SPRINGER BRIEFS IN STATISTICS

Chin-Diew Lai

Generalized Weibull Distributions

Springer

SpringerBriefs in Statistics

For further volumes:
http://www.springer.com/series/8921

Chin-Diew Lai

Generalized Weibull Distributions

Springer

Chin-Diew Lai
Statistics and Bioinformatics Manawatu Campus
Massey University Institute of Fundamental Sciences
Palmerston North
New Zealand

ISSN 2191-544X ISSN 2191-5458 (electronic)
ISBN 978-3-642-39105-7 ISBN 978-3-642-39106-4 (eBook)
DOI 10.1007/978-3-642-39106-4
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013950387

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

This monograph is dedicated to the memory of my late wife Ai Ing
I wish to thank my wife Hayley Fang for her loving support

Chin-Diew Lai

Contents

Chapter 1
Weibull Distribution

1.1 Introduction

In probability theory and statistics, the Weibull distribution is a continuous probability distribution named after Waloddi Weibull who described it in detail in 1951, although it was first identified by Fréchet (1927) and first applied by Rosin and Rammler (1933) to describe the size distribution of particles.

The Weibull distribution has since become one of the most cited lifetime distributions in reliability engineering and other disciplines. It adequately describes observed failure times of many different types of components and phenomena.

Over the last three decades, numerous articles have been written on this distribution. Hallinan (1993) presented an insightful review by presenting a number of historical facts, and many forms of this distribution as used by the practitioners and possible confusions and errors that arise due to this non-uniqueness. Johnson et al. (1994) devoted a comprehensive chapter on a systematic study of this distribution. A monograph written by Murthy et al. (2004) contains nearly every facet concerning the Weibull distribution and its extensions. Since 2004, there has been significant development of generalized Weibull models, thus there is real need for an update on such models. Rinne (2008) is another valuable handbook on the Weibull distribution. Lai et al. (2011) also provide a bird's eye view of this important subject.

In this chapter, we first define a two-parameter Weibull distribution and then consider some basic properties such as the moments, Weibull probability plots, parameter estimation and the hazard rate function. A three-parameter Weibull distribution is then introduced in Sect. 1.3. We study some graphical methods for estimating parameters, reliability properties and order statistics of this model. Section 1.4 briefly discusses the applications of the Weibull models in diverse fields. A sample list of selected applications with references are given.

The main theme of this mini-monograph is on generalized Weibull distributions. These Weibull related distributions have arisen out of the need to provide more flexibility in modeling lifetime data in diverse disciplines. Many empirical hazard rate plots exhibit non-monotonic shapes including unimodal and bathtub shapes. In

C.-D. Lai, *Generalized Weibull Distributions*, SpringerBriefs in Statistics,
DOI: 10.1007/978-3-642-39106-4_1,

Chap. 2, we study over twenty five such generalizations. Chapter 3 considers those generalizations constructed from two or more Weibull distribution. Lastly in Chap. 4, we study some discrete distributions constructed from the Weibull or generalized Weibull distribution.

A Preliminary

Let T be the lifetime random variable with $f(t)$, $F(t)$ being its probability density function (pdf) and cumulative distribution function (cdf), respectively. The reliability or survival function is given by $\overline{F}(t) = 1 - F(t)$.

The hazard (failure) rate function is defined as

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{\overline{F}(t)}; \tag{1.1}$$

$h(t)\Delta t$ gives (approximately) the probability of failure in $(t, t + \Delta t]$ given the 'unit' has survived uptill time t.

We say that F is IFR (DFR) if $h(t)$ is increasing (decreasing) in t.

F has a bathtub shape failure (hazard) rate if $h(t)$ is initially decreasing during the 'infant mortality' phase, approximately constant during the 'useful period', and finally, increasing during the 'wearout' phase. For most bathtub shaped hazard rate distributions, $h(t)$ is decreasing and then increasing. The hazard rate function $h(t)$ is said to be unimodal if it has a unique mode, i.e., it has an upside-down bathtub shape.

The cumulative hazard rate function is defined as

$$H(t) = \int_0^t h(x)dx. \tag{1.2}$$

It is easy to show that the reliability function can be represented as

$$\overline{F}(t) = e^{-H(t)}. \tag{1.3}$$

Obviously, the cumulative hazard function completely determines the lifetime (ageing) distribution and it must satisfy the following three conditions in order to yield a proper lifetime (ageing) distribution: (i) $H(t)$ is nondecreasing for all $t \geq 0$ (ii) $H(0) = 0$ and (iii) $\lim_{t\to\infty} H(t) = \infty$.

Since the reliability function $\overline{F}(t)$ and the hazard rate function $h(t)$ can be uniquely determined from each other, a new ageing distribution can therefore be derived by constructing one of them first.

The mean residual life (MRL) of F is defined as

$$\mu(t) = E(T - t \mid T > t) = \left[\int_t^\infty \overline{F}(x) dx \right] / \overline{F}(t).$$

We say that F is IMRL (DMRL) if $\mu(t)$ is increasing (decreasing) in t.

Readers should note that the *acronyms* that are defined throughout the text can be navigated via the *index* section of this monograph.

1.2 Two-Parameter Weibull Distribution

The Weibull distribution is one of the best known distributions and has wide applications in diverse disciplines. It has undeniably received most attention since 1970s because of its simplicity and flexibility. A report by Weibull (1977) lists over one thousand references to the applications of the basic Weibull model.

In the forthcoming subsections, we will discuss its properties in some details.

1.2.1 Derivations

The Weibull model can be derived from the following transformations:

- Let $Y = -X$ where X has a type 3 extreme value distribution (Johnson et al. 1995, Chap. 22)
- Let $T = X^{1/\alpha}$ where X has the exponential distribution with parameter λ.

1.2.2 Distribution Function

The distribution function of the standard two-parameter Weibull distribution (Weibull 1951) is given as

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\beta} \right)^\alpha \right], \ \alpha, \beta > 0, t \geq 0. \tag{1.4}$$

The distribution function has another form due to a different parametrization but the present one seems more natural when a location parameter is added to the model as given in the next section. The parameters α and β are usually called the shape and scale parameters, respectively. The corresponding reliability function is given by

$$\overline{F}(t) = \exp\left[-\left(\frac{t}{\beta} \right)^\alpha \right], \ t \geq 0. \tag{1.5}$$

We observe that both the reliability (survival) function and the hazard rate function as given in (1.14) below have simple forms which give the Weibull a head start over other lifetime models.

For some applications, one may find it be more convenient to re-parameterize the above as

$$\overline{F}(t) = \exp\left\{-\lambda t^{\alpha}\right\},\ \alpha, \lambda > 0. \tag{1.6}$$

Two well known special cases are the exponential distribution ($\alpha = 1$) and the Rayleigh distribution ($\alpha = 2$).

If X has a two-parameter Weibull distribution, then $\log X$ is an extreme value distribution with location parameter $\log(\beta)$ and scale parameter $1/\alpha$, i.e., the pdf of $Y = \log X$ is:

$$f(y) = \alpha \exp\left\{\alpha\left(y - \ln \beta\right) - \exp\left[\alpha\left(y - \ln \beta\right)\right]\right\}. \tag{1.7}$$

1.2.3 Skewness and Kurtosis

The distribution is positively skewed for small value of α. The skewness index $\sqrt{\beta_1}$ decreases and equals zero for $\alpha = 3.6$ (approximately). Thus, for values of α in the vicinity of 3.6, the Weibull distribution is similar in shape to a normal distribution. The coefficient of kurtosis β_2 also decreases with α and then increases, reaching a minimum value of about 2.71 when $\alpha = 3.35$ (approximately). See, e.g., Johnson et al. (1994, pp. 631–635) and Mudholkar and Kollia (1994).

1.2.4 Probability Density Function

The probability density function that corresponds to (1.4) is given by

$$f(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1} \cdot \exp\left[-\left(\frac{t}{\beta}\right)^{\alpha}\right]. \tag{1.8}$$

The value of α has strong effects on the shape of the probability density function. For $0 < \alpha \leq 1$, the probability density function is a monotonic decreasing function and is convex as t increases. For $\alpha > 1$, the density function has a unimodal shape. Plots of density functions are given in Fig. 1.1.

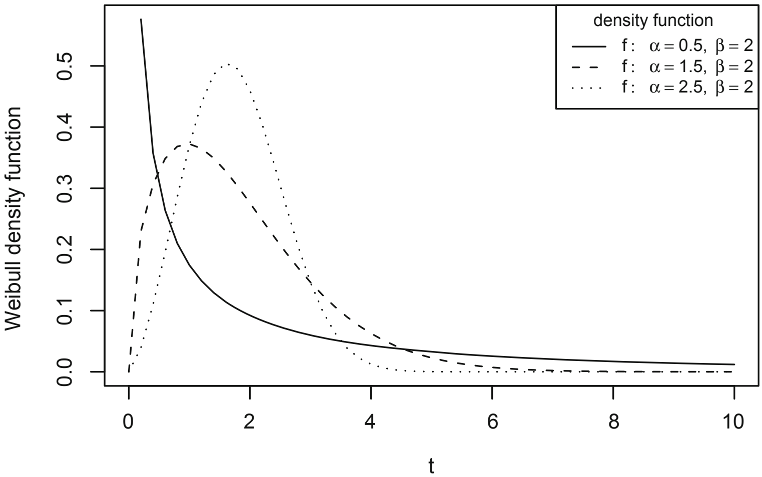

Fig. 1.1 Weibull density functions

1.2.5 The Mean, Variance and Moments

The kth moment about the origin may be obtained via its special case without a scale parameter defined by $X' = X/\beta$. Now it is easy to show that

$$\mu'_k = E(X'^k) = \Gamma\left(\frac{k}{\alpha} + 1\right),\ k = 1, 2, \ldots \tag{1.9}$$

In particular, the mean and variance are, respectively;

$$E(X') = \Gamma\left(\frac{1}{\alpha} + 1\right) \tag{1.10}$$

and

$$\mathrm{var}(X') = \Gamma\left(\frac{2}{\alpha} + 1\right) - \left\{\Gamma\left(\frac{1}{\alpha} + 1\right)\right\}^2 \tag{1.11}$$

(Johnson et al. 1994, p. 632). Here $\Gamma(\cdot)$ denotes the usual gamma function. The coefficient of variation is:

$$\nu = \sqrt{\frac{\Gamma\left(2/\alpha + 1\right)}{\Gamma^2\left(1/\alpha + 1\right)} - 1}. \tag{1.12}$$

The kth moment of X is obtained by

$$E(X^k) = \beta^k \mu'_k = \beta^k \Gamma\left(\frac{k}{\alpha} + 1\right). \tag{1.13}$$

1.2.6 Hazard Rate Function

The hazard rate is defined as the ratio of the density function to its survival function, so the hazard rate function of the two-parameter Weibull distribution is given by

$$h(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1}. \tag{1.14}$$

The shape parameter α also has a strong influence on the shape of the Weibull hazard rate. It is obvious to see that $h(t)$ is a decreasing function in t when $0 < \alpha < 1$, constant when $\alpha = 1$ (the exponential case), and an increasing function when $\alpha > 1$. In view of the monotonic behaviour of its hazard rate function, the Weibull distribution often becomes suitable when the conditions for 'strict randomness' of the exponential distribution are not satisfied, with the shape parameter α having a value depending upon the fundamental nature being considered. Thus the Weibull model is flexible and can be used to model IFR (increasing failure rate) or DFR (decreasing failure rate) ageing distributions. A moderate value of α within 1 to 3 is appropriate in most situations (Lawless 2003). However, the very monotonic shape of its hazard rate has also become a limitation in reliability applications because for many real life data $h(t)$ exhibits some form of non-monotonic behavior. For this reason, several generalizations and modifications of the Weibull distribution have been proposed to meet the need of having various shapes.

The hazard rate of the alternative form (1.6) together with its derivatives are, respectively,

$$h(t) = \alpha\,\lambda(\lambda\,t)^{\alpha-1},$$

$$h'(t) = \alpha(\alpha-1)\overset{2}{\lambda}(\lambda\,t)^{\alpha-2}, \quad h''(t) = \alpha(\alpha-1)(\alpha-2)\overset{3}{\lambda}(\lambda\,t)^{\alpha-3}.$$

It follows from the above that for $1 < \alpha < 2$, $h'(t) > 0$ and $h''(t) < 0$ for all t so the hazard rate of the Weibull distribution is (ultimately increasing but decelerating). However, $h(t)$ does not converge to an asymptote.

The plots of the Weibull hazard rate functions are given in Fig. 1.2.

1.2.7 Mean Residual Life Function

Nassar and Eissa (2003) have shown that the MRL (mean residual life) function has the following form

$$\mu(t) = \beta e^{\tau}\Gamma\,(1 + 1/\alpha)\,[1 - \Gamma_{\tau}(1/\alpha)/\,\Gamma(1/\alpha)],\ \tau = (t/\beta)^{\alpha}. \tag{1.15}$$

Here $\Gamma_{\tau}(r) = \int_0^{\tau} x^{r-1}e^{-x}dx$ is the incomplete gamma function and $\Gamma(\cdot)$ is defined by $\Gamma(r) = \int_0^{\infty} x^{r-1}e^{-x}dx$.

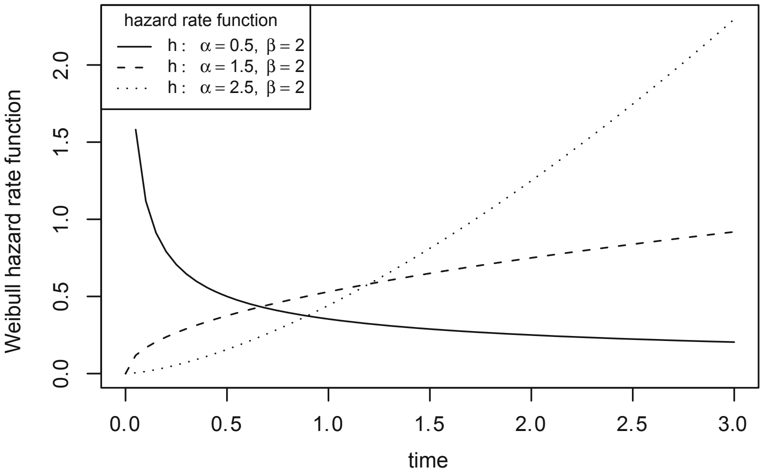

Fig. 1.2 Weibull hazard rate functions

For $\beta = \sqrt{2}$ and $\alpha = 2$,

$$\mu(t) = \sqrt{2\pi}e^{t^2/2}\left(1 - \Phi(t)\right),\ t > 0 \tag{1.16}$$

where $\Phi(\cdot)$ denote the cdf of a standard normal variable.

As IFR (DFR) implies DMRL (IMRL), it follows that $\mu(t)$ decreases (increases) in t for $\alpha > 1$ ($0 < \alpha \le 1$).

1.2.8 TTT Plot for Two-parameter Weibull Distribution

In reliability and survival analysis, we often see the use of the scaled total time on test statistic:

$$\phi(r/n) = T(X_r)/(X_1 + X_2 + \cdots + X_n), \tag{1.17}$$

derived from the total time on test (TTT) transform of F given by

$$H_F^{-1}(p) = \int_0^{F^{-1}(p)} \overline{F}(x)dx,\ 0 < p < 1. \tag{1.18}$$

Here

$$T(X_r) = nX_1 + (n-1)(X_2 - X_1) + \cdots + (n-r+1)(X_r - X_{r-1}), \tag{1.19}$$

where $r = 1, 2, \ldots, n$ and n is the total number of failure times ordered as $X_1 < X_2 < \cdots < X_n$. The shape characteristics of the scaled TTT plot $(r/n, \phi(r/n))$ are related to the shape of the empirical hazard rate which in turn will assist us to select a probability model. If the empirical TTT plot is close to the graph of the scaled transform based on the standard Weibull model, then the model can be accepted as an adequate model.

The TTT plot of the Weibull model is a straight line if $\alpha = 1$. It is convex for $\alpha < 1$ and concave for $\alpha > 1$.

1.2.9 Methods of Estimating Parameters

Many different methods can be applied to estimate the parameters of the Weibull distribution. Generally, these methods can be classified into two main categories, the graphical techniques and the statistical methods. Some frequently used graphical methods include methods using the empirical cumulative distribution plot (Nelson 1982), Weibull probability plot (Nelson 1982; Kececioglu 1991; Lawless 2003), hazard rate plot (Nelson 1982) and so on.

1.2.10 Maximum Likelihood Estimation

Suppose there are r components in a sample of N components failed in a sample testing. Given the failure data following the Weibull distribution, $t_1, t_2, \ldots, t_r$ are the lifetime of r failed components; let t_r be the censoring time for the rest $n - r$ components. The likelihood function of the standard two-parameter Weibull distribution has the form:

$$L(\alpha, \beta) = \frac{N!}{(N-r)!} \left(\frac{\alpha}{\beta^\alpha}\right)^r \prod_{i=1}^{r} t_i^{\alpha-1} \exp\left\{-\frac{1}{\beta^\alpha}\left[\sum_{i=1}^{r} t_i^\alpha + (N-r)t_r^\alpha\right]\right\}. \tag{1.20}$$

Thus the log-likelihood function is given by:

$$\log L(\alpha, \beta) = \log\left[\frac{N!}{(N-r)!}\right] + r\,(\log\alpha - \alpha\log\beta) + (\alpha - 1)\sum_{i=1}^{r}\log t_i - \left\{\frac{1}{\beta^\alpha}\left[\sum_{i=1}^{r} t_i^\alpha + (N-r)t_r^\alpha\right]\right\}. \tag{1.21}$$

Take the first derivative with respect to both parameters and then set them to zero, we obtain

$$\frac{\sum_{i=1}^{r} t_i^{\alpha} \log t_i + (N-r)t_r^{\alpha} \log t_r}{\sum_{i=1}^{r} t_i^{\alpha} + (N-r)t_r^{\alpha}} - \frac{1}{\alpha} - \frac{1}{r}\sum_{i=1}^{r} \log t_i = 0 \tag{1.22}$$

and

$$\beta = \left\{ \frac{1}{r} \left[\sum_{i=1}^{r} t_i^{\alpha} + (N-r)t_r^{\alpha} \right] \right\}^{1/\alpha}. \tag{1.23}$$

Solving the Eq. (1.22), we can find the maximum likelihood estimate (MLE) of the shape parameter α, and the estimation of the scale parameter β can then be obtained from (1.23). For estimation procedures for grouped data, see, e.g., Nelson (1982); Lawless (2003); Cheng and Chen (1988) and Rao et al. (1994).

We note that for the MLEs, the likelihood equation (1.22) needs to be solved numerically and related software programs need to be applied. Balakrishnan and Kateri (2008) proposed an alternative approach based on a very simple and easy-to-apply graphical method which also readily shows the existence and uniqueness of the maximum likelihood estimates. Furthermore, for censored data from a large sample size, a closed-form estimator for the shape parameter was also obtained.

1.2.11 Other Estimation Methods

The least squares estimation (LSE) and the MLE are two common methods to estimate the Weibull parameters. While the MLE method is preferred by many researchers because of its good theoretical properties, the LSE method, especially when used in conjunction with graphical methods such as the Weibull probability plot (WPP) to be discussed in the next section, is most widely used by practitioners. This is due in part to its computational simplicity.

A minimax optimization procedure for estimating the Weibull parameters with the Kolmogorov-Smirnov distance used as the objective was proposed by Ling and Pan (1998).

There are several other parameter estimation methods for the Weibull and we refer our readers to the monograph by Murthy et al. (2004) for details.

1.2.12 Confidence Interval and Prediction Interval for Weibull Mean

Krishnamoorthy et al. (2009) considered the problems of constructing confidence interval for a Weibull mean and setting prediction limits for future samples. Specifically, they constructed upper prediction limits that include at least l of m samples from a Weibull distribution at each of r locations. The methods are based on the generalized variable approach. The procedures can be easily extended to the type II censored samples. The generalized variable methods for estimating a Weibull mean when the samples are type I censored are also outlined in their paper. The proposed methods are thought to be conceptually simple and easy to use.

Yang et al. (2007) proposed a general method called 'analytically adjusted naive' method for constructing confidence intervals for their functions which eliminated the need for the extensive tables. The method is applied to obtain confidence intervals for the scale parameter, the mean-time-to-failure, the percentile function, and the reliability functions. Monte-Carlo simulations showed that these intervals possess finite sample properties, having coverage probabilities very close to their nominal levels, irrespective of the sample size and the degree of censorship.

1.2.13 Bias

It is worth noting that when dealing with complete and small samples, the MLE and LSE estimates of the Weibull parameters, especially the shape parameter, are long known to be significantly biased (Montanari et al. 1997). The bias is also significant for the heavy censoring cases which are common in field conditions.

There are several methods for correcting the bias of the MLE of the Weibull parameters, mostly in the area of dielectric breakdowns studies. We refer our readers to Ross (1994, 1996) and Hirose (1999) for a review of various bias correction formulas for the shape parameter.

1.3 Three-Parameter Weibull Distribution

Introducing a location parameter to a two-parameter Weibull distribution results a three-parameter Weibull distribution. This more general but also more flexible distribution has cumulative distribution function given by

$$F(t) = 1 - \exp\left\{-\left[\frac{t-\tau}{\beta}\right]^{\alpha}\right\},\ t \geq \tau. \tag{1.24}$$

The three parameters of the distribution are given by the set $\theta = \{\alpha, \beta, \tau\}$ with $\alpha > 0, \beta > 0$ and $\tau \geq 0$; where β is a scale parameter, α is the shape parameter that determines the appearance or shape of the distribution and τ is the location parameter. The survival function is

$$\overline{F}(t) = \exp\left\{-\left[\frac{t-\tau}{\beta}\right]^{\alpha}\right\} \tag{1.25}$$

An alternative form of a three-parameter Weibull distribution can be expressed as

$$F(t) = 1 - \exp\left\{-\lambda\,(t-\tau)^{\alpha}\right\},\; t \geq \tau. \tag{1.26}$$

Here, the parameter λ combines both features of scale and shape. Clearly, $\lambda = \beta^{-\alpha}$. For $\tau = 0$, this becomes a two-parameter Weibull distribution. Note that we referred to this special case as the *standard* Weibull model; however, Johnson et al. (1994) called a standard Weibull when $\beta = 1$ (or $\lambda = 1$) and the location parameter $\tau = 0$ in the above equations.

1.3.1 Density Function

The probability density function of the three-parameter Weibull distribution is

$$f(t) = \alpha\beta^{-\alpha}(t-\tau)^{\alpha-1}\exp\left\{-\left[\frac{t-\tau}{\beta}\right]^{\alpha}\right\}, t \geq \tau. \tag{1.27}$$

1.3.2 Mode and Median

The mode is at $t = \beta\left(\frac{\alpha-1}{\alpha}\right)^{1/\alpha} + \tau$ for $\alpha > 1$ and at τ for $0 < \alpha \leq 1$. The median of the distribution is at $\beta(\log 2)^{1/\alpha} + \tau$.

1.3.3 Moments

The kth moment of X defined by the density function (1.27) can be easily obtained from the relationship $X = \beta X' + \tau$ with $\mu_k' = E(X'^k)$ given by Eq. (1.9). In particular, the mean and variance of the three-parameter Weibull random variable are, respectively,

$$E(X) = \beta\Gamma\left(\frac{1}{\alpha}+1\right) + \tau \tag{1.28}$$

and

$$\mathrm{var}(X) = \beta^2\Gamma\left(\frac{2}{\alpha}+1\right) - \beta^2\left\{\Gamma\left(\frac{1}{\alpha}+1\right)\right\}^2. \tag{1.29}$$

1.3.4 Weibull Probability Plot

The Weibull probability plot (WPP) can be constructed in several ways (Nelson and Thompson 1971). In the early 1970s a special paper was developed for plotting the data in the form $F(t)$ versus t on a graph paper with log-log scale on the vertical axis and log scale on the horizontal axis. A WPP plotting of data involves computing the empirical distribution function which can be estimated in different ways with the two standard ones being

- $\hat{F}(t_i) = i/(n+1)$, the "mean rank" estimator, and
- $\hat{F}(t_i) = (i-0.5)/n$, the "median rank" estimator.

Here, the data set consists of successive failure times t_i, $t_1 < t_2 < \ldots < t_n$. For censored data (right censored or interval), the approach to obtain the empirical distribution functions needs to be modified, see for example, Nelson (1982).

These days, most statistical software packages contain programs to produce these plots automatically from a given data set. A well known statistical package MINITAB provides a Weibull probability plot under **Graph** menu >> **Probability Plot.**

We may use ordinary graph paper or spreadsheet software with unit scale for plotting. Taking logarithms twice of both sides of the survival function (1.25) yields

$$\log(-\log \overline{F}(t)) = \alpha \log(t-\tau) - \alpha \log \beta. \tag{1.30}$$

Let $y = \log(-\log \overline{F}(t))$ and $x = \log(t-\tau)$. Then we have a linear equation

$$y = \alpha x - \alpha \log \beta. \tag{1.31}$$

The plot is now on a linear scale. We can now see that a WPP can indicate a straight line if the assumption of a Weibull population for the data set concerned is plausible. The least squares estimates derived from (1.31) can be used as an initial estimates of the Weibull parameters with $\hat{\alpha}$ = regression coefficient and $\hat{\beta} = \exp\{-(y\text{–intercept}/\hat{\alpha})\}$.

Thus the Weibull probability plot is a favoured tool by many reliability engineers.

Example: Suppose a data set consists of the times-to-failure of a product: 15, 32, 61, 67, 75, 116, 148, 178, 181, 183.

Figure 1.3 on the next page gives the two-parameter Weibull probability plot with the linear regression line being fitted. The figure is plotted using R[1]:

>lm(formula = $y \sim$ log(lifetime))

Coefficients:

(Intercept)	log(lifetime)
−6.892	1.437

[1] R is a statistical package, see http://www.r-project.org/: for further information.

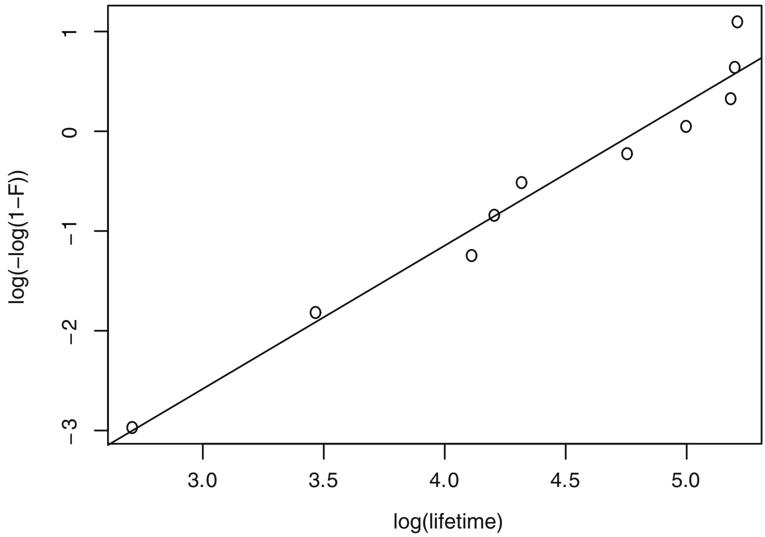

Fig. 1.3 Weibull probability plot with fitted regression line

Comparing Two Estimation Approaches for Parameters of Weibull Distribution Based on WPP

Note that in the above WPP, we have assigned $Y = \log(-\log(\overline{F}(t))$ and $X = \log(T - \tau)$. For simplicity, let us assume $\tau = 0$. The least squares estimates of the parameters can be obtained from (1.31). Thus this method is using the least-squares regression of Y on X, i.e. minimizing the sum of squares of the vertical residuals, to fit a straight line to the data points on WPP and then calculate the least squares estimates. This method is known to be biased.

In the existing literature the least-squares regression of X on Y, i.e. minimizing the sum of squares of the horizontal residuals, has been used by some Weibull authors. This motivated Zhang and Xie (2007) to carry out a comparison between the estimators of the two least squares regression methods using intensive Monte Carlo simulations. Both complete and censored data are examined. Surprisingly, the result shows that least squares regression of Y on X performs better for small and complete samples; whereas the least squares regression of X on Y performs better in other cases in view of bias of the estimators. The two methods are also compared in terms of other model statistics. In general, when the shape parameter is less than one, least squares regression of Y on X provides a better model; otherwise, the least squares regression of X on Y tends to be better.

1.3.5 Weibull Hazard Plot

The hazard plot is analogous to the probability plot, the principal difference being that the observations are plotted against the cumulated hazard (failure) rate rather than the cumulated probability value. Moreover, this is designed for censored data.

Let $H(t)$ denote the cumulative hazard rate , then it is easy to show that $\overline{F}(t) = \exp(-H(t))$ so

$$H(t) = -\log \overline{F}(t) = \left(\frac{t-\tau}{\beta}\right)^{\alpha} \tag{1.32}$$

or equivalently

$$H(t)^{1/\alpha}/\beta = (t-\tau). \tag{1.33}$$

Let $y = \log(t-\tau)$ and $x = \log H(t)$, then we have

$$y = \log \beta + \frac{1}{\alpha}x. \tag{1.34}$$

Rank the n survival times (including the censored) in ascending order and let K denote the reverse ranking order of the survival time, i.e., $K = n$ for the smallest survival time and $K = 1$ for the largest survival time. The hazard (rate) is estimated from $100/K$ (a missing value symbol is entered at a censored failure time). The cumulative hazard is obtained by cumulating the hazard rates. The Weibull hazard plot is simply the plot arising from (1.32). See Nelson (1972) for further details.

Both WPP and the Weibull hazard plot, in addition to providing simple straight line fitting for parameter estimation, also have a role in model validation which is important in any engineering analysis.

1.3.6 Estimation of Parameters

The parameters can be estimated using the equations for the mean (1.28), the variance (1.29), and the third central moment given by

$$\mu_3 = \beta^3 \left\{\Gamma(1+3/\alpha) - 3\Gamma(1+1/\alpha)\Gamma(1+2/\alpha) + 2[\Gamma(1+1/\alpha)]^3\right\}. \tag{1.35}$$

Solving the three equations simultaneously yields the parameter estimates by the method of moments.

Maximum likelihood estimation for the parameters in the three-parameter Weibull distribution needs an initial value in the iteration scheme for likelihood equations. In certain cases, maximum likelihood estimation can break down for the three-parameter Weibull model, as no local maximum of the likelihood exists. To avoid these difficulties, Some alternatives were proposed.

- Islam et al. (2001) explicitly obtained the modified maximum likelihood estimators of τ and β. They also showed that the modified maximum likelihood estimates are asymptotically equivalent to the maximum likelihood estimates.
- Ahmad (1994) suggested a modification of the weighted least-squares to estimate the location parameter τ first and then find the weighted least-squares estimators for the scale parameter β and shape parameter α.

