue*), even if updates are not rejected, note that the spins have to stay $+$ for all time due to the majority rule

resp. $(\mathbf{f}_1, \mathbf{f}_2)$. We want to modify our dynamics a little bit to get in the context of Theorem 6 or 7.

The idea to prove each line of Eq. 48 is to replace the dynamic $\sigma$ by another one that creates more – spins, but that we can handle better.

Let us start with the case of zone $B_1$. We look at the dynamics restricted to the rectangle like in Fig. 6 (where the case of $B_4$ is treated), and we decide to cancel all the updates that concern spins of the lower or right side of the rectangle, and remark that the spins on the two opposite sides remain $+$ for all time. The obtained dynamics $\sigma^{(1)}$ has more – spins than the original one (it can be seen from the graphical construction of Sect. 3.1) and falls in the setup described in Sect. 3.4. More precisely, we have a rectangle with mixed boundary condition instead of a square, but Theorem 6 also applies in that case).

If $\eta^{(1)}$ denotes the interface function corresponding to $\sigma^{(1)}$ (in the setup of Sect. 3.4) and $q(\cdot,t)$ denotes the function whose graph in the coordinate frame $(\mathbf{f}_1, \mathbf{f}_2)$ is the boundary of $\mathcal{D}(t)$ restricted to the zone $B_1$ (we chose $t$ small enough so that $q$ remains defined on some "large interval" $I$). The first line of Eq. 48 is proved if one can show that

$$\frac{1}{L}\eta^{(1)}(Lx, L^2\varepsilon) \leq q(x, \varepsilon(1+\delta)), \forall x \in I, \tag{49}$$

where $I$ is an interval that is strictly included in the domain of definition $I_0$ of $q(\cdot, t)$. $I$ is chosen sufficiently large to have a control on the whole region $B_1$.

Thus using Theorem 6, it is sufficient to prove that

$$u(x, \varepsilon) < q(x, \varepsilon(1-\delta)), \forall x \in I, \tag{50}$$

where $u$ is the solution of Eq. 39 with initial condition $u_0 = q(\cdot, 0)$ corresponding to the initial position of the interface. Continuity properties of the heat equation allow us to say that if the interval $I$ is fixed and $u_0$ is smooth enough, one has uniformly in $x \in I$ (uniformity concerns the Taylor rest $O(\varepsilon^2)$)

$$u(x, \varepsilon) = u_0(x) + \frac{1}{2}\partial_x^2 u_0(x)\varepsilon + O(\varepsilon^2). \tag{51}$$

This does not hold uniformly in the full interval $I_0$ because of the Dirichlet boundary condition that makes the drift equal to zero at the extremities of the interval. The anisotropic motion by curvature is sufficiently regular to obtain something similar for $q(x, \varepsilon)$: uniformly on $x \in I$

$$q(x, \varepsilon) - q(x, 0) = \varepsilon\frac{1}{2}\partial_x^2 q(x, 0) + O(t^2). \tag{52}$$

At an informal level, this is just a Taylor formula combined with Eq. 12 and some trigonometry (for the projections).

As curvature remains positive everywhere for all time and $q(x, 0) = u(x, 0)$, it is straightforward to deduce Eq. 50 from Eqs. 51 and 52 for small enough $\varepsilon$.

The treatment of a zone around the pole (say $A_1$) is more delicate but follows the same line. The difference is that we have to perform a longer chain of modification of the dynamics. Each modification makes the dynamics have more $-$ spins, or result in a dynamics that coincide with the previous one with large probability.

We look at a dynamic restricted to a rectangle around the pole that includes $A_1$, e.g., take the rectangle twice as large as $A_1$ (see Fig. 7). We want to modify the dynamics by fixing the boundary condition on the rectangle (or to freeze updates). The problem here is that to get to the setup of interface dynamics of Sect. 3.5 we need to freeze some sites with $+$ spins as well as sites with $-$ spins so that the modified dynamics does not compare well with the original one.

A way to get around this problem is to prove first that sites at a macroscopic distance (i.e., positive distance on the rescaled picture) from $\mathcal{D}(0)$ do not change sign ever with large probability (this is for instance the upper inclusion bound in Proposition 1).

Knowing this, we can freeze spins to $+$ in the upper part of our rectangle provided it does not touch $\mathcal{D}(0)$: the modified dynamics that we obtain coincide with the original with large probability.

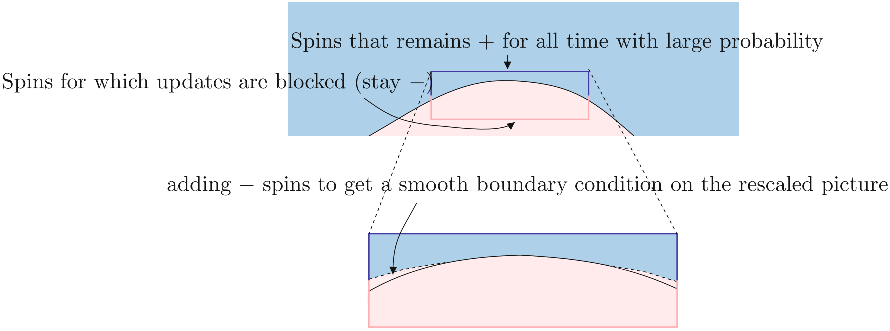


**Fig. 7** In order to control the dynamics on a zone of type $B$, we look at the dynamic restricted to a rectangle around the pole. In order to cut dependence from what happens outside the rectangle, we block the update on a set of spins that have to stay – at all time (*dark red*). On the rest of the boundary, we note that the spins have to stay + for all time with large probability so that we can couple the dynamics that reject updates on those site so that it coincides with the original dynamics with large probability

Then if we freeze the spins in the lower part of the rectangle to $-$, we get dynamics that compare well with the previous one (it has more $-$ spins, see upper part of Fig. 7).

Once the whole boundary has been frozen, we add $-$ spins to our initial condition so that on the macroscopic scale, our initial condition is a smooth interface: we cancel the irregularity at the boundary (see Fig. 7 lower part).

The dynamics restricted to the rectangle are not exactly interface dynamics, so we have to modify it once again. We cancel all the moves that break the interface between $+$ and $-$ into several connected components. With our modified initial condition (an interface that has a unique local maximum), a disconnection of the interface can only occur by adding a $+$ spins so the modified dynamics has one again more $-$ spins than the original one.

Let $\sigma^{(2)}$ denote the last mentioned modified dynamics and $\eta^{(2)}(x,t)$ denote the interface function that corresponds to it. Let $r$ denote the function whose graph in $(\mathbf{e}_1, \mathbf{e}_2)$ is the boundary of $\mathcal{D}(t)$ in the zone $A_1$.

We are left with proving that

$$\frac{1}{L}\eta^{(2)}(Lx, L^2\varepsilon) \leq r(x, \varepsilon(1-\delta)), \forall x \in J, \tag{53}$$

for a small interval $J$ around the pole.

Using Theorem 7, one gets that for small $\alpha > 0$, for any positive $t$ w.h.p.

$$\frac{1}{L}\eta^{(2)}(Lx, L^2 t) \leq w(x, (1-\alpha)t), \tag{54}$$

where $w$ is the solution of $\partial_t w = (1/2)\partial_x^2 w$ with Dirichlet boundary condition, and initial condition given by the the interface. The number $\alpha$ can be taken arbitrarily small by reducing the size of the zone around the pole that is considered.

With this information, proving Eq. 53 is reduced to show that

$$w(x,(1+\alpha)\varepsilon) < r(x,\varepsilon(1+\delta)) \quad \forall x \in J, \tag{55}$$

where $I$ is an interval corresponding to the zone $A_1$. Equation 55 is proved in the same manner in which we proved Eq. 50, we choose $\alpha = \delta/2$, and perform Taylor expansion at first order on both sides:

$$\begin{aligned} w(x,(1+\delta/2)\varepsilon) &= w(x,0) + \varepsilon(1-\delta/2)/2(\partial_x)^2\, w(x,0) + O(\varepsilon^2), \\ r(x,\varepsilon(1+\delta)) &= r(x,0) + \varepsilon(1-\delta)\frac{(\partial_x)^2 r(x,0)}{2(1+|\partial_x r(x,0)|)^2} + O(\varepsilon^2), \end{aligned} \tag{56}$$

where $O(\varepsilon^2)$ is uniform in $x \in I$. Again, the second line is formally Eq. 12 and some trigonometry. Then, using the fact that $w(\cdot,0) = r(\cdot,0)$ on an interval around the pole and that $|\partial_x r(x,0)|$ is uniformly small on $J$ if $J$ is chosen small enough, we can deduce Eq. 55 for Eq. 56 for $\varepsilon$ small enough, with a small interval $J$.

Proving inclusion 47 relies on the same ingredients: first separate different zones that one wants to control, and then, for each zone reduce to an interface dynamics via a chain of modification. After doing this, we use the scaling results, Theorems 6 and 7 to conclude. Then one derives the result for the original dynamics using monotonicity. The chain of dynamic modification is a bit longer and tedious in this case that for the upper-bound 46.

## 5 Positive Magnetic Fields: Proof of Theorem 3 when the Initial Condition is a Square

The proof of the scaling limit of the – droplet in this case uses several techniques in common with the zero-magnetic field case, but several features of the model makes it much simpler in a sense:

- The scaling limit, although being nonsmooth, is a much simpler object than anisotropic motion by curvature. In particular, there is no instantaneous travel of information on the rescaled picture: if one modifies one side of the droplet, it will take a positive time to have an effect on regions that are at a positive distance of where the modification took place.
- $+$ spins remain $+$ for all times, making the set of – a decreasing set in time.

Recall that the aim of this section is to prove Theorem 3 in the special case where $\sigma(0)$ is equal to – in $[-L,L]^2 \cap (\mathbb{Z}^*)^2$ and $+$ outside of it. Recall that in that case $\mathcal{D}(t) = \cap_{i=1}^4 \mathcal{D}_i(t)$ where $\mathcal{D}_i(t)$ is defined in Eq. 22.

The upper inclusion $\frac{1}{L}\mathcal{A}_L(t) \subset \mathcal{D}^{(\delta)}(t)$ is an easy consequence of Theorem 4: the idea is to couple the dynamics with four corner growth dynamics using the same clock process for updates. What can be rather surprising is that this simple method gives a sharp bound.

Indeed, for the model based on simple exclusion, one can get an upper bound in the same manner by comparing with four corner growth dynamics, replacing $g(x,t)$ by the solution of the heat equation with initial condition $|x|$ (and time scaling $L$ by $L^2$), but this would not be sharp. Indeed, the shape obtained as an upper-bound with such a technique presents some angles on points with extremal coordinate (poles), whereas the true scaling limit is smooth (at least $C^2$). The reason is that the drift of the interface at the pole is positive when the pole is convex, and one needs to apply Theorem 7 to take the drift into account.

Here on the contrary, nothing happens around the poles due to the singularities of the function $b(\theta)$ (recall Eq. 17) that is equal to zero for $\theta = i\pi/2$, $i = 1, \dots, 4$. The main thing to prove to get the lower-bound $\frac{1}{L}\mathcal{A}_L(t) \supset \mathcal{D}^{(-\delta)}(t)$ is to show that the interaction between the four quadrants of our shape is quite limited.

It is quite difficult to control directly what happens around the pole when the interface is not completely flat, but one finds a way to bypass this problem. Similarly to what is done in [14] in the case of dynamics for polymer with an attractive substrate, one adds, in a quite artificial manner, a flat part of interface around the pole, and modify a bit the statement that has to be proved (see Lemma 2 below). Adding this flat part makes the evolution of the four corners almost independent of one another for some positive time, and thus allows to use Theorem 5 to control the evolution of each corner and thus of the total shape.

## 5.1 The Upper-Bound

The upper-inclusion of Theorem 3 follows quite easily from Theorem 4. We prove it in this section.

**Proposition 3** *For any $\delta > 0$, for the dynamic with $h = \infty$ starting from the initial condition – in $[-L, L]^2 \cap (\mathbb{Z}^*)^2$ and + outside of it, one has w.h.p.*

$$\frac{1}{L}\mathcal{A}_L(Lt) \subset \mathcal{D}^{(\delta)}(t), \forall t \geq 0, \tag{57}$$

*where $\mathcal{D}(t)$ is defined by Eqs.* 22 and 23.

*Proof* Using the graphical construction of Sect. 3.1, we can couple the dynamics $\sigma(t)$ with initial condition $\sigma_0$: – in $[-L, L]^2$ (we drop intersection with $(\mathbb{Z}^*)^2$ in the notation for conciseness) and + elsewhere with other dynamics using the same clock process:

- The dynamics $\sigma_1$ with initial condition – in $[-L, \infty)^2$ and + elsewhere.
- The dynamics $\sigma_2$ with initial condition – in $[-L, \infty) \times [-\infty, L)$ and + elsewhere.
- The dynamics $\sigma_3$ with initial condition – in $[-\infty, L)^2$ and + elsewhere.
- The dynamics $\sigma_4$ with initial condition – in $[-\infty, L) \times [-L, \infty)$ and + elsewhere.

We define $\mathcal{A}^1_L(t)$, $\mathcal{A}^2_L(t)$, $\mathcal{A}^3_L(t)$, $\mathcal{A}^4_L(t)$ analogously to $\mathcal{A}_L(t)$ of Eq. 8 for these four dynamics. According to monotonicity properties of the dynamics in the initial condition, one has $\mathcal{A}_L(t) \subset \mathcal{A}_L{}^i(t)$ for all $i \in [1,4]$ so that:

$$\mathcal{A}_L(t) \subset \mathcal{A}_L^1(t) \cap \mathcal{A}_L^2(t) \cap \mathcal{A}_L^3(t) \cap \mathcal{A}_L^4(t). \tag{58}$$

The dynamics $\sigma^1$ is the same as the one considered in Theorem 4 with a space shift of the initial condition, and thus by Theorem 4,

$$\mathcal{A}_L{}^1(t) \subset \{x\mathbf{f}_1 + y\mathbf{f}_2 \mid y \geq g(x,t) - 2 - \delta\} =: A_1^\delta(t). \tag{59}$$

Defining $A_i^\delta(t)$, $i = 2,3,4$ as rotations of $A_1^\delta(t)$ by angle $(i-1)\pi/2$, one gets analogous inclusion for $\mathcal{A}_L{}^i(t)$ by symmetry, and using Eq. 58 we get that w.h.p.

$$\mathcal{A}_L(t) \subset A_1^\delta(t) \cap A_2^\delta(t) \cap A_3^\delta(t) \cap A_4^\delta(t) \subset \mathcal{D}^{(\delta)}(t). \tag{60}$$

## 5.2 *The Lower-Bound*

The aim of this section is to prove the other inclusion of Theorem 3.

**Proposition 4** *For any $\delta > 0$, for the dynamic with $h = \infty$ starting from the initial condition – in $[-L, L]^2 \cap (\mathbb{Z}^*)^2$ and + outside of it, one has w.h.p.*

$$\frac{1}{L}\mathcal{A}_L(Lt) \supset \mathcal{D}^{(-\delta)}(t), \quad \forall t \geq 0, \tag{61}$$

*where $\mathcal{D}(t)$ is defined by Eqs.* 22 and 23.

Let us explain how the proof goes. Consider the sets $\mathcal{A}_L{}^i(t)$ defined in the previous section. The idea in our proof of the lower-bound it to show that the inclusion used in Eq. 58 is almost an equality. We will decompose the proof in two steps:

- First, we prove that when the rescaled time $t$ is smaller than one, the inclusion 58 is indeed an equality with large probability.
- Second, when the rescaled time $t$ is larger than one, we use a special strategy involving adding portions of straight line on the interface around the pole in order to reduce oneself to a case where one can treat the four corner dynamics independently.

The control of the dynamics for times $t \leq 1$ is summarized in the following lemma.

**Lemma 1** *For any $\delta > 0$ one has with high probability, for all $t \leq 1 - \delta$ one has*

$$\mathcal{D}_t^{(-\delta)} \subset \frac{1}{L}\mathcal{A}_L(Lt). \tag{62}$$

*Proof* Define $\mathcal{K}_L$ to be the sets of sites in $[-L, L]^2 \cap (\mathbb{Z}^*)^2$ that are neighboring either the horizontal or vertical axis,

$$\mathcal{K}_L := \{x \in (\mathbb{Z}^*)^2 \cap [-L, L]^2 \mid \min(|\mathrm{x}_1|, |\mathrm{x}_2|) = 1/2\}. \tag{63}$$

Set

$$\tau := \inf\{t \geq 0 \mid \exists x \in \mathcal{K}_L, \sigma_x(Lt) = +\}. \tag{64}$$

The reader can check that for $t \leq \tau$, the evolutions of the four corners do not interact with one another and that

$$\mathcal{A}_L(t) = \mathcal{A}_L^1(t) \cap \mathcal{A}_L^2(t) \cap \mathcal{A}_L^3(t) \cap \mathcal{A}_L^4(t). \tag{65}$$

Moreover, as a consequence, one has

$$\tau := \inf_{i=1\ldots 4} \tau_i := \inf\{t \geq 0 \mid \exists x \in \mathcal{K}_L, \exists i \in \{1,\ldots,4\}, \sigma_x^i(Lt) = +\}. \tag{66}$$

What remains to check is that $\tau_1$ is roughly equal to $L$. If one uses the particle system description from Sect. 3.2, the dynamic $\sigma_1$ corresponds to TASEP with *step* initial condition: i.e., with the negative half-line full of particles and the positive half-line empty.

With this description, $\tau_1$ is the first time that either the rightmost particle, which starts from 0, reaches $L$, or the leftmost empty space (or antiparticle) hits –$L$. As the jump rate of the particles (and thus of antiparticles) is equal to one, the central limit theorem for a sum of independent exponential variables gives us that these time are equal to $L + O(L^{1/2})$ where the second term includes the random normal correction. Thus one can infer that w.h.p.:

$$\tau_1 \geq L - L^{3/4},$$

so that the same is true for $\tau$.

From Theorem 4, one gets that with high probability for all $t \leq 1$,

$$\mathcal{A}_L^1(Lt) \supset \{x\mathbf{f}_1 + y\mathbf{f}_2 \mid y \geq g(x,t) - 2 + \delta\} = A_1^{-\delta}(t). \tag{67}$$

so that combined with Eq. 65 one gets that w.h.p. for all $t \leq 1 - \delta$,

$$\mathcal{A}_L(Lt) \supset A_1^{-\delta}(t) \cap A_2^{-\delta}(t) \cap A_3^{-\delta}(t) \cap A_4^{-\delta}(t) \supset \mathcal{D}^{(-\delta)}(t). \tag{68}$$

We address now the issue of time larger than one. At this time, the different corners start to interact so that one has to find a trick to regain independence. For $t \in [1,4]$, set $d(t) = 2\sqrt{t} - t$ the positive solution of $g(t,x) = -x + 2$. The reader can check that for all $t \geq 1$, $\mathcal{D}(t)$ is inscribed in the square $[-d(t), d(t)]^2$, i.e.,

$$\max_{\mathbf{x}\in\mathcal{D}(\mathbf{t})} \mathbf{x}\cdot\mathbf{e}_1 = -\min_{\mathbf{x}\in\mathcal{D}(t)} \mathbf{x}\cdot\mathbf{e}_1 = d(t), \tag{69}$$

the same being valid for the second coordinate.

We will intersect $\mathcal{D}(t)$ with a smaller square, which gets us a shape with vertical and horizontal edges around the poles. More precisely, given $\delta$, for $t \leq 4(1-\delta)$ (we have in that case $d(t) \geq \delta$ if $\delta$ is small enough) we define the function

$$\bar{g}(x,t) := \begin{cases} g(x,t) - 2 \text{ if } |x| \leq d(t) - \delta, \\ |x| - (d(t) - \delta) + g(d(t) - \delta, t) - 2 \text{ if } |x| \geq d(t) - \delta. \end{cases} \tag{70}$$

It is continuous in $x$, convex in $x$ and has slope equal to $\pm 1$ outside of $[-d(t)+\delta, d(t)-\delta]$. Define $\bar{\mathcal{D}}_1(t)$ to be the epigraph of $\bar{g}(x,t)$ in $(0,\mathbf{f}_1,\mathbf{f}_2)$ and $\bar{\mathcal{D}}_i(t)$ its rotation by an angle $(i-1)\pi/2$, and set

$$\bar{\mathcal{D}}(t) = \bigcap_{i=1}^{4} \bar{\mathcal{D}}_i(t). \tag{71}$$

The reader can check that

$$\bar{\mathcal{D}}(t) = \mathcal{D}(t) \cap [-r(t), r(t)]^2, \tag{72}$$

where $r(t) \in (d(t) - \delta/2, d(t))$ is the unique solution of the equations

$$\bar{g}(x,t) = -x. \tag{73}$$

To check that $r(t) > d(t) - \delta/2$ it is sufficient to observe that

$$\bar{g}(d(t) - \delta/2, t) = \delta/2 + g(d(t) - \delta, t) - 2 < -d(t) + \delta/2, \tag{74}$$

where the last inequality comes from the fact that

$$g(x,t) < -d(t) + 2, \forall x \in (0, d(t)).$$

Hence, $\bar{\mathcal{D}}(t)$ has flat parts around the pole. Thus, if one starts dynamics from a shape approximating $\bar{\mathcal{D}}(t)$ the four corners will not interact instantaneously like in the proof of Lemma 1. We will use this fact to prove the following result.

**Lemma 2** *For any $\delta > 0$ small enough and $\varepsilon \le \delta/4$, one has for all $k \ge 0$ such that $\varepsilon k \le 3 - 4\delta$ w.h.p.,*

$$\frac{1}{L}\mathcal{A}_L((1-\delta)(1+\varepsilon k)L) \supset \bar{\mathcal{D}}(1+\varepsilon k). \tag{75}$$

*As a consequence, for all $t \ge 1-\delta$ one has*

$$\frac{1}{L}\mathcal{A}_L(Lt) \supset \mathcal{D}(t)^{(-4\delta)}. \tag{76}$$

*Proof* The main part of the job is proving Eq. 75, as Eq. 76 is deduced from it by monotonicity in $t$ of $\mathcal{A}_L(t)$. We proceed by induction on $k$. For $k=0$, the result is just a consequence of Lemma 1 for $t = 1-\delta$.

Now we suppose that the result is true for $k$ and and prove it for $k+1$. Using the graphical construction, one can couple $(\sigma((1-\delta)(1+\varepsilon k)+t))_{t\ge 0}$ with $\sigma^k(t)$ a dynamic starting with initial condition – in $L\bar{\mathcal{D}}(1+\varepsilon k)$ and + elsewhere, in such a way that

$$\mathcal{A}_L((1-\delta)(1+\varepsilon k)+t) \supset \mathcal{A}_L{}^k(t), \quad \forall t \ge 0, \tag{77}$$

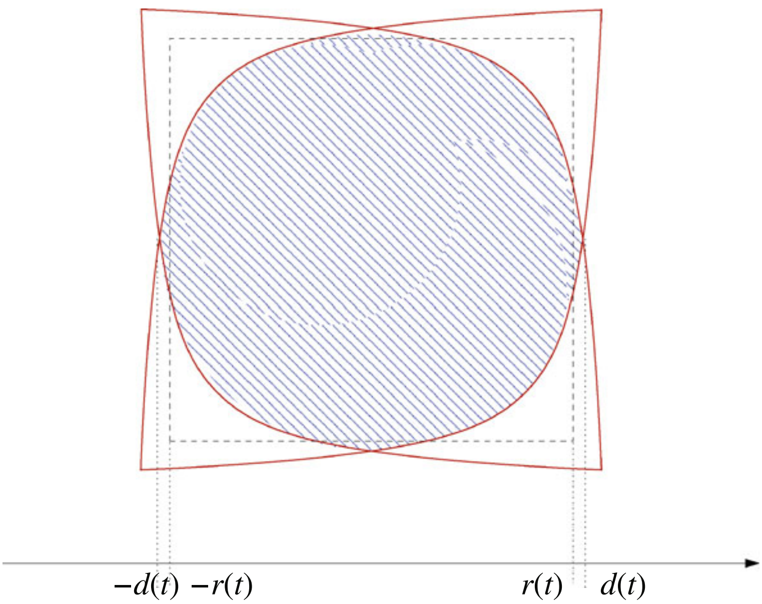


**Fig. 8** The shape $\mathcal{D}(t)$ (*darkened on the figure*) is obtained by intersection four rotated version of the epigraph of the function $x \mapsto g(x,t) - 2$ (recall Eq. 21). For $t \in [1,4]$ (as above) this shape is inscribed in $[-d(t), d(t)]^2$ where $d(t) = 2\sqrt{t} - t$. We define $\bar{\mathcal{D}}(t)$ to be the intersection of $\mathcal{D}(t)$ with $[-r(t), r(t)]^2$ defined in Eq. 73

whenever Eq. 75 holds for $k$, where ${\mathcal{A}_L}^k$ is defined as in Eq. 8 with $\sigma$ replaced by $\sigma^k$.

