Advances in Delays and Dynamics 6

Alexandre Seuret
Laurentiu Hetel
Jamal Daafouz
Karl H. Johansson *Editors*

Delays and Networked Control Systems

Springer

Advances in Delays and Dynamics

Volume 6

About this Series

Delay systems are largely encountered in modeling propagation and transportation phenomena, population dynamics, and representing interactions between interconnected dynamics through material, energy, and communication flows. Thought as an open library on delays and dynamics, this series is devoted to publish basic and advanced textbooks, explorative research monographs as well as proceedings volumes focusing on delays from modeling to analysis, optimization, control with a particular emphasis on applications spanning biology, ecology, economy, and engineering. Topics covering interactions between delays and modeling (from engineering to biology and economic sciences), control strategies (including also control structure and robustness issues), optimization, and computation (including also numerical approaches and related algorithms) by creating links and bridges between fields and areas in a delay setting are particularly encouraged.

More information about this series at http://www.springer.com/series/11914

Alexandre Seuret · Laurentiu Hetel
Jamal Daafouz · Karl H. Johansson
Editors

Delays and Networked Control Systems

Springer

Editors
Alexandre Seuret
LAAS—UPR CNRS
Université de Toulouse
Toulouse
France

Laurentiu Hetel
CRISTAL—UMR CNRS
École centrale de Lille
Villeneuve-d'Ascq Cedex
France

Jamal Daafouz
Automatic Control-Identification Diagnosis
CRAN—UMR CNRS, Institut National Polytechnique de Lorraine, Université de Lorraine
Vandoeuvre-lès-Nancy Cedex
France

Karl H. Johansson
School of Electrical Engineering
KTH—Royal Institute of Technology
Stockholm
Sweden

ISSN 2197-117X ISSN 2197-1161 (electronic)
Advances in Delays and Dynamics
ISBN 978-3-319-32371-8 ISBN 978-3-319-32372-5 (eBook)
DOI 10.1007/978-3-319-32372-5

Library of Congress Control Number: 2016937345

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Preface

The purpose of this volume in the newly established series Advances in Delays and Dynamics (ADD@S) is to provide a collection of recent results on the design and analysis of *Delays and Networked Control Systems*.

Networked systems represent today a general paradigm for describing phenomena in various domains such as Biology (for genes transcription networks or models of schools of fishes), Robotics (in teleoperated manipulators for medical applications or in coordinated unmanned vehicles), Computer Sciences (congestion control and load balancing in the Internet management), Energy Management (in electric grids), Traffic Control (fluid models of traffic flow), etc. The analysis and design of networked systems represents nowadays an important challenge in the Automatic Control community.

An enormous scientific and industrial interest has been shown in Networked Control Systems, which are ubiquitous in most of modern control devices. A Networked Control System (NCS) is a control system wherein its components (plants, sensors, embedded control algorithms and actuators) are spatially distributed. The defining feature of an NCS is that control and feedback signals are exchanged among the system's components in the form of digital information packages. The primary advantages of an NCS are reduced wiring, ease of diagnosis, and maintenance and increased flexibility. Despite these advantages, the use of communication networks also introduces several imperfections: limited information bandwidth, communication delays, complex interactions between control algorithms, real-time scheduling protocols, etc. Such imperfections may lead to poor system performances and even instability if not appropriately taken into account. The practical and theoretical challenges brought in the context of NCS rely on the consideration of the imperfections induced by the use of communication networks in control loops.

Several models have been proposed as abstractions to the complex phenomena occurring in NCS. The use of time-delay models is unavoidable in NCS since the transmission of information through a network is not instantaneous. The challenge here is to deal with the infinite dimensional nature of the obtained system.

Network links with limited capacity and bandwidth can be modeled in various manners. The first model consists of including into the control loop a *quantization* process, which basically constrains a signal evolving in a continuous set of values to a relatively small and possibly *saturated* discrete set. Another model has been proposed to describe the discretization in time of the exchanged information. Embedded control algorithms use *sampled versions* of the system state or output. The difficulty lies in fact that, in real time applications, sampling is often generated in an asynchronous manner. This issue makes the analysis and design of NCS a complex task. On one side, this asynchronism may represent an undesired phenomena (jitter or packet dropout) and it may be a source of instability. From the control theory point of view, it must be taken into account in a robust manner. On the other side, one may deliberately introduce asynchronism in the control loop via scheduling algorithms, in order to reduce the number of data transmissions and, therefore, optimize the computational costs. This corresponds to the recent research trend of *event-based control* where a data is transmitted only if a particular event has occurred.

The new challenges for control design of networked systems are particularly evident in large-scale interconnection of multiagent systems. For example, in formation and cooperation control, it is not reasonable, from the practical point of view, to allow all-to-all communication. Each agent has only a *local view* of the overall network and he may not be able to store and manipulate the complete state of the system. Hence, an agent is able to exchange information only with its neighbors in the space or in the communication graph.

The chapters in this volume deal with several aspects of time delays and networked systems. In the literature many different techniques have been proposed for the analysis and design of such systems. Widely used techniques include *Lyapunov*-based analysis and design in the time domain, and *spectral methods* in the frequency domain. The reader will find examples of these techniques in this volume. The main ideas of the individual papers included here were presented and discussed at a workshop organized by the International Scientific Coordination Network on Delay Systems (DelSys) in November 2013 at LAAS, Toulouse, France. The International scientific coordination network on Delay Systems DelSys, supported by the French Center Scientific Research (CNRS) gathers several European research teams working in the field of time-delay systems. The main objectives of "DelSys" are twofold: first, to better organize the European research on such topics and second to better emphasize the research trends in the field.

The book is collected under the following parts:

Part I Delays in Large Scale and Infinite Dimensional Systems

The first part of the book is concerned with the presentation of recent tools for the analysis of time-delay systems. In the following chapters, a particular attention will be paid on several classical analyses and control problems of large scale or infinite dimensional systems with delays.

The first chapter by *Islam Boussaada* and *Silviu-Iulian Niculescu* deals with the study of the multiplicity of imaginary crossing roots in some networks with delays. The proposed method is based on the properties of the Vandermonde matrices, which is proved to be an appropriate method to solve this problem. The second chapter by *Michaël di Loreto, Sérine Damak*, and *Sabine Mondié* addresses the problem of stability, stabilization, and control of a network of conservation laws. The method is based on the construction of a state-space realization for networks of linear hyperbolic conservation laws for deriving sufficient conditions for stability and stabilization of the systems. This part ends with a chapter by *Igor Pontes Duff, Pierre Vuillemin, Charles Poussot-Vassal, Corentin Briat,* and *Cédric Seren*. The chapter introduces a novel method for the model reduction of large scale time-delay systems using norm approximation. The method detailed in this work aims at overcoming some limitations by exploiting the recent $\mathscr{H}_2$ model reduction results for linear time invariant systems, which is well adapted to approximate infinite dimensional models.

Part II Control Systems Under Asynchronous Sampling

The second part of the book focuses on the analysis and design of systems with asynchronous sampling intervals which occur in Networked Control Systems. Both linear and nonlinear systems are being considered.

The first chapter, by *Sylvain Durand*, *Nicolas Marchand* and *José Fermi Guerrero-Castellanos*, revisits the classical "universal stabilizer formula" in an event-triggered control configuration. The case of nonlinear systems affine in the control with delay in the state is considered using the existence of Control Lyapunov–Krasovskii/-Razumikhin functionals. The second chapter, by *Hassan Omran*, *Laurenţiu Hetel*, *Jean-Pierre Richard* and *Françoise Lamnabhi-Lagarrigue*, addresses the stability of bilinear systems with aperiodic sampled-data state-feedback controllers. The analysis is based on a hybrid system model and the use of linear matrix inequalities (LMI) criteria that allow to asses the local stability of the closed-loop system. The third chapter, by *Alexandre Seuret* and *Corentin Briat*, provides new LMI conditions for the stability of asynchronous sampled-data linear systems with input delay. The analysis is performed by the use of the Lyapunov–Krasovskii method and of loop functionals that allow to take into account the variations of the sampling interval. Chapter 4, by *Mahmoud Abdelrahim*, *Romain Postoyan*, *Jamal Daafouz* and *Dragan Nešić*, presents results for the synthesis of output event-triggered controllers for nonlinear systems. The key idea of the approach is to combine techniques from event- and time-triggered control, in order to turn the sampling mechanism on only after a fixed amount of time has elapsed since the last transmission.

Part III Time-Delay Approaches in Networked Control Systems

The third part is dedicated to the use of time-delay models for the analysis and design of Networked Control Systems.

In the first chapter, by *Francesco Ferrante*, *Frédéric Gouaisbaut* and *Sophie Tarbouriech*, the effect of quantization is analyzed for control of linear systems with input delays. The saturating quantizer is studied locally, using convex optimization methods, using some modified sector conditions. The second chapter, by *Kun Liu*, *Emilia Fridman* and *Karl H. Johansson*, investigates the use of the time-delay approach for the analysis of Networked Control Systems under scheduling protocols. The key idea is to tackle the stability problem using delay dependent Lyapunov–Krasovskii methods. In the third chapter, by *Xu-Guang Li*, *Arben Cela*, and *Silviu-Iulian Niculescu*, networked control systems with hypersampling periods are analyzed. Stability regions are computed for linear systems using parameters sweeping techniques.

Part IV Cooperative Control

The last part of the book exposes several contributions dealing with the design of cooperative control and observation laws for networked control systems. In these chapters, several discussions on the application of consensus algorithms are provided.

The first chapter of this part by *Alicia Arce Rubio, Alexandre Seuret, Yassine Ariba*, and *Alessio Mannisi* aims at presenting distributed control laws, which allow a fleet of two drones to carry a load in a cooperative manner. Two approaches based on the linear quadratic regulator method and on model predictive control are discussed. The second chapter by *Pablo Millan, Luis Orihuela Isabel Jurado, Carlos Vivas*, and *Francesco R. Rubio* exposes some recent tools which aim at taking into account delays in distributed estimation and control. Indeed, the influence delays and packet dropouts induced by communication are analyzed and robust control and estimation laws are provided. The third chapter by *Constantin Morarescu, Pierre Riedinger, Marcos C. Bragagnolo* and *Jamal Daafouz* exposes a novel method for the design and the analysis of a reset strategy for consensus in networks with cluster pattern. The next chapter by *Paresh Deshpande, Christopher Edwards*, and *Prathyush P. Menon* presents a synthesis of distributed control laws for multiagent systems. The key issue of this chapter is to design distributed control laws which are based on delayed relative measurements among the agents. The last chapter by *Constantin Morarescu* and *Mirko Fiacchini* concerns the implementation strategies for topology preservation in multiagent systems.

Toulouse Alexandre Seuret
Lille Laurentiu Hetel
Nancy Jamal Daafouz
Stockholm Karl H. Johansson
June 2014

Contents

Acronyms

ASF	Average Sampling Frequency
CLF	Control Lyapunov Function
CLKF	Control Lyapunov–Krasovskii Functional
CLRF	Control Lyapunov–Razumikhin Function
GUES	Globally Uniformly Exponentially Stable
IQC	Integral Quadratic Constraints
ISS	Input-to-State Stability
LDI	Linear Difference Inclusion
LKF	Lyapunov–Krasovskii Functional
LMI	Linear Matrix Inequality
LQR	Linear Quadratic Regulator
LTI	Linear-Time-Invariant (system)
LUF	Lower Upper Factorization
MASP	Maximum Allowable Sampling Period
MATI	Maximum Allowable Transmission Interval
MPC	Model Predictive Control
MSI	Minimum Inter-sampling Interval
NCS	Network Control Systems
PD	Proportional Derivative (controller)
PDE	Partial Differential Equation
RR	Round Robin (protocol)
SISO	Single Input–Single Output (systems)
TDS	Time-Delay Systems
TF-IRKA	Iterative Rational Krylov Algorithm using Transfer Function Evaluation
TOD	Try-Once-Discard (protocol)
UAV	Unmanned Areal Vehicles
VTOL	Vertical TakeOff and Landing
WN	Wireless Network

Part I
Delays in Large Scale and Infinite Dimensional Systems

Chapter 1
On the Codimension of the Singularity at the Origin for Networked Delay Systems

Dina-Alina Irofti, Islam Boussaada and Silviu-Iulian Niculescu

Abstract Continuous-time dynamical networks with delays have a wide range of application fields from biology, economics to physics, and engineering sciences. Usually, two types of delays can occur in the communication network: internal delays (due to specific internal dynamics of a given node) and external delays (related to the communication process, due to the information transmission and processing). Besides, such delays can be totally different from one node to another. It is worth mentioning that the problem becomes more and more complicated when a huge number of delays has to be taken into account. In the case of constant delays, this analysis relies much on the identification and the understanding of the spectral values behavior with respect to an appropriate set of parameters when crossing the imaginary axis. There are several approaches for identifying the imaginary crossing roots, though, to the best of the authors' knowledge, the bound of the multiplicity of such roots has not been deeply investigated so far. This chapter provides an answer for this question in the case of time-delay systems, where the corresponding quasi-polynomial function has non-spare polynomials and no coupling delays. Furthermore, we will also show the link between this multiplicity problem and Vandermonde matrices, and give the upper bound for the multiplicity of an eigenvalue at the origin for such a time-delay system modeling network dynamics in the presence of time-delay.

D.-A. Irofti (✉) · I. Boussaada · S.-I. Niculescu
Laboratoire des Signaux et Systèmes, CNRS-CentraleSupélec-UPSUD,
3 rue Joliot-Curie, 91192 Gif-sur-Yvette Cedex, France
e-mail: dina.irofti@l2s.centralesupelec.fr

S.-I. Niculescu
e-mail: Silviu.Niculescu@l2s.centralesupelec.fr

I. Boussaada
IPSA, 15 rue Maurice Grandcoing, Ivry sur Seine, France
e-mail: islam.boussaada@l2s.centralesupelec.fr

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_1

1.1 Introduction

In the case of a n-dimensional linear system of ordinary differential equations $\dot{x} = Ax$, where $x \in \mathbb{R}^n$ and $A \in \mathscr{M}_n(\mathbb{R})$, the n eigenvalues of the matrix A are at the same time the spectral values of the system. Since the corresponding characteristic equation is a polynomial of degree n in the Laplace variable, it has at most n complex roots—the eigenvalues of the system. So we can say that the codimension of a given spectral value (its multiplicity) can be at most equal to the dimension of the state-space.

On the other hand, the case of time-delay systems is slightly different as, in this case, the associated characteristic equation has an infinite number of roots. Consider a networked system described by the following delay-differential equations:

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - \tau_i), \tag{1.1}$$

under appropriate initial conditions belonging to the Banach space of continuous $\mathscr{C}([-\tau_N, 0], \mathbb{R}^n)$. Here $x = (x_1, \ldots, x_n)$ denotes the state-vector, matrices $\{A_i \in \mathscr{M}_n(\mathbb{R})\}$ for $i = 0 \ldots N$ (see [1, 3]). The N constant delays τ_i, $i = 1 \ldots N$ with $\tau_1 < \tau_2 < \cdots < \tau_N$ are positive and without any loss of generality, we consider $\tau_0 = 0$. In the associated characteristic equation (a transcendental equation in the Laplace variable λ), some exponential terms appear due to the delays. Thus, the system (1.1) has the characteristic function $\Delta : \mathbb{C} \times \mathbb{R}_+^N \to \mathbb{C}$ defined as:

$$\Delta(\lambda, \tau) = P_0(\lambda) + \sum_{i=1}^{N} P_i(\lambda)\, e^{-\tau_i \lambda}. \tag{1.2}$$

Assume that the polynomial P_0 is a monic polynomial of degree n in λ and the polynomials P_i are such that $\{\mathrm{deg(P_i)} \leq (\mathrm{n} - 1)\}$, $\{\forall 1 \leq \mathrm{i} \leq \mathrm{N}\}$. The study of zeros of the quasi-polynomial (1.2) plays a crucial role in the analysis of asymptotic stability of the zero solution of system (1.1). Indeed, the zero solution is asymptotically stable if all the zeros of (1.2) are in the open left-half complex plane [18]. According to this definition, the parameter space which is spanned by the coefficients of the polynomials P_i, can be split into stability and instability domains (nothing else that the so-called D-decomposition, see [18] and references therein). These two domains are separated by a boundary corresponding to a spectra consisting in roots with zero real parts and roots with negative real part. Moreover, under appropriate algebraic restrictions, a given root associated to that boundary can have high multiplicity.

Roughly speaking, continuous-time networks with delays can imply some internal delays on one hand (due to specific internal dynamics of the nodes), but also some information transmission and processing delays on the other hand (delays caused by the communication between nodes). Thus, in [2] a network dealing with information transmission delays is studied with respect to the consensus problem, and it is proved that for the continuous-time system to reach consensus, the existence of a simple eigenvalue at the origin represents some necessary condition.

Inspired by such a fact, this chapter focuses on the codimension of the zero spectral value for linear time-delay systems. A typical example for a nonsimple zero spectral value is the Bogdanov–Takens singularity which is characterized by an algebraic multiplicity two and a geometric multiplicity one. Cases with higher order multiplicities of the zero spectral value are known as generalized Bogdanov–Takens singularities and could be involved in concrete applications. For instance, the Bogdanov–Takens singularity is identified in [12] where the case of two coupled scalar delay equations modeling a physiological control problem is studied. In [17], this type of singularity is also encountered in the study of coupled axial-torsional vibrations of an oilwell rotary drilling system. Moreover, the paper [7] is dedicated to this type of singularities, where codimensions two and three are studied and the associated center manifold are explicitly computed.

Commonly, the time-delay induces desynchronizing and/or destabilizing effect on the dynamics. However, new theoretical developments in control of finite-dimensional dynamical systems suggest the use of delays in the control laws as controller parameters for stabilization purposes. For instance, the papers [5, 21] are concerned with the stabilization of the inverted pendulum by delayed control laws and provide concrete situations where the codimension of the zero spectral value exceeds the number of the scalar equations modeling the inverted pendulum on cart. In [21], the authors prove that some appropriate delayed proportional-derivative (PD) controller stabilizes the inverted pendulum by identifying a codimension three singularity for a system of two coupled delayed equations. In [5], the same singularity is characterized using a particular delay block configuration. It is shown that two delay blocks offset a PD delayed controller. Although the algebraic structure of the multiplicity problem makes the finite aspect of such a codimension evident, to the best of the authors' knowledge, the question *on the upper bound of the codimension of the zero spectral value* did not receive a complete characterization.

In this chapter, which is based on some of the authors' previous results of [6], we investigate this type of singularity and give an answer to the question above. This work is motivated by the fact that the knowledge of such an information is crucial when dealing with nonlinear analysis and center manifold computations. Indeed, when the zero spectral value is the only eigenvalue with zero real part, then the center manifold dimension is none other than the codimension of the generalized Bogdanov–Takens singularity [8, 11, 13, 14].

The effective method elaborated in this chapter emphasizes the connections between the codimension problem and functional confluent Vandermonde matrices. To the best of the authors' knowledge, the use of Vandermonde matrix in control area concerns more structural properties of control systems, for instance, in [15] the controllability of a finite-dimensional dynamical system is guaranteed by the invertibility of such a matrix. Next, in the context of time-delay systems, the use of Vandermonde matrix properties was proposed by [18, 19] when controlling one chain of integrators by delay blocks. Here we further explore the algebraic properties of such matrices into a different context.

The remaining chapter is organized as follows. Section 1.2 includes some motivating example. Next the main results are proposed in Sects. 1.3 and 1.4. Various illustrative examples are presented in Sect. 1.5. Finally, some concluding remarks end the chapter.

1.2 Motivating Example: Network with Information Delays

The multiplicity of an eigenvalue at the origin could also be studied in synchronization problems on networks with time-delays (see [2], where the approach is based on graph theory). For example, consider a network with n agents where the evolution of the state (also called *opinion*) of the agent i at time t, $x_i(t)$, is of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} \left(x_j(t-\tau) - x_i(t-\tau)\right). \tag{1.3}$$

This means that the agent i receives some information from the agent j, the communication process needs some time τ, and after receiving the information the agent i also needs some time τ to process it. The two periods of time could be different or not. In Eq. (1.3), a_{ij} denotes some “measures” of the influence of agent j on the agent i and, by convention, is always positive. If the agent i receive the information from the agent j and is immediately able to use it (without processing it), then the Eq. (1.3) above becomes:

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} \left(x_j(t-\tau) - x_i(t)\right). \tag{1.4}$$

If, in addition, the agent x_i has also a sort of internal dynamics, Eq. (1.4) becomes

$$\dot{x}_i(t) = b_i x_i(t) + c_i x_i(t-\tau_i) + \sum_{j=1}^{n} a_{ij} \left(x_j(t-\tau) - x_i(t)\right). \tag{1.5}$$

To conclude, Eq. (1.5) represents a general expression, we can use in the case of a network where both internal and external delays are taken into account.

We consider now the simple case of a network of the form of a complete, regular graph, including four nodes, that is to say that from every node we can reach any other node in the network. Suppose that the *opinion* of each node evolves under the law (1.3), where a_{ij} represents the corresponding element of the adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Applying Eq. (1.3) to each agent i, we can rewrite the system as $\dot{x}(t) = -Lx(t-\tau)$, where the matrix L, also called *Laplacian*, is of the form

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

It is easy to see that $L = D - A$, where A is the adjacency matrix and D is the degree matrix, $D := diag\{d_1, \ldots, d_4\}$, with d_i the number of incoming/outgoing arcs to x_i. Since we supposed that our network is regular, all the d_is are equals.

It is proved in [2] that if τ is sufficiently small (see the so-called *delay margin*) and the Laplacian has a simple eigenvalue at the origin, then the system reaches consensus. Indeed, if we take a look at the characteristic equation of the system, $\det\left(\lambda I + Le^{-\lambda\tau}\right) = 0$, we can easily see that its characteristic function becomes:

$$\Delta(\lambda, \tau) = \lambda\left(\lambda^3 + 12\lambda^2 e^{-\lambda\tau} + 48\lambda e^{-2\lambda\tau} + 64 e^{-3\lambda\tau}\right),$$

and hence $\lambda = 0$ is a simple root.

1.3 Main Results

Let us first recall some useful definitions and notations:

First, a polynomial P of degree n is said to be *sparse* when $P(x) = \sum_{k=0}^{n} a_k x^k$ and $\prod_{k=0}^{n-1} a_k = 0$ (see, for instance, [10]).
Second, by *regular* quasi-polynomial we understand the situation when the polynomials P_i in (1.2) for $i = 1, \ldots, N$ are not sparse i.e., $\forall i = 1, \ldots, N$ and $\{\forall k = 0, \ldots, \deg(P_i)\}$ we assume that $a_{i,k} \neq 0$.

The main result can be summarized as follows:

Proposition 1 *The codimension of zero singularity of the regular characteristic quasi-polynomial function Δ given by (1.2) cannot be larger than $N(n_0 + 1) + n$, where N is the number of the nonzero distinct delays, n is the degree of P_0 and n_0 is the upper degree of the polynomial family P_i, for $1 \leq i \leq N$.*

Remark 2 Several other notions can be encountered in the literature describing sparsity. Among others we mention the lacunary polynomials [16].

We need first to introduce some notations. Let denote by $\Delta^{(k)}(\lambda)$ the k-th derivative of $\Delta(\lambda)$ with respect to the variable λ. We say that zero is an eigenvalue of algebraic multiplicity $m \geq 1$ for (1.1) if $\Delta(0) = \Delta^{(k)}(0) = 0$ for all $k = 1, \ldots, m-1$ and $\Delta^{(m)}(0) \neq 0$.

Consider the nonzero distinct delays such that $0 < \tau_1 < \tau_2 < \cdots < \tau_N$ and the polynomials P_i such that P_0 is an unitary polynomial with $\deg(P_0) = n$ and $\deg(P_i) \leq n-1$ for $1 \leq i \leq N$ and let $n_0 = \max_{1 \leq i \leq N} \deg(P_i)$. We denote by $a_{i,k}$ the coefficient of the monomial λ^k for the polynomial P_i, thus $a_{0,n} = 1$.

Since we are dealing only with the values of $\Delta_k(0)$, we suggest to translate the problem into the parameter space (the space of the coefficients of the P_i), this will be more appropriate and will consider parametrization by τ.
The following lemma allows to establish an m-set of multivariate algebraic functions (polynomials) vanishing at zero when the multiplicity of the zero root of the transcendental equation $\Delta(\lambda, \tau) = 0$ is equal to m.

Lemma 3 *Zero is a root of $\Delta^{(k)}(\lambda)$ for $k \geq 0$ if and only if the coefficients of P_i for $0 \leq i \leq N$ satisfy the following assertion*

$$a_{0,k} = -\sum_{i=1}^{N} \left[a_{i,k} - \sum_{l=0}^{k-1} \frac{(-1)^{l+k+1} a_{i,l} {\tau_i}^{k-l}}{(k-l)!} \right]. \tag{1.6}$$

The proof of the above Lemma 3 can be found in [6].

Example 1 To illustrate Lemma 3, as well as the resulting Proposition 1, consider the scalar delay-differential equation:

$$\dot{x}(t) + a_{0,0}x(t) + a_{1,0}x(t-\tau_1) + a_{2,0}x(t-\tau_2) = 0,$$

where $a_{0,0},\ a_{1,0},\ a_{2,0} \in \mathbb{R}$ The corresponding characteristic quasi-polynomial function is given by

$$\Delta(\lambda, \tau) = \lambda + a_{0,0} + a_{1,0}e^{-\tau_1\lambda} + a_{2,0}e^{-\tau_2\lambda}.$$

For $k = 0$, Equality (1.6) gives the first sufficient condition guaranteeing a multiplicity at least one for the zero singular value. Indeed, $\{\Delta(0, \tau) = 0 = a_{0,0} + a_{1,0} + a_{2,0}\}$ which gives as $a_{0,0} = -(a_{1,0} + a_{2,0})$.

A sufficient condition for a multiplicity at least two for the zero singular value is obtained by computing the first partial derivative of Δ with respect to λ, leading to the condition $a_{0,1} = 1 = \tau_1 a_{1,0} + \tau_2 a_{2,0}$, which satisfy Equality (1.6) where $k = 1$. Finally, one can easily show that if Eq. (1.6) is satisfied for all $k = 0,\ 1,\ 2$, then the zero singularity is of multiplicity three (which is the maximal multiplicity) if and only if

$$a_{0,0} = -\frac{\tau_1 + \tau_2}{\tau_1\tau_2}, a_{1,0} = -\frac{\tau_2}{(\tau_1 - \tau_2)\,\tau_1}, a_{2,0} = \frac{\tau_1}{\tau_2\,(\tau_1 - \tau_2)}.$$

Such a result is consistent with respect to Proposition 1.

Introduce

$$\nabla_k(\lambda) = \sum_{i=0}^{N} \frac{d^k}{d\lambda^k} P_i(\lambda) + \sum_{l=0}^{k-1} \left((-1)^{l+k} \binom{k}{l} \sum_{i=1}^{N} {\tau_i}^{k-l} \frac{d^l}{d\lambda^l} P_i(\lambda) \right),$$

with the notations and the remarks above, we are able to prove the main result.

Proof (Proof of Proposition 1). We shall consider the variety associated with the vanishing of the polynomials ∇_k (defined in Lemma 3), that is $\{\nabla_0(0) = \cdots = \nabla_{m-1}(0) = 0\}$ and $\nabla_m(0) \neq 0$ and we aim to find the maximal m (codimension of the zero singularity).

Consider the first elements from the family ∇_k

$$\nabla_0(0) = 0 \Leftrightarrow \sum_{i=0}^{N} a_{i,0} = 0,$$

$$\nabla_1(0) = 0 \Leftrightarrow \sum_{i=0}^{N} a_{i,1} - \sum_{i=1}^{N} a_{i,0}\,\tau_i = 0,$$

$$\nabla_2(0) = 0 \Leftrightarrow 2! \sum_{i=0}^{N} a_{i,1} - 2! \sum_{i=1}^{N} a_{i,0}\,\tau_i + \sum_{i=1}^{N} a_{i,0}\,\tau_i^2 = 0,$$

if we consider $a_{i,k}$ and τ_l as variables, the obtained algebraic system is nonlinear and solving it in all generality (without attributing values for n and N) becomes a very difficult task. Indeed, even by using Gröbner basis methods [9], this task is still complicated since the set of variables depends on N, n and n_0. Since our aim is to establish an upper bound, we assume here that all the polynomials P_i satisfy the condition $\deg(P_i) = n_0$ for all $1 \leq i \leq N$. We consider $a_{i,k}$ as variables and τ_l as parameters, and we adopt the following notation $a_0 = (a_{0,0},\ a_{0,1},\ \ldots,\ a_{0,n-1})^T$ and $a_i = (a_{i,0},\ a_{i,1},\ \ldots,\ a_{i,n_0})^T$ for $1 \leq i \leq N$ and denote by $\tau = (\tau_1,\ \tau_2,\ \ldots,\ \tau_N)$ and $a = (a_1,\ a_2,\ \ldots,\ a_N)^T$.

Consider the ideal I_1 generated by the n polynomials $\{\langle \nabla_0(0),\ \nabla_1(0),\ \ldots,\ \nabla_{n-1}(0) \rangle\}$. As it can be seen from Lemma 3, the variety V_1 associated with the ideal I_1 has the following linear representation $a_0 = M_1(\tau)\, a$, where $M_1 \in \mathscr{M}_{n,N(n_0+1)}(\mathbb{R}[\tau])$. In some sense, in this variety there are no any "restriction" on the components of a when a_0 is left "free". Since $a_{0,k} = 0$ for all $k > n$, the remaining equations consist in an algebraic system only in a and parametrized by τ. Consider now the ideal I_2 generated by the $N(n_0 + 1)$ polynomials:

$$I_2 = \langle \nabla_{n+1}(0),\ \nabla_{n+2}(0),\ \ldots,\ \nabla_{n+N(n_0+1)}(0) \rangle.$$

It can be observed that the variety V_2 associated with I_2 can be written as $M_2(\tau)a = 0$, which is nothing else that an homogeneous linear system with $\{M_2 \in \mathscr{M}_{N(n_0+1)}(\mathbb{R}[\tau])\}$. More precisely, M_2 is a functional Vandermonde matrix:

$$M_2(\tau) = (V(\tau_1), \frac{d}{d\tau_1}V(\tau_1), \ldots, \frac{d^{n_0}}{d\tau_1^{n_0}}V(\tau_1), \ldots, V(\tau_N), \ldots, \frac{d^{n_0}}{d\tau_N^{n_0}}V(\tau_N)), \text{ where}$$

$$V(x) = ((-x)^{n+1},\ (-x)^{n+2},\ \ldots,\ (-x)^{n+N(n_0+1)})^T. \tag{1.7}$$

Obviously, every subset of vectors $F_k = (V(\tau_k),\ \ldots,\ \frac{d^{n_0}}{d\tau_k^{n_0}}V(\tau_k))$ is a family of vectors in $\mathbb{R}^{N(n-1)}([\tau_k])$, which are linearly independent since, as it can be seen in (1.7) that for any $i \neq l$, $\deg(\mathrm{V_i}) \neq \deg(\mathrm{V_l})$, where V_k is the k-th component of the vector V and $\deg(\mathrm{V_k})$ denotes the degree of the polynomial $V_k(x)$ in x. Moreover, no any element from F_l (the family of vectors in $\mathbb{R}^{N(n-1)}([\tau_l])$) can be written as a linear combination of elements of F_k with $l \neq k$, which proves that $\det(\mathrm{M})$ can not vanish. Furthermore, the direct computation of the determinant of the matrix M gives

$$|\det(M)| = \left|\prod_{1\le k\le n-2}(n_0+1-k)!^N\right| \cdot \left|\prod_{1\le i<l\le N}(\tau_i-\tau_l)^{(n_0+1)^2}\prod_{1\le h\le N}\tau_h^{(n_0+1)(n+1)}\right|.$$

Since we are concerned only with nonzero distinct delays, this determinant cannot vanish. Thus, the only solution for this subsystem is the zero solution, that is $a = 0$.

Now consider the polynomial defined by $\nabla_n(0)$, by Lemma 3

$$\nabla_n(0) = 0 \Leftrightarrow 1 = \sum_{i=1}^{N}\sum_{l=0}^{n-1}\frac{(-1)^{l+n+1}\, a_{i,l}\tau_i^{n-l}}{(n-l)!}$$

substituting the unique solution of V_2 into the last equality leads to an incompatibility result. In conclusion, it follows straightforwardly that the maximal codimension of the zero singularity is less or equal to $N(n_0+1)+n$.

Remark 4 In the light of the results of proposition 1, we are able to establish the *codimension's upper bound* of the *zero singularity* of the characteristic quasi-polynomial function Δ when all its parameters are left "free". Indeed, in such a case, it is assumed that $n_0 = \max_{1\le i\le N}\deg(P_i)$ is exactly $n-1$.

The codimension of zero singularity of the characteristic quasi-polynomial function Δ given by (1.2) cannot be larger than $(N+1)n$, where N is the number of the nonzero distinct delays and n is the degree of P_0.

This bound is the same as the one from the Polya-Szegö result in [20] which gives a bound for the number of quasi-polynomial roots that are contained in a given horizontal strip. Note that the proof of Polya-Szegö result is based on Rouché's lemma, however, in this chapter, we proposed a different approach.

Remark 5 It is important to emphasize that the codimension's upper bound may exceed the number of free parameters involved in the quasi-polynomial function Δ. Indeed, the number of free parameters is $N(n+1)+n$ which is greater than $(N+1)n$.

1.4 Link with Vandermonde Matrices

In the sequel, by *functional confluent Vandermonde matrix* W, we associate to a given positive integer $s \geq 0$ the square matrix defined by

$$W = [W_1 \; W_2 \; \dots \; W_M] \in \mathscr{M}_\delta(\mathbb{R}), \tag{1.8}$$

$$\text{where } W_i = [f(x_i) \; f^{(1)}(x_i) \; \dots \; f^{(d_i-1)}(x_i)] \tag{1.9}$$

such that $\sum_{i=1}^{M} d_i = \delta$, and

$$f(x_i) = [x_i^s \; \dots \; x_i^{\delta+s-1}]^T, \quad \text{for} \quad 1 \leq i \leq M. \tag{1.10}$$

When $s = 0$, the matrix W simply defines a confluent Vandermonde matrix and thus $f(x_i) = [1 \, x_i \dots x_i^{\delta-1}]^T$. If in addition, $d_i = 1$ for $i = 1 \dots N$ then W is nothing else than a Vandermonde matrix and, in this case, $M = \delta$ since W is assumed to be a square matrix.

Let ξ stands for the vector composed from x_i counting their repetition d_i through columns of W, that is

$$\xi = (\underbrace{x_1, \dots, x_1}_{d_1}, \dots, \underbrace{x_M, \dots, x_M}_{d_M}).$$

For instance, one has $\xi_1 = x_1$ and $\xi_{d_1+d_2+1} = \xi_{d_1+d_2+d_3} = x_3$. In the light of the above notations, and under the setting $d_0 = 0$, without any loss of generality: $\xi_k = \xi_{d_0+\dots+d_r+\alpha} = \xi_{\sum_{l=0}^{\rho(k)-1} d_l + \varkappa(k)}$, where $0 \leq r \leq M-1$ and $\alpha \leq d_{r+1}$. Here $\rho(k)$ denotes the index of component of x associated with ξ_k, that is $x_{\rho(k)} = \xi_k$ and by $\varkappa(k)$ the order of ξ_k in the sequence of ξ composed only by $x_{\rho(k)}$. Obviously, $\rho(k) = r+1$ and $\varkappa(k) = \alpha$.

The following theorem provides the LU-factorization of the matrix W:

Theorem 6 *Given the functional confluent Vandermonde matrix (1.8)–(1.10), the unique LU-factorization with unitary diagonal elements $L_{i,i} = 1$ is given by the formulae:*

$$\begin{cases} L_{i,1} = x_1^{i-1} \quad for \quad 1 \le i \le \delta, \\ U_{1,l} = W_{1,l} \quad for \quad 1 \le l \le \delta, \\ L_{i,l} = L_{i-1,l-1} + L_{i-1,l}\, \xi_l \quad for \quad 2 \le l \le i, \\ U_{i,l} = (\varkappa(l) - 1)\, U_{i-1,l-1} \\ \qquad\quad + \; U_{i-1,l}\, \left(x_{\rho(l)} - \xi_{i-1}\right) \quad for \quad 2 \le i \le l. \end{cases}$$

The proof of the above theorem can be found in [4].

1.5 Illustrative Examples

As mentioned before, we can find in [2] a graph theory approach for an appropriate study of synchronization on networks. In this section, we shall use two oriented graphs to represent a network with three agents where only one delay is involved (corresponding to both information transport and process), but also a four agents network with two different delays, one corresponding to the transmission, the other to the processing of the information.

1.5.1 Three Agents Network with One Delay

This example consists in a three agents network, as depicted in Fig. 1.1. The dynamics of this system is given by

$$\begin{cases} \dot{x}_1(t) = \alpha_{11}x_1(t) + \gamma_{11}x_1(t-\tau), \\ \dot{x}_2(t) = \alpha_{22}x_2(t) + \gamma_{21}x_1(t-\tau) + \gamma_{23}x_3(t-\tau), \\ \dot{x}_3(t) = \alpha_{33}x_1(t) + \gamma_{11}x_1(t-\tau). \end{cases}$$

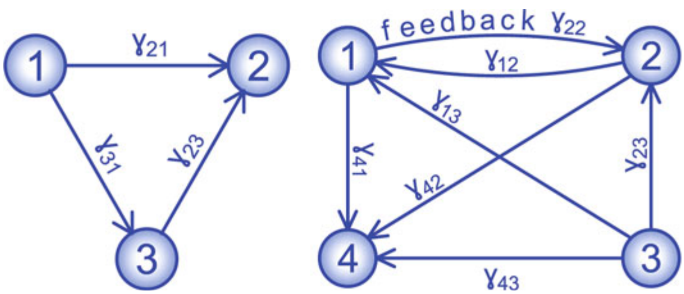

Fig. 1.1 Network consisting of 3 agents (*left*) and 4 agents (*right*)

This can be read as: the evolution of the first node x_1 is a function of its current state and its past state, while the behavior of the second agent x_2 depends not only on its current state, but also on the information sent by the agents 1 and 3, transmission that requires a time τ to take place. Finally, the third agent uses its own information and the information sent by the first agent (information that also needs a time τ to travel). In all cases, the influence of one agent on itself is weighted by some coefficients α_{ii}, while the coefficients γ_{ij} denote the impact of other agents. We write the state-space representation of this system as $\dot{x}(t) = A_0x(t) + A_1x(t-\tau)$, where $x = (x_1\, x_2\, x_3)^T$ belongs to $\mathbb{R}^3$, τ is a constant positive delay, and

$$A_0 = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix}, \quad A_1 = \begin{bmatrix} \gamma_{11} & 0 & 0 \\ \gamma_{21} & 0 & \gamma_{23} \\ \gamma_{31} & 0 & 0 \end{bmatrix}.$$

The characteristic transcendental function is given by $\{\Delta(\lambda, \tau) = P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau}\}$, with

$$P_0(\lambda) = \lambda^3 + (-\alpha_{11} - \alpha_{22} - \alpha_{33})\,\lambda^2 + (\alpha_{11}\alpha_{22} + (\alpha_{11} + \alpha_{22})\,\alpha_{33})\,\lambda - \alpha_{11}\alpha_{22}\alpha_{33},$$
$$P_1(\lambda) = -\gamma_{11}\lambda^2 + (\gamma_{11}\alpha_{22} + \gamma_{11}\alpha_{33})\,\lambda - \gamma_{11}\alpha_{22}\alpha_{33}.$$

We can easily prove that the multiplicity of the zero spectral value for this network is at most two, i.e., $\Delta(\lambda = 0, \tau) = \Delta'(\lambda = 0, \tau) = 0$ and $\Delta''(\lambda = 0, \tau) \neq 0$), and this maximum multiplicity is reached when the condition $\alpha_{11} = -\gamma_{11} = \frac{1}{\tau}$ holds, with $\tau > 0$ and other real coefficients left free.

1.5.2 Four Agents Network with Two Delays

This second example illustrates the case of a network including four agents (see Fig. 1.1). If agents 1, 3, and 4 use their own information and the one sent by others, weighted by some real coefficients γ_{ij} (as the previous example, the communication process needs some time τ_2), the agent 2 is particular: it uses the information sent by the third agent that has an impact denoted by γ_{23}, its own information, but only after receiving a feedback from the agent 1 (the feedback, weighted by γ_{22}, also needs some time τ_2), and finally it uses all this information after processing it. The information processing, weighted by β_{22}, needs some time τ_1 to take place. The mathematical model of this system can be written as $\dot{x}(t) = A_0x(t) + A_1x(t-\tau_1) + A_2x(t-\tau_2)$, with

$$A_0 = \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{33} & 0 \\ 0 & 0 & 0 & \alpha_{44} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \gamma_{12} & \gamma_{13} & 0 \\ 0 & \gamma_{22} & \gamma_{23} & 0 \\ 0 & 0 & 0 & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 0 \end{bmatrix}.$$

where $x = \left(x_1\ x_2\ x_3\ x_4\right)^T$, and τ_1, τ_2 are constant, positive, distinct delays. We write the characteristic transcendental function

$$\Delta(\lambda, \tau) = P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1} + P_2(\lambda)e^{-\lambda\tau_2}, \quad \text{with}$$

$$\begin{aligned}
P_0(\lambda) =& \lambda^4 - \left(\alpha_{1,1}\alpha_{3,3}\alpha_{4,4}\right)\lambda^3 + \left(\alpha_{1,1}\alpha_{3,3} + \left(\alpha_{1,1} + \alpha_{3,3}\right)\alpha_{4,4}\right)\lambda^2 - \alpha_{1,1}\alpha_{3,3}\alpha_{4,4}\lambda,\\
P_1(\lambda) =& -\beta_{2,2}\lambda^3 + \left(\alpha_{1,1}\beta_{2,2} + \beta_{2,2}\alpha_{4,4} + \beta_{2,2}\alpha_{3,3}\right)\lambda^2\\
& - \left(\alpha_{1,1}\beta_{2,2}\alpha_{3,3} + \left(\alpha_{1,1}\beta_{2,2} + \beta_{2,2}\alpha_{3,3}\right)\alpha_{4,4}\right)\lambda + \alpha_{1,1}\beta_{2,2}\alpha_{3,3}\alpha_{4,4},\\
P_2(\lambda) =& -\gamma_{2,2}\lambda^3 + \left(\alpha_{1,1}\gamma_{2,2} + \gamma_{2,2}\alpha_{4,4} + \gamma_{2,2}\alpha_{3,3}\right)\lambda^2\\
& + \left(-\alpha_{1,1}\gamma_{2,2}\alpha_{3,3} + \left(-\alpha_{1,1}\gamma_{2,2} - \gamma_{2,2}\alpha_{3,3}\right)\alpha_{4,4}\right)\lambda + \alpha_{1,1}\gamma_{2,2}\alpha_{3,3}\alpha_{4,4}.
\end{aligned}$$

It is easy to see that the multiplicity at origin is at least 1 when the condition $\beta_{22} = -\gamma_{22}$ holds, as $\Delta(\lambda = 0, \tau) = 0$ is equivalent with $\alpha_{11}\alpha_{33}\alpha_{44}\left(\beta_{22} + \gamma_{22}\right) = 0$. Next we compute the first derivative of $\Delta(\lambda, \tau)$ with respect to λ in $\lambda = 0$ and we can see that we have a double root at the origin if the condition $\gamma_{22} = 1/(\tau_1 - \tau_2)$ is satisfied. Furthermore, we are not able to find any other restrictions on the parameters for the second derivative of $\Delta(\lambda, \tau)$ to vanish, such that the delays τ_1 and τ_2 are positive and distinct, and all other coefficients are real numbers. Thus, we conclude that the maximum multiplicity at the origin for this network is 2.

1.6 Conclusion

This chapter introduces the problem of identifying the maximal dimension of the generalized eigenspace associated with a zero singularity for a class of quasi-polynomials. Under the assumption that all the imaginary roots are located at the origin, our result gives the relation between d (the maximal dimension of the projected state on the center manifold associated with the generalized Bogdanov–Takens singularities) from one side and N (the number of the delays) and n (the degree of the polynomial P_0) from the other side. When n_0, the maximal degree of the polynomial family $(P_i)_{1 \le i \le N}$, is less than $n - 1$, then a sharper upper bound for the dimension of the state in the center manifold can be derived.

The material in this chapter was partially presented at the 13th European Control Conference 24–27th of June 2014 in Strasbourg, France, [6]. More general systems, involving sparse polynomials with coupled delays will be considered in future works.

References

1. L.V. Ahlfors, in *Complex Analysis* (McGraw-Hill Inc., 1979)
2. F. Atay, The consensus problem in networks with transmission delays. Philos. Trans. Royal Soc. A: Math. Phys. Eng. Sci. **371**, 2013 (1999)
3. R. Bellman. K.L. Cooke, *Differential-Difference Equations* (Academic Press, 1963)

4. I. Boussaada, S.-I. Niculescu, Computing the codimension of the singularity at the origin for delay systems: The missing link with Birkhoff incidence matrices, in *International Symposium on Mathematical Theory of Networks and Systems (MTNS)* (2014)
5. I. Boussaada, I.-C. Moraresc u, S.-I. Niculescu, Inverted pendulum stabilization: characterisation of codimension-three triple zero bifurcation via multiple delayed proportional gains. Syst. Control Lett. **82**, 1–9 (2012)
6. I. Boussaada, D.I. Irofti, S.-I. Niculescu, Computing the codimension of the singularity at the origin for time-delay systems in the regular case: a vandermonde-based approach, in *European Control Conference (ECC)* (2014)
7. S. Campbell, Y. Yuan, Zero singularities of codimension two and three in delay differential equations. Nonlinearity **21**(11), 2671–2691 (2008)
8. J. Carr, *Application of Center Manifold Theory* (Springer, 1981)
9. D. Cox, J. Little, D. O'Shea, *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics (Springer, 2007)
10. M. Giesbrecht, D.-S. Roche, Interpolation of shifted-lacunary polynomials. Comput. Complex. **19**(3), 333–354 (2010)
11. J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*. vol. 42 (Springer, 2002)
12. J.K. Hale, W. Huang, Period doubling in singularly perturbed delay equations. J. Differ. Equ. **114**(1), 1–23 (1994)
13. J.K. Hale, S.M.V. Lunel, *Introduction to Functional Differential Equations*. Applied Mathematics Sciences, vol. 99 (Springer, 1993)
14. Y. Kuznetsov, *Elements of Applied Bifurcation Theory*. Second edition of Applied Mathematics Sciences, vol. 112 (Springer, 1998)
15. T. Kailath, *Linear Systems*. Prentice-Hall information and system sciences series, vol. 1 (Prentice Hall International, 1980)
16. M. Marden, *Geometry of Polynomials*. Mathematical Surveys 3 (Providence, AMS, 1966)
17. M.B.S. Marquez, I. Boussaada, H. Mounier, S.-I. Niculescu, *Analysis and Control of Oilwell Drilling Vibrations*. Advances in Industrial Control (Springer, 2015)
18. W. Michiels, S.-I. Niculescu, *Stability and Stabilization of Time-Delay Systems*. Advances in Design and Control, Society for Industrial and Applied Mathematics (SIAM), vol. 12 (2007)
19. S.-I. Niculescu, W. Michiels, Stabilizing a chain of integrators using multiple delays. IEEE Trans. Autom. Control **49**(5), 802–807 (2004)
20. G. Polya, G. Szegö, *Problems and Theorems in Analysis*. Vol. I: Series, Integral Calculus, Theory of Functions (Springer, 1972)
21. J. Sieber, B. Krauskopf, Bifurcation analysis of an inverted pendulum with delayed feedback control near a triple-zero eigenvalue singularity. Nonlinearity **17**(1), 85–103 (2007)

Chapter 2
Stability and Stabilization for Continuous-Time Difference Equations with Distributed Delay

Michael Di Loreto, Sérine Damak and Sabine Mondié

Abstract Motivated by linear hyperbolic conservation laws, we investigate in this chapter new conditions for stability and stabilization for linear continuous-time difference equations with distributed delay. For this, we propose first a state-space realization of networks of linear hyperbolic conservation laws via continuous-time difference equations. Then, based on some recent works, we propose sufficient conditions for exponential stability, which appear also to be necessary and sufficient in some particular cases. Then, the stabilization problem as well as the closed-loop performances are analyzed with constructive methods for state feedback synthesis.

2.1 Introduction

Linear 1-D conservation laws are linear one-dimensional first-order hyperbolic systems of partial differential equations (PDEs) [6]. Conservation laws appear in many engineering applications, as for instance in electrical lines, gas flow in pipelines, open water channel flow, road traffic or heat exchangers. See for instance [2, 3, 24, 30, 42]. When the conservation laws are linearized, they are reduced to transport or propagation equations [5, 8, 15, 21, 43].

From this well known fact, we propose to investigate the stability and performance properties for a set of linear coupled conservation laws with boundary control from

---

This chapter represents an extended version of the conference proceedings paper [14].

M. Di Loreto (✉) · S. Damak
Université de Lyon, Laboratoire Ampère, UMR CNRS 5005, INSA-Lyon,
20 Avenue Albert Einstein, 69621 Villeurbanne, France
e-mail: michael.di-loreto@insa-lyon.fr

S. Damak
e-mail: serine.damak@insa-lyon.fr

S. Mondié
Departamento de Control Automatico, Av. Instituto Politecnico Nacional 2508,
Col. San Pedro Zacatenco, 07360 Del. G.A. Madero, Mexico D.F., Mexico
e-mail: smondie@ctrl.cinvestav.mx

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_2

their transport phenomena, using an exact state-space time-delay realization. The time-delay model is governed by linear continuous-time difference equations, in the form

$$x(t) = Ax(t-r) + Bu(t),\ t \geq 0, \tag{2.1}$$

where $u(t)$ is the input control, $x(t)$ the state, A a constant $n \times n$ matrix, B a constant $n \times m$ matrix and $r > 0$ the transport delay. Based on this model, we investigate exponential stability conditions, as well as boundary control for exponential stabilization and closed-loop performances achievement.

Feedback control for (2.1) may involve static delayed-state feedback, which yields a continuous-time difference equation in closed-loop. More generally, feedback control may involve integral operators in time (and in space), like distributed delay operators. The system (2.1) may be then transformed into the more general class of integral-difference equations in the form

$$x(t) = Ax(t-r) + \int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta,\ t \geq 0, \tag{2.2}$$

where the matrix function $G(\theta)$ has piecewise continuous bounded elements defined for $\theta \in [0, r]$ [5, 9, 31].

Stability and stabilization problems for conservation laws with boundary control have received renewed attention in recent years, related to challenging applications. See for instance [4, 10, 11, 16, 18, 25, 33, 41] and the references therein for Lyapunov-based approaches, [34] for a frequency approach, or [45] for a functional approach. On the other side, stability and stabilization of difference equations in the form (2.1) were studied in various works. In [1, 17, 27–29, 35], sensitivity in the delays for stability and stabilization were analyzed, via delay-independent spectral conditions, Lyapunov-Krasovskii approach was investigated in [7, 12, 20, 22, 40, 44, 46] with numerical tractable conditions for testing stability. Note also that the difference equation (2.1) appears fundamental in neutral time-delay systems. See for instance [23, 26, 32] and in references therein. Integral-difference equations in the form (2.2) have received recently a renewed interest. With constructive methods, stability conditions using Lyapunov-Krasovskii approach were proposed in [13, 37–39].

In this chapter, we propose to analyze linear 1-D conservation laws using continuous-time difference equations (with distributed delay). From a Lyapunov-Krasovskii approach, we give necessary and sufficient conditions for exponential stability in the case of purely difference equations, and sufficient conditions for exponential stability of integral-difference equations in the form (2.2). Then, state-feedback synthesis is studied via constructive numerical algorithms, in order to achieve closed-loop stability or L_2-gain performance.

The chapter is organized as follows: An exact time-delay state-space realization for linear 1-D hyperbolic conservation laws is derived in Sect. 2.2. Essential definitions and stability results are given in Sect. 2.3. In Sect. 2.4, state-feedback synthesis is analyzed. Some concluding remarks and a perspective discussion for large-scale network of conservation laws end the chapter.

Notations: The transpose of a matrix P is denoted by P^T while the smallest and the largest eigenvalues of a symmetric positive (semi)definite matrix P are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively. The standard notation $P \succ 0$ and $P \succeq 0$ are used for symmetric positive definite and positive semidefinite matrices, respectively. Similarly, $P \prec 0$ and $P \preceq 0$ stand for symmetric negative definite and negative semi-definite matrices, respectively.

The space of continuous and bounded functions defined on $[-r, 0)$ is $\mathscr{C}([-r, 0), \mathbb{R}^n)$, while the space of piecewise right-continuous and bounded functions defined on $[-r, 0)$ is denoted by $\mathscr{P}_{\mathscr{C}}([-r, 0), \mathbb{R}^n)$. These spaces are equipped with the uniform norm $\|\varphi\|_r = \sup_{-r \le \theta \le 0} \|\varphi(\theta)\|$ and with the L_2 norm $\|\varphi\|^2_{L_2} = \int_{-r}^0 \|\varphi(\theta)\|^2 \, \mathrm{d}\theta$, where $\|\varphi(\theta)\|$ stands for the Euclidean norm. For a matrix $G \in \mathbb{R}^{n\times n}$, $\|G\|$ denotes the spectral norm, that is $\|G\|^2 = \lambda_{\max}(G^T G)$. I_n stands for the $n \times n$ identity matrix, and $\rho(A)$ denotes the spectral radius for the matrix A. We denote by $x(t, \varphi)$ the solution of the system under consideration with the initial condition φ and by $x_t(\varphi) = \{x(t + \theta, \varphi) \mid \theta \in [-r, 0)\}$ the partial trajectory of the system. When the initial function is clear from the context, the argument φ is dropped.

2.2 Linear Hyperbolic Conservation Laws

In this section, starting from linear one-dimensional first-order hyperbolic partial differential equations with boundary conditions, we derive an exact state-space realization as a time-delay model.

In order to fix the ideas, let us consider first the simplest example of a scalar linear conservation law, governed by

$$\lambda \partial_t \psi(t, z) + \partial_z \psi(t, z) = 0, \;\; t > 0, z \in (0, \ell], \tag{2.3}$$

with $\lambda > 0$, an initial condition $\psi(0, z) = \psi_0(z)$ for $z \in (0, \ell]$ and a (time-dependent) boundary condition $\psi(t, 0) = u(t)$ for $t > 0$. For any $z \in (0, \ell]$, the unique solution of (2.3) is

$$\psi(t, z) = \begin{cases} \psi_0(z - \frac{t}{\lambda}) & , \;\text{for}\;\; t \le \lambda z \\ \psi(t - \lambda z, 0) = u(t - \lambda z) & , \;\text{for}\;\; t > \lambda z \end{cases} . \tag{2.4}$$

Indeed, the Laplace transform applied on (2.3) leads to $\hat{\psi}(s, z) = \mathrm{e}^{-\lambda z s} \hat{\phi}(s)$, for some function $\phi(\cdot)$. In other words, $\psi(t, z) = \phi(t - \lambda z)$, for any $t \ge 0$. At time $t = 0$, we obtain $\psi_0(z) = \phi(-\lambda z)$, so that

$$\phi(t - \lambda z) = \psi_0(z - \frac{t}{\lambda}), \;\; \text{for}\; t \le \lambda z.$$

Furthermore, at $z = 0$, $u(t) = \psi(t, 0) = \phi(t)$, which implies that $\psi(t, z) = \psi(t - \lambda z, 0)$ for $t > \lambda z$. It is then a routine to verify that $\psi(t, z)$ is continuous at $t = \lambda z$, and is indeed the solution of (2.3).

The main interest in the formulation (2.4) is the delay-realization of the solution, which unifies the time-space coupled effects in (2.3) into a space-dependent time-delay. For the more general case, the 1-D linear (lossless) conservations laws are of the form [6]

$$\Lambda \partial_t \psi(t, z) + \partial_z \psi(t, z) = 0, \ t > 0, z \in (0, \ell), \tag{2.5}$$

where $\psi : \mathbb{R}^+ \times [0, \ell] \to \mathbb{R}^n$, $\psi(t, z) = \begin{bmatrix} \psi^+(t,z) \\ \psi^-(t,z) \end{bmatrix}$ with $\psi^+(\cdot, \cdot) \in \mathbb{R}^{n^+}$, $\psi^-(\cdot, \cdot) \in \mathbb{R}^{n^-}$, $n^+ + n^- = n$ and $\Lambda = \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix}$. The matrix Λ is assumed, without loss of generality, to be diagonal, that is $\Lambda^+ = \mathrm{diag}(\lambda_1^+, \ldots, \lambda_{n^+}^+)$, $\Lambda^- = \mathrm{diag}(\lambda_1^-, \ldots, \lambda_{n^-}^-)$, where $\lambda_i^+ > 0$ for $i = 1, \ldots, n^+$ and $\lambda_i^- < 0$ for $i = 1, \ldots, n^-$. The initial condition for (2.5) is $\psi(0, z) = \psi_0(z)$ for $z \in (0, \ell)$, and the boundary conditions are of the form [16]

$$\begin{bmatrix} \psi^+(t, 0) \\ \psi^-(t, \ell) \end{bmatrix} = \begin{bmatrix} \Gamma_{00} & \Gamma_{01} \\ Ł\Gamma_{10} & \Gamma_{11} \end{bmatrix} \begin{bmatrix} \psi^+(t, \ell) \\ \psi^-(t, 0) \end{bmatrix} + \begin{bmatrix} B_+ \\ B_- \end{bmatrix} u(t), \tag{2.6}$$

for constant matrices Γ_{ij} with appropriate size, $B_+ \in \mathbb{R}^{n^+ \times m}$, $B_- \in \mathbb{R}^{n^- \times m}$ and $u(\cdot) \in \mathbb{R}^m$ some exogenous input signal. From (2.4), it is trivial to see that

$$\psi_i^+(t, z) = \psi_i^+(t - \lambda_i^+ z, 0), \ \text{for } t > \lambda_i^+ z, \tag{2.7}$$

$$\psi_i^-(t, z) = \psi_i^-(t + \lambda_i^-(\ell - z), \ell), \ \text{for } t > \lambda_i^-(z - \ell), \tag{2.8}$$

where ψ_i^+ and ψ_i^- stand for the entries of ψ^+ and ψ^-, respectively. Introduce the time-delay operators $\sigma_i^+ : f(t) \mapsto f(t - \lambda_i^+ \ell)$ for $i = 1, \ldots, n^+$, $\sigma_i^- : f(t) \mapsto f(t + \lambda_i^- \ell)$ for $i = 1, \ldots, n^-$. With $\sigma^+ = \mathrm{diag}(\sigma_1^+, \ldots, \sigma_{n^+}^+)$ and $\sigma^- = \mathrm{diag}(\sigma_1^-, \ldots, \sigma_{n^-}^-)$, the system (2.5) and (2.6) is transformed into the delay-operator realization

$$\begin{bmatrix} \psi^+(t, 0) \\ \psi^-(t, \ell) \end{bmatrix} = \begin{bmatrix} \Gamma_{00} & \Gamma_{01} \\ Ł\Gamma_{10} & \Gamma_{11} \end{bmatrix} \begin{bmatrix} \sigma^+ \psi^+(t, 0) \\ \sigma^- \psi^-(t, \ell) \end{bmatrix} + \begin{bmatrix} B_+ \\ B_- \end{bmatrix} u(t). \tag{2.9}$$

For the instantaneous state $x(t) = \begin{bmatrix} \psi^+(t,0) \\ \psi^-(t,\ell) \end{bmatrix}$, and if the delays $\lambda_i^+ \ell$ and $-\lambda_i^- \ell$ are rationally dependent, the realization (2.9) is a state-space realization with difference equations in the form (2.1), where the matrix A depends on the length of the delays and on the matrices Γ_{ij}. Remark that (2.9) holds for any $t \geq \max_i \{\lambda_i^+ \ell, -\lambda_i^- \ell\}$. For times less than this lower bound, the solution is determined as in (2.4) from its initial condition. We can then reduce this realization for any $t \geq 0$ with a time translation. From Eqs. (2.7) and (2.8), note also that the stability for $x(t)$ is equivalent to the stability of $\psi(t, z)$ for any $t \geq 0$ and $z \in [0, \ell]$.

2.3 Exponential Stability Analysis

2.3.1 Definitions and Main Properties

Let us consider the linear system governed by difference equations with pointwise commensurate delays and distributed delays

$$x(t) = Ax(t-r) + \int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta \tag{2.10}$$

where $A \in \mathbb{R}^{n\times n}$, $r > 0$, and the matrix function $G(\theta) \in \mathbb{R}^{n\times n}$ has piecewise continuous bounded elements defined for $\theta \in [0, r]$. For any piecewise right-continuous and bounded initial condition $\varphi \in \mathscr{P}_{\mathscr{C}}([-r, 0), \mathbb{R}^n)$, there exists a unique solution $x(t, \varphi)$ of (2.10), for all $t \geq 0$. Such a solution is called the system response of (2.10). This solution is piecewise continuous, in general, and is determined by an iterative scheme in time, with

$$x(0, \varphi) = A\varphi(-r) + \int_0^r G(\theta)\varphi(-\theta)\mathrm{d}\theta.$$

Such equality defines the step discontinuity of the system response at time $t = 0$, which will be propagated in time. In the case of linear 1-D conservation laws, the identity $x(0, \varphi) = \varphi(0^-)$ holds, leading to the so-called initial matching condition. In such a case, if the initial condition $\varphi \in \mathscr{C}([-r, 0), \mathbb{R}^n)$, the system response $x(t, \varphi)$ is also continuous.

Let us remind definitions and some results concerning estimates of this solution. Obviously, these definitions apply also in the case of difference equations (2.1) where $G(\cdot) = 0$ in (2.10).

Definition 1 System (2.10) is said to be exponentially stable if there exists $\gamma \geq 0$ and $\mu > 0$ such that any solution $x(t, \varphi)$ of the system (2.10) satisfies

$$\|x(t, \varphi)\| \leq \gamma \, \|\varphi\|_r \, \mathrm{e}^{-\mu t}, t \geq 0.$$

In such a case, the system response $x(t, \varphi)$ is said to be exponentially stable, with decay rate greater than μ. In the present contribution, we also use the concept of L_2-exponential stability, which is clearly weaker than exponential stability, defined as follows.

Definition 2 System (2.10) is said to be L_2-exponentially stable if there exists $\gamma \geq 0$ and $\mu > 0$ such that any solution $x(t, \varphi)$ of the system (2.10) satisfies

$$\|x_t(\varphi)\|_{L_2} \leq \gamma \, \|\varphi\|_r \, \mathrm{e}^{-\mu t}, t \geq 0.$$

Considering the distributed delay in (2.10) as a perturbation term, a general result for exponential stability was obtained in [37]. Exponential estimate for the system response was retrieved from the analysis of some difference equations with only pointwise delays. In this contribution, while the arguments are indeed similar, another characterization for exponential stability is obtained. For the construction of exponential estimates, we will use the following central result on L_2-exponential stability.

Theorem 1 [12] *Let $x_t(\varphi)$ be a partial trajectory for (2.10), and assume that there exists a continuous functional $v : \mathscr{P}_{\mathscr{C}}([-r, 0), \mathbb{R}^n) \to \mathbb{R}$ such that $v(x_t(\varphi))$ is upper right-hand side differentiable with respect to t along the trajectories of (2.10) and satisfies the following conditions:*

(i) $\alpha_1 \|\varphi\|_{L_2}^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_r^2$, *for some constants* $0 < \alpha_1$, $0 \leq \alpha_2$,
(ii) $\frac{\mathrm{d}}{\mathrm{d}t} v(x_t(\varphi)) + 2\mu\, v(x_t(\varphi)) \leq 0$ *for some* $\mu > 0$.

Then system (2.10) is L_2-exponentially stable, and the following exponential estimate

$$\|x_t(\varphi)\|_{L_2} \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\, \|\varphi\|_r \, \mathrm{e}^{-\mu t}$$

holds for all $t \geq 0$.

2.3.2 Stability for Difference Equations

For continuous-time linear difference equations of the form (2.1) with $u(\cdot) = 0$, the exponential stability analysis is trivial. Indeed, its system response is

$$x(t, \varphi) = A^k \varphi(t - kr),\ \forall t \in [(k-1)r, kr),\ k \in \mathbb{N}. \tag{2.11}$$

It follows that the system (2.1) is exponentially stable if and only if $\rho(A) < 1$ (see for instance [19]), or equivalently if and only if there exists a symmetric $n \times n$ positive definite matrix P such that

$$A^T P A - P \prec 0. \tag{2.12}$$

These stability conditions are equivalent to those related to discrete-time linear systems, and show the fruitful realization as a time-delay system for linear conservation laws.

2.3.3 Stability for Difference Equations with Distributed Delay

For integral-difference equations in the form (2.10), our strategy for the exponential stability analysis is based on the following result, leading to a delay-dependent stability condition [14]. Such a condition does not require the assumption that the matrix A is stable.

Theorem 2 *The system (2.10) is exponentially stable if there exist an $n \times n$ symmetric real positive definite matrix $\tilde{P}$, an $n \times n$ symmetric real positive semi-definite matrix $S(\theta)$, for all $\theta \in [0, r]$, and a positive constant $\mu > 0$ such that*

$$P = \tilde{P} - \int_0^r S(\theta)\,\mathrm{d}\theta \succ 0, \tag{2.13}$$

and $M_\mu(\theta) \succeq 0$, for all $\theta \in [0, r]$, where $M_\mu(\theta)$ is given by

$$-M_\mu(\theta) = r\begin{bmatrix} A^T\tilde{P}A - \mathrm{e}^{-2\mu r}P & A^T\tilde{P}G(\theta) \\ G^T(\theta)\tilde{P}A & G^T(\theta)\tilde{P}G(\theta) - \frac{\mathrm{e}^{-2\mu\theta}S(\theta)}{r} \end{bmatrix}. \tag{2.14}$$

Moreover, for any $\varepsilon \in (0, \mu)$, the following exponential estimate of the system response holds

$$\|x(t, \varphi)\| \le \gamma\, \|\varphi\|_r\, \mathrm{e}^{-(\mu-\varepsilon)t}, \tag{2.15}$$

for all $t \ge 0$, where

$$\gamma = \sqrt{\frac{\lambda_{\max}(P) + \alpha + \dfrac{\alpha}{2r\varepsilon \mathrm{e}}}{\lambda_{\min}(P)}}. \tag{2.16}$$

The positive constant α is given by

$$\alpha = \frac{\alpha_2}{\alpha_1} \cdot \sup_{0\le\theta\le r} \lambda_{\max}(S(\theta)),$$

where $\alpha_1 = \mathrm{e}^{-2\mu r}\lambda_{\min}(P)$ and $\alpha_2 = r\lambda_{\max}(P) + \frac{r^2}{2} \cdot \sup_{0\le\theta\le r} \lambda_{\max}(S(\theta))$.

Proof Assume that the conditions of the theorem are fulfilled, and consider the Lyapunov-Krasovskii functional

$$\begin{aligned} v_\mu(x_t(\varphi)) &= \int_{t-r}^{t} \mathrm{e}^{-2\mu(t-\theta)} x^T(\theta) P x(\theta)\,\mathrm{d}\theta \\ &\quad + \int_0^r \int_{t-\theta}^{t} \mathrm{e}^{-2\mu(t-\xi)} x^T(\xi) S(\theta) x(\xi)\,\mathrm{d}\xi\,\mathrm{d}\theta, \end{aligned} \tag{2.17}$$

where P, $S(\theta)$ and μ are described in (2.13) and (2.14). This functional $v_\mu(\cdot)$ is continuous and admits the following lower and upper bounds

$$\alpha_1 \, \|\varphi\|_{L_2}^2 \le v_\mu(\varphi) \le \alpha_2 \, \|\varphi\|_r^2 \,, \tag{2.18}$$

where α_1 and α_2 are given in Theorem 2. The fact that P is symmetric positive definite implies that $\alpha_1 > 0$ and $\alpha_2 > 0$. Furthermore, $v_\mu(x_t(\varphi))$ is upper right-hand differentiable, and its upper right-hand side time derivative along the trajectories of (2.10) satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} v_\mu(x_t(\varphi)) =& -2\mu \, v_\mu(x_t(\varphi)) + x^T(t)\tilde{P}x(t) - \mathrm{e}^{-2\mu r} x^T(t-r)Px(t-r)\\ &- \int_0^r \mathrm{e}^{-2\mu\theta} x^T(t-\theta)S(\theta)x(t-\theta)\,\mathrm{d}\theta.\end{aligned} \tag{2.19}$$

Substituting (2.10) into (2.19) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} v_\mu(x_t(\varphi)) + 2\mu \, v_\mu(x_t(\varphi)) =& \, 2x^T(t-r)A^T\tilde{P}\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta\\ &+ x^T(t-r)[A^T\tilde{P}A - \mathrm{e}^{-2\mu r}P]x(t-r)\\ &- \int_0^r \mathrm{e}^{-2\mu\theta} x^T(t-\theta)S(\theta)x(t-\theta)\mathrm{d}\theta\\ &+ \left(\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta\right)^T \tilde{P}\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta.\end{aligned} \tag{2.20}$$

Applying Jensen's integral inequality to the last term in (2.20),

$$\begin{aligned}&\left(\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta\right)^T \tilde{P}\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta\\ &\qquad \le r\int_0^r x^T(t-\theta)G^T(\theta)\tilde{P}G(\theta)x(t-\theta)\mathrm{d}\theta,\end{aligned}$$

and substituting it into (2.20), we finally obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\mu(x_t(\varphi)) + 2\mu \, v_\mu(x_t(\varphi)) \le -\int_0^r \chi^T(\theta)M_\mu(\theta)\chi(\theta)\,\mathrm{d}\theta,$$

where $\chi^T(\theta) = \left[(1/r)x^T(t-r) \; x^T(t-\theta)\right]$ and $M_\mu(\theta)$ is given in (2.14). Since $M_\mu(\theta) \succeq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\mu(x_t(\varphi)) + 2\mu \, v_\mu(x_t(\varphi)) \le 0, \;\; \forall t \ge 0. \tag{2.21}$$

From Theorem 1 and (2.18), we conclude that $x_t(\varphi)$ is L_2-exponentially stable, with

$$\|x_t(\varphi)\|_{L_2} \le \sqrt{\frac{\alpha_2}{\alpha_1}}\, \|\varphi\|_r\, \mathrm{e}^{-\mu t}. \tag{2.22}$$

The inequality (2.22) implies that

$$\int_0^r \mathrm{e}^{-2\mu\theta} x^T(t-\theta)S(\theta)x(t-\theta)\mathrm{d}\theta \le \alpha\, \|\varphi\|_r^2\, \mathrm{e}^{-2\mu t}, \tag{2.23}$$

where α is given in Theorem 2. From (2.19) and (2.21), we see that the system response $x(t)$ satisfies

$$\begin{aligned} x^T(t)\tilde{P}x(t) \le{} & \mathrm{e}^{-2\mu r} x^T(t-r)Px(t-r) \\ & + \int_0^r \mathrm{e}^{-2\mu\theta} x^T(t-\theta)S(\theta)x(t-\theta)\,\mathrm{d}\theta, \end{aligned} \tag{2.24}$$

or in other words, from (2.13) and (2.23),

$$x^T(t)Px(t) \le \mathrm{e}^{-2\mu r} x^T(t-r)Px(t-r) + \alpha\, \|\varphi\|_r^2\, \mathrm{e}^{-2\mu t}.$$

This inequality is equivalent to

$$\psi(t) = \mathrm{e}^{-2\mu r}\psi(t-r) + f(t),$$

for $\psi(t) = x^T(t)Px(t)$ and some piecewise continuous function $f(t)$, with $|f(t)| \le \alpha\, \|\varphi\|_r^2\, \mathrm{e}^{-2\mu t}$, for all $t \ge 0$. Let $n \in \mathbb{N}$, and take $t = nr + \xi$ where $\xi \in [-r, 0)$. From the standard arguments in [32] (Lemma 6, p. 797), we obtain

$$\begin{aligned} \psi(t) &= \mathrm{e}^{-2\mu nr}\psi(t-nr) + \sum_{k=0}^{n-1} \mathrm{e}^{-2\mu kr} f(t-kr) \\ &\le \mathrm{e}^{-2\mu nr}\psi(t-nr) + n\alpha\, \|\varphi\|_r^2\, \mathrm{e}^{-2\mu t}. \end{aligned}$$

Substituting $\psi(t)$ and considering that $t \in [(n-1)r, nr)$, this last inequality leads to

$$\lambda_{\min}(P)\, \|x(t,\varphi)\|^2 \le \left[\lambda_{\max}(P) + \alpha + \frac{\alpha}{r}t\right] \|\varphi\|_r^2\, \mathrm{e}^{-2\mu t}.$$

For any $\varepsilon \in (0, \mu)$ (which is possible since $\mu > 0$), we observe that

$$t\, \mathrm{e}^{-2\mu t} = t\, \mathrm{e}^{-2\varepsilon t}\mathrm{e}^{-2(\mu-\varepsilon)t} \le \frac{1}{2\varepsilon \mathrm{e}}\mathrm{e}^{-2(\mu-\varepsilon)t}.$$

As $\mathrm{e}^{-2\varepsilon t} \leq 1$, we finally get

$$\|x(t,\varphi)\|^2 \leq \frac{\lambda_{\max}(P) + \alpha + \frac{\alpha}{2r\varepsilon \mathrm{e}}}{\lambda_{\min}(P)} \|\varphi\|_r^2 \, \mathrm{e}^{-2(\mu-\varepsilon)t},$$

which gives the desired exponential estimate for the system response of (2.10), for all $t \geq 0$, since such a bound is independent from $n \in \mathbb{N}$.

Remark 1 The conditions described in Theorem 2 ensure that A is a Schur stable matrix. Indeed, it follows from (2.14) that $A^T \tilde{P} A - \mathrm{e}^{-2\mu r} P \preceq 0$. From (2.13), this in turn implies that

$$\begin{aligned} A^T P A - P + (1 - \mathrm{e}^{-2\mu r}) P &= A^T P A - \mathrm{e}^{-2\mu r} P \\ &\preceq A^T \tilde{P} A - \mathrm{e}^{-2\mu r} P \\ &\preceq 0. \end{aligned}$$

Since P is symmetric positive definite and $\mu > 0$, we conclude that $A^T P A - P \prec 0$, which is equivalent to the assertion that A is Schur stable. Clearly, the Schur stability assumption in the Lyapunov-type Theorem 3 of [37] is fulfilled, as well as the functional conditions 1 and 2 therein. Hence Theorem 3 in [37] allows to conclude that system (2.10) is exponentially stable.

Remark 2 Note that the matrix $S(\theta)$, for $\theta \in [0, r]$, is required to be only symmetric positive semi-definite. This fact should be compared to time-delay systems with differential equations [23], where $S(\theta)$ is required, in general, to be positive definite. This remark allows us to apply Theorem 2 in the particular case of difference equations with only pointwise delays, as in [7] or [22]. Indeed, for such systems, $G(\theta) = 0$ for all $\theta \in [0, r]$. Taking $S(\theta) = 0$ on the whole interval, the conditions given in Theorem 2 lead to the existence of $P \succ 0$ and $\mu > 0$ such that $A^T P A - \mathrm{e}^{-2\mu r} P \preceq 0$.

Remark 3 A virtue of Theorem 2 is that, unlike in [37], there is no additional assumption on the stability of the difference operator $\mathscr{A} : \mathscr{P}_{\mathscr{C}}([-r, 0), \mathbb{R}^n) \mapsto \mathbb{R}^n$ defined by

$$\mathscr{A}(x_t) = x(t) - Ax(t - r).$$

It is worthy of mention that a similar feature appears in [20], in the case of neutral type systems. Furthermore, exponential estimates for the system response described in [37] require some knowledge about the decreasing properties of $\|A^k\|$, when $k \to \infty$, $k \in \mathbb{N}$. This fact is not used a priori in the estimate of Theorem 2.

The conditions presented in Theorem 2 are not easy to verify as they depend on the continuous parameter θ in the bounded interval $[0, r]$. Similar conditions to those in Proposition 4 of [37] can be obtained from Theorem 2. But in order to reduce the conservatism of these sufficient conditions, we propose the following tractable conditions.

Lemma 1 *The system (2.10) is exponentially stable if there exist an $n \times n$ symmetric real positive definite matrix $\tilde{P}$, a symmetric positive semi-definite matrix $\tilde{S}$, and some constants $\mu > 0$ and $\beta > 0$ such that*

$$\beta \int_0^r \|G(\theta)\|^2 \,\mathrm{d}\theta \cdot I_n \prec \tilde{P}, \tag{2.25}$$

$$\tilde{S} \preceq \beta \cdot I_n, \tag{2.26}$$

$$0 \preceq \tilde{M}_\mu, \tag{2.27}$$

where $\tilde{M}_\mu$ is given by

$$-\tilde{M}_\mu = r \begin{bmatrix} A^T \tilde{P} A - \mathrm{e}^{-2\mu r}(\tilde{P} - \beta \int_0^r \|G(\theta)\|^2 \,\mathrm{d}\theta \cdot I_n) & A^T \tilde{P} \\ \tilde{P} A & \tilde{P} - \frac{\mathrm{e}^{-2\mu r}}{r} \tilde{S} \end{bmatrix}. \tag{2.28}$$

Moreover, for any $\varepsilon \in (0, \mu)$, the following exponential estimate of the system response holds

$$\|x(t, \varphi)\| \leq \gamma \, \|\varphi\|_r \, \mathrm{e}^{-(\mu-\varepsilon)t}, \tag{2.29}$$

for all $t \geq 0$, where

$$\gamma = \sqrt{\frac{\lambda_{\max}(P) + \alpha + \dfrac{\alpha}{2r\varepsilon \mathrm{e}}}{\lambda_{\min}(P)}} \tag{2.30}$$

and $P = \tilde{P} - \int_0^r G^T(\theta)\tilde{S}G(\theta)\mathrm{d}\theta$. The positive constant α is obtained by

$$\alpha = \frac{\alpha_2 \beta}{\alpha_1} \cdot \sup_{0 \leq \theta \leq r} \lambda_{\max}(G^T(\theta)G(\theta)),$$

where $\alpha_1 = \mathrm{e}^{-2\mu r}\lambda_{\min}(P)$ and $\alpha_2 = r\lambda_{\max}(P) + \frac{r^2}{2}\beta \cdot \sup_{0 \leq \theta \leq r} \lambda_{\max}(G^T(\theta)G(\theta))$.

Proof The proof is divided into two steps. In the first step, we show that (2.25)–(2.27) imply $M_\mu(\theta) \succeq 0$, for all $\theta \in [0, r]$, in Theorem 2. The second step will be devoted to the exponential estimate (2.29).

Assume that conditions (2.25)–(2.27) are satisfied. For any $\theta \in [0, r]$, we have $\mathrm{e}^{-2\mu r}\tilde{S} \preceq \mathrm{e}^{-2\mu\theta}\tilde{S}$. Furthermore, from (2.26),

$$-\beta \int_0^r \|G(\theta)\|^2 \,\mathrm{d}\theta \cdot I_n \preceq -\int_0^r G^T(\theta)\tilde{S}G(\theta)\mathrm{d}\theta.$$

Using these two inequalities as upper bounds in the block-diagonal terms of $\tilde{M}_\mu$ in (2.28), we see that the matrix $N_\mu(\theta)$ defined by

$$- N_\mu(\theta) = r \begin{bmatrix} A^T \tilde{P} A - \mathrm{e}^{-2\mu r}(\tilde{P} - \int_0^r G^T(\theta)\tilde{S}G(\theta)\mathrm{d}\theta) & A^T \tilde{P} \\ \tilde{P} A & \tilde{P} - \frac{\mathrm{e}^{-2\mu\theta}}{r}\tilde{S} \end{bmatrix}, \quad (2.31)$$

is a symmetric real positive semi-definite matrix, for all θ in $[0, r]$. For any θ in $[0, r]$, we define $S(\theta) = G^T(\theta)\tilde{S}G(\theta)$ and

$$P = \tilde{P} - \int_0^r S(\theta)\,\mathrm{d}\theta, \quad (2.32)$$

where $\tilde{P} \succ 0$ is the solution of (2.25)–(2.27). The matrix $S(\theta)$ is symmetric positive semi-definite, while P in (2.32) is positive definite since (2.25) and (2.26) are fulfilled. We see that, with the notations in Theorem 2,

$$-M_\mu(\theta) = \begin{bmatrix} I_n & 0 \\ 0 & G^T(\theta) \end{bmatrix} (-N_\mu(\theta)) \begin{bmatrix} I_n & 0 \\ 0 & G(\theta) \end{bmatrix}.$$

Then, we conclude that $M_\mu(\theta) \succeq 0$, for all $\theta \in [0, r]$. The exponential stability of (2.10) and the estimate (2.29) for the system response follow straightforwardly from Theorem 2, since, by construction,

$$\begin{aligned} \lambda_{\max}(S(\theta)) &= \lambda_{\max}(G^T(\theta)\tilde{S}G(\theta)) \\ &\leq \beta\,\lambda_{\max}(G^T(\theta)G(\theta)) \\ &= \beta\,\|G(\theta)\|^2 . \end{aligned}$$

2.3.4 Conservatism Analysis

The conditions for exponential stability of (2.10) presented in Lemma 1 are less conservative than those given in [37]. In order to prove this assertion, we consider the two possible cases.

First, let us assume that

$$\sup_{0\leq\theta\leq r} \|G(\theta)\| > 0, \quad (2.33)$$

and the conditions of Proposition 1 in [37] are fulfilled. Under such assumptions, we construct, in what follows, the solutions of (2.25)–(2.27).

Since

$$\frac{1}{r}\|G_r\|_{L_2}^2 = \frac{1}{r}\int_0^r \|G(\theta)\|^2\,\mathrm{d}\theta \leq \sup_{0\leq\theta\leq r} \|G(\theta)\|^2$$

holds, Proposition 1 in [37] tells us that there exist symmetric positive definite matrices W_0, W_1 and P_1, such that

$$\|G_r\|_{L_2}^2 (P_1 + P_1 A W_0^{-1} A^T P_1) \prec \lambda_{\min}(W_1) \cdot I_n, \tag{2.34}$$

and

$$A^T P_1 A - P_1 = -(W_0 + r W_1). \tag{2.35}$$

By definition, $G(\cdot)$ has piecewise continuous and bounded elements, so that (2.33) implies that $0 < \|G_r\|_{L_2}^2$. By Schur complement, we see from (2.34) that

$$\begin{bmatrix} W_0 & -A^T P_1 \\ -P_1 A & \frac{\lambda_{\min}(W_1)}{\|G_r\|_{L_2}^2} I_n - P_1 \end{bmatrix} \succ 0. \tag{2.36}$$

Moreover, from (2.35),

$$\begin{aligned} W_0 &= -A^T P_1 A + P_1 - r W_1 \preceq -A^T P_1 A + P_1 - r\lambda_{\min}(W_1) \cdot I_n \\ &= -A^T P_1 A + P_1 - r \frac{\lambda_{\min}(W_1)}{\|G_r\|_{L_2}^2} \|G_r\|_{L_2}^2 \cdot I_n. \end{aligned}$$

From this, define

$$\tilde{S} = r \frac{\lambda_{\min}(W_1)}{\|G_r\|_{L_2}^2} \cdot I_n \overset{\triangle}{=} \beta \cdot I_n. \tag{2.37}$$

The matrix $\tilde{S}$ is symmetric positive definite, and satisfies (2.26). Defining $\tilde{P} = P_1$ the symmetric positive definite matrix solution of (2.35), we see that

$$r W_1 = -W_0 + \tilde{P} - A^T \tilde{P} A \prec \tilde{P}.$$

This implies $r\lambda_{\min}(W_1) \cdot I_n \prec \tilde{P}$, or equivalently

$$\beta \int_0^r \|G(\theta)\|^2 \, \mathrm{d}\theta \cdot I_n \prec \tilde{P}.$$

Then (2.25) holds. We next show that (2.27) holds for some $\mu > 0$. From (2.36) and the upper bound on W_0, the matrix

$$\tilde{M}_0 = r \begin{bmatrix} -A^T \tilde{P} A + \tilde{P} - \beta \|G_r\|_{L_2}^2 \cdot I_n & -A^T \tilde{P} \\ -\tilde{P} A & \frac{1}{r}\tilde{S} - \tilde{P} \end{bmatrix}$$

is symmetric positive definite, and $\lambda_{\max}(\tilde{P}) \leq \frac{1}{r}\lambda_{\max}(\tilde{S})$. Remarking that $\tilde{M}_0$ satisfies

$$\frac{1}{r}\tilde{M}_0 \preceq \begin{bmatrix} \tilde{P} & 0 \\ 0 & \frac{1}{r}\tilde{S} \end{bmatrix} - \begin{bmatrix} A^T & 0 \\ I_n & 0 \end{bmatrix} \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{P} \end{bmatrix} \begin{bmatrix} A & I_n \\ 0 & 0 \end{bmatrix}$$
$$\preceq \begin{bmatrix} \tilde{P} & 0 \\ 0 & \frac{1}{r}\tilde{S} \end{bmatrix} \preceq \frac{1}{r}\lambda_{\max}(\tilde{S}) \cdot I_{2n},$$

we conclude that $\lambda_{\min}(\tilde{M}_0) \leq \lambda_{\max}(\tilde{S})$. In other words, $1 - \frac{\lambda_{\min}(\tilde{M}_0)}{\lambda_{\max}(\tilde{S})} \in [0, 1)$. Define $\mu > 0$ such that

$$0 < \mu \leq -\frac{1}{2r}\ln\left(1 - \frac{\lambda_{\min}(\tilde{M}_0)}{\lambda_{\max}(\tilde{S})}\right).$$

It is then a routine to verify that, for such μ,

$$-\lambda_{\min}(\tilde{M}_0) \cdot I_{2n} \preceq r \begin{bmatrix} (\mathrm{e}^{-2\mu r} - 1)\tilde{P} & 0 \\ 0 & \frac{(\mathrm{e}^{-2\mu r}-1)}{r}\tilde{S} \end{bmatrix},$$

holds, that is

$$-\tilde{M}_\mu = -\tilde{M}_0 - r \begin{bmatrix} (\mathrm{e}^{-2\mu r} - 1)\tilde{P} & 0 \\ 0 & \frac{(\mathrm{e}^{-2\mu r}-1)}{r}\tilde{S} \end{bmatrix}$$
$$- r \begin{bmatrix} (1 - \mathrm{e}^{-2\mu r})\beta \, \|G_r\|_{L_2}^2 \cdot I_n & 0 \\ 0 & 0 \end{bmatrix}$$
$$\preceq -\tilde{M}_0 - r \begin{bmatrix} (\mathrm{e}^{-2\mu r} - 1)\tilde{P} & 0 \\ 0 & \frac{(\mathrm{e}^{-2\mu r}-1)}{r}\tilde{S} \end{bmatrix} \preceq 0.$$

We have then proved that the conditions of Lemma 1 are implied by Proposition 1 in [37]. The converse is false, as shown in the following counter-example.

