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Potential Game Theory

Applications in Radio Resource Allocation

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To my parents and sister—Q.D. Lã

To my parents, elder sister (in memory), sisters, wife, and children—Y.H. Chew

To my parents, wife, and children—B.-H. Soong

Preface

Game theory is the formal studies of decision-making among multiple autonomous players who have common or conflicting interests and whose interactions influence the outcomes obtained by each participant. It has found its applications in the areas of telecommunication engineering since the early 1990s, for solving problems such as flow control and routing. In recent years, game-theoretical approach to radio resource allocation is one of the most extensively investigated research topics in wireless communications. A significant number of scientific papers and books, as well as special journal issues (e.g., the *IEEE Journal on Selected Areas in Communications* and the *IEEE Journal on Selected Topics in Signal Processing*) have been dedicated to this topic.

In some interactive scenarios involving selfish decentralized individuals, game theory helps us formulate analytical models so that we can examine the possible final outcomes and study the stability conditions of these outcomes. This is important in devising good strategies, which we ultimately hope will lead the system to these stable, efficient states at which the overall performance is improved and sustainable compared to a randomized, uncontrolled operating state. To this end, it is essential to emphasize the need to establish the existence of Nash equilibria in wireless communication games, one of which will be chosen as the final stable operating point of the system. The system exhibits a preferable and desirable property, if only one unique Nash equilibrium exists and any initially adopted strategy profile of players is able to converge to this Nash equilibrium by applying some iterative dynamics.

Existence and convergence of Nash equilibrium do not always apply to any arbitrary utility functions and strategy sets. However, there are special types of games, including *potential games*, where at least a pure-strategy Nash equilibrium is guaranteed to exist and can be reached with certain classes of learning dynamics such as the best responses. In a potential game formulation, one can identify a special function called the *potential function*, which changes values whenever there is a change in the utility of any single player due to his/her own strategy deviation, according to some predefined relationships. As such, the game's equilibria can often be associated with the optimum points of this potential function. Potential games are first studied by Monderer and Shapley [11]. Due to their desirable properties, they have been adopted to model radio resource allocation problems. Despite the promising number of applications in several wireless communications problems, it seems that the means to formulate a problem as a potential game is still vastly done through the process of trials and errors, with most applications being limited to a few known utility functions. In fact, there lacks a unifying framework in order for us to gain an in-depth insight in order to better exploit this very useful technique. For example, the overarching question of whether or not there is a systematic method to identify and to define the potential function of a game is still unanswered. Alternatively, how one can generalize and establish new potential game models for existing practical problems is another complicated and challenging problem. To the best of the authors' knowledge, there are dozens of textbooks that present excellent accounts of the use of game theory for wireless communications. Nevertheless, potential games often only receive a one-chapter treatment at best. No books or monographs are available to address the aforementioned concerns.

In this monograph, we attempt at a complete treatment of potential game theory and its applications in radio resource management for wireless communications systems and networking. We hope to pave the way to more extensive and rigorous research findings on a topic whose capacity for practical applications is potentially huge but yet not fully exploited. First and foremost, a generalized and rigorous mathematical framework on potential games will be presented. Consequently, we will discuss new as well as existing findings on the formulation of potential games and their applications in solving a variety of wireless communications problems.

The monograph is comprised of five chapters and is divided into two parts:

- In *Part I—Theory*, the purpose is to introduce the necessary background, as well as the notations and concepts used in game theory. In particular, we document our studies of a class of games known as potential games, which have found useful applications in the context of radio resource allocation. The materials covered in Part I will lay the fundamentals for the actual applications presented in Part II. Part I consists of two chapters:
  - Chapter 1 serves as a concise introductory text to game theory. It reviews the most elemental concepts and building blocks in game theory. We put an emphasis on the use of iterative decision dynamics in myopic computation of Nash equilibria, which is a process often employed in practical applications. The discussion is facilitated with a series of toy examples in order to have a better understanding of the abstract concepts.
  - Chapter 2 is the focal point of the monograph where theoretical treatments on potential games and our contributions to the literature on this topic are presented. Besides theoretical definition and characterization, we also give a very detailed and rigorous discussion on the questions of how to identify whether a game is a potential game, how to find the corresponding potential function, and how to formulate the utility function so that the resulting game is a potential game. The chapter is a cornerstone of the monograph, which serves as a basis for all subsequent discussions.

- *Part II—Applications* looks into a variety of practical problems in wireless resource allocation which can be formulated as potential games. In this part, we present our own results as well as summarize existing related works. It includes three chapters:
  - Chapter 3 uses game-theoretical approaches to achieve fair and efficient spectrum access schemes for the distributed OFDMA network consisting of transmit-receive pairs which exploits spatial frequency reuse. We discuss how a potential game can be formulated, the behaviors of strategy domination, and how it can be overcome, as well as an analysis of the price of anarchy. The system performance when best-response algorithm is used will be evaluated.
  - Chapter 4 looks at the subcarrier allocation problem for a downlink multicell multiuser OFDMA network where a potential game is also formulated. We propose our iterative algorithm for obtaining the Nash equilibria and address several performance issues such as fairness for edge-users as well as when the system is overloaded. Numerical results show the improvement in efficiency and fairness of this approach over existing schemes.
  - Chapter 5 gives a summary of existing approaches that apply potential games in solving wireless communications and networking problems, focusing on the formulations using exact potential games and pseudo-potential games. A non-exhaustive list of selected applications discussed in this chapter includes Menon et al. [9], Buzzi et al. [2], Neel et al. [12], Babadi et al. [1], Scutari et al. [14], Perlaza et al. [13], Mertikopoulos et al. [10], Xu et al. [16], Heikkinen [3], and Xiao et al. [15], to name a few.

This monograph is helpful for engineering students at the graduate and advanced undergraduate levels to learn and understand the fundamentals of potential game theory. It is also intended to introduce researchers and practitioners on how this theory can be used to solve the practical radio resource allocation problems. Researchers, scientists, and engineers in the fields of telecommunication, wireless communications, computer sciences, and others will certainly benefit from the contents of the book.

Chapters 3 and 4 of this monograph make use of materials that have been published in the authors' earlier papers [5–8], as well as the first author's Ph.D. thesis [4].

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2015

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Contents

Acronyms

3G	Third generation
3GPP	Third-Generation Partnership Project
4G	Fourth generation
ANSS	Autonomous number of subcarrier selection
AWGN	Additive white Gaussian noise
BER	Bit-error rate
BS	Base station
BSI	Binary symmetric interaction
CCI	Co-channel interference
CDF	Cumulative distribution function
CDMA	Code-division multiple access
FA	Fixed allocation scheme
FFT	Fast Fourier transform
GSO	General symmetric observations
IEEE	Institute of Electrical and Electronic Engineers
ISM	Industrial, scientific, and Medical
LTE	Long-term evolution
MC	Mesh client
MR	Mesh router
MS	Mobile station
OFDM	Orthogonal frequency-division multiplexing
OFDMA	Orthogonal frequency-division multiple access
OFDMA/WF	OFDMA iterative waterfilling scheme
PG-best	Potential game with best-response scheme
PG-better	Potential game with better-response scheme
PoA	Price of anarchy
QAM	Quadrature amplitude modulation
QoS	Quality of service
RRM	Radio resource management
SBRC	Strict best-response compatible
SINR	Signal-to-interference-and-noise ratio

SM/PP	SINR maximization with power pricing scheme
WF/PP	Waterfilling with power pricing scheme
WiMAX	Worldwide interoperability for microwave access
WLAN	Wireless local area network
WSC-A	Weak strategic complements with aggregation
WSS-A	Weak strategic substitutes with aggregation

Part I
Theory

Chapter 1
An Introduction to Game Theory

Abstract This chapter presents introductory materials on game theory. It is written mainly for readers who have some basic mathematical background on algebra, probability and set theory but may or may not have any prior exposure to game theory. It summarizes the important concepts and definitions for readers to understand the content in the next few chapters, which are the main theme of this book—potential games. Unlike what most game-theoretic texts often do, we limit our theoretical exposition to only elemental concepts which are sufficient to comprehend the materials in this monograph. We focus on the discussion of strategic-form non-cooperative games and Nash equilibrium which are central to game theory. In addition, as a means of bridging between game theory and their applications in wireless communications, we also spend part of the chapter discussing the issue of computing Nash equilibria via iterative gameplay. The use of myopic decision dynamics, which fall under such iterative processes, has been often adopted when applying game-theoretical approaches to solving wireless communications problems. The approach will be explained here through a series of simple exercises, with the convergence issue demonstrated and discussed.

1.1 Overview

What did game theory evolve from? Researchers often trace the history of game theory back to von Neumann's 1928 paper, *Zur Theorie der Gesellschaftspiele* (On the Theory of Parlor Games) [15] which laid the first solid mathematical formulation of a game. His groundbreaking work focused on zero-sum two-player games and proved the existence of mixed strategy equilibria via the application of fixed-point theorems. Even though this theory was initially applied on parlor games such as chess and poker, the deep connection between game theory and economics was eventually forged after the revolutionary book *Theory of Games and Economic Behavior* [16], co-authored by von Neumann and Morgenstern, was published in 1944. Nowadays, this book is still regarded as "the book upon which modern game theory is based".[1] What makes the book revolutionary is the introduction of the

[1] According to description from Princeton University Press at http://press.princeton.edu/titles/7802.html

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expected utility theory, which proposed that a *rational* individual whose preferences satisfy some axioms, will always take actions to maximize his/her expected utility function. This result was immediately adopted by economists and mathematicians in studying decision making under uncertainties. Shortly after, John F. Nash in 1950 proved the existence of mixed-strategy equilibria in non-zero-sum games [8, 9], thereby known as the Nash equilibrium, thus providing the missing piece to the puzzle and subsequently allowing full-fledged development and evolution of the theory into what is known today. Nash's work earned him a Nobel Prize in Economics in 1994, with him being one of the first among eleven game theorists who were awarded the prize, as of 2015. At presence, game theory is a well-recognized and well-discussed subject which finds its applications in a wide range of fields and disciplines, including not only economics but also political sciences, philosophy, biology, computer sciences and engineering.

What is game theory about? In every situation which can be modeled by a game, one will always find multiple *decision makers*—those who are assumed to be rational in the von Neumann sense as aforementioned—interact in the presence of conflict of interest. Game theory assists in the decision making process, by formulating mathematical model to predict how those participants, i.e., *players*, should behave. Thus, the question that game theory tries to answer is so intuitive—how are rational individuals going to act under a certain circumstance?—that it easily appeals to scientists studying human behaviors. However, the players in games are not only limited to human beings but are also extended to animals or computerized machines. There has been countless literature extensively developing and detailing this theory. For example, [3, 4, 6, 7, 13, 14] are among the recommended textbooks on game theory suitable for students of various disciplines and degrees, ranging from undergraduate to graduate levels.

Owing to these enormous and readily available materials, the authors do not wish to fully cover every single detail about game theory in this monograph. We will only provide a quick and self-contained introduction which is sufficient for readers to understand the contents presented in the next few chapters. The most fundamental concepts, which serve as building blocks for the subsequent discussion, will be introduced in this chapter. In Sect. 1.2, we mainly focus our discussions on non-cooperative games and the concept of Nash equilibrium, of which any beginners exposed to game theory would probably first encounter. In Sect. 1.3, we discuss iterative gameplay and myopic decision dynamics, which are mechanisms to compute Nash equilibrium for any practical application employing game theoretic approaches.

1.2 Fundamentals of Non-cooperative Game Theory

Non-cooperative game theory is one of the most elemental branches of game theory, and arguably the most important one. Non-cooperative games are also often encountered in wireless resource allocation problems. They will be frequently referred to simply as *games* throughout this monograph for brevity. Such a game arises whenever multiple players get engaged in an interactive decision making situation in which individuals have to act independently. Its name often implies a competitive nature where no forced cooperation among players are allowed, as opposed to *cooperative games* wherein there are external mechanisms (such as contracts or coalitions) enforcing cooperation among players. In this short exposition, we aim to introduce non-cooperative games including their strategic form's representation and associated notions, such as strategies, the Nash equilibrium, Pareto optimality, as well as some toy examples in order to understand the concepts.

1.2.1 Strategic Non-cooperative Games

The most basic form to denote a non-cooperative game is the strategic-form representation. Strategic form is used whenever the game does not require the notion of time, i.e., outcomes of the game can be analyzed as if players make decision simultaneously. In the case of sequential actions, i.e., in a dynamic context, extensive-form games can be used. This is one of several other classes of games that deal with the dynamic settings and often introduced as an alternative to strategic-form games. For introductory materials on extensive-form games, readers can refer to [3, 13].

Formally, a game can always be established if the three fundamental elements are clearly defined:

- *The set of players.*
- *The strategies associated with the players,* i.e., all the actions that a player can possibly select from.
- *The utilities (payoffs) for the players,* i.e., a function/rule that governs the payoff that a player will be awarded for taking a certain strategy given the other players' strategies.

A game refers to the scenario where every participating player interacts by choosing strategies to influence the final outcome of the game. An important, implicit assumption in game theory is the rationality assumption: *Players are always rational*. A player is rational if he or she makes decisions which are consistent to the purpose of maximizing his/her own utility function. Consequently, a rational player always favors strategy A over strategy B if A gives him/her a better payoff than B.

Mathematically, let us denote by $\mathscr{N} = \{1, 2, \ldots, N\}$ the set of N participating players. For every ith player in $\mathscr{N}$, his/her collection of all possible actions or strategies forms a *strategy set*, which can be informally expressed as

$$\mathbf{S}_i = \{S_i | \, S_i \text{ is a valid strategy for the } i\text{th player}\}. \quad (1.1)$$

The strategy set of player i, $\mathbf{S}_i$, can be either a continuum set such as subsets of a finite-dimension Euclidean space, or a finite set consisting of a finite number of discrete elements. Both types—discrete or continuous strategy sets—arise naturally in communications scenarios. For example, in a distributed multi-user multi-channel system, a mobile user's strategy can contain discrete actions such as choosing the operating frequency among a set of wireless frequencies, or contain continuous actions such as determining the player's transmission power level within a continuous feasible range.

Given all players' strategy sets, the *strategy space* of a game can be defined as follows.

Definition 1.1. The strategy space $\mathbb{S}$ is defined as the Cartesian products of all individual strategy sets, i.e.,

$$\mathbb{S} = \mathbf{S}_1 \times \ldots \times \mathbf{S}_N. \tag{1.2}$$

Each element $S = (S_1, S_2, \ldots, S_n) \in \mathbb{S}$ is said to be a *strategy profile*.

Often, if one is referring to a single player (e.g., the ith player), then S can be rewritten as $S = (S_i, S_{-i})$ where S_{-i} refers to the joint strategy adopted by player i's opponents. The domain of S_{-i} is denoted by $\mathbf{S}_{-i}$.

Definition 1.2. For each player i, his/her *utility function* U_i is a function that maps each strategy profile S to a real number, that is, $U_i(S) : \mathbb{S} \mapsto \mathbb{R}$, where $\mathbb{R}$ is the set of real number.

The notations $U_i(S)$ and $U_i(S_i, S_{-i})$ are both frequently used in standard texts on game theory.

Definition 1.3. The game $\mathscr{G}$ that consists of the player set $\mathscr{N}$, the strategy space $\mathbb{S}$ and the utility functions U_i, $\forall i \in \mathscr{N}$ will be denoted as

$$\mathscr{G} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i \in \mathscr{N}}]\,. \tag{1.3}$$

1.2.2 Pure and Mixed Strategies

In game theory, there is often a need to classify a player's strategy into *pure* strategies and *mixed* strategies. A pure strategy requires a player to play a certain action with certainty, i.e., with probability 1. Meanwhile, a mixed strategy is defined as a collection of pure strategies with a predetermined probability distribution assigned to each pure strategy. Given player i's strategy set, we define pure and mixed strategies as follows.

Definition 1.4. A pure strategy for player i is simply an element S_i of the set $\mathbf{S}_i$ in the set-theoretic sense.

Definition 1.5. Given $\mathbf{S}_i$, a mixed strategy ρ_i is a probability distribution over $\mathbf{S}_i$.

For example, if $\mathbf{S}_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,K}\}$ is the set of K pure strategies for player i, then ρ_i can be expressed as a probability distribution $(p_{i,1}, p_{i,2}, \ldots, p_{i,K})$. Here, $p_{i,k} = \Pr(a_{i,k})$ represents the probability that player i should choose strategy k, where $p_{i,k} \in [0, 1], \forall k$ and $\sum_{k=1}^{K} p_{i,k} = 1, \forall i$. We often use the notation $p_{i,k} = \rho_i(a_{i,k})$ to indicate that ρ_i assigns strategy $a_{i,k}$ probability $p_{i,k}$. A pure strategy $a_{i,k}$ is therefore considered a special case where $p_{i,k} = 1$ and $p_{i,m} = 0, \forall m \neq k$.

We can denote the *mixed strategy profile* by $\rho = (\rho_1, \rho_2, \ldots, \rho_N)$. Hence, we can think of the inclusion of mixed strategies as a *mixed-strategy extension* to a game, as defined in Definition 1.3.

Definition 1.6. A game $\mathscr{G}$ has a mixed-strategy extension $\left[\mathscr{N}, \bar{\Delta}, \{U_i\}_{i\in\mathscr{N}}\right]$. Here, $\bar{\Delta} = \Delta_1 \times \Delta_2 \times \ldots \times \Delta_N$ where each Δ_i refers to the set of all probability distributions over $\mathbf{S}_i$, with $\rho_i \in \Delta_i$. The (extended) utility function $U_i : \bar{\Delta} \mapsto \mathbb{R}$ now maps each mixed strategy profile ρ to the expected value computed from the portfolio of selected pure strategies.

In the above definition, the expected payoff for player i due to a mixed strategy profile ρ is calculated as

$$U_i(\rho) = \sum_{S\in\mathbb{S}} \left(\prod_{j=1}^{N} \rho_j(S_j) \right) U_i(S). \tag{1.4}$$

1.2.3 Dominant and Dominated Strategies

In some games, there are situations in which a certain strategy always gives a player better utility than another strategy, no matter what actions his/her opponents take. Such a strategy is called a *dominant strategy*. On the other hand, a strategy that always yields a worse utility compared to another dominant strategy regardless of what opponents may do is known as a *dominated* strategy.