As Eq. 75 holds w.h.p. from the induction hypothesis, we can prove Eq. 75 w.h.p. for $k+1$ if one proves that w.h.p.

$$\mathcal{A}_L^k((1-\delta)\varepsilon) \supset L\bar{\mathcal{D}}(1+\varepsilon(k+1)). \tag{78}$$

Now, we couple $\sigma^k(t)$ with four interface dynamics $\sigma^{(i,k)}(t)$ starting with initial condition with initial condition – in $L\bar{\mathcal{D}}_i(1+\varepsilon k)$ and + elsewhere using the same clock process.

As we remarked in the proof of Lemma 1, one has

$${\mathcal{A}_L}^k(t) \subset \bigcap_{i=1}^{4} {\mathcal{A}_L}^{(i,k)}(t), \tag{79}$$

and this inclusion is an equality up to the first time a site near the axes changes spin. More precisely set

$$\begin{aligned} \bar{\mathcal{K}}_L &:= \{x \in (\mathbb{Z}^*)^2 \cap [-Lr(1+\varepsilon k), Lr(1+\varepsilon k)]^2 \mid \min(|\mathrm{x}_1|, |\mathrm{x}_2|) = 1/2\} \\ &= \mathcal{K}_L \cap L\bar{\mathcal{D}}(1+\varepsilon k), \end{aligned} \tag{80}$$

and

$$\bar{\tau} := \inf\{t \geq 0 \mid \exists x \in \bar{\mathcal{K}}_L, \sigma_x^k(t) = +\}. \tag{81}$$

One has from the definition of our dynamics that

$$\mathcal{A}_L^k(t) = \bigcap_{i=1}^{4} \mathcal{A}_L{}^{(i,k)}(t), \quad \forall t \leq \bar{\tau}. \tag{82}$$

Our first task is to show that w.h.p. $\bar{\tau} \geq \varepsilon$ in order to be allowed to use Eq. 82. We check that

$$\bar{\tau}_1 := \{t \geq 0 \mid \exists x \in \bar{\mathcal{K}}_L, \sigma_x^{(1,k)}(Lt) = -\} \geq \varepsilon, \tag{83}$$

which by symmetry is sufficient. Indeed

$$\bar{\tau} = \min_{i \in \{1,\ldots,4\}} = \bar{\tau}_i$$

where the $\bar{\tau}_i$ are defined similarly to $\bar{\tau}_1$.

We use again the correspondence with particle system of Sect. 3.2 to do that: the dynamics $\sigma^{(1,k)}$ correspond to a particle system with an initial condition where the rightmost particle is located at $L(d(1 + \varepsilon k) - \delta)$ and the leftmost empty space at $-L(d(1 + \varepsilon k) - \delta)$.

The time $\bar{\tau}_1$ is the first time where either the rightmost particle hits $Lr(1 + \varepsilon k)$ or the leftmost antiparticle hits $-Lr(1 + \varepsilon k)$. Using central limit theorem for the sums of exponential variables one has w.h.p.,

$$\bar{\tau}_1 \geq L((r - d)(1 + \varepsilon k) + \delta) - L^{3/4}. \tag{84}$$

As $r(t) \geq d(t) - \delta/2$, one has

$$\bar{\tau}_1 \geq \delta/4\, L \geq \varepsilon L \quad \text{w.h.p.}$$

provided that $\varepsilon \leq \delta/4$.

Now, having shown that Eq. 82 holds for $t = \varepsilon$ w.h.p., to get Eq. 78, it is sufficient to prove that w.h.p. (recall the definition of $\bar{\mathcal{D}}(t)$ in Eq. 71)

$$\mathcal{A}_L^{(1,k)}((1 - \delta)\varepsilon L) \supset \bar{\mathcal{D}}_1(1 + (k + 1)\varepsilon). \tag{85}$$

As the two sets are epigraphs of functions in $(0, \mathbf{f}_1, \mathbf{f}_2)$, it is sufficient to prove an inequality between functions. Let $\eta^1(x, t)$ denote the interface function corresponding to $\sigma^{(1,k)}$. Theorem 5 give us the scaling limit for the evolution of $\eta^1(x, t)$, which is given by $u(x, t)$ the solution of Eq. 34 with initial condition

$$u_0(x) := \bar{g}(x, 1 + \varepsilon k). \tag{86}$$

Thus, Eq. 85 is proved if we can show that

$$u(x, (1 - \delta)\varepsilon) < \bar{g}(x, 1 + \varepsilon(k + 1)), \quad \forall x \in \mathbb{R}. \tag{87}$$

The reader can readily check, using Eq. 35 that

$$u(x,(1-\delta)\varepsilon) = g(x, 1+\varepsilon(k+1)-\delta\varepsilon) - 2 < \bar{g}(x, 1+\varepsilon(k+1)), \tag{88}$$

$$\forall x \in [-d(1+\varepsilon k)+\delta, d(1+\varepsilon k)-\delta]. \tag{89}$$

What remains to be shown is that the inequality is also valid outside of the interval $[-d(1+\varepsilon k)+\delta, d(1+\varepsilon k)-\delta]$. Set $x > d(1+\varepsilon k)-\delta$ (by symmetry of the functions it is sufficient to check this case), as $u(\cdot, t)$ is 1-Lipshitz for all $t$

$$\begin{aligned} u(x,(1-\delta)\varepsilon) &\le u(d(1+\varepsilon k)-\delta,(1-\delta)\varepsilon) + (x - d(1+\varepsilon k)+\delta) \\ &< \bar{g}(d(1+\varepsilon k)-\delta, 1+\varepsilon(k+1)) + (x - d(1+\varepsilon k)+\delta) \\ &= \bar{g}(x, 1+\varepsilon(k+1)), \end{aligned} \tag{90}$$

where the last inequality comes from the definition of $\bar{g}$.

This concludes the proof of Eq. 85, and thus, of Eq. 75.

We can now detail the proof of Eq. 76. For $t \ge 4-4\delta$, we note that that $\mathcal{D}^{(-4\delta)}(t) = \emptyset$ because $d(t) \le 4\delta/\sqrt{2}$ so that the inclusion is trivial.

For $t \in (1-\delta, 4-4\delta]$, note that, with high probability, Eq. 75 holds for all $k$ with $\varepsilon k \le 3-4\delta$. Let $k_t$ be the smallest integer such that $(1-\delta)(1+\varepsilon k) \ge t$. As $\mathcal{A}_L(t)$ decreases in time, from Eq. 75, for all $t \in (1-\delta, 4-4\delta]$ one has w.h.p.

$$\frac{1}{L}\mathcal{A}_L(Lt) \supset \frac{1}{L}\mathcal{A}_L((1-\delta)(1+\varepsilon k_t)) \supset \bar{\mathcal{D}}(1+\varepsilon k_t). \tag{91}$$

Then we note that $\bar{\mathcal{D}}(t) \supset \mathcal{D}^{(-\delta)}(t)$ and that

$$1+\varepsilon k_t \le \frac{t}{1-\delta} + \varepsilon, \tag{92}$$

so that Eq. 91 implies (as $b(\theta) \le 1$ for all $\beta$)

$$\frac{1}{L}\mathcal{A}_L(Lt) \supset \mathcal{D}^{(-\delta)}\left(\frac{t}{1-\delta}+\varepsilon\right) \supset \mathcal{D}^{(-4\delta)}, \tag{93}$$

if $\delta$ is chosen small enough.

Proposition 4 is just the concatenation of Lemma 1 with Lemma 2.

## 6 Proof of Theorem 3 with General Initial Condition (Sketch)

In this section, we briefly expose what modifications are needed to adapt the proof above to general condition. The upper-bound part is quite simple. Given a convex shape $\mathcal{D}$, it is possible to identify four poles, which are the points where one of the coordinates takes their extremal value (the pole can degenerate into a segment in case there is a flat part, or two of them might coincide, but this does not create any problem for our definitions).

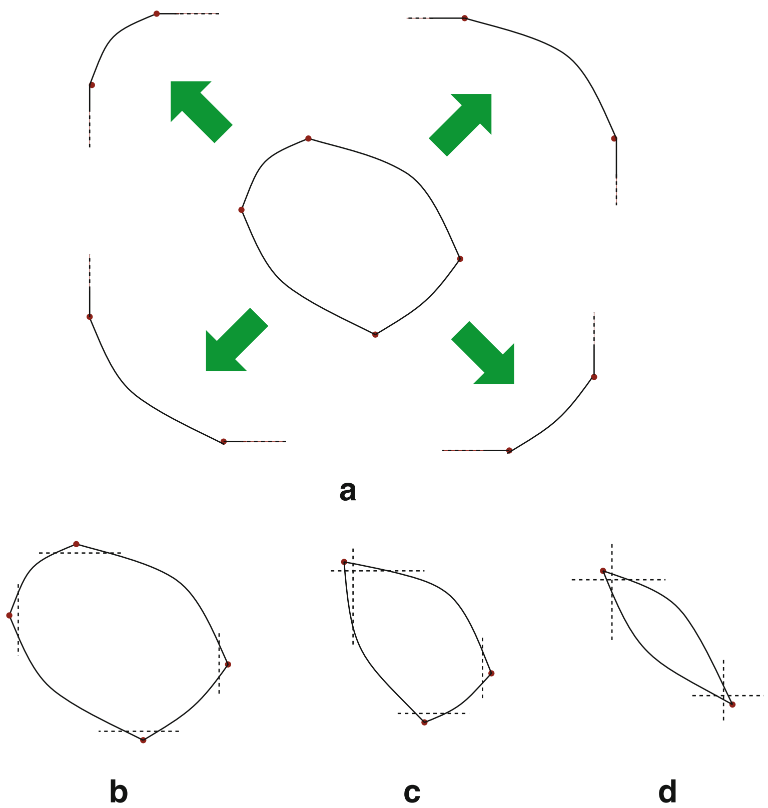


**Fig. 9** In **a,** we show how the initial condition of the dynamics $\sigma^i$, i.e., the functions $g_0^{(i)}$ are extracted from the initial condition. When two poles coincide like in **c** or **d**, the corresponding $g_0^{(i)}$ is just a corner. Items **b–d** show how $\bar{\mathcal{D}}(t)$ is constructed from $\mathcal{D}(t)$: The convex is cut at a distance $\delta$ from where the coordinate take their extremal values

Then when they are not degenerate, the parts of the boundary between two poles can be described as the graph of a convex 1-Lipshitz function in an appropriate frame of coordinate: either $(\mathbf{f}_1, \mathbf{f}_2)$ or a rotated version of it. These functions are defined on an interval of finite length (reduced to a point in case of degeneration) but can be extended on $\mathbb{R}$ by adding semiinfinite lines of slope 1 and $-1$ on the left resp. right of their graph (see Fig. 9). Let us call $(g_0^{(i)})_{i=1}^4$ the obtained functions.

We call $g^{(i)}$ the solution of Eq. 34 given by Eq. 35 with initial condition $g_0^{(i)}$ and let us call $A_i(t)$ the epigraph of $g^{(i)}(t, \cdot)$ in the frame of coordinate used to define $g_0^{(i)}$. Then one has

$$\mathcal{D}(t) := \cap_{i=1}^4 A_i(t), \tag{94}$$

where $\mathcal{D}(t)$ is defined by Eq. 19.

## 6.1 Upper-Bound

To prove the upper-bound, we adopt the proof of Sect. 5.1. We define $\sigma^i$ as the dynamics started with initial condition – on $LA_i(0) \cap (\mathbb{Z}^*)^2$ and + elsewhere, and from Theorem 5 we get that $g_0^{(i)}$ gives the scaling limit of the interface. Hence, we can deduce out of it that w.h.p. $\mathcal{A}^i_L(t)$, the renormalized set of – spins of $\sigma^i$ is contained in the epigraph of $g^i(t,\cdot) - \delta$ that we call $A_i^\delta(t)$ (we use Theorem 5 on a large interval and use the fact that with large probability, there is no particle motion outside of a large interval until time $t$ because of our choice the initial condition).

The we conclude using monotonicity, with Eqs. 58 and 60.

## 6.2 Lower-Bound

For the lower-bound, as in the case of the square, we want to modify our initial condition so that the different corner dynamics that we have defined do not interact immediately. As the shape we start with is arbitrary, this time, it is not guaranteed that things go well until $t = 1$.

Let $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ be defined such that $[x_1(t), x_2(t)] \times [y_1(t), y_2(t)]$ is the smallest rectangle in which $\mathcal{D}(t)$ is inscribed. Given $\delta$ let us define

$$\bar{\mathcal{D}}(t) := \mathcal{D}(t) \cap ([x_1(t) + \delta, x_2(t) - \delta] \times [y_1(t) + \delta, y_2(t) - \delta]). \tag{95}$$

Then one wants to show an result analogous to Lemma 2. Set $T^* := \inf\{t | \mathcal{D}(t) = \emptyset\}$.

**Lemma 3** *For any $\delta > 0$ and $\varepsilon$ small enough, one has for all $k \geq 0$ such that $\varepsilon k \leq T^* - 4\delta$ w.h.p.,*

$$\frac{1}{L}\mathcal{A}_L((1-\delta)\varepsilon k L) \supset \bar{\mathcal{D}}(\varepsilon k). \tag{96}$$

*As a consequence, for all $t \geq 0$ one has*

$$\frac{1}{L}\mathcal{A}_L(Lt) \supset \mathcal{D}(t)^{(-4\delta)}. \tag{97}$$

The proof of the lemma is really similar to the one of Lemma 2. One proves the result by induction over $k$. To perform the induction step, one needs to control the evolution after a time $(1-\delta)\varepsilon$ starting from $\bar{\mathcal{D}}(\varepsilon k)$. One uses the flat parts of $\bar{D}(\varepsilon k)$ to make the four corner dynamics independent of one another until time $(1-\delta)\varepsilon$, and using Theorem 5 to have an approximation of the shape of the domain of – after that time.

Afterwards, one needs to show that the shape obtained with the use of Theorem 5 contains $\bar{\mathcal{D}}(\varepsilon(k+1))$, which is done by proving inequalities between function, as in the case of the square. The details are left to the interested and motivated reader.

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# Airy Processes and Variational Problems

**Jeremy Quastel and Daniel Remenik**

*We thank J. Baik and Z. Liu for pointing out the missing $\sqrt{2}$ on the right-hand side of this equality in an earlier version of this manuscript. See [10] for more details.*

**Abstract** We review the Airy processes—their formulation and how they are conjectured to govern the large time, large distance spatial fluctuations of 1-D random growth models. We also describe formulae which express the probabilities that they lie below a given curve as Fredholm determinants of certain boundary value operators, and the several applications of these formulae to variational problems involving Airy processes that arise in physical problems, as well as to their local behaviour.

## 1 Introduction

### *1.1 Airy Processes and the Kardar–Parisi–Zhang (KPZ) Universality Class*

The *Airy processes* are a collection of stochastic processes which are expected to govern the long time, large scale, spatial fluctuations of random growth models in the 1-D *Kardar-Parisi-Zhang (KPZ) universality class* for wide classes of initial data. Although there is no precise definition of the KPZ class, it can be identified at the roughest level by the unusual $t^{1/3}$ scale of fluctuations. It is expected to contain a large class of random growth processes, as well as randomly stirred 1-D fluids, polymer chains directed in 1-D and fluctuating transversally in the other due to a random potential (with applications to domain interfaces in disordered crystals), driven lattice gas models, reaction-diffusion models in 2-D random media (including biological

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models such as bacterial colonies), randomly forced Hamilton–Jacobi equations, etc. The model giving its name to the universality class is the *KPZ equation*, which was introduced by [59] as a model of randomly growing interfaces, and is given by

$$\partial_t h = -\frac{1}{2}(\partial_x h)^2 + \frac{1}{2}\partial_x^2 h + \xi,$$

where $\xi(t,x)$ is Gaussian space–time white noise, $\mathbb{E}\left(\xi(t,x)\xi(s,y)\right) = \delta_{s=t}\delta_{x=y}$.

A combination of non-rigorous methods (renormalization, mode-coupling, replicas) and mathematical breakthroughs on a few special models has led to very precise predictions of universal scaling exponents and exact statistical distributions describing the long time properties. These predictions have been repeatedly confirmed through Monte-Carlo simulation as well as experiments; in particular, recent spectacular experiments on turbulent liquid crystals by Takeuchi and Sano [80, 81] have been able to even confirm some of the predicted fluctuation statistics in a physical system.

The conjectural picture that has developed is that the universality class is divided into subuniversality classes which depend on the class of initial data (or boundary conditions), but not on other details of the particular models. There are three classes of initial data which stand out because of their self-similarity properties: Dirac $\delta_0$, corresponding to curved, or droplet type initial data; 0, corresponding to growth off a flat substrate; and $e^{B(x)}$ where $B(x)$ is a two-sided Brownian motion, corresponding to growth in equilibrium. As we will see later, each of these three classes correspond to concrete initial (or boundary) conditions for the discrete models in the KPZ class. In addition to these three basic initial data, there are three non-homogeneous subuniversality classes corresponding roughly to starting with one of the basic three on one side of the origin, and another on the other side. For one specific discrete model (last passage percolation or, equivalently, the totally asymmetric exclusion process), the asymptotic spatial fluctuations have been computed exactly for these six basic classes of initial data, and are given by the Airy processes: the three basic Airy processes, $\mathrm{Airy}_2$, $\mathrm{Airy}_1$ and $\mathrm{Airy}_{\mathrm{stat}}$, and the crossover Airy processes $\mathrm{Airy}_{2\to1}$, $\mathrm{Airy}_{2\to\mathrm{BM}}$ and $\mathrm{Airy}_{1\to\mathrm{BM}}$. Although these processes have been proved to arise as the limiting spatial fluctuations only for one model (and actually several others in the case of the $\mathrm{Airy}_2$ process), as a consequence of the universality conjecture for the KPZ class it is expected that the same should hold for the other models in the class.

The purpose of this review is two-fold. First, we will explain in detail in the introduction, the conjectural picture that we have just sketched from two different points of view: last passage percolation (or, more generally, directed random polymers) and the KPZ equation (or, more precisely, the stochastic heat equation). Along the way we will survey known results for these models.

Our second purpose is to survey a collection of results for the Airy processes which express the probability that they lie below a given curve as Fredholm determinants of certain boundary value operators. These expressions have turned out to be very useful in obtaining some exact distributions through certain variational formulae, and in addition have allowed one to study some local properties of these processes. This will be the subject of Sects. 2–4,

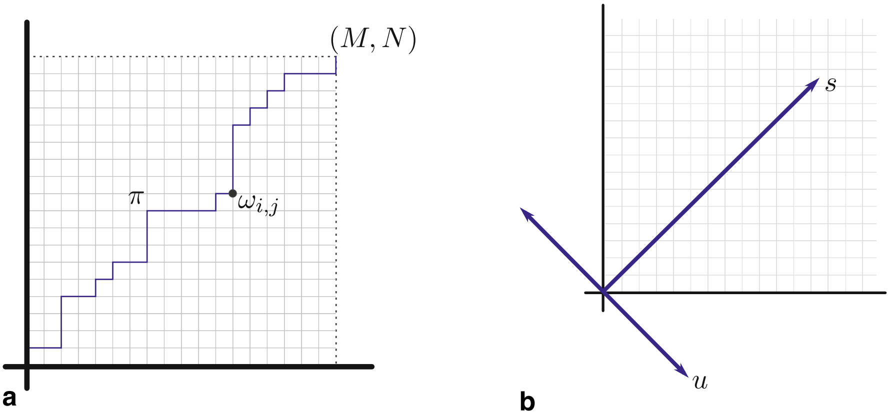


**Fig. 1** **a** A polymer/LPP path $\pi$ connecting the origin to $(M, N)$. **b** Time $s$ and space $u$ axes in LPP

## *1.2 Directed Random Polymers and Last Passage Percolation*

### Polymers

Consider the following model of a *directed polymer in a random environment*. A *polymer path* is an up-right path $\pi = (\pi_0, \pi_1, \dots)$ in $(\mathbb{Z}_+)^2$ started at the origin, that is, $\pi_0 = (0,0)$ and $\pi_k - \pi_{k-1} \in \{(1,0),(0,1)\}$ (see Fig. 1a). On $(\mathbb{Z}_+)^2$ we place a collection of independent random weights $\{\omega_{i,j}\}_{i,j>0}$. The *energy* of a polymer path segment $\pi$ of length $N$ is

$$H_N(\pi) = -\sum_{k=1}^{N} \omega_{\pi_k}.$$

We define the *weight* of such a polymer path segment as

$$W_N(\pi) = e^{-\beta H_N(\pi)} = e^{\beta \sum_{k=1}^{N} \omega_{\pi_k}} \tag{1}$$

for some fixed $\beta > 0$ which is known as the *inverse temperature* of the model. Let $\Pi_{M,N}$ denote the set of upright paths going from the origin to $(M, N) \in (\mathbb{Z}_+)^2$. If we restrict our attention to such paths, then we talk about a *point-to-point polymer*, defined through the following path measure on $\Pi_{M,N}$:

$$Q^{\text{point}}_{M,N}(\pi) = \frac{1}{Z^{\text{point}}(M,N)} W_{M+N}(\pi) \tag{2}$$

The normalizing constant,

$$Z^{\text{point}}(M,N) = \sum_{\pi \in \Pi_{M,N}} W_{M+N}(\pi)$$

is known as the *point-to-point partition function*. Similarly, if we consider all possible paths of length $2N$, then we talk about a *point-to-line polymer*, defined through the following path measure on $\bigcup_{k=-N,\dots,N} \Pi_{N+k,N-k}$ (that is, all paths of length $2N$):

$$Q_N^{\text{line}}(\pi) = \frac{1}{Z^{\text{line}}(N)} W_{2N}(\pi), \tag{3}$$

with the *point-to-line partition function*

$$Z^{\text{line}}(N) = \sum_{k=-N}^{N} Z^{\text{point}}(N+k, N-k).$$

A main quantity of interest in each case is the *free energy*, defined as the logarithm of the partition function. In the point-to-line case, another important quantity of interest is the position of the endpoint of the randomly chosen path, which we will denote by $\kappa_N$. It is widely believed that these quantities should satisfy the scalings

$$\log\left(Z^{\text{point}}(N,N)\right) \sim a_2 N + b_2 N^{\chi} \zeta_2, \tag{4}$$

$$\log\left(Z^{\text{line}}(N)\right) \sim a_1 N + b_1 N^{\chi} \zeta_1, \tag{5}$$

$$\kappa_N \sim N^{\xi} \mathcal{T} \tag{6}$$

as $N \to \infty$, where the constants $a_1, a_2$ and $b_1, b_2$ may depend on the distribution of the $\omega_{i,j}$ and $\beta$, but $\zeta_1, \zeta_2$ and $\mathcal{T}$ should be universal up to some fairly generic assumptions on the $\omega_{i,j}$'s, while the fluctuation exponent

$$\chi = 1/3$$

and wandering exponent

$$\xi = 2/3.$$

Here, and in the rest of this chapter, whenever we write a relation like

$$Z_N \sim aN + bN^{\kappa} \zeta$$

as $N \to \infty$, what we mean is that

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{Z_N - aN}{bN^{\kappa}} \leq m\right) = \mathbb{P}(\zeta \leq m).$$

One can also have higher, $d+1$ dimensional versions of the model, with the paths directed in 1-D, and wandering in the other $d$. In all dimensions, the scaling exponents $\chi$ and $\xi$ are conjectured to satisfy the *KPZ scaling relation*

$$\chi = 2\xi - 1, \tag{7}$$

while the universality of the limiting distributions is unclear except in $d = 1$. For recent progress on (7), see [6, 7, 34].

Although there are few results available in the general case described above, the zero-temperature limit $\beta \to \infty$, known as *last passage percolation*, is very well understood, at least for some specific choices of the environment variables $\omega_{i,j}$. Before introducing this model, we will briefly introduce the Tracy–Widom distributions from random matrix theory, which will, somewhat surprisingly, play an important role in the sequel.