1.3.7 Order Statistics

Let $X_1, X_2, \ldots, X_n$ denote n independent and identically distributed three-parameter Weibull random variables. Furthermore, let $X_{(1)} < X_{(2)} < \cdots \leq X_{(n)}$ denote the order statistics from these n variables. The kth order statistic $X_{(k)}$ from a sample of n observations corresponds to the lifetime of a $(n-k+1)$-out-of-n system of n independent and identically distributed Weibull components. The probability density function of $X_{(1)}$, is given by

$$\begin{aligned} f_1(t) &= n[1-F(t)]^{n-1} f(x) \\ &= \frac{n\alpha}{\beta}\left(\frac{t-\tau}{\beta}\right)^{\alpha-1} e^{-n[(t-\tau)/\beta]^{\alpha}}, \; t \geq \tau \geq 0. \end{aligned} \tag{1.36}$$

It is obvious that $X_{(1)}$ is also distributed as a Weibull random variable, except that α is replaced by $\beta n^{-1/\alpha}$. The density function of $X_{(r)}$ $(1 \leq r \leq n)$ is

$$\begin{aligned} f_r(t) = &\frac{n!}{(r-1)!(n-r)!}\left(1-e^{-[(t-\tau)/\beta]^{\alpha}}\right)^{r-1} e^{-[(t-\tau)/\beta]^{\alpha}(n-r+1)} \\ &\times \alpha\beta^{-\alpha}(t-\tau)^{\alpha-1}, \; t \geq \tau \geq 0. \end{aligned} \tag{1.37}$$

It can be shown that

$$E\left[(X_{(r)})^k\right] = \sum_{i=0}^{k} \tau^i \beta^{k-i} \omega_{(r)}^{k-i} \tag{1.38}$$

where

$$\omega_{(r)}^k = \frac{n!}{(r-1)!(n-r)!}\Gamma\left(1+\frac{k}{\alpha}\right)\sum_{i=0}^{r-1}\frac{(-1)^r \binom{r-1}{i}}{(n-r+i+1)^{1+(k/\alpha)}}.$$

1.3.8 Hazard Rate Function

The hazard rate function for the three-parameter Weibull is

$$h(t) = \frac{f(t)}{\overline{F}(t)} = \frac{\alpha}{\beta}\left(\frac{t-\tau}{\beta}\right)^{\alpha-1}, \ t \geq \tau. \tag{1.39}$$

It is obvious that $h(t)$ given above is similar to its corresponding function for the two-parameter case as given in (1.14) except the function is now defined over $[\tau, \infty)$ instead of $[0, \infty)$.

There are many extensions, generalizations and modifications of the Weibull distribution. They arise out of the need to model some empirical data sets which cannot be adequately described by a three-parameter Weibull model. For example, the monotonic property of the Weibull's hazard rate function which is unable to capture the behavior of a data set that has a bathtub shaped hazard rate. Lai et al. (2001) and Xie et al. (2003) reviewed several Weibull-related distributions that exhibit bathtub shaped hazard rates. Plots of mean residual life from several of these generalized Weibull derived models were given in Lai et al. (2004). For simplicity, we sometimes simply refer these Weibull-related models as Weibull models.

1.4 Applications of Weibull Models

There are numerous applications for Weibull and Weibull related distributions in all aspects of life so it would be futile, perhaps unhelpful, to list them all here. Most of the applications are pitched towards reliability, survival analysis, product warranty, medicine, biology, agriculture, forestry (tree diameter growth) and industrial engineering (manufacturing and delivery times, reliability control charts), to name just a few. Since the Weibull distribution is also an extreme value distribution it is frequently used to model environmental data such as rains, floods and wind speeds (wind speeds in most parts of the world can be modeled using the Weibull distribution). The following are a selected few applications. These are arranged in alphabetical order as follows:

1. Adhesive wear in metals—Queeshi and Sheikh (1997).
2. Aircraft windshield hazards data—Murthy et al. (2004).
3. Analysis of survival data from clinical trials—Carroll (2003).
4. Annual flood discharge rates—Mudholkar and Hutson (1996).
5. Bus motor failures—Davis (1952), Mudholkar et al. (1995)
6. Carbon fibers and composites hazards—Durham and Padgett (1997).
7. Cavitation erosion resistance—Meged (2004).
8. Cleaning web hazard times in photocopiers—Murthy et al. (2004).
9. Device hazard times—Aarset (1987).
10. Die cracking in the assembly and reliability testing of flip-chip (FC) packages—Zhao (2004).
11. Earthquake-probability of occurrence—Hagiwaraa (1974).
12. Earthquakes interoccurrence times—Hasumi et al. (2009).

13. Fatigue of bearings—Cohen et al. (1984).
14. Flood frequency—Heo et al. (2001).
15. Food safety—Fernández et al. (2002); Marabi et al. (2003).
16. Food preservation—Corzo et al. (2008), Oms-Oliu et al. (2009).
17. Food drying process—Miranda et al. (2010).
18. Forecasting technological change—Sharif and Islam (1980).
19. Fracture strength of glass—Keshevan et al. (1980).
20. Forest fire—Grissino-Mayer (1999).
21. Fracture strength data obtained from ASTM D3039 tension tests of 19 identical carbon-epoxy composite specimens—Birgoren and Dirikolu (2004).
22. Hazard of coatings recoil compressive hazard in high performance polymers—Almeida (1999); Newell et al. (2002).
23. Hazards of brittle materials—Fok et al. (2001).
24. Hazard probability prediction of concrete components—Li et al. (2003).
25. Inventory control—Tadikamalla (1978); Lo et al. (2007); Pan et al. (2009); Yang et al. (2011).
26. Large-scale multiprocessor systems—Al-Rousan and Shaout (2004).
27. Material strength—Weibull (1939).
28. Pharmacoepidemiology—Suissa (2008).
29. Product warranty—Blischke and Murthy (1994, 1996); Murthy and Djamaludin (2002).
30. Pitting corrosion and pipeline reliability—Sheikh et al. (1990).
31. Reliability control charts—Batson et al. (2006); Chen and Cheng (2007); Sürücü and Sazak (2009).
32. Reliability sampling plans—Tsai et al. (2008).
33. Retailer’s pricing and ordering strategy—Shah and Raykundaliya (2010).
34. Software faults—Zhang (2008).
35. Strength of brittle materials—Basu et al. (2009).
36. Strength of dental materials—Nakamura et al. (2010).
37. Tree diameter growth—Bailey and Dell (1973); Maltamo et al. (1995).
38. Throttle hazard times—Carter (1986); Murthy et al. (2004).
39. Wind velocity—Justus et al. (1978); Tuller and Brett (1984).
40. Wind speed data—Al-Hasan and Nigmatullin (2003); Rehman et al. (1994); Lun and Lam (2000).
41. Wind energy—Ojosu and Salawu (1990); Yeh and Wang (2008); Akda and Dinlerb (2009); Akdağ and Güler (2009).
42. Yield strength of steel, fatigue life of steel—Weibull (1951).

We refer our readers to Chap. 7 of Rinne (2008) for other applications of the Weibull model.

References

Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, *36*, 106–108.

Ahmad, K. E. (1994). Modified weighted least-squares estimators for the three-parameter Weibull distribution. *Applied Mathematics Letters*, *7*(5), 53–56.

Akda, S. A., & Dinlerb, A. (2009). A new method to estimate Weibull parameters for wind energy applications. *Energy Conversion and Management*, *50*(7), 1761–1766.

Akdağ, S. A. and Güler, Ö (2009), Calculation of wind energy potential and economic analysis by using Weibull distribution—A case study from Turkey. Part 2: Economic analysis, *Energy Sources, Part B: Economics, Planning, and Policy, 4*(1), 9–16.

Al-Hasan, M., & Nigmatullin, R. R. (2003). Identification of the generalized Weibull distribution in wind speed data by the Eigen-coordinates method. *Renewable Energy*, *28*(1), 93–110.

Almeida, J. B. (1999). Application of Weilbull statistics to the failure of coatings. *Journal of Material Processing and Technology*, *93*, 257–263.

Al-Rousan, M., & Shaout, A. (2004). Closed-form solution for reliability of SCI-based multi-processor systems using Weibull distribution and selfhealing rings. *Computers and Electrical Engineering*, *30*(4), 309–329.

Bailey, R. L., & Dell, T. R. (1973). Quantifying diameter distributions with the Weibull function. *Forest Science*, *19*(2), 97–104.

Balakrishnan, N., & Kateri, M. (2008). On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data. *Statistics and Probability Letters*, *78*, 2971–2975.

Basu, B., Tiwari, D., Kundu, D., & Prasad, R. (2009). Is Weibull distribution the most appropriate statistical strength distribution for brittle materials? *Ceramics International*, *35*, 237–246.

Batson, R. G., Jeong, Y., Fonseca, D. J., & Ray, P. S. (2006). Control charts for monitoring field failure data. *Quality and Reliability Engineering International*, *22*(7), 733–755.

Birgoren, B., & Dirikolu, M. H. (2004). A computer simulation for estimating lower bound fracture strength of composites using Weibull distribution. *Composites Part B-Engineering*, *35*(3), 263–266.

Blischke, W. R., & Murthy, D. N. P. (1994). *Product warranty handbook*. New York: Marcel Dekker.

Blischke, W. R., & Murthy, D. N. P. (1996). *Warranty cost analysis*. New York: Marcel Dekker.

Carroll, K. J. (2003). On the use and utility of the Weibull model in the analysis of survival data. *Controlled Clinical Trials*, *24*, 682–701.

Carter, A. D. S. (1986). *Mechanical Reliability*. New York: Wiley.

Chen, H. F., & Cheng, Y. Y. (2007). Non-normality effects on the economic-statistical design of $\bar{X}$ charts with Weibull in-control time. *European Journal of Operations Research*, *176*(2), 986–998.

Cheng, K. F., & Chen, C. H. (1988). Estimation of the Weibull parameters with grouped data. *Communications in Statistics-Theory and Methods*, *17*, 325–341.

Corzo, O., Bracho, N., Pereira, A., & Alberto Vásquez, A. (2008). Weibull distribution for modeling air drying of coroba slices. *Food Science and Technology*, *41*, 2023–2028.

Durham, S. D., & Padgett, W. J. (1997). Cumulative damage model for system failure with application to carbon fibers and composites. *Technometrics*, *39*, 34–44.

Fernández, A., Collado, J., Cunha, L. M., Ocio, M. J., & Martinez, A. (2002). Empirical model building based on Weibull distribution to describe the joint effect of pH and temperature on the thermal resistance of Bacillus cereus in vegetable substrate. *International Journal of Food Microbiology*, *77*(1–2), 147–153.

Fok, S. L., Mitchell, B. C., Smart, J., & Marsden, B. J. (2001). A numerical study on the application of the Weibull theory to brittle materials. *Engineering Fracture Mechanics*, *68*, 1171–1179.

Fréchet, M. (1927). Sur la loi de probabilité de lécart maximum. *Annales de la Société Polonaise de Mathematique, Cracovie*, *6*, 93–116.

Grissino-Mayer, H. D. (1999). Modeling fire interval data from the American southwest with the Weibull distribution. *International Journal of Wildland Fire*, *9*(1), 37–50.

Hagiwaraa, Y. (1974). Probability of earthquake occurrence as obtained from a Weibull distribution analysis of crustal strain. *Tectonophysics*, *23*(3), 313–318.
Hallinan, A. J. (1993). A review of Weibull distribution. *Journal of Quality Technology*, *25*(2), 85–93.
Hasumi, T., Akimoto, T., & Aizawa, Y. (2009). The Weibull-log Weibull distribution for interoccurrence times of earthquakes. *Physica A: Statistical Mechanics and its Applications*, *388*(4), 491–498.
Heo, J. H., Boes, D. C., & Salas, J. D. (2001). Regional flood frequency analysis based on a Weibull model: Part 1. *Estimation and asymptotic variances, Journal of Hydrology*, *242*, 157–170.
Islam, M. Q., Tiku, M. L., & Yildirim, F. (2001). Non-normal regression: Part 1: Skewed distributions. *Communications in Statistics-Theory and Methods*, *30*, 993–1020.
Johnson, N. L., Kotz, S., & Balakrishan, N. (1994). *Continuous univariate distributions, Vol. 1 (2nd Edn.)*. New York: Wiley.
Justus, C. G., Hargraves, W. R., Mikhail, A., & Graber, D. (1978). Methods for estimating wind speed frequency distributions. *Journal of Applied Meteorology*, *17*(3), 350–385.
Kececioglu, D. (1991). *Reliability engineering handbook*. Englewood Cliffs, New Jersey: Prentice-Hall.
Keshevan, K., Sargent, G., & Conrad, H. (1980). Statistical analysis of the Hertzian fracture of pyrex glass using the Weibull distribution function. *Journal of Material Science*, *15*, 839–844.
Krishnamoorthy, K., Lin, Y., & Xia, Y. (2009). Confidence limits and prediction limits for a Weibull distribution based on the generalized variableapproach. *Journal of Statistical Planning and Inference*, *139*(8), 2675–2684.
Lai, C. D., Xie, M., & Murthy, D. A. P. (2001). Bathtub shaped life distributions, Handbook of Statistics. In N. Balakrishnan and C. R. Rao (Eds.) (Chapter 3, Vol. 20, pp 69–104) . Amsderdam: Elsevier.
Lai, C. D., Zhang, L. Y., & Xie, M. (2004). Mean residual life and other properties of Weibull related bathtub shape failure rate distributions. *International Journal of Reliability, Quality and Safety Engineering*, *11*(2), 113–132.
Lai, C. D., Murthy, D. N. P., & Xie, M. (2011). Weibull distributions. *Wiley Interdisciplinary Reviews*, *3*(3), 282–287.
Lawless, J. F. (2003). *Statistical models and methods for lifetime data*. Hoboken, New Jersey: John Wiley & Sons.
Li, Q. S., Fang, J. Q., Liu, D. K., & Tang, J. (2003). Failure probability prediction of concrete components. *Cement and Concrete Research*, *33*, 1631–1636.
Ling, J., & Pan, J. (1998). A new method for selection of population distribution and parameter estimation. *Reliability Engineering and System Safety*, *60*, 247–255.
Lo, S. T., Wee, H. M., & Huang, W. C. (2007). An integrated production-inventory model with imperfect production processes and Weibull distribution deterioration under inflation. *International Journal of Production Economics*, *106*(1), 248–260.
Lun, I. Y. F., & Lam, J. C. (2000). A study of Weibull parameters using long-term wind observations. *Renewable Energy*, *20*(2), 145–153.
Maltamo, M., Puumalainen, J., & Päivinen, R. (1995). Comparison of beta and Weibull functions for modelling basal area diameter distribution in stands of pinus sylvestris and picea abies. *Scandinavian Journal of Forest Research*, *10*(1&4), 284–295.
Marabi, A., Livings, S., Jacobson, M., & Saguy, I. S. (2003). Normalized Weibull distribution for modeling rehydration of food particulates. *European Food Research and Technololoy*, *217*, 311–318.
Meged, Y. (2004). An improved method for determination of the cavitation erosion resistance by a Weibull distribution. *Journal of Testing and Evaluation*, *32*(5), 373–382.
Miranda, M., Vega-Gálvez, A., Garcia, P., Scala, K. D., Shi, J., & Xue, S. (2010). Effect of temperature on structural properties of Aloe vera (Aloe barbadensis Miller) gel and Weibull distribution for modelling drying process. *Food and Bioproducts Processing*, *88*(2–3), 138–144.

Montanari, G. C., Mazzanti, G., Cacciari, M., & Fothergill, J. C. (1997). In search of convenient techniques for reducing bias in the estimation of Weibull parameter for uncensored tests. *IEEE Transactions on Dielectrics and Electrical Insulation*, *4*(4), 306–313.

Mudholkar, G. S., & Kollia, G. D. (1994). Generalized Weibull family: A structural analysis. *Communications in Statistics-Theory and Methods*, *23*, 1149–1171.

Mudholkar, G. S., Srivastava, D. K., & Freimer, M. (1995). The exponentiated Weibull family: A reanalysis of bus-motor-failure data. *Technometrics*, *37*, 436–445.

Mudholkar, G. S., & Hutson, A. D. (1996). The exponentiated Weibull family: Some properties and a flood data application. *Communications in Statistics-Theory and Methods*, *25*(12), 3059–3083.

Murthy, D. N. P., & Djamaludin, I. (2002). Product warranty: A review. *International Journal of Production Economics*, *79*, 231–260.

Murthy, D. N. P., Xie, M., & Jiang, R. (2004). *Weibull models*. Hoboken, New Jersey: John Wiley & Sons, Inc.

Nakamura, Y., Hojo, S., & Sato, H. (2010). The effect of surface roughness on the Weibull distribution of porcelain strength. *Dental Materials Journal*, *29*(1), 30–34.

Nassar, M. M., & Eissa, F. H. (2003). On the exponentiated Weibull distribution. *Communications in Statistics-Theory and Methods*, *32*(7), 1317–1336.

Nelson, W., & Thompson, V. C. (1971). Weibull probability papers. *Journal of Quality Technology*, *3*, 140–146.

Nelson, W. (1972). Theory and application of hazard plotting for censored failure data. *Journal of Quality Technology*, *14*, 935–966.

Nelson, W. (1982). *Applied life data analysis*. New York: Wiley.

Newell, J. A., Kurzeja, T., Spence, M., & Lynch, M. (2002). Analysis of recoil compressive failure in high performance polymers using two- and four-parameter Weibull models. *High Performance Polymers*, *14*(4), 425–434.

Ojosu, J. O., & Salawu, R. I. (1990). *An evaluation of wind energy potential as a power generation source in Nigeria, Solar & Wind Technology*, *7*(6), 663–673.

Oms-Oliu, G., Odriozola-Serrano, I., Soliva-Fortuny, R., & Martín-Belloso, O. (2009). Use of Weibull distribution for describing kinetics of antioxidant potential changes in fresh-cut watermelon. *Journal of Food Engineering*, *95*, 99–105.

Pan, W., Hua, G., Yu, L., Zhang, J., Xie, G., & Wang, S. (2009). Determining optimal selling price, order size and the number of price changes with Weibull distribution deterioration. *Journal of Computers*, *4*(12), 1294–1300.

Queeshi, F. S., & Sheikh, A. K. (1997). Probabilistic characterization of adhesive wear in metals. *IEEE Transactions on Reliability*, *46*, 38–44.

Rao, A. V., Rao, A. V. D., & Narasimham, V. L. (1994). Asymptotically optimal grouping for maximum likelihood estimation of Weibull parameters. *Communications in Statistics, Series B: Simulation and Computation*, *23*, 1077–1096.

Rehman, S., Halawani, T. O., & Husain, T. (1994). Weibull parameters for wind speed distribution in Saudi Arabia. *Solar Energy*, *53*(6), 473–479.

Rinne, H. (2008). *The Weibull distribution: A handbook*. Chapman and Hall/CRC.

Rosin, P., & Rammler, E. (1933). The laws governing the fineness of powdered coal. *Journal of the Institute of Fuel*, *7*, 29–36.

Ross, R. (1994). Formulas to describe the bias and standard deviation of the ML-estmated Weibull shape parameters. *IEEE Transactions on Dielectrics and Electrical Insulation*, *1*(2), 247–253.

Ross, R. (1996). Bias and standard deviation due to Weibull parameter estimation for small data sets. *IEEE Transactions on Dielectrics and Electrical Insulation*, *3*(1), 28–42.

Shah, N. H., & Raykundaliya, N. (2010). Retailer's pricing and ordering strategy for Weibull distribution deterioration under trade credit in declining market. *Applied Mathematical Sciences*, *4*(21), 1011–1020.

Sharif, M. N., & Islam, M. N. (1980). The Weibull distribution as a general model for forecasting technological change. *Technological Forecasting and Social Change*, *19*(3), 247–256.

Sheikh, A. K., Boah, J. K., & Hansen, D. A. (1990). Statistical modelling of pitting corrosion and pipeline reliability. *Corrosion*, *46*, 190–196.

Suissa, S. (2008). Immortal time bias in pharmacoepidemiology. *American Journal of Epidemiology*, *167*(4), 492–499.

Sürücü, B., & Sazak, H. S. (2009). Monitoring reliability for a three-parameter Weibull distribution. *Reliability Engineering and System Safety*, *94*, 503–508.

Tadikamalla, P. R. (1978). Applications of the Weibull distribution in inventory control. *The Journal of the Operational Research*, *29*(1), 77–83.

Tsai, T. R., Lu,Y. T., & Wu, S. J. (2008). Reliability sampling plans for Weibull distribution with limited capacity of test facility. *Computers & Industrial Engineering*, *55*(3), 721–728.

Tuller, S. E., & Brett, A. C. (1984). The characteristics of wind velocity that favor the fitting of a Weibull distribution in wind speed analysis. *Journal of Applied Meteorology*, *23*(1), 124–134.

Weibull, W. (1939). A statistical theory of the strength of material. *Ingeniors Vetenskapa Acadamiens Handlingar, Stockholm*, *151*, 1–45.

Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics-Transactions of the American Society of Mechanical Engineers*, *18*(3), 293–297.

Weibull, W. (1977). *References on Weibull distribution*. A Report, Forsvarets Teletekniska Laboratorium, Stockholm: FTL.

Xie, M., Lai, C. D., & Murthy, D. N. P. (2003). Weibull-related distributions for modelling of bathtub shaped failure rate functions. In H. Lindqvist 330 References and K. A. Doksum, (Eds.), *Mathematical and statistical methods in reliability B* (pp. 283–297). Singapore: World Scientific publishing.

Yang, Z., Xie, M., & Wong, A. C. M. (2007). A unified confidence interval for relaibility-related quantities of two-parameter Weibull distribution. *Journal of Statistical Computations and Simulation*, *77*(5), 365–378.

Yang, G. K., Lin, R., Lin, J., Hung, K. C., Chu, P., & Chouhuang, W. (2011). Note on inventory models with Weibull distribution deterioration. *Production Planning & Control: The Management of Operations*, *22*(4), 437–444.

Yeh, T. H., & Wang, L. (2008). A study on generator capacity for wind turbines under various tower heights and rated wind speeds using Weibull distribution. *IEEE Transactions on Energy Conversion*, *23*(2), 592–602.

Zhang, T., & Xie, M. (2007). Failure rate analysis with extended Weibull distribution. *Communications in Statistics-Simulation and Computation*, *36*, 579–592.

Zhang, H. (2008). On the distribution of software faults. *IEEE Transactions on Software Engineering*, *34*(2), 301–302.

Zhao, J. H. (2004). A three-parameter Weibull-like fitting function for flip-chip die strength data. *Microelectronics and Reliability*, *44*(3), 459–470.

Chapter 2
Generalized Weibull Distributions

2.1 Introduction

Since 1970s, many extensions of the Weibull distribution have been proposed to enhance its capability to fit diverse lifetime data and Murthy et al. (2004) proposes a scheme to classify these distributions. Over the last 40 years papers dealing with various extensions of the Weibull distribution and their applications number well over 4000.

A common factor among the generalized models considered in this chapter is that the Weibull distribution is a special case of theirs. Either $F(t)$ or $h(t)$ of these modified models are related to the corresponding function of the Weibull distribution in some way. The means of these distributions usually do not have a simple expression although their hazard rate functions $h(t)$ are more capable to model diverse problems than the Weibull model does.

2.2 Methods of Constructions

Generalized Weibull distributions can be constructed in many ways. Members of this family usually contain the standard Weibull model as a special case. The following is a brief sketch of some of the common methods. A new generalized distribution may be obtained by:

- adding a constant λ to the existing hazard rate $h(t)$ of the Weibull distribution or a generalized Weibull model,
- transformations (linear, inverse or log) of the Weibull random variable,
- transformations of the distribution $F(t)$ or the survival function $\overline{F}(t)$ of the Weibull distrbution in such a way that the new function remains a distribution or survival function,
- competing risk approach (minimum of two or more Weibull variables),

C.-D. Lai, *Generalized Weibull Distributions*, SpringerBriefs in Statistics,
DOI: 10.1007/978-3-642-39106-4_2,

- mixtures of two or more Weibull variables, mixtures of two or more generalized Weibull variables, mixing a Weibull distribution with a generalized Weibull distribution, etc.
- method of compounding such as letting $T = \min(X_1, X_2, \ldots, X_N)$ where X's are independent and identically distributed Weibull variables and N is the compounded variable,
- probability integral transform, e.g., let a new density be defined as $f(t) = \psi(G(t))g$ where G is cdf of the Weibull variable with its density function g, ψ is a probability density function having support on the unit interval, or
- by adding a frailty parameter or a tilt parameter, see Chap. 7 of Marshall and Marshall and Olkin (2007).

For a more detailed discussion on constructing life-time distributions, see Lai et al. (2011).

2.3 Four-Parameter Weibull Distribution

This model was proposed by Kies (1958) and its survival function is given by

$$\overline{F}(t) = \exp\left\{-\lambda\left[\frac{(t-a)}{(b-t)}\right]^{\alpha}\right\}; \quad 0 < a \le t \le b < \infty, \lambda, \alpha > 0. \tag{2.1}$$

We note that the support of this distribution is a finite interval. Smith and Hoeppner (1990) also presented a similar four-parameter Weibull model. They also applied the model to fit fatigue and compliance calibration data.

2.4 Five-Parameter Weibull Distribution

Phani (1987) extended the model due to Kies (1958) with survival function given by

$$\overline{F}(t) = \exp\left\{\frac{-\lambda(t-a)^{\alpha_1}}{(b-t)^{\alpha_2}}\right\}; \quad 0 \le a < t < b < \infty, \alpha_1, \alpha_2, \lambda > 0. \tag{2.2}$$

The tensile strength of two groups of fused silica optical fibers has been analyzed by Phani (1987) using the above Weibull distribution function.

2.5 Truncated Weibull Distribution

A simple way to construct a new distribution is by truncating the density function, either from above or from below or both. Let G denote the two-parameter Weibull distribution function: $G(t) = 1 - \exp\{-(t/\beta)^{\alpha}\}$ with its corresponding density function

$g = (\alpha/\beta)(t/\beta)^{\alpha-1} \exp[-(t/\beta)^{\alpha}]$. Then a doubly truncated Weibull distribution can be specified by letting

$$f(t) = \frac{g(t)}{G(b) - G(a)} \tag{2.3}$$

and

$$F(t) = \frac{G(t) - G(a)}{G(b) - G(a)}, \tag{2.4}$$

respectively, for $0 \leq a < t < b < \infty$. The density function in (2.3) can be written explicitly as

$$f(t) = \frac{(\alpha/\beta)(t/\beta)^{\alpha-1} \exp[-(t/\beta)^{\alpha}]}{\exp\{-(a/\beta)^{\alpha}\} - \exp\{-(b/\beta)^{\alpha}\}}. \tag{2.5}$$

Two special cases are as follows:

1. $a = 0$ and $b < \infty$, a right truncated Weibull model emerges,
2. $a > 0$ and $b \to \infty$, a left truncated Weibull distribution results.

Note: The right truncated and left truncated Weibull distributions are also known as the upper truncated and lower truncated Weibull distributions, respectively.

We now restrict ourselves to discuss the upper truncated Weibull as the related properties of the lower truncated case can be obtained similarly.

2.5.1 Mean and Variance

General moment expressions were derived in McEwen and Parresol (1991). The mean is

$$\mu = \frac{\beta\Gamma(1 + 1/\alpha)}{1 - \exp[-(b/\beta)^{\alpha}]}$$

and the variance can be shown to be

$$\sigma^2 = \beta^2 \left\{\Gamma(1 + 2/\alpha)(1 - \exp[-(b/\beta)^{\alpha}]) - \Gamma^2(1 + 1/\alpha)\right\} /(1-\exp[-(b/\beta)^{\alpha}])^2.$$

2.5.2 Hazard Rate Function

The hazard rate function is given by

$$h(t) = \frac{(\alpha/\beta)(t/\beta)^{\alpha-1} \exp[-(t/\beta)^{\alpha}]}{G(b) - G(t)}, \quad 0 \leq t \leq b. \tag{2.6}$$

Because of the right truncation at $t = b$, it is clear from the above equation that $h(t) \to \infty$ as $t \to b$.

Like the standard Weibull case, the shape of $h(t)$ depends only on the shape parameter α. Zhang and Xie (2011) showed the following results:

Case 1: $\alpha \geq 1$.

(i) In this case, $h'(t) > 0$ for $0 \leq t \leq b$, and hence $h(t)$ is an increasing function;
(ii) $h(0) = 0$ if $\alpha > 0$ and $h(0) = 1/\{\beta - \beta \exp[-(b/\beta)]\}$, if $\beta = 1$.

Case 2: $\alpha < 1$.

(i) In this case, $h(t) \to \infty$ for $t \to 0$ and $t \to b$;
(ii) the hazard rate function has a bathtub shape.

In fact, McEwen and Parresol (1991) have already shown by heuristical arguments that the corresponding hazard rate function can be bathtub-shaped.

Zhang and Xie (2011) computed and presented the minimum values of $h(t)$ for different values of α, β and b.

2.5.3 Estimation of Model Parameters

Zhang and Xie (2011) presented two methods: (i) The graphical approach based on the Weibull probability plot for the truncated Weibull, and (ii) the maximum likelihood method.

2.5.4 Applications

Several applications were reviewed in Zhang and Xie (2011). The applications of the truncated Weibull distribution include:

- engineering fields,
- tree diameter and height distributions in forestry,
- fire size,
- high-cycle fatigue strength prediction, and
- seismological data analysis for earthquakes.

2.6 Inverse Weibull Distribution

The inverse Weibull model is also known as the reverse Weibull model in the literature.

Let X denote the 2-parameter Weibull model with distribution function $1 - e^{-(t/\beta)^{\alpha}}$. Define T by the inverse transformation:

$$T = \frac{\beta^2}{X}. \tag{2.7}$$

Then T has a distribution function given by

$$F(t) = \exp(-(\beta/t)^{\alpha}), \quad \alpha, \beta > 0, \ t \geq 0. \tag{2.8}$$

Alternatively, we may express (2.8) as

$$F(t) = \exp(-(t/\beta)^{-\alpha}), \quad \alpha, \beta > 0, \ t \geq 0. \tag{2.9}$$

The inverse Weibull is also known as type 2 extreme value or the Fréchet distribution. (Johnson et al. 1995, Chap. 22) whereas the distribution of the negative of the Weibull random variable is a type 3 extreme value distribution. Weibull and Freéchet (inverse Weibull) distributions are both special cases of the generalized extreme value distribution.

Erto (1989) has discussed the properties of this distribution and its potential use as a lifetime model. The maximum likelihood estimation and the least squares estimation of the parameters of the inverse Weibull distribution have been discussed by Calabria and Pulcini (1990).

2.6.1 Density Function

The density function of the inverse Weibull model is

$$f(t) = \alpha\beta^{\alpha}t^{-\alpha-1}e^{-(\beta/t)^{\alpha}}. \tag{2.10}$$

The density plots are given in Fig. 2.1.

2.6.2 Moments

The kth moment about 0 is

$$\mu'_k = \beta^k\Gamma(1 - k\alpha^{-1})$$

which does not have a finite value for $k \geq \alpha$. The reason for this is that the inverse Weibull distribution has a heavy right tail.

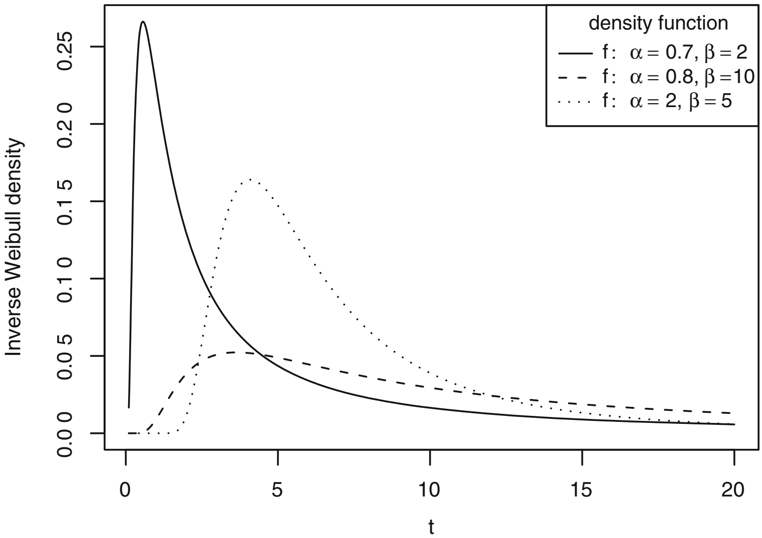

Fig. 2.1 Inverse Weibull density functions

2.6.3 Hazard Rate Function

The hazard rate function is given by

$$h(t) = \frac{\alpha\beta^{\alpha}t^{-\alpha-1}e^{-(\beta/t)^{\alpha}}}{1 - e^{-(\beta/t)^{\alpha}}}. \tag{2.11}$$

It can be shown that the inverse Weibull distribution generally exhibits a long right tail and its hazard rate function is similar to that of the log-normal and inverse Gaussian distributions.