Definition 1.7. For player i, $S_i \in \mathbf{S}_i$ is strictly dominant if and only if

$$\forall\, T_i \in \mathbf{S}_i,\, T_i \neq S_i : \quad U_i(S_i, S_{-i}) > U_i(T_i, S_{-i}) \quad \forall S_{-i} \in \mathbf{S}_{-i}. \tag{1.5}$$

If one replaces the ">" sign in the above with the "≥", "<" and "≤" signs, the definitions refer to *weakly dominant*, *strictly dominated* and *weakly dominated* strategies, respectively.

1.2.4 The Concept of Nash Equilibrium

Nash equilibrium is a crucial concept in predicting a game's outcome. By definition, a Nash equilibrium is a strategy profile such that if the opponents' strategies remain unaltered, no player would be tempted to move away from his/her current strategy. The definitions of Nash equilibrium in pure strategies and in mixed strategies are given separately.

Definition 1.8. The (pure) strategy profile $S^* \in \mathbb{S}$ is a pure-strategy Nash equilibrium if and only if

$$U_i(S_i^*, S_{-i}^*) \geq U_i(S_i', S_{-i}^*) \quad \forall S_i' \in \mathbf{S}_i,\ \forall i \in \mathcal{N}. \tag{1.6}$$

Definition 1.9. The mixed strategy profile $\rho^* \in \bar{\Delta}$ is a mixed-strategy Nash equilibrium if and only if

$$U_i(\rho_i^*, \rho_{-i}^*) \geq U_i(\rho_i', \rho_{-i}^*) \quad \forall \rho_i' \in \Delta_i,\ \forall i \in \mathcal{N}. \tag{1.7}$$

Nash equilibrium is perhaps the most important concept in game theory. At a Nash equilibrium, no player is able to gain by deviating from the current point. Thus, it can be seen as a "stable operating point" from the system perspective. Obtaining a Nash equilibrium state is often the ultimate objective in a resource allocation game. The proofs of Nash equilibrium existence for certain classes of games are important landmarks in game theory. These results are established in the works by Nash [8], Debreu [1], Fan [2] and Glicksberg [5]; and are stated in the following theorems.

Theorem 1.1 (Nash). *Every finite strategy game has a Nash equilibrium in mixed strategies.*

Proof. Refer to Fudenberg et al. [3], p. 29–30. □

Theorem 1.2 (Debreu-Fan-Glicksberg). *A game has a pure-strategy Nash equilibrium if for all $i \in \mathcal{N}$, the strategy set $\mathbf{S}_i$ is a nonempty, convex and compact subset of a Euclidean space, and the utility function U_i is continuous and quasi-concave in each S_i.*

Proof. Refer to Fudenberg et al. [3], p. 34. □

Theorem 1.3 (Glicksberg). *A game has a mixed-strategy Nash equilibrium if for all $i \in \mathcal{N}$, the strategy set $\mathbf{S}_i$ is a nonempty compact subset of a metric space and the utility function U_i is continuous.*

Proof. Refer to Fudenberg et al. [3], p. 36. □

1.2.5 Best-Response Correspondence

In game theory, the *best-response correspondence* of a player given his opponents' strategies [3] is what he/she should play in order to maximize his/her utility.

Definition 1.10. Consider a game $\mathcal{G}$ as in (1.3). For each player i, the best-response correspondence for him/her is a set-valued mapping $\mathcal{B}_i(S_{-i}) : \mathbf{S}_{-i} \mapsto \mathbf{S}_i$ such that

$$\mathcal{B}_i(S_{-i}) = \{S_i^* \mid S_i^* \in \arg\max_{S_i \in \mathbf{S}_i} U_i(S_i, S_{-i})\}. \tag{1.8}$$

It is straightforward to see that in a Nash equilibrium, every player plays a best response to the other players' strategies.

Theorem 1.4. *A (pure or mixed) strategy profile S^* is a Nash equilibrium if and only if*

$$S_i^* \in \mathcal{B}_i(S_{-i}^*), \quad \forall i \in \mathcal{N}. \tag{1.9}$$

Proof. Clearly, (1.9) is equivalent to (1.6) (or to (1.7) for mixed strategies). □

1.2.6 Pareto Optimal Strategies

Another significant concept encountered in game theory is *Pareto optimality*. A strategy profile is said to be Pareto optimal if improving the utilities of some players would lead to a decrease in utilities for some other players. No strategy deviation from this point can *mutually* improve the payoffs of all players.

Definition 1.11. The strategy profile $\sigma \in \mathbb{S}$ is Pareto optimal if and only if

$$\nexists S \in \mathbb{S} : \quad \forall i \in \mathcal{N}, \quad U_i(S) \geq U_i(\sigma) \tag{1.10}$$

with strict inequality for at least one i.

Remark 1.1. Both the Nash equilibrium and Pareto optimal strategy profiles represent desirable outcomes in certain senses, depending on the context. However, in most of the games, it is often inconclusive as to whether a Nash equilibrium is Pareto optimal and whether a Pareto optimal strategy profile is a Nash equilibrium. In the next section, we will encounter examples where these two are distinctly separated, like the prisoner's dilemma (Example 1.1); and where they coincide, such as in the BoS game (Example 1.2).

1.2.7 Examples

We give a few examples of strategic games to illustrate the concepts presented before.

Example 1.1 (The Prisoner's Dilemma). The game of prisoner's dilemma is very often used as an introductory example to illustrate the basic concepts in game theory. The story associated with it was due to Tucker [17]. The description of the game is as follows.

Suppose that two suspects are arrested for a crime, and questioned separately. If they both keep quiet (i.e., strategy C, which stands for cooperating *with each other), they will both go to prison for a year. If one suspect supplies evidence (i.e., D which means* defecting*) then he will be freed, and the other one who plays C will be imprisoned for six years. If both* defect *then they will both be imprisoned for four years.*

Figure 1.1 depicts a *payoff matrix* for the outcomes of this game in its non-cooperative strategic-form version. The game elements are as follows:

- Players: $\mathscr{N} = \{1,2\}$,
- Strategy sets: $\mathbf{S}_1 = \mathbf{S}_2 = \{C,D\}$,
- Strategy space: $\mathbb{S} = \{(C,C),(C,D),(D,C),(D,D)\}$,
- Utility functions:

$$U_1 : \mathbb{S} \mapsto \{-1,-6,0,-4\} \tag{1.11}$$

$$U_2 : \mathbb{S} \mapsto \{-1,0,-6,-4\}. \tag{1.12}$$

Note that the name "prisoner's dilemma" can also be generalized to any 2-player games where the set of four distinct payoff values $\{-6,-4,-1,0\}$ used in the example are replaced by any set of real numbers $\{a,b,c,d\}$ provided that $a < b < c < d$.

Lemma 1.1. *In the prisoner's dilemma above, the following statements hold:*

- *D is the strictly dominant strategy for both players,*
- *C is the strictly dominated strategy for both players,*
- (D,D) *is the only Nash equilibrium, representing the case where both non-cooperatively defect,*
- *However,* (D,D) *is not Pareto optimal. Meanwhile,* (C,C)*,* (C,D) *and* (D,C) *are all Pareto optimal.*

Fig. 1.1 Payoff matrix for prisoners' dilemma

		Player 2: C	Player 2: D
Player 1	C	$-1,-1$	$-6,\ 0$
	D	$0,-6$	$-4,-4$

Proof. The proof directly follows from the definitions. In short,

- For player 1, D is strictly dominant as $U_1(D,C) > U_1(C,C)$ and $U_1(D,D) > U_1(C,D)$, i.e., player 1 prefers D over C regardless of what player 2 chooses. For player 2, the same argument applies.
- (D,D) is a Nash equilibrium, as at this profile, both players cannot achieve a better utility by deviating. For instance, player 1 is worse off if he/she switches to C; and so is player 2. It is the unique Nash equilibrium as no other strategy profile satisfies this property.
- However, (D,D) is not Pareto optimal, as there exists another profile, namely (C,C), at which players have mutually better payoffs. At (C,C), no further mutual improvement is possible. Although player 1 can gain from switching to D , it will result in a loss for player 2, and vice versa. A Pareto optimal strategy is thus one that does not permit mutual improvement in the payoffs of all players. By the same reasoning, both (C,D) and (D,C) are Pareto optimal as well.

□

The prisoner's dilemma is an example of games with a unique pure-strategy Nash equilibrium. Our next example is a game with multiple pure-strategy as well as mixed-strategy Nash equilibria.

Example 1.2 (BoS). The game BoS describes a scenario where two players find it better to align their interests rather than not to. Its most common version and associated story is known as the *Battles of the Sexes*, which was given by Luce and Raiffa [6]. Other authors, such as Osborne [13], used an alternative name *Bach or Stravinsky*. Here, we refer to the game simply by the abbreviation BoS. Its description is as follows.

Two people wish to go to a concert together. There are two concerts available, one featuring music by Bach (B) and the other by Stravinsky (S). Each person prefers a different concert, but would rather attend the same concert together than go alone. They do not communicate and do not know which concert the other decides to go.

Its payoff matrix is given in Fig. 1.2.

Lemma 1.2. *In the BoS game above, the following statements hold:*

- *There are two pure-strategy Nash equilibria:* (B,B) *and* (S,S)*, both of which are Pareto optimal.*
- *There is one mixed-strategy Nash equilibrium* $\rho^* = (\rho_1^*, \rho_2^*)$ *where* $\rho_1^* = (\frac{3}{5}, \frac{2}{5})$ *and* $\rho_2^* = (\frac{2}{5}, \frac{3}{5})$.

Proof. To verify the two pure-strategy equilibria and their optimality is straightforward. The readers are invited to try it out, using the definitions in (1.6) and (1.10).

Fig. 1.2 Payoff matrix for BoS

		Player 2: B	Player 2: S
Player 1	B	$3,2$	$0,0$
	S	$0,0$	$2,3$

To compute the mixed-strategy Nash equilibrium, suppose that player 1 assigns probability p to B and $1-p$ to S, while for player 2 it is $(q, 1-q)$.

For player 1, his/her expected payoff for choosing B is

$$U_1(B, \rho_2) = (q)(3) + (1-q)(0) = 3q.$$

His/her expected payoff for choosing S is

$$U_1(S, \rho_2) = (q)(0) + (1-q)(2) = 2(1-q).$$

For player 2, his/her expected payoff for choosing B is

$$U_2(\rho_1, B) = (p)(2) + (1-p)(0) = 2p.$$

His/her expected payoff for choosing S is

$$U_1(S, \rho_2) = (p)(0) + (1-p)(3) = 3(1-p).$$

Now, for (ρ_1^*, ρ_2^*) to be a valid equilibrium, player 1 must not have an incentive to deviate (to a pure strategy) when player 2 plays ρ_2^*. Player 2 must also not have an incentive to deviate when player 1 plays ρ_1^*. In other words, in the mixed-strategy Nash equilibrium, for each player, *none of the expected payoffs for choosing a pure strategy dominates another* and hence, players are indifferent among them. Thus, the following *indifference equations* hold: $U_1(B, \rho_2^*) = U_1(S, \rho_2^*)$ and $U_2(\rho_1^*, B) = U_2(\rho_1^*, S)$. This leads to

$$\begin{cases} 3q = 2(1-q) \\ 2p = 3(1-p) \end{cases} \tag{1.13}$$

which is solved by $p = \frac{3}{5}$ and $q = \frac{2}{5}$. □

Exercise 1.1. Find the best responses for players in the BoS game. Check if in the Nash equilibria, both players mutually play best responses.

Solution 1.1. The best responses in pure strategies are highlighted by (*) for player 1 and (†) for player 2 in Fig. 1.3. Clearly, at the two pure-strategy Nash equilibria (B, B) and (S, S), both play best responses mutually.

In mixed-strategies, the best-response correspondences can be computed by determining which expected utility is dominant for a given p or q.

Fig. 1.3 BoS with best responses for pure strategies in (*) and (†)

		Player 2: B	Player 2: S
Player 1	B	$3^*, 2^\dagger$	$0, 0$
	S	$0, 0$	$2^*, 3^\dagger$

For player 1, $U_1(B, \rho_2)$ is dominant to $U_1(S, \rho_2)$ when $3q > 2(1-q)$, or $q > \frac{2}{5}$. Else, $U_1(S, \rho_2)$ is dominant when $q < \frac{2}{5}$; and when $q = \frac{2}{5}$, player 1 is indifferent as shown before. That is, any $p \in [0, 1]$ is equally acceptable. Thus, his/her best-response correspondence is

$$p \in \mathscr{B}_1(q) = \begin{cases} \{0\} & q < \frac{2}{5} \\ [0, 1] & q = \frac{2}{5} \\ \{1\} & q > \frac{2}{5}. \end{cases} \tag{1.14}$$

Similarly, player 2's best-response correspondence is

$$q \in \mathscr{B}_2(p) = \begin{cases} \{0\} & p < \frac{3}{5} \\ [0, 1] & p = \frac{3}{5} \\ \{1\} & p > \frac{3}{5}. \end{cases} \tag{1.15}$$

Both players' best-response correspondences can then be plotted together, as shown in Fig 1.4. Their intersections correspond to Nash equilibria. There are three intersections, equal to the total number of equilibria aforementioned.

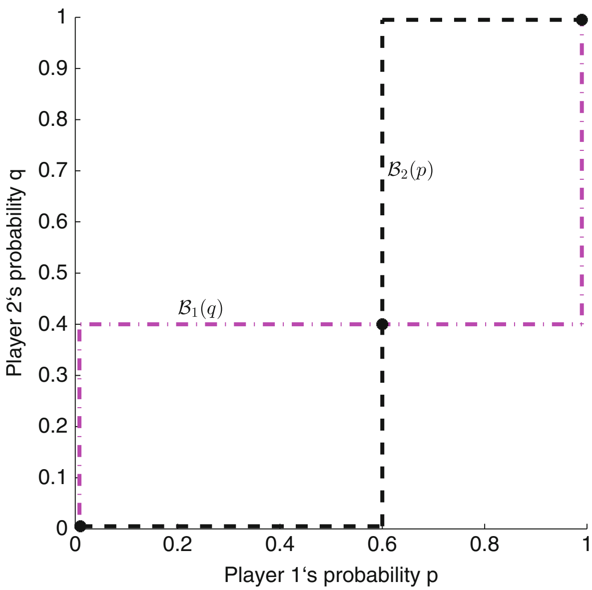

Fig. 1.4 Best-response correspondences in the BoS game: $p = 1$ (or $q = 1$) corresponds to pure-strategy B; and $p = 0$ (or $q = 0$) to S for player 1 (or player 2)

Fig. 1.5 Payoff matrix for matching pennies

		Player 2	
		H	T
Player 1	H	$1,-1$	$-1,\ 1$
	T	$-1,\ 1$	$1,-1$

Exercise 1.2. Is the mixed-strategy Nash equilibrium above Pareto optimal?

A game can also have no pure-strategy Nash equilibrium but has equilibrium in mixed strategies, as shown in the next example.

Example 1.3 (Matching Pennies). The game of matching pennies is a game where two players have totally opposing interest. It is also a zero-sum game and was first considered by von Neumann in [15]. Here is its description.

Two players simultaneously flip a coin and reveal to one another. If the coins match, i.e., they both show heads *(H) or* tails *(T), player 2 pays player 1 one dollar. If the coins do not match, i.e., one is* head *and one is* tail, *player 1 pays player 2 one dollar.*

The payoff matrix is given in Fig. 1.5.

Readers can try the following exercise.

Exercise 1.3. Find any dominant strategies and Pareto optimal strategy profiles for Matching pennies. Hence, deduce that there is no Nash equilibrium in pure strategy. Find a mixed-strategy Nash equilibrium by (1) the indifference equations and (2) plotting the best-response correspondences.

In general, games may have multiple, unique or no pure-strategy Nash equilibria as the examples have shown. Theorem 1.1 guarantees that in finite games, at least one Nash equilibrium (which can be in mixed strategy) exists.

In the following section, some game-theoretical mechanisms related to Nash equilibrium computation in practical applications are discussed.

1.3 Computation of Nash Equilibrium via Iterative Gameplay

1.3.1 Iterative Gameplay vs. Repeated Games

From previous examples, we have had a glimpse at the method to compute Nash equilibria for simple games, both in pure strategies and mixed strategies. This section looks into the mechanisms through which Nash equilibrium can often be computed in games of larger dimensions where the previous method is deemed too complicated or difficult. We may limit our attention to only pure-strategy Nash equilibria, as most practical applications would like their systems to operate at one single stable point rather than oscillating among various states.

The procedures discussed in the previous section also make use of an implicit assumption—we must know the complete payoff matrix of a game in order to identify players' best responses and/or assign probabilities and to solve the indifference equations. However, in reality, distributed non-cooperative players often have no means of communications and are likely to have no knowledge of the opponents' payoff information. In fact, their strategic decisions are often made *myopically*, i.e., the decision is based only on *one's own information and current observation of the outcome of the game*, not on past knowledge or future speculation. The challenge is, for a strategic game with incomplete information, Nash equilibrium may not be reached in just one play. The game may need to be *iterated* so that players update their strategies based on their observations of opponents' actions until a (pure-strategy) Nash equilibrium is reached. This process is referred to as *iterative gameplay*.

It is worth mentioning that iterative gameplay is not a formal concept in game theory, but instead a technique frequently adopted in most game-theoretical algorithms in engineering problems. Its main objective is to help in achieving a Nash equilibrium outcome, under the assumption of incomplete information and myopic behaviors. During an iterative gameplay, based on the current game outcome, players may gradually improve their utilities towards a more stable and optimal solution by employing a certain procedure to adapt their strategies. The procedures for strategy adaptation are called *decision dynamics* or *decision rules*. Our focus in this book will be on decision rules that are myopic.

The underlying principles of iterative play are somehow different from another branch in game theory known as *repeated games*. In repeated games, an original one-shot game is also played across several stages and the strategies adopted over time can also be learned from history. However, rigorous mathematical treatment of repeated games often introduces a discount factor $0 < \alpha < 1$ into the original utility function in such a way that at a later tth stage, the expected payoff for player i corresponding to a strategy profile S becomes $U_{i,t}(S) = \alpha^t U_i(S)$ where $U_i(S)$ is the payoff that player i would get in the one-shot version. Therefore, the accumulated payoff if the game is played from stage 0 to stage T is $U_i = \sum_{t=0}^{T} U_{i,t}$. In repeated games, one can define *subgame perfect Nash equilibrium* and invoke *Folk theorem* [3] to find the solutions.