### Tracy–Widom Distributions

We will restrict our attention to the distributions arising from the Gaussian Unitary Ensemble (GUE) and the Gaussian Orthogonal Ensemble (GOE), although these are by no means the only distributions coming from random matrix theory which appear in the study of models in the KPZ universality class. The reader can consult [5, 63] for good expositions on random matrix theory.

We start with the unitary case. Let $\mathcal{N}(a, b)$ denote a Gaussian random variable with mean $a$ and variance $b$. An $N \times N$ *GUE matrix* is an (complex-valued) Hermitian matrix $A$ such that $A_{i,j} = \mathcal{N}(0, N/\sqrt{2}) + \mathrm{i}\mathcal{N}(0, N/\sqrt{2})$ for $i > j$ and $A_{i,i} = \mathcal{N}(0, N)$. Here, we assume that all the Gaussian variables appearing in the different entries are independent (subject to the Hermitian condition). The variance normalization by $N$ was chosen here to make the connection with models in the KPZ class more transparent. An alternative way to describe the Gaussian Unitary Ensemble is as the probability measure on the space of $N \times N$ Hermitian matrices $A$ with density (with respect to the Lebesgue measure on the $N^2$ independent parameters corresponding to the real entries on the diagonal and the real and imaginary components of the entries above the diagonal)

$$\frac{1}{Z_N} e^{-\frac{1}{2}\frac{1}{N}\operatorname{tr} A^2}$$

for some normalization constant $Z_N$. If $\lambda_1^N, \dots, \lambda_N^N$ are the eigenvalues of such a matrix, then the Wigner semicircle law states that the empirical eigenvalue density $N^{-1}\sum_{i=1}^N \delta_{\lambda_i^N}$ has approximately a semicircle distribution on the interval $[-2N, 2N]$. The *Tracy–Widom GUE distribution* [82] arises from studying the fluctuations of the eigenvalues of a GUE matrix at the edge of the spectrum: if we denote by $\lambda_{\mathrm{GUE}}^{\max}(N)$ the largest eigenvalue of an $N \times N$ GUE matrix then [82],

$$\lambda_{\mathrm{GUE}}^{\max}(N) \sim 2N + N^{1/3}\zeta_2$$

as $N \to \infty$, where $\zeta_2$ has the GUE Tracy–Widom distribution, which is defined as follows:

$$F_{\mathrm{GUE}}(s) := \mathbb{P}(\zeta_2 \le s) = \det(I - P_s K_{\mathrm{Ai}} P_s)_{L^2(\mathbb{R})}, \tag{8}$$

where $K_{\mathrm{Ai}}$ is the *Airy kernel*

$$K_{\mathrm{Ai}}(x,y)=\int_0^\infty d\lambda \mathrm{Ai}\,(x+\lambda)\mathrm{Ai}\,(y+\lambda), \tag{9}$$

$\mathrm{Ai}(\cdot)$ is the Airy function, $P_a$ denotes the projection onto the interval $(a,\infty)$, and the determinant means the Fredholm determinant on the Hilbert space $L^2(\mathbb{R})$. We will talk at length about the Airy kernel and related operators in later sections, so for now we will postpone the discussion. Fredholm determinants can be regarded as the natural generalization of the usual determinant to operators on infinite dimensional spaces. We will review their definition and properties in Sect. 2. Since these determinants will appear often during the rest of this introduction, the reader who is not familiar with them may want to read Sect. 2 before continuing.

Before continuing to the $F_{\mathrm{GOE}}$, we quickly note that one of the key contributions of Tracy and Widom [82] was to connect (8) to integrable systems. Let $q(s)$ be the Hastings–McLeod solution of the Painlevé II equation

$$q''(s)=2q(s)^3+sq(s), \tag{10}$$

defined by the additional boundary condition

$$q(s)\sim \mathrm{Ai}(s) \quad \text{as } s\to\infty. \tag{11}$$

Then,

$$F_{\mathrm{GUE}}(s)=e^{-\int_s^\infty dx\,(x-s)^2q^2(x)}.$$

The story for the Gaussian Orthogonal Ensemble is similar. An $N\times N$ *GOE matrix* is a (real-valued) symmetric matrix $A$ such that $A_{i,j}=\mathcal{N}(0,N)$ for $i>j$ and $A_{i,i}=\mathcal{N}(0,\sqrt{2}N)$, where as before we assume that all the Gaussian variables appearing in the different entries are independent (subject to the symmetry condition). Analogously to the GUE case, the Gaussian Orthogonal Ensemble can be regarded as the probability measure on the space of $N\times N$ real symmetric matrices $A$ with density

$$\frac{1}{Z_N}e^{-\frac{1}{4N}\mathrm{tr}A^2}$$

for some normalization constant $Z_N$. As for the GUE, the Wigner semicircle law states that the empirical eigenvalue density for the GOE has approximately a semicircle distribution on the interval $[-2N,2N]$. The fluctuations of the spectrum at its edge now give rise to the *Tracy–Widom GOE distribution*: we denote by $\lambda_{\mathrm{GOE}}^{\max}(N)$ the largest eigenvalue of an $N\times N$ GOE matrix, then [83]

$$\lambda_{\mathrm{GOE}}^{\max}(N)\sim 2N+N^{1/3}\zeta_1$$

as $N\to\infty$, where $\zeta_1$ has the GOE Tracy–Widom distribution, defined as

$$F_{\mathrm{GOE}}(s):=\mathbb{P}(\zeta_1\le m)=\det(I-P_0B_sP_0)_{L^2(\mathbb{R})}, \tag{12}$$

where $B_s$ is the kernel

$$B_s(x, y) = \mathrm{Ai}(x + y + s). \tag{13}$$

This Fredholm determinant formula for $F_{\mathrm{GOE}}$ is essentially due to [76], and was proved in [47]. The original formula derived by Tracy and Widom is

$$F_{\mathrm{GOE}}(s) = e^{-\frac{1}{2}\int_s^\infty dx\, q(x)} \sqrt{F_{\mathrm{GUE}}(s)} \tag{14}$$

with $q$ as above.

### Last Passage Percolation

We come back now to our discussion about directed random polymers, and in particular their zero-temperature limit. We will restrict the discussion to *geometric last passage percolation (LPP)*, where one considers a family $\{\omega_{i,j}\}_{i,j>0}$ of independent geometric random variables with parameter $q$ (i.e. $\mathbb{P}(\omega_{i,j} = k) = q(1-q)^k$ for $k \geq 0$). For convenience, we also set for now $\omega_{i,j} = 0$ if $i$ or $j$ is 0. As $\beta \to \infty$, the random path measures in (2) and (3) assign an increasingly larger mass to the path $\pi$ of length $K > 0$ which maximizes the weight $W_K(\pi)$. In the limit, the path measures $Q^{\mathrm{point}}_{M,N}$ and $Q^{\mathrm{line}}_N$ concentrate on the maximizing path, and the quantities which play the role of the free energy are the *point-to-point last passage time*,

$$L^{\mathrm{point}}(M, N) = \max_{\pi \in \Pi_{M,N}} \sum_{i=0}^{M+N} \omega_{\pi_i}$$

and the *point-to-line last passage time* by

$$L^{\mathrm{line}}(N) = \max_{k=-N,\dots,N} L^{\mathrm{point}}(N + k, N - k). \tag{15}$$

Observe that these last passage times are random, as they depend on the random environment defined by the $\omega_{i,j}$.

The breakthrough, which in a sense got the whole field started, was the surprising 1999 result by Baik, Deift, and Johansson [11] which proved that the asymptotic fluctuations of the longest increasing subsequence of a random permutation have the Tracy–Widom GUE distribution. There is an intimate (and simple) connection between this model and LPP which we will not discuss, instead we will state the companion result by Johansson [56] for the point-to-point LPP case:

$$L^{\mathrm{point}}(N, N) \sim c_1 N + c_2 N^{1/3} \zeta_2, \tag{16}$$

where $c_1, c_2$ are some explicit constants which depend only on $q$ and can be found in [56] and $\zeta_2$ has the Tracy–Widom GUE distribution. A similar result holds for point-to-line LPP. The longest increasing subsequence version goes back to Baik and Rains [9], while the analogue for LPP which we state here was first proved in [26] (see also [25, 76]):

$$L^{\mathrm{line}}(N) \sim c'_1 N + c'_2 N^{1/3} \zeta_1, \tag{17}$$

where $\zeta_1$ now has the GOE Tracy–Widom distribution.

The reason why these exact results (and others we will discuss below) can be obtained for geometric last passage percolation and other related models is that the LPP has an extremely rich algebraic structure which allows one to write explicit formulae for the distribution of the last passage times. The algebraic structure arises from regarding the model as a randomly growing Young tableau, where the cell $(i, j)$ is added at time $L^{\text{point}}(i, j)$. This shift of perspective relates the problem to the representation theory of the symmetric group, and in particular to the Robinson–Schensted–Knuth (RSK) correspondence, which is the main combinatorial tool used in [56] to prove the following remarkable formula:

$$\mathbb{P}(L^{\text{point}}(M, N) \leq s) = \det(I - P_s K_N^{\text{Meix}} P_s)_{L^2(\mathbb{R})}$$

for $M \leq N$, where the *Meixner kernel* $K_N^{\text{Meix}}$ is given by

$$K_N^{\text{Meix}}(x, y) = \frac{\kappa_N}{\kappa_{N-1}} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x - y} \sqrt{w(x)w(y)},$$

$w(x) = \binom{M-N+x}{x}$, and the functions $p_N(x)$ are the normalized Meixner polynomials, i.e. the normalized family of discrete orthogonal polynomials $p_N(x)$ with respect to the weight $w(x)$, with $p_N(x)$ of degree $N$ and leading coefficient $\kappa_N$. A non-trivial asymptotic analysis of this kernel allowed Johansson to deduce that the above Fredholm determinant converges as $N \to \infty$ to the Fredholm determinant appearing in the definition (8) of the Tracy–Widom GUE distribution. A more detailed discussion of these facts is beyond the scope of this review; what the reader should keep from this discussion is that the exact results which we are discussing depend crucially on what is usually referred to as *exact solvability* or *integrability*: the availabilty of (extremely non-trivial) exact formulae for quantities of interest. These formulae arise from the very rich algebraic structure present in some (but by no means all) models in the KPZ class. For a recent survey on this subject, see [23].

As part of the general KPZ universality conjecture, one expects that (16) and (17) hold not only for LPP, but in general for any $\beta > 0$. In other words, the belief is that in (4) and (5), the random variables $\zeta_2$ and $\zeta_1$ have respectively the Tracy–Widom GUE and GOE distributions. There has been only partial progress in proving this conjecture for point-to-point directed polymers (and virtually none in the point-to-line case), the difficulty lying in the lack of exact solvability. Versions of this conjecture have been proved for two related models in the point-to-point case: the continuum random polymer in [4] (building on results of [87–89]) and the semi-discrete polymer of O'Connell and Yor in [22, 30] (see also [66]). In the setting of discrete directed random polymers, [40] showed that if the weights are chosen so that $-w_{i,j}$ is distributed as the logarithm of a Gamma random variable with parameter $\theta_i + \hat{\theta}_j$ (for some fixed $\theta_i$'s and $\hat{\theta}_j$'s) then the model is exactly solvable in the sense explained above. This was later used in [31] to prove that the asymptotic fluctuations of the free energy of the point-to-point polymer (at least for low enough temperature) have the conjectured Tracy–Widom GUE distribution.

### Spatial Fluctuations and the Airy Processes

The Airy processes arise from LPP when we look not only at the fluctuations of the free energy at a single site, but instead at several sites. To this end, we define the rescaled point-to-point process $u \mapsto H_N^{\text{point}}(u)$ by linearly interpolating the values given by scaling $L^{\text{point}}(M, N)$ through the relation

$$L^{\text{point}}(N+u, N-u) = c_1 N + c_2 N^{1/3} H_N^{\text{point}}(c_3 N^{-2/3} u) \tag{18}$$

for $u = -N, \dots, N$, where the constants $c_i$ have explicit expressions which depend only on $q$ and can be found in [57]. Observe this corresponds to looking at the free energy at a line of slope $-1$ passing through $(N, N)$. The limiting behaviour of $H_N^{\text{point}}$ is described by the Airy$_2$ process $\mathcal{A}_2$ (minus a parabola, see Theorem 1). This process was introduced by [68], and is defined through its finite-dimensional distributions, which are given by a Fredholm determinant formula: given $x_0, \dots, x_n \in \mathbb{R}$ and $u_1 < \cdots < u_n$ in $\mathbb{R}$,

$$\mathbb{P}(\mathcal{A}_2(u_1) \le x_1, \dots, \mathcal{A}_2(u_n) \le x_n) = \det\left(I - \mathrm{f}^{1/2} K_{\mathrm{Ai}}^{\mathrm{ext}} \mathrm{f}^{1/2}\right)_{L^2(\{u_1,\dots,u_n\}\times\mathbb{R})}, \tag{19}$$

where we have counting measure on $\{u_1, \dots, u_n\}$ and Lebesgue measure on $\mathbb{R}$, f is defined on $\{u_1, \dots, u_n\} \times \mathbb{R}$ by

$$\mathrm{f}(u_j, x) = \mathbf{1}_{x \in (x_j, \infty)}, \tag{20}$$

and the *extended Airy kernel* [49, 62, 68] is defined by

$$K_{\mathrm{Ai}}^{\mathrm{ext}}(u, \xi; u', \xi') = \begin{cases} \int_0^\infty d\lambda\, e^{-\lambda(u-u')} \mathrm{Ai}(\xi+\lambda)\mathrm{Ai}(\xi'+\lambda), & \text{if } u \ge u' \\ -\int_{-\infty}^0 d\lambda\, e^{-\lambda(u-u')} \mathrm{Ai}(\xi+\lambda)\mathrm{Ai}(\xi'+\lambda), & \text{if } u < u'. \end{cases} \tag{21}$$

Although it is not obvious from the definition, the Airy$_2$ process is stationary (this will become clear in Sect. 3.1), and as should be expected from (16), $\mathbb{P}(\mathcal{A}_2(u) \le m) = F_{\mathrm{GUE}}(m)$ for all $u$. There is a close connection, which we will explain in Sect. 1.5, between the Airy kernel $K_{\mathrm{Ai}}$ appearing in the definition (8) of the Tracy–Widom GUE distribution and the extended kernel $K_{\mathrm{Ai}}^{\mathrm{ext}}$.

The precise result linking the point-to-point LPP spatial fluctuations to the Airy$_2$ process is due to [57] (see also [68]):

**Theorem 1** (Johansson [57]). *There is a continuous version of $\mathcal{A}_2$, and*

$$H_N^{\text{point}}(u) \xrightarrow[N\to\infty]{} \mathcal{A}_2(u) - u^2$$

*in distribution in the topology of uniform convergence of continuous functions on compact sets.*

In the LPP picture, the 'time' variable (which we will denote by $s$) flows in the $(1, 1)$ direction of the plane, while 'space' (which, as above, we will denote by $u$) corresponds to the direction $(1, -1)$ (see Fig. 1b). In this sense, Theorem 1 describes the spatial fluctuations of the point-to-point last passage times as time $s \to \infty$.

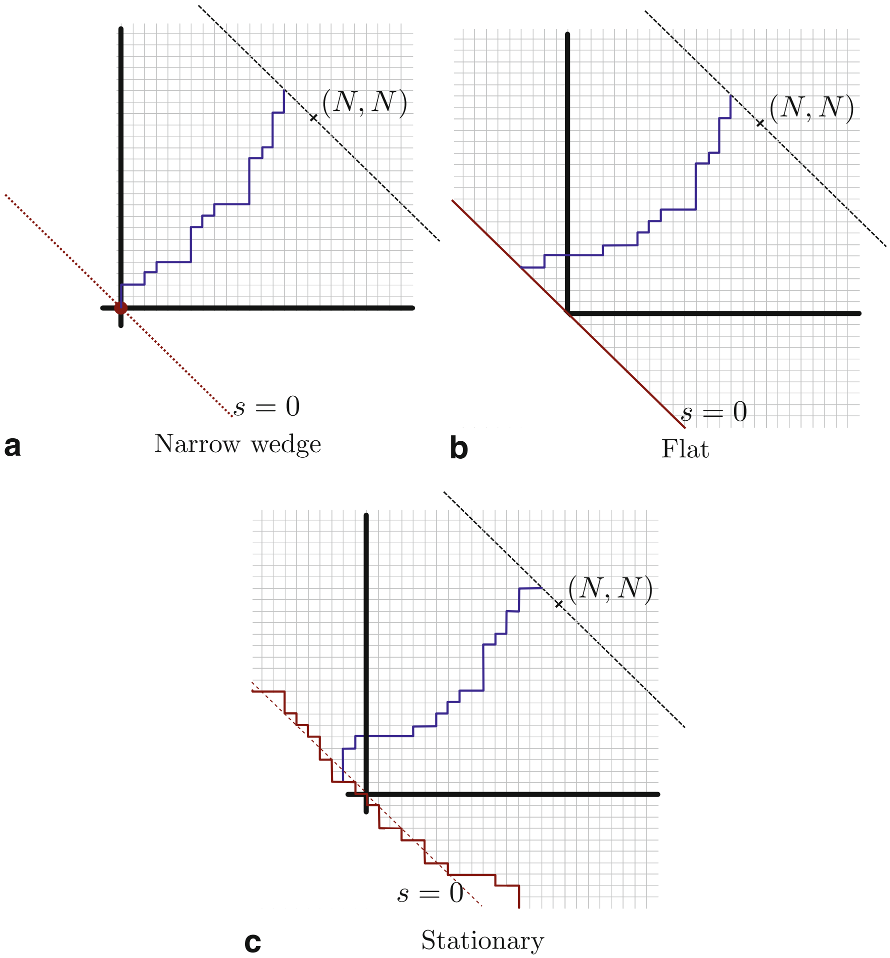


**Fig. 2** Schematic representation of the LPP models with asymptotic spatial fluctuations given by: (a) $\mathrm{Airy}_2$; (b) $\mathrm{Airy}_1$; (c) $\mathrm{Airy}_{\mathrm{stat}}$

One can think of extending the LPP model to paths starting at $s = 0$ with any space coordinate, i.e. paths which start at any point of the form $(k, -k)$, $k \in \mathbb{Z}$. To recover point-to-point LPP, one simply sets $\omega_{i,j} = 0$ whenever $i \leq 0$ or $j \leq 0$, which is easily seen to be equivalent (from the point of view of last passage times) to forcing our paths to start at the origin. In this sense, point-to-point LPP and the $\mathrm{Airy}_2$ process correspond to the $\delta_0$ (also known as *delta*, *narrow wedge* or *curved*) initial data (see Fig. 2a). Note that in this case, we only assign positive weights to sites such that $s > |u|$. To recover the flat, stationary and mixed initial data which we introduced earlier, we need to assign weights to sites such that $s \leq |u|$.

*Remark* 1 The results for the flat, stationary and mixed initial data have been proved in settings which differ slightly from the one introduced here. To avoid additional notation and complications, we will state the results on the present setting. We refer the reader to the corresponding references for more details on the differences. In the case of multi-point results, one can translate between the various settings by using the slow decorrelation result proved of [39, 45], as done in [13, 38].

We start with the *flat* initial data. It corresponds to extending the weights $\omega_{i,j}$ to be independent geometric random variables with parameter $q$ whenever $i + j > 0$ and setting $\omega_{i,j} = 0$ otherwise (see Fig. 2b). This corresponds to letting our paths start at any site in the line $s = 0$ but not attaching any additional weights along that line, which explains the name, 'flat'. The corresponding point-to-line rescaled process may be defined as follows: First we extend the definition of last passage times to accommodate the flat initial data,

$$L^{\text{point}}_{\text{flat}}(M, N) = \max_{i \in \mathbb{Z}} \max_{\pi \in \Pi_{(i,-i)\to(M,N)}} \sum_{j=0}^{2i+M+N} \omega_{\pi_j}$$

with self-explanatory notation, and then we define the rescaled process $u \mapsto H^{\text{line}}_N(u)$ by linearly interpolating the values given the relation

$$L^{\text{point}}_{\text{flat}}(N + u, N - u) = c_1 N + c_2 N^{1/3} H^{\text{line}}_N(c_3 N^{-2/3} u)$$

for $u = -N, \ldots, N$. The flat initial data give rise to the Airy$_1$ process $\mathcal{A}_1$, which was introduced by [76], and is defined through its finite-dimensional distributions,

$$\mathbb{P}(\mathcal{A}_1(u_1) \le \xi_1, \ldots, \mathcal{A}_1(u_n) \le \xi_n) = \det(I - \mathrm{f} K^{\text{ext}}_1 \mathrm{f})_{L^2(\{u_1,\ldots,u_n\}\times\mathbb{R})}, \tag{22}$$

with f as in (20) and

$$K^{\text{ext}}_1(u, \xi; u', \xi') = -\frac{1}{\sqrt{4\pi(u' - u)}} \exp\left(-\frac{(\xi' - \xi)^2}{4(u' - u)}\right) \mathbf{1}_{u'>u} \tag{23}$$
$$+ \operatorname{Ai}(\xi + \xi' + (u' - u)^2) \exp\big((u' - u)(\xi + \xi') + \tfrac{2}{3}(u' - u)^3\big).$$

The Airy$_1$ process is stationary, and as should be expected from (17), its marginals are given by the Tracy–Widom GOE distribution: $\mathbb{P}(\mathcal{A}_1(u) \le m) = F_{\text{GOE}}(2\,m)$ for all $u$.

**Theorem 2** ([25–27]).

$$H^{\text{line}}_N(u) \xrightarrow[N\to\infty]{} 2^{1/3} \mathcal{A}_1(2^{-2/3} u)$$

*in the sense of convergence of finite-dimensional distributions (on a slightly different setting than the one presented here, see Remark 1).*

The powers of $2^{1/3}$ in the above limit should be regarded as an arbitrary normalization (in fact, one could have defined the Airy$_1$ process as this scaled version of it). The appearance of these factors has to do with the fact that the natural scaling in the definition of these quantities differs between random matrix models and models such as LPP or directed polymers (as, for instance, in (25) below). See Sect. 2 of [41] for a related discussion.