Example 1 Let us consider the difference equation

$$x(t) = \mathrm{e}^{-2}x(t-1) - \int_0^1 \mathrm{e}^{-2\theta}x(t-\theta)\mathrm{d}\theta. \tag{2.38}$$

The solutions of the characteristic equation of this system,

$$\lambda = -3, \;\; \lambda_k = -2 + j2k\pi, k \in \mathbb{Z},$$

lie in the open left-half complex plane, so this system is exponentially stable. Proposition 1 requires to find $p > 0$, $w_0 > 0$ and $w_1 > 0$ such that

$$p(1 - \mathrm{e}^{-4}) = w_0 + w_1 \;\text{ and }\; pw_0 + p^2\mathrm{e}^{-4} < w_1 w_0.$$

These conditions lead to $p^2 + pw_0 + w_0^2 e^4 < 0$, which is impossible. However, the conditions (2.25)–(2.27) admit a solution

$$\tilde{P} = 10.1540,\ \tilde{S} = 32.4619,\ \beta = 32.5292,\ \mu = 0.46.$$

Hence, from Lemma 1, we conclude that the system response of (2.38) is exponentially stable, that is, for $t \geq 0$,

$$\|x(t,\varphi)\| \leq \sqrt{\frac{609.1465 + \frac{111.6439}{\varepsilon}}{2.1871}}\, \|\varphi\|_r\, \mathrm{e}^{-(0.46-\varepsilon)t},$$

for any $\varepsilon \in (0, 0.46)$.

If we assume that

$$\sup_{0\leq\theta\leq r} \|G(\theta)\| = 0,$$

Proposition 1 leads to the existence of symmetric positive definite matrices W_0, W_1 and P_1 such that

$$A^T P_1 A - P_1 = -(W_0 + rW_1).$$

Take any symmetric positive definite matrix Q, and define the symmetric positive definite matrix

$$\tilde{S} = r(P_1 + P_1 A(W_0 + rW_1)^{-1} A^T P_1) + Q,$$

and $\tilde{P} = P_1$. It is readily seen that (2.25) and (2.26) are fulfilled, with $\beta = \lambda_{\max}(\tilde{S}) > 0$. The rest of the proof follows step by step the previous case.

Example 2 Consider the linear hyperbolic system of balance laws described by

$$\xi_t(z,t) + \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}^{-1} \xi_z(z,t) = 0,\ z \in (0,1],\ t > 0, \tag{2.39}$$

with $r > 0$. The initial condition is $\xi(z, 0) = \xi_0(z)$, for any $z \in [0, 1]$, where $\xi_0(\cdot)$ is some continuous function. The boundary condition is $\xi(0, t) = u(t)$ with $u(t)$ the boundary input control given by

$$u(t) = K_1 \int_0^1 K(\nu)\xi(\nu,t)\mathrm{d}\nu - K_2\xi(1,t),$$

where K_1 and K_2 are two 2×2 constant matrices, and $K(\nu)$ is a 2×2 matrix with piecewise continuous elements in $[0, 1]$. The solution of (2.39) satisfies $\xi(z, t) = \xi(0, t - rz)$, for $t \geq rz$. In closed-loop, the state variable $x(t) = \xi(0, t)$ is then governed by

$$x(t) = -K_2 x(t-r) + K_1 \int_0^r K(r^{-1}\theta)x(t-\theta)\mathrm{d}\theta,$$

for $t \geq r$, with some continuous initial condition on $[0, r)$. With

$$K_2 = -\begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \; K(\theta) = \begin{bmatrix} 0.07 & 0 \\ 0 & \mathrm{e}^{-\theta} \end{bmatrix}, \; K_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix},$$

and $r = \sqrt{2}$, a solution of Lemma 1 is

$$\tilde{P} = \begin{bmatrix} 58.3791 & -10.9932 \\ -10.9932 & 68.3308 \end{bmatrix}, \; \tilde{S} = \begin{bmatrix} 333.8131 & 20.7117 \\ 20.7117 & 266.6636 \end{bmatrix}$$

$$\beta = 339.7947, \; \mu = 0.2109.$$

The exponential estimate of the system solution is described by

$$\|x(t,\varphi)\| \leq \sqrt{\frac{200.4865 + \frac{17.5405}{\varepsilon}}{42.4658}} \, \|\varphi\|_r \; \mathrm{e}^{-(0.2109-\varepsilon)t}$$

for any $\varepsilon \in (0, 0.2109)$, $t \geq 0$.

2.4 State-Feedback Synthesis

In the previous section, we were interested with stability analysis. For stabilizing controller synthesis, the previous conditions turn to be useful.
Indeed, let us consider the system (2.1) with the state-feedback

$$u(t) = Fx(t-r). \tag{2.40}$$

The closed-loop system (2.1)–(2.40) is

$$x(t) = (A + BF)x(t-r), \; t \geq 0. \tag{2.41}$$

The stabilizing state-feedback gain F can then be synthesized by the following (necessary and) sufficient condition: If there exist a symmetric positive definite matrix Q and a matrix K such that

$$\begin{bmatrix} Q & QA^T + K^T B^T \\ AQ + BK & Q \end{bmatrix} \succ 0, \tag{2.42}$$

then the state-feedback gain $F = KQ^{-1}$ stabilizes (2.41). Note also that the difference equation (2.1) is useful for performance analysis. To see this, consider the disturbed system

$$x(t) = Ax(t-r) + Bu(t) + Ew(t), \tag{2.43}$$

with the controlled output equation

$$y(t) = Cx(t) + Du(t). \tag{2.44}$$

The input $w(t)$ is a disturbance, and is assumed to be in $L_2([0, \infty), \mathbb{R}^w)$, while the matrices E, C and D are real with appropriate size. The controller synthesis consists in determining a static state-feedback $u(t) = Fx(t-r)$, such that the autonomous closed-loop system

$$x(t) = (A + BF)x(t-r) \tag{2.45}$$

is exponentially stable, and the closed-loop controlled output $y(t)$ satisfies, for null initial conditions in the state $x(t)$, the L_2-gain performance specification

$$\|y(t)\|_{L_2} \leq \gamma \, \|w(t)\|_{L_2}, \tag{2.46}$$

for a given $\gamma \geq 0$. This problem admits an immediate answer.

Theorem 3 *Assume that there exist a symmetric positive definite matrix P and a matrix Y such that the matrix*

$$M = \begin{bmatrix} P & 0 & (AP+BY)^T & (C(AP+BY)+DY)^T \\ * & \gamma^2 \cdot I & E^T & (CE)^T \\ * & * & P & 0 \\ * & * & * & I \end{bmatrix}$$

is positive definite. Then, for the static-state feedback $F = YP^{-1}$, the closed-loop system (2.45) is exponentially stable. Furthermore, for a null state initial condition, the closed-loop controlled output $y(t)$ in (2.44) satisfies (2.46).

For distributed-delay synthesis in (2.10) to get exponential stability, Lemma 1 can be used. Indeed, assume that there exist an $n \times n$ symmetric real positive definite matrix $\tilde{P}$, a symmetric positive semi-definite matrix $\tilde{S}$, and some constants $\mu > 0$ and $\alpha \geq 0$ such that

$$\alpha \cdot I_n \prec \tilde{P}, \tag{2.47}$$
$$0 \preceq \tilde{M}_\mu, \tag{2.48}$$

where $\tilde{M}_\mu$ is given by

$$-\tilde{M}_\mu = r \begin{bmatrix} A^T\tilde{P}A - \mathrm{e}^{-2\mu r}(\tilde{P} - \alpha \cdot I_n) & A^T\tilde{P} \\ \tilde{P}A & \tilde{P} - \frac{\mathrm{e}^{-2\mu r}}{r}\tilde{S} \end{bmatrix}. \tag{2.49}$$

Then, we define $\beta = \lambda_{\max}(\tilde{S})$ and $\frac{\alpha}{\beta} = \int_0^r \|G(\theta)\|^2 \, d\theta$. Note that if $\tilde{S} = 0$, β can be taken as an arbitrary positive constant. With this construction, Lemma 1 is satisfied, so that exponential stability holds. For a construction of $G(\cdot)$, take for instance $G(\theta) = e^{M\theta}$ with M some symmetric matrix. It follows that $\|G(\theta)\|^2 = \rho(e^{D\theta})$, where

$$M + M^T = PDP^{-1},\ D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

$\lambda_i \in \mathbb{R}$ and for some invertible matrix P. If $\lambda_n = \rho(D)$, it follows by definition that

$$\lambda_n \frac{\alpha}{\beta} = e^{\lambda_n r} - 1. \tag{2.50}$$

It is a routine to see that (2.50) admits a positive solution $\lambda_n > 0$ if $\frac{\alpha}{\beta} > r$. Then, it is possible to determine arbitrary $\lambda_i \leq \lambda_n$ for $i = 1, \ldots, n-1$ and to define $G(\cdot)$ (in a non unique way) as above. Finally, note that the decay rate μ can be obtained by an optimization procedure, or can be imposed in the feedback problem to achieve a specified time-response performance.

2.5 Conclusion

In this chapter, linear 1-D conservation laws with boundary control are analyzed as linear continuous-time difference equations with input control. Exponential stability conditions are proposed, for static state-feedback or state-feedback with distributed delay. A brief discussion on feedback synthesis for performance achievement is also mentioned. A conservatism analysis is also made to show the improvement with respect to conditions which already appear in the literature.

This work has not addressed the difficult question of uncertain constant delays, where the constructive conditions for stability in the literature appear to be quite conservative. A first tentative is proposed in [13]. This is one of the two challenging perspectives of this chapter. The other perspective is to apply the results issued from [36] for large-scale interconnected systems of conservation laws. The main interest in this extension is to obtain conditions for stability which are independent of the size of the interconnection structure. For this, the delay-realization used in this chapter seems to be well suitable for such a generalization.

References

1. C.E. Avellar, J.K. Hale, On the zeros of exponentials polynomials. J. Math. Anal. Appl. **73**(2), 434–452 (1980)
2. A. Aw, M. Rascle, Resurection of second-order models for traffic flow. SIAM J. Appl. Math. **60**(3), 916–938 (2000)

3. P.S. Barnett, The analysis of traveling waves on power system transmission lines. Ph.D. Thesis, University of Canterbury, Christchurch, New Zealand (1974)
4. G. Bastin, B. Haut, J.M. Coron, B. d'Andrea-Novel, Lyapunov stability analysis of networks of scalar conservation laws. Netw. Heterogen. Media **2**(4), 749–757 (2007)
5. R. Bellman, K.L. Cooke, *Differential-Difference Equations.* (Academic Press, 1963)
6. A. Bressan, in *Hyperbolic Systems of Conservation Laws*, The One Dimensional Cauchy Problem (Oxford University Press, 2000)
7. L.A.V. Carvalho, On quadratic Lyapunov functionals for linear difference equations. Linear Algebra Appl. **240**, 41–64 (1996)
8. K.L. Cooke, D.W. Krumme, Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations. J. Math. Anal. Appl. **24**(2), 372–387 (1968)
9. C. Corduneanu, *Integral Equations and Applications.* (Cambridge University Press, 1991)
10. J.M. Coron, B. d'Andrea-Novel, G. Bastin, A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. IEEE Trans. Autom. Control **52**(1), 2–11 (2006)
11. J.M. Coron, G. Bastin, B. d'Andrea-Novel, Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. SIAM J. Control Optim. **47**(3), 1460–1498 (2008)
12. S. Damak, M. Di Loreto, W. Lombardi, V. Andrieu, Exponential L_2-stability for a class of linear systems governed by continuous-time difference equations. Automatica **50**(12), 3299–3303 (2014)
13. S. Damak, M. Di Loreto, S. Mondié, Stability of linear continuous-time difference equations with distributed delay: constructive exponential estimates. Int. J. Robust Nonlinear Control **25**(17), 3195–3209 (2015)
14. S. Damak, M. Di Loreto, S. Mondié, Difference equations in continuous time with distributed delay: exponential estimates, in *IEEE American Control Conference (ACC)* (2014)
15. R. Datko, Representation of solutions and stability of linear differential-difference equations in a Banach space. J. Differ. Equ. **29**(1), 105–166 (1978)
16. A. Diagne, G. Bastin, J.M. Coron, Lyapunov exponential stability of 1-D linear hyperbolic systems of balance laws. Automatica **48**(1), 109–114 (2012)
17. M. Di Loreto, J.J. Loiseau, in *On the Stability of Positive Difference Equations*, ed. by R. Sipahi, T. Vyhlidal, S.I. Niculescu, P. Pepe. Time Delay Systems - Methods, Applications and New Trends, Series LNCIS, vol. 423. (Springer, 2012), pp. 125–147
18. F. Di Meglio, R. Vazquez, M. Krstic, Stabilization of a system of $n+1$ coupled first-order hyperbolic linear PDEs with a single boundary input. IEEE Trans. Autom. Control **58**(12), 3097–3111 (2013)
19. S. Elaydi. *An Introduction to Difference Equations.*, 3rd edn (Springer, 2005)
20. E. Fridman, Stability of linear descriptor systems with delay: a Lyapunov-based approach. J. Math. Anal. Appl. **273**(1), 24–44 (2002)
21. P. Grabowski, F.M. Callier, Boundary control systems in factor form: transfer functions and input-output maps. Integr. Eqn. Oper. Theory **41**(1), 1–37 (2001). Birkhauser, Basel
22. K. Gu, Stability problem of systems with multiple delay channels. Automatica **46**(4), 743–751 (2010)
23. K. Gu, V.L. Kharitonov, J. Chen, *Stability of Time-Delay Systems.* (Birkhauser, 2003)
24. M. Gugat, M. Herty, Existence of classical solutions and feedback stabilization for the flow in gas network. ESAIM Control Optim. Calc. Var. **17**(1), 28–51 (2011)
25. A. Halanay, V. Răsvan, Stability radii for some propagation models. IMA J. Math. Control Inf. **14**(1), 95–107 (1997)
26. J.K. Hale, S.M. Verduyn Lunel, *Introduction to Functional Differential Equations.* (Springer, 1993)
27. J.K. Hale, S.M. Verduyn Lunel, in *Effects of Small Delays on Stability and Control.*, ed. by H. Bart, I. Gohberg, A. Ran. Operator Theory and analysis, vol. 122 (Birkhauser, 2001), pp. 275–301
28. J.K. Hale, S.M. Verduyn, Lunel, Strong stabilization of neutral functional differential equations. IMA J. Math. Control Inf. **19**(1–2), 5–23 (2002)

29. J.K. Hale, S.M. Verduyn, Lunel stability and control of feedback systems with time-delays. Int. J. Syst. Sci. **34**(8–9), 497–504 (2003)
30. J. de Halleux, C. Prieur, J.M. Coron, B. d'Andrea-Novel, G. Bastin, Boundary feedback control in networks of open-channels. Automatica **39**(8), 1365–1676 (2003)
31. A.J. Jerri, *Introduction to Integral Equations with Applications*, 2nd edn (Wiley, 1999)
32. V.L. Kharitonov, Lyapunov functionals and Lyapunov matrices for neutral type time delay systems: a single delay case. Int. J. Control **78**(11), 783–800 (2005)
33. M. Krstic, A. Smyshlyaev, Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. Syst. Control Lett. **57**(9), 750–758 (2008)
34. X. Litrico, V. Fromion, Boundary control of hyperbolic conservation laws using a frequency domain approach. Automatica **45**(3), 647–659 (2009)
35. H. Logemann, S. Townley, The effect of small delays in the feedback loop on the stability of neutral systems. Syst. Control Lett. **27**(5), 267–274 (1996)
36. P. Massioni, M. Verhaegen, Distributed control for identical dynamically coupled systems: a decomposition approach. IEEE Trans. Autom. Control, **54**(1), 124–135 (2009)
37. D. Melchor-Aguilar, Exponential stability of some linear continuous time difference systems. Syst. Control Lett. **61**(1), 62–68 (2012)
38. D. Melchor-Aguilar, Exponential stability of linear continuous time difference systems with multiple delays. Syst. Control Lett. **62**(10), 811–818 (2013)
39. S. Mondié, D. Melchor-Aguilar, Exponential stability of integral delay systems with a class of analytical kernels. IEEE Trans. Autom. Control **57**(2), 484–489 (2012)
40. P. Pepe, The Lyapunov's second method for continuous time difference equations. Int. J. Robust Nonlinear Control **13**(15), 1389–1405 (2003)
41. C. Prieur, Control of systems of conservation laws with boundary errors. Netw. Heterogen. Media **4**(2), 393–407 (2009)
42. V. Răsvan, S.I. Niculescu, Oscillations in lossless propagation models: a Liapunov-Krasovskii approach. IMA J. Math. Control Inf. **19**(1–2), 157–172 (2002)
43. V. Răsvan, in Delays, Propagation, Conservation Laws, ed. by R. Sipahi, T. Vyhlidal, S.I. Niculescu, P. Pepe. Time Delay Systems: Methods, Applications and New Trends LNCIS, vol. 423. (Springer, 2012), pp. 147–159
44. L. Shaikhet, About Lyapunov functionals construction for difference equations with continuous time. Appl. Math. Lett. **17**(8), 985–991 (2004)
45. J. Valein, E. Zuazua, Stabilization of the wave equation on 1-D networks. SIAM J. Control Optim. **48**(4), 2771–2797 (2009)
46. E. Verriest, New qualitative methods for stability of delay systems. Kybernetika **37**(3), 225–228 (2001)

Chapter 3
Model Reduction for Norm Approximation: An Application to Large-Scale Time-Delay Systems

Igor Pontes Duff, Pierre Vuillemin, Charles Poussot-Vassal, Corentin Briat and Cédric Seren

Abstract The computation of $\mathcal{H}_2$ and $\mathcal{H}_{2,\Omega}$ norms for LTI Time-Delay Systems (TDS) are important challenging problems for which several solutions have been provided in the literature. Several of these approaches, however, cannot be applied to systems of large dimension because of the inherent poor scalability of the methods, e.g., LMIs or Lyapunov-based approaches. When it comes to the computation of frequency-limited norms, the problem tends to be even more difficult. In this chapter, a computationally feasible solution using $\mathcal{H}_2$ model reduction for TDS, based on the ideas provided in [3], is proposed. It is notably demonstrates on several examples that the proposed method is suitable for performing both accurate model reduction and norm estimation for large-scale TDS.

3.1 Introduction

Forewords: Modeling is an essential step to well understand and interact with a physical dynamical phenomena. It, among other, permits to analyze, simulate, optimize, and control dynamical processes. The values of a model lies on its ability to describe the reality as accurately as possible. In general, dynamical models are described by equations and their complexity is somehow linked to its number of equations and

I. Pontes Duff (✉) · P. Vuillemin · C. Poussot-Vassal · C. Seren
Onera-French Aerospace Lab, 31055 Toulouse, France
e-mail: ipontes@onera.fr

P. Vuillemin
e-mail: Pierre.Vuillemin@onera.fr

C. Poussot-Vassal
e-mail: Charles.Poussot-Vassal@onera.fr

C. Seren
e-mail: Cedric.Seren@onera.fr

C. Briat
Department of Biosystems Science and Engineering (D-BSSE), The Swiss Federal Institute of Technology–Zürich (ETH-Z), Zurich, Switzerland
e-mail: corentin@briat.info, cbriat@bsse.ethz.ch

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_3

variables. Although complex models have a high degree of likeness with reality, in practice, due to numerical limitations, they are problematic to manipulate. Actually, complex models are difficult to analyze and to control, due to limited computational capabilities, storage constraints, and finite machine precision. Therefore, a good model has to reach a trade-off between its accuracy and complexity.

In addition to high state-space complexity, we will be interested in how any delay may affect the system. This kind of models fall then in the class of infinite-dimensional systems. As a matter of consequence, classical analysis and control methods are not applicable as it. Even if many dedicated approaches have been derived to handle TDS problems (see [18]), most of them are limited to delay systems with low-order state-space vector and associated methods are not scalable when the number of state variables is increasing. Many examples can be found in the context of network systems, where delays appear naturally as the amount of time necessary to transmit some information between different systems (communication lag). In other systems, they are intrinsic part of the natural phenomena as it can be seen in chemical reactions, traffic jam, and heating systems.

This context justifies the search of simpler models in order to avoid numerical issues and to apply the classical methods of analysis and control. This is the philosophy of model approximation methods and they will be the main tool of this chapter. These methods permit to obtain a simpler model which well approximates the original one, even of infinite dimension.

Contribution: In this chapter, we will work in the framework of linear time-invariant dynamical systems and the complexity will be associated with the dimension of the state space of the systems. The aim of this chapter is to investigate the approximation of large-scale TDS and to estimate their norms. We will be particularly interested in the problem of the $\mathscr{H}_2$ (and $\mathscr{L}_2$) optimal model approximation using interpolatory methods and norm computation applied in the scenario of large-scale TDS.

In practice, in order to compute the $\mathscr{H}_2$ and $\mathscr{H}_{2,\Omega}$ norms for a finite dimensional system, poles or the Gramian-based system realization is required. In the case of TDS, those techniques are no longer applicable because of their infinite-dimensional nature and dedicated approaches should be rather developed, see e.g., [11]. In this work, modal truncation applied to TDS is firstly presented in order to show an intuitive approach for approximation of the norms. Since this approach is restricted to the case of single-delay TDS, optimal model reduction is introduced and used in order to obtain a finite dynamical system for which the classical tools can be efficiently applied. The obtained reduced model will be used to estimate the norm of the original one.

Outline: This work is organized in four sections and a conclusion. Section 3.2 recalls the definitions of the $\mathscr{H}_2$ and $\mathscr{H}_{2,\Omega}$-norms and presents some recent results associated with their computations. Section 3.3 aims at presenting an intuitive approach which allows to compute the norm of a single TDS via the modal truncation/representation and the Lambert function. Although it is a simple method, easy to implement, it is not applicable in the case of multiple-delay systems and is not accurate enough in the context of large-scale TDS. Thus, Sect. 3.4 is devoted to the introduction of the rational interpolation problem using Loewner matrices as well as the $\mathscr{H}_2$-optimal approximation conditions. In the end of this section, a realization-free algorithm will be presented (see [3]), allowing, from a time-delay system, to obtain a simple LTI finite dimension system that well approximates the original large-scale TDS. The norm computation of this simpler model will be used to estimate the norm of the original TDS. Performance of the proposed method on is finally assessed on numerical TDS study cases.

Notations: In this chapter let us denote by $\mathbb{R}^{n_y \times n_u}$ and $\mathbb{C}^{n_y \times n_u}$ the set of real and complex matrices with n_y rows and n_u columns, respectively. The transpose matrix of $A \in \mathbb{C}^{n_y \times n_u}$ is denoted by $A^T \in \mathbb{C}^{n_u \times n_y}$, where $(A^T)_{i,j} = A_{j,i}$. $\mathbb{R}_+$ denotes the nonnegative real number, $\mathbb{R}_-$ denotes the nonpositive real numbers, and $\mathbb{R}^* = \mathbb{R} \backslash \{0\}$. $\mathbb{C}^+$ denotes the open right-half plane and $\mathbb{C}^-$ denotes the open left-half plane. Notice that $H(s) \in \mathbb{C}^{n_y \times n_u}$ represents the transfer function with n_y outputs and n_u inputs of a real linear dynamical system which will be denoted by $\mathbf{H}$. Let us denote $H'(s)$ the derivative of the transfer function with respect to $s \in \mathbb{C}$.

3.2 Preliminaries

3.2.1 System Definition

In this chapter, we consider the following general class of *linear time-invariant time-delay systems with constant and discrete-delays* [5, 15],

$$\mathbf{H_d} := \begin{cases} \dot{x}(t) = A_0 x(t) + Bu(t) + \sum_{k=1}^{d} A_k x(t - h_k) \\ y(t) = Cx(t) \end{cases}, \tag{3.1}$$

where $h_k \in \mathbb{R}_+^*$, $k = 1, \dots, d$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$ are the delays, the state of the system, the input and the output, respectively. The matrices of the system are assumed to be constant and of appropriate dimensions.

3.2.2 Norms of Linear Time-Invariant Systems

Let $\mathbf{H} = (E, A, B, C)$ be a descriptor finite-dimensional LTI system with state-space representation

$$\mathbf{H} := \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ \quad y(t) = Cx(t), \end{cases} \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$ are the state of the system, the input and the output, respectively. The matrices of the system are assumed to be constant and of appropriate dimensions. The corresponding transfer function is simply given by

$$H(s) = C(sE - A)^{-1}B \in \mathbb{C}^{n_y \times n_u}. \quad (3.3)$$

The poles of $H(s)$ are the solutions of the generalized eigenvalue problem over the pencil (E, A). When $E = I_n$, then the poles are simply given by the eigenvalues of the matrix A and if, moreover, A has semisimple eigenvalues, then we also have [1]

$$H(s) = C(sI_n - A)^{-1}B = \sum_{i=1}^{n} \frac{\phi_k}{s - \lambda_k}, \quad (3.4)$$

where $\lambda_k \in \mathbb{C}$ and $\lim_{s \to \lambda_k} (s - \lambda_k)H(s) = \phi_k \in \mathbb{C}^{m_y \times m_u}$ are the poles and the residues of the transfer function, respectively. This formulation is known as the *modal representation* and is the core of the method proposed in Sect. 3.3. Note that in this representation, residues ϕ_k associated to poles λ_k reflect the importance and the contribution of each modal content.

We also recall the $\mathscr{L}_2$ and $\mathscr{L}_\infty$ norm definitions of a real continuous signal $\mathbf{u}(t) \in \mathbb{R}^n$ as follows:

$$\|\mathbf{u}\|_{L_2}^2 = \int_{-\infty}^{\infty} \mathbf{u}(t)^T \mathbf{u}(t) dt \qquad \text{and} \qquad \|\mathbf{u}\|_{L_\infty} = \text{ess}\sup_t \mathbf{u}(t).$$

3.2.2.1 The $\mathscr{L}_2$ and $\mathscr{H}_2$ Norms

Let us denote by $\mathscr{L}_2(i\mathbb{R})$ the Hilbert space of matrix-valued meromorphic functions $F : \mathbb{C} \to \mathbb{C}^{n_y \times n_u}$ for which the integral $\int_{\mathbb{R}} \text{trace}[\overline{F(i\omega)}F(i\omega)^T]d\omega < \infty$. The space $\mathscr{L}_2(i\mathbb{R})$ is the space of matrix-valued functions $H : \mathbb{C} \to \mathbb{C}^{n_y \times n_u}$ having no poles on the imaginary axis.

Definition 1 ($\mathscr{L}_2(i\mathbb{R})$*-inner product*) The inner product for this space is defined as

$$\langle \mathbf{H}, \mathbf{G} \rangle_{\mathscr{L}_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\left(\overline{H(i\omega)}G(i\omega)^T\right) d\omega,$$

for $\mathbf{H}, \mathbf{G} \in \mathscr{L}_2(i\mathbb{R})$ and its induced norm is defined as $\|\mathbf{H}\|_{\mathscr{L}_2} = \langle \mathbf{H}, \mathbf{H} \rangle_{\mathscr{L}_2}^{\frac{1}{2}}$.

Let $\mathscr{H}_2(\mathbb{C}^+)$ be the closed subspace of $\mathscr{L}_2(i\mathbb{R})$ which contains the matrix functions $F(s)$, analytic in the open right-half plane and $\mathscr{H}_2(\mathbb{C}^-)$, the closed subspace of $\mathscr{L}_2(i\mathbb{R})$ which contains the matrix functions $F(s)$, analytic in the open left-half plane. The following proposition stands (note that it will be an useful result to characterize the stability of a linear system from $\mathscr{L}_2(i\mathbb{R})$).

Proposition 1 ($\mathscr{L}_2(i\mathbb{R})$ decomposition) *We have*

$$\mathscr{L}_2(i\mathbb{R}) = \mathscr{H}_2(\mathbb{C}^-) \bigoplus \mathscr{H}_2(\mathbb{C}^+)$$

and this sum is, moreover, orthogonal.

Proof See e.g., [13]. ◇

In other words, the above proposition states that a system $\mathbf{H} \in \mathscr{L}_2(i\mathbb{R})$ can be written as the sum of its purely stable part located inside $\mathscr{H}_2(\mathbb{C}^+)$ and its purely unstable parts located inside $\mathscr{L}_2(i\mathbb{R})\backslash\mathscr{H}_2(\mathbb{C}^+)$. Let us now define the $\mathscr{H}_2$-norm:

Definition 2 (*$\mathscr{H}_2$-norm*) The $\mathscr{H}_2$-norm is defined for $\mathbf{H} \in \mathscr{H}_2(\mathbb{C}^+) := \mathscr{H}_2$, a stable and strictly proper system, and is given by

$$\|\mathbf{H}\|_{\mathscr{H}_2}^2 := \frac{1}{2\pi}\int_{-\infty}^{+\infty} \text{trace}\left(\overline{H(i\omega)}H(i\omega)^T\right)d\omega. \tag{3.5}$$

We then have the following proposition:

Proposition 2 ($\mathscr{H}_2$-norm for systems with semisimple eigenvalues) *Let us consider the system* (3.2) *with $E = I_n$ and assume further that A has semisimple eigenvalues. Then we have that*

$$\|\mathbf{H}\|_{\mathscr{H}_2}^2 = \sum_{k=1}^{n} \text{trace}\left(\phi_k H(-\lambda_k)^T\right), \tag{3.6}$$

where $\lambda_k \in \mathbb{C}$ and $\phi_k \in \mathbb{C}^{n_y \times n_u}$, $k = 1, \ldots, n$ are the poles and the associated residues of the transfer function, respectively.

It is important to stress that formula (3.6) can be readily extended to infinite-dimensional systems (with discrete spectrum) with simple poles. In such a case, the sum becomes infinite. However, while this method is easily applicable for finite-dimensional systems, the case of infinite-dimensional dynamical systems, like TDS, is more involved due to the necessity of computing first the eigenvalues. Note, on the other hand, that a lower bound on the $\mathscr{H}_2$-norm can be computed by evaluating (3.5) on a finite number of poles, only. As the truncation order increases, the lower bound will tend to the true $\mathscr{H}_2$-norm.

3.2.2.2 The $\mathscr{H}_{2,\Omega}$-Norm

We generalize here the $\mathscr{H}_2$-norm, which is computed for $\omega \in \mathbb{R}_+$, to its frequency-limited version, denoted by the $\mathscr{H}_{2,\Omega}$-norm, which is computed over the compact support Ω.

Definition 3 (*$\mathcal{H}_{2,\Omega}$-norm*) The $\mathcal{H}_{2,\Omega}$-norm with $\Omega = [0, \omega_0]$, defined for any stable system $\mathbf{H} \in \mathcal{H}_\infty$, is given by

$$\|\mathbf{H}\|^2_{\mathcal{H}_{2,\Omega}} := \frac{1}{2\pi} \int_{-\omega_0}^{+\omega_0} \text{trace}\left(\overline{H(i\omega)}H(i\omega)^T\right) d\omega. \tag{3.7}$$

Proof See [24]. ◇

Similarly to the $\mathcal{H}_2$-norm case, the following result holds:

Proposition 3 ($\mathcal{H}_{2,\Omega}$-norm for systems with semisimple eigenvalues) *Let us consider the system* (3.2) *with $E = I_n$ and assume further that A is diagonalizable with simple eigenvalues. Then for $\Omega = [0, \omega_0]$, we have:*

$$\|\mathbf{H}\|^2_{\mathcal{H}_{2,\Omega}} = -\frac{2}{\pi} \sum_{k=1}^{n} \text{trace}\left(\phi_k H(-\lambda_k)^T\right) \arctan\left(\frac{\omega_0}{\lambda_k}\right), \tag{3.8}$$

where $\arctan(z)$ *denotes the principal value of the complex arc-tangent of $z \neq \pm i$, and $\lambda_k \in \mathbb{C}$ and $\phi_k \in \mathbb{C}^{m_y \times m_u}$ are the poles and associated residues of the transfer function, respectively.*

3.3 Modal Truncation for Norm Approximation

A simple and intuitive approach, referred to as *modal truncation*, for computing norms of a certain class of TDS is presented in the sequel. Modal truncation is indeed simple to use, easy to implement, and can be made arbitrarily accurate. However, it may lead to poor results if significant poles are neglected. A limitation of the approach, which is based on Lambert's function, is that it cannot be applied to systems with multiple delays. This case will be treated in the next section. Following the above discussion, let us then consider following the system:

$$\mathbf{H_1} = \begin{cases} \dot{x}(t) = Ax(t-h) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{3.9}$$

where we assume that A is diagonalizable, i.e., there exists $X \in \mathbb{R}^{n \times n}$ nonsingular such that we have

$$A = X^{-1} \Sigma X \quad \text{where} \quad \Sigma = \text{diag}(\gamma_1, \ldots, \gamma_n),$$

where the γ_i's are the eigenvalues of A. The following result stands:

Proposition 4 (Modal representation of single-delay TDS) *The transfer function associated with* (3.9) *can be written as*

$$H_1(s) = \sum_{i=1}^{n} \frac{c_i^T b_i}{s - \gamma_i e^{-hs}}, \tag{3.10}$$

where $c_i = (CX^{-1}e_i)^T \in \mathbb{C}^{1\times m_y}$, $b_i = e_i^T XB \in \mathbb{C}^{1\times m_u}$ *and* e_i *is the vector with* i*-th component equal to 1 and zero otherwise.*

Proof The proof follows from simple algebraic manipulations. Indeed, we have

$$\begin{aligned} H_1(s) &= C(sI - Ae^{-sh})^{-1}B \\ &= CX(sI - \Sigma e^{-sh})^{-1}X^{-1}B \\ &= \sum_{i=1}^{n} \frac{c_i^T b_i}{s - \gamma_i e^{-hs}}, \end{aligned} \tag{3.11}$$

where c_i and b_i are defined as in Proposition 4. ◇

We then have the following result:

Theorem 1 (Single-delay TDS $\mathscr{H}_2$ and $\mathscr{H}_{2,\Omega}$-norms) *Let* $\Omega = [0, \omega_0]$. *Then the* $\mathscr{H}_2$*-norm and* $\mathscr{H}_{2,\Omega}$*-norm of the system* (3.9) *are given by*

$$\begin{aligned} \|H_1\|^2_{\mathscr{H}_2} &= \sum_{i=1}^{n} \sum_{k\in\mathbb{Z}} \frac{c_i^T b_i}{1 + \lambda_k^{(i)} h} H_1(-\lambda_k^{(i)}), \\ \|H_1\|^2_{\mathscr{H}_{2,\Omega}} &= -\frac{2}{\pi} \sum_{i=1}^{n} \sum_{k\in\mathbb{Z}} \mathrm{trace}\left(\frac{c_i^T b_i}{1 + \lambda_k^{(i)} h} H_1(-\lambda_k^{(i)})^T \right) \arctan\left(\frac{\omega_0}{\lambda_k^{(i)}} \right), \end{aligned} \tag{3.12}$$

where $\lambda_k^{(i)}$ *are* $H_1(s)$*'s poles,* c_i *and* b_i *are defined as in Proposition 4.*

Proof Each canonical rational term $\frac{c_i^T b_i}{s - \gamma_i e^{-hs}}$ in (3.10) has an infinite number of poles $\lambda_k^{(i)}$ solutions of the equations

$$s - \gamma_i e^{-hs} = 0. \tag{3.13}$$

This equation can be solved for all $i = 1, \ldots, n$ using the Lambert function denoted by $W(z)$ and defined as $z = W(z)e^{W(z)}$ for all $z \in \mathbb{C}$. Since this function is multivalued, we denote $W_k(z)$ the k-th branch of $W(z)$, for $k \in \mathbb{Z}$. The eigenvalues are then defined as:

$$\lambda_k^{(i)} = \frac{1}{h} W_k(\gamma_i)h, \; k \in \mathbb{Z}, \; i = 1, \ldots, n. \tag{3.14}$$

For simple eigenvalues, the residue associated with $\lambda_k^{(i)}$ is thus defined as

$$\phi_k^{(i)} := \lim_{s\to\lambda_k^{(i)}} (s - \lambda_k^{(i)}) H_1(s). \tag{3.15}$$

Letting $d_i(s) := s - \gamma_i e^{-sh}$ and $d_i'(s)$ its derivative at $s \in \mathbb{C}$, one can write

$$\phi_k^{(i)} = \frac{c_i^T b_i}{d_i'(\lambda_k^{(i)})} = \frac{c_i^T b_i}{1 + \lambda_k^{(i)} h}. \tag{3.16}$$

Substituting this in the infinite-dimensional versions of (3.6) and (3.8) given by Proposition 2 and 3 leads to,

$$\begin{aligned} \|H_1\|^2_{\mathscr{H}_2} &= \sum_{i=1}^{n} \sum_{k\in\mathbb{Z}} \phi_k^{(i)} H_1(-\lambda_k^{(i)}), \\ \|H_1\|^2_{\mathscr{H}_{2,\Omega}} &= \sum_{i=1}^{n} \sum_{k\in\mathbb{Z}} -\tfrac{2}{\pi} \operatorname{trace}\left(\phi_k^{(i)} H_1(-\lambda_k^{(i)})^T\right) \arctan\left(\frac{\omega}{\lambda_k^{(i)}}\right). \end{aligned} \tag{3.17}$$

which concludes the proof. ◇

Since this result involves infinite sums that cannot be computed exactly, we then rely on a truncation scheme to evaluate them. By choosing a finite subset D of $\mathbb{Z}$ with cardinal M, we get the following approximants for the norms:

$$\|H_1\|^2_{\mathscr{H}_2} \approx \sum_{i=1}^{n} \sum_{k\in D} \phi_k^{(i)} H_1(-\lambda_k^{(i)}), \tag{3.18}$$

and

$$\|H_1\|^2_{\mathscr{H}_{2,\Omega}} \approx -\frac{2}{\pi} \sum_{i=1}^{n} \sum_{k\in D} \operatorname{trace}\left(\phi_k^{(i)} H(-\lambda_k^{(i)})^T\right) \arctan\left(\frac{\omega_0}{\lambda_k^{(i)}}\right), \tag{3.19}$$

where $n \times M$ is the truncation order. Note that an approximate upper bound on the $\mathscr{H}_\infty$-norm can be obtained by differentiating (3.19) (see [25]). It is also important to stress that truncation may yield complex valued norms if the branches of the Lambert function are not carefully chosen (note that the chosen truncation poles have to be closed by conjugation, i.e., if $\lambda_k^{(i)}$ is chosen, $\overline{\lambda_k}^{(i)}$ has to be also chosen, too).

As stated in the introductory part of this section, the approximation can be quite poor if significant poles are discarded. It is however expected that poles that are far away from the imaginary axis contribute in a negligible way to the overall value of the norm; see, e.g., [27]. The poles that must be privileged are therefore the dominant ones; i.e., those that have larger real parts (see [19]). We have the following example illustrating the points made in this section.

Example 1 We illustrate here the above results on a simple TDS of the form (3.1) of dimension $n = 10$ with a single-delay h (i.e., $d = 1$). The performance of the modal truncation is evaluated by the relative errors defined as :

$$\mathscr{J} = \left| 100 \frac{\|H_1\|_{\mathscr{H}_2} - \|\hat{H}\|_{\mathscr{H}_2}}{\|H_1\|_{\mathscr{H}_2}} \right| \tag{3.20}$$

and

$$\mathscr{J}_\Omega = \left| 100 \frac{\|H_1\|_{\mathscr{H}_{2,\Omega}} - \|\hat{H}\|_{\mathscr{H}_{2,\Omega}}}{\|H_1\|_{\mathscr{H}_{2,\Omega}}} \right|. \tag{3.21}$$

We then consider the following scenarios:

Scenario 1. We generate 100 random systems with random delay $h \in [0, 0.1]$.
Scenario 2. We generate 100 random systems with random delay $h \in [0, 1]$.

For both scenarios, the $\mathscr{H}_2$-norm and $\mathscr{H}_{2,\Omega}$-norm are computed by numerical integration and approximated using the results in Theorem 1 for various truncation orders, i.e., $n \times M = 20, 40, 60, 80$ and 100. We get the results summarized in Tables 3.1 and 3.2 where we can see that, as expected, the relative error decreases as the truncation order increases.

Note that since the Lambert's function can only be applied in particular (restrictive) cases, the modal truncation approach cannot be directly generalized to the multiple-delay case. The poles calculation step must indeed be replaced first by a nonlinear eigenvalue problem; see e.g., [10, 21]. The generalization and the improvement of this method are disregarded for a future research.

In what follows, an approximation approach, introduced by [3], is presented to handle more complex TDS.

Table 3.1 Performance of the truncation method in Scenario 1 for various truncation orders

$n \times M$ (number of poles)	20	40	60	80	100
E($\mathscr{J}$)(%)	0.62	0.16	0.11	0.08	0.07
max($\mathscr{J}$)(%)	18.73	0.69	0.49	0.37	0.0029
E($\mathscr{J}_\Omega$)(%)	9.7815	0.0584	0.0151	0.0071	0.0064
max($\mathscr{J}_\Omega$)(%)	392.1953	1.4407	0.3902	0.1986	0.2405

Table 3.2 Performance of the truncation method in Scenario 2 for various truncation orders

$n \times M$ (number of poles)	20	40	60	80	100
E($\mathscr{J}$)(%)	5.39	0.80	0.60	0.50	0.45
max($\mathscr{J}$)(%)	44.55	23.86	23.99	24.07	24.10
E($\mathscr{J}_\Omega$)(%)	10.012	0.0458	0.0103	0.0041	0.0028
max($\mathscr{J}_\Omega$)(%)	190.5209	0.7071	0.1043	0.0383	0.0330

3.4 $\mathscr{H}_2/\mathscr{L}_2$ Realization-Free Model Approximation

The model reduction problem is concerned with the determination of a model that approximates a more complex one, i.e., with higher dimension. The rationale behind model reduction is to try to reduce the computational burden of complex models for simulation and design. Reduced-order models can be found using many different techniques such as Lyapunov equations [8, 20, 26], Krylov spaces based projections [1, 2, 7, 22] and Loewner framework [3, 9, 14]. A good way for selecting a "good" reduced-order model is by optimization. That is why we seek a model that minimizes a certain distance to the original one. To this aim, several distances can be considered (e.g., $\mathscr{H}_2$, $\mathscr{H}_\infty$ and ν-gap). The model obtained by this process has to give similar output for the same input compared to the original model.

The $\mathscr{H}_2$-norm, which induces a metric on $\mathscr{H}_2 \times \mathscr{H}_2$, is considered here. With this in mind, we can state the following problem:

Problem 1 *($\mathscr{H}_2$ Approximation problem)*
Given an LTI system $\mathbf{H} \in \mathscr{H}_2$ and a scalar $r \in \mathbb{N}^*$, find a solution to the optimization problem

$$\hat{\mathbf{H}} := \underset{G \in \mathscr{H}_2, \dim(G) \leq r}{\operatorname{argmin}} \|\mathbf{H} - \mathbf{G}\|_{\mathscr{H}_2}, \tag{3.22}$$

where $\hat{\mathbf{H}}$ is the reduced-order model of dimension r.

The $\mathscr{H}_2$ approximation problem actually amounts to find a reduced-order model that minimizes the L_2-to-L_∞ gain of the error system $\mathbf{H} - \hat{\mathbf{H}}$ as we have

$$\|\mathbf{y} - \hat{\mathbf{y}}\|_{L_\infty} := \|(\mathbf{H} - \hat{\mathbf{H}})\mathbf{u}\|_{L_\infty} \leq \|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathscr{H}_2} \|\mathbf{u}\|_{L_2}, \tag{3.23}$$

where $\|.\|_{L_2}$ denotes the L_2-norm and $\|.\|_{L_\infty}$ the L_∞-norm.

Moreover, the $\mathscr{H}_2$ and $\mathscr{H}_{2,\Omega}$-norms of a given system $\mathbf{H}$ can be approximated by the norm of $\hat{\mathbf{H}}$ since, from the triangle inequality, we have

$$\left| \|\mathbf{H}\|_{\mathscr{H}_2} - \|\hat{\mathbf{H}}\|_{\mathscr{H}_2} \right| \leq \|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathscr{H}_2}$$

and

$$\left| \|\mathbf{H}\|_{\mathscr{H}_{2,\Omega}} - \|\hat{\mathbf{H}}\|_{\mathscr{H}_{2,\Omega}} \right| \leq \|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathscr{H}_{2,\Omega}} \leq \|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathscr{H}_2}.$$

We can see that the value of $\|\mathbf{H}\|_{\mathscr{H}_2}$ is contained in a ball of radius $\|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathscr{H}_2}$ centered around $\|\hat{\mathbf{H}}\|_{\mathscr{H}_2}$. Using the fact that finite-dimensional stable LTI systems are dense in $\mathscr{H}_2$, we can conclude that $\|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathscr{H}_2}$ can be made as small as desired by increasing the order of the reduced model.

In practice, the main issue here is that Problem 1 is nonconvex and is not tractable when large-scale systems or infinite-dimensional systems are considered. For the finite-dimensional case, many algorithms have been derived, e.g., $\mathscr{H}_2$ model reduction [4, 6] and frequency-limited $\mathscr{H}_2$ model reduction [23]. The approaches mentioned in these references are based on the fact that a realization of the model to be reduced is available. The approach that we consider here, which does not involve any system realization, is more suitable for large-scale TDS (or more generally infinite-dimensional ones) since the evaluations of the transfer function at certain points are required.

When the system to be reduced is unstable, then we can consider the following alternative:

Problem 2 *($\mathscr{L}_2$ Approximation problem)*
Given an LTI system $\mathbf{H} \in \mathscr{L}_2$ and a scalar $r \in \mathbb{N}^*$, find a solution to the optimization problem

$$\hat{\mathbf{H}} = \underset{G \in \mathscr{L}_2, \dim(G) \leq r}{\operatorname{argmin}} \|\mathbf{H} - \mathbf{G}\|_{\mathscr{L}_2}, \tag{3.24}$$

where $\hat{\mathbf{H}}$ is the reduced-order model of dimension r.

3.4.1 First-Order Optimality Conditions for $\mathscr{H}_2/\mathscr{L}_2$ Model Approximation

The first step for solving the optimization Problems 1 and 2 consists in deriving the first-order optimality conditions. To this aim, let us consider that the finite-dimensional system $\hat{\mathbf{H}}$ of order r and define its transfer function as

$$\hat{H}(s) = \sum_{k=1}^{r} \frac{\hat{c}_k \hat{b}_k^T}{s - \hat{\lambda}_k}, \tag{3.25}$$

where $\hat{c}_k \hat{b}_k^T = \operatorname{res}(\hat{H}(s), \hat{\lambda}_k) \in \mathbb{C}^{n_y \times n_u}$ are the residues of $\hat{H}$ at $\hat{\lambda}_k \in \mathbb{C}$, $\hat{c}_k$ and $\hat{b}_k$ are called the left and the right tangential directions. One need now to find the values for $\hat{\lambda}_k$, $\hat{c}_k$ and $\hat{b}_k$ for which $\hat{\mathbf{H}}$ locally minimizes $\|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathscr{H}_2}$. The next result provides the first-order optimality conditions for solving Problem 1; see, e.g., [7, 22].

Theorem 2 (Necessary condition of $\mathscr{H}_2$ optimality) *Let $\mathbf{H}, \hat{\mathbf{H}} \in \mathscr{H}_2$ be two systems with semisimple poles. Assume further that the reduced-system $\hat{H}$, given by (3.25), locally minimizes $\|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathscr{H}_2}$, then the conditions*

$$H(-\hat{\lambda}_k)\hat{b}_k = \hat{H}(-\hat{\lambda}_k)\hat{b}_k, \;\; \hat{c}_k^T H(-\hat{\lambda}_k) = \hat{c}_k^T \hat{H}(-\hat{\lambda}_k) \tag{3.26}$$

and

$$\hat{c}_k^T \left.\frac{dH}{ds}\right|_{-\hat{\lambda}_k} \hat{b}_k = \hat{c}_k^T \left.\frac{d\hat{H}}{ds}\right|_{-\hat{\lambda}_k} \hat{b}_k \tag{3.27}$$

hold, where the $\hat{\lambda}_k$ *are the poles of* $\hat{\mathbf{H}}$ *and* $\hat{b}_k$ *and* $\hat{c}_k$ *are its tangential directions, respectively.*

The above theorem states that a local minimum is a bitangential Hermite interpolant of the original model evaluated at the mirror image of the low-order model poles with respect to its tangential directions $\hat{b}_k$ and $\hat{c}_k$, given by its residues (see [7, 22]). Note also that the poles and residues are not known a priori and must be computed using the iterative algorithm proposed in [3]. This procedure will be discussed more in detail in Sect. 3.4.3.

Let us now consider the optimality conditions for the problem $\mathscr{L}_2$ in the SISO case. We then have the following result initially proposed in [13]:

Theorem 3 (Condition for $\mathscr{L}_2$ optimality) *Let* $\mathbf{H} \in \mathscr{L}_2(i\mathbb{R})$ *be a given system and define its decomposition as* $\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^-$ *where* $\mathbf{H}^+ \in \mathscr{H}(\mathbb{C}^+)$ *and* $\mathbf{H}^- \in \mathscr{H}(\mathbb{C}^-)$. *Assume that the system* $\hat{\mathbf{H}}$, *given by* (3.25)*, has semisimple eigenvalues and that it locally minimizes* $\|\mathbf{H} - \mathbf{G}\|_{\mathscr{L}_2}$, *then the conditions*

$$H^+(-\hat{\lambda}_i) = \hat{H}^+(-\hat{\lambda}_i), \; \left.\frac{dH^+}{ds}\right|_{s=-\hat{\lambda}_i} = \left.\frac{d\hat{H}^+}{ds}\right|_{s=-\hat{\lambda}_i}, \tag{3.28}$$

$$H^-(-\hat{\lambda}_j) = \hat{H}^-(\hat{\lambda}_j), \; \left.\frac{dH^-}{ds}\right|_{s=-\hat{\lambda}_j} = \left.\frac{d\hat{H}^-}{ds}\right|_{s=-\hat{\lambda}_j}, \tag{3.29}$$

hold for all $i = 1, \ldots, k$ *and all* $j = k + 1, \ldots, r$.

The conditions presented in Theorem 3 have the particularity that one should know the decomposition of a transfer function $H(s)$ into its stable and purely unstable parts $H^+(s)$ and $H^-(s)$. Finding this decomposition is not a trivial problem and that why those conditions will not be applicable in what follows.

3.4.2 Rational Interpolation

In this section, we present an algorithm introduced in [3] allowing us to find a local minimizer to Problem 1 without requiring any realization of the original system. This feature makes the realization-free approach very appealing in the context of large-scale and infinite-dimensional systems. The approach is mainly based on [14] where an interpolation framework, based on simple successive evaluations of the transfer function of the original system, is provided. The following theorem, proved in [14], is fundamental for our purpose :

Theorem 4 (Interpolatory Lowner framework) *Let* $H(s)$ *be a given transfer function associated with a system* $\mathbf{H}$. *Let also us denote* r *shifts points by* $s_1, \ldots, s_r \in \mathbb{C}$ *and* r *tangential directions by* $c_1, \ldots c_r \in \mathbb{C}^{n_y \times 1}$, $b_1, \ldots, b_r \in \mathbb{C}^{n_u \times 1}$. *Then, the* r*-dimensional descriptor model* $\hat{\mathbf{H}} = (\hat{E}, \hat{A}, \hat{B}, \hat{C})$ *given, for* $i, j = 1, \ldots, r$, *by*

$$(\hat{E})_{ij} = \begin{cases} -\frac{c_i^T(H(s_i)-H(s_j))b_j}{s_i-s_j}, & i \neq j, \\ -c_i^T H'(s_i) b_i, & i = j, \end{cases}$$

$$(\hat{A})_{ij} = \begin{cases} -\frac{c_i^T(s_i H(s_i)-s_j H(s_j))b_j}{s_i-s_j}, & i \neq j, \\ -c_i^T (sH(s))'|_{s=s_i} b_i, & i = j, \end{cases}$$

$$\hat{C} = [H(s_1)b_1 \dots H(s_r)b_r] \ \ and \ \hat{B} = \begin{bmatrix} c_1^T H(s_1) \\ \vdots \\ c_r^T H(s_r) \end{bmatrix}$$

interpolates $H(s)$ in the sense of equations (3.26).

Consequently, the obtained reduced-order model given by $\hat{H}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$ satisfies, for all $k = 1, \dots r$, the conditions (3.27) for the given shift points $\{s_1, \dots, s_r\}$, and the given tangential directions $\{c_1, \dots c_r\}$ and $\{b_1, \dots, b_r\}$. To finally compute the (sub)optimal approximation, the currently unknown triplet $\{\hat{\lambda}_i, \hat{b}_i, \hat{c}_i\}$ has to be determined. This problem is addressed in the next section, thanks to a dedicated algorithm.

3.4.3 Realization-Free Model Approximation

An algorithm, named **TF-IRKA** (Iterative Rational Krylov Algorithm using Transfer Function Evaluation), has been proposed in [3] in order to obtain a reduced-order model that locally satisfies the optimality conditions, i.e., find suboptimal values for $\{\hat{\lambda}_i, \hat{b}_i, \hat{c}_i\}$, $i = 1, \dots, r$. This algorithm is recalled below for completeness.

Algorithm 5 (TF-IRKA [3])

1: **Initialization**: Transfer function $H(s)$, dimension r, $\sigma^0 = \{\sigma_1^0, \dots, \sigma_r^0\} \in \mathbb{C}^{r\times 1}$ initial interpolation points and tangential directions $b_1, \dots, b_r \in \mathbb{C}^{n_u \times 1}$ and $c_1, \dots, c_r \in \mathbb{C}^{n_y \times 1}$.
2: **while** not convergence **do**
3: Build $\hat{E}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$ using Theorem 4.
4: Find the triplet $(x_i^{(k)}, y_i^{(k)}, \lambda_i^{(k)})$ solution of the generalized eigenvalue problem

$$\hat{A}^{(k)} x_i^{(k)} = \lambda_i^{(k)} \hat{E}^{(k)} x_i^{(k)} \text{ and } y_i^{(k)*} \hat{E}^{(k)} x_j^{(k)} = \delta_{ij}, \tag{3.30}$$

where δ_{ij} is the Kronecker delta function.
5: Set $\sigma_i^{(k+1)} \leftarrow -\lambda_i^{(k)}$, $b_i^{(k+1)T} \leftarrow y_i^{(k)} \hat{B}^{(k)}$ and $c_i^{(k+1)} \leftarrow \hat{C}^{(k)} x_i^{(k)}$, for $i = 1, \dots, r$.
6: **end while**

7: Check that the conditions (3.26) are satisfied.
8: Build matrices $\hat{E}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$.

This algorithm can be applied to any system for which the corresponding transfer function is available, regardless its dimension. When the transfer function is unknown, finite difference methods can be used to evaluate the derivative of the transfer function [16]. A first advantage of the above algorithm lies in its wide applicability, as opposed to modal truncation (see Sect. 3.3) which is only applicable to a certain class of TDS. A second advantage is that an suboptimal model is automatically found by the algorithm, whereas for modal truncation, certain parameters, such as the truncation order, have to be chosen by hand. However, there is neither any guarantee of convergence of the algorithm nor any certificate that the reduced-order model is stable. Fortunately, extensive numerical applications of the algorithm tend to suggest that most of the time, it converges toward a stable system; see, e.g., [16]. Moreover, numerical mechanism can be applied to enhance its convergence [17].

3.4.4 Numercial Examples

3.4.4.1 Single-Delay Model

We consider here the second scenario of Example 1, i.e., we generate 100 stable systems of dimension $n = 10$ with a randomly chosen delay $h \in [0, 1]$. We then compute reduced-order models for various r using **TF-IRKA**. We then compute the $\mathscr{H}_2$-norm of both the original and the reduced-order model. The results on the statistics of the relative error given by (3.20) are summarized in Table 3.3 where we can observe that, even for a small reduction order, the estimate of the $\mathscr{H}_2$-norm is quite accurate. For comparison, when $r = 6$, the error is comparable to the one obtained using a modal truncation with an order 100. This demonstrates the usefulness of the proposed optimization-based approach.

Table 3.3 Performance of the model approximation in Exemple 1, Scenario 2 for various reduction order r

reduction order r	2	4	6	8	10
E($\mathscr{J}$)(%)	6.4999	1.4561	**0.3956**	0.0347	0.0201
max($\mathscr{J}$)(%)	327.8265	32.7315	24.2551	0.9224	0.4530

3.4.4.2 Multiple-Delay Model

Let us now consider the following time-delay system

$$\mathbf{H_2} := \begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -3 & 2 \\ 0 & 0 & -1 \end{bmatrix} x(t) + \frac{1}{5} \begin{bmatrix} -1 & 1 & 2 \\ 1 & -3 & 2 \\ 0 & 0 & -1 \end{bmatrix} (\nabla x)(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x(t), \end{cases} \tag{3.31}$$

where $(\nabla x)(t) := x(t - h_1) + x(t - h_2)$, $h_1 = 0.3$ and $h_2 = 0.9$. We then apply **TF-IRKA** in order to obtain a reduced-order model of dimension $r = 10$. The Bode plots of both the original system and the reduced-order one, depicted in Fig. 3.1, illustrates the accurate matching of the two frequency responses in amplitude.

Let us now address the $\mathscr{H}_2$-norm computation. The results are depicted in Fig. 3.2 where we compare the actual $\mathscr{H}_2$-norm of the original system computed using the method in [11] with the one computed from the reduced-order model. We can see, as expected, that as the reduction order increases the $\mathscr{H}_2$-norm of the reduced model converges to the actual one. Note that the relative error is rather very small even for $r = 2$.

The results obtained show that this approach is appropriate and relevant to estimate $\mathscr{H}_2$-norm of a multiple TDS as well as to obtain a good finite dimension approximation.

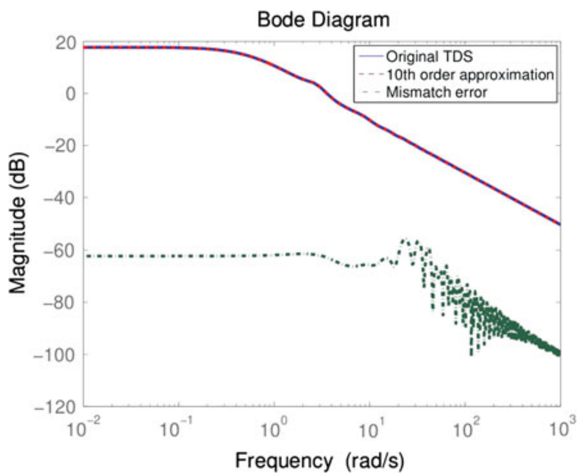

Fig. 3.1 Magnitude Bode plots of the model (3.31), the corresponding reduced-order model with $r = 10$ and the mismatch error between both models

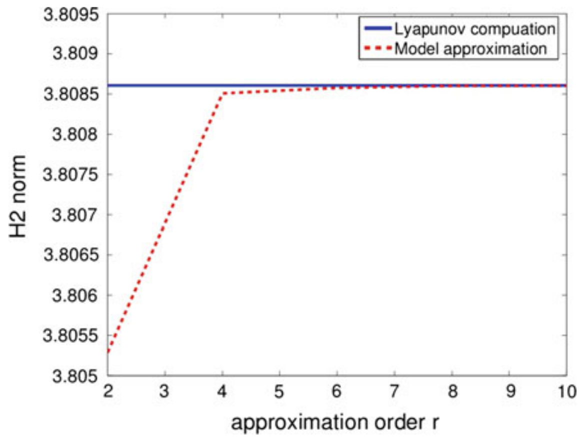

Fig. 3.2 $\mathscr{H}_2$-error of approximation as function of order r of the reduced-order model

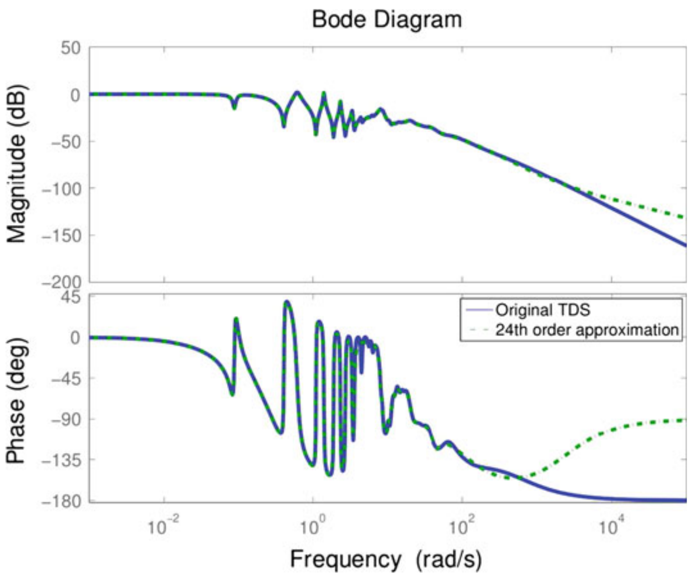

Fig. 3.3 Bode plots of the model (3.32) and the corresponding reduced-order model with $r = 24$

3.4.4.3 Norm Estimation for Large-Scale Systems

Let us now consider a beam model, denoted by $\mathbf{H}_{BEAM}$, taken from the COMP$l_e ib$ library [12] whose order is 348. Note that this system does not involve any delay but has large dimension. This system is assumed to be controlled using a PID controller $\mathbf{H}_{PID}$ with input lag $\tau = 0.1$. The resulting closed-loop system has the following transfer function representation

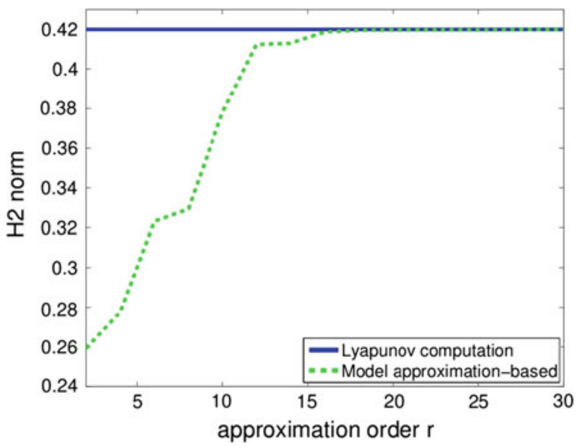

Fig. 3.4 $\mathscr{H}_2$-error of approximation as function of order from reduced model

$$H_{\text{feedback}}(s) = \frac{H_{\text{PID}}(s)H_{\text{BEAM}}(s)}{1 + H_{\text{PID}}(s)H_{\text{BEAM}}(s)e^{-\tau s}}. \tag{3.32}$$

We then apply **TF-IRKA** in order to obtain a reduced-order model of dimension $r = 24$ and we get the Bode plots of Fig. 3.3. We can see that the bode plots match quite well until the pulsation 100 rad/s. Let us consider now the $\mathscr{H}_2$-norm computation from the reduced-order model. The results are depicted in Fig. 3.4 where we can see that a fairly good estimate is obtained for a reduction order of $r = 16$. As before, the norm for the delay system has been computed using the method in [11].

3.5 Conclusion

In this chapter, two approaches for approximating norms of TDS have been described and analyzed. The first approach, modal truncation, is a direct extension of the delay-free case. It is an intuitive approach limited to systems with only one single delay. Then a second approach proving to be applied in more general cases has been presented. It is the optimal model approximation approach and it has been assessed on many different TDS (single delay, multiple delays) and on very large-scale models as well, making this approach very powerful for complex systems subject to delays. Therefore, the approach provides promising perspectives for estimating the $\mathscr{H}_2$-norm and the frequency behavior of TDS systems, whatever the dimension of the system.

References

1. A.-C. Antoulas, *Approximation of Large-Scale Dynamical Systems* (SIAM, 2005)
2. A.-C. Antoulas, An overview of model reduction methods and a new result, in *IEEE Conference on Decision and Control (CDC)* (2009)
3. C. Beattie, S. Gugercin, Realization-independent $\mathscr{H}_2$-approximation, in *IEEE Conference on Decision and Control (CDC)* (2012)
4. G. Flagg, C. Beattie, S. Gugercin, Convergence of the iterative rational Krylov algorithm. Syst. Control Lett. **61**(6), 688–691 (2012)
5. K. Gu, V.L. Kharitonov, J. Chen, *Stability of Time-Delay Systems* (Birkhäuser, 2003)
6. S. Gugercin, A.C. Antoulas, C. Beattie, $\mathscr{H}_2$ model reduction for large-scale linear dynamical systems. SIAM J. Matrix Anal. Appl. **30**(2), 609–638 (2008)
7. S. Gugercin, C. Beattie, A.-C. Antoulas, Rational Krylov methods for optimal $\mathscr{H}_2$ model reduction. ICAM Technical report (2006)
8. Y. Halevi, Frequency weighted model reduction via optimal projection. IEEE Trans. Autom. Control **37**(10), 1537–1542 (1992)
9. A. Ionita, A. Antoulas, Data-driven parametrized model reduction in the Loewner framework. SIAM J. Sci. Comput. **36**(3), A984–A1007 (2014)
10. E. Jarlebring, K. Meerbergen, W. Michiels, Computing a partial Schur factorization of nonlinear eigenvalue problems using the infinite Arnoldi method. SIAM J. Matrix Anal. Appl. **35**(2), 411–436 (2014)
11. E. Jarlebring, J. Vanbiervliet, W. Michiels, Characterizing and computing the norm of time-delay systems by solving the delay Lyapunov equation. IEEE Trans. Autom. Control **56**(4), 814–825 (2011)
12. F. Leibfritz, W. Lipinski, Description of the benchmark examples in compleib 1.0. Technical report. Department Mathematics, University of Trier, Germany (2003)
13. C. Magruder, C. Beattie, S. Gugercin, Rational Krylov methods for optimal $\mathscr{L}_2$ model reduction, in *IEEE Conference on Decision and Control (CDC)* (2010)
14. A.-J. Mayo, A.-C. Antoulas, A framework for the solution of the generalized realization problem. Linear Algebra Appl. **425**(2), 634–662 (2007)
15. S.I. Niculescu, *Delay Effects on Stability. A Robust Control Approach* (Springer, Heidelberg, 2001)
16. I. Pontes Duff, P. Vuillemin, C. Poussot-Vassal, C. Seren, C. Briat, Large-scale time delay systems stability analysis by model approximation techniques (submitted)
17. C. Poussot-Vassal, P. Vuillemin, Introduction to MORE: a MOdel REduction toolbox, in *IEEE International Conference on Control Applications (ICCA)* (2012)
18. J.-P. Richard, Time-delay systems: an overview of some recent advances and open problems. Automatica **39**(10), 1667–1694 (2003)
19. J. Rommes, N. Martins, Efficient computation of multivariable transfer function dominant poles using subspace acceleration. IEEE Trans. Power Syst. **21**(4), 1471–1483 (2006)
20. J.T. Spanos, M.H. Milman, D.L. Mingori, A new algorithm for $\mathscr{L}_2$ optimal model reduction. Automatica **28**(5), 897–909 (1992)
21. R. Van Beeumen, K. Meerbergen, W. Michiels, A rational Krylov method based on hermite interpolation for nonlinear eigenvalue problems. SIAM J. Sci. Comput. **35**(1), A327–A350 (2013)
22. P. Van Dooren, K.-A. Gallivan, P.-A. Absil, $\mathscr{H}_2$-optimal model reduction of MIMO systems. Appl. Math. Lett. **21**(12), 1267–1273 (2008)
23. P. Vuillemin, C. Poussot-Vassal, D. Alazard, Poles residues descent algorithm for optimal frequency-limited $\mathscr{H}_2$ model approximation, in *European Control Conference (ECC)* (2014)
24. P. Vuillemin, C. Poussot-Vassal, D. Alazard, Spectral expression for the frequency-limited $\mathscr{H}_2$-norm of LTI dynamical systems with high order poles, in *European Control Conference (ECC)* (2014)

25. P. Vuillemin, C. Poussot-Vassal, D. Alazard, Two upper bounds on the $\mathcal{H}_\infty$-norm of LTI dynamical systems, in *IFAC World Congress* (2014)
26. D.A. Wilson, Optimum solution of model-reduction problem. Proc. Inst. Electr. Eng. **117**(6), 1161–1165 (1970)
27. P. Zitek, T. Vyhlídal, Rating the significance of infinite chains of poles in time-delay systems, in *IFAC World Congress* (2011)

Part II
Control Systems Under Asynchronous Sampling

Chapter 4
General Formula for Event-Based Stabilization of Nonlinear Systems with Delays in the State

Sylvain Durand, Nicolas Marchand and J. Fermi Guerrero-Castellanos

Abstract In this chapter, a universal formula is proposed for event-based stabilization of nonlinear systems affine in the control and with delays in the state. The feedback is derived from the seminal law proposed by E. Sontag (1989) and then extended to event-based control of affine nonlinear undelayed systems. Under the assumption of the existence of a control Lyapunov–Krasovskii functional (CLKF), the proposal enables smooth (except at the origin) asymptotic stabilization while ensuring that the sampling intervals do not contract to zero. Global asymptotic stability is obtained under the small control property assumption. Moreover, the control can be proved to be smooth anywhere under certain conditions. Simulation results highlight the ability of the proposed formula. The particular linear case is also discussed.

4.1 Introduction

The control synthesis problem is quite complex for systems with nonlinearities, particularly when the control laws have to be implemented on a real-time platform. Different techniques exist. The most classical way to address a discrete-time feedback for nonlinear systems is (i) to implement a (periodic) continuous-time control algorithm with a sufficiently small sampling period. This procedure is denoted as *emulation*. However, the hardware used to sample and hold the plant measurements

---

S. Durand (✉)
ICube, INSA Strasbourg, University of Strasbourg, CNRS (UMR 7357),
Strasbourg, France
e-mail: sylvain@durandchamontin.fr

N. Marchand
GIPSA-Lab, University of Grenoble Alpes, CNRS (UMR 5216),
Grenoble, France
e-mail: nicolas.marchand@gipsa-lab.fr

J.F. Guerrero-Castellanos
Faculty of Electronics, Autonomous University of Puebla (BUAP),
Puebla, Mexico
e-mail: fguerrero@ece.buap.mx

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_4

or compute the feedback control action may make impossible the reduce of the sampling period to a level that guarantees acceptable closed-loop performance, as demonstrated in [15]. Furthermore, although periodicity simplifies the design and analysis, it results in a conservative usage of resources. Other methods are (ii) the application of *sampled-data control* algorithms based on an approximated discrete-time model of the process, like in [26], or (iii) the modification of a continuous-time stabilizing control using a general formula to obtain a *redesigned control* suitable for sampled-data implementation, as done in [25]. However, all these techniques are not generic enough for engineering applications. Finally, (iv) *event-triggered control* approaches have also been suggested as a solution in recent decades, where the control law is event-driven. These novel alternatives are resource-aware implementations, they overcome drawbacks of emulation, redesigned control and complexity of the underlying nonlinear sampled-data models.

Whereas the control law is computed and updated at the same rate regardless whether is really required or not in the classical time-triggered approaches, the *event-based paradigm* relaxes the periodicity of computations and communications in calling for resources whenever they are indeed necessary (for instance when the dynamics of the controlled system varies). This is clearly an opportunity for embedded and networked control systems. Nevertheless, although event-based control is well-motivated, only few works report theoretical results about stability, convergence, and performance. Typical event-detection mechanisms are functions on the variation of the state (or at least the output) of the system, like in [3, 4, 6, 7, 9, 14, 23, 29, 30]. It has notably been shown in [4] that the control law can be updated less frequently than with a periodic scheme while still ensuring the same performance. Stabilization of linear and nonlinear systems is analyzed in [1, 8, 24, 34, 35], where the events are related to the variation of a Lyapunov function or the time derivative of a Lyapunov function (and consequently to the state too). On the other hand, only few works deal with time-delay systems (which are of high concern in networked systems and in general for cyber-physical systems). One can refer to [6, 13, 21, 22] for linear systems for instance. As evidenced by the above reviewed literature, very little attention has been dedicated to the stabilization of nonlinear time-delayed systems using an event-based approach. To the authors' knowledge, this is the first time that an event-based control strategy is proposed.

Technically, it has been shown that if a *control Lyapunov function* (CLF) is known for a nonlinear system that is affine in the control, then the CLF and the system equations can be used to redesign the feedback by means of so-called universal formulas. These formulas are called *universal* because they depend only upon the CLF and the system equations, and not on the structure of those equations. The concept of CLF is therefore a useful tool for synthesizing robust control laws for nonlinear systems. In particular, the present work is based on the *Sontag's universal formula* [33], which event-based version was recently proposed in [24] for undelayed systems. The combination of (i) an event function (based on the time derivative of the CLF) and (ii) a feedback function (that is only updated when the event function vanishes) ensures the strict decrease of the CLF and consequently the asymptotic stability of the closed-loop system. For time-delay systems, the idea of CLF has been extended in the form of *control Lyapunov–Razumikhin functions* (CLRF) and *control Lyapunov–Krasovskii functionals* (CLKF), see [16–18]. The latter form is more flexible and easier to construct than CLRFs. Moreover, if a CLKF is known for a nonlinear time-delay system, several stabilizing control laws can be constructed using universal formulas derived for CLFs (such as the *Sontag*'s one for instance) to achieve global asymptotic stability of the closed-loop system. Accordingly, the universal event-based formula developed in [24] for undelayed systems is extended here for the stabilization of affine nonlinear time-delay systems using CLKF. The present work extends the results previously presented in [8]. The class of time-delay systems under consideration is restricted to depend on some discrete delays and a distributed delay. Moreover, only state delays are considered (delays in the control signal, i.e., input delays, are not concerned).

The rest of the document is organized as follows. In Sect. 4.2, preliminaries on classical (time-triggered) stabilization of nonlinear time-delay systems are presented. CLF and CLKF definitions are recalled as well as well-known universal formulas. The main contribution is then detailed in Sect. 4.3. The event-based paradigm is first introduced and then a universal event-based formula, based on the *Sontag*'s formula, is proposed for the stabilization of affine nonlinear systems with delays in the state. The smooth control particular case is also treated. Illustrative examples are given for both nonlinear and linear cases. A discussion finally concludes the chapter. Proofs are given in Appendix.

4.2 Preliminaries on Nonlinear Time-Delay System Stabilization

Stability is an important issue in control theory. For nonlinear dynamical systems, this is mainly treated with the theory of Lyapunov: if the derivative of a Lyapunov function candidate (a scalar positive definite function of the states) can be shown to be negative definite along the trajectories of a given system, then the system is guaranteed to be asymptotically stable [19]. For closed-loop systems, this means to propose a feedback function and then search for an appropriate Lyapunov function or, inversely, propose a Lyapunov function candidate and then find a feedback strategy that renders its derivative negative [19]. Nevertheless, it can be difficult to find a Lyapunov function candidate or even to determine whether or not one exists. Obviously, some techniques can help for such Lyapunov-based control synthesis.

4.2.1 Control Lyapunov Function

The (Lyapunov-based) control synthesis problem was made more formal with the introduction of *control Lyapunov function* (CLF) [2, 32] for systems affine in the control input. A CLF is a (smooth) positive definite, radially unbounded function, which derivative can be made negative definite at each state (except possibly at the origin) by some feasible input. In addition, one may require that the CLF fulfills the *small control property* for global stability.

To summarize, let us consider the affine nonlinear dynamical system

$$\begin{aligned} &\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)\, u(t), \\ &\text{with} \quad x(0) := x_0, \end{aligned} \tag{4.1}$$

with $x(t) \in \mathscr{X} \subset \mathbb{R}^n$ and $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ the state and input (control) space vectors. $f : \mathscr{X} \to \mathscr{X}$ and $g : \mathscr{X} \to \mathbb{R}^{n\times m}$ are smooth functions with f vanishing at the origin. Also, let define $\mathscr{X}^* := \mathscr{X} \backslash \{0\}$ hereafter. Note that only null stabilization is considered here and the dependence on t can be omitted in the sequel for the sake of simplicity.

Definition 1 (*Control Lyapunov function* [33]). A smooth and positive definite functional $V : \mathscr{X} \to \mathbb{R}$ is a *control Lyapunov function (CLF)* for system (4.1) if for each $x \neq 0$ there is some $u \in \mathscr{U}$ such that

$$\begin{aligned} &\alpha(x) + \beta(x)u < 0, \\ &\text{with} \left| \begin{aligned} &\alpha(x) := L_f V(x) = \frac{\partial V}{\partial x} f(x), \\ &\beta(x) := L_g V(x) = \frac{\partial V}{\partial x} g(x), \end{aligned} \right. \end{aligned} \tag{4.2}$$

where $L_f V$ and $L_g V$ are the Lie derivatives of f and g functions respectively.

Property 1 (Small control property [33]). *If for any $\mu > 0$, $\varepsilon > 0$ and x in the ball $\mathscr{B}(\mu)\backslash\{0\}$, there is some u with $\|u\| \leq \varepsilon$ such that inequality* (4.2) *holds, then it is possible to design a feedback control that asymptotically stabilizes the system.*

Furthermore, it has been shown that if a CLF is known for a nonlinear system that is affine in the control, then the CLF and the system equations can be used to find some so-called *universal formulas* that render the system asymptotically stable. Several known universal formulas exist, in particular *Sontag*'s [33] and *Freeman*'s [10] formulas are presented in the sequel. Other methods, like the *domination redesign* formula [31] is also discussed but it will not be treated in details here.

4.2.2 Control Lyapunov–Krasovskii Functionals

For (nonlinear) time-delay systems, there exist two main Lyapunov techniques, called the *Krasovskii* method of Lyapunov functionals [20] and the *Razumikhin* method of Lyapunov functions [28]. Motivated by the concept of CLF and the role it plays in robust stabilization of nonlinear systems, these methods have also been extended in the form of *control Lyapunov–Razumikhin functions* (CLRF) [16] and *control Lyapunov–Krasovskii functionals* (CLKF) [17].

Several stabilizing control laws can be constructed to achieve global asymptotic stability of the closed-loop system using one of the universal formulas derived for CLFs. For instance, *Sontag*'s [33] and *Freeman*'s [10] formulas apply for CLKF [17], whereas the *domination redesign* formula [31] applies for CLRF [16]. Note that this latter formula also applies for an augmented CLKF, as shown in [18]. Moreover, the CLKF form is more flexible and easier to construct than CLRFs. For these reasons, only *Krasovskii* methods are detailed in the sequel (but the proposal can be easily extended to the *Razumikhin* version).

Hereafter, the state of a time-delay system is described by $x_d : [-r, 0] \to \mathscr{X}$ defined by $x_d(t)(\theta) = x(t + \theta)$. This notation, used in [17] in particular, seems more convenient than the more conventional $x_t(\theta)$. Note that the dependence on t and θ can be omitted in the sequel for the sake of simplicity, writing $x_d(\theta)$ – or only x_d – instead of $x_d(t)(\theta)$ for instance. Let consider the affine (in the control) nonlinear dynamical time-delay system

$$\begin{aligned} \dot{x} &= f(x_d) + g(x_d)u, \\ \text{with} \quad & x_d(0)(\theta) := \chi_0(\theta), \end{aligned} \tag{4.3}$$

where $f : \mathscr{X} \to \mathscr{X}$, $g : \mathscr{X} \to \mathbb{R}^{n\times m}$ are smooth functions and $\chi_0 : [-r, 0] \to \mathscr{X}$ is a given initial condition.

Remark 1 Input delays of the form $u(t - \tau)$ are not considered in this chapter. However, the control law is computed using the state x_d of the time-delay system.

Note that the class of time-delay systems under consideration in this paper is restricted to depend on l discrete delays and a distributed delay in the form

$$\begin{aligned} \dot{x} &= \Phi(x_\tau) + g(x_\tau)u, \\ \text{with} \quad & \Phi(x_\tau) := f_0(x_\tau) + \int_{-r}^{0} \Gamma(\theta) F\big(x_\tau, x(t+\theta)\big) d\theta, \\ \text{and} \quad & x_\tau := \big[x, x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_l)\big], \end{aligned} \tag{4.4}$$

where $f_0 : \mathscr{X} \to \mathscr{X}$, $g : \mathscr{X} \to \mathbb{R}^{n\times m}$ and $F : \mathbb{R}^{(l+2)n} \to \mathbb{R}^{\Gamma}$ are smooth functions of their arguments. Without loss of generality, it is assumed that $F(x_\tau, 0) = 0$ and the matrix $\Gamma : [-r, 0] \to \mathbb{R}^{n\times\Gamma}$ is piecewise continuous (hence, integrable) and bounded.

Definition 2 (*Control Lyapunov–Krasovskii functional* [17]). Let defined a smooth functional $V : \mathscr{X} \to \mathbb{R}$ of the particular form

$$V(x_d) = V_1(x) + V_2(x_d) + V_3(x_d), \tag{4.5}$$
$$\text{with} \quad \left| \begin{array}{l} V_2(x_d) = \displaystyle\sum_{j=1}^{l} \int_{-\tau_j}^{0} S_j(x(t-\varsigma))d\varsigma, \\ V_3(x_d) = \displaystyle\int_{-r}^{0} \int_{t+\theta}^{t} L(\theta, x(\varsigma))d\varsigma d\theta, \end{array} \right.$$

where V_1 is a smooth, positive definite, radially unbounded function of the current state x (i.e., the classical control Lyapunov function for undelayed systems, as defined in Definition 1), V_2 and V_3 are nonnegative functionals respectively due to the discrete delays and the distributed delay in (4.4), $S_j : \mathscr{X} \to \mathbb{R}$ and $L : \mathbb{R}^+ \times \mathscr{X} \to \mathbb{R}$ are nonnegative integrable functions, smooth in the x-argument. Then V in (4.5) is a *control Lyapunov–Krasovskii functional (CLKF)* for system (4.4) if there exists a function λ, with $\lambda(s) > 0$ for $s > 0$, and two class $\mathscr{K}_\infty$ functions κ_1 and κ_2 such that

$$\kappa_1(|\chi_0|) \leq V(\chi_d) \leq \kappa_2(\|\chi_d\|),$$

and (see Remark 3 for the definition of $L_f^* V$)

$$\beta_d(\chi_d) = 0 \quad \Rightarrow \quad \alpha_d(\chi_d) \leq -\lambda(|\chi_0|), \tag{4.6}$$
$$\text{with} \quad \left| \begin{array}{l} \alpha_d(x_d) := L_f^* V(x_d), \\ \beta_d(x_d) := L_g V_1(x_d), \end{array} \right.$$

for all piecewise continuous functions $\chi_d : [-r, 0] \to \mathscr{X}$, where χ_0 is defined in (4.3).

Remark 2 The restriction on the class of delay systems (4.4) and the corresponding particular CLKF (4.5) is needed to avoid the problems that arise due to non-compactness of closed bounded sets in the space $\big(C([-r, 0], \mathscr{X}\big), \|\cdot\|)$, where $C([-r, 0], \mathscr{X})$ denotes the space of continuous functions from $[-r, 0]$ into $\mathscr{X}$. This is discussed in [16, 17].

Remark 3 Whereas the classical Lie derivative notation is used in $L_g V_1(x) = \frac{\partial V_1}{\partial x} g(x)$ for the CLKF part V_1 which is function of the current state x, an extended Lie derivative is required for functionals of the form (4.5). $L_f^* V$, initially defined in [17], comes from the time derivative of the CLKF V in (4.5) along trajectories of the system (4.4), that is

$$\dot{V} = L_f^* V(x_d) + L_g V_1(x_d)u = \alpha_d(x_d) + \beta_d(x_d)u, \tag{4.7}$$
$$\text{with} \quad L_f^* V(x_d) := \frac{\partial V_1}{\partial x} \Phi + \sum_{j=1}^{l} \Big(S_j(x) - S_j(x(t-\tau_j)) \Big)$$
$$+ \int_{-r}^{0} \Big(L(\theta, x) - L(\theta, x(t+\theta)) \Big) d\theta,$$

where Φ is defined in (4.4).

4.2.3 Universal Formulas for the Stabilization of Affine Nonlinear Time-Delay Systems

Universal formulas derived for CLFs have been extended for the stabilization of affine nonlinear time-delay systems (4.4) with a CLKF of the form (4.5). In particular, the *Sontag*'s [33] and *Freeman*'s [10] versions are detailed here.

Theorem 1 (Sontag's universal formula with CLKF [17]). *Assume that system* (4.4) *admits a CLKF of the form* (4.5)*. For any real analytic function* $q : \mathbb{R} \to \mathbb{R}$ *such that* $q(0) = 0$ *and* $bq(b) > 0$ *for* $b \neq 0$*, let* $\phi_s : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$\phi_s(a, b) := \begin{cases} \dfrac{a + \sqrt{a^2 + bq(b)}}{b} & \text{if } \ b \neq 0, \\ 0 & \text{if } \ b = 0. \end{cases} \tag{4.8}$$

Then, the feedback $u : \mathscr{X} \to \mathscr{U}$*, smooth on* $\mathscr{X}^*$*, defined by*

$$u(x_d) := -\beta_d(x_\tau)\,\phi_s\Big(\alpha_d(x_d),\, \|\beta_d(x_d)\|^2\Big), \tag{4.9}$$

with x_τ *and* α_d*,* β_d *defined in* (4.4) *and* (4.6) *respectively, is such that* (4.6) *is satisfied for all nonzero piecewise continuous functions* $\chi_d : [-r, 0] \to \mathscr{X}$*.*

Theorem 2 (Freeman's universal formula with CLKF [17]). *Assume that system* (4.4) *admits a CLKF of the form* (4.5)*. For any continuous and positive definite function* $\eta : \mathbb{R}^2 \to \mathbb{R}$*, let* $\phi_f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$\phi_f(a, b) := \begin{cases} \dfrac{a + \eta(a, b)}{b} & \text{if } \ a + \eta(a, b) > 0, \\ 0 & \text{if } \ a + \eta(a, b) \leq 0. \end{cases} \tag{4.10}$$

Then, the feedback $u : \mathscr{X} \to \mathscr{U}$*, smooth on* $\mathscr{X}^*$*, defined by*

$$u(x_d) := -\beta_d(x_\tau)\,\phi_f\Big(\alpha_d(x_d),\, \|\beta_d(x_d)\|^2\Big), \tag{4.11}$$

with x_τ *and* α_d*,* β_d *defined in* (4.4) *and* (4.6) *respectively, is such that* (4.6) *is satisfied for all nonzero piecewise continuous functions* $\chi_d : [-r, 0] \to \mathscr{X}$*.*

Property 2 (Small control property with CLKF [17]). *If the CLKF* V *in Theorem 1 of Theorem 2 satisfies the* small control property, *then the control is continuous at the origin and so is globally asymptotically stable the closed-loop system.*

Remark 4 Choosing the function η (4.10) as the particular form

$$\eta(a, b) = \sqrt{a^2 + bq(b)},$$

where q is a continuous, positive semidefinite function, gives the same function (4.8) as originally proposed by *Sontag*.