As a matter of fact, most wireless resource allocation problems in the literature do have the game iterated through multiple stages. However, as myopic behaviors are considered, the lack of past memories and future speculation will render the complex multi-stage strategies in the studies of repeated games inapplicable. Folk theorem does not hold for these *myopic games* [11]; and the discount factor and the accumulated utilities will not be used. As such, we make a point in distinguishing the concept of *iterative gameplay* in wireless resource allocation context against the *repeated games* traditionally known in game theory.

Next, we shall look at a few important decision rules often encountered in practice.

1.3.2 Decision Rules, Best-Response and Better-Response Dynamics

In the context of iterative gameplay, decision rules govern the way players update their strategies. The decision rules can be classified according to the timing of decisions, as well as the manner in which new strategies are updated.

In terms of timing, Neel [10] defined the following decision rules according to their timing: synchronous, asynchronous, sequential (or round-robin) and random timing. We assume that strategy adaptation decisions are made in a series of time instances, indexed by $t = 0, 1, 2, \ldots$. Then, the various timing notions are informally described as follows.

Definition 1.12. In terms of decision timing, a decision rule is said to be

- *Synchronous*: if decisions by all players occur at the same time instance t, $\forall t$. That is, players update their strategies simultaneously.
- *Asynchronous*: if at a time instance t, there are a random number of players making decisions, $\forall t$. For example, players can choose to update to a new strategy at a probability p and stay inactive with probability $1-p$, which leads to a random number of them updating at a time.
- *Sequential*: if there is only one player making decision at a time instance t, $\forall t$. That is, players take turn to act in sequence via a predefined order.
- *Random-timing*: if there is only one random player $i = \mathsf{rand}(\mathcal{N})$ making decision at a time instance t, $\forall t$. Gameplay is still sequential but the order is random.

Here, we write $\mathsf{rand}(\{.\})$ or $\mathsf{rand}(A)$ in order to denote a randomized selection among elements of a set $A = \{.\}$.

In terms of how to determine the new strategy to be adopted by a player, the best-response dynamics and better-response dynamics are the two most important and commonly used decision rules. The best-response dynamics, as the name suggests, are based on the requirement that players adapt their strategies from their best responses given the opponents' actions. Best-response dynamics are defined formally through the following rule [10]:

Definition 1.13. In the *best-response dynamics*, every player i will select a new strategy $\hat{S}_i$ such that

$$\hat{S}_i \in \mathscr{B}_i(S_{-i}), \ \forall i \in \mathcal{N}. \tag{1.16}$$

Best-response dynamics require player i to do an exhaustive search over the strategy set $\mathbf{S}_i$ for the best option, which may incur a high order of computational complexity in games with large strategy sets. Alternatively, another mechanism which requires less searching effort, known as the better-response dynamic [10], is also frequently adopted.

Definition 1.14. In the *better-response dynamics*, player i will choose a new strategy T_i over the current strategy S_i if and only if T_i is any randomly selected strategy that improves his/her payoff, given the opponents' strategy S_{-i}. We formally write it as

$$T_i = \mathsf{rand}(\{S_i' | \ S_i' \in \mathbf{S}_i, U_i(S_i', S_{-i}) > U_i(S_i, S_{-i})\}), \ \forall i \in \mathcal{N}. \tag{1.17}$$

The combination of these classifications can give rise to more specific decision dynamics, such as synchronous best-response dynamics, sequential better-response dynamics, random-timing better-response dynamics, and so on. In this monograph, we will not deal exhaustively with all types of decision dynamics listed above. Instead, a great deal of our discussion will focus on the sequential decision rules, which have been prominently adopted in wireless communications problems.

All decision rules above only require players to use their current information from the game, i.e., the observed strategies of their opponents, and thus are myopic. The ultimate purpose of using such myopic decisions is to drive the game towards a Nash equilibrium only based on locally available information. We allow each player to individually improve his/her utility and hope that the process will *eventually converge* to a point when such improvements are stabilized. If it happens, at this point, all players are virtually playing their mutual best responses which is by definition a Nash equilibrium. The Nash equilibrium obtained via this process thus represents a stable operating point for the distributed system.

1.3.3 Using Best-Response Dynamics to Find Nash Equilibrium

In this section, we give some examples of using iterative best responses to search for a Nash equilibrium.

Example 1.4. We consider a dynamic spectrum access game. Suppose that there are two available frequency bands A and B. Two players 1 and 2, which are mobile users, compete for the spectrum bandwidth to transmit their data. Each player has three strategies: transmitting in band A, transmitting in band B and transmitting in both bands A and B. The three strategies are denoted by A, B and AB, respectively. The payoff matrix is assumed to be computed from some performance metrics and tabulated as in Fig. 1.6. Note that although our example is given with only two players and simplified payoff values in order to illustrate the concepts, in actual applications, we must deal with a much larger number of players and the payoff information may contain time-varying channel parameters instead of static values as shown.

The best responses (in pure strategies) are again highlighted by (*) for player 1 and (†) for player 2. We see that there is a unique Nash equilibrium in which player 1 transmits on both bands and player 2 transmits only on band B. If both players have

Fig. 1.6 Payoff matrix for the example spectrum access game

		Player 2		
		A	*B*	*AB*
Player 1	*A*	$3,3$	$5,9^{\dagger}$	$3,8$
	B	$9^{*},5$	$2,2$	$2,6^{\dagger}$
	AB	$6,3$	$6^{*},6^{\dagger}$	$5^{*},5$

Fig. 1.7 Sequential best-response dynamics, at $t = 0$

		Player 2		
		A	*B*	*AB*
Player 1	*A*	$3,3$	$5,9^{\dagger}$	$3,8$
	B	$9^{*},5$	$2,2$	$2,6^{\dagger}$
	AB	$6,3$	$6^{*},6^{\dagger}$	$5^{*},5$

Fig. 1.8 Sequential best-response dynamics, at $t = 1$

		Player 2		
		A	*B*	*AB*
Player 1	*A*	$3,3$	$5,9^{\dagger}$	$3,8$
	B	$9^{*},5$	$2,2$	$2,6^{\dagger}$
	AB	$6,3$	$6^{*},6^{\dagger}$	$5^{*},5$

Fig. 1.9 Sequential best-response dynamics, at $t = 2$

		Player 2		
		A	*B*	*AB*
Player 1	*A*	$3,3$	$5,9^{\dagger}$	$3,8$
	B	$9^{*},5$	$2,2$	$2,6^{\dagger}$
	AB	$6,3$	$6^{*},6^{\dagger}$	$5^{*},5$

Fig. 1.10 Sequential best-response dynamics, at $t = 3$

		Player 2		
		A	*B*	*AB*
Player 1	*A*	$3,3$	$5,9^{\dagger}$	$3,8$
	B	$9^{*},5$	$2,2$	$2,6^{\dagger}$
	AB	$6,3$	$6^{*},6^{\dagger}$	$5^{*},5$

complete information, both can easily figure out that (AB, B) is a Nash equilibrium and enact this strategy profile.

However, if players only have their own payoff information and can only observe the action of the opponent, then myopic dynamics, such as sequential best-response dynamic can be used for strategy adaptation. At the beginning ($t = 0$), assume that players randomly select their strategies, which results in the strategy profile (B, A). This is shown in Fig. 1.7.

Player 2 is first to make a decision. Its best response for the current situation is to switch to *AB* which gives the highest reward of 6. Thus, at $t = 1$, the game is at strategy profile (B, AB) as shown in Fig. 1.8.

Player 1 is next to move. Similarly, it decides to switch to its best response *AB* as it gets the highest payoff of 5. Thus, at $t = 2$, the game is at strategy profile (AB, AB) as shown in Fig. 1.9.

Now, player 2 finds its new best response to be *B* and updates accordingly. The game is now in the state shown in Fig. 1.10.

Fig. 1.11 Modified payoff matrix for the spectrum access game in Example 1.5

Player 2

Player 1	A	B	AB
A	$3,3$	$5,9^{\dagger}$	$3,8$
B	$9^{*},5$	$2,2$	$5^{*},9^{\dagger}$
AB	$6,3$	$6^{*},6^{\dagger}$	$4,5$

Fig. 1.12 Modified payoff matrix for the spectrum access game in Example 1.6

Player 2

Player 1	A	B	AB
A	$3,3$	$5,9^{\dagger}$	$3,8$
B	$9^{*},5$	$2,2$	$2,6^{\dagger}$
AB	$6,7^{\dagger}$	$6^{*},6$	$5^{*},5$

Finally, both players are happy to stay at (AB,B) which is the Nash equilibrium. Thus, in this example, the sequential best-response dynamic is able to converge to a stable state.

Exercise 1.4. Apply the dynamic, assuming various different starting strategy profiles, for the above example and verify that the dynamic still converges.

Example 1.5. Suppose that we modify the payoff matrix in Example 1.4 to that shown in Fig. 1.11.

By identifying best responses, this game is shown to have multiple pure-strategy Nash equilibria at (AB,B) and (B,AB). With complete information, some coordination is also needed so that both players select the same equilibrium.

Exercise 1.5. Apply the sequential best-response dynamic, assuming various different starting strategy profiles, for Example 1.5. Verify that the dynamic converges to either one of the two pure-strategy Nash equilibria.

The previous two examples show games that are stable under the sequential best-response dynamic and Nash equilibrium convergence can be obtained. However, there are cases where convergence is not guaranteed, as the next examples show.

Example 1.6. Suppose that we modify the payoff matrix in Example 1.4 to that shown in Fig. 1.12. This game does not possess any pure-strategy Nash equilibrium, as none of the profiles shows mutual best-responses. If one is to apply the myopic dynamics, one will find that it will fail to converge and the game is not stable.

Exercise 1.6. Apply the sequential best-response dynamic, assuming various different starting strategy profiles, for Example 1.6. Verify that the dynamic will end up with an infinite loop among (B,A), (B,AB), (AB,AB), (AB,A) and back to (B,A).

This example shows that convergence is not possible where no pure-strategy Nash equilibrium exists. However, equilibrium existence does not guarantee convergence either, as the next example shows.

Example 1.7. Suppose that we modify the payoff matrix in Example 1.4 to that shown in Fig. 1.13. The game has a single pure-strategy Nash equilibrium (A,B), due to the mutual best-responses. However, the dynamic may still fall into a loop.

Fig. 1.13 Modified payoff matrix for the spectrum access game in Example 1.7

		Player 2		
		A	B	AB
Player 1	A	$3,3$	$8^*,9^\dagger$	$3,8$
	B	$9^*,5$	$2,2$	$2,6^\dagger$
	AB	$6,7^\dagger$	$6,6$	$5^*,5$

Exercise 1.7. Apply the sequential best-response dynamic, assuming various different starting strategy profiles, for Example 1.6. Verify that the dynamic will end up with an infinite loop among (B,A), (B,AB), (AB,AB), (AB,A) and back to (B,A), if one begins the game from any of these four strategy profiles at $t = 0$.

In conclusion, in a formulated strategic game, convergence to Nash equilibrium using myopic dynamics is always desired as the system can finally achieve stability. However, as illustrated above, there is no guarantee that the method will converge, even if a Nash equilibrium exists. Fortunately, convergence can be ensured for games with special properties, among which is the class of potential games. Potential games are the main topic of this book and will be examined in depth in Chap. 2.

1.3.4 Price of Anarchy

Price of anarchy (PoA) is a concept in algorithmic game theory, which literally means the amount of damage suffered by the members of a system (i.e., the players) due to the absence of a central authority. Specifically, it measures the discrepancy in efficiency of the game behaviors between the socially Pareto optimal point in the presence of a centralized controller and the one due to distributed selfish behaviors among players (which can be the Nash equilibrium, or any other allocation point obtained by a distributed method) [12]. Often, the ratio between the performance measures of the two points is computed. In that sense, PoA is seen as a metric to evaluate how efficient an implemented allocation is, such as one obtained via myopic dynamics, compared to the social optimum.

The performance measure is also referred to in the literature as the *welfare function*. Assuming that to each strategy profile $S = (S_i, S_{-i}) \in \mathbb{S}$, one can assign a real number $\Theta(S)$ as a measure of efficiency, then the welfare function is mathematically a mapping $\Theta : \mathbb{S} \mapsto \mathbb{R}$. A common choice for the welfare function is the *utilitarian* welfare, defined as follows.

Definition 1.15. Given $S \in \mathbb{S}$, the utilitarian welfare function $\Theta : \mathbb{S} \mapsto \mathbb{R}$ is simply the sum of the utility (or cost) functions of all players, i.e.,

$$\Theta(S) \triangleq \sum_{i=1}^{N} U_i(S). \tag{1.18}$$

Profile S is preferred to T if $\Theta(S) > \Theta(T)$ for a utility maximization game, or $\Theta(S) < \Theta(T)$ for a cost minimization game.

Now, let $\hat{S} \in \mathbb{S}$ be the socially optimal point with respect to the welfare function Θ, i.e., $\hat{S} = \arg\max_{S\in\mathbb{S}} \Theta(S)$ for maximization, or $\hat{S} = \arg\min_{S\in\mathbb{S}} \Theta(S)$ for minimization. Also, let $S^* \in \mathbb{S}$ be a particular outcome of the game, e.g., a Nash equilibrium. By convention, the PoA can be defined separately for these two scenarios as follows.

Definition 1.16. Let $PoA_{\max}(S^*)$ and $PoA_{\min}(S^*)$ be the PoAs of strategy profile S^* for utility-maximization and cost-minimization games, respectively. Then,

$$PoA_{\max}(S) = \frac{\Theta(\hat{S})}{\Theta(S^*)}, \tag{1.19a}$$

$$PoA_{\min}(S) = \frac{\Theta(S^*)}{\Theta(\hat{S})}. \tag{1.19b}$$

Example 1.8. As a simple example of computing PoA, let us look at the prisoner's dilemma (Example 1.1) whose payoff matrix is given in Fig. 1.1.

The utilitarian welfare function Θ can be computed for each profile. For example,

$$\Theta(D, D) = -4 - 4 = -8.$$

Similarly, we obtain $\Theta(D, C) = \Theta(C, D) = -6$ and $\Theta(C, C) = -2$. Here, (D, D) is the unique Nash equilibrium which can be reached via best-response dynamics, while (C, C) is the socially optimal point that maximizes Θ.

As this is a utility-maximization game, the PoA of the Nash equilibrium (D, D) is given by

$$PoA(D, D) = \frac{\Theta(C, C)}{\Theta(D, D)} = \frac{-8}{-2} = 4, \tag{1.20}$$

which is relatively far from its optimal value of 1.

We can arrive at the well-known conclusion that in the prisoner's dilemma, the Nash equilibrium is fairly inefficient.

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Chapter 2
Potential Games

Abstract This chapter deals with theories related to the class of games known as potential games. What make potential games attractive are their useful properties concerning the existence and attainability of their Nash equilibria. These properties have direct consequences: the straightforward applications of equilibrium-seeking dynamics to the gameplay of potential games can lead to a Nash equilibrium solution. However, before being able to successfully apply such rewarding results to practical problems, one needs to answer the important question of *how to identify if a game is a potential game*. To put the question in another way is *how to formulate a problem so that the resulting game is a potential game.* We hope that this chapter can successfully address these questions and generalize techniques in identifying and formulating a potential game. Through a systematic examination over existing work, we are able to account for the methodologies involved, and provide readers with useful and unified insights. In the identification problems, we examine the structures of the game's strategy space and utility functions, and their properties upon which the conditions of potential games are satisfied. Meanwhile in the formulation problem, we suggest two distinct approaches. For the forward approach, we examine the methods to design utility functions with certain properties so that a potential function can be derived, and hence a potential game is formulated. In the reverse approach, we begin by defining a potential function whereby the utility functions of players can later be obtained. We will also discuss practical examples in the context of wireless communications and networking in order to illustrate the ideas.

2.1 Definition

The seminal paper by Monderer and Shapley in 1996 [31] coined the term "potential games". It presented the first systematic investigation and fundamental results for a certain type of games for which potential functions exist. However, the first concept of potential games can actually be traced back to the work by Rosenthal in 1973 [40], about games having pure-strategy Nash equilibria known as "congestion games". Today, the theory of potential games has been further developed by many authors. They have also grown out of their pure mathematical realm and found successful applications in solving engineering problems.

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Mathematically, there can be various types of potential games. In all these types of games, however, the common thread is the existence of an associated function—the *potential function*—that maps the game's strategy space $\mathbb{S}$ to the space of real number $\mathbb{R}$. Their classifications depend on the specific relationship between the potential function and the utility functions of players. The potential function is therefore the most important element in the studies of potential games. The origin of the term "potential" was drawn from analogies to the similarly-named concept of potential in vector field analysis, whose leading examples include gravitational potential and electric potential in physics.

Monderer and Shapley [31] listed four types of potential games: *ordinal potential games*, *weighted potential games*, *exact potential games*, and *generalized ordinal potential games*. Other extensions also exist in the literature. Of interest in this monograph are *best-response potential games* proposed by Voorneveld [49], and *pseudo-potential games* proposed by Dubey et al. [13]. For completeness, we will present here the definitions for all these types of potential games.

2.1.1 Exact Potential Games

Definition 2.1 (Exact Potential Game). The game $\mathcal{G}$ [1] is an exact potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) = F(T_i, S_{-i}) - F(S_i, S_{-i}),$$
$$\forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \tag{2.1}$$

In exact potential games, the change in a single player's utility due to his/her own strategy deviation results in exactly the same amount of change in the potential function.

Assuming that each strategy set $\mathbf{S}_i$ is a continuous interval of $\mathbb{R}$ and each utility function U_i is everywhere continuous and differentiable, we say $\mathcal{G}$ is a continuous game. For such a game to be an exact potential game, an equivalent definition to (2.1) states that, $\forall i \in \mathcal{N}$:

$$\frac{\partial U_i(S_i, S_{-i})}{\partial S_i} = \frac{\partial F(S_i, S_{-i})}{\partial S_i}, \quad \forall S_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \tag{2.2}$$

Among the various types of potential games, exact potential games are those whose definition requires the strictest condition of exact equality. Other types of potential games are defined by loosening this condition. Exact potential games are however the most important and have received the highest level of interest in both theoretical research and practical applications.

[1]In this chapter, a game $\mathcal{G}$ will be understood as $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i\in\mathcal{N}}]$, unless otherwise stated.