The *stationary* initial data is slightly more cumbersome to introduce. The name stationary comes from the fact that for the closely related totally asymmetric exclusion process (TASEP), this initial condition corresponds to starting with particles placed according to a product Bernoulli measure with parameter $1/2$, which is stationary for the process. Translated to LPP, this initial condition corresponds to the following: Let $(S_n)_{n\in\mathbb{Z}}$ be the path of a double-sided simple random walk on $\mathbb{Z}$ with $S_0 = 0$, which we assume to be independent of the weights $\omega_{i,j}$. We rotate this random walk path by an angle of $-\pi/4$ and then put it along the $s = 0$ line by defining the (random) discrete curve $\gamma_0 = \{(\frac{1}{2}(S(i)+i), \frac{1}{2}(S(i)-i)),\ i \in \mathbb{Z}\}$. We then extend the weights $\omega_{i,j}$ to be independent geometric random variables with parameter $q$ whenever $(i, j)$ lies above $\gamma_0$ and $\omega_{i,j} = 0$ otherwise (see Fig. 2c). The corresponding stationary rescaled process $H_N^{\text{stat}}(u)$ can be defined analogously to the previous cases, by maximizing over paths starting at $\gamma_0$ and going to the anti-diagonal line passing through $(N, N)$. It gives rise to the Airy$_{\text{stat}}$ process, $\mathcal{A}_{\text{stat}}$. Its definition is also given in terms of finite-dimensional distributions involving Fredholm determinants, but the formulae are a lot more cumbersome. We will not need the exact formulae, so we refer the reader to [13] for the details. Despite its name, $\mathcal{A}_{\text{stat}}$ is not stationary as a process. In fact, due to the connection with stationary TASEP, $\mathcal{A}_{\text{stat}}$ is just a standard double-sided Brownian motion, but with a non-trivial random height shift at the origin given by the Baik–Rains distribution, see [8]. The convergence result in this case is the following:

**Theorem 3** ([13]).

$$H_N^{\text{stat}}(u) \xrightarrow[N\to\infty]{} \mathcal{A}_{\text{stat}}(u)$$

*in the sense of convergence of finite-dimensional distributions (on a slightly different setting than the one presented here, see Remark 1).*

The mixed initial conditions can be obtained by placing one condition on each half of the line $u = 0$. We will explain how this is done in the case of the *half-flat*, or *wedge→flat* initial data, and leave the examples leading to $\mathcal{A}_{2\to\text{BM}}$ and $\mathcal{A}_{1\to\text{BM}}$ to the interested reader (see [29, 38]). To obtain the Airy$_{2\to1}$ process, we extend the weights $\omega_{i,j}$ to be independent geometric random variables with parameter $q$ whenever $i, j > 0$, or $i + j > 0$ with $i < 0$, setting $\omega_{i,j} = 0$ for all other sites. The half-flat rescaled process $H_N^{\text{half-line}}(u)$ is obtained as in the previous cases, and gives rise to the Airy$_{2\to1}$ process, $\mathcal{A}_{2\to1}$. It was introduced by [28], and is given by

$$\mathbb{P}(\mathcal{A}_{2\to1}(u_1) \le \xi_1, \dots, \mathcal{A}_{2\to1}(u_m) \le \xi_m) = \det(I - \mathrm{f} K_{2\to1}^{\text{ext}} \mathrm{f})_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})},$$

with f as in (20) and

$$K_{2\to1}^{\text{ext}}(u,\xi;u',\xi') = -\frac{1}{\sqrt{4\pi(\xi'-\xi)}}\exp\left(-\frac{(\tilde{\xi}'-\tilde{\xi})^2}{4(u'-u)}\right)\mathbf{1}_{u'>u} + \frac{1}{(2\pi\mathrm{i})^2}\int_{\gamma_+} dw \int_{\gamma_-} dz\, \frac{e^{w^3/3+u'w^2-\tilde{\xi}'w}}{e^{z^3/3+uz^2-\tilde{\xi}z}}\frac{2\,w}{(z-w)(z+w)}, \qquad (24)$$

where $\tilde{\xi} = \xi - u^2 \mathbf{1}_{u \le 0}$, $\tilde{\xi}' = \xi' - (u')^2 \mathbf{1}_{u' \le 0}$ and the paths $\gamma_+, \gamma_-$ satisfy $-\gamma_+ \subseteq \gamma_-$ with $\gamma_+ : e^{\mathrm{i}\phi_+}\infty \to e^{-\mathrm{i}\phi_+}\infty$, $\gamma_- : e^{-\mathrm{i}\phi_-}\infty \to e^{\mathrm{i}\phi_-}\infty$ for some $\phi_+ \in (\pi/3, \pi/2)$, $\phi_- \in (\pi/2, \pi - \phi_+)$. As could be expected from the above description, the Airy$_{2\to 1}$ process crosses over between the Airy$_2$ and the Airy$_1$ processes in the sense that $\mathcal{A}_{2\to 1}(u + v)$ converges to $2^{1/3}\mathcal{A}_1(2^{-2/3}u)$ as $v \to \infty$ and $\mathcal{A}_2(u)$ when $v \to -\infty$. The convergence result is the following:

**Theorem 4** ([28]).

$$H_N^{\text{half-line}}(u) - u^2 \mathbf{1}_{u \le 0} \xrightarrow[N\to\infty]{} \mathcal{A}_{2\to 1}(u)$$

*in the sense of convergence of finite-dimensional distributions (on a slightly different setting than the one presented here, see Remark 1).*

Some of these results have been extended to the case where the points at which one computes the corresponding finite-dimensional distributions do not all lie in the same anti-diagonal line, but instead fall in certain *space-like curves* lying close enough to such a line, see e.g. [24, 27] for more details and [38] for further extensions.

From the definitions it is clear that the basic three Airy processes $\mathcal{A}_2$, $\mathcal{A}_1$, and $\mathcal{A}_{\text{stat}}$ are invariant under $\mathcal{A}(u) \mapsto \mathcal{A}(-u)$, but the mixed cases are not.

Since all initial data are superpositions of Dirac masses, there is a sense in which the Airy$_2$ process is the most basic. For example, using the fact that point-to-line last passage times are computed simply as the maximum of point-to-point last passage times, [57] obtained the following celebrated formula as a corollary of 16 and Theorem 1:

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}} \{\mathcal{A}_2(x) - x^2\} \le m\right) = F_{\text{GOE}}(4^{1/3}m). \tag{25}$$

A direct proof of this formula was later provided in [41]. The argument used in this second proof starts with a different expression for the finite-dimensional distributions of $\mathcal{A}_2$ in terms of the Fredholm determinant of a certain boundary value operator. This type of formula, and their extensions to continuum statistics, are the starting point of most of the results we will survey in Sects. 3.2 and 4. For example, as described in Sect. 4, they allow to compute the asymptotic distribution of $\kappa_N$, the position of the endpoint in the maximizing path in point-to-line LPP.

Extrapolating from (25) leads to a conjecture that the one-point marginals of the other Airy processes should be obtained through certain variational problems involving the Airy$_2$ process. To state the precise conjectures, we turn to the stochastic heat equation, whose logarithm is the solution of the KPZ equation. The advantage of this model over LPP and other discrete models is that it is linear in the initial data, and hence, the heuristics are more easily stated in that context. The disadvantage is that most of the argument relies on conjectures based on universality.

### 1.3 The Continuum Random Polymer and the Stochastic Heat Equation

We now consider the continuum version of the finite temperature discrete random polymers (2) and (3). The (point-to-point) continuum random polymer is a random probability measure $P^{\beta,\xi}_{T,x}$ on continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$ and $x(T)=x$ with formal weight

$$e^{-\beta\int_0^T dt\,\xi(t,x(t))-\frac{1}{2}\int_0^T dt\,|\dot{x}(t)|^2}$$

given to the path $x(\cdot)$, where $\xi(t,x)$, $t\geq 0$, $x\in\mathbb{R}$ is space–time white noise, i.e. the distribution-valued Gaussian variable, such that for smooth functions $\varphi$ of compact support in $\mathbb{R}_+\times\mathbb{R}$, $\langle\varphi,\xi\rangle := \int_{\mathbb{R}_+\times\mathbb{R}} dt\,dx\,\varphi(t,x)\xi(t,x)$ are mean zero Gaussian random variables with covariance structure $E[\langle\varphi_1,\xi\rangle\langle\varphi_1,\xi\rangle]=\langle\varphi_1,\varphi_2\rangle$. One can also think of the continuum random polymer as having a density

$$e^{-\beta\int_0^T dt\,\xi(t,x(t))}$$

with respect to the Brownian bridge. Neither prescription makes mathematical sense, but the second one does if one smooths out the white noise $\xi(t,x)$ in space. Removing the smoothing, one find that there is indeed a limiting measure supported on continuous functions $C[0,T]$ which we call $P^{\beta,\xi}_{T,x}$. In fact, it is a Markov process, and one can define it directly as follows: Let $z(s,x,t,y)$ denote the solution of the stochastic heat equation after time $s\geq 0$ starting with a delta function at $x$,

$$\partial_t z=\tfrac{1}{2}\partial_y^2 z-\beta\xi z,\qquad t>s,\ y\in\mathbb{R},\qquad z(s,x,s,y)=\delta_x(y). \tag{26}$$

It is important that they are all using the same noise $\xi$. Note that the stochastic heat equation is well posed [92]. The solutions look locally like exponential Brownian motion in space. They are Hölder $\frac{1}{2}-\delta$ for any $\delta>0$ in $x$ and $\frac{1}{4}-\delta$ for any $\delta>0$ in $t$. In fact, exponential Brownian motion $e^{B(x)}$ is invariant up to multiplicative constants, i.e. if one starts (26) with $e^{B(x)}$ where $B(x)$ is a two-sided Brownian motion, then there is a (random) $C(t)$ so that $C(t)z(t,x)$ is an exponential of another two-sided Brownian motion [15]. $P^{\beta,\xi}_{T,x}$ is then defined to be the probability measure on continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$ and $x(T)=x$ and finite-dimensional distributions

$$P^{\beta,\xi}_{T,x}(x(t_1)\in dx_1,\dots,x(t_n)\in dx_n) = \frac{z(0,0,t_1,x_1)z(t_1,x_1,t_2,x_2)\cdots z(t_{n-1},x_{n-1},t_n,x_n)z(t_n,x_n,T,x)}{z(0,0,T,x)}dx_1\cdots dx_n$$

for $0<t_1<t_2<\cdots<t_n<T$. One can check these are a.s. a consistent family of finite-dimensional distributions. This holds basically because of the Chapman–Kolmogorov equation

$$\int_{-\infty}^{\infty} du\,z(s,x,\tau,u)z(\tau,u,t,y)=z(s,x,t,y)$$

for $s < \tau \leq t$, which is a consequence of the linearity of the stochastic heat equation.

Note that the construction is for each $T > 0$ fixed. Unlike the usual case of diffusions, the measures are very inconsistent for varying $T$. One should imagine that the polymer paths are peeking into the future to see the best route, so the measure depends considerably on all the noise in the time interval $[0, T]$. We can also define the joint measure $\mathbb{P}^{\beta}_{T,x} = P^{\beta,\xi}_{T,x} \otimes Q(\xi)$ where $Q$ is the distribution of the $\xi$, i.e. the probability measure of the white noise.

**Theorem 5** ([3]).

(i) *The measures* $P^{\beta,\xi}_{T,x}$ and $\mathbb{P}^{\beta}_{T,x}$ *are well-defined (the former, $Q$-almost surely).*

(ii) $P^{\beta,\xi}_{T,x}$ *is a Markov process supported on Hölder continuous functions of exponent* $\frac{1}{2} - \delta$ *for any* $\delta > 0$, *for $Q$-almost every $\xi$.*

(iii) *Let* $t^n_k = \frac{k}{2^n}$. *Then with* $\mathbb{P}^{\beta}_{T,x}$ *probability one, we have that for all* $0 \leq t \leq 1$

$$\sum_{k=1}^{\lfloor 2^n t \rfloor} \left(x(t^n_k) - x(t^n_{k-1})\right)^2 \xrightarrow[n \to \infty]{} t,$$

*i.e. the quadratic variation exists, and coincides with the one obtained for* $\mathbb{P}^{0}_{T,x}$ *(the Brownian bridge measure).*

(iv) $P^{\beta,\xi}_{T,x}$ *is singular with respect to* $\mathbb{P}^{0}_{T,x}$ *(the Brownian bridge measure) for $Q$-almost every $\xi$.*

So the continuum random polymer looks locally like (but is singular with respect to) Brownian motion. One can also define the point-to-line continuum random polymer $\mathbb{P}^{\beta}_{T}$, in the same way as in the discrete case. For large $T$, one expects $\mathrm{Var}_{\mathbb{P}^{\beta}_{T}}(x(T)) \sim T^{4/3}$ in the point-to-line case or $\mathrm{Var}_{\mathbb{P}^{\beta}_{T,0}}(x(T/2)) \sim T^{4/3}$ in the point-to-point case. Here, the variance is over the random background as well as $P^{\beta,\xi}_{T,x}$. The conditional variance given $\xi$ should be much smaller.

If $z(t,x)$ is the solution of (26), then $h(t,x) = -\beta^{-1} \log z(t,x)$ can be thought of as either the (renormalized) free energy of the point-to-point continuum random polymer, or the *Hopf–Cole solution of the KPZ equation*,

$$\partial_t h = -\tfrac{\beta}{2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + \xi, \tag{27}$$

for random interface growth. Since $\log z(t,x)$ looks locally like Brownian motion, (27) is not well posed (see [54] for recent progress on this question). If $\xi$ were smooth, then the Hopf–Cole transformation takes (26) to (27). For white noise $\xi$, we take $h(t,x) = -\beta^{-1} \log z(t,x)$ with $z(t,x)$ a solution of (26) to be the *definition* of the solution of (27). It is known [15] that these are the solutions one obtains if one smooths the noise, solves the equation, and takes a limit as the smoothing is removed (and after subtraction of a diverging constant). They are also the solutions obtained as the limit of discrete models like asymmetric exclusion in the weakly asymmetric limit [15], or directed polymers in the *intermediate disorder limit* [2].

To understand the intermediate disorder limit, we consider how the KPZ Eq. (27) rescales. Let

$$h_\epsilon(t,x) = \epsilon^a h(\epsilon^{-z}t, \epsilon^{-1}x).$$

Recall the white noise has the distributional scale invariance

$$\xi(t,x) \overset{\text{dist}}{=} \epsilon^{\frac{z+1}{2}} \xi(\epsilon^z t, \epsilon^1 x).$$

Hence, setting $\beta = 1$ for clarity,

$$\partial_t h_\epsilon = -\tfrac{1}{2}\epsilon^{2-z-a}(\partial_x h_\epsilon)^2 + \tfrac{1}{2}\epsilon^{2-z}\partial_x^2 h_\epsilon + \epsilon^{a-\frac{1}{2}z+\frac{1}{2}}\xi.$$

Because the paths of $h$ are locally Brownian in $x$, we are forced to take $a = 1/2$ to see non-trivial limiting behaviour. This forces us to take

$$z = 3/2$$

The non-trivial limiting behaviours of models in the KPZ universality class are all obtained in this scale.

On the other hand, if we started with KPZ with noise of order $\epsilon^{1/2}$,

$$\partial_t h = -\tfrac{1}{2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2\, h + \epsilon^{1/2}\xi,$$

then a diffusive scaling,

$$h_\epsilon(t,x) = h(\epsilon^{-2}t, \epsilon^{-1}x),$$

would bring us back to the standard KPZ Eq. (27). This is the intermediate disorder scaling in which KPZ and the continuum random polymer can be obtained from discrete directed polymers. It tells us that if we set

$$\beta = \epsilon^{1/2}\tilde{\beta}$$

in (1), then the distribution of the rescaled polymer path

$$x_\epsilon(t) := \epsilon x_{\lfloor \epsilon^{-2} t \rfloor} \qquad 0 \le t \le T$$

will converge to the continuum random polymer, with temperature $c\tilde{\beta}$ (see [2] for details).

## 1.4 General Conjectural Picture for the SHE

Define $A_t$ from the solution of (26) by

$$z(0,y;t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t} - \frac{t}{24} + 2^{-1/3}t^{1/3}A_t(2^{-1/3}t^{-2/3}(x-y))}. \tag{28}$$

$A_t(\,\cdot\,)$ is called the *crossover Airy process*, the key conjecture being

$$A_t(x) \to \mathcal{A}_2(x), \tag{29}$$

This is known in the sense of 1-D distributions (see [4], where (29) is Conjecture 1.5). A non-rigorous derivation based on a factorization approximation for the Bethe eigenfunctions of the $\delta$-Bose gas can be found in [69]. Note however, the factorization assumption is almost certainly false.

Now one tries to use the linearity of the stochastic heat equation to solve for general initial data $z(0,x) = z_0(x)$,

$$z(t,x) = \int_{-\infty}^{\infty} dy\, \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t} - \frac{t}{24} + 2^{-1/3}t^{1/3}A_t(2^{-1/3}t^{-2/3}(x-y))} z_0(y). \tag{30}$$

It is not hard to see that the equality is correct in the sense of 1-D distributions, but not more. If one wants, for example, joint distributions of $z(t,x_i)$ for more than one $x_i$, then one needs to enhance the crossover Airy process in (28) to a two parameter process $A_t(2^{-1/3}t^{-2/3}x, 2^{-1/3}t^{-2/3}y)$. The conjectural limit of this is a two parameter process, we call the *Airy sheet*. However, we do not even have a full conjecture for its finite-dimensional distributions, though some properties can be described (see [36]).

Calling $\tilde{x} = 2^{-1/3}t^{-2/3}x$ and $\tilde{y} = 2^{-1/3}t^{-2/3}y$ and starting with initial data $z_0(x) = \exp\{2^{-1/3}t^{1/3}f(2^{-1/3}t^{-2/3}x)\}$, we can rewrite the exponent in (30) as

$$2^{-1/3}t^{1/3}[A_t(\tilde{x}-\tilde{y}) - (\tilde{x}-\tilde{y})^2 - f(\tilde{y})] - \tfrac{1}{24}t$$

so that for large $t$, the fluctuation field $2^{1/3}t^{-1/3}[\log z(t,x) + \frac{1}{24}t + \log(\sqrt{2\pi t})]$ is well approximated by

$$\sup_{\tilde{y}\in\mathbb{R}} \{\mathcal{A}_2(\tilde{x}-\tilde{y}) - (\tilde{x}-\tilde{y})^2 - \tilde{f}(\tilde{y})\}.$$

The type of initial data would appear to be quite restrictive, but actually this picks out the appropriate self-similar classes. The easiest example is the flat case $f = 0$. We obtain the statement,

$$\mathcal{A}_1(x) \overset{\text{(d)}}{=} \sup_{y\in\mathbb{R}} \{\mathcal{A}_2(y-x) - (y-x)^2\} \tag{31}$$

in the sense of 1-D distributions. Since the left-hand side is just the GOE Tracy–Widom law, this is the well known theorem of Johansson (25) once again.

If one starts with a two-sided Brownian motion, then the required self-similarity of this initial data is just the Brownian scaling and one arrives at

$$\mathcal{A}_{\mathrm{stat}}(x) \overset{\text{(d)}}{=} \sup_{y\in\mathbb{R}} \{\mathcal{A}_2(y-x) - (y-x)^2 - \sqrt{2}B(y)\}.$$

The mixed cases require a tiny bit more care. Let's explain the heuristics first for the case of the Airy$_{2\to1}$ process. Starting from the step initial data $z(0,x) = \mathbf{1}_{x>0}$, the prediction is

$$-\log z(t,x) \approx \tfrac{1}{2t}x^2\mathbf{1}_{x<0} + \tfrac{1}{24}t + \log(\sqrt{2\pi t}) - 2^{-1/3}t^{1/3}\mathcal{A}_{2\to1}(2^{-1/3}t^{-2/3}x). \tag{32}$$

On the other hand, by linearity, we have for each fixed $x$ in distribution:

$$z(t,x) = \int_0^\infty dy\, z(0,y;t,x) = \int_0^\infty dy\, \tfrac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t} - \frac{t}{24} + 2^{-1/3} t^{1/3} A_t(2^{-1/3} t^{-2/3}(x-y))}. \tag{33}$$

Comparing with (32), we deduce that the processes $\sup_{y\ge 0} (\mathcal{A}_2(x-y) - (x-y)^2)$ and $\mathcal{A}_{2\to 1}(x) - x^2 \mathbf{1}_{x<0}$ should have the same 1-D distribution or, equivalently, that

$$\mathcal{A}_{2\to 1}(x) - x^2 \mathbf{1}_{x<0} \overset{\text{(d)}}{=} \sup_{y\le x} \{\mathcal{A}_2(y) - y^2\} \tag{34}$$

for each fixed $x\in\mathbb{R}$. This distributional identity has actually been proved rigourously, and its proof is based on the methods we will survey in Sects. 3 and 4 (see Theorem 15).

The same heuristic argument works for the other two crossover cases. If we let $z(0,x) = e^{B(x)} \mathbf{1}_{x\ge 0}$, where $B(x)$ is a standard Brownian motion, then (32) and (33) are replaced respectively by

$$-\log z(t,x) \approx \tfrac{1}{2t} x^2 \mathbf{1}_{x<0} + \tfrac{1}{24} t + \log(\sqrt{2\pi t}) - 2^{-1/3} t^{1/3} \mathcal{A}_{2\to \mathrm{BM}}(2^{-1/3} t^{-2/3} x)$$

and

$$\begin{aligned} z(t,x) &= \int_0^\infty dy\, z(0,y;t,x) \\ &= \int_0^\infty dy\, \tfrac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t} - \frac{t}{24} + B(y) + 2^{-1/3} t^{1/3} A_t(2^{-1/3} t^{-2/3}(x-y))}, \end{aligned}$$

and now the same scaling argument allows to conjecture that

$$\mathcal{A}_{2\to \mathrm{BM}}(x) - x^2 \mathbf{1}_{x<0} \overset{\text{(d)}}{=} \sup_{y\le x} (\mathcal{A}_2(y) + \widetilde{B}(x-y) - y^2)$$

for each fixed $x \in \mathbb{R}$, where now $\widetilde{B}(y)$ is a Brownian motion with diffusion coefficient 2. An analogous argument with $z(0,x) = \mathbf{1}_{x\le 0} + e^{B(x)} \mathbf{1}_{x\ge 0}$ translates into conjecturing that

$$\mathcal{A}_{1\to \mathrm{BM}}(x) \overset{\text{(d)}}{=} \sup_{y\in\mathbb{R}} (\mathcal{A}_2(y) + \widetilde{B}(x-y) \mathbf{1}_{y\le x} - y^2)$$

for each fixed $x \in \mathbb{R}$. As we explained, these equalities in distribution will only hold in the sense of 1-D distributions, i.e. for each fixed $x$.

The strategy used in the proof of (34) is considerably more difficult to implement for the other two crossover cases (see Sect. 4.4 and the discussion at the end of Sect. 1.2 in [73]), and in fact these identities remain conjectures for now. In work in progress [42], obtain an improved version of the slow decorrelation result proved in [39], which should allow to prove a general version of formulae for last passage times in last passage percolation in terms of variational problems for the Airy$_2$ process. In particular, such a result would give a proof of these conjectural formulae.

## *1.5 Determinantal Formulae and Extended Kernels*

As we already mentioned, the results we will survey in Sects. 3 and 4 are based on alternative Fredholm determinant formulae for the finite-dimensional distributions of the Airy processes. We will introduce these formulae in Sect. 3, but before doing that let us explain why the original extended kernel formulae are natural. We will first do this in a simpler setting, namely random point processes on finite sets. Then we will explain how similar arguments can be used to derive the formula (19) for the finite-dimensional distributions of the $\text{Airy}_2$ process.

### Extended Kernels and the Eynard–Mehta Theorem

Let $\mathcal{X}$ be a finite set. A *random point process* on $\mathcal{X}$ is a probability measure on the family $2^{\mathcal{X}}$ of subsets of $\mathcal{X}$, which we think of as point configurations. A random point process is called *determinantal* if there exists a $|\mathcal{X}| \times |\mathcal{X}|$ matrix $K$ with rows and columns indexed by the elements of $\mathcal{X}$ such that

$$\rho(A) := \mathbb{P}(\{X \in 2^{\mathcal{X}} : A \subseteq X\}) = \det(K|_A),$$

where $\mathbb{P}$ is the probability measure underlying the point process and $K|_A$ is the submatrix of $K$ indexed by $A$,

$$K_A = [K(x, x')]_{x,x' \in A}.$$

The function $\rho$ is called the *correlation function* of the process, and $K$ is called its *correlation kernel*. For more details see [18] and references therein. The term determinantal was introduced in [19].

We are interested in a particular type of random point processes. Let $\mathcal{X}^1, \dots, \mathcal{X}^n$ be $n$ disjoint finite sets. We consider a point process supported on $kn$-point configurations with the property that there are exactly $k$ points in each $\mathcal{X}^i$. The probability of such a configuration is given as follows: given collections of points $\{x_i^j\}_{i=1,\dots,k} \subseteq \mathcal{X}^j$ for $j = 1, \dots, n$, we set

$$\begin{aligned}
&\mathbb{P}\big(\big\{\{x_i^1\}_{i=1,\dots,k} \cup \dots \cup \{x_i^n\}_{i=1,\dots,k}\big\}\big) \\
&= Z^{-1} \det[\phi_i(x_j^1)]_{i,j=1}^k \det[W_1(x_i^1, x_j^2)]_{i,j=1}^k \\
&\cdots \det[W_{n-1}(x_i^{n-1}, x_j^n)]_{i,j=1}^k \det[\psi_i(x_j^n)]_{i,j=1}^k, \qquad (35)
\end{aligned}$$

where the $\phi_i$'s are some functions on $\mathcal{X}^1$, the $\psi_i$'s are some functions on $\mathcal{X}^n$ and the $W_i$'s are matrices with rows indexed by $\mathcal{X}^i$ and columns indexed by $\mathcal{X}^{i+1}$. The normalization constant $Z$ is chosen so that the total mass of the measure is 1. We are assuming implicitly that the right-hand side above is non-negative for any admissible point configuration.

Write $\Phi$ for the $k \times |\mathcal{X}^1|$ matrix with $k$ rows and columns indexed by elements of $\mathcal{X}^1$ which is defined by $\Phi_{i,x} = \Phi_i(x)$ for $1 \leq i \leq k$ and $x \in \mathcal{X}^1$. Similarly, write

$\Psi$ for the $|\mathcal{X}^n| \times k$ matrix with $k$ columns and rows indexed by $\mathcal{X}^k$ which is defined by $\Psi_{x,i} = \psi_i(x)$ for $1 \le i \le k$ and $x \in \mathcal{X}^n$. Furthermore, define the $k \times k$ matrix

$$M = \Phi W_1 \cdots W_{n-1} \Psi.$$

We will assume that $\det(M) \neq 0$. Under this assumption, it can be shown (see e.g. [21]) that the normalization constant $Z$ in (35) equals $\det(M)$.