Remark 5 As already said, the *domination redisign* formula [31] has also been extended for time-delay systems using CLRF in [16] and CLKF in [18]. The feedback $u : \mathscr{X} \to \mathscr{U}$, smooth on $\mathscr{X}^*$, takes the more general form

$$u(x_d) := -\beta_d(x_\tau)\,\phi_d\big(V(x_d)\big), \tag{4.12}$$

where the scalar function ϕ_d is called the *dominating function*. Also, a particular choice of this function can lead to the original *Sontag*'s function (4.8).

4.3 Event-Based Stabilization of Nonlinear Time-Delay Systems

The idea behind extending the (time-triggered) universal formulas to event-driven versions is to obtain equivalent but resource-aware strategies, because the control signal will be computed and updated only when a certain condition is satisfied in the event-based case. This was already done in [24] for the undelayed case and it is extended here for time-delay systems. The event-based paradigm is first introduced. Then, an *event-based formula* for the stabilization of affine nonlinear time-delay systems admitting a CLKF is then detailed, derived from the *Sontag*'s formula [33]. Other universal formulas are not concerned but the extension is trivial since they are all similarly constructed. An illustrative example highlights the ability of the proposal. Finally, the particular case of linear systems is discussed.

4.3.1 Event-Based Formalization

The classical discrete-time framework of controlled systems consists in sampling the system uniformly in time with a constant sampling period. Although periodicity simplifies the design and analysis, it results in a conservative usage of resources (computation, communication, energy) since the control law is computed and updated at the same rate regardless it is really required or not. Fortunately, some innovative works addressed resource-aware implementations of the control law, where the control law is event driven (when a certain condition is satisfied).

Definition 3 (*Event-based feedback*) By *event-based feedback* we mean a set of two functions, that are

(i) *an* ***event function*** $\varepsilon : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ that indicates if one needs (when $\varepsilon \leq 0$) or not (when $\varepsilon > 0$) to recompute the control law;
(ii) *a* ***feedback function*** $\upsilon : \mathscr{X} \to \mathscr{U}$.

The solution of (4.1) with event-based feedback (ε, υ) starting in x_0 at $t = 0$ is then defined as the solution of the differential system

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)\, \upsilon(t_i) \quad \forall t \in [t_i, t_{i+1}[, \tag{4.13}$$

where the time instants t_i, with $i \in \mathbb{N}$, are considered as *events* (they are determined when the event function ε vanishes and denote the sampling time instants). Also let define x_i the memory of the state value at the last event, that is

$$x_i := x(t_i). \tag{4.14}$$

With such a formalization, the control value is updated each time ε becomes negative. Usually, one tries to design an event-based feedback so that ε cannot remain negative (and so is updated the control only punctually). In addition, one also wants that two events are separated with a nonvanishing time interval avoiding the *Zeno* phenomenon. All these properties are encompassed with the *Minimal Inter-Sampling Interval* (MSI) property introduced in [24]. In particular:

Property 3 (Semi-uniformly MSI). *An event-triggered feedback is said to be* semi-uniformly MSI *if and only if the inter-execution times can be below bounded by some nonzero minimal sampling interval* $\underline{\tau}(\delta) > 0$ *for any* $\delta > 0$ *and any initial condition* x_0 *in the ball* $\mathscr{B}(\delta)$ *centered at the origin and of radius* δ.

Remark 6 A semi-uniformly MSI event-driven control is a piecewise constant control with nonzero sampling intervals (useful for implementation purpose).

A particular event-based feedback has already been proposed in [24] for the stabilization of affine nonlinear undelayed systems, based on the *Sontag*'s universal formula [33]. The idea is to have a control law υ quite similar to the one in the classical approach and an event function ε related to the time derivative of the CLF in order to ensure a (global) asymptotic stability of the closed-loop system. In the present chapter, such an event-based feedback is extended for the stabilization of affine nonlinear systems with time delay using CLKF. In the sequel, let

$$x_{di} := x_d(t_i) \tag{4.15}$$

be the memory of the delayed state value at the last event, by analogy with (4.14).

4.3.2 Event-Based Stabilization of Nonlinear Time-Delay Systems

Based on the *Sontag's universal formula with CLKF* previously introduced in Theorem 1, an event-based feedback (see Definition 3) that asymptotically stabilizes affine nonlinear time-delay systems is proposed here.

Theorem 3 (Event-based universal formula with CLKF). *If there exists a CLKF V of the form* (4.5) *for system* (4.4)*, then the event-based feedback* (ε, υ) *defined by*

$$\upsilon(x_d) = -\beta_d(x_\tau)\Delta(x_\tau)\gamma(x_d), \tag{4.16}$$

$$\begin{aligned}\varepsilon(x_d, x_{di}) &= -\alpha_d(x_d) - \beta_d(x_d)\upsilon(x_{di}) \\ &\quad -\sigma\sqrt{\alpha_d(x_d)^2 + \Omega(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T},\end{aligned} \tag{4.17}$$

with

- α_d *and* β_d *as defined in* (4.6)*;*
- $\Delta : \mathscr{X}^* \to \mathbb{R}^{m\times m}$ *(a tunable parameter) and* $\Omega : \mathscr{X} \to \mathbb{R}$ *are smooth positive definite functions;*
- $\gamma : \mathscr{X} \to \mathbb{R}$ *defined by*

$$\gamma(x_d) := \begin{cases} \frac{\alpha_d(x_d)+\sqrt{\alpha_d(x_d)^2+\Omega(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T} & \text{if } x_d \in \mathscr{S}_d, \\ 0 & \text{if } x_d \notin \mathscr{S}_d, \end{cases} \tag{4.18}$$

with $\mathscr{S}_d := \{x_d \in \mathscr{X} \mid \|\beta_d(x_d)\| \neq 0\}$*;*

- $\sigma \in [0, 1[$ *a tunable parameter;*

where x_{di} *and* x_τ *are defined in* (4.15) *and* (4.4) *respectively, is semi-uniformly MSI, smooth on* $\mathscr{X}^*$ *and such that the time derivative of* V *satisfies* (4.6) $\forall x \in \mathscr{X}^*$.

Remark 7 The simplification made with respect to the original result in [24] (for the stabilization of nonlinear undelayed systems) resides in the assumptions made for the functions Ω and Δ, that are more restrictive here whereas they are assumed to be definite only on the set $\mathscr{S}_d$ in the original work.

Remark 8 The idea behind the construction of the event-based feedback (4.16)–(4.17) is to compare the time derivative of the CLKF V (i) in the event-based case, that is when applying the piecewise feedback $\upsilon(x_{di})$, and (ii) in the classical case, that is, when applying $\upsilon(x_d)$ instead of $\upsilon(x_{di})$. The event function is the weighted difference between both, where σ is the weighted value. By construction, an event is enforced when the event function ε vanishes to zero, that is, hence when the stability of the event-based scheme does not behave as the one in the classical case. Also, the convergence will be faster with higher σ but with more frequent events in return. $\sigma = 0$ means updating the control when $\dot{V} = 0$.

Property 4 (Global asymptotic stability). *If the CLKF V in Theorem 3 satisfies the* small control property*, then the event-based feedback* (4.16)–(4.17) *is continuous at the origin and so is globally asymptotically stable the closed-loop system.*

Property 5 (Smooth control). *If there exists some smooth function* $\omega : \mathscr{X} \to \mathbb{R}^+$ *such that on* $\mathscr{S}_d^* := \mathscr{S}_d \backslash \{0\}$

$$\omega(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T - \alpha_d(x_d) > 0, \tag{4.19}$$

then the control is smooth on $\mathscr{X}$ *as soon as* $\Omega(x_d)\|\Delta(x_d)\|$ *vanishes at the origin with*

$$\Omega(x_d) := \omega(x_d)^2\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T - 2\alpha_d(x_d)\omega(x_d). \tag{4.20}$$

Proof All proofs are given in the Appendix section.

4.3.2.1 Example

Consider the nonlinear time-delay system

$$\begin{aligned} &\dot{x}_1 = u, \\ &\dot{x}_2 = -x_2 + x_{2d} + x_1^3 + u, \\ &\text{with} \quad x_{2d} := x_2(t - \tau), \end{aligned} \tag{4.21}$$

that admits a CLKF (proposed in [17])

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}\int_{-\tau}^{0} x_{2d}^2(\theta)d\theta, \tag{4.22}$$

$$\text{with} \quad \left| \begin{aligned} &\alpha_d = x_2(-x_2 + x_{2d} + x_1^3) + \tfrac{1}{2}(x_2^2 - x_{2d}^2), \\ &\beta_d = x_1 + x_2. \end{aligned} \right.$$

Indeed, setting $\lambda(|x|) = \frac{1}{4}|x|^4$ yields

$$\begin{aligned} &\beta_d = 0 \Rightarrow x_1 = -x_2, \\ &\Rightarrow \alpha_d = -\frac{1}{2}(x_2 - x_{2d})^2 - x_2^4 \le -x_2^4 \le -\lambda(|x|), \end{aligned}$$

which proves that (4.22) is a CLKF for (4.21) using Definition 2.

The time evolution of x, $\upsilon(x)$ and the event function $\varepsilon(x, x_i)$ is depicted in Fig. 4.1, for $\Delta = I_n$ (the identity matrix), $\Omega(x)$ is as defined in (4.20) (for smooth control everywhere), with $\omega = 0.1$, $\sigma = 0.6$, $x_0 = \begin{bmatrix} 0.5 & -1 \end{bmatrix}^T$ and a time delay $\tau = 2\,s$. One could remark that only 7 events occurs in the 50 s simulation time (including the first event at $t = 0$) when applying the proposed event-based approach (4.16)–(4.17). Furthermore, x_1 and x_2 rapidly converge to 0 with the first 4 events.

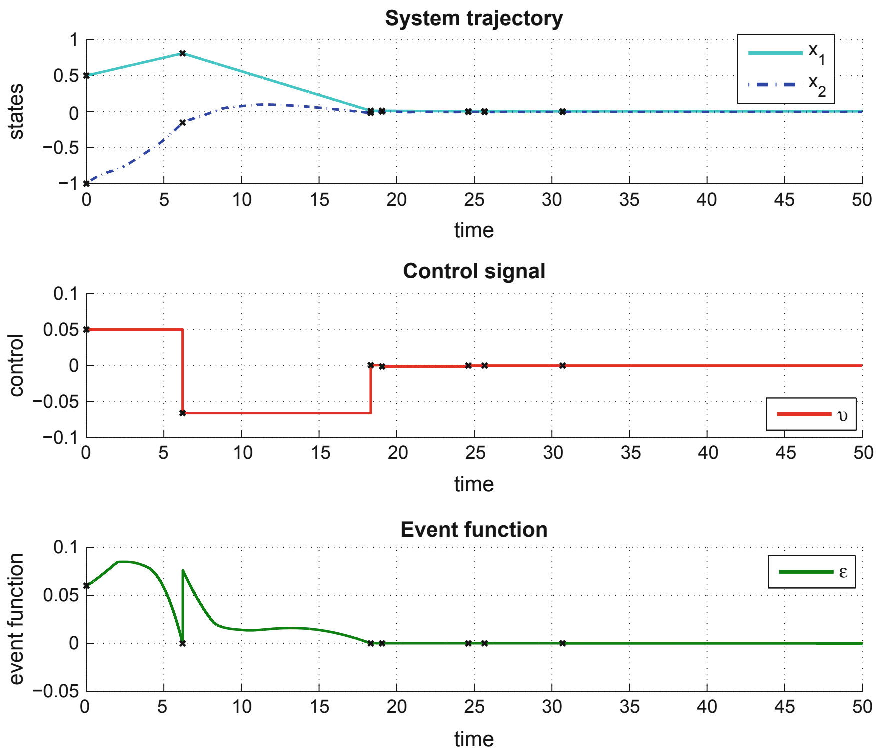

Fig. 4.1 Simulation results of system (4.21) with CLKF as in (4.22) and event-based feedback (4.16)–(4.17)

4.3.3 Particular Case of Linear Systems

Consider the simple linear system with single delay τ

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bu(t). \tag{4.23}$$

Take P and S the positive definite matrices solution of the linear matrix inequality (LMI) given by

$$\begin{bmatrix} A^T P + PA - 4\rho PBR^{-1}B^T P + S & PA_d - 4\rho PBR^{-1}B^T P \\ A_d^T P - 4\rho PBR^{-1}B^T P & -S \end{bmatrix} < 0, \tag{4.24}$$

where R is positive definite matrix, and $\rho > 0$, are tunable parameters. Then the Lyapunov–Krasovskii functional V defined by

$$V(x_d) = x^T(t)Px(t) + \int_{\tau}^{0} x_d(\theta)^T Sx_d(\theta)d\theta \tag{4.25}$$

is a CLKF for system (4.23) since for all x, $u = -2\rho B^T Px$ renders the time derivative of V strictly negative for $x \neq 0$.

Remark 9 The particular *delay-independent* form (4.25) has been proposed for system (4.23) without control input. More complex *delay-dependent* forms also exist in the literature but are not concerned here, see [11, 12, 27] for instance for further details.

Remark 10 Remember the first right-hand term in (4.25) is the classical CLF for a linear system without delay, whereas the second term is added for a single delay. The third term in the general CLKF form (4.5) is not needed in the present case without distributed delay.

The (extended) Lie derivatives are then obtained from the expressions in (4.6)–(4.7), that yields

$$\begin{aligned} \alpha_d(x_d) &= \begin{bmatrix} x \\ x_d \end{bmatrix}^T \begin{bmatrix} A^T P + PA + S & PA_d \\ A_d^T P & -S \end{bmatrix} \begin{bmatrix} x \\ x_d \end{bmatrix}, \\ \beta_d(x_d) &= 2 \begin{bmatrix} x \\ x_d \end{bmatrix}^T PB. \end{aligned} \tag{4.26}$$

Then, with $\Omega(x_d)$ according to (4.20) for the tunable parameters defined by $\Delta = R^{-1}$ and $\omega = \rho$, the control given by

$$\upsilon(x_d) = -\omega \Delta \begin{bmatrix} \beta(x)^T \\ \beta(x_d)^T \end{bmatrix} \tag{4.27}$$

is smooth everywhere and linear. The event function given by

$$\varepsilon(x_d, x_{di}) = (\sigma - 1)\alpha_d(x_d) + \omega\beta_d(x_d)\Delta \begin{bmatrix} \beta(x_i - \sigma x)^T \\ \beta(x_{di} - \sigma x_d)^T \end{bmatrix} \tag{4.28}$$

is linear.

4.4 Conclusion

In this chapter, an extension of the *Sontag*'s universal formula was proposed for event-based stabilization of affine nonlinear systems with delays in the state. Whereas the original work deals with control Lyapunov functions for the case of undelayed systems, some *control Lyapunov–Krasovskii functionals* (CLKF) are now required for a global (except at the origin) asymptotic stabilization of time-delay systems. The sampling intervals do not contract to zero, avoiding *Zeno* phenomena. Moreover, the control is continuous at the origin if the CLKF fulfills the *small control property*. With additional assumption, the control can be proved to be smooth everywhere.

Simulation results were provided, highlighting the low frequency of control updates. The linear case was also discussed.

Next step is to test the proposal in a real-time implementation and also consider input delays. Another way of investigation could be to develop general universal event-based formulas for nonlinear (time-delay) systems, in the spirit of [5].

Appendix

Proofs of the present contribution were previously presented in [8]. They are recalled here.

Proof of Theorem 3

The proof follows the one developed in [24] for event-based control of systems without delays (4.1). First, let define hereafter

$$\psi(x) := \sqrt{\alpha_d(x)^2 + \Omega(x)\beta_d(x)\Delta(x)\beta_d(x)^T}. \tag{4.29}$$

Let begin establishing γ is smooth on $\mathscr{X}^*$. For this, consider the algebraic equation

$$P(x_d, \zeta) := \beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T\zeta^2 - 2\alpha_d(x_d)\zeta - \Omega(x_d) = 0. \tag{4.30}$$

Note first that $\zeta = \gamma(x)$ is a solution of (4.30) for all $x_d \in \mathscr{X}$. It is easy to prove that the partial derivative of P with respect to ζ is always strictly positive on $\mathscr{X}^*$

$$\frac{\partial P}{\partial \zeta} := 2\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T\zeta - 2\alpha_d(x_d). \tag{4.31}$$

Indeed, when $\|\beta_d(x_d)\| = 0$, (4.6) gives $\frac{\partial P}{\partial \zeta} = -2\alpha_d(x_d) \geq 2\lambda(|\chi_0|) > 0$ and when $\|\beta_d(x_d)\| \neq 0$, (4.18) gives $\frac{\partial P}{\partial \zeta} = 2\sqrt{\alpha_d(x_d)^2 + \Omega(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T} > 0$ replacing ζ in (4.31) by the expression of γ (since $\zeta = \gamma(x)$ is a solution of (4.30)). Therefore $\frac{\partial P}{\partial \zeta}$ never vanishes at each point of the form $\{(x_d, \gamma(x_d))|x_d \in \mathscr{X}^*\}$. Furthermore, P is smooth w.r.t. x_d and ζ since so are α_d, β_d, Ω and Δ. Hence, using the implicit function theorem, γ is smooth on $\mathscr{X}^*$.

The decrease of the CLKF of the form (4.5) when applying the event-based feedback (4.16)–(4.17) is easy to prove. For this, let consider the time interval $[t_i, t_{i+1}]$, that is the interval separating two successive events. Recall that x_{di} denotes the value of the state when the i^{th} event occurs and t_i the corresponding time instant, as defined in (4.15). At time t_i, when the event occurs, the time derivative of the CLKF, i.e., (4.7), after the update of the control, is

$$\frac{dV}{dt}(x_{di}) = \alpha_d(x_{di}) + \beta_d(x_{di})\upsilon(x_{di}) = -\psi(x_{di}) < 0$$

when substituting (4.18) in (4.16), where ψ is defined in (4.29). More precisely, defining a compact set not containing the origin, that is $\Sigma = \{x_d \in CP([-r, 0], \mathscr{X}) : d \leq \|x_d\| \leq D\}$, where $CP([-r, 0], \mathscr{X})$ denotes the space of piecewise continuous functions from $[-r, 0]$ into $\mathscr{X}$, d and D are some constant in $\mathbb{R}^+$. If V is a CLKF for the system of the form (4.4) then for all $0 < \delta < D$ there exists $\varepsilon > 0$ such that $\alpha_d(\chi_d) \geq -\frac{1}{2}\lambda(|\chi_0|) \Rightarrow |\beta_d(\chi_d)| \geq \varepsilon$ for $\chi_d \in \Sigma$. This gives

$$\dot{V} \leq -\lambda(|x|).$$

One can refer to Lemma 1 in [17], and [16], for further details. With this updated control, the event function (4.17) hence becomes strictly positive

$$\varepsilon(x_{di}, x_{di}) = (1 - \sigma)\psi(x_{di}) > 0,$$

since $\sigma \in [0, 1[$, where ψ is defined in (4.29). Furthermore, the event function necessarily remains positive before the next event by continuity, because an event will occur when $\varepsilon(x_d, x_{di}) = 0$ (see Definition 3). Therefore, on the interval $[t_i, t_{i+1}]$, one has

$$\begin{aligned}\varepsilon(x_d, x_{di}) &= -\alpha_d(x_d) - \beta_d(x_d)\upsilon(x_{di}) - \sigma\psi(x_d), \\ &= -\frac{dV}{dt}(x_d) - \sigma\psi(x_d) \geq 0,\end{aligned}$$

which ensures the decrease of the CLKF on the interval since $\sigma\psi(x_d) \geq 0$, where ψ is defined in (4.29). Moreover, t_{i+1} is necessarily bounded since, if not, V should converge to a constant value where $\frac{dV}{dt} = 0$, which is impossible thanks to the inequality above. The event function precisely prevents this phenomena detecting when $\frac{dV}{dt}$ is *close to vanish* and updates the control if it happens, where σ is a tunable parameter fixing how "close to vanish" has to be the time derivative of V.

To prove that the event-based control is MSI, one has to prove that for any initial condition in an a priori given set, the sampling intervals are below bounded. First of all, notice that events only occur when ε becomes negative (with $x_d \neq 0$). Therefore, using the fact that when $\beta_d(x_d) = 0$, $\alpha_d(x_d) < -\lambda(|\chi_0|)$ (because V is a CLKF as defined in Definition 2), it follows from (4.17), on $\{x_d \in \mathscr{X}^* \mid \|\beta_d(x_d)\| = 0\}$, that

$$\varepsilon(x_d, x_{di}) = -\alpha_d(x_d) - \sigma|\alpha_d(x_d)| = (1 - \sigma)\lambda(|\chi_0|) > 0,$$

because $\sigma \in [0, 1[$ and $\lambda(s) > 0$ for $s > 0$. Therefore, there is no event on the set $\{x_d \in \mathscr{X} | \|\beta_d(x_d)\| = 0\} \cup \{0\}$. The study is then restricted to the set $\mathscr{S}_d^* = \{x_d \in \mathscr{X}^* | \|\beta_d(x_d)\| \neq 0\}$, where Ω and Δ are strictly positive by assumption. Rewriting the time derivative of the CLKF along the trajectories yields

$$\begin{aligned}\frac{dV}{dt}(x_d) &= \alpha_d(x_d) + \beta_d(x_d)\upsilon(x_{di}),\\ &= -\psi(x_d) + \beta_d(x_d)\big(\upsilon(x_{di}) - \upsilon(x_d)\big),\end{aligned} \tag{4.32}$$

when using the definition of $\upsilon(x_d)$ in (4.16) and (4.18), where ψ is defined in (4.29). Let respectively define the level and the set

$$\begin{aligned}\vartheta_i &:= V(x_{di}), \qquad \forall x_{di} \in \mathscr{S}_d,\\ \mathscr{V}_{\vartheta_i} &:= \{x_d \in \mathscr{X} \,|\, V(x_d) \leq \vartheta_i\}.\end{aligned}$$

From the choice of the event function, it follows from (4.32) that x_d belongs to $\mathscr{V}_\vartheta \subset \mathscr{V}_{\vartheta_i}$. Note that if x_{di} belongs to $\mathscr{S}_d$, this is not necessarily the case for x_d that can escape from this set. First see that, since (i) $\Omega(x_d)$ is such that $\alpha_d(x_d)^2 + \Omega(x_d)\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T > 0$ for all $x_d \in \mathscr{S}_d^*$, and (ii) $\alpha_d(x_d)$ is necessarily nonzero on the frontier of $\mathscr{S}_d$ (except possibly at the origin)

$$\frac{dV}{dt}(x_{di}) = -\psi(x_{di}) \leq -\inf_{\substack{x_{di}\in\mathscr{S}_d\\ \text{s.t.}\, V(x_{di})=\vartheta_i}} \psi(x_{di}) =: -\varphi(\vartheta_i) < 0. \tag{4.33}$$

Considering now the second time derivative of the CLKF

$$\begin{aligned}\ddot{V}(x_d) &= \left(\frac{\partial\alpha_d}{\partial x_d}(x_d) + \upsilon(x_{di})^T\frac{\partial\beta_d^T}{\partial x_d}(x_d)\right)\Theta(x_d, x_{di}),\\ \text{with}\quad &\Theta(x_d, x_{di}) := \Phi(x_\tau) + g(x_\tau)\upsilon(x_{di}),\end{aligned} \tag{4.34}$$

where Φ is defined in (4.4). By continuity of all the involved functions (except for Γ in Φ which is piecewise continuous but bounded by assumption), both terms can be bounded for all $x_d \in \mathscr{V}_{\vartheta_i}$ by the following upper bounds $\rho_1(\vartheta_i)$ and $\rho_2(\vartheta_i)$ such that

$$\begin{aligned}\rho_1(\vartheta_i) &:= \sup_{\substack{x_{di}\in\mathscr{S}_d\ \text{s.t.}\ V(x_{di})=\vartheta_i\\ x_d\in\mathscr{V}_{\vartheta_i}}} \left\|\frac{\partial\alpha_d}{\partial x_d}(x_d) + \upsilon(x_{di})^T\frac{\partial\beta_d^T}{\partial x_d}(x_d)\right\|,\\ \rho_2(\vartheta_i) &:= \sup_{\substack{x_{di}\in\mathscr{S}_d\ \text{s.t.}\ V(x_{di})=\vartheta_i\\ x_d\in\mathscr{V}_{\vartheta_i}}} \|\Theta(x_d, x_{di})\|,\end{aligned}$$

where Θ is defined in (4.34). Therefore, $\dot{V}$ is strictly negative at any event instant t_i and cannot vanish until a certain time $\underline{\tau}(\vartheta_i)$ is elapsed (because its slope is positive). This minimal sampling interval is only depending on the level ϑ_i. A bound on $\underline{\tau}(\vartheta_i)$ is given by the inequality

$$\frac{dV}{dt}(x_d) \leq \frac{dV}{dt}(x_{di}) + \rho_1\rho_2(t - t_i), \qquad \forall x_d \in \mathscr{V}_{\vartheta_i},$$

that yields

$$\underline{\tau}(\vartheta_i) \geq \frac{\varphi(\vartheta_i)}{\rho_1(\vartheta_i)\rho_2(\vartheta_i)} > 0,$$

where φ is defined in (4.33). As a consequence, the event-based feedback (4.16)–(4.17) is semi-uniformly MSI. This ends the proof of Theorem 3.

Proof of Property 4

To prove the continuity of υ at the origin, one only needs to consider the points in $\mathscr{S}$ since $\upsilon(x_d) = 0$ if $\|\beta_d(x_d)\| = 0$. Then (4.16) gives

$$\begin{aligned}\|\upsilon(x_d)\| \leq & \frac{|\alpha_d(x_d)|}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\|\Delta(x_d)\beta_d(x_d)^T\| \\ & + \frac{\psi(x_d)}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\|\Delta(x_d)\beta_d(x_d)^T\|, \\ \leq & \frac{2|\alpha_d(x_d)|}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T}\|\Delta(x_d)\beta_d(x_d)^T\| \\ & + \sqrt{\Omega(x_d)\|\Delta(x_d)\|}.\end{aligned} \tag{4.35}$$

With the small control property (see Property 1), for any $\varepsilon > 0$, there is $\mu > 0$ such that for any $x_d \in \mathscr{B}(\mu)\backslash\{0\}$, there exists some u with $\|u\| \leq \varepsilon$ such that $L_f^* V(x_d) + [L_g V_1(x_d)]^T u = \alpha_d(x_d) + \beta_d(x_d)u < 0$ and therefore $|\alpha_d(x_d)| < \|\beta_d(x_d)\|\varepsilon$. It follows

$$\|\upsilon(x_d)\| \leq \frac{2\varepsilon\|\beta_d(x_d)\|\|\Delta(x_d)\beta_d(x_d)^T\|}{\beta_d(x_d)\Delta(x_d)\beta_d(x_d)^T} + \sqrt{\Omega(x_d)\|\Delta(x_d)\|}.$$

Since the function $(v_1, v_2) \rightarrow \frac{\|v_1\|\|v_2\|}{v_1^T v_2}$ is continuous w.r.t. its two variables at the origin where it equals 1, since Ω and Δ are also continuous, since $\Omega(x_d)\|\Delta(x_d)\|$ vanishes at the origin, for any ε', there is some μ' such that $\forall x_d \in \mathscr{B}(\mu')\backslash\{0\}$, $\|\upsilon(x_d)\| \leq \varepsilon'$ which ends the proof of continuity.

Proof of Property 5

With Ω defined as in (4.20), the feedback in (4.16) becomes

$$\upsilon(x_d) = -\beta_d(x_d)\Delta(x_d)\omega(x_d)$$

if the condition (4.19) is satisfied, which is obviously smooth on $\mathscr{X}$. Note that the expression of Ω in (4.20) comes from the solution of (4.30), where ω only has to be smooth.

References

1. A. Anta, P. Tabuada, To sample or not to sample: self-triggered control for nonlinear systems. IEEE Trans. Autom. Control **55**(9), 2030–2042 (2010)
2. Z. Artstein, Stabilization with relaxed controls. Nonlinear Anal.: Theory Methods Appl. **7**(11), 1163–1173 (1983)
3. K. E. Årzén, A simple event-based PID controller, in *IFAC World Congress* (1999)
4. K. J. Åström, B. Bernhardsson, Comparison of Riemann and Lebesque sampling for first order stochastic systems, in *IEEE Conference on Decision and Control (CDC)* (2002)
5. J. Curtis, R. Beard, Satisficing: a new approach to constructive nonlinear control. IEEE Trans. Autom. Control **49**(7), 1090–1102 (2004)
6. S. Durand, Event-based stabilization of linear system with communication delays in the measurements, in *American Control Conference (ACC)* (2013)
7. S. Durand, N. Marchand, Further results on event-based PID controller. *European Control Conference (ECC)* (2009)
8. S. Durand, N. Marchand, J.F. Guerrero Castellanos, Event-based stabilization of nonlinear time-delay systems, in *IFAC World Congress* (2014)
9. A. Eqtami, D.V. Dimarogonas, K.J. Kyriakopoulos, Event-triggered control for discrete-time systems. *IEEE American Control Conference (ACC)* (2010)
10. R. Freeman, P. Kokotovic, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques* (Birkhäuser, 1996)
11. E. Fridman, Introduction to time-delay and sampled-data systems, in *European Control Conference (ECC)* (2014)
12. K. Gu, S. Niculescu, Survey on recent results in the stability and control of time-delay systems. J. Dyn. Syst. Meas. Control **125**(2), 158–165 (2003)
13. M. Guinaldo, D. Lehmann, J. Sánchez, S. Dormido, K.H. Johansson, Distributed event-triggered control with network delays and packet losses, in *IEEE Conference on Decision and Control (CDC)* (2012)
14. W.P.M.H. Heemels, J.H. Sandee, P.P.J. van den Bosch, Analysis of event-driven controllers for linear systems. Int. J. Control **81**(4), 571–590 (2009)
15. P. Hsu, S. Sastry, The effect of discretized feedback in a closed loop system, in *IEEE Conference on Decision and Control (CDC)* (1987)
16. M. Jankovic, Control Lyapunov-Razumikhin functions to time-delay systems, in *IEEE Conference on Decision and Control (CDC)* (1999)
17. M. Jankovic, Extension of control Lyapunov functions to time-delay systems, in *IEEE Conference on Decision and Control (CDC)* (2000)
18. M. Jankovic, Control of nonlinear systems with time-delay, in *IEEE Conference on Decision and Control (CDC)* (2003)
19. H. Khalil, *Nonlinear Systems*, 3rd edn (Prentice Hall, 2002)
20. N. Krasowskii, *Stability of Motion* (Stanford University Press, 1963) (in Russian)
21. D. Lehmann, J. Lunze, Event-based control with communication delays, in *IFAC World Congress* (2011)
22. D. Lehmann, J. Lunze, Event-based control with communication delays and packet losses. Int. J. Control **85**(5), 563–577 (2012)
23. J. Lunze, D. Lehmann, A state-feedback approach to event-based control. Automatica **46**(1), 211–215 (2010)

24. N. Marchand, S. Durand, J.F. Guerrero-Castellanos, A general formula for event-based stabilization of nonlinear systems. IEEE Trans. Autom. Control **58**(5), 1332–1337 (2013)
25. D. Nešić, L. Grüne, Lyapunov-based continuous-time nonlinear controller redesign for sampled-data implementation. Automatica **41**(7), 1143–1156 (2005)
26. D. Nešić, A. Teel, A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. IEEE Trans. Autom. Control **49**(7), 1103–1122 (2004)
27. A. Papachristodoulou, M. Peet, S. Lall, Constructing Lyapunov-Krasovskii functionals for linear time delay systems, in *American Control Conference (ACC)* (2005)
28. B. Razumikhin, On the stability of systems with a delay. J. Appl. Math. Mech. (trans: Prikl. Mat. Mekh.) (1956) (in Russian)
29. J. Sánchez, M. Guarnes, S. Dormido, On the application of different event-based sampling strategies to the control of a simple industrial process. Sensors **9**(9), 6795–6818 (2009)
30. J.H. Sandee, W.P.M.H. Heemels, P.P.J. van den Bosch, Event-driven control as an opportunity in the multidisciplinary development of embedded controllers, in *American Control Conference (ACC)* (2005)
31. R. Sepulchre, M. Jankovic, P. Kokotovic, *Constructive Nonlinear Control* (Springer, 1997)
32. E.D. Sontag, A Lyapunov-like characterization of asymptotic controllability. SIAM J. Control Optim. **21**(3), 462–471 (1983)
33. E.D. Sontag, A “universal” construction of Artstein’s theorem on nonlinear stabilization. Syst. Control Lett. **13**(2), 117–123 (1989)
34. P. Tabuada, Event-triggered real-time scheduling of stabilizing control tasks. IEEE Trans. Autom. Control **52**(9), 1680–1685 (2007)
35. M. Velasco, P. Martí, E. Bini, On Lyapunov sampling for event-driven controllers, in *IEEE Conference on Decision and Control (CDC)* (2009)

Chapter 5
Analysis of Bilinear Systems with Sampled-Data State Feedback

Hassan Omran, Laurentiu Hetel, Jean-Pierre Richard and Françoise Lamnabhi-Lagarrigue

Abstract In this chapter we consider the stability analysis of bilinear systems controlled via a sampled-data state feedback controller. Sampling periods may be time-varying and subject to uncertainties. The goal of this study is to find a constructive manner to estimate the maximum allowable sampling period (MASP) that guarantees the local stability of the system. Stability criteria are proposed in terms of linear matrix inequalities (LMI).

5.1 Introduction

This chapter is dedicated to the local stability analysis of bilinear sampled-data systems, controlled via a linear state feedback static controller, using a hybrid systems methodology. When a continuous-time controller is emulated, intuitively the stability will be preserved if the sampling intervals are sufficiently small. Nevertheless, this issue has been rarely addressed in a formal quantitative study for bilinear systems. Our purpose is to find a constructive way to calculate the Maximum Allowable Sampling Period (MASP).

H. Omran (✉)
École Centrale de Lyon, Écully, France
e-mail: hassan.omran@ec-lyon.fr

L. Hetel · J.-P. Richard
CRIStAL (CNRS UMR 9189), École Centrale de Lille,
59651 Villeneuve D'Ascq, France
e-mail: laurentiu.hetel@ec-lille.fr

J.-P. Richard
INRIA, Project NON-A, Lille, France
e-mail: jean-pierre.richard@ec-lille.fr

F. Lamnabhi-Lagarrigue
CNRS, Laboratoire des Signaux et Systèmes, European Embedded Control
Institute (EECI), SUPELEC, 3 Rue Joliot Curie, 91192 Gif-sur-Yvette, France
e-mail: lamnabhi@lss.supelec.fr

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_5

Sampled-data systems have been receiving a considerable attention in the past decades [15, 23, 27, 38]. One of the most frequently used methods in the literature is the emulation approach [39]. In this method a stabilizing controller is designed in continuous-time, without the sampling effect. Next, the controller is implemented digitally. However, even when the sampling period is small, it is in general a challenging problem to ensure whether the system remains stable after the sampled-data controller implementation. Moreover, it is very important to provide a quantitative estimation of the MASP which guarantees the stability, especially from the numerical implementation point of view.

The case of linear time-invariant sampled-data systems has been studied extensively in the literature (see for example [13, 15–17, 23]). For the nonlinear case, see the works in [27, 31, 39], where explicit formulas for MASP have been provided. The works in [7, 38, 49], consider the more general networked control systems case. We remark in particular the work in [39] where the authors use results from the hybrid systems theory, to provide generic local and global stability conditions. Other results on the bilinear case have been presented in [41, 42].

Two constructive methods are considered. They are both based on the hybrid systems framework. The first method is a specialization of the result used for the general nonlinear case [39]. The contribution here is to find a constructive way to apply this generic method, for the particular case of bilinear systems. The second method is based on a direct search of a Lyapunov function using LMIs. The novelty here is to avoid some conservative upper bounds on the derivative of a Lyapunov function in the first method.

The chapter is organized as follows. First, bilinear systems are introduced in Sect. 5.2. In Sect. 5.3, some results on the stabilization of bilinear systems are presented. In Sect. 5.4, we formulate the problem under study. Section 5.5 is dedicated to system modeling. In Sect. 5.6, we introduce the main results, where sufficient conditions for the local stability of sampled-data bilinear systems are provided. Finally, the results are illustrated by means of a numerical example in Sect. 5.7.

5.2 Bilinear Systems

Bilinear systems are considered as the "simplest" class of nonlinear systems. They are linear separately with respect to the state and the control, but not to both of them jointly. Since the beginning of 1970s, they have attracted the attention of many researchers [5, 11, 32, 43]. The associated state-space model is:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} [u(t)]_i N_i x(t) + B_0 u(t), \quad \forall t \geq t_0, \tag{5.1}$$

where the state vector is $x(t) \in \mathbb{R}^n$, and the control input is $u(t) \in \mathbb{R}^m$. The term $A_0 x$ is called the *drift*, $B_0 u$ is the *additive control* and $\sum_{i=1}^{m} [u]_i N_i x$ is the *multiplicative control* [11].

Bilinear systems have applications in various domains since many processes can be modeled by this way. Examples of these processes are found in engineering application such as power electronics [24, 46], a.c. transmission systems [35], controlled hydraulic systems [21] and chemical processes [12]. Bilinear systems can also be encountered in domains such as ecology, socioeconomics, biology and immunology [32, 33], only to cite a few.

From the point of view of nonlinear systems theory, the study of bilinear systems is very interesting since such models offer a more accurate approximation to nonlinear systems than the classical linear ones. This can be seen in the added bilinear terms, in state and control, which may come from a Taylor series truncation: [36, 43] and the references therein give more insight to the approximation of more highly nonlinear systems by bilinear models. As a matter of fact, bilinear systems have also an interesting variable structure characteristic. For example, it has been shown in [32] that bilinear models have more powerful controllability properties than the linear ones. For information about structural properties, system characterization and solutions, see [34].

5.3 Stabilization of Bilinear Systems

Even for such “simplest” class of nonlinear systems, the feedback stabilization of bilinear systems is a challenging problem, and several controller structures can be found in the literature [1, 22, 28, 30, 34, 44, 45]. We mention as follows some of the notable approaches. Linear state feedback $u = Kx$ has been proposed in several works [1, 34]. Quadratic controller has been considered in [22, 34, 44], and improvements have been provided in the literature (see [8, 45] for normalized quadratic control methods). In [30, 34] a discontinuous bang–bang controller has been proposed. In the special case of dyadic bilinear systems $\dot{x} = A_0 x + \sum_{i=1}^{m} b_i (c_i^T x + 1)u$, several authors have considered stabilization using the so-called division controllers [9, 22]. In [25], necessary and sufficient conditions for the global asymptotic stabilization by using a homogeneous feedback is provided for a class of bilinear systems (with scalar multiplicative control and no additive control). Sliding mode control has also been applied, see for example [47]. In [26], a polynomial static output feedback controller has been proposed, with a guaranteed upper bound of a performance index. Global asymptotic stabilization using a hybrid controller has been proposed in [3]. Finally, stabilization of bilinear discrete-time systems using polyhedral Lyapunov functions has been considered in [4].

The linear state feedback is an interesting solution because of its simplicity [1]. It is easily implemented, and several results address the problem of finding such controllers. Unfortunately, in nontrivial cases it has been shown that it is usually impossible to stabilize globally the bilinear systems with linear feedback control [34] (page. 39). As a matter of fact, in the scalar case ($n = 1$), it is impossible. For planar single-input systems ($n = 2, m = 1$), necessary and sufficient conditions are given in [29]. To our best knowledge, the problem is not fully analyzed yet for $n > 2$.

Recently in [1, 2, 48], numerically tractable conditions have allowed for the design of a linear state feedback controller that ensures local asymptotic stabilization.

Theorem 1 *([1]) Given the system (5.1) and the polytope containing the origin:*

$$\mathscr{P}_c = conv\{x_1, x_2, \ldots, x_p\} \tag{5.2}$$
$$= \{x \in \mathbb{R}^n : a_j^T x \leq 1, \; j = 1, 2, \ldots, r\}. \tag{5.3}$$

Then, a controller $u(t) = Kx(t)$, with $K \in \mathbb{R}^{m\times n}$, which guarantees the asymptotic stability of the resulting closed-loop system, can be found if there exist scalars $\gamma \in (0, \; 1)$ and $c > 0$, a symmetric matrix $P > 0$, and a matrix $W \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} 1 & \gamma a_j^T Pc \\ cPa_j\gamma & Pc \end{bmatrix} \geq 0, \quad j = 1, 2, \ldots, r,$$
$$\begin{bmatrix} 1 & x_i^T \\ x_i & cP \end{bmatrix} \geq 0, \quad i = 1, 2, \ldots, p,$$

$$\gamma(A_0 P + P A_0^T) + \gamma(B_0 W + W^T B_0^T) + \begin{bmatrix} x_i^T N_1 \\ x_i^T N_2 \\ \vdots \\ x_i^T N_m \end{bmatrix} W$$
$$+ W^T \begin{bmatrix} N_1^T x_i & N_2^T x_i & \cdots & N_m^T x_i \end{bmatrix} < 0,$$
$$i = 1, 2, \ldots, p.$$

The controller is given by $K = WP^{-1}$ and $\mathscr{P}_c$ belongs to the domain of attraction of the equilibrium.

The LMI conditions depend on the vertices of the convex polytope $\mathscr{P}_c$ (5.2), and the dual representation (5.3) where the polytope is presented by r hyperplanes. The proposed conditions are sufficient only for the local stabilization. Note that the above LMI conditions require the pair (A_0, B_0) to be asymptotically stabilizable. However, this condition is not necessary for the stabilization of bilinear systems. This can be seen in the following example.

Example 1 Consider the bilinear system

$$\dot{x} = A_0 x + B_0 u + uNx, \qquad u = Kx,$$

with

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; K = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

which is equivalent to:

$$\begin{cases} \dot{x}_1 = -x_1 - x_2^2, \\ \dot{x}_2 = \quad x_1 x_2. \end{cases}$$

Even though the pair (A_0, B_0) is not stabilizable, the system is still shown to be asymptotically stable using center manifold method.[1]

In spite of this academic example, this state feedback design strategy has shown its interest in practical applications [1, 40]. The question now is how to guarantee the stability of the closed loop with a discrete controller implementation.

5.4 Problem Formulation

Consider the bilinear system (5.1). We suppose that the following assumptions hold:

- The control is a piecewise-constant control law

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}),$$

  with a set of sampling instants $\{t_k\}_{k \in \mathbb{N}}$ satisfying:

$$0 < \varepsilon \leq t_{k+1} - t_k \leq \overline{h}, \qquad \forall k \in \mathbb{N}, \tag{5.4}$$

  where $\overline{h}$ is a given MASP.
- The pair A_0, B_0 is stabilizable, and the linear feedback gain $K \in \mathbb{R}^{m \times n}$ is calculated so that the system (5.1) with the continuous state feedback $u(t) = Kx(t)$ has a locally asymptotically stable equilibrium point at $x = 0$. The actual domain of attraction (a connected neighborhood of $x = 0$, see [20]) is denoted $\mathscr{D}_0$.
- The state variables are subject to constraints defined by a polytopic set $\mathscr{P} \subset \mathscr{D}_0$:

$$\mathscr{P} = conv\{x_1, x_2, \ldots, x_p\} \tag{5.5}$$
$$= \{x \in \mathbb{R}^n : a_j^T x \leq 1, \; j = 1, 2, \ldots, r\} \tag{5.6}$$

  corresponding to an admissible set in the state-space.[2]

---

[1] Jean-Pierre Richard, Lecture Notes: Systèmes Dynamiques, http://researchers.lille.inria.fr/~jrichard/pdfs/SystDynJPR2009_part3.pdf.

[2] The equivalence between the representations in (5.5) and (5.6) is given in [10] (Theorem 1.29).

Under these assumptions, we obtain the closed-loop sampled-data system:

$$\dot{x}(t) = \big(A_0 + \sum_{i=1}^{m} [Kx(t_k)]_i N_i\big)x(t) + B_0 K x(t_k),\ \forall t \in [t_k, t_{k+1}),\ \forall k \in \mathbb{N}. \quad (5.7)$$

System (5.7) may also be written as follows

$$\dot{x}(t) = \tilde{A}[x(t), e(t)]x(t) + Be(t), \quad \forall t \in [t_k, t_{k+1}) \quad (5.8)$$

with

$$e(t) = x(t_k) - x(t),$$

$$\tilde{A}[x, e] := A_0 + B_0 K + \sum_{i=1}^{m} [K(x+e)]_i N_i, \quad (5.9)$$

and

$$B = B_0 K. \quad (5.10)$$

The goal of the chapter is twofold. First, we would like to ensure that the obtained sampled-data system satisfies the state-space constraints (5.5) or (5.6) for any $x_0 \in \mathscr{P}$. Secondly, we would like to provide conditions that guarantee the asymptotic convergence of the system solutions to the origin.

Problem *Find a criterion for the local asymptotic stability of the equilibrium point* $x = 0$ *of the bilinear sampled-data system* (5.7)*, together with an estimate* $\mathscr{E} \subset \mathscr{P}$ *of the domain of attraction, such that for any initial condition* $x(t_0) \in \mathscr{E}$ *the system solutions satisfy* $x(t) \in \mathscr{P}$, $\forall t > t_0$*, and* $x(t) \to 0$.

5.5 Hybrid System Framework

Several works about sampled-data systems [7, 14, 39] adopt the hybrid systems framework [18, 19]. A hybrid system $\mathscr{H}$ is a tuple $(\mathscr{A}, \mathscr{B}, F, G)$, where $\mathscr{A} \subseteq \mathbb{R}^n$ and $\mathscr{B} \subseteq \mathbb{R}^n$ are, respectively, the *flow set* and the *jump set*, while $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^n$ are, respectively, the *flow map* and *jump map*. The hybrid system is usually represented by:

$$\mathscr{H} : \begin{cases} \dot{\xi} &= F(\xi), \quad \xi \in \mathscr{A}, \\ \xi^+ &= G(\xi), \quad \xi \in \mathscr{B}. \end{cases}$$

The dynamics given by F, determines the continuous-time evolution (flow) of the state through $\mathscr{A}$, while G determines the discrete-time evolution (jumps) in $\mathscr{B}$. See [18] for more details about hybrid dynamical systems.

In [38], L_p-stability properties have been studied for NCS with scheduling protocols. The results are based on the hybrid modeling approach and the small gain theorem, and they can be applied to the sampled-data case to calculate the MASP. In [6, 7], the bound on the MASP has been improved, using a Lyapunov-based method, which result has been particularized to the sampled-data case in [39]. Consider the plant:

$$\dot{x}_p = f_p(x_p, u), \quad y = g_p(x_p),$$

where x_p is the plant state, u is the control input, y is the measured output. Suppose that asymptotic stability is guaranteed by the continuous-time output feedback:

$$\dot{x}_c = f_c(x_c, y), \quad u = g_c(x_c),$$

where x_c is the controller state. The sampled-data implementation of the controller can be written in the following form:

$$\begin{aligned} \dot{x}_p &= f_p(x_p, \hat{u}), & t \in [t_k, t_{k+1}), \\ y &= g_p(x_p), & \\ \dot{x}_c &= f_c(x_c, \hat{y}), & t \in [t_k, t_{k+1}), \\ u &= g_c(x_c), & \\ \dot{\hat{y}} &= 0, & t \in [t_k, t_{k+1}), \\ \dot{\hat{u}} &= 0, & t \in [t_k, t_{k+1}), \end{aligned} \tag{5.11}$$

with

$$\begin{aligned} \hat{y}(t_k^+) &= y(t_k), \\ \hat{u}(t_k^+) &= u(t_k), \end{aligned} \tag{5.12}$$

where x_p and x_c are respectively the states of the plant and of the controller, y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the most recently transmitted plant and controller output values. In between sampling instants, the values of $\hat{y}$ and $\hat{u}$ are held constant. Define the augmented state vector $x(t)$ and the network-induced error $e(t)$:

$$e(t) = \begin{bmatrix} e_y(t) \\ e_u(t) \end{bmatrix} := \begin{bmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{bmatrix} \in \mathbb{R}^{n_e}, \quad x(t) := \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix} \in \mathbb{R}^{n_x}. \tag{5.13}$$

Note that the error vector is subject to resets at each sampling instant. The sampled-data system (5.11) can be written as a system with jumps:

$$\begin{aligned} \dot{x} &= f(x, e) \quad t \in [t_k, t_{k+1}), \\ \dot{e} &= g(x, e) \quad t \in [t_k, t_{k+1}), \\ e(t_k^+) &= 0, \end{aligned} \tag{5.14}$$

with $0 < \varepsilon \leq t_{k+1} - t_k \leq \overline{h}$, for all $k \in \mathbb{N}$, $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$. The functions f and g are obtained by direct calculations from the sampled-data system (5.11) (see [39] and [38]):

$$f(x,e) := \begin{bmatrix} f_p(x_p, g_c(x_c) + e_u) \\ f_c(x_c, g_p(x_p) + e_y) \end{bmatrix}; \; g(x,e) := \begin{bmatrix} -\frac{\partial g_p}{\partial x_p} f_p(x_p, g_c(x_c) + e_u) \\ -\frac{\partial g_c}{\partial x_c} f_c(x_c, g_p(x_p) + e_y) \end{bmatrix}.$$

It should be noted that $\dot{x} = f(x,0)$ is the closed loop system without the sampled-data implementation. Considering a clock τ which evolves with respect to the sampling instants, system (5.14) can be written as the following hybrid system:

$$\left.\begin{aligned} \dot{x} &= f(x,e) \\ \dot{e} &= g(x,e) \\ \dot{\tau} &= 1 \end{aligned}\right\} \quad \tau \in [0, \overline{h}], \qquad \left.\begin{aligned} x^+ &= x \\ e^+ &= 0 \\ \tau^+ &= 0 \end{aligned}\right\} \quad \tau \in [\varepsilon, \overline{h}], \tag{5.15}$$

with $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$, $\tau \in \mathbb{R}_+$, $\overline{h} \geq \varepsilon > 0$. The following theorem provides a quantitative method to estimate the MASP, using the model (5.15).

In a similar way, we fit the sampled-data system (5.7) into a hybrid model. The system (5.7) is formulated similarly to (5.11) as follows:

$$\begin{aligned} \dot{x} &= A_0 x + \sum_{i=1}^{m} u_i N_i x + B_0 u, \;\; t \in [t_k, t_{k+1}), \\ y &= x, \\ u &= K\hat{y}, \\ \dot{\hat{y}} &= 0, \qquad\qquad\qquad\qquad t \in [t_k, t_{k+1}), \\ \hat{y}(t_k^+) &= y(t_k). \end{aligned} \tag{5.16}$$

The hybrid model for this case is determined by

$$\begin{aligned} &\left.\begin{aligned} \dot{x} &= f(x,e) = \tilde{A}[x,e]x + Be \\ \dot{e} &= g(x,e) = -\tilde{A}[x,e]x - Be \\ \dot{\tau} &= 1 \end{aligned}\right\} \quad \tau \in [0, \overline{h}] \\ &\left.\begin{aligned} x^+ &= x \\ e^+ &= 0 \\ \tau^+ &= 0 \end{aligned}\right\} \quad \tau \in [\varepsilon, \overline{h}] \end{aligned} \tag{5.17}$$

with $\tilde{A}[x,e]$ and B given in (5.9) and (5.10), ε and $\overline{h}$ given in (5.4). Note that in contrast to the general case model, there is no $\hat{u}$ in (5.16). This is due to the fact that the considered controller is a static one. In this case, we may consider only one ZOH mechanism in the input side of the controller.

For the hybrid system (5.17), we are only interested in stability with respect to the variables x and e. We consider the set $\{(x,e,\tau) : x = 0, e = 0, \tau \in [0\ \overline{h}]\}$, and the classical definition of stability from [18, 19].

5.6 Local Stability and MASP Estimation

In this section, we provide sufficient stability conditions for the considered case of sampled-data bilinear systems (5.7), or equivalently (5.17). The conditions are used to estimate an upper bound on the MASP. Two methods are to be considered. First, we introduce a method that is based on the application of results for general nonlinear sampled-data systems in [39] (Method 1). Next, to avoid the use of conservative bounds in the previous method, we look directly for an underlying Lyapunov function by formalizing the conditions as LMIs (Method 2). In both of these methods, we will be dealing with local asymptotic stability. A preliminary version of these approaches has been presented in [41]. Consider the polytope $\mathscr{P}$ defined in (5.5). If $x(t_k)$ is in the polytope $\mathscr{P}$, then

$$A[x(t_k)] := \tilde{A}[x(t), e(t)] \in conv\{A_1, A_2, \ldots, A_p\},$$

with

$$A_q = A[x_q] \quad \forall q \in \{1, 2, \ldots, p\}. \tag{5.18}$$

Note that the set of barycentric coordinates that determine $x(t_k)$ with respect to the vertex of the polytope $\mathscr{P}$, determine also $A[x(t_k)]$ with respect to the vertices in (5.18). This is due to the linearity of $A[x(t_k)]$ in $x(t_k)$, and it can be seen as follows. If $x(t_k) \in \mathscr{P}$, then there exist positive scalars

$$\{\lambda_q(t_k)\}_{q=1}^{p}, \quad \sum_{q=1}^{p} \lambda_q(t_k) = 1 \tag{5.19}$$

such that

$$x(t_k) = \sum_{q=1}^{p} \lambda_q(t_k) x_q.$$

Hence

$$\begin{aligned}
\sum_{q=1}^{p} \lambda_q(t_k) A_q &= \sum_{q=1}^{p} \lambda_q(t_k) \Big(A_0 + B_0 K + \sum_{i=1}^{m} [K x_q]_i N_i \Big) \\
&= A_0 + B_0 K + \sum_{i=1}^{m} \Big[K \big(\sum_{q=1}^{p} \lambda_q(t_k) x_q \big) \Big]_i N_i \\
&= \tilde{A}[x(t), e(t)].
\end{aligned} \tag{5.20}$$

5.6.1 Method 1: Adaptation of a Result on General Nonlinear Sampled-Data Systems

The following theorem proposes stability conditions using an adaptation of the results in [39] for the case of bilinear systems.

Theorem 2 *Consider the bilinear sampled-data system* (5.17)*, the polytope* $\mathscr{P}$ *in* (5.5)*, and a function*

$$\mathscr{T}(\gamma, L) := \begin{cases} \frac{1}{Lr}arctan(r) & \gamma > L \\ \frac{1}{L} & \gamma = L \\ \frac{1}{Lr}arctanh(r) & \gamma < L \end{cases} \tag{5.21}$$

with

$$r = \sqrt{\left|\frac{\gamma^2}{L^2} - 1\right|} \tag{5.22}$$

where L *is given by*

$$L = \frac{1}{2}\max\{-\lambda_{min}(B^T + B), 0\}. \tag{5.23}$$

The parameter γ *is given by* $\gamma = \sqrt{\chi}$*, with* χ *the solution to the following optimization problem:*

$$\chi = \min \chi', \tag{5.24}$$

satisfying the constraints $\exists P \in \mathbb{R}^{n\times n}$ *a symmetric positive definite matrix,* $\exists P_2, P_3 \in \mathbb{R}^{n\times n}$*, and* $\exists \alpha > 0$*, such that* $\chi' > 0$ *and*

$$\begin{bmatrix} -P_3 - P_3^T + I & -P_2 + P_3^T A_j + P & P_3^T B - B \\ * & P_2^T A_j + A_j^T P_2 + \alpha I & P_2^T B \\ * & * & (\alpha - \chi')I + B^T B \end{bmatrix} < 0,$$
$$\forall j \in \{1, 2, ..., p\}, \tag{5.25}$$

where A_j *are the vertices given in* (5.18)*. Assume that the MASP is strictly bounded by* $\mathscr{T}(\gamma, L)$*, i.e.* $\overline{h} < \mathscr{T}(\gamma, L)$*. Then, for the bilinear sampled-data system* (5.17)*, the set* $\{(x, e, \tau) : x = 0, e = 0\}$ *is locally uniformly asymptotically stable.*

Proof This proof is mainly based on an adaptation of Theorem 1 in [39] to the bilinear case.

Let $\phi : [0, \tilde{T}] \to \mathbb{R}$ be the solution to

$$\dot{\phi} = -2L\phi - \gamma(\phi^2 + 1) \qquad \phi(0) = \lambda^{-1} \tag{5.26}$$

where $\lambda \in (0, 1)$. We recall the following result.

Claim *[7] $\phi(\tau) \in [\lambda, \lambda^{-1}]$ for all $\tau \in [0, \tilde{T}]$. Moreover, we have that $\phi(\tilde{T}) = \lambda$ for $\tilde{T}$ given by*

$$\tilde{T}(\lambda, \gamma, L) := \begin{cases} \frac{1}{Lr} arctan\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}(\frac{\gamma}{L}-1)+1+\lambda}\right) & \gamma > L \\ \frac{1}{L}\frac{1-\lambda}{1+\lambda} & \gamma = L \\ \frac{1}{Lr} arctanh\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}(\frac{\gamma}{L}-1)+1+\lambda}\right) & \gamma < L \end{cases} \tag{5.27}$$

with r is given in (5.22).

Consider the following notations

$$\xi := [x^T, e^T, \tau]^T, \tag{5.28}$$

$$F(\xi) := [f(x,e)^T, g(x,e)^T, 1]^T. \tag{5.29}$$

Note that $\tilde{T}(\lambda, \gamma, L)$ in (5.27) and $\mathscr{T}(\gamma, L)$ in (5.21) satisfy $\mathscr{T}(\gamma, L) = \tilde{T}(0, \gamma, L)$, and for a fixed L and γ we have that $\tilde{T}(\cdot, \gamma, L)$ is strictly decreasing. Hence, since the conditions of the theorem require $\overline{h}$ to be strictly smaller than $\mathscr{T}(\gamma, L)$, there exists $\lambda \in (0, 1)$ such that $\overline{h} = \tilde{T}(\lambda, \gamma, L)$. For the considered value of λ, define the function

$$U(\xi) = V(x) + \gamma\phi(\tau)W^2(e) \tag{5.30}$$

with a quadratic function $V(x) = x^T P x$, and $W(e) = |e|$. The function $U(\xi)$ will be used as a Lyapunov function. Note that

$$\lambda_{min}(P)|x|^2 + \lambda\gamma|e|^2 \le U(\xi) \le \lambda_{max}(P)|x|^2 + \lambda^{-1}\gamma|e|^2, \ \forall\tau \in [0, \overline{h}]. \tag{5.31}$$

The Lyapunov function is nonincreasing at sampling instants as it can be seen from the following

$$\begin{aligned} U(\xi^+) &= V(x^+) + \gamma\phi(\tau^+)W^2(e^+) \\ &= V(x) \\ &\le V(x) + \gamma\phi(\tau)W^2(e) = U(\xi), \quad \forall\tau \in [\varepsilon, \overline{h}]. \end{aligned} \tag{5.32}$$

In order to treat the quantity $\langle\nabla U(\xi), F(\xi)\rangle$ we need two inequalities that correspond to both $\langle\nabla W(e), g(x,e)\rangle$ and $\langle\nabla V(x), f(x,e)\rangle$. We get the first inequality as follows:

$$\begin{aligned} \left\langle\nabla W(e), g(x,e)\right\rangle &= \frac{e^T}{W(e)}(-\tilde{A}[x,e]x - Be) \\ &= -\frac{1}{2W(e)}e^T(B^T + B)e - \frac{1}{W(e)}e^T\tilde{A}[x,e]x \\ &\le \frac{1}{2}\max\{-\lambda_{min}(B^T + B), 0\}W(e) + |\tilde{A}[x,e]x| \end{aligned}$$

$$\left\langle \nabla W(e), g(x, e) \right\rangle \leq LW(e) + H(x, e) \tag{5.33}$$

with

$$H(x, e) = \left| \tilde{A}[x, e]x \right|, \tag{5.34}$$

and L given in (5.23). The second inequality is found as follows. Consider P, P_2, P_3, γ, and α from the solution to the optimization problem in (5.24), (5.25). For all $(x + e) \in \mathscr{P}$, by multiplying the LMIs in (5.25) each by the appropriate coefficients from (5.20), and then taking the sums over $j \in \{1, 2, \cdots, p\}$ we obtain

$$\begin{bmatrix} -P_3 - P_3^T + I & -P_2 + P_3^T \tilde{A}[x, e] + P & P_3^T B - B \\ * & P_2^T \tilde{A}[x, e] + \tilde{A}^T[x, e] P_2 + \alpha I & P_2^T B \\ * & * & (\alpha - \gamma^2) I + B^T B \end{bmatrix} < 0. \tag{5.35}$$

Define the continuous, positive definite function $\rho(s) = \alpha s^2$. Multiplying the LMI in (5.35) by $[\dot{x}^T \;\; x^T \;\; e^T]$ from the left, and by its transpose from the right implies

$$\begin{aligned} &\langle \nabla V(x), f(x, e) \rangle + \rho(|x|) + \rho(W(e)) + H^2(x, e) - \gamma^2 W^2(e) \\ &+ 2(x^T P_2^T + \dot{x}^T P_3^T)(-\dot{x} + \tilde{A}[x, e]x + Be) \leq 0, \end{aligned} \tag{5.36}$$

where the term

$$2(x^T P_2^T + \dot{x}^T P_3^T)(-\dot{x} + \tilde{A}[x, e]x + Be) = 0, \tag{5.37}$$

has been obtained from the descriptor method [15]. Note that from (5.35) the following inequality will be satisfied locally inside the addressed polytopic region

$$\langle \nabla V(x), f(x, e) \rangle \leq -\rho(|x|) - \rho(W(e)) - H^2(x, e) + \gamma^2 W^2(e). \tag{5.38}$$

From (5.33) and (5.38) we have

$$\begin{aligned} \langle \nabla U(\xi), F(\xi) \rangle \leq & -\rho(|x|) - \rho(W(e)) - H^2(x, e) + \gamma^2 W^2(e) \\ & + 2\gamma\phi(\tau) W(e)(LW(e) + H(x, e)) \\ & - \gamma W^2(e)(2L\phi(\tau) + \gamma(\phi^2(\tau) + 1)) \\ \leq & - \rho(|x|) - \rho(W(e)) - H^2(x, e) \\ & + 2\gamma\phi(\tau) W(e) H(x, e) - \gamma^2 W^2(e)\phi^2(\tau) \end{aligned}$$

yielding

$$\langle \nabla U(\xi), F(\xi) \rangle \leq -\rho(|x|) - \rho(W(e)), \; \forall \tau \in [0, \overline{h}], \; \forall (x + e) \in \mathscr{P}. \tag{5.39}$$

Finally, the local stability is straightforward using the results from [19] (Theorem 3.18, page 52).

Remark 1 In this method, the MASP is calculated by the expression (5.21), based on L and γ. L is calculated analytically, whereas γ is found by solving LMI conditions. The optimization problem is a minimization of γ because for any constant L, $\mathscr{T}(\cdot, L)$ is a strictly decreasing function. Note that since γ does not depend on L, and from the continuity of $\mathscr{T}(\gamma, \cdot)$:

$$\mathscr{T}(\gamma, 0) = \lim_{L \to 0} \mathscr{T}(\gamma, L) = \lim_{L \to 0} \frac{\arctan(\sqrt{\left|\frac{\gamma^2}{L^2} - 1\right|})}{\sqrt{\left|\gamma^2 - L^2\right|}} = \frac{\pi}{2\gamma}.$$

Remark 2 The stability conditions presented in this theorem are based on the generic inequalities (5.38), (5.33) for nonlinear system presented in [39]. Our contribution is to provide a constructive manner to apply this result to the case of bilinear systems. We provide explicit forms of $H(x, e)$, $W(e)$, $V(x)$, and we find L, γ that gives the upper bound on MASP. We provide as well, an LMI formulation that allows us to obtain sufficient stability condition. Note that in order to obtain LMI based stability conditions the approach has been adapted to the bilinear case: the function $H(\cdot, \cdot)$ used here has been modified to depend both on the error $e(t)$ and the state $x(t)$, while in [39] it is only a function of x. Finally, note that the number of LMIs in (5.25) is p (number of vertices in (5.18)), while in the preliminary version of the result [41], the number of LMIs is p^2.

5.6.2 Method 2: Direct Lyapunov Function Approach

In the previous method, the stability conditions are obtained using upper estimations of the derivative of a Lyapunov function in (5.33) and (5.38). Such upper estimations may be found conservative. In order to avoid them, we provide as follows a second method which evaluates directly the derivative of a Lyapunov function.

Theorem 3 *Consider the bilinear sampled-data system* (5.17)*. Suppose that MASP is bounded by a value $\mathscr{T}$, i.e., $\overline{h} \leq \mathscr{T}$. Assume that there exist symmetric positive definite matrices $P, X, Y \in \mathbb{R}^{n \times n}$, such that the following LMIs are satisfied*

$$\begin{bmatrix} A_j^T P + P A_j & PB - aA_j^T X - bA_j^T Y \\ * & -a(\frac{X}{\mathscr{T}} + B^T X + XB) - \frac{Y}{\mathscr{T}} - b(B^T Y + YB) \end{bmatrix} < 0,$$
$$\forall j \in \{1, 2, ..., p\}, \quad \forall a \in \{exp(1), 1\}, \quad \forall b \in \{1, 0\}, \tag{5.40}$$

where A_j are the vertices in given in (5.18)*.*

Then the set $\{(x, e, \tau) : x = 0, e = 0\}$ of the bilinear sampled-data system (5.17) *is locally uniformly asymptotically stable.*

Proof First, we recall the notations ξ and $F(\xi)$ defined in (5.28), (5.29). We consider the function

$$\overline{U}(\xi) = \overline{V}(x) + \overline{W}(\tau, e) \tag{5.41}$$

with $\overline{V}(x) = x^T P x$, and $\overline{W}(\tau, e) = \exp(\frac{\mathscr{T}-\tau}{\mathscr{T}})e^T X e + (\frac{\mathscr{T}-\tau}{\mathscr{T}})e^T Y e$. This Lyapunov function will be used to prove the stability of the hybrid system (5.17). It is inspired by the Lyapunov functions from [7, 37, 39].

From the fact that $P > 0$, $X > 0$ and $Y > 0$ we have that $\overline{U}(\xi)$ satisfies

$$\overline{U}(\xi) \geq \lambda_{min}(P)|x|^2 + \lambda_{min}(X)|e|^2, \ \forall \tau \in [0, \overline{h}], \tag{5.42}$$
$$\overline{U}(\xi) \leq \lambda_{max}(P)|x|^2 + (\lambda_{max}(X)\exp(1) + \lambda_{max}(Y))|e|^2, \ \forall \tau \in [0, \overline{h}]. \tag{5.43}$$

At sampling instants, $\overline{U}(\xi)$ is nonincreasing

$$\begin{aligned} \overline{U}(\xi^+) &= \overline{V}(x^+) + \overline{W}(\tau^+, e^+) \\ &= x^T P x \\ &\leq x^T P x + \overline{W}(\tau, e) = \overline{U}(\xi), \quad \forall \tau \in [\varepsilon, \overline{h}]. \end{aligned} \tag{5.44}$$

For all $(x + e) \in \mathscr{P}$, by multiplying the LMIs in (5.40) by the appropriate coefficients from (5.20), and taking the sums over the resulting inequalities, we have that $\exists \varepsilon > 0$ such that

$$\begin{bmatrix} \tilde{A}^T[x, e]P + P\tilde{A}[x, e] & PB - a\tilde{A}^T[x, e]X - b\tilde{A}^T[x, e]Y \\ * & -a(\frac{X}{\mathscr{T}} + B^T X + XB) - \frac{Y}{\mathscr{T}} - b(B^T Y + YB) \end{bmatrix} \leq -\varepsilon I,$$
$$\forall a \in \{exp(1), 1\}, \quad \forall b \in \{1, 0\}. \tag{5.45}$$

Note that for any $\tau \in [0, \mathscr{T}]$, we have that $exp((\mathscr{T} - \tau)/\mathscr{T}) \in [1, exp(1)]$ and $(\mathscr{T} - \tau)/\mathscr{T} \in [0, 1]$. Thus, multiplying the LMI in (5.45) by $[x^T \ \ e^T]$ from the left, and by its transpose from the right implies

$$\begin{aligned} &2x^T P\tilde{A}[x, e]x + 2x^T \Big(PB - \exp(\frac{\mathscr{T} - \tau}{\mathscr{T}})\tilde{A}^T[x, e]X - (\frac{\mathscr{T} - \tau}{\mathscr{T}})\tilde{A}^T[x, e]Y\Big)e \\ &+ 2e^T \Big(- \exp(\frac{\mathscr{T} - \tau}{\mathscr{T}})(\frac{X}{2\mathscr{T}} + XB) - \frac{Y}{2\mathscr{T}} - (\frac{\mathscr{T} - \tau}{\mathscr{T}})(YB)\Big)e \\ &\leq -\varepsilon x^T x - \varepsilon e^T e, \end{aligned}$$

which implies that

$$\langle \nabla \overline{U}(\xi), F(\xi) \rangle \leq -\varepsilon x^T x - \varepsilon e^T e, \ \forall \tau \in [0, \overline{h}], \ \forall (x + e) \in \mathscr{P}. \tag{5.46}$$

Finally, the local stability is straightforward using the results from [19] (Theorem 3.18, page 52).

Remark 3 In this method the MASP is found by solving a set of LMIs for the maximum possible value of $\mathscr{T}$. The existence of a solution to the LMI conditions, guarantees the existence of a Lyapunov function that will yield the asymptotic stability. Note that the proposed conditions directly study the derivative of the Lyapunov function. Numerical examples will show the conservatism reduction in comparison with the approach in Method 1. Note that both the approach of Method 1 and Method 2 are robust not only to the sampled-data implementation but also to variations of the sampling intervals.

Remark 4 Note that the local asymptotic stability of the hybrid system (5.17) implies the local asymptotic stability of (5.7). As a matter of fact, the established asymptotic stability is local in both Method 1 and Method 2, since the inequalities in (5.39) and (5.46) are satisfied only inside the studied polytope $\mathscr{P}$. Moreover, one can find an invariant set $\mathscr{E} \in \mathscr{P}$, such that for $x(t_0) \in \mathscr{E}$ one has $|(x(t_0), e(t_0))| = |(x(t_0), 0)| \leq \Delta$ for some $\Delta > 0$, for which all solutions converge to the origin.

5.7 Numerical Example

In this section we present a numerical comparison of the two proposed methods. Consider the following bilinear systems, described by the matrices

$$A_0 = \begin{bmatrix} -0.5 & 3 & 6.5 \\ 2.95 & 9.4 & 3.9 \\ 1.6 & 6.8 & 3.8 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} -2.8 & -5.2 \\ 0.8 & -10.1 \\ 1.6 & -3 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A continuous-time state feedback controllers has been computed in order to locally stabilize the origin of the bilinear system. The linear state feedback controller defined by

$$K = \begin{bmatrix} 0.0004 & 0.0018 & 0.0008 \\ 0.5601 & 1.6338 & 2.1431 \end{bmatrix}$$

was proven to establish the local stability for the bilinear system (in the continuous-time case). We consider a local polytopic region

$$\mathscr{P} = [-1, +1] \times [-0.5, +0.5] \times [-0.5, +0.5].$$

Using Method 1, we found that the system is locally stable if $\overline{h} < \mathscr{T} = 7 \times 10^{-3}$. This was calculated from (5.21) for $L = 29.09$, $\gamma = 204.79$ and $\alpha = 68.55$. Using Method 2, we found that the sampled-data system is locally stable for a larger MASP

$\overline{h} \leq \mathscr{T} = 18 \times 10^{-3}$. The results illustrate the reduction of conservatism in Method 2 with respect to Method 1, and with respect to the results in [41] where stability is guaranteed for $\overline{h} \leq 14.7 \times 10^{-3}$.

5.8 Conclusion

In this chapter, we have provided sufficient conditions for the local stability of bilinear sampled-data systems, controlled via a linear state feedback controller. We presented results for estimating the MASP that guarantees the local stability of the system. Two methods which are based on a hybrid system approach were considered. The first method is an adaptation of results on the general nonlinear case, while the second one is based on a direct search of a Lyapunov function for the hybrid model. The stability conditions, in both methods, were given in the form of LMIs, which are easily computationally tractable. The results were illustrated by a numerical example.

References

1. F. Amato, C. Cosentino, A. Fiorillo, A. Merola, Stabilization of bilinear systems via linear state-feedback control. IEEE Trans. Circuits Syst. II: Express Briefs **56**(1), 76–80 (2009)
2. F. Amato, C. Cosentino, A. Merola, Stabilization of bilinear systems via linear state feedback control, in *15th Mediterranean Conference on Control Automation, MED '07* (2007)
3. V. Andrieu, S. Tarbouriech, Global asymptotic stabilization for a class of bilinear systems by hybrid output feedback. IEEE Trans. Autom. Control **58**(6), 1602–1608 (2013)
4. N. Athanasopoulos, G. Bitsoris, Unconstrained and constrained stabilisation of bilinear discrete-time systems using polyhedral Lyapunov functions. Int. J. Control **83**(12), 2483–2493 (2010)
5. C. Bruni, G.D. Pillo, G. Koch, Bilinear systems: An appealing class of "nearly linear" systems in theory and applications. IEEE Trans. Autom. Control **19**(4), 334–348 (1974)
6. D. Carnevale, A. Teel, D. Nešić, Further results on stability of networked control systems: a Lyapunov approach, in *American Control Conference* (2007)
7. D. Carnevale, A. Teel, D. Nešić, A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems. IEEE Trans. Autom. Control **52**(5), 892–897 (2007)
8. M.-S. Chen, Y.-Z. Chen, Normalised quadratic controls for a class of bilinear systems. IEE Proc. Control Theory Appl. **149**(6), 520–524 (2002)
9. M.-S. Chen, Y.-R. Hwang, Y.-J. Kuo, Exponentially stabilizing division controllers for dyadic bilinear systems. IEEE Trans. Autom. Control **48**(4), 106–110 (2003)
10. F. Christophersen, *Optimal Control of Constrained Piecewise Affine Systems* (Springer, Lecture Notes in Control and Information Sciences, 2007)
11. D. Elliott, *Bilinear Control Systems Matrices in Action*, vol.169 of Applied Mathematical Sciences (Springer, 2009)
12. M. Espana, I. Landau, Reduced order bilinear models for distillation columns. Automatica **14**(4), 345–355 (1978)
13. C. Fiter, L. Hetel, W. Perruquetti, J.-P. Richard, A state dependent sampling for linear state feedback. Automatica **48**(8), 1860–1867 (2012)
14. F. Forni, S. Galeani, D. Nešić, L. Zaccarian, Event-triggered transmission for linear control over communication channels. Automatica **50**(2), 490–498 (2013)

15. E. Fridman, A refined input delay approach to sampled-data control. Automatica **46**(2), 421–427 (2010)
16. E. Fridman, A. Seuret, J.-P. Richard, Robust sampled-data stabilization of linear systems: an input delay approach. Automatica **40**(8), 1441–1446 (2004)
17. H. Fujioka, Stability analysis of systems with aperiodic sample-and-hold devices. Automatica **45**(3), 771–775 (2009)
18. R. Goebel, R. Sanfelice, A. Teel, Hybrid dynamical systems. IEEE Control Syst. **29**(2), 28–93 (2009)
19. R. Goebel, R. Sanfelice, A. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness* (Princeton University Press, 2012)
20. L. Gruyitch, J.-P. Richard, P. Borne, J.-C. Gentina, *Stability domains* (Chapman&Hall\CRC, 2004)
21. L. Guo, H. Schwarz, A control scheme for bilinear systems and application to a secondary controlled hydraulic rotary drive, in *IEEE Conference on Decision and Control (CDC)* (1989)
22. P.O. Gutman, Stabilizing controllers for bilinear systems. IEEE Trans. Autom. Control **26**(4), 917–922 (1981)
23. L. Hetel, A. Kruszewski, W. Perruquetti, J.P. Richard, Discrete and intersample analysis of systems with aperiodic sampling. IEEE Trans. Autom. Control **56**(7), 696–1701 (2011)
24. T. Hu, A nonlinear-system approach to analysis and design of power-electronic converters with saturation and bilinear terms. IEEE Trans. Power Electron. **26**(2), 399–410 (2011)
25. H. Jerbi, Global feedback stabilization of new class of bilinear systems. Syst. Control Lett. **42**(4), 313–320 (2001)
26. D. Kang, S. Won, Y.J. Jang, Guaranteed cost control for bilinear systems by static output feedback. Appl. Math. Comput. **219**(14), 7398–7405 (2013)
27. I. Karafyllis, C. Kravaris, Global stability results for systems under sampled-data control. Int. J. Robust Nonlinear Control **19**(10), 1105–1128 (2008)
28. M. Khlebnikov, Quadratic stabilization of bilinear control systems, in *European Control Conference (ECC)* (2015)
29. D. Koditschek, K. Narendra, Stabilizability of second-order bilinear systems. IEEE Trans. Autom. Control **28**(10), 987–989 (1983)
30. R. Longchamp, Stable feedback control of bilinear systems. IEEE Trans. Autom. Control **25**(2), 302–306 (1980)
31. F. Mazenc, M. Malisoff, T.N. Dinh, Robustness of nonlinear systems with respect to delay and sampling of the controls. Automatica **49**(6), 1925–1931 (2013)
32. R. Mohler, Natural bilinear control processes. IEEE Trans. Syst. Sci. Cybern. **6**(3), 192–197 (1970)
33. R. Mohler, *Bilinear Control Processes: With Applications to Engineering, Ecology, and Medicine* (Academic Press, Mathematics in Science and Engineering, 1974)
34. R. Mohler, *Nonlinear Systems*, vol. 2: Applications to Bilinear Control (Prentice Hall, 1991)
35. R. Mohler, A. Khapalov, Bilinear control and application to flexible a.c. transmission systems. J. Optim. Theory Appl. **105**(3), 621–637 (2000)
36. R. Mohler, W. Kolodziej, An overview of bilinear system theory and applications. IEEE Trans. Syst. Man Cybern. **10**(10), 683–688 (1980)
37. P. Naghshtabrizi, J.P. Hespanha, A.R. Teel, Exponential stability of impulsive systems with application to uncertain sampled-data systems. Syst. Control Lett. **57**(5), 378–385 (2008)
38. D. Nešić, A. Teel, Input-output stability properties of networked control systems. IEEE Trans. Autom. Control **49**(10), 1650–1667 (2004)
39. D. Nešić, A. Teel, D. Carnevale, Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems. IEEE Trans. Autom. Control **54**(3), 619–624 (2009)
40. C. Olalla, I. Queinnec, R. Leyva, A. El Aroudi, Optimal state-feedback control of bilinear dc-dc converters with guaranteed regions of stability. IEEE Trans. Ind. Electron. **59**(10), 3868–3880 (2012)

41. H. Omran, L. Hetel, J.-P. Richard, F. Lamnabhi-Lagarrigue, Stability of bilinear sampled-data systems with an emulation of static state feedback, in *IEEE Conference on Decision and Control (CDC)* (2012)
42. H. Omran, L. Hetel, J.-P. Richard, F. Lamnabhi-Lagarrigue, Stability analysis of bilinear systems under aperiodic sampled-data control. Automatica **50**(4), 1288–1295 (2014)
43. P. Pardalos, V. Yatsenko, *Optimization and Control of Bilinear Systems*, vol.11 of Springer Optimization and Its Applications (Springer, 2008)
44. E. Ryan, N. Buckingham, On asymptotically stabilizing feedback control of bilinear systems. IEEE Trans. Autom. Control **28**(8), 863–864 (1983)
45. J. Shen, Y. Zhu, Global feedback stabilization of multi-input bilinear systems. Appl. Math. Lett. **26**(8), 820–825 (2013)
46. H. Sira-Ramírez, R. Silva-Ortigoza, *Control Design Techniques in Power Electronics Devices* (Springer, 2006)
47. C.W. Tao, W.-Y. Wang, M. Chan, Design of sliding mode controllers for bilinear systems with time varying uncertainties. IEEE Trans. Syst. Man Cybern.-Part B: Cybern. **34**(1), 639–645 (2004)
48. S. Tarbouriech, I. Queinnec, T. Calliero, P.L.D. Peres. Control design for bilinear systems with a guaranteed region of stability: an LMI-based approach, in *17th Mediterranean Conference on Control and Automation, MED '09* (2009)
49. W. Zhang, *Stability analysis of networked control systems*. PhD thesis. Departement of Electrical Engineering and Computer Science, Case Western Reserve University (2001)

Chapter 6
On the Stability Analysis of Sampled-Data Systems with Delays

Alexandre Seuret and Corentin Briat

Abstract Controlling a system through a network amounts to solve certain difficulties such as, among others, the consideration of aperiodic sampling schemes and (time-varying) delays. In most of the existing works, delays have been involved in the input channel through which the system is controlled, thereby delaying in a continuous way the control input computed by the controller. We consider here a different setup where the delay acts in a way that the current control input depends on past state samples, possibly including the current one, which is equivalent to considering a discrete-time delay, at the sample level, in the feedback loop. An approach based on the combination of a discrete-time Lyapunov–Krasovskii functional and a looped-functional is proposed and used to obtain tailored stability conditions that explicitly consider the presence of delays and the aperiodic nature of the sampling events. The stability conditions are expressed in terms of linear matrix inequalities and the efficiency of the approach is illustrated on an academic example.

6.1 Introduction

Sampled-data systems are an important class of systems that have been extensively studied in the literature [9] as they arise, for instance, in digital control [23] and networked control systems [16, 38]. The aperiodic nature of the sampling schemes creates additional difficulties in the analysis and the control of such systems as those schemes are much less understood than their periodic counterpart. Several approaches have been proposed in order to characterize the behavior of such systems. Those based

A. Seuret (✉)
LAAS-CNRS, Université de Toulouse, CNRS,
7 avenue du Colonel Roche,
31077 Toulouse, France
e-mail: aseuret@laas.fr

C. Briat
Department of Biosystems Science and Engineering (D-BSSE),
Swiss Federal Institute of Technology Zurich (ETH-Z),
Mattenstrasse 26, 4058 Basel, Switzerland
e-mail: briatc@bsse.ethz.ch; corentin@briat.info

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_6

on the discretization of the sampled-data system have been discussed, for instance, in [10, 17, 27, 35]. In these works, the sampling-period-dependent matrices of the discrete-time system are embedded in a convex polytope and the analysis is carried out using standard robust analysis techniques. This approach leads to efficient and tractable conditions that can be easily used for control design. A limitation of the approach, however, is that it only applies to unperturbed linear time-invariant systems. A second approach is based on the so-called "input-delay approach" which consists of reformulating the original sampled-data system into a time-delay system subject to a sawtooth input-delay [11, 12, 20, 29]. This framework allows for the application of well-known analysis techniques developed for time-delay systems, such as those based on the Lyapunov-Krasovskii theorem. Its main advantage is its applicability to uncertain, time-varying and even nonlinear systems. A limitation, however, is the difficulty of designing controllers with such an approach. Robust analysis techniques based, for instance, on small-gain results [24], Integral Quadratic Constraints [14, 18, 19] or well-posedness theory [1] have also been successfully applied. Approaches based on impulsive systems using Lyapunov functionals [25] or clock-dependent Lyapunov functions [3] also exist. Notably, the latter approach is able to characterize the stability of periodic and aperiodic sampled-data systems subject to both time-invariant and time-varying uncertainties. Even more interestingly, convex robust stabilization conditions for sampled-data systems can also be easily obtained using this approach. In this regard, this framework combines the advantages of discrete-time and functional-based approaches. Finally, approaches based on looped-functionals have been proposed in [7, 8, 31] in order to obtain stability conditions for sampled-data and impulsive systems. This particular type of functional has the interesting property of relaxing the positivity requirement which is necessary in Lyapunov-based approaches. Instead of that, one demands the fulfillment of a "looping condition", a certain boundary condition that can be made structurally satisfied while constructing the functional. In this regard, this class of functionals is therefore more general than Lyapunov(-Krasovsksii) functionals as the looping condition turns out to be a weaker condition than the positive definiteness condition; see e.g. [7, 8, 31].

We propose to derive here stability conditions for (uncertain) aperiodic sampled-data systems with discrete-time input delay. While the delayed sampled-data systems considered in [22, 30] are subject to a continuous-time delay (the delay is expressed in seconds), the systems we are interested in here involve a discrete-time delay (the delay is expressed in a number of samples). A solution to this problem, based on state augmentation, has been proposed in [32] for the constant delay case. This approach yields quite accurate results at the expense of a rather high computational cost, restricting then its application to small delay values. In order to remove this limitation, an alternative approach relying on the use of a mixture of a Lyapunov–Krasovskii and a looped-functional has been proposed in [33] in the case of constant time-delays. The objective of the current chapter is to extend these conditions to the case of time-varying delays. These conditions are expressed in terms of LMIs and illustrated on a simple example.

The chapter is organized as follows. Section 6.2 states the considered problem while Sect. 6.3 presents several preliminary results on looped-functionals. The main results of the chapter are proved in Sect. 6.4 where the cases of constant and time-varying discrete delays are considered. An illustrative example and a discussion of the results are finally treated in Sect. 6.5.

Notations: Throughout the chapter, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$ and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The sets $\mathbb{S}_n$ and $\mathbb{S}_n^+$ represent the set of symmetric and symmetric positive definite matrices of dimension n, respectively. Moreover, for two matrices $A, B \in \mathbb{S}_n$, the inequality $A \prec B$ means that $A - B$ is negative definite. In symmetric matrices, the $*$'s are a shorthand for symmetric terms. For any square matrix $A \in \mathbb{R}^{n\times n}$, we also define $\text{He}(A) := A + A^T$. Finally, I represents the identity matrix of appropriate dimension while 0 stands for the zero-matrix.

6.2 Problem Formulation

Let us consider here linear continuous-time systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t),\ t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{6.1}$$

where $x, x_0 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state of the system, the initial condition and the control input, respectively. Above, the matrices A and B are not necessarily perfectly known but may be uncertain and/or time-varying. The control input u is assumed to be given by the following equation

$$u(t) = Kx(t_{k-h}),\ t \in [t_k, t_{k+1}),\ k \in \mathbb{N}, \tag{6.2}$$

where $K \in \mathbb{R}^{m\times n}$ is a controller gain and the sequence $\{t_k\}_{k\in\mathbb{N}}$ is the sequence of sampling instants. It is assumed that this sequence is strictly increasing and does not admit any accumulation point, that is, we have that $t_k \to \infty$ as $k \to \infty$. We also make the additional assumption that the difference $T_k := t_{k+1} - t_k$ belongs, for all $k \in \mathbb{N}$, to the interval $[T_{min}, T_{max}]$ where $0 \leq T_{min} \leq T_{max}$. The delay h will either be considered to be constant or bounded and time-varying. In the latter case, the delay will be denoted by h_k to emphasize its time-varying nature.