Jiang et al. (2001) showed that the hazard rate function is unimodal (upside-down bathtub shaped) with the mode at $t = t_M$ given by the solution of the following equation

$$\frac{z(t_M)}{1 - e^{-z(t_M)}} = 1 + 1/\alpha \tag{2.12}$$

where $z(t) = (\beta/t)^{\alpha}$ and

$$\lim_{t\to 0} h(t) = \lim_{t\to\infty} h(t) = 0. \tag{2.13}$$

This is in contrast to the standard Weibull model for which the hazard rate is either decreasing (if $0 < \alpha < 1$), constant (if $\alpha = 1$) or increasing (if $\alpha > 1$).

Figure 2.2 gives the hazrad rate plots of the inverse Weibull distribution.

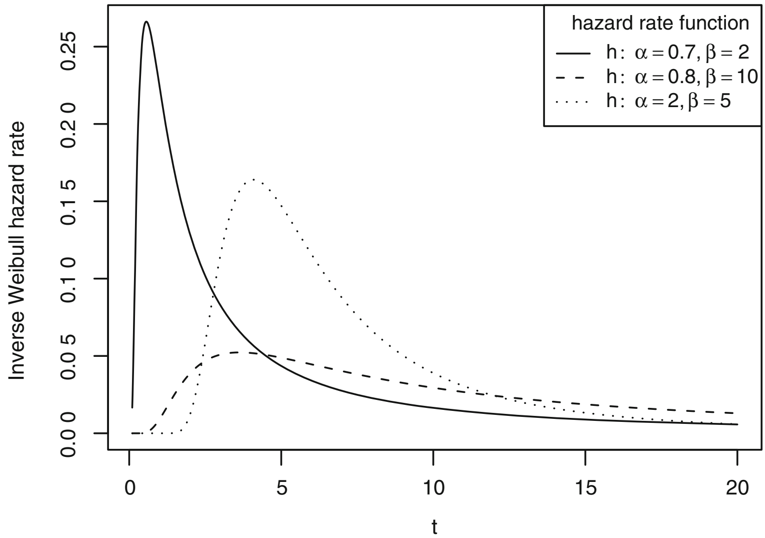

Fig. 2.2 Inverse Weibull hazard rate functions

We note with interest that the density functions and the hazard rate functions of the inverse Weibull model have similar shapes for a given pair of α and β.

2.6.4 Inverse Weibull Probability Plot

The inverse Weibull transform is given by

$$x = \log t, \quad y = -\log\left(-\log(F(t))\right). \tag{2.14}$$

The above transformation was first proposed by Drapella (1993). The plot y versus x is called the inverse Weibull probability plot (IWPP) plot. Under this transform, the inverse Weibull model as given in (2.9) also yields a straight line relationship

$$y = \alpha(x - \log \beta). \tag{2.15}$$

2.6.5 Applications

Keller et al. (1985) used the inverse Weibull distribution for reliability analysis of commercial vehicle engines.

2.6.6 Models Involving Two Inverse Weibull Distributions

This was investigated by Jiang et al. (2001), see also Chap. 3 of this monograph for a more detailed discussion.

2.6.7 The Generalized Inverse Weibull Distribution

de Gusmão et al. (2009) proposed a generalized inverse Weibull distribution by adding another parameter γ to the standard inverse Weibull:

$$\overline{F}(t) = 1 - \exp\left[-\gamma(\beta/t)^{\alpha}\right]. \tag{2.16}$$

When $\gamma = 1$, it clearly reduces to the inverse Weibull distribution.

Density Function

$$f(t) = \gamma\alpha\beta^{\alpha}t^{-(\alpha+1)}\exp\left[-\gamma(\beta/t)^{\alpha}\right]. \tag{2.17}$$

Moments

The kth moment about 0 is

$$\mu'_k = \gamma^{\frac{k}{\alpha}}\beta^k\Gamma(1 - k\alpha^{-1})$$

which does not have a finite value for $k \geq \alpha$ just as the inverse Weibull distribution.

Hazard Rate Function

The hazard rate function is given by

$$h(t) = \gamma\alpha\beta^{\alpha}t^{-(\alpha+1)}\exp\left[-\gamma(\beta/t)^{\alpha}\right]\left\{1 - \exp\left[-\gamma(\beta/t)^{\alpha}\right]\right\}^{-1}. \tag{2.18}$$

By differentiating the above $h(t)$ with respect to t, we can easily show that $h(t)$ is unimodal (upside-down bathtub shaped) with a maximum value at t^*, where t^* satisfies the nonlinear equation:

$$\gamma(\beta/t^*)\left\{1 - \exp\left[-\gamma(\beta/t^*)^{\alpha}\right]\right\} = 1 + \alpha^{-1}.$$

Estimate of Parameter

The maximum likelihood estimates of the parameters with censored data were obtained and studied in de Gusmão et al. (2009).

Further Extensions

de Gusmão et al. (2009) also considered the mixture of two generalized inverse Weibull distributions. Further, they also proposed the so called 'The log-generalized inverse Weibull distribution'.

2.7 Reflected Weibull Distribution

Suppose X has a three-parameter Weibul distribution, then $T = -X$ has a reflected Weibull whose distribution function is

$$F(t) = \exp\left\{-\left(\frac{\tau - t}{\beta}\right)^{\alpha}\right\}, \quad \alpha, \beta > 0, -\infty < t < \tau. \tag{2.19}$$

This is also known as type 3 extreme value distribution (Johnson et al. 1995, Chap. 22). The density function is given by

$$f(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{\tau - t}{\beta}\right)^{\alpha-1} \exp\left\{-\left(\frac{\tau - t}{\beta}\right)^{\alpha}\right\}, \quad \alpha, \beta > 0, -\infty < t < \infty. \tag{2.20}$$

The hazard rate function is given by

$$h(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{\tau - t}{\beta}\right)^{\alpha-1} \frac{\exp\left\{-(\frac{\tau-t}{\beta})^{\alpha}\right\}}{1 - \exp\left\{-(\frac{\tau-t}{\beta})^{\alpha}\right\}}. \tag{2.21}$$

Strictly speaking, the reflected Weibull is not suitable for reliability modelling unless $\tau > 0$ and $(\tau/\beta)^{\alpha} \geq 9$ so that $\Pr(0 < T < \tau) \approx 1$.

The model is fitted to an observed age distribution of holders of a certain type of life insurance policy by Cohen (1973).

2.8 Log Weibull Distribution

This is an extreme value distribution derived from the logarithmic transformation of the two-parameter Weibull having distribution function as given in (1.4). The transformed variable has distribution function given by

$$F(t) = 1 - \exp\left\{-\exp\left(\frac{t-a}{b}\right)\right\}, \quad -\infty < t < \infty, \tag{2.22}$$

where we have let $a = \log\beta$, $b = 1/\alpha$. This is also known as type 1 extreme value distribution or the Gumbel distribution. In fact, it is the most commonly referred to in discussions of extreme value distributions (Johnson et al. 1995, Chap. 22). The density function is already given in (1.7)—though in a different parametrization, i.e.,

$$f(t) = \frac{1}{b}\exp\left(\frac{t-a}{b}\right)\exp\left\{-\exp\left(\frac{t-a}{b}\right)\right\}, \quad -\infty < t < \infty. \tag{2.23}$$

The hazard rate functions is given by

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{1}{b}\exp\left(\frac{t-a}{b}\right). \tag{2.24}$$

2.8.1 Applications

The log Weibull (Gumbel) distribution is an extreme value distribution. Thus it has been applied to many extreme value data such as flood flows, wind speeds, radioactive emissions, brittle strength of crystals, and etc. Chapter 22 Sect. 14 of Johnson et al. (1995) lists many applications of this distribution.

2.9 Stacy's Weibull Distribution

Stacy (1962) proposed a distribution which he called the 'generalized gamma model' with the pdf and cdf, given by

$$f(t) = \frac{\alpha\beta^{-\alpha c}}{\Gamma(c)} t^{\alpha c-1}\exp\left\{-(t/\beta)^{\alpha}\right\} \tag{2.25}$$

and

$$F(t) = \Gamma(c)^{-1}\gamma\left(c, (t/\beta)^{\alpha}\right), \tag{2.26}$$

respectively, for $\alpha, \beta, c > 0$, where $\gamma(\cdot, \cdot)$ denotes the incomplete gamma function defined by

$$\gamma(a, t) = \int_0^t x^{a-1}e^{-x}dx.$$

2.9.1 Special Cases

For $c = 1$, it reduces to the standard Weibull distribution. For $\alpha = 1$, it becomes the two-parameter gamma distribution. If $c \to \infty$, it becomes the lognormal distribution. For $c = 1/2$ and $\alpha = 2$, it reduces to the half-normal distribution. Further, the chi-square distribution and the Levy distribution are also included as special cases.

2.9.2 Hazard Rate Function

The hazard rate function $h(t)$ can be expressed as

$$h(t) = \frac{\alpha\beta^{-\alpha c} t^{\alpha c-1} \exp\{-(t/\beta)^\alpha\}}{\Gamma(c) - \gamma\,(c, (t/\beta)^\alpha)}. \tag{2.27}$$

Glaser (1980) showed that the hazard rate function can exhibit various shapes as given below:

Case 1: $\alpha c < 1$.

(a) If $\alpha < 1$, then $h(t)$ is decreasing.
(b) If $\alpha > 1$, then $h(t)$ has a bathtub shape.

Case 2: $\alpha c > 1$.

(a) If $\alpha > 1$, then $h(t)$ is increasing.
(b) If $\alpha < 1$, then $h(t)$ has a upside-down bathtub shape.

Case 3: $\alpha c = 1$.

(a) If $\alpha = 1, c = 1$, then $h(t)$ is a constant.
(b) If $\alpha < 1$, then $h(t)$ is decreasing.
(c) If $\alpha > 1$, then $h(t)$ is increasing.

McDonald and Richards (1987) and Pham and Almhana (1995) also considered the shape of the hazard rate function $h(t)$.

2.9.3 Estimation of Parameters

Parameter estimation for Stacy's generalized Weibull (gamma) distribution has been widely treated in the literature. The maximum likelihood method and the method of moment are the two common approaches but it has been known that there are various difficulties in implementing these methods to this distribution. Other methods such as the heuristics and graphical methods were also proposed. For a literature review of these methods see Gomes et al. (2008).

Gomes et al. (2008) proposed a new method of estimation through the power transformation $T = X^c$ where X has Stacy's Weibull distribution as given in (2.25) above. Then T has a gamma distribution with shape parameter c and scale parameter β^α. Based on this they constructed a heuristic method called algorithm I.E.R.V. which involves looping around the shape parameter α.

2.9.4 Applications

Stacy's Weibull distribution has applications in many fields such as health costs, civil engineering (flood frequency, e.g, Pham and Almhana 1995), economics (income distribution, e.g., Klieber and Kotz 2003) and others.

2.10 Exponentiated Weibull Distribution

Mudholkar and Srivastava (1993) proposed a modification to the standard Weibull model through the introduction of an additional parameter ν $(0 < \nu < \infty)$. The distribution function is

$$F(t) = [G(t)]^\nu = [1 - \exp\{-(t/\beta)^\alpha\}]^\nu, \quad \alpha, \beta > 0, t \geq 0, \tag{2.28}$$

where $G(t)$ is the standard two-parameter Weibull distribution. The support for F is $[0, \infty)$.

When $\nu = 1$, the model reduces to the standard two-parameter Weibull model. When ν is an integer, the model is a special case of the multiplicative model to be discussed in Sect. 3.3 . The distribution has been studied extensively by Mudholkar and Hutson (1996), Jiang and Murthy (1999) and more recently Nassar and Eissa (2003).

2.10.1 Density Function

The density function is given by

$$f(t) = \nu\{G(t)\}^{\nu-1} g(t), \tag{2.29}$$

where $g(t)$ is the density function of the standard two-parameter Weibull distribution. So

$$f(t) = \frac{\alpha\nu}{\beta^\alpha} t^{\alpha-1} e^{-(t/\beta)^\alpha} \left(1 - e^{-(t/\beta)^\alpha}\right)^{\nu-1}. \tag{2.30}$$

Two special cases worth noting:

(i) For $\alpha = 1$, the probability density function is

$$f(t) = \frac{\nu}{\beta} e^{-t/\beta} \left(1 - e^{-t/\beta}\right)^{\nu-1} \tag{2.31}$$

which is the exponentiated exponential distribution studied by Gupta et al. (1998).

(ii) For $\alpha = 2$, we obtain the probability density function of the two-parameter Burr type X distribution:

$$f(t) = \frac{2\nu}{\beta^2} e^{-(t/\beta)^2} \left(1 - e^{-(t/\beta)^2}\right)^{\nu-1}. \tag{2.32}$$

We also note that from (2.30),

$$f(0) = \begin{cases} 0 & \text{if } \alpha\nu > 1, \\ \beta^{-\alpha} & \text{if } \alpha\nu = 1, \\ \infty & \text{if } \alpha\nu < 1. \end{cases} \tag{2.33}$$

The value of $f(0)$ will determine the value of the hazard rate function at $t = 0$ which in turn has an impact on the shape of the mean residual life MRL $\mu(t)$ of a lifetime distribution, see for example Theorem 4.2 of Lai and Xie (2006).

The density function is plotted for different parameter values in Fig. 2.3.

2.10.2 Hazard Rate Function

The hazard rate function was given in Mudholkar et al. (1995) and Mudholkar and Hutson (1996):

$$h(t) = \frac{\nu(\alpha/\beta)(t/\beta)^{\alpha-1}[1 - \exp(-(t/\beta)^\alpha)]^{\nu-1} \exp(-(t/\beta)^\alpha)}{1 - [1 - \exp(-(t/\beta)^\alpha)]^\nu}. \tag{2.34}$$

For small t, Jiang and Murthy (1999) have shown that

$$h(t) \approx \left(\frac{\alpha\nu}{\beta}\right) \left(\frac{t}{\beta}\right)^{\alpha\nu-1}. \tag{2.35}$$

In other words, for small t, $h(t)$ can be approximated by the hazard rate of a two-parameter Weibull distribution with shape parameter $(\alpha\nu)$ and scale parameter β.

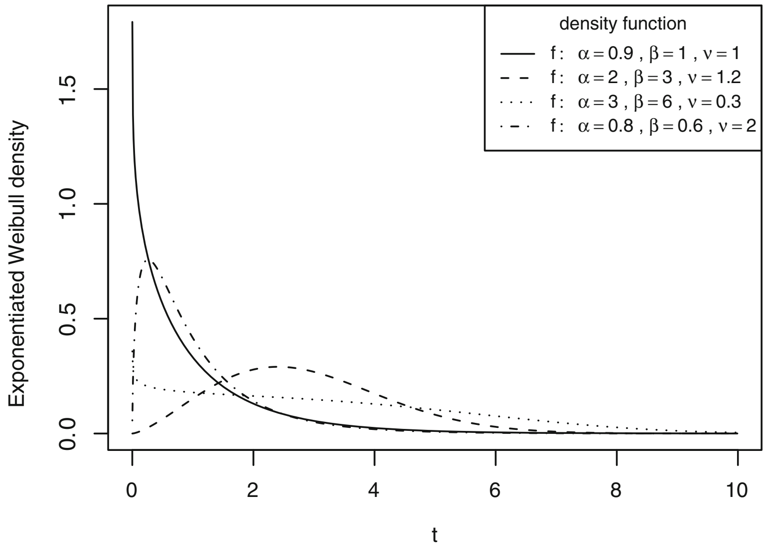

Fig. 2.3 Exponentiated Weibull density functions

For large t , i.e., $t \to \infty$, the term $\dfrac{\exp(-(t/\beta)^{\alpha})}{1-[1-\exp(-(t/\beta)^{\alpha})]^{\nu}}$ in (2.34) converges to $1/\nu$ by applying the L'Hospital's rule. It is now clear that (2.34) converges to

$$h(t) \approx \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1} \tag{2.36}$$

for large t.

In other words, for large t, $h(t)$ can be approximated by the hazard rate of a two-parameter Weibull distribution with shape parameter α and scale parameter β.

Mudholkar et al. (1995), Mudholkar and Hutson (1996) and Jiang and Murthy (1999) have all considered the shapes of $h(t)$ and its characterization in the parameter space. The shape of $h(t)$ does not depend on β and varies with α and ν. The characterization on the (α, ν)- plane is as follows:

- $\alpha \leq 1$ and $\alpha\nu \leq 1$: $h(t)$ monotonically decreasing.
- $\alpha \geq 1$ and $\alpha\nu \geq 1$: $h(t)$ monotonically increasing.
- $\alpha < 1$ and $\alpha\nu > 1$: $h(t)$ has an upside-down bathtub shape.
- $\alpha > 1$ and $\alpha\nu < 1$: $h(t)$ has a bathtub shape.

Figure 2.4 contains four plots of the the hazard rate of the exponentiated Weibull distribution.

The exponentiated Weibull was first introduced by Mudholkar and Srivastava (1993) for modeling bathtub shaped hazard rate data.

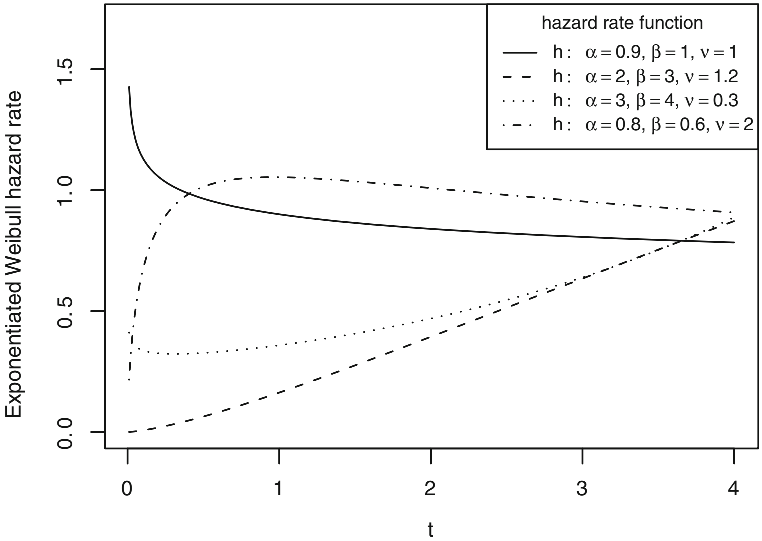

Fig. 2.4 Exponentiated Weibull hazard rate functions

2.10.3 Mean Residual Life Function

The mean residual life of the exponentiated Weibull has been given by Nassar and Eissa (2003) for positive integer ν as:

$$\mu(t) = \nu \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} / \left[(j+1)\overline{F}(t)\right] \left\{ \beta\Gamma\left(\frac{1}{\beta}+1\right)\left[(j+1)^{-1/\alpha} - (j+1)^{-1}\Gamma_{(j+1)\tau}\left(\frac{1}{\beta}\right)/\Gamma\left(\frac{1}{\beta}\right)\right] + te^{-(j+1)\tau}\right\} - t, \tag{2.37}$$

where $\tau = (t/\beta)^{\alpha}$. Thus, the mean of the distribution for $\nu \in \mathcal{N}^+$ is

$$\mu = \mu(0) = \nu\Gamma\left(\frac{1}{\alpha}+1\right)\sum_{j=0}^{\nu-1}(-1)^j\binom{\nu-1}{j}/\left[(j+1)^{-1/\alpha}\right]. \tag{2.38}$$

From the general theory that connecting bathtub (inverted bathtub) shaped hazard rate distributions with increasing (decreasing) then decreasing (increasing) mean residual life distributions (see Sect. 4.5 of Lai and Xie (2006) for details), Nassar and Eissa (2003) have established the shape of $\mu(t)$ as follows:

Let $\mu(t)$ be given as (2.38), then

(i) $\mu = \alpha$ if and only if $\alpha = \nu = 1$.

(ii) $\mu(t)$ is decreasing (increasing) if $\alpha \geq 1$ and $\alpha\nu \geq 1$ ($\alpha \leq 1$ and $\alpha\nu \leq 1$).
(iii) $\mu(t)$ is decreasing and then increasing with a change point t_m if $\alpha < 1$ and $\alpha\nu > 1$.
(iv) $\mu(t)$ is increasing and then decreasing with a change point t_m if $\alpha > 1$ and $\alpha\nu < 1$.

Nadarajah and Gupta (2005) obtained the kth moment about the origin as

$$\mu'_k = \nu\beta^k \Gamma\left(\frac{k}{\alpha}\right) \sum_{i=0}^{\infty} \frac{(1-\nu)_i}{i!(i+1)^{(k+\alpha)/\alpha}}, \quad k > -\alpha, \tag{2.39}$$

where $(1-\nu)_i = (a)_i = a(a+1)\ldots a(a+i-1)$.

If ν is an integer, then

$$\mu'_k = \nu\beta^k \Gamma\left(\frac{k}{\alpha}\right) \sum_{i=0}^{\nu-1} \frac{(1-\nu)_i}{i!(i+1)^{(k+\alpha)/\alpha}}, \quad k > -\alpha, \tag{2.40}$$

which was established by Nassar and Eissa (2003). Equation (2.40) follows from (2.39) because $(1-\nu)_i = 0$ for all $i \geq \nu$.

Xie et al. (2004) have studied the change points of $h(t)$ and $\mu(t)$ in terms of individual model parameters. The difference D of two change points is tabulated for various combinations of the parameters.

2.10.4 Graphical Study

Jiang and Murthy (1999) discussed the shape of the WPP for the exponentiated Weibull family and gave a parametric characterization of its probability density and hazard rate functions. Their paper also deals with the issues relating to modeling a given data set and the problem of estimating the model parameters.

2.10.5 Applications

The model has been applied

- to model the bathtub failure rate behavior of the data in Aarset (1987) (Mudholkar and Srivastava 1993)
- to reanalyse the bus motor failure data (Mudholkar et al. 1995), and
- to analyse a flood data (Mudholkar and Hutson 1996).

2.11 Beta-Weibull Distribution

The distribution is a generalization of the exponentiated Weibull distribution discussed in the preceding section. Let $G(t)$ denote the standard two-parameter Weibull distribution function given as $G(t) = 1 - \exp\{-(t/\beta)^{\alpha}\}$.

The beta-Weibull was first proposed by Famoye et al. (2005) by coupling the beta density and the Weibull distribution function such that the distribution function of the new distribution is

$$F(t) = \frac{1}{B(a,b)} \int_0^{G(t)} u^{a-1}(1-u)^{b-1} du; \quad 0 < a, b < \infty, \tag{2.41}$$

where $B(a, b)$ denotes the beta function defined by $\Gamma(a)\Gamma(b)/\Gamma(a+b) = \int_a^b x^{a-1} (1-x)^{b-1} dx$. The same distribution was also proposed by Wahed et al. (2009). Clearly, if $b = 1$, (2.41) reduces to the exponentiated Weibull distribution. In general, the distribution function cannot be expressed explicitly.

The basic properties are given in Famoye et al. (2005).

2.11.1 Density and Hazard Rate Function

The density function is rather simple:

$$f(t) = \frac{1}{B(a,b)} \frac{\alpha}{\beta} \left(\frac{t}{\beta}\right)^{\alpha-1} \left[1 - e^{-(t/\beta)^{\alpha}}\right]^{a-1} e^{-b(t/\beta)^{\alpha}}. \tag{2.42}$$

The hazard rate function can be obtained from the preceding two equations as

$$h(t) = \frac{f(t)}{\overline{F}(t)}.$$

It cannot be expressed explicitly.

Lee et al. (2007) examined the shapes of the hazard rate function and we now summarize them below:

(a) $h(t)$ is a constant b/β if $a = \alpha = 1$,
(b) $h(t)$ is a decreasing function if $a\alpha \le 1$ and $\alpha \le 1$,
(c) $h(t)$ is an increasing function if $a\alpha \ge 1$ and $\alpha \ge 1$,
(d) $h(t)$ has a bathtub shape if $a\alpha < 1$ and $\alpha > 1$, and
(e) $h(t)$ has an upside-down bathtub shape if $a\alpha > 1$ and $\alpha < 1$.

2.11.2 Comparison with Exponentiated Weibull Distribution

Lee et al. (2007) carried out a simulation study to compare the beta-Weibull model with the exponentiated Weibull model. They concluded that the bias of the MLE from the beta-Weibull distribution is smaller than the bias of the exponentiated Weibull model with comparable standard errors when $b < 1$. The bias and the standard errors are, in general, smaller for the exponentiated Weibull distribution when $b \geq 1$. (Recall, the beta-Weibull becomes the exponentiated Weibull when $b = 1$.) For the three data sets analyzed in Lee et al. (2007), the estimates for the parameter b are less than 1 which suggests the usefulness of the beta-Weibull for describing survival data sets.

2.11.3 Applications of Beta-Weibull Model

The model is applied by Lee et al. (2007) to two censored data sets of bus-motor failures and a censored data set of Arm A (Efron 1988) of the head-and-neck cancer clinical trial.

Wahed et al. (2009) also fitted the model to a breast cancer data set and they concluded that the beta-Weibull family is a reasonable candidate for modeling survival data.

Log Beta-Weibull Model

Let $Y = \log T$ where T has the beta-Weibull distribution given above . Then Y has a log beta-Weibull distribution studied by Ortega et al. (2011). Based on Y, they also proposed and studied the log-beta Weibull regression model which they considered to be very suitable for modeling censored and uncensored data.

2.12 Extended Weibull Model of Marshall and Olkin

Marshall and Olkin (1997) proposed a modification to the standard Weibull model through the introduction of an additional parameter ν $(0 < \nu < \infty)$. The model is given through its survival function function

$$\overline{F}(t) = \frac{\nu \bar{G}(t)}{1 - (1 - \nu)\bar{G}(t)} = \frac{\nu \bar{G}(t)}{G(t) + \nu \bar{G}(t)} \tag{2.43}$$

where $G(t)$ is the distribution function of the two-parameter Weibull and $\overline{F}(t) = 1 - F(t)$.

The parameter ν is called a tilt parameter in Marshall and Marshall and Olkin (2007).

The case when G is an exponential distribution function has been considered as the exponential-geometric distribution studied by Adamidis and Loukas (1998) and Marshall and Olkin (1997).

When $\nu = 1$, $\overline{F}(t) = \bar{G}(t)$ so the model reduces to the standard Weibull model.

2.12.1 *Extended Weibull*

Using (1.4) as G in (2.43), we then have the distribution function given by

$$F(t) = 1 - \frac{\nu \exp[-(t/\beta)^{\alpha}]}{1 - (1 - \nu)\exp[-(t/\beta)^{\alpha}]}. \tag{2.44}$$

Marshall and Olkin (1997) called this the extended Weibull distribution. The mean and variance of the distribution cannot be given in a closed form, but they can be obtained numerically. The model may be considered as a competitor to the three-parameter Weibull distribution defined in (1.19).

Ghitany et al. (2005) showed that this distribution can be obtained by compounding the Weibull extension model of Xie et al. (2002) with the exponential distribution. That is, the extended Weibull is the result of mixing the Weibull extension model of Xie et al. with the exponential density. Zhang et al. (2007) also investigated some reliability properties of this model.

The resulting density function associated with (2.44) is given by

$$f(t) = \frac{(\alpha\nu/\beta)(t/\beta)^{\alpha-1}\exp[-(t/\beta)^{\alpha}]}{\{1 - (1 - \nu)\exp[-(t/\beta)^{\alpha}]\}^2} \tag{2.45}$$

The plots of the density function for different parameter values are given in Fig. 2.5.

2.12.2 *Shapes of the Hazard Rate Function*

The hazard rate function that corresponds to (2.45) is

$$h(t) = \frac{(\alpha/\beta)(t/\beta)^{\alpha-1}}{1 - (1 - \nu)\exp[-(t/\beta)^{\alpha}]}. \tag{2.46}$$

Marshall and Olkin (1997) carried out a partial study on the shape of the hazard rate. They found that $h(t)$ is increasing when $\nu \geq 1, \alpha \geq 1$ and decreasing when $\nu \leq 1, \alpha \leq 1$. If $\alpha > 1$, then the hazard rate function is initially increasing and

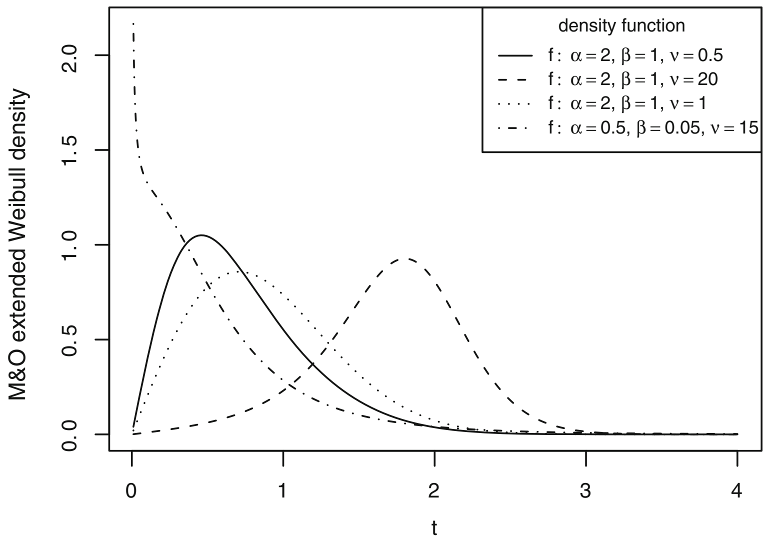

Fig. 2.5 M&O extended Weibull density functions

eventually increasing, but there may be an interval where it is decreasing. Similarly, when $\alpha < 1$, the hazard rate function is initially decreasing and eventually decreasing, but there may be an interval where it is increasing.

It is shown in Marshall and Olkin (1997) that $h(t)$ has modified bathtub shape (MBT, or N shape) for $\beta = 1, \alpha = 2, \nu = 0.05$ or $\nu = 0.1$.

Gupta et al. (2010) gave a complete study concerning the shapes of the hazard rate and their results were given in their Theorems 4–6 which we will summarize them below. In order to present these results, the shape parameter α is re-parameterized through the transformation $\psi = \frac{\alpha-1}{\alpha}e^{1/\alpha}$.

Case (i): $\nu > 0, \alpha > 1$.

1. $h(t)$ is initially increasing.
2. If $\nu \geq 1 - \psi$, then $h(t)$ is increasing.
3. If $\nu > 1 - \psi$, then $h(t)$ is (strictly) increasing and followed by a bathtub shape (i.e, it has an N shape). Such a shape is also called a modified bathtub shape.

Case (ii): $\nu > 0,\ 0 < \alpha < 1$.

1. $h(t)$ is initially decreasing.
2. If $\nu \leq 1 - \psi$, then $h(t)$ is decreasing.
3. If $\nu < 1 - \psi$, then $h(t)$ is initially (strictly) decreasing and followed by an upside-down bathtub shape (i.e., it has a reflected N shape).