The Eynard–Mehta Theorem states that a random point process defined as in (35) is determinantal. Moreover, the theorem gives an explicit formula for the correlation kernel. The precise statement is the following:

**Theorem 6** ([43]). *The random point processes defined by* (35) *is determinantal. Its correlation kernel is the block matrix $K$ with $n \times n$ blocks, such that the $(i, j)$-block has rows indexed by $\mathcal{X}^i$ and columns indexed by $\mathcal{X}^j$, and is given by*

$$K_{i,j} = W_i \cdots W_{n-1} \Psi M^{-1} \Phi W_1 \cdots W_{j-1} - W_i \cdots W_{j-1}.$$

For a simple proof of this result see [21]. Remarkably, the inverse $M^{-1}$ can be computed, or at least approximated, in many cases of interest.

The connection with the models we have discussed so far is through certain families of non-intersecting paths. The Airy$_2$ process can be obtained directly as a limit of the top line of several different families of non-intersecting paths, one of which is presented in Sect. 1.5.2 (for some others see [58]). For the other Airy processes presented in Sect. 1.2.4, the connection with non-intersecting paths is less immediate (see for instance the discussion preceding Lemma 3.4 in [26]), but in any case enough of the above structure remains, and the proofs still rely crucially on a version of Theorem 6. On the other hand, we may think of the $kn$-point configurations where the measure defined in (35) is supported as defining a family of $k$ (in principle not necessarily non-intersecting) paths. For example, the first path would be expressed by $(x_1^1, x_1^2, \ldots, x_1^n)$. It turns out, as we will see below, that probability measures on families of $k$ non-intersecting paths on $\mathcal{X}^1 \cup \cdots \cup \mathcal{X}^n$ are naturally given by expressions like (35), and hence have a determinantal structure. If the sets $\mathcal{X}^i$ are endowed with some total order and we assume that our non-intersecting paths are arranged so that $(x_1^1, x_1^2, \ldots, x_1^n)$ is the top path, then one can prove (see e.g. [57]) that

$$\mathbb{P}\big(x_1^1 \le z_1, \ldots, x_1^n \le z_n\big) = \det(I - PKP), \tag{36}$$

where $z_i \in \mathcal{X}^i$, $K$ is the correlation kernel given by the Eynard–Mehta Theorem and $P$ is block-diagonal matrix with $n$ diagonal blocks defined so that, for $i = 1, \ldots, n$, $P_{i,i}$ has rows and columns indexed by $\mathcal{X}^i$ and is given by $(P_{i,i}v)_j = \mathbf{1}_{x_j^i > z_i} v_j$. This should be compared with an expression like (19).

If we go back to thinking about these paths as defining a random point process, then they are given by a measure on $kn$-point configurations on $\mathcal{X}^1 \cup \cdots \cup \mathcal{X}^n$. Therefore, if the process is determinantal, its correlation kernel necessarily has to be a matrix with rows and columns indexed by $\mathcal{X}^1 \cup \cdots \cup \mathcal{X}^n$. The Eynard–Mehta Theorem implies moreover that the correlation matrix is partitioned naturally into $n \times n$ blocks, with the $(i, j)$ block having rows indexed by $\mathcal{X}^i$ and columns indexed by $\mathcal{X}^j$. To see how this structure relates with the extended kernels introduced in Sect. 1.2.4 for the Airy processes, we make the following observation. An operator $T$ acting on $L^2(\{t_1, \dots, t_n\} \times \mathbb{R})$ can be regarded as an operator-valued matrix $(T_{i,j})_{i,j=1,\dots,n}$ with entries $T_{i,j}$ (acting on $L^2(\mathbb{R})$), which acts on $f \in L^2(\mathbb{R})^n$ as $(Tf)_i = \sum_{j=1}^n T_{i,j} f_j$ (more precisely, we are using the fact that $L^2(\{t_1, \dots, t_n\} \times \mathbb{R})$ and $\bigoplus_{t \in \{t_1,\dots,t_n\}} L^2(\mathbb{R})$ are isomorphic as Hilbert spaces). Hence an extended kernel formula like (19) can be thought of as the determinant of an $n \times n$ matrix whose entries are operators on $L^2(\mathbb{R})$. Similarly, we may think of (36) as the determinant of an $n \times n$ matrix whose $(i, j)$ entry maps $L^2(\mathcal{X}^i)$ to $L^2(\mathcal{X}^j)$. Since the Airy processes live on the real line instead of finite sets, these latter spaces are replaced by $L^2(\mathbb{R})$.

In the next section, we will explain how these ideas can be used to derive the extended kernel formula for $\text{Airy}_2$.

#### Derivation of the $\text{Airy}_2$ Process from Dyson Brownian Motion

The original derivation of the $\text{Airy}_2$ process was done in [68] using quantum statistical mechanical arguments, while Johansson's proof of Theorem 1 relies crucially on the connection between LPP and the Robinson–Schensted–Knuth algorithm, which provides a family of discrete non-intersecting paths, the top line of which converges to $\mathcal{A}_2$. We will briefly explain the derivation of the Fredholm determinant formula for the $\text{Airy}_2$ process using a different model, the Dyson Brownian motion. We refer the reader to [57, 58, 84, 86] for more details on the derivation of $\text{Airy}_2$ from non-intersecting paths.

Consider the evolving eigenvalues of an $N \times N$ GUE matrix with each (algebraically independent) entry diffusing according to a stationary Ornstein–Uhlenbeck process. We write the eigenvalues at time $t$ as $\lambda^N(t) = (\lambda_1^N(t), \dots, \lambda_N^N(t))$ so that $\lambda_i(t)$ decreases in $i$. This eigenvalue diffusion, called the *stationary GUE Dyson Brownian motion*, can be written as the solution of a certain $N$-dimensional SDE, and it can be shown that it is stationary, with distribution given by the eigenvalue distribution of an $N \times N$ GUE matrix. Moreover, the paths followed by the $N$ eigenvalues almost surely form an ensemble of non-intersecting curves.

Suppose we look at this eigenvalue diffusion at times $t_1 < \cdots < t_n$, and we condition the $N$ paths to be pairwise non-intersecting. To investigate the transitions between $t_m$ and $t_{m+1}$, suppose we condition this eigenvalue diffusion to start at time $t_m$ at $\lambda_i^N(t_m) = x_i$ for some fixed $x_1 < \cdots < x_N$, and we also fix destination points $y_1 < \cdots < y_N$. Let $p_t(x, y)$ be the transition probability density of an (1-D) Ornstein–Uhlenbeck process from $x$ at time 0 to $y$ at time $t$. Then, in this setting, the Karlin–McGregor Theorem [60] implies that the transition probability density for these $N$ non-intersecting paths to end at the prescribed destination points $y_1, \dots, y_N$ is given by a constant times

$$\det[p_{t_{m+1}-t_m}(x_i, y_j)]_{i,j=1}^N. \tag{37}$$

The transition function $p_{t_{m+1}-t_m}$ corresponds then to the matrix $W_m$ in (35). Of course, since our paths take values in $\mathbb{R}$ now, we no longer have a matrix, but the Eynard–Mehta Theorem still holds in this setting (see e.g. [84]). The functions $\phi_i$ and $\psi_i$ in 35 are related to the (stationary) marginals for $\lambda_t^N$, and in this case are equal (due to stationarity) and expressed simply in terms of Hermite polynomials. The result after further computations and using (36) is the following [84]: given $x_1, \dots, x_n \in \mathbb{R}$,

$$\mathbb{P}\big(\lambda_1^N(t_1) \le x_1, \dots, \lambda_1^N(t_n) \le x_n\big) = \det(I - \mathrm{f} K_{\mathrm{Hrm},N}^{\mathrm{ext}} \mathrm{f})_{L^2(\{t_1,\dots,t_n\}\times\mathbb{R})},$$

where f is defined as in (20) and $K_{\mathrm{Hrm},N}^{\mathrm{ext}}$ is the *extended Hermite kernel*

$$K_{\mathrm{Hrm},N}^{\mathrm{ext}}(s,x;t,y) = \begin{cases} \sum_{k=0}^{N-1} e^{k(s-t)} \varphi_k(x)\varphi_k(y) & if s \ge t, \\ -\sum_{k=N}^{\infty} e^{k(s-t)} \varphi_k(x)\varphi_k(y) & if s < t, \end{cases}$$

and where $\varphi_k(x) = e^{-x^2/2} p_k(x)$ with $p_k$ the $k$-th normalized Hermite polynomial (so that $\|\varphi_k\|_2 = 1$). Note that $\lambda_1^N$ is the top line of our family of non-intersecting paths, so this probability is the same as the probability that all paths stay below the $x_i$'s. Note also the similarity between this formula and the formula (19) for $\mathcal{A}_2$. We remark that the scaling of the eigenvalues appearing in the last formula differs by a factor of $\sqrt{N}$ with the one introduced in Sect. 1.2.2; the present choice is the one that is naturally associated with the operator $D$ introduced next.

The kernel $K_{\mathrm{Hrm},N}^{\mathrm{ext}}$ has a nice algebraic structure. Writing

$$D = -\tfrac{1}{2}(\Delta - x^2 + 1),$$

(i.e. $Df(x) = -\frac{1}{2}(f''(x) - (x^2-1)f(x))$), the *harmonic oscillator functions* $\varphi_k$ satisfy $D\varphi_k = k\varphi_k$, and moreover $\{\varphi_k\}_{k\ge0}$ forms a complete orthonormal basis of $L^2(\mathbb{R})$. Define the *Hermite kernel* as

$$K_{\mathrm{Hrm},N}(x,y) = \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(y),$$

which is then just the projection onto span $\{\varphi_0, \dots, \varphi_{N-1}\}$. Then the following formula holds:

$$K_{\mathrm{Hrm},N}^{\mathrm{ext}}(s,x;t,y) = \begin{cases} e^{(s-t)D} K_{\mathrm{Hrm},N}(x,y) & \text{if } s \ge t, \\ -e^{(s-t)D}(I - K_{\mathrm{Hrm},N})(x,y) & \text{if } s < t. \end{cases}$$

Now introduce the rescaled process

$$\widetilde{\lambda}_i^N(t) = \sqrt{2} N^{1/6}(\lambda_i^N(N^{-1/3}t) - \sqrt{2N}).$$

Changing variables $x \mapsto \frac{1}{\sqrt{2}N^{1/6}}x + \sqrt{2\,N}$, $y \mapsto \frac{1}{\sqrt{2}N^{1/6}}y + \sqrt{2\,N}$ in the kernel accordingly, a calculation gives

$$\mathbb{P}\big(\widetilde{\lambda}_1^N(t_1) \le x_1, \dots, \widetilde{\lambda}_1^N(t_n) \le x_n\big) = \det(I - \mathrm{f} \widetilde{K}_{\mathrm{Hrm},N}^{\mathrm{ext}} \mathrm{f})_{L^2(\{t_1,\dots,t_n\}\times\mathbb{R})}$$

with

$$\widetilde{K}^{\text{ext}}_{\text{Hrm},N}(s,x;t,y) = \begin{cases} e^{(s-t)H_N}\widetilde{K}_{\text{Hrm},N}(x,y) & if\, s \geq t, \\ -e^{(s-t)H_N}(I - \widetilde{K}_{\text{Hrm},N})(x,y) & if\, s < t, \end{cases}$$

where $\widetilde{K}_{\text{Hrm},N}(x,y) = \frac{1}{\sqrt{2}N^{1/6}} K_{\text{Hrm},N}\Big(\frac{x}{\sqrt{2}N^{1/6}} + \sqrt{2\,N}, \frac{y}{\sqrt{2}N^{1/6}} + \sqrt{2\,N}\Big)$ and the operator $H_N = -\Delta + x + \frac{x^2}{2\,N^{2/3}}$.

The above rescaling corresponds to focusing in on the top curves of the Dyson Brownian motion. It is known that in the limit $N \to \infty$, $\widetilde{K}_{\text{Hrm},N}$ converges to the Airy kernel $K_{\text{Ai}}$, while it is clear that $H_N$ converges to the *Airy Hamiltonian* $H$:

$$H = -\Delta + x \tag{38}$$

(i.e. $Hf(x) = -f''(x) + xf(x)$). Putting aside precise convergence issues, the result is:

$$\lim_{N\to\infty} \widetilde{K}^{\text{ext}}_{\text{Hrm},N}(s,x;t,y) = \begin{cases} e^{(s-t)H}K_{\text{Ai}}(x,y) & if\, s \geq t, \\ -e^{(s-t)H}(I - K_{\text{Ai}})(x,y) & if\, s < t. \end{cases} \tag{39}$$

The obvious question at this point is what is the relationship between this limit and the extended Airy kernel (21). It turns out that they are the same. This is a consequence of the following remark, which implies that the nice structure we saw in $\widetilde{K}_{\text{Hrm},N}$ survives in the limit:

**Remark 2** The shifted Airy functions $\phi_\lambda(x) = \text{Ai}(x - \lambda)$ are the generalized eigenfunctions of the Airy Hamiltonian, as $H\phi_\lambda = \lambda\phi_\lambda$ (we say generalized because $\phi_\lambda \notin L^2(\mathbb{R})$). The Airy kernel $K_{\text{Ai}}$ is the projection of $H$ onto its negative generalized eigenspace. This is seen by observing that if we define the operator $A$ to be the *Airy transform*, $Af(x) := \int_{-\infty}^{\infty} dz\, \text{Ai}(x - z)f(z)$, then $K_{\text{Ai}} = A\bar{P}_0A^*$, where $\bar{P}_0 f(x) = \mathbf{1}_{x<0} f(x)$.

In particular, $e^{tH}$ is defined spectrally. Formally, its integral kernel is given by $e^{tH}(x,y) = \int_{-\infty}^{\infty} d\lambda\, e^{-t\lambda}\text{Ai}(x+\lambda)\,\text{Ai}(y+\lambda)$. The integral converges when $t < 0$ by the decay properties of the Airy function, but it diverges when $t > 0$ (it can be interpreted as $\delta_{x=y}$ when $t = 0$). Nevertheless, in our formulae, $e^{tH}$ will always appear after $K_{\text{Ai}}$ when $t > 0$. This has the effect of restricting the integral to $\lambda > 0$, which converges because the Airy function is bounded.

As a consequence of the above discussion, we obtain the following result, see [84]:

**Theorem 7**

$$\widetilde{\lambda}^N_1(t) \xrightarrow[N\to\infty]{} \mathcal{A}_2(t)$$

*in the sense of convergence of finite-dimensional distributions.*

The extended kernels which define the other Airy processes do not have exactly the same structure. One reason behind this is that, apart from $\mathcal{A}_2$ and $\mathcal{A}_1$, the other processes are not stationary.

The other reason is that, as we mentioned, in some cases (for example $\mathrm{Airy}_1$), the processes are obtained after further limiting procedures, which destroy part of the structure. Nevertheless, as we will see later, enough of this structure remains for our purposes.

We have wandered a bit far from the main subject of this survey in the hope that the reader will get a feeling about why extended kernels appear naturally for our processes. In the rest of this chapter, we will deal with formulae which are given as Fredholm determinants of certain operators acting on $L^2(\mathbb{R})$, as opposed $L^2(\{t_1,\ldots,t_n\}\times\mathbb{R})$. In light of the above discussion, it is slightly surprising that such formulae should exist.

## 2 Fredholm Determinants

If $K$ is an integral operator acting on $H = L^2(X, d\mu)$ through its kernel

$$(Kf)(x) = \int_X K(x,y)f(y)d\mu(y), \tag{40}$$

we define the *Fredholm determinant* by

$$\det(I+\lambda K) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_X \cdots \int_X \det\left[K(x_i,x_j)\right]_{i,j=1}^n d\mu(x_1)\cdots d\mu(x_n). \tag{41}$$

If $|K(x,y)| \le B$ for all $x, y$, and $\mu$ is a finite measure, the *Fredholm series* (41) converges by Hadamard's inequality,

$$|\det(C_1,\ldots,C_n)| \le \|C_1\|\cdots\|C_n\|$$

where $\|C_i\|$ denotes the Euclidean length of the column vector $C_i$, since the length of the column vector in $\left[K(x_i,x_j)\right]_{i,j=1}^n$ is bounded by $Bn^{1/2}$, and hence the $n$-th summand in (41) is bounded by $\frac{\lambda^n}{n!}B^n n^{n/2}$.

If one is not familiar with the definition (41) one might even wonder what it has to do with determinants. Take a matrix $K = \left[K_{ij}\right]_{i,j=1}^d$, $d < \infty$, and consider the $d\times d$ determinant $\det(I+\lambda K)$. Clearly, it is a polynomial of degree $d$ in $\lambda$, $\sum_{n=0}^d a_n\lambda^n$, and its coefficients are given by the rule $a_n = \frac{1}{n!}\partial_\lambda^n \det(I+\lambda K)\,|_{\lambda=0}$. To compute this, use the rule for differentiating determinants,

$$\partial_\lambda \det(C_1,\ldots,C_d) = \sum_{n=1}^{d} \det(C_1,\ldots,\partial_\lambda C_n,\ldots,C_d)$$

and the fact that, in our particular case, $C_n(\lambda) = e_n + \lambda K_{\cdot,n}$ is linear in $\lambda$ and $C_n(0) = e_n$, the $n$-th unit vector. The result is

$$\det(I + \lambda K) = 1 + \lambda \sum_{1\le i\le d} K_{ii} + \lambda^2 \sum_{1\le i<j\le d} \det \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix} + \lambda^3 \sum_{1\le i<j<k\le d} \det \begin{bmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{bmatrix} + \cdots + \lambda^d \det K.$$

Replacing the ordered sums with unordered sums gives a factor $1/n!$, and setting $\lambda = 1$, we can see that this is a special case of (41). Von Koch's idea [91] was that this formula for the determinant was the natural one to extend to $d = \infty$. Fredholm replaced the integral operator (40) on $L^2([0,1], dx)$ by its discretization $[\frac{1}{n}K(\frac{i}{n}, \frac{j}{n})]_{i,j=1}^n$ to obtain (41), which he then used to characterize the solvability of the integral equation $(I + K)u = f$ via the non-vanishing of the determinant of $I + K$.

One can of course imagine other, more intuitive definitions of the determinant. Perhaps

$$\det(I + K) = \prod_n (1 + \lambda_n), \tag{42}$$

where $\lambda_n$ are the eigenvalues of $K$, counted with multiplicity. Or

$$\det(I + \lambda K) = e^{\operatorname{tr}\log(1+\lambda A)} \tag{43}$$

with the trace

$$\operatorname{tr} K = \int d\mu(x)\, K(x, x). \tag{44}$$

Of course, these definitions require some smallness condition on $K$, but at least they make apparent the important fact that the determinant is invariant under conjugation $\det(I + M^{-1}KM) = \det(I + K)$, or

$$\det(I + K_1K_2) = \det(I + K_2K_1), \tag{45}$$

(usually referred to as the cyclic property of determinants) as well as the formula

$$\partial_\beta \det(I + K(\beta)) = \det(I + K(\beta)) \operatorname{tr}((I + K(\beta))^{-1}\partial_\beta K(\beta)) \tag{46}$$

for $K(\beta)$ depending smoothly on a parameter $\beta$.

A more modern way to write (41) is

$$\det(I + \lambda K) = \sum_{n=0}^{\infty} \lambda^n \operatorname{tr} \Lambda^n(K) \tag{47}$$

where $\Lambda^n(K)$ denotes the action of the tensor product $A \otimes \cdots \otimes A$ on the anti-symmetric subspace of $H \otimes \cdots \otimes H$. If $P_n$ denotes the projection onto that subspace and $C_n = P_n \Lambda^n(K) P_n$ then,

$$C_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A f_{\sigma(1)} \otimes \cdots \otimes A f_{\sigma(n)}$$
$$= \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \int \cdots \int d\mu(y_1) \cdots d\mu(y_n)\, K(x_1, y_{\sigma(1)})$$
$$\cdots K(x_n, y_{\sigma(n)}) f_1(y_1) \cdots f_n(y_n)$$

which shows that $C_n$ is an integral operator with kernel $\det\left[K(x_i, x_j)\right]_{i,j=1}^n$ and hence (47) is just a slick way to write (41). The advantage of (41) is that it can be used directly to define the Fredholm determinant for operators on a general separable Hilbert space, but we will not need this point of view here (see [79] for more details).

The natural notion of smallness for Fredholm determinants turns out to be the trace norm on operators

$$\|K\|_1 := \operatorname{tr}|K|,$$

where $|K| = \sqrt{K^* K}$ is the unique positive square root of the operator $K^* K$. An (necessarily compact) operator with finite trace norm is called *trace class*. Using the Parseval relation, one can check that for such operators the trace can be defined as

$$\operatorname{tr} K = \sum_{n=1}^{\infty} \langle e_n, K e_n \rangle,$$

as it is basis independent. This works for operators on any separable Hilbert space, and in our setting it can be shown that this definition of trace coincides with (44) for $K$ of trace class. The *Hilbert–Schmidt norm* $\|K\|_2 = \sqrt{\operatorname{tr}(|K|^2)}$ is easier to compute,

$$\|K\|_2 = \left( \int dx\, dy\, |K(x, y)|^2 \right)^{1/2},$$

and the relation between these norms and the more common operator norm $\|K\|_{\mathrm{op}}$ is

$$\|K\|_{\mathrm{op}} \leq \|K\|_2 \leq \|K\|_1,$$

as well as

$$\|K_1 K_2\|_1 \leq \|K_1\|_2 \|K_2\|_2, \quad \|AK\|_1 \leq \|A\|_{\mathrm{op}} \|K\|_1, \quad \text{and} \quad \|AK\|_2 \leq \|A\|_{\mathrm{op}} \|K\|_2,$$

all of which can be checked easily. Of course, in the latter two, $A$ need not be compact. The reason the trace norm is so useful is:

**Lemma 1**

1. (*Lidskii's Theorem*) If $K$ *is trace class, then* $\operatorname{tr} K = \sum_n \lambda_n$, where $\lambda_n$ *are the eigenvalues* of $K$. It *follows that the three definitions* (41), (42) and (43) are equivalent.
2. $A \mapsto \det(I + A)$ *is continuous in trace norm. Explicitly,*

$$|\det(I + K_1) - \det(I + K_2)| \leq \|K_1 - K_2\|_1 \exp(\|K_1\|_1 + \|K_2\|_1 + 1). \quad (48)$$

Lidskii's theorem is non-trivial and its proofs use heavy function theory, but 48 can be explained easily. Let $f(z) = \det(I + \frac{1}{2}(K_1 + K_2) + z(K_1 - K_2))$, so that the left-hand side of (48) is $|f(\frac{1}{2}) - f(-\frac{1}{2})| \leq \sup_{-1/2\leq t\leq 1/2} |f'(t)|$. Cauchy's integral formula $f'(z) = \frac{1}{2\pi i}\oint \frac{f(z')}{z'-z}dz'$ shows that $\sup_{-1/2\leq t\leq 1/2}|f'(t)| \leq \frac{1}{R}\sup_{|z|\leq R+\frac{1}{2}} |f(z)|$. The eigenvalues of $\Lambda^n(K)$ are $\lambda_{i_1}\cdots\lambda_{i_n}$, $i_1 < \cdots < i_n$, so $\operatorname{tr}\Lambda^n(K) = \sum_{i_1<\cdots<i_n} \lambda_{i_1}\cdots\lambda_{i_n}$ and hence $|\operatorname{tr}\Lambda^n(K)| \leq \frac{1}{n!}\|K\|_1^n$, implies

$$|\det(I + \lambda K)| \leq e^{\lambda\|K\|_1}.$$

Therefore, $\sup_{|z|\leq R+\frac{1}{2}} |f(z)| \leq \exp(\frac{1}{2}\|K_1+K_2\|_1 + (R+\frac{1}{2})\|K_1-K_2\|_1)$ and taking $R = \|K_1 - K_2\|_1^{-1}$ gives (48).