Fig. 2.1 Prisoners' dilemma with payoff matrix in (**a**) and a potential function in (**b**)

a

	C	D
C	$-1,-1$	$-6,\ 0$
D	$0,-6$	$-4,-4$

b

	C	D
C	1	2
D	2	4

Example 2.1. The prisoner's dilemma in Sect. 1.2.7 is an exact potential game. The payoff table is reproduced in Fig. 2.1a and a corresponding potential function is given alongside in Fig. 2.1b. It is not difficult to verify this by simply stepping through all possible unilateral strategy changes. For instance, for the switch from (C,C) to (D,C) due to player 1, $U_1(C,C) - U_1(D,C) = (-1) - 0 = -1$. Correspondingly, $F(C,C) - F(D,C) = 1 - 2 = -1$. The other strategy changes can be verified similarly.

Please note that the potential function is not unique. How to obtain such a function is one of the objectives of this chapter and will become clearer after more contents are introduced. The actual procedures will be discussed in Sect. 2.3.2.

2.1.2 Weighted Potential Games

Definition 2.2 (Weighted Potential Game). The game $\mathcal{G}$ is a weighted potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) = w_i \left(F(T_i, S_{-i}) - F(S_i, S_{-i}) \right), \quad \forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i} \tag{2.3}$$

where $(w_i)_{i \in \mathcal{N}}$ constitutes a vector of positive numbers, known as the weights.

In weighted potential games, a player's change in payoff due to his/her unilateral strategy deviation is equal to the change in the potential function (also known as w-potential function in [31]) but scaled by a weight factor. Clearly, all exact potential games are weighted potential games with all players having identical weights of 1.

Similarly, (2.3) is equivalent to the following condition for continuous games. That is, $\forall i \in \mathcal{N}$:

$$\frac{\partial U_i(S_i, S_{-i})}{\partial S_i} = w_i \frac{\partial F(S_i, S_{-i})}{\partial S_i}, \quad \forall S_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \tag{2.4}$$

Although defined separately, weighted potential games and exact potential games can be made equivalent by scaling the utility functions appropriately.

Fig. 2.2 A weighted potential game with payoff matrix in (**a**) and a potential function in (**b**)

a

	C	D
C	$-2, -3$	$-12, 0$
D	$0, -18$	$-8, -12$

b

	C	D
C	1	2
D	2	4

Lemma 2.1. *$\mathscr{G} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i \in \mathscr{N}}]$ is a weighted potential game with potential function $F(S)$ and weights $(w_i)_{i \in \mathscr{N}}$ if and only if $\mathscr{G}' = \left[\mathscr{N}, \mathbb{S}, \{V_i = \frac{1}{w_i} U_i\}_{i \in \mathscr{N}}\right]$ is an exact potential game with potential function $F(S)$.*

Proof. Clearly, the following conditions

$$U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) = w_i \left(F(T_i, S_{-i}) - F(S_i, S_{-i})\right)$$

and

$$V_i(T_i, S_{-i}) - V_i(S_i, S_{-i}) = F(T_i, S_{-i}) - F(S_i, S_{-i})$$

are equivalence. Thus, necessity and sufficiency are apparent. □

Due to their equivalence, in our subsequent discussion, we will focus our discussion on exact potential games. However, equivalent results should be equally available for weighted potential games as well.

Example 2.2. The prisoner's dilemma in Example 2.1 with scaled utility functions, $(w_1, w_2) = (2, 3)$, is a weighted potential game (Fig. 2.2). Note that these two games can have the same potential function. The validation for the potential function is also very straightforward.

2.1.3 Ordinal Potential Games

Definition 2.3 (Ordinal Potential Game). The game $\mathscr{G}$ is an ordinal potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathscr{N}$:

$$U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) > 0 \Leftrightarrow F(T_i, S_{-i}) - F(S_i, S_{-i}) > 0,$$
$$\forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \quad (2.5)$$

Note that (2.5) can be equivalently rewritten as follows. $\forall i \in \mathscr{N}$:

$$\operatorname{sgn}[U_i(T_i, S_{-i}) - U_i(S_i, S_{-i})] = \operatorname{sgn}[F(T_i, S_{-i}) - F(S_i, S_{-i})],$$
$$\forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i} \quad (2.6)$$

where sgn() is the signum function.

Fig. 2.3 An ordinal potential game with payoff matrix in (**a**) and a potential function in (**b**)

a

	C	D
C	$-1, -2$	$-9, 0$
D	$0, -8$	$-4, -5$

b

	C	D
C	0	1
D	1	2

Unlike in exact potential games, ordinal potential games only require that the change in the potential function due to a unilateral strategy deviation only needs to be of the same sign as the change in the player's utility function. In other words, if player i gains a better (worse) utility from switching his/her strategy, this should lead to an increase (decline) in the potential function F, and vice versa.

For continuous games, $\forall i \in \mathscr{N}$:

$$\operatorname{sgn}\left[\frac{\partial U_i(S_i, S_{-i})}{\partial S_i}\right] = \operatorname{sgn}\left[\frac{\partial F(S_i, S_{-i})}{\partial S_i}\right], \ \forall S_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \tag{2.7}$$

Example 2.3. The following ordinal potential game (Fig. 2.3) is a variant of the prisoner's dilemma (see Example 2.1) with modified payoffs. The procedures for obtaining the associated potential function will be discussed in Sect. 2.3.1.

2.1.4 Generalized Ordinal Potential Games

Generalized ordinal potential games are an extension from ordinal potential games, as defined in [31]. We include their definition for completeness.

Definition 2.4 (Generalized Ordinal Potential Game). The game $\mathscr{G}$ is a generalized ordinal potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathscr{N}$:

$$\begin{aligned} &U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) > 0 \Rightarrow F(T_i, S_{-i}) - F(S_i, S_{-i}) > 0, \\ &\quad \forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \end{aligned} \tag{2.8}$$

Basically, an increase (decrease) in a player's utility due to his/her unilateral strategy deviation implies an increase (decrease) in the potential function. But the reverse is not true, unlike in ordinal potential games.

Example 2.4. The game presented in Fig. 2.4 is a generalized ordinal potential game. Note that $F(1A, 2A) - F(1A, 2B) > 0$ does not imply $U_2(1A, 2A) - U_2(1A, 2B) > 0$. Hence, the game is not an ordinal potential game and the example indicates that ordinal potential games are a subset of generalized ordinal potential games.

Fig. 2.4 A generalized ordinal potential game with payoff matrix in (**a**) and a potential function in (**b**)

a

	$2A$	$2B$
$1A$	4, 3	3, 3
$1B$	3, 4	4, 3

b

	$2A$	$2B$
$1A$	3	0
$1B$	2	1

2.1.5 Best-Response Potential Games

Best-response potential games were introduced by Voorneveld [49]. We include their definition for completeness.

Definition 2.5 (Best-Response Potential Game). The game $\mathcal{G}$ is a best-response potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$\mathcal{B}_i(S_{-i}) = \arg\max_{S_i \in \mathbf{S}_i} F(S_i, S_{-i}),\ \forall S_{-i} \in \mathbf{S}_{-i} \tag{2.9}$$

where $\mathcal{B}_i(S_{-i})$ is player i's best-response correspondence which is defined in (1.8).

Note that the equality in (2.9) should be interpreted as the two sets are equal. The notion of best-response potential games deviates considerably from the previous notions in Sects. 2.1.1–2.1.4. It requires that all strategies that maximize player i's utility must also maximize the potential function, and vice versa.

The next lemma discusses the relationship between best-response potential games and ordinal potential games.

Lemma 2.2. *Every ordinal potential game is also a best-response potential game.*

Proof. We shall prove by contradiction. Let us consider an ordinal potential game $\mathcal{G}$ with potential function F. Then, for an arbitrary player i, we consider some of his/her best responses $\hat{S}_i \in \mathcal{B}_i(S_{-i})$. Now, assuming $\mathcal{G}$ is not a best-response potential game, then there exists at least one $\hat{S}_i$ which does not maximize $F(\hat{S}_i, S_{-i})$. Consequently, it implies that there exists S_i' such that $F(S_i', S_{-i}) > F(\hat{S}_i, S_{-i})$ which results in $U(S_i', S_{-i}) > U(\hat{S}_i, S_{-i})$ from the definition given in (2.5). This contradicts the fact that $\hat{S}_i$ is a best-response strategy. Hence, the set of maximizers for every player's utility function should also be identical to the set of maximizers for the potential function and the lemma holds. □

On the other hand, a best-response potential game may not necessarily be an ordinal or generalized ordinal potential game as the following example shows.

Example 2.5. The following game by Voorneveld [49] shown in Fig. 2.5 is a best-response potential game. However, since $F(1A, 2B) - F(1A, 2C) > 0$ while $U_2(1A, 2B) - U_2(1A, 2C) < 0$, the game cannot be an ordinal or generalized ordinal potential game.

Fig. 2.5 A best-response potential game with payoff matrix in (**a**) and a potential function in (**b**)

a

	$2A$	$2B$	$2C$
$1A$	2,2	1,0	0,1
$1B$	0,0	0,1	1,0

b

	$2A$	$2B$	$2C$
$1A$	4	3	0
$1B$	0	2	1

2.1.6 Pseudo-Potential Games

The concept of pseudo-potential games was introduced by Dubey et al. [13].

Definition 2.6 (Pseudo-Potential Game). The game $\mathcal{G}$ is a pseudo-potential game if and only if a continuous function $F : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$\mathcal{B}_i(S_{-i}) \supset \arg\max_{S_i \in \mathbf{S}_i} F(S_i, S_{-i}), \ \forall S_{-i} \in \mathbf{S}_{-i}. \tag{2.10}$$

The above implies that the set of maximizers of the function F with respect to the strategy of player i, while keeping opponents' strategies constant, is *included* in player i's best-response correspondence. It suffices to say that in order for player i to obtain one of his/her best responses, he/she might do so by maximizing the pseudo-potential function F.

Example 2.6. Consider again the game in Example 2.4 (see Fig. 2.4a). The game is also a pseudo-potential game. Its potential function is also given by Fig. 2.4b. We note that it is not a best-response potential game as $\arg\max_{S_2} F(1A, S_2) = 2A$ while $\arg\max_{S_2} U_2(1A, S_2) = \{2A, 2B\}$.

Pseudo-potential games are included because they have applications in distributed power control for wireless networks. Specifically, two special classes of pseudo-potential games known as games of weak strategic substitutes and/or weak strategic complements with aggregation (WSC-A/WSS-A) are applied to analyze power control problems. We will return to these applications in Sect. 5.2.

2.1.7 Relations Among Classes of Potential Games

Several classes of potential games have been defined. The following theorem sums up their inter-relationships.

Theorem 2.1. *Let* E, W, O, G, B *and* P *denote the classes of* finite *exact, weighted, ordinal, generalized ordinal, best-response, and pseudo- potential games, respectively. Then*

(i) $\mathsf{E} \subset \mathsf{W} \subset \mathsf{O} \subset \mathsf{G} \subset \mathsf{P}$
(ii) $\mathsf{E} \subset \mathsf{W} \subset \mathsf{O} \subset \mathsf{B} \subset \mathsf{P}$
(iii) $\mathsf{G} \cap \mathsf{B} \neq \emptyset$, $\mathsf{G} \setminus \mathsf{B} \neq \emptyset$, *and* $\mathsf{B} \setminus \mathsf{G} \neq \emptyset$.

Proof. The results were concluded due to several works such as [13, 31, 43, 49].

From their definitions and the results presented in (2.1), (2.3), (2.5), and (2.8), it is obvious that $\mathsf{E} \subset \mathsf{W} \subset \mathsf{O} \subset \mathsf{G}$. To see that $\mathsf{G} \subset \mathsf{P}$, Schipper [43] argued that if $\mathscr{G} \in \mathsf{G}$, then $\mathscr{G}$ has no strict improvement cycle which means it also has no strict best-response cycle; and hence, we also have $\mathscr{G} \in \mathsf{P}$. However, $\mathscr{G} \in \mathsf{P}$ does not imply $\mathscr{G} \in \mathsf{G}$, which means $\mathsf{P} \not\subset \mathsf{G}$. Hence, (i) is proven. The concept of improvement cycles and strict best-response cycles will be defined in details in Sect. 2.3.1.

In Sect. 2.1.5, Lemma 2.2 shows that $\mathsf{O} \subset \mathsf{B}$, while Example 2.5 shows that $\mathsf{B} \not\subset \mathsf{O}$ Also by definition, $\mathsf{B} \subset \mathsf{P}$. Meanwhile, Example 2.6 shows that $\mathsf{P} \not\subset \mathsf{B}$. Hence, (ii) is proven.

To establish (iii), [49] gave examples of games that are both in G and B, in G but not in B, and in B but not in G. □

2.2 Fundamental Properties of Potential Games

In this section, we discuss the properties possessed by potential games. These include two results of paramount importance, which are the *existence* of pure-strategy Nash equilibria and the *convergence* to these equilibria in potential games. In the literature, Monderer and Shapley [31] established the key existence and convergence results for ordinal potential games. According to Theorem 2.1, these results should also apply to exact and weighted potential games. Later works [13, 43] extended these existence and convergence properties to pseudo-potential games; however, the results are more restrictive than those of ordinal potential games. Again, from Theorem 2.1, results for pseudo-potential games directly apply to generalized ordinal and best-response potential games, both of which are subsets of pseudo-potential games. Thus, we will present our discussion according to two main types of games, i.e., ordinal potential games and pseudo-potential games, separately.

2.2.1 Nash Equilibrium Existence

The key idea to show that a Nash equilibrium exists in potential games is the observation that *the set of equilibria in such a game is tied to that of an identical-interest game*, where every player maximizes the common potential function. We begin our discussion with the case of ordinal potential games.

Theorem 2.2 (Monderer and Shapley). *If F is a potential function for the ordinal potential game $\mathscr{G} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i\in\mathscr{N}}]$, then the set of Nash equilibria of $\mathscr{G}$ coincides with the set of Nash equilibria for the identical interest game $\mathscr{G}^\dagger = [\mathscr{N}, \mathbb{S}, \{F\}_{i\in\mathscr{N}}]$. That is,*

$$\mathsf{NESet}(\mathscr{G}) \equiv \mathsf{NESet}(\mathscr{G}^\dagger) \tag{2.11}$$

where NESet *denotes the set of Nash equilibria of a game.*

Proof. First, assume that S^* is a Nash equilibrium for $\mathcal{G}$. Then, $\forall i$:

$$U_i(S_i^*, S_{-i}^*) - U_i(S_i, S_{-i}^*) \geq 0 \quad \forall S_i \in \mathbf{S}_i. \tag{2.12}$$

Also by the definition of ordinal potential game (2.5), this leads to, $\forall i$:

$$F(S_i^*, S_{-i}^*) - F(S_i, S_{-i}^*) \geq 0 \quad \forall S_i \in \mathbf{S}_i. \tag{2.13}$$

Hence, S^* is also a Nash equilibrium for $\mathcal{G}^\dagger$. Thus, $\mathsf{NESet}(\mathcal{G}) \subseteq \mathsf{NESet}(\mathcal{G}^\dagger)$.

Similarly, we can show that $\mathsf{NESet}(\mathcal{G}^\dagger) \subseteq \mathsf{NESet}(\mathcal{G})$. Thus, $\mathsf{NESet}(\mathcal{G}^\dagger) \equiv \mathsf{NESet}(\mathcal{G})$. □

Corollary 2.1. *If F has a maximum point in $\mathbb{S}$ then $\mathcal{G}$ has a pure-strategy Nash equilibrium.*

Clearly, every maximum point S^* for F has to satisfy (2.13) and thus coincides with a (pure-strategy) Nash equilibrium for $\mathcal{G}$. Note also that S^* can either be a local or a global optimum. The set of global maximizers for F therefore is a subset of $\mathsf{NESet}(\mathcal{G})$. However, one may only consider these global maxima more "desirable" in terms of social optimality if the potential function itself represents a meaningful measure of such optimality, such as the utilitarian welfare function (1.18). In Chaps. 3 and 4, we will discuss applications where the potential function in fact coincides with the utilitarian welfare function.

The next two theorems characterize Nash equilibrium existence for ordinal and pseudo-potential games, according to the properties of their strategy spaces and potential functions.

Theorem 2.3. *The following statements are true.*

- *Every finite (ordinal) potential game admits at least one pure-strategy Nash equilibrium.*
- *Every continuous (ordinal) potential game whose strategy space $\mathbb{S}$ is compact (i.e., closed and bounded) and potential function F is continuous admits at least one pure-strategy Nash equilibrium. Moreover, if F is strictly concave, the Nash equilibrium is unique.*

Proof. For finite games, $\mathbb{S}$ is bounded and a maximum of $F(S)$ always exists. Hence, a Nash equilibrium exists.

For continuous games with compact $\mathbb{S}$ and continuous F, the same argument holds. If F is also strictly concave, it has a unique, global maximum.

Note that all the results apply to all exact potential games equally well. Several engineering applications of potential games only need to invoke those results for exact potential games. □

For pseudo-potential games, similar, albeit weaker, results are also obtained.

Theorem 2.4 (Dubey and Schipper). *Consider the pseudo-potential game* $\mathscr{G} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i \in \mathscr{N}}]$ *with potential function F, and* $\mathscr{G}^{\dagger} = [\mathscr{N}, \mathbb{S}, \{F\}_{i \in \mathscr{N}}]$. *If F has a maximum, then* $\mathscr{G}$ *has a pure-strategy Nash equilibrium; and*

$$\mathsf{NESet}(\mathscr{G}^{\dagger}) \subseteq \mathsf{NESet}(\mathscr{G}). \tag{2.14}$$

Proof. Every $S^* \in \arg\max F(S)$ is, by definition, a Nash equilibrium of $\mathscr{G}^{\dagger}$. From the definition of pseudo-potential games in (2.10), we derive that $S^* \in \mathsf{NESet}(\mathscr{G})$. Therefore, $\mathsf{NESet}(\mathscr{G}^{\dagger}) \subseteq \mathsf{NESet}(\mathscr{G})$ and $\mathscr{G}$ has at least one (pure-strategy) Nash equilibrium as long as F has a maximum. □

As a direct consequence, we have:

Corollary 2.2. *Any pseudo-potential game which is either finite, or has a compact strategy space and a continuous potential function, possesses a pure-strategy Nash equilibrium.*

2.2.2 Nash Equilibrium Convergence

Previously, we have established the existence of at least one pure-strategy Nash equilibrium. This section looks at how the players can achieve a Nash equilibrium in potential games. The main idea is via sequential decision dynamics in which *players take turn to act in sequence or in a round-robin manner*. Each player in turn selects a new strategy based on a certain decision rule, thus creating a unilateral strategy deviation and inducing a corresponding change in the potential function. If the change represents an improvement in the value of the function, one expects a *series of improvement that drives the game toward one of its equilibria*.