Case (iii): $\alpha = 1$.

1. If $0 < \nu < 1$, then $h(t)$ is decreasing.
2. If $\alpha > 1$, then $h(t)$ is increasing.

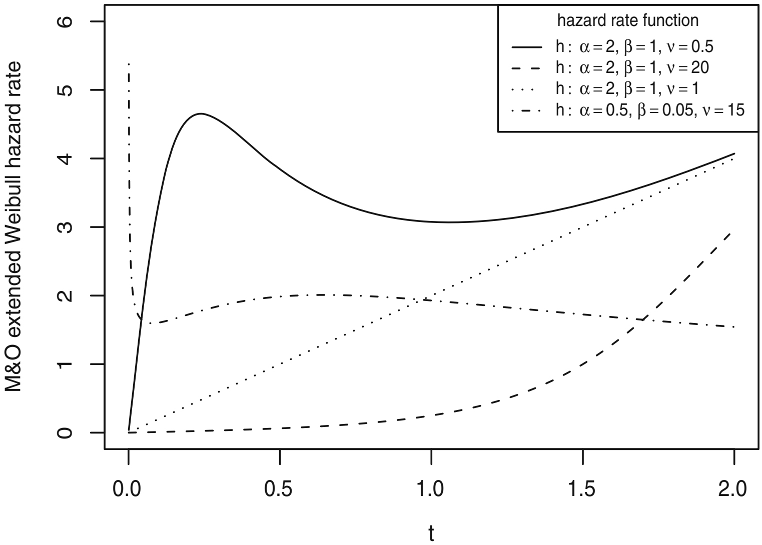

Fig. 2.6 M&O extended Weibull hazard rate functions

Of course, Case (iii) has already been established by Marshall and Olkin (1997). We also note that for $\nu = \alpha = 1$, the model reduces to the exponential distribution and thus $h(t)$ is a constant.

Figure 2.6 gives the plots of $h(t)$ from the extended Weibull model for various combination of parameter values.

2.12.3 Turning Points of $h(t)$

For non-monotonic types, the estimates of the turning points of $h(t)$ and their confidence intervals were provided by Gupta et al. (2010).

2.12.4 Mean Residual Life

In view of the complexity of the mean residual life function $\mu(t)$ of the Weibull distribution, we anticipate that the mean residual life function of the extended Weibull is worse in terms of complexity. Lai et al. (2004) have provided several plots of MRL for different combinations of parameter values from this distribution.

2.12.5 Application

Zhang et al. (2007) fitted the model to the failure times of a sample of devices from a field-tracking study of a large system.

2.13 The Weibull-Geometric Distribution

The Weibull-geometric distribution, abbreviated as WG, was proposed and studied by Barreto-Souza et al. (2010).

Let $\{X_i\}$ be independent and identically distributed Weibull random variables having shape and scale parameters α and β, respectively.

Consider $T = \min\{X_1, X_2, \ldots, X_Z\}$ where Z is discrete random variable having a geometric distribution with probability function:

$$p(z) = (1-p)p^{z-1}; \quad 0 < p < 1, z = 1, 2, \ldots \tag{2.47}$$

Then T has the Weibull-geometric distribution with density function

$$f(t) = \alpha\beta^{-\alpha}(1-p)t^{\alpha-1}e^{-(t/\beta)^{\alpha}}\left\{1 - pe^{-(t/\beta)^{\alpha}}\right\}^{-2}; \ \alpha, \beta > 0. \tag{2.48}$$

The above model may arise from a situation where failure (of a device for example) occurs due to the presence of an unknown number, say Z, of initial defects of the same kind. The random variables $X_i's$ represent their lifetimes (with constant, increasing or decreasing hazard rate) and each defect can be detected only after causing failure, in which case it is repaired perfectly. Thus, the distributional assumptions given earlier lead to the WG distribution for modeling the time of the first failure.

In the derivation of the WG distribution, it is required that $0 < p < 1$. However, for negative value of p, Eq. (2.48) is still a proper density function. Thus we can extend the definition of the WG distribution in (2.48) for any $p < 1$.

Figure 2.7 gives the density plots with $\alpha = 5$ and $\beta = 1.111$ for different values of p.

2.13.1 Distribution Function

The cdf of the Weibull-geometric model has a simple form:

$$F(t) = \frac{1 - e^{-(t/\beta)^{\alpha}}}{1 - pe^{-(t/\beta)^{\alpha}}}, \tag{2.49}$$

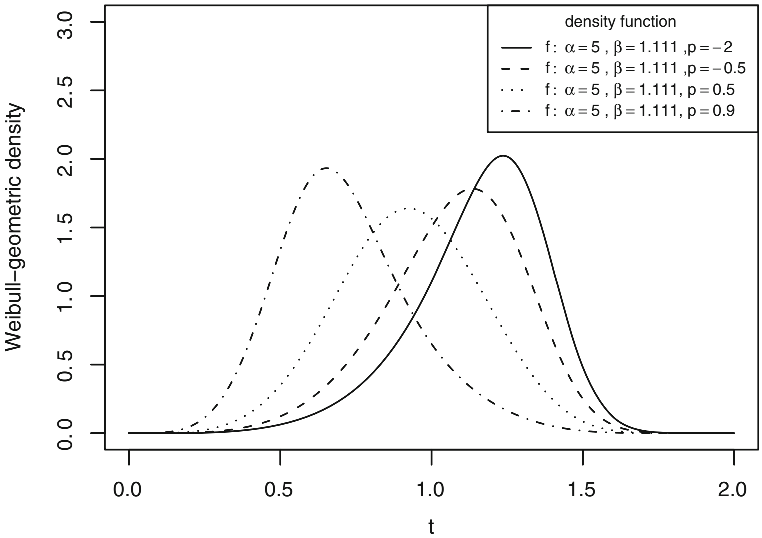

Fig. 2.7 Weibull-geomeric density functions

and the corresponding survival function is

$$\overline{F}(t) = \frac{(1-p)e^{-(t/\beta)^{\alpha}}}{1-pe^{-(t/\beta)^{\alpha}}}. \tag{2.50}$$

2.13.2 Hazard Rate Function

The hazard rate function of the Weibull-geometric distribution is

$$h(t) = \alpha\beta^{-\alpha}t^{\alpha-1}\left\{1-pe^{-(t/\beta)^{\alpha}}\right\}^{-1}. \tag{2.51}$$

The above function is decreasing for $0 < \alpha < 1$ and $p < 1$. However, for other parameter values, it can take different shapes. We use two figures to illustrate some of the possible shapes of the hazard function for selected values of α and β and $p = 0.2, 0.5, 0, 0.5, 0.9$.

Figure 2.8 gives the plots with $\alpha = 1.5$ and $\beta = 2$.

Figure 2.9 gives the plot for $\alpha = 5$ and $\beta = 1.111$.

These preceding two plots show that the hazard rate function of the WG distribution is much more flexible than some other generalized Weibull distributions.

We note that for $\alpha = 1.5$, $\beta = 2$ and $p = 0.9$, the hazard rate function is first increasing and then followed by a bathtub curve so it has a modified bathtub shape.

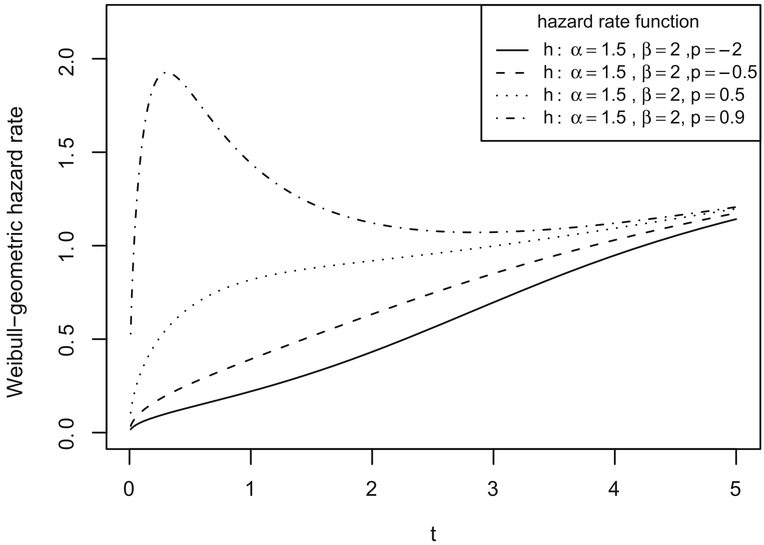

Fig. 2.8 Weibull-geomeric hazard rate functions—I

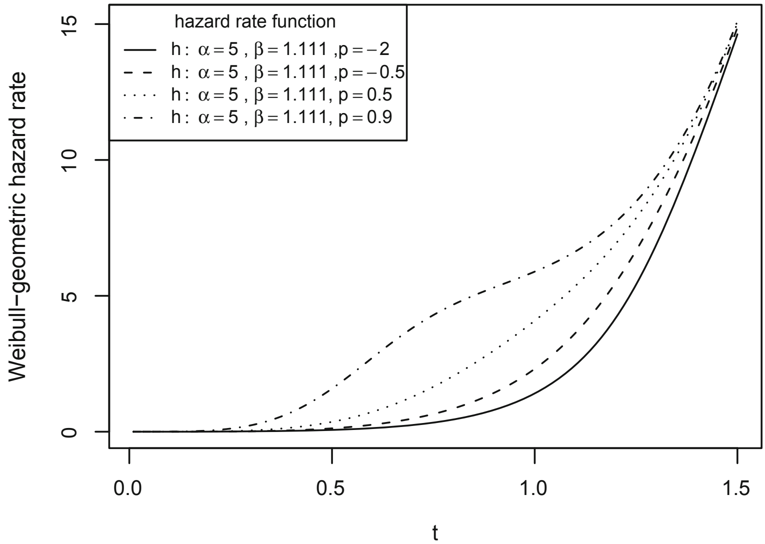

Fig. 2.9 Weibull-geomeric hazard rate functions—II

2.13.3 Quantile Function and Moments

The quantile function is

$$Q(u) = \beta \left\{ \log \left(\frac{1 - pu}{1 - u} \right) \right\}^{1/\alpha} \tag{2.52}$$

and the kth moment about the origin 0 is

$$\mu_k' = (1 - p)\beta^k \Gamma(k/\alpha + 1)\Psi(p, k/\alpha, 1) \tag{2.53}$$

where $\Psi(z, s, a) = \{\Gamma(s)\}^{-1} \int_0^\infty t^{s-1} e^{-at}(1 - ze^t)^{-1} dt;\ z < 1, a, s > 0$ is known as the Lerch's transcendent function which can be computed via Mathematica or Maple.

2.13.4 Estimation of Parameters

Maximum likelihood estimates and other inferential properties were derived by Barreto-Souza et al. (2010).

2.13.5 Applications

Barreto-Souza et al. (2010) fitted the Weibull-geometric, extended exponential-geometric (EEG), and Weibull models to a real data set given in Meeker and Escobar (1998, p. 149). The data present the fatigue life (rounded to the nearest thousand cycles) for 67 specimens of Alloy T7987 that failed before having accumulated 300 thousand cycles of testing:

94, 118, 139, 159, 171, 189, 227, 96, 121, 140, 159, 172, 190, 256, 99, 121, 141, 159, 173, 196, 257, 99, 123, 141, 159, 176, 197, 269, 104, 129, 143, 162, 177, 203, 271, 108, 131, 144, 168, 180, 205, 274, 112, 133, 149, 168, 180, 211, 291, 114, 135, 149, 169, 184, 213, 117, 136, 152, 170, 187, 224, 117, 139, 153, 170, 188, 226.

It was found that the WG model gives a better fit than its competitors.

2.14 Weibull-Poisson Distribution

Hemmati and Khorram (2011) proposed a three-parameter ageing distribution using the same construction technique as in the preceding section.

Again, let X_i denote the standard Weibull variable and define $T = \min\{X_1, X_2, \ldots, X_Z\}$ where Z denotes the zero truncated Poisson random variable having probability function given by

$$p(z) = e^{-\lambda}\lambda^{z}(1-e^{-\lambda})^{-1}/z!;\ \lambda > 0, z = 1, 2, \ldots \tag{2.54}$$

It is easy to show that the resulting density function is given by

$$f(t) = \frac{\lambda\alpha}{\beta(1-e^{-\lambda})}(t/\beta)^{\alpha-1}\exp\{-\lambda-(t/\beta)^{\alpha}+\lambda e^{-(t/\beta)^{\alpha}}\};\quad \lambda, \alpha, \beta > 0. \tag{2.55}$$

The survival function is reasonably straightforward and it is given by

$$\overline{F}(t) = \left(1-\exp\left\{\lambda e^{-(t/\beta)^{\alpha}}\right\}\right)/(1-e^{\lambda}). \tag{2.56}$$

It is clear from either (2.55) or (2.56) that the Weibull-Poisson distribution reduces to the Weibull distribution as $\lambda \to 0$.

2.14.1 Hazard Rate Function

The hazard rate function of the Weibull-Poisson model is given by

$$h(t) = \frac{\alpha\lambda(1-e^{\lambda})(t/\beta)^{\alpha-1}\exp\{-\lambda-(t/\beta)^{\alpha}+\lambda e^{-(t/\beta)^{\alpha}}\}}{\beta(1-e^{-\lambda})(1-\exp\{\lambda e^{-(t/\beta)^{\alpha}}\})}. \tag{2.57}$$

Hemmati and Khorram (2011) showed that $h(t)$ is increasing, decreasing or has a modified bathtub shape (N-shape, i.e., $h(t)$ strictly increasing then then followed by a bathtub shape). Not that $h(t)$ is unable to achieve a bathtub shape.

2.15 Modified Weibull Distribution

Lai et al. (2003) proposed a three-parameter generalized Weibull model which they called the modified Weibull distribution. The distribution function is given by

$$F(t) = 1-\exp(-at^{\alpha}e^{\lambda t}),\quad t \geq 0, \tag{2.58}$$

where the parameters $\lambda > 0$, $\alpha > 0$ and $a > 0$. For $\lambda = 0$, (2.58) reduces to a Weibull distribution. This simple generalization of the Weibull model is able to exhibit a bathtub shaped hazard rate function. We now study this model in some detail.

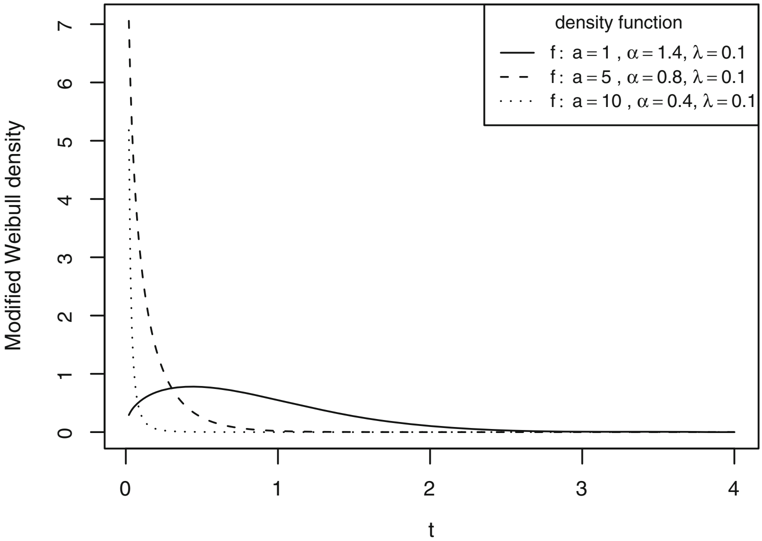

Fig. 2.10 Modified Weibull density functions

2.15.1 Density Function

The density function of the modified Weibull is given by

$$f(t) = a(\alpha + \lambda t)t^{\alpha-1}e^{\lambda t}\exp(-at^{\alpha}e^{\lambda t}). \tag{2.59}$$

Some density plots from this model are given in Fig. 2.10.

2.15.2 Moments

The moments of the distribution can be found in Nadarajah (2005).

2.15.3 Hazard Rate Function

The hazard rate function of this generalized Weibull model also has a simple form:

$$h(t) = a(\alpha + \lambda t)t^{\alpha-1}e^{\lambda t}. \tag{2.60}$$

Clearly the shape of $h(t)$ depends only on α because of through the term $t^{\alpha-1}$ and the remaining two parameters have no direct effect on the shapes.

Case (i): $\alpha \geq 1$.

1. $h(t)$ is increasing in t, implying an increasing hazard rate function, thus F is IFR.
2. $h(0) = 0$ if $\alpha > 1$ and $h(0) = a$ if $\alpha = 1$.
3. $h(t) \to \infty$ as $t \to \infty$.

Case (ii): For $0 < \alpha < 1$.

1. $h(t)$ initially decreases and then increases, implying it has a bathtub shape.
2. $h(t) \to \infty$ as $t \to 0$ and $h(t) \to \infty$ as $t \to \infty$.
3. The change point t^*, the turning point of the hazard rate function, is

$$t^* = \frac{\sqrt{\alpha} - \alpha}{\lambda}. \tag{2.61}$$

The interesting feature of this hazard curve is that t^* increases as λ decreases. The limiting case when $\lambda = 0$ reduces to the standard Weibull distribution.

Figure 2.11 gives the hazard plots for the modified Weibull distribution.

Useful Periods

The 'useful period' of this lifetime distribution when its $h(t)$ has a bathtub shape is studied in detail by Bebbington et al. (2006).

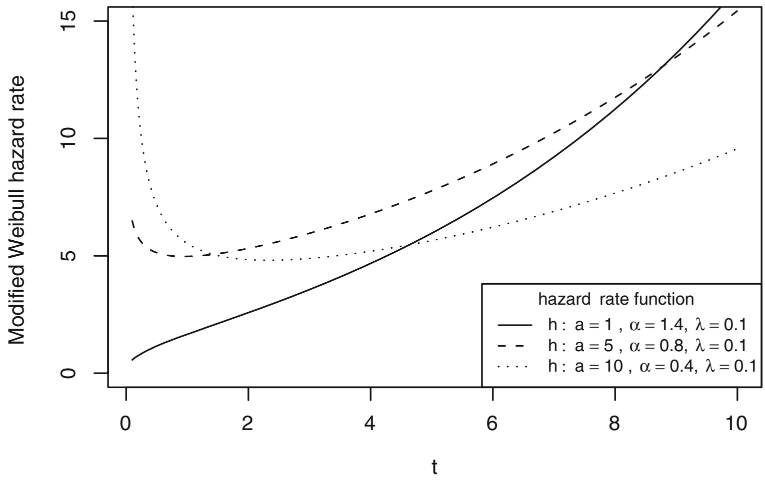

Fig. 2.11 Modified Weibull hazard rate functions

2.15.4 Mean Residual Life

In common with many other generalized Weibull distributions, the mean of this distribution does not have a closed form. Plots of the mean residual life function $\mu(t)$ can be found in Lai et al. (2004) or Xie et al. (2004) for some selected parameter values. The latter paper also computed the distance between the change points of $h(t)$ and $\mu(t)$.

2.15.5 Estimation of Parameters

A simple method for estimating the parameters of the modified Weibull model through a WPP was given in Lai et al. (2003). Bebbington et al. (2008) have suggested an empirical estimator for the turning point t^* and the theory was illustrated by means of real data set. A simulation study was conducted to assess the performance of the estimators in practice.

2.15.6 Estimates of Parameters for Progressively Type-II Censored Samples

Progressively type-II censoring is often used in life testing. Consider an experiment in which n units are placed on a life test. At the time of the first failure, k_1 units are randomly removed from the remaining $n-1$ units. At the second failure, k_2 units from the remaining $n - 2 - k_1$ units are randomly removed. The life test continues until the mth failure at which time, all the remaining $k_m = n - m - k_1 - k_2 - \cdots - k_{m-1}$ units are removed.

Ng (2005) studied the estimation of parameters based on a progressively type-II censored sample from the modified Weibull model specified by (2.58). He first obtained the maximum likelihood estimates of the model parameters. The estimators based on a least-squares fit of a multiple linear regression on a Weibull probability paper plot were also obtained and compared with the MLE via Monte Carlo simulations.

Recently, Jiang et al. (2010) also studied maximum likelihood estimation of the model parameters of the modified Weibull distribution with progressively type-2 censored samples. The property of the log-likelihood function was investigated by introducing a simple transformation to decrease the dimension of the parameter vector. Existence and uniqueness of the MLEs of the model parameters were proved. Several examples were presented to illustrate the uniqueness and existence property of the MLEs.

2.15.7 Application

Failure times data from Aarset (1987) was fitted by this modified Weibull model.

2.15.8 Competing Risk Models Involving Two Modified Weibull Distributions

Alwasel (2009) proposed a competing risk model involving two modified Weibull distributions. The maximum likelihood estimates of the six parameters (since each of modified Weibull models has three parameters) were also derived although not in a closed form. So a numerical method is required for computing the MLEs of these parameters.

2.15.9 Beta Modified Weibull Distribution

Silva et al. (2010) proposed a new distribution called the beta modified Weibull based on the following construction scheme:

$$F(t) = \frac{1}{B(a,b)} \int_0^{G(t)} u^{a-1}(1-u)^{b-1} du \tag{2.62}$$

where $G(t)$ denotes the cdf of the modified Weibull distribution. The density function is given by

$$f(t) = \frac{1}{B(a,b)} G(t)^{a-1}(1-G)^{b-1} g(u) du \tag{2.63}$$

where $g(t) = G'(t)$.

The model contains several well known sub-models including the beta-Weibull studied by Lee et al. (2007).

The hazard function $h(t)$ of the beta modified Weibull can be bathtub shaped, increasing, decreasing or inverted bathtub shaped (UBT) depending on the values of the parameters.

2.15.10 Generalized Modified Weibull Family

The survival function of the distribution proposed and studied by Carrasco et al. (2008) is

$$\overline{F}(t) = 1 - \left(1 - \exp\left\{-at^{\alpha}e^{\lambda t}\right\}\right)^{\beta}, \quad \lambda \geq 0, \alpha, \beta, a > 0; t \geq 0. \tag{2.64}$$

Clearly, the distribution is a simple extension of the modified Weibull distribution of Lai et al. (2003) since (2.64) reduces to (2.58) when $\beta = 1$. In fact, it includes several other distributions such as type 1 extreme value, the exponentiated Weibull of Mudholkar and Srivastava (1993) as specified in (2.28) above, and others. An important feature of this lifetime (ageing) distribution is its considerable flexibility in providing hazard rates of various shapes.

2.16 Generalized Weibull Family

The so called generalized Weibull model was derived by Mudholkar and Kollia (1994) and Mudholkar and Hutson (1996) from the basic two-parameter Weibull distribution by appending an additional parameter. The quantile function for the new model is given by

$$\begin{aligned} Q(u) &= \beta\left[\theta 1 - (1-u)^{1/\theta}\right]^{1/\alpha}, \quad \theta < \infty \\ &= \beta[-\log(1-u)]^{1/\alpha}, \quad \theta \to \infty \end{aligned} \tag{2.65}$$

where the new parameter θ is unconstrained so that $-\infty < \theta < \infty$. It follows immediately that the survival function is

$$\overline{F}(t) = \left[1 - \left(\frac{t}{\beta}\right)^{\alpha} / \theta\right]^{\theta}; \quad \alpha, \beta > 0, \tag{2.66}$$

where the support is for $F(t)$ is $(0, \infty)$ for $\theta \le 0$ and $(0, \beta\theta^{1/\alpha})$ for $\theta > 0$. So the support of the distribution can be a finite interval or unbounded.

Indeed the model reduces to the basic two-parameter Weibull when $\theta \to \infty$. Nikulin and Haghighi (2006) observed that the generalized Weibull distribution turns into the exponential if $\alpha = 1$ and $\theta \to \infty$, and the log-logistic distribution if $\theta = -1$, which is often used as a model in survival studies. Further, common distributions such as the lognormal and gamma distributions may be well approximated by this family. They also noted that if $\alpha \ge 0$ and $\theta < 0$, then the family coincides with Burr XII distributions.

2.16.1 *Characterization of Hazard Rate*

This generalized family not only contains distributions with bathtub and inverted (upside-down bathtub) hazard rate shapes, but also allows for a broader class of montonic hazard rates. The model hazard rate is given by

$$h(t) = \frac{\alpha(t/\beta)^{\alpha-1}}{\beta\left(1 - (t/\beta)^{\alpha}/\theta\right)}. \tag{2.67}$$

The following classification was obtained by Mudholkar and Hutson (1996):

1. $\alpha < 1$ and $0 < \theta < \infty$: $h(t)$ has a bathtub shape.
2. $\alpha \leq 1$ and $\theta \leq 0$: $h(t)$ is decreasing in t.
3. $\alpha > 1$ and $-\infty < \theta < 0$: $h(t)$ has an inverted bathtub shape.
4. $\alpha \geq 1$ and $\theta \geq 0$: $h(t)$ is increasing in t.
5. $\alpha = 1$ and $\theta \to \infty$: F is exponential (i.e., $h(t)$ is a constant).

It was also shown that the generalized Weibull family (2.68) is closed under proportional hazards relationships, that is, for any $\nu > 0$, $\overline{F}(t)^{\nu}$ is also a member of the family (2.68).

Furthermore, for $\theta \leq 0$, (2.68) reduces to the hazard rate of the Burr XII distribution, see for example, Sect. 2.3.13 of Lai and Xie (2006).

2.16.2 Estimation of Parameters

The maximum likelihood estimates of the model parameters were obtained by Mudholkar and Hutson (1996).

2.16.3 Applications

Mudholkar and Hutson (1996) have successfully fitted the model to the two-arm clinical trials data (Head-and-Neck cancer trials) considered by Efron (1988).

2.17 Jeong's Extension of Generalized Weibull

Jeong (2006) extended the above generalized Weibull model by incorporating another parameter with the survival function given as

$$\overline{F}(t) = \exp\left[-\frac{\theta^{1-\tau}\{(t/\beta)^{\alpha} + \theta\}^{\tau} - \theta}{\tau}\right]; \quad 0 < \alpha, \beta, \theta < \infty, \ -\infty < \tau < \infty. \tag{2.68}$$

The distribution becomes the standard Weibull distribution when $\tau = 1$. It reduces to the Mudholkar's generalized Weibull of (2.66) as $\tau \to 0$.

This new parameter family was motivated to parameterize the cumulative incidence function (in the context of survival analysis) completely.

The model has an application to breast cancer data.

2.17.1 Hazard Rate Function

The hazard rate function of Jeong's extension is

$$h(t) = \frac{\alpha(t/\beta)^{\alpha}\theta^{1-\tau}}{t\,[(t/\beta)^{\alpha} + \theta]^{1-\tau}}. \tag{2.69}$$

By differentiating $\log h(t)$ with respect to t and then setting the derivative to zero, we find the turning point $\beta\left\{\frac{\theta(1-\alpha)}{\alpha\tau-1}\right\}^{1/\alpha}$ exists on the real axis only if (a) $\alpha < 1$ and $\alpha\tau > 1$ or (b) $\alpha > 1$ and $\alpha\tau < 1$.

When (a) holds, $h(t)$ has a bathtub shape and if (b) holds then it has an upside-down bathtub shape (unimodal). Otherwise, $h(t)$ can be a constant, decreasing or increasing.

2.17.2 Application

The model was applied to two breast cancer data sets from the National Surgical Adjuvant Breast and Bowel Project.

2.18 Generalized Power Weibull Family

Nikulin and Haghighi (2006) introduced a family of distributions with survival function

$$\overline{F}(t) = \exp\left\{1 - \left(1 + (t/\beta)^{\alpha}\right)^{\theta}\right\}; \quad \alpha, \beta, \theta > 0. \tag{2.70}$$

Obviously, the case $\theta = 1$ reduces it to the standard Weibull distribution.

The quantile function is given as below:

$$Q(p) = \beta\{(1 - \log(1-p))^{1/\theta} - 1\}^{1/\alpha}; \quad 0 < p < 1. \tag{2.71}$$

2.18.1 Probability Density Function

The probability density function is given by

$$f(t) = \frac{\alpha\theta}{\beta^{\alpha-1}} t^{\alpha}\left\{1 + (t/\beta)^{\alpha}\right\}^{\theta-1} \exp\left\{1 - (1 + (t/\beta)^{\alpha})^{\theta}\right\}. \tag{2.72}$$

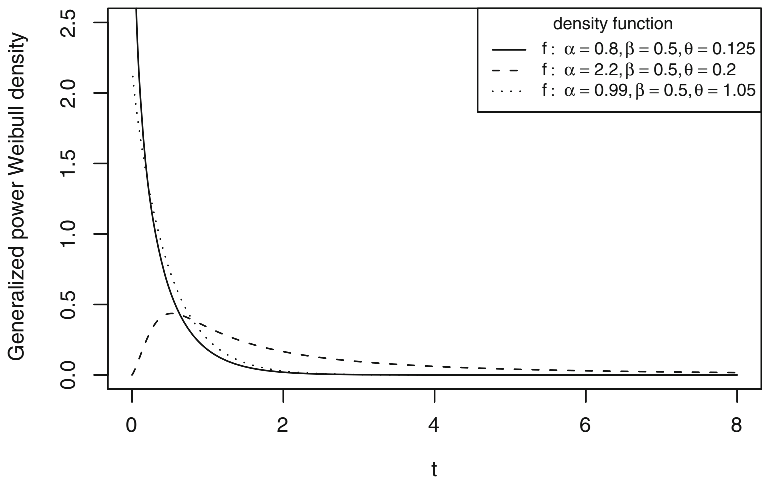

Fig. 2.12 Generalized power Weibull density functions

The density plots of the generalized power Weibull for three combinations of the parameters α, β and θ are given in Fig. 2.12.

2.18.2 Hazard Rate Function

The hazard rate of the generalized power Weibull family is

$$h(t) = \frac{\alpha\theta}{\beta^{\alpha-1}} t^{\alpha} \left\{1 + (t/\beta)^{\alpha}\right\}^{\theta-1}. \tag{2.73}$$

It has been shown in Nikulin and Haghighi (2006) that the hazard rate has nice and flexible properties. Depending on the values of the parameters, $h(t)$ can be constant, monotone (increasing, decreasing), unimodal or bathtub shaped.

More specifically, it has been shown that the hazard rate curve is

(a) motone increasing if either $\alpha > 1$ and $\alpha\theta > 1$ or $\alpha = 1$ and $\theta > 1$;
(b) monotone decreasing if either $0 < \alpha < 1$ and $\alpha\theta < 1$ or $0 < \alpha < 1$ and $\alpha\theta = 1$;
(c) unimodal (inverted bathtub shaped) if $\alpha > 1$ and $0 < \alpha\theta < 1$;
(d) bathtub shaped $0 < \alpha < 1$ and $\alpha\theta > 1$.

The plots of hazard rates are given in given in Fig. 2.13.

Using a chi-squared goodness-of-fit test, Nikulin and Haghighi (2006) showed that the model provides a good fit to the well known randomly censored survival times data for patients at ARM A of the Head-and-Neck Cancer Trail.