**Examples** 1. *(Gaussian distribution)* A trivial example is $K(x, y) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$. The operator is rank one, so if $P_s$ is the orthogonal projection from $L^2(\mathbb{R}) \to L^2(s,\infty)$ then by (43) we have

$$\det(I - P_s K P_s)_{L^2(\mathbb{R},dx)} = 1 - \operatorname{tr} P_s K P_s = \int_{-\infty}^{s} dx\, \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}.$$

Of course, the Gaussian here could be replaced by an arbitrary density.
2. *(GUE)* Consider the Airy kernel $K_{\mathrm{Ai}}(x, y) = \int_0^\infty dt\,\mathrm{Ai}(x + t)\mathrm{Ai}(y + t)$ and let $\mathrm{Ai}_t(x) = \mathrm{Ai}(x + t)$ and $H = -\partial_x^2 + x$. Then $H\mathrm{Ai}_t = -t\mathrm{Ai}_t$, the $\mathrm{Ai}_t$, $t \in \mathbb{R}$ are generalized eigenfunctions of $H$, and $K_{\mathrm{Ai}}$ is the orthogonal projection onto the negative eigenspace of $H$ (see Remark 2). Using $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$, we have $\partial_t \frac{\mathrm{Ai}(x+t)\mathrm{Ai}'(y+t)-\mathrm{Ai}'(x+t)\mathrm{Ai}(y+t)}{y-x} = \mathrm{Ai}(x + t)\mathrm{Ai}(y + t)$, which yields the Christoffel–Darboux formula,

$$K_{\mathrm{Ai}}(x, y) = \frac{\mathrm{Ai}'(x + t)\mathrm{Ai}(y + t) - \mathrm{Ai}(x + t)\mathrm{Ai}'(y + t)}{y - x}.$$

To show $P_s K_{\mathrm{Ai}} P_s$ is trace class, write $K_{\mathrm{Ai}} = B_0 P_0 B_0$ where

$$B_0(x, y) = \mathrm{Ai}(x + y). \quad (49)$$

Then use $\|K_1K_2\|_1 \leq \|K_1\|_2\|K_2\|_2$ to get

$$\|P_s K_{\mathrm{Ai}} P_s\|_1 \leq \|P_s B_0 P_0\|_2^2 \leq \int_0^\infty \int_s^\infty \mathrm{Ai}^2(x + y)dxdy, \quad (50)$$

which is finite by the following well-known estimates for the Airy function (see (10.4.59–60) in [1]):

$$|\mathrm{Ai}(x)| \le Ce^{-\frac{2}{3}x^{3/2}} \quad \text{for } x > 0, \qquad |\mathrm{Ai}(x)| \le C \quad \text{for } x \le 0. \tag{51}$$

The GUE Tracy–Widom distribution is given by

$$F_{\mathrm{GUE}}(s) = \det(I - P_s K_{\mathrm{Ai}} P_s)_{L^2(\mathbb{R},dx)}.$$

On the face of it, it is not so obvious why such an expression would define a probability distribution function. From 50 it is clear that $\lim_{s\to\infty}\det(I - P_s K_{\mathrm{Ai}} P_s) = 1$. Since $P_s K_{\mathrm{Ai}} P_s$ is a composition of projections, its eigenvalues satisfy $1 \ge \lambda_1(s) \ge \lambda_2(s) \ge \cdots \ge 0$. Recall the min-max characterization of eigenvalues

$$\lambda_k(s) = \max_{\dim U = k}\ \min_{f\in U} \frac{\langle f, P_s K_{\mathrm{Ai}} P_s f\rangle}{\langle f, f\rangle},$$

from which it is apparent that $\lambda_i(s)$ is non-decreasing as $s$ decreases, and hence $\det(I - P_s K_{\mathrm{Ai}} P_s) = \prod_i (1 - \lambda_i(s))$ is non-increasing with decreasing $s$. In fact, $\lambda_1(s) \nearrow 1$ as $s \searrow -\infty$ since if $f$ is in the negative eigenspace of $H$, $\langle P_s f, K_{\mathrm{Ai}} P_s f\rangle \to \langle f, K_{\mathrm{Ai}} f\rangle = \langle f, f\rangle$. This shows that $\det(I - P_s K_{\mathrm{Ai}} P_s) \searrow 0$ as $s \searrow -\infty$ (for an asymptotic expansion of $F_{\mathrm{GUE}}(s)$ as $s \searrow -\infty$ see [12]).

3. *(GOE)* $F_{\mathrm{GOE}}(s) = \det(I - P_s B_0 P_s)_{L^2(\mathbb{R},dx)}$ where $B_0(x,y)$ is as in 49. The key to show that $B_0$ is trace class in this case is the identity

$$\int_{-\infty}^{\infty} dx\, \mathrm{Ai}(a+x)\mathrm{Ai}(b-x) = 2^{-1/3}\mathrm{Ai}(2^{-1/3}(a+b)) \tag{52}$$

(see, for example, (3.108) in [90]). One defines $G_1(x,z) = 2^{1/3}\mathrm{Ai}(2^{1/3}x + z)e^z$ and $G_2(z,y) = e^{-z}\mathrm{Ai}(2^{1/3}y - z)$ and notes that $P_s B_0 P_s = (P_s G_1)(G_2 P_s)$. Then (51) allows to show that each of the last two factors has finite Hilbert–Schmidt norm, yielding that $P_s B_0 P_s$ is trace class.

4. *(Airy$_1$ process)* Recall the Fredholm determinant formula (22) for the finite-dimensional distributions of the Airy$_1$ process. It turns out that the kernel $\mathrm{f} K_1^{\mathrm{ext}} \mathrm{f}$ inside the determinant is not trace class, basically because the heat kernel is not even Hilbert–Schmidt on $L^2([s,\infty))$ for $s \in \mathbb{R}$. Nevertheless, the series (41) defining the Fredholm determinant is finite in this case, because one can conjugate the kernel $\mathrm{f} K_1^{\mathrm{ext}} \mathrm{f}$ to something which can be proved to be trace class (see [25]).

The situation in the last example, where the natural expression for a kernel defines an operator which is not trace class, but which is conjugate to a trace class operator, arises often. Here by conjugacy, we mean the following: two operators $K$ and $\tilde{K}$ are conjugate if there exists some invertible linear mapping $U$ acting on measurable functions on $X$ such that $K = U\tilde{K}U^{-1}$. Observe that such a pair of operators has the same Fredholm series expansion (41), i.e. $\det(I + K) = \det(I + \tilde{K})$. This allows to extend the manipulations on Fredholm determinants to operators which are conjugate to trace class operators, provided that one is careful in keeping track of the needed conjugations.

The reason we start with the Fredholm expansion (41) is that this is the way the determinant usually arises from combinatorial expressions. Sometimes the kernels are not trace class, but this should not bother us so much as long as some version of the formal expression can be shown to converge, for instance as in Example 4 above. Often, it is genuinely difficult to show that the resulting expressions define a probability distribution, and we only know it because they arose this way.

## 3 Boundary Value Kernels and Continuum Statistics of Airy Processes

### *3.1 Boundary Value Kernel Formulae for Finite-Dimensional Distributions*

Recall the formula (19) for the finite-dimensional distributions of the $\text{Airy}_2$ process. It is given in terms of the Fredholm determinant of what we call an *extended kernel*, that is, (the kernel of) an operator acting on the “extended space” $L^2(\{t_1, \ldots, t_n\} \times \mathbb{R})$. Although such formulae have been very useful in the study of models in the KPZ class, they suffer from two problems. First, if one wants to take the number $n$ of times $t_i$ to infinity, a big difficulty appears in the fact that these formulae involve Fredholm determinants on the Hilbert space $L^2(\{t_1, \ldots, t_n\} \times \mathbb{R})$, and thus the space itself is changing as $n$ grows. Second, these formulae are useful for computing long range properties of the processes (for instance an asymptotic expansion of the covariance of $\mathcal{A}_2(s)$ and $\mathcal{A}_2(t)$ as $|t-s| \to \infty$, see [93]), but are not suitable for studying short range properties such as regularity of the sample paths.

The second type of Fredholm determinant formula, which is the one we will use for most of the rest of this chapter, was actually introduced as the original definition of the $\text{Airy}_2$ process by [68]. It is given as follows: for $t_1 < \cdots < t_n$ and $x_1, \ldots, x_n \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{P}\left(\mathcal{A}_2(t_1) \le x_1, \ldots, \mathcal{A}_2(t_n) \le x_n\right) \\ &\quad = \det\left(I - K_{\text{Ai}} + \bar{P}_{x_1} e^{(t_1-t_2)H} \bar{P}_{x_2} e^{(t_2-t_3)H} \cdots \bar{P}_{x_n} e^{(t_n-t_1)H} K_{\text{Ai}}\right)_{L^2(\mathbb{R})}, \end{aligned} \tag{53}$$

where $K_{\text{Ai}}$ is Airy kernel (9), $H$ is the Airy Hamiltonian (38) and $\bar{P}_a$ denotes the projection onto the interval $(-\infty, a]$:

$$\bar{P}_a f(x) = \mathbf{1}_{x \le a} f(x).$$

Note that the Fredholm determinant is now computed on the Hilbert space $L^2(\mathbb{R})$ instead of $L^2(\{t_1, \ldots, t_n\} \times \mathbb{R})$, which makes taking $n \to \infty$ at least feasible. Note also that the time increments $t_i - t_{i+1}$ appear explicitly in the formula, which explains why this formula will be more suitable for the study of short-range properties. Another advantage of this formula is that it makes apparent that $\mathcal{A}_2$ is a stationary process.

The equivalence of (19) and (53) was derived formally in [68] and [69]. The proof in [69] is based in the following idea. As we explained in Sect. 1.5.1, the extended kernel formula (19) can be thought of as the determinant of an $n \times n$ matrix whose entries are operators acting on $L^2(\mathbb{R})$. By rewriting this operator as a sum of an upper-triangular part and lower-triangular part and using algebraic properties of the determinant and the algebraic relationships between the different entries of this matrix, [68] showed that the determinant equals the determinant of an operator-valued matrix $I + G$ such that only the first column of $G$ is non-zero. Therefore $\det(I+G)_{L^2(\{t_1,\dots,t_n\}\times\mathbb{R})} = \det(I+G_{1,1})_{L^2(\mathbb{R})}$ (to see this, simply pretend the operators in the determinants are matrices), and an explicit calculation of $G_{1,1}$ yields (53).

The argument given in [69] which we just sketched is almost a complete proof. There are nevertheless some subtleties. For example, it is not a priori obvious that for $s, t > 0$, $e^{-sH}$ can be applied to the image of $\bar{P}_a e^{-tH}$. Moreover, in order to manipulate Fredholm determinants, one needs to check that certain analytical conditions are satisfied (see Sect. 2). The technical details are discussed in [72], which in fact shows that a formula analogous to 53 holds for the Airy$_1$ process as well. It is given as follows: for $t_1 < \cdots < t_n$ and $x_1, \dots, x_n \in \mathbb{R}$,

$$\begin{aligned}&\mathbb{P}(\mathcal{A}_1(t_1) \le x_1, \dots, \mathcal{A}_1(t_n) \le x_n) \\ &= \det\big(I - B_0 + \bar{P}_{x_1} e^{(t_2-t_1)\Delta} \bar{P}_{x_2} e^{(t_3-t_2)\Delta} \cdots \bar{P}_{x_n} e^{(t_1-t_n)\Delta} B_0\big)_{L^2(\mathbb{R})},\end{aligned} \tag{54}$$

where $B_0$ is given by the kernel $B_0(x, y) = \mathrm{Ai}(x + y)$ defined in (13) and $\Delta$ is the Laplacian operator. Observe that in all but the last factor of the form $e^{s\Delta}$ in the above formula holds that $s > 0$, in which case $e^{s\Delta}$ is the usual heat kernel. This kernel is ill-defined for $s < 0$, but it turns out that in this case the operator $e^{s\Delta} B_0$ makes sense if defined via the integral kernel

$$e^{s\Delta} B_0(x, y) = e^{2s^3/3+s(x+y)} \mathrm{Ai}(x + y + s^2). \tag{55}$$

What we mean by this is that if $s, t > 0$ then, with this definition the semigroup property $e^{t\Delta} e^{-s\Delta} B_0 = e^{(t-s)\Delta} B_0$ holds.

As we will see in Sect. 3.2, it is fruitful to think of the operator appearing in (53) as the solution of certain boundary value problem, so we will refer to formulae like this as *boundary value kernel* formulae. By using (55) one can rewrite the definition (23) of the extended kernel for $\mathcal{A}_1$ as

$$K_1^{\mathrm{ext}}(s, x; t, y) = \begin{cases} e^{(t-s)\Delta} B_0(x, y) & \text{if } s \ge t, \\ -e^{(t-s)\Delta}(I - B_0)(x, y) & \text{if } s < t. \end{cases}$$

It becomes clear then that both the extended kernel formula and the boundary value kernel formula for Airy$_1$ are obtained from the corresponding formulae for Airy$_2$ by substituting $H$ with $-\Delta$ and $K_{\mathrm{Ai}}$ with $B_0$. It turns out, as shown in [32], that the necessary structure behind these formulae hold for a much wider class of processes, including for instance, the stationary GUE Dyson Brownian motion and non-stationary processes like the $\mathrm{Airy}_{2\to1}$ process, and the Pearcey process [85]. For example, for $\mathrm{Airy}_{2\to1}$ one has [32]

$$\begin{aligned}&\mathbb{P}(\mathcal{A}_{2\to1}(t_1)\le x_1,\ldots,\mathcal{A}_{2\to1}(t_n)\le x_n)\\&=\det(I-K^{t_1}_{2\to1}+\bar{P}_{\tilde{x}_1}e^{(t_2-t_1)\Delta}\bar{P}_{\tilde{x}_2}\cdots e^{(t_n-t_{n-1})\Delta}\bar{P}_{\tilde{x}_n}e^{(t_1-t_n)\Delta}K^{t_1}_{2\to1})_{L^2(\mathbb{R})},\qquad(56)\end{aligned}$$

where $\tilde{x}_i=x_i-t_i^2\mathbf{1}_{t_i\le0}$ and $K^t_{2\to1}(x,y)=K^{\mathrm{ext}}_{2\to1}(t,x;t,y)$ with $K^{\mathrm{ext}}_{2\to1}$ as in (24).

Interestingly, it is shown in [32] that in a setting corresponding to discrete non-intersecting paths, analogous boundary value kernel formulae can be obtained directly from applying the Karlin–McGregor formula (37) (or rather its combinatorial analogue, the Lindström–Gessel–Viennot Theorem [51, 61]), bypassing the direct application of the Eynard–Mehta Theorem. In the case of the $\mathrm{Airy}_2$ process, a suitable limit of a discrete family of non-intersecting should lead to (53) (cf. (39)). Such a procedure does not seem to work for the $\mathrm{Airy}_1$ process. In fact, in that case the determinantal process used to derive (23) is signed (in the sense that the measure defined by the analog of (35) is signed), see [26], and hence it is not clear how to associate directly to it a family of non-intersecting paths.

As we will see below, the integral kernels of the operators appearing inside the Fredholm determinants in (53), (54) and (56) can be expressed simply in terms of hitting probabilities of Browian motion. In other words, hitting probabilities of curves by $\mathcal{A}_2$, $\mathcal{A}_1$ and $\mathcal{A}_{2\to1}$ can be expressed in terms of Fredholm determinants of the analogous hitting probabilities for Brownian motion. Given the above discussion (and the discussion in Sect. 1.5.2), this is not entirely surprising in the case of $\mathcal{A}_2$, as it follows from the non-intersecting nature of systems of Brownian paths that can be used to approximate $\mathcal{A}_2$. For the same reason, it is surprising in a sense that the same structure is present in $\mathcal{A}_1$.

## 3.2 Continuum Statistics and Boundary Value Problems

Consider the following problem: compute the probability that inside a finite interval $[\ell,r]$, the $\mathrm{Airy}_2$ process lies below a given function $g$. The obvious way to proceed is to take a fine mesh $\ell=t_1<t_2<\cdots<t_n=r$ of the interval $[\ell,r]$, take $x_i=g(t_i)$, and attempt to take a limit as $n\to\infty$ in the formula for the finite-dimensional distributions of $\mathcal{A}_2$,

$$\begin{aligned}&\mathbb{P}(\mathcal{A}_2(t_1)\le g(t_1),\ldots,\mathcal{A}_2(t_n)\le g(t_n))\\&=\det\big(I-K_{\mathrm{Ai}}+\bar{P}_{g(t_1)}e^{(t_1-t_2)H}\bar{P}_{g(t_2)}e^{(t_2-t_3)H}\cdots\bar{P}_{g(t_n)}e^{(t_n-t_1)H}K_{\mathrm{Ai}}\big).\qquad(57)\end{aligned}$$

Here the Fredholm determinant is computed on the Hilbert space $L^2(\mathbb{R})$, which we will omit from the subscript in the sequel. By Theorem 1, $\mathcal{A}_2$ has a continuous version, and hence $\lim_{n\to\infty}\mathbb{P}(\mathcal{A}_2(t_1)\le g(t_1),\ldots,\mathcal{A}_2(t_n)\le g(t_n))=\mathbb{P}(\mathcal{A}_2(t)\le g(t)\mathrm{for}t\in[\ell,r])$. To study the right-hand side of (57), we need to compute the limit of the operator appearing inside the determinant. Observe that the last exponential equals $e^{(r-\ell)H}$, and hence does not depend on $n$. On the other hand, for $s < t$, the operator $e^{(s-t)H}$ can be thought of as mapping a function $f$ to the solution $u(t,\cdot)$ at time $t$ of the PDE $\partial_t u + Hu = 0$ with initial condition $u(s,\cdot) = f(\,\cdot\,)$. Therefore the operator,

$$\bar{P}_{g(t_1)} e^{(t_1-t_2)H} \bar{P}_{g(t_2)} e^{(t_2-t_3)H} \cdots e^{(t_{n-1}-t_n)H} \bar{P}_{g(t_n)} \tag{58}$$

can be thought of as solving the same PDE (backwards in time) on the interval $[\ell, r]$ with the additional condition that all the mass above $g(t_i)$ is removed at each of the discrete times $t_i$. Note that the PDE is solved backwards because, if we apply this operator to a function on its right, we first apply $\bar{P}_{g(t_n)}$, then $e^{(t_{n-1}-t_n)H}$, then $\bar{P}_{g(t_{n-1})}$, and so on. Since solving the PDE $\partial_t u + Hu = 0$ forward or backwards in time gives the same answer, if we want to think of (58) as being solved forward in time, all we need to do is reverse the order in which the $g(t_i)$ appear. The result is the following: Given $g \in H^1([\ell, r])$ (i.e. both $g$ and its derivative are in $L^2([\ell, r])$), define an operator $\Theta^g_{[\ell,r]}$ acting on $L^2(\mathbb{R})$ as follows: $\Theta^g_{[\ell,r]} f(\,\cdot\,) = u(r,\cdot)$, where $u(r,\cdot)$ is the solution at time $r$ of the boundary value problem

$$\begin{aligned} \partial_t u + Hu &= 0 \quad \text{for}\, x < g(t),\ t \in (\ell, r) \\ u(\ell, x) &= f(x)\mathbf{1}_{x<g(\ell)} \\ u(t,x) &= 0 \quad \text{for}\, x \geq g(t). \end{aligned}$$

Further, define $\hat{g}(t) = g(\ell + r - t)$. Then

$$\left\| \bar{P}_{g(t_1)} e^{(t_1-t_2)H} \bar{P}_{g(t_2)} e^{(t_2-t_3)H} \cdots \bar{P}_{g(t_n)} e^{(t_n-t_1)H} K_{\mathrm{Ai}} - \Theta^{\hat{g}}_{[\ell,r]} e^{(t_n-t_1)H} K_{\mathrm{Ai}} \right\|_1 \xrightarrow[n\to\infty]{} 0. \tag{59}$$

Since the convergence holds in trace class norm, (59) can be used to answer the question with which we started this subsection:

**Theorem 8** ([41], Theorem 2).

$$\mathbb{P}(\mathcal{A}_2(t) \leq g(t)\,\text{for}\, t \in [\ell, r]) = \det\!\big(I - K_{\mathrm{Ai}} + \Theta^g_{[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}}\big)\,. \tag{60}$$

Observe that we have written $g$ instead of $\hat{g}$ in (60). We may do this because the Airy$_2$ is invariant under time reversal, so we can replace $g$ by $\hat{g}$ on the left-hand side.

The limit (59) is proved in Proposition 3.2 of [41] (in fact only along the dyadic sequence $n_k = 2^k$, but this is enough for deducing Theorem 8). The proof is based on the following probabilistic representation of the solutions of the above boundary value problem: if $\Theta^g_{[\ell,r]}(x,y)$ denotes the integral kernel of $\Theta^g_{[\ell,r]}$, then

$$\begin{aligned} \Theta^g_{[\ell,r]}(x,y) = e^{\ell x - r y + (r^3-\ell^3)/3} \frac{e^{-(x-y)^2/4(r-\ell)}}{\sqrt{4\pi(r-\ell)}} \\ \cdot\, \mathbb{P}_{\hat{b}(\ell)=x-\ell^2,\hat{b}(r)=y-r^2}\Big(\hat{b}(s) \leq g(s) - s^2 \text{on}[\ell, r]\Big)\,, \end{aligned} \tag{61}$$

where the probability is computed with respect to a Brownian bridge $\hat{b}(s)$ from $x - \ell^2$ at time $\ell$ to $y - r^2$ at time $r$ and with diffusion coefficient 2. This formula is Theorem 3 of [41], its proof is based on an application of the Feynman–Kac and Cameron–Martin–Girsanov formulae.

The argument that proves Theorem 8 can be adapted to obtain a similar result for Airy$_1$. Fix $\ell < r$. Given $g \in H^1([\ell, r])$, define an operator $\Lambda^g_{[\ell,r]}$ acting on $L^2(\mathbb{R})$ as follows: $\Lambda^g_{[\ell,r]} f(\cdot) = u(r, \cdot)$, where $u(r, \cdot)$ is the solution at time $r$ of the boundary value problem

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{for } x < g(t),\ t \in (\ell, r) \\ u(\ell, x) &= f(x)\mathbf{1}_{x<g(\ell)} \\ u(t, x) &= 0 \quad \text{for } x \geq g(t). \end{aligned}$$

**Theorem 9** ([72], Theorem 4)

$$\mathbb{P}(\mathcal{A}_1(t) \leq g(t) \text{ for } t \in [\ell, r]) = \det\left(I - B_0 + \Lambda^g_{[\ell,r]} e^{-(r-\ell)\Delta} B_0\right). \tag{62}$$

Although the proof of this result is similar to the proof for the Airy$_2$ case, the argument is a bit more involved because, as written, the operator in the determinant is not trace class, so one needs to conjugate appropriately. Of course, similar arguments should allow one to obtain continuum statistics formulae for other processes for which boundary value kernel formulae are available (see [32] for the case of stationary GUE Dyson Brownian motion).

The operator $\Lambda^g_{[\ell,r]}$ also has a simple representation in terms of Brownian motion (see [72]), which has recently been used in [46] to verify numerically the experimental values obtained in [81] for the persistence probabilities of Airy$_1$. The negative persistence exponent is defined by

$$\mathbb{P}(\mathcal{A}_1(t) \leq m,\ 0 \leq t \leq L) \sim e^{-\kappa_- L}$$

where $m$ is the mean of $F_{\mathrm{GOE}}$. Takeuchi has measured $\kappa_- \approx 3.2 \pm 0.2$ in computer simulations of the Eden model [81]. Ferrari and Frings [46] have computed numerically (62) finding

$$\kappa \approx 2.9,$$

which is fairly close. Note that Takeuchi has also measured the positive persistence probabilities $\mathbb{P}(\mathcal{A}_1(t) \geq m,\ 0 \leq t \leq L) \sim e^{-\kappa_+ L}$. An interesting question is whether there exists a simple enough mathematical formula to check such a thing.

## 4 Applications

In this section we will describe some applications of the boundary value kernel formulae for Airy processes which were introduced in the previous section. The first two applications refer to asymptotic statistics for directed polymers and LPP, while the next two involve respectively the Airy$_1$ and Airy$_{2\to1}$ processes.

### 4.1 Point-to-Line LPP and GOE

Recall the variational formula (25) relating the Airy$_2$ process with the Tracy–Widom GOE distribution:

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}}\{\mathcal{A}_2(x)-x^2\}\le m\right)=F_{\rm GOE}(4^{1/3}m). \tag{63}$$

As we explained in Sect. 1.2.3, Johansson's proof [57] was very indirect, relying on the convergence of the spatial fluctuations of point-to-point LPP to $\mathcal{A}_2$ together with (15) and (17).