We can formalize the aforementioned idea by introducing the concept of improvement path.

Definition 2.7. A sequence of strategy profile $\rho = (S^0, S^1, S^2, \ldots)$ such that for every index $k \geq 0$, S^{k+1} is obtained from S^k by allowing a player $i(k)$ (the single deviator in step k) to change his/her strategy, is called a *path*. A path ρ is an *improvement path* if in each step k, the deviator $i(k)$ experiences a gain in his/her utility, i.e., $U_{i(k)}(S^{k+1}) > U_{i(k)}(S^k)$. Moreover, ρ is called a *cycle* if ρ is of finite length and its terminal element S^K coincides with its initial element S^0.

An improvement path ρ is allowed to terminate if no further possible improvement can be obtained. Some paths might not terminate (i.e., are infinite or become a cycle). We are interested in *finite* improvement paths.

Theorem 2.5. *For any strategic game* $\mathscr{G}$, *if a finite improvement path exists, its end point corresponds to a Nash equilibrium.*

Proof. We prove by contradiction. Let ρ be a finite improvement path whose end point is S^K. If we assume that S^K is not a Nash equilibrium, then there exists a player

$i(K)$ who can deviate from his/her current strategy $S^K_{i(K)}$ to a new strategy $T_{i(K)}$ in order to improve his/her utility. We could now add $S^{K+1} = (T_{i(K)}, S^K_{-i(K)})$ to ρ to extend this path. This contradicts the initial assumption that ρ must terminate at S^K. □

The above result implies that any decision dynamic that can generate a *finite* improvement path will eventually end up at a Nash equilibrium. It has an immense consequence—*most practical applications of game theoretic formulations in general, and potential games in particular, apply this principle in finding a Nash equilibrium*. This result does not require a game to be a potential game; but a potential game will guarantee this, which we will show shortly.

One might now ask, what kinds of decision dynamics, among those we introduced in Sect. 1.3.2, will generate finite improvement paths for potential games?

Clearly, myopic best-response and better-response (random or deterministic) dynamics create improvement paths, by their definitions in (1.16) and (1.17). Thus, they are prime candidates for further investigation.

We will discuss the answer for different types of potential games subsequently.

2.2.2.1 Finite Ordinal Potential Games

We look at an important theorem by Monderer and Shapley [31].

Theorem 2.6 (Monderer and Shapley). *For finite ordinal potential games, every improvement path is finite. This is known as the* finite improvement path *property.*

Proof. It is obvious that every improvement path must terminate as the increment of the potential function is finite and bounded. □

Corollary 2.3. *For finite ordinal potential games, every sequence of better and best responses converges to a Nash equilibrium, regardless of its starting point.*

2.2.2.2 Continuous Ordinal Potential Games

In continuous context, absolute convergence may or may not be realized in a finite number of steps. A classic example is the convergent sequence $\{1 - \frac{1}{2^n}\}_{n=1,2,\ldots}$ which ultimately converges but goes on infinitely. This is due to the sequence's infinitesimal stepsizes as $n \to \infty$.

One however can control the stepsize by defining the concept of ϵ-improvement path.

Definition 2.8. A path $\rho = (S^0, S^1, S^2, \ldots)$ is an ϵ-improvement path if in each step k, the deviating player $i(k)$ experiences $U_{i(k)}(S^{k+1}) > U_{i(k)}(S^k) + \epsilon$, for some $\epsilon \in \mathbb{R}_+$.

This also facilitates the concept of ϵ-equilibrium, which is a strategy profile that is approximately close to an actual Nash equilibrium.

Definition 2.9. The strategy profile $\tilde{S} \in \mathbb{S}$ is an ϵ-equilibrium if and only if $\exists \epsilon \in \mathbb{R}_+$ such that, $\forall i \in \mathcal{N}$:

$$U_i(\tilde{S}_i, \tilde{S}_{-i}) \geq U_i(S_i, \tilde{S}_{-i}) - \epsilon, \quad \forall S_i \in \mathbf{S}_i. \tag{2.15}$$

The ϵ-equilibrium is a refinement of the original Nash equilibrium and is sometimes preferred as a solution concept especially in situations which require less computational complexity.

Theorem 2.7 (Monderer and Shapley). *For continuous ordinal potential games with bounded utility functions, every ϵ-improvement path is finite. This is known as the* approximate finite improvement path *property.*

Proof. For ordinal potential games whose utility functions are bounded, their potential functions must also be bounded. That is, $\exists L \in \mathbb{R}, L < \infty$ such that $L = \sup_{S \in \mathbb{S}} F(S)$.

Now suppose that $\rho = (S^0, S^1, \ldots, S^k, \ldots)$ is an ϵ-improvement path which is also infinite. By definition, $U_{i(k-1)}(S^k) - U_{i(k-1)}(S^{k-1}) > \epsilon, \forall k$. As the game is an ordinal potential game, there exists a sufficiently small constant ϵ' such that $F(S^k) - F(S^{k-1}) > \epsilon', \forall k$. This implies $F(S^k) - F(S^0) > k\epsilon'$ or

$$F(S^k) > F(S^0) + k\epsilon', \forall k. \tag{2.16}$$

Clearly, $\lim_{k \to \infty} F(S^k) = \infty$ which is a contradiction. □

Thus, any ϵ-improvement path ρ must terminate after a certain K steps, at which point $F(S^k) \leq L < F(S^k) + \epsilon'$ or $F(S^k) > L - \epsilon'$. This suggests that the end point of such a path is an ϵ-equilibrium, which we state in the following corollary.

Corollary 2.4. *For continuous ordinal potential games, every better-response sequence that is compatible with ϵ-improvement converges to an ϵ-equilibrium in a finite number of steps.*

Note that although traditional best-response and better-response dynamics still advance towards a Nash equilibrium in continuous games, whether they will terminate in a finite number of steps is not guaranteed. However, in case this happens, ϵ-improvement path can be used to approximate the solution.

2.2.2.3 Pseudo-Potential Games

For pseudo-potential games, the following results hold.

Theorem 2.8 (Dubey and Schipper). *For finite pseudo-potential games, sequential best-response dynamics converges to a Nash equilibrium in a finite number of steps.*

Proof. Convergence in finite games is established from Proposition 2 of [43]. We omit the details. □

In summary, convergence to a Nash equilibrium is guaranteed in all finite ordinal potential games by using best-response and better-response dynamics. For continuous ordinal potential games, these dynamics are able to converge to an ϵ-Nash equilibrium. On the other hand, convergence results in pseudo-potential games are only guaranteed for best-response dynamics, as the definition of pseudo-potential games is strongly tied to best responses.

2.3 Identification of Potential Games

In this section and Sect. 2.4 that follows, we present our studies to address the challenges when applying potential games to practical problems. We have seen that being able to know if a game is a potential game is important as it guarantees that at least one equilibrium solution exists. This section specifically provides an answer to this crucial question of *how to identify that a game is a potential game*. We hope to develop a set of rules which allow us to achieve this purpose. Specifically, we look into characterizing the necessary and sufficient conditions in the strategy space and utility functions of players, for a game to be a potential game. We will first present the results for ordinal and pseudo-potential games in Sect. 2.3.1. Exact potential games, both continuous and finite, will be tackled in Sect. 2.3.2.

2.3.1 Ordinal and Pseudo- Potential Game Identification

2.3.1.1 Ordinal Potential Game Identification

For ordinal potential games, Voorneveld et al. [50] derived two necessary and sufficient conditions which will be stated in Theorem 2.9 (Theorem 3.1 of [50]). Before introducing the theorem, a few concepts need to be defined. Recall the definitions of paths and cycles previously (Definition 2.7).

Definition 2.10. A path $\rho = (S^0, S^1, S^2, \ldots, S^K)$ is *non-deteriorating* if $U_{i(k)}(S^k) \leq U_{i(k)}(S^{k+1})$, $\forall k$ where $i(k)$ is the deviating player in step k. For two arbitrary strategy profiles S and T, we write $S \to T$ if there exists a non-deteriorating path from S to T.

Definition 2.11. The cycle $\rho = (S^0, S^1, S^2, \ldots, S^K = S^0)$ is called a *weak improvement cycle* if it is non-deteriorating; and $U_{i(k)}(S^k) < U_{i(k)}(S^{k+1})$ at some k.

We now define a binary relation $\approx$ on $\mathbb{S}$ based on the $\to$ relation, such that for $S, T \in \mathbb{S}$, $S \approx T$ if both $S \to T$ and $T \to S$ hold. Then, this binary relation $\approx$ can easily be verified to be an equivalence relation.[2] The equivalent class of an element $S \in \mathbb{S}$ is now defined as the set $[S]$ where $[S] = \{T \in \mathbb{S} \,|S \approx T\}$. Subsequently, one can define the set of equivalence classes on $\mathbb{S}$ induced by $\approx$, denoted by

$$\mathbb{S}_{\approx} = \{[S], \forall S \in \mathbb{S}\}\,. \tag{2.17}$$

On $\mathbb{S}_{\approx}$, we then define another preference relation $\prec$ such that $[S] \prec [T]$ if $[S] \not\equiv [T]$ and $S \to T$. Moreover, $\prec$ can be shown to be irreflexive and transitive.

Finally, we define the proper order as follows.

Definition 2.12. The tuple $(\mathbb{S}_{\approx}, \prec)$ is said to be *properly ordered* if there exists a function $F : \mathbb{S}_{\approx} \mapsto \mathbb{R}$ such that $\forall S, T \in \mathbb{S}$, $[S] \prec [T] \Rightarrow F([S]) < F([T])$.

For a better understanding of various concepts here, we refer the readers to some textbooks in abstract algebra, such as [14].

The following theorem provides characterization of ordinal potential games. We omit the proof due to its technicalities which are out of the scope of our book.

Theorem 2.9 (Voorneveld). *The game $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i\in\mathcal{N}}]$ is an ordinal potential game if and only if the two following conditions hold:*

(1) $\mathbb{S}$ has no weak improvement cycle.
(2) $(\mathbb{S}_{\approx}, \prec)$ is properly ordered.

Corollary 2.5 (Voorneveld). *If $\mathbb{S}$ is finite (or countably infinite), condition (2) in Theorem 2.9 can be omitted.*

Remark 2.1. For finite/countable games, a crude method of identifying ordinal potential games is to *exhaustively check for lack of weak improvement cycles*.

The following example involves verifying whether a finite game is an ordinal potential game using this method.

Example 2.7. Consider the 3-player game in Fig. 2.6.

In this game, strategy space is finite and can be represented by a *graph*. Vertices of the graph correspond to $S1, S2, \ldots, S8$. An edge connects two vertices if a single

3A

	2A	2B
1A	(S1) 1,0,1	(S2) 1,1,1
1B	(S4) 6,6,4	(S3) 4,4,6

3B

	2A	2B
1A	(S5) 6,6,6	(S6) 6,4,6
1B	(S8) 4,6,6	(S7) 4,4,4

Fig. 2.6 A 3-player strategic game. Strategy profiles are labeled $S1$–$S8$

[2] In mathematics, a binary relation between two members of a set is an equivalence relation if and only if it is reflexive, symmetric and transitive [14].

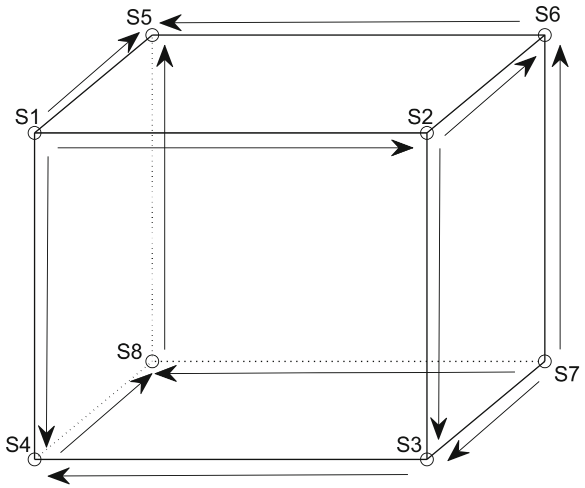

Fig. 2.7 Graphical representation of the game in Fig. 2.6

player's deviation causes a switch between the two corresponding profiles. The full graph is depicted in Fig. 2.7.

Moreover, we use an *arrow* superimposed on an edge to indicate that a non-deteriorating path exists from one vertex to another. Any resulting *directed cycle* is a weak improvement cycle. Clearly, by searching within this graph, we find no such cycle. Therefore, this is an ordinal potential game according to Corollary 2.5.

2.3.1.2 Pseudo-Potential Game Identification

For pseudo-potential games, similar results were provided by Schipper [43]. Note that concurrently, [49] also presented the characterization for best-response potential games but these results are not repeated here.

Analogous to ordinal potential games, the conditions involve a lack of *strict best-response cycles*, and *proper order on* $(\mathbb{S}_{\approx}, \prec)$. Note that the requirement of path improvement is now restricted to best-response moves only.

Definition 2.13. A path $\rho = (S^0, S^1, S^2, \ldots, S^K)$ is *strict best-response compatible* (SBRC) if $\forall k = 0, \ldots, K$:

$$S_{i(k)}^{k+1} = \begin{cases} S_{i(k)}^{k} & \text{if } S_{i(k)}^{k} \in \mathscr{B}_{i(k)}(S_{-i(k)}^{k}) \\ \hat{S}_{i(k)} \in \mathscr{B}_{i(k)}(S_{-i(k)}^{k}) & \text{otherwise,} \end{cases} \tag{2.18}$$

where $i(k)$ is the deviating player in step k. That is, the deviator either deviates to his/her best-response strategy, or stays at the current strategy if it is already a best response. For two arbitrary profiles S and T, we write $S \rhd T$ if there exists a SBRC path from S to T.

Definition 2.14. The cycle $\rho = (S^0, S^1, S^2, \ldots, S^K = S^0)$ is called a *strict best-response cycle* if it is SBRC; and $U_{i(k)}(S^k) < U_{i(k)}(S^{k+1})$ for at some k.

Similarly, the binary equivalence relation $S \approx T$ is then defined on $\mathbb{S}$ if $S \triangleright T$ and $T \triangleright S$. The corresponding set of equivalence classes is denoted by $\mathbb{S}_{\approx}$ and the preference relation $\prec$ is similarly defined on $\mathbb{S}_{\approx}$.

Main results are stated in the next theorem and corollary (Theorems 1–4 from [43]). Once again we omit the proofs.

Theorem 2.10 (Schipper). *Consider a game* $\mathscr{G} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i \in \mathscr{N}}]$ *and the following two conditions:*

(1) $\mathbb{S}$ *has no strict best-response cycle,*
(2) $(\mathbb{S}_{\approx}, \prec)$ *is properly ordered.*

Then,

- ***Sufficiency:*** $\mathscr{G}$ *is a pseudo-potential game if (1) and (2) hold.*
- ***Necessity:*** *If* $\mathscr{G}$ *is a pseudo-potential game with potential function F, then (1) and (2) hold for the game* $\mathscr{G}^{\dagger} = [\mathscr{N}, \mathbb{S}, \{F\}_{i \in \mathscr{N}}]$.

Corollary 2.6 (Schipper). *If either* $\mathbb{S}$ *is countable or each* $\mathbf{S}_i \subseteq \mathbb{R}$, *Theorem 2.10 is valid without considering condition (2).*

Finally, a method of knowing if a finite game is *not* a pseudo-potential game is by *exhaustively checking for existence of strict best-response cycles*. This is a weaker result than the case of ordinal potential games.

The following example shows a game which is not a pseudo-potential game.

Example 2.8. Consider the 2-player game in Fig. 2.8.

It is easy to see that $(S1, S2, S5, S4, S1)$ is a strict best-response cycle. Thus, this is not a pseudo-potential game (as well as ordinal potential game).

2.3.1.3 Construction of Potential Functions

For finite ordinal and pseudo-potential games, we observe that there is a simple method to construct the ordinal and pseudo-potential functions. The method was first proposed by Schipper [43] for finite pseudo-potential games. We find that it can be extended to finite ordinal potential games. The following discussion for ordinal potential games may be an unreported result as the authors are not aware of any existing relevant articles in the literature.

Fig. 2.8 A 2-player 3 × 3 strategic game

		Player 2	
	$2A$	$2B$	$2C$
Player 1 $1A$	$(S1)$ 3,6	$(S2)$ 0,10	$(S3)$ 1,7
Player 1 $1B$	$(S4)$ 1,4	$(S5)$ 2,0	$(S6)$ 5,2
Player 1 $1C$	$(S7)$ 2,2	$(S8)$ 1,5	$(S9)$ 3,3

The idea is to define a potential function by assigning a strength ranking to each strategy profile S in $\mathbb{S}$. This rank is measured by *counting the number of other strategy profiles* $S' \neq S$ *such that from* S' *there exists a path leading to* S. We require this path to be a non-deteriorating path for ordinal potential games.

Recall the notation $S' \rightarrow S$ when there is a non-deteriorating path from S' to S. We denote $r : \mathbb{S} \mapsto \mathbb{N}$ the rank function of a strategy profile $S \in \mathbb{S}$ (where $\mathbb{N}$ is the set of natural numbers). The rank function is thereby defined as follows

$$r(S) = \sum_{S' \in \mathbb{S}} \mathbf{1}(S' \rightarrow S) \tag{2.19}$$

where $\mathbf{1}(S' \rightarrow S)$ is the indicator function for the relation $\rightarrow$, i.e.,

$$\mathbf{1}(S' \rightarrow S) = \begin{cases} 1 & \text{if } S' \rightarrow S, \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

Theorem 2.11. *If the game* $\mathscr{G}$ *is a finite ordinal potential game, then the rank function* $r(S)$ *defines a potential function for* $\mathscr{G}$.

Proof. We prove by contradiction.