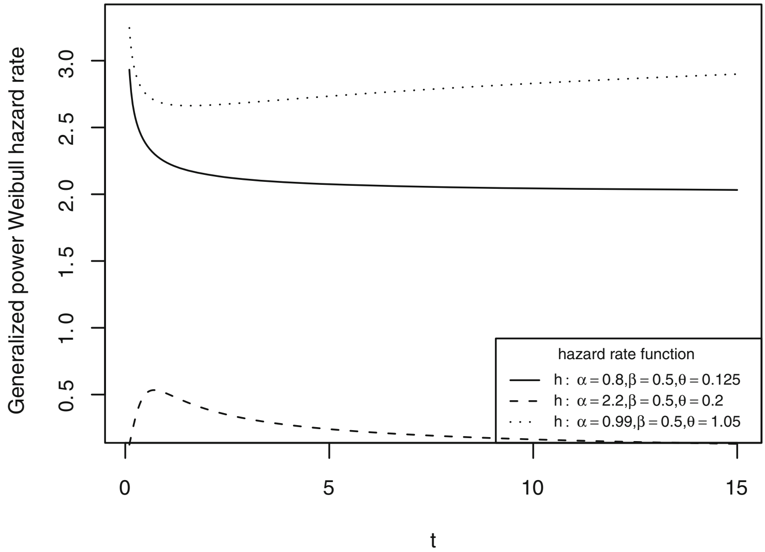

Fig. 2.13 Generalized power Weibull hazard rate functions

2.19 Slymen-Lachenbruch Modified Weibull Distribution

Slymen and Lachenbruch (1984) proposed a modified Weibull model with survival function given by

$$\overline{F}(t) = \exp\left\{-\exp\left[c + \frac{\beta(t^{\alpha} - t^{-\alpha})}{2\alpha}\right]\right\}; \quad \alpha, \beta, c > 0. \tag{2.74}$$

Thus

$$\log[-\log \overline{F}(t)] = c + \frac{\beta(t^{\alpha} - t^{-\alpha})}{2\alpha}. \tag{2.75}$$

We can see that a Weibull type probability plot can be constructed from the last equation.

Hazard Rate Function

The hazard rate function can be expressed as:

$$h(t) = \frac{\beta}{2}\left(t^{\alpha-1} - t^{-\alpha-1}\right) \exp\left[c + \frac{\beta(t^{\alpha} - t^{-\alpha})}{2\alpha}\right]. \tag{2.76}$$

Slymen and Lachenbruch (1984) showed that if $2\{(\alpha+1)t^{-\alpha-2} - (\alpha-1)t^{\alpha-2}\} \times (t^{\alpha-1} - t^{-\alpha-1})^{-2}$ is a bounded function then $h(t)$ can have a bathtub shape.

2.20 Flexible Weibull

Bebbington et al. (2007a) introduced a distribution which is quite simple and yet very flexible to model reliability data. The proposed distribution has survival function given as

$$\overline{F}(t) = \exp\left(-e^{\alpha t - \beta/t}\right). \tag{2.77}$$

For $\beta = 0$, it reduces to type 1 extreme value distribution (Johnson et al. 1995, Chap. 22) so that it can be transformed to the standard Weibull and hence the name flexible Weibull was given.

When $\beta = \alpha$, it reduces to a special case of the Slymen-Lachenbruch modified Weibull model considered in the preceding section.

It follows easily from (2.77) that

$$\log[-\log(\overline{F}(t))] = \alpha t - \beta/t \tag{2.78}$$

showing that the two parameters α and β can be initially estimated through a probability plot.

2.20.1 *Density Function*

The density function corresponding to (2.77) is

$$f(t) = (\alpha + \beta/t^2)\exp(\alpha t - \beta/t)\exp\left(-e^{\alpha t - \beta/t}\right). \tag{2.79}$$

The plots of three density curves are given in Fig. 2.14.

2.20.2 *Hazard Rate Function*

Unlike many other generalized Weibull distribution, the hazard rate of this distribution is rather simple:

$$h(t) = (\alpha + \beta/t^2)\exp(\alpha t - \beta/t). \tag{2.80}$$

A distribution F is said to be is IFRA (increasing failure rate average) if and only if $\int_0^t h(s)ds/t = -\log\overline{F}(t)/t$ is increasing in t. IFR (increasing average failure rate) is a subclass of IFRA. Note that the IFRA ageing class is featured prominently in Barlow and Proschan (1981).

It was shown by Bebbington et al. (2007a) that the distribution is IFR if $\alpha\beta \geq 27/64$, IFRA if and only if $\alpha\beta \geq 1/4$, which includes the IFR case, as required. We

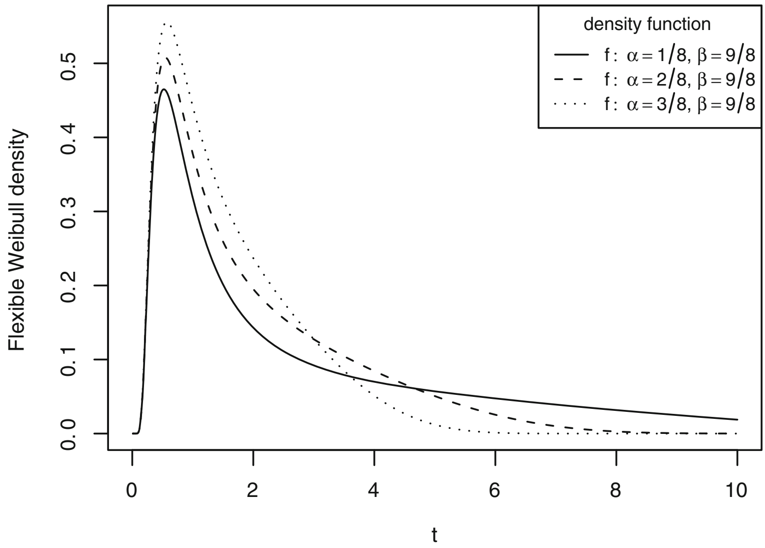

Fig. 2.14 Flexible Weibull density functions

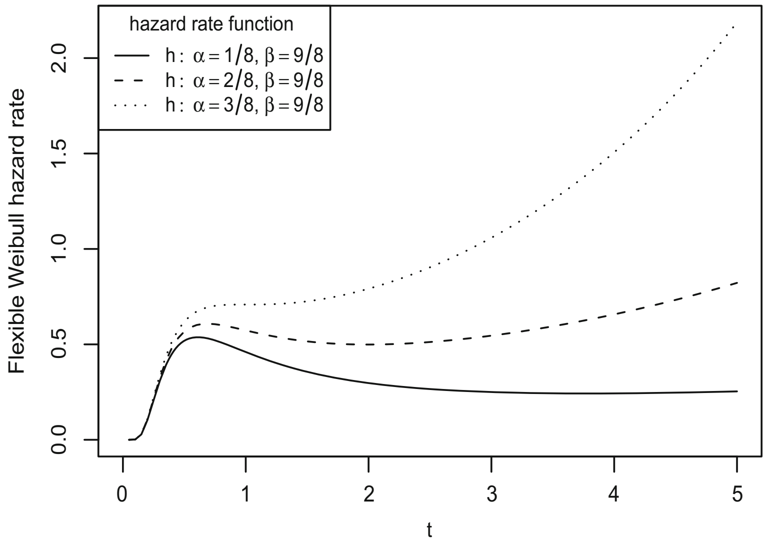

Fig. 2.15 Flexible Weibull hazard rate functions

see that if $1/4 < \alpha\beta \leq 27/64$, then the distribution is IFRA, but not IFR. Also, if $\alpha\beta < 27/64$, the hazard rate has a modified bathtub shape (i.e., h is first increasing followed by a bathtub shape).

The hazard rate plots are given in Fig. 2.15.

2.20.3 Parameter Estimation

Bebbington et al. (2007a) showed that α and β can be estimated through the Weibull-type probability plot using (2.78) above.

The maximum likelihood estimates of two parameters can also be obtained numerically through maximizing the likelihood function.

2.20.4 Applications

The flexible Weibull distribution has applications in:

- survival analysis for secondary reactor pumps (Bebbington et al. 2007a),
- human mortality study (Bebbington et al. 2007b),
- stop over duration of animals in the presence of trap-effects (Choqueta et al. 2013).

2.21 Weibull Extension Model

Another generalization of Weibull was introduced by Xie et al. (2002) and a detailed statistical analysis was given in Tang et al. (2003). In the latter, the authors simply referred their distribution as ‘the Weibull extension model’.

The distribution is in fact a generalization of the model studied by Chen (2000). The cumulative distribution function is given by

$$F(t) = 1 - \exp\left\{-\lambda\beta\left[e^{(t/\beta)^{\alpha}} - 1\right]\right\}, \quad t \geq 0, \alpha, \beta, \lambda > 0. \tag{2.81}$$

We see that the distribution approaches to a two-parameter Weibull distribution when $\lambda \to \infty$ with β in such a manner that $\beta^{\alpha-1}/\lambda$ is held constant.

For $\lambda = 1$, the above distribution is the exponential power distribution considered and studied by Smith and Bain (1975, 1976). The case with scale parameter $\beta = 1$ was considered by Chen (2000) who also considered the estimation of parameters.

2.21.1 Density Function

The probability density function is

$$f(t) = \lambda\alpha(t/\beta)^{\alpha-1} \exp\left\{(t/\beta)^{\alpha} + \lambda\beta(1 - e^{(t/\beta)^{\alpha}})\right\}. \tag{2.82}$$

Figure 2.16 gives the density plots.

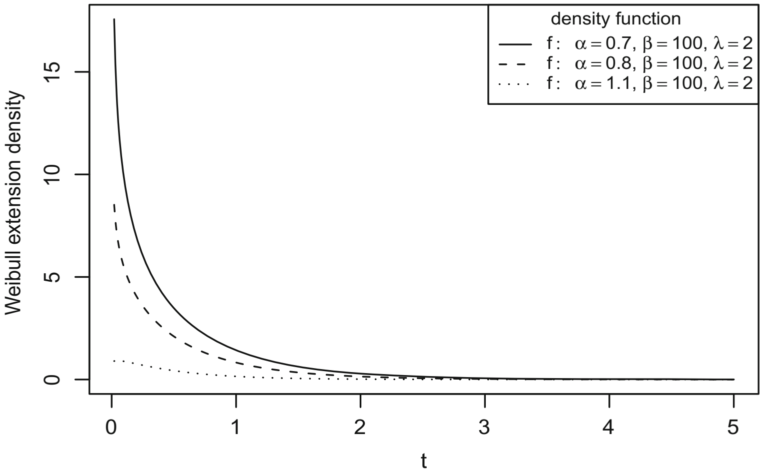

Fig. 2.16 Weibull extension density functions

2.21.2 Hazard Rate Function

The hazard rate function that corresponds to (2.81) is

$$h(t) = \lambda\alpha(t/\beta)^{\alpha-1}\exp\left[(t/\beta)^{\alpha}\right]. \tag{2.83}$$

The shape of $h(t)$ depends only on the shape parameter α.

For $\alpha \geq 1$:

(i) $h(t)$ is an increasing function;
(ii) $h(0) = 0$ if $\alpha > 1$ and $r(0) = \lambda$, if $\alpha = 1$;
(iii) $h(t) \to \infty$ as $t \to \infty$.

For $0 < \alpha < 1$:

(i) $h(t)$ is decreasing for $t < t^*$ and increasing for $t > t^*$ with

$$t^* = \beta(1/\alpha - 1)^{1/\alpha}. \tag{2.84}$$

This implies that the hazard rate function has a bathtub shape;
(ii) $h(t) \to \infty$ for $t \to 0$ or $t \to \infty$;
(iii) The change point t^* increases as the shape parameter α decreases.

The hazard rate plots of the distribution are given in Fig. 2.17.

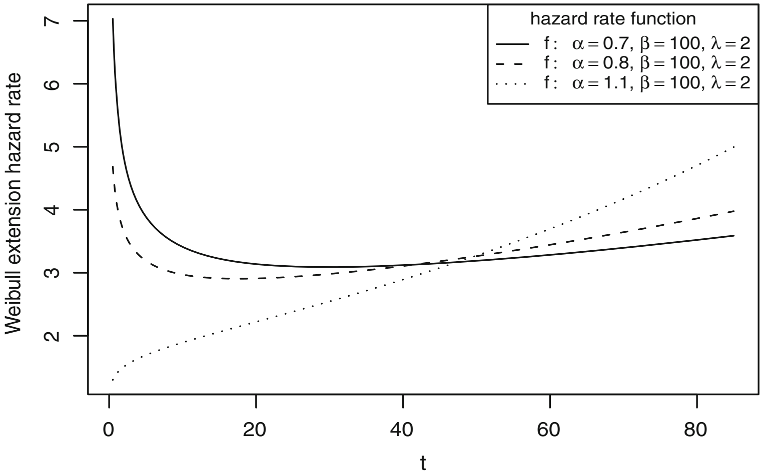

Fig. 2.17 Weibull extension hazard rate functions

2.21.3 Mean Residual Life

For this model, the mean residual life function $\mu(t)$ does not have a closed form although its turning point μ_0 can be obtained numerically. In fact, $\mu(t)$ attains its maximum value at $t = \mu_0$. Xie et al. (2004) have given plots of $\mu(t)$ and $h(t)$ for $\alpha = 0.45, 0.5$, and 0.6 with $\beta = 100$, $\lambda = 2$.

2.21.4 Other Properties

Quantile function, skewness and kurtosis, estimation of parameters, etc., were given in Tang et al. (2003).

Nadarajah (2005) derived explicit algebraic formulae for the kth moment (about zero) of the distribution when $1/\alpha$ is a non-negative integer.

2.21.5 Model Adequacy

Complete Bayesian analysis of the model has been provided by Gupta et al. (2008) using Markov chain Monte Carlo simulations. A thorough study has also been conducted for checking adequacy of the model for a given data using some of the graphical and numerical methods based on predictive simulation ideas. A real data set was considered for an illustration of their methods.

2.21.6 Applications

The model has been fitted to:

- the failure times of Aarset data (Aarset 1987)
- the failure times of a sample of devices in a large system (Gupta et al. 2008). The detailed description of the data set can be found in Meeker and Escobar (1998).

2.22 The Odd Weibull Distribution

Cooray (2006) has constructed a generalized Weibull family called the odd Weibull family based on the idea of evaluating the distribution of the 'odd of death' of a lifetime variable.

2.22.1 Distribution Function

The distribution function of the odd Weibull is given as

$$F(t) = 1 - \left(1 + (e^{(t/\beta)^{\alpha}} - 1)^{\theta}\right)^{-1}, \tag{2.85}$$

with $\beta > 0$ be the scale parameter and $\alpha\theta > 0$ as the shape parameter.

When $\theta = 1$, $F(t)$ is the cdf of the Weibull distribution. On the other hand, when $\theta = -1$, $F(t)$ has the inverse Weibull distribution given by $F(t) = e^{-(t/\beta)^{\alpha}}$.

Jiang et al. (2008) showed that the logit function, the logarithm of the odds, of the odd Weibull distribution can be written as the product of the logit function of the standard Weibull and θ. Similarly, it can also be written as the product of the logit function of the inverse Weibull and $-\theta$.

The density function is

$$f(t) = \left(\frac{\alpha\theta}{t}\right)\left(\frac{t}{\beta}\right) e^{(t/\beta)^{\alpha}} \left(e^{(t/\beta)^{\alpha}} - 1\right)^{\theta-1} \left[1 + \left(e^{(t/\beta)^{\alpha}} - 1\right)^{\theta}\right]^{-2}. \tag{2.86}$$

2.22.2 Quantile Function

The quantile function can be shown to be

$$Q(u) = \beta \left\{ \log\left[1 + \left(\frac{u}{1-u}\right)^{1/\theta}\right]\right\}. \tag{2.87}$$

2.22.3 Hazard Rate Function

The hazard rate function of the odd Weibull distribution is

$$h(t) = \left(\frac{\alpha\theta}{t}\right)\left(\frac{t}{\beta}\right)e^{(t/\beta)^{\alpha}}\left(e^{(t/\beta)^{\alpha}} - 1\right)^{\theta-1}\left[1 + \left(e^{(t/\beta)^{\alpha}} - 1\right)^{\theta}\right]^{-1}. \quad (2.88)$$

Cooray (2006) has shown that the odd Weibull family can model various hazard shapes (increasing, decreasing, bathtub, and unimodal); thus the family is proved to be flexible for fitting reliability and survival data. More precisely, he established that

(1) for $\alpha < 0, \theta < 0$ or $\alpha < 1, \alpha\theta \geq 1$, $h(t)$ is unimodal,
(2) for $\alpha > 1, \alpha\theta > 1$, $h(t)$ is increasing,
(3) for $\alpha < 1, \alpha\theta < 1$, $h(t)$ is decreasing, and
(4) for $\alpha > 1, \alpha\theta \leq 1$, $h(t)$ has a bathtub shape.

Cooray (2006) also indicated that in the regions where $(\alpha > 1, \alpha\theta > 1)$ and $(\alpha < 1, \alpha\theta < 1)$, $h(t)$ may have some other shapes. Jiang et al. (2008) found, using numerical analysis, that the 'other shapes' are N (modified bathtub) and reflected N shapes when the model parameters are near the boundary line $\alpha\theta = 1$. The latter authors also studied the tail behaviours of $h(t)$ in detail.

2.22.4 Weibull Probability Plot Parameter Estimation

The shapes of the Weibull probability plot with different parameters were presented in Jiang et al. (2008) and the steps of the graphical estimation were also iterated. In estimating parameters of the odd Weibull model by a graphical approach, it is necessary to determine if the shape parameters are positive or negative. Cooray (2006) suggested to employ the total-time-test (TTT) for this purpose whereas Jiang et al. (2008) concluded that it is easy to check the sign of the shape parameters using a Weibull probability plot.

2.22.5 Optimal Burn-In Time and Useful Period

For a product lifetime exhibiting a bathtub shaped hazard rate, an important issue is to determine the optimal burn-in time. As seen from an earlier subsection, the odd Weibull model can indeed display such a shape. Jiang et al. (2008) observed that the second portion (stable period) of the odd Weibull model, when exhibiting a bathtub shape, could be quite long and flat which is a good property in application. They discussed those issues concerning the optimal burn-in time and the useful period for the model.

2.22.6 Application

The model has been fitted to a sample of 208 data points, which represent the ages at death in weeks for male mice exposed to 240r of gamma radiation (Kimball 1960).

2.23 Generalized Logistic Frailty Model

Vaupel (1990) proposed and examined a logistic frailty model to correct an inherent deficiency in the Gompertz-Makeham law often used in a mortality study. The Vaupel model has incorporated a deceleration parameter $s \geq 0$ in such a way that the resulting hazard is a logistic function given by

$$h(t) = \frac{Ae^{t/\beta}}{1 + sA\beta(e^{t/\beta} - 1)}. \tag{2.89}$$

The survival function corresponding to (2.89) is

$$\overline{F}(t) = \left[1 + sA\beta(e^{t/\beta} - 1)\right]^{-\frac{1}{s}} \tag{2.90}$$

which is relatively simple. On differentiation of (2.89), we have

$$h'(t) = \frac{Ae^{(t/\beta)}(1/\beta - sA)}{\left[1 + sA\beta(e^{t/\beta} - 1)\right]^2}. \tag{2.91}$$

From the preceding equation, we see that $h(t)$ is increasing (decreasing) for $\beta sA < 1 (sA\beta > 1)$ and it converges to a constant as $t \to \infty$. For human mortality, the case $(sA\beta > 1)$ is obviously unrealistic as it would imply the immortality of man.

2.23.1 Generalized Logistic Frailty Distribution

In view of its hazard rate function being monotonic, the logistic frailty model has limited applications in reliability and survival analysis. To provide a more flexible model that is applicable to various disciplines associated with lifetime data, Lai and Izadi (2012) generalized the logistic frailty model in (2.90) by incorporating a shape parameter:

$$\overline{F}(t) = \left[1 + sA\beta(e^{(t/\beta)^{\alpha}} - 1)\right]^{-\frac{1}{s}} \tag{2.92}$$

where $A, \alpha, \beta > 0$ and $s \geq 0$. Here α and β are clearly the shape and scale parameter, respectively. As mentioned earlier, s is a deceleration parameter whereas A is some kind of 'normalizing constant', particularly if $s = 0$.

We will see how this distribution is related to the Weibull in the next subsection.

2.23.2 Limiting Case

As $s \to 0$, (2.92) is reduced to

$$\begin{aligned} \lim_{s\to 0} \overline{F}(t) &= \lim_{s\to 0} \left[1 + sA\beta(e^{(t/\beta)^\alpha} - 1)\right]^{-\frac{1}{s}} \\ &= \exp\left\{-A\beta(e^{(t/\beta)^\alpha} - 1)\right\}. \end{aligned} \tag{2.93}$$

which is the Weibull extension model of Chen (2000) and Xie et al. (2002). It has been shown in Xie et al. (2002) that (2.93) is either IFR (increasing failure rate), DFR (decreasing failure rate) or a bathtub shaped hazard rate distribution.

2.23.3 Probability Density Function

The density function of the generalized logistic frailty model can be obtained easily by differentiating (2.92) with respect to t and it is given by

$$f(t) = A\alpha(t/\beta)^{\alpha-1} e^{(t/\beta)^\alpha} \left[1 + sA\beta(e^{(t/\beta)^\alpha} - 1)\right]^{-\frac{s+1}{s}}. \tag{2.94}$$

2.23.4 Hazard Rate Function

The hazard rate function that corresponds to (2.92) is

$$h(t) = \frac{A\beta(t/\beta)^{\alpha-1} e^{(t/\beta)^\alpha}}{1 + sA\beta(e^{(t/\beta)^\alpha} - 1)}. \tag{2.95}$$

Rewriting the above in a slightly different form,

$$h(t) = \frac{A\alpha(t/\beta)^{\alpha-1} e^{(t/\beta)^\alpha}/(sA\beta)}{e^{(t/\beta)^\alpha} + (1 - sA\beta)/(sA\beta)}. \tag{2.96}$$

From the preceding equation, it is clear that $h(t)$ is increasing if $sA\beta < 1$ and $\alpha > 1$. We have also mentioned that for $s = 0$, (2.96) can yield IFR (increasing failure rate), DFR (decreasing failure rate) or a bathtub shape.

Other possible shapes are displayed in Fig. 2.18.

Note that the one of the above curves almost has an S shape which may be useful for describing human late life mortality deceleration phenomenon as observed in Greenwood and Irwin (1931).

The shapes of the hazard rate of the generlized logistic frailty model are summarized in terms β and c ($c = aA\alpha$) by Lai and Izadi (2012).

2.23.5 Applications

The model has been fitted to:

- the data set given in Wang (2000) which consists of the lifetime failures of 18 electronic devices, and
- the data of active repair times (in hours) for an airborne communication transceiver which was reported in Von Alven (1964).

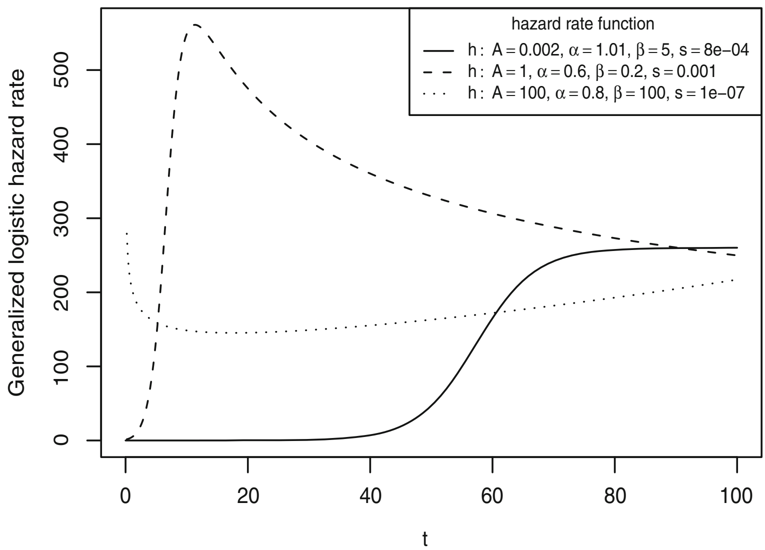

Fig. 2.18 Generalized logistic hazard rate functions

2.24 Generalized Weibull-Gompertz Distribution

Nadarajah and Kotz (2005) proposed a generalization of the standard Weibull model with four parameters having survival function given as below:

$$R(t) = \exp\left\{-at^b\left(e^{ct^d} - 1\right)\right\}, a, d > 0;\ b, c \geq 0; t \geq 0. \tag{2.97}$$

Since (2.97) includes the Gompertz (1825) distribution as its special case when $b = 0$ and $d = 1$. For this reason we may refer it as the generalized Weibull-Gompertz distribution. For $b = 0$ it contains the Weibull extension model of Xie et al. (2002) as given in Sect. 2.21 above. Since the models have four parameters, it may be considered as an over-parameterized model.

2.25 Generalized Weibull Distribution of Gurvich et al.

It has been pointed by Nadarajah and Kotz (2005) that several of the modifications of the Weibull distribution discussed in this section can arise from a representation suggested by Gurvich et al. (1997). This distribution did not arise from a reliability perspective but from the context of modeling random length of brittle materials. The distribution function of this class is given by

$$F(t) = 1 - \exp\{-aG(t)\}, \tag{2.98}$$

where $G(t)$ is a monotonically increasing function in t such that $G(t) \geq 0$.

Several distributions we presented in this chapter can be derived by assigning appropriate expressions to $G(t)$. For example,

- the modified Weibull model of Lai et al. (2003) is obtained by setting $G(t) = t^\alpha \exp(\lambda t)$,
- the Weibull extension model of Xie et al. (2002) is obtained by letting $G(t) = \alpha \exp(t/\beta)^\alpha - 1$,
- the log Weibull model is derived by assigning $G(t) = \exp((t-a)/b))$, and
- the model considered by Nadarajah and Kotz (2005) follows from taking $G(t) = t^b\left(e^{ct^d} - 1\right)$.

Several other Weibull related distributions are also contained in this family.

Pham and Lai (2007) have noted that Gurvich's model is so general in which $G(t)$ is in fact the cumulative hazard function of an arbitrary lifetime distribution. We note that any survival function $\overline{F}$ can be expressed as $\overline{F}(t) = \exp\{-\int_0^t h(t)dt\} = \exp\{-H(t)\}$. Letting $aG(t) = H(t)$, we see that the Gurvich's model includes all the continuous lifetime distributions.

2.26 Weibull Models with Varying Parameters

The parameters of the models discussed so far are held constants. This section deals with models where some of parameters are

(i) functions of the variable t,
(ii) function of some supplementary variables (denoted by S), or
(iii) random variables (covariates).

2.26.1 Time Varying Parameters

In these models the scale parameter (β) and/or the shape parameter (α) of the standard Weibull model given by (1.4) are function of the variable t.

Examples

(i) Only one of the two parameters is varying with time.
(ii) Both α and β are functions of time. For example,

$$\alpha(t) = a\left(1 + \frac{1}{t}\right)^{b} e^{c/t}, \quad \beta(t) = a' t^{b'} e^{c'/t}. \tag{2.99}$$

(iii) $F(t) = 1 - \exp\{-\Lambda(t)\}$ where $\Lambda(t)$ is a nondecreasing function with $\Lambda(0)$=0 and $\Lambda(\infty) = \infty$. For example,

$$\Lambda(t) = \sum_{i=1}^{m} \lambda_i \phi_i(t), \quad 1 \le i \le m. \tag{2.100}$$

2.26.2 Weibull Accelerated Models

In these models the scale parameter β is a function of some supplementary variable (covariate) S. In reliability applications S represents the stress on the item so the life of the item (a random variable with distribution F) is a function of S. The shape parameter is unaffected by S and hence a constant. See Sect. 2.6.2 of Murthy et al. (2004).

Arrhenius Model

The relationship is given by

$$\beta(S) = \exp(\gamma_0 + \gamma_1 S). \tag{2.101}$$

(See Jensen (1995) for a summary this model). We see that $\log(\beta(S))$ is linear in S.

Power Model

The relationship is given by

$$\beta(S) = \frac{e^{\gamma_0}}{S^{\gamma_1}}. \tag{2.102}$$

The power model is also briefly in Jensen (1995).

A more general formulation is one where the scale parameter is expressed as

$$\beta(S) = \exp\left(b_0 + \sum_{i=1}^{k} b_i s_i \right) \tag{2.103}$$

where $S = (s_1, s_2, \ldots, s_k)'$ is a k-dimensional vector of supplementary variables.

One may note that these types of models have been used extensively in accelerated life testing in reliability, see for example, Nelson (1990) or Chap. 6 of Lawless (2003).

2.26.3 *Weibull Proportional Hazard Models*

An alternative approach to modeling the effect of the supplementary variables (covariates) on the survival function $\overline{F}(t)$ is through the hazard rate function. The relationship is given by

$$h(t|S) = \psi(S)h_0(t), \tag{2.104}$$

where $h_0(t)$ is called the baseline hazard for a two-parameter Weibull distribution. The only restriction on the scalar function $\psi(\cdot)$ is that it be positive. The last equation tells us $h(t|S)$ is proportional to the baseline hazard rate h_0 and thus the name 'proportional hazard' model.

Many different forms for $\psi(\cdot)$ have been proposed in the literature. One such is the following:

$$\psi(S) = \exp\left(\sum_{i=1}^{k} b_i s_i \right). \tag{2.105}$$

(See for e.g., Kalbfleisch and Prentice 2002 and Lawless 2003). It now follows from (2.104) that the survival function

$$\overline{F}(t|S) = \overline{F}(t)^{\psi(S)}. \tag{2.106}$$

(Note that in the current context, $\overline{F}(t)$ is the Weibull survival function, but it could denote any other survival functions in a general setting).

When ψ is given by (2.105) and (2.104) becomes

$$h(t|S) = h_0(t) \exp\left(\sum_{i=1}^{k} b_i s_i\right) \tag{2.107}$$

The hazard model above is generally known as the Cox proportional hazard model. Taking the natural logarithm of both sides of the preceding equation, we have

$$\log(h(t|S)/h_0(t)) = \sum_{i=1}^{k} b_i s_i \tag{2.108}$$

We now have a fairly 'simple' linear model where the parameters can be readily estimated.

2.26.4 Applications

In the recent years, the Cox proportional model is a popular survival model when covariates are involved. Examples of applications include:

- epidemiology and biostatistics, e.g., multiple infection data,
- financial analysis, e.g., stock exchange market,
- clinical trials, and
- survival analysis, e.g., cancer relapse data.

References

Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, *36*, 106–108.

Adamidis, K., & Loukas, S. (1998). A lifetime with decreasing failure rate. *Statistics & Probability Letters*, *39*, 35–42.

Alwasel, I. A. (2009). Statistical inference of a competing risks model with modified Weibull Distributions. *International Journal of Mathematical Analysis*, *3*(19), 905–918.

Barlow, R. E., & Proschan, F. (1981). *Statistical theory of reliability and life testing*. To Begin With, Silver Spring.

Barreto-Souza, W., de Morais, A. L., & Cordeiro, G. M. (2010). The Weibull-geometric distribution. *Journal of Statistical Computation and Simulation* (first published on line on June 11, 2010) 1–13. doi:10.1080/00949650903436554.

Bebbington, M., Lai, C. D., & Zitikis, R. (2006). Useful periods for lifetime distributions with bathtub shaped hazard rate functions. *IEEE Transactions on Reliability*, *55*(2), 245–251.

Bebbington, M., Lai, C. D., & Zitikis, R. (2007a). A flexible Weibull extension. *Reliability Engineering and System Safety*, *92*, 719–726.

Bebbington, M., Lai, C. D., & Zitikis, R. (2007b). Modeling human mortality using mixtures of bathtub shaped failure rate distributions. *Journal of Theoretical Biology*, *245*, 528–538.

Bebbington, M., Lai, C. D., & Zitikis, R. (2008). Estimating the turning point of a bathtub shaped failure distribution. *Journal of Statistical Planning and Inference*, *138*(4), 1157–1166.