A direct proof of this variational formula was provided in [41], based on the continuum statistics formula given in Theorem 8. An interesting consequence of this derivation was that it allowed to identify the factor of $4^{1/3}$ on the right-hand side of the identity, which had been lost in Johansson's argument in the process of translating between the available results at the time (see Sect. 2 for an account of how to get the correct factor directly from LPP).

We will explain next the derivation of the formula, skipping some details. We rewrite the desired probability as

$$\lim_{L\to\infty}\mathbb{P}\big(\mathcal{A}_2(t)\le m+t^2\ \ \forall\, t\in[-L,L]\big)\,.$$

For fixed $L>0$, Theorem 8 implies that this probability is given by

$$\det\big(I-K_{\rm Ai}+\Theta_L e^{2LH}K_{\rm Ai}\big)\,, \tag{64}$$

where

$$\Theta_L=\Theta_{[-L,L]}^{g(t)=t^2+m}.$$

The nice thing is that the choice of $g(t)=t^2+m$ is the simplest possible from the point of view of explicit calculations, because it cancels exactly the parabola appearing on the right-hand side of (61). The probability appearing in that formula is then reduced to the probability of a Brownian bridge staying below level $m$, and this is easy to compute using the reflection principle (method of images):

$$\begin{aligned}\mathbb{P}_{\hat b(-L)=x-L^2,\hat b(L)=y-L^2}&\Big(\hat b(s)\le m\ \text{on}[-L,L]\Big)\\&=1-\mathbb{P}_{\hat b(-L)=x-L^2,\hat b(L)=y-L^2}\Big(\hat b(s)>m\ \text{for some}\ s\in[-L,L]\Big)\\&=1-e^{-(x-m-L^2)(y-m-L^2)/2\,L}\end{aligned} \tag{65}$$

(we leave the simple computation to the reader, alternatively see [20], p. 67). Putting this back in $\Theta_L$ gives

$$\Theta_L=\bar P_{m+L^2}e^{-2LH}\bar P_{m+L^2}-\bar P_{m+L^2}R_L\bar P_{m+L^2}, \tag{66}$$

where $R_L$ is the reflection term

$$R_L(x,y) = \frac{1}{\sqrt{8\pi L}} e^{-(x+y-2m-2L^2)^2/8L-(x+y)L+2L^3/3}. \tag{67}$$

The $e^{-2LH}$ in the first term in $\Theta_L$ comes from the 1 in (65) and appears by either reversing the use of the Cameron–Martin–Girsanov and Feynman–Kac formulas in the derivation of (66) or by an explicit computation of the integral kernel of $e^{-(r-\ell)H}$ as

$$e^{-(r-\ell)H}(x,y) = e^{\ell x - ry + (r^3-\ell^3)/3} \frac{e^{-(x-y)^2/4(r-\ell)}}{\sqrt{4\pi(r-\ell)}}.$$

Referring to (64), we have by the cyclic property of determinants (45) and the identity $e^{2LH}K_{\mathrm{Ai}} = (e^{LH}K_{\mathrm{Ai}})^2$ (which follows from Remark 2) that

$$\mathbb{P}\big(\mathcal{A}_2(t) \le t^2 + m \text{ for } t \in [-L, L]\big) = \det\big(I - K_{\mathrm{Ai}} + e^{LH}K_{\mathrm{Ai}}\Theta_L e^{LH}K_{\mathrm{Ai}}\big). \tag{68}$$

To obtain the $L \to \infty$ asymptotics, we decompose $\Theta_L$ so as to expose the two limiting terms, as well as a remainder term $\Omega_L$:

$$\Theta_L = e^{-2LH} - R_L + \Omega_L, \tag{69}$$

where $\Omega_L = (R_L - \bar{P}_{m+L^2} R_L \bar{P}_{m+L^2}) - (e^{-2LH} - \bar{P}_{m+L^2} e^{-2LH} \bar{P}_{m+L^2})$. It is shown in [41] that

$$\big\| e^{LH}K_{\mathrm{Ai}}\Omega_L e^{LH}K_{\mathrm{Ai}} \big\|_1 \xrightarrow[L\to\infty]{} 0. \tag{70}$$

The proof amounts essentially to asymptotic analysis involving the Airy function. In view of this fact and the decomposition (69), and since $e^{LH}K_{\mathrm{Ai}}e^{-2LH}e^{LH}K_{\mathrm{Ai}} = K_{\mathrm{Ai}}$, we see that the key point is the limiting behaviour in $L$ of

$$e^{LH}K_{\mathrm{Ai}}R_L e^{LH}K_{\mathrm{Ai}}.$$

To explain how this last product can be computed, we will proceed in a slightly formal manner through an argument based on the Baker–Campbell–Hausdorff formula, as done for a related problem in [73] (see Sect. 4.4). Since $K_{\mathrm{Ai}}$ is a projection and $H$ leaves $K_{\mathrm{Ai}}$ invariant, we will pretend that $e^{LH}$ and $K_{\mathrm{Ai}}$ commute, so we have to compute the limit of $e^{LH}R_L e^{LH}$. Define the *reflection operator* $\varrho_m$ by

$$\varrho_m f(x) = f(2\,m - x).$$

Then the operator $R_L$ defined in (67) can be rewritten as

$$R_L = e^{(2L^3)/3} e^{-L\xi} \varrho_{m+L^2} e^{2L\Delta} e^{-L\xi} = e^{(2L^3)/3} e^{-L\xi} e^{L\Delta} \varrho_{m+L^2} e^{L\Delta} e^{-L\xi}. \tag{71}$$

Here $e^{r\xi}$ ($\xi$ stands for a generic variable) denotes the multiplication operator $(e^{r\xi} f)(x) = e^{rx} f(x)$. The second equality follows from the reflection principle applied to the heat kernel.

The following identities will be useful, where $[\cdot,\cdot]$ denotes commutator:

$$[H,\Delta]=[\xi,\Delta]=-2\nabla,\qquad [H,\nabla]=[\xi,\nabla]=-I,\qquad [H,\xi]=-2\nabla.$$

If $A$ and $B$ are two operators such that $[A,[A,B]]=c_1 I$ and $[B,[A,B]]=c_2 I$ for some $c_1,c_2\in\mathbb{R}$, then the Baker–Campbell–Hausdorff formula reads

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]}.$$

Using this we have

$$e^{-L\xi}e^{L\Delta}=e^{L^3/6}e^{L\Delta+L^2\nabla-L\xi}.$$

Using the Baker–Campbell–Hausdorff formula again, we deduce that

$$e^{LH}e^{-L\xi}e^{L\Delta}=e^{L^3/6}e^{LH}e^{L\Delta+L^2\nabla-L\xi}=e^{-L^3/3}e^{L^2\nabla},$$

while an analogous computation yields

$$e^{L\Delta}e^{-L\xi}e^{LH}=e^{-L^3/3}e^{-L^2\nabla}.$$

Employing these identities on the right-hand side of (71) yields

$$e^{LH}R_L e^{LH}=e^{L^2\nabla}\varrho_{m+L^2}e^{-L^2\nabla}.$$

Since $e^{r\nabla}$ is the shift operator $(e^{r\nabla}f)(x)=f(x+r)$, we have $e^{r\nabla}\varrho_m=\varrho_m e^{-r\nabla}=\varrho_{m-r/2}$, and we obtain

$$e^{LH}R_L e^{LH}=\varrho_m.$$

Remarkably, the result does not depend on $L$. The conclusion from using this, (70) and (68) in (69) and taking $L\to\infty$ is

$$\mathbb{P}\big(\mathcal{A}_2(t)\le t^2+m \text{ for all } t\in\mathbb{R}\big)=\det(I-K_{\mathrm{Ai}}\varrho_m K_{\mathrm{Ai}})\,. \tag{72}$$

The use of the Baker–Campbell–Formula in the derivation of this identity can be replaced by an explicit integral calculation (see the proof of Proposition 1.3 of [41]).

To finish our proof of (63) we need to show that the right-hand side of (72) equals $F_{\mathrm{GOE}}(4^{1/3}m)$. Recall the definition of the kernel $B_0(x,y)=\mathrm{Ai}(x+y)$ and observe that $K_{\mathrm{Ai}}=B_0P_0B_0$. Recall also that the shifted Airy functions form a generalized orthonormal basis of $L^2(\mathbb{R})$ (see Remark 2), which implies that $B_0^2=I$. Therefore we can use the cyclic property of determinants (45) to deduce that

$$\det(I-K_{\mathrm{Ai}}\varrho_m K_{\mathrm{Ai}})=\det(I-P_0B_0\varrho_m B_0P_0)\,.$$

Now

$$B_0\varrho_m B_0(x,y)=\int_{-\infty}^{\infty}d\lambda\,\mathrm{Ai}(x+\lambda)\mathrm{Ai}(2\,m-\lambda+y),$$

and using the identity (52) we deduce that

$$B_0 \varrho_m B_0(x, y) = \widetilde{B}_m(x, y) := 2^{-1/3}\mathrm{Ai}(2^{-1/3}(x + y + 2\,m), \tag{73}$$

and thus

$$\det(I - K_{\mathrm{Ai}} \varrho_m K_{\mathrm{Ai}}) = \det\big(I - P_0 \widetilde{B}_m P_0\big)\,.$$

Performing the change of variables $x \mapsto 2^{1/3}x$, $y \mapsto 2^{1/3}y$, in the series defining the last Fredholm determinant, shows that the determinant on the right-hand side of (72) equals $\det\,(I - P_0 B_{4^{1/3}m} P_0)$, which is $F_{\mathrm{GOE}}(4^{1/3}m)$ by 1.14.

## *4.2 Endpoint Distribution of Directed Polymers*

In the setting of geometric LPP (see Sect. 1.2.3), consider the random variable

$$\kappa_N = \min\left\{\mathrm{k} \in \{-\mathrm{N}, \ldots, \mathrm{N}\} \colon \sup_{\mathrm{j}=-\mathrm{N},\ldots,\mathrm{k}} \mathrm{L}_{\mathrm{N}}^{\mathrm{point}}(\mathrm{j}) = \sup_{\mathrm{j}=-\mathrm{N},\ldots,\mathrm{N}} \mathrm{L}_{\mathrm{N}}^{\mathrm{point}}(\mathrm{j})\right\}.$$

$\kappa_N$ corresponds to the location of the endpoint of the maximizing path in point-to-line LPP.

Interest in the scaling properties and distribution of this random variable goes back at least to the early 1990s. One can also consider the analogous random variable in the setting of directed random polymers, but due to the KPZ universality conjecture, one expects that the asymptotic behaviour and statistics are the same as in LPP. [64] considered the polymer case and derived non-rigorously the scaling relation

$$|\kappa_N| \sim N^{2/3}$$

(c.f. (6)). In view of this, we define the rescaled endpoint

$$\mathcal{T}_N = c_3^{-1} N^{-2/3} \kappa_N,$$

where $c_3$ is the constant appearing in (18). Recalling the definition of the rescaled point-to-point last passage time (18) as the linear interpolation of the values given by

$$H_N^{\mathrm{point}}(t) = \frac{1}{c_2 N^{1/3}}\left[L^{\mathrm{point}}(N + c_3^{-1}N^{-2/3}t, N - c_3^{-1}N^{-2/3}t) - c_1 N\right]$$

for $t$ such that $c_3^{-1}N^{-2/3}t \in \{-N, \ldots, N\}$ we deduce that

$$\mathcal{T}_N = \min\left\{\mathrm{t} \in \mathbb{R} \colon \sup_{\mathrm{s}\le\mathrm{t}} \mathrm{H}_{\mathrm{N}}^{\mathrm{point}}(\mathrm{s}) = \sup_{\mathrm{s}\in\mathbb{R}} \mathrm{H}_{\mathrm{N}}^{\mathrm{point}}(\mathrm{s})\right\}.$$

Recalling that $H_N(t)$ converges to $\mathcal{A}_2(t) - t^2$ by Theorem 1, it becomes clear that $\mathcal{T}_N$ should converge to the point where $\mathcal{A}_2(t) - t^2$ attains it maximum. In fact, this is what Johansson proved, although he had to make a (very reasonable) technical assumption on the Airy$_2$ process which he was not able to prove with the tools available at the time.

**Theorem 10** ([57]). *Assume that the process $\mathcal{A}_2(t) - t^2$ attains its maximum at a unique point and let*

$$\mathcal{T} = \arg\max_{t\in\mathbb{R}}\{\mathcal{A}_2(t) - t^2\}.$$

*Then*

$$\mathcal{T}_N \xrightarrow[N\to\infty]{} \mathcal{T}$$

*in the sense of convergence in distribution.*

Although the result is of course very interesting, as it shows that the limiting end-point distribution exists (under the technical assumption), it gives no information on the distribution of $\mathcal{T}$. Quoting [58], for all we know $\mathcal{T}$ could be Gaussian. Nevertheless, from KPZ universality, one expects that this is not the case. For example, [55] conjectured on the basis of analogy with the arg max of a Brownian motion minus a parabola (for which one has a complete analytical solution, see [52]), that the tails of $\mathcal{T}$ decay like $e^{-ct^3}$, which of course rules out Gaussian behaviour.

It turns out that the distribution of $\mathcal{T}$ can be computed explicitly through an argument based on the continuum statistics formula of Theorem 8. This was done in [65], where in fact the joint density of

$$\mathcal{T} = \arg\max_{t\in\mathbb{R}}\{\mathcal{A}_2(t) - t^2\} \qquad \text{and} \qquad \mathcal{M} = \max_{t\in\mathbb{R}}\{\mathcal{A}_2(t) - t^2\}$$

was computed. Moreover, the argument implies that the maximum of $\mathcal{A}_2(t) - t^2$ is attained at a unique point, thus completing the proof of Theorem 10. The uniqueness of the maximum was also proved slightly earlier by [37] using completely different techniques, and a proof for general stationary processes is now available [67].

The computation is as follows. For simplicity we will assume the uniqueness of the maximizing point of $\mathcal{A}_2(t) - t^2$, and will explain later how the uniqueness can actually be obtained from this argument. Let $(\mathcal{M}_L, \mathcal{T}_L)$ denote the maximum and the location of the maximum of $\mathcal{A}_2(t) - t^2$ restricted to $t \in [-L, L]$, and let $f_L$ be the joint density of $(\mathcal{M}_L, \mathcal{T}_L)$. By results of [35], the joint density $f(m,t)$ of $\mathcal{M}, \mathcal{T}$ is well approximated by $f_L(m,t)$,

$$f(t,m) = \lim_{L\to\infty} f_L(t,m).$$

By definition,

$$f_L(t,m) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\delta}\mathbb{P}(\mathcal{M}_L \in [m, m+\varepsilon],\ \mathcal{T}_L \in [t, t+\delta]),$$

provided that the limit exists. The main contribution in the above expression comes from paths entering the space–time box $[t, t+\delta] \times [m, m+\varepsilon]$ and staying below the level $m$ outside the time interval $[t, t+\delta]$. More precisely, if we denote by $\underline{D}_{\varepsilon,\delta}$ and $\overline{D}_{\varepsilon,\delta}$ the sets

$$\underline{D}_{\varepsilon,\delta} = \big\{\mathcal{A}_2(s) - s^2 \le m, \ \ s \in [t, t+\delta]^{\mathrm{c}}, \mathcal{A}_2(s) - s^2 \le m + \varepsilon, \ \ s \in [t, t+\delta], \\ \mathcal{A}_2(s) - s^2 \in [m, m+\varepsilon] \text{ for some } s \in [t, t+\delta]\big\},$$

and

$$\overline{D}_{\varepsilon,\delta} = \big\{\mathcal{A}_2(s) - s^2 \le m + \varepsilon, \ \ s \in [-L, L], \\ \mathcal{A}_2(s) - s^2 \in [m, m+\varepsilon] \text{ for some } s \in [t, t+\delta]\big\},$$

then

$$\underline{D}_{\varepsilon,\delta} \subseteq \{\mathcal{M}_L \in [m, m+\varepsilon], \mathcal{T}_L \in [t, t+\delta]\} \subseteq \overline{D}_{\varepsilon,\delta}.$$

Letting $\underline{f}(t,m) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon\delta}\mathbb{P}(\underline{D}_{\varepsilon,\delta})$ and defining $\overline{f}(t,m)$ analogously (with $\overline{D}_{\varepsilon,\delta}$ instead of $\underline{D}_{\varepsilon,\delta}$), we deduce that $\underline{f}(t,m) \le f(t,m) \le \overline{f}(t,m)$. In what follows we will compute $\underline{f}(t,m)$. It will be clear from the argument that for $\overline{f}(t,m)$ we get the same limit. The conclusion is that

$$f_L(t,m) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon\delta}\mathbb{P}(\underline{D}_{\varepsilon,\delta}).$$

We rewrite this last equation as

$$f_L(t,m) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon\delta}\big[\,\mathbb{P}\big(\mathcal{A}_2(s) \le h_{\varepsilon,\delta}(s),\ s \in [-L, L]\big) \\ - \mathbb{P}\big(\mathcal{A}_2(s) \le h_{0,\delta}(s),\ s \in [-L, L]\big)\,\big],$$

where

$$h_{\varepsilon,\delta}(s) = s^2 + m + \varepsilon\mathbf{1}_{s\in[t,t+\delta]}.$$

These two probabilities have explicit Fredholm determinant formulae by Theorem 8. We get, using the cyclic property of determinants as in (68),

$$f_L(t,m) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon\delta}\Big[\det\Big(I - K_{\mathrm{Ai}} + e^{LH}K_{\mathrm{Ai}}\Theta^{h_{\varepsilon,\delta}}_{[-L,L]}e^{LH}K_{\mathrm{Ai}}\Big) \\ - \det\Big(I - K_{\mathrm{Ai}} + e^{LH}K_{\mathrm{Ai}}\Theta^{h_{0,\delta}}_{[-L,L]}e^{LH}K_{\mathrm{Ai}}\Big)\Big].$$

The limit in $\varepsilon$ becomes a derivative

$$f_L(t,m) = \lim_{\delta\to 0}\frac{1}{\delta}\,\partial_\beta \det\Big(I - K_{\mathrm{Ai}} + e^{LH}K_{\mathrm{Ai}}\Theta^{h_{\beta,\delta}}_{[-L,L]}e^{LH}K_{\mathrm{Ai}}\Big)\Big|_{\beta=0},$$

which in turn gives a trace by (46),

$$f_L(t,m) = \det\Big(I - K_{\mathrm{Ai}} + e^{LH} K_{\mathrm{Ai}} \Theta^{h_{0,\delta}}_{[-L,L]} e^{LH} K_{\mathrm{Ai}}\Big)$$

$$\cdot \lim_{\delta\to 0} \frac{1}{\delta} \mathrm{tr}\Big[(I - K_{\mathrm{Ai}} + e^{LH} K_{\mathrm{Ai}} \Theta^{h_{0,\delta}}_{[-L,L]} e^{LH} K_{\mathrm{Ai}})^{-1} e^{LH} K_{\mathrm{Ai}} \left[\partial_\beta \Theta^{h_{\beta,\delta}}_{[-L,L]}\right]_{\beta=0} e^{LH} K_{\mathrm{Ai}}\Big]. \tag{74}$$

One has to check here that the required limits hold in trace class norm, see [65]. Note that $h_{0,\delta} = g_m$, where $g_m$ is the parabolic barrier

$$g_m(s) = s^2 + m,$$

so in particular the determinant and the first factor inside the trace do not depend on $\delta$. We know moreover from the arguments in Sect. 4.1 that

$$\lim_{L\to\infty} \Big(I - K_{\mathrm{Ai}} + e^{LH} K_{\mathrm{Ai}} \Theta^{h_{0,\delta}}_{[-L,L]} e^{LH} K_{\mathrm{Ai}}\Big) = I - K_{\mathrm{Ai}} \varrho_m K_{\mathrm{Ai}}$$

in trace norm. In particular, we have

$$\lim_{L\to\infty} \det\Big(I - K_{\mathrm{Ai}} + e^{LH} K_{\mathrm{Ai}} \Theta^{h_{0,\delta}}_{[-L,L]} e^{LH} K_{\mathrm{Ai}}\Big) = F_{\mathrm{GOE}}(4^{1/3} m).$$

The next step is to compute $\partial_\beta \Theta^{h_{\beta,\delta}}_{[-L,L]}|_{\beta=0}$. Recalling that $h_{0,\delta}(s) = g_m(s) = s^2 + m$ and also $h_{\varepsilon,\delta}(s) = g_{m+\varepsilon}(s)$ for $s \in [t, t+\delta]$, we have, by the semigroup property,

$$\Theta^{h_{\varepsilon,\delta}}_{[-L,L]} - \Theta^{h_{0,\delta}}_{[-L,L]} = \Theta^{g_m}_{[-L,t]} \left[\Theta^{g_{m+\varepsilon}}_{[t,t+\delta]} - \Theta^{g_m}_{[t,t+\delta]}\right] \Theta^{g_m}_{[t+\delta,L]}.$$

Computing the desired derivative involves just the middle bracket, which we note corresponds to the same boundary value problem as in Sect. 4.1, only at two different levels $m$ and $m+\varepsilon$. Since we have explicit formulae, the derivative can be computed explicitly. The computation is slightly tedious, and the only delicate part is to justify that the necessary limits occur in trace class norm. We refer to [65] for the details.

Going back to (74), we recall that the trace is linear and continuous under the trace class norm topology, so in view of the preceding discussion we have

$$\lim_{L\to\infty} f_L(t,m) = F_{\mathrm{GOE}}(4^{1/3} m) \mathrm{tr}\Big[(I - K_{\mathrm{Ai}} \varrho_m K_{\mathrm{Ai}})^{-1}$$

$$\lim_{L\to\infty} \lim_{\delta\to 0} \frac{1}{\delta} e^{LH} K_{\mathrm{Ai}} \left[\partial_\beta \Theta^{h_{\beta,\delta}}_{[-L,L]}\right]_{\beta=0} e^{LH} K_{\mathrm{Ai}}\Big]. \tag{75}$$

Once again we need to compute limits, again taking care that they hold in trace class norm as necessary. We skip the details and just write down the result, (76):

$$\lim_{L\to\infty} \lim_{\delta\to 0} \frac{1}{\delta} e^{LH} K_{\mathrm{Ai}} \left[\partial_\beta \Theta^{h_{\beta,\delta}}_{[-L,L]}\right]_{\beta=0} e^{LH} K_{\mathrm{Ai}} = \Psi, \tag{76}$$

where

$$\Psi(x,y) = B_0 P_0 \psi_{t,m}(x) B_0 P_0 \psi_{-t,m}(y)$$

and

$$\psi_{t,m}(x) = 2e^{t^3+(m+x)t}\Big[\mathrm{Ai}'(m+t^2+x) + t\mathrm{Ai}(m+t^2+x)\Big]$$

(we remark that we have written these formulae in a slightly different way compared to [65], but the reader should have no problem translating between the formulae). The limit in $\delta$ is relatively straightforward, while the limit in $L$ involves an argument similar to the one used in Sect. 4.1. Using this formula in (75), the trace becomes

$$\mathrm{tr}\big[(I - K_{\mathrm{Ai}}\varrho_m K_{\mathrm{Ai}})^{-1}\Psi\big] = \big\langle (I - K_{\mathrm{Ai}}\varrho_m K_{\mathrm{Ai}})^{-1} B_0 P_0 \psi_{t,m}, B_0 P_0 \psi_{-t,m}\big\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes inner product in $L^2(\mathbb{R})$.