Assume that $\mathscr{G}$ is an ordinal potential game. Along any $S' \rightarrow S$ path, the stepwise change in the utility function of any deviating player must be non-negative. Hence, if $F()$ is a potential function it must satisfy $F(S') \leq F(S)$. Now supposing that $r(S)$ is not a potential function, then there exists a $S' \rightarrow S$ path such that $r(S') > r(S)$. By definition of $r()$, S' has more strategy profiles that can be led to it via a non-deteriorating path than S does. This means,

$$\exists S'' \in \mathbb{S} : \; S'' \rightarrow S', \text{ but } S'' \not\rightarrow S. \tag{2.21}$$

However, since $\rightarrow$ is transitive, $S'' \rightarrow S'$ and $S' \rightarrow S$ imply $S'' \rightarrow S$. This contradicts (2.21) which requires $S'' \not\rightarrow S$. Hence, $r(S)$ must be a potential function. □

An intuitive interpretation of the above theorem is that, given any finite ordinal potential game, the number of strategy profiles that have a non-deteriorating path leading to S can be set as the potential value of S. We further observe that if $S \rightarrow S'$ and $S' \rightarrow S$, or S and S' belong to the same equivalence class $[S]$, i.e., $[S] = [S']$, then $r(S) = r(S')$. In addition, if $[S] \prec [S']$ then $r(S) < r(S')$. Thus, the function $r()$ ranks all equivalence classes of $\mathscr{G}$ in accordance with the relation $\prec$. This is essentially an implication of the *proper order* on $(\mathbb{S}_{\approx}, \prec)$ as stated in Theorem 2.9.

Our next theorem shows that there can be infinitely many ways of constructing potential functions, as long as all the strategy profiles are assigned with numbers which preserve the order as that given by $r()$.

Theorem 2.12. *Let* $\mathscr{G}$ *be a finite ordinal potential game and assume that the rank function* $r()$ *assigns values* $\{0, 1, \ldots, M\}$ *to its strategy profiles. Also, let*

$(x_0, x_1, \ldots, x_M)$ *be an arbitrary* $(M+1)$*-tuple of real numbers such that* $x_0 < x_1 < \ldots < x_M$. *Then* $\mathscr{G}$ *admits any function* F *as its potential function where*

$$F(S) = x_i \quad \text{if} \quad r(S) = i, \ \forall i = 0, 1, \ldots, M. \tag{2.22}$$

Proof. The proof is left as an exercise. □

Example 2.9. Consider the ordinal potential game in Example 2.7. For this game, the possible non-deteriorating paths can be deduced from Fig. 2.7. A potential function therefore can be computed from the rank function as follows.

Profile	Non-deteriorating paths from	Rank $r(S)$
$S1$	None	0
$S2$	$S1$	1
$S3$	$S1, S2, S7$	3
$S4$	$S1, S2, S3, S7$	4
$S5$	$S1, S2, S3, S4, S5, S6, S7$	7
$S6$	$S1, S2, S7$	3
$S7$	None	0
$S8$	$S1, S2, S3, S4, S7$	5

One can verify that this function satisfy the ordinal potential function definition.

Exercise 2.1. Use the rank function to compute a potential function for the ordinal potential game in Example 2.3.

For pseudo-potential games, by replacing the requirement of non-deteriorating paths by strict best-response compatible (SBRC) paths, we can have similar results. Recall the notation $S' \triangleright S$ if there is a SBRC path from S' to S. The indicator function $\mathbf{1}(S' \triangleright S)$ is also defined similarly.

Theorem 2.13 (Schipper). *Let* $\mathscr{G}$ *be a finite pseudo-potential game. Then,* $\mathscr{G}$ *admits the following rank function* $r : \mathbb{S} \mapsto \mathbb{N}$ *as its potential function:*

$$r(S) = \sum_{S' \in \mathbb{S}} \mathbf{1}(S' \triangleright S). \tag{2.23}$$

Proof. Similar to the proof of Theorem 2.11. □

Analogous to ordinal potential games, more generalized constructions can be realized.

Corollary 2.7. *Let* $\mathscr{G}$ *be a finite pseudo-potential game and the rank function* $r()$ *assigns values* $\{0, 1, \ldots, M\}$ *to its strategy profiles. Also, let* $(x_0, x_1, \ldots, x_M)$ *be an arbitrary* $(M+1)$*-tuple of real numbers such that* $x_0 < x_1 < \ldots < x_M$. *Then* $\mathscr{G}$ *admits any function* F *as its potential function where*

$$F(S) = x_i \quad \textit{if} \quad r(S) = i, \ \forall i = 0, 1, \ldots, M. \tag{2.24}$$

Exercise 2.2. Use the rank function in (2.24) to compute a potential function for the pseudo-potential game in Example 2.6.

2.3.2 Exact Potential Game Identification

2.3.2.1 Continuous Exact Potential Game Identification

This section looks into the issue of identifying continuous exact potential games. In our discussion, we will assume that each strategy set of the game $\mathscr{G}$ is a continuous interval of real numbers, i.e., $\mathbf{S}_i \subseteq \mathbb{R}$; and that the utility function $U_i : \mathbf{S}_i \mapsto \mathbb{R}$ is continuous and differentiable everywhere on $\mathbf{S}_i$. In accordance with previous notations, we denote $F : \mathbb{S} \mapsto \mathbb{R}$ the possible potential function.

The condition for continuous potential games is relatively straightforward.

Theorem 2.14 (Monderer and Shapley). *The game $\mathscr{G}$ is a continuous exact potential game with potential function F if and only if*

$$\frac{\partial^2 U_i}{\partial S_i \partial S_j} = \frac{\partial^2 U_j}{\partial S_i \partial S_j}, \quad \forall i, j \in \mathscr{N}. \tag{2.25}$$

Proof. Equation (2.25) follows directly from our alternative definition (2.2). □

A benefit of the condition (2.25) is that it allows us to identify a continuous exact potential game without knowing its potential function.

The next question is how to find the potential function, assuming that the game is known to be a potential game. Fortunately, Monderer and Shapley [31] also provided us with a useful formula. Their result is restated as follows.

Theorem 2.15 (Monderer and Shapley). *If $\mathscr{G}$ is a continuous exact potential game then its potential function F satisfies*

$$F(S) - F(T) = \sum_{i \in \mathscr{N}} \int_0^1 \left(\gamma'(z) \frac{\partial U_i}{\partial S_i}(\gamma(z)) \right) \mathrm{d}z \tag{2.26}$$

where $\gamma(z) : [0, 1] \mapsto \mathbb{S}$ is a continuously differentiable path in $\mathbb{S}$ that connects two strategy profiles S and T; such that $\gamma(0) = T$ and $\gamma(1) = S$.

Proof. The proof comes directly from the gradient theorem in vector calculus [51]. For any smooth curve C from T to S in $\mathbb{S} \subseteq \mathbb{R}^{|\mathscr{N}|}$ and any function F whose gradient vector ∇F is continuous on $\mathbb{S}$, the gradient theorem allows us to evaluate the line integral along $\gamma(z)$ as

$$F(S) - F(T) = \int_{C[T \to S]} \nabla F(\mathbf{s}) \cdot d\mathbf{s}, \tag{2.27}$$

where $\mathbf{s}$ is a vector variable representing points along C.

After introducing $\mathbf{s} = \gamma(z)$ such that $\mathbf{s} = T$ when $z = 0$ and $\mathbf{s} = S$ when $z = 1$, by chain rule, $d\mathbf{s} = \gamma'(z)dz$ and therefore

$$\begin{aligned} F(S) - F(T) &= \int_0^1 \big(\gamma'(z) \cdot \nabla F(\gamma(z))\big)\, dz \\ &= \sum_{i=1}^{|\mathcal{N}|} \int_0^1 \left(\gamma_i'(z) \frac{\partial F_i}{\partial S_i}(\gamma(z))\right) dz \end{aligned} \tag{2.28}$$

Then, since F is a potential function, $\frac{\partial F_i}{\partial S_i} = \frac{\partial U_i}{\partial S_i}$ and (2.26) follows. □

In conclusion, for a given continuous game, we can theoretically verify its exact potential property and evaluate its potential function. However, (2.26) is often too general and tedious to evaluate. In most practical applications of potential games, derivation of potential functions may be obtained through much simpler procedures. For example, Sect. 2.4 discusses potential games with utility functions having certain properties where the potential functions can be automatically derived.

We give an example of finding the potential function using (2.26). This example is introduced in [31].

Example 2.10 (A Cournot Competition). In a Cournot competition, players are the N firms, which compete in the market for a certain product.

Player i's strategy is to produce q_i products. The strategy space is $\mathbb{S} = \mathbb{R}^N_{\geq 0}$. If player i produces q_i products, it bears a cost $c_i(q_i)$. We can assume the function c_i is differentiable and $c_i(0) = 0$.

All generated products are sold at a common price p, determined by the total supplies $Q = \sum_{i \in \mathcal{N}} q_i$ via an inverse demand function $p = f(Q)$. We assume a linear inverse demand $p = f(Q) = a - bQ$, where $a, b > 0$.

Each player's utility function equals its profit given by

$$U_i = pq_i - c_i(q_i) = \left(a - b\sum_{j=1}^{N} q_j\right) q_i - c_i(q_i). \tag{2.29}$$

We can check the utility function against (2.25). Our game is a continuous exact potential game as

$$\frac{\partial^2 U_i}{\partial q_i \partial q_j} = \frac{\partial^2 U_j}{\partial q_i \partial q_j} = -b. \tag{2.30}$$

To find the potential function using (2.26), we select $S = (q_1, q_2, \ldots, q_N)$ and $T = \mathbf{0}$, the origin. Take the path γ to be the straight line from T to S.

Then, at point $\gamma(z)$ along the path, its projection on the ith-axis is zq_i; and its gradient is $(q_1, q_2, \ldots, q_N)$.

We assume further that $F(\mathbf{0}) = 0$. Then (2.26) becomes

$$F(S) = \sum_{i=1}^{N} \left(\int_0^1 q_i \frac{\partial U_i(zq_i, zq_{-i})}{\partial q_i} \mathrm{d}z \right). \tag{2.31}$$

Here,

$$\frac{\partial U_i}{\partial q_i} = a - b \sum_{j \neq i} q_j - 2bq_i - c_i'(q_i) \tag{2.32}$$

so

$$\begin{aligned} \int_0^1 q_i \frac{\partial U_i(zq_i, zq_{-i})}{\partial q_i} \mathrm{d}z &= q_i \int_0^1 \left[a - b \sum_{j \neq i} zq_j - 2bzq_i - c_i'(zq_i) \right] \mathrm{d}z \\ &= q_i \left[a - \frac{1}{2} b \sum_{j \neq i} q_j - bq_i - \frac{1}{q_i} (c_i(q_i) - c_i(0)) \right] \\ &= aq_i - \frac{1}{2} b \sum_{j \neq i} q_i q_j - bq_i^2 - c_i(q_i). \end{aligned} \tag{2.33}$$

Finally, from (2.31) and (2.33),

$$F(S) = a \sum_{i=1}^{N} q_i - \frac{1}{2} b \sum_{i=1}^{N} \sum_{j \neq i} q_i q_j - b \sum_{i=1}^{N} q_i^2 - \sum_{i=1}^{N} c_i(q_i). \tag{2.34}$$

This is the potential function we obtain, which is identical to the one given in [31].

From the example above, one sees that the potential function depends on the assigned value of $F(T)$. The next result addresses its uniqueness.

Theorem 2.16 (Monderer and Shapley). *If F_1 and F_2 are two possible potential functions of an exact potential game $\mathscr{G}$ (both continuous or finite) then they differ only by a constant c, i.e.,*

$$F_1(S) - F_2(S) = c \quad \forall S \in \mathbb{S}. \tag{2.35}$$

Proof. See [31] (Lemma 2.7). □

Thus, unlike ordinal and pseudo-potential games, the potential functions in exact potential games are unique up to addition of a constant.

2.3.2.2 Finite Exact Potential Game Identification

Previously, conditions for continuous games are established due to the smoothness property of the strategy space. Similarly, for finite games, we will also examine their strategy spaces. However, the property we look at now involves cycles, i.e., paths that start and end at the same point. Recall that identifying finite ordinal and pseudo-potential games involves checking for lack of cycles having weak improvement or strict best-response properties, respectively. For exact potential games, interestingly, it happens that the total change in the utility functions along every cycle is 0.

Theorem 2.17 (Monderer and Shapley). *Let $\rho = (S^0, S^1, S^2, \ldots, S^K = S^0)$ be an arbitrary finite cycle in a finite game $\mathscr{G}$. Denote $i(k)$ the deviating player from S^k to S^{k+1}. Then, $\mathscr{G}$ is an exact potential game if and only if*

$$\sum_{k=0}^{K-1} \left[U_{i(k)}(S^{k+1}) - U_{i(k)}(S^k) \right] = 0. \tag{2.36}$$

Proof. See Appendix A of [31]. □

Naturally, the above necessary and sufficient condition leads to the method of *exhaustively checking all finite cycles* to verify if a given game is also an exact potential game. The next corollary makes the task less tedious by restricting our search to *all cycles of length 4* only.

Corollary 2.8 (Monderer and Shapley). *Suppose that ρ is a cycle of length 4 in a game $\mathscr{G}$,[3] as described below:*

$$\rho = \begin{array}{ccc} A & \longleftarrow & D \\ \downarrow & & \uparrow \\ B & \longrightarrow & C \end{array} \tag{2.37}$$

where $A = (S_i, S_j, S_{-\{i,j\}})$, $B = (T_i, S_j, S_{-\{i,j\}})$, $C = (T_i, T_j, S_{-\{i,j\}})$, and $D = (S_i, T_j, S_{-\{i,j\}})$ are the 4 strategy profiles forming the cycle, due to two deviating players i and j.

Then, $\mathscr{G}$ is an exact potential game if and only if, $\forall i, j \in \mathscr{N}$:

$$\underbrace{[U_i(B) - U_i(A)]}_{i\ from\ A\ to\ B} + \underbrace{[U_j(C) - U_j(B)]}_{j\ from\ B\ to\ C} + \underbrace{[U_i(D) - U_i(C)]}_{i\ from\ D\ to\ C} + \underbrace{[U_j(A) - U_j(D)]}_{j\ from\ D\ to\ A} = 0,$$

$$\forall S_i, T_i \in \mathbf{S}_i; \forall S_j, T_j \in \mathbf{S}_j \forall S_{-\{i,j\}}. \tag{2.38}$$

[3] Although this section deals with finite games, this corollary is valid also for continuous games.

Although this result has reduced the amount of computation significantly, the verification process is still very time consuming. For example, if $\mathcal{G}$ has N players and each player has $|\mathbf{S}_i| = M$ strategies, then the number of times (2.38) needs to be checked is

$$\binom{N}{2}\left[\binom{M}{2}\right]^2 = \frac{N(N-1)}{2}\left[\frac{M(M-1)}{2}\right]^2. \tag{2.39}$$

The complexity for this checking method is thus $O(N^2M^4)$. Recently, there have been efforts to simplify the procedure and reduce this computation complexity.

Remark 2.2. Hino [17] observed that instead of checking all 2×2 combinations of strategy profiles, we only need to check for those cycles consisting of adjacent rows and columns when the game is represented in payoff matrix. If there are N players and each has M strategies, then the number of equations to be checked is reduced to

$$\binom{N-1}{1}\left[\binom{M-1}{1}\right]^2 = (N-1)(M-1)^2. \tag{2.40}$$

This proposed method will reduce the checking time complexity from $O(N^2M^4)$ to $O(NM^2)$.

Remark 2.3. In a recent paper, Cheng [10] used the technique of semi-tensor product of matrices to verify a potential game by obtaining the *potential equation*. The game $\mathcal{G}$ is an exact potential game if and only if the potential equation has a solution. The potential function can be constructed from the solution of potential equation. Refer to [10] for more details.

We conclude this discussion with the construction of potential function for finite exact potential games. Two previous results come in handy. First, Theorem 2.17 shows that from one point S, any path that comes back to S gives zero change in the sum of utilities as well as potential function. In other words, any two possible paths from S to T yield *exactly the same sum*. Our second result is Theorem 2.16 allowing us to obtain a unique potential function up to addition of a constant. Thus, we present an algorithm which calculates the potential function for every strategy profile of a finite exact potential game $\mathcal{G}$, given a starting strategy profile S and an arbitrary initial potential value α assigned to S. The idea is to walk through all strategy profiles in $\mathcal{G}$ starting from S; and at each point we accumulate to α the sum of utility changes and assign this value to the potential function at the current point. Algorithm 2.1 gives the procedures.

In Algorithm 2.1, walking through all points in strategy space $\mathbb{S}$ starting from S is performed using the procedure traverse($\mathcal{G}$,S). Meanwhile, traverse($\mathcal{G}$,S) itself returns a Boolean value, which is true if there are unvisited nodes, and false otherwise. Assuming $\mathbb{S}$ can be represented by a graph structure with vertices corresponding to strategy profiles and edges corresponding to possible strategy deviation by a single player, traverse($\mathcal{G}$,S) might make use of classic graph traversal

Algorithm 2.1 Exact potential function computation for finite games.

Require: Finite exact potential game $\mathscr{G}$, initial profile S, constant α

$F(S) \leftarrow \alpha$
while traverse($\mathscr{G}$,S) **do** ▷ Visiting all profiles in $\mathscr{G}$ starting from S
  $i \leftarrow$ deviating player
  $A \leftarrow$ previous strategy profile
  $B \leftarrow$ new strategy profile
  $F(B) \leftarrow F(A) + U_i(B) - U_i(A)$
end while
return F

Fig. 2.9 A generalized prisoner's dilemma game where $a > b > c > d$

		Player 2	
		C	D
Player 1	C	$(M)\, b, b$	$(N)\, d, a$
	D	$(Q)\, a, d$	$(P)\, c, c$

algorithms such as the *breadth-first search* or *depth-first search* algorithms [19] to exhaustively visit all nodes in an optimal manner. At each unvisited node, say B, it will identify its predecessor A and the deviating player i in the current move. Subsequently, it calculates the potential function for the new node by adding the utility change of that player to the predecessor's potential value. When traverse($\mathscr{G}$,S) ends, all nodes have been visited and the final potential function F is returned.

We shall go through a simple example where Algorithm 2.1 is used to compute the potential function for the Prisoner's dilemma.

Example 2.11. Consider the Prisoners' dilemma with a generalized payoff matrix in Fig. 2.9. Its graphical representation has a simple structure of 4 vertices (M, N, P, Q) and 4 edges (MN, NP, PQ, QM).

We assume that our algorithm starts at N and an initial value $\alpha = 1$ is used. We assume the particular graph traversal order $N \to M \to Q \to P$ (note that the order is not important). The computation is as follows.

Profile	Path	Deviating player	Potential
N	None	None	1
M	$N \to M$	2	$1 + b - a$
Q	$M \to Q$	1	$1 + b - a + a - b = 1$
P	$Q \to P$	2	$1 + c - d$

Exercise 2.3. Use Algorithm 2.1 to compute a potential function for Example 2.1.