Calabria, R., & Pulcini, G. (1990). On the maximum likelihood and least-squares estimation in the inverse Weibull distribution. *Statistica Applicata*, *2*(1), 53–66.

Carrasco, J. M. F., Ortega, E. M. M., & Cordeiro, G. M. (2008). A generalized modified Weibull distribution for lifetime modeling. *Computational Statistics and Data Analysis*, *53*, 450–462.

Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Statistics and Probability Letters*, *49*, 155–161.

Choqueta, R., Guédonb, Y., Besnarda, A., Guillemainc, M., & Pradel, R. (2013). Estimating stop over duration in the presence of trap-effects R. *Ecological Modelling*, *250*, 111–118.

Cohen, A. C. (1973). The reflected Weibull distribution. *Technometrics*, *15*(4), 867–873.

Cooray, K. (2006). Generalization of the Weibull distribution: The odd Weibull family. *Statistical Modelling*, *6*, 265–277.

de Gusmão, F. R. S., Ortega, E. M. M., & Cordeiro, G. M., (2009). The generalized inverse Weibull distribution. *Statistical Papers*. doi:10.1007/s00362-009-0271-3.

Drapella, A. (1993). The complementary Weibull distribution: Unknown or just forgotten. *Quality and Reliability Engineering International*, *9*, 383–385.

Efron, B. (1988). Logistic regression, survival analysis, and the Kaplan-Meier curve. *Journal of the American Statistical Association*, *83*, 415–425.

Erto, P. (1989). Genesis, properties and identification of the inverse Weibull lifetime model. *Staistica Applicata*, *1*(2), 117–128.

Famoye, F., Lee, C., & Olumolade, O. (2005). The beta-Weibull distribution. *Journal of Statistical Theory and Applications*, *4*, 121–136.

Ghitany, M. E., Al-Hussaini, E. K., & Al-Jarallah, R. A. (2005). Marshall-Olkin extended Weibull distribution and its application to censored data. *Journal of applied Statistics*, *32*(10), 1025–1034.

Glaser, R. E. (1980). Bathtub and related failure rate characterizations. *Journal of the American Statistical Association*, *75*, 667–672.

Gomes, O., Combes, C., & Dussauchoy, A. (2008). Parameter estimation of the generalized gamma distribution. *Mathematics and Computers in Simulation*, *79*, 963–995.

Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality, and on the mode of determining the value of life contingencies. *Philosophical Transactions of the Royal Society*, *115*, 513–585.

Greenwood, M., & Irwin, J. O. (1931). The biostatistics of senility. *Human Biology*, *11*(1), 1–23.

Gupta, R. C., Gupta, R. D., & Gupta, P. L. (1998). Modelling failure time data by Lehman alternatives. *Communications in Statistics-Theory and Methods*, *27*(4), 887–904.

Gupta, A., Mukherjee, B., & Upadhyay, S. K. (2008). Weibull extension: A Bayes study using Markov chain Monte Carlo simulation. *Reliability Engineering and System Safety*, *93*, 1434–1443.

Gupta, R. C., Lvin, S., & Peng, C. (2010). Estimating turning points of the failure rate of the extended Weibull distribution. *Computational Statistics and Data Analysis*, *54*, 924–934.

Gurvich, M. R., Dibenedetto, A. T., & Rande, S. V. (1997). A new statistical distribution for characterizing the random length of brittle materials. *Journal of Material Science*, *32*, 2559–2564.

Hemmati, F., & Khorram, E. (2011). A new three-parameter ageing distribution. *Journal of Statistical Planning and Inference*, *141*(7), 2266–2275.

Jensen, F. (1995). *Electronic Component Reliability*. New York: Wiley.

Jeong, J. H. (2006). A new parametric family for modelling cumulative incidence function: application to breast cancer data. *Journal of the Royal Statistical Society: Series A*, *169*(2), 289–303.

Jiang, R., & Murthy, D. N. P. (1999). The exponentiated Weibull family: A grapgical approach. *IEEE Transactions on Reliability*, *48*(1), 68–72.

Jiang, R., Murthy, D. N. P., & Ji, P. (2001). Models involving two inverse Weibull distributions. *Reliability Engineering and System Safety*, *73*, 73–81.

Jiang, H., Xie, M., & Tang, L. C. (2008). On the odd Weibull distribution. *Proceedings of the Institute of Mechnical Engineers, Part O: Journal of Risk and Reliability*, *22*, 583–594.

Jiang, H., Xie, M., & Tang, L. C. (2010). On MLEs of the parameters of a modified Weibull distribution for progressively type-2 censored samples. *Journal of Applied Statistics*, *37*(4), 617–627.

Johnson, N. L., Kotz, S., & Balakrishan, N. (1995). *Continuous Univariate Distributions* (2nd ed., Vol. 2). New York: Wiley.

Kalbfleisch, J. D., & Prentice, R. L. (2002). *The statistical analysis of failure time data* (2nd ed.). New York: John Wiley & Sons.

Keller, A. Z., Giblin, M. T., & Farnworth, N. R. (1985). Reliability analysis of commercial vehicle engines. *Reliability Engineering*, *10*(1), 15–25.

Kies, J. A. (1958). The strength of glass, Naval Research Lab. Report No.5093, Washington D.C.

Kimball A. W. (1960). Estimation of mortality intensities in animal experiments. *Biometrics*, *16*, 505–521.

Klieber, C., & Kotz, S. (2003). *Statistical size distributions in economic and actuarial sciences*. New York: Wiley-Interscience.

Lai, C. D., & Izadi, M. (2012). Generalized logistic frailty model. *Statistics and Probability Letters*, *82*, 1969–1977.

Lai, C. D., Xie, M., & Murthy, D. N. P. (2003). Modified Weibull model. *IEEE Transactions on Reliability*, *52*, 33–37.

Lai, C. D., Zhang, L. Y., & Xie, M. (2004). Mean residual life and other properties of Weibull related bathtub shape failure rate distributions. *International Journal of Reliability, Quality and Safety Engineering*, *11*(2), 113–132.

Lai, C. D., & Xie, M. (2006). *Stochastic ageing and dependence for reliability*. New York: Springer.

Lai, C. D., Murthy, D. N. P., & Xie, M. (2011). Weibull distributions. *Wiley Interdisciplinary Reviews*, *3*(3), 282–287.

Lawless, J. F. (2003). *Statistical models and methods for lifetime data*. Hoboken: John Wiley & Sons.

Lee, C., Famoye, F., & Olumolade, O. (2007). Beta-Weibull distribution: Some properties and applications to censored data. *Journal of Modern Applied Statistical Methods*, *6*, 173–186.

Marshall, A. W., & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, *84*, 641–652.

Marshall, A. W., & Olkin, I. (2007). *Life distributions: Structure of nonparametric, semiparametric, and parametric families*. New York: Springer.

McDonald, J. B., & Richards, D. O. (1987). Model selection: Some generalized distributions. *Communications in Statistics-Theory and Methods*, *16*(4), 1049–1074.

McEwen, R. P., & Parresol, B. R. (1991). Moment expressions and summary statistics for the complete and truncated Weibull distribution. *Communications in Statistics-Theory and Methods*, *20*, 1361–1372.

Meeker, W. Q., & Escobar, L. A. (1998). *Statistical methods for reliability data*. New York: Wiley & Sons.

Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, *42*, 299–302.

Mudholkar, G. S., & Kollia, G. D. (1994). Generalized Weibull family: A structural analysis. *Communications in Statistics-Theory and Methods*, *23*, 1149–1171.

Mudholkar, G. S., Srivastava, D. K., & Freimer, M. (1995). The exponentiated Weibull family: A reanalysis of bus-motor-failure data. *Technometrics*, *37*, 436–445.

Mudholkar, G. S., & Hutson, A. D. (1996). The exponentiated Weibull family: Some properties and a flood data application. *Communications in Statistics-Theory and Methods*, *25*(12), 3059–3083.

Murthy, D. N. P., Xie, M., & Jiang, R. (2004). *Weibull models*. Hoboken: John Wiley & Sons, Inc.

Nadarajah, S. (2005). On the moments of the modified Weibull distribution. *Reliability Engineering and System Safety*, *90*, 114–117.

Nadarajah, S., & Gupta, A. K. (2005). On the moments of the exponentiated Weibull distributions. *Communications in Statistics-Theory and Methods*, *34*, 253–256.

Nadarajah, S., & Kotz, S. (2005). On some recent modifications of Weibull distribution. *IEEE Transactions on Reliability*, *54*(4), 561–562.

Nassar, M. M., & Eissa, F. H. (2003). On the exponentiated Weibull distribution. *Communications in Statistics-Theory and Methods*, *32*(7), 1317–1336.

Nelson, W. (1990). *Accelerated life testing: Statistical models, test plans, and data analyses*. New York: Wiley.

Ng, H. K. T. (2005). Parameter estimation for a modified Weibull distribution, for progressively type-II censored samples. *IEEE Transactions on Reliability*, *54*(3), 374–380.

Nikulin, M., & Haghighi, F. (2006). A Chi-squared test for the generalized power Weibull family for the head-and-neck cancer censored data. *Journal of Mathematical Sciences*, *133*(3), 1333–1341.

Ortega, E. M. M., Cordeiro, G. M., & Hashimoto, E. M. (2011). A log-linear regression model for the Beta-Weibull distribution. *Communications in Statistics: Simulation and Computation*, *40*(8), 1206–1235.

Pham, T., & Almhana, J. (1995). The generalized Gamma-distribution-its hazard rates and stress-strength model. *IEEE Transactions on Reliability*, *44*, 392–397.

Pham, H., & Lai, C. D. (2007). On recent generalizations of the Weibull distribution. *IEEE Transactions on Reliability*, *56*(3), 454–459.

Phani, K. K. (1987). A new modified Weibull distribution function. *Communications of the American Ceramic Society*, *70*, 182–184.

Silva, G. O., Ortega, E. M. M., & Cordeiro, G. M. (2010). The beta modified Weibull distribution. *Lifetime Data Analysis*, *16*, 409–430.

Slymen, D. J., & Lachenbruch, P. A. (1984). Survival distributions arising from two families and generated by transformations. *Communications in Statistics-Theory and Methods*, *13*, 1179–1201.

Smith, R. M., & Bain, L. J. (1975). An exponential power life-testing distribution. *Communications in Statistics-Theory and Methods*, *4*, 469–481.

Smith, R. M., & Bain, L. J. (1976). Correlation-type goodness of fit statistics with censored sampling, life-testing distribution. *Communications in Statistics-Theory and Methods*, *5*, 119–132.

Smith, F., & Hoeppner, D. W. (1990). Use of the four parameter Weibull function for fitting fatigue and compliance calibration data. *Engineering Fracture Mechanics*, *36*, 173–178.

Stacy, E. W. (1962). A generalization of gamma distribution. *Annals of Mathematical Statistics*, *33*, 1187–1192.

Tang, Y., Xie, M., & Goh, T. N. (2003). Statistical analysis of a Weibull extension model. *Communicationsin Statistics-Theory and Methods*, *32*(5), 913–928.

Vaupel, J. W. (1990). Kindred lifetimes: Frailty models in population genetics. In J. Adams, D. A. Lam, A. I. Hermalin, & P. E. Smouse (Eds.), *Convergent issues in genetics and demography* (pp. 155–170). Oxford: Oxford University Press.

Von Alven, W. H. (1964). *Reliability engineering*. New Jersey: Prentice Hall.

Wahed, A. S., Luong, T. M., & Jeong, J. H. (2009). A new generalization of Weibull distribution with application to breast cancer data set. *Statistics in Medicine*, *28*, 2077–2094.

Wang, F. K. (2000). A new model with bathtub-shaped failure rate using an additive Burr XII distribution. *Reliability Engineering and System Safety*, *70*, 305–312.

Xie, M., Tang, Y., & Goh, T. N. (2002). A modified Weibull extension with bathtub-shaped failure rate function. *Reliability Engineering and System Safety*, *76*, 279–285.

Xie, M., Goh, T. N., & Tang, Y. (2004). On changing points of mean residual life and failure rate function for some generalized Weibull distributions. *Reliability Engineering and System Safety*, *84*, 293–299.

Zhang, L. F., Xie, M., & Tang, L. C. (2007). A study of two estimation approaches for parameters of Weibull distribution based on WPP. *Reliability Engineering and System Safety*, *92*, 360–368.

Zhang, T., & Xie, M. (2011). On the upper truncated Weibull distribution and its reliability implications. *Reliability Engineering and System Safety*, *96*, 194–200.

Chapter 3
Models Involving Two or More Weibull Distributions

In this chapter we discuss several univariate statistical models that are derived from two or more independent Weibull distributions that interact with each other under some governing 'operations'. For example, Barlow and Proschan (1981) raised a general question regarding under what reliability operations a given class of life distributions is preserved.

In a reliability setting, a system generally consists of several components. We assume that each component lifetime X_i, $i = 1, 2, \ldots, n$ has a Weibull distribution with cdf F_i having scale parameter α_i and shape parameter β_i. We further assume that these components are statistically independent although this assumption may not hold in many practical situations. Generally a reliability system has a structure that determines how these components are linked (arranged) together. For example, the components can be arranged in series, parallel or series-parallel hybrid. It is well known that the reliability of a system depend on (a) the system structure and (b) reliability functions of all the components.

In general, a survival model that involves a larger number of subpopulations (components) is likely to be hampered by a larger number of model parameters to be estimated.

Four such models are now presented in this chapter: (A) Competing risk models, (B) multiplicative models, (C) sectional models and (D) mixture models. In addition, we also consider these models when the underlying distribution is the inverse Weibull distribution instead of Weibull itself. Structures A and B correspond to the series and parallel systems, respectively. After a general introduction of these models, we will present several examples of generalized Weibull models that arise from these structures.

3.1 *n*-fold Mixture Model

The literature on Weibull mixture models has grown at an increasing pace and there are many applications of such models. A mixture aries because of the overall population is not homogeneous. Formally a mixture model corresponds to the

C.-D. Lai, *Generalized Weibull Distributions*, SpringerBriefs in Statistics,
DOI: 10.1007/978-3-642-39106-4_3, © The Author(s) 2014

mixture distribution that represents the probability distribution of observations in the overall population.

Let p_i be the mixing proportion of the ith subpopulation so the distribution of the mixture model is represented by

$$F(t) = \sum_{i=1}^{n} p_i F_i(t), \quad p_i \geq 0, \sum_{i=1}^{n} p_i = 1. \tag{3.1}$$

Here we assume the n Weibull random variables involved in the mixture are statistically independent.

The hazard rate function of a finite mixture is given by

$$h(t) = \sum w_i(t) h_i(t), \tag{3.2}$$

where $h_i(t)$ denotes the hazard rate function of the ith subpopulation (or ith component), and

$$w_i(t) = \frac{p_i \overline{F}_i(t)}{\sum_{i=1}^{n} p_i \overline{F}_i(t)}; \;\; 0 \leq p_i \leq 1, \;\; 1 \leq i \leq n \tag{3.3}$$

such that $\sum_{i=1}^{n} w_i(t) = 1$. The preceding two equations show that the hazard rate function of the mixture model is simply a weighted mean of the hazard rates for the subpopulations with the weights varying with time t.

In most applications in the literature, only two-fold mixtures were considered to minimize the number of parameters to be estimated. We will consider this special case in a Sect. 3.6 below.

Gurland and Sethuraman (1995) have considered a mixture of the Weibull distribution with hazard rate $\lambda \alpha t^{\alpha-1}$ and the exponential distribution with hazard rate λ_1. For $\alpha > 1$, the Weibull distribution is IFR. They found that the resulting mixture distribution is 'ultimately' DFR. In fact, the mixture with the exponential has resulted a hazard rate with an upside-down bathtub shape.

A basic feature of the finite mixture model is that the mixing weights are all positive. However, Titteringto et al. (1985) suggested that it is possible to have mixture model with negative weights as long as the density function for the model is valid.

Let us consider a life distribution $G(t, \theta)$ with a parameter θ that itself is random variable with distribution function Ψ. Then the resulting distribution F can be expressed as

$$F(t) = \int G(t|\theta) d\Psi(\theta).$$

Such an infinite mixture is also known a compound distribution.

3.1.1 Reliability Approximation Using a Finite Weibull Mixture Distributions

Bŭcar et al. (2004) have shown that the reliability of an arbitrary system can be approximated well by a finite Weibull mixture with positive component weights only, without knowing the structure of the system, on condition that the unknown parameters of the mixture can be estimated. To support the main idea, they have presented five examples for demonstration. In order to estimate the unknown component parameters and components weights of the Weibull mixture, some of the already known methods were applied and the EM algorithm for the m-fold mixture was derived. The fitted distributions obtained by different methods were compared to the empirical ones by calculating the AIC and δ_c values. The authors concluded that the suggested Weibull mixture with an arbitrary but finite number of components is suitable for lifetime data approximation.

For other ageing characteristics of Weibull mixtures, see Murthy et al. (2004).

3.2 n-fold Competing Risk Model

The ‘competing risk’ problem arises in the study of any failure process in which there is more than one distinct cause of failure. This often occurs in clinical, epidemiologic, demographic, basic science and industrial literature.

Again we assume here the n Weibull random variables involved in the n-fold competing risks model are independent. The distribution function is given by

$$F(t) = 1 - \prod_{i=1}^{n} (1 - F_i(t)). \tag{3.4}$$

The above model represents the lifetime distribution of a series system of n independent but not necessarily identical Weibull components. The mean of the n-fold competing risk model is

$$\mu = \int_0^\infty \prod_{i=1}^{n} (1 - F_i(t))dt.$$

It is easy to show that the hazard rate function of a series system is the sum of the hazard rates of its n components, namely

$$h(t) = \sum_{i=1}^{n} h_i(t). \tag{3.5}$$

The special case $n = 2$ is more widely studied in the literature, see for example, Jiang and Murthy (1997a). A more detailed discussion is given in Sect. 3.5.

For a detailed study of the n-fold competing risk models, see Chap. 9 of Murthy et al. (2004).

3.2.1 Common Shape Parameter Case

Nelson (1982, Chap. 5) considered the case where the shape parameters are the same for all subpopulations. He discussed parameter estimation based on a Weibull hazard plot. Since the shape parameters are the same, this model can be reduced to a simple univariate Weibull distribution.

3.3 *n*-fold Multiplicative Model

The n-fold multiplicative model also known as the complementary risk model in the literature, has the distribution function given by

$$F(t) = \prod_{i=1}^{n} F_i(t). \tag{3.6}$$

It is now obvious how the name 'multiplicative model' has arrived. The readers are referred to Chap. 10 of Murthy et al. (2004) for details model structure, distribution function, the shapes of the hazard rate function, etc.

If all the n component lifetimes are independent and identically distributed as a Weibull distribution, then (3.6) is equivalent to the exponentiated Weibull distribution given in Sect. 2.10. It is easy to see that (3.6) corresponds to the distribution of a system of n independent components that are arranged in parallel.

The multiplicative model involving two Weibull distribution was studied in detail by Jiang and Murthy (1997e). It was shown that there are only four possible shapes of the hazard rate function: (i) decreasing, (ii) increasing (iii) UBT or (iv) MBT depending on the values of the scale and shape parameters of the two Weibull variables. It is interesting to note that, like the case of mixture, a BT shaped hazard rate cannot be achieved under a multiplicative Weibull model.

3.4 *n*-fold Sectional Model

The distribution function in a n-fold sectional model has n segments joined together as follows:

$$F(t) = \begin{cases} k_1 F_1(t), & 0 \le t \le t_1, \\ 1 - k_2 \overline{F}_2(t), & t_1 < t \le t_2, \\ \cdots\cdots & \\ 1 - k_n \overline{F}_n(t), & t > t_{n-1}. \end{cases} \tag{3.7}$$

where the sub-populations $F_i(t)$ are the two- or three-parameter Weibull distributions and the t_i's (called partition points) are an increasing sequence. Jiang and Murthy (1997b) and Jiang et al. (1999a) considered the case where $n = 2$ in detail whereas Jiang and Murthy (1997c) considered the case where $n = 3$. Both cases can achieve a bathtub shaped hazard rate; the former has a 'V' shape bathtub whereas the latter can attain a flat bottom provided $\alpha_2 = 1$ where α_i is the shape parameter of the ith Weibull distribution. In the case $n = 3$, Jiang and Murthy (1997c) reported that the hazard rate can be one of twenty different shapes which can be classified into five types—(i) I (ii) D (iii) BT (iv) UBT and (v) roller-coaster shape. Thus, the sectional models involving three Weibull distributions can be used to model a variety of reliability data.

3.4.1 Sectional Weibull Hazard Rate Models

Instead of a Weibull sectional model involving $F(t)$ or $\overline{F}(t)$ as in (3.7), one can construct a sectional Weibull hazard rate model involving two or more truncated Weibull hazard rate functions, see for example, Ananda and Singh (1993), Mukherjee, and Roy (1993), Griffith (1982), and Kunitz (1989).

In what follows we restrict ourselves to discuss only the cases involving two Weibull or generalized Weibull populations.

3.5 Additive Weibull Model

Earlier, Xie and Lai (1996) proposed a generalized Weibull model which they called the additive Weibull model in view of its additive property of the two hazard rate functions.

The additive Weibull model is effectively a twofold competing risks model involving two Weibull distributions as studied later by Jiang and Murthy (1997d).

The reliability function has a very simple form and is now given below:

$$\overline{F}(t) = \exp\left\{-(t/\beta_1)^{\alpha_1} - (t/\beta_2)^{\alpha_2}\right\}, \ \alpha_1, \alpha_2, \beta_1, \beta_2 > 0. \tag{3.8}$$

It is easy to see that the mean of the additive Weibull model

$$\mu = \int_0^\infty \overline{F}(t)dt = \int_0^\infty \exp\left\{-(t/\beta_1)^{\alpha_1} - (t/\beta_2)^{\alpha_2}\right\} dt$$

does not have a closed form.

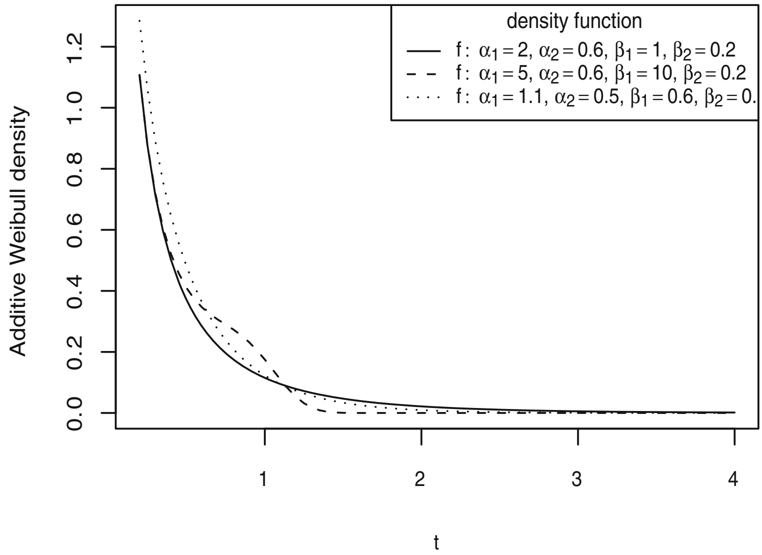

Fig. 3.1 Additive Weibull density functions

3.5.1 Density Function

The density function is simply given by

$$f(t) = f_1(t)\overline{F}_2(t) + f_2(t)\overline{F}_1(t)$$

where $\overline{F}_i(t) = \exp\{-(t/\beta_i)^{\alpha_i}\}$ and $f_i(t) = (\alpha_i/\beta_i)(t/\beta_i)^{\alpha_i - 1}\exp\{-(t/\beta_i)_i^{\alpha}\}$.

The densities with various parameter values are plotted in Fig. 3.1.

3.5.2 Hazard Rate Function

The hazard rate function of this additive model also has very simple expression:

$$h(t) = (\alpha_1/\beta_1)(t/\beta_1)^{\alpha_1 - 1} + (\alpha_2/\beta_2)(t/\beta_2)^{\alpha_2 - 1}, \ t \geq 0. \tag{3.9}$$

By inspection, it is easy to see that F is IFR if both shape parameters are greater than 1, i.e., if $\alpha_1 > 1$ and $\alpha_2 > 1$; and DFR if $\alpha_1 < 1$ and $\alpha_2 < 1$.

For $\alpha_1 < 1$ and $\alpha_2 > 1$, $h(t)$ then has a bathtub shape. This is because the second term in the preceding equation which dominates for small t is decreasing. For large t, the first term of $h(t)$ dominates and is an increasing function of t. An important feature of this model is that $h(t)$, with an appropriate choice of parameter

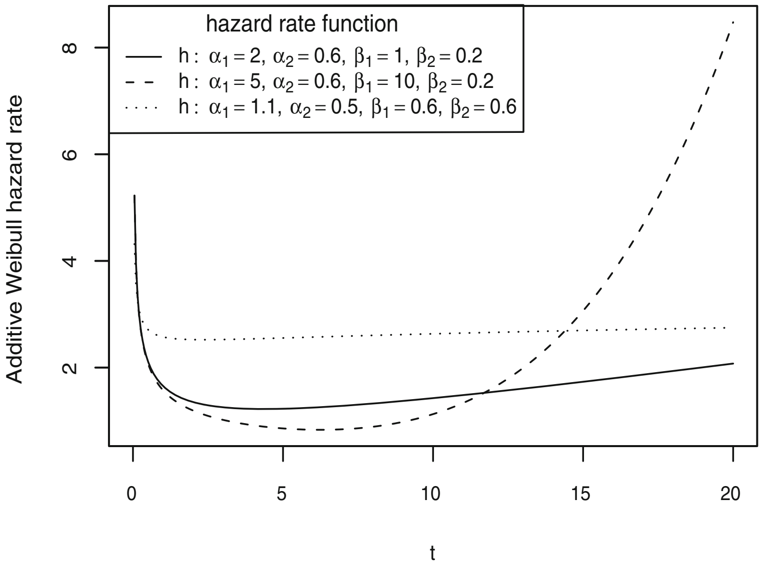

Fig. 3.2 Additive Weibull hazard rate functions

values, can achieve a fairly flat middle part of the life phase so it is able to model a population with a stable useful period of a life span. Thus, the model establishes itself an important place in the reliability literature.

See the plots in Fig. 3.2 given on the next page:

There is however, an undesirable aspect of this model which yields rather low values of $h(t)$ in the middle period of the lifetime. This can be rectified by adding a 'lift' factor h_0 to $h(t)$ as was proposed in Lai et al. (2004). See also Bebbington et al. (2008). The resulting new system corresponds to a series system of two Weibull components augmented by an exponential component with scale parameter h_0. This additional component is arranged in series along with the first two components. Since the exponential is a special case of the Weibull distribution, the modified additive model corresponds to a threefold competing risks model involving Weibull distributions.

It is obvious that $h(t)$ cannot achieve an UBT shape.

3.5.3 Special Case: The Reduced Model

In order to reduce the number of parameters, we construct a reduced model by letting $\alpha_1 = \alpha =, \alpha_2 = 1/\alpha, \beta_1 = \beta_2 = \beta$ in (3.8) and (3.9). The hazard rate of the reduced model has only two parameters:

$$h(t) = \frac{\alpha}{\beta}(t/\beta)^{\alpha-1} + \frac{1}{\alpha\beta}(t/\beta)^{1/\alpha-1},\ t \geq 0. \tag{3.10}$$

To enhance its applicability, a 'lift' parameter h_0 may be added to give

$$h(t) = \frac{\alpha}{\beta}(t/\beta)^{\alpha-1} + \frac{1}{\alpha\beta}(t/\beta)^{1/\alpha-1} + h_0,\ h_0 > 0, t \geq 0. \tag{3.11}$$

Xie and Lai (1996) showed that the simplified model is also able to give rise to a bathtub shaped hazard rate.

Stable period: Bebbington et al. (2006) proposed computationally tractable formal mathematical definitions for the 'useful period' of lifetime distributions with bathtub shaped hazard rate functions. Detailed analysis of the reduced additive Weibull hazard rate function illustrates its utility for identifying such useful periods.

We note that the whereabouts of the useful period is useful information in determining an optimal burn-in time. Burn-in is a widely used engineering protocol to weed out weak items before the product is released for field use. see Bebbington et al. (2005) for a discussion.

3.5.4 Probability Plots

Xie and Lai (1996) proposed a graphical method for estimating the parameters of (3.8) using the following two equations assuming $\alpha_1 > 1$ and $\alpha_2 < 1$:

$$\overline{F}(t) \approx \exp\left\{-(t/\beta_2)^{\alpha_2}\right\} \text{ as } t \to \infty$$

and

$$\overline{F}(t) \approx \exp\left\{-(t/\beta_1)^{\alpha_1}\right\} \text{ as } t \to 0.$$

Taking natural logarithm of the last two equations, we have

$$\ln[-\ln(\overline{F}(t)] \approx \alpha_2 \ln(t) + \alpha_2 \ln(\beta_2), \text{ for small } t \tag{3.12}$$

and

$$\ln[-\ln(\overline{F}(t)] \approx \alpha_1 \ln(t) + \alpha_1 \ln(\beta_1),\ \text{ for large } t. \tag{3.13}$$

On this probability plot, Eqs. (3.12) and (3.13) are straight lines with α_1 and α_2 as respective slopes, and $\alpha_2 \ln(\beta_2)$ and $\alpha_1 \ln(\beta_1)$ as intercepts respectively.

To implement this plot, the failure times are ranked in the ascending order and then split them into two subsets. The first subset will be used for Eq. (3.12) and then the second for Eq. (3.13). The parameters can also be estimated from this generalized Weibull plot and these estimates may also be used as the initial values for other formal methods of estimate.

The above probability plot can be simplified if we reparameterize the model as

$$\overline{F}(t) = \exp\left\{-\lambda_1 t^{\alpha_1} - \lambda_2 t^{\alpha_2}\right\}, \tag{3.14}$$

where $\lambda_1 = 1/\beta_1^{\alpha_1}$ and $\lambda_2 = 1/\beta_2^{\alpha_2}$ as was given in Xie and Lai (1996) when they implemented the probability plot.

3.5.5 Estimation

The initial estimates of the parameters can be achieved using Weibull-type probability plots as demonstrated by Xie and Lai (1996) and Usgaonkar and Mariappan (2009).

The maximum likelihood estimation of the parameters of the additive Weibull models can be implemented via Matlab as in Bebbington et al. (2010). Kanie and Nonaka (1985) presented an analytical technique for estimating the two shape parameters with $\alpha_1 \neq \alpha_2$ based on two cumulative hazard functions. For other methods see the discussion given in Sect. 9.2.3 of Murthy et al. (2004).

3.5.6 Mean Residual Life

The mean residual life of the model is plotted in Lai et al. (2004) when the hazard rate has a bathtub shape.

Bebbington et al. (2006) defined and considered analogous 'stable periods' in the case of the corresponding upside-down bathtub shaped mean residual life functions. The useful period of the mean residual life $\mu(t)$ of the reduced additive Weibull model can be determined.

Bebbington et al. (2008) showed the mean residual life is reduced in the presence of a constant competing risk.

3.5.7 Applications

- Xie and Lai (1996) fitted the distribution to an actual set of failure time data collected during unit testing.
- Usgaonkar and Mariappan (2009) applied the additive Weibull model to three real reliability data sets.

3.6 Mixtures of Two Weibull Distributions

Jiang and Murthy (1995, 1998) have studied the the twofold mixture model in detail and their results were summarized in Chap. 8 of Murthy et al. (2004).