It only remains to simplify the expression. We first use (73) and the facts that $K_{\mathrm{Ai}} = B_0 P_0 B_0$, $B_0^2 = I$ and $B_0^* = B_0$ to write

$$\begin{aligned}
&\big\langle (I - K_{\mathrm{Ai}}\varrho_m K_{\mathrm{Ai}})^{-1} B_0 P_0 \psi_{t,m}, B_0 P_0 \psi_{-t,m}\big\rangle \\
&\qquad = \big\langle (I - B_0 P_0 B_m P_0 B_0)^{-1} B_0 P_0 \psi_{t,m}, B_0 P_0 \psi_{-t,m}\big\rangle \\
&\qquad = \big\langle B_0 (I - P_0 B_m P_0)^{-1} P_0 \psi_{t,m}, B_0 P_0 \psi_{-t,m}\big\rangle \\
&\qquad = \big\langle (I - P_0 B_m P_0)^{-1} P_0 \psi_{t,m}, P_0 \psi_{-t,m}\big\rangle .
\end{aligned}$$

Next we introduce the scaling operator $Sf(x) = f(2^{1/3}x)$. One can check easily that $S^{-1} = 2^{1/3}S^*$ and that $P_0$ commutes with $S$ and $S^{-1}$. We also have $SB_mS^{-1} = B_{4^{1/3}m}$. Thus writing $\tilde{m} = 2^{-1/3}m$ we get

$$\begin{aligned}
&\big\langle (I - P_0 B_m P_0)^{-1} P_0 \psi_{t,m}, P_0 \psi_{-t,m}\big\rangle \\
&\qquad = \big\langle (I - S^{-1} P_0 B_{2\tilde{m}} P_0 S)^{-1} P_0 \psi_{t,m}, P_0 \psi_{-t,m}\big\rangle \\
&\qquad = \big\langle S^{-1}(I - P_0 B_{2\tilde{m}} P_0)^{-1} P_0 S \psi_{t,m}, P_0 \psi_{-t,m}\big\rangle \\
&\qquad = 2^{1/3}\big\langle (I - P_0 B_{2\tilde{m}} P_0)^{-1} P_0 S \psi_{t,m}, P_0 S \psi_{-t,m}\big\rangle .
\end{aligned}$$

which is equal to $2^{1/3}\gamma(t, 4^{1/3}m)$.

Using this formula in (75) yields the joint density of $\mathcal{T}$ and $\mathcal{M}$. Define the resolvent kernel

$$\varsigma_m(x,y) = (I - P_0 B_m P_0)^{-1}(x,y)$$

and, for $t,m \in \mathbb{R}$, define

$$\Psi_{t,m}(x,y) = 2^{1/3}\psi_{t,m}(2^{1/3}x)\psi_{-t,m}(2^{1/3}y)$$

and

$$\gamma(t,m) = 2^{1/3}\int_0^\infty dx \int_0^\infty dy\, \psi_{-t,4^{-1/3}m}(2^{1/3}x)\varsigma_m(x,y)\psi_{t,4^{-1/3}m}(2^{1/3}y).$$

**Theorem 11** ([65], Theorem 2) *The joint density* $f(t,m)$ *of* $\mathcal{T}$ and $\mathcal{M}$ *is given by*

$$\begin{aligned} f(t,m) &= \gamma(t, 4^{1/3}m) F_{\mathrm{GOE}}(4^{1/3}m) \\ &= \det(I - P_0 B_{4^{1/3}m} P_0 + P_0 \Psi_{t,m} P_0) - F_{\mathrm{GOE}}(4^{1/3}m). \end{aligned} \tag{77}$$

To see where the second equality in 77 comes from, observe that $\gamma(t, 4^{1/3}m)$ equals the trace of the operator $(I - P_0 B_{4^{1/3}m} P_0)^{-1} P_0 \Psi_{t,m} P_0$ and that $\Psi_{t,m}$ is a rank one operator. The identity that follows now from the general fact that for two operators $A$ and $B$ such that $B$ is rank one, one has $\det(I - A + B) = \det(I - A)[1 + \mathrm{tr}((I - A)^{-1} B)]$.

Integrating over $m$, one obtains a formula for the probability density $f_{\mathrm{end}}(t)$ of $\mathcal{T}$. Unfortunately, it does not appear that the resulting integral can be calculated explicitly, so the best formula one has is

$$f_{\mathrm{end}}(t) = \int_{-\infty}^{\infty} dm\, f(t,m).$$

One can readily check nevertheless that $f_{\mathrm{end}}(t)$ is symmetric in $t$. The second formula for $f(t,m)$ is suitable for numerical computations, using the numerical scheme and Matlab toolbox developed by Bornemann in [16, 17] for the computation of Fredholm determinants. Figure 3 shows a contour plot of the joint density of $\mathcal{M}$ and $\mathcal{T}$, while Fig. 4 shows a plot of the marginal $\mathcal{T}$ density.

As we mentioned, interest in this problem dates back at least 2 decades. In particular, there has been a resurgence of interest in the last couple of years. An alternative way to obtain the Airy$_2$ process is as a limit in large $N$ of the top path in a system of $N$ non-intersecting random walks, or Brownian motions, the so called vicious walkers [48] (this is of course related to the setting presented in Sect. 1.5.2). [44, 74, 75, 78] obtained various expressions for the joint distributions of $\mathcal{M}$ and $\mathcal{T}$ in such a system at finite $N$. [50] obtained the $F_{\mathrm{GOE}}$ distribution from large $N$ asymptotics non-rigorously, and furthermore made connections between these problems and Yang–Mills theory. But for several years people were not able to perform asymptotic analysis on the formulae obtained for $\mathcal{T}$ at finite $N$.

After [65] appeared, [77] succeeded in extracting asymptotics from the vicious walkers formula, and obtained an alternative formula for $f(t,m)$. His formula is given as follows. The Painlevé II Eq. (10) and (11) has a Lax pair formulation

$$\frac{\partial}{\partial \zeta}\Phi = A\Phi, \qquad \frac{\partial}{\partial s}\Phi = B\Phi \tag{78}$$

for a 2-D vector $\Phi = \Phi(\zeta, s)$, where the $2 \times 2$ matrices $A = A(\zeta, s)$ and $B = B(\zeta, s)$ are given by

$$A(\zeta,s) = \begin{pmatrix} 4\zeta q & 4\zeta^2 + s + 2q^2 + 2q' \\ -4\zeta^2 - s - 2q^2 + 2q' & -4\zeta q \end{pmatrix}$$
$$\text{and} \quad B(\zeta,s) = \begin{pmatrix} q & \zeta \\ -\zeta & -q \end{pmatrix}.$$

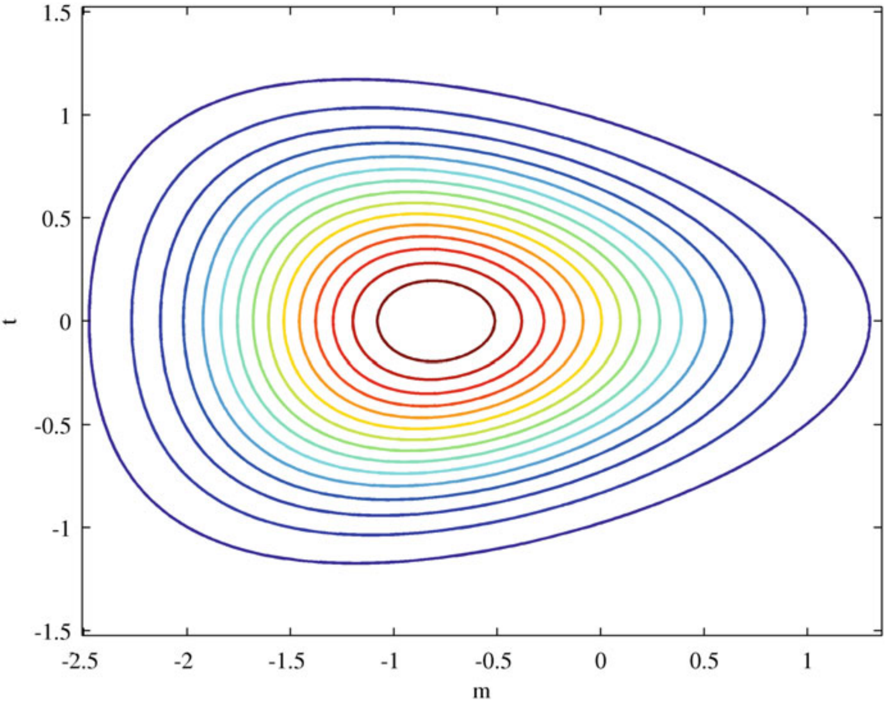


**Fig. 3** Contour plot of the joint density of $\mathcal{M}$ and $\mathcal{T}$

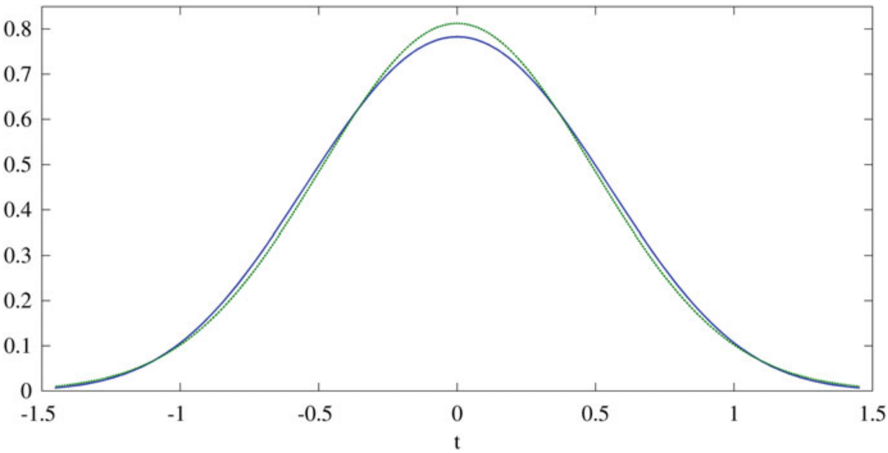


**Fig. 4** Plot of the density of $\mathcal{T}$ compared with a Gaussian density with the same variance 0.2409 (*dashed line*). The excess kurtosis $\mathbb{E}(\mathcal{T}^4)/\mathbb{E}(\mathcal{T}^2)^2 - 3$ is $-0.2374$

The compatibility of this overdetermined system implies that $q(s)$ solves Painlevé II. Now let $\Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}$ be the unique solution of (78) satisfying

$$\Phi_1(\zeta;s) = \cos\left(\frac{4}{3}\zeta^3 + s\zeta\right) + O(\zeta^{-1}), \quad \Phi_2(\zeta;s) = -\sin\left(\frac{4}{3}\zeta^3 + s\zeta\right) + O(\zeta^{-1}),$$

as $\zeta \to \pm\infty$ for $s \in \mathbb{R}$. The formula of [77] is

$$\gamma(t,m) = \tfrac{16}{\pi^2}\langle h_{4^{2/3}t}, h_{-4^{2/3}t}\rangle_{L^2(m,\infty)} \tag{79}$$

where

$$h_t(x) = \int_0^\infty d\zeta\, \zeta\, \Phi_2(\zeta,x)e^{-t\zeta^2}.$$

Although Schehr's argument is non-rigourous, a later chapter of [14] proved directly the equivalence of the formula of [77] and 77, thus establishing the validity of (79) based on Theorem 11.

Before turning to the tail behaviour of $\mathcal{T}$, let us briefly explain how the uniqueness of the maximizer of $\mathcal{A}_2(t) - t^2$ can be established directly from the argument we described above. In the derivation of the formula, we assumed that the maximum of $\mathcal{A}_2(t)-t^2$ is obtained at a unique point. However, it is not necessary to do this. In fact, if one follows the argument without this assumption, one ends up with a formula for what is in principle a super-probability density, i.e. a non-negative function $f(t,m)$ on $\mathbb{R}\times\mathbb{R}$ with $\int_{\mathbb{R}\times\mathbb{R}} dm\, dt\, f(t,m) \geq 1$, and in fact one can see from the argument that

$$\int_{\mathbb{R}\times\mathbb{R}} dm\, dt\, f(t,m) = \text{expected number of maxima of } \mathcal{A}_2(t) - t^2.$$

Recall that from (63), the distribution of $\mathcal{M}$ is given by a scaled version of $F_{\mathrm{GOE}}$. A non-trivial computation (see Sect. 3) gives

$$\int_{-\infty}^{\infty} dt\, f(t,m) = 4^{1/3}F'_{\mathrm{GOE}}(4^{1/3}m).$$

This shows that $f(t,m)$ has total integral 1, which can only be true if the maximum is unique almost surely, since the global maximum is attained at at least one point.

We mentioned earlier in the conjecture that $\mathcal{T}$ should have tails which decay like $e^{-ct^3}$ (see e.g. [55]). This can be proved using the techniques described in this review:

**Theorem 12** ([14, 35, 71, 77]). There is a $c > 0$, such that for every $\kappa > \frac{32}{3}$ and large enough $t$,

$$e^{-\kappa t^3} \leq \mathbb{P}(|\mathcal{T}| > t) \leq c e^{-\frac{4}{3}t^3 + 2t^2 + \mathcal{O}(t^{3/2})}.$$

[35] had obtained the $e^{-ct^3}$ decay for some $c > 0$. The statement we included here is the one appearing in [71]. In fact, Schehr's formula and its validation in [14]

later yielded a lower bound that matches the $e^{-\frac{4}{3}t^3}$ behaviour of the upper bound, so we know now that $\frac{4}{3}$ is the correct exponent. A precise asymptotic expansion of $\mathbb{P}(|\mathcal{T}| > t)$ based on that formula has recently been obtained in [33]. The reason why [71] obtained a slightly worse lower bound is technical, and arises from the fact that the explicit formula (77) for $f(t, m)$ is not useful for providing a lower bound, and instead one needs to use a different argument. On the other hand, the upper bound can be obtained directly from (77). In fact, the second formula expresses this joint density as the difference of two Fredholm determinants, so we may use (48) to estimate the difference, and then all that remains is to show that this estimate can be integrated in $m$. See [71] for more details.

### 4.3 Local Behaviour of Airy$_1$

As we mentioned, the boundary value kernel formulae introduced in Sect. 3.1 are better adapted than the standard extended kernel formulae to study short-range properties of the processes. An interesting application is the following:

**Theorem 13** ([72], Theorem 2). *The Airy$_1$ process $\mathcal{A}_1$ and the Airy$_2$ process $\mathcal{A}_2$ have versions with Hölder continuous paths with exponent $\frac{1}{2} - \delta$ for any $\delta > 0$.*

Continuity was known for $\mathcal{A}_2$ (see Theorem 1) but not for $\mathcal{A}_1$. The Hölder $\frac{1}{2}-$ continuity for $\mathcal{A}_2$ also follows from the work of [35]. Their proof is based on a certain Brownian Gibbs property for the Airy$_2$ line ensemble (an infinite collection of continuous, non-intersecting paths, the top line of which is $\mathcal{A}_2$), and as such it cannot be extended to Airy$_1$, given that no analog of the Airy$_2$ line ensemble is known in the flat case. This regularity is expected to hold in fact for all the Airy processes in view of the fact that they are believed to look locally like a Brownian motion (see Sect. 1.3). Analogous results have recently become available for the solutions of the KPZ equation at finite times with certain initial conditions [37, 54, 70].

The proof of Theorem 13 is based on an application of a suitable version of the Kolmogorov criterion. In the Airy$_1$ case, it involves studying a truncated version of the process, $\mathcal{A}_1{}^M(t) = \mathcal{A}_1(t)\mathbf{1}_{|\mathcal{A}_1(t)|\leq M} + M\mathbf{1}_{\mathcal{A}_1(t)>M} - M\mathbf{1}_{\mathcal{A}_1(t)<-M}$ and then proving the following estimate: for fixed $\delta > 0$, there is a $t_0 \in (0, 1)$ and an $n_0 \in \mathbb{N}$ such that for $0 < t < t_0$, $n \geq n_0$ and $M = (3 \log(t^{-(1+n)}))^{1/3}$ we have

$$\mathbb{E}\big([\mathcal{A}_1{}^M(t) - \mathcal{A}_1{}^M(0)]^{2n}\big) \leq ct^{1+(1-\delta)n}$$

where the constant $c > 0$ is independent of $\delta$, $n_0$ and $t_0$. The proof of this estimate can be reduced to obtaining a suitable estimate on the difference

$$\left|\det(I - B_0 + \bar{P}_a e^{t\Delta} \bar{P}_b e^{-t\Delta} B_0) - \det(I - B_0 + \bar{P}_a B_0)\right|$$

for $b \geq a \geq -M$. An important technical problem is that the kernels appearing inside these determinants are not trace class, so one needs to conjugate appropriately. We refer to [72] for the details. The argument for Airy$_2$ is similar.

As we mentioned, the Airy processes are expected to look locally like a Brownian motion. In this direction, it can be shown using the boundary value kernel formulae that the finite-dimensional distributions of the $\mathrm{Airy}_1$ process converge under diffusive scaling to those of a Brownian motion. The same result was proved earlier by [53] for $\mathrm{Airy}_2$ using different techniques. In fact, for $\mathrm{Airy}_2$, a stronger statement is now available ([35]), namely that it is locally absolutely continuous with respect to Brownian motion.

**Theorem 14** ([72], Theorem 3). *For any fixed* $s \in \mathbb{R}$, *let* $B_\varepsilon(\,\cdot\,)$ *be defined by* $B_\varepsilon(t) = \varepsilon^{-1/2}(\mathcal{A}_1(s+\varepsilon t) - \mathcal{A}_1(s))$, $t > 0$. *Then* $B_\varepsilon(\,\cdot\,)$ *converges to Brownian motion in the sense of convergence of finite-dimensional distributions. The same holds for* $\tilde{B}_\varepsilon(\,\cdot\,)$ *defined by* $\tilde{B}_\varepsilon(t) = B_\varepsilon(-t)$, $t > 0$.

The proof of this result follows from an explicit computation of

$$\mathbb{P}\big(\mathcal{A}_1(\varepsilon t_1) \le x + \sqrt{\varepsilon} y_1, \ldots, \mathcal{A}_1(\varepsilon t_n) \le x + \sqrt{\varepsilon} y_n \,|\, \mathcal{A}_1(0) = x\big)$$

and its limit as $\varepsilon \to 0$, see [72] for the details. The same proof works for the $\mathrm{Airy}_2$ process and, in view of (56), it should be simple to adapt it to the $\mathrm{Airy}_{2\to 1}$ process.

## 4.4 Marginals of $\mathrm{Airy}_{2\to 1}$

The last application of the results of Sect. 3.2 that we will discuss is a proof of the conjecture (34) that the marginals of the $\mathrm{Airy}_{2\to 1}$ process can be obtained from a variational problem for $\mathcal{A}_2(t) - t^2$ on a half-line. The result is the following:

**Theorem 15** ([73], Theorem 1). *Fix* $\alpha \in \mathbb{R}$. *For every* $m \in \mathbb{R}$,

$$\mathbb{P}\Big(\sup_{t \le \alpha}\,(\mathcal{A}_2(t) - t^2) \le m - \min\{0,\alpha\}^2\Big) = \mathbb{P}(\mathcal{A}_{2\to 1}(\alpha) \le m)\,.$$

The right-hand side can be expressed in terms of a Fredholm determinant. Define the *crossover distributions* $G_\alpha^{2\to 1}$, for $\alpha \in \mathbb{R}$, as

$$G_\alpha^{2\to 1}(m) = \mathbb{P}(\mathcal{A}_{2\to 1}(\alpha) \le m)\,.$$

We claim that

$$G_\alpha^{2\to 1}(m) = \det(I - P_m K_\alpha P_m), \tag{80}$$

where $K_\alpha = K_\alpha^1 + K_\alpha^2$ and the kernels $K_\alpha^1$ and $K_\lambda^2$ are given by

$$K_\alpha^1(x,y) = \int_0^\infty d\lambda\, e^{2\alpha\lambda} \mathrm{Ai}(x - \lambda + \max\{0,\alpha\}^2)\mathrm{Ai}(y + \lambda + \max\{0,\alpha\}^2)$$

and

$$K_\alpha^2(x,y) = \int_0^\infty d\lambda \mathrm{Ai}(x + \lambda + \max\{0,\alpha\}^2)\mathrm{Ai}(y + \lambda + \max\{0,\alpha\}^2).$$

As noted in Appendix A of [28], the kernel $K^{\rm ext}_{2\to1}$ defined in (24) can be expressed in terms of Airy functions:

$$K^{\rm ext}_{2\to1}(s,t;x,y) = L_0(s,x;t,y) + e^{2t^3/3-2\,s^3/3+t\tilde{y}-s\tilde{x}}[L_1+L_2](s,x;t,y),$$

where

$$L_0(s,x;t,y) = -e^{(s-t)\Delta}(\tilde{x},\tilde{y}) = -\frac{1}{\sqrt{4\pi(t-s)}}e^{-(\tilde{x}-\tilde{y})^2/4(t-s)},$$
$$L_1(s,x;t,y) = \int_0^\infty d\lambda\, e^{\lambda(s+t)}\mathrm{Ai}(\hat{x}-\lambda)\mathrm{Ai}(\hat{y}+\lambda),$$
$$L_2(s,x,t,y) = \int_0^\infty d\lambda\, e^{\lambda(t-s)}\mathrm{Ai}(\hat{x}+\lambda)\mathrm{Ai}(\hat{y}+\lambda)$$

with $\tilde{x} = x - s^2\mathbf{1}_{s\le0}$, $\tilde{y} = y - t^2\mathbf{1}_{t\le0}$, $\hat{x} = x + s^2\mathbf{1}_{s\ge0}$ and $\hat{y} = y + t^2\mathbf{1}_{t\ge0}$. Using this for $s = t = \alpha$, it is straightforward to check that $K^{\rm ext}_{2\to1}(t,\cdot;t,\cdot)$ is just a conjugation of the kernel $K_\alpha$, and (80) follows.

The fact that $G^{2\to1}_\alpha$ crosses over between the GUE and GOE distributions is of course a particular case of the crossover property of the Airy$_{2\to1}$ process, but can be easily obtained from (80) as well (see the discussion after Theorem 1 in [73]).

The proof of Theorem 15 is similar to (and, in fact, somewhat simpler than) the proof of (25). Basically, one applies Theorem 8 and the cyclic property of determinants to compute the desired probability as

$$\begin{aligned}&\lim_{L\to\infty}\mathbb{P}(\mathcal{A}_2(t)\le g(t)\text{ for }t\in[-L,\alpha])\\&=\lim_{L\to\infty}\det\big(I-K_{\rm Ai}+e^{(\alpha+L)H}K_{\rm Ai}\Theta^g_{[-L,\alpha]}K_{\rm Ai}\big)\end{aligned}$$

with $g(t) = t^2 + \overline{m}$ and $\overline{m} = m - \min\{0,\alpha\}^2$. An argument similar to the one used in Sect. 4.1 (applying the Baker–Campbell–Hausdorff formula and later checking the result rigorously, plus some asymptotic analysis to show that an error term goes to 0 in trace class norm as $L\to\infty$) yields

$$\begin{aligned}&\mathbb{P}\Big(\sup_{t\le\alpha}(\mathcal{A}_2(t)-t^2)\le m-\min\{0,\alpha\}^2\Big)\\&=\det\big(I-K_{\rm Ai}P_{\overline{m}+\alpha^2}K_{\rm Ai}-K_{\rm Ai}e^{\alpha\xi}\varrho_{\overline{m}+\alpha^2}e^{-\alpha\xi}\bar{P}_{\overline{m}+\alpha^2}K_{\rm Ai}\big)\,.\end{aligned}\tag{81}$$

Since $K_{\rm Ai} = B_0P_0B_0$ and $B_0^2 = I$, we have by the cyclic property of determinants that the right-hand side of (81) equals

$$\det\big(I-P_0B_0P_{\overline{m}+\alpha^2}B_0P_0-P_0B_0e^{\alpha\xi}\varrho_{\overline{m}+\alpha^2}e^{-\alpha\xi}\bar{P}_{\overline{m}+\alpha^2}B_0P_0\big)\,.$$

Shifting the variables in the last determinant by $-m$, we deduce that

$$\mathbb{P}\Big(\sup_{t\ge\alpha}(\mathcal{A}_2(t)-t^2)\le\overline{m}\Big)=\det(I-P_mE_1P_m-P_mE_2P_m)\,,$$

where

$$E_1(x,y) = \int_{-\infty}^{\overline{m}+\alpha^2} d\lambda \mathrm{Ai}(x - m + 2\overline{m} + 2\alpha^2 - \lambda) e^{-2(\lambda-\overline{m}-\alpha^2)\alpha} \mathrm{Ai}(y - m + \lambda)$$

and

$$E_2(x,y) = \int_{\overline{m}+\alpha^2}^{\infty} d\lambda \mathrm{Ai}(x - m + \lambda)\mathrm{Ai}(y - m + \lambda).$$

Shifting $\lambda$ by $\overline{m} + \alpha^2$ in both integrals and changing $\lambda$ to $-\lambda$ shows that $E_1(x,y) = K_\alpha^1(y,x)$ and $E_2 = K_\alpha^2$, whence the equality in Theorem 15 follows since $E_1^* = K_\alpha^1$ and $E_2^* = K_\alpha^2$.

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