2.4 Formulation of Exact Potential Games

In the previous section, we have tackled the question of how to identify whether a game is a potential game. We specified the conditions for a given game to be a certain type of potential games and how to compute the potential function. The reversed question, which is the second objective of this chapter, is *how to construct potential games*. This is of equal importance, because if one knows the techniques to formulate potential games, including how to define the utility functions of players and the potential functions, we believe that potential games can become a more effective problem-solving tool and can find a much wider range of applications.

For a given engineering problem, especially in the field of wireless communications, it may sometimes be more desirable to formulate this problem as a potential game in order to have solutions which are stable and achievable. However, how to formulate such a game is still a great challenge. In most existing problems, potential games are often established by introducing their associated potential functions and verifying them with the definition. Moreover, generally known models are limited to a few known forms of potential functions as well as players' utility functions. The unanswered questions are, is there an effective method to design utility functions and potential functions so that one is able to construct potential games? Are there any special properties or structures in these functions that can help us identify potential games? Some formulation principles may already exist or have implicitly been used in the literature, but there is still a lack of systematic studies to formalize the underlying rules. Thus, our studies aim to uncover them and hopefully inspire researchers to fully exploit the "potential" in using the potential game technique. In this section, our focus will therefore shift towards the *design* aspect—the generalized method which are useful in constructing new potential games. Our discussion focuses especially on *exact potential games* which have gained the most attention and practical applications.

We start our investigation by firstly identifying useful properties of the utility functions of players based on which the resulting games are potential games. A number of such properties are presented in Sect. 2.4.1. Subsequently, we propose two formal approaches of formulating potential games in Sect. 2.4.2 which are the forward and the backward methods. The forward method defines the utility functions of players such that they satisfy aforementioned properties, and how a potential function can be obtained from these utility functions efficiently. In the backward method, the potential function for the game is first defined and we will use it to derive the utility functions of players.

2.4.1 Utility Function Considerations

A few properties are identified to be useful in constructing the utility functions of games which turn out to be a potential game. They are *separability*, *symmetry of observations* as well as *linear combinations among utility functions*. Additionally, we discuss the impact of imposing *constraints* on a given potential game.

2.4.1.1 Linear Combination of Utility Functions

Our first observation is that we can derive new exact potential games from existing ones by using a new utility function which is a linear combination of existing utility functions. This property stems from a much more powerful result which states that the set of all potential games with the same player set and strategy space forms a *linear space*. Fachini et al. [15] mentioned this property and a proof was available in [32]. The following result holds.

Theorem 2.18. *Let* $\mathscr{G}_1 = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i\in\mathscr{N}}]$ *and* $\mathscr{G}_2 = [\mathscr{N}, \mathbb{S}, \{V_i\}_{i\in\mathscr{N}}]$ *be two exact potential games. Then* $\mathscr{G}_3 = [\mathscr{N}, \mathbb{S}, \{\alpha U_i + \beta V_i\}_{i\in\mathscr{N}}]$ *is also an exact potential game,* $\forall \alpha, \beta \in \mathbb{R}$.

Proof. Suppose that $F(S)$ and $G(S)$ are potential functions of $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively. It is straightforward to show that $\alpha F(S) + \beta G(S)$ is a potential function for $\mathscr{G}_3$. □

This property is useful when we would like to jointly maximize two objectives via a weighted sum of the two. Knowing that using each objective separately as the utility function results in an exact potential game, we are guaranteed that the combined objective also leads to another exact potential game.

Remark 2.4. Only exact potential games and weighted potential games have the linear combination property [15]. For ordinal potential games, counter-examples were given in [32].

2.4.1.2 Separability

Next, we observe that if every player's utility function is separable into multiple terms with certain structures, the game can be shown to be an exact potential game.

Strategic Separability

The first notion of separability is what we term *strategic separability*, meaning that one's utility function can be decomposed into the summation of a term contributed solely by one's own strategy, and another term contributed solely by the opponents' joint strategy.

Definition 2.15. The game $\mathscr{G}$ is strategically separable if $\forall i$, $\exists P_i : \mathbf{S}_i \mapsto \mathbb{R}$ and $\exists Q_i : \mathbf{S}_{-i} \mapsto \mathbb{R}$ such that

$$U_i(S_i, S_{-i}) = P_i(S_i) + Q_i(S_{-i}). \tag{2.41}$$

Theorem 2.19. *If $\mathcal{G}$ is strategically separable, then it is also an exact potential game with the following potential function*

$$F(S) = \sum_{i\in\mathcal{N}} P_i(S_i). \tag{2.42}$$

Proof. For any unilateral strategy deviation of an arbitrary player i from S_i to T_i, we have

$$\begin{aligned} U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) &= P_i(T_i) + Q_i(S_{-i}) - P_i(S_i) - Q_i(S_{-i}) \\ &= P_i(T_i) - P_i(S_i). \end{aligned} \tag{2.43}$$

At the same time,

$$\begin{aligned} F_i(T_i, S_{-i}) - F_i(S_i, S_{-i}) &= P_i(T_i) + \sum_{j\neq i} P_j(S_j) - P_i(S_i) - \sum_{j\neq i} P_j(S_j) \\ &= P_i(T_i) - P_i(S_i). \end{aligned} \tag{2.44}$$

Hence, $F(S)$ is a potential function for $\mathcal{G}$. □

The following example from wireless communications shows a formulation which, by making some approximations, exhibits strategic separability and thus is a potential game.

Example 2.12 (Potentialized Game for CDMA Power Control). In code-division multiple access (CDMA) networks, all mobile stations (MSs) share the wireless medium and the transmission power of any MS can cause interference to the other MSs. The CDMA power control game studies the power adaptation for distributed, selfish MSs under such conflict of interest.

The players are N MSs, each of which transmits to a pre-assigned base station (BS). Note that some BSs can be shared by multiple players. Player i's strategy is its power level p_i which is bounded in a certain range, e.g., $[0, P_{\max}]$. We use (p_i, p_{-i}) to refer to strategies in this game.

The signal-to-noise-and-interference ratio (SINR) γ_i of player i is calculated as

$$\gamma_i(p_i, p_{-i}) = \frac{p_i g_{ii}}{\sum_{j=1, j\neq i}^{N} p_j g_{ji} + \sigma^2} \tag{2.45}$$

where g_{ji} is the channel gain between player j's transmitter and player i's BS; and σ^2 is the noise power.

The achievable rate for player i is given by

$$R_i(p_i, p_{-i}) = \log_2(1 + G\gamma_i) \tag{2.46}$$

where G is the spreading gain of the CDMA system [7].

A feasible optimization objective often incorporates rate R_i as the reward obtained by player i, as well as a cost for spending power. Linear pricing is often used for CDMA systems [42] where the cost is expressed as $c_i p_i$ for player i. The positive constant c_i indicates the price per unit power used. Thus, the following utility function can be considered:

$$U_i(p_i, p_{-i}) = R_i(p_i, p_{-i}) - c_i p_i, \forall i \in \mathcal{N}. \tag{2.47}$$

The resulting game is $\mathcal{G} = \left[\mathcal{N}, [0, P_{\max}]^N, \{U_i\}_{i \in \mathcal{N}}\right]$. Candogan et al. [7] proposed the following approximation to the utility function, so that a "potentialized" game is obtained.

The approximated utility function at high SINR is proposed to be

$$\tilde{U}_i(p_i, p_{-i}) = \log_2 \left(G \frac{p_i g_{ii}}{\sum_{j=1, j \neq i}^{N} p_j g_{ji} + \sigma^2} \right) - c_i p_i, \forall i \in \mathcal{N} \tag{2.48}$$

where the term 1 has been dropped within rate calculation.

We can see that this function is strategically separable. In fact,

$$\tilde{U}_i(p_i, p_{-i}) = \log_2 (G p_i g_{ii}) - c_i p_i - \log_2 \left(\sum_{j=1, j \neq i}^{N} p_j g_{ji} + \sigma^2 \right), \tag{2.49}$$

where the first two terms only depend on player i's strategy and the last term only depends on the opponents' strategies. According to Theorem 2.19, the game $\mathcal{G}' = \left[\mathcal{N}, [0, P_{\max}]^N, \{\tilde{U}_i\}_{i \in \mathcal{N}}\right]$ is an exact potential game.

Note that strategic separability is a sufficient condition for exact potential games. However, not all potential games are separable in this manner.

Coordination-Dummy Separability

The second notion of separability is known as *coordination-dummy separability*. It was first discussed by Slade [45], and was later linked to potential games by Fachini et al. [15] and Ui [46].

Definition 2.16. The game $\mathcal{G}$ is coordination-dummy separable if $\exists P : \mathbb{S} \mapsto \mathbb{R}$ and $\exists Q_i : \mathbf{S}_{-i} \mapsto \mathbb{R}, \ \forall i$ such that

$$U_i(S_i, S_{-i}) = P(S) + Q_i(S_{-i}). \tag{2.50}$$

Basically, one's utility function is a linear combination of two terms: $P(S)$ which is common and identical to all players, and $Q_i(S_{-i})$ which only depends on joint actions of one's opponents. With $P(S)$ alone, one effectively plays an identical-interest game, also known as a perfect coordination game. On the other hand, $Q_i(S_{-i})$ is said to be a dummy function because it solely depends on other players' strategies. Altogether, such a utility function $U_i = P(S) + Q_i(S_{-i})$ is called coordination-dummy separable. The term coordination-dummy was suggested in [15].

Theorem 2.20 (Slade, Fachini, Ui). *$\mathscr{G}$ is coordination-dummy separable if and only if it is an exact potential game with potential function $P(S)$.*

Proof. To prove the sufficiency, suppose $\mathscr{G}$ is coordination-dummy separable. Then (2.50) holds. By checking the definition (2.1) on $P(S)$, clearly it is a potential function and $\mathscr{G}$ is an exact potential game.

To prove the necessity, we see that if $\mathscr{G}$ is an exact potential game with some potential function $P(S)$, then for each i we can let $Q_i(S) = U_i(S) - P(S)$. Then for any $S_i, T_i \in \mathbf{S}_i$, since $P(S)$ is a potential function, by definition,

$$U_i(S_i, S_{-i}) - P(S_i, S_{-i}) = U_i(T_i, S_{-i}) - P(T_i, S_{-i}), \tag{2.51}$$

which implies

$$Q_i(S_i, S_{-i}) = Q_i(T_i, S_{-i}), \ \forall S_i, T_i \in \mathbf{S}_i. \tag{2.52}$$

Thus, Q_i depends only on S_{-i}. Thus, $\mathscr{G}$ is coordination-dummy separable. □

Coordination-dummy separability provides both necessary and sufficient conditions for exact potential games. In constructing potential games, this notion and the previous strategic separability serve as useful rules—as long as we can design utility functions that are separable, our games will be exact potential games.

Remark 2.5. Separability notions give rise to special types of exact potential games. For example,

1. Identical-interest games: We encounter these games before (e.g., Theorem 2.2). It is a special case of coordination-dummy separability where all the dummy terms vanish.
2. No-Conflict Games: Here, $U_i \equiv U_i(S_i), \forall i$. As such, a player's utility function only depends on his/her own actions and is not affected by other players' actions.
3. Dummy Games: On the contrary, when the coordination term vanishes we have a dummy game.

We shall look at an example from wireless communications where an identical-interest game is considered.

Example 2.13 (Cooperative Players of Identical Interest in Wireless Mesh Networks). A wireless mesh network [1] is a self-configured wireless ad-hoc communications network where mobile nodes are organized in a mesh topology. There are two types of nodes: mesh routers (MR) and mesh clients (MC). The MRs form the backbone infrastructure of the network and are equipped with functionality to carry out the tasks of resource allocation. In [12], Duarte et al. investigated the problem of decentralized channel assignment among players which are MRs in a wireless mesh network. Because the backbone networks of MRs are partially connected, [12] assumed that the MRs can play cooperatively; hence, all the MRs share an *identical objective function*.

In a wireless mesh network of arbitrary topology, there are N MRs which form the set of players $\mathcal{N}$ in the game, whose decisions are to assign channels to their associated MCs. There are K available channels. Let $\mathbf{A} \in \{0, 1\}^{N \times K}$ be the channel assignment matrix whose element a_{ik} equals 1 when channel k is assigned to one of player i's MCs, and 0 otherwise. Hence, player i's strategy is expressed by $S_i = \mathbf{a}_i^T$, the $1 \times K$ i^{th} row vector of $\mathbf{A}$. The game's strategy space is therefore $\mathbb{S} = \{0, 1\}^{N \times K}$. As usual, we denote a strategy profile by S.

The following metric is defined in [12] which characterizes the performance of a player in the game:

$$M_i = \frac{\alpha_i}{\beta_i} \sum_{k=1}^{K} a_{ik} \frac{R}{l_{ik}}, \quad \forall i \in \mathcal{N} \tag{2.53}$$

where

- α_i is a connectivity coefficient. If MR i can reach the network gateway, $\alpha_i = 1$. Otherwise, $\alpha_i = 0$.
- β_i is the hop count from MR i to the gateway.
- R is the link data rate, which is determined by the modulation and coding schemes.
- l_{ik} counts the number of interfering links sharing channel k with player i.

As the players are cooperative, a common network objective is defined which are jointly maximized among all players as follows.

$$U_i(S) = F(S) = \sum_{i=1}^{N} M_i, \quad \forall i \in \mathcal{N}. \tag{2.54}$$

The resulting strategic game is $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$. As $\mathcal{G}$ is a game of identical interest, it is also an exact potential game with potential function $F(S)$.

An implication of this identical-interest game formulation is that cooperative players can play in a distributed manner via well-known methods such as best-response and better-response dynamics, at the same time ensuring convergence to a Nash equilibrium.

2.4.1.3 Symmetry of Observations

We next examine the situations where the utility functions of players exhibit symmetries across the variables (strategies). Potential games can arise in these circumstances.

Bilateral Symmetric Interaction

One straightforward notion of symmetric observations is that due to *bilateral* or *pairwise* strategic interactions. To be precise, for player i, the utility function U_i contains a term $w_{ij}(S_i, S_j)$ which takes place solely due to the pairwise interaction between him/her and another player j, and does not depend on the actions of the rest of the players. We interpret $w_{ij}(S_i, S_j)$ as the observation seen by i due to the strategy of j. Symmetry of observations occurs when, for all S_i and S_j, we have $w_{ij}(S_i, S_j) = w_{ji}(S_j, S_i)$. That is, the observations are said to be symmetric across the pair i and j.

Games where observations are symmetric across all pairs of players were termed *bilateral symmetric interaction* (BSI) games in Ui [46]. That is, $\forall i, j \in \mathcal{N}, i \neq j$, there exist functions $w_{ij} : \mathbf{S}_i \times \mathbf{S}_j \mapsto \mathbb{R}$ such that $w_{ij}(S_i, S_j) = w_{ji}(S_j, S_i)$ for all $S_i \in \mathbf{S}_i$ and $S_j \in \mathbf{S}_j$. Moreover, the utility function of player i is assumed to be of the form

$$U_i(S_i, S_{-i}) = \sum_{j \neq i} w_{ij}(S_i, S_j), \forall i \in \mathcal{N}. \tag{2.55}$$

The next theorem from [46] shows that BSI games are exact potential games.

Theorem 2.21 (Ui). *Assume $\mathcal{G}$ is a BSI game. Then it is also an exact potential game with the following potential function*

$$F(S) = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}, j < i} w_{ij}(S_i, S_j). \tag{2.56}$$

Proof. For $S_i, T_i \in \mathbf{S}_i$, it is straightforward to see that

$$U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) = \sum_{j \in \mathcal{N}, j \neq i} w_{ij}(T_i, S_j) - \sum_{j \in \mathcal{N}, j \neq i} w_{ij}(S_i, S_j), \tag{2.57}$$

and at the same time,

$$F(T_i, S_{-i}) - F(S_i, S_{-i}) = \sum_{j \in \mathcal{N}, j \neq i} w_{ij}(T_i, S_j) - \sum_{j \in \mathcal{N}, j \neq i} w_{ij}(S_i, S_j). \tag{2.58}$$

□

Remark 2.6. Coupled with the linear combination property, we can combine BSI games with no-conflict games. The resulting exact potential game has utility functions of the form $U_i(S) = \sum_{j \neq i} w_{ij}(S_i, S_j) + P_i(S_i)$. This is the original utility function proposed in [46].

We shall look at an example from network engineering where one can enjoy a potential game formulation owing to BSI property.

Example 2.14 (Routing Game). In computer and telecommunications networks, the routing problem is where multiple players need to decide how to split their traffic loads from their sources to their destinations through various links in the network. The players, for example, can be various service providers sending data to their subscribers. Altman et al. [3] proposed a game-theoretic approach to this problem.

In the model, there are N players and a network of several nodes, connected by directed links. There are a total of L links. Each player can allocate a certain amount of their traffic loads to each link, i.e., λ_i^l is the load allocated to link l by player i. Thus, the strategy of player i is described by a L-tuple $S_i = (\lambda_i^l)_{l=1,\dots,L}$.

We denote the total traffic loads on a link l as $q_l = \sum_{i=1}^N \lambda_i^l$. It is assumed that for each link, a usage cost is present. The cost per unit traffic f_l is linearly proportional to the link usage and is given by

$$f_l(q_l) = a_l q_l + b_l, \quad a_l, b_l < 0 \tag{2.59}$$

where $a_l, b_l < 0$ are assumed so that players will maximize the negative sum of costs, equivalent to minimizing their actual costs.

Thus, the cost incurred by player i for link l is $\lambda_i^l f_l(q_l)$. Its utility function is thus given by

$$U_i(S_i, S_{-i}) = \sum_{l=1}^{L} \lambda_i^l (a_l q_l + b_l) \triangleq \sum_{l=1}^{L} u_i^l(S_i, S_{-i}). \tag{2.60}$$

Thus, we have a load allocation routing game. In [3], the authors first introduced a potential function and established the exact potential property through it. However, we can naturally deduce this property without knowing the existence of this function. We can verify using (2.25) but it can be tedious differentiating U_i with respect to a vector variable. Instead, we will check for BSI.