The closed-loop system obtained from the interconnection of (6.1) and (6.2) is given, for all k in $\mathbb{N}$, by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKx(t_{k-h}),\ t \in [t_k, t_{k+1}), \\ x(\theta) &= x_0,\ \theta \leq 0. \end{aligned} \tag{6.3}$$

The discretized version of the previous system is given by

$$x(t_{k+1}) = A_d(T_k)x(t_k) + B_d(T_k)x(t_{k-h}),\ k \geq 0, \tag{6.4}$$

where $A_d(T_k) = e^{AT_k}$ and $B_d(T_k) = \int_0^{T_k} e^{A\tau} BK \mathrm{d}\tau$.

When the sampling period is fixed and known, the stability of the system (6.4) can be established by either augmenting the model with past state values and using then a quadratic discrete-time Lyapunov function, or by using a discrete-time Lyapunov–Krasovskii functional directly on the delayed system [13, 36]. When the sampling is aperiodic, however, discrete-time methods can still be used by embedding the uncertain matrices $A_d(T_k)$ and $B_d(T_k)$ into a polytope [10, 15, 17]. Unfortunately, this approach is only applicable when the matrices (A, B) of the system are constant and perfectly known. To overcome this limitation, several methods can be applied. The first one is the so-called input-delay approach [11, 12] and is based on the reformulation of the sampled state into a delayed state with sawtooth delay. The analysis is then carried out using, for instance, Lyapunov–Krasovskii functionals. The second one is based on the reformulation of a sampled-data system into an impulsive system. The stability of the underlying impulsive system can then be established out using Lyapunov functionals [25], looped-functionals [8, 31] or clock-dependent Lyapunov functions [3, 4, 6]. In this chapter, we will opt for an approach based on a combination of a looped-functional and a discrete-time Lyapunov–Krasovskii functional, and demonstrate its applicability. Note that an approach based on a looped-functional combined with a Lyapunov function has been considered in [32] together with a state-augmentation approach for the system. A major drawback is that the dimension of the augmented system is hn and, therefore, LMI-based methods will not scale very well with the delay size. The consideration of a Lyapunov–Krasovskii functional in the current chapter aims at overcoming such a difficulty by working directly on the original system.

Remark 1 It is worth mentioning that the class of systems considered in this chapter differs from the class of systems described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKx(t_k - \bar{h}),\ t \in [t_k, t_{k+1}), \\ x(\theta) &= \phi(\theta),\ \theta \in [-\bar{h},\ 0], \end{aligned} \tag{6.5}$$

where $\bar{h}$ in a positive scalar. Such systems have been extensively studied in the literature; see e.g. [22, 26, 30]. Note, however, that when the sampling-period T is constant and the delay $\bar{h}$ satisfies $\bar{h} = hT$, then the two classes of systems coincide with each other. In this regard, none of these classes is included in the other meaning, therefore, that distinct methods need to be developed for each class.

6.3 Preliminaries

6.3.1 An Appropriate Modeling Using Lifting

The looped-functional approach relies on the characterization of the trajectories of system (6.3) in a lifted domain [8, 37]. Therefore, we view the entire state trajectory as a sequence of functions $\{x(t_k+\tau),\ \tau\in(0,T_k]\}_{k\in\mathbb{N}}$ with elements having a unique continuous extension to $[0, T_k]$ defined as

$$\chi_k(\tau) := x(t_k+\tau) \text{ with } \chi_k(0) = \lim_{s\downarrow t_k} x(s). \tag{6.6}$$

Finally we define $\mathbb{K}_{[T_{min},T_{max}]}$ as the set defined by

$$\mathbb{K}_{[T_{min},T_{max}]} := \bigcup_{T\in[T_{min},\ T_{max}]} \mathscr{C}([0,\ T],\mathbb{R}^n)$$

where $\mathscr{C}([0,\ T],\mathbb{R}^n)$ denotes the set of continuous functions mapping $[0, T]$ to $\mathbb{R}^n$. Using this notation, system (6.3) can be rewritten as

$$\dot{\chi}_k(\tau) = A\chi_k(\tau) + BK\chi_{k-h}(0),\ \tau\in[0,\ T_k),\quad \forall k\in\mathbb{N}. \tag{6.7}$$

Looped-functionals consider this state definition for assessing stability in an efficient and flexible manner. Notably, the positivity requirement of the functional can be shown to be relaxed and the resulting stability condition can be generally written as a convex expression of the system data, see e.g. [7, 8], allowing then for an easy application of these results to time-varying systems.

Up to now, looped-functionals have not been used to obtain stability conditions for sampled-data systems with discrete time-delay h. We, therefore, propose to extend the results initially proposed in [7, 8, 31] to this case. In what follows, we will denote by χ_k^h the function collecting the sampled and delayed values of the state, i.e.,

$$\forall\theta = -h, -h+1, \ldots, 0,\quad \chi_k^h(\theta) = \chi_{k+\theta}(0) = x(t_{k+\theta}). \tag{6.8}$$

We, finally, define the set $\mathscr{D}_h$ as

$$\mathscr{D}_h = \left\{X\ :\ \{-h,\ldots,0\}\to\mathbb{R}^n\right\}$$

which contains all possible sequences from $\{-h,\ldots,0\}$ to $\mathbb{R}^n$.

6.3.2 Functional-Based Results

The following technical definition is necessary before stating the main general result about looped-functionals result.

Definition 1 [8] Let $0 < T_{min} \leq T_{max} < +\infty$. A functional

$$f : [0, T_{max}] \times \mathbb{K}_{[T_{min},T_{max}]} \times [T_{min}, T_{max}] \to \mathbb{R}$$

is said to be a **looped-functional** if the following conditions hold

1. the equality

$$f(0, z, T) = f(T, z, T) \tag{6.9}$$

   holds for all functions $z \in C([0, T], \mathbb{R}^n) \subset \mathbb{K}_{[T_{min}, T_{max}]}$ and all $T \in [T_{min}, T_{max}]$, and
2. it is differentiable with respect to the first variable with the standard definition of the derivative.

The set of all such functionals is denoted by $\mathfrak{LF}([T_{min}, T_{max}])$.

The idea for proving stability of (6.3) is to look at a positive definite quadratic form $V(x)$ such that the sequence $\{V(\chi_k(T_k))\}_{k\in\mathbb{N}}$ is monotonically decreasing. This is formalized below through a functional existence result:

Theorem 1 *Let $T_{min} \le T_{max}$ be two finite positive scalars and $V : \mathbb{R}^n \times \mathscr{D}_{h_{max}} \to \mathbb{R}_+$ be a form verifying*

$$X \in \mathscr{D}_{h_{max}}, \quad \mu_1 ||X||^2_{h_{max}} \le V(X(0), X) \le \mu_2 ||X||^2_{h_{max}}, \tag{6.10}$$

for some scalars $0 < \mu_1 \le \mu_2$. Assume that one of the following equivalent statements hold:

(i) The sequence $\{V(\chi_k(T_k), \chi_k^h)\}_{k\in\mathbb{N}}$ is decreasing
(ii) There exists a looped-functional $\mathscr{V} \in \mathfrak{LF}([T_{min}, T_{max}])$ such that the functional $\mathscr{W}_k$ as

$$\mathscr{W}_k(\tau, \chi_k, \chi_k^h) := \frac{\tau}{T_k}\Lambda_k + V(\chi_k(\tau), \chi_k^h) + \mathscr{V}(\tau, \chi_k, T_k), \tag{6.11}$$

where $\Lambda_k = V(\chi_k(T_k), \chi_{k+1}^h) - V(\chi_k(T_k), \chi_k^h)$, has a derivative along the trajectories of system $\dot{\chi}_k(\tau) = A\chi_k(\tau) + BK\chi_{k-h(k)}(0)$, $\tau \in [0, T_k]$

$$\frac{d}{d\tau}\mathscr{W}_k(\tau, \chi_k, \chi_k^h) := \frac{1}{T_k}\Lambda_k + \frac{d}{d\tau}V(\chi_k(\tau), \chi_k^h) + \frac{d}{d\tau}\mathscr{V}(\tau, \chi_k, T_k), \tag{6.12}$$

which is negative definite for all $\tau \in (0, T_k)$, $T_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}$.

Then, the solutions of system (6.3) are asymptotically stable for any sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfying $t_{k+1} - t_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}$.

Proof The proof is omitted but is similar to the proof provided in [8, 31].

In the remainder of the chapter, we will propose three stability conditions addressing the cases of constant and time-varying delay h for both certain and uncertain aperiodic sampled-data systems.

6.4 Main Results

6.4.1 Stability Analysis for Constant Delay h

This section provides a stability result for aperiodic sampled-data systems with a constant delay h. We have the following result:

Theorem 2 *The sampled-data system (6.7) with the delay h and* $T_k := t_{k+1} - t_k \in [T_{min},\ T_{max}]$, $k \in \mathbb{N}$, *is asymptotically stable if there exist matrices* $R, Q, Z \in \mathbb{S}^n_+$, $P, X \in \mathbb{S}^{2n}$, $S \in \mathbb{S}^n$, $U \in \mathbb{R}^{n\times n}$ *and a matrix* $Y \in \mathbb{R}^{4n\times n}$ *such that the LMIs*

$$\begin{aligned} \Phi_0 \quad &:= \begin{bmatrix} I \\ I \end{bmatrix} P \begin{bmatrix} I \\ I \end{bmatrix} \quad \succ 0, \\ \Phi_1(\theta) &:= \begin{bmatrix} F_0(\theta) & \theta Y \\ \star & -\theta Z \end{bmatrix} \prec 0, \\ \Phi_2(\theta) &:= F_0(\theta) + \theta F_1 \quad \prec 0, \end{aligned} \tag{6.13}$$

hold for all $\theta \in \{T_{min}, T_{max}\}$ *where*

$$\begin{aligned} F_0(\theta) &= F_{00} + F_{01} + \theta\,\mathrm{He}\left(\begin{bmatrix} M_1 \\ M_3 \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} M_0 \\ 0 \end{bmatrix}\right), \\ F_{00} &= M_\Delta^{\mathsf{T}} S M_\Delta + \theta M_T^{\mathsf{T}} X M_T + \mathrm{He}(M_\Delta^{\mathsf{T}} U M_T + M_\Delta Y), \\ F_{01} &= M_2^{\mathsf{T}} \Phi_0 M_2 - M_T^{\mathsf{T}} P M_T + M_3^{\mathsf{T}} Q M_3 - M_4^{\mathsf{T}} Q M_4 + h^2 M_\delta^{\mathsf{T}} R M_\delta - M_h^{\mathsf{T}} R M_h, \\ F_1 &= M_0^{\mathsf{T}} Z M_0 + \mathrm{He}(M_0^{\mathsf{T}}(S M_\Delta + U M_T)) - 2 M_T^{\mathsf{T}} X M_T, \end{aligned} \tag{6.14}$$

with

$$\begin{aligned} M_0 &= \begin{bmatrix} A & 0 & 0 & BK \end{bmatrix}, & M_1 &= \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}, & M_2 &= \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}, \\ M_3 &= \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix}, & M_4 &= \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}, & M_\Delta &= \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}, \\ M_\delta &= \begin{bmatrix} 0 & I & -I & 0 \end{bmatrix}, & M_h &= \begin{bmatrix} 0 & 0 & I & -I \end{bmatrix}, & M_T &= \begin{bmatrix} M_2^{\mathsf{T}} & M_3^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}. \end{aligned}$$

Proof Consider a Lyapunov function for the discrete-time system (6.4) given by

$$\begin{aligned} V(\chi_k(\tau), \chi_k^h) = \begin{bmatrix} \chi_k(\tau) \\ \chi_k(0) \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} \chi_k(\tau) \\ \chi_k(0) \end{bmatrix} + \sum_{i=k-h}^{k-1} \chi_i^{\mathsf{T}}(0) Q \chi_i(0) \\ + h \sum_{i=-h}^{-1} \sum_{j=k+i}^{k-1} \delta_i^{\mathsf{T}}(0) R \delta_i(0), \end{aligned} \tag{6.15}$$

where $\delta_i(0) = \chi_{i+1}(0) - \chi_i(0)$. On the other hand, we define the functional $\mathscr{V}$ as follows

$$
\begin{aligned}
T_k \mathscr{V}(\tau, \chi_k, T_k) = \tau(T_k - \tau) \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}^{\mathsf{T}} X \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} \\
+ \tau(\chi_k(\tau) - \chi_k(T_k))^{\mathsf{T}} S(\chi_k(\tau) - \chi_k(T_k)) \\
+ 2\tau(\chi_k(\tau) - \chi_k(T_k))^{\mathsf{T}} U \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} - \tau \int_{\tau}^{T_k} \dot{\chi}_k^{\mathsf{T}}(s) Z \dot{\chi}_k(s) ds,
\end{aligned} \tag{6.16}
$$

where the matrices above are such that $Z \in \mathbb{S}_+^n$, $S \in \mathbb{S}^n$, $X \in \mathbb{S}^{2n}$ and $U \in \mathbb{R}^{n \times 2n}$. This functional has been build in order to satisfy the looped condition. Indeed, one can easily verify that

$$
\mathscr{V}(0, \chi_k, T_k) = \mathscr{V}(T_k, \chi_k, T_k) = 0,
$$

for all $T_k \in [T_{min}, \ T_{max}]$. As already highlighted in [8, 31], the consideration of looped-functionals allows to enlarge the set of acceptable functionals in comparison to Lyapunov–Krasovskii functionals. Firstly, the matrices S, X and U are sign-indefinite in the current setting while they would have been required to be positive definite in usual Lyapunov approaches such as the one in [11, 25]. Second, the proposed functional includes more components than it is usually proposed in the literature (see for instance [11, 25, 31]). Indeed, looped-functionals allow one to include terms like $\chi_k(T_k)$ which would have been difficult to consider in the Lyapunov–Krasovskii framework. Following Theorem 1, let us consider

$$
\dot{\mathscr{W}}_k(\tau, \chi_k, \chi_k^h) = \frac{1}{T_k} \left(\Lambda_k + T_k \dot{V}(\chi_k(\tau), \chi_k^h) + T_k \dot{\mathscr{V}}(\tau, \chi_k, T_k) \right), \tag{6.17}
$$

where Λ_k is defined in Theorem 1. By virtue of the same theorem, the asymptotic stability of the system (6.7) is then proved if $V(\chi_k(0), \chi_k^h)$ is positive definite and $\dot{\mathscr{W}}_k$ is negative definite. Note that the necessity is lost by choosing specific forms for the functionals (6.15)–(6.16). Regarding the first condition, we have that

$$
\begin{aligned}
V(\chi_k(0), \chi_k^h) = \chi_k^{\mathsf{T}}(0) \Phi_0 \chi_k(0) + \sum_{i=k-h}^{k-1} \chi_i^{\mathsf{T}}(0) Q \chi_i(0) \\
+ h \sum_{i=-h}^{-1} \sum_{j=k+i}^{k-1} \delta_i^{\mathsf{T}}(0) R \delta_i(0)
\end{aligned}
$$

which is positive definite provided that the matrices Φ_0, Q and R are positive definite as well. Let us focus now on the condition on $\mathscr{W}_k$ for which we will provide an upper-bound expressed in terms of the augmented vector

$$
\xi_k(\tau) := \mathrm{col}(\chi_k(\tau), \chi_k(T_k), \chi_k(0), \chi_{k-h}(0)).
$$

By virtue of the above definition, Λ_k can be rewritten as

$$\Lambda_k = \begin{bmatrix} \chi_{k+1}(0) \\ \chi_{k+1}(0) \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} \chi_{k+1}(0) \\ \chi_{k+1}(0) \end{bmatrix} - \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}$$
$$+ \sum_{i=k-h+1}^{k} \chi_i^{\mathsf{T}}(0) Q \chi_i(0) - \sum_{i=k-h}^{k-1} \chi_i^{\mathsf{T}}(0) Q \chi_i(0)$$
$$+ h \sum_{i=-h}^{-1} \left(\sum_{j=k+i+1}^{k} \delta_i^{\mathsf{T}}(0) R \delta_i(0) - \sum_{j=k+i}^{k-1} \delta_i^{\mathsf{T}}(0) R \delta_i(0) \right).$$

Since $\chi_{k+1}(0) = \chi_k(T_k)$, the previous expression can be easily expressed in terms of the augmented vector $\xi_k(\tau)$. Applying then Jensen's inequality yields

$$\begin{aligned} \Lambda_k &\le \xi_k^{\mathsf{T}}(\tau) \left(M_2^{\mathsf{T}} \Phi_0 M_2 - \begin{bmatrix} M_2 \\ M_3 \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} M_2 \\ M_3 \end{bmatrix} + M_3^{\mathsf{T}} Q M_3 - M_4^{\mathsf{T}} Q M_4 \right. \\ &\quad \left. + h^2 M_\delta^{\mathsf{T}} R M_\delta - M_h^{\mathsf{T}} R M_h \right) \xi_k(\tau) \\ &= \xi_k^{\mathsf{T}}(\tau) F_{01} \xi_k(\tau). \end{aligned} \tag{6.18}$$

Let us focus now on the second term of $\dot{\mathscr{W}}$, as defined in (6.17), given by

$$\begin{aligned} T_k \dot{V}(\chi_k(\tau), X) &= 2T_k \begin{bmatrix} \chi_k(\tau) \\ \chi_k(0) \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} \dot{\chi}_k(\tau) \\ 0 \end{bmatrix} \\ &= T_k \xi_k^{\mathsf{T}}(\tau)\, \mathrm{He} \left(\begin{bmatrix} M_1 \\ M_3 \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} M_0 \\ 0 \end{bmatrix} \right) \xi_k(\tau). \end{aligned} \tag{6.19}$$

Finally, the last term of $\dot{\mathscr{W}}$, as defined in (6.17), is given by

$$\begin{aligned} T_k \dot{\mathscr{V}}(\tau, \chi_k, T_k) &= (T_k - 2\tau) \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}^{\mathsf{T}} X \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} \\ &\quad + (\chi_k(\tau) - \chi_k(T_k))^{\mathsf{T}} S (\chi_k(\tau) - \chi_k(T_k)) + 2\tau \dot{\chi}_k^{\mathsf{T}}(\tau) S (\chi_k(\tau) - \chi_k(T_k)) \\ &\quad + 2(\chi_k(\tau) - \chi_k(T_k))^{\mathsf{T}} U \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} + 2\tau \dot{\chi}_k^{\mathsf{T}}(\tau) U \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} \\ &\quad + \tau \dot{\chi}_k^{\mathsf{T}}(\tau) Z \dot{\chi}_k(\tau) - \int_\tau^{T_k} \dot{\chi}_k^{\mathsf{T}}(s) Z \dot{\chi}_k(s) ds. \end{aligned} \tag{6.20}$$

We can rewrite the above expression in terms of the matrices F_{00}, F_1, F_2 defined in Theorem 2 to get

$$T_k \dot{\mathscr{V}}(\chi_k, \tau) = \xi_k^{\mathsf{T}}(\tau) \left[F_{00} + \tau F_1 - \mathrm{He}(Y M_\Delta) \right] \xi_k(\tau) - \int_\tau^{T_k} \dot{\chi}_k^{\mathsf{T}}(s) Z \dot{\chi}_k(s) ds. \tag{6.21}$$

In order to find a convenient upper-bound on the last integral term, we propose to consider the affine version of Jensen's inequality, discussed in [2], to get that

$$-\int_{\tau}^{T_k} \dot{\chi}_k^{\mathsf{T}}(s) Z \dot{\chi}_k(s) ds \le \xi_k^{\mathsf{T}}(\tau) \left[\mathrm{He}(Y M_{\Delta}) + (T_k - \tau) Y Z^{-1} Y^{\mathsf{T}}\right] \xi_k(\tau),$$

where $Y \in \mathbb{R}^{4n \times n}$ is a free matrix. The benefit of the affine version of Jensen's inequality is, in essence, only of computational nature. It has indeed been discussed in [2] that when the interval of integration is uncertain or time-varying, it is preferable to use the affine version to limit the increase of conservatism. The price to pay is a moderate increase of the computational complexity through the presence of the additional matrix Y.

Substituting then this inequality back into (6.21), leads to

$$T_k \dot{\mathscr{V}}(\chi_k, \tau) \le \xi_k^{\mathsf{T}}(\tau) \left[F_{00} + \tau F_1 + (T_k - \tau) Y Z^{-1} Y^{\mathsf{T}}\right] \xi_k(\tau), \tag{6.22}$$

where F_{00} and F_1 are given in Theorem 2. Summing then (6.18), (6.19) and (6.22) all together, we get that $\dot{\mathscr{W}}_k$ is negative definite if

$$F_0(T_k) + \tau F_1 + (T_k - \tau) Y Z^{-1} Y^{\mathsf{T}} \tag{6.23}$$

is negative definite for all $(\tau, T_k) \in \mathscr{S}$ where

$$\mathscr{S} := \{(\tau, T) \in \mathbb{R}_+^2 : \tau \in [0, T],\ T \in [T_{min}, T_{max}]\},$$

and where $F_0(T_k)$ is defined in Theorem 2. Exploiting the fact that the matrix (6.23) is affine in τ and T_k, hence convex in these variables, allows us to easily conclude that the matrix (6.23) is negative definite for all $\tau \in [0, T_k]$ and all $T_k \in [T_{min}, T_{max}]$ if and only if it is negative definite at the vertices of the set $\mathscr{S}$ or, equivalently, negative definite on the set $\{(T_{min}, T_{min}), (T_{max}, T_{max}), (0, T_{min}), (0, T_{max})\}$. Each one of these points leads to one of the following LMI conditions:

$$\begin{array}{rcll} \Phi_1(T_{min}) & = & F_{00} + T_{min} F_1 & \prec 0, \\ \Phi_1(T_{max}) & = & F_{00} + T_{max} F_1 & \prec 0, \\ \tilde{\Phi}_2(T_{min}) & := & F_{00} + T_{min} Y Z^{-1} Y^{\mathsf{T}} & \prec 0, \\ \tilde{\Phi}_2(T_{max}) & := & F_{00} + T_{max} Y Z^{-1} Y^{\mathsf{T}} & \prec 0. \end{array}$$

Applying finally the Schur complement with respect to the last term in $\tilde{\Phi}_2(\cdot)$ yields $\Phi_2(\cdot)$. The proof is complete.

A similar approach is considered in [33] with the difference that another looped-functional $\mathscr{V}$ is used. Another notable difference is the use of the reciprocally convex combination lemma of [28] yielding less conservative conditions without the introduction of the slack variable Y.

6.4.2 Stability Analysis for Time-Varying Delay h_k

Interestingly, Theorem 2 can be easily extended to cope with time-varying delays. In this respect, we now consider that the delay is time-varying and belongs to $\{0,\ldots,\bar{h}\}$, $\bar{h}\in\mathbb{N}$. This leads to the following result:

Corollary 1 *The sampled-data system (6.7) with time-varying delay $h_k\in\{0,\ldots,\bar{h}\}$ and $T_k := t_{k+1}-t_k\in[T_{min},\ T_{max}]$, $k\in\mathbb{N}$, is asymptotically stable if there exist matrices $R, Z\in\mathbb{S}^n_+$, $P, X\in\mathbb{S}^{2n}$, $S\in\mathbb{S}^n$, $U\in\mathbb{R}^{n\times n}$ and a matrix $Y\in\mathbb{R}^{4n\times n}$ such that the LMIs*

$$\begin{aligned}
\Psi_0 \quad &:= \begin{bmatrix} I\\ I\end{bmatrix} P \begin{bmatrix} I\\ I\end{bmatrix} \quad \succ 0,\\
\Psi_1(\theta) &:= \begin{bmatrix} G_0(\theta) & \theta Y\\ \star & -\theta Z\end{bmatrix} \prec 0,\\
\Psi_2(\theta) &:= G_0(\theta)+\theta G_1 \quad \prec 0,
\end{aligned} \tag{6.24}$$

hold for all $\theta\in\{T_{min}, T_{max}\}$ where $G_{00}=F_{00}$, $G_1=F_1$ and

$$\begin{aligned}
G_0(\theta) &= G_{00}+G_{01}+\theta\,\mathrm{He}\left(\begin{bmatrix} M_1\\ M_3\end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} M_0\\ 0\end{bmatrix}\right),\\
G_{01} &= M_2^{\mathsf{T}}\Phi_0 M_2 - M_T^{\mathsf{T}} P M_T + \bar{h}^2 M_\delta^{\mathsf{T}} R M_\delta - M_h^{\mathsf{T}} R M_h.
\end{aligned} \tag{6.25}$$

Proof As in the proof of Theorem 2, we consider the looped-functional $\mathscr{V}$ given in (6.15). However, we shall consider here the Lyapunov–Krasovskii functional V given by

$$V(\chi_k(\tau),\chi_k^{\bar{h}}) \;=\; \begin{bmatrix}\chi_k(\tau)\\ \chi_k(0)\end{bmatrix}^{\mathsf{T}} P \begin{bmatrix}\chi_k(\tau)\\ \chi_k(0)\end{bmatrix} + \bar{h}\sum_{i=-\bar{h}}^{-1}\sum_{j=k+i}^{k-1}\delta_i^{\mathsf{T}}(0)R\delta_i(0), \tag{6.26}$$

where $\delta_i(0)=\chi_{i+1}(0)-\chi_i(0)$. This functional is nothing else but the one we considered for establishing Theorem 2 in which the matrix Q has been set to zero. The proof is now very similar to the one of Theorem 2 and, therefore, only the part pertaining on Λ_k is detailed because of its dissimilarity. Simple calculations show that

$$\begin{aligned}
\Lambda_k = &\begin{bmatrix}\chi_k(T_k)\\ \chi_k(T_k)\end{bmatrix}^{\mathsf{T}} P \begin{bmatrix}\chi_k(T_k)\\ \chi_k(T_k)\end{bmatrix} - \begin{bmatrix}\chi_k(T_k)\\ \chi_k(0)\end{bmatrix}^{\mathsf{T}} P \begin{bmatrix}\chi_k(T_k)\\ \chi_k(0)\end{bmatrix}\\
&+\bar{h}^2(\chi_{k+1}(0)-\chi_k(0))^{\mathsf{T}} R(\chi_{k+1}(0)-\chi_k(0)) - \bar{h}\sum_{i=k-\bar{h}}^{k-1}\delta_i^{\mathsf{T}}(0)R\delta_i(0).
\end{aligned}$$

Since $h_k \leq \bar{h}$ it holds that

$$\Lambda_k \leq \begin{bmatrix} \chi_k(T_k) \\ \chi_k(T_k) \end{bmatrix}^\mathsf{T} P \begin{bmatrix} \chi_k(T_k) \\ \chi_k(T_k) \end{bmatrix} - \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}^\mathsf{T} P \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} \\ + \bar{h}^2(\chi_{k+1}(0) - \chi_k(0))^\mathsf{T} R(\chi_{k+1}(0) - \chi_k(0)) - h_k \sum_{i=k-h_k}^{k-1} \delta_i^\mathsf{T}(0) R \delta_i(0).$$

Applying Jensen's inequality to the last summation term, and using the definition of the matrices M_2, M_3, M_δ yields

$$\begin{aligned} \Lambda_k &= \xi_k^\mathsf{T}(\tau) \left(M_2^\mathsf{T} \Phi_0 M_2 - \begin{bmatrix} M_2 \\ M_3 \end{bmatrix}^\mathsf{T} P \begin{bmatrix} M_2 \\ M_3 \end{bmatrix} + \bar{h}^2 M_\delta^\mathsf{T} R M_\delta - M_h^\mathsf{T} R M_h \right) \xi_k(\tau) \\ &= \xi_k^\mathsf{T}(\tau) G_{01} \xi_k(\tau). \end{aligned} \tag{6.27}$$

The rest of the proof is identical to the proof of Theorem 2.

6.4.3 Robust Stability Analysis

One of the main advantages of the proposed method based lies in the possibility of extending the stability conditions to the case of uncertain systems. Assume now that the matrices of the system are time-varying/uncertain and can be written as

$$\begin{bmatrix} A(t) & B(t) \end{bmatrix} = \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} A_i & B_i \end{bmatrix}, \tag{6.28}$$

where N is a positive integer, A_i and B_i, $i = 1, \ldots, N$, are some matrices of appropriate dimensions and the vector $\lambda(t)$ evolves in the N unit simplex defined as

$$\mathscr{U} := \left\{ \lambda \in \mathbb{R}_+^N : \sum_{i=1}^{N} \lambda_i = 1 \right\}. \tag{6.29}$$

This leads us to the following result:

Corollary 2 *The sampled-data system (6.7)–(6.28) with constant delay h and $T_k := t_{k+1} - t_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}$, is asymptotically stable if there exist matrices $R, Q, Z \in \mathbb{S}_+^n$, $P, X \in \mathbb{S}^{2n}$, $S \in \mathbb{S}^n$, $U \in \mathbb{R}^{n\times n}$ and some matrices $Y_i \in \mathbb{R}^{4n\times n}$ such that the LMIs*

$$\begin{aligned} \Phi_0 \quad &:= \begin{bmatrix} I \\ I \end{bmatrix} P \begin{bmatrix} I \\ I \end{bmatrix} \succ 0, \\ \Phi_1^i(\theta) &:= \begin{bmatrix} F_0^i(\theta) & \theta Y_i \\ \star & -\theta Z \end{bmatrix} \prec 0, \\ \Phi_2^i(\theta) &:= F_0^i(\theta) + \theta F_1^i \prec 0, \end{aligned} \tag{6.30}$$

hold for all $\theta \in \{T_{min}, T_{max}\}$ *where*

$$\begin{aligned}
F_0^i(\theta) &= F_{00}^i + F_{01} + \theta\,\mathrm{He}\left(\begin{bmatrix} M_1 \\ M_3 \end{bmatrix}^\mathsf{T} P \begin{bmatrix} M_0^i \\ 0 \end{bmatrix}\right), \\
F_{00}^i &= M_\Delta^\mathsf{T} S M_\Delta + \theta M_T^\mathsf{T} X M_T + \mathrm{He}(M_\Delta^\mathsf{T} U M_T + M_\Delta Y_i) \\
F_{01}^i &= M_2^\mathsf{T} \Phi_0 M_2 - M_T^\mathsf{T} P M_T + M_3^\mathsf{T} Q M_3 - M_4^\mathsf{T} Q M_4 + h^2 M_\delta^\mathsf{T} R M_\delta - M_h^\mathsf{T} R M_h, \\
F_1 &= M_0^{i\,\mathsf{T}} Z M_0^i + \mathrm{He}(M_0^{i\,\mathsf{T}}(S M_\Delta + U M_T)) - 2 M_T^\mathsf{T} X M_T,
\end{aligned} \tag{6.31}$$

with $M_0^i = \begin{bmatrix} A_i & 0 & 0 & B_i K \end{bmatrix}$.

Proof The proof is straightforward by noting that the LMI conditions in Theorem 2 are convex in the matrix M_0. Remarking also that

$$M_0 = \begin{bmatrix} A(t) & 0 & 0 & B(t)K \end{bmatrix} = \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} A_i & 0 & 0 & B_i K \end{bmatrix}$$

implies that the LMI conditions are convex in the matrices of the system $A(t)$ and $B(t)$. By virtue of standard results on systems with polytopic uncertainties (see e.g. [5]), it is enough to check the feasibility of the LMI at the vertices of the set $\mathscr{U}$ and the result directly follows.

Remark 2 The above result can be easily extended to the time-varying delay case by setting the matrix Q to 0. This is not presented for brevity.

6.5 Example

Example 1 ([39]) Let us consider the sampled-data system (6.3) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \text{ and } K = \begin{bmatrix} 3.75 & 11.5 \end{bmatrix}.$$

Using an eigenvalue-based analysis, theoretical stability-preserving upper bounds for the constant sampling period can be determined for any fixed delay h. These values can be understood as a theoretical limit for the upper bounds obtained in the aperiodic case. These theoretical upper bounds and the results computed by solving the conditions of Theorem 2 are given in Tables 6.1 and 6.2. More particularly, Table 6.1 compares the maximal allowable sampling period $T = T_{max} = T_{min}$ obtained using Theorem 2 and previous results of the literature. We can immediately see that the numerical values obtained using Theorem 2 are slightly more conservative than those previously obtained by the same authors. Yet, the obtained numerical values are close to the theoretical value.

On the other hand, Table 6.2 compares the obtained results with those obtained with the methods developed in [32, 33]. It can be seen again that the results of [32]

Table 6.1 Maximal allowable sampling period $T_{max} = T_{min}$ for Example 1 with periodic samplings for several values of h (the symbol -* means "untested because of a too high computational complexity")

h	0	1	2	5	10
Theoretical bounds	1.729	0.763	0.463	0.216	0.112
[26] (with $\tau = hT$)	1.278	0.499	0.333	0.166	0.090
[21] (with $\tau = hT$)	1.638	0.573	0.371	0.179	0.096
[30] (with $\tau = hT$)	1.721	0.701	0.431	0.197	0.103
[33]	1.728	0.761	0.448	0.199	0.103
[32]	1.729	0.763	0.463	-*	-*
Theorem 2	1.720	0.536	0.318	0.146	0.077

Table 6.2 Maximal allowable sampling period T_{max} for Example 1 with $T_{min} = 10^{-2}$ and for several values of h (the symbol -* means "untested because of a too high computational complexity")

h	0	1	2	5	10
[33]	1.708	0.618	0.377	0.176	0.094
[32]	1.729	0.763	0.463	-*	-*
Theorem 2	1.245	0.460	0.283	0.132	0.071

Table 6.3 Maximal allowable sampling period T_{max} for Example 1 with $T_{min} = 10^{-2}$ and for several values of the upper bound $\bar{h}$ of the time-varying delay h

$\bar{h}$	0	1	2	5	10
Corollary 1, $T_{min} = T_{max}$	–	0.465	0.264	0.115	0.059
Corollary 1, $T_{min} < T_{max}$	–	0.402	0.240	0.109	0.057

are less conservative for small values of the delay h. The main reason for this is that the current chapter uses Jensen's inequality, which is more conservative than the integral inequality considered in [32] for large values of the delay. However, the computational burden of the approach of [32] increases exponentially with the delay h, making it inapplicable for systems with large delays.

Finally, Table 6.3 shows the results obtained using Corollary 1, which addresses the case of time-varying delay h_k. We can observe a notable decrease of the maximal allowable sampling period.

6.6 Conclusions

In this chapter, a way for analyzing stability of periodic and aperiodic uncertain sampled-data systems with discrete-time delays is presented. Instead of using a discrete-time criterion that would prevent the generalization of the approach to uncertain systems with time-varying uncertainties, an alternative approach based on looped-functionals has been preferred. The main novelty of the method relies on the stability analysis, which merges the continuous-time and discrete-time criteria at the same time. This is combination of discrete- and continuous-time approach has been possible by the introduction of a lifted version of the state vector. Further extensions aims at reducing the conseratism of the stability conditions by employing recent and more efficient inequalities such as the reciprocally convex combination lemma [28] and Wirtinger-based integral inequality [34].

References

1. Y. Ariba, C. Briat, K.H. Johansson, Simple conditions for L_2 stability and stabilization of networked control systems, in *IFAC World Congress*, 2011
2. C. Briat, Convergence and equivalence results for the Jensen's inequality: application to time-delay and sampled-data systems. IEEE Trans. Autom. Control **56**(7), 1660–1665 (2011)
3. C. Briat, Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints. Automatica **49**(11), 3449–3457 (2013)
4. C. Briat, Convex conditions for robust stabilization of uncertain switched systems with guaranteed minimum and mode-dependent dwell-time. Syst. Control Lett. **78**, 63–72 (2015)
5. C. Briat, Linear Parameter-Varying and Time-Delay Systems—Analysis, Observation, Filtering and Control. Advances on Delays and Dynamics, vol. 3 (Springer, Heidelberg, 2015)
6. C. Briat, Theoretical and numerical comparisons of looped functionals and clock-dependent Lyapunov functions—the case of periodic and pseudo-periodic systems with impulses. Int. J. Robust Nonlinear Control (2016) (to appear)
7. C. Briat, A. Seuret, Convex dwell-time characterizations for uncertain linear impulsive systems. IEEE Trans. Autom. Control **57**(12), 3241–3246 (2012)
8. C. Briat, A. Seuret, A looped-functional approach for robust stability analysis of linear impulsive systems. Syst. Control Lett. **61**(10), 980–988 (2012)
9. T. Chen, B.A. Francis, *Optimal Sampled-Data Control Systems* (Springer, 1995)
10. M.B.G. Cloosterman, N. van de Wouw, W.P.M.H. Heemels, H. Nijmeijer, Stability of networked control systems with uncertain time-varying delays. IEEE Trans. Autom. Control **54**(7), 1575–1580 (2009)
11. E. Fridman, A refined input delay approach to sampled-data control. Automatica **46**(2), 421–427 (2010)
12. E. Fridman, A. Seuret, J.-P. Richard, Robust sampled-data stabilization of linear systems: an input delay approach. Automatica **40**(8), 1141–1446 (2004)
13. E. Fridman, U. Shaked, Stability and guaranteed cost control of uncertain discrete delay systems. Int. J. Control **78**(4), 235–246 (2005)
14. H. Fujioka, Stability analysis of systems with aperiodic sample-and-hold devices. Automatica **45**(3), 771–775 (2009)
15. W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, D. Nesic, Networked control systems with communication constraints: tradeoffs between transmission intervals, delays and performance. IEEE Trans. Autom. Control **55**(8), 1781–1796 (2010)
16. J.P. Hespanha, P. Naghshtabrizi, Y. Xu, A survey of recent results in networked control systems. Proc. IEEE **95**(1), 138–162 (2007)
17. L. Hetel, J. Daafouz, C. Iung, Stabilization of arbitrary switched linear systems with unknown time-varying delays. IEEE Trans. Autom. Control **51**(10), 1668–1674 (2006)
18. C.-Y. Kao, H. Fujioka, On stability of systems with aperiodic sampling devices. IEEE Trans. Autom. Control **58**(3), 2085–2090 (2013)

19. C.-Y. Kao, D.-R. Wu, On robust stability of aperiodic sampled-data systems: an integral quadratic constraint approach, in *IEEE American Control Conference (ACC)*, 2014
20. K. Liu, E. Fridman, Discontinuous Lyapunov functionals for linear systems with sawtooth delays, in *IFAC Workshop on Time-Delay Systems (TDS)*, 2009
21. K. Liu, E. Fridman, Networked-based stabilization via discontinuous Lyapunov functionals. Int. J. Robust Nonlinear Control **22**(4), 420–436 (2012)
22. K. Liu, E. Fridman, Wirtinger's inequality and Lyapunov-based sampled-data stabilization. Automatica **48**(1), 102–108 (2012)
23. R.H. Middleton, G.C. Goodwin, *Digital Control and Estimation: A Unified Approach* (Prentice-Hall, 1990)
24. L. Mirkin, Some remarks on the use of time-varying delay to model sample-and-hold circuits. IEEE Trans. Autom. Control **52**(6), 1009–1112 (2007)
25. P. Naghshtabrizi, J.P. Hespanha, A.R. Teel, Exponential stability of impulsive systems with application to uncertain sampled-data systems. Syst. Control Lett. **57**(5), 378–385 (2008)
26. P. Naghshtabrizi, J.P. Hespanha, A.R. Teel, Stability of delay impulsive systems with application to networked control systems. Trans. Inst. Meas. Control **32**(5), 511–528 (2010)
27. Y. Oishi, H. Fujioka, Stability and stabilization of aperiodic sampled-data control systems: an approach using robust linear matrix inequalities, in *Joint IEEE Conference on Decision and Control and Chinese Control Conference (CDC-CCC)*, 2009
28. P.G. Park, J.W. Ko, C. Jeong, Reciprocally convex approach to stability of systems with time-varying delays. Automatica **47**(1), 235–238 (2011)
29. A. Seuret, Stability analysis for sampled-data systems with a time-varying period, in *Joint IEEE Conference on Decision and Control and Chinese Control Conference (CDC-CCC)*, 2009
30. A. Seuret, Stability analysis of networked control systems with asynchronous sampling and input delay, in *IEEE American Control Conference (ACC)*, 2011
31. A. Seuret, A novel stability analysis of linear systems under asynchronous samplings. Automatica **48**(1), 177–182 (2012)
32. A. Seuret, C. Briat, Stability analysis of uncertain sampled-data systems with incremental delay using looped-functionals. Automatica **55**, 274–278 (2015)
33. A. Seuret, C. Briat, F. Gouaisbaut. Stability analysis of asynchronous sampled-data systems with discrete-time constant input delay, in *IEEE Conference on Decision and Control (CDC)*, 2014
34. A. Seuret, F. Gouaisbaut, Wirtinger-based integral inequality: application to time-delay systems. Automatica **49**(9), 2860–2866 (2013)
35. Y.S. Suh, Stability and stabilization of nonuniform sampling systems. Automatica **44**(12), 3222–3226 (2008)
36. S. Xu, J. Lam, C. Yang, Quadratic stability and stabilization of uncertain linear discrete-time systems with state delay. Syst. Control Lett. **43**(2), 77–84 (2001)
37. Y. Yamamoto, New approach to sampled-data control systems: a function space method, in *IEEE Conference on Decision and Control (CDC)*, 1990
38. S. Zampieri, A survey of recent results in networked control systems, in *IFAC World Congress*, 2008
39. W. Zhang, M.S. Branicky, S.M. Phillips, Stability of networked control cystems. IEEE Control Syst. Mag. **21**(1), 84–99 (2001)

Chapter 7
Output Feedback Event-Triggered Control

Mahmoud Abdelrahim, Romain Postoyan, Jamal Daafouz and Dragan Nešić

Abstract Event-triggered control has been proposed as an alternative implementation to conventional time-triggered approach in order to reduce the amount of transmissions. The idea is to adapt transmissions to the state of the plant such that the loop is closed only when it is needed according to the stability or/and the performance requirements. Most of the existing event-triggered control strategies assume that the full state measurement is available. Unfortunately, this assumption is often not satisfied in practice. There is therefore a strong need for appropriate tools in the context of output feedback control. Most existing works on this topic focus on linear systems. The objective of this chapter is to first summarize our recent results on the case where the plant dynamics is nonlinear. The approach we follow is emulation as we first design a stabilizing output feedback law in the absence of sampling; then we consider the network and we synthesize the event-triggering condition. The latter combines techniques from event-triggered and time-triggered control. The results are then proved to be applicable to linear time-invariant (LTI) systems as a particular case. We then use these results as a starting point to elaborate a co-design method, which allows us to jointly construct the feedback law and the triggering condition for LTI systems where the problem is formulated in terms of linear matrix

R. Postoyan—This work is partially supported by the ANR under the grant COMPACS (ANR-13-BS03-0004-02).
D. Nešić—This work is also supported by the Australian Research Council under the Discovery Projects.

M. Abdelrahim (✉) · R. Postoyan · J. Daafouz
Université de Lorraine, CRAN, UMR 7039 and CNRS, CRAN, UMR 7039, Vandoeuvre-les-Nancy, France
e-mail: othmanab1@univ-lorraine.fr

R. Postoyan
e-mail: romain.postoyan@univ-lorraine.fr

J. Daafouz
e-mail: jamal.daafouz@univ-lorraine.fr

D. Nešić
Department of Electrical and Electronic Engineering,
University of Melbourne, Parkville, VIC 3010, Australia
e-mail: dnesic@unimelb.edu.au

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_7

inequalities (LMI). We then exploit the flexibility of the method to maximize the guaranteed minimum amount of time between two transmissions. The results are illustrated on physical and numerical examples.

7.1 Introduction

In many control applications, nowadays, the plant and the controller communicate with each other via a shared communication network. This architecture is referred to as *Networked Control Systems* (NCS) and offers great benefits compared to the conventional point-to-point connection in terms of lighter wiring, lower installation costs, flexible reconfiguration and ease of maintenance. A major challenge in NCS is to achieve the control objectives despite the communication constraints induced by the network (like time-varying sampling, delay, packet dropout, etc.). Since the network may be shared by other applications, it is desirable in practice to reduce the usage of the network.

In conventional setups, data transmissions are time-driven and two successive transmission instants are constrained to be less than a fixed constant, called the *maximum allowable transmission interval* (MATI) (see e.g. [16, 26]). Although this strategy is appealing from the analysis and the implementation point of views, it is not obvious that time-triggering is always appropriate for NCS. Indeed, the same amount of transmissions per unit of time is generated under this paradigm, even when transmissions are not necessary in view of the control objectives. To overcome this shortcoming, event-triggered control has been proposed as an alternative [6, 7].

Event-triggered control is an implementation technique in which the transmission instants are defined based on a state-dependent criterion. The idea is to adapt the amount of transmissions according to the system state such that the feedback loop is closed only when it is needed in view of the stability and/or performance requirements. This may significantly reduce the amount of transmissions compared to the time-triggering paradigm, see e.g. [12, 20, 24] and the references therein. A fundamental issue in the event-triggered implementation is to ensure the existence of a uniform strictly positive lower bound on the inter-transmission times. The existence of such a lower bound on the inter-transmission times is not only useful to prove stability but this requirement is also essential to prevent the occurrence of Zeno phenomenon, i.e. to avoid the generation of an infinite number of transmissions in a finite time. Moreover, the existence of this lower bound is required in practice in order to respect the hardware constraints.

Most of the existing results on event-triggered control assume that the full state measurement is available and can be used for feedback, see e.g. [12, 15, 21]. This is not realistic in many applications since in practice we often have access to an output of the plant and not to the full state. The design of output feedback event-triggered controllers is much more challenging, in particular because it is more difficult to ensure the existence of a minimum amount of time between two control input updates compared to the state feedback case, see [8]. Few results in the literature have addressed this problem and mostly for linear systems. To the best of our knowledge, this problem has been first investigated in [14] and then in e.g. [8, 9, 18, 25, 28] for LTI systems and only in [27] for nonlinear systems.

The purpose of this chapter is to explain how to synthesize stabilizing output feedback event-triggered controllers for a class of nonlinear systems. We first design the event-triggered controllers by emulation, i.e. a stabilizing feedback law is first constructed in the absence of network and then the triggering condition is synthesized to preserve stability. The design objectives are to guarantee a global asymptotic stability property and to ensure the existence of a uniform strictly positive lower bound on the inter-transmission times. The proposed strategy combines the event-triggering condition of [24] adapted to output measurements and the results on time-driven sampled-data systems in [17]. Indeed, the event-triggering condition is only (continuously) evaluated after T units of times have elapsed since the last transmission, where T corresponds to the MATI given by [17]. This two-step procedure is justified by the fact that the adaptation of the event-triggering condition of [24] to output feedback on its own can lead to the Zeno phenomenon. It has to be noted that the event-triggering mechanism that we propose is different from the periodic event-triggered control paradigm, see e.g. [11, 19], where the triggering condition is verified only at some periodic sampling instants. In our case, the triggering mechanism is *continuously* evaluated, once T units of time have elapsed since the last transmission. For LTI systems, the required conditions are reformulated as an LMI which is shown to be always feasible for LTI systems that are stabilizable and detectable. To further reduce transmissions, we start from the results obtained by emulation to develop an LMI-based co-design procedure to simultaneously design the output feedback law and the event-triggering condition for LTI systems. We have then exploited these LMIs to enlarge the guaranteed minimum inter-transmission time. The results are demonstrated on illustrative examples. This chapter summarizes our results in [1–4], where the interested reader will find the proofs as well as additional results and examples.

7.2 Emulation Design for Nonlinear Systems

In this section, we explain how to synthesize stabilizing output feedback event-triggered controllers for a class of nonlinear systems by emulation. After deriving the hybrid model of the closed-loop system, we recall on an illustrative example taken from [8] the main issue with output feedback event-triggered controllers which does not allow for straightforward extension of the existing results on state feedback control. Then, we present the technical assumptions we impose on the nonlinear system and we introduce the triggering condition. We then state the main stability result. Finally, we demonstrate the technique on a single-link robot arm model.

7.2.1 Hybrid Model

Consider the nonlinear plant model

$$\dot{x}_p = f_p(x_p, u), \qquad y = g_p(x_p), \tag{7.1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output of the plant. We first ignore the communication constraints and we focus on general dynamic controllers of the form

$$\dot{x}_c = f_c(x_c, y), \qquad u = g_c(x_c, y), \tag{7.2}$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state. We emphasize that the x_c-system is not necessarily an observer. Moreover, (7.2) captures static feedback laws as a particular case by setting $u = g_c(y)$. We follow an emulation approach in this section. Hence, we assume that the controller (7.2) renders the origin of system (7.1) globally asymptotically stable in the absence of network. Afterwards, we take into account the communication constraints in the sense that the plant output and the control input are sent only at transmission instants t_i, $i \in \mathbb{Z}_{\geq 0}$. We are interested in an event-triggered implementation in the sense that the sequence of transmission instants is determined by a criterion based on the output measurement, see Fig. 7.1. At each transmission instant, the plant output is sent to the controller which computes a new control input that is instantaneously transmitted to the plant. We assume that this process is performed in a synchronous manner and we ignore the computation times and the possible transmission delays. In that way, we obtain

$$\begin{array}{lll} \dot{x}_p = f_p(x_p, \hat{u}), & \dot{x}_c = f_c(x_c, \hat{y}) & t \in [t_i, t_{i+1}] \\ \dot{\hat{y}} = 0, & \dot{\hat{u}} = 0 & t \in [t_i, t_{i+1}] \\ \hat{y}(t_i^+) = y(t_i), & \hat{u}(t_i^+) = u(t_i), & u = g_c(x_c, \hat{y}), \end{array}$$

where $\hat{y}$ and $\hat{u}$, respectively, denote the last transmitted values of the plant output and the control input. We assume that zero-order-hold devices are used to generate the sampled values $\hat{y}$ and $\hat{u}$, which leads to $\dot{\hat{y}} = 0$ and $\dot{\hat{u}} = 0$. We introduce the network-induced error $e := (e_y, e_u) \in \mathbb{R}^{n_e}$, where $e_y := \hat{y} - y$ and $e_u := \hat{u} - u$ which are

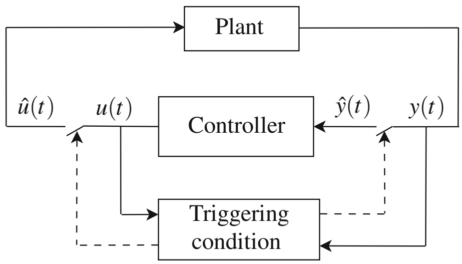

Fig. 7.1 Event-triggered control schematic [8]

reset to 0 at each transmission instant. We notice that the closed-loop system is a hybrid dynamical model since it combines continuous-time evolutions (the plant and the controller dynamics) and discrete phenomena (transmissions). We model the event-triggered control system using the hybrid formalism of [10] as in e.g. [8, 9, 20, 23] for which a jump corresponds to a transmission. In that way, the system is modelled as

$$\begin{bmatrix} \dot{x} \\ \dot{e} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} f(x,e) \\ g(x,e) \\ 1 \end{bmatrix} \quad (x,e,\tau) \in C, \qquad \begin{bmatrix} x^+ \\ e^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \quad (x,e,\tau) \in D, \quad (7.3)$$

where $x := (x_p, x_c) \in \mathbb{R}^{n_x}$ and $\tau \in \mathbb{R}_{\geq 0}$ is a clock variable which describes the time elapsed since the last jump, $f(x,e) = [f_p(x_p, g_c(x_c, y+e_y) + e_u),\ f_c(x_c, y + e_y)]$ and $g(x,e) = [-\frac{\partial}{\partial x_p} g_p(x_p) f_p(x_p, g_c(x_c, y+e_y) + e_u),\ -\frac{\partial}{\partial x_c} g_c(x_c, y+e_y)$ $f_c(x_c, y+e_y)]$.

The flow and the jump sets of (7.3) are defined according to the triggering condition we will define. As long as the triggering condition is not violated, the system flows on C and a jump occurs when the state enters in D. When $(x, e, \tau) \in C \cap D$, the solution may flow only if flowing keeps (x, e, τ) in C, otherwise the system experiences a jump. The functions f and g are assumed to be continuous and the sets C and D will be closed (which ensure that system (7.3) is well-posed, see Chap. 6 in [10]). We briefly recall some basic notions related to the hybrid formalism of [10].

The solutions to system (7.3) are defined on so-called hybrid time domains. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact hybrid time domain* if $E = \cup([t_j, t_{j+1}], j)$ for $j \in \{0, ..., J-1\}$ and for some finite sequence of times $0 = t_0 \leq t_1 \leq ... \leq t_J$, and it is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ..., J\})$ is a compact hybrid time domain. A function $\phi : E \to \mathbb{R}^n$ is a hybrid arc if E is a hybrid time domain and if for each $j \in \mathbb{Z}_{\geq 0}$, $t \mapsto \phi(t, j)$ is locally absolutely continuous on $I^j := \{t : (t, j) \in E\}$. A hybrid arc ϕ is a solution to system (7.3) if: (i) $\phi(0,0) \in C \cup D$; (ii) for any $j \in \mathbb{Z}_{\geq 0}$, $\phi(t, j) \in C$ and $\dot{\phi}(t, j) = F(\phi(t, j))$ for almost all $t \in I^j$; (iii) for every $(t, j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$, $\phi(t, j) \in D$ and $\phi(t, j+1) = G(\phi(t, j))$. A solution ϕ to system (7.3) is *maximal* if it cannot be extended, *complete* if its domain, $\operatorname{dom}\phi$, is unbounded, and it is *Zeno* if it is complete and $\sup_t \operatorname{dom}\phi < \infty$.

7.2.2 Motivational Example [8]

Before presenting our results, we first explain the issue with output-based event-triggered controllers which prevents the direct extension of state feedback results. To clarify the problem, we recall the numerical example in [8] where an LTI plant model is given by

$$\dot{x}_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = [-1 \ \ 4] \ x_p, \tag{7.4}$$

where $x \in \mathbb{R}^2$ is the plant state, $u \in \mathbb{R}$ is the control input and $y \in \mathbb{R}$ is the output of the plant. Let us first consider the state feedback case as in [24]. In the absence of network, the closed-loop system can be stabilized by the state feedback controller $u = [1 \quad -4]x_p$. By taking into account the effect of the network, we define the network-induced error as $e(t) = x(t_i) - x(t)$ for almost all $t \in [t_i, t_{i+1}]$. It has been shown in [24] that the triggering condition

$$|e| \leq \sigma |x|, \tag{7.5}$$

for some sufficiently small $\sigma > 0$, guarantees an asymptotic stability property for the closed-loop while the inter-transmission times are lower bounded by a strictly positive lower bound, under some conditions as illustrated in Figs. 7.2 and 7.3.

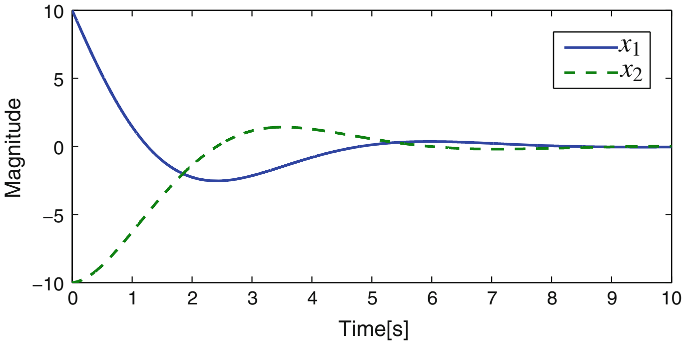

Fig. 7.2 State trajectories of the plant

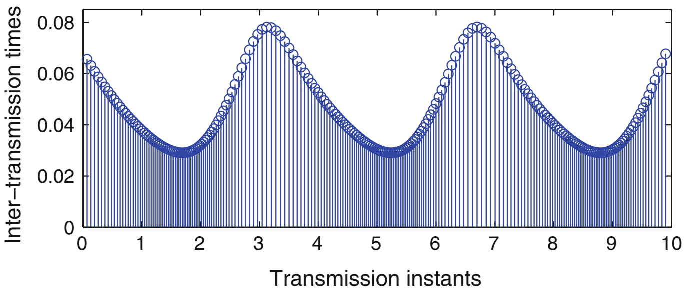

Fig. 7.3 Inter-transmission times

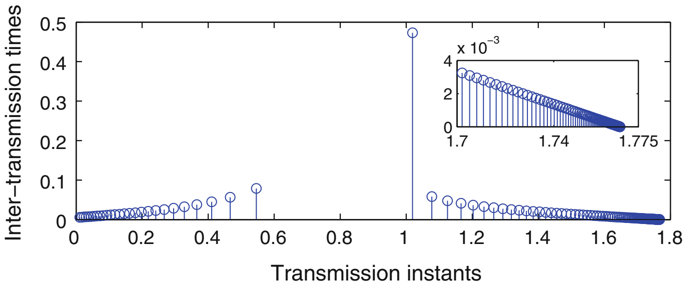

Fig. 7.4 Inter-transmission times with a zoom-in of the last transmissions

We recall now the output feedback case in [8] where system (7.4) is stabilized by the following dynamic controller

$$\dot{x}_c = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y, \qquad u = [1 \quad -4]\; x_c, \tag{7.6}$$

where $x_c \in \mathbb{R}^2$ is the state of the dynamic controller. Define the network-induced error as $e_y(t) = y(t_i) - y(t)$ for almost all $t \in [t_i, t_{i+1}]$. The straightforward extension of the triggering condition (7.5) yields

$$|e_y| \leq \sigma |y|. \tag{7.7}$$

Unfortunately, this triggering rule is not suitable since when $y = 0$, an infinite number of jumps occurs for any value of $x_p \neq 0$. This situation is shown in Fig. 7.4 where we note that the transmission instants accumulate at $t = 1.7674$. In [8], this issue was overcome by adding a constant to the triggering condition which leads to

$$|e_y| \leq \sigma |y| + \varepsilon, \tag{7.8}$$

for some $\varepsilon > 0$, from which a practical stability property is derived, i.e. the state trajectory converges to a neighbourhood to the origin whose 'size' depends on the parameter ε.

In this chapter, we aim to design the flow and the jump sets of system (7.3), i.e. the triggering condition, such that a global asymptotic stability property is guaranteed and the number of transmissions is reduced, while ensuring the existence of a strictly positive lower bound on the inter-transmission times.

7.2.3 Stability Results

We first make the following assumption on system (7.3), which is inspired by [17].

Assumption 1 There exist locally Lipschitz positive definite functions $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ and $W : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, real numbers

$L \geq 0$, $\gamma > 0$, $\underline{\alpha}, \overline{\alpha} \in \mathscr{K}_\infty$[1] and continuous, positive definite functions $\delta : \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}$ and $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathbb{R}^{n_x}$

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|), \tag{7.9}$$

for all $e \in \mathbb{R}^{n_e}$ and almost all $x \in \mathbb{R}^{n_x}$

$$\langle \nabla V(x), f(x,e) \rangle \leq -\alpha(|x|) - H^2(x) - \delta(y) + \gamma^2 W^2(e) \tag{7.10}$$

and for all $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$

$$\langle \nabla W(e), g(x,e) \rangle \leq LW(e) + H(x). \tag{7.11}$$

□

Conditions (7.9) and (7.10) imply that the system $\dot{x} = f(x,e)$ is $\mathscr{L}_2$-gain stable from W to $(H, \sqrt{\delta})$. This property can be analysed by investigating the robustness property of the closed-loop system (7.1) and (7.2) with respect to input and/or output measurement errors in the absence of sampling. Note that, since W is positive definite and continuous (since it is locally Lipschitz), there exists $\chi \in \mathscr{K}_\infty$ such that $W(e) \leq \chi(|e|)$ (according to Lemma 4.3 in [13]) and hence (7.9), (7.10) imply that the system $\dot{x} = f(x,e)$ is input-to-state stable (ISS). We also assume an exponential growth condition of the e-system on flows in (7.11) which is similar to the one used in [17].

Under Assumption 1, the adaptation of the idea of [24] leads to a triggering condition of the form

$$\gamma^2 W^2(e) \leq \delta(y). \tag{7.12}$$

The problem is that Zeno phenomenon may occur with this type of triggering conditions as explained in Sect. 7.2.2. We propose instead to evaluate the event-triggering condition only after T units have elapsed since the last transmission, where T corresponds to the MATI given by [17]. We thus redesign the triggering condition as follows:

$$\gamma^2 W^2(e) \leq \delta(y) \text{ or } \tau \in [0, T], \tag{7.13}$$

where we recall that $\tau \in \mathbb{R}_{\geq 0}$ is the clock variable introduced in (7.3). Consequently, the flow and the jump sets of system (7.3) are

$$\begin{aligned} C &= \Big\{(x,e,\tau) : \gamma^2 W^2(e) \leq \delta(y) \text{ or } \tau \in [0,T]\Big\} \\ D &= \Big\{(x,e,\tau) : \Big(\gamma^2 W^2(e) = \delta(y) \text{ and } \tau \geq T\Big) \text{ or } \Big(\gamma^2 W^2(e) \geq \delta(y) \text{ and } \tau = T\Big)\Big\}. \end{aligned} \tag{7.14}$$

[1]A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{K}$ if it is zero at zero, strictly increasing, and it is of class $\mathscr{K}_\infty$ if in addition $\gamma(s) \to \infty$ as $s \to \infty$.

Hence, the inter-jump times are uniformly lower bounded by T. This constant is selected such that $T < \mathcal{T}(\gamma, L)$, where

$$\mathcal{T}(\gamma, L) := \begin{cases} \frac{1}{Lr}\arctan(r), & \gamma > L, \\ \frac{1}{L}, & \gamma = L, \\ \frac{1}{Lr}\text{arctanh}(r), & \gamma < L, \end{cases} \tag{7.15}$$

with $r := \sqrt{\left|(\frac{\gamma}{L})^2 - 1\right|}$ and L, γ come from Assumption 1 as in [17]. We are ready to state the main result.

Theorem 2 *Suppose that Assumption 1 holds and consider system (7.3) with the flow and the jump sets (7.14), where the constant T is such that $T \in (0, \mathcal{T}(\gamma, L))$. There exists*[2] *$\beta \in \mathcal{KL}$ such that any solution $\phi = (\phi_x, \phi_e, \phi_\tau)$ with $(\phi_x(0,0), \phi_e(0,0)) \in \mathbb{R}^{n_x+n_e}$ satisfies*

$$|\phi_x(t, j)| \leq \beta(|(\phi_x(0,0), \phi_e(0,0))|, t + j) \quad , \forall (t, j) \in dom\, \phi, \tag{7.16}$$

furthermore, if ϕ is maximal, then it is complete. □

Property (7.16) indicates that the state trajectory of the x-system asymptotically converges to the origin while the completeness property implies that the hybrid time domains of the maximal solutions to (7.3) are unbounded. That is equivalent to forward completeness for continuous-time ordinary differential equations [5].

7.2.4 *Illustrative Example*

Consider the dynamics of a single-link robot arm

$$\dot{x}_{p1} = x_{p2}, \qquad \dot{x}_{p2} = -\sin(x_{p1}) + u, \qquad y = x_{p1}, \tag{7.17}$$

where x_{p1} denotes the angle, x_{p2} the rotational velocity and u the input torque. The system can be written as

$$\dot{x}_p = Ax_p + Bu - \phi(y), \qquad y = Cx_p, \tag{7.18}$$

[2] A continuous function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each $t \in \mathbb{R}_{\geq 0}$, $\gamma(., t)$ is of class $\mathcal{K}$, and, for each $s \in \mathbb{R}_{\geq 0}$, $\gamma(s, .)$ is decreasing to zero.

where $x_p = (x_{p1}, x_{p2})$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T$, $\phi(y) = \begin{bmatrix} 0 \\ \sin(y) \end{bmatrix}$. In order to stabilize system (7.18), we first construct a state feedback controller of the form $u = Kx_p + B^T\phi(y)$. Hence, system (7.17) reduces to

$$\dot{x}_p = (A + BK)x_p, \qquad y = Cx_p. \tag{7.19}$$

We design the gain K such that the eigenvalues of the closed-loop system (7.19) are $(-1, -2)$ (which is possible since the pair (A, B) is controllable). Hence, the gain K is selected to be $K = [-2 \quad -3]$. Next, since only the measurement of y is available, we construct a state-observer of the following form

$$\begin{aligned} \dot{x}_c &= Ax_c + Bu - \phi(y) + M(y - Cx_c) \\ &= (A - MC)x_c + Bu - \phi(y) + My, \end{aligned} \tag{7.20}$$

where $x_c \in \mathbb{R}^2$ is the estimated state and M is the observer gain matrix. We design the gain matrix M such that the eigenvalues of $(A - MC)$ are $(-5, -6)$ (which is possible since the pair (A, C) is observable). Thus, the observer gain is selected to be $M = [11 \quad 30]^T$. As a result, the closed-loop system in the absence of sampling is given by

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu - \phi(y), & y &= Cx_p \\ \dot{x}_c &= (A - MC)x_c + Bu - \phi(y) + My, & u &= Kx_c + B^T\phi(y). \end{aligned} \tag{7.21}$$

We now take into account the effect of the network. We consider the scenario where the controller receives the output measurements only at transmission instants t_i, $i \in \mathbb{Z}_{\geq 0}$ while the controller is directly connected to the plant actuators. We design a triggering condition of the form (7.13). As a consequence, the network-induced error is $e = e_y = \hat{y} - y$ and we obtain, for almost all $t \in [t_i, t_{i+1}]$

$$\begin{aligned} \dot{x}_p &= Ax_p + B\Big(Kx_c + B^T\phi(\hat{y})\Big) - \phi(y) \\ \dot{x}_c &= (A - MC + BK)x_c + MCx_p + Me_y. \end{aligned} \tag{7.22}$$

Let $x = (x_p, x_c)$. Then, system (7.22) can be written as follows

$$\begin{aligned} \dot{x} &= \begin{bmatrix} A & BK \\ MC & A - MC + BK \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ M \end{bmatrix} e + \begin{bmatrix} \phi(y + e) - \phi(y) \\ 0 \end{bmatrix} \\ &=: \mathscr{A}x + \mathscr{B}e + \psi(y, e). \end{aligned} \tag{7.23}$$

Since $e = \hat{y} - y$ and in view of (7.17), we have $\dot{e} = -\dot{y} = -x_{p2}$. Hence, the functions f, g in (7.3) are $f(x, e) = \mathscr{A}x + \mathscr{B}e + \psi(y, e)$ and $g(x, e) = -x_{p2}$. By taking $W(e) = |e|$ and $V(x) = x^T Px$, all conditions in Assumption 1 are satisfied. We obtain the numerical values $L = 0$, $\gamma = 26.5333$, which give, in view of

(7.15), $\mathscr{T} = 0.0592$. We take $T = 0.059$. Figure 7.5 shows that the plant and the estimated state asymptotically converge to the origin as expected. The generated inter-transmission times are shown in Fig. 7.6 where we can observe the interaction between the time-triggered and the event-triggered criteria. We ran simulations for 200 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$. The obtained minimum and average inter-transmission times, respectively, denoted as $\tau_{\min}$ and τ_{avg} are $\tau_{\min} = 0.059$ and $\tau_{\text{avg}} = 0.0625$. The constant τ_{avg} serves as a measure of the amount of transmissions (the bigger τ_{avg}, the less transmissions). Figure 7.7 presents the inter-transmission times with the triggering condition $\gamma^2 W^2(e) \leq \delta(y)$ without enforcing a constant time T between transmissions, (i.e. $T = 0$ in (7.13), (7.14)). We note that Zeno phenomenon occurs in this case as discussed in Sect. 7.2.2.

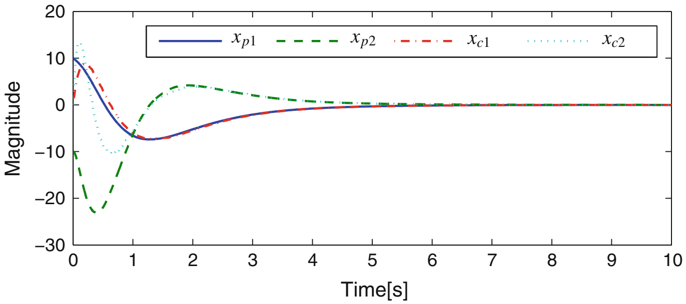

Fig. 7.5 Actual and estimated states of the plant

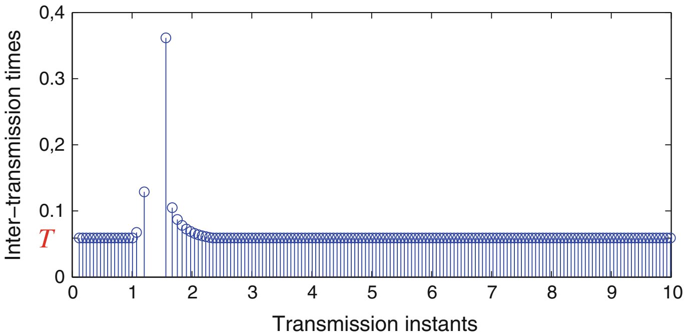

Fig. 7.6 Inter-transmission times

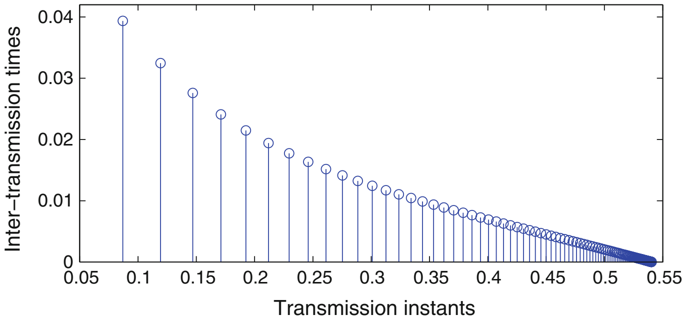

Fig. 7.7 Inter-transmission times with [24]

7.2.5 Application to Linear Time-Invariant Systems

We now focus on the particular case of linear systems. We formulate the required conditions in Assumption 1 as an LMI constraint. Consider the LTI plant model

$$\dot{x}_p = A_p x_p + B_p u, \qquad y = C_p x_p, \tag{7.24}$$

where $x_p \in \mathbb{R}^{n_p}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$ and A_p, B_p, C_p are matrices of appropriate dimensions. We design the following dynamic controller to stabilize (7.24) in the absence of sampling

$$\dot{x}_c = A_c x_c + B_c y, \qquad u = C_c x_c + D_c y, \tag{7.25}$$

where $x_c \in \mathbb{R}^{n_c}$ and A_c, B_c, C_c, D_c are matrices of appropriate dimensions. Afterwards, we take into account the communication constraints. Then, the hybrid model (7.3) is

$$\begin{bmatrix} \dot{x} \\ \dot{e} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} \mathscr{A}_1 x + \mathscr{B}_1 e \\ \mathscr{A}_2 x + \mathscr{B}_2 e \\ 1 \end{bmatrix} \quad (x, e, \tau) \in C, \qquad \begin{bmatrix} x^+ \\ e^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \quad (x, e, \tau) \in D, \tag{7.26}$$

where

$$\mathscr{A}_1 := \begin{bmatrix} A_p + B_p D_c C_p & B_p C_c \\ B_c C_p & A_c \end{bmatrix}, \qquad \mathscr{B}_1 := \begin{bmatrix} B_p D_c & B_p \\ B_c & 0 \end{bmatrix},$$
$$\mathscr{A}_2 := \begin{bmatrix} -C_p(A_p + B_p D_c C_p) & -C_p B_p C_c \\ -C_c B_c C_p & -C_c A_c \end{bmatrix}, \quad \mathscr{B}_2 := \begin{bmatrix} -C_p B_p D_c & -C_p B_p \\ -C_c B_c & 0 \end{bmatrix}.$$

We obtain the following result.

Proposition 1 *Consider system (7.26). Suppose that there exist* $\varepsilon_1, \varepsilon_2, \mu > 0$ *and a positive definite symmetric real matrix* P *such that*

$$\begin{bmatrix} \mathscr{A}_1^T P + P\mathscr{A}_1 + \varepsilon_1 \mathbb{I}_{n_x} + \mathscr{A}_2^T \mathscr{A}_2 + \varepsilon_2 \overline{C}_p^T \overline{C}_p & P\mathscr{B}_1 \\ \mathscr{B}_1^T P & -\mu \mathbb{I}_{n_e} \end{bmatrix} \leq 0, \tag{7.27}$$

where $\overline{C}_p = [C_p \quad 0]$ *and 0 represents the matrix of zeros of size* $n_y \times n_c$. *Then Assumption 1 globally holds with* $V(x) = x^T P x$, $\underline{\alpha}(s) = \lambda_{\min}(P)s^2$, $\overline{\alpha}(s) = \lambda_{\max}(P)s^2$, $W(e) = |e|$, $H(x) = |\mathscr{A}_2 x|$, $L = |\mathscr{B}_2|$, $\gamma = \sqrt{\mu}$, $\alpha(s) = \varepsilon_2 s^2$ *and* $\delta(y) = \varepsilon_1 |y|^2$, *for* $s \geq 0$. □

Note that, in view of (7.14) and Proposition 1, the flow and the jump sets become

$$\begin{aligned} C &= \left\{(x, e, \tau) : \gamma^2|e|^2 \leq \varepsilon_1|y|^2 \text{ or } \tau \in [0, T]\right\} \\ D &= \left\{(x, e, \tau) : \left(\gamma^2|e|^2 = \varepsilon_1|y|^2 \text{ and } \tau \geq T\right) \text{ or } \left(\gamma^2|e|^2 \geq \varepsilon_1|y|^2 \text{ and } \tau = T\right)\right\}. \end{aligned} \tag{7.28}$$

7.3 Co-design for LTI Systems

In Sect. 7.2, we have assumed that the feedback control law was known in the absence of network, then we synthesized the triggering condition. This sequential order of design may prevent an efficient usage of the computation and communication resources as we are restricted by the initial choice of the feedback law. To overcome this limitation, in this section, we use the triggering condition designed in Sect. 7.2.5 for LTI systems as a starting point to simultaneously design the event-triggering condition and the feedback law.

Consider the LTI plant model (7.24) and the dynamic controller (7.25). For the sake of simplicity, we design the dynamic controller (7.25) with $D_c = 0$ and we obtain the hybrid model (7.26). Our objective is to design the dynamic controller (7.25) and the flow and the jump sets (7.28) of the hybrid system (7.26) such that the conclusions of Theorem 2 hold. The idea is to start from LMI (7.27) to establish an LMI-based co-design procedure of both the flow and the jump sets (7.28) and the dynamic controller (7.25). It is important to note that the derivation of LMI for co-design from (7.27) is not trivial as the nonlinear term $\mathscr{A}_2^T \mathscr{A}_2$ depends on the controller matrices. This term does not appear in the classical output feedback design problems and cannot be directly handled by congruence transformations like in standard output feedback design problems [22].

7.3.1 Analytical Result

The following theorem reduces the co-design problem of the output feedback law (7.25) and the parameters of the flow and the jump sets (7.28) to the solution of LMI. We use boldface symbols to emphasize the LMI decision variables.

Theorem 3 *Consider system (7.26) with the flow and the jump sets (7.28). Suppose that there exist symmetric positive definite real matrices* $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n_p \times n_p}$, *real matrices* $\boldsymbol{M} \in \mathbb{R}^{n_p \times n_p}$, $\boldsymbol{Z} \in \mathbb{R}^{n_p \times n_y}$, $\boldsymbol{N} \in \mathbb{R}^{n_u \times n_p}$ *and* $\boldsymbol{\varepsilon}, \boldsymbol{\mu} > 0$ *such that*[3]

$$\begin{bmatrix} \Sigma(\boldsymbol{Y}A_p + \boldsymbol{Z}C_p) & \star & \star & \star & \star & \star & \star \\ A_p + \boldsymbol{M}^T & \Sigma(A_p\boldsymbol{X} + B_p\boldsymbol{N}) & \star & \star & \star & \star & \star \\ \boldsymbol{Z}^T & 0 & -\boldsymbol{\mu}\mathbb{I}_{n_y} & \star & \star & \star & \star \\ B_p^T\boldsymbol{Y} & B_p^T & 0 & -\boldsymbol{\mu}\mathbb{I}_{n_u} & \star & \star & \star \\ \boldsymbol{Y}A_p + \boldsymbol{Z}C_p & \boldsymbol{M} & 0 & 0 & -\boldsymbol{Y} & \star & \star \\ A_p & A_p\boldsymbol{X} + B_p\boldsymbol{N} & 0 & 0 & -\mathbb{I}_{n_p} & -\boldsymbol{X} & \star \\ C_p & C_p\boldsymbol{X} & 0 & 0 & 0 & 0 & -\boldsymbol{\varepsilon}\mathbb{I}_{n_y} \end{bmatrix} < 0, \tag{7.29}$$

$$\begin{bmatrix} -\mathbb{I}_{n_y} & \star & \star & \star \\ 0 & -\mathbb{I}_{n_u} & \star & \star \\ -C_p^T & 0 & -\boldsymbol{Y} & \star \\ -\boldsymbol{X}C_p^T & -\boldsymbol{N}^T & -\mathbb{I}_{n_p} & -\boldsymbol{X} \end{bmatrix} < 0. \tag{7.30}$$

Take $\gamma = \sqrt{\boldsymbol{\mu}}$, $L = |\mathscr{B}_2|$, $\varepsilon_1 = \boldsymbol{\varepsilon}^{-1}$ *and*

$$\begin{aligned} A_c &= V^{-1}(\boldsymbol{M} - \boldsymbol{Y}A_p\boldsymbol{X} - \boldsymbol{Y}B_p\boldsymbol{N} - \boldsymbol{Z}C_p\boldsymbol{X})U^{-T} \\ B_c &= V^{-1}\boldsymbol{Z}, \qquad C_c = \boldsymbol{N}U^{-T}, \end{aligned} \tag{7.31}$$

where $U, V \in \mathbb{R}^{n_p \times n_p}$ *are any invertible matrices such that*[4] $UV^T = \mathbb{I}_{n_p} - \boldsymbol{X}\boldsymbol{Y}$.

Then, there exists $\chi \in \mathcal{KL}$ *such that any solution* $\phi = (\phi_x, \phi_e, \phi_\tau)$ *satisfies*

$$|\phi_x(t, j)| \leq \chi(|(\phi_x(0, 0), \phi_e(0, 0))|, t + j) \quad \forall (t, j) \in dom\,\phi \tag{7.32}$$

and, if ϕ *is maximal, it is also complete.* □

We note that by solving the LMI (7.29), (7.30), which are computationally tractable, we obtain the feedback law, see (7.31), and the parameters of the triggering condition $\gamma^2|e|^2 \leq \varepsilon_1|y|^2$ or $\tau \in [0, T]$. We note also that the nonstandard

---

[3] The symbol $\star$ denotes symmetric blocks while $\Sigma(.)$ stands for $(.) + (.)^T$.

[4] In view of the Schur complement of LMI (7.30), we deduce that $\begin{bmatrix} \boldsymbol{Y} & \mathbb{I}_{n_p} \\ \mathbb{I}_{n_p} & \boldsymbol{X} \end{bmatrix} > 0$ which implies that $\boldsymbol{X} - \boldsymbol{Y}^{-1} > 0$ and thus, $\mathbb{I}_{n_p} - \boldsymbol{X}\boldsymbol{Y}$ is nonsingular. Hence, the existence of nonsingular matrices U, V is always ensured.

term $\mathscr{A}_2^T\mathscr{A}_2$ in (7.27) is the reason why the constructed LMI (7.29) differs from the classical one and why the additional convex constraint (7.30) is needed in Theorem 3.

7.3.2 Optimization

Although the existence of strictly positive lower bound on the inter-transmission times is guaranteed by different techniques in the literature, the available expressions are often subject to some conservatism. It is therefore unclear whether the event-triggered controller has a dwell-time which is compatible with the hardware limitations. We investigate in this section how to employ the LMI conditions (7.29), (7.30) to maximize the guaranteed minimum inter-transmission time in order to increase the implementability of the event-triggered controller. We first state the following lemma to motivate our approach.

Lemma 1 *Let $\mathscr{S}$ be the set of solutions to system (7.26), (7.28). It holds that*

$$T = \inf_{\phi\in\mathscr{S}}\{t'-t : \exists j \in \mathbb{Z}_{>0},\ \ (t,j),(t,j+1),(t',j+1),(t',j+2)\in dom\,\phi\}. \tag{7.33}$$

□

Lemma 1 implies that the lower bound T on the inter-transmission times guaranteed by (7.28) corresponds to the actual minimum inter-transmission time as defined by the right-hand side of (7.33). Hence, by maximizing T, we enlarge the minimum inter-transmission time.

To maximize T, we will maximize $\mathscr{T}(\gamma, L)$ in (7.15). We see that $\mathscr{T}$ increases as γ and L decrease. Hence, our objective is to minimize γ and L. Since γ corresponds to $\sqrt{\mu}$ and μ enters linearly in the LMI (7.29), we can directly minimize γ under the LMI (7.29), (7.30). The minimization of L, on the other hand, requires more attention. We recall that $L = |\mathscr{B}_2| = \sqrt{\lambda_{\max}(\mathscr{B}_2^T\mathscr{B}_2)}$, where

$$\mathscr{B}_2^T\mathscr{B}_2 = \begin{bmatrix} B_c^TC_c^TC_cB_c & 0 \\ 0 & B_p^TC_p^TC_pB_p \end{bmatrix}, \tag{7.34}$$

hence,

$$L = \max\left(\sqrt{\lambda_{\max}(B_c^TC_c^TC_cB_c)}, \sqrt{\lambda_{\max}(B_p^TC_p^TC_pB_p)}\right). \tag{7.35}$$

Therefore, L can be minimized up to $\sqrt{\lambda_{\max}(B_p^TC_p^TC_pB_p)}$ which is fixed as it only depends on the plant matrices. In view of (7.31), we have that

$$B_c^TC_c^TC_cB_c = \boldsymbol{Z}^TV^{-T}U^{-1}\boldsymbol{N}^T\boldsymbol{N}U^{-T}V^{-1}\boldsymbol{Z}. \tag{7.36}$$

Thus, L depends nonlinearly on the LMI variables $\boldsymbol{N}$ and $\boldsymbol{Z}$ and it can a priori not be directly minimized. To overcome this issue, we impose the following upper bound

$$B_c^T C_c^T C_c B_c < \alpha\beta\mathbb{I}_{n_y}, \tag{7.37}$$

for some $\alpha, \beta > 0$. As a result, minimizing α and β may help to minimize L. We translate inequality (7.37) into an LMI and we state the following claim.

Claim *Assume that LMI (7.29), (7.30) are verified. Then, there exist* $\boldsymbol{\alpha}, \boldsymbol{\beta} > 0$ *such that*

$$\begin{bmatrix} \boldsymbol{\alpha}\mathbb{I}_{n_y} & \star & \star & \star \\ 0 & \boldsymbol{\beta}\mathbb{I}_{n_u} & \star & \star \\ 0 & \boldsymbol{N}^T & \boldsymbol{X} & \star \\ \boldsymbol{Z} & 0 & \mathbb{I}_{n_p} & \boldsymbol{Y} \end{bmatrix} > 0, \tag{7.38}$$

which implies that inequality (7.37) holds. □

We note that (7.38) does not introduce additional constraints on system (7.26) compared to (7.29), (7.30). This comes from the fact that there always exist $\alpha, \beta > 0$ (eventually large) such that (7.38) holds, in view of Schur complement of (7.38).

In conclusion, we formulate the problem as a multiobjective optimization problem as we want to minimize μ, α, β under the constraint (7.29), (7.30) and (7.38). Several approaches have been proposed in the literature to handle such problems. We choose the weighted sum strategy among others and we formulate the LMI optimization problem as follows

$$\begin{array}{c} \min \lambda_1\mu + \lambda_2\alpha + \lambda_3\beta \\ \text{subject to (7.29), (7.30), (7.38)} \end{array} \tag{7.39}$$

for some weights $\lambda_1, \lambda_2, \lambda_3 \geq 0$.

7.3.3 Illustrative Example

Consider the LTI plant model

$$\dot{x}_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & -5 & -6 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 10 \\ -50 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_p. \tag{7.40}$$

First, we solve the optimization problem (7.39) to seek for the largest possible lower bound on the inter-transmission times. We set $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and we obtain $T = 0.005$. Table 7.1 gives the minimum and the average inter-sampling times for 100 randomly distributed initial conditions such that $|(x(0, 0), e(0, 0))| \leq 100$ and $\tau(0, 0) = 0$. We observe from the corresponding entries in Table 7.1 that $\tau_{\min} = \tau_{\text{avg}}$ which implies that generated transmission instants are periodic. This may be

Table 7.1 Minimum and average inter-transmission times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$ for a simulation time of 10 s

	Guaranteed dwell-time	$\tau_{\min}$	τ_{avg}
Optimization problem (7.39) $\lambda_1 = \lambda_2 = \lambda_3 = 1$	0.0049	0.0049	0.0049
Optimization problem (7.41) $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 10^4$	0.0047	0.0047	0.0052
Emulation (7.27)	7.3389×10^{-6}	7.3389×10^{-6}	8.5396×10^{-4}

explained by the fact that the output-dependent part in (7.28), i.e. $\mu|e|^2 \leq \varepsilon^{-1}|y|^2$, is 'quickly' violated. To avoid that phenomenon, we optimize the parameters μ, ε such that the rule is violated after the longest possible time since the last transmission instant, see [4] for more detailed explanations. Thus, the problem can be formulated as follows

$$\begin{array}{c} \min \lambda_1\mu + \lambda_2\alpha + \lambda_3\beta + \lambda_4\varepsilon \\ \text{subject to (7.29), (7.30), (7.38)} \end{array} \tag{7.41}$$

for some weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

By playing with the weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we found that the best tradeoff between $\tau_{\min}$ and τ_{avg} is obtained with $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 10^4$, as shown in Table 7.1. Note that when we consider the same dynamic controller as in the case (7.41) and we design the triggering condition by emulation according to (7.27), we obtain the results shown in the last row of Table 7.1. We note that the co-design procedure yields larger lower bound on the inter-transmission times as well as larger average inter-transmission time, i.e. less amount of transmissions. Figures 7.8, 7.9, respectively, shows the state trajectory and a close up of the inter-transmission times generated by the event-triggered controller obtained by (7.41).

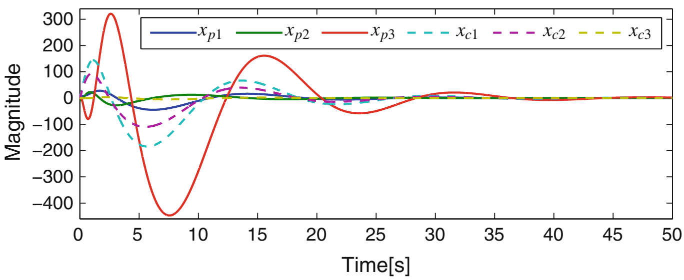

Fig. 7.8 State trajectories of the plant and of the dynamic controller

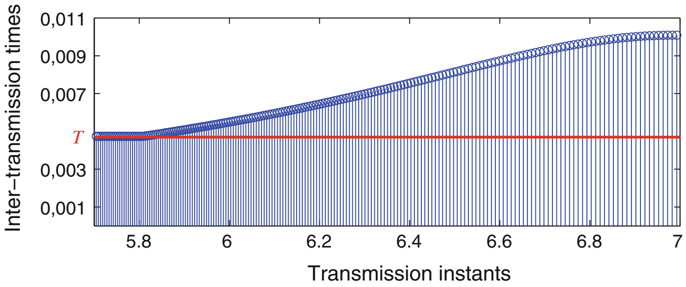

Fig. 7.9 Inter-transmission times

7.4 Conclusion

In this chapter, we have first developed output feedback event-triggered controllers to stabilize a class of nonlinear systems by following the emulation design approach. In emulation, the sequential order of the synthesis of event-triggered controllers restricts us with the initial choice of the stabilizing feedback law. One solution to increase the design flexibility is to simultaneously establish the feedback law and the event-triggering condition so that the stabilization and the communication constraints are handled at the same time. For this purpose, we have proposed an LMI-based co-design algorithm for LTI systems and we have then discussed how the resulted LMI can be exploited to optimize the parameters of the event-triggering condition. Future work will focus on the robustness of the triggering mechanism with respect to measurement noise and to consider some performance requirements.