Let $\overline{F}_i(t) = \exp\{-(t/\beta_i)^{\alpha_i}\}$ be the survival function of the ith Weibull population with density function $f_i(t)$ and its corresponding hazard rate function $h_i(t), i = 1, 2$.

Then the distribution function of the mixture of two Weibull distributions is given by

$$F(t) = pF_1(t) + (1-p)F_2(t) \tag{3.15}$$

with $F_i(t), i = 1, 2$ are given above. As a result, the model is characterized by five parameters—the shape and scale parameters for the two subpopulations and the mixing proportion parameter p, where $0 < p < 1$.

The density and hazard rate functions of the mixture model are given by

$$f(t) = pf_1(t) + (1-p)f_2(t) \tag{3.16}$$

and

$$h(t) = \frac{p\overline{F}_1(t)}{p\overline{F}_1(t) + (1-p)\overline{F}_2(t)} h_1(t) + \frac{(1-p)\overline{F}_2(t)}{p\overline{F}_1(t) + (1-p)\overline{F}_2(t)} h_2(t). \tag{3.17}$$

The moments of the mixture model are simply the mixture of the component moments.

3.6.1 Shape of the Density Function

The shape of the density function $f(t)$ depends on the parameter values. Murthy et al. (2004) pointed out out that although the twofold mixture model is characterized by five parameters, the shape of the density function is only a function of the two shape parameters, the ratio of the two scale parameters, and the mixing proportion.

The possible shapes are given in Chap. 8 of Murthy et al. (2004):

- Decreasing
- Unimodal
- Decreasing followed by unimodal
- Bimodal

Figure 3.3 gives some density plots of the Weibull mixtures.

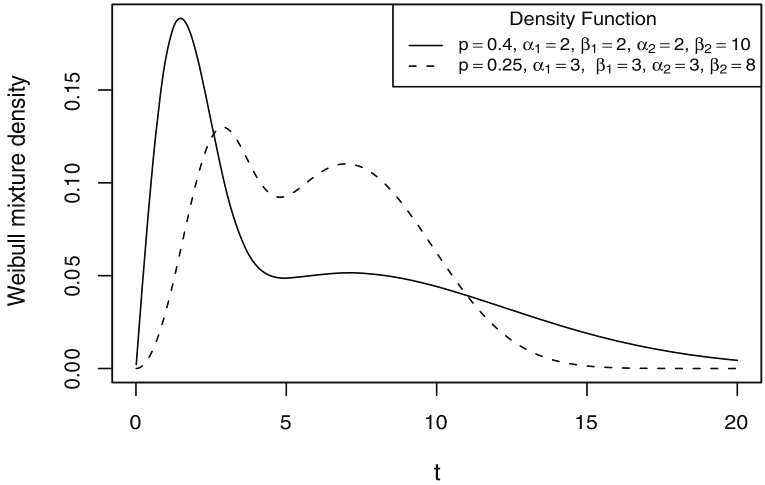

Fig. 3.3 Weibull mixture density

3.6.2 Shape of the Hazard Rate Function

The shape of the hazard rate depends on the parameter values. Chap. 8 of Murthy et al. (2004) also list the possible shapes which include, but not limited to the following:

- Decreasing
- Increasing
- Unimodal followed by increasing (*N* shape)
- Decreasing followed by unimodal (reflected *N* shape)
- Bimodal followed by increasing

The next figure (Fig. 3.4) gives two of the many possible hazard rate shapes of the Weibull mixture model.

We also note that a mixture of two Weibull distribution cannot give rise to a BT shaped hazard rate. See, for example, Glaser (1980) for details.

Wondmagegnehu (2004) has fine-tuned the results of Jiang and Murthy (1998) with a definite discriminant between the two classes based on the value of the mixing proportion p with $p_1 = p$ and $p_2 = 1 - p$ as given in (3.15). In the case where the two Weibull shape parameters are different, i.e., $\alpha_1 \neq \alpha_2$ and both $\alpha_1, \alpha_2 > 1$, he used several examples to illustrate all possible shapes that the mixture hazard rate can encounter.

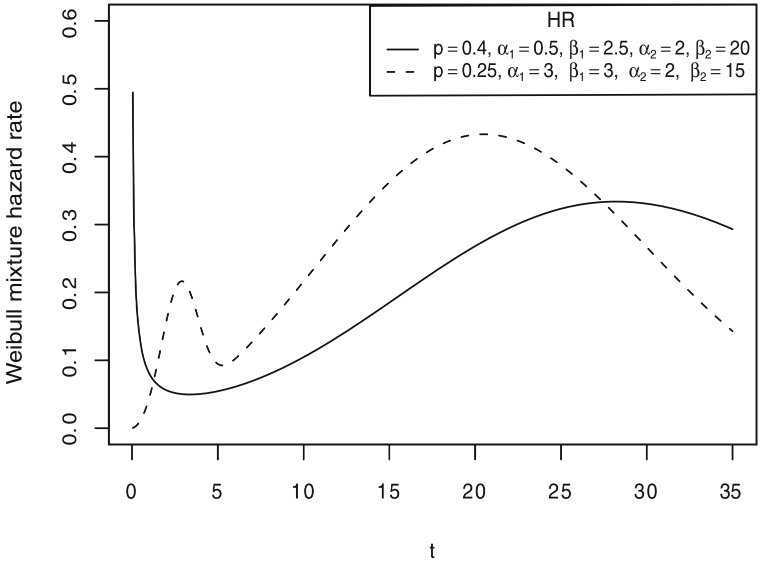

Fig. 3.4 Weibull mixture HR

3.6.3 Generalized Mixtures

Generalized mixtures are mixtures that represent the distribution function as a linear combination of the distribution functions of components in the mixture, and the coefficients may be any real number. By contrast, the coefficients in usual mixtures are proportions that lie between 0 and 1. The generalized mixtures appear in characterization problems, in inference procedures and in some representations of systems. For example, the distribution function of the lifetime of a parallel system can be written as a generalized mixture of the distributions of series system lifetimes (see, e.g. Navarro et al. 2007). For a comprehensive review, see Navarro et al. (2009).

For Weibull and inverse Weibull mixture models allowing negative weights, see Jiang et al. (1999b).

3.6.4 Other Properties

Other properties of the twofold Weibull mixture such as the Weibull probability plots, estimation of parameters, etc, see Chap. 8 of Murthy et al. (2004).

3.7 Model Involving Two or More Inverse Weibull Distributions

Recall in Chap. 2, the inverse Weibull is obtained by inverting the Weibull random variable. Its distribution function is given by

$$F(t) = \exp\left\{-(\beta/t)^{\alpha}\right\},\ \alpha, \beta, t > 0. \tag{3.18}$$

Jiang et al. (2001) have studied three models (mixture, competing risk and multiplicative) involving two inverse Weibull distributions.

3.7.1 Mixture of Two Inverse Weibull Distributions

The distribution function is given by

$$F(t) = pF_1(t) + (1-p)F_2(t),\ 0 < p < 1, \tag{3.19}$$

where $F_1(t)$ and $F_2(t)$ are the two basic inverse Weibull distributions of the form given by (3.18) and α_i and β_i, $(i = 1, 2)$ are the shape and scale parameters of population i, respectively.

The density function of the mixture is simply given by

$$f(t) = pf_1(t) + (1-p)f_2(t) \tag{3.20}$$

where $f_i(t) = \alpha_i {\beta_i}^{\alpha_i} t^{-\alpha_i - 1} e^{-(\beta_i/t)^{\alpha_i}}$, $i = 1, 2$.

Density Function

The density function for the mixture model can be one of the following two shapes: (i) unimodal and (ii) bimodal. The figure on the next page (Fig. 3.5) gives two density plots; one each of the two possible shapes.

Hazard Rate Function

The hazard rate function has a similar shape as its density function, namely it is either unimodal or bimodal as illustrated by Fig. 3.6 given on the next page. This is all together expected since the hazard rate of the two sub-populations are unimodal (UBT).

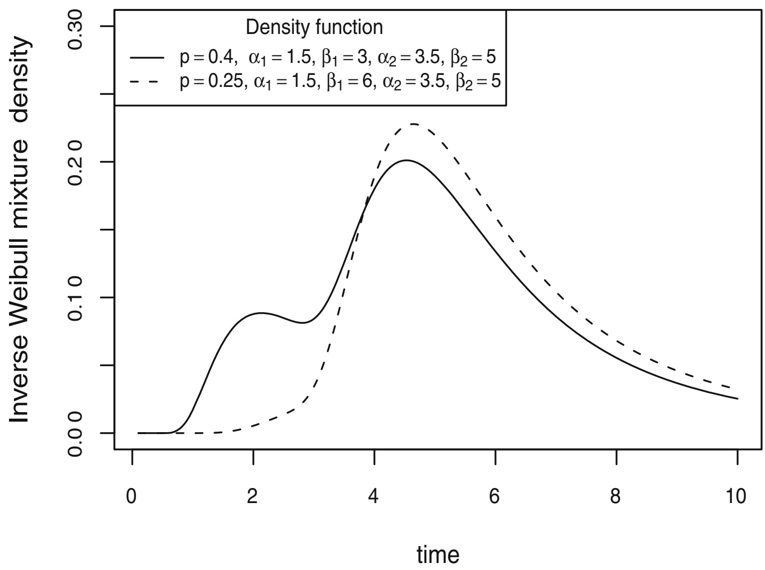

Fig. 3.5 Inverse Weibull mixture density functions

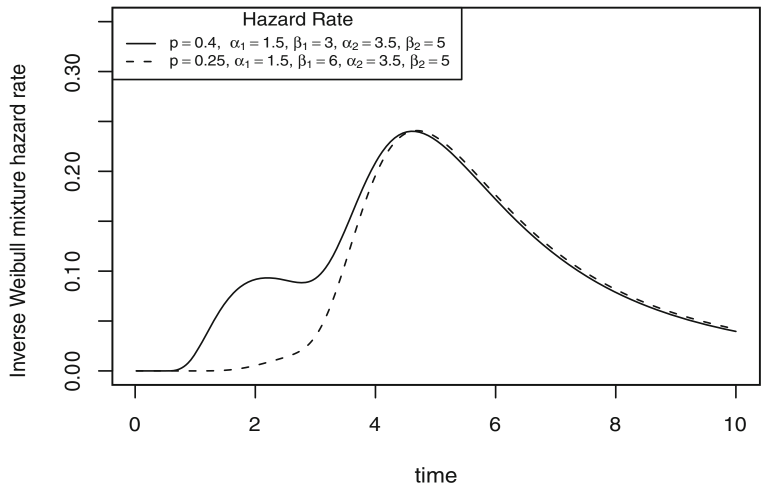

Fig. 3.6 Inverse Weibull mixture hazard rate functions

Inverse Weibull Probability Plots

A detailed study of the inverse Weibull probability plots and estimation of parameter is given in Jiang et al. (2001).

3.7.2 Competing Risk Model

The model is given by

$$\overline{F}(t) = \overline{F}_1(t)\overline{F}_2(t) \tag{3.21}$$

or equivalently,

$$F(t) = F_1(t) + F_2(t) - F_1(t)F_2(t).$$

where the two subpopulations are inverse Weibull distributions of the form given by (3.18). The density function of the model is given by

$$f(t) = f_1(t)\overline{F}_2(t) + f_2(t)\overline{F}_1(t). \tag{3.22}$$

The hazard rate of the competing risk model has a simple form, that is,

$$h(t) = h_1(t) + h_2(t) \tag{3.23}$$

where h_i is the hazard rate of the ith population, $i = 1, 2$.

Interesting enough, the shapes of the hazard rates in a competing risks model are almost the same as those for the mixture model, i.e., either $h(t)$ is UBT or bimodal.

3.7.3 Multiplicative Model

The model is given by

$$F(t) = F_1(t)F_2(t) \tag{3.24}$$

where the two subpopulations are inverse Weibull distributions of the form given by (3.18).

$$f(t) = f_1(t)F_1(t) + f_2(t)F_2(t). \tag{3.25}$$

Although one would expect the density function of the multiplicative model to be either unimodal or bimodal, an examination of the density plots for a range of parameter values, Jiang et al. (2001) found that the hazard rate function is always UBT, so is its density function.

3.8 Mixtures of Two Generalized Weibull Distributions

In this section, we consider mixtures of two generalized Weibull distributions which could be very useful in modeling human mortality data. It is obvious that a host of distributions can be constructed through mixtures of this kind.

For illustrative purpose, let F_1 and F_2 denote the flexible Weibull distribution and the reduced additive Weibull distribution, respectively. The former is discussed in Sect. 2.20 whereas the latter is specified by (3.8).

Bebbington et al. (2007) showed, that a mixture, assigning mortality to exogenous or endogenous causes, using the reduced additive and flexible Weibull distributions, models well human mortality over the entire life span.

Specifically, the two survival functions are

$$\overline{F}_1(t) = \exp\{-e^{\alpha t - \beta/t} - \gamma t\} \tag{3.26}$$

and

$$\overline{F}_2(t) = \exp\{-(at)^b - (at)^{1/b} - ct\}. \tag{3.27}$$

The components of the mixture distribution are asymptotically consistent with the reliability and biological theories of ageing. The relative simplicity of the mixing distribution makes feasible a technique where the curvature functions of the corresponding survival and hazard rate functions are used to identify the beginning and the end of various life phases, such as infant mortality, the end of the force of natural selection, and the late life mortality deceleration. The results are illustrated with a comparative analysis of Canadian and Indonesian mortality data.

3.9 Composition of Two Cumulative Weibull Hazard Rates

The so called ‘ ’ model proposed by Stoyan et al. (2013) is essentially the sectional Weibull hazard rate model discussed in Sect. 3.4.1. Instead of using the hazard rates directly, the model is specified through a cumulative hazard rate defined by

$$H(t) = H_1(t) + H_2(t) \tag{3.28}$$

where

$$H_1(t) = \begin{cases} \left(\frac{t}{\beta_1}\right)^{\alpha_1} & \text{for } t \le t_1 \\ \left(\frac{t_1}{\beta_1}\right)^{\alpha_1} & \text{otherwise} \end{cases} \tag{3.29}$$

and

$$H_2(t) = \begin{cases} 0 & \text{for } t \le t_2 \\ \left(\frac{t-t_2}{\beta_2}\right)^{\alpha_2} & \text{otherwise} \end{cases} \tag{3.30}$$

The component Weibull hazard rates are right truncated as:

$$h_1(t) = \begin{cases} \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha_1 - 1} & \text{for } t \le t_1 \\ 0 & \text{otherwise} \end{cases} \tag{3.31}$$

and

$$h_2(t) = \begin{cases} 0 & \text{for } t \le t_2 \\ \frac{\alpha_2}{\beta_2}\left(\frac{t - t_2}{\beta_2}\right)^{\alpha_2 - 1} & \text{otherwise.} \end{cases} \tag{3.32}$$

Thus the cumulative hazard rate function is increasing on $[0, t_1]$, constant on $[t_1, t_2]$ and again increasing for $t > t_2$.

The Weibull probability plot is constructed by Stoyan et al. (2013) and the least squares method is used to estimate the model parameters. The model was applied as a flaw distribution to fit the Weibull's porcelain data.

3.10 Relative Ageing of Two 2-Parameter Weibull Distributions

Although this section does not discuss a generalized Weibull distribution, it nevertheless does fall within the scope of this chapter.

We say that X ages faster than Y if the ratio of the hazard rate of X over the hazard rate of Y is an increasing function of t. The concept of relative ageing is a particular form of partial ordering between two lifetime random variables. This concept was defined in Sect. 2.10 of Lai and Xie (2006).

Suppose we have two independent Weibull random variables X and Y with distribution functions $F(x)$ and $G(y)$, respectively, given by

$$F(x) = 1 - \exp\left\{-(x/\beta_2)^{\alpha_2}\right\}, \quad G(y) = 1 - \exp\left\{-(y/\beta_1)^{\alpha_1}\right\}. \tag{3.33}$$

This ratio of the hazard rate of X to the hazard rate of Y is given by

$$\frac{\beta_1 \alpha_2}{\beta_2 \alpha_1} \times \frac{\beta_1^{\alpha_1 - 1}}{\beta_2^{\alpha_2 - 1}} t^{\alpha_2 - \alpha_1} \tag{3.34}$$

which is an increasing function of t if $\alpha_2 > \alpha_1$.

Suppose $E(X) = E(Y)$, i.e., $\beta_2 \Gamma(1 + 1/\alpha_2) = \beta_1 \Gamma(1 + 1/\alpha_1)$. Lai and Xie (2003) have shown that $\mathrm{var}(X) \le \mathrm{var}(Y)$. Two Weibull distributions can also be partially ordered with respect to the shape parameter in 'convex ordering' in the sense as in Barlow and Proschan (1981, p. 105) (See also Sect 10.3.2 of Lai and Xie (2006) for a brief definition).

References

Ananda, M. M., & Singh, A. K. (1993). Estimation of a composite hazard rate model. *Microelectronics and Reliability*, *33*, 2013–2019.

Barlow, R. E., & Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing*. Silver Spring: To Begin With.

Bebbington, M., Lai, C. D., & Zitikis, R. (2005). Estimating the turning points of failure rate and mean residual life functions. *Communication of Dependability and Quality Management: An International Journal*, *8*(3), 11–15.

Bebbington, M., Lai, C. D., & Zitikis, R. (2007). Modeling human mortality using mixtures of bathtub shaped failure distributions. *Journal of Theoretical Biology*, *245*, 528–538.

Bebbington, M., Lai, C. D., Wellington, M., & Zitikis, R. (2010). Discrete bathtub distributions: Mind the step function!, advanced reliability modeling IV: Beyond traditional reliability and maintainability approaches. In S. Chukova, J. Haywood & T. Dohi (Eds), *Proceedings 4th Asia-Pacific International Symposium (APARM2010)* (pp. 57–64). Taiwan: McGraw Hill.

Bebbington, M., Lai, C. D., & Zitikis, R. (2006). Useful periods for lifetime distributions with bathtub shaped hazard rate functions. *IEEE Transactions on Reliability*, *55*(2), 245–251.

Bebbington, M., Lai, C. D., & Zitikis, R. (2008). Reduction in mean residual life in the presence of a constant competing risk. *Applied Stochastic Models in Business and Industry*, *24*, 51–63.

Bucar, T., Nagode, M., & Fajdiga, M. (2004). Reliability approximation using finite Weibull mixture distributions. *Reliability Engineering and System Safety*, *84*, 241–251.

Glaser, R. E. (1980). Bathtub and related failure rate characterizations. *Journal of the American Statistical Association*, *75*, 667–672.

Griffith, W. S. (1982). Representation of distributions having monotone or bathtub-shaped failure rates. *IEEE Transactions on Reliability*, *31*, 95–96.

Gurland, J., & Sethuraman, J. (1995). How pooling failure data may reverse increasing failure rates. *Journal of the American Statistical Association*, *90*, 1416–1423.

Jiang, R., & Murthy, D. N. P. (1995). Modeling failure data by mixture of two Weibull distributions. *IEEE Transactions on Reliability*, *44*, 477–488.

Jiang, R., & Murthy, D. N. P. (1997a). Parametric study of competing risk model involving two Weibull distributions. *International Journal of Reliability, Quality and Safety Engineering*, *4*(1), 17–34.

Jiang, R., & Murthy, D. N. P. (1997b). Parametric study of sectional models involving two Weibull distributions. *Reliability Engineering and System Safety*, *56*, 151–159.

Jiang, R., & Murthy, D. N. P. (1997c). Two sectional models involving three Weibull distributions. *Quality and Reliability Engineering International*, *13*, 83–96.

Jiang, R., & Murthy, D. N. P. (1997d). Parametric study of competing risk model involving two Weibull distributions. *International Journal of Reliability, Quality and Safety Engineering*, *4*(1), 17–34.

Jiang, R., & Murthy, D. N. P. (1997e). Parametric study of multiplicative model involving two Weibull distributions. *Reliability Engineering and System Safety*, *55*(1), 217–226.

Jiang, R., & Murthy, D. N. P. (1998). Mixture of Weibull distributions—parametric characterization of failure rate function. *Applied Stochastic Models and Data Analysis*, *14*, 47–65.

Jiang, R., Zuo, M. J., & Li, H. X. (1999a). Two sectional models involving two Weibull distributions. *International Journal of Reliability, Quality and Safety Engineering*, *6*, 103–122.

Jiang, R., Zuo, M. J., & Li, H. X. (1999b). Weibull and inverse Weibull mixture models allowing negative weights. *Reliability Engineering and System Safety*, *64*, 227–234.

Jiang, R., Murthy, D. N. P., & Ji, P. (2001). Models involving two inverse Weibull distributions. *Reliability Engineering and System Safety*, *73*, 73–81.

Kanie, H., & Nonaka, Y. (1985). Estimation of Weibull shape-parameters for two independent competing risks. *IEEE Transactions on Reliability*, *34*, 53–56.

Kunitz, H. (1989). A new class of bathtub-shaped hazard rates and its application in a comparison of two test-statistics. *IEEE Transactions on Reliability*, *38*, 351–353.

Lai, C. D., & Xie, M. (2006). *Stochastic Ageing and Dependence for Reliability*. New York: Springer.
Lai, C. D., & Xie, M. (2003). Relative ageing for two parallel systems and related problems. *Mathematical and Computer Modelling*, *38*, 1339–1345.
Lai, C. D., Zhang, L. Y., & Xie, M. (2004). Mean residual life and other properties of Weibull related bathtub shape failure rate distributions. *International Journal of Reliability, Quality and Safety Engineering*, *11*(2), 113–132.
Mukherjee, S. P., & Roy, D. (1993). A versatile failure time distribution. *Microelectronics and Reliability*, *33*, 1053–1056.
Murthy, D. N. P., Xie, M., & Jiang, R. (2004). *Weibull Models*. Hoboken, New Jersey:Wiley.
Navarro, J., Ruiz, J. M., & Sandoval, C. J. (2007). Properties of coherent systems with dependent components. *Communications in Statistics-Theory and Methods*, *36*, 1–17.
Navarro, J., Guillam, A., & Ruiz, M. (2009). Generalized mixtures in reliability modelling: Applications to the construction of bathtub shaped hazard models and the study of systems. *Stochastic Models in Business and Industry*, *25*, 323–337.
Nelson, W. (1982). *Applied Life Data Analysis*. New York: Wiley.
Stoyan, D., Funke, C., & Rasche, S. (2013). Back and beyond Weibull—the hazard function approach. *Computational Materials Science*, *68*, 181–188.
Titteringto, M., Smith, A. F. M., & Markov, U. E. (1985). *Statistical Analysis of Finite Mixture Distributions*. New York: Wiley.
Usgaonkar, S. G. G., & Mariappan, V. (2009). Additive Weibull model for reliability analysis. *International of Performability Engineering*, *5*(3), 243–250.
Wondmagegnehu, E. T. (2004). On the behavior and shape of mixture failure rates from a family of IFR Weibull distributions. *Naval Research Logistics*, *51*, 491–500.
Xie, M., & Lai, C. D. (1996). Reliability analysis using additive Weibull model with bath-tubed shape failure rate function. *Reliability Engineering and System Safety*, *52*, 87–93.

Chapter 4
Discrete Weibull Distributions and Their Generalizations

4.1 Introduction

In lifetime modeling, it is common to treat failure data as being continuous, implying some degree of precision in measurement. Too often in practice, however, failures are either noted at regular inspection intervals, occur in a discrete process or are simply recorded in bins. In life testing experiments, it is sometimes impossible or inconvenient to measure the life length of a device, on a continuous scale. For example, in the case of an on/off- switching device, the lifetime of the switch is a discrete random variable. In many practical situations, reliability data are measured in terms of the number of runs, cycles, or shocks the device sustains before it fails. In survival analysis, we may record the number of days of survival for lung cancer patients since therapy, or the times from remission to relapse are also usually recorded in number of days. In this context, the geometric and negative binomial distributions are known discrete alternatives for the exponential and gamma distributions, respectively. It is well known that these discrete distributions have monotonic hazard rate functions and thus they are unsuitable for some situations.

Fortunately, many continuous distributions can be 'discretized'. As mentioned above, the geometric and the negative binomial can be derived by 'discretizing' the exponential and gamma, respectively. There are three discrete versions of the continuous Weibull distribution (Murthy et al. 2004, pp. 33–34). The discrete versions of the normal and Rayleigh distributions were also proposed by Roy (2003) and (2004) respectively. Recently, Burr XII and Pareto distributions were also discretized by Krishna and Pundir (2009).

In this chapter, we limit ourselves to discuss the discrete Weibull-type distributions.

4.2 Discrete Distribution

Let the random variable X with support $\mathcal{N}^{+} = \{1, 2, \ldots\}$ be the discrete lifetime of a component and denote by $p(k)$ the probability of a failure occurs at time k, i.e.,

C.-D. Lai, *Generalized Weibull Distributions*, SpringerBriefs in Statistics,
DOI: 10.1007/978-3-642-39106-4_4, © The Author(s) 2014

$$p(k) = \Pr\{X = k\},\ k = 1, 2, \ldots \tag{4.1}$$

The above definition can be extended to a random variable Y with support $\mathcal{N} = \{0, 1, 2, \ldots\}$ by a simple translation $Y = X - 1$.

A convention is given for this chapter to avoid any possible confusion concerning the nature of a function. Whenever a function, say g, is defined on $\mathcal{N}$ or $\mathcal{N}^+$, its argument is denoted by i, thus writing $g(i)$. If the function g is defined on a continuous scale (e.g., $[0, \infty)$ or $[1, \infty)$), then its argument is denoted by t, thus writing $g(t)$.

As an exception to the above rule, we use a common notation $p(i)$ to denote the probability mass function of the discretized version of the continuous random variable having density function $f(t)$.

The reliability (survival) function that corresponds to X is given by

$$\overline{F}(k) = \Pr\{X > k\} = \sum_{j=k+1}^{\infty} p(j),\ \ k = 1, 2, \ldots, \tag{4.2}$$

noting that $\overline{F}(0) = 1$. The survival function may be defined over the whole non-negative real line by

$$\overline{F}(t) = \overline{F}(k) \quad \text{for } 0 \le k \le t < k + 1, \tag{4.3}$$

where $t \in [0, \infty)$. Here $\overline{F}(t)$ is referred to the right continuous survival function.

The hazard (failure) rate function $h(k)$ is defined as

$$h(k) = \Pr(X = k \mid X \ge k) = \frac{\Pr(X = k)}{\Pr(X \ge k)} = \frac{p(k)}{\overline{F}(k-1)}, \tag{4.4}$$

provided $\Pr(X \ge i) > 0$. The above equation may be expressed as

$$h(k) = \frac{\overline{F}(k-1) - \overline{F}(k)}{\overline{F}(k-1)}. \tag{4.5}$$

The preceding Eq. (4.5) may be considered as the ‘classical discrete hazard rate function’. Although this definition is widely used in the literature, there are a few problems associate with this definition. We will discuss this issue in some details and provide an alternative definition of discrete hazard rate over the next two subsections.

Shaked et al. (1995) gave the necessary and sufficient conditions for a sequence $\{h(k), k \ge 1\}$ to be a failure (hazard) rate:

(a) For all $k < m$, $h(k) < 1$ and $h(m) = 1$. The distribution is defined over $\{1, 2, \ldots, m\}$, or
(b) For all $k \in \mathcal{N}^+ = \{1, 2, \ldots\}$, $0 \le h(k) \le 1$ and $\sum_{i=1}^{\infty} h(i) = \infty$. The distribution is defined over $k \in \mathcal{N}^+$ in this case.

Further Relationships

It follows from (4.5) that

$$\overline{F}(k) = \prod_{1 \le i \le k} (1 - h(i))$$
$$= (1 - h(1))\,(1 - h(2)) \cdots (1 - h(k))\,. \tag{4.6}$$

Moreover, it can be shown easily that

$$p(k) = \overline{F}(k-1)h(k) = h(k)\,(1 - h(1))\,(1 - h(1)) \cdots (1 - h(k-1))\,. \tag{4.7}$$

Ageing classifications of discrete lifetimes are given in Roy and Gupta (1992). Gupta et al. (1997) discussed monotonic properties of discrete failure (hazard) rates.

4.2.1 *Some Problems of Usual ('Classical') Definition of Discrete Hazard Rate*

The definition of the failure rate $h(k)$ given in (4.4) is different from its continuous counterpart in several aspects. It is the conditional probability of failure at $X = k$ given the 'device' has not failed by $k - 1$ and thus $h(k) \le 1$. In contrast, it is the product $h(t)\Delta$ (Δ small) that is approximately the probability of immediate failure conditional on $X > t$ for the continuous case. Hence $h(t)$ of a continuous lifetime can be unbounded in some situation but $h(k)$ of a discrete lifetime model is always finite.

The hazard rate defined by (4.4) is not additive for a competing risk model which is equivalent to a system with components arranged in series.

In addition, the cumulative hazard function $H(k) = \sum_{i=1}^{k} h(i)$ is not equivalent to $-\log \overline{F}(k)$ as in the continuous case. Thus,

$$H(k) = \sum_{i=1}^{k} r(i) \neq -\log \overline{F}(k). \tag{4.8}$$

For a more detailed discussion on the problems associated with this definition, see for example, Sect. 6.10 of Lai and Xie (2006) or Xie et al. (2002).

4.2.2 *Alternative Definition of Discrete Hazard Rate*

Because of the problems associated with the classical definition of the failure rate $h(k)$, several authors have presented an alternative definition of a discrete failure rate function, see Sect. 6.11 of Lai and Xie (2006) and the references therein. In order to make a distinction, the alternative failure rate function will be denoted by $h_1(k)$.

$$h_1(k) = \log\left\{\frac{\overline{F}(k-1)}{\overline{F}(k)}\right\}, \quad k = 1, 2, .. \tag{4.9}$$

Equation (4.9) may be considered as a 'nonclassical' discrete hazard rate function.

Earlier, we have established that

$$-\log\overline{F}(k) \neq H(k), \;\; H(k) = h(1) + h(2) + \cdots + h(k).$$

However, if we alternatively define the cumulative hazard function as:

$$H_1(k) = h_1(1) + h_1(2) + \cdots + h_1(k). \tag{4.10}$$

Then it follows from (4.9) that

$$H_1(k) = -\log\overline{F}(k). \tag{4.11}$$

which has an analogous relation between the survival function and the cumulative hazard function as for the continuous case.

A battery of discrete distributions are used in Xie et al. (2002) to illustrate a 'non-classical' definition of the discrete hazard rate function, which we shall also investigate in the current paper, alongside the 'classical' definition of the discrete hazard rate function.

Relationships Between $h(k)$ and $h_1(k)$

Fortunately, there is a simple relationship between the two definitions of failure (hazard) rates $h(k)$ and $h_1(k)$::

$$\begin{aligned} h_1(k) &= -\log\left[\overline{F}(k)/\overline{F}(k-1)\right] \\ &= -\log\left\{\frac{\overline{F}(k-1) + \overline{F}(k) - \overline{F}(k-1)}{\overline{F}(k-1)}\right\} = \log[(1-h(k)] \end{aligned} \tag{4.12}$$

$$h(k) = 1 - e^{-h_1(k)}, \tag{4.13}$$

showing that the two concepts $h(k)$ and $h_1(k)$ have the same monotonic property, i.e., $h_1(k)$ is increasing (decreasing) if and only if $h(k)$ is increasing (decreasing).