References

1. M. Abdelrahim, Output feedback event-triggered control. Ph.D. thesis, Centre de Recherche en Automatique de Nancy (CRAN), Université de Lorraine (2014)
2. M. Abdelrahim, R. Postoyan, J. Daafouz, D. Nešić, Stabilization of nonlinear systems using event-triggered output feedback laws, in *International Symposium on Mathematics Theory of Networks and Systems (MTNS)* (2014)
3. M. Abdelrahim, R. Postoyan, J. Daafouz, D. Nešić, Stabilization of nonlinear systems using event-triggered output feedback laws. IEEE Trans. Autom. Control, Accepted in 2016
4. M. Abdelrahim, R. Postoyan, J. Daafouz, D. Nešić, Co-design of output feedback laws and event-triggering conditions for linear systems, in *IEEE Conference on Decision and Control (CDC)* (2014)
5. A. Angeli, E.D. Sontag, Forward completeness, unboundedness observability, and their Lyapunov characterizations. Syst. Control Lett. **38**(4), 209–217 (1999)
6. K.E. Årzén, A Simple event-based PID controller, in *IFAC World Congress* (1999)
7. K.J. Åström, B.M. Bernhardsson, Comparison of periodic and event based sampling for first order stochastic systems, in *IFAC World Congress* (1999)

8. M.C.F. Donkers, W.P.M.H. Heemels, Output-based event-triggered control with guaranteed $\mathscr{L}_\infty$-gain and improved and decentralised event-triggering. IEEE Trans. Autom. Control **57**(6), 1362–1376 (2012)
9. F. Forni, S. Galeani, D. Nešić, L. Zaccarian, Event-triggered transmission for linear control over communication channels. Automatica **50**(2), 490–498 (2014)
10. R. Goebel, R.G. Sanfelice, A.R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness* (Princeton University Press, 2012)
11. W.P.M.H. Heemels, M.C.F. Donkers, A.R. Teel, Periodic event-triggered control for linear systems. IEEE Trans. Autom. Control **58**(4), 847–861 (2013)
12. W.P.M.H. Heemels, K.H. Johansson, P. Tabuada, An introduction to event-triggered and self-triggered control, in *IEEE Conference on Decision and Control (CDC)* (2012)
13. H.K. Khalil, *Nonlinear Systems*, 3rd edn. (Prentice Hall, 2002)
14. E. Kofman, J.H. Braslavsky, Level crossing sampling in feedback stabilization under data-rate constraints, in *IEEE Conference on Decision and Control (CDC)* (2006)
15. Q. Liu, Z. Wang, X. He, Z.H. Zhou, A survey of event-based strategies on control and estimation. Syst. Sci. Control Eng. **2**(1), 90–97 (2014)
16. D. Nešić, A.R. Teel, Input-output stability properties of networked control systems. IEEE Trans. Autom. Control **49**(10), 1650–1667 (2004)
17. D. Nešić, A.R. Teel, D. Carnevale, Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems. IEEE Trans. Autom. Control **54**(3), 619–624 (2009)
18. C. Peng, Q. Han, Output-based event-triggered $\mathscr{H}_\infty$ control for sampled-data control systems with nonuniform sampling, in *American Control Conference (ACC)* (2013)
19. R. Postoyan, A. Anta, W.P.M.H. Heemels, P. Tabuada, D. Nešić, Periodic event-triggered control for nonlinear systems, in *IEEE Conference on Decision and Control (CDC)* (2013)
20. R. Postoyan, A. Anta, D. Nešić, P. Tabuada, A unifying Lyapunov-based framework for the event-triggered control of nonlinear systems, in *IEEE Conference on Decision and Control (CDC) and European Control Conference (ECC)* (2011)
21. R. Postoyan, P. Tabuada, D. Nešić, A. Anta, Event-triggered and self-triggered stabilization of distributed networked control systems, in *IEEE Conference on Decision and Control (CDC) and European Control Conference (ECC)* (2011)
22. C. Scherer, P. Gahinet, M. Chilali, Multiobjective output-feedback control via LMI optimization. IEEE Trans. Autom. Control **42**(7), 896–911 (1997)
23. A. Seuret, C. Prieur, N. Marchand, Stability of non-linear systems by means of event-triggered sampling algorithms. IMA Math. Control Inf. **31**(3), 415–433 (2014)
24. P. Tabuada, Event-triggered real-time scheduling of stabilizing control tasks. IEEE Trans. Autom. Control **52**(9), 1680–1685 (2007)
25. P. Tallapragada, N. Chopra, Event-triggered decentralized dynamic output feedback control for LTI systems. Estim. Control Netw. Syst. **3**(1), 31–36 (2012)
26. G.C. Walsh, O. Beldiman, L.G. Bushnell, Asymptotic behavior of nonlinear networked control systems. IEEE Trans. Control Syst. Technol. **46**(7), 1093–1097 (2001)
27. H. Yu, P.J. Antsaklis, Event-triggered output feedback control for networked control systems using passivity: achieving $\mathscr{L}_2$ stability in the presence of communication delays and signal quantization. Automatica **49**(1), 30–38 (2012)
28. X. Zhang, Q. Han, Event-based dynamic output feedback control for networked control systems, in *American Control Conference (ACC)* (2013)

Part III
Time-Delay Approaches in Networked Control Systems

Chapter 8
Stabilization by Quantized Delayed State Feedback

Francesco Ferrante, Frédéric Gouaisbaut and Sophie Tarbouriech

Abstract This chapter is devoted to the design of a static-state feedback controller for a linear system subject to saturated quantization and delay in the input. Due to quantization and saturation, we consider, for the closed-loop system, a weaker notion of stability, namely local ultimate boundedness. The closed-loop system is then modeled as a stable linear system subject to discontinuous perturbations. Then by coupling a certain Lyapunov–Krasovskii functional via S-procedure to adequate sector conditions, we derive sufficient conditions to ensure for the trajectories of the closed-loop system finite time convergence into a compact S_u surrounding the origin, from every initial condition belonging to a compact set S_0. Moreover, the size of the initial condition set S_0 and the ultimate set S_u are then optimized by solving a convex optimization problem over linear matrix inequality (LMI) constraints. Finally, an example extracted from the literature shows the effectiveness of the proposed methodology.

S. Tarbouriech—This work has been funded by the ANR under grant LIMICOS ANR-12-BS03-0005-01.

F. Ferrante (✉) · F. Gouaisbaut · S. Tarbouriech
CNRS, LAAS, 7 avenue du colonel Roche, F-31400 Toulouse, France
e-mail: francesco.ferrante.2011@ieee.org

F. Ferrante
Univ de Toulouse, ISAE, LAAS, F-31400 Toulouse, France

F. Gouaisbaut
Univ de Toulouse, UPS, LAAS, F-31400 Toulouse, France
e-mail: gouaisbaut@laas.fr

S. Tarbouriech
Univ de Toulouse, LAAS, F-31400 Toulouse, France
e-mail: tarbouriech@laas.fr

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_8

8.1 Introduction

Networked control systems have been extensively studied due to their practical interest (see for instance [12]) and also because the introduction of communication networks in control loops poses new theoretical challenges. Indeed, the constraints induced by networks can seriously degrade the closed-loop system performances. Among all these constraints, two of them are particularly important: quantization and delay. For a complete overview of the issues arising in networked control systems see; [10, 12, 15].

Quantization is a phenomenon occurring in all data networks, where a real-valued signal is transformed into a piecewise constant signal. Whenever quantization is uniform, global asymptotic stability of the closed-loop cannot be established; see [6]. In particular, for large signals, the quantizer saturates and as a consequence, global stabilization cannot be obtained except for open-loop stable systems; see [17]. Furthermore, as uniform quantizers manifest a dead-zone around the zero, in general, local asymptotic stability for the origin cannot be guaranteed; see [2, 3]. Several methods have been proposed in the literature to deal with quantized systems. One first method to cope with quantized closed-loop systems consists of adopting a robust control point of view. Namely, the closed-loop system is modeled as a nominal system perturbed by a perturbation, i.e., the quantization error. Then, by using sector conditions and classical tools like small gain theorem or Lyapunov functions coupled with Input-to-State stability (ISS) properties [2, 3, 14] or S-procedure [1, 8], closed-loop stability can be assessed [10, 18].

Time delays naturally occur in networked control systems due to finite-time propagation of signals in communication media. Clearly, the presence of time delays can induce an additional degradation of the closed-loop performances; see [11]. In that case, the idea generally proposed is to extend classical results (without delay) to account time delay by considering either a Lyapunov–Razumikhin function [14] or a Lyapunov–Krasovskii functional [5, 9]. Using Lyapunov–Razumikhin function leads to very conservative results [11] but may deal with time-varying delays. Concerning Lyapunov–Krasovskii-based approach, the applicability of this methodology requires to properly define a Lyapunov–Krasovskii functional. Such a choice may entail some conservatism and some technical problems; see [16].

In this chapter, we focus on the stabilization problem for linear systems subject to constant time delay and saturated quantization in the input channel. The resulting closed-loop system is modeled as a linear time-delay system perturbed by two different nonlinearities allowing to describe more precisely the saturation and the quantization phenomena separately. Following the work of [19], the saturation function is embedded into a local sector condition, while the nonlinearity associated to the quantization error is encapsulated into a sector via the use of certain sector conditions. At this point, due to the presence of the uniform quantizer, (at least for unstable open-loop plants), it is impossible to prove asymptotic stability of the origin; see [18]. To overcome this problem, we focus on local ultimate boundedness for the closed-loop system. In particular, we design the controller gain K in a way such that there exists a set (inner set) which is a finite time attractor for every initial condition belonging to a compact set (outer set) containing the previous one. Then, by relying on a Lyapunov–Krasovskii functional, we turn the solution to the considered control design problem into the solution to a quasi-LMI optimization problem. Finally, a procedure based on convex optimization is proposed to optimize the size of the outer and the inner sets.

Notation: The notation $P > \mathbf{0}$, for $P \in \mathbb{R}^{n\times n}$, means that P is symmetric and positive definite. The sets $\mathbb{S}_n$ and $\mathbb{S}_n^+$ represent, the set of symmetric and symmetric positive definite matrices of $\mathbb{R}^{n\times n}$, respectively. Given a vector x (a matrix A), $x'(A')$ denotes the transpose of $x(A)$. The symmetric matrix $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ stands for $\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$. The matrix $diag(A, B)$ stands for the diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Moreover, for any square matrix $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{He}(A) = A + A'$. The matrix $\mathbf{1}$ represents the identity matrix of appropriate dimension. The notation $\mathbf{0}_{n,m}$ stands for the matrix in $\mathbb{R}^{n\times m}$ whose entries are zero and, when no confusion is possible, the subscript will be omitted. For any function $x : [-h, +\infty) \to \mathbb{R}^n$, the notation $x_t(\theta)$ stands for $x(t+\theta)$, for all $t \geq 0$ and all $\theta \in [-h, 0]$. $|x|$ refers to the classical Euclidean norm and $\|x_t\|$ refers to the induced norm defined by $\|x_t\| = \sup_{\theta\in[-h,0]} |x(t+\theta)|$. For $P > 0$, $|x|_P$ refers to the modified Euclidean norm $|x|_P = \sqrt{x'Px}$. Its induced norm is then denoted $\|x_t\|_P$. For a generic positive ρ, we define the functional sets $L_V(\rho) = \left\{x \in \mathscr{C}^1 : V(x_t, \dot{x}_t) \leq \rho\right\}$ and $\mathrm{int}\, L_V(\rho) = \left\{x \in \mathscr{C}^1 : V(x_t, \dot{x}_t) < \rho\right\}$. For a generic positive ρ, we define the sets $S(\rho) = \{x \in \mathbb{R}^n : \eta(|x|, 0) \leq \rho\}$ and $\mathrm{int}\, S(\rho) = \{x \in \mathbb{R}^n : \eta(|x|, 0) < \rho\}$, where η is a class $\mathscr{K}$ function.

8.2 Problem Statement

Consider the following continuous-time system with delayed saturated quantized input:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + B\,\mathrm{sat}(\mathrm{q}(u(t-h))) \\ &x(0) \in \mathbb{R}^n \end{aligned} \tag{8.1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $x(0)$ the initial state, u the control input and $h \geq 0$ the constant input delay. A, B are real constant matrices of appropriate dimensions. The saturation map, $\mathrm{sat}(u) : \mathbb{R}^m \to \mathbb{R}^m$ is classically defined from the symmetric saturation function having as level the positive vector $\bar{u}$: $\mathrm{sign}(\bar{u}_{(i)}) \min \{\bar{u}_{(i)}, |\bar{u}_{(i)}|\}$. The quantization function, depicted in Fig. 8.1, is defined as

$$\mathrm{q}\colon \begin{cases} \mathbb{R}^p & \to \mathbb{R}^p \\ u & \mapsto \Delta \mathrm{sign}(u) \lfloor \frac{|u|}{\Delta} \rfloor. \end{cases} \tag{8.2}$$

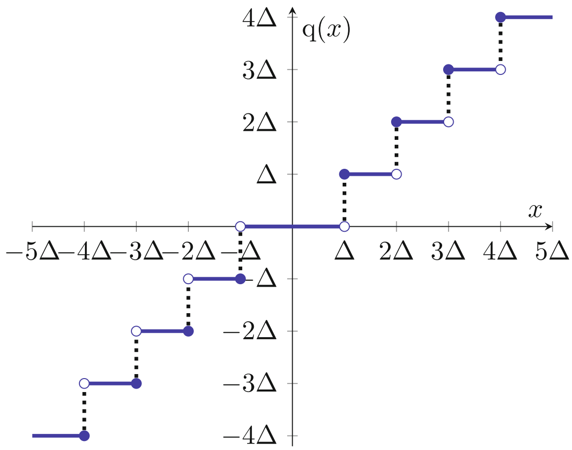

Fig. 8.1 Quantization function

Assuming to fully measure the state x, we want to stabilize system (8.1) (in a sense to be specified) by a static-state feedback controller, that is

$$u = Kx.$$

Plugging the above controller yields the following dynamics for the closed-loop system

$$\dot{x}(t) = Ax(t) + B\,\mathrm{sat}(\mathrm{q}(Kx(t-h))). \tag{8.3}$$

Now, due to the delayed input, the closed-loop system turns into a time-delay system, thus a suitable initial condition needs to be selected for the controller. Notice that, actually the infinite dimensionality is introduced in the closed-loop system by the controller. Thus, to make (8.3) an effective description of the closed-loop system, the initial condition of the delayed-system (8.3) needs to be chosen as

$$x(t) = x_0 = x(0) \quad \forall t \in [-h, 0].$$

In particular this choice avoids jumps in the plant state at time $t = h$. Furthermore, note that x_0 is assumed to be a constant given vector of $\mathbb{R}^n$. Then, the closed-loop system can be described by the following functional differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\,\mathrm{sat}(\mathrm{q}(Kx(t-h))) \\ x(t) &= x_0 \quad \forall t \in [-h, 0]. \end{aligned} \tag{8.4}$$

By defining

$$\Psi(y) = \mathrm{q}(y) - y$$
$$\phi(y) = \mathrm{sat}(y) - y,$$

system (8.3) can be rewritten equivalently as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKx(t-h) + B\Psi(Kx(t-h)) + B\phi\left(\Psi(Kx(t-h)) + Kx(t-h)\right) \\ x(t) &= x_0 \qquad \forall t \in [-h, 0] \end{aligned} \tag{8.5}$$

Due to quantization, whenever the open-loop system (8.1) is unstable, asymptotic stabilization of the origin cannot be achieved via any gain K; see [15]. To overcome this drawback, we rest on ultimate boundedness for the closed-loop system trajectories, whose definition is recalled below (see [13] for more details):

Definition 1 Let S_0 and S_u be two compact sets containing the origin, the solutions to (8.4) with x_0 starting from S_0 are ultimately bounded in the set S_u if there exists a time $T = T(|x_0|) \geq 0$ such that for every $t \geq T$

$$x_0 \in S_0 \Rightarrow x_t \in S_u.$$

Hence, the problem we solve can be then summarized as follows:

Problem 1 Let A, B real matrices of adequate dimension, determine $K \in \mathbb{R}^{m\times n}$, and two compact sets S_0 and S_u, with $0 \in S_u \subset S_0$, such that for every $x_0 \in S_0$, the resulting trajectory to (8.5) is ultimately bounded in S_u.

8.3 Main Results

To solve Problem 1, we propose a Lyapunov-based result, which establishes local ultimate boundedness of a time-delay system, provided that certain inequalities involving a Lyapunov–Krasovskii functional hold.

Theorem 1 *Given a functional $V : \mathscr{C} \times \mathscr{C} \to \mathbb{R}^+$, assume that there exist κ, η and ω three class $\mathscr{K}$ functions such that for every positive t*

$$\kappa(|x(t)|) \leq V(x_t, \dot{x}_t) \leq \eta(\|x_t\|, \|\dot{x}_t\|) \tag{8.6}$$

and there exist two positive scalars γ and β, with $\beta < \gamma$ such that for every positive t and for every $x_t \in L_V(\gamma) \setminus \mathrm{int}\, L_V(\beta)$ one gets:

$$\dot{V}(x_t, \dot{x}_t) \leq -\omega(|x_t(0)|). \tag{8.7}$$

Then,

- *for every initial condition* $x(0) \in S(\gamma)$ *the solutions to system (8.3) are bounded,*
- *for every* $x(0) \in S(\gamma)$ *there exists a positive time* T *such that for every* $t \geq T$*,*

$$x(t) \in S(\kappa^{-1}(\beta)).$$

Proof First, notice that

$$V(\varphi_0(t), \dot{\varphi}_0(t)) = V(x_0, 0) \leq \eta(|x_0|, 0) \leq \gamma.$$

Thus, thanks to (8.7), for every positive t one has

$$V(x_t, \dot{x}_t) < V(x_0, \dot{x}_0).$$

Furthermore due to (8.6) one has

$$\kappa(|x(t)|) \leq V(x_t, \dot{x}_t) < V(x_0, 0) \leq \eta(|x_0|, 0) \leq \gamma$$

which implies that

$$|x(t)| \leq \kappa^{-1}(\gamma), \quad \forall t \geq 0$$

and then boundedness is proven.

To prove finite time convergence, denote $T = \inf\{t > 0 \colon x_t \in L_V(\beta)\}$, for every $t \in [0, T]$ we have $x_t \in L_V(\gamma)$, then by integration of (8.7) along the trajectories of (8.5) yields

$$V(x_T, \dot{x}_T) \leq -\int_0^T \omega(|x_\tau(0)|)d\tau + V(x_0, 0).$$

Now, observe that whenever $x(\tau) \in L_V(\gamma)$ due to (8.6) we have

$$\kappa(|x(\tau)|) \leq \gamma.$$

Thus, one gets

$$\omega(|x(\tau)|) \leq \omega(\kappa^{-1}(\gamma)), \forall \tau \in [0, T]$$

and then

$$V(x_T, \dot{x}_T) \leq -T\omega(\kappa^{-1}(\gamma)) + \gamma.$$

From (8.6), to require that $\kappa(|x(T)|) \leq \beta$ if suffices to impose that

$$-T\omega(\kappa^{-1}(\gamma)) + \gamma \leq \beta$$

which leads to

$$T \geq \frac{\gamma - \beta}{\omega(\kappa^{-1}(\gamma))}.$$

Moreover since (8.7) holds also on the boundary of $L_V(\beta)$, trajectories cannot leave such a set and for all $t \geq T$

$$\kappa(|x(t)|) \leq V(\varphi_t, \dot{\varphi}_t) \leq \beta$$

and this concludes the proof. ■

Remark 1 In the previous result, any assumption on the considered norm was given. In particular, the norms involved in the above result may be chosen differently from each other. For example, one can assume relation (8.6) and (8.7) hold as follows

$$\kappa(|x(t)|_a) \leq V(x_t, \dot{x}_t) \leq \eta(\|x_t\|_b, \|\dot{x}_t\|_c) \tag{8.8}$$

$$\dot{V}(x_t, \dot{x}_t) \leq -\omega(|x_t(0)|_d) \tag{8.9}$$

where $|\cdot|_a$, $|\cdot|_b$, $|\cdot|_c$ and $|\cdot|_d$ are any vector norms and $\|\cdot\|_b$ and $\|\cdot\|_c$ the corresponding induced function norms. The proof of Theorem 1 is almost the same, except that combining the left-hand side of (8.8) and (8.9) one should deal with different norms, i.e. $|\cdot|_a$ and $|\cdot|_d$. However, since $|\cdot|_a$ and $|\cdot|_d$ are equivalent, there exists a positive scalar δ such that

$$|x_t(0)|_d \leq \delta |x_t(0)|_a,$$

allowing to obtain a similar result to Theorem 1.

Now, we present two lemmas that will be useful in the sequel:

Lemma 1 ([7]) *For every $u \in \mathbb{R}^m$ the following conditions hold:*

$$\Psi(u)' T_1 \Psi(u) - \Delta^2 \operatorname{trace}(T_1) \leq 0, \tag{8.10}$$

$$\Psi(u)' \, T_2 \, (\Psi(u) + u) \leq 0, \tag{8.11}$$

for any diagonal positive definite matrices $T_1, T_2 \in \mathbb{R}^{m \times m}$.

Lemma 2 ([19]) *Consider a matrix $G \in \mathbb{R}^{m \times m}$, the nonlinearity $\phi(u) = \operatorname{sat}(u) - u$ satisfies*

$$\phi(u)' T_3 (\operatorname{sat}(u) + Gx) \leq 0, \tag{8.12}$$

for every diagonal positive definite matrix $T_3 \in \mathbb{R}^{m \times m}$, if $x \in \mathscr{S}(\bar{u})$ defined by

$$\mathscr{S}(\bar{u}) = \left\{ x \in \mathbb{R}^n : |G_{(i)} x| \leq \bar{u}, \forall i \in \{1, \ldots, m\} \right\}.$$

Based on the use of a Lyapunov–Krasovskii functional, the following theorem provides sufficient conditions to solve Problem 1.

Theorem 2 *If there exist a matrix $W \in \mathbb{R}^{n \times n}$, three symmetric positive define matrices $J, L, U \in \mathbb{R}^{n \times n}$, two matrices $Y, Z \in \mathbb{R}^{m \times n}$, three positive definite diagonal matrices $T_1, T_2, T_3 \in \mathbb{R}^{m \times m}$, and two positive scalars σ, σ_2 such that $\sigma_2 < \sigma$ and*

$$\begin{bmatrix} h^2 J - \mathrm{He}(W) & L + AW' - W & BY - W & \mathbf{0} & B & B \\ \star & \sigma L + U - e^{-\sigma h} J + \mathrm{He}(WA') & e^{-\sigma h} J + BY + WA' & \mathbf{0} & B & B - Z'T_3 \\ \star & \star & -e^{-\sigma h}(J+U) + \mathrm{He}(BW) & \mathbf{0} & B - Y'T_2 & B - Y'T_3 \\ \star & \star & \star & (\Delta^2 \,\mathrm{trace}(T_1) - \sigma_2)\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -T_1 - 2T_2 & -T_3 \\ \star & \star & \star & \star & \star & -2T_3 \end{bmatrix} < \mathbf{0} \tag{8.13}$$

$$\begin{bmatrix} L & Z_{(i)} \\ \star & \bar{u}^2_{(i)} \end{bmatrix} \geq \mathbf{0} \qquad i \in \{1, \ldots, m\} \tag{8.14}$$

then the gain $K = YW'^{-1}$, the sets $S_0 = L_v(1)$ and $S_u = L_v(\beta)$ with $\frac{\sigma_2}{\sigma} < \beta < 1$ are a solution to Problem 1, where $P = W^{-1}LW'^{-1}$ and $Q = W^{-1}UW'^{-1}$.

Proof Consider the following Krasovskii–Lyapunov functional

$$V(x_t, \dot{x}_t) = x(t)'Px(t) + \int_{t-h}^{t} e^{\sigma(s-t)} x(s)'Qx(s)ds + h\int_{-h}^{0}\int_{t+\theta}^{t} e^{\sigma(s-t)}\dot{x}(s)'R\dot{x}(s)dsd\theta \tag{8.15}$$

where P, Q, R are symmetric positive definite matrices. Notice that for $t > 0$ the functional satisfies

$$x(t)'Px(t) \leq V(x_t, \dot{x}_t) \leq x(t)'Px(t) + h \sup_{\tau\in[t-h,t]} x(\tau)'Qx(\tau) + \frac{h^3}{2} \sup_{\vartheta\in[t-h,t]} \dot{x}(\vartheta)'R\dot{x}(\vartheta). \tag{8.16}$$

and then

$$|x(t)|_P^2 \leq V(x_t, \dot{x}_t) \leq \|x_t\|_P^2 + \|x_t\|_{hQ}^2 + \frac{h^3}{2}\|\dot{x}_t\|_R^2. \tag{8.17}$$

Now, by computing the time-derivative along the solutions to system (8.5) of the above functional, one gets

$$\begin{aligned} \dot{V}(x_t, \dot{x}_t) = {}& 2\dot{x}(t)'Px(t) + x(t)'Qx(t) - e^{-\sigma h}x(t-h)'Qx(t-h) \\ & - \sigma\int_{t-h}^{t} e^{\sigma(s-t)}x(s)'Qx(s)ds - h\sigma\int_{-h}^{0}\int_{t+\theta}^{t} e^{\sigma(s-t)}\dot{x}(s)'R\dot{x}(s)dsd\theta \\ & + h^2\dot{x}(t)'R\dot{x}(t) - h\int_{-h}^{0} e^{\sigma\theta}\dot{x}(t+\theta)'R\dot{x}(t+\theta)d\theta, \end{aligned} \tag{8.18}$$

and

$$\begin{aligned} \dot{V}(x_t, \dot{x}_t) = {}& 2\dot{x}(t)'Px(t) + x(t)'Qx(t) - e^{-\sigma h}x(t-h)'Qx(t-h) \\ & - \sigma\int_{t-h}^{t} e^{\sigma(s-t)}x(s)'Qx(s)ds - h\sigma\int_{-h}^{0}\int_{t+\theta}^{t} e^{\sigma(s-t)}\dot{x}(s)'R\dot{x}(s)dsd\theta \\ & + h^2\dot{x}(t)'R\dot{x}(t) - he^{\sigma h}\int_{-h}^{0} \dot{x}(t+\theta)'R\dot{x}(t+\theta)d\theta. \end{aligned} \tag{8.19}$$

Moreover, by Jensen's inequality, it follows

$$\begin{aligned}\dot{V}(x_t, \dot{x}_t) \leq & 2\dot{x}(t)'Px(t) + x(t)'Qx(t) - e^{-\sigma h}x(t-h)'Qx(t-h) \\ & - \sigma \int_{t-h}^{t} e^{\sigma(s-t)}x(s)'Qx(s)ds + h\left(-\sigma \int_{-h}^{0}\int_{t+\theta}^{t} e^{\sigma(s-t)}\dot{x}(s)'R\dot{x}(s)dsd\theta\right) \\ & + h^2\dot{x}(t)'R\dot{x}(t) - e^{-\sigma h}\left(x(t)' - x(t-h)'\right)R\left((x(t) - x(t-h)\right).\end{aligned} \tag{8.20}$$

We want to show that if (8.13) and (8.14) hold, then there exists a positive small enough constant ε such that, along the solutions to system (8.5),

$$\dot{V}(x_t, \dot{x}_t) \leq -\varepsilon x(t)'x(t), \tag{8.21}$$

whenever $x_t \in L_V(1) \setminus \text{int}\, L_V(\beta)$. To prove (8.21), by following S-procedure arguments, it suffices to prove that there exists a positive scalar θ, such that

$$\mathscr{L}_1 = \dot{V}(x_t, \dot{x}_t) - \theta(\beta - V(x_t, \dot{x}_t)) \leq -\varepsilon x(t)'x(t).$$

From (8.20), one can write

$$\mathscr{L}_1 \leq \mathscr{L}_0 - \theta(\beta - V(x_t, \dot{x}_t)), \tag{8.22}$$

where $\mathscr{L}_0$ denotes the right-hand side of (8.20). Define

$$\zeta = \left[\dot{x}(t)'\ x(t)'\ x(t-h)'\ \mathbf{1}_m'\right]'. \tag{8.23}$$

Then, inequality (8.22) reads:

$$\mathscr{L}_1 \leq \zeta' M \zeta + (-\sigma + \theta)V(x_t, \dot{x}_t) + \sigma_2 - \theta\beta. \tag{8.24}$$

where

$$M = \begin{bmatrix} h^2R & P & \mathbf{0} & \mathbf{0} \\ \star & \sigma P + Q - e^{-\sigma h}R & e^{-\sigma h}R & \mathbf{0} \\ \star & \star & -e^{-\sigma h}(R+Q) & \mathbf{0} \\ \star & \star & \star & -\frac{\sigma_2}{m}\mathbf{1} \end{bmatrix}. \tag{8.25}$$

Moreover, by selecting $\theta = \sigma$ and $\beta = \frac{\sigma_2}{\sigma}$, from the above expression we obtain

$$\mathscr{L}_1 \leq \zeta' M \zeta. \tag{8.26}$$

Now define

$$\begin{aligned}\Omega = &\zeta' M\zeta - \Psi(Kx(t-h))'T_1\Psi(Kx(t-h)) + \Delta^2\,\mathrm{trace}(T_1)\\ &- 2\Psi(Kx(t-h))'T_2\Psi(Kx(t-h)) - 2\Psi(Kx(t-h))'T_2Kx(t-h)\\ &- 2\phi\,(\Psi(Kx(t-h)) + Kx(t-h))'\,T_3\Big(\phi\,(\Psi(Kx(t-h)) + Kx(t-h))\\ &+ \Psi(Kx(t-h)) + Kx(t-h) + Gx\Big).\end{aligned} \tag{8.27}$$

The satisfaction of (8.14), with the change of variables $P = W^{-1}LW'^{-1}$ and $Z = YW'^{-1}$ guarantees that the set $\mathscr{E}(P,1) = \{x \in \mathbb{R}^n : x'Px \le 1\}$ is contained in the polyhedral set $\mathscr{S}(\bar{u})$ defined in Lemma 2. Moreover, due to (8.16), it follows that if $x_t \in L_v(1)$ then $x(t) \in \mathscr{E}(P,1)$ and therefore also $x(t) \in \mathscr{S}(\bar{u})$, that guarantees the satisfaction of relation (8.12) in Lemma 2. Furthermore, note that from (8.15), one can write:

$$V(x_0, \dot{x}_0) = x_0'Px_0 + \int_{-h}^{0} e^{\sigma s}x_0'Qx_0ds \le x_0'(P + hQ)x_0,$$

since x_0 is supposed to be constant on $[-h, 0]$. Then, from (8.16), for any $x_0 \in \mathscr{E}(P + hQ, 1)$ then it follows that $x_0 \in Lv(1)$. Thus, by using Lemmas 1 and 2, for every $x_t \in Lv(1)$, one has

$$\zeta' M\zeta \le \Omega = \xi'\mathscr{N}\xi \tag{8.28}$$

where

$$\mathscr{N} = \begin{bmatrix} h^2R & P & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \star & \sigma P + Q - e^{-\sigma h}R & e^{-\sigma h}R & \mathbf{0} & \mathbf{0} & -G'T_3\\ \star & \star & -e^{-\sigma h}(R+Q) & \mathbf{0} & -K'T_2 & -K'T_3\\ \star & \star & \star & \mathscr{N}_4 & \mathbf{0} & \mathbf{0}\\ \star & \star & \star & \star & -T_1 - 2T_2 & -T_3\\ \star & \star & \star & \star & \star & -2T_3 \end{bmatrix},$$

with $\mathscr{N}_4 = \frac{1}{m}(-\sigma + \Delta^2\,\mathrm{trace}(T_1))\mathbf{1}$ and

$$\xi' = \left[\zeta'\;\Psi(Kx(t-h))'\;\phi\,(\Psi(Kx(t-h)) + Kx(t-h))'\right].$$

To prove our claim, we show that having (8.13) satisfied implies that (8.21) holds. To this end, in light of (8.24), (8.26), and (8.28), it suffices to show that along the trajectories of (8.5) one has

$$\xi'\mathscr{N}\xi < -\varepsilon x'x.$$

By Finsler lemma [18], the above inequality is equivalent to find a matrix X of appropriate dimensions, such that

$$\mathscr{Q} = \mathscr{N} + X\bar{B} + \bar{B}'X' < \mathbf{0},$$

where $\bar{B} = \begin{bmatrix} -\mathbf{1} & A & BK & \mathbf{0} & B & B \end{bmatrix}$. Now, select X as follows

$$X = \begin{bmatrix} H' & H' & H' & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}'.$$

Then, by pre- and post-multiplying $\mathcal{Q}$, respectively, by diag$\{H^{-1}, H^{-1}, H^{-1}, \sqrt{m}\mathbf{1}, \mathbf{1}, \mathbf{1}\}$ and diag$\{H'^{-1}, H'^{-1}, H'^{-1}, \sqrt{m}\mathbf{1}, \mathbf{1}, \mathbf{1}\}$ with

$$\begin{aligned} H^{-1} &= W, & WRW' &= J, & WQW' &= U, \\ WPW' &= V, & KW' &= Y, & GW' &= Z, \end{aligned}$$

we obtain the left-hand side of (8.13). Then the satisfaction of (8.13) implies that the inequality

$$\xi' \mathcal{N} \xi < -\varepsilon x' x$$

holds along the trajectories of system (8.5). Thus, by invoking Theorem 1, thanks to Remark 1, setting

$$|\cdot|_a = |\cdot|_P, \qquad |\cdot|_b = \sqrt{|\cdot|_P^2 + |\cdot|_{hQ}^2}, \quad |\cdot|_c = |\cdot|_R, \qquad |\cdot|_d = |\cdot|_2,$$

and

$$\kappa(s) = s^2, \qquad \eta(s, y) = s^2 + \frac{h^3}{2} y^2, \quad \gamma = 1, \qquad \beta = \frac{\sigma_2}{\sigma},$$

establishes the result. ■

Remark 2 The above result is based on the use of a classical Lyapunov functional. As pointed by [9], this result can be extended to the more realistic case of time-varying delay by considering a more complex functional and by following analogous arguments as the ones shown in the proof of Theorem 2.

8.4 Optimization Issues

In solving Problem 1, the implicit objective is to obtain a set S_0 as large as possible and a set S_u as small as possible. The problem of maximizing the size of S_0 and minimizing the size of S_u relies on the choice of a good measure of such sets. At this stage, it is interesting to remark that as mentioned in the proof of Theorem 2, if $x(0) \in \mathscr{E}(P + hQ, 1)$, then $x_0 \in S_0 = L_v(1)$. This implies that we can use the ellipsoid $\mathscr{E}(P + hQ, 1)$ to implicitly maximize S_0. Hence, when considering ellipsoidal sets, several measures and therefore associated criteria can be considered. Volume, minor axis, directions of interest, inclusion of a given shape set defined through the extreme points v_r , $r = 1, \ldots, n_r$ (see for example [1]) are some of the tools usually adopted to indirectly optimize the size of such sets. The natural choice to maximize the size of S_0 in an homogeneous way, consists in minimizing the trace of the matrix $P + hQ$.

By the same way, if $x_t \in Lv(\beta)$ then from (8.16), $x(t) \in \mathscr{E}(P, \beta)$. Then, to minimize the size of S_u, one can simultaneously minimize β and maximize the trace of P. Therefore the minimization of trace$(P + hQ)$ clashes with the minimization of the set S_u. Thus a trade-off between the two objectives needs to be considered. Moreover, the matrices P and Q do not directly appear in (8.13) and (8.14), thus the trace minimization problem needs to be rewritten in the decision variables. To this end, it is worthwhile to notice that the minimization of trace$(P + hQ)$, due to positive definiteness of $P + hQ$, can be implicitly performed by maximizing the trace$(P + hQ)^{-1}$. In particular, since

$$Q = W^{-1}UW'^{-1}, \quad P = W^{-1}LW'^{-1},$$

it turns out that $(P + hQ)^{-1} = W'(hU + L)^{-1}W$. Then, thanks to [4], we get

$$W'(hU + L)^{-1}W \geq W + W' - (hU + L).$$

Thus, let δ_1 and δ_2 be two tuning parameters, we can consider the following criterion

$$\text{minimize } \delta_1\sigma_2 - \delta_2 \operatorname{trace}(2W - hU - L).$$

At this point, it is important to note that conditions provided by Theorem 2 are nonlinear in the decision variables, which prevents from solving directly a convex optimization problem. This is more specifically a problem in case of products of decision matrices. On the other hand, products of a decision matrix with a scalar are numerically tractable if the scalar is considered either as a tuning parameter (but there is no guarantee that the problem remains feasible) or fixed via an iterative search. Next, we consider the following additional constraints in the decision variables:

- $T_2 = \tau_2\mathbf{1}$,
- $T_3 = \tau_3\mathbf{1}$.

Considering σ, τ_2, τ_3 as tuning parameters. Problem 1 can be solved by solving the following LMI optimization problem:

$$\begin{aligned}
&\underset{L,J,U,W,\sigma_2,T_1}{\text{minimize}} \quad \delta_1\sigma_2 - \delta_2 \operatorname{trace}(2W - hU - L) \\
&\text{subject to} \\
&(8.13),\ (8.14),\ \sigma_2 < \sigma,\ L > \mathbf{0},\ J > \mathbf{0},\ U > \mathbf{0},\ T_1 > \mathbf{0}.
\end{aligned}$$

8.5 Example

Consider the system described by [9]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{sat}(u(t-h)).$$

The time delay is fixed to $h = 0.2$ and the level of saturation is $\bar{u} = 5$. We choose the quantization error bound as $\Delta = 1$. Notice that, since the open-loop system is unstable ($spec(A) = \{-0.5,\ 1\}$), the closed-loop trajectories cannot converge to the origin due to the quantizer. Using iterative research for scalars σ, τ_2, τ_3, we obtain the gain

$$K = \begin{bmatrix} -1.2773 & -2.5541 \end{bmatrix}.$$

For different initial conditions, several trajectories are depicted in Fig. 8.2. The ellipsoid $\mathscr{E}(P + hQ, 1)$ and $\mathscr{E}(P, \beta)$ are depicted in blue and dashed blue, respectively. The two red crosses are the two unstable equilibrium points due to the saturation. Furthermore, notice that a trajectory starting from an initial point outside the ellipsoid $\mathscr{E}(P, 1)$ and necessary such that x_0 is outside $L_v(1)$ still converges into $\mathscr{E}(P, \beta)$ which indicates clearly the conservatism of our technique. For $x_0 = [-7.7074\ 6.5731]'$, Fig. 8.3 shows the evolution of the control u. Its value saturates for less than half a second and does not converge to zero.

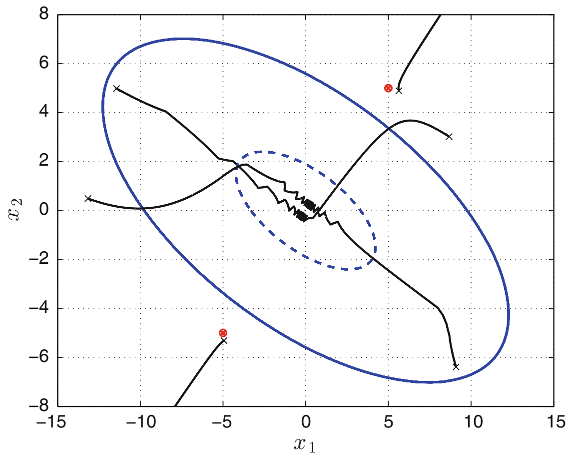

Fig. 8.2 Trajectories of the closed-loop system

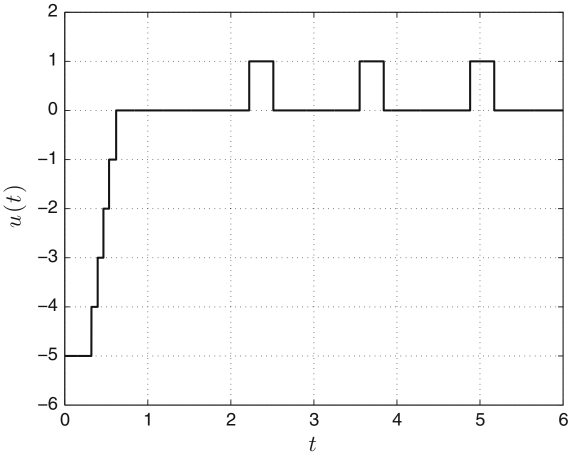

Fig. 8.3 Evolution of u for $x_0 = [-7.7074, 6.5731]'$

8.6 Conclusion

In this chapter, we tackled the stabilization problem of linear systems with saturated, quantized, and delayed input. Specifically, the proposed methodology allows to design the controller to achieve local ultimate boundedness for the closed-loop system. Moreover, the control design problem is turned into an optimization problem aiming at the optimization of the size of the outer set S_0 and of the inner set S_u. The solution to such an optimization problem can be performed in a convex optimization setup, providing a computer-oriented solution. The effectiveness of the proposed strategy is supported by a numerical example. Future works should be devoted to the use of more complex Lyapunov functionals depending on extra states and slack variables, as well as the extension to these results to the case of time-varying time delay.

References

1. S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. (Society for Industrial and Applied Mathematics, 1997)
2. R.W. Brockett, D. Liberzon, Quantized feedback stabilization of linear systems. IEEE Trans. Autom. Control **45**(7), 1279–1289 (2000)
3. F. Ceragioli, C. De Persis, P. Frasca, Discontinuities and hysteresis in quantized average consensus. Automatica **47**(9), 1919–1928 (2011)
4. J. Daafouz, J. Bernussou, Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties. Syst. Control Lett. **43**(5), 355–359 (2001)
5. C. De Persis, F. Mazenc, Stability of quantized time-delay nonlinear systems: a Lyapunov-Krasowskii-functional approach. Math. Control Signals Syst. **21**(4), 337–370 (2010)
6. D.F. Delchamps, Stabilizing a linear system with quantized state feedback. IEEE Trans. Autom. Control **35**(8), 916–924 (1990)

7. F. Ferrante, F. Gouaisbaut, S. Tarbouriech, Observer-based control for linear systems with quantized output, in *European Control Conference.* (2014)
8. F. Ferrante, F. Gouaisbaut, S. Tarbouriech, Stabilization of continuous-time linear systems subject to input quantization. Automatica **58**, 167–172 (2015)
9. E. Fridman, M. Dambrine, Control under quantization, saturation and delay: an LMI approach. Automatica **45**(10), 2258–2264 (2009)
10. M. Fu, L. Xie, The sector bound approach to quantized feedback control. IEEE Trans. Autom. Control **50**(11), 1698–1711 (2005)
11. K. Gu, V.L. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Control Engineering (Birkhäuser Boston, 2003)
12. I.L. Hurtado, C.T. Abdallah, C. Canudas-de Wit, Control under limited information: special issue (part I). Int. J. Robust Nonlinear Control **19**(16), 1767–1769 (2009)
13. H.K. Khalil, *Nonlinear Systems*, 3rd edn. (Prentice-Hall, 2002)
14. D. Liberzon, Quantization, time delays, and nonlinear stabilization. IEEE Trans. Autom. Control **51**(7), 1190–1195 (2006)
15. D. Liberzon, Nonlinear control with limited information. Commun. Inf. Syst. **9**(1), 41–58 (2009)
16. A. Seuret, F. Gouaisbaut, Wirtinger-based integral inequality: application to time-delay systems. Automatica **49**(9), 2860–2866 (2013)
17. S. Tarbouriech, G. Garcia, J.M. Gomes da Silva Jr., I. Queinnec, *Stability and Stabilization of Linear Systems with Saturating Actuators* (Springer, 2011)
18. S. Tarbouriech, F. Gouaisbaut, Control design for quantized linear systems with saturations. IEEE Trans. Autom. Control **57**(7), 1883–1889 (2012)
19. S. Tarbouriech, C. Prieur, J.M. Gomes da Silva Jr., Stability analysis and stabilization of systems presenting nested saturations. IEEE Transactions on Automatic Control **51**(8), 1364–1371 (2006)

Chapter 9
Discrete-Time Networked Control Under Scheduling Protocols

Kun Liu, Emilia Fridman and Karl Henrik Johansson

Abstract This chapter analyzes the exponential stability of discrete-time networked control systems viadelay-dependent Lyapunov-Krasovskii methods. The time-delay approach has been developed recently for the stabilization of continuous-time networked control systems under a Round-Robin protocol and a weighted Try-Once-Discard protocol, respectively. In the present chapter, the time-delay approach is extended to the stability analysis of discrete-time networked control systems under both these scheduling protocols. First, the closed-loop system is modeled as a discrete-time switched system with multiple and ordered time-varying delays under the Round-Robin protocol. Then, a discrete-time hybrid system model for the closed-loop system is presented under these protocols. It contains time-varying delays in the continuous dynamics and in the reset conditions. The communication delays are allowed to be larger than the sampling intervals. Polytopic uncertainties in the system model can be easily included in our analysis. The efficiency of the time-delay approach is illustrated in an example of a cart-pendulum system.

9.1 Introduction

Network control systems (NCSs) are spatially distributed systems in which the communication between sensors, actuators, and controllers occurs through a communication network [1, 22]. In many such systems, only one node is allowed to use

K. Liu (✉)
School of Automation,
Beijing Institute of Technology, 100081 Beijing, China
e-mail: kunliubit@bit.edu.cn

E. Fridman
School of Electrical Engineering, Tel Aviv University, 69978 Tel Aviv, Israel
e-mail: emilia@eng.tau.ac.il

K.H. Johansson
ACCESS Linnaeus Centre and School of Electrical Engineering, KTH Royal Institute of Technology, SE-100 44, Stockholm, Sweden
e-mail: kallej@kth.se

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_9

the communication channel at each time instant. In the present chapter, we focus on the stability analysis of discrete-time NCSs with communication constraints. The scheduling of sensor information towards the controller is defined by a Round-Robin (RR) protocol or by a weighted Try-Once-Discard (TOD) protocol. A linear system with distributed sensors is considered. Three recent approaches for NCSs are based on discrete-time systems [2, 3, 7], impulsive/hybrid systems [10, 18, 19] and time-delay systems [4, 6, 8, 12, 21].

The time-delay approach was developed for the stabilization of continuous-time NCSs under a RR protocol in [15] and under a weighted TOD protocol in [16]. The closed-loop system was modeled as a switched system with multiple and ordered time-varying delays under RR protocol or as a hybrid system with time-varying delays in the dynamics and in the reset equations under TOD protocol. Differently from the existing hybrid and discrete-time approaches on the stabilization of NCS with scheduling protocols, the time-delay approach allows treating the case of large communication delays.

In the present chapter, the time-delay approach is extended to the stability analysis of discrete-time NCSs under RR and weighted TOD scheduling protocols. First, the closed-loop system is modeled as a discrete-time switched system with multiple and ordered time-varying delays under RR protocol. Then, a discrete-time hybrid system model for the closed-loop system is presented that contains time-varying delays in the continuous dynamics and in the reset conditions under TOD and RR protocols. Differently from [14], the same conditions are derived for the exponential stability of the resulting hybrid system model under both these scheduling protocols. The communication delays are allowed to be larger than the sampling intervals. The conditions are given in terms of Linear Matrix Inequalities (LMIs). Polytopic uncertainties in the system model can be easily included in the analysis. The efficiency of the presented approach is illustrated by a cart-pendulum system.

Notation: Throughout the chapter, the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. $\mathbb{Z}^+$, $\mathbb{N}$ and $\mathbb{R}^+$ denote the set of non-negative integers, positive integers integers and non-negative real numbers, respectively.

9.2 Discrete-Time Networked Control Systems Under Round-Robin Scheduling: A Switched System Model

9.2.1 Problem Formulation

Consider the system architecture in Fig. 9.1 with plant

$$x(t+1) = Ax(t) + Bu(t), \quad t \in \mathbb{Z}^+, \tag{9.1}$$

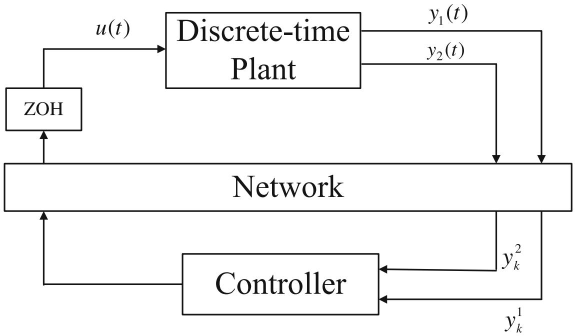

Fig. 9.1 Discrete-time NCSs under RR scheduling

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, A and B are system matrices with appropriate dimensions. The initial condition is given by $x(0) = x_0$.

The NCS has several nodes connected via networks. For the sake of simplicity, we consider two sensor nodes $y_i(t) = C_i x(t), i = 1, 2$ and we denote $C = \left[C_1^T \; C_2^T \right]^T$, $y(t) = \left[y_1^T(t) \; y_2^T(t) \right]^T \in \mathbb{R}^{n_y}, \; t \in \mathbb{Z}^+$. We let s_k denote the unbounded and monotonously increasing sequence of sampling instants, i.e.,

$$0 = s_0 < s_1 < \cdots < s_k < \cdots , \; \lim_{k\to\infty} s_k = \infty, \; s_{k+1} - s_k \le \text{MATI}, \; k \in \mathbb{Z}^+, \tag{9.2}$$

where $\{s_0, s_1, s_2, \dots\}$ is a subsequence of $\{0, 1, 2, \dots\}$ and MATI denotes the Maximum Allowable Transmission Interval. At each sampling instant s_k, one of the outputs $y_i(t) \in \mathbb{R}^{n_i} (n_1 + n_2 = n_y)$ is sampled and transmitted via the network. First, we consider the RR scheduling protocol for the choice of the active output node: the outputs are transmitted one after another, i.e., $y_i(t) = C_i x(t), t \in \mathbb{Z}^+$ is transmitted only at the sampling instant $t = s_{2p+i-1}, \; p \in \mathbb{Z}^+, \; i = 1, 2$. After each transmission and reception, the values in $y_i(t)$ are updated with the newly received values, while the values of $y_j(t)$ for $j \neq i$ remain the same, as no additional information is received. This leads to the constrained data exchange expressed as

$$y_k^i = \begin{cases} y_i(s_k) = C_i x(s_k), & k = 2p + i - 1, \\ y_{k-1}^i, & k \neq 2p + i - 1, \end{cases} \quad p \in \mathbb{Z}^+.$$

It is assumed that no packet dropouts and no packet disorders will happen during the data transmission over the network. The transmission of the information over the network is subject to a variable delay $\eta_k \in \mathbb{Z}^+$, which is allowed to be larger than the sampling intervals. Then $t_k = s_k + \eta_k$ is the updating time instant of the ZOH device.

Assume that the network-induced delay η_k and the time span between the updating and the current sampling instants are bounded:

$$t_{k+1} - 1 - t_k + \eta_k \le \tau_M, \; 0 \le \eta_m \le \eta_k \le \eta_M, \; k \in \mathbb{Z}^+, \tag{9.3}$$

where τ_M, η_m and η_M are known non-negative integers. Then

$$\begin{aligned}(t_{k+1}-1)-s_k &= s_{k+1}-s_k+\eta_{k+1}-1 \le \text{MATI}+\eta_M-1=\tau_M,\\ (t_{k+1}-1)-s_{k-1} &= s_{k+1}-s_{k-1}+\eta_{k+1}-1 \le 2\,\text{MATI}+\eta_M-1\\ &= 2\tau_M-\eta_M+1 \triangleq \bar{\tau}_M,\\ t_{k+1}-t_k &\le \tau_M-\eta_m+1.\end{aligned} \tag{9.4}$$

Note that the first updating time t_0 corresponds to the first data received by the actuator, which means that $u(t)=0, t\in[0,t_0-1]$. Then for $t\in[0,t_0-1]$, (9.1) is given by

$$x(t+1)=Ax(t),\ t=0,1,\ldots,t_0-1,\quad t\in\mathbb{Z}^+. \tag{9.5}$$

In [15], a time-delay approach was developed for the stability and L_2-gain analysis of continuous-time NCSs with RR scheduling. In this section, we consider the stability analysis of discrete-time NCSs under RR scheduling protocol.

It is assumed that the controller and the actuator are event-driven. Suppose that there exists a matrix $K=\begin{bmatrix}K_1 & K_2\end{bmatrix}$, $K_1\in\mathscr{R}^{n_u\times n_1}$, $K_2\in\mathscr{R}^{n_u\times n_2}$ such that $A+BKC$ is Schur. Consider the static output feedback of the form:

$$u(t)=K_1y_k^1+K_2y_k^2,\ t\in\left[t_k,t_{k+1}-1\right],\ t\in\mathbb{N},\ k\in\mathbb{N},$$

Following [15], the closed-loop system with RR scheduling is modeled as a switched system:

$$\begin{aligned}x(t+1)&=Ax(t)+A_1x(t_{k-1}-\eta_{k-1})+A_2x(t_k-\eta_k),\ t\in[t_k,t_{k+1}-1],\\ x(t+1)&=Ax(t)+A_1x(t_{k+1}-\eta_{k+1})+A_2x(t_k-\eta_k),\ t\in[t_{k+1},t_{k+2}-1],\end{aligned} \tag{9.6}$$

where $k=2p-1,\ p\in\mathbb{N},\ A_i=BK_iC_i,\ i=1,2$. For $t\in[t_k,t_{k+1}-1]$, we can represent

$$t_k-\eta_k=t-\tau_1(t),\quad t_{k-1}-\eta_{k-1}=t-\tau_2(t),$$

where

$$\begin{aligned}&\tau_1(t)=t-t_k+\eta_k<\tau_2(t)=t-t_{k-1}+\eta_{k-1},\\ &\tau_1(t)\in[\eta_m,\tau_M],\quad \tau_2(t)\in[\eta_m,\ \bar{\tau}_M],\quad t\in[t_k,t_{k+1}-1].\end{aligned} \tag{9.7}$$

Therefore, (9.6) for $t\in[t_k,t_{k+1}-1]$ can be considered as a system with two time-varying interval delays, where $\tau_1(t)<\tau_2(t)$. Similarly, for $t\in[t_{k+1},t_{k+2}-1]$, (9.6) is a system with two time-varying delays, one of which is less than another.

For $t\in[t_0,t_1-1]$, the closed-loop system is reduced to the following form

$$x(t+1)=Ax(t)+A_1x(t_0-\eta_0),\ t\in[t_0,t_1-1].$$

9.2.2 Stability Analysis of the Switched System Model

Applying the following discrete-time Lyapunov-Krasovskii functional (LKF) to system (9.6) with time-varying delay from the maximum delay interval $[\eta_m, \bar{\tau}_M]$ [5]:

$$\begin{aligned}
V_{RR}(t) &= x^T(t)Px(t) + \sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1}x^T(s)S_0x(s) \\
&\quad + \eta_m \sum_{j=-\eta_m}^{-1}\sum_{s=t+j}^{t-1} \lambda^{t-s-1}\eta^T(s)R_0\eta(s) + \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \lambda^{t-s-1}x^T(s)S_1x(s) \\
&\quad + (\bar{\tau}_M - \eta_m)\sum_{j=-\bar{\tau}_M}^{-\eta_m-1}\sum_{s=t+j}^{t-1} \lambda^{t-s-1}\eta^T(s)R_1\eta(s), \\
\eta(t) &= x(t+1) - x(t),\ P > 0,\ S_i > 0,\ R_i > 0,\ i = 0, 1,\ 0 < \lambda < 1,\ t \geq 0,
\end{aligned}$$

where following [13], we define (for simplicity) $x(t) = x_0,\ t \leq 0$.

Similar to [15], by taking advantage of the ordered delays and using convex analysis of [20], we arrive to the following sufficient conditions for the stability of the switched system:

Theorem 1 *Given scalars* $0 < \lambda \leq 1$, *positive integers* $0 \leq \eta_m \leq \eta_M < \tau_M$, *and* K_1, K_2, *let there exist scalars* $n \times n$ *matrices* $P > 0$, $S_\vartheta > 0$, $R_\vartheta > 0$ $(\vartheta = 0, 1)$, G_1^i, G_2^i, G_3^i $(i = 1, 2)$ *such that the following matrix inequalities are feasible:*

$$\Omega_i = \begin{bmatrix} R_1 & G_1^i & G_2^i \\ * & R_1 & G_3^i \\ * & * & R_1 \end{bmatrix} \geq 0, \tag{9.8}$$

$$(F_0^i)^T P F_0^i + \Sigma + (F_{01}^i)^T H F_{01}^i - \lambda^{\eta_m} F_{12}^T R_0 F_{12} - \lambda^{\bar{\tau}_M} F^T \Omega_i F < 0, \tag{9.9}$$

where

$$\begin{aligned}
&F_0^i = [A\ 0\ A_i\ A_{3-i}\ 0], \\
&F_{01}^i = [A - I\ 0\ A_i\ A_{3-i}\ 0],\ i = 1, 2, \\
&F_{12} = [I\ -I\ 0\ 0\ 0], \\
&F = \begin{bmatrix} 0 & I & -I & 0 & 0 \\ 0 & 0 & I & -I & 0 \\ 0 & 0 & 0 & I & -I \end{bmatrix}, \\
&\Sigma = \mathrm{diag}\{S_0 - \lambda P, -\lambda^{\eta_m}(S_0 - S_1), 0, 0, -\lambda^{\bar{\tau}_M} S_1\}, \\
&H = \eta_m^2 R_0 + (\bar{\tau}_M - \eta_m)^2 R_1.
\end{aligned}$$

Then, the closed-loop system (9.6) is exponentially stable with the decay rate λ.

Proof Consider $t \in [t_k, t_{k+1} - 1]$ and define $\xi(t) = \mathrm{col}\{x(t), x(t - \eta_m), x(t_{k-1} - \eta_{k-1}), x(t_k - \eta_k), x(t - \bar{\tau}_M)\}$. Along (9.6), we have

$$
\begin{aligned}
V_{RR}(t+1) - \lambda V_{RR}(t) \leq \xi^T(t)[(F_0^i)^T P F_0^i + \Sigma + (F_{01}^i)^T H F_{01}^i]\xi(t) \\
-\lambda^{\eta_m}\eta_m \sum_{s=t-\eta_m}^{t-1} \eta^T(s) R_0 \eta(s) \\
-\lambda^{\bar{\tau}_M}(\bar{\tau}_M - \eta_m) \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \eta^T(s) R_1 \eta(s).
\end{aligned}
$$

By Jensen's inequality [9], we have

$$
\begin{aligned}
\eta_m \sum_{s=t-\eta_m}^{t-1} \eta^T(s) R_0 \eta(s) &\geq \sum_{s=t-\eta_m}^{t-1} \eta^T(s) R_0 \sum_{s=t-\eta_m}^{t-1} \eta(s) \\
&= \xi^T(t) F_{12}^T R_0 F_{12} \xi(t).
\end{aligned}
$$

Taking into account that $t_{k-1} - \eta_{k-1} < t_k - \eta_k$ (i.e. that the delays satisfy the relation (9.7)) and applying further Jensen's inequality, we obtain

$$
\begin{aligned}
&-(\bar{\tau}_M - \eta_m) \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \eta^T(s) R_1 \eta(s) \\
&= -(\bar{\tau}_M - \eta_m)\Big[\sum_{s=t_k-\eta_k}^{t-\eta_m-1} \eta^T(s) R_1 \eta(s) + \sum_{s=t_{k-1}-\eta_{k-1}}^{t_k-\eta_k-1} \eta^T(s) R_1 \eta(s) \\
&\qquad + \sum_{s=t-\bar{\tau}_M}^{t_{k-1}-\eta_{k-1}-1} \eta^T(s) R_1 \eta(s)\Big] \\
&\leq -\frac{1}{\alpha_1} f_1(t) - \frac{1}{\alpha_2} f_2(t) - \frac{1}{\alpha_3} f_3(t),
\end{aligned}
$$

where

$$
\begin{aligned}
&\alpha_1 = \frac{t - \eta_m - t_k + \eta_k}{\bar{\tau}_M - \eta_m},\ \alpha_2 = \frac{t_k - \eta_k - t_{k-1} + \eta_{k-1}}{\bar{\tau}_M - \eta_m},\ \alpha_3 = \frac{\bar{\tau}_M - t + t_{k-1} - \eta_{k-1}}{\bar{\tau}_M - \eta_m}, \\
&f_1(t) = [x(t-\eta_m) - x(t_k - \eta_k)]^T R_1 [x(t-\eta_m) - x(t_k - \eta_k)], \\
&f_2(t) = [x(t_k-\eta_k) - x(t_{k-1} - \eta_{k-1})]^T R_1 [x(t_k-\eta_k) - x(t_{k-1} - \eta_{k-1})], \\
&f_3(t) = [x(t_{k-1}-\eta_{k-1}) - x(t - \bar{\tau}_M)]^T R_1 [x(t_{k-1}-\eta_{k-1}) - x(t - \bar{\tau}_M)].
\end{aligned}
$$

Denote

$$
\begin{aligned}
g_{1,2}(t) &= [x(t-\eta_m) - x(t_k - \eta_k)]^T G_1^1 [x(t_k - \eta_k) - x(t_{k-1} - \eta_{k-1})], \\
g_{1,3}(t) &= [x(t-\eta_m) - x(t_k - \eta_k)]^T G_2^1 [x(t_{k-1} - \eta_{k-1}) - x(t - \bar{\tau}_M)], \\
g_{2,3}(t) &= [x(t_k-\eta_k) - x(t_{k-1} - \eta_{k-1})]^T G_3^1 [x(t_{k-1} - \eta_{k-1}) - x(t - \bar{\tau}_M)].
\end{aligned}
$$

Note that (9.8) with $i = 1$ guarantees $\begin{bmatrix} R_1 & G_j^1 \\ * & R_1 \end{bmatrix} \geq 0$ $(j = 1, 2, 3)$, and, thus,

$$\begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \geq 0, \; i \neq j, \; i = 1, 2, \; j = 2, 3.$$

Then, we arrive to

$$-(\bar{\tau}_M - \eta_m) \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \eta^T(s) R_1 \eta(s) \leq -\frac{1}{\alpha_1} f_1(t) - \frac{1}{\alpha_2} f_2(t) - \frac{1}{\alpha_3} f_3(t)$$
$$\leq -f_1(t) - f_2(t) - f_3(t) - 2g_{1,2}(t) - 2g_{1,3}(t) - 2g_{2,3}(t) = -\xi^T(t) F^T \Omega_1 F \xi(t),$$

where Ω_1 is given by (9.8) with $i = 1$. The latter inequality holds if (9.8) is feasible [20]. Hence, (9.9) with $i = 1$ guarantees that $V_{RR}(t+1) - \lambda V_{RR}(t) \leq 0$ for $t \in [t_k, t_{k+1} - 1]$.

Similarly, for $t \in [t_{k+1}, t_{k+2} - 1]$, (9.8) and (9.9) with $i = 2$ guarantee $V_{RR}(t + 1) - \lambda V_{RR}(t) \leq 0$. Thus, (9.6) is exponentially stable with the decay rate λ.

9.3 Discrete-Time Networked Systems Under the Try-Once-Discard and Round-Robin Scheduling: A Hybrid Time-Delay Model

9.3.1 Problem Formulation

In [16], a weighted TOD protocol was analyzed for the stabilization of continuous-time NCSs. In this section, we consider discrete-time NCSs under TOD and RR scheduling via a hybrid delayed model.

Consider (9.1) with two sensor nodes $y_i(t) = C_i x(t), i = 1, 2$ and a sequence of sampling instants (9.2). At each sampling instant s_k, one of the outputs $y_i(t) \in \mathbb{R}^{n_i}$ $(n_1 + n_2 = n_y)$ is sampled and transmitted via the network. Denote by $\hat{y}(s_k) = \left[\hat{y}_1^T(s_k) \; \hat{y}_2^T(s_k) \right]^T \in \mathbb{R}^{n_y}$ the output information submitted to the scheduling protocol. At each sampling instant s_k, one of $\hat{y}_i(s_k)$ values is updated with the recent output $y_i(s_k)$.

It is assumed that no packet dropouts and no packet disorders will happen during the data transmission over the network. The transmission of the information over the network is subject to a variable delay η_k. Then $t_k = s_k + \eta_k$ is the updating time instant. As in the previous section, we allow the delays to be large provided that the old sample cannot get to the destination (to the controller or to the actuator) after the current one. Assume that the network-induced delay η_k and the time span between the updating and the current sampling instants satisfy (9.3).

Following [16], consider the error between the system output $y(s_k)$ and the last available information $\hat{y}(s_{k-1})$:

$$\begin{aligned}&e(t) = \text{col}\{e_1(t), e_2(t)\} \equiv \hat{y}(s_{k-1}) - y(s_k),\ t \in [t_k, t_{k+1} - 1],\\ &t \in \mathbb{Z}^+, k \in \mathbb{Z}^+,\ \hat{y}(s_{-1}) \stackrel{\Delta}{=} 0,\ e(t) \in \mathbb{R}^{n_y}.\end{aligned}$$

The control signal to be applied to the system (9.1) is given by

$$u(t) = K_{i_k^*} y_{i_k^*}(t_k - \eta_k) + K_i \hat{y}_i(t_{k-1} - \eta_{k-1})_{|i \neq i_k^*},\ t \in [t_k, t_{k+1} - 1],$$

where $K = [K_1\ K_2]$, $K_1 \in \mathbb{R}^{n_u \times n_1}$, $K_2 \in \mathbb{R}^{n_u \times n_2}$ such that $A + BKC$ is Schur. The closed-loop system can be presented as

$$\begin{aligned}&x(t+1) = Ax(t) + A_1 x(t_k - \eta_k) + B_i e_i(t)_{|i \neq i_k^*},\\ &e(t+1) = e(t),\ t \in [t_k, t_{k+1} - 2],\ t \in \mathbb{Z}^+,\end{aligned} \tag{9.10}$$

with the delayed reset system for $t = t_{k+1} - 1$

$$\begin{aligned}&x(t_{k+1}) = Ax(t_{k+1} - 1) + A_1 x(t_k - \eta_k) + B_i e_i(t_k)_{|i \neq i_k^*},\\ &e_i(t_{k+1}) = C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})],\ i = i_k^*,\\ &e_i(t_{k+1}) = e_i(t_k) + C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})],\ i \neq i_k^*,\end{aligned} \tag{9.11}$$

where $A_1 = BKC,\ B_i = BK_i,\ K = [K_1\ K_2],\ i = 1, 2$. The initial condition for (9.10), (9.11) has the form of $e(t_0) = -Cx(t_0 - \eta_0) = -Cx_0$ and (9.5). We will consider stability analysis of the discrete-time hybrid system (9.10), (9.11) under TOD and RR protocols described next.

9.3.2 *Scheduling Protocols*

9.3.2.1 TOD Protocol

Let $Q_1 > 0,\ Q_2 > 0$ be some weighting matrices. At the sampling instant s_k, the weighted TOD protocol is a protocol for which the active output node is defined as any index i_k^* that satisfies

$$|\sqrt{Q}_{i_k^*} e_{i_k^*}(t)|^2 \geq |\sqrt{Q}_i e_i(t)|^2,\ t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}^+,\ i = 1, 2. \tag{9.12}$$

A possible choice of i_k^* is given by

$$i_k^* = \min\{\arg \max_{i \in \{1,2\}} |\sqrt{Q}_i \left(\hat{y}_i(s_{k-1}) - y_i(s_k)\right)|^2\}.$$

9.3.2.2 RR Protocol

The active output node is chosen periodically:

$$i_k^* = i_{k+2}^*, \text{ for all } k \in \mathbb{Z}^+, \; i_0^* \neq i_1^*. \tag{9.13}$$

9.3.3 Stability Analysis of Discrete-Time Hybrid Delayed System Under TOD and RR Protocols

Consider the LKF of the form:

$$V_e(t) = V(t) + \frac{t_{k+1} - t}{\tau_M - \eta_m + 1} e_i^T(t_k) Q_i e_i(t_k)_{|i \neq i_k^*},$$

where

$$\begin{aligned} V(t) &= \tilde{V}(t) + V_Q(t), \\ V_Q(t) &= (\tau_M - \eta_m) \sum_{s=t_k-\eta_k}^{t-1} \lambda^{t-s-1} \zeta^T(s) Q \zeta(s), \\ \tilde{V}(t) &= x^T(t) P x(t) + \sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1} x^T(s) S_0 x(s) + \sum_{s=t-\tau_M}^{t-\eta_m-1} \lambda^{t-s-1} x^T(s) S_1 x(s) \\ &\quad + \eta_m \sum_{j=-\eta_m}^{-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \zeta^T(s) R_0 \zeta(s) \\ &\quad + (\tau_M - \eta_m) \sum_{j=-\tau_M}^{-\eta_m-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \zeta^T(s) R_1 \zeta(s), \end{aligned}$$

and

$$\begin{aligned} &\zeta(t+1) = x(t+1) - x(t), \; P > 0, \; S_i > 0, \; R_i > 0, \; Q > 0, \; Q_j > 0, \\ &0 < \lambda < 1, \; i = 0, 1, \; j = 1, 2, \; t \in [t_k, t_{k+1} - 1], \; t \in \mathbb{Z}^+, \; k \in \mathbb{Z}^+, \end{aligned}$$

where we define $x(t) = x_0, \; t \leq 0$. Our objective is to guarantee that

$$V_e(t+1) - \lambda V_e(t) \leq 0, \; t \in [t_k, t_{k+1} - 1], \; t \in \mathbb{Z}^+ \tag{9.14}$$

holds along (9.10), (9.11). The inequality (9.14) implies the following bound

$$\begin{aligned} &V(t) \leq V_e(t) \leq \lambda^{t-t_0} V_e(t_0), \; t \geq t_0, \; t \in \mathbb{Z}^+, \\ &V_e(t_0) \leq V(t_0) + \min_{i=1,2} \{|\sqrt{Q_i} e_i(t_0)|^2\}, \end{aligned} \tag{9.15}$$

for the solution of (9.10), (9.11) with the initial condition (9.5) and $e(t_0) \in \mathbb{R}^{n_y}$. Here we took into account that for the case of two sensor nodes

$$|\sqrt{Q_i}e_i(t_0)|^2_{|i\neq i_k^*} = \min_{i=1,2}\{|\sqrt{Q_i}e_i(t_0)|^2\}.$$

From (9.15), it follows that the system (9.10), (9.11) is exponentially stable with respect to x. The novel term $V_Q(t)$ of the LKF is inserted to cope with the delays in the reset conditions

$$\begin{aligned}
&V_Q(t_{k+1}) - \lambda V_Q(t_{k+1}-1)\\
&= (\tau_M - \eta_m)[\sum_{s=t_{k+1}-\eta_{k+1}}^{t_{k+1}-1} \lambda^{t_{k+1}-s-1}\zeta^T(s)Q\zeta(s) - \sum_{s=t_k-\eta_k}^{t_{k+1}-2}\lambda^{t_{k+1}-s-1}\zeta^T(s)Q\zeta(s)]\\
&\le (\tau_M - \eta_m)[\zeta^T(t_{k+1}-1)Q\zeta(t_{k+1}-1) - \lambda^{\tau_M}\sum_{s=t_k-\eta_k}^{t_{k+1}-\eta_{k+1}-1}\zeta^T(s)Q\zeta(s)]\\
&\le (\tau_M - \eta_m)\zeta^T(t_{k+1}-1)Q\zeta(t_{k+1}-1) - \lambda^{\tau_M}|\sqrt{Q}[x(t_{k+1}-\eta_{k+1}) - x(t_k-\eta_k)]|^2,
\end{aligned}$$

where we applied Jensen's inequality (see e.g., [9]). The term $\frac{t_{k+1}-t}{\tau_M-\eta_m+1}e_i^T(t_k)$ $Q_ie_i(t_k)$ is inspired by the similar construction of the LKF for the sampled-data systems [4]. We have

$$\begin{aligned}
&V_e(t_{k+1}) - \lambda V_e(t_{k+1}-1)\\
&= \tilde V(t_{k+1}) - \lambda\tilde V(t_{k+1}-1) + \frac{t_{k+2}-t_{k+1}}{\tau_M-\eta_m+1}e_i^T(t_{k+1})Q_ie_i(t_{k+1})_{|i\neq i_{k+1}^*}\\
&\quad - \frac{\lambda}{\tau_M-\eta_m+1}e_i^T(t_k)Q_ie_i(t_k)_{|i\neq i_k^*} + (\tau_M-\eta_m)\zeta^T(t_{k+1}-1)Q\zeta(t_{k+1}-1)\\
&\quad - \lambda^{\tau_M}|\sqrt{Q}[x(t_{k+1}-\eta_{k+1}) - x(t_k-\eta_k)]|^2.
\end{aligned}$$

Note that under TOD protocol for $i_{k+1}^* = i_k^*$

$$e_i^T(t_{k+1})Q_ie_i(t_{k+1})_{|i\neq i_{k+1}^*} \le e_{i_k^*}^T(t_{k+1})Q_{i_k^*}e_{i_k^*}(t_{k+1}), \tag{9.16}$$

whereas for $i_{k+1}^* \neq i_k^*$ the latter relation holds with equality. Under RR protocol we have $i_{k+1}^* \neq i_k^*$. Hence

$$\begin{aligned}
\frac{t_{k+2}-t_{k+1}}{\tau_M-\eta_m+1}e_i^T(t_{k+1})Q_ie_i(t_{k+1})_{|i\neq i_{k+1}^*} &\le e_{i_k^*}^T(t_{k+1})Q_ie_{i_k^*}(t_{k+1})\\
&= |\sqrt{Q_{i_k^*}}C_{i_k^*}[x(t_{k+1}-\eta_{k+1}) - x(t_k-\eta_k)]|^2.
\end{aligned}$$

Assume that

$$\lambda^{\tau_M}Q > C_i^TQ_iC_i,\ \ i=1,2. \tag{9.17}$$

Then for $t = t_{k+1} - 1$, we obtain

$$V_e(t+1) - \lambda V_e(t) \le \tilde{V}(t+1) - \lambda\tilde{V}(t) + (\tau_M - \eta_m)\zeta^T(t)Q\zeta(t) - \frac{\lambda}{\tau_M - \eta_m + 1} e_i^T(t_k)Q_i e_i(t_k)_{|i\ne i_k^*}.$$

Furthermore, due to

$$-\frac{\lambda}{\tau_M - \eta_m + 1} = -\frac{1}{\tau_M - \eta_m + 1} + \frac{1-\lambda}{\tau_M - \eta_m + 1} \le -\frac{1}{\tau_M - \eta_m + 1} + 1 - \lambda,$$

for $t = t_{k+1} - 1$, we arrive at

$$\begin{aligned} V_e(t+1) - \lambda V_e(t) &\le \tilde{V}(t+1) - \lambda\tilde{V}(t) + (\tau_M - \eta_m)\zeta^T(t)Q\zeta(t) \\ &-[\frac{1}{\tau_M - \eta_m + 1} - (1-\lambda)]e_i^T(t_k)Q_i e_i(t_k)_{|i\ne i_k^*} \stackrel{\Delta}{=} \Psi(t). \end{aligned} \tag{9.18}$$

For $t \in [t_k, t_{k+1} - 2]$, we have

$$V_e(t+1) - \lambda V_e(t) \le \tilde{V}(t+1) - \lambda\tilde{V}(t) + (\tau_M - \eta_m)\zeta^T(t)Q\zeta(t) + \left[\frac{t_{k+1} - t - 1}{\tau_M - \eta_m + 1} - \lambda\frac{t_{k+1} - t}{\tau_M - \eta_m + 1}\right] e_i^T(t_k)Q_i e_i(t_k)_{|i\ne i_k^*}.$$

Since

$$\begin{aligned} \frac{t_{k+1} - t - 1}{\tau_M - \eta_m + 1} - \lambda\frac{t_{k+1} - t}{\tau_M - \eta_m + 1} &= -\frac{1}{\tau_M - \eta_m + 1} + (1-\lambda)\frac{t_{k+1} - t}{\tau_M - \eta_m + 1} \\ &\le -\frac{1}{\tau_M - \eta_m + 1} + 1 - \lambda, \end{aligned}$$

we conclude that (9.18) is valid also for $t \in [t_k, t_{k+1} - 2]$. Therefore, (9.14) holds if

$$\Psi(t) \le 0, \quad t \in [t_k, t_{k+1} - 1]. \tag{9.19}$$

Note that $i \ne i_k^*$ for $i = 1, 2$ is the same as $i = 3 - i_k^*$. By using the standard arguments for the delay-dependent analysis [20], we derive the following conditions for (9.19) (and, thus for (9.15)):

Theorem 2 *Given scalar* $0 < \lambda < 1$, *positive integers* $0 \le \eta_m \le \eta_M < \tau_M$, *and* K_1, K_2, *if there exist* $n \times n$ *matrices* $P > 0$, $Q > 0$, $S_j > 0$, $R_j > 0$ $(j = 0, 1)$, S_{12}, $n_i \times n_i$ *matrices* $Q_i > 0$ $(i = 1, 2)$ *such that (9.17) and*

$$\Omega = \begin{bmatrix} R_1 & S_{12} \\ * & R_1 \end{bmatrix} \ge 0,$$

$$\hat{F}_0^T P \hat{F}_0 + \hat{\Sigma} + \hat{F}_{01}^T W \hat{F}_{01} - \lambda^{\eta_m} F_{12}^T R_0 F_{12} - \lambda^{\tau_M} \hat{F}^T \Omega \hat{F} < 0,$$

are feasible, where

$$\begin{aligned}
&\hat{F}_0 = [A\ 0\ A_1\ 0\ B_{3-i}],\\
&\hat{F}_{01} = [A - I\ 0\ A_1\ 0\ B_{3-i}],\\
&\hat{F} = \begin{bmatrix} 0 & I & -I & 0 & 0 \\ 0 & 0 & I & -I & 0 \end{bmatrix},\\
&\hat{\Sigma} = \text{diag}\{S_0 - \lambda P, -\lambda^{\eta_m}(S_0 - S_1), 0, -\lambda^{\tau_M} S_1, \varphi\},\\
&\varphi = -[\frac{1}{\tau_M - \eta_m + 1} - (1 - \lambda)]Q_{3-i},\\
&W = \eta_m^2 R_0 + (\tau_M - \eta_m)^2 R_1 + (\tau_M - \eta_m)Q, \quad i = 1, 2.
\end{aligned}$$

Then the solutions of the hybrid system (9.10), (9.11) satisfy the bound (9.15), implying exponential stability of (9.10), (9.11) with respect to x. Moreover, if the aforementioned inequalities are feasible with $\lambda = 1$, then the bound (9.15) holds with $\lambda = 1 - \varepsilon$, where $\varepsilon > 0$ is small enough.

Remark 1 The inequality $V_e(t) \le \lambda^{t-t_0} V_e(t_0),\ t \ge t_0,\ t \in \mathbb{Z}^+$ in (9.15) guarantees that

$$\frac{t_{k+1} - t_k}{\tau_M - \eta_m + 1} e_i^T(t_k) Q_i e_i(t_k)_{|i \neq i_k^*}$$

is bounded, and it does not guarantee that $e(t_k)$ is bounded. That is why (9.15) implies only partial stability with respect to x.

Remark 2 Note that for the stability analysis of discrete-time systems with time-varying delay in the state, a switched system transformation approach can be used in addition to a Lyapunov-Krasovskii method. See more details in [11].

Remark 3 Application of Schur complement leads the matrix inequalities of Theorems 1 and 2 to LMIs that are affine in the system matrices. Therefore, for the case of system matrices from the uncertain time-varying polytope

$$\begin{aligned}
&\tilde{\Theta} = \sum_{j=1}^{N} \tilde{g}_j(t)\tilde{\Theta}_j,\ 0 \le \tilde{g}_j(t) \le 1,\\
&\sum_{j=1}^{N} \tilde{g}_j(t) = 1,\ \tilde{\Theta}_j = \left[\, A^{(j)}\ B^{(j)} \,\right],
\end{aligned}$$

the LMIs need to be solved simultaneously for all N vertices $\tilde{\Theta}_j$, using the same decision matrices.

9.4 Example: Discrete-Time Cart-Pendulum

Consider the following linearized model of the inverted pendulum on a cart [15]:

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{a}{M} \\ 0 \\ \dfrac{-a}{Ml} \end{bmatrix} u(t), \ t \in \mathbb{R}^+$$

with $M = 3.9249\,\text{kg}, m = 0.2047\,\text{kg}, l = 0.2302\,\text{m}, g = 9.81\,\text{N/kg}, a = 25.3\,\text{N/V}$. In the model, x and θ represent cart position coordinate and pendulum angle from vertical, respectively. Such a model discretized with a sampling time $\text{T}_\text{s} = 0.001$ s:

$$\begin{bmatrix} x(t+1) \\ \Delta x(t+1) \\ \theta(t+1) \\ \Delta\theta(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.001 & 0 & 0 \\ 0 & 1 & -0.0005 & 0 \\ 0 & 0 & 1.00 & 0.001 \\ 0 & 0 & 0.0448 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ \Delta x(t) \\ \theta(t) \\ \Delta\theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0064 \\ 0 \\ -0.0280 \end{bmatrix} u(t), \ t \in \mathbb{Z}^+.$$

The pendulum can be stabilized by a state feedback $u(t) = K \begin{bmatrix} x & \Delta x & \theta & \Delta\theta \end{bmatrix}^T$ with the gain $K = [K_1 \ \ K_2]$

$$K_1 = \begin{bmatrix} 5.825 & 5.883 \end{bmatrix}, \ \ K_2 = \begin{bmatrix} 24.941 & 5.140 \end{bmatrix},$$

which leads to the closed-loop system eigenvalues $\{0.8997, 0.9980 + 0.0020i, 0.9980 - 0.0020i, 0.9980\}$. Suppose the variables θ, $\Delta\theta$ and x, Δx are not accessible simultaneously. We consider measurements $y_i(t) = C_i x(t), t \in \mathbb{Z}^+$, where

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the values of η_m given in Table 9.1, we apply Theorem 2 with $\lambda = 1$ and find the maximum values of MATI that preserve the stability of hybrid system (9.10), (9.11) with respect to x (see Table 9.1). From Table 9.1, it is observed that the presented TOD protocol, which possesses less decision variables in the LMI conditions, stabilizes the system for larger MATI than the RR protocol in Theorem 1. Moreover, when $\eta_m >$ MATI ($\eta_m \geq 4$), our method is still feasible (communication delays are larger than the sampling intervals).

Table 9.1 Example: maximum values of MATI for different $\eta_m = \eta_M$

MATI \ $\eta_m = \eta_M$	0	2	4	5	8	11	Decision variables
Theorem 1 (RR)	4	3	3	2	1	Infeasible	146
Theorem 2 (TOD/RR)	4	4	3	3	2	1	82

9.5 Conclusions

In this chapter, a time-delay approach has been developed for the stability analysis of discrete-time NCSs under the RR or under a weighted TOD scheduling. Polytopic uncertainties in the system model can be easily included in the analysis. A numerical example illustrated the efficiency of our method. It was assumed that no packet dropouts will happen during the data transmission over the network. Note that the time-delay approach has been developed for NCSs under stochastic protocols in [17], where the network-induced delays are allowed to be larger than the sampling intervals in the presence of collisions. For application of the presented approach in this chapter to discrete-time NCSs under scheduling protocols and actuator constraints, see [14].

Acknowledgments This work was partially supported by the Knut and Alice Wallenberg Foundation (grant no. Dnr KAW 2009.088), the Swedish Research Council (grant no. VR 621- 2014-6282), and the National Natural Science Foundation of China (grant no. 61503026, 61440058).

References

1. P. Antsaklis, J. Baillieul, Special issue on technology of networked control systems. Proc. IEEE **95**(1), 5–8 (2007)
2. M.B.G. Cloosterman, L. Hetel, N. van de Wouw, W.P.M.H. Heemels, J. Daafouz, H. Nijmeijer, Controller synthesis for networked control systems. Automatica **46**(10), 1584–1594 (2010)
3. M.C.F. Donkers, W.P.M.H. Heemels, N. van de Wouw, L. Hetel, Stability analysis of networked control systems using a switched linear systems approach. IEEE Trans. Autom. Control **56**(9), 2101–2115 (2011)
4. E. Fridman, A refined input delay approach to sampled-data control. Automatica **46**(2), 421–427 (2010)
5. E. Fridman, U. Shaked, Delay-dependent stability and H_∞ control: constant and time-varying delays. Int. J. Control **76**(1), 48–60 (2003)
6. E. Fridman, A. Seuret, J.-P. Richard, Robust sampled-data stabilization of linear systems: an input delay approach. Automatica **40**(8), 1441–1446 (2004)
7. H. Fujioka, A discrete-time approach to stability analysis of systems with aperiodic sample-and-hold devices. IEEE Trans. Autom. Control **54**(10), 2440–2445 (2009)
8. H. Gao, T. Chen, J. Lam, A new system approach to network-based control. Automatica **44**(1), 39–52 (2008)
9. K. Gu, V. Kharitonov, J. Chen, *Stability of Time-delay Systems* (Birkhauser, Boston, 2003)
10. W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, D. Nesic, Networked control systems with communication constraints: tradeoffs between transmission intervals, delays and performance. IEEE Trans. Autom. Control **55**(8), 1781–1796 (2010)
11. L. Hetel, J. Daafouz, C. Iung, Equivalence between the Lyapunov-Krasovskii functionals approach for discrete delay systems and that of the stability conditions for switched systems. Nonlinear Anal. Hybrid Syst. **2**(3), 697–705 (2008)
12. K. Liu, E. Fridman, Wirtinger's inequality and Lyapunov-based sampled-data stabilization. Automatica **48**(1), 102–108 (2012)
13. K. Liu, E. Fridman, Delay-dependent methods and the first delay interval. Syst. Control Lett. **64**, 57–63 (2014)
14. K. Liu, E. Fridman, Discrete-time network-based control under scheduling and actuator constraints. Int. J. Robust Nonlinear Control **25**, 1816183 (2014)

15. K. Liu, E. Fridman, L. Hetel, Stability and L_2-gain analysis of networked control systems under Round-Robin scheduling: a time-delay approach. Syst. Control Lett. **61**(5), 666–675
16. K. Liu, E. Fridman, L. Hetel, Network-based control via a novel analysis of hybrid systems with time-varying delays, in *IEEE Conference on Decision Control (CDC)* (2012)
17. K. Liu, E. Fridman, K.H. Johansson, Networked control with a stochastic scheduling: a time-delay approach, in *World Congress of the IFAC* (2014)
18. D. Nesic, D. Liberzon, A unified framework for design and analysis of networked and quantized control systems. IEEE Trans. Autom. Control **54**(4), 732–747 (2009)
19. D. Nesic, A.R. Teel, Input-output stability properties of networked control systems. IEEE Trans. Autom. Control **49**(10), 1650–1667 (2004)
20. P.G. Park, J.W. Ko, C. Jeong, Reciprocally convex approach to stability of systems with time-varying delays. Automatica **47**(1), 235–238 (2011)
21. A. Seuret, A novel stability analysis of linear systems under asynchronous samplings. Automatica **48**(1), 177–182 (2012)
22. W. Zhang, M.S. Branicky, S.M. Phillips, Stability of networked control systems. IEEE Control Syst. Mag. **21**(1), 84–99 (2001)

Chapter 10
Stabilization of Networked Control Systems with Hyper-Sampling Periods

Xu-Guang Li, Arben Çela and Silviu-Iulian Niculescu

Abstract This chapter considers the stabilization of Networked Control Systems (NCSs) under the hyper-sampling mode. Such a sampling mode, recently proposed in the literature, appears naturally in the scheduling policies of real-time systems under constrained (calculation and communication) resources. Meanwhile, as expected, the stabilization problem under the hyper-sampling mode is much more complicated than in the case of single-sampling mode. In this chapter, we propose a procedure to design the feedback gain matrix such that we can obtain a stabilizable region as large as possible. In the first step, we determine the stabilizable region under the single-sampling period. This step can be easily obtained by solving some linear matrix inequalities (LMIs) and from the result we may obtain a stabilizable region for the hyper-sampling period. Then, in the second step, we further detect the stabilizable region, based on the one found in the first step, by adjusting the feedback gain matrices based on the asymptotic behavior analysis. By this step, a larger stabilizable region may be found and this step can be used in an iterative manner. The proposed procedure will be illustrated by a numerical example. We can see from the example that the stabilizable region under the hyper-sampling period may lead to a smaller average sampling frequency (ASF) guaranteeing the stability of the NCS than the single-sampling period, i.e., less system resources are required by the hyper-sampling period.

X.-G. Li (✉)
School of Information Science and Engineering, Northeastern University,
Shenyang 110004, Liaoning, China
e-mail: masdanlee@163.com

A. Çela
UPE, ESIEE Paris, Computer Science and Telecommunication Department,
93162 Noisy-Le-Grand, France
e-mail: a.cela@esiee.fr

S.-I. Niculescu
Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506),
CNRS-CentraleSupélec-Université Paris-Sud, 91192 Gif-Sur-Yvette, France
e-mail: Silviu.Niculescu@l2s.centralesupelec.fr

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_10

10.1 Introduction

The stability of Networked Control Systems (NCSs) is an important research topic in the area of control, see e.g., [1, 11, 21]. Calculation and communication resources limitation, the distribution of calculation, actuation, and sensing devices, and the related scheduling render the analysis and design complex and challenging.

A simple model of an NCS is given by a sampled-data one with the single-sampling period (that is, an NCS essentially has one constant or time-varying sampling period). For such NCSs with single-sampling periods, there exists an abundant literature devoted to the stability and related problems.

In particular, a lot of recent studies focus on the robust stability and stabilization problems when NCSs are subject to uncertain sampling periods and/or network-induced delays (see, for instance, the delayed-input method used in [9], the small gain approach adopted in [17], and the convex-embedding approach used in [7, 10]).

For nominal NCS models (i.e., when the sampling periods, network-induced delays, and the system matrices are all fixed), an eigenvalue-based approach was introduced in [16] for characterizing the stability domain for the sampling period or network-induced delay. Such an approach represents a novelty in the domain. Recently, a hyper-sampling period, from a perspective of real-time systems, was proposed in [2, 5] (see also [4] for a more detailed introduction). Compared with the single-sampling counterpart, a hyper-sampling period consists of multiple sub-sampling periods and hence provides with a more flexible and realistic sampling mechanism for NCSs.

In the hyper-sampling context, stability conditions were studied in [13], where a discrete-time model and some robust analysis techniques similar to the ones used in [8, 18, 19] were adopted. It was shown in [13] that an NCS can be asymptotically stable with less system resources consumption. In [13] the feedback gain matrix is supposed to be designed a priori and the stability region with respect to the sub-sampling periods are explicitly studied. One may naturally predict that system resources may be further saved if both the hyper-sampling period and the feedback gain matrix are considered as *free* design parameters. This motivates us to consider in this chapter the following *co-design problem*:

For an NCS under the hyper-sampling period, design the feedback gain matrices such that the obtained stabilizable region in the space of sub-sampling parameters is as large as possible.

For simplicity, we assume in this chapter that the hyper-sampling period has two sub-sampling periods T_1 and T_2 which practically means that a control task is executed twice in each hyper-sampling period. We believe that the method proposed and the results obtained here may be extended to the case involving more sub-sampling periods. When the hyper-sampling period is assumed to have two sub-sampling periods T_1 and T_2, the value of $1/(T_1 + T_2)$ corresponds to an index *average sampling frequency* (ASF). It is not hard to see that if an NCS can be stabilized by a hyper-sampling period with a smaller ASF, then less calculation and communication resources are consumed (the consumption of resources is in general proportional to the ASF). Therefore, in this chapter, we will consider how to find a stabilizable region for an NCS in the $T_1 - T_2$ plane with the values $T_1 + T_2$ as large as possible.

Due to the complexity of the problem, we start with the specific case where $T_1 = T_2$ (instead of direct studying the general case) and calculate the maximal stabilizable bound $\overline{T}$, i.e., the NCS can be stabilized by some (not necessary one) feedback gain matrices if $0 < T_1 = T_2 < \overline{T}$. We will show that for this specific case (corresponding to the single-sampling case), the maximal stabilizable bound $\overline{T}$ can be easily obtained through a *necessary and sufficient condition* in terms of linear matrix inequalities (LMIs) . We next seek the stabilizable region in the hyper-sampling case. It should be emphasized that, unlike for the single-sampling case, for a given hyper-sampling period (T_1, T_2) there does not exist a direct way so far to find a stabilizing feedback gain matrix K (the corresponding conditions are in terms of nonlinear matrix inequalities).

In this chapter, we will propose an eigenvalue-based procedure to iteratively adjust the feedback gain matrix K. First, from the results for the single-sampling case, we may obtain a stabilizable region for the hyper-sampling period in the $T_1 - T_2$ plane. This stabilizable region is dented by $S^{(0)}$, whose boundary is denoted by $B^{(0)}$ and can be detected by parameter sweeping. Each point on $B^{(0)}$ must correspond to a feedback gain matrix with which the NCS has characteristic roots located on the unit circle, called critical characteristic roots or eigenvalues.[1] Then, we study the asymptotic behavior of these critical characteristic roots with respect to the elements of K such that we know the way to adjust K in order to have a larger stabilizable region. In this way, we will have some new feedback gain matrices leading to a new stabilizable region $S^{(1)}$, whose boundary is $B^{(1)}$. Next, the above step may be applied to the points on $B^{(1)}$ and then obtain one more stabilizable region $S^{(2)}$. By repeating this method in an iterative manner, we may obtain new stabilizable regions $S^{(3)}$, $S^{(4)}$, ... Finally, the combination $S^{(0)} \cup S^{(1)} \cup \cdots$ is the overall stabilizable region we detect.

The asymptotic behavior analysis is a relatively new approach for the analysis and design of NCSs (see e.g., [16]), and, to the best of the authors' knowledge, was not sufficiently exploited in the community of NCSs. In this chapter, we will only study the case with only simple critical characteristic roots (i.e., we suppose that no multiple critical characteristic roots appear). One may refer to [14] for a general method for asymptotic behavior analysis. The proposed procedure will be illustrated by a numerical example. In addition, we can see from the example that, compared with the single-sampling period, a smaller ASF guaranteeing the NCS stability can be obtained by the hyper-sampling period. That is to say, calculation and communication resources may be saved by adopting the hyper-sampling period.

This chapter is organized as follows. In Sect. 10.2, some preliminaries are given. In Sect. 10.3, the stabilization for NCSs under the single-sampling mode is considered. A procedure for designing the feedback gain matrices under the hyper-sampling mode

[1] In this chapter, the words "eigenvalue" and "characteristic root" may be used alternatively.

is proposed in Sect. 10.4. An illustrative example is given in Sect. 10.5. Finally, some concluding remarks end this chapter in Sect. 10.6.

Notations: In this chapter, the following standard notations will be used: $\mathbb{R}$ ($\mathbb{R}_+$) is the set of (positive) real numbers; $\mathbb{N}$ is the set of non-negative integers and $\mathbb{N}_+$ is the set of positive integers. Next, I is the identity matrix with appropriate dimensions. For a matrix A, A' denotes its transpose. We denote by $\rho(A)$ the spectral radius of matrix A. Finally, $A > 0$ implies that A is positive definite.

10.2 Preliminaries

The controlled plant of a networked control system (NCS) is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{10.1}$$

where $x(t)$ and $u(t)$ denote, respectively, the system state and control input at time t, and A and B are constant matrices with appropriate dimensions. It is a trivial assumption that A is not Hurwitz, otherwise the system is open-loop stable and less interesting for study. At a sampling instant t_k ($k \in \mathbb{N}$), the control input to the plant (1) is updated to $u(t_k)$. Implemented with the Zero-Order-Hold (ZOH) devices, the control signal is

$$u(t) = u(t_k), \ t_k \le t < t_{k+1}. \tag{10.2}$$

We employ the commonly used state feedback control:

$$u(t_k) = Kx(t_k), \tag{10.3}$$

where K is the feedback gain matrix, to be designed in this chapter. The closed-loop of the NCS can be expressed by the following discrete-time model

$$x(t_{k+1}) = \Phi(T(k), K)x(t_k), \ k \in \mathbb{N}, \tag{10.4}$$

where $T(k) \overset{\Delta}{=} t_{k+1} - t_k$ denote the sampling periods and $\Phi(T(k), K)$ is the transition matrix function defined by

$$\Phi(\alpha, \beta) = e^{A\alpha} + \int_0^{\alpha} e^{A\theta} d\theta B\beta. \tag{10.5}$$

Introducing $\widetilde{A}(T(k)) = e^{AT(k)}$ and $\widetilde{B}(T(k)) = \int_0^{T(k)} e^{A\theta} d\theta B$, we may rewrite $\Phi(T(k), K)$ as

$$\Phi(T(k), K) = \widetilde{A}(T(k)) + \widetilde{B}(T(k))K. \tag{10.6}$$

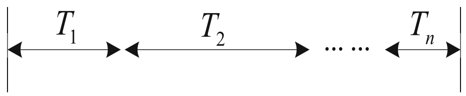

Fig. 10.1 A hyper-sampling period

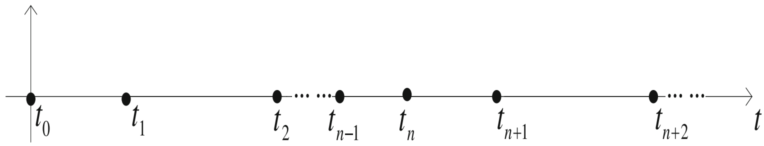

Fig. 10.2 Sampling instants under the hyper-sampling mode

In the control area, the most used sampling mode is the *standard*[2] single-sampling mode. In particular, throughout this chapter we focus on the nominal cases and the sampling periods $T(k) \triangleq t_k - t_{k-1}$ are a constant value, denoted by T, for all $k \in \mathbb{N}_+$ under the single-sampling mode.

A well-known necessary and sufficient stability condition for the NCS under the standard single-sampling mode is as follows (see e.g., [1, 6]).

Lemma 1 *The networked control system described by (10.1)–(10.3) with a constant sampling period T is asymptotically stable if and only if the transition matrix $\Phi(T, K)$ is Schur.*

The stabilization problem in the single-sampling case is relatively simple to solve (details will be given in Sect. 10.3). As earlier mentioned, the main objective of this chapter is to study the stabilization under the hyper-sampling mode. In general, a hyper-sampling period is composed of $n \in \mathbb{N}_+$ sub-sampling periods $T_i \in \mathbb{R}_+, i = 1, \ldots, n$, as depicted in Fig. 10.1. The n sub-sampling periods are allowed to be different from each other.

Under the hyper-sampling mode, the sampling instants t_k ($k \in \mathbb{N}$) are as depicted in Fig. 10.2. It follows that $t_1 - t_0 = T_1$ ($t_0 = 0$), $t_2 - t_1 = T_2, \ldots, t_n - t_{n-1} = T_n$, $t_{n+1} - t_n = T_1$, $t_{n+2} - t_{n+1} = T_2, \ldots$. That is, the sampling periods $T(k)$ are generated periodically according to hyper-sampling period. More precisely, they are generated by the following rule:

$$T(k) = t_{k+1} - t_k = \begin{cases} T_{k+1 \bmod n}, k+1 \mod n \neq 0, k \in \mathbb{N}, \\ T_n, k+1 \mod n = 0, k \in \mathbb{N}, \end{cases} \tag{10.7}$$

where the notation of *modulo operation* "$a \mod b$" denotes the remainder of division of a by b.

[2] We use the term "standard" simply to differ with the hyper-sampling mode discussed in this chapter.

Remark 1 It is easy to see that the hype-sampling mode reduces to the single-sampling mode when $n = 1$. Thus, we may treat the single-sampling mode as a special case of the hyper-sampling mode. In general, a larger n offers more design flexibility of the hyper-sampling period. Meanwhile, the price to be paid for a larger n is the increased solution complexity of the stabilization problem.

For a hyper-sampling period with n sub-sampling periods $T_1, \ldots, T_n$, we define the *Average Sampling Frequency* (ASF) as follows:

$$f_{\mathscr{A}} = \frac{n}{\sum_{i=1}^{n} T_i}. \tag{10.8}$$

Remark 2 The concept of ASF is easy to understand: On average, in a unit of time the system state is sampled $f_{\mathscr{A}}$ times (or, equivalently, the system state is sampled once in $1/f_{\mathscr{A}}$ units of time). The value of $f_{\mathscr{A}}$ corresponds to the calculation and communication resources consumed by a control task, since a sample instant is associated with a series of actions including state sampling by sensor, data transition over network, calculation of control input by controller, and updating of control input by actuator. The higher the ASF is, more resources are consumed.

For the sake of simplicity, in this chapter we study the case $n = 2$. That is, the hype-sampling period is assumed to contain two sub-sampling periods T_1 and T_2. In this context, a hyper-sampling period can be denoted by a pair (T_1, T_2). In our opinion, the results of this chapter can be extended to the case with more sub-sampling periods. Following the analysis in [13], we have the following necessary and sufficient stability condition:

Lemma 2 *The networked control system described by (10.1)–(10.3) with two sub-sampling periods T_1 and T_2 is asymptotically stable if and only if $\Phi(T_1, K)\Phi(T_2, K)$ (or, equivalently $\Phi(T_2, K)\Phi(T_1, K)$) is Schur.*

Lemma 2 is based on the discrete-time expression of the NCS:

$$x(t_{k+2}) = \Phi(T_2, K)\Phi(T_1, K)x(t_k),$$

if k is even while $x(t_{k+2}) = \Phi(T_1, K)\Phi(T_2, K)x(t_k)$ if k is odd. Next, the equivalence (from the stability point of view) between $\Phi(T_1, K)\Phi(T_2, K)$ and $\Phi(T_2, K)\Phi(T_1, K)$ is due to the following immediate yet important property:

Propertry 1 For two square matrices Q_1 and Q_2, the matrices Q_1Q_2 and Q_2Q_1 have the same eigenvalues.

In view of Lemma 2, we define

$$\Phi_H(T_1, T_2, K) = \Phi(T_1, K)\Phi(T_2, K),$$

and we know that the NCS is asymptotically stable if and only if $\Phi_H(T_1, T_2, K)$ is Schur (it is equivalent if we define $\Phi_H(T_1, T_2, K)$ as $\Phi(T_2, K)\Phi(T_1, K)$). Next, we clarify the notions "*stabilizable hyper-sampling period*", "*stabilizable point*", and "*stabilizable region*", to be frequently used in this chapter.

A hyper-sampling period (T_1, T_2) is called a stabilizable one, if there exists a feedback gain matrix K stabilizing the closed-loop NCS (i.e., there exists a K such that $\Phi_H(T_1, T_2, K)$ is Schur). The corresponding point, with coordinate (T_1, T_2) in the $T_1 - T_2$ parameter plane, is called a *stabilizable point*. A stabilizable region refers to the set of stabilizable points in the $T_1 - T_2$ parameter plane.

Note that a stabilizable region generally corresponds to multiple different feedback gain matrices. For instance, if for a feedback gain matrix K_α (K_β), the NCS is asymptotically stable when (T_1, T_2) lies in a region S_{K_α} (S_{K_β}), then S_{K_α} (S_{K_β}) is a stabilizable region. Furthermore, $S_{K_\alpha} \cup S_{K_\beta}$ is a larger stabilizable region and a stabilizing K for all $(T_1, T_2) \in S_{K_\alpha} \cup S_{K_\beta}$ does not necessarily exist.

Remark 3 According to Property 1, if (T_1^*, T_2^*) is a stabilizable point if and only if (T_2^*, T_1^*) is a stabilizable point. Therefore, it suffices to consider only the domain with $T_1 \leq T_2$ ($T_1 > 0, T_2 > 0$). For an obtained stabilizable region S^* therein, there must be a stabilizable region $S^\sharp$ in the domain with $T_2 \leq T_1$ ($T_1 > 0, T_2 > 0$), such that S^* and $S^\sharp$ are symmetric with respect to the line $T_1 = T_2$.

In the sequel, we will first study the stabilization problem in the case of single-sampling period. Next, we will study the stabilization problem in the case of hyper-sampling period based on the obtained results for the single-sampling period case.

10.3 Stabilization of NCS Under Single-Sampling Mode

Although the result given below is not new, it will represent the staring point of our study in handling the case of hyper-sampling period.

Lemma 3 *Consider a networked control system described by (10.1)–(10.3) under the single-sampling mode. For a given sampling period T, the networked control system is stabilizable if and only if there exist a positive-definite matrix P and a matrix Y such that the following linear matrix inequality (LMI) is feasible*

$$\begin{pmatrix} -P & \widetilde{A}(T)P + \widetilde{B}(T)Y \\ (\widetilde{A}(T)P + \widetilde{B}(T)Y)' & -P \end{pmatrix} < 0. \quad (10.9)$$

If the LMI (10.9) is feasible, we have a feedback gain matrix $K = YP^{-1}$ with which the networked control system is asymptotically stable.

Proof The condition of Lemma 3 can be easily developed from a standpoint of discrete-time system. For a given T, the NCS is stabilizable if and only if there exists a feedback gain matrix K and a positive-definite matrix P such that:

$$(\widetilde{A}(T) + \widetilde{B}(T)K)P(\widetilde{A}(T) + \widetilde{B}(T)K)' - P < 0,$$

which is equivalent to the condition:

$$(\widetilde{A}(T) + \widetilde{B}(T)Y)P^{-1}(\widetilde{A}(T) + \widetilde{B}(T)Y)' - P < 0, \tag{10.10}$$

where $Y = KP$. The condition (10.10) can be equivalently transformed into the LMI form (10.9) by using the Schur complement properties (see [3]). □

Lemma 3 can be easily implemented by using the LMI toolbox in MATLAB. Thus, for any single-sampling period T, we may precisely determine if the NCS is stabilizable and obtain a corresponding feedback gain matrix K if stabilizable.

Furthermore, by sweeping T and using Lemma 3, we may accurately find the stabilizable interval $T \in (0, \overline{T})$ under the single-sampling mode (note that this result is without conservatism). Then, we choose some $T_{0,i}$ such that $0 < T_{0,1} < T_{0,2} < \cdots < \overline{T}$ and for each $T_{0,i}$ we have a stabilizing feedback gain matrix, denoted by $K_i^{(0)}$. Each $K_i^{(0)}$ provides with a stabilizable region in the $T_1 - T_2$ plane near $(T_1 = T_{0,i}, T_2 = T_{0,i})$, denoted by $S_i^{(0)}$. Note that $S_i^{(0)}$ can be obtained by parameter sweeping.

The combination of all $S_i^{(0)}$, $S_1^{(0)} \cup S_2^{(0)} \cup \cdots$, constitute a (larger) stabilizable region $S^{(0)}$ in the $T_1 - T_2$ plane. The boundary of $S^{(0)}$ is denoted by $B^{(0)}$. Note that $S^{(0)}$ is a stabilizable region for the hyper-sampling mode, though it is obtained from the results for the stabilization of the single-sampling case.

In the next section, we will further enlarge the stabilizable region for the hyper-sampling mode based on $S^{(0)}$.

10.4 Stabilization of NCS Under Hyper-sampling Mode

First of all, it should be noticed that, unlike for the single-sampling case, it is difficult to determine if an NCS is stabilizable for a given hyper-sampling period (T_1, T_2) and to find the corresponding stabilizing feedback gain matrix K (if stabilizable). If we straightforwardly follow the idea in Sect. 10.3, we need to find a positive-definite matrix P and a feedback gain matrix K such that the following matrix inequality holds:

$$\Phi_H(T_1, T_2, K)P\Phi_H^{'}(T_1, T_2, K) - P < 0. \tag{10.11}$$

However, the condition (10.11) is a nonlinear matrix inequality and it is not easy to equivalently transform it into a linear one. To the best of the authors' knowledge, it is difficult to directly find a K satisfying condition (10.11). The problem will become more involved when $n > 2$.

Instead of trying to give a direct procedure, in the sequel, we will take advantage of the results obtained for the single-sampling case to design the feedback gain matrix

K for the case of hyper-sampling period. From the results proposed in Sect. 10.3, we have a stabilizable region $S^{(0)}$ with its boundary $B^{(0)}$ in the $T_1 - T_2$ plane. For every point (T_1, T_2) on $B^{(0)}$, there is a K with which $\Phi_H(T_1, T_2, K)$ has an eigenvalue located on the unit circle (such an eigenvalue is called a "critical" one).

To simplify the analysis of this chapter, we adopt the following assumption.

Assumption 1 All the critical eigenvalues of the closed-loop system are simple.

Remark 4 If multiple critical eigenvalues appear, the problem will generally become more complicated and we may invoke the Puiseux series to treat such a case (see e.g., [14] for the analysis of multiple critical roots for time-delay systems).

Remark 5 If λ^* is a critical eigenvalue, its conjugate $\overline{\lambda^*}$ is also a critical eigenvalue and the variations of λ^* and $\overline{\lambda^*}$ as K varies are symmetric with respect to the real axis. Thus, it is sufficient to analyze either of them.

In the sequel, we will design K through analyzing the asymptotic behavior of the critical eigenvalues with respect to the elements of K. Without any loss of generality, suppose K has $m \in \mathbb{N}_+$ elements. For instances, a 1×3 K has 3 elements and it can be expressed by $(k_1\ k_2\ k_3)$; a 2×2 K has 4 elements and it can be expressed by $\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$. The elements of K are expressed by k_γ, $\gamma = 1, \ldots, m$. The characteristic function for the transition matrix of an NCS under the hyper-sampling mode can be denoted by:

$$f(\lambda, T_1, T_2, k_1, \ldots, k_m) = \det(\lambda I - \Phi_H(T_1, T_2, K)). \tag{10.12}$$

By the implicit function theorem (see e.g., [12, 20]), we have the following theorem:

Theorem 2 *Suppose when* $\lambda = \lambda^*, T_1 = T_1^*, T_2 = T_2^*, k_1 = k_1^*, \ldots, k_m = k_m^*$,

$$f(\lambda, T_1, T_2, k_1, \cdots k_m) = 0$$

and $f_\lambda \neq 0$. *As* k_γ *vary near* k_γ^* $(\gamma = 1, \ldots, m)$, $f(\lambda, T_1, T_2, k_1, \ldots, k_m) = 0$ *uniquely determines a characteristic root* $\lambda(k_1, \ldots, k_m)$ *with* $\lambda(k_1^*, \ldots, k_m^*) = \lambda^*$ *and* $\lambda(k_1, \ldots, k_m)$ *has continuous partial derivatives*

$$\frac{\partial \lambda}{\partial k_1} = -\frac{f_{k_1}}{f_\lambda}, \ldots, \frac{\partial \lambda}{\partial k_m} = -\frac{f_{k_m}}{f_\lambda}.$$

According to Theorem 2, we may express the asymptotic behavior of λ with respect to the elements of the feedback gain matrix K by the following (first-order) Taylor series

$$\Delta\lambda = C_1 \Delta(k_1) + \cdots + C_m \Delta(k_m) + o(\Delta(k_1), \ldots, \Delta(k_m)), \tag{10.13}$$

where

$$C_\gamma = \frac{\partial \lambda}{\partial k_\gamma}, \gamma = 1, \ldots, m.$$

Remark 6 In this chapter, we only invoke the first-order terms of the Taylor series. If needed, we may further invoke higher-order terms. One may refer to [15] concerning the degenerate case for time-delay systems, where invoking the first-order terms is not sufficient for the stability analysis.

For a critical characteristic root λ (i.e., $|\lambda| = 1$), from the stability point of view, we are interest in the direction of $\Delta\lambda$ with respect to the unit circle. If the direction points inside the unit circle, the variation makes the system asymptotically stable. Equivalently, we may consider the variation of the norm of the critical characteristic root λ, i.e., $\Delta(|\lambda|)$. Such an analysis can be fulfilled by computing the projection of $\Delta\lambda$ on the normal line of the unit circle at λ. We have the following theorem.

Theorem 3 *Suppose when* $\lambda = \lambda^*, T_1 = T_1^*, T_2 = T_2^*, k_1 = k_1^*, \ldots, k_m = k_m^*$, $f(\lambda, T_1, T_2, k_1, \cdots k_m) = 0$ *and* $f_\lambda \neq 0$. *As* k_γ *vary near* k_γ^* *(*$\gamma = 1, \ldots, m$*), it follows that*

$$\Delta(|\lambda|) = \left(\mathrm{Re}(\lambda^*)\ \mathrm{Im}(\lambda^*)\right) \cdot \left(\mathrm{Re}(\Delta\lambda)\ \mathrm{Im}(\Delta\lambda)\right).$$

We now apply Theorem 3 to adjust K in order to have a larger stabilizable region.

We choose some points on $B^{(0)}$, denoted by $(T_{1,i}^{(0)}, T_{2,i}^{(0)})$ $(i = 1, 2, \ldots)$. Each $(T_{1,i}^{(0)}, T_{2,i}^{(0)})$ corresponds to a $K_i^{(0)}$ (whose elements are denoted by $k_{i,1}^{(0)}, \ldots, k_{i,m}^{(0)}$) and $\lambda_i^{(0)}$ with $\left|\lambda_i^{(0)}\right| = 1$ such that $f(\lambda_i^{(0)}, T_{1,i}^{(0)}, T_{2,i}^{(0)}, k_{i,1}^{(0)}, \ldots, k_{i,m}^{(0)}) = 0$. Then, we may adjust $K_i^{(0)}$ according to Theorem 3 to find a new feedback gain matrix, denote by $K_i^{(1)}$ such that the NCS with $K_i^{(1)}$ is asymptotically stable near $(T_{1,i}^{(0)}, T_{2,i}^{(0)})$.

More precisely, for each element k_γ we may know from Theorem 3 the following. Suppose other elements of K are fixed, a sufficient small increase (decrease) of k_γ at $k_{i,\gamma}^{(0)}$ makes $\lambda_i^{(0)}$ move inside the unit circle if

$$\left(\mathrm{Re}(\lambda^*)\ \mathrm{Im}(\lambda^*)\right) \cdot \left(\mathrm{Re}(C_\gamma)\ \mathrm{Im}(C_\gamma)\right) > 0 \quad (< 0).$$

With this property, we may adjust all elements $k_{i,\gamma}^{(0)}, \gamma = 1, \ldots, m$, appropriately to find a new stabilizing feedback gain matrix.

From each $K_i^{(1)}$ we have a new stabilizable region near $(T_{1,i}^{(0)}, T_{2,i}^{(0)})$, denoted by $S_i^{(1)}$. The combination of all $S_i^{(1)}$, $S_1^{(1)} \cup S_2^{(1)} \cup \cdots$, constitute a larger stabilizable region $S^{(1)}$ in the $T_1 - T_2$ plane. The boundary of $S^{(1)}$ is denoted by $B^{(1)}$.

The above step can be used iteratively such that a sequence of new stabilizable regions $S^{(2)}, S^{(3)}, \ldots$ can be obtained. This is the procedure, proposed in this chapter, for solving the stabilization problem in the hyper-sampling case and can be summarized as below.

Procedure for stabilization of NCSs with hyper-sampling period:

- Step 1: Using the method proposed in Sect. 10.3, we find the stabilizable region $S^{(0)}$, and the boundary of $S^{(0)}$, $B^{(0)}$, in the $T_1 - T_2$ plane. Let $l = 0$.
- Step 2: Choose some points on $B^{(l)}$, $(T_{1,i}^{(l)}, T_{2,i}^{(l)})$, $i = 1, 2, \ldots$ Each $(T_{1,i}^{(l)}, T_{2,i}^{(l)})$ corresponds to a $K_i^{(l)}$ such that $\rho(\Phi_H(T_{1,i}^{(l)}, T_{2,i}^{(l)}, K_i^{(l)})) = 1$. Then, we adjust the elements of $K_i^{(l)}$ according to Theorem 3 to find a new feedback gain matrix $K_i^{(l+1)}$ associated with a new stabilizable region $S_i^{(l+1)}$. The combination of all $S_i^{(l+1)}$ form a new stabilizable region $S^{(l+1)}$, whose boundary is $B^{(l+1)}$.
- Step 3: If we want to further detect the stabilizable region in the $T_1 - T_2$ plane, let $l = l + 1$ and return to Step 2. Otherwise or when it is hard to find a larger stabilizable region by Step 2, the procedure stops. The combination $S^{(0)} \cup \cdots \cup S^{(l)}$ is the overall stabilizable region we find.

Remark 7 The above procedure is not very simple to use and the computational effort may further increase if we choose more points ont the boundaries. However, as the procedure can be implemented off-line, the computational complexity is not an important issue here.

10.5 Illustrative Example

In the sequel, the procedure proposed in Sect. 10.4 will be illustrated by a numerical example.

Example 1 Consider an NCS with the controlled plant (10.1) with

$$A = \begin{bmatrix} 12 & 1 \\ 1 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}.$$

We first employ Step 1 to find the stabilizable region $S^{(0)}$. The stabilizable interval under the single-sampling mode is $T \in (0, 1.26)$. That is, the minimal stabilizable ASF under the single-sampling mode is $f_{\mathscr{A}} = \frac{1}{1.26} = 0.79$. The stabilizable region $S^{(0)}$ is shown in Fig. 10.3.

Next, on the boundary of $S^{(0)}$, $B^{(0)}$, we choose some $(T_{1,i}^{(0)}, T_{2,i}^{(0)})$ and apply Step 2 to adjust the corresponding $K_i^{(0)}$. As a consequence, we may find some new feedback gain matrices $K_i^{(1)}$ and a new stabilizable region $S^{(1)}$ with the boundary $B^{(1)}$.

We may repeat the above step (Step 2), and, as a consequence, we find a sequence of new stabilizable regions $S^{(2)}$, $S^{(3)}$, $S^{(4)}$, as shown in Fig. 10.3, with the boundaries $B^{(2)}$, $B^{(3)}$, $B^{(4)}$. If needed, we may obtain more stabilizable regions by repeating step 2 for more times.

We see from Fig. 10.3 that each time a new stabilizable region (with larger values of $T_1 + T_2$) is found, a smaller ASF can be obtained. For instance, we

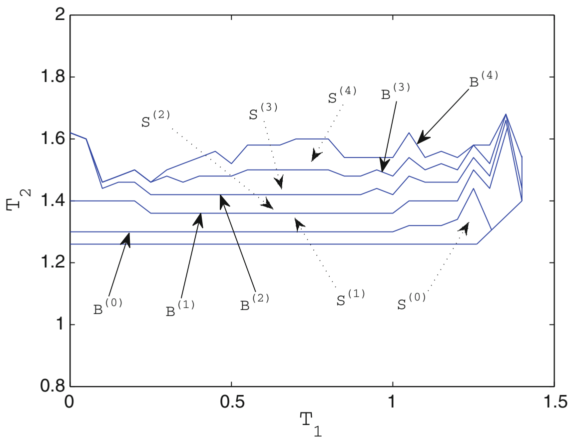

Fig. 10.3 Stabilizable region found for Example 1

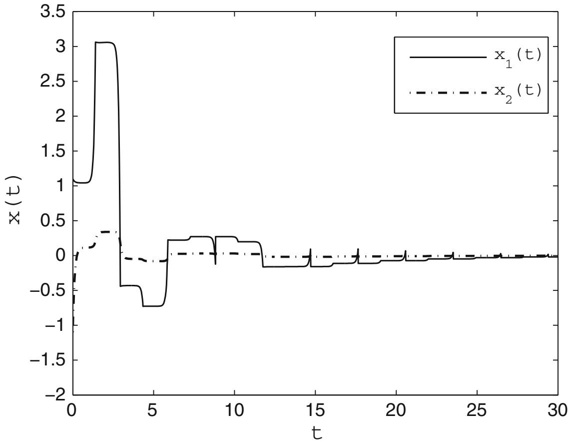

Fig. 10.4 State evolution $x(t)$ for Example 1 (initial condition $x(0) = (1.1\ -1.1)'$)

find a hyper-sampling period ($T_1 = 1.40$, $T_2 = 1.54$) with the corresponding $K = (-120.475116\ -5.723971)$, in the obtained stabilizable region. That is, the ASF corresponding to this hyper-sampling period is $f_{\mathscr{A}} = \frac{2}{1.40+1.54} = 0.68$, smaller than the minimal ASF under the single-sampling mode, 0.79. To illustrate the asymptotic stability of the NCS under this hyper-sampling period, we give the state response with an initial state $x(0) = (1.1\ -1.1)'$ in Fig. 10.4.

10.6 Concluding Remarks

In this chapter, we proposed a procedure for the stabilization of networked control systems (NCSs) under the hyper-sampling mode. The procedure consists of two steps.

Step 1 is to solve the stabilization problem in the case of single-sampling period and we can obtain a stabilizable region for the hyper-sampling period from this step. Step 2 is to find a larger stabilizable region based on the results of Step 1 by using a method for asymptotic behavior analysis. Step 2 can be used in an iterative manner such that the stabilizable region can be further detected in the parameter plane.

An example illustrates the proposed procedure and shows that the hyper-sampling period may lead to a smaller average sampling frequency (ASF) guaranteeing the asymptotic stability of the NCS than the single-sampling period, which means that calculation and communication resources of an NCS can be saved by using the hyper-sampling mode.

Acknowledgments Xu-Guang LI's work is partially supported by National Natural Science Foundation of China (No. 61473065). The illustrative example of this chapter is finished with the help of Xian-Feng DENG, Yan-Jun DONG, Dun-Jian LIU, and Shan-Shan JIANG (postgraduate students in Northeastern University, China).

References

1. K.J. Åström, B. Wittenmark, *Computer-Controlled Systems*, 3rd edn. (Prentice Hall, 1997)
2. M. Ben Gaid, A. Çela, Y. Hamam, Optimal integrated control and scheduling of networked control systems with communication constraints: application to a car suspension system. IEEE Trans. Control Syst. Technol. **14**(4), 776–787 (2006)
3. S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory* (SIAM, 1994)
4. A. Çela, M. Ben Gaid, X.-G. Li, S.-I. Niculescu, *Optimal Design of Distributed Control and Embedded Systems* (Springer, 2014)
5. A. Cervin, Integrated Control and Real-Time Scheduling. Ph.D. thesis, Department of Automatic Control, Lund Institute of Technology, Sweden (2003)
6. T. Chen, B. Francis, *Optimal Sampled-Data Control Systems* (Springer, 1995)
7. M.B.G. Cloosterman, L. Hetel, N. Van deWouw, W.P.M.H. Heemels, J. Daafouz, H. Nijmeijer, Controller synthesis for networked control systems. Automatica **46**(10), 1584–1594 (2010)
8. H. Fujioka, A discrete-time approach to stability analysis of systems with aperiodic sample-and-hold devices. IEEE Trans. Autom. Control **54**(10), 2440–2445 (2009)
9. E. Fridman, A. Seuret, J.-P. Richard, Robust sampled-data stabilization of linear systems: an input delay approach. Automatica **40**(8), 1441–1446 (2004)
10. R.H. Gielen, S. Olaru, M. Lazar, W.P.M.H. Heemels, N. van de Wouw, S.-I. Niculescu, On polytopic inclusions as a modelling framework for systems with time-varying delays. Automatica **46**(3), 615–619 (2010)
11. D. Hristu-Varsakelis, W.S. Levine (eds.), *Handbook of Networked and Embedded Control Systems* (Birkhäuser, 2005)
12. S.G. Krantz, H.R. Parks, *The Implicit Function Theorem: History, Theory, and Applications* (Springer, 2002)
13. X.-G. Li, A. Çela, S.-I. Niculescu, A. Reama, Some problems in the stability of networked-control systems with periodic scheduling. Int. J. Control **83**(5), 996–1008 (2010)
14. X.-G. Li, S.-I. Niculescu, A. Çela, H.-H. Wang, T.-Y. Cai, On computing Puiseux series for multiple imaginary characteristic roots of LTI systems with commensurate delays. IEEE Trans. Autom. Control **58**(5), 1338–1343 (2013)
15. X.-G. Li, S.-I. Niculescu, A. Çela, H.-H. Wang, T.-Y. Cai, Invariance properties for a class of quasipolynomials. Automatica **50**(3), 890–895 (2014)

16. C.-F. Mendez-Barrios, S.-I. Niculescu, J. Chen, M. Maya-Mendez, Output feedback stabilisation of single-input single-output linear systems with I/O network-induced delays. An eigenvalue-based approach. Int. J. Control **87**(2), 346–362 (2014)
17. L. Mirkin, Some remarks on the use of time-varying delay to model sample-and-hold circuits. IEEE Trans. Autom. Control **52**(6), 1109–1112 (2007)
18. J. Skaf, S. Boyd, Analysis and synthesis of state-feedback controllers with timing jitters. IEEE Trans. Autom. Control **54**(3), 652–657 (2009)
19. Y.S. Suh, Stability and stabilization of nonuniform sampling systems. Automatica **44**(12), 3222–3226 (2008)
20. M.M. Vainberg, V.A. Trenogin, *Theory of Branching of Solutions of Non-linear Equations* (Noordhoff International Publishing, 1974)
21. W. Zhang, M.S. Branicky, S.M. Phillips, Stability of networked control systems. IEEE Control Syst. Mag. **21**(1), 84–99 (2001)

Part IV
Cooperative Control

Chapter 11
Optimal Control Strategies for Load Carrying Drones

Alicia Arce Rubio, Alexandre Seuret, Yassine Ariba and Alessio Mannisi

Abstract This chapter studies control strategies for load carrying drones. Load carrying drones not only have to fly in a cooperative way, but also are mechanically interconnected. Due to these characteristics, the control problem is an interesting and challenging issue to deal with. Throughout this chapter, a dynamic model based on first principle is developed. To that end, it is proposed to model this system as a ball and beam system lifted by two drones. Afterwards, different control techniques are implemented and compared by simulations. Specifically, linear-quadratic regulator (LQR) and model predictive control (MPC) are studied. Both control techniques belong to the optimal control methodology. This comparison is interesting since LQR permits to perform an optimal control law with short execution times, while MPC deals with physical constraints and predictions, being the execution time and the physical constraints important issues to handle in this kind of systems. Finally, simulation results and open issues are discussed.

A.A. Rubio (✉) · A. Seuret · Y. Ariba · A. Mannisi
CNRS, LAAS, 7 avenue du colonel Roche, 31400 Toulouse, France
e-mail: aarce@laas.fr

A. Seuret
e-mail: aseuret@laas.fr

Y. Ariba
e-mail: ariba@icam.fr

A. Mannisi
e-mail: amanissi@laas.fr

A.A. Rubio · A. Seuret · Y. Ariba · A. Mannisi
LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France

Y. Ariba
Icam Toulouse, Toulouse, France

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_11

11.1 Introduction

Interest in using drones (that is, unmanned aerial vehicles—UAVs—with the capacity to fly semi- or fully autonomously thanks to an on-board computer and sensors [1]) for scientific investigations dates back to the 1970s. Since then, billions of dollars have been poured into research and development of military and experimental drones. Indeed, during the last years, an increased use of flying drones has been noticed. The invention of light materials, low energy consumption machines, and high performance processing units led to the construction of flexible flying robots. They can be used in a variety of applications such as vehicle tracking, traffic management and fire detection [2, 3]. Within the family of the vertical take-off and landing (VTOL) drones, unmanned quadrotor helicopters [4] that base their operation in the appropriate control of four rotors have received a growing attention, mainly due to their capability to outperform most of other types of helicopters on the issues of maneuver ability, survivability, simplicity of mechanics, and increased payloads [5]. In fact, there are several advantages to quadcopters over comparably-scaled helicopters: the simplicity of their mechanical structure, the use of four small propellers resulting in a more fault-tolerant mechanical design capable of aggressive maneuvers at low altitude, good maneuver ability, and increased payload [6]. Untapping the potential of quadrotors requires, however, advanced control designs so as to achieve precise trajectory tracking combined with effective disturbance attenuation, particularly since quadrotor's model is highly nonlinear and their flight performance can be influenced by sudden wind gusts especially during flights in low altitudes. Moreover, the application studied in this chapter, which is the control of multiple quadrotor robots that cooperatively grasp and transport a payload in two dimensions, adds difficulty to the problem. Although the problem associated to quadrotor control has been addressed by many publications (such as those focused in PID control [7], sliding mode control [8], H_∞ control [9] and bounded control [10]), the novelty of the work presented herein is the application and subsequent comparison of model predictive control (MPC) [11] and linear-quadratic regulator (LQR) control techniques. To the best knowledge of the authors of this article this has not been realized before, a fact which further supports the interest of this work. Such a comparison is valuable since LQR permits to perform an optimal control law with short execution times while MPC deals with physical constraints and predictions. Execution time and physical constraints being important issues to take into account, while facing the control problem discussed in this chapter, the proposed application and comparison of MPC and LQR techniques therefore represents a useful framework aimed to provide researchers in the area with additional control possibilities.

To reach these ends, the chapter is organized as follows: Sect. 11.2 describes the system under study, while Sect. 11.3 presents the dynamic model. In Sect. 11.4, the control problem is motivated and control methodologies are developed. Section 11.5 shows and discusses the simulation results. Finally, the conclusions are exposed in Sect. 11.6.

11.2 System Under Study

The system under study is composed of two drones which aim to carry a load. The main feature of this system is that load carrying drones present mechanic links. These mechanic links depend on the way the drones carry the load. Therefore, it is proposed to describe this kind of system as a ball and beam system lifted by the drones. The mass center of the ball and beam models the load mass center. Figure 11.1 shows a scheme of the proposed system.

As observed in Fig. 11.1, drones are assumed to be quadrotors. The quadrotors comprises four propellers each one. The quadrotor trajectory is regulated by the angular speeds of the propellers resulting in a lift force which is referred to as f_1 for drone 1 and f_2 for drone 2 in Fig. 11.1. The ball and beam systems are lifted by the couple of drones by means of rigid cables with a fixed length equal to h. The beam length is equal to $2L$. In addition, it is supposed that the beam is nondeformable.

For sake of simplicity, in this work it is assumed that the drones only move in the XZ plane. Specifically, the x position of the drones is fixed, while the degree of freedom is the altitude z_1 for drone 1 and z_2 for drone 2. Thus, the longitudinal distance between the drones is fixed to the value of the beam length ($2L$). The angles formed by the vertical axis and the rigid cables are denoted as ϕ_1 for drone 1 and ϕ_2 for drone 2 and the angle formed by the beam and the horizontal axis is denoted θ.

It is defined two different coordinate reference systems x_0Oz_0 and $x_BO_Bz_B$. The global coordinate reference system x_0Oz_0 is located in the ground fixed in the x position corresponding to the middle distance between the drones. The local frame $x_BO_Bz_B$ is located in the beam mass center.

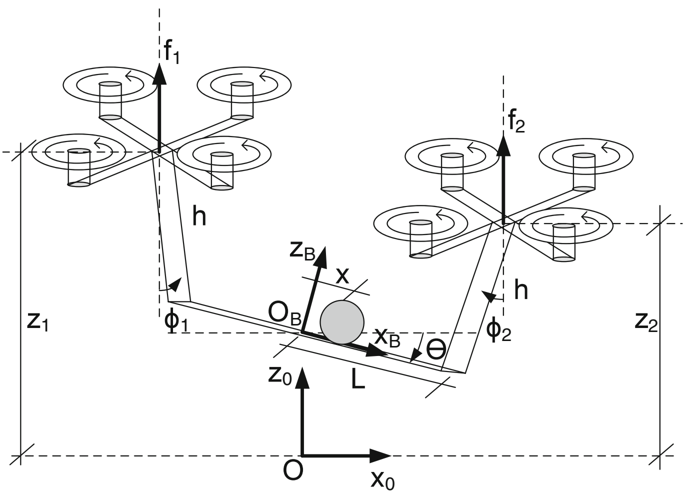

Fig. 11.1 Drone ball and beam system

Table 11.1 System parameters

Parameter	Value
Ball mass	0.1 kg
Beam mass	4 kg
Beam length	2 m
Drone 1	10 kg
Drone 2	10 kg
Rigid cable length	1 m
Damping factor	0.5

Variables related to the gyroscopic effects are not included in this study, since the control is divided in two levels. The control structure based on two control levels has been previously proposed for tracking positioning of quadrotors [9]. In our case, the high-level control calculates the references for the lift forces f_1 and f_2, while the low-level control is dedicated to the drone stabilization. Herein, the drone stabilization is assumed to be perfectly controlled to be focused on the high-level control.

The main parameters of the system are listed in Table 11.1.

11.3 Dynamic Model

The system under study presgvented in the previous section is modeled with first principles equations. To that end, the kinematics equations are developed and afterwards the Lagrange–Euler equations are obtained.

11.3.1 Kinematics Equations

As previously mentioned, two different coordinate reference systems are defined. The local frame position, O_B, at the global frame is

$$O_B = \begin{bmatrix} L - hsin\phi_2 - Lcos\theta \\ z_2 - hcos\phi_2 - Lsin\theta \end{bmatrix}. \tag{11.1}$$

This matrix represents the transformation from local frame to global frame. The local frame speed, $\dot{O}_B$, at the global frame is obtained by deriving the position with respect to the time:

$$\dot{O}_B = \begin{bmatrix} -h\,\dot{\phi}_2\,cos\phi_2 + L\,\dot{\theta}\,sin\theta \\ \dot{z}_2 + h\,\dot{\phi}_2\,sin\phi_2 - L\,\dot{\theta}\,cos\theta \end{bmatrix}. \tag{11.2}$$

The ball position and speed at local frame are denoted x and $\dot{x}$ and using the transformation matrices Eqs. (11.1) and (11.2), the position and speed at global frame, x_0 and z_0, are:

$$\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} L - hsin\phi_2 - Lcos\theta - xcos\theta \\ z_2 - hcos\phi_2 - Lsin\theta - xsin\theta \end{bmatrix}, \\ \begin{bmatrix} \dot{x}_0 \\ \dot{z}_0 \end{bmatrix} = \begin{bmatrix} -h\,\dot{\phi}_2 cos\phi_2 + (L+x)\dot{\theta} sin\theta - \dot{x} cos\theta \\ \dot{z}_2 + h\dot{\phi}_2 sin\phi_2 - (L+x)\dot{\theta} cos\theta - \dot{x} sin\theta \end{bmatrix}. \tag{11.3}$$

Given that the longitudinal distance between the drones is fixed, the angle θ can be expressed as a function of angles ϕ_1 and ϕ_2:

$$\begin{bmatrix} sin\theta \\ cos\theta \end{bmatrix} = \frac{1}{2L} \begin{bmatrix} z_2 - z_1 + hcos\phi_1 - hcos\phi_2 \\ 2L - hsin\phi_1 - hsin\phi_2 \end{bmatrix}. \tag{11.4}$$

Replacing θ in Eq. (11.3), the ball position and speed at global frame result in:

$$\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x + \dfrac{hs\phi_1 - hs\phi_2}{2} - \dfrac{x(hs\phi_1 + hs\phi_2)}{2L} \\ \dfrac{z_2 + z_1 - hc\phi_1 - hc\phi_2}{2} + \dfrac{x(z_2 - z_1 + hc\phi_1 - hc\phi_2)}{2L} \end{bmatrix},$$

$$\begin{bmatrix} \dot{x}_0 \\ \dot{z}_0 \end{bmatrix} = \begin{bmatrix} \dfrac{(x-L)h\dot{\phi}_1 c\phi_1 - (x+L)h\dot{\phi}_2 c\phi_2}{2L} - \dfrac{\dot{x}(2L + hs\phi_1 + hs\phi_2)}{2L} \\ \left(\dfrac{(L+x)(\dot{z}_2 + h\dot{\phi}_2 s\phi_2) + (L-x)(\dot{z}_1 + h\dot{\phi}_1 s\phi_1)}{2L} + \right. \\ \left. + \dfrac{\dot{x}(z_2 - z_1 + hc\phi_1 - hc\phi_2)}{2L} \right) \end{bmatrix}, \tag{11.5}$$

where $c\theta$ and $s\theta$ correspond to $cos\theta$ and $sin\theta$, respectively.

11.3.2 Lagrange–Euler Equations

The motion equations can be expressed by the Lagrange-Euler formulation based on the kinetic and potential energy concepts:

$$\begin{bmatrix} \dfrac{d}{dt}\dfrac{\partial \mathscr{L}}{\partial \dot{x}} - \dfrac{\partial \mathscr{L}}{\partial x} \\ \dfrac{d}{dt}\dfrac{\partial \mathscr{L}}{\partial \dot{z}_1} - \dfrac{\partial \mathscr{L}}{\partial z_1} \\ \dfrac{d}{dt}\dfrac{\partial \mathscr{L}}{\partial \dot{z}_2} - \dfrac{\partial \mathscr{L}}{\partial z_2} \\ \dfrac{d}{dt}\dfrac{\partial \mathscr{L}}{\partial \dot{\phi}_1} - \dfrac{\partial \mathscr{L}}{\partial \phi_1} \\ \dfrac{d}{dt}\dfrac{\partial \mathscr{L}}{\partial \dot{\phi}_2} - \dfrac{\partial \mathscr{L}}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ f_2 \\ 0 \\ 0 \end{bmatrix}, \tag{11.6}$$

where $\mathscr{L}$ is the Lagragian of the system. The Lagragian is calculated as the difference between kinetic and potential energies. The equations are obtained by the software MAXIMA.

The system kinetic and potential energies are the addition of the ball, beam, and drone kinetic and potential energies. The ball kinetic energy, T_b, beam kinetic energy, T_B, and drone kinetic energies, T_{d1} and T_{d2} are:

$$\begin{aligned} T_b &= \frac{1}{2} m_b \begin{bmatrix} x_0 y_0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} & T_B &= \frac{1}{2} \rho \int_{-l}^{l} \|\dot{S}_0(s)\|^2 \, ds, \\ T_{d1} &= \frac{1}{2} m_{d1} \dot{z}_1^2, & T_{d2} &= \frac{1}{2} m_{d2} \dot{z}_2^2, \end{aligned} \tag{11.7}$$

being m_b the ball mass, l the beam volume, m_{d1} the drone 1 mass, m_{d2} the drone 2 mass and ρ the beam density. The beam density ρ is equal to the ratio between the beam mass, m_B, and the beam volume l. The variable $\dot{S}_O$ corresponds to the speed of a generic point of the beam at the global coordinate reference system. The ball potential energy, U_b, the beam potential energy, U_B, and the drone potential energies, U_{d1} and U_{d2}, are expressed as

$$\begin{aligned} U_b &= m_b \; g \; y_o, & U_B &= m_B \; g \; O_{B,z}, \\ U_{d1} &= m_{d1} \; g \; z_1, & U_{d2} &= m_{d2} \; g \; z_2, \end{aligned} \tag{11.8}$$

where g is the acceleration of gravity and $O_{B,z}$ is the z position of the local frame which is placed at the beam mass center.

11.3.3 State-Space Model

The previous model is linearized at an operating point for control purposes. The operating point corresponds to a ball position equal to (0, 0), drone altitudes z_1, z_2 equal to 0 and null system speeds (ball, beam, and drones). Moreover, it is considered as new variables F_1 and F_2 to represent the lift forces. These variables are equal to zero at equilibrium, and are calculated as

$$\begin{aligned} F_1 &= f_1 - m_{d1} \; g - \frac{m_b + m_B}{2} \; g, \\ F_2 &= f_2 - m_{d2} \; g - \frac{m_b + m_B}{2} \; g. \end{aligned} \tag{11.9}$$

As a result, the linear model in matrix form is

$$\underbrace{\begin{bmatrix} \ddot{x}(t) \\ \ddot{z}_1(t) \\ \ddot{z}_2(t) \\ \ddot{\phi}_1(t) \\ \ddot{\phi}_2(t) \end{bmatrix}}_{\dot{\Gamma}} = \mathbf{A} \underbrace{\begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \\ \phi_1(t) \\ \phi_2(t) \end{bmatrix}}_{\Gamma} + \mathbf{B} \underbrace{\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}}_{\mathbf{U}}, \tag{11.10}$$

where matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{g(2m_B+m_b)}{4Lm_B} & -\frac{g(2m_B+m_b)}{4Lm_B} & \frac{g(m_B+m_b)}{4m_B} & -\frac{g(m_B+m_b)}{4m_B} \\ a_1 & 0 & 0 & 0 & 0 \\ -a_2 & 0 & 0 & 0 & 0 \\ 0 & -\frac{gm_b}{4Lhm_B} & \frac{gm_b}{4Lhm_B} & -\frac{g(m_B+m_b)}{hm_B} & -\frac{g(m_B+m_b)}{2hm_B} \\ 0 & \frac{gm_b}{4Lhm_B} & -\frac{gm_b}{4Lhm_B} & -\frac{g(m_B+m_b)}{2hm_B} & -\frac{g(m_B+m_b)}{hm_B} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \frac{8m_B+3m_b+12m_{d2}}{p} & -\frac{4m_B+3m_b}{p} \\ -\frac{4m_B+3m_b}{p} & \frac{8m_B+3m_b+12m_{d1}}{p} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and where

$$a_1 = \frac{3gm_b(2m_B+m_b+2m_{d2})}{p}, \quad a_2 = \frac{3gm_b(2m_B+m_b+2m_{d1})}{p},$$
$$p = L(2m_Bm_b + (8m_B+3m_b)(m_{d1}+m_{d2}) + 12m_{d1}m_{d2} + 4m_B^2),$$

From Eq. (11.10), the following state-space model is obtained:

$$\underbrace{\begin{bmatrix} \ddot{\Gamma} \\ \dot{\Gamma} \end{bmatrix}}_{\dot{\mathbf{X}}} = \underbrace{\begin{bmatrix} \mathbf{M} & \mathbf{A}^T \\ \mathbf{I} & 0 \end{bmatrix}}_{\mathbf{A_{sys}}} \underbrace{\begin{bmatrix} \dot{\Gamma} \\ \Gamma \end{bmatrix}}_{\mathbf{X}} + \underbrace{\begin{bmatrix} \mathbf{B}^T \\ 0 \end{bmatrix}}_{\mathbf{B_{sys}}} \underbrace{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}}_{\mathbf{U}}. \tag{11.11}$$

Vector **X** is the state vector which contains the speeds and positions of the ball and drones and the system is represented by matrices $\mathbf{A_{sys}}$ and $\mathbf{B_{sys}}$. Matrix **M** includes damping factors which affects the angular movement of the rigid cables modeled by variables ϕ_1 and ϕ_2. The damping factors avoid infinitive bouncing associated to ideal pendulum problem. Thus, this matrix is

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} -\mu & 0 \\ 0 & -\mu \end{bmatrix} \end{bmatrix}. \tag{11.12}$$

11.4 Control Problem

The control objective of this system is to maintain the ball in the equilibrium point on the beam by means of regulating the drone altitudes. As previously mentioned, the system control is divided in two levels. The high-level control calculates the lift force to track an altitude reference that depends on the ball position and the altitude of the other drone, while the low-level control is dedicated to the drone stabilization. The low-level control is integrated in each drone and calculates the angular speeds of the four rotors to obtain a total lift force equal to the reference provided by the high-level control. The control problem is schemed in Fig. 11.2.

As observed in Fig. 11.2, it is considered that the stabilization controllers are feedback control strategies. The angular speeds, ω_i, of the four rotors are measured and the lift force, f_1 or f_2, is estimated. Then the control loop is closed by obtaining the error between the reference of the lift force and the estimated lift force. This work focuses on the high-level control and it is assumed that the drone stabilization is perfectly controlled. In addition, the closed-loop scheme of the high-level control receives a reference vector, $\mathbf{Y}_{ref}$, to be tracked. The system output vector, $\mathbf{Y}$, comprises all the states that are measured. We assumed that all the states included in vector $\mathbf{X}$ are measured, that is, the ball position, x, and speed, $\dot{x}$, the drone altitudes, z_1 and z_2, and speeds, $\dot{z}_1$ and $\dot{z}_2$, and the angles ϕ_1 and ϕ_2 and its angular speeds $\dot{\phi}_1$ and $\dot{\phi}_2$.

It is proposed herein to compare different optimal control techniques to evaluate the difficulties associated to this system. In particular, linear-quadratic regulator (LQR) and model predictive control (MPC) are developed. LQR allows to solve online optimization control problems with fast execution time and low computational effort, while MPC deals with physical constraints and predictions. Both control methodologies present interesting features for this control problem. For aerial application, short execution times with low computation effort are demanding but at the same time, handling physical constraints is required to avoid collisions and instable scenarios caused by disturbances. These controllers are detailed in the next subsections.

Moreover, controllers are implemented in CPUs on-board, and therefore discrete control laws are studied. The sampling time is a design parameter which has to be appropriately chosen. It is important to remark that in this control scenario composed

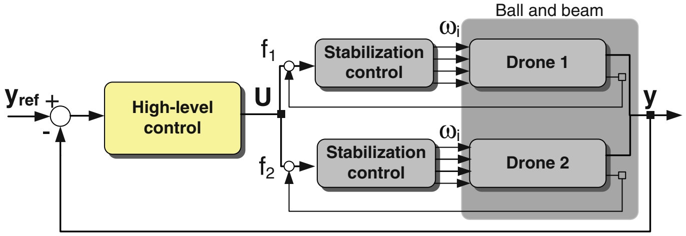

Fig. 11.2 Control scheme

by two control levels, each control level may present different sampling time. Specifically, the low-level control is performed faster than the high-level control for this application. Accounting for possible hardware limitations, a sampling time of 200 ms is set for this study. Note that the optimal controllers require the system model for design. The dynamic model presented in Eq. (11.11) is in continuous time and it has to be discretized for a 200-ms sampling time resulting in

$$\mathbf{X}(k+1) = \mathbf{A_d}\ \mathbf{X}(k) + \mathbf{B_d}\ \mathbf{U}(k), \tag{11.13}$$

$$\mathbf{Y}(k) = \mathbf{C_d}\ \mathbf{X}(k) + \mathbf{D_d}\ \mathbf{U}(k), \tag{11.14}$$

11.4.1 Linear-Quadratic Regulator

LQR is an optimal and feedback control law which minimizes every sampling time the following objective function J:

$$J = \sum_{k=0}^{\infty} \left(\mathbf{X}(k)^T \mathbf{Q} \mathbf{X}(k) + \mathbf{U}(k)^T \mathbf{R} \mathbf{U}(k) \right), \tag{11.15}$$

where **X**(k) and **U**(k) are the state and input vectors at instant k and matrices **Q** and **R** are weighting matrices. Given that the system is modeled by the discrete-time state-space model presented in Eqs. (11.13), (11.14), the analytical optimal control sequence results in

$$\mathbf{U}(k) = -\mathbf{F}\ \left(\mathbf{X}(k) - \mathbf{X_{ref}}(k)\right), \tag{11.16}$$

where

$$\mathbf{F} = \left(\mathbf{R} + \mathbf{B_d}^T\ \mathbf{P}\ \mathbf{B_d}\right)^{-1}\ \mathbf{B_d}^T\ \mathbf{P}\ \mathbf{A_d}, \tag{11.17}$$

being **P** the unique positive definite solution to the discrete-time algebraic Riccati equation (DARE):

$$\mathbf{P} = \mathbf{Q} + \mathbf{A_d}^T \left(\mathbf{P} - \mathbf{P}\mathbf{B_d} \left(\mathbf{R} + \mathbf{B_d}^T \mathbf{P}\mathbf{B_d}\right)^{-1} \mathbf{B_d}\mathbf{P}\right) \mathbf{A_d}. \tag{11.18}$$

11.4.2 Model Predictive Control

Model predictive control has been successfully applied to many industrial processes [11] and drone applications such as [9]. The controller calculates the optimal control action taking future predictions and constraints into account. The optimization is repeated each sampling time with a moving horizon. MPC is generally formulated with state-space models as

$$\min_{\varepsilon(k+m+1),\mathbf{U}(k+m)} \sum_{m=1}^{N-1} \varepsilon(k+m+1)^T \,\mathbf{Q}\, \varepsilon(k+m+1) + \mathbf{U}(k+m)^T \,\mathbf{R}\, \mathbf{U}(k+m), \tag{11.19}$$

where

$$\varepsilon(k+m+1) = \mathbf{X}(k+m+1) - \mathbf{X_{ref}}(k+m+1), \tag{11.20}$$

being N the prediction horizon, $\mathbf{Q}$ and $\mathbf{R}$ the weighting matrices. The error ε is defined as the difference between the state vector $\mathbf{X}$ and the reference $\mathbf{X_{ref}}$ since the output vector $\mathbf{Y}$ is equal to the state vector. The optimization problem is subject to the system dynamic model presented in Eq. (11.14) and the following system constraints:

$$\underline{\mathbf{X}} \le \mathbf{X}(k) \le \overline{\mathbf{X}},\ \underline{\mathbf{U}} \le \mathbf{U}(k) \le \overline{\mathbf{U}},\ \underline{\mathbf{Y}} \le \mathbf{Y}(k) \le \overline{\mathbf{Y}}. \tag{11.21}$$

The variable vectors denoted with an over line contain the upper limits, while the variable vectors with an under line contains the lower limits. Particularly, the constraints included in this problem are the beam length that limits the ball position, x, and maximum and minimum values for the lift forces, f_1 and f_2, imposed by rotor physical constraints. Values of 30 and −35 N are chosen for the upper and lower lift force bounds, respectively.

11.5 Simulation Results

This section is dedicated to compare and discuss the simulation results for LQR and MPC. First, preliminary simulations are performed in order to tune the weighting matrices, $\mathbf{Q}$ and $\mathbf{R}$, for the controllers with the objective to achieve fast performance with nonaggressive control actions. The best values for both matrices and both controllers correspond to

$$\begin{aligned} \mathbf{Q} &= \mathbf{I}_{n_x \times n_x}, \\ \mathbf{R} &= 0.004\, \mathbf{I}_{n_u \times n_u}. \end{aligned} \tag{11.22}$$

Furthermore, for the MPC the prediction horizon, N, is another design parameter. The prediction horizon directly influences on the computational demand, that is, the higher the prediction horizon the higher the computational demand is. Otherwise, MPC and LQR have the same performance for a sufficiently higher N. Then it is achieved a tradeoff with a prediction horizon set at 12 samplings.

Figure 11.3 compares LQR and constrained MPC simulated for initial drones altitudes equal to 4.5 and 3.5 m for drone 1 and drone 2.

Figure 11.3a shows the performance of the ball position for LQR in red, for MPC in blue, and bounds in magenta dash lines. Note that both controllers are able to

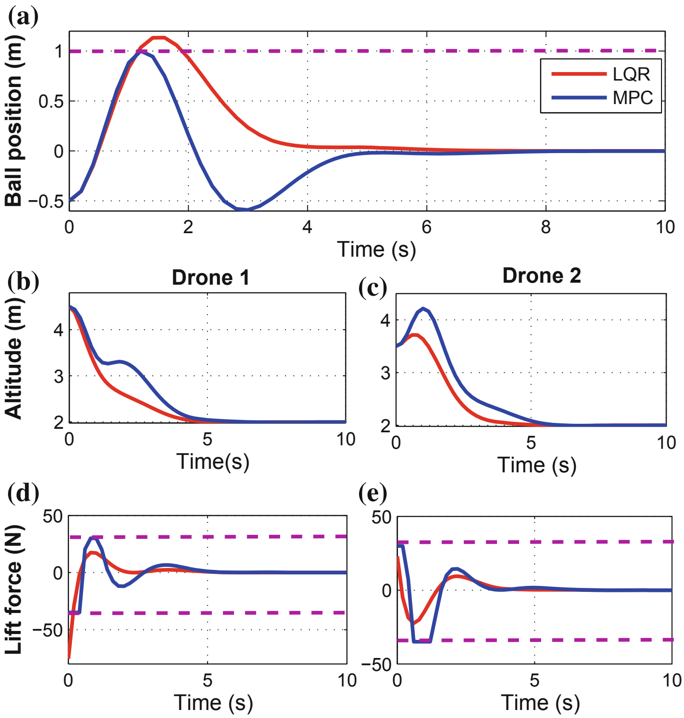

Fig. 11.3 Comparison of LQR and MPC controllers on simulations

regulated the ball position and stabilized the ball in the equilibrium point set at $x = 0$. However, LQR violates the constraints imposed by the beam length and at time 1.75 s the ball falls from the beam. Figure 11.3b, c show the drone altitudes, z_1 and z_2. The altitude references for both altitudes are maintained constant at 2 m during all the simulation. LQR and MPC appropriately regulate the altitudes. As seen in the figures, MPC is slightly slower due to the constraints. Figure 11.3d, e present the simulation results for the drone lift forces. MPC performs inside the bounded region for all the simulation, while LQR violates the constraints at the beginning of the simulation imposed to the drone 1 lift force.

In order to test in more detail the LQR, several simulations are performed and presented in Fig. 11.4. The weighting matrices are modified and the lift forces are saturated to the maximum and minimum values only for the case of the LQR. In Fig. 11.4a, the ball position performances are presented where red, magenta, and green dash lines correspond to the LQR performance with a weighting matrix $\mathbf{R}$ equal to $0.008\,\mathbf{I}_{n_u \times n_u}$, $0.004\,\mathbf{I}_{n_u \times n_u}$, and $0.001\,\mathbf{I}_{n_u \times n_u}$, respectively. Matrix $\mathbf{Q}$ is kept constant and equal to the identity matrix. Moreover, red, magenta, and green solid lines are the LQR performances for the previous weighting matrices including saturations on the maximum and minimum values of the lift forces. As observed in the figure, only

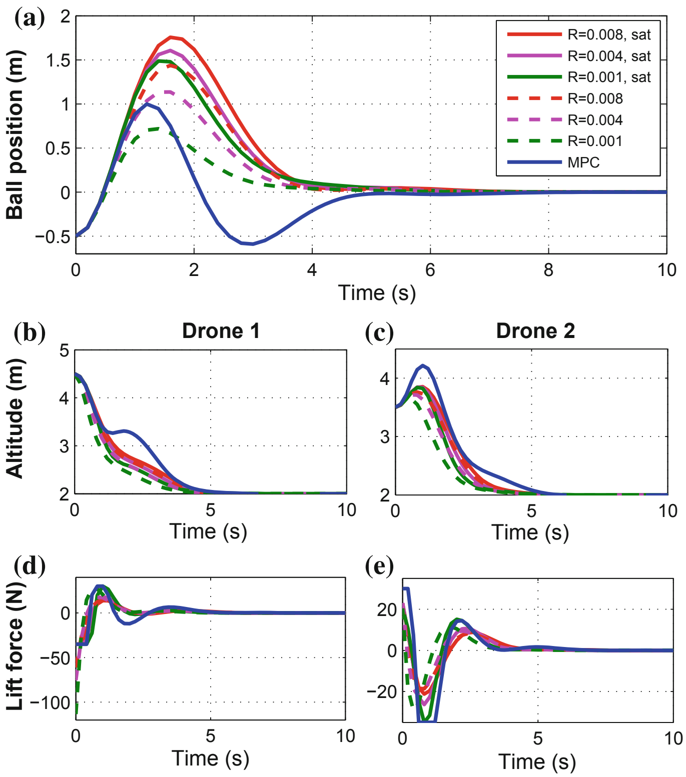

Fig. 11.4 Saturated and nonsaturated LQR with different weighting matrices

the LQR performance for a weighting matrix $\mathbf{R}$ of $0.004\,\mathbf{I}_{n_u \times n_u}$ without saturation avoids the ball to fall. However, after studying Fig. 11.4d, e, LQR with a weighting matrix $\mathbf{R}$ of $0.004\,\mathbf{I}_{n_u \times n_u}$ requires to implement lift forces for drone 1 out of bounds. Therefore, it is demonstrated that even though LQR is a good candidate to regulate this system due to its fast execution time, the system performance under the bounds is not guaranteed.

11.5.1 Disturbances

LQR and MPC are tested under disturbances. To that end, simulations are performed with disturbances in the ball position and drone altitudes. In addition, changes in the references are also included in the simulation. The results are compared in Fig. 11.5. The initial conditions are the same as presented in the previous simulations. The ball position is modified to a value of x equal to 0.5 m at time 15 s as seen in Fig. 11.5a. The altitude of drone 1 is modified to values of z_1 equal to 1.2 and 2.7 at time 20

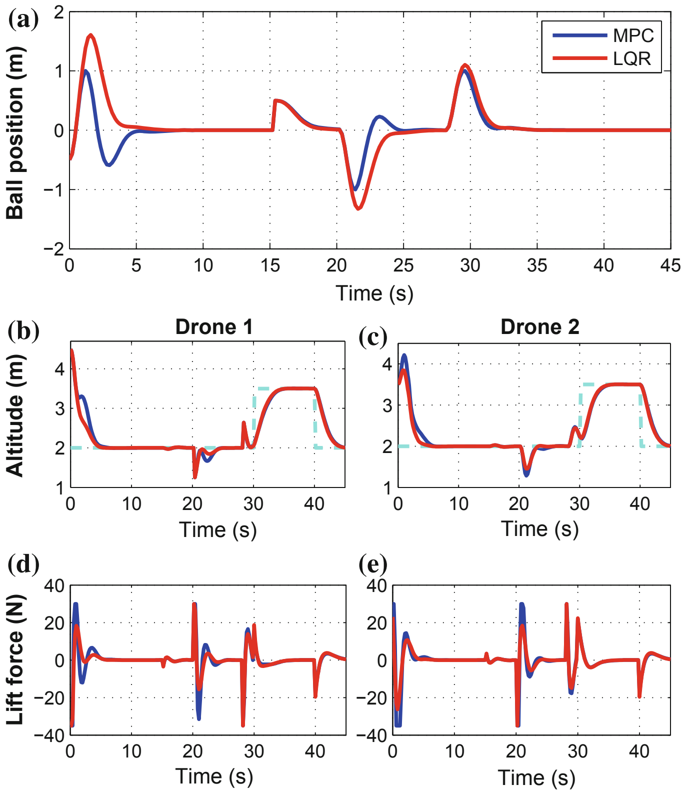

Fig. 11.5 LQR and MPC with disturbances

and 28 s as shown in Fig. 11.5b. The altitude references are at the beginning of the simulation equal to 2 m and simultaneously change to a value of 3.5 m at time 30 s and again to 2 m at time 40 s. The references are shown in Fig. 11.5b, c as cyan dash lines. In Fig. 11.5, blue solid lines correspond to the MPC, while red solid lines correspond to the LQR with lift forces saturated. As mentioned in the previous subsection, LQR is not able to perform under the bounded region and the ball falls from the beam at times 1.75, 21, and 29 s. MPC performs under bounds during all the simulation and faster than the saturated LQR as seen in Fig. 11.5a at time 21 s. For both cases, the altitude references are perfectly tracked with a similar rise time.

11.5.2 Execution Times

Finally, executions times are obtained during simulation to test the suitability of the real-time implementation of MPC for this application. Note that the controllers

Table 11.2 Execution times

Controller	Average (ms)	Maximum (ms)	Minimum (ms)
LQR	0.01225	0.0338	0.00798
MPC	5.8	6.8	4.9

are simulated using MATLAB in PC with i7-260M CPU under 64 bits Windows platform. Table 11.2 lists the average, maximum and minimum execution times for LQR and MPC. The average time for LQR is 0.01225 ms, while MPC is 5.8 ms. Both average times are far from 200 ms which is the sampling time, and thus a robust online implementation is feasible. Note that the hardware on-board with a real-time software may execute the control laws faster. Due to the fast execution times, the sampling time could be reduced in order to obtain better performance in terms of response speed. Then the prediction horizon for MPC needs to be recalculated for the new sampling time. The prediction horizon depends on the system rise time. Therefore, if the sampling time is reduced, the prediction horizon increases. Also, the MPC execute time increases with the prediction horizon. The reduction of the sampling time needs to be carefully studied.

11.6 Conclusion

In the present chapter we have presented, modeled, and analyzed a novel application of multiagent drone system dealing with load carrying. The main problem related to this application comes from the mechanical links between the load and the drones which carry it. This system was proposed to be modeled as a ball and beam lifted by to drones and a mathematical model based on first principles was developed. Under the assumption that all the states are measured, LQR and MPC controllers have been analyzed by simulations. LQR presented very fast execution times but it is not guaranteed a performance in the bounded area, that is, that the ball does not fall from the beam as seen in different simulations. On the other hand, MPC deals with constraints and regulates the ball position without violating the constraints imposed in the problem. Moreover, execution times are short enough for this application to guarantee the online implementation.

This work has presented only a preliminary study with centralized controllers which have the knowledge of all the states variables. In future work, we aim at considering a more general setup with a higher number of drones and where only part of the states are locally measured on each drone. In order to reduce the execution time of the MPC and to make the problem scalable in an easy way, distributed MPC strategies should be studied. In addition, a uncertainties and aerodynamic disturbances need to be included in future studies.

References

1. P. McBride, B. Orwell, The application of unmanned aircraft systems in domestic surveillance operations. J. Air Law Comm. Summer **74**(3), 627–628 (2009)
2. D. Zorbas, T. Razafindralambo, D.P. Luigi, F. Guerriero, Energy efficient mobile target tracking using flying drones. Proc. Comput. Sci. **19**, 80–87 (2013)
3. H. Chen, X. Wang, Y. Li, A survey of autonomous control for uav, in *International Conference on Artificial Intelligence and Computational Intelligence*, vol. 2, 2009, pp. 267–271
4. G. Hoffmann, H. Huang, S. Waslander, C. Tomlin, Quadrotor helicopter flight dynamics and control: theory and experiment, in *The AIAA Guidance, Navigation, and Control Conference*, 2007
5. S. Bouabdallah, M. Becker, R. Siegwart, Autonomous miniature flying robots: coming soon-research, development, and results. IEEE Robot. Autom. Mag. **14**(3), 88–98 (2007)
6. K. Alexis, G. Nikolakopoulos, A. Tzes, Switching model predictive attitude control for a quadrotor helicopter subject to atmospheric disturbances. Control Eng. Pract. **19**, 1195–1207 (2011)
7. S. Bouabdallah, A. Noth, R. Siegwart, PID-LQ control techniques applied to an indoor micro quadrotor, in *The IEEE/RSJ International Conference on Intelligent Robots and Systems*, 2004, pp. 2451–2456
8. S. Waslander, G. Hoffmann, J. Jang, C. Tomlin, Multi-agent quadrotor test bed control design: integral sliding mode vs. reinforcement learning, in *The IEEE/RSJ International Conference on Intelligent Robotics and Systems*, 2005, pp. 468–473
9. G.V. Raffo, M.G. Ortega, F.R. Rubio, An integral predictive/nonlinear $\mathscr{H}_\infty$ control structure for a quadrotor helicopter. Automatica **46**, 29–39 (2010)
10. S. Bouabdallah, Design and control of quadrotors with application to autonomous flying. Ph.D. dissertation, EPFL STI School of Engineering, Lausanne, Switzerland, 2007
11. E. Camacho, C. Bordons, *Model Predictive Control* (Springer, London, 2004)

Chapter 12
Delays in Distributed Estimation and Control over Communication Networks

Pablo Millán, Luis Orihuela and Isabel Jurado

Abstract This chapter introduces a distributed estimation and control technique with application to networked systems. The problem consists of monitoring and controlling a large-scale plant using a network of agents which collaborate exchanging information over an unreliable network. We propose an agent-based scheme based on an estimation structure that combines local measurements of the plant with remote information received from neighboring agents. We discuss the design of stabilizing distributed controllers and observers when the interagent communication is affected by delays and packet dropouts. Some simulations will be shown to illustrate the performance of this approach.

12.1 Introduction

Wireless communication network is a technology that has been attracting interest in the past decade due to its large variety of applications and utilities. One of the most important characteristic of this kind of systems is that allows the integration of different devices, providing flexibility, robustness, and ease of configuration of the system.

The devices interconnected in the wireless network (WN) are agents that may have sensing and actuation interfaces, as well as computation and communication capabilities. These particular systems are very useful in applications as process control systems [31, 41], mobile vehicles [4, 6, 10], tracking and surveillance [35, 38], or water delivery control [3, 17].

Among many advantages of this kind of systems [30], the capability of each agent to cooperate makes WNs a powerful network for facing complex problems.

P. Millán (✉) · L. Orihuela · I. Jurado
Dpto. de Ingeniería, Universidad Loyola Andalucía, Sevilla, Spain
e-mail: pmillan@uloyola.es

L. Orihuela
e-mail: dorihuela@uloyola.es

I. Jurado
e-mail: ijurado@uloyola.es

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_12

Due to the complexity of these problems and to the fact that WNs are usually large-scale systems, it is not feasible or advisable to control these systems in a centralized manner. Very often, decentralized techniques are not recommendable either since they do not include communication between agents. On the contrary, distributed schemes can provide suitable solutions to be implemented over WNs [26].

The goal of distributed control techniques is to make all the agents in the network seek for the same system-wide objective [22]. A fundamental aspect of this scheme is that the agents have to act according to partial measurements of the state and ignoring the actual control signal that is being applied to the plant. As the performance of the closed-loop system depends on the decisions of all the agents, communication is a very important issue in this approach [19].

One of the major difficulties of distributed solutions over WNs is the fact that transmission channels are not completely reliable due to noises, limited bandwidth, and large number of concurrent transmitters over the same channel. The most common consequences of network congestion are packet dropouts and time delays that can degrade the performance or even destabilize the systems.

On the one hand, many research has been developed to study these effects and propose centralized solutions, see [14, 15, 23, 34, 39, 40] and references therein. On the other hand, there exists a vast literature in the field of distributed control considering ideal networks including MPC-based approaches [2, 5, 8, 18, 25, 32, 33, 37], techniques for large-scale plants [11, 21, 24], and distributed versions of the Kalman filter [1, 16, 20, 27, 28].

In this chapter, the problem of distributed control and estimation is addressed together with network-induced delays and dropouts. This work is an extension of the papers [24, 30], dealing this time with the problem of network-induced delays and dropouts within this distributed paradigm.

The objective is to control a discrete linear time-invariant (LTI) system using a set of agents connected through a communication network. These agents must be able to estimate the state of the system, as well as to control it. However, each one has access only to some outputs of the plant, which makes the interconnection between agents (with its associated delays and dropouts) an essential issue to achieve the system-wide objective. The specific estimation structure implemented in the agents merges a local Luenberger-like observer with consensus strategies. Since the Separation Principle does not hold, it is necessary to design the controllers and the observers in a unique centralized offline step. The stability of the system and estimation errors is ensured by using a Lyapunov–Krasovskii framework. The synthesis problem is posed as a matrix inequality which can be solved using the well known cone complementary algorithm [9].

The chapter is organized as follows. Section 12.2 describes the different elements involved in the problem, namely: plant, agents, and the communication network. The dynamics of the state and estimation errors is studied in Sect. 12.3. Section 12.4 presents the design method based on the Lyapunov–Krasovskii theorem. Section 12.5 illustrates the effectiveness of the approach with some simulations. Finally, Sect. 12.6 outlines the main conclusions and future work.

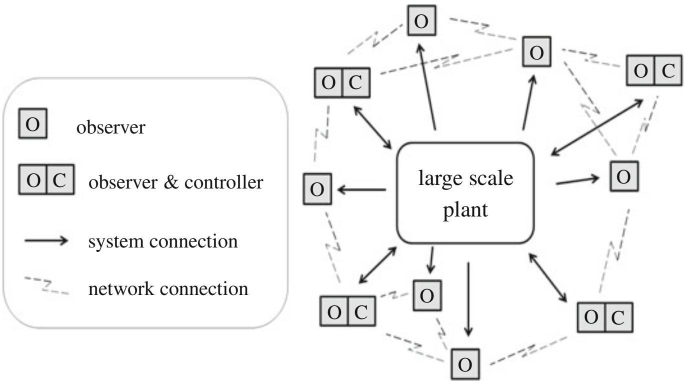

Fig. 12.1 Distributed scheme for the control of a large-scale plant

12.2 System Description and Problem Formulation

This chapter considers an estimation and control scheme composed by a large-scale plant, a communication network, and a set of distributed agents, as depicted in Fig. 12.1. In the following, the different elements composing the distributed system are described in detail.

12.2.1 Plant

We consider a discrete LTI system described in state-space representation. As Fig. 12.1 illustrates, the plant is controlled and/or observed by a set of p agents, each one possibly managing a different control signal. The dynamics of the system can be described as

$$x(k+1) = Ax(k) + \sum_{i=1}^{p} B_i u_i(k), \tag{12.1}$$

where $x \in \mathbb{R}^n$ is the state of the plant and $u_i \in \mathbb{R}^{d_i}$ $(i = 1, \ldots, p)$ is the control signal that agent i applies to the system, and $A \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times d_i}$ are known matrices. For those agents with no direct access to plant inputs, matrices B_i are set to zero.

Defining an augmented control matrix as

$$B \triangleq \begin{bmatrix} B_1 \; B_2 \; \ldots \; B_p \end{bmatrix} \tag{12.2}$$

and an augmented control vector

$$\mathscr{U}(k) \triangleq \left[u_1^T(k)\ u_2^T(k)\ \ldots\ u_p^T(k) \right]^T, \tag{12.3}$$

Equation (12.1) can be compactly rewritten as

$$x(k+1) = Ax(k) + B\mathscr{U}(k), \tag{12.4}$$

where $\mathscr{U}(k) \in \mathbb{R}^d$, with $d = \sum_{i=1}^{p} d_i$. The pair (A, B) in (12.4) is required to be stabilizable.

12.2.2 Network

In the proposed scheme, the agents are linked using a communication network to make possible the information exchange in real time. Each agent is restricted to receive information only from neighboring agents.

The resulting communication topology can be represented using a directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, with $\mathscr{V} = 1, 2, \ldots, p$ being the set of nodes (agents) of the graph (network), and $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$, being the set of links. Assuming that the cardinality of $\mathscr{E}$ is equal to l, and defining $\mathscr{L} = 1, 2, \ldots, l$, it is obvious that a bijective function $g : \mathscr{E} \to \mathscr{L}$ can be built so that a given link can be either referenced by the pair of nodes it connects $(i, j) \in \mathscr{E}$ or the link index $r \in \mathscr{L}$, so that $r = g(i, j)$. The set of nodes connected to node i is named the neighborhood of i, and denoted as $\mathscr{N}_i \triangleq \{j \in \mathscr{V} | (i, j) \in \mathscr{E}\}$. Directed communications are considered so that link (i, j) implies that node i receives information from node j.

Network links are not assumed to be completely reliable. This way, the packets that the agents exchange may be dropped or delayed. Figure 12.2 illustrates a possible time scheduling in which both effects appear.

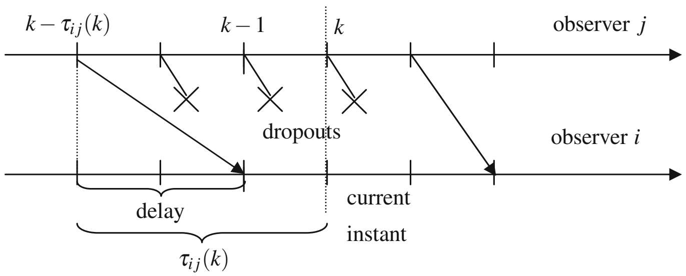

Fig. 12.2 Time scheduling

The input delay approach [12] makes possible to define artificial delays $\tau_{ij} \in \mathbb{N}$ that include the effect of sampling, communication delays, and packet dropouts. Concretely, τ_{ij} is the difference between the current time instant k and the last instant in which a packet from node j was received by node i. It is assumed that the number of consecutive data dropouts is bounded by n_p, and the maximum network-induced delay is bounded by $\bar{d}$. Under these assumptions, the artificial delay can be bounded as $\tau_{ij}(k) \leq \bar{d} + n_p \triangleq \tau_M, \ \forall k$.

Note that each delay τ_{ij} is directly associated to a link, in such a way that the following equivalent notation for the delays can be used:

$$\tau_r(k) = \tau_{ij}(k), \ \ r = 1, \ldots, l, \tag{12.5}$$

where $r = g(i, j)$. That is, we can either refer the delays to a pair of nodes (τ_{ij}) or to a link (τ_r).

12.2.3 Agents

As said before, the large-scale plant (12.1) is collectively monitored and controlled by a network of agents. Each of these agents can be endowed with all or part of the following capabilities:

- sensing plant outputs,
- computing estimations of the plant state,
- applying control actions,
- communicating with neighboring agents.

The approach adopted in this chapter is a distributed scheme in which every agent builds its own estimations of the plant's states based on the information locally collected by the agent (plant outputs) and that received from neighboring agents. Based on these estimations, those agents with access to a control channel compute the control actions to be applied.

Let us define y_i as the plant output measured by agent i:

$$y_i(k) = C_i x(k) \in \mathbb{R}^{r_i}, \tag{12.6}$$

where matrices $C_i \in \mathbb{R}^{r_i \times n}$ are known. If an agent j has no sensing capabilities, then its corresponding matrix C_j is set to zero. Let C denote an augmented output matrix defined as

$$C \triangleq \left[\, C_1^T \ C_2^T \ \ldots C_p^T \,\right]^T .$$

It is assumed that the pair and (A, C) is detectable.