4.3 Discrete Weibull Models

The distribution function, survival function and the hazard rate functions of the continuous Weibull distribution, are respectively given by

$$F(t) = 1 - \exp\{-\lambda t^{\alpha}\}, \tag{4.14}$$

$$\overline{F}(t) = \exp\{-\lambda t^{\alpha}\} \tag{4.15}$$

and

$$h(t) = \alpha\lambda e^{\alpha-1}. \tag{4.16}$$

Here the lifetime variable X can only assume non-negative integer values and this defines the support for $F(t)$. There are several versions of discretized Weibull distributions but we consider only three relatively well known discrete analogues of the Weibull distribution.

Essentially, one can construct a discrete Weibull distribution by

- discretizing $F(t)$ of the Weibull distribution, or
- discretizing the hazard rate function $h(t)$ of the Weibull distribution.

4.3.1 Model-1

This model was proposed by Nakagawa and Osaki (1975). Its distribution function is given by

$$F(k) = \begin{cases} 1 - q^{k^{\alpha}} & k = 0, 1, 2, 3 \ldots, \\ 0 & k < 0. \end{cases} \tag{4.17}$$

This was derived by discretizing the distribution function of the continuous Weibull distribution given in (4.14) by letting $t = k$ and $q = e^{-\lambda}$, we obtain (4.17).

It follows from (4.17) that

$$p(k) = q^{(k-1)^{\alpha}} - q^{k^{\alpha}}. \tag{4.18}$$

4.3.2 Model-2

This model was derived by Stein and Dattero (1984). The hazard rate function of the model is given by

$$h(k) = \begin{cases} ck^{\alpha-1} & k = 1, 2, \ldots, m, \\ 0 & k < 0. \end{cases} \tag{4.19}$$

with $\alpha > 0$ and $0 < c \leq 1$ and m is given by

$$m = \begin{cases} [c^{-\{1/(\alpha-1)\}}]^+ & \text{if } \alpha > 1, \\ \infty & \text{if } \alpha \leq 1; \end{cases} \tag{4.20}$$

and $[\ \ ]^+$ represents the integer part of the quantity inside the square brackets.

Thus, the distribution is obtained by discretizing the hazard rate function of the continuous Weibull distribution subject to the conditions given by Shaked et al. (1995).

The probability mass function is

$$p(k) = ck^{\alpha-1} \times \prod_{j=1}^{k-1}(1 - cj^{\alpha-1});\ \ k = 1, 2, \ldots, m. \tag{4.21}$$

4.3.3 Model-3

This model was proposed by Padgett and Spurrier (1985) with distribution function specified as

$$F(k) = 1 - \exp\left\{-\sum_{i=0}^{k} h(i)\right\} = 1 - \exp\left\{-\sum_{i=0}^{k} ci^{\alpha-1}\right\}, \quad c > 0, k = 0, 1, 2, \ldots \tag{4.22}$$

Note that the condition $h(k) \leq 1$ is no longer required in this case. The hazard rate $h(k) = ck^{\alpha-1}$ is in fact the alternative (nonclassical) discrete hazard rate $h_1(k)$ defined by (4.9).

4.3.4 Estimation of Parameters

The traditional methods (method of moments and the maximum likelihood) are not easy to implement when estimating the parameters of the discrete Weibull distributions.

Khan et al. (1989) discussed estimation of parameters of the type I (i.e., Model-1) discrete Weibull distribution (4.17) and proposed the method of proportions to estimate the parameters. They compared this method with the method of moments based on simulated observations and studied its basic properties. The authors stated that their method can be easily applied to the type II discrete Weibull model given by (4.19). Santos and Alves (2013) proposed a new estimate of the shape parameter.

4.3.5 Applications

The discrete Weibull (model-1) is applied by Santos and Alves (2013) to a financial data set dealing with durations between violations in a quantitative risk management environment.

4.4 Discrete Inverse Weibull Distribution

Recall, the inverse Weibull distribution is given by

$$F(t) = e^{-\lambda t^{-\alpha}},\ t > 0. \tag{4.23}$$

The inverse Weibull distribution has received considerable attention in the literature. Keller and Kamath (1982) studied the shapes of the density and failure (hazard) rate functions for the basic inverse model and Keller et al. (1985) used the model for the reliability analysis of commercial vehicle engines. Erto (1989) introduced further properties and identification of the model. Calabria and Pulcini (1989, 1990) dealt with parameter estimation of the model. Jiang and Murthy (1999) considered Weibull and inverse Weibull mixture models with negative weights. Also, Drapella (1993) and Jiang et al. (2001) introduced graphical plotting techniques, known as the inverse Weibull probability paper (IWPP) plot and the Weibull probability paper (WPP) plot to determine the suitability of the Weibull and the inverse Weibull models for fitting a given data set. They showed that if IWPP plot is roughly a straight line, then the inverse Weibull model may be used to fit the given data set. Similarly if WPP plot is roughly a straight line, then the Weibull model may be used to fit the data set concerned.

Drapella (1993) called (4.23) as the complementary Weibull distribution whereas both Bergstrom and Edin (1992) and Mudholkar and Kollia (1994) called it the reciprocal Weibull distribution. Bergstrom and Edin (1992) has found the distribution a good fit to an employment duration data set. A detailed study of the inverse Weibull distribution including its model structure, hazard plot, parameter estimation and application was given in Sect. 6.6 of Murthy and Djamaludin (2002). See also Sect. 2.6 of this monograph.

Aghababaei Jazi et al. (2010) proposed and studied a discrete version of the inverse Weibull distribution and the some of the findings are given in the following subsections.

4.4.1 Discrete Inverse Weibull Model

A discrete analogue of the continuous inverse Weibull is given by the cdf

$$F(k) = q^{k^{-\alpha}}, \quad k = 1, 2, \ldots. \tag{4.24}$$

and its corresponding probability function is

$$p(k) = \begin{cases} q, & k = 1 \\ q^{k^{-\alpha}} - q^{(k-1)^{-\alpha}}, & k = 2, 3, \ldots \end{cases} \tag{4.25}$$

where $0 < q < 1$ and $\alpha > 0$.

4.4.2 Derivation

It is easy to show that if X is a r.v. from distribution (4.23), then the distribution function of $[X] + 1$ is given by (4.24). It now follows that the proposed discrete distribution may also be obtained from a continuous Weibull random variable through the process of 'inversion' and then 'discretization'.

Note that in (4.24), $q = e^{-\lambda}$ where λ is the scale parameter that appears in (4.23).

4.4.3 Properties of Its Probability Function

Figure 4.1 gives the plots of the probability function $f(k)$ specified in (4.25) for (i) $q = 0.05$, $\alpha = 3$ and $q = 0.2$, $\alpha = 2$ (ii). The scale parameter q completely determines the probability function $f(k)$ at $k = 1$. When $\log q \geq$

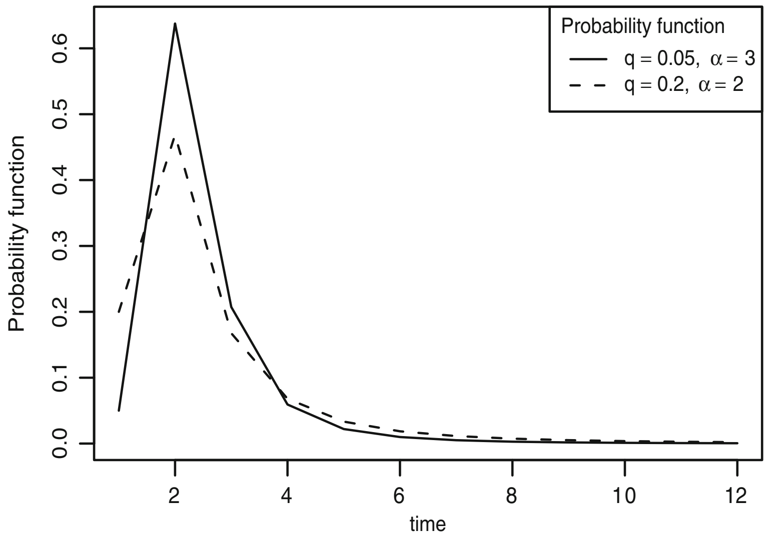

Fig. 4.1 Discrete inverse Weibull probability function

$\log(4.2)/(2^{-\alpha} - 1)$, $f(k)$ is monotone decreasing. Otherwise, it is no longer monotone decreasing but it is unimodal (with mode at 2). The shape parameter α is more influential on $f(k)$ than q after $k = 1$. When α becomes smaller, the tail of $f(k)$ becomes longer, i.e., the probability mass on smaller values of k are then shifted to the larger values of k.

Also the probability function $p(k)$ in (4.25) is not infinitely divisible because there is no positive probability mass at $k = 0$; and hence the corresponding random variable is not self-decomposable (see Steutel and van Harn 2004, for a proof).

4.4.4 Moments

The moments cannot be expressed in the closed form. The mean is given as

$$\mu = E(X) = \sum_{i=1}^{\infty} (1 - q^{i^{-\beta}}) \tag{4.26}$$

4.4.5 Inverse Weibull Probability Paper Plot

Drapella (1993) proposed the inverse Weibull transform

$$x = \log(t), \qquad y = -\log(-\ln(F(t)))$$

and called the plot y versus x the "inverse Weibull probability paper (IWPP) plot". He showed that IWPP plot is linear for the continuous inverse Weibull model.

Applying the same set of transformations to our discrete inverse Weibull model (4.24), we then obtain a straight line given below:

$$y = \alpha x - \log(-\log q). \tag{4.27}$$

4.4.6 Hazard Rate Functions

For the discrete inverse Weibull distribution, the hazard rate and the alternative hazard rate, are given by

$$h(k) = \frac{q^{n^{-\alpha}} - q^{(k-1)^{-\alpha}}}{1 - q^{(n-1)^{-\alpha}}}; \ k = 1, 2, \ldots \tag{4.28}$$

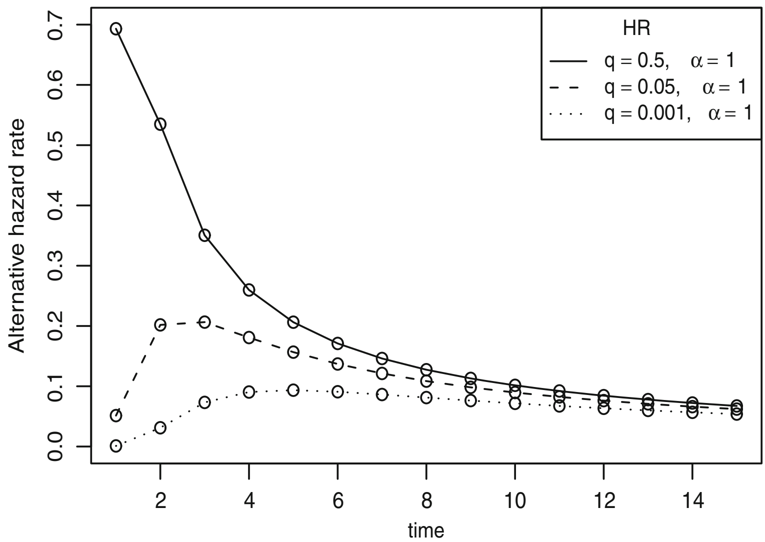

Fig. 4.2 Discrete inverse Weibull hazard rate function

and

$$h_1(k) = \log\left(\frac{1 - q^{(k-1)^{-\alpha}}}{1 - q^{k^{-\alpha}}}\right);\ k = 1, 2, \ldots, \tag{4.29}$$

respectively.

Figure 4.2 illustrates the alternative hazard rate function of the discrete inverse Weibull for $(q, \alpha) = (0.5, 1), (0.05, 1), (0.001, 1)$. For $q = 0.5$, the alternative hazard rate is a monotone decreasing function. For smaller values of q, it is unimodal and hence non-monotone.

It is easy to show that if $A = (1 - q)^{-1} \geq B = \left(1 - q^{2^{-\beta}}\right)^{1/2}$, then the alternative (nonclassical) hazard rate is monotone decreasing, is unimodal otherwise. For example, if $A \leq \min\left\{B, \frac{1-q^{3^{-\alpha}}}{B^4}\right\}$, the alternative hazard rate is unimodal with mode at $k = 2$.

In contrast to the discrete inverse Weibull, the hazard rate of the discrete Weibull is monotone (increasing for $\alpha > 1$ and decreasing for $\alpha < 1$).

For a discussion on how to estimate hazard rate functions for discrete lifetimes see Grimshaw et al. (2005).

4.4.7 Estimation of Parameters

Aghababaei Jazi et al. (2010) presented four methods for estimating the parameters q and β. These are (i) method of proportions, (ii) method of moments, (iii) heuristic algorithm, and (iv) inverse Weibull probability plot (IWPP).

The first two are the traditional estimation methods, the third is a numerical optimization algorithm and the last is a graphical method followed by the least squares estimates.

4.4.8 Application

Aghababaei Jazi et al. (2010) fitted the discrete inverse Weibull to 18 lifetimes of certain electronic devices given in Wang (2000).

4.5 Discrete Additive Weibull Distribution

In reliability modeling, it is common practice to treat failure data as being continuous, implying some degree of precision in measurement. Too often in practice however, failures are either noted at regular inspection intervals, occur in a discrete process, such as switching instants, or are simply recorded in bins. How does this degree of imprecision affect the models fitted, and the inferences to be drawn from them?

For example, an important concept in reliability is the bathtub shaped hazard rate function, which consists of an initially decreasing hazard rate (the 'wear-in' phase) followed by an approximately constant hazard (the 'useful' phase) and finally an increasing hazard rate (the 'wear-out' phase). The point of minimum hazard is termed the 'turning point'. This obviously provides an opportunity to improve the expected operating time by means of 'burn-in' (using up the initial wear-in phase).

The very concept of bathtub-shaped of a discrete hazard rate function is murky, with several possible definitions of the hazard rate as discussed in Sect. 4.2, It is unclear whether these all produce the same shaped hazard rate function when fitted to data, much less the same turning point and burn-in properties.

Most of the results presented below can be found in Bebbington et al. (2010, 2012).

The continuous additive Weibull distribution defined in (3.14) has survival function

$$\overline{F}(t) = \exp\left\{-\lambda_1 t^{\alpha} - \lambda_2 t^{\beta}\right\} \quad \text{for all} \quad t \in [0, \infty), \tag{4.30}$$

where $\lambda_1, \lambda_2, \alpha, \beta \in (0, \infty)$ are parameters. Intuitively, $\overline{F}(t)$ is the survival function of $X = \min\{W_1, W_2\}$, where W_1 and W_2 are independent Weibull random variables with parameters (λ_1, α) and (λ_2, β), respectively. Importantly, its hazard rate function:

$$h_0(t) = \frac{-\overline{F}'(t)}{\overline{F}(t)} = \alpha\lambda_1 t^{\alpha-1} + \beta\lambda_2 t^{\beta-1} \tag{4.31}$$

can attain a bathtub shape, meaning that it is initially decreasing but ultimately increasing. It has been shown in Xie and Lai (1996) that $h_0(t)$ is bathtub shaped when $\alpha < 1 < \beta$, or when $\beta < 1 < \alpha$. The turning point of the hazard rate function $h_0(t)$, i.e., the point t where the bathtub shaped hazard rate function achieves its minimum, is given by (Xie and Lai 1996; Eq. 9)

$$t = \left(\frac{\alpha(1-\alpha)\lambda_1}{\beta(\beta-1)\lambda_2}\right)^{1/(\beta-\alpha)} \tag{4.32}$$

when $\alpha < 1 < \beta$, and by an analogous formula when $\beta < 1 < \alpha$.

In this section we develop and explore a discrete analogue of the above hazard rate function and in particular determine its shape and, when it exists, a (unique) turning point.

4.5.1 Hazard Rates of the Discrete Additive Weibull Model

As noted earlier, several discrete lifetime distributions have been introduced in the literature. Many such distributions have been derived by discretizing their continuous counterparts. A number of discrete distributions are discussed in Xie et al. (2002), whose hazard rate functions are explored with the help of the ratio of two consecutive probabilities. We shall also encounter these ratios later in this section, especially when dealing with the aforementioned 'non-classical' definition of the discrete hazard rate function.

Throughout this section we concentrate on the discrete version of the additive Weibull distribution due to its ability to produce increasing, decreasing, and bathtub shaped hazard rate functions. The latter has also been produced by a geometric-exponentiated Poisson competing risk model by Jiang (2010).

When the survival function is additive Weibull (4.30), then with the notation $q_1 = \exp\{-\lambda_1\}$ and $q_2 = \exp\{-\lambda_2\}$ we have from (4.5) that

$$h(i) = 1 - q_1^{i^\alpha-(i-1)^\alpha} q_2^{i^\beta-(i-1)^\beta}. \tag{4.33}$$

The survival function that corresponds to (4.33) can be obtained from (4.6).

The following theorem given in Bebbington et al. (2010) specifies the possible shapes of $h(i)$.

Theorem 4.1 *We have the following statements:*

- *If either (i)* $\alpha \geq 1$ *and* $\beta > 1$ *or (ii)* $\alpha > 1$ *and* $\beta \geq 1$*, then* $h : \mathcal{N}^+ \to [0, 1)$ *is strictly increasing.*
- *If* $\alpha = 1$ *and* $\beta = 1$*, then* $h : \mathcal{N}^+ \to [0, 1)$ *is constant.*
- *If either (iii)* $\alpha \leq 1$ *and* $\beta < 1$ *or (iv)* $\alpha < 1$ *and* $\beta \leq 1$*, then* $h : \mathcal{N}^+ \to [0, 1)$ *is strictly decreasing.*
- *If* $\alpha < 1 < \beta$*, then* $h : \mathcal{N}^+ \to [0, 1)$ *is bathtub shaped with the minimum achieved at one of the three points* $\lfloor t_{\alpha,\beta} \rfloor$, $1 + \lfloor t_{\alpha,\beta} \rfloor$, *and* $2 + \lfloor t_{\alpha,\beta} \rfloor$*, where*

$$t_{\alpha,\beta} = \left(\frac{\alpha(1-\alpha)\lambda_1}{\beta(\beta-1)\lambda_2} \right)^{1/(\beta-\alpha)}, \tag{4.34}$$

  where $\lfloor \cdot \rfloor$ *is the floor function, that is,* $\lfloor x \rfloor$ *is the largest integer* k *such that* $k \leq x$*.*
- *For* $\beta < 1 < \alpha$*, similar results are obtained as the preceding case.*

Alternative Hazard Rate

The definition of the ‘non-classical’ hazard rate function $h_1 : \mathcal{N}^+ \to [0, \infty)$ is given (4.9) by the formula

$$h_1(i) = \log[S(i-1)/S(i)] = -\log[1 - h(i)], \tag{4.35}$$

where the second equality follows from (4.13), and is the definition used in Jiang (2010). In the case of the additive Weibull survival function,

$$h_1(i) = \lambda_1 \Big(i^{\alpha} - (i-1)^{\alpha} \Big) + \lambda_2 \Big(i^{\beta} - (i-1)^{\beta} \Big). \tag{4.36}$$

The survival function that corresponds to (4.36) can be obtained from (4.11), i.e.,

$$F_1(k) = \prod_{i=1}^{k} e^{-h_1(i)}. \tag{4.37}$$

It appears that the overall shapes of the two hazard rate functions $h(i)$ and $h_1(i)$ are same, and we formulate this as the next theorem.

Theorem 4.2 *The statements of Theorem 4.1 hold with* $h : \mathcal{N}^+ \to [0, 1)$ *replaced by* $h_1 : \mathcal{N}^+ \to [0, \infty)$.

Theorem 4.2 can be explained by first extending Eq. (4.35) to the continuous scale (with t), then differentiating the result with respect to t, and in this way arriving at the equation $h_1'(t) = h'(t)/(1 - h(t))$. Obviously, the signs of the derivatives $h_1'(t)$ and $h'(t)$ coincide, and hence so do the monotonicity properties of $h_1(t)$ and $h(t)$ (see, e.g., Xie et al. 2002; Property 1) including their turning points.

4.5.2 Discretizing the Hazard Rate of Continuous Weibull Model

Note that the factors $i^\alpha - (i-1)^\alpha$ and $i^\beta - (i-1)^\beta$ on the right-hand side of Eq. (4.36) are discrete analogues of the derivatives $(d/dt)t^\alpha$ and $(d/dt)t^\beta$, which are of course equal to $\alpha t^{\alpha-1}$ and $\beta t^{\beta-1}$, respectively. In turn, discrete analogues of the latter functions are $\alpha i^{\alpha-1}$ and $\beta i^{\beta-1}$, respectively. Making these substitutions on the right-hand side of Eq. (4.31), we arrive at the discrete version

$$h_0(i) = \alpha\lambda_1 i^{\alpha-1} + \beta\lambda_2 i^{\beta-1} \tag{4.38}$$

of the continuous hazard rate function $h_0(t)$ given by Eq. (4.31).

Equation (4.38) can be derived simply by 'discretizing' the hazard rate of the continuous additive Weibull model (4.31) by simply replacing t by i and $[0, \infty)$ by $\mathcal{N}^+$.

As the new hazard rate is not bounded above by 1, we regard $h_0(k)$ is a nonclassical discrete hazard rate so Eq. (4.11) applies and the survival function that corresponds to (4.38) is given by

$$\overline{F}_0(k) = \prod_{i=1}^{k} e^{-h_0(i)}, \tag{4.39}$$

which is not the same as (4.37).

Note Type-III discrete Weibull distribution was also obtained this way.

Summary

We can obtain a discrete hazard rate at least in three ways via:

1. discretize the continuous survival function and find the classical hazard rate $h(k)$ using (4.5),
2. find the nonclassical hazard rate $h_1(t)$ using the relationship between $h(k)$ and $h_1(k)$ as given by (4.12) or (4.13), or
3. discretize the hazard rate function $h(t)$ of the continuous lifetime distribution and denote it by $h_0(k)$. Then consider the latter as a nonclassical discrete hazard rate.

4.5.3 Examples

Several discrete survival data sets are fitted by the three versions of discrete additive Weibull model and the readers are referred to Bebbington et al. (2010) for details.

The following are fitted hazard rate plots for Aarset (1987) data where $h(k)$, $h_0(k)$ and $h_1(k)$ are given by (4.33), (4.39) and (4.37), respectively. The parameters of each of the models were estimated by the maximum likelihood method.

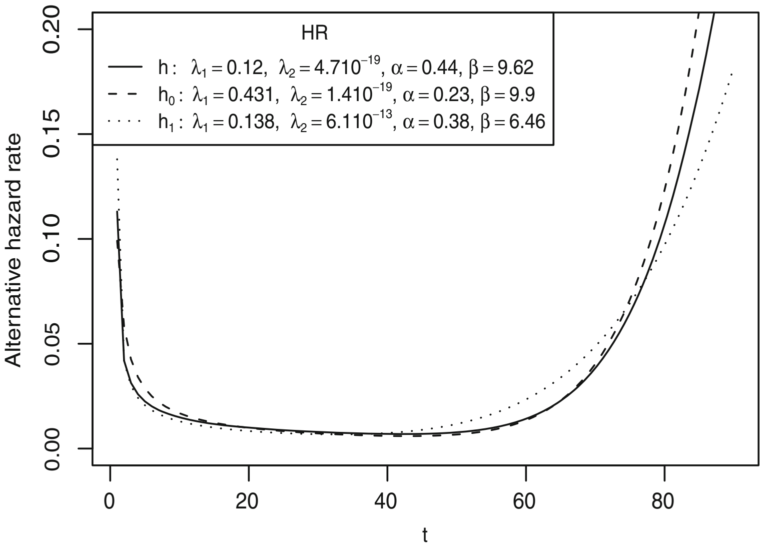

Fig. 4.3 Discrete additive Weibull hazard rate functions

Note that in Fig. 4.3, the discrete hazard rate functions were drawn as continuous curves rather than step functions to improve readability.

4.5.4 Applications

Bebbington et al. (2012) fitted the discrete additive Weibull distribution to the following data sets:

- Time to failure of 18 electronic devices (Wang 2000)
- Failure times of 50 devices (Aarset 1987)
- Lifetimes of 20 batteries (Ansell and Ansell 1987)
- Unit testing failure data (Xie and Lai 1996)
- BT-Serum reversal times (Carrasco et al. 2008).

4.5.5 Remark

While both the continuous and the discrete versions of the additive Weibull hazard rate function have same shapes for any given set of parameters $\lambda_1, \lambda_2, \alpha$ and β, application of the models to a range of data sets has shown that the behavior of the fitted continuous model is not necessarily a strong guide to the behavior of the fitted discrete versions. This has implications for burn-in, in particular. Further, different definitions of the discrete hazard rate functions can have different tail behavior.

4.6 Conclusion

This monograph gives an update on the work by Murthy et al. (2004) on generalized Weibull distributions. The aim is to afford the readers a number of families of distributions in which the Weibull family is a special case. By and large, these extensions/generalization provide more flexible hazard rate shapes for fitting real life data than the Weibull distribution can deliver. For each generalized family, we briefly describe it's construction and present its reliability properties in some detail. Furthermore, shapes of the density functions and the hazard rate functions are provided whenever deemed to be appropriate.

In recent years, we witness the proliferation of lifetime distributions many of which were derived via transformations of random variables. Despite their mathematical elegance, these models are often overparameterized and they lack motivation. In our view, the sample size of a reliability data set is usually small so a simpler but a flexible reliability model is really what a researcher is looking for. One would hope this could become a focus for future research on reliability modeling.

References

Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, *36*, 106–108.

Aghababaei Jazi, M., Lai, C. D., & Alamatsaz, M. H. (2010). A discrete inverse Weibull distribution and estimation of its parameters. *Statistical Methodology*, *7*, 121–132.

Ansell, R. O., & Ansell, J. I. (1987). Modelling the reliability of sodium sulfur cells. *Reliability Engineering*, *17*, 127–137.

Bebbington, M, Lai, C. D., Wellington, M. & Zitikis, R. (2010). Discrete bathtub distributions: Mind the step function!, *Advanced Reliability Modeling IV: Beyond Traditional Reliability and Maintainability Approaches*. In Chukova, S., Haywood, J. & Dohi, T. (Eds.), *Proceedings 4th Asia-Pacific International Symposium (APARM2010)* (pp. 57–64). Taiwan: McGraw Hill.

Bebbington, M., Lai, C. D., Wellington, M., & Zitikis, R. (2012). The discrete additive Weibull distribution: A bathtub-shaped hazard for discontinuous failure data. *Reliability Engineering and System Safety*, *106*, 37–44.

Bergstrom, R., & Edin, P.-A. (1992). Time aggregation and the distributional shape of unemployment duration. *Journal of Applied Econometrics*, *7*, 5–30.

Calabria, R., & Pulcini, G. (1989). Confidence limits for reliability and tolerance limits in the inverse Weibull distribution. *Reliability Engineering and System Safety*, *24*, 77–85.

Calabria, R., & Pulcini, G. (1990). On the maximum likelihood and least-squares estimation in the inverse Weibull distribution. *Statistica Applicata*, *2*(1), 53–66.

Carrasco, J. M. F., Ortega, E. M. M., & Cordeiro, G. M. (2008). A generalized modified Weibull distribution for lifetime modeling. *Computational Statistics and Data Analysis*, *53*, 450–462.

Drapella, A. (1993). The complementary Weibull distribution: Unknown or just forgotten. *Quality and Reliability Engineering International*, *9*, 383–385.
Erto, P. (1989). Genesis, properties and identification of the inverse Weibull lifetime model. *Staistica Applicata*, *1*(2), 117–128.
Grimshaw, S. D., McDonald, J., & McQueen, G. R. (2005). Estimating hazard functions for discrete lifetimes. *Communications in Statistics–Simulation and Computation*, *34*, 451–463.
Gupta, P. L., Gupta, R. C., & Tripathi, R. C. (1997). On the monotonic properties of discrete failure rates. *Journal of Statistical Planning and Inference*, *65*, 255–268.
Jiang, R., & Murthy, D. N. P. (1999). The exponentiated Weibull family: A grapgical approach. *IEEE Transactions on Reliability*, *48*(1), 68–72.
Jiang, R., Murthy, D. N. P., & Ji, P. (2001). Models involving two inverse Weibull distributions. *Reliability Engineering and System Safety*, *73*, 73–81.
Jiang, R. (2010). Discrete competing risk model with application to modeling bus-motor failure data. *Reliability Engineering and System Safety*, *95*, 981–988.
Keller, A. Z. & Kamath, A. R. R. (1982). Alternative reliability models for mechanical systems. In *Proceedings of the Third International Conference on Reliability and Maintainability*, pp. 411–415.
Keller, A. Z., Giblin, M. T., & Farnworth, N. R. (1985). Reliability analysis of commercial vehicle engines. *Reliability Engineering*, *10*(1), 15–25.
Khan, M. S. A., Khalique, A., & Abouammoh, A. M. (1989). On estimating parameters in a discrete Weibull distribution. *IEEE Transactions on Reliability*, *38*(3), 348–350.
Krishna, H., & Pundir, P. S. (2009). Discrete Burr and discrete Pareto distributions. *Statistical Methodology*, *6*, 177–188.
Lai, C. D., & Xie, M. (2006). *Stochastic Ageing and Dependence for Reliability*. New York: Springer.
Mudholkar, G. S., & Kollia, G. D. (1994). Generalized Weibull family: A structural analysis. *Communications in Statistics–Theory and Methods*, *23*, 1149–1171.
Murthy, D. N. P., & Djamaludin, I. (2002). Product warranty: A review. *International Journal of Production Economics*, *79*, 231–260.
Murthy, D. N. P., Xie, M., & Jiang, R. (2004). *Weibull Models*. Hoboken, NJ: Wiley.
Nakagawa, T., & Osaki, S. (1975). The discrete Weibull distribution. *IEEE Transactions on Reliability*, *24*, 300–301.
Padgett, W. J., & Spurrier, J. D. (1985). On discrete failure models. *IEEE Transactions on Reliability*, *R-34(3)*, 253–255.
Roy, D., & Gupta, R. P. (1992). Classifications of discrete lives. *Microelectronics and Reliability*, *32*, 1459–1473.
Roy, D. (2003). The discrete normal distribution. *Communications in Statistics–Theory and Methods*, *32*(10), 1871–1883.
Roy, D. (2004). Discrete Rayleigh distribution. *IEEE Transactions on Reliability*, *53*(2), 255–260.
Santos, P. A., & Alves, M. I. F. (2013). Improved shape parameter estimation in a discrete Weibull model. In P. E. Oliveira, M. da Graa Temido, C. Henriques, & M. Vichi (Eds.), *Recent Developments in Modeling and Applications in Statistics* (pp. 71–80). Berlin: Springer.
Shaked, M., Shanthikumar, J. G., & Valdez-Torres, J. B. (1995). Discrete hazard rate functions. *Computers and Operations Research*, *22*(4), 391–402.
Stein, W. E. & Dattero, R. (1984). A new discrete Weibull distribution. *IEEE Transactions on Reliability, R-33*, 196–197.
Steutel, F. W., & van Harn, K. (2004). *Infinite Divisibility of Probability of Distributions on the Real Line*. New York: Marcel Dekker.
Wang, F. K. (2000). A new model with bathtub-shaped failure rate using an additive Burr XII distribution. *Reliability Engineering and System Safety, 70*, 305–312.
Xie, M., & Lai, C. D. (1996). Reliability analysis using additive Weibull model with bath-tubed shape failure rate function. *Reliability Engineering and System Safety*, *52*, 87–93.
Xie, M., Gaudoin, O., & Bracquemond, C. (2002). Redefining failure rate function for discrete distribution. *International Journal of Reliability, Quality and Safety Engineering*, *9*(3), 275–286.

Index

C.-D. Lai, *Generalized Weibull Distributions*, SpringerBriefs in Statistics,
DOI: 10.1007/978-3-642-39106-4, © The Author(s) 2014