On the other hand, the control counterpart of each agent generates an estimation-based control input to the plant, $u_i(k)$, in the form

$$u_i(k) = K_i\hat{x}_i(k) \in \mathbb{R}^{d_i}, \tag{12.7}$$

where $\hat{x}_i \in \mathbb{R}^n$ denotes the estimation of the plant state computed by agent i, and $K_i \in \mathbb{R}^{d_i \times n}$ $(i \in \mathscr{V})$ are local controllers to be designed. Let K denote the augmented control matrix, defined by

$$K = \left[K_1^T \; K_2^T \; \dots \; K_p^T \right]^T.$$

Every agent $i \in \mathscr{V}$ implements an estimator of the plant's state based on the following structure:

$$\begin{aligned} \hat{x}_i(k+1) = {} & A\hat{x}_i(k) + B\mathscr{U}_i(k) \\ & + M_i(y_i(k) - C_i\hat{x}_i(k)) \quad \textit{local information} \\ & + \sum_{j \in \mathscr{N}_i} N_{ij}[\hat{x}_j(k - \tau_{ij}(k)) - \hat{x}_i(k - \tau_{ij}(k))], \quad \textit{remote information} \end{aligned} \tag{12.8}$$

where $\mathscr{U}_i(k) = K\hat{x}_i(k) \in \mathbb{R}^d$ is the estimation of the whole control action applied to the plant.

Looking at Eq. (12.8), each agent has two different sources of information to correct its estimates. The first one is the output measured from the plant, $y_i(k)$, which is used similarly to a classical Luenberger observer, $M_i(y_i(k) - \hat{y}_i(k))$, being $M_i, \; i \in \mathscr{V}$, the observers matrices to be designed. The second source of information comes from the estimates received from neighboring nodes, which are also used to correct estimations through the terms $N_{ij}(\hat{x}_j(k - \tau_{ij}(k)) - \hat{x}_i(k - \tau_{ij}(k))), \; \forall j \in \mathscr{N}_i$, where $N_{ij}, \; (i, j) \in \mathscr{E}$, are consensus gains to be synthesized. Please notice that the estimations are sent through the communication network, and thus they are affected by delays and dropouts modeled through the extended delays $\tau_{ij}(k)$.

It is worth recalling that the individual agents cannot access to all the control actions being applied to the plant, as each agent implements different control actions based on its particular state estimation (12.7), that is, $B\mathscr{U}_i(k) \neq B\mathscr{U}(k)$.

Ideally, Eq. (12.8) should be implemented using the augmented control vector $\mathscr{U}(k)$ that the network, as a whole, applies to the plant. However, this information is not available to the agents. To circumvent this difficulty and make Eq. (12.8) realizable, the proposed solution consists, roughly speaking, in letting each agent to run its observer with the augmented control vector obtained from its particular estimate. In general, estimated and actual control inputs are different, but if the observers are properly designed and the nodes estimations converge to the plant states, these differences progressively vanish.

12.2.4 Problem Formulation

Once all the elements of the control scheme have been introduced, this section ends with the formal definition of the problem to be solved in this chapter. First, let us define the following sets:

$$\mathscr{M} \triangleq \{M_i, i \in \mathscr{V}\}, \tag{12.9}$$

$$\mathscr{N} \triangleq \{N_{ij}, (i, j) \in \mathscr{E}\}, \tag{12.10}$$

$$\mathscr{K} \triangleq \{K_i, i \in \mathscr{V}\}. \tag{12.11}$$

The goal is to design the set of distributed observers $\mathscr{M}$, consensus matrices $\mathscr{N}$, and controllers $\mathscr{K}$, in such a way that all the estimation errors of each agent $e_i(k) \triangleq x(k) - \hat{x}_i(k)$ and the plant $x(k)$ are stabilized in spite of the delays and packet dropouts affecting the communication.

12.3 Dynamics of the State and Estimation Errors

In order to provide a solution for the previous problem, the dynamics of the state and of the estimation errors are studied in detail.

Proposition 1 *The dynamics of the plant state $x(k)$ is given by*

$$x(k + 1) = (A + BK)\, x(k) + \Upsilon(\mathscr{K})e(k), \tag{12.12}$$

where

$$\Upsilon(\mathscr{K}) = \begin{bmatrix} -B_1K_1 & -B_2K_2 & \cdots & -B_pK_p \end{bmatrix}.$$

The proof is immediate from Eq. (12.4) and the definition of the estimation errors.

The following proposition studies the evolution of the error vector defined as $e(k) \triangleq [e_1^T(k), \ldots, e_p^T(k)]^T \in \mathbb{R}^{np}$.

Proposition 2 *The dynamics of the error vector $e(k)$ is given by*

$$e(k + 1) = (\Phi(\mathscr{M}) + \Psi(\mathscr{K}))\, e(k) + \Lambda(\mathscr{N})d(k), \tag{12.13}$$

where $d(k) \triangleq \left[e^T(k - \tau_1(k)), \ldots, e^T(k - \tau_r(k))\right]^T$ is a vector stacking l delayed versions of the error vector and

$$\Phi(\mathscr{M}) = diag\{(A - M_1C_1), \ldots, (A - M_pC_p)\},$$

$$\Psi(\mathscr{K}) = diag\{BK, \ldots, BK\} + \begin{bmatrix} -B_1K_1 & \ldots & -B_pK_p \\ \vdots & \ddots & \vdots \\ -B_1K_1 & \ldots & -B_pK_p \end{bmatrix},$$

$$\Lambda(\mathscr{N}) = \left[\, \Theta_1(N_{ij}) \, \dots \, \Theta_r(N_{ij} \, \dots \, \Theta_l(N_{ij}) \,\right],$$

being $\Theta_r(N_{ij})$ $(r = 1, \dots, l)$ *a matrix associated with link* r *and a couple of agents* $(i, j) = g^{-1}(r)$ *with the following structure:*

$$\Theta(N_{ij}) = \begin{array}{c} \begin{matrix} \textit{column} & i & & j & \end{matrix} \\ \begin{bmatrix} 0 \cdots & 0 & \cdots & 0 & \cdots 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 \cdots & -N_{ij}C_{ij} & \cdots & N_{ij}C_{ij} & \cdots 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 \cdots & 0 & \cdots & 0 & \cdots 0 \end{bmatrix} \end{array} \textit{row } i.$$

Proof The observation error at instant $k + 1$ can be obtained using Eq. (12.8) and Proposition 1:

$$\begin{aligned} e_i(k+1) &= x(k+1) - \hat{x}_i(k+1) \\ &= (A + BK)x(k) + \Upsilon(\mathscr{K})e(k) - A\hat{x}_i(k) \\ &\quad - B\mathscr{U}_i(k) - M_iC_i(x(k) - \hat{x}_i(k)) \\ &\quad - \sum_{j \in \mathscr{N}_i} N_{ij}C_{ij}(\hat{x}_j(k - \tau_{ij}(k)) - \hat{x}_i(k - \tau_{ij}(k))). \end{aligned} \tag{12.14}$$

After some mathematical manipulations, Eq. (12.14) can be rewritten as

$$\begin{aligned} e_i(k+1) = (A - M_iC_i)e_i(k) + BKe_i(k) + \Upsilon(\mathscr{K})e(k) \\ - \sum_{j \in \mathscr{N}_i} N_{ij}C_{ij}(e_i(k - \tau_{ij}(k)) - e_j(k - \tau_{ij}(k))). \end{aligned}$$

Finally, since the error vector has been defined as $e^T(k) = \left[\, e_1^T(k) \, \dots \, e_p^T(k) \,\right]$, it is easy to see that the dynamics of $e(k)$ is (12.13). □

Remark 1 The structure of (2) reveals that, even in the absence of time delays, the Separation Principle does not hold, for matrix $\Psi(\mathscr{K})$ depends on the controllers to be designed. This can be easily justified if we recall that the agents ignore the actual control signal being applied to the plant, and resort to estimations based on the knowledge of the distributed controllers. However, despite this drawback, it will be shown that it is possible to propose an unified design in which all the elements, namely controllers and observers, can be designed to guarantee the overall stability of the system.

12.4 Controller and Observer Design

This sections introduces a result to solve the design problem introduced in Sect. 12.2. The design method resorts to a Lyapunov–Krasovskii approach to prove asymptotic stability of the plant state and the estimation errors when network-induced delays and dropouts exist.

The following theorem proposes a centralized design method through an optimization problem subject to a nonlinear matrix inequality.

Theorem 1 *The problem formulated in Sect. 12.2 can be solved by finding positive definite matrices P_x, P_e, Z_1, Z_2, and sets $\mathscr{M}$, $\mathscr{N}$, $\mathscr{K}$ in (12.9)–(12.11) of observers, consensus matrices, and controllers in such a way that the following matrix inequality is satisfied:*

$$\begin{bmatrix} W & S^T \\ * & -H^{-1} \end{bmatrix} < 0, \tag{12.15}$$

where:

$$W = \begin{bmatrix} -P_x & 0 & 0 & 0 \\ * & -P_e + Z_1 - lZ_2 & \bar{1} \otimes Z_2 & 0 \\ * & * & -2I \otimes Z_2 & \bar{1}^T \otimes Z_2 \\ * & * & * & -Z_1 - lZ_2 \end{bmatrix}, \tag{12.16}$$

$$S = \begin{bmatrix} A + BK & \Upsilon(\mathscr{K}) & 0 & 0 \\ 0 & \Phi(\mathscr{M}) + \Psi(\mathscr{K}) & \Lambda(\mathscr{N}) & 0 \\ 0 & \Phi(\mathscr{M}) + \Psi(\mathscr{K}) - I & \Lambda(\mathscr{N}) & 0 \end{bmatrix}, \tag{12.17}$$

$$H^{-1} = \begin{bmatrix} P_x^{-1} & 0 & 0 \\ * & P_e^{-1} & 0 \\ * & * & \frac{1}{l\tau_M^2} Z_2^{-1} \end{bmatrix}. \tag{12.18}$$

Proof Consider the following quadratic Lyapunov–Krasovskii functional:

$$V(x(k), e(k)) = x^T(k)P_x x(k) + e^T(k)P_e e(k) + \sum_{i=k-\tau_M}^{k-1} e^T(i)Z_1 e(i) \tag{12.19}$$

$$+ l \times \tau_M \sum_{j=-\tau_M+1}^{0} \sum_{i=k+j-1}^{k-1} \Delta e^T(i) Z_2 \Delta e(i), \tag{12.20}$$

where P_x and P_e are positive definite matrices and $\Delta e(k) \triangleq e(k+1) - e(k)$.

The forward difference of the functional (12.19) can be expressed in the following way:

$$
\begin{aligned}
\Delta V(x(k), e(k)) = {} & x^T(k+1)P_x x(k+1) - x^T(k)P_x x(k) + e^T(k+1)P_e e(k+1) \\
& - e^T(k)P_e e(k) + e^T(k)Z_1 e(k) - e^T(k-\tau_M)Z_1 e(k-\tau_M) \\
& + l \times \tau_M^2 \Delta e^T(k) Z_2 \Delta e(k) - l \times \tau_M \sum_{j=k-\tau_M}^{k-1} \Delta e^T(j) Z_2 \Delta e(j).
\end{aligned}
$$

Using Propositions 1 and 2 to substitute the evolution of the plant state and the estimation error in last equation, it yields

$$
\begin{aligned}
\Delta V(x(k), e(k)) = {} & x^T \left[(A+BK)^T P_x (A+BK)\right] x(k) \\
& + e^T(k) \left[\Upsilon^T(\mathscr{K}) P_x \Upsilon(\mathscr{K})\right] e(k) \\
& + 2x^T(k) \left[(A+BK)^T P_x \Upsilon(\mathscr{K})\right] e(k) - x^T(k) P_x x(k) \\
& + e^T(k) \left[(\Phi(\mathscr{M}) + \Psi(\mathscr{K}))^T P_e (\Phi(\mathscr{M}) + \Psi(\mathscr{K}))\right] e(k) \\
& + d^T(k) \Gamma^T(\mathscr{N}) P_e \Gamma(\mathscr{N}) d(k) \\
& + 2e^T(k) (\Phi(\mathscr{M}) + \Psi(\mathscr{K}))^T P_e \Gamma(\mathscr{N}) d(k) - e^T(k) P_e e(k) \\
& + e^T(k) Z_1 e(k) - e^T(k-\tau_M) Z_1 e(k-\tau_M) \\
& + l \times \tau_M^2 \Delta e^T(k) Z_2 \Delta e(k) - l \times \tau_M \sum_{j=k-\tau_M}^{k-1} \Delta e^T(j) Z_2 \Delta e(j).
\end{aligned}
$$

Note that the last term is included l times, one for each link. To take into account the delay of each different communication link ($\tau_r(k), \forall r = 1, \ldots, l$), we split it in l terms, each one considering the delay in each specific link:

$$
\begin{aligned}
-\tau_M \sum_{j=k-\tau_M}^{k-1} \Delta e^T(j) Z_2 \Delta e(j) = {} & -\tau_M \sum_{j=k-\tau_M}^{k-\tau_r(k)-1} \Delta e^T(j) Z_2 \Delta e(j) \\
& - \tau_M \sum_{j=k-\tau_r(k)}^{k-1} \Delta e^T(j) Z_2 \Delta e(j),
\end{aligned}
$$

The resulting terms can be bounded using the Jensen inequality:

$$
-\tau_M \sum_{j=k-\tau_M}^{k-\tau_r(k)-1} \Delta e^T(j) Z_2 \Delta e(j) \le -\left[\sum_{j=k-\tau_M}^{k-\tau_r(k)-1} \Delta e(j)\right]^T Z_2 \left[\sum_{j=k-\tau_M}^{k-\tau_r(k)-1} \Delta e(j)\right],
$$

$$
-\tau_M \sum_{j=k-\tau_r(k)}^{k-1} \Delta e^T(j) Z_2 \Delta e(j) \le -\left[\sum_{j=k-\tau_r(k)}^{k-1} \Delta e(j)\right]^T Z_2 \left[\sum_{j=k-\tau_r(k)}^{k-1} \Delta e(j)\right].
$$

Consider now the term $l \times \tau_M^2 \Delta e^T(k) Z_2 \Delta e(k)$ in (12.21). Using Proposition 2, this term can be rewritten as

$$\begin{aligned} l \times \tau_M^2 \Delta e^T(k) Z_2 \Delta e(k) = & \, e^T(k)(\Phi^T(\mathscr{M}) + \Psi(\mathscr{K}) - I)^T \left[l \times \tau_M^2 Z_2 \right] \\ & \times \, (\Phi(\mathscr{M}) + \Psi(\mathscr{K}) - I) e(k) \\ & + \, 2e^T(k)(\Phi^T(\mathscr{M}) + \Psi(\mathscr{K}) - I)^T l \times \tau_M^2 Z_2 \Lambda(\mathscr{N}) d(k). \end{aligned}$$

Thus, it is possible to bound the forward difference of the Lyapunov–Krasovkii functional as follows:

$$\begin{aligned} \Delta V(x(k), e(k)) \le & \, x^T \left[(A + BK)^T P_x (A + BK) \right] x(k) \\ & + \, e^T(k) \left[\Upsilon^T(\mathscr{K}) P_x \Upsilon(\mathscr{K}) \right] e(k) \\ & + \, 2x^T(k) \left[(A + BK)^T P_x \Upsilon(\mathscr{K}) \right] e(k) - x^T(k) P_x x(k) \\ & + \, e^T(k) \left[(\Phi(\mathscr{M}) + \Psi(\mathscr{K}))^T P_e (\Phi(\mathscr{M}) + \Psi(\mathscr{K})) \right] e(k) \\ & + \, d^T(k) \Gamma^T(\mathscr{N}) P_e \Gamma(\mathscr{N}) d(k) \\ & + \, 2e^T(k)(\Phi(\mathscr{M}) + \Psi(\mathscr{K}))^T P_e \Gamma(\mathscr{N}) d(k) - e^T(k) P_e e(k) \\ & + \, e^T(k) Z_1 e(k) - e^T(k - \tau_M) Z_1 e(k - \tau_M) \\ & + \, e^T(k)(\Phi^T(\mathscr{M}) + \Psi(\mathscr{K}) - I)^T \left[l \times \tau_M^2 Z_2 \right] \\ & \times \, (\Phi(\mathscr{M}) + \Psi(\mathscr{K}) - I) e(k) \\ & + \, 2e^T(k)(\Phi^T(\mathscr{M}) + \Psi(\mathscr{K}) - I)^T l \times \tau_M^2 Z_2 \Lambda(\mathscr{N}) d(k) \\ & - \sum_{r=1}^{l} \left(\left[\sum_{j=k-\tau_M}^{k-\tau_r(k)-1} \Delta e(j) \right]^T Z_2 \left[\sum_{j=k-\tau_M}^{k-\tau_r(k)-1} \Delta e(j) \right] \right) \\ & - \sum_{r=1}^{l} \left(\left[\sum_{j=k-\tau_r(k)}^{k-1} \Delta e(j) \right]^T Z_2 \left[\sum_{j=k-\tau_r(k)}^{k-1} \Delta e(j) \right] \right). \end{aligned}$$

Defining an augmented state vector as

$$\xi(k) = \left[x^T(k) \quad e^T(k) \quad d^T(k) \quad e^T(k - \tau_M) \right]^T,$$

the bound of $\Delta V(x(k), e(k))$ can be rewritten in a compact way using the matrices W, S, and T defined in Theorem 1:

$$\Delta V(x(k), e(k)) \le \xi^T(k) W \xi(k) - \xi^T(k) S^T H S \xi(k).$$

Therefore, if matrix $\xi^T(k) W \xi(k) - \xi^T(k) S^T H S \xi(k)$ is negative definite, the state of the system and the estimation errors are asymptotically stable. Using Schur complement, this matrix inequality if equivalent to (12.15), and thus the proof is completed. □

The main hindrance of the design method proposed in Theorem 1 is the nonlinearity of the the matrix inequality (12.15) because of the presence of the matrix H^{-1}. Nonetheless, it is possible to adapt the cone complementary algorithm in (see [9]), which let us address the nonlinearities H^{-1} by introducing some new matrix variables and constraints.

First, define a new matrix variable T. Then replace the matrix H^{-1} in (12.15) by the term T and add the additional LMI $H^{-1} \geq T$, which is equivalent to:

$$\begin{bmatrix} H^{-1} & I \\ I & T^{-1} \end{bmatrix} \geq 0.$$

Then introducing variables $\hat{T}, \hat{H}$, the original matrix inequality (12.15) can be substituted by

$$\begin{bmatrix} W & S^T \\ * & T \end{bmatrix} < 0, \ \begin{bmatrix} \hat{H} & I \\ I & \hat{T} \end{bmatrix} \geq 0, \ \hat{T} = T^{-1}, \ \hat{H} = H^{-1}.$$

Using a cone complementarity algorithm, it is possible to obtain feasible solutions for the optimization problem in Theorem 1 by solving the following problem:

$$\text{Minimize } \mathbf{Tr}\left(\hat{H}H + \hat{T}T\right)$$

subject to

$$\begin{cases} \begin{bmatrix} W & S^T \\ * & T \end{bmatrix} < 0, \\ \begin{bmatrix} \hat{H} & I \\ I & \hat{T} \end{bmatrix} \geq 0, \begin{bmatrix} T & I \\ I & \hat{T} \end{bmatrix} \geq 0, \begin{bmatrix} H & I \\ I & \hat{H} \end{bmatrix} \geq 0. \end{cases} \quad (12.21)$$

In order to find a solution for this problem, the iterative algorithm introduced in [9] can be applied. See [23] for further details.

Remark 2 Once the observers and controllers are designed, the implementation is fully distributed, and each agent requires only available local information to operate. Nonetheless, the design method that stems from Theorem 1 needs to be performed offline prior to the implementation, which requires that some information is known a priori, namely: network topology, outputs that every agent can measure, and control channels they has access to.

As regards the computational complexity in the design phase, the relevant figure here is the number of variables to be computed, which is $N_{\#} = n^2(6p^2 + l + \frac{1}{2}) + n\big(6p + \sum_{i=1}^{p}(r_i + d_i) + \frac{1}{2}\big)$. Thus, the number of variables grows rapidly with the number of agents and the number of states of the plant, which makes it hard to solve for large systems.

As an illustrative example, the application example introduced in the next section has 8 states and outputs, 4 agents, 4 control inputs, and a cycle graph. The number of variables is $N_{\#} = 6980$ and the computation time to solve the design problem is approximately 150 min, using Matlab LMI Toolbox on a PC with a 2.5 GHz Intel Core i5 processor and 8GB RAM.

12.5 Application Example

In this section, the proposed design method is tested on a simulated plant consisting of a set of coupled oscillators. First, the plant will be described, giving the necessary considerations with respect to the agents, their observability and control capacities, and communication delays. Finally, a set of different simulations will be shown.

12.5.1 System Description

We consider a set of N inverted pendulums coupled by springs, as Fig. 12.3 shows. The pendulums have all the same characteristics, that is, mass m, length l. The springs are characterized by the same elastic constant k. This mechanical system has been used as a testbed in engineering and control, see [13]. However, what is more interesting of this plant is the fact that it represents the dynamics of a set of coupled oscillators, which has numerous applications in fields as physics, medicine, or communications, see [7, 36]. The objective is to maintain all the pendulums or oscillators in their upright unstable equilibrium points.

In the following, we will consider that the pendulums are being controlled around the upright unstable equilibrium point. Each pendulum is described by two state variables: angular position θ_i and angular velocity $\omega_i \triangleq \dot{\theta}_i$. It is assumed that the control signal is a torque applied to the base of the pendulum. With the hypothesis of small angles, the dynamics of a single pendulum is given by

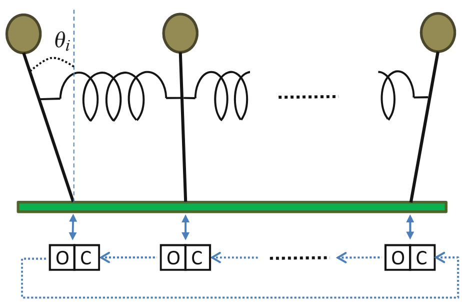

Fig. 12.3 Set of pendulums coupled by springs. Agents and communication graph

$$\begin{bmatrix} \dot{\theta}_i \\ \ddot{\theta}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} - \frac{a_i}{ml^2} & 0 \end{bmatrix} \begin{bmatrix} \theta_i \\ \dot{\theta}_i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u_i + \sum_j \begin{bmatrix} 0 & 0 \\ \frac{h_{ij}}{ml^2} & 0 \end{bmatrix} \begin{bmatrix} \theta_j \\ \dot{\theta}_j \end{bmatrix}, \quad (12.22)$$

where a_i is the number of springs connected to pendulum i and $h_{ij} = 1$ if pendulum i is connected to pendulum j with a spring, and 0 otherwise. Therefore, the third term represents the influence of the neighborhood in the dynamics of the pendulum i.

The state of the complete system will be the vector stacking all the angular position and velocities of all the agents, that is, $x = [\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \ldots, \theta_N, \dot{\theta}_N]^T$. Finally, the system dynamics are discretized with sampling period T_s to obtain an equivalent equation to the one in (12.1).

12.5.1.1 Network of Agents

For this system, we will consider a simple network of N agents. Each agent measures the angular position and velocity of a pendulum and applies a torque to its base. The communication graph is a cycle, as Fig. 12.3 shows. In order to stabilize the whole set of pendulums, the agents must apply coordinated control actions. To do so, it becomes essential for an agent to know the rest of the states.[1]

The communication between agents is affected by delays. Not only do these delays come for communication drawbacks (congestion, dropouts, etc.), but also due to the sampling period. The inverted pendulum is, in general, a system with fast dynamics that needs very short sampling periods. It is fairly possible that the agents are not equipped with powerful communication devices to achieve the required rates. Anyway, even in the case they are, the sampling rate could be artificially enlarged pursuing a reduction of the energy consumption.

12.5.2 Simulation Results

For the simulations, we have chosen the set of parameters given in Table 12.1.

In the first experiment, it is shown that the distributed controllers achieve the stabilization of the system for an arbitrary initial condition close to the unstable equilibrium point (Fig. 12.4).

The figure below presents the estimation performance of agent 1. Concretely, it shows the angular velocity of pendulums 2, 3, and 4, together with the estimation of these states from agent 1. As we can see, the agent achieves nice estimations in spite of the communication delays and the distance in the network (Fig. 12.5).

---

[1]The reader may think that, for this particular system, it is only necessary to know the state of the pendulums in the neighborhood. If the agents do not need the estimations of the whole augmented state, we could implement here a sort of reduced-order distributed observer, as the one proposed in [29] for non-delayed systems.

Table 12.1 List of parameters

Parameter	Value	Unit	Description
N	4		Number of pendulums
p	4		Number of agents
m	1	kg	Mass of the pendulum
l	2	m	Length of the pendulum bar
k	5	N/m	Elastic constant of the string
g	9.8	m/s^2	Gravity
T_s	0.05	s	Sampling period
τ_M	2		Maximum delay

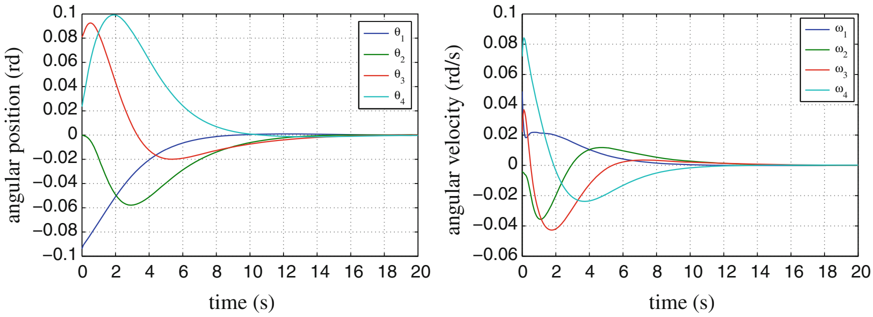

Fig. 12.4 *Left* Evolution of the angular positions. *Right* Evolution of the angular velocities

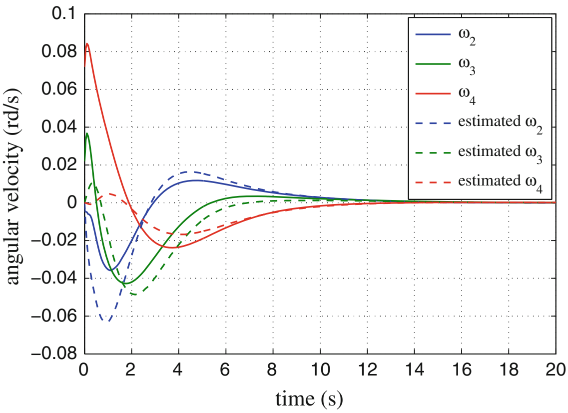

Fig. 12.5 Estimation from agent 1 and actual evolution of the angular velocities of pendulums 2, 3 and 4

The second experiment illustrates the response of the system to external disturbances. Consider that, starting from the equilibrium point, the third pendulum is affected by a disturbance that abruptly changes its position between seconds 5 and 6.

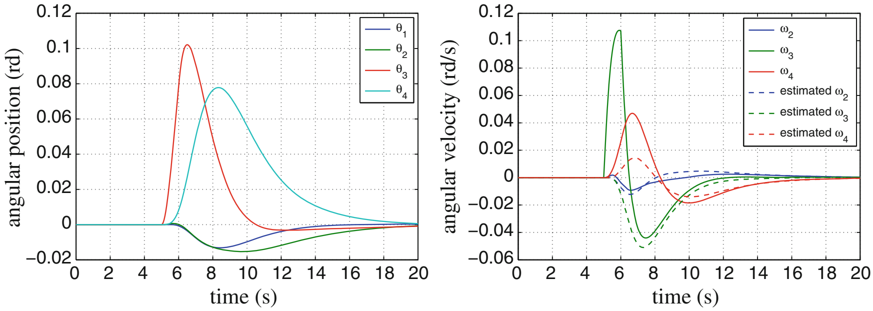

Fig. 12.6 *Left* Evolution of angular positions. *Right* Estimation of angular velocities from agent1

Because of the couplings, both pendulums 2 and 4 are affected as well. Figure 12.6 shows that the response of the controllers and observers are fairly good, despite they have not been designed to reject any disturbances. As expected, the state of pendulum 2 is estimated faster than the others.

12.6 Conclusions

This chapter has studied the problem of stabilizing a large-scale plant with an agent-based distributed paradigm. Unreliable networks affected by time-varying delays and dropouts have been considered. The observers' structure merges a Luenberger-like structure with consensus matrices.

The solution presented ensures the stabilization of both the system state and the observation errors using a Lyapunov–Krasovskii functional. As it has been shown, the design of the controllers and observers must be done in a unique centralized step, which constitutes the weak point of the solution. However, once the controllers and observers are designed, they work in a completely distributed fashion, requiring minimum computation and memory resources. The authors are currently working toward the development of a distributed design method.

Some simulations have been presented to show the performance of the obtained solution.

References

1. P. Alriksson, A. Rantzer, Distributed Kalman filtering using weighted averaging, in *International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, 2006
2. I. Alvarado, D. Limon, D. Muñoz de la Peña, J.M. Maestre, M.A. Ridao, H. Scheu, W. Marquardt, R.R. Negenborn, B. De Schutter, F. Valencia, J. Espinosa, A comparative analysis of distributed MPC techniques applied to the HD-MPC four-tank benchmark. J. Process Control **21**(5), 800–815 (2011)

3. L. Bencini, G. Collodi, D. Di Palma, A. Manes, G. Manes, Advanced distributed monitoring system for agriculture based on wireless sensor network technology, in *World Congress on Computers in Agriculture and Natural Resources*, 2009
4. L. Briñón Arranz, A. Seuret, C. de Canudas Wit, Translation control of a fleet circular formation of AUVs under finite communication range, in *IEEE Conference on Decision and Control (CDC) and Chinese Control Conference (CCC)*, 2009
5. E. Camponogara, D. Jia, B.H. Krogh, S. Talukdar, Distributed model predictive control. IEEE Control Syst. **22**(1), 44–52 (2002)
6. J. Cortés, S. Martínez, T. Karatas, F. Bullo, Coverage control form mobile sensing networks. IEEE Trans. Robot. Autom. **20**(2), 243–255 (2004)
7. A. Díaz-Guilera, A. Arenas, Phase patterns of coupled oscillators with application to wireless communication. Lectures Notes on Computer Science: Bio-inspired Computing and Communication **5151**, 184–191 (2008)
8. W.B. Dunbar, Distributed receding horizon control of dynamically coupled nonlinear systems. IEEE Trans. Autom. Control **52**(7), 1249–1263 (2007)
9. L. El Ghaoui, F. Oustry, M. AitRami, A cone complementary linearization algorithm for static output-feedback and related problems. IEEE Trans. Autom. Control **42**(8), 1171–1176 (1997)
10. D. Estrin, R. Govindan, J. Heidemann, S. Kumar, Next century challenges: scalable coordination in sensor networks, in *ACM/IEEE International Conference on Mobile Computing and Networking*, 1999
11. M. Farina, G. Ferrari-Trecate, R. Scattolini, Distributed moving horizon estimation for linear constrained systems. IEEE Trans. Autom. Control **55**(11), 2462–2475 (2010)
12. E. Fridman, A. Seuret, J.P. Richard, Robust sampled-data stabilization of linear systems: an input delay approach. Automatica **40**(9), 1441–1446 (2004)
13. M. Guinaldo, D.V Dimarogonas, K.H. Johansson, J. Sánchez, S. Dormido, Distributed event-based control for interconnected linear systems, in *IEEE Conference on Decision and Control (CDC) and European Control Center (ECC)*, 2011
14. X. Jiang, Q.L. Han, S. Liu, A. Xue, A new H_∞ stabilization criterion for networked control systems. IEEE Trans. Autom. Control **53**(4), 1025–1032 (2008)
15. I. Jurado, M.G. Ortega, D.E. Quevedo, F.R. Rubio, An H_∞ suboptimal robust control approach for systems with uncertainties and data dropouts. Int. J. Syst. Sci. **46**(11), 1971–1981 (2014)
16. M. Kamgarpour, C. Tomlin, Convergence properties of a decentralized Kalman filter, in *IEEE Conference on Decision and Control (CDC)*, 2008
17. M. Lin, Y. Wu, I. Wassell, Wireless sensor network: Water distribution monitoring system, in *IEEE Radio and Wireless Symposium (RWS)*, 2008
18. J.P Lynch, K.H. Law, J.A. Blume, Decentralized control techniques for large-scale civil structural systems, in *International Modal Analysis Conference*, 2002
19. J.M. Maestre, D.M. de la Peña, E.F. Camacho, A distributed mpc scheme with low communication requirements, in *American Control Conference (ACC)*, 2009
20. J.M. Maestre, P. Giselsson, A. Rantzer, Distributed receding horizon Kalman filter, in *IEEE Conference on Decision and Control (CDC)*, 2010
21. J.M. Maestre, D. Muñoz de la Peña, E.F. Camacho, Wireless sensor network analysis through a coalitional game: Application to a distributed Kalman filter, in *IEEE International Conference on Networking, Sensing and Control (ICNSC)*, 2011
22. J.M. Maestre, R.R. Negenborn, *Distributed Model Predictive Control Made Easy* (Springer, 2014)
23. P. Millán, L. Orihuela, G. Bejarano, C. Vivas, T. Alamo, F.R. Rubio, Design and application of suboptimal mixed H_2/H_∞ controllers for networked control systems. IEEE Trans. Control Syst. Technol. **20**(4), 1057–1065 (2012)
24. P. Millán, L. Orihuela, C. Vivas, F.R. Rubio, Distributed consensus-based estimation considering network induced delays and dropouts. Automatica **48**(10), 2726–2729 (2012)
25. R.R. Negemborn, De Shutter B., J. Hellendoorn, Multi-agent model predictive control for transportation networks: Serial versus parallel schemes. Eng. Appl. Artif. Intell. **21**(3), 353–366 (2008)

26. R. Olfati-Saber, Distributed Kalman filter with embedded consensus filters, in *IEEE Conference on Decision and Control (CDC) and the European Control Conference (ECC)*, 2005
27. R. Olfati-Saber, Distributed Kalman filtering for sensor networks, in *IEEE Conference on Decision and Control (CDC)*, 2007
28. R. Olfati-Saber, Kalman-consensus filter: Optimality, stability, and performance, in *IEEE Conference on Decision and Control (CDC) and Chinese Control Conference (CCC)*, 2009
29. L. Orihuela, P. Millán, C. Vivas, F.R. Rubio, Reduced-order H_2/H_∞ distributed observer for sensor networks. Int. J. Control **86**(10), 1870–1879 (2013)
30. L. Orihuela, P. Millán, C. Vivas, F.R. Rubio, Suboptimal distributed control and observation: application to a four coupled tanks system. Int. J. Syst. Sci. **47**(8), 1755–1771 (2016)
31. H. Ramamurthy, B.S. Prabhu, R. Gadh, A.M. Madni, Wireless industrial monitoring and control using a smart sensor platform. IEEE Sens. J. **7**(5), 611–617 (2007)
32. S. Roshany-Yamchi, M. Cychowski, R.R. Negenborn, B. De Schutter, K. Delaney, J. Connell, Kalman filter-based distributed predictive control of large-scale multi-rate systems: application to power networks. IEEE Trans. Control Syst. Technol. **21**(1), 27–39 (2012)
33. R. Scattolini, Architectures for distributed and hierarchical model predictive control: a review. J. Process Control **19**(5), 723–731 (2009)
34. A. Seuret, Networked control under synchronization errors, in *American Control Conference (ACC)*, 2008
35. G. Simon, M. Maroti, A. Ledeczi, G. Balogh, B. Kusy, A. Nadas, G. Pap, J. Sallai, K. Frampton, Sensor network-based countersniper system, in *International Conference on Embedded Networked Sensor Systems (SenSys)*, 2004
36. A. Stefanovska, Coupled oscillators: complex but not complicated cardiovascular and brain interactions. IEEE Eng. Med. Biol. Mag. **26**(6), 25–29 (2007)
37. A.N. Venkat, J.B. Rawlings, S.J. Wright, Stability and optimality of distributed model predictive control, in *IEEE Conference on Decision and Control (CDC) and European Control Conference (ECC)*, 2005
38. J. Yick, B. Mukherjee, D. Ghosal, Analysis of a prediction-based mobility adaptive tracking algorithm, in *International Conference on Broadband Networks (BWCCA)*, 2005
39. M. Yu, L. Wang, T. Chu, F. Hao, Stabilization of networked control systems with data packet dropout and transmission delays: continuous-time case. Eur. J. Control **11**(1), 4049 (2005)
40. D. Yue, Q.L. Han, J. Lam, Network-based robust H_∞ control of systems with uncertainty. Automatica **41**(6), 999–1007 (2005)
41. G. Zhao, Wireless sensor networks for industrial process monitoring and control: a survey. Netw. Protocols Algorithms **3**(1), 46–63 (2011)

Chapter 13
Design and Analysis of Reset Strategy for Consensus in Networks with Cluster Pattern

Marcos Cesar Bragagnolo, Irinel-Constantin Morărescu, Jamal Daafouz and Pierre Riedinger

Abstract This chapter addresses the problem of consensus in networks partitioned in several disconnected clusters. Each cluster is represented by a fixed, directed, and strongly connected graphs. In order to enforce the consensus, we assume that each cluster poses a leader that can reset its state by taking into account other leaders state. First, we characterize the consensus value of this model. Second, we provide sufficient condition in LMI form for the stability of the consensus. Finally, we perform a decay rate analysis and design the interaction network of the leaders which allows to reach a prescribed consensus value.

13.1 Introduction

The dynamical systems appearing in diverse areas of science and engineering are obtained by interconnecting many simpler systems. Doing so we are getting dynamical networks whose links are fixed, time-dependent or event-dependent. Different models coming from sociology [1], biology [2], or physics [3] have been widely analyzed. As in many works in the literature, we call agents the constitutive elements of the network and their number will define the network dimension. During the last decades, an increasing interest has been given to large dimension networks

I.-C. Morărescu—This work was funded by the ANR project COMPACS—"Computation Aware Control Systems", ANR-13-BS03-004.

M.C. Bragagnolo (✉) · I.-C. Morărescu · J. Daafouz · P. Riedinger
Université de Lorraine, CRAN, UMR 7039, 2 Avenue de la Forêt de Haye, Vandoeuvre-lès-Nancy, France
e-mail: marcos-cesar.bragagnolo@univ-lorraine.fr

I.-C. Morărescu
e-mail: constantin.morarescu@univ-lorraine.fr

J. Daafouz
e-mail: jamal.daafouz@univ-lorraine.fr

P. Riedinger
e-mail: pierre.riedinger@univ-lorraine.fr

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_13

[4–6] which are obtained interconnecting several networks of smaller dimension. In other words, inside large networks we can detect groups of agents called clusters or communities. The agents belonging to a community are better connected [7–9] and they agree/synchronize faster on some quantity of interest called state.

The coordination behavior in networks has been studied under diverse hypothesis such as: directed or undirected interactions, fixed or time-varying interaction graph, delayed or un-delayed, synchronized or desynchronized interactions, linear or non-linear, and continuous or discrete agent dynamics [10–15]. The agreement speed in various frameworks has also been quantified (see for instance [16, 17]). Some works have been oriented toward networks in which the global agreement cannot be reached and only local ones are obtained [7, 18]. Some others, designed controllers that are able to maintain the network connectivity in order to ensure the global coordination is achieved [19, 20]. Finally, the agreement in networks of agents with discrete dynamics in which any agent is linked with not more than one neighbor at each time [21–24] has also been considered.

In [25], we considered networks that are partitioned in several clusters. The agents are able to continuously interact only with neighbors belonging to the same cluster. Moreover, each cluster contains an agent with powerful communication capacity called leader. Each leader can interact with some other leaders via a network with communication constraints. Thus, the leaders will interact only at specific isolated instants that will be defined in the next section as a nearly periodic sequence. In other words, we address the problem of consensus for agents subject to both continuous and discrete dynamics. Precisely, we are focusing on agreement in networks of reset systems, which are a particular class of hybrid systems (see [26–29]). The existing literature on reset systems treats mostly linear dynamics. Two types of reset rules may be encountered, those defined by a time condition and those defined by a state one. The former type is usually defined by a periodic or quasi-periodic reset rule [29]. The later type assumes the stability of the system inside a given set and controls the evolution of the system to remain inside this set [28]. In other words, the reset occurs when the trajectory is about to get out of the stability set.

Resetting control structure in consensus protocol has been studied before, although not in the same manner as treated in [25]. In [30], the authors revisited the Clegg integrator in order to propose a quasi-reset control law for multiagent systems . Mainly, they reinterpret the example in [27] in the framework of interconnected systems. This leads to a lesser magnitude of the control effort, while maintaining a settling time equivalent to the standard consensus protocol. This approach differs from the approach in [25] because in [30] the interconnection graph is still connected so the system can still achieve consensus, while in [25] a general consensus cannot be reached without the reset protocol.

The first result of this work evaluates and study the stability of the possible consensus value when the leaders interact and reset their state in a near-periodic manner. The second objective is to design the network of leaders in order to reach a priori specified consensus value. In order to reach this goal, we consider that each cluster has a fixed and known interconnection topology. Therefore, we modify the consensus value of the whole network by changing the weights in the network of leaders. The set of consensus values that can be reached is contained in the convex hull of the initial local agreement values, which, for scalar states, is just the interval defined by the minimum and maximum initial local agreements.

Notation. The following standard notation will be used throughout the paper. The sets of nonnegative integers, real and nonnegative real numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively. For a vector x, we denote by $\|x\|$ its Euclidian norm. The transpose of a matrix A is denoted by $A^\top$. Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, notation $A > 0$ ($A \geq 0$) means that A is positive (semi-)definite. By I_k we denote the $k \times k$ identity matrix. $\mathbb{1}_k$ and $\mathbf{0}_k$ are the column vectors of size k having all the components equal 1 and 0, respectively. We also use $x(t_k^-) = \lim_{t \to t_k, t \leq t_k} x(t)$.

13.2 Problem Formulation

We consider a network of n agents described by the digraph (i.e., directed graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the vertex set $\mathcal{V}$ represents the set of agents and the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represents the interactions.

Definition 1 In this chapter, we will use the following terms:

- A **path** in a given digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a union of directed edges $\bigcup_{k=1}^{p} (i_k, j_k)$ such that $i_{k+1} = j_k$, $\forall k \in \{1, \ldots, p-1\}$.
- Two nodes i, j are **connected** in a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if there exists at least a path in $\mathcal{G}$ joining i and j (i.e., $i_1 = i$ and $j_p = j$).
- A **strongly connected digraph** is such that any two distinct nodes are connected. A **strongly connected component** of a digraph is a maximal subset such that any of its two distinct nodes are connected.

In the sequel, we consider that the agent set $\mathcal{V}$ is partitioned in m strongly connected clusters/communities $\mathcal{C}_1, \ldots, \mathcal{C}_m$ and no link between agents belonging to different communities exists. Each community possesses one particular agent called leader and denoted in the following by $l_i \in \mathcal{C}_i$, $\forall i \in \{1, \ldots, m\}$. The set of leaders will be referred to as $\mathcal{L} = \{l_1, \ldots, l_m\}$. At specific time instants t_k, $k \geq 1$, called reset times, the leaders interact between them following a predefined interaction map $\mathcal{E}_l \subset \mathcal{L} \times \mathcal{L}$. We also suppose that $\mathcal{G}_l = (\mathcal{L}, \mathcal{E}_l)$ is strongly connected. The rest of the agents will be called as followers and denoted by f_j. For the sake of clarity, we consider that the leader is the first element of its community:

$$\mathcal{C}_i = \{l_i, f_{m_{i-1}+2}, \ldots, f_{m_i}\}, \ \forall i \in \{1, \ldots, m\} \tag{13.1}$$

where $m_0 = 0$, $m_m = n$ and the cardinality of $\mathcal{C}_i$ is given by $|\mathcal{C}_i| \triangleq n_i = m_i - m_{i-1}, \forall i \geq 1$. In order to keep the presentation simple, each agent will have a scalar state also denoted by l_i for the leader l_i and f_j for the follower f_j. We also introduce the vectors $x = (l_1, f_2, \ldots, f_{m_1}, \ldots, l_m, \ldots, f_{m_m})^\top \in \mathbb{R}^n$ and $x_l = (l_1, l_2, \ldots, l_m)^\top \in \mathbb{R}^m$ collecting all the states of the agents and all the leaders' states, respectively.

We are ready now to introduce the linear reset system describing the overall network dynamics:

$$\begin{cases} \dot{x}(t) &= -Lx(t), \qquad \forall t \in \mathbb{R}_+ \setminus \mathscr{T}, \\ x_l(t_k) &= P_l x_l(t_k^-), \qquad \forall t_k \in \mathscr{T}, \\ x(0) &= x_0 \end{cases} \tag{13.2}$$

where $\mathscr{T} = \{t_k \in \mathbb{R}_+ \mid t_k < t_{k+1},\ \forall k \in \mathbb{N}, t_k \text{ reset time}\}$, $L \in \mathbb{R}^{n\times n}$ is a generalized Laplacian matrix associated to the graph $\mathscr{G}$ and $P_l \in \mathbb{R}^{m\times m}$ is a Perron matrix associated to the graph $\mathscr{G}_l = (\mathscr{L}, \mathscr{E}_l)$. Precisely, the entries of L and P_l satisfies the following relations:

$$\begin{cases} L_{(i,j)} = 0, & \text{if } (i,j) \notin \mathscr{E}, \\ L_{(i,j)} < 0, & \text{if } (i,j) \in \mathscr{E},\ i \neq j, \\ L_{(i,i)} = -\sum_{j\neq i} L_{i,j}, & \forall i = 1, \ldots, n, \end{cases} \tag{13.3}$$

$$\begin{cases} P_{l(i,j)} = 0, & \text{if } (i,j) \notin \mathscr{E}_l, \\ P_{l(i,j)} > 0, & \text{if } (i,j) \in \mathscr{E}_l,\ i \neq j, \\ \sum_{j=1}^{m} P_{l(i,j)} = 1, & \forall i = 1, \ldots, m. \end{cases} \tag{13.4}$$

The values $L_{(i,j)}$ and $P_{l(i,j)}$ represent the weight of the state of the agent j in the updating process of the state of agent i when using the continuous and discrete dynamics, respectively. These values describe the level of democracy inside each community and in the leaders' network. In particular, L has the following block diagonal structure $L = diag(L_1, L_2, ..., L_m)$, $L_i \in \mathbb{R}^{n_i}$ with $L_i \mathbb{1}_{n_i} = \mathbf{0}_{n_i}$ and $P_l \mathbb{1}_m = \mathbb{1}_m$. Due to the strong connectivity of $\mathscr{C}_i$, $i = 1, m$ and $\mathscr{G}_l$, 0 is simple eigenvalue of each L_i and 1 is simple eigenvalue of P_l.

13.3 Agreement Behavior

In this section, we assume that system (13.2) achieves consensus and we characterize the possible values for it. First, we show that each agent tracks a local agreement function which is piecewise constant. In the second subsection, we prove that the vector of local agreements lies in a subspace defined by the system's dynamics and initial condition. Therefore, if the consensus is achieved and the corresponding consensus value is x^* then $x^*\mathbb{1}_m$ belongs to the same subspace. Moreover, this value is determined only by the initial condition of the network and by the interconnection structure.

As we have noticed, $\mathbb{1}_{n_i}$ is the right eigenvector of L_i associated with the the eigenvalue 0 and $\mathbb{1}_m$ is the right eigenvector of P_l associated with the eigenvalue 1. In the sequel, we denote by w_i the left eigenvector of L_i associated with the the eigenvalue 0 such that $w_i^\top \mathbb{1}_{n_i} = 1$. Similarly, let $v = (v_1, \ldots, v_m)^\top$ be the left eigenvector of P_l associated with the eigenvalue 1 such that $v^\top \mathbb{1}_m = 1$. Due to the structure (13.1) of the communities, we emphasize that each vector w_i can be decomposed in its first component $w_{i,l}$ and the rest of its components grouped in the vector $w_{i,f}$. Let us introduce the matrix of the left eigenvectors of the communities:

$$W = \begin{bmatrix} w_1^\top & 0 & \cdots & 0 \\ 0 & w_2^\top & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_m^\top \end{bmatrix} \in \mathbb{R}^{m\times n}. \tag{13.5}$$

13.3.1 Local Agreements

Let us first recall a well-known result concerning the consensus in networks of agents with continuous time dynamics (see [14] for instance).

Lemma 1 *Let $\mathscr{G}$ be a strongly connected digraph and L the corresponding Laplacian matrix. Consider a network of agents whose collective dynamics is described by $\dot{x}(t) = -Lx(t)$. Let us also consider $L\mathbb{1} = \mathbf{0}$, $u^\top L = \mathbf{0}$ and $u^\top \mathbb{1} = 1$. Then the agents asymptotically reach a consensus and the consensus value is given by $x^* = u^\top x(0)$. Moreover, the vector u defines an invariant subspace for the collective dynamics: $u^\top x(t) = u^\top x(0), \forall t \geq 0$.*

Remark 1 When dynamics (13.2) is considered, Lemma 1 implies that between two reset instants t_k and t_{k+1}, the agents belonging to the same community converge to a local agreement defined by $x_i^*(k) = w_i^\top x_{\mathscr{C}_i}(t_k)$ where $x_{\mathscr{C}_i}(\cdot)$ is the vector collecting the states of the agents belonging to the cluster $\mathscr{C}_i$. Nevertheless, at the reset times the value of the local agreement can change. Thus,

$$\begin{aligned} &w_i^\top x(t) = w_i^\top x_{\mathscr{C}_i}(t_k), \;\; \forall t \in (t_k, t_{k+1}) \text{ and possibly} \\ &w_i^\top x_{\mathscr{C}_i}(t) \neq w_i^\top x_{\mathscr{C}_i}(t_k), \quad \text{for } t \notin (t_k, t_{k+1}). \end{aligned}$$

Therefore, the agents whose collective dynamics is described by the hybrid system, (13.2), may reach a consensus only if the local agreements converge one to each other.

13.3.2 Consensus Value

Before presenting our next result, let us introduce the following vectors:

$$\begin{aligned} x^*(t) &= (x_1^*(t), x_2^*(t), ..., x_m^*(t))^\top \in \mathbb{R}^m \\ u &= (v_1/w_{1,l}, v_2/w_{2,l}, ..., v_m/w_{m,l})^\top \in \mathbb{R}^m \end{aligned} \tag{13.6}$$

represents the local agreement of the cluster $\mathscr{C}_i$ and $v \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^{n_i}$ are defined at the beginning of the section as left eigenvectors associated with the matrices describing the reset dynamics of the leaders and the continuous dynamics of each cluster, respectively. It is noteworthy that $x^*(t)$ is time-varying but piecewise constant: $x^*(t) = x^*(k)\ \forall t \in (t_k, t_{k+1})$.

Proposition 1 *Consider the system (13.2) with L and P_l defined by (13.3) and (13.4), respectively. Then*

$$u^\top x^*(t) = u^\top x^*(0),\ \forall t \in \mathbb{R}_+. \tag{13.7}$$

Corollary 1 *Consider the system (13.2) with L and P_l defined by (13.3) and (13.4), respectively. Assuming the agents of this system reach a consensus, the consensus value is*

$$x^* = \frac{u^\top W x(0)}{\sum_{i=1}^m u_i}. \tag{13.8}$$

In order to simplify the presentation and without loss of generality, in what follows, we consider that $\sum_{i=1}^{m} u_i = 1$. A trivial result which may be seen as a consequence of Corollary 1 is the following.

Corollary 2 *If the matrices L, P_l are symmetric (i.e., ith agent takes into account the state of jth agent as far as jth takes into account the ith one and they give the same importance one to another) the consensus value is the average of the initial states.*

13.4 Stability Analysis

13.4.1 Prerequisites

The stability analysis of the equilibrium point x^* will be given by means of some LMI conditions. Precisely, we recall and adapt some results presented in [29]. Since the consensus value is computed in the previous section, we can first define the disagreement vector $y = x - x^* \mathbb{1}_n$. We also introduce an extended stochastic matrix

$P_{ex} = T^\top \begin{bmatrix} P_l & 0 \\ 0 & I_{n-m} \end{bmatrix} T$ where T is a permutation matrix allowing to recover the cluster structure of L. It is noteworthy that $L\mathbb{1}_n = \mathbf{0}_n$ and $P_{ex}\mathbb{1}_n = \mathbb{1}_n$. Thus, the disagreement dynamics is exactly the same as the system one:

$$\begin{cases} \dot{y}(t) = -Ly(t), & \forall t \in \mathbb{R}_+ \setminus \mathscr{T} \\ y(t_k) = P_{ex}y(t_k^-) & \forall t_k \in \mathscr{T} \\ y(0) = y_0 \end{cases} . \tag{13.9}$$

Now we have to analyze the stability of the equilibrium point $y^* = \mathbf{0}_n$ for the system (13.9). We note that Theorem 2 in [29] cannot be directly applied due to the marginal stability of the matrices L and P_{ex}.

The reset sequence is defined such that $t_{k+1} - t_k = \delta + \delta'$ where $\delta \in \mathbb{R}_+$ is fixed and $\delta' \in \Delta$ with $\Delta \subset \mathbb{R}_+$ a compact set. Thus the set of reset times $\mathscr{T}$ belongs to the set $\Phi(\Delta) \triangleq \left\{\{t_k\}_{k\in\mathbb{N}}, t_{k+1} - t_k = \delta + \delta'_k, \delta'_k \in \Delta, \forall k \in \mathbb{N}\right\}$ of all admissible reset sequences.

Definition 2 We say that the equilibrium $y^* = \mathbf{0}_n$ of the system (13.9) is Globally Uniformly Exponentially Stable (GUES) with respect to the set of reset sequences $\Phi(\Delta)$ if there exist positive scalars c, λ such that for any $\mathscr{T} \in \Phi(\Delta)$, any $y_0 \in \mathbb{R}^n$, and any $t \geq 0$

$$\|\varphi(t, y_0)\| \leq ce^{-\lambda t}\|y_0\|. \tag{13.10}$$

The following theorem is instrumental:

Theorem 1 (Theorem 1 in [29]) *Consider the system (13.9) with the set of reset times $\mathscr{T} \in \Phi(\Delta)$. The equilibrium $y^* = \mathbf{0}_n$ is GUES if and only if there exists a positive function $V : \mathbb{R}^n \mapsto \mathbb{R}_+$ strictly convex,*

$$V(y) = y^\top S_{[y]} y,$$

homogeneous (of second order), $S_{[\cdot]} : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, $S_{[y]} = S_{[y]}^\top = S_{[ay]} > 0$, $\forall x \neq 0, a \in \mathbb{R}$, $a \neq 0$, $V(0) = 0$, such that $V(y(t_k)) > V(y(t_{k+1}))$ for all $y(t_k) \neq 0$, $k \in \mathbb{N}$ and any of the possible reset sequences $\mathscr{T} \in \Phi(\Delta)$.

13.4.2 Parametric LMI Condition

In the sequel, we define a quasi-quadratic Lyapunov function satisfying Theorem 1 by means of some LMI. Therefore, the following result gives sufficient conditions for the stability of the equilibrium point $y^* = \mathbf{0}_n$ for the system (13.9) or equivalently of $x^*\mathbb{1}_n$ for the system (13.2).

Theorem 2 *Consider the system (13.2) with $\mathscr{T}$ in the admissible reset sequences $\Phi(\Delta)$. If there exist matrices $S(\delta')$, $S(\cdot) : \Delta \mapsto \mathbb{R}^{n\times n}$ continuous with respect to δ', $S(\delta') = S^\top(\delta') > 0,\ \delta' \in \Delta$ such that the LMI*

$$\begin{aligned}&\Big(I_n - \mathbb{1}_n u^\top W\Big)^\top S(\delta_a)\Big(I_n - \mathbb{1}_n u^\top W\Big) - \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top S(\delta_b)\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) > 0,\\ &Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)}\end{aligned} \tag{13.11}$$

is satisfied on span$\{\mathbb{1}_n\}^\perp$ for all $\delta_a, \delta_b \in \Delta$, then x^* is GUES for (13.2). Moreover, the stability is characterized by the quasi-quadratic Lyapunov function $V(t) = V(x(t)) \triangleq \max_{\delta'\in\Delta}(x(t) - x^*\mathbb{1}_n)^\top S(\delta')(x(t) - x^*\mathbb{1}_n)$ satisfying $V(t_k) > V(t_{k+1})$.

Due to the space limitations, we do not provide the proofs of the results above. However, we point out that Theorem 1 requires to solve a parametric LMI which can be approximated by a finite number of LMIs using polytopic embeddings. The set $\{X \in \mathbb{R}^{n\times n} \mid X = e^{-L\delta_a},\ \delta_a \in \Delta\}$ can be embedded into the polytopic set defined by the vertices $Z_1, \ldots, Z_{h+1}$ where

$$\begin{aligned}Z_1 &= I_n\\ Z_i &= \sum_{l=0}^{i-1} \frac{(-L)^l}{l!}\delta_{max}^l,\ \forall i \in \{2, \ldots, h+1\}\end{aligned}$$

with $\delta_{max} = \max_{\delta'\in\Delta} \delta'$, $(-L)^0 = I_n$ and $0! = 1$. Then, Theorem 2 can be replaced by the following result.

Theorem 3 *Consider the system (13.2) with $\mathscr{T}$ in the admissible reset sequences $\Phi(\Delta)$. If there exist symmetric positive definite matrices S_i, $1 \le i \le h+1$ such that the LMI*

$$\begin{aligned}&\Big(I_n - \mathbb{1}_n u^\top W\Big)^\top S_i \Big(I_n - \mathbb{1}_n u^\top W\Big) - \Big(Y(\delta)Z_i - \mathbb{1}_n u^\top W\Big)^\top S_j \Big(Y(\delta)Z_i - \mathbb{1}_n u^\top W\Big) > 0,\\ &Y(\delta) \triangleq P_{ex} e^{-L(\delta)}\end{aligned} \tag{13.12}$$

is satisfied on span$\{\mathbb{1}_n\}^\perp$ for all $i, j \in \{1, \ldots, h+1\}$, then x^* is GUES for (13.2).

13.5 Complementary Results

13.5.1 Decay Rate Analysis

Once the global uniform exponential stability of x^* ensured by Theorem 2, we can compute the convergence speed of the state of system (13.2). In other words, we are searching to evaluate λ in (13.10). From the proof of Theorem 1, one has $\lambda = \dfrac{\ln \lambda_d}{\delta + \delta_{max}}$ where $\delta_{max} = \max\limits_{\delta' \in \Delta} \delta'$ and λ_d defined as the decay rate of the linear difference inclusion (LDI)

$$x(t_{k+1}) \in \mathscr{F}(x(t_k)), \quad k \in \mathbb{N}, \tag{13.13}$$

where

$$\mathscr{F}(x) = \left\{ P_{ex} e^{-L(\delta+\delta')},\ \delta' \in \Delta \right\}.$$

Precisely, for the LDI (13.13) there exist $M > 0$ and $\xi \in [0, 1]$ such that

$$\|x(t_k) - x^* \mathbb{1}_n\| \leq M\xi^k \|x(0) - x^* \mathbb{1}_n\|, \quad \forall k \in \mathbb{N}, \tag{13.14}$$

and λ_d is defined as the smaller ξ satisfying (13.14).

Thus in order to quantify the convergence speed of system (13.2), we only have to evaluate λ_d. Let us denote again $y = x - x^* \mathbb{1}_n$ and note that $V(y)$ defined by Theorem 2 is a norm. That implies there exist $\alpha, \beta > 0$ such that

$$\alpha \|y\|^2 \leq V(y) \leq \beta \|y\|^2.$$

Consequently, one obtains that the decay rate λ_d coincides with the decay rate of V. Thus, the following result can be derived directly from Theorem 2.

Proposition 2 *Assume there exist* $\alpha > 0,\ \beta > 0,\ \xi \in (0, 1]$ *and the matrices* $S(\delta')$, $S\cdot) : \Delta \mapsto \mathbb{R}^{n \times n}$ *continuous with respect to* δ', $S(\delta') = S^\top(\delta') > 0,\ \delta' \in \Delta$ *fulfilling the following constraints*

$$\begin{aligned} &\alpha I_n \leq S(\delta') \leq \beta I_n, \quad \forall \delta' \in \Delta \\ &\xi^2 \left(I_n - \mathbb{1}_n u^\top W\right)^\top S(\delta_a) \left(I_n - \mathbb{1}_n u^\top W\right) - \left(Y(\delta_a) - \mathbb{1}_n u^\top W\right)^\top S(\delta_b) \left(Y(\delta_a) - \mathbb{1}_n u^\top W\right) > 0, \\ &Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)} \end{aligned} \tag{13.15}$$

on $span\{\mathbb{1}_n\}^\perp$ *for all* $\delta_a, \delta_b \in \Delta$. *Then the decay rate is defined as*

$$\lambda_d = \min_{\xi \ satisfies\ (13.15)} \xi$$

and

$$\|x(t_k) - x^* \mathbb{1}_n\| \leq \frac{\beta}{\alpha} (\lambda_d)^k \|x(0) - x^* \mathbb{1}_n\|, \quad \forall k \in \mathbb{N}.$$

Remark 2 It is noteworthy that $0 < \lambda_d \leq 1$ and for a priori fixed values of $\alpha,\ \beta$, we can use the bisection algorithm to approach as close as we want the value of λ_d.

Remark 3 To complete the decay rate analysis, we can consider that $P_{ex},\ L$ and λ_d are fixed and perform a line search to find the nominal reset period δ that ensures the convergence speed constraint. In other words, we check if (13.15) has solutions for $\xi = \lambda_d$ and δ heuristically sweeping the positive real axis. Moreover, we can progressively decrease λ_d and re-iterate the line search in order to find the smaller reachable decay rate.

13.5.2 Convergence Toward a Prescribed Value

In what follows, we assume that the value x^* is a priori fixed and at least a vector u satisfying (13.8) exists. Under this assumption, we are wondering if there exists a matrix P_l that allows system (13.2) to reach the consensus value x^*. It is worth noting that the network topology is considered fixed and known for each cluster. Under these assumptions, a consensus value is imposed by a certain choice of v such that $v^\top \mathbb{1}_m = 1$ and v left eigenvector of P_l associated with the eigenvalue 1. In other words, we arrive to a joint design of P_l and the Lyapunov function V guaranteeing the trajectory of (13.2) ends up on x^*.

Theorem 4 *Let us consider the system (13.2) with $\mathscr{T}$ in the admissible reset sequences $\Phi(\Delta)$ and let x^* be a priori fixed by a certain choice of v. If there exist matrices $R(\delta'),\ R(\cdot) : \Delta \mapsto \mathbb{R}^{n\times n}$ continuous with respect to δ', $R(\delta') = R^\top(\delta') > 0,\ \delta' \in \Delta$ and P_l stochastic such that the LMI*

$$\begin{aligned} &\begin{bmatrix} Z(\delta_a) & \left(Y(\delta_a) - \mathbb{1}_n u^\top W\right)^\top \\ \left(Y(\delta_a) - \mathbb{1}_n u^\top W\right) & R(\delta_b) \end{bmatrix} > 0, \\ &Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)} \\ &Z(\delta_a) \triangleq \left(I_n - \mathbb{1}_n u^\top W\right)^\top + \left(I_n - \mathbb{1}_n u^\top W\right) - R(\delta_a) \end{aligned} \tag{13.16}$$

with the constraint

$$v^\top P_l = v^\top$$

is satisfied on $span\{\mathbb{1}_n\}^\perp$ for all $\delta_a, \delta_b \in \Delta$, then x^* is GUES for (13.2). Moreover, the stability is characterized by the quasi-quadratic Lyapunov function $V(t) = \max_{\delta' \in \Delta} (x(t) - x^* \mathbb{1}_n)^\top R(\delta')^{-1} (x(t) - x^* \mathbb{1}_n)$ satisfying $V(t_k) > V(t_{k+1})$.

13.5.3 Convergence Toward a Prescribed Value with a Prescribed Decay Rate

Combining the results of Proposition 2 and Theorem 4, we can design the matrix P_l that allows to reach an a priori given consensus value x^* with a decay rate inferior to an a priori fixed value. Precisely, the following result holds.

Theorem 5 *Let us consider the system (13.2) with $\mathscr{T}$ in the admissible reset sequences $\Phi(\Delta)$ and let x^* be a priori fixed by a certain choice of v. Let us also consider $\bar{\lambda} \in (0, 1)$ a priori fixed. If there exist matrices $R(\delta')$, $R(\cdot) : \Delta \mapsto \mathbb{R}^{n\times n}$ continuous with respect to δ', $R(\delta') = R^\top(\delta') > 0$, $\delta' \in \Delta$ and P_l stochastic such that the LMI*

$$\begin{aligned}
&\begin{bmatrix} Z(\delta_a) & \left(Y(\delta_a) - \mathbb{1}_n u^\top W\right)^\top \\ \left(Y(\delta_a) - \mathbb{1}_n u^\top W\right) & \bar{\lambda}^2 R(\delta_b) \end{bmatrix} > 0, \\
&Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)} \\
&Z(\delta_a) \triangleq \left(I_n - \mathbb{1}_n u^\top W\right)^\top + \left(I_n - \mathbb{1}_n u^\top W\right) - R(\delta_a)
\end{aligned} \tag{13.17}$$

with the constraint

$$v^\top P_l = v^\top$$

is satisfied on $span\{\mathbb{1}_n\}^\perp$ for all $\delta_a, \delta_b \in \Delta$, then x^* is GUES for (13.2) and (13.14) is satisfied for $\xi = \bar{\lambda}$ and $M = \beta/\alpha$ where β and α are the minimum and the maximum eigenvalue of $R(\delta')$, $\delta' \in \Delta$, respectively.

Remark 4 It is noteworthy that LMI (13.17) implies LMI (13.15) but they are not equivalent. Therefore, the decay rate λ_d is often strictly smaller than $\bar{\lambda}$.

13.6 Illustrative Example

An academic example consisting in a network of 5 agents partitioned in 2 clusters ($n_1 = 3, n_2 = 2$) is used in the sequel to illustrate the theoretical results. We consider the dynamics (13.2) with

$$L = \begin{bmatrix} 4 & -2 & -2 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad P_l = \begin{bmatrix} 0.45 & 0.55 \\ 0.25 & 0.75 \end{bmatrix}, \tag{13.18}$$

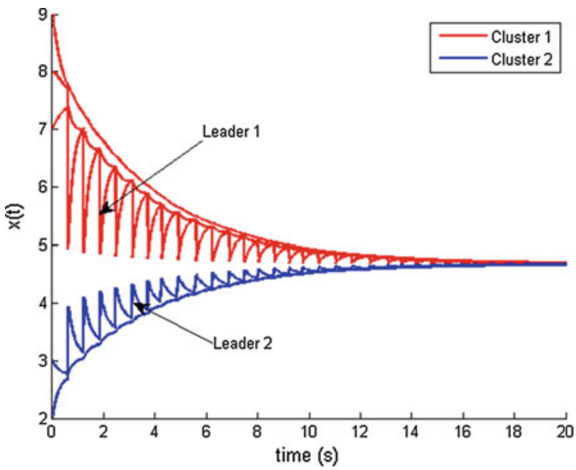

Fig. 13.1 The state-trajectories of the agents converging to the calculated consensus value

and the reset sequence given by $\delta = 0.5$ and δ'_k randomly chosen in $\Delta = [0, 0.2]$. The initial condition of the system is $x(0) = (8, 7, 9, 2, 3)$ and the corresponding consensus value computed by (13.8) is $x^* = 4.6757$. The convergence of the 5 agents toward x^* is illustrated in Fig. 13.1 emphasizing that the leaders trajectories are non-smooth while the followers trajectories are.

To find the decay rate λ_d, we make use the bisection algorithm as stated in Remark 2. The value of λ_d obtained was $\lambda_d = 0.855$. Number of iterations of the bisection algorithm is $k = 30$ on all three cases.

We understand that the consensus value is always a convex combination of the initial agreement values of the clusters. In our case, any consensus value can be imposed between the two initial agreements 2.75 and 7.5. In Fig. 13.2 the consensus value was fixed at $x^* = 6.5$.

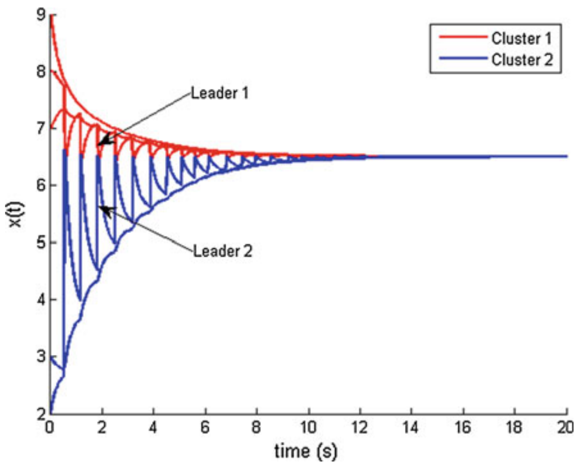

Fig. 13.2 The states of a system ($x^* = 6.5$)

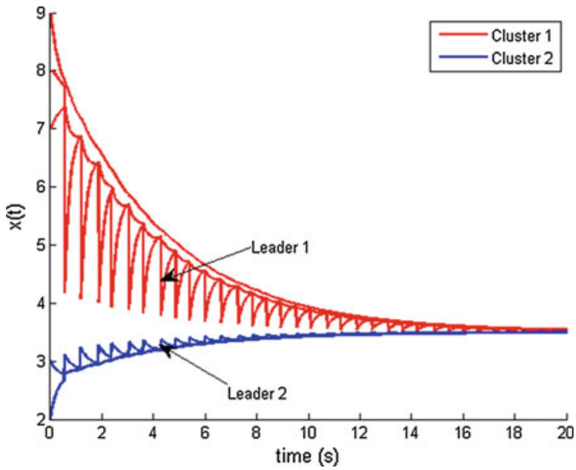

Fig. 13.3 The states of a system ($x^* = 3.5$)

The obtained P_l matrix was

$$P_l = \begin{bmatrix} 0.6870 & 0.3130 \\ 0.7825 & 0.2175 \end{bmatrix}.$$

The decay rate associated with this P_l is $\lambda_d = 0.782$ and can be ameliorated by using Theorem 5. Imposing $\bar{\lambda} = 0.82$ in (13.17) one gets

$$P_l = \begin{bmatrix} 0.6425 & 0.3575 \\ 0.8937 & 0.1063 \end{bmatrix}. \tag{13.19}$$

and for this P_l, the corresponding decay rate is $\lambda_d = 0.756 < \bar{\lambda}$ as noticed in Remark 4. Similar analysis has been done for $x^* = 6$. When P_l is designed without decay rate constraint one gets $\lambda_d = 0.799$ and is improved to $\lambda_d = 0.747$ designing P_l based on Theorem 5.

In Fig. 13.3 the consensus value was fixed as $x^* = 3.5$. The obtained P_l matrix was

$$P_l = \begin{bmatrix} 0.3010 & 0.6990 \\ 0.0874 & 0.9126 \end{bmatrix}. \tag{13.20}$$

13.7 Conclusions and Perspectives

In this work, we have considered networks of linear agents partitioned in several clusters disconnected one of each other. Each cluster has a linear impulsive leader that resets its state nearly periodically by taking into account the state of some neighboring leaders. On one hand we have characterized the consensus value for this type of networks and we have performed its stability analysis. On the other hand, we have provided the convergence speed toward consensus and we have designed the interconnection network between the leaders allowing to reach a prescribed consensus value. Our results are computationally oriented since they are given in LMI form. Two academic examples illustrate the entire theoretical developments. Future investigations may consider the influence of the leaders centrality on the convergence speed. Other interesting issue would be related to the influence of the nominal reset period δ on the decay rate λ_d. Finally, we consider that networks with impulsive leaders having event-based reset rules may be of particular interest.

References

1. R. Hegselmann, U. Krause: Opinion dynamics and bounded confidence models, analysis, and simulation. J. Artif. Soc. Soc. Simul. **5**(3) (2002)
2. O. Ratmann, C. Wiuf, J.W. Pinney, From evidence to inference: probing the evolution of protein interaction networks. HFSP J. **3**(5), 290–306 (2009)
3. D. Gfeller, P. De Los, Rios: spectral coarse graining and synchronization in oscillator networks. Phys. Rev. Lett. **100**, 174104 (2008)
4. V.D. Blondel, J.-L. Guillaume, R. Lambiotte, E. Lefebvre, Fast unfolding of communities in large networks. J Stat. Mech: Theory Exp. P10008(10) (2008)
5. S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D.-U. Hwang, Complex networks: structure and dynamics. Phys. Rep. **424**(4–5), 175–308 (2006)
6. I. Hanski, Metapopulation dynamics. Nature **396**, 41–49 (1998)
7. I.-C. Morărescu, A. Girard, Opinion dynamics with decaying confidence: application to community detection in graphs. IEEE Trans. Autom. Control **56**(8), 1862–1873 (2011)
8. M.E.J. Newman, Modularity and community structure in networks. Proc. Natl. Acad. Sci. USA **103**(23), 8577–8582 (2006)
9. M.E.J. Newman, M. Girvan, Finding and evaluating community structure in networks. Phys. Rev. E **69**(2), 026113 (2004)
10. A. Jadbabaie, J. Lin, A.S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Trans. Autom. Control **48**(6), 988–1001 (2003)
11. L. Moreau, Stability of multi-agent systems with time-dependent communication links. IEEE Trans. Autom. Control **50**(2), 169–182 (2005)
12. I.-C. Morărescu, S.-I. Niculescu, A. Girard, Consensus with constrained convergence rate and time-delays. in *IFAC Worksop on Time-Delay Systems (TDS)* (2010)
13. R. Olfati-Saber, J.A. Fax, R.M. Murray, Consensus and cooperation in networked multi-agent systems. Proc. IEEE **95**(1), 215–233 (2007)
14. R. Olfati-Saber, R.M. Murray, Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Autom. Control **49**(9), 1520–1533 (2004)
15. W. Ren, R.W. Beard, Consensus seeking in multiagent systems under dynamically changing interaction topologies. IEEE Trans. Autom. Control **50**(5), 655–661 (2005)
16. A. Olshevsky, J.N. Tsitsiklis, Convergence speed in distributed consensus and averaging. SIAM J. Control Optim. **48**(1), 33–55 (2009)
17. L. Xiao, S. Boyd, Fast linear iterations for distributed averaging. Syst. Control Lett. **53**, 65–78 (2004)
18. B. Touri, A. Nedic, On approximations and ergodicity classes in random chains. IEEE Trans. Autom. Control **57**(11), 2718–2730 (2012)

19. M. Fiacchini, I.-C. Morărescu, Set theory based condition in LMI form for network topology preservation for decentralized control. in *IEEE Conference on Decision and Control (CDC)* (2012)
20. M.M. Zavlanos, G.J. Pappas, Distributed connectivity control of mobile networks. IEEE Trans. Robot. **24**(6), 1416–1428 (2008)
21. B.D.O. Anderson, C. Yu, A.S. Morse, in *Convergence of Periodic Gossiping Algorithms* (Springer, 2010)
22. T.C. Aysal, M.E. Yildriz, A.D. Sarwate, A. Scaglione, Broadcast gossip algorithms for consensus. IEEE Trans. Sig. Process. **57**, 2748–2761 (2009)
23. S. Boyd, A. Ghosh, B. Prabhakar, D. Shah, Randomized gossip algorithms. IEEE Trans. Inf. Theory **52**(6), 2508–2530 (2006)
24. J. Liu, B.D.O. Anderson, M. Cao, A.S. Morse, Analysis of accelerated gossip algorithms. in*IEEE Conference on Decision and Control (CDC)* (2009)
25. M.C. Bragagnolo, I.C. Morărescu, J. Daafouz, P. Riedinger, LMI sufficient conditions for the consensus of linear agents with nearly-periodic resets. in *IEEE American Control Conference (ACC)* (2014)
26. A. Banos, J. Carrasco, A. Barreiro, Reset times-dependent stability of reset control systems. IEEE Trans. Autom. Control **56**(1), 217–223 (2011)
27. O. Beker, C.V. Hollot, Y. Chait, Plant with an integrator: an example of reset control overcoming limitations of linear feedback. IEEE Trans. Autom. Control **46**(11), 1797–1799 (2001)
28. D. Nešić, L. Zaccarian, A.R. Teel, Stability properties of reset systems. Automatica **44**, 2019–2026 (2008)
29. L. Hetel, J. Daafouz, S. Tarbouriech, C. Prieur, Stabilization of linear impulsive systems through a nearly-periodic reset. Nonlinear Anal: Hybrid Syst. **7**, 4–15 (2013)
30. T. Yucelen, W.M. Haddad, E.N. Johnson, Consensus protocols for networked multiagent systems with a uniformly continuous quasi-resetting architecture. in *IEEE American Control Conference (ACC)* (2013)

Chapter 14
Synthesis of Distributed Control Laws for Multi-agent Systems Using Delayed Relative Information with LQR Performance

Paresh Deshpande, Prathyush P. Menon and Christopher Edwards

Abstract In this chapter, a multiagent system composed of linear identical dynamical agents is considered. The agents are assumed to share relative state information over a communication network. This exchange of relative information is assumed to be subject to delays. New methods to synthesize distributed state feedback control laws for the multiagent system, using delayed relative information along with local state information with guaranteed LQR performance, are presented in this chapter. Two types of delays are considered in the relative information exchange: fixed and time-varying. Existing delay-dependent stability criteria are modified to incorporate LQR performance guarantees while retaining convex LMI representations to facilitate the synthesis of the control gains.

14.1 Introduction

Research in consensus and coordination of multiagent systems has received a great deal of attention over the past decade. One problem which is addressed in many of these papers involves ensuring a collection of multiple agents, interconnected over an information network, and operate in agreement or in a synchronized manner. Often the topology of the interconnections is captured as a graph, and in recent years many researchers have obtained novel results by combining graph theory along with systems and control ideas. See [1, 6, 16, 20, 21, 25, 26, 32, 37] and the references therein for further details and examples.

P. Deshpande (✉)
ASML B.V., Veldhoven, The Netherlands
e-mail: pd101@alumni.leicester.ac.uk

P.P. Menon · C. Edwards
University of Exeter, Exeter, UK
e-mail: p.m.prathyush@exeter.ac.uk

C. Edwards
e-mail: C.Edwards@exeter.ac.uk

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_14

Recently, progress has been made in terms of stabilization and consensus in a network of dynamical systems subject to performance guarantees such as the rate of convergence and LQR/$\mathscr{H}_2$ performance. The rate of convergence can be enhanced by optimizing the weights associated with the consensus algorithm which essentially improves the algebraic connectivity associated with the Laplacian matrix of the graph formed according to the underlying communication topology. In [41], the weights of the Laplacian matrix of the graph are optimized to attain faster convergence to a consensus value: this is posed as a convex optimization problem and solved using LMI tools. The algebraic connectivity, characterized by the second smallest eigenvalue of the Laplacian matrix, is maximized in [18] to improve the convergence performance. An optimal communication topology for multiagent systems is sought in [5] to achieve a faster rate of convergence. A distributed control methodology ensuring LQR performance in the case of a network of linear homogenous systems is presented in [2]. The robust stability of the collective dynamics with respect to the robustness of the local node level controllers and the underlying topology of the interconnections is also established in [2]. A decentralized receding horizon controller with guaranteed LQR performance for coordinated problems is proposed in [17] and the efficacy is demonstrated by an application to attain coordination among a flock of unmanned air vehicles. In [19], the relationship between the interconnection graph and closed-loop performance in the design of distributed control laws is studied using an LQR cost function. In [24], decentralized static output feedback controllers are used to stabilize a homogeneous network comprising a class of dynamical systems with guaranteed $\mathscr{H}_2$ performance, where an upper bound on the collective performance is given, depending only on the node level quadratic performance. LQ optimal control laws for a wide class of systems, known as spatially distributed large scale systems, are developed in [28] by making use of an approximation method. In [4], LQR optimal algorithms for continuous as well as discrete time consensus are developed, where the agent dynamics are restricted to be single integrators. However, interesting relations between the optimality in LQR performance and the Laplacian matrix of the underlying graph are developed. In [23], procedures to design distributed controllers with $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance have been proposed for a certain class of decomposable systems. Although delays are an ubiquitous factor associated with network interconnections as a result of information exchange over a communication medium, in all the above research work [2, 4, 17, 19, 23, 24, 28] no attempt is made to explicitly address or exploit the effects of the measurement delay.

Significant research efforts analyzing the stability and performance of collective dynamics (at network level) in the face of different types of delays have taken place in the recent past: Refs. [3, 22, 29–31, 33, 35, 38, 42] are few examples, although this list is not exhaustive.[1] Necessary and sufficient conditions for average consensus problems in networks of linear agents in the presence of communication delays have been derived in [31]. Stability criteria associated with the consensus dynamics in networks of agents in the presence of communication delays was subse-

[1]Another research area involving the stabilization of time-delay systems is networked control systems [15, 40]. This is not the class of problems considered in this chapter.

quently developed in [38] using Lyapunov–Krasovskii-based techniques. Moreover, the strong dependency of the magnitude of delay and the initial conditions on the consensus value was also established in [38]. In [29], a network of second-order dynamical systems with heterogeneously delayed exchange of information between agents is considered, where flocking or rendezvous is obtained using decentralized control. This can also be tuned locally, based only on the delays to the local neighbors. Both frequency and time domain approaches are utilized in [29] to establish delay-dependent and delay-independent collective stabilities. Subsequently, the theory was extended in [33] to the case of a network formed from a certain class of nonlinear systems. The robustness of linear consensus algorithms and conditions for convergence subject to node level self delays and relative measurement delays are developed and reported in [30] building on the research described in [29, 33]. 'Scalable' delay-dependent synthesis of consensus controllers for linear multiagent networks making use of delay-dependent conditions is proposed in [30]. Reference [42] reports an independent attempt to achieve second-order consensus using delayed position and velocity information. Recently another methodology, based on a cluster treatment of characteristic roots, has been proposed in [3] to study the effect of large and uniform delays in second-order consensus problems with undirected graphs. In [35], the performance of consensus algorithms in terms of providing a fast convergence rate involving communication delays, was studied for second-order multiagent systems. In [22], using methods based on Lyapunov–Krasovskii theory and an integral inequality approach, sufficient conditions for robust $\mathscr{H}_\infty$ consensus are developed for the case of directed graphs consisting of linear dynamical systems to account for node level disturbances, uncertainties and time-varying delays.

The main contributions of the present chapter are as follows: While exchanging information among agents over a network, delays are common. Previous efforts to investigate stability and robustness in the face of delays clearly emphasizes the need to account for these delays explicitly. However research in this direction is limited when compared to the available voluminous research in the case of 'delay free' consensus algorithms. Motivated by this fact, the idea of designing delay-dependent distributed optimal LQR control laws for homogeneous linear multi agent networks is put forward in this chapter. At a collective network level, a certain level of guaranteed cost is attained, which takes into account the control effort. Fixed, as well as time-varying delays are accounted for in the synthesis process. A Lyapunov–Krasovskii functional approach is used for synthesizing control laws in the presence of fixed delays, whereas a method from [7] is exploited to synthesize distributed control laws in the presence of time-varying delays employing a descriptor system representation. The efficacy of the proposed approaches are demonstrated by considering a homogeneous linear multi agent network (cyclic) where the node level dynamics are represented as double integrators as in [29, 35, 42].

14.2 Preliminaries

In this chapter, the set of real numbers is denoted by $\mathbb{R}$. Real-valued vectors of length m are denoted as $\mathbb{R}^m$. Real-valued matrices of dimension $m \times n$ are denoted as $\mathbb{R}^{m\times n}$. A column vector is denoted by $\mathscr{C}ol(.)$ and a diagonal matrix is denoted by $\mathscr{D}iag(.)$. The notation $P = P^{\mathrm{T}} > 0$ is used to describe a symmetric positive definite(s.p.d) matrix. An identity matrix of dimension $n \times n$ is denoted by I_n. Finally, the Kronecker product is denoted by the symbol $\otimes$.

Basic concepts from graph theory are described in this section. Standard texts such as [11] can be referred to for further reading on graph theory. An undirected graph $\mathscr{G}$ is described by a set of vertices $\mathscr{V}$ and a set of edges $\mathscr{E} \subset \mathscr{V}^2$, where an edge is denoted by $e = (\alpha, \beta) \in \mathscr{V}^2$, i.e., an unordered pair. A finite graph which consists of N vertices along with k edges for a network is represented as $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. In this chapter, bidirectional communication is assumed and hence the graphs for the network considered are undirected. The graph is assumed to contain no loops and no multiple edges between two nodes. The adjacency matrix, for the graph $\mathscr{A}(\mathscr{G}) = [a_{ij}]$, is defined by $a_{ij} = 1$ if i and j are adjacent nodes of the graph, and $a_{ij} = 0$ otherwise. The adjacency matrix thus defined is symmetric. The degree matrix is represented by the symbol $\Delta(\mathscr{G}) = [\delta_{ij}]$. $\Delta(\mathscr{G})$ is a diagonal matrix, and each element δ_{ii} is the degree of the i^{th} vertex. The difference $\Delta(\mathscr{G}) - \mathscr{A}(\mathscr{G})$ defines the Laplacian of $\mathscr{G}$, written as $\mathscr{L}$. For an undirected graph, $\mathscr{L}$ is symmetric positive semidefinite. $\mathscr{L}$ has a smallest eigenvalue of zero and the corresponding eigenvector is given by $\mathbf{1} = \mathscr{C}ol(1, \ldots 1)$. $\mathscr{L}$ is always rank deficient and the rank of $\mathscr{L}$ is $n - 1$ if and only if $\mathscr{G}$ is connected.

14.3 Problem Formulation

Consider a network of N identical linear systems given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{14.1}$$

for $i = 1, \ldots, N$, where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ represent the states and the control inputs. The constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ and it is assumed that the pair (A, B) is controllable. Each agent is assumed to have knowledge of its local state information along with delayed relative state information. The relative information communicated to each agent (node) is given by

$$z_i(t) = \sum_{j\in\mathscr{J}_i} (x_i(t-\tau) - x_j(t-\tau)),$$

where τ is a delay in communication of relative information. The dynamical systems for which the ith dynamical system has information is denoted by $\mathscr{J}_i \subset \{1, 2, \ldots N\}/\{i\}$. Two cases for the delay τ are considered in this chapter:

- a known fixed delay;
- a bounded time-varying delay with a known maximum bound.

The intention is to design control laws of the form

$$u_i(t) = -Kx_i(t) - Hz_i(t - \tau),$$

where $K \in \mathbb{R}^{m\times n}$ is designed to achieve consensus and $H \in \mathbb{R}^{m\times n}$, the relative information scaling matrix, is fixed a priori. The closed-loop system at a node level is given by

$$\dot{x}_i(t) = (A - BK)x_i(t) - BHz_i(t - \tau), \tag{14.2}$$

and using Kronecker products, the system in (14.1) at a network level is given by

$$\dot{X}(t) = (I_N \otimes A)X(t) + (I_N \otimes B)U(t), \tag{14.3}$$

where the augmented states and control inputs are $X(t) = \mathscr{C}ol(x_1(t), \ldots, x_N(t))$ and $U(t) = \mathscr{C}ol(u_1(t), \ldots, u_N(t))$, respectively. The relative information in (14.3) at a network level can be written as

$$Z(t) = (\mathscr{L} \otimes I_n)X(t - \tau), \tag{14.4}$$

where $\mathscr{L}$ is the Laplacian matrix associated with the sets $\mathscr{J}_i$. Using (14.4) the control law is given by

$$U(t) = -(I_N \otimes K)X(t) - (\mathscr{L} \otimes BH)X(t - \tau).$$

Substituting the previous expression into (14.3), the closed-loop system at a network level is given by

$$\dot{X}(t) = (I_N \otimes (A - BK))X(t) - (\mathscr{L} \otimes BH)X(t - \tau). \tag{14.5}$$

Since $\mathscr{L}$ is symmetric positive semidefinite, by spectral decomposition $\mathscr{L} = V\Lambda V^T$ where $V \in \mathbb{R}^{N\times N}$ is an orthogonal matrix formed from the eigenvectors of $\mathscr{L}$ and $\Lambda = diag(\lambda_1, \ldots \lambda_N)$ is the matrix of the eigenvalues of $\mathscr{L}$. Consider an orthogonal state transformation

$$X \mapsto (V^T \otimes I_n)X = \tilde{X}.$$

The closed-loop system (14.5) in the new coordinates is given by

$$\dot{\tilde{X}}(t) = (I_N \otimes (A - BK))\tilde{X}(t) - (\Lambda \otimes BH)\tilde{X}(t - \tau). \tag{14.6}$$

Because Λ is a diagonal matrix, system (14.6) is equivalent to

$$\dot{\tilde{x}}_i(t) = A_0\tilde{x}_i(t) + A_i\tilde{x}_i(t-\tau), \tag{14.7}$$

for $i = 1, \ldots, N$ where $A_0 := A - BK$ and $A_i := -\lambda_i BH$.

In this chapter, it is assumed the initial condition $\tilde{x}(\theta) = \tilde{x}(0)$ for $-\tau \leq \theta \leq 0$. The transformed system in (14.7) can be equivalently thought of as

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) - \lambda_i BH\tilde{x}_i(t-\tau) + Bu_i(t), \tag{14.8}$$

where $u_i(t) = -K\tilde{x}_i(t)$.

Remark 1 The decomposition in (14.8) can be implemented when the delay τ is identical across all communication links. In reality, the time-delays across the communication links will be unequal. One possible way to overcome this problem is to introduce delay buffers to equalize the delays.

Remark 2 Though the implementation of the controllers is decentralized the computation of the control gains K is obtained from the decomposition of (14.6). This requires the full information of the Laplacian $\mathscr{L}$ and hence the method is not applicable to scale free networks.

The objective is to design the gain matrix K under the following scenarios

- a delay-dependent design for a known fixed delay τ;
- a delay-dependent design for bounded time-varying delays $\tau(t)$.

In both cases, a suboptimal level of LQR performance must be enforced on the overall system.

The stabilization of linear systems with delays, with the structure given in (14.8), has been studied extensively in the control literature. Various stability analysis and control design methods have been proposed. In [7, 8], a descriptor representation along with Lyapunov–Krasovskii functionals are used to obtain stability criteria for linear time-delay systems. In [12] a Lyapunov–Krasovskii functional approach based on the partitioning of the delay is proposed for linear time-delay systems. In [9], bounds on the derivative of delays are considered to derive delay-dependent stability criteria. Most recent methods involve establishing LMI feasibility problems (with varying levels of complexity in terms of the number of decision variables). The reader is referred to [14, 27, 34, 36, 39] for further reading in this area. In this chapter, the systems in (14.8) are to be stabilized simultaneously in the presence of delays while guaranteeing an LQR performance. This is achieved by building on the existing analysis techniques [7, 13]. The techniques in [7, 13], while not necessarily the most recent in the literature, have been found to yield tractable LMI representations under certain mild simplifications. This is important because of the large number of decision variables involved resulting from the multiple agents. The results in [7, 13] are shown to provide a good trade-off between unnecessary conservatism and tractability of LMI formulations.

14.4 Delay-Dependent Control Design for a Fixed Delay

Prior to stating the objective of delay-dependent control design, anexplicit model transformation from [13] for the system in (14.7) is first performed. For the system given in (14.7), the following observation holds

$$\tilde{x}_i(t-\tau) = \tilde{x}_i(t) - \int_{t-\tau}^{t} \dot{\tilde{x}}_i(\theta)d\theta = \tilde{x}_i(t) - \int_{t-\tau}^{t} (A_0\tilde{x}_i(\theta) + A_i\tilde{x}_i(\theta-\tau))\, d\theta,$$

for $t \geq \tau$. Using the previous expression, the system in (14.7) can be represented as

$$\dot{\tilde{x}}_i(t) = (A_0 + A_i)\tilde{x}_i(t) + \int_{t-\tau}^{t} (-A_iA_0\tilde{x}_i(\theta) - A_iA_i\tilde{x}_i(\theta-\tau))d\theta, \tag{14.9}$$

for all $i = 1, \ldots, N$. As argued in [13], system (14.9) can be transformed, by shifting the time axis and lifting the initial conditions, into the system

$$\dot{y}_i(t) = \bar{A}_{0i}y_i(t) + \int_{t-2\tau}^{t} \bar{A}_i(\theta)y_i(\theta)d\theta, \tag{14.10}$$

where

$$\bar{A}_{0i} := A_0 + A_i,$$
$$\bar{A}_i(\theta) := \begin{cases} -A_iA_0 & \theta \in [t-\tau, t], \\ -A_iA_i & \theta \in [t-2\tau, t-\tau), \end{cases}$$

with the new initial condition $y(\theta) = \phi(\theta)$ for $-2\tau \leq \theta \leq 0$. According to [13], stability of (14.10) implies stability of (14.7) but not vice-versa. The control design objective for delay-dependent control design can now be stated as the design of gain matrix K for the systems in (14.10) such that the cost functions

$$J_i = \int_{0}^{\infty} (y_i^T(t)Qy_i(t) + u_i^T(t)Ru_i(t))dt \tag{14.11}$$

are minimized for all $i = 1, \ldots, N$, where

$$u_i(t) = -Ky_i(t) \tag{14.12}$$

and $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices.

Remark 3 In [13] it is shown that the transformed system in (14.10) has all the poles of the original system in (14.8) plus additional poles due to the transformation. Consequently stability of the transformed system implies stability of the original system but not vice-versa, due to the lifting of the initial conditions of the original system. In this chapter, LQR control design has been employed on the transformed system. This will also guarantee a level of performance for the original system in (14.8).

Theorem 1 *For a known fixed delay τ, a given scaling matrix $H \in \mathbb{R}^{m\times n}$, selected weighting matrices Q and R and scalars α_0 and α_1, the control laws in (14.12) simultaneously stabilize the transformed systems in (14.10) if there exist matrices $Z > 0$, W in $\mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$ such the following LMI conditions are satisfied*

$$\begin{bmatrix} -Z & I_n \\ * & -W \end{bmatrix} < 0, \tag{14.13}$$

$$\begin{bmatrix} \bar{\Phi}_i & -A_i AW + A_i BY & -A_i^2 W & \frac{WQ^{1/2}}{\tau} & \frac{Y^T}{\tau} \\ * & -\alpha_0 W & 0 & 0 & 0 \\ * & * & -\alpha_1 W & 0 & 0 \\ * & * & * & -\frac{I}{\tau} & 0 \\ * & * & * & * & -\frac{R^{-1}}{\tau} \end{bmatrix} < 0, \tag{14.14}$$

where

$$\bar{\Phi}_i = \frac{1}{\tau}(((A + A_i)W - BY) + ((A + A_i)W - BY)^T) + (\alpha_0 + \alpha_1)W$$

for all $i = 1, \ldots, N$. The state feedback gain matrix is then given by $K = YW^{-1}$.

Furthermore since the J_i from (14.11) satisfy, for $i = 1, \ldots, n$

$$J_i < y_i^T(0)Py(0)(1 + \frac{1}{2}\alpha_0\tau^2 + \frac{3}{2}\alpha_1\tau^2),$$

minimizing $Trace(Z)$ subject to (14.13)–(14.14) minimizes a bound on the LQR cost.

Proof This proof uses a restricted Lyapunov–Krasovskii functional as suggested in *Proposition 5.16* from [13]. For the system in (14.10) consider a Lyapunov–Krasovskii functional of the form

$$V_i(y(t)) = y_i^T(t)Py_i(t) + \int_{t-2\tau}^{t}\int_{\theta}^{t} \alpha(\theta)y_i^T(s)Py_i(s)dsd\theta, \tag{14.15}$$

where $P > 0$ and $P \in \mathbb{R}^{n\times n}$ for all $i = 1, \ldots, N$. In (14.15) $\alpha(\theta) > 0$ is a positive scalar function defined over the interval $t - 2\tau \le \theta \le t$. Consider the inequality

$$y_i^T(t)\left(P\bar{A}_{0i}+\bar{A}_{0i}^TP+\int\limits_{t-2\tau}^{t}\alpha(\theta)Pd\theta\right)y_i(t)\quad+\int\limits_{t-2\tau}^{t}2y_i^T(t)P\bar{A}_i(\theta)y_i(\theta)d\theta$$
$$-\int\limits_{t-2\tau}^{t}\alpha(\theta)y_i^T(\theta)Py_i(\theta)d\theta<-y_i^T(t)Qy_i(t)-u_i^T(t)Ru_i(t). \tag{14.16}$$

By adding and subtracting terms involving a symmetric matrix function $M(\theta) \in \mathbb{R}^{n\times n}$, the inequality in (14.16) is equivalent to

$$y_i^T(t)\left(P\bar{A}_{0i}+A_{0i}^TP+Q+K^TRK+\int\limits_{t-2\tau}^{t}M(\theta)d\theta\right)y_i(t)$$
$$+\int\limits_{t-2\tau}^{t}\bar{y}_i^T(t,\theta)\begin{pmatrix}\alpha(\theta)P-M(\theta) & P\bar{A}_i(\theta)\\ * & -\alpha(\theta)P\end{pmatrix}\bar{y}_i(t,\theta)d\theta<0. \tag{14.17}$$

where
$$\bar{y}_i^T(t,\theta)=\left(y_i^T(t)\ y_i^T(\theta)\right).$$

Define the symmetric matrix function $M(\theta)$ as

$$M(\theta)=\begin{cases}M_0, & t-\tau\le\theta\le t\\ M_1, & t-2\tau\le\theta<t-\tau,\end{cases}$$

where M_0 and M_1 are symmetric matrices $\in \mathbb{R}^{n\times n}$ and the scalar function $\alpha(\theta)$ as

$$\alpha(\theta)=\begin{cases}\alpha_0, & t-\tau\le\theta\le 0\\ \alpha_1, & t-2\tau\le\theta<t-\tau,\end{cases}$$

where $\alpha_0>0$ and $\alpha_1>0$. Then as argued in [13] inequality in (14.17) is satisfied for $P>0$ and

$$\begin{aligned}P(A_0+A_i)+(A_0+A_i)^TP+Q+K^TRK+\tau(M_0+M_1)&<0,\\ \begin{bmatrix}cc\alpha_0P-M_0-PA_iA_0\\ *-\alpha_0P\end{bmatrix}&<0,\\ \begin{bmatrix}cc\alpha_1P-M_1-PA_iA_i\\ *-\alpha_1P\end{bmatrix}&<0.\end{aligned} \tag{14.18}$$

Using the Schur complement and eliminating M_0 and M_1 the inequalities in (14.18) are satisfied if

$$\begin{bmatrix} \Phi_i & -PA_iA_0 & -PA_iA_i \\ * & -\alpha_0 P & 0 \\ * & * & -\alpha_1 P \end{bmatrix} < 0, \tag{14.19}$$

where

$$\Phi_i = \frac{1}{\tau}(P(A_0 + A_i) + (A_0 + A_i)^T P + Q + K^T RK) + (\alpha_0 + \alpha_1)P, \tag{14.20}$$

and $A_0 = A - BK$ for all $i = 1, \ldots, N$. To develop a convex representation define $W = P^{-1}$. Pre- and postmultiplying (14.19) by $diag(W, W, W)$ means (14.19) is equivalent to

$$\begin{bmatrix} \hat{\Phi}_i & -A_iA_0W & -A_iA_iW \\ * & -\alpha_0 W & 0 \\ * & * & -\alpha_1 W \end{bmatrix} < 0, \tag{14.21}$$

where

$$\hat{\Phi}_i = \tfrac{1}{\tau}\left((A_0 + A_i)W + W(A_0 + A_i)^T + WQW\right) \\ +\tfrac{1}{\tau}(WK^T RKW) + (\alpha_0 + \alpha_1)W,$$

for all $i = 1, \ldots, N$. Define an auxiliary symmetric matrix $Z \in \mathbb{R}^{n\times n}$ and employ the change of decision variables $KW = Y$ where $Y \in \mathbb{R}^{m\times n}$. From applying the Schur complement to (14.21), the inequalities in (14.18) become the LMI stated in the theorem statement in (14.14). Inequality (14.16) is equivalent to

$$\dot{V}_i(y_i(t)) < -y_i^T(t)Qy_i(t) - u_i^T(t)Ru_i(t).$$

Integrating both sides the previous equation from 0 to ∞ yields

$$-y_i^T(0)Py_i(0) - \int_{-2\tau}^{0}\int_{\theta}^{0} \alpha(\theta)y_i^T(s)Py_i(s)dsd\theta < -J_i. \tag{14.22}$$

Assuming new initial conditions $y_i(s) = y_i(0)$ for $s < 0$, an upper bound for J_i is given by

$$J_i < y_i^T(0)Py(0)(1 + \frac{1}{2}\alpha_0\tau^2 + \frac{3}{2}\alpha_1\tau^2), \tag{14.23}$$

by explicitly evaluating the integration on the L.H.S of (14.22). Minimization of $Trace(P)$ ensures minimizations of the cost J_i for all $i = 1, \ldots, N$. From (14.13), it can be shown by Schur complement that $Z > W^{-1} = P$ and hence minimizing $Trace(Z)$ ensures cost minimization from (14.23). ■

Remark 4 Note that the matrix H and the scalars α_0 and α_1 are fixed in (14.14) which renders the matrix A_i fixed for all $i = 1, \ldots, N$ and hence (14.14) is an LMI which can be easily solved using modern convex optimization techniques.

14.5 Delay-Dependent Control Design for Time-Varying Delays

In the previous section, state feedback control laws were designed based on the assumption of relative information having a fixed delay. Assuming fixed delays in a network is somewhat idealistic and hence a need arises for control design involving time-varying delays. In [7], a control design methodology has been presented for time-varying delay where the delay $\tau(t)$ is a bounded continuous function satisfying $0 \leq \tau(t) \leq \tau_m$ for $t \geq 0$ where τ_m is known. The equivalent systems to (14.8) with time-varying delays $\tau(t)$ are given by

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) - \lambda_i BH\tilde{x}_i(t - \tau(t)) + Bu_i(t), \tag{14.24}$$

for $i = 1, \ldots, N$. Again the objective is to minimize a cost function of the form

$$J_i = \int_0^\infty (\tilde{x}_i^T(t)Q\tilde{x}_i(t) + u_i^T(t)Ru_i(t))dt,$$

where

$$u_i(t) = -K\tilde{x}_i(t), \tag{14.25}$$

for all $i = 1, \ldots, N$.

Theorem 2 *Assume the bound on the delay τ_m is known, then for a given scaling matrix H from (14.24) and given weighting matrices Q and R, the control laws in (14.25) simultaneously stabilize the systems in (14.8) if there exist matrices $W_1 > 0$, W_2, W_3, $Z > 0$, $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$, $\bar{S} > 0 \in \mathbb{R}^{n\times n}$ such that the following LMI conditions are satisfied:*

$$\begin{bmatrix} -Z & I_{n\times n} \\ I_{n\times n} & -W_1 \end{bmatrix} < 0, \tag{14.26}$$

$$\begin{bmatrix} \bar{S} & 0 & \bar{S}A_i^T \\ * & \bar{F}_1 & \bar{F}_2 \\ * & * & \bar{F}_3 \end{bmatrix} > 0, \tag{14.27}$$

$$\begin{bmatrix} \tilde{W}_{2F} & \Phi_i & \tau_m W_2^T & W_1 Q^{1/2} & Y^T \\ * & \tilde{W}_{3F} & \tau_m W_3^T & 0 & 0 \\ * & * & -\tau_m \bar{S} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & R^{-1} \end{bmatrix} < 0, \tag{14.28}$$

where

$$\begin{aligned} \Phi_i &= W_3 - W_2^T + W_1(A + A_i) + \tau_m \bar{F}_2 - Y^T B^T, \\ \tilde{W}_{2F} &= W_2 + W_2^T + \tau_m \bar{F}_1, \\ \tilde{W}_{3F} &= -W_3 - W_3^T + \tau_m \bar{F}_3, \end{aligned}$$

for all $i = 1, \ldots, N$. *The state feedback gain matrix is then given by*

$$K = Y W_1^{-1}.$$

Furthermore $J_i < \tilde{x}_i^T(0) P_1 \tilde{x}_i(0)$ *and so minimizing* $Trace(Z)$ *subject to (14.26)–(14.28) minimize a bound on the LQR cost.*

Proof This proof uses concepts from *Corollary 3* from [7] to design control laws for the system in (14.8) with a certain level of performance. As in [7] represent the system in (14.7) as a descriptor system given by

$$\dot{\tilde{x}}_i(t) = \tilde{y}_i(t), \qquad 0 = -\tilde{y}_i(t) + (A_0 + A_i)\tilde{x}(t) - A_i \int_{t-\tau(t)}^{t} \tilde{y}_i(s) ds,$$

for $i = 1, \ldots, N$. The previous system can be written as

$$E\dot{\bar{x}}_i(t) = \begin{pmatrix} \dot{\tilde{x}}_i(t) \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ (A_0 + A_i) & -I \end{pmatrix}}_{\tilde{A}_{0i}} \bar{x}_i(t) - \begin{pmatrix} 0 \\ A_i \end{pmatrix} \int_{t-\tau(t)}^{t} \tilde{y}_i(s) ds,$$

for all $i = 1, \ldots, N$. In the previous equation, $\bar{x}_i^T(t) = \begin{pmatrix} \tilde{x}_i^T & \tilde{y}_i^T \end{pmatrix}$ and $E = diag(I, 0)$. Consider a Lyapunov–Krasovskii functional of the form

$$V_i(t) = V_{1i}(t) + V_{2i}(t),$$

where

$$V_{1i}(t) = \bar{x}_i^T E P \bar{x}_i, \qquad V_{2i}(t) = \int_{-\tau_m}^{0} \int_{t+\theta}^{t} \tilde{y}_i^T(s) S \tilde{y}_i(s) ds d\theta.$$

For this functional, we assume that the matrix $S \in \mathbb{R}^{n\times n}$ is symmetric positive definite and the matrix P is such that $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$ with $P_1 > 0$. The cost functions J_i, for $i = 1, \ldots N$, can be represented as

$$J_i = \int_0^{\infty} \bar{x}_i^T(t) \begin{bmatrix} Q + K^T RK & 0 \\ 0 & 0 \end{bmatrix} \bar{x}_i(t)dt. \quad (14.29)$$

Then, the objective is to ensure the inequality

$$\dot{V}_i(t) = \dot{V}_{1i}(t) + \dot{V}_{2i}(t) < -\bar{x}_i^T(t) \begin{bmatrix} Q + K^T RK & 0 \\ 0 & 0 \end{bmatrix} \bar{x}_i(t) \quad (14.30)$$

hold. In the left-hand side of (14.30), we have

$$\dot{V}_{1i}(t) = P^T \tilde{A}_{0i} + \tilde{A}_{0i}^T P - 2 \int_{t-\tau(t)}^{t} G(\bar{x}_i(t), \tilde{y}_i(s))ds, \quad (14.31)$$

where

$$G(\bar{x}_i(t), \tilde{y}_i(s)) = \bar{x}_i^T(t) P^T \begin{pmatrix} 0 \\ A_i \end{pmatrix} \tilde{y}_i(s),$$

and

$$\dot{V}_{2i}(t) = \tau_m \tilde{y}_i^T(t) S \tilde{y}_i(t) - \int_{t-\tau_m}^{t} \tilde{y}_i^T(s) S \tilde{y}_i(s)ds. \quad (14.32)$$

Select a matrix $F \in \mathbb{R}^{2n\times 2n}$ such that

$$\tilde{S}_F = \begin{bmatrix} S & \begin{bmatrix} 0 & A_i^T \end{bmatrix} P \\ P^T \begin{bmatrix} 0 \\ A_i^T \end{bmatrix} & F \end{bmatrix} > 0 \quad (14.33)$$

for all $i = 1 \ldots, N$. It follows that

$$\int_{t-\tau}^{t} \begin{bmatrix} \tilde{y}_i(s) \\ \bar{x}_i(t) \end{bmatrix}^T \tilde{S}_F \begin{bmatrix} \tilde{y}_i(s) \\ \bar{x}_i(t) \end{bmatrix} ds > 0. \quad (14.34)$$

By rearranging (14.34), the integral term in (14.31) is

$$
\begin{aligned}
-2\int_{t-\tau(t)}^{t} G(\bar{x}_i(t),\tilde{y}_i(s))ds &< \int_{t-\tau}^{t}\left(\tilde{y}_i^T(s)S\tilde{y}_i(s)+\bar{x}_i^T(t)F\bar{x}_i(t)\right)ds \\
&= \int_{t-\tau}^{t}(\tilde{y}_i^T(s)S\tilde{y}_i(s))ds+(\bar{x}_i^T(t)F\bar{x}_i(t))\int_{t-\tau}^{t}ds \\
&\leq \int_{t-\tau_m}^{t}\tilde{y}_i^T(s)S\tilde{y}_i(s)ds+\tau_m\bar{x}_i^T(t)F\bar{x}_i(t). \qquad (14.35)
\end{aligned}
$$

Substituting (14.35) in (14.31) and using (14.32), inequality (14.30) is satisfied if

$$
\begin{bmatrix} S & [0\ \ A_i^T]P \\ * & F \end{bmatrix} > 0 \qquad (14.36)
$$

and

$$
P^T\tilde{A}_{0i}+\tilde{A}_{0i}^TP+\tau_mF+\begin{bmatrix} Q+K^TRK & 0 \\ 0 & \tau_mS \end{bmatrix} < 0 \qquad (14.37)
$$

hold for all $1 = 1, \ldots, N$. Define

$$
P^{-1}=W=\begin{bmatrix} W_1 & 0 \\ W_2 & W_3 \end{bmatrix}. \qquad (14.38)
$$

To create convex LMI representations from the matrix inequalities (14.36) and (14.37) define $Z > 0 \in \mathbb{R}^{n\times n}$. Pre and post multiply (14.36) by $diag(S^{-1}, W^T)$ and $diag(S^{-1}, W)$ respectively. Also pre and post multiply (14.37) by W^T and W respectively. Using the linearizations

$$
W^TFW=\bar{F}=\begin{bmatrix} \bar{F}_1 & \bar{F}_2 \\ * & \bar{F}_3 \end{bmatrix}
$$

and $\bar{S} = S^{-1}$, and $KW_1 = Y$, the inequalities in (14.36) and (14.37) can be represented by the LMIs in (14.27) and (14.28). An expression for the maximum bound on the cost J_i can be obtained by integrating (14.30) as

$$
J_i < \tilde{x}_i^T(0)P_1\tilde{x}_i(0)+\int_{-\tau_m}^{0}\int_{\theta}^{0}\tilde{y}_i^T(s)S\tilde{y}_i(s)dsd\theta,
$$

for all $i = 1, \ldots, N$. Assuming the initial condition $\tilde{x}(\theta) = \tilde{x}(0)$ for $-\tau_m \le \theta \le 0$, $\tilde{y}_i(\theta) = \dot{\tilde{x}}(\theta) = 0$ for $-\tau_m \le \theta \le 0$. Hence the maximum bound on the cost J_i is given by

$$J_i < \tilde{x}_i^T(0) P_1 \tilde{x}_i(0). \tag{14.39}$$

From (14.26) it can be shown that $Z > W_1^{-1} = P_1$ and hence minimizing $Trace(Z)$ minimizes the $Trace(P_1)$ ensuring cost minimization from (14.39). ■

Remark 5 Note that τ and the matrix H are fixed which renders the matrix A_i fixed for all $i = 1, \ldots, N$ and hence (14.27) and (14.28) are LMI representations. Consequently the conditions of Theorem 2 can be easily tested via modern convex optimization solvers

14.6 Numerical Example

To illustrate the design methodologies, a cyclic nearest neighbor configuration of 5 vehicles moving in a x–y plane, with each vehicle described by two decoupled double integrators is considered. The linear system model is given by

$$\dot{\zeta}_i = A\zeta_i + Bu_i$$

where ζ_i represents the x and y plane positions and velocities which constitute the states of the ith vehicle. The matrices A and B are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix H associated with the relative information exchange in (14.2) is given by

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

For the method proposed in Sect. 14.4, it is assumed that a fixed communication delay of $\tau = 0.1$ s is present in the exchange of relative information. The matrices Q and R for the cost functions given in (14.11) and (14.29) have been chosen as

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the delay-dependent LMIs in (14.13) and (14.14), the scalars α_0 and α_1 have been chosen as $\alpha_0 = 3$ and $\alpha_1 = 3$. For $\tau = 0.1$ s, the gain matrix K obtained from this approach is

$$K = \begin{bmatrix} 2.3931 & 0 & 2.8428 & 0 \\ 0 & 2.3931 & 0 & 2.8428 \end{bmatrix}. \tag{14.40}$$

Figure 14.1 shows that a rendezvous occurs at around $t = 5$ s with the gain matrix K obtained in (14.40).

In [13], it is stated that the stability criteria of *Proposition 5.16* is conservative. Hence the gain matrix K in (14.40) should be able to cope with larger delays. Simulations show that the agents do not attain a rendezvous and diverge once τ exceeds $0.6s$. This is shown in Fig. 14.2.

If there is no communication of relative information, i.e., the control law in (14.2) is replaced by $u_i(t) = -Kx_i(t)$, for the system in (14.1), a standard LQR problem

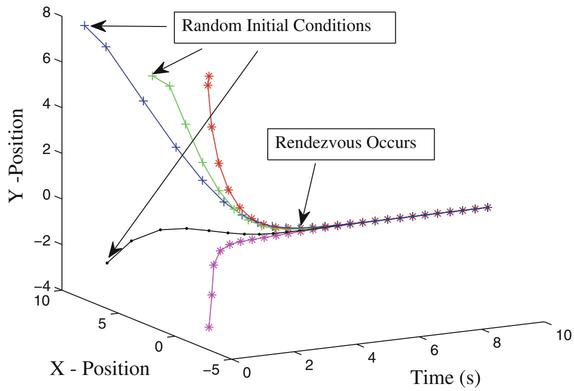

Fig. 14.1 Delay-dependent control with delay of $\tau = 0.1$ s

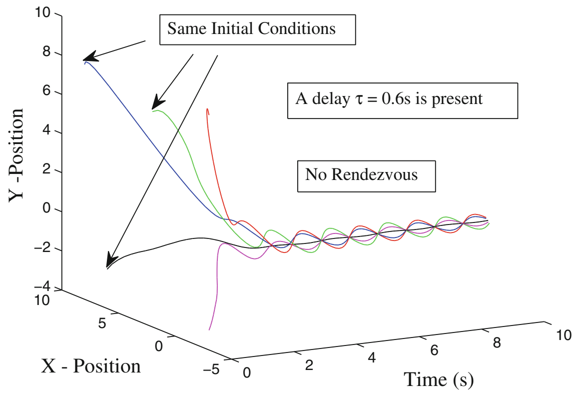

Fig. 14.2 No Rendezvous after $\tau = 0.6$ s

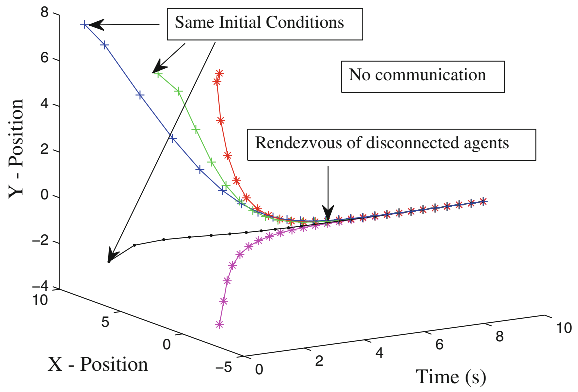

Fig. 14.3 Rendezvous of individual agents

results for each agent. With the same set of matrices (A, B, Q, R) using the MATLAB command `'lqr'`, the local state feedback gain matrix K obtained is

$$K = \begin{bmatrix} 3.1623 & 0 & 4.0404 & 0 \\ 0 & 3.1623 & 0 & 4.0404 \end{bmatrix}. \tag{14.41}$$

Figure 14.3 shows 5 disconnected agents attaining a rendezvous at $t = 4.8$ s.

Comparing the plots in Figs. 14.1 and 14.3, it can be seen that the performance achieved both with and without the relative information is similar. One distinct advantage that can be observed is the reduction of the magnitude of the gains for velocity and position feedback in (14.40) as compared to (14.41).

For the method proposed in Sect. 14.6, the bounded time-varying delay is given by $0 \leq \tau(t) \leq 0.1$ s. The gain matrix obtained from this approach is

$$K = \begin{bmatrix} 2.9779 & 0 & 4.4182 & 0 \\ 0 & 2.9779 & 0 & 4.4182 \end{bmatrix}. \tag{14.42}$$

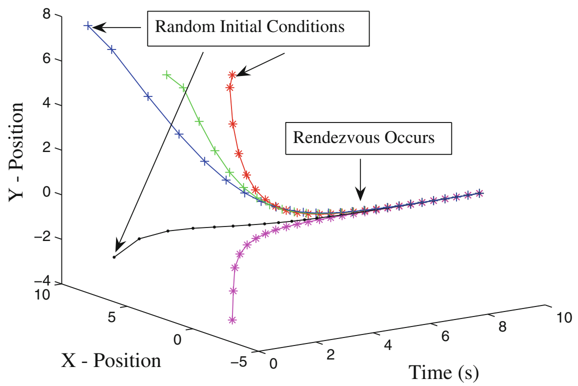

Fig. 14.4 Rendezvous with time-varying delay

Figure 14.4 shows 5 agents attaining a rendezvous at around $t = 6$ s for time-varying delay at $\tau(t) = 0.07$ s.

14.7 Conclusions and Future Work

In this chapter a multiagent system composed of linear identical dynamical agents was considered. The agents are assumed to share relative state information over a communication network. This exchange of relative information was assumed to be subject to delays. New methods to synthesize distributed state feedback control laws for the multiagent system, using delayed relative information along with local state information with guaranteed LQR performance, were developed. Two types of delays were considered in the relative information exchange: fixed and time-varying. For the double integrator system considered, the gains with fixed delay in relative information are lower in magnitude as compared to the gains obtained by solving an individual LQR problem for a single agent with no delays with similar performance. Thus the use of relative information maybe advantageous in terms of distributing the control effort. In the case of a time-varying delay, the method which has been proposed guarantees a bound on the LQR performance for delays with a known maximum bound.

The stabilization techniques which were used to incorporate LQR performance, while not necessarily the most recent in terms of the time-delay literature, were shown to yield tractable LMI representations under certain mild simplifications. This is important because of the large number of decision variables involved resulting from the multiple agents. These techniques provided a good trade-off between unnecessary conservatism and tractability of LMI formulations. Ongoing research efforts are attempting to use methods based on discretized Lyapunov–Krasovskii functionals and some of the new delay-dependent stabilization techniques which have been developed to reduce the conservatism.

References

1. B. Bamieh, F. Paganini, M.A. Dahleh, Distributed control of spatially invariant systems. IEEE Trans. Autom. Control **47**(7), 1091–1107 (2002)
2. F. Borrelli, T. Keviczky, Distributed LQR design for identical dynamically decoupled systems. IEEE Trans. Autom. Control **53**(8), 1901–1912 (2008)
3. R. Cepeda-Gomez, N. Olgac, Consensus analysis with large and multiple communication delays using spectral delay space concept. Int. J. Control **84**(12), 1996–2007 (2011)
4. Y. Cao, W. Ren, LQR-based optimal linear consensus algorithms, in *IEEE American Control Conference (ACC)* (2009)
5. J.C. Delvenne, R. Carli, S. Zampieri, Optimal strategies in the average consensus problem, in *IEEE Conference on Decision and Control (CDC)* (2007)
6. J.A. Fax, R.M. Murray, Information flow and cooperative control of vehicle formations. IEEE Trans. Autom. Control **49**(9), 1465–1476 (2004)

7. E. Fridman, U. Shaked, An improved stabilization method for linear time-delay systems. IEEE Trans. Autom. Control **47**(11), 1931–1937 (2002)
8. E. Fridman, U. Shaked, Delay-dependent stability and $\mathscr{H}^{\text{inf}}$ control: constant and time-varying delays. Int. J. Control **76**(1), 48–60 (2003)
9. E. Fridman, U. Shaked, K. Liu, New conditions for delay-derivative-dependent stability. Automatica **45**(11), 2723–2727 (2009)
10. P. Gahinet, A. Nemirovski, A.J. Laub, M. Chilali, *LMI Control ToolBox* (The MathWorks Inc., 1995)
11. C. Godsil, G. Royle, *Algebraic Graph Theory* (Springer, 2001)
12. F. Gouaisbaut, D. Peaucelle, Delay-dependent stability analysis of linear time delay systems, in *IFAC Workshop on Time Delay Systems (TDS)* (2006)
13. K. Gu, V.L. Kharitonov, J. Chen, *Stability of Time-Delay Systems* (Birkhauser, 2003)
14. Y. He, Q. Wang, C. Lin, M. Wu, Delay-range-dependent stability for systems with time-varying delay. Automatica **43**(2), 371–376 (2007)
15. S. Hirche, T. Matiakis, M. Buss, A distributed controller approach for delay-independent stability of networked control systems. Automatica **45**(8), 828–1836 (2009)
16. A. Jadbabaie, J. Lin, A.S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbour rules, in *IEEE Conference on Decision and Control (CDC)* (2002)
17. T. Keviczky, F. Borrelli, K. Fregene, D. Godbole, G. Balas, Decentralized receding horizon control and coordination of autonomous vehicle formations. IEEE Trans. Autom. Control **16**(1), 19–33 (2008)
18. Y.S. Kim, M. Mesbahi, On maximizing the second smallest eigenvalue of a state-dependent graph Laplacian. IEEE Trans. Autom. Control **51**(1), 116–120 (2006)
19. C. Langbort, V. Gupta, Minimal interconnection topology in distributed control design. SIAM J. Control Optim. **48**(1), 397–413 (2009)
20. C. Langbort, R.S. Chandra, R. D'Andrea, Distributed control design for systems interconnected over an arbitrary graph. IEEE Trans. Autom. Control **49**(9),1502–1519 (2004)
21. Z. Lin, B. Francis, M. Maggiore, State agreement for continuous time coupled nonlinear systems. SIAM J. Control Optim. **46**(1), 288–307 (2007)
22. Y. Liu, Y. Jia, Robust $\mathscr{H}_\infty$ consensus control of uncertain multi-agent systems with time-delays. Int. J. Control Autom. Syst. **9**(6), 1086–1094 (2011)
23. P. Massioni, M. Verhagen, Distributed control for identical dynamically coupled systems: A decomposition approach. IEEE Trans. Autom. Control **54**(1), 124–135 (2009)
24. P.P. Menon, C. Edwards, Static output feedback stabilisation and synchronisation of complex networks with H2 performance. Int. J. Robust Nonlinear Control **20**(6), 703–718 (2010)
25. M. Mesbahi, State-dependent graphs, in *IEEE Conference on Decision and Control (CDC)* (2003)
26. L. Moreau, Stability of multiagent systems with time dependent communication links. IEEE Trans. Autom. Control **50**(2), 169–182 (2005)
27. Y.S. Moon, P. Park, W.H. Kwon, Y.S. Lee, Delay-dependent robust stabilization of uncertain state-delayed systems. Int. J. Control **74**(14), 1447–1455 (2001)
28. N. Motee, A. Jadbabaie, Approximation method and spatial interpolation in distributed control systems, in *IEEE American Control Conference (ACC)* (2009)
29. U. M*ü*nz, A. Papachristodoulou, F. Allg*ö*wer, Delay-dependent rendezvous and flocking of large scale multi-agent systems with communication delays, in *IEEE Conference on Decision and Control (CDC)* (2008)
30. U. M*ü*nz, A. Papachristodoulou, F. Allg*ö*wer, Delay robustness in consensus problems. Automatica **46**(8), 1252–1265 (2010)
31. R. Olfati-Saber, R.M. Murray, Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Autom. Control **49**(9), 1520–1533 (2004)
32. R. Olfati-Saber, J.A. Fax, R.M. Murray, Consensus and cooperation in networked multi-agent systems. Proc. IEEE **95**(1), 215–133 (2007)
33. A. Papachristodoulou, A. Jadbabaie, U. M*ü*nz, Effects of delay in multi-agent consensus and oscillator synchronization. IEEE Trans. Atuom. Control **55**(6), 1471–1477 (2010)

34. P. Park, J.W. Ko, Stability and robust stability for systems with a time-varying delay. Automatica **43**(10), 1855–1858 (2007)
35. J. Qin, H. Gao, W.X. Zheng, Second order consensus for multi-agent systems with switching topology and communication delay. Syst. Control Lett. **60**(6), 390–397 (2011)
36. J.P. Richard, Time-delay systems: an overview of some recent advances and open problems. Automatica **39**(10), 1667–1694 (2003)
37. W. Ren, E.M. Atkins, Distributed multi-vehicle coordinated control via local information exchange. Int. J. Robust Nonlinear Control **17**(10–11), 1002–1033 (2007)
38. A. Seuret, D.V. Dimarogonas, K.H. Johanson, Consensus under communication delays, in *IEEE Conference on Decision and Control (CDC)* (2008)
39. J. Sun, G.P. Liu, J. Chen, D. Rees, Improved delay-range-dependent stability criteria for linear systems with time-varying delays. Automatica **46**(2), 466–470 (2009)
40. E. Tian, D. Yue, Z. Gu, Robust $\mathscr{H}_\infty$ control for nonlinear system over a network: A piecewise analysis method. Fuzzy Sets Syst. **161**(21), 2731–2745 (2010)
41. L. Xiao, S. Boyd, Fast linear iterations for distributed averaging. Syst. Control Lett. **53**(1), 65–78 (2004)
42. W. Yang, A.L. Bertozzi, X.F. Wang, Stability of a second order consensus algorithm with time delay, in *IEEE Conference on Decision and Control (CDC)* (2008)

Chapter 15
Topology Preservation for Multi-agent Networks: Design and Implementation

Irinel-Constantin Morărescu and Mirko Fiacchini

Abstract We consider a network of interconnected systems with discrete-time dynamics. Each system is called agent and we assume that two agents can interact as far as their states are close in a sense defined by an algebraic relation. In this work, we present several implementation strategies answering to different classical problems in multiagent systems. The primary goal of our methodology is to characterize the controllers that preserve a given interconnection subgraph that makes possible the global coordination. The second goal is to choose among these controllers those that ensure an agreement. This is done by solving a convex optimization problem associated to the minimization of a well-chosen cost function. Examples concerning full or partial consensus of agents with double-integrator dynamics illustrate the implementation of the proposed methodology.

15.1 Introduction

The consensus problem has been extensively studied in the last decade. Depending on the application, the framework can assume directed or undirected interaction graph, connections affected or not by delays, discrete or continuous, linear or nonlinear agent dynamics, fixed or dynamical interaction graph, synchronized or desynchronized

The work of I.-C. Morărescu was partially funded by the ANR project "Computation Aware Control Systems", ANR-13-BS03-004-02.

I.-C. Morărescu (✉)
Université de Lorraine, CRAN, UMR 7039 and CNRS, CRAN, UMR 7039,
54500 Vandoeuvre-lès-Nancy, France
e-mail: constantin.morarescu@univ-lorraine.fr

M. Fiacchini
Université Grenoble Alpes, GIPSA-lab, F-38000 Grenoble, France
e-mail: mirko.fiacchini@gipsa-lab.fr

M. Fiacchini
CNRS, GIPSA-lab, F-38000 Grenoble, France

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5_15

interactions [12, 14, 16, 17, 19]. Controlling the network in a decentralized way, by modeling it as a multiagent system, results in computation and communication cost reduction [15, 18, 20]. On the other hand, the coordination and performances of interconnected systems are related to the network topology. Most of existing works assume the connectivity of the interaction graph in order to guarantee the coordination behavior. However, some works have been oriented toward networks in which the global agreement cannot be reached and only local ones are obtained [15, 21]. Others propose controllers that are able to maintain the network connectivity in order to ensure the global coordination [6, 8, 9, 22].

Here we briefly recall the main results provided in our previous work [8, 9] and we show how they can be implemented. Precisely, we consider a multiagent system with discrete-time dynamics and state-dependent interconnection topology. Two agents are able to communicate if an algebraic relation between their states is satisfied. The connected agents are called neighbors. The agents update their state in a decentralized manner by taking into account their neighbors state. A connection is preserved as far as the algebraic relation is verified. The design of the decentralized controllers satisfying the algebraic constraint can be done either by minimizing a cost function [8], or by negotiations through the network at each step [7]. Our approach use invariance-based techniques (see [1–3] for the use of invariance in control theory) to characterize the conditions assuring that the algebraic constraint holds. The resulting topology preservation conditions rewrites as a convex constraint that may be posed in LMI form [4, 5]. Thus, we not only propose a new tool for decentralized control but also an easy implementable one. The practical implementation of this set theory-based control strategy [10, 11] requires a minimal number of interconnections ensuring the network connectivity. It should be noted that our procedure is quite flexible and, as we shall see, additional global objectives can be addressed. Precisely, we focus on the implementation of the topology preservation, presented in [8], to tackle specific problems concerning multiagent systems. The subsystems composing the network are mobile agents moving on the plane and whose communication capability is subject to constraints on their distances. Different coordination tasks, as flocking, consensus, and predictive control, are considered and solved employing the LMI conditions for avoiding the connections loss. Numerical illustrative examples allow us to analyze the results and to compare the different control strategies.

The chapter is organized as follows. Section 15.2 contains the main theoretical results. First, we formulate the decentralized control problem under analysis. Second, we derive the LMI conditions for network topology preservation in general settings as well as in the case of common feedback gains. Control design strategies for full or partial state consensus of identical systems with double-integrator dynamics are discussed in Sect. 15.3. In Sect. 15.4, we present some numerical examples illustrating the control strategies proposed in Sect. 15.3. Some conclusions and remarks on further works are provided at the end of the chapter.

15.1.1 Notation

The set of positive integers smaller than or equal to the integer $n \in \mathbb{N}$ is denoted as $\mathbb{N}_n$, i.e., $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. Given the finite set $\mathscr{A} \subseteq \mathbb{N}_n$, $|\mathscr{A}|$ is its cardinality. Given a symmetric matrix $P \in \mathbb{R}^{n \times n}$, notation $P > 0$ ($P \geq 0$) means that P is positive (semi-)definite. By $A^\dagger$ we denote the left pseudoinverse of the matrix A. Given the matrix $T \in \mathbb{R}^{n \times m}$ and $N \in \mathbb{N}$, $\mathscr{D}_N(T) \in \mathbb{R}^{nN \times mN}$ is the block-diagonal matrix whose N block-diagonal elements are given by T, while $\mathscr{D}(A, B, \ldots, Z)$ is the block-diagonal matrix, of adequate dimension, whose block-diagonal elements are the matrices $A, B, \ldots, Z$. Given a set of N matrices A_k with $k \in \mathbb{N}_N$, denote by $\{A_k\}_{k \in \mathbb{N}_N}$ the matrix obtained concatenating A_k in column.

15.2 Set Theory Results for Topology Preservation

15.2.1 Problem Formulation

Throughout the chapter, we consider a multiagent system with $V \geq 2$ interconnected agents assumed identical:

$$x_i^+ = Ax_i + Bu_i, \tag{15.1}$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^m$ is the control input of the i-th agent and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

In order to pursue collaborative tasks in a decentralized way, the agents exchange some information. The information available to every agent is supposed to be partial, as only a portion of the overall system is assumed accessible to every agent. We suppose that any agent has access to the state of a neighbor only if a constraint on the distance between them is satisfied. If communication network looses its connectivity the system may not be able to reach the global objective. Thus, the primary problem underlying any cooperative task in the multiagent context is the connection topology preservation. Theoretical results on this topic, presented in [8], are recalled hereafter and applied in the following sections.

15.2.2 Feedback Design for Topology Preservation

Let us suppose that the initial interconnection topology is given by the graph $G = (\mathscr{V}, \mathscr{E})$ where the vertex set is $\mathscr{V} = \mathbb{N}_V$ and the connecting edge set $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ represents the set of pairs of agents that satisfy a distance-like condition. Precisely, given the real scalar $r > 0$, $d \in \mathbb{N}$ with $d \leq n$ and $T \in \mathbb{R}^{d \times n}$ such that $TT^\top$ is invertible, the initial edge set is given by

$$\mathscr{E} = \{(i, j) \in \mathbb{N}_V \times \mathbb{N}_V \mid \|T(x_i(0) - x_j(0))\|_2 \leq r\}.$$

The set of edges that must be preserved is denoted by $\mathscr{N} \subseteq \mathscr{E}$. We suppose that every agent i knows the state of the j-th one if and only if $(i, j) \in \mathscr{N}$.

Definition 1 For all $i \in \mathscr{V}$, we define the set of connected neighbors of the i-th agent as

$$\mathscr{N}_i = \{j \in \mathbb{N}_V : (i, j) \in \mathscr{N}\}.$$

Given the set of connections $\mathscr{N}$, the objective is to design a decentralized control law ensuring that none of these connections are lost. In other words, the aim is to design the state-dependent control actions $u_i(k)$ independently from $u_j(k)$, for all $i, j \in \mathbb{N}_V$ and $k \in \mathbb{N}$, such that every connection in $\mathscr{N}$ is maintained.

As usual in multiagent systems, we consider the i-th input to be the sum of terms proportional to the distances between agent i and its neighbors. That is, denoting $e_{l,m} = x_l - x_m$ for all $l, m \in \mathbb{N}_V$, we define

$$u_i = \sum_{j \in \mathscr{N}_i} K_{i,j}(x_i - x_j) = \sum_{j \in \mathscr{N}_i} K_{i,j} e_{i,j}. \tag{15.2}$$

The design of each u_i is reduced to the design of the controller gains $K_{i,j}$ chosen such that the link (i, j) is preserved where the dynamics of the ij system results in

$$e_{i,j}^+ = (A + BK_{i,j} + BK_{j,i})e_{i,j} + \sum_{k \in \mathscr{N}_i}^{k \neq j} BK_{i,k} e_{i,k} - \sum_{k \in \mathscr{N}_j}^{k \neq i} BK_{j,k} e_{j,k}, \tag{15.3}$$

for all $i, j \in \mathbb{N}_V$. It is not difficult to see that, in the centralized case the dynamics of the error can be imposed by an adequate choice u_i, for all $i \in \mathbb{N}_V$, provided that the agents dynamics is stabilizable.

The dynamics of the ij system is given by the matrix $A + BK_{i,j} + BK_{j,i}$ if no perturbations due to the presence of other agents are present. Such perturbations, which complicate the decentralized control design, can be bounded within a set depending on the radius r and on the information on the neighbors common to the i-th and j-th agents.

Consider the sets

$$\begin{aligned} \mathscr{N}_{i,j} &= \mathscr{N}_i \cap \mathscr{N}_j, \\ \bar{\mathscr{N}}_{i,j} &= \mathscr{N}_i \setminus (\mathscr{N}_{i,j} \cup \{j\}), \\ \bar{\mathscr{N}}_{j,i} &= \mathscr{N}_j \setminus (\mathscr{N}_{i,j} \cup \{i\}), \end{aligned} \tag{15.4}$$

then $\mathscr{N}_{i,j}$ denotes the common neighbors of the i-th and the j-th agents and $\bar{\mathscr{N}}_{i,j}$ the neighbors of the i-th one which are neither j nor one of its neighbors, analogously for $\bar{\mathscr{N}}_{j,i}$. We define the cardinalities

$$N = 2|\mathscr{N}_{i,j}| + 1, \qquad \bar{N} = |\bar{\mathscr{N}}_{i,j}| + |\bar{\mathscr{N}}_{i,j}|,$$

where the indices are avoided here and in the following definitions to improve the readability. The dynamics of the ij system, perturbed by the noncommon neighbors, is

$$e^{+}_{i,j} = (A + BK_{i,j} + BK_{j,i})e_{i,j} + \sum_{k\in\mathcal{N}_{i,j}} (BK_{i,k}e_{i,k} - BK_{j,k}e_{j,k}) + w_{i,j}, \qquad (15.5)$$

with the bounded perturbation described by

$$w_{i,j} = \sum_{k\in\bar{\mathcal{N}}_{i,j}} (BK_{i,k}e_{i,k}) - \sum_{l\in\bar{\mathcal{N}}_{j,i}} (BK_{j,l}e_{j,l}). \qquad (15.6)$$

The problem addressed in this work can be state as follows:

Problem 1 Design a procedure to find at each step a condition on the decentralized control gains $K_{i,j}$, with $i, j \in \mathbb{N}_V$ such that the following algebraic relation is satisfied

$$\|Te^{+}_{i,j}\|_2 < r, \qquad \forall (i, j) \in \mathcal{N}, \qquad (15.7)$$

if the constraints

$$\begin{aligned} &\|Te_{i,k}\|_2 \le r, \ \forall k \in \bar{\mathcal{N}}_{i,j}, \\ &\|Te_{j,k}\|_2 \le r, \ \forall k \in \bar{\mathcal{N}}_{j,i}, \end{aligned} \qquad (15.8)$$

hold.

In order to ease the presentation, we introduce different notations for the controller gains.

Definition 2 Denote with $\mathrm{e} \in \mathbb{R}^{nN}$ the vector obtained concatenating $e_{i,j}$ with all $e_{i,k}$ and $e_{j,k}$ where $k \in \mathcal{N}_{i,j}$. Denote with $\check{\mathbf{K}}_{i,j} \in \mathbb{R}^{m\times n(N-1)}$ the matrix obtained concatenating $K_{i,k}$ and $-K_{j,k}$ where $k \in \mathcal{N}_{i,j}$ and with $\hat{\mathbf{K}}_{i,j} \in \mathbb{R}^{m\times n\bar{N}}$ the vector obtained concatenating all $K_{i,k}$ where $k \in \bar{\mathcal{N}}_{i,j}$ and $-K_{j,k}$ where $k \in \bar{\mathcal{N}}_{j,i}$. We also define

$$\begin{aligned} &\Delta = T\ [A + B\check{K}_{i,j},\ B\check{\mathbf{K}}_{i,j}]\ \mathcal{D}_N(T)^{\dagger} \in \mathbb{R}^{d\times dN}, \\ &\Gamma = TB\,\hat{\mathbf{K}}_{i,j}\ \mathcal{D}_{\bar{N}}(T)^{\dagger} \in \mathbb{R}^{d\times d\bar{N}}, \\ &Z = \mathcal{D}_N(T)\,\mathrm{e} \in \mathbb{R}^{dN}, \end{aligned} \qquad (15.9)$$

where $\check{K}_{i,j} = K_{i,j} + K_{j,i}$.

We recall here an important contribution presented in [8], namely the sufficient condition for the constraint (15.7) to hold.

Theorem 1 *Problem 1 admits solutions if there exists* $\Lambda = \mathcal{D}(\lambda_1 I_d, ..., \lambda_{\bar{N}} I_d)$ *with* $\lambda_k \geq 0$, *for all* $k \in \mathbb{N}_{\bar{N}}$ *such that*

$$\begin{bmatrix} r^2 - r^2 \sum_{k\in\mathbb{N}_{\bar{N}}} \lambda_k & 0 & Z^\top \Delta^\top \\ 0 & \Lambda & \Gamma^\top \\ \Delta Z & \Gamma & I_d \end{bmatrix} > 0. \tag{15.10}$$

Furthermore, any solution (Δ, Γ) *of the previous LMI defines admissible controller gains for the Problem 1.*

Proof First notice that every solution of (15.10) satisfies also

$$\sum_{k\in\mathbb{N}_{\bar{N}}} \lambda_k < 1, \qquad \Gamma^\top \Gamma - \Lambda < 0, \tag{15.11}$$

as the principal minors of a positive definite matrix are positive definite. Since (15.11) is a necessary condition for Problem 1 to admit a solution, there is no loss of generality in assuming it satisfied. Condition (15.7) is equivalent to

$$[Z^\top \; \bar{Z}^\top] \begin{bmatrix} \Delta^\top \Delta & \Delta^\top \Gamma \\ \Gamma^\top \Delta & \Gamma^\top \Gamma \end{bmatrix} \begin{bmatrix} Z \\ \bar{Z} \end{bmatrix} < r^2. \tag{15.12}$$

This condition must be satisfied for every $\bar{Z}$ such that

$$\bar{Z}^\top D_k \bar{Z} \leq r^2, \qquad \forall k \in \mathbb{N}_{\bar{N}}, \tag{15.13}$$

with

$$D_k = diag(0_d, \ldots, 0_d, I_d, 0_d, \ldots, 0_d) \in \mathbb{R}^{d\bar{N}\times d\bar{N}},$$

holds. Applying the S-procedure, a sufficient condition for (15.7) to hold for every $\bar{Z} \in \mathbb{R}^{d\bar{N}}$ satisfying (15.13) is the existence of $\lambda_k \geq 0$, for all $k \in \mathbb{N}_{\bar{N}}$, such that

$$Z^\top \Delta^\top \Delta Z + 2\bar{Z}^\top \Gamma^\top \Delta Z + \bar{Z}^\top [\Gamma^\top \Gamma - \Lambda] \bar{Z} < r^2 - r^2 \delta, \tag{15.14}$$

for every $\bar{Z} \in \mathbb{R}^{d\bar{N}}$, with $\delta = \sum_{k\in\mathbb{N}_{\bar{N}}} \lambda_k$. From (15.11) and Z being known, the left-hand side of (15.14) is a concave function in $\bar{Z}$ whose maximum is attained at

$$\bar{Z} = -(\Gamma^\top \Gamma - \Lambda)^{-1} \Gamma^\top \Delta Z. \tag{15.15}$$

Hence condition (15.14) holds for every $\bar{Z} \in \mathbb{R}^{d\bar{N}}$ if and only if it is satisfied for the maximum of the function at left-hand side, that is if and only if

$$Z^\top \Delta^\top \Delta Z - Z^\top \Delta^\top \Gamma (\Gamma^\top \Gamma - \Lambda)^{-1} \Gamma^\top \Delta Z < r^2 - r^2 \delta, \tag{15.16}$$

which is given by (15.14) at (15.15). Hence every Λ, Δ and Γ satisfying conditions (15.11) and (15.16) ensure the satisfaction of $\|Te^+_{i,j}\|_2 < r$ for all $\bar{Z}$ such that (15.13) holds. The condition (15.16) is equivalent to

$$
\begin{aligned}
&\begin{bmatrix} Z^\top \Delta^\top \Delta Z - r^2 + r^2\delta & Z^\top \Delta^\top \Gamma \\ \Gamma^\top \Delta Z & \Gamma^\top \Gamma - \Lambda \end{bmatrix} < 0 \\
\Leftrightarrow &\begin{bmatrix} Z^\top \Delta^\top \Delta Z & Z^\top \Delta^\top \Gamma \\ \Gamma^\top \Delta Z & \Gamma^\top \Gamma \end{bmatrix} < \begin{bmatrix} r^2 - r^2\delta & 0 \\ 0 & \Lambda \end{bmatrix} \\
\Leftrightarrow &\begin{bmatrix} Z^\top \Delta^\top \\ \Gamma^\top \end{bmatrix} \begin{bmatrix} \Delta Z & \Gamma \end{bmatrix} < \begin{bmatrix} r^2 - r^2\delta & 0 \\ 0 & \Lambda \end{bmatrix} \\
\Leftrightarrow &\begin{bmatrix} r^2 - r^2\delta & 0 & Z^\top \Delta^\top \\ 0 & \Lambda & \Gamma^\top \\ \Delta Z & \Gamma & I_d \end{bmatrix} > 0.
\end{aligned}
$$

Thus (15.10) is equivalent to (15.14), sufficient condition for (15.7) to hold.

The quantity $\delta = \sum_{k \in \mathbb{N}_{\bar{N}}} \lambda_k$ can be geometrically interpreted as a bound on the perturbation generated in the ij dynamics by the noncommon neighbors. Precisely, the effect of the noncommon neighbors can be modeled as a perturbation on the ij system bounded by an ellipsoid determined by $T^\top T$ and of radius $\sqrt{\delta} r$. Therefore, the condition $\delta < 1$, implicitly imposed by (15.10), is necessary to ensure the preservation of the connection (i, j).

15.2.3 Network Preservation with Common Feedback Gains

The condition presented in the previous subsection ensures that the algebraic constraint related to the ij system is satisfied at the successive time instant. No insurance on its satisfaction along the evolution of the overall system can be guaranteed, unless proper choices of $K_{i,j}$ are done. In case the feedback gains are assumed to be the same for every agent and every ij system, a sufficient condition for guaranteeing the network topology preservation at every future time instant can be posed.

Assumption 2 Given the system (15.1) with control input (15.2), assume that $K_{i,j} = \bar{K}$ for all $(i, j) \in \mathcal{N}$.

The objective is to characterize the set of common feedback gains such that, if applied to control the multiagent system, they ensure that the value of $\|Te_{i,j}\|_2$ does not increase for all $(i, j) \in \mathcal{N}$. In fact, this would clearly implies that if the connection condition is satisfied by the initial condition, i.e., $\|Te_{i,j}(0)\|_2 \le r$ for all $(i, j) \in \mathcal{N}$, it holds also at every successive instant, then the network topology is preserved. Given the sets as in (15.4), define

$$
N_M = \max_{(i,j)\in\mathcal{N}} \{|\mathcal{N}_i| + |\mathcal{N}_j| - 2\}.
$$

Then, roughly speaking, $N_M \in \mathbb{N}$ is the maximum over $(i, j) \in \mathscr{N}$ of the sum of neighbors of the agents i-th and j-th, apart from the agents themselves.

Proposition 1 *Let Assumption 2 hold. If there exists* $\lambda \in [0, 1]$ *such that*

$$\begin{bmatrix} \lambda T^\top T & (A + 2B\bar{K})^\top T^\top \\ T(A + 2B\bar{K}) & \lambda I_d \end{bmatrix} \geq 0, \\ \begin{bmatrix} (1-\lambda)T^\top T & N_M \bar{K}^\top B^\top T^\top \\ N_M T B \bar{K} & (1-\lambda) I_d \end{bmatrix} \geq 0, \tag{15.17}$$

then the systems given by (15.5) and (15.6) are such that $\|Te^+_{i,j}\|_2 \leq r$ *for all* $(i, j) \in \mathscr{N}$ *if* $e_{l,k} \in \mathbb{R}^n$ *satisfies* $\|Te_{l,k}\|_2 \leq r$ *for all* $(l, k) \in \mathscr{N}$.

Proof Define the set $\mathscr{B}_T = \{e \in \mathbb{R}^n : \|Te\|_2 \leq r\}$, then $e \in \mathscr{B}_T$ if and only if $e^\top T^\top T e \leq r^2$. The first condition in (15.17) is equivalent to $(A + 2B\bar{K})^\top T^\top T(A + 2B\bar{K}) \leq \lambda^2 T^\top T$, which implies that $(A + 2B\bar{K})\mathscr{B}_T \subseteq \lambda\mathscr{B}_T$. From Assumption 2 one have that $K_{i,j} = K_{j,i} = \bar{K}$, which means that $A + 2B\bar{K}$ is the dynamics of any i, j system in the absence of the perturbation of the neighbors. Then the set $\mathscr{B}_T$ is mapped in $\lambda\mathscr{B}_T$ if no perturbation is present, that is $(A + BK_{i,j} + BK_{j,i})e_{i,j} \in \lambda\mathscr{B}_T$, for all $e_{i,j} \in \mathscr{B}_T$. Analogously, the second condition in (15.17) is equivalent to $N_M^2 \bar{K}^\top B^\top T^\top T B \bar{K} \leq (1-\lambda)^2 T^\top T$, which leads to $\sum_{k \in \mathbb{N}_{N_M}} B\bar{K}\mathscr{B}_T = N_M B\bar{K}\mathscr{B}_T \subseteq (1-\lambda)\mathscr{B}_T$. This means that if $e_{i,k} \in \mathscr{B}_T$ for all $k \in \mathscr{N}_i \setminus \{j\}$ and $e_{k,j} \in \mathscr{B}_T$ for all $k \in \mathscr{N}_j \setminus \{i\}$, as implicitly assumed, then

$$\sum_{k \in \mathscr{N}_{i,j}} (B\bar{K}e_{i,k} - B\bar{K}e_{j,k}) + \sum_{k \in \bar{\mathscr{N}}_{i,j}} (B\bar{K}e_{i,k}) - \sum_{l \in \bar{\mathscr{N}}_{j,i}} (B\bar{K}e_{j,l}) \in (1-\lambda)\mathscr{B}_T,$$

for all $(i, j) \in \mathscr{N}$. From properties of the Minkowski set addition, we have $e^+_{i,j} \in \lambda\mathscr{B}_T + (1-\lambda)\mathscr{B}_T = \mathscr{B}_T$, if $e_{l,k} \in \mathscr{B}_T$ for all $(l, k) \in \mathscr{N}$, which ends the proof.

Corollary 1 *Let Assumption 2 hold. If there exist* $\lambda \in [0, 1]$ *and* $\bar{\lambda} > 0$ *such that*

$$\begin{bmatrix} (\lambda - \bar{\lambda})T^\top T & (A + 2B\bar{K})^\top T^\top \\ T(A + 2B\bar{K}) & (\lambda - \bar{\lambda}) I_d \end{bmatrix} \geq 0, \\ \begin{bmatrix} (1-\lambda)T^\top T & N_M \bar{K}^\top B^\top T^\top \\ N_M T B \bar{K} & (1-\lambda) I_d \end{bmatrix} \geq 0,$$

then the systems given by (15.5) and (15.6) are such that

$$\|Te^+_{i,j}\|_2 \leq (1 - \bar{\lambda})\|Te_{i,j}\|_2,$$

for all $(i, j) \in \mathscr{N}$ *if* $e_{l,k} \in \mathbb{R}^n$ *satisfies* $\|Te_{l,k}\|_2 \leq r$ *for all* $(l, k) \in \mathscr{N}$.

15.3 Applications to Decentralized Control of Multiagent Systems

In this section we illustrate the application of our results, published in [8, 9] and recalled above, for controlling the multiagent system presented in the first part of Sect. 15.2. Different strategies (based on optimal and predictive control) to achieve the collaborative objectives are presented hereafter and numerically implemented.

Let us consider the double-integrator dynamics on the plane, that is, (15.1) with $x_i = [p_i^x(k),\ v_i^x(k),\ p_i^y(k),\ v_i^y(k)]^\top$, the input $u_i = [u_i^x,\ u_i^y]^\top$ and

$$A = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{bmatrix}, \quad B = \begin{bmatrix} \bar{B} & 0 \\ 0 & \bar{B} \end{bmatrix}, \text{ where } \bar{A} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We denote $p_{i,j}^y = p_i^x - p_j^x$, $v_{i,j}^y = v_i^x - v_j^x$, $p_{i,j}^y = p_i^y - p_j^y$, $v_{i,j}^y = v_i^x - v_j^x$ and

$$e_{i,j} = [p_{i,j}^y,\ v_{i,j}^y,\ p_{i,j}^y,\ v_{i,j}^y]^\top = x_i^\top - x_j^\top, \tag{15.18}$$

and $u_{i,j} = [u_i^x - u_j^x,\ u_i^y - u_j^y]^\top$. The control inputs are given by (15.2) and Definition 2 with feedback gains

$$\check{K}_{i,j} = \begin{bmatrix} k_{i,j}^{p^x} & k_{i,j}^{v^x} & 0 & 0 \\ 0 & 0 & k_{i,j}^{p^y} & k_{i,j}^{v^y} \end{bmatrix}, \tag{15.19}$$

for all $(i, j) \in \mathcal{N}$. Once obtained a value for $\check{K}_{i,j}$, we define the nominal selection $K_{i,j} = K_{j,i} = 0.5\check{K}_{i,j}$ for all $(i, j) \in \mathcal{N}$.

Moreover, the following constraint on the norm of $\check{K}_{i,j}$ is imposed

$$\check{K}_{i,j}^\top \check{K}_{i,j} \leq I_n, \tag{15.20}$$

to limit the effect of the control of the ij nominal system on the neighbors. Recall, in fact, that the perturbation on the neighbors of the agents i and j depends on their states and on the gains $K_{i,j}$ and $K_{j,i}$.

15.3.1 Topology Preservation Constraint

We suppose that the distance between two agents must be smaller than or equal to r to allow them to communicate. Thus the topology preservation problem consists of upper-bounding by r the euclidean distance between the connected neighbors. The constraint on the state of the ij system to preserve is

$$p_{i,j}^y(k)^2 + p_{i,j}^y(k)^2 \leq r^2. \tag{15.21}$$

Notice that the effect of the inputs u_i^x and u_i^y at time k has no influence on p_i^x and p_i^y at time $k+1$. Thus, any algebraic condition involving the positions $p_i^x,\ p_i^y$ of the systems at $k+1$ would not depend on the control action $u_i^x,\ u_i^y$ at time k. From the computational point of view, every constraint concerning only the agents positions, would lead to LMI conditions independent on the variable $\check{K}_{i,j}$. Then the results provided in Theorem 1 are not applicable directly in this case for the state at time $k+1$. On the other hand, the controls $u_i^x(k), u_i^y(k)$ affect the position (and the velocity) at time $k+2$ and a condition on the feedback gain $\check{K}_{i,j}$ to ensure the preservation of the (i,j) connection at time $k+2$ can be posed. The distance constraint can be imposed on the states at $k+2$, as nothing can be done at time k in order to prevent its violation at time $k+1$. Then a constraint on $e_{i,j}(k)$ can be determined characterizing the region of the state space such that $p_{i,j}^y(k)^2 + p_{i,j}^y(k)^2 \leq r^2$ and $p_{i,j}^y(k+1)^2 + p_{i,j}^y(k+1)^2 \leq r^2$ in terms of matrix T. Since the former constraint does not involve the input, only $p_{i,j}^y(k+1)^2 + p_{i,j}^y(k+1)^2 \leq r^2$ might be taken into account for the control design.

Proposition 2 *The condition (15.21) holds at time $k+2$ if and only if we have that $\|Te_{i,j}(k+1)\|_2 \leq r$ with*

$$T = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 0 & 1 & t \end{bmatrix}. \tag{15.22}$$

Proof The region of the space of $e_{ij}(k)$ such that the topology constraint (15.21) is satisfied at $k+1$ is given by $p_{ij}^x(k+1)^2 + p_{ij}^y(k+1)^2 \leq r^2$, which is equivalent to $\|Te_{ij}(k)\|_2 \leq r$ for T as in (15.22). Hence imposing that the system error state belongs to such a region at $k+1$ implies assuring that the distance between the agents i-th and j-th is smaller than or equal to r at $k+2$, preserving the topology at $k+2$. Then $p_{ij}^x(k+2)^2 + p_{ij}^y(k+2)^2 \leq r^2$ if and only if

$$\|Te_{ij}(k+1)\|_2 = \|T(Ae_{ij}(k) + Bu_{ij}(k))\|_2 \leq r,$$

with T as in (15.22).

Proposition 2, then, implies that the topology preservation constraint for time $k+2$ can be expressed in terms of $e_{i,j}(k)$ and the input $u_{i,j}(k)$. The results presented in Theorem 1, with T as in (15.22), allow to characterize the sets of feedback gains ensuring the satisfaction of the distance constraint at $k+2$, for every pair of connected neighbors i and j. Such set would depend on the current state $e_{i,j}(k)$ and on the gains designed to compensate the errors and enforce the topology preservation.

15.3.2 Relevant Multiagents Applications

Among the local feedback gains which guarantee the connection preservation, different selection criteria can be applied, depending on the collaborative task to be achieved. Hereafter three popular criteria are illustrated and analyzed.

15.3.2.1 Full State Consensus

The first criterion is to select the feedback gain, among those satisfying (15.10), to achieve the full state agreement. In other words, the objective in this case is to both steer all the agents at the same point and align all the velocities without loosing any connection. One possibility is to compute at any sampling instant the matrix $\check{K}_{i,j}$ minimizing a sum of nominal values of the position distance at $k+2$ and of the speed difference at $k+1$. By nominal values, we mean the values of positions and speeds in absence of the perturbation on the ij system due to the other agents. Then, given the positive weighting parameters $q_p,\ q_v \in \mathbb{R}$, the cost to minimize is

$$\begin{aligned} Q_c(e_{i,j}(k), \check{K}_{i,j}) = q_p(p^y_{i,j}(k+2)^2 + p^y_{i,j}(k+2)^2) \\ + q_v(v^y_{i,j}(k+1)^2 + v^y_{i,j}(k+1)^2). \end{aligned} \quad (15.23)$$

Proposition 3 *Any optimal solution of the convex optimization problem*

$$\begin{aligned} \min_{\Delta,\, \Gamma,\, \Lambda,\, \check{K}_{i,j}, M} \quad & e_{i,j}(k)^\top M^\top M e_{i,j}(k) \\ s.t. \quad & (15.9), (15.10), (15.19), \\ & \begin{bmatrix} I_n & \check{K}^\top_{i,j} \\ \check{K}_{i,j} & I_m \end{bmatrix} \geq 0, \end{aligned} \quad (15.24)$$

with

$$M = \begin{bmatrix} q_p & q_p t & 0 & 0 \\ 0 & q_v & 0 & 0 \\ 0 & 0 & q_p & q_p t \\ 0 & 0 & 0 & q_v \end{bmatrix} (A + B\check{K}_{i,j}), \quad (15.25)$$

and T as in (15.22), minimizes the cost (15.23) subject to the norm gain constraint (15.20) and the distance constraints (15.21) at $k+2$.

15.3.2.2 Partial State Consensus: Flocking

An alternative objective, often considered in the framework of decentralized control, is partial state consensus. That is, to steer a part of the state $e_{i,j}$ to zero, for all $(i,\ j) \in \mathcal{N}$. To preserve the connectivity of $G = (\mathcal{V}, \mathcal{N})$ while speed differences converge to zero, the cost to minimize is a measure of the difference between neighbors speeds, for instance

$$Q_f(e_{i,j}(k), \check{K}_{i,j}) = v^y_{i,j}(k+1)^2 + v^y_{i,j}(k+1)^2. \quad (15.26)$$

This is achieved by solving a convex optimization problem analogous to (15.24), as stated in the proposition below. The proof is avoided since similar to the one of Proposition 3.

Proposition 4 *Any optimal solution of the convex optimization problem (15.24) with*

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} (A + B\check{K}_{i,j}), \tag{15.27}$$

and T as in (15.22), minimizes the cost (15.26) subject to the norm gain constraint (15.20) and the distance constraints (15.21) at $k+2$.

Clearly, changing opportunely the matrix M would permit to regulate different part of the state of the ij system and also any linear combination of the state.

15.3.2.3 Predictive Control

Finally, we present another interesting optimization criterion. One of the most popular control technique suitable for dealing with control in presence of hard constraints is the predictive control. These control strategies exploit the prediction of the system evolution and the receding horizon strategy to react in advance in order to prevent the constraint violations and to avoid the potentially dangerous regions of the state space. Moreover, the control input that would generate the optimal trajectory, among the admissible ones, is usually computed and applied. In general, the longer is the prediction horizon, the higher is the capability of prevent unsafe regions and constraint violations. Based on this idea, we propose to optimize a measure of the future state position, in order to react in advance and prevent the states to approach the limits of the distance constraints. In particular, we minimize a measure of the nominal distance between the positions of the i-th and j-th agents at time $k+3$ in function of the input gain at time k, that is, $(p^y_{i,j}(k+2) + tv^y_{i,j}(k+1))^2 + (p^y_{i,j}(k+2) + tv^y_{i,j}(k+1))^2$. The control horizon can be extended to values higher than 3, but the predicted state $e_{i,j}(k+N)$ would depend on the future inputs and the cost would result in a nonconvex function of $\check{K}_{i,j}$. A simplifying hypothesis can be posed to obtain a suboptimal control strategy but with greater prediction capability. Let us denote the horizon $N_p \in \mathbb{N}$ and suppose that only the nominal control action $u_{i,j}(k) = \check{K}_{i,j}e_{i,j}(k)$ is applied, i.e., $u_{i,j}(k+p) = 0$ for $p \in \mathbb{N}_{N_p}$. The minimization of the nominal position at $k + N_p$, i.e.,

$$Q_p(e_{i,j}(k), \check{K}_{i,j}) = p^y_{i,j}(k+N_p)^2 + p^y_{i,j}(k+N_p)^2, \tag{15.28}$$

leads to a suboptimal control with high predictive power.

Proposition 5 *Any optimal solution of the convex optimization problem (15.24) with*

$$M = T + (N_p - 1)t \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \check{K}_{i,j} \right), \tag{15.29}$$

and T as in (15.22), minimizes the cost (15.28) subject to the norm gain constraint (15.20) and the distance constraints (15.21) at $k+2$.

The benefits of the prediction-based strategy will be highlighted in the numerical examples section.

15.4 Numerical Examples

Consider the six interconnected agents with the initial conditions given in [13] and connected by the minimal robust graph computed in the same work. That is, $\mathscr{N} = \{(1, 2),\ (2, 3),\ (3, 4),\ (4, 5),\ (5, 6)\}$, $r = 3.2$ and initial conditions:

$$\begin{aligned} x_1(0) &= [-4\ \ -v_0\ \ 3\ \ 0]^\top, & x_6(0) &= [4\ \ v_0\ \ 3\ \ 0]^\top, \\ x_2(0) &= [-2\ \ -v_0\ \ 2\ \ 0]^\top, & x_5(0) &= [2\ \ v_0\ \ 2\ \ 0]^\top, \\ x_3(0) &= [-1\ \ -v_0\ \ 0\ \ 0]^\top, & x_4(0) &= [1\ \ v_0\ \ 0\ \ 0]^\top, \end{aligned}$$

where v_0 is used as a parameter to analyze the maximal initial speed that may be dealt with by different control strategy. It is noteworthy that, as shown in [13], for the classical consensus algorithm the preservation of the minimal robust graph is guaranteed for a critical speed value $v_c \simeq 0.23$. Nevertheless, it is numerically shown that the sufficient condition is conservative since for $v_0 = 1.5v_c$ (generating approximately a 4 times higher global velocity disagreement) the robust graph is not broken. We also note that the classical consensus algorithm is not able to preserve the connectivity when the global disagreement is 5 times superior to the one guaranteeing the consensus (i.e., $v_0 > 2.1v_c$).

In the sequel, we show that our design allows to increase considerably the initial speed value (and consequently the initial global disagreement) avoiding the loss of connections. Let us first give the initial error vectors between the states of the neighbors:

$$\begin{aligned} e_{1,2}(0) &= [-2\ \ 0\ \ 1\ \ 0]^\top, & e_{5,6}(0) &= [-2\ \ 0\ \ -1\ \ 0]^\top, \\ e_{2,3}(0) &= [-1\ \ 0\ \ 2\ \ 0]^\top, & e_{4,5}(0) &= [-1\ \ 0\ \ -2\ \ 0]^\top, \end{aligned}$$
$$e_{3,4}(0) = [-2\ \ -2v_0\ \ 0\ \ 0]^\top.$$

15.4.1 Flocking

The control problem formulated in Sect. 15.3 has admissible solutions for $v_0 = 19v_c$ and the connection between the third and the fourth agent is lost for $v_0 = 20v_c$ as shown in Fig. 15.2. It is worth noting that the control acts like springs between agents' velocities (compare the bottom of Figs. 15.1, 15.2 and 15.3). First, the control cancels the speed difference between neighbors with opposite velocities creating a speed disagreement in both symmetric branches of the graph. Next, it cancels the disagreement between 2-nd and the 3-rd agent and between the 4-th and 5-th one, mimicking a gossiping procedure where the choice of active communication link

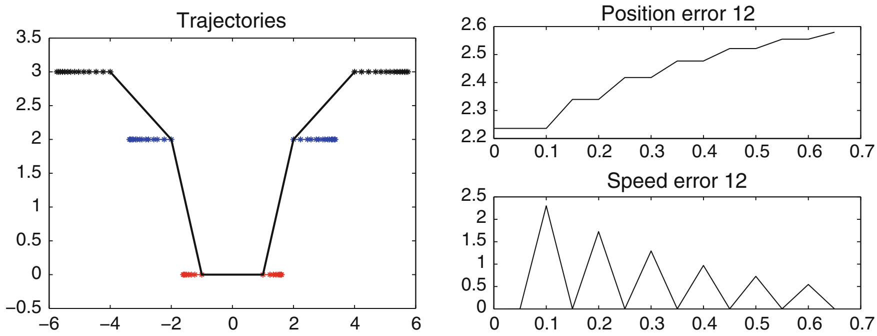

Fig. 15.1 Flocking: trajectories and errors of the 12 system

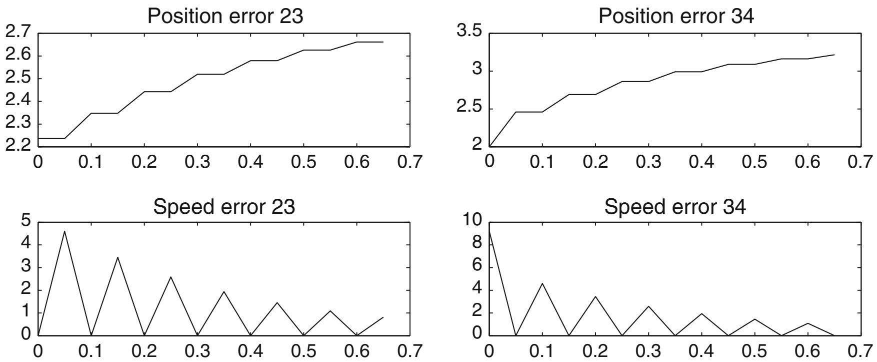

Fig. 15.2 Flocking: errors of the 23 and the 34 systems

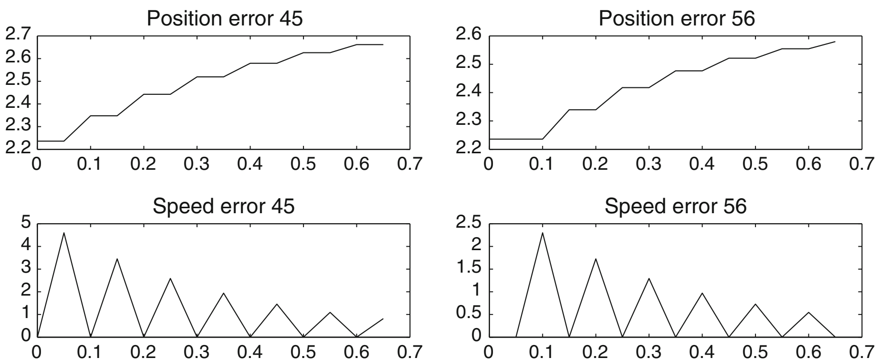

Fig. 15.3 Flocking: errors of the 45 and the 56 systems

is given by the error between neighbors speeds. Doing so, either the flocking is reached before the connectivity is lost, or the graph splits into two groups that will independently agree to two different velocity values.

15.4.2 Full State Consensus

The control problem formulated in Sect. 15.3 with $q_x = 10,\ q_v = 1$ has admissible solutions for $v_0 = 23v_c$ as shown in Fig. 15.4.

15.4.3 Predictive Control Strategies

The control problem formulated in Sect. 15.3 with $N_p = 3$ works for $v_0 = 21v_c$, but the trajectories are far from ideal. The behaviour is largely improved with $N_p = 21$, see Fig. 15.5 representing the trajectories and the time evolution of the 34 dynamics for $v_0 = 28v_c$. Notice how the position error of the critical system, the 34, approaches the bound avoiding the constraint violation, also for an initial speed much higher than those used for the other approaches, i.e., $v_0 = 28v_c$. Furthermore, the evolutions

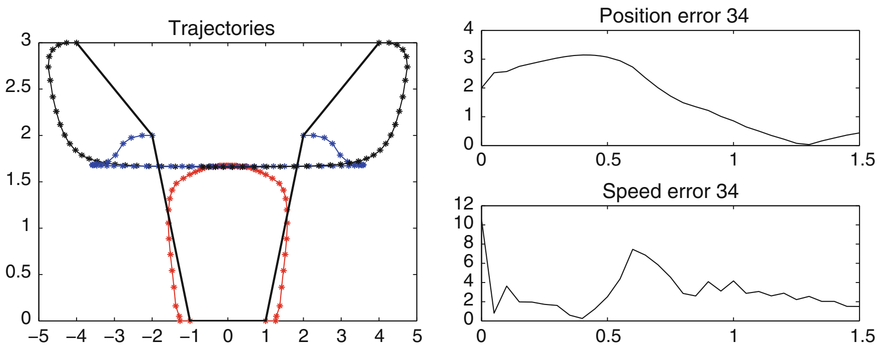

Fig. 15.4 Consensus: trajectories and errors of the 34 system

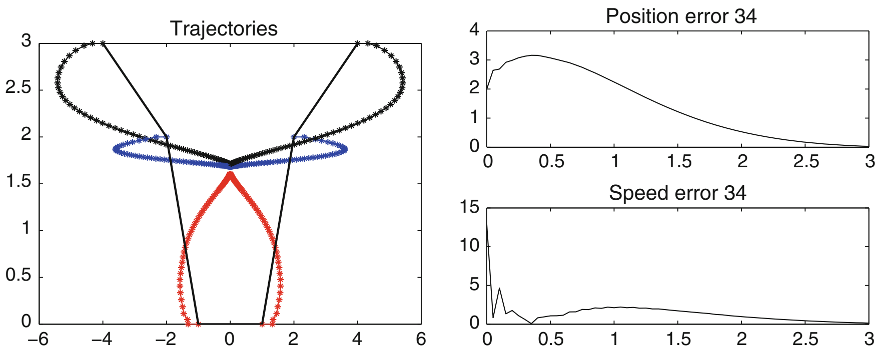

Fig. 15.5 Predictive control: trajectories and errors of the 34 system

and trajectories present a much smoother and regular behavior. All these desirable properties are due to the predictive capability of the approach which permits the control to react to possible violations and to prevent undesired situations in advance.

15.5 Conclusion and Further Works

This chapter provides an LMI-based methodology to design the controllers that preserve a given network topology in multiagent applications. Precisely, we suppose that the agents have limited communication capability and they have to stay in a given range in order to preserve their neighbors. We show that each agent can preserve all its neighbors by applying a controller obtained by solving a specific LMI. On the other hand, different convex optimization problems have been posed in order to pursue several classical objectives in the multiagent context, as consensus, flocking, and predictive control. The numerical simulations show the effectiveness of the method with respect to other existing techniques.

We note that the main applications provided in the chapter concern fleets of autonomous vehicles. Thus, the size of this associated network does not represent and obstacle for the numerical treatments by LMIs. Moreover, we can choose the network to be preserved as a very sparse one. Consequently, the number of low order LMIs to be solved is of the same order as the network size.

References

1. G. Basile, G. Marro, *Controlled and Conditioned Invariants in Linear system Theory* (Prentice Hall, 1992)
2. D.P. Bertsekas, Infinite-time reachability of state-space regions by using feedback control. IEEE Trans. Autom. Control **17**(5), 604–613 (1972)
3. F. Blanchini, S. Miani, *Set-Theoretic Methods in Control* (Birkhäuser, 2008)
4. S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory* (SIAM, 1994)
5. S. Boyd, L. Vandenberghe, *Convex Optimization* (Cambridge University Press, 2004)
6. F. Bullo, J. Cortés, S. Martinez, *Distributed Control of Robotic Networks*, A Mathematical Approach to Motion Coordination Algorithms (Princeton University Press, 2009)
7. G. Ferrari-Trecate, L. Galbusera, M. Marciandi, R. Scattolini, Model predictive control schemes for consensus in multi-agent systems with single- and double-integrator dynamics. IEEE Trans. Autom. Control **54**(11), 2560–2572 (2009)
8. M. Fiacchini, I.-C. Morărescu, Set theory based condition in LMI form for network topology preservation for decentralized control, in *IEEE Conference on Decision and Control (CDC)* (2012)
9. M. Fiacchini, I.-C. Morărescu, Convex conditions on decentralized control for graph topology preservation. IEEE Trans. Autom. Control **59**(6), 1640–1645 (2014)
10. M. Fiacchini, T. Alamo, E.F. Camacho, On the computation of convex robust control invariant sets for nonlinear systems. Automatica **46**(8), 1334–1338 (2010)
11. M. Fiacchini, S. Tarbouriech, C. Prieur, Polytopic control invariant sets for continuous-time systems: a viability theory approach. in *American Control Conference (ACC)* (2011)

12. A. Jadbabaie, J. Lin, A.S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Trans. Autom. Control **48**(6), 988–1001 (2003)
13. S. Martin, A. Girard, Sufficent conditions for flocking via graph robustness analysis, in *IEEE Conference on Decision and Control (CDC)* (2010)
14. L. Moreau, Stability of multiagent systems with time-dependent communication links. IEEE Trans. Autom. Control **50**(2), 169–182 (2005)
15. I.-C. Morărescu, A. Girard, Opinion dynamics with decaying confidence: Application to community detection in graphs. IEEE Trans. Autom. Control **56**(8), 1862–1873 (2011)
16. I.-C. Morărescu, S.-I. Niculescu, A. Girard, Consensus with constrained convergence rate and time-delays, in *IFAC Worksop on Time-Delay Systems (TDS)* (2010)
17. R. Olfati-Saber, R.M. Murray, Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Autom. Control **49**(9), 1520–1533 (2004)
18. W. Ren, R.W. Beard, Consensus seeking in multiagent systems under dynamically changing interaction topologies. IEEE Trans. Autom. Control **50**(5), 655–661 (2005)
19. W. Ren, R.W. Beard, E.M. Atkins, A survey of consensus problems in multi-agent coordination, in *American Control Conference (ACC)* (2005)
20. S. Shakkottai, R. Srikant, *Network Optimization and Control*, Foundations and Trends in Networking (NoW Publishers, 2007)
21. B. Touri, A. Nedic, On approximations and ergodicity classes in random chains. IEEE Trans. Autom. Control **57**(11), 2718–2730 (2012)
22. M.M. Zavlanos, G.J. Pappas, Distributed connectivity control of mobile networks. IEEE Trans. Robot. **24**(6), 1416–1428 (2008)

Index

A. Seuret et al. (eds.), *Delays and Networked Control Systems*,
Advances in Delays and Dynamics 6, DOI 10.1007/978-3-319-32372-5