For an arbitrary link l and player i, we see that

$$u_i^l(S_i, S_{-i}) = a_l(\lambda_i^l)^2 + b_l \lambda_i^l + a_l \sum_{j \neq i} \lambda_i^l \lambda_j^l. \tag{2.61}$$

Thus, $a_l \lambda_i^l \lambda_j^l$ is a symmetric observation across player i and j on link l, for any j. Subsequently, one can appropriately express the overall utility into the form $U_i = \sum_{j \neq i} w_{ij}(S_i, S_j) + P_i(S_i)$ where

$$w_{ij}(S_i, S_j) = \sum_{l=1}^{L} a_l \lambda_i^l \lambda_j^l \tag{2.62}$$

and

$$P_i(S_i) = \sum_{l=1}^{L} (a_l (\lambda_i^l)^2 + b_l \lambda_i^l). \tag{2.63}$$

The game is an exact potential game as it is a linear combination of a BSI objective and a no-conflict term. Furthermore, by Theorems 2.21 and 2.18, a potential function can be automatically found, given by

$$\begin{aligned} F(S) &= \sum_{i=1}^{N} \sum_{j<i} w_{ij}(S_i, S_j) + \sum_{i=1}^{N} P_i(S_i) \\ &= \sum_{l=1}^{L} \left[a_l \sum_{i=1}^{N} \sum_{j<i} \lambda_i^l \lambda_j^l + \sum_{i=1}^{N} \left(a_l (\lambda_i^l)^2 + b_l \lambda_i^l \right) \right]. \end{aligned} \tag{2.64}$$

One can easily verify that this function coincides with the one introduced in [3].

General Symmetric Observations

Previously, we see that in practical scenarios with BSI structures, we can formulate a potential game. Next, we propose and investigate a more general notion of symmetric observations.

Definition 2.17. Consider a game $\mathcal{G}$. If for any pair of players $i \neq j \in \mathcal{N}$, there exist functions $g_{ij} : \mathbb{S} \mapsto \mathbb{R}$, $g_{ji} : \mathbb{S} \mapsto \mathbb{R}$, $Q_{ij} : \mathbf{S}_{-j} \mapsto \mathbb{R}$ and $Q_{ji} : \mathbf{S}_{-i} \mapsto \mathbb{R}$ such that $g_{ji}(S) = g_{ij}(S)$, $\forall S$ and

$$\begin{aligned} U_i(S) &= g_{ij}(S) + Q_{ij}(S_{-j}), \\ U_j(S) &= g_{ji}(S) + Q_{ji}(S_{-i}), \end{aligned} \tag{2.65}$$

then $\mathcal{G}$ is said to have *general symmetric observations* (GSO) across all players.[4]

Thus, in GSO games, any pair of players i and j share a common observation $g_{ij}(S)$ due to their strategic interactions. The utility function can be decomposed into $g_{ij}(S)$ and a second term $Q_{ij}(S_{-j})$ which represents the contribution to player i's utility function due to all players except j.

[4] To the best of our knowledge, this GSO investigation has not been reported in the literature.

Games with BSI structures can be seen as a special case of GSO. In fact, for any $j \neq i$, one can rewrite the utility function of a BSI game (2.55) into

$$U_i(S_i, S_{-i}) = w_{ij}(S_i, S_j) + \underbrace{\sum_{k \neq i,j} w_{ik}(S_i, S_k)}_{Q_{ij}(S_{-j})} . \tag{2.66}$$

Meanwhile, unlike in BSI games, the symmetric observation in GSO games needs not involve only two players i and j. That is, in (2.65), g_{ij} can in fact be expressed as $g_{ij}(S_i, S_j, S_{-\{i,j\}})$. In addition, we do not require U_i to be the summation of all pairs of observations. A game is a GSO game as long as we can decompose the utility functions for any pair of players i and j according to our definition.

Our main result is as follows.

Theorem 2.22. *GSO games are exact potential games.*

Proof. We will make use of Corollary 2.8 and prove that (2.38) holds for any arbitrary cycle of length 4, say $\rho = (A, B, C, D, A)$ as in (2.37). Here, i and j are two active players. Moreover, $A = (S_i, S_j, R)$, $B = (T_i, S_j, R)$, $C = (T_i, T_j, R)$, and $D = (S_i, T_j, R)$ where $R = S_{-\{i,j\}}$.

In fact, using (2.65),

$$U_i(B) - U_i(A) = g_{ij}(B) - g_{ij}(A) + Q_{ij}(T_i, R) - Q_{ij}(S_i, R) \tag{2.67}$$

Similarly,

$$U_j(C) - U_j(B) = g_{ij}(C) - g_{ij}(B) + Q_{ji}(T_j, R) - Q_{ji}(S_j, R), \tag{2.68}$$

$$U_i(D) - U_i(C) = g_{ij}(D) - g_{ij}(C) + Q_{ij}(S_i, R) - Q_{ij}(T_i, R), \tag{2.69}$$

$$U_j(A) - U_j(D) = g_{ij}(A) - g_{ij}(D) + Q_{ji}(S_j, R) - Q_{ji}(T_j, R). \tag{2.70}$$

By summing up (2.67)–(2.70), we obtain

$$[U_i(B) - U_i(A)] + [U_j(C) - U_j(B)] + [U_i(D) - U_i(C)] + [U_j(A) - U_j(D)] = 0. \tag{2.71}$$

Thus, Corollary 2.8 guarantees that the defined GSO game is an exact potential game. □

A well-known class of games in the literature, the *congestion games* proposed by Rosenthal [40], turn out to exhibit the GSO property.

Example 2.15 (Congestion Game). In a congestion game, there are N players. In addition, there are K resources which are indexed by $k \in \mathscr{K} = \{1, 2, \ldots, K\}$. Each player is supposed to select a subset of the K available resources and their choices can be overlapped.

In practical scenarios, constraints may be imposed so that certain combinations of resources are infeasible or invalid for a particular player. For example, the resources can be a collection of roads where players need to take to get to a destination. Some road segments are physically separated and cannot be validly combined. Such constraints limit the strategy space of player i to only a subset of the complete strategy space $2^{\mathscr{K}}$, the power set of $\mathscr{K}$. Thus, we denote the strategy set of player i as $\mathbf{S}_i \subseteq 2^{\mathscr{K}}, \forall i$. Each strategy $S_i \in \mathbf{S}_i$ corresponds to a feasible *set of selected resources*.[5]

It is assumed that shared resources incur costs, which depend on the *number* of users occupying that resource. For resource k, denote this cost by $c_k(x_k)$ (assuming $c_k(x_k) < 0$), where x_k is the number of players choosing k. The same cost $c_k(x_k)$ is incurred on every sharing player. The utility function of player i is the total costs over all individual resources he/she selects, given by

$$U_i(S) = \sum_{k \in S_i} c_k(x_k(S)), \tag{2.72}$$

where the notion $x_k(S)$ indicates that the number of players choosing resource k can be determined from the joint strategies of all players.

In a congestion game, congestion occurs as multiple players simultaneously choose a resource. Thus, players try to minimize their total costs (or maximize the negative total costs). Congestion games are well-known to be potential games [31]. We will alternatively show that congestion games are indeed GSO games.

Between players i and j, let $\sigma_{ij} = S_i \cap S_j$ which represents the set of common resources shared between i and j, and define

$$g_{ij} = g_{ji} = \begin{cases} \sum_{k \in \sigma_{ij}} c_k(x_k(S)) & \sigma_{ij} \neq \emptyset \\ 0 & \sigma_{ij} = \emptyset. \end{cases} \tag{2.73}$$

Then, $\forall i$ and $\forall j \neq i$:

$$U_i(S) = g_{ij}(S) + \sum_{k \in S_i \setminus \sigma_{ij}} c_k(x_k(S_{-j})). \tag{2.74}$$

where $Q_{ij}(S_{-j}) \triangleq \sum_{k \in \mathbf{S}_i \setminus \sigma_{ij}} c_k(x_k(S_{-j}))$ represents the remaining costs from player i's resources that are not selected by player j. By Theorem 2.22, congestion games are exact potential games.

Due to their practical considerations, congestion games have found applications in several networking problems. One example is the network formation game introduced in Chap. 19 of Nisan et al. [39], in which players try to build a network

[5]The use of set-valued strategies here is originally considered in Rosenthal [40].

by forming edges across network nodes to connect their sources to destinations. The cost for an edge is evenly shared by players using it and thus is a function of the number of sharing players.

The readers are invited to work out the potential function for the above congestion game.

Exercise 2.4. Find a potential function for the congestion game.

Solution 2.4. The potential function for this game will be derived in the Appendix of this chapter.

In conclusion, symmetric observation across players' utility functions is another criterion to establish exact potential games.

2.4.1.4 External Constraints and Utility Functions

In the congestion games presented in Example 2.15, we see that imposing constraints can make some actions infeasible to some players and the question is how that affects the feasible strategy space. This happens commonly for practical systems. For example, in downlink cellular systems, two users within a cell might not be allocated the same frequency bands; and there may be a constraint for total transmitted power level. On the other hand, the system quality-of-service (QoS) requirements may specify some performance metrics such as minimum throughput or maximum delay, which similarly restricts the set of feasible actions. As such, these constraints reduce the feasible strategy space as compared to the game without any constraint.

The question we investigate here is how the constraints affect the formulation of potential games. In doing so, we look at our game-theoretic problem from the viewpoint of mathematical optimization. Suppose that the original game $\mathscr{G} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i\in\mathscr{N}}]$ is formulated before incorporating the constraints. This can be expressed as a collection of N optimization problems,

$$(\mathscr{G}): \qquad \forall i \in \mathscr{N}: \quad \max_{S_i \in \mathbf{S}_i} U_i(S_i, S_{-i}), \tag{2.75}$$

where the set of all strategy profiles feasible for (2.75) is the strategy space $\mathbb{S}$ of $\mathscr{G}$.

Now, let us assume that there are a number of constraints to be imposed on the game. Adopting the standard notations of a mathematical optimization problem [5], the new game with the imposed constraints can be written as

$$(\mathscr{G}'): \qquad \forall i \in \mathscr{N}: \quad \max_{S_i \in \mathbf{S}_i} U_i(S_i, S_{-i}),$$
$$\text{subject to (s.t.)} \begin{cases} g_k(S_i, S_{-i}) \leq 0, & k = 1, 2, \ldots, K \\ h_m(S_i, S_{-i}) = 0, & m = 1, 2, \ldots, M. \end{cases} \tag{2.76}$$

Here, we assume that there are K inequality constraints in the form $g_k(S) \leq 0$, as well as M equality constraints in the form $h_m(S) = 0$, where the strategy profile S is treated as a decision variable. The constraints commonly encountered in wireless communications problems can easily be expressed into one of the two forms above. For examples,

- In a wireless power control application, the players (wireless radios)' strategies are assumed to be power levels p_i, $i = 1, 2, \ldots, N$. It is required that $0 \leq p_i \leq P_{max}$, $\forall i \in \mathcal{N}$. Equivalently, we can express these constraints into $2N$ inequality constraints of the form $g_k(p) \leq 0$, $k = 1, 2, \ldots, 2N$, where

$$g_k(p) \triangleq \begin{cases} -p_k, & k = 1, 2, \ldots, N \\ p_{k-N} - P_{\max}, & k = N+1, N+2, \ldots, 2N. \end{cases} \tag{2.77}$$

- In a wireless channel assignment problem, there are N players and M channels for data transmission. Here we define $a_{im} \in \{0, 1\}$ as the channel assignment indicator between player i and channel m. Player i's strategy is represented by the $1 \times M$ vector $S_i = [a_{i1}\ a_{i2}\ \ldots\ a_{iM}]$. Each of the M channels can be assigned to exactly one player. These constraints can then be expressed as

$$h_m(S) \triangleq \sum_{i=1}^{N} a_{im} - 1 = 0, \quad m = 1, 2, \ldots, M. \tag{2.78}$$

We introduce some notations as follows. For each of the constraints, we can define the set of feasible strategy profiles that satisfy the particular constraint as follows.

$$\mathbb{G}_k = \{S |\ g_k(S) \leq 0\}, \quad \forall k = 1, 2, \ldots, K \tag{2.79}$$

$$\mathbb{H}_m = \{S |\ h_m(S) = 0\}, \quad \forall m = 1, 2, \ldots, M. \tag{2.80}$$

Thus, all feasible strategy profiles comprising the strategy space $\mathbb{S}'$ for the constrained game $\mathcal{G}'$ in (2.76) belong to the intersection of all the sets defined above. That is,

$$\mathbb{S}' \triangleq \left(\bigcap_{k=1}^{K} \mathbb{G}_k \right) \cap \left(\bigcap_{m=1}^{M} \mathbb{H}_m \right) \cap \mathbb{S}. \tag{2.81}$$

The set $\mathbb{S}'$ is assumed, for non-triviality, that $\mathbb{S}' \neq \emptyset$. This leads to a new game

$$\mathcal{G}' = \left[\mathcal{N}, \mathbb{S}', \{U_i\}_{i \in \mathcal{N}} \right]. \tag{2.82}$$

We claim the following result.

Theorem 2.23. *If $\mathcal{G}$ is an exact potential game then so is $\mathcal{G}'$.*

Proof. The equality $U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) = F(T_i, S_{-i}) - F(S_i, S_{-i})$ holds for any S_i, T_i such that $(S_i, S_{-i}), (T_i, S_{-i}) \in \mathbb{S}$, if $\mathcal{G}$ is an exact potential game. Thus, they remain valid if we restrict (S_i, S_{-i}) and (T_i, S_{-i}) to the new strategy space $\mathbb{S}'$, which is a subset of $\mathbb{S}$. □

This theorem is also a useful property for deriving new potential games from existing ones. Suppose that practical considerations require additional constraints. If we are certain that such constraints only modify the strategy space, we can guarantee that the new game is still an exact potential game. In subsequent chapters (Chaps. 3 and 4), we will introduce applications where addition of constraints is encountered.

Remark 2.7. This result is based on the fact that the condition (2.1) for exact potential games holds universally across the strategy space. Then, it automatically holds within any of its subset. We note that sometimes this equality may even hold for a superset of the original strategy space. For example, in the Cournot competition (Example 2.10), we can hypothetically enlarge the strategy space by allowing it to take negative quantities (i.e., removing the constraints $q_i \geq 0, \forall i$) and still retain the equality relationship of the potential function, even if such a relaxation may not have practical meanings in this case. However, extreme care needs to be taken when considering an enlarged strategy space. It is always advisable to examine if the condition (2.1) still holds for supersets of the strategy space.

Remark 2.8. Note that there are other types of constraints that not only modify our strategy space but also require us to make more significant alterations, e.g., redefine our objective function or reformulate the problem. In such scenarios, this theorem may not apply.

To conclude this section, we look at an example from wireless communications.

Example 2.16 (Power Control with Coupled Constraints). Let us revisit the uplink of a CDMA wireless network similar to Example 2.12. A power control game for the N MSs, with power level p_i as the strategy, is considered. The SINR of player i is again given by $\gamma_i(p_i, p_{-i}) = \gamma_i(\mathbf{p})$ in (2.45). Scutari et al. [44] proposed a power minimization game where the objective is for each player to minimize its transmit power p_i, subject to a coupled constraint f_i on the SINR, given by

$$f_i(\gamma_i(\mathbf{p})) \geq \phi_i, \quad \forall i \tag{2.83}$$

where $f_i(.)$ is a continuous function on $\mathbb{R}_+$, whose choice depends on the respective QoS requirements; and ϕ_i are real constants.

Here, we start with the following game

$$\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i = -\log(p_i)\}_{i\in\mathcal{N}}] \tag{2.84}$$

whose feasible strategy space is

$$\mathbb{S} = \{\mathbf{p} | \, p_i > 0, f_i(\gamma_i(\mathbf{p})) \geq \phi_i, \forall i\} \tag{2.85}$$

where **p** denotes the network power vector which also is a strategy profile. Note that 0 is not a feasible strategy for any player as the log() function is undefined at 0. It is straightforward to see that $\mathscr{G}$ is a game of no conflict; as such, it is an exact potential game and admits the following potential function $F(\mathbf{p}) = -\sum_{i\in\mathscr{N}} \log(p_i)$.

Now, let us consider a new game $\mathscr{G}'$ derived from $\mathscr{G}$ by imposing a maximum power constraint $p_i \leq P_{\max}, \forall i$. Its strategy space is therefore

$$\mathbb{S}' = \{\mathbf{p}|\ \mathbf{p} \in (0, P_{\max}]^N, f_i(\gamma_i(\mathbf{p})) \geq \phi_i, \forall i\}. \tag{2.86}$$

Clearly, we have $\mathbb{S}' \subset \mathbb{S}$. Thus, according to Theorem 2.23, $\mathscr{G}'$ is also an exact potential game with the same potential function $F(\mathbf{p})$.

In [44], it was also assumed that $\mathbb{S}' \neq \emptyset$ and $\mathbb{S}'$ is convex and compact. As a result, $F(\mathbf{p})$ has a unique maximum on $\mathbb{S}'$ and $\mathscr{G}'$ admits a unique Nash equilibrium.

The above is a simple example to illustrate the use of Theorem 2.23. In Chaps. 3 and 4, we will again encounter more non-trivial examples which demonstrate the usefulness of this result.

2.4.2 *Game Formulation Principles*

Having introduced several properties that can serve as useful guidelines in designing potential games, we now consolidate them into design principles. From our survey, we have observed that most of the formulations of exact potential games applied to wireless communications and networking can be classified under either one of two design principles: the *forward method* and the *backward method*.

Note that the described methods are primarily concerned with the design of utility functions and/or potential functions. Prior to formulating a game-theoretic problem for a practical scenario, one still needs to identify the players, their means of interactions and possible strategies, as well as the constraints, objectives and assumptions. There is no clear-cut process; and in this book we focus on generalizing methods for potential game and utility function design while assuming the parameters are already in place.

2.4.2.1 Forward Method

In the forward method, the utility functions are purposely designed to have one of the desired properties: separability, symmetric observations, or linear combination of utilities, which are known to lead to potential games. The potential function can then be associated with one of the known forms specified by these properties.

To be precise, we denote $\mathscr{D}$ as the set of utility functions having one or more of the above desirable properties. The forward method is as follows.

Algorithm 2.2 Forward method

1: $\forall i: \quad U_i(S) \leftarrow \mathscr{D}$
2: $F(S) \leftarrow$ associated potential function according to the structure of $U_i(S)$

Several authors have proposed potential game models with utility functions adhering to one of these properties.

- In [35, 36], Neel et al. discussed game models for cognitive radio networks. One of their models has the utility function of the form $U_i(S) = f_{i,1}(p_i) - f_{i,2}(I_i)$, where $f_{i,1}(p_i)$ is a function of the player i's received signal strength which depends on its power level p_i. $f_{i,2}(I_i)$ is another function of the player i's received interference which only depends on the opponents' transmit power p_{-i}. This design is based on the strategic separability.
- In [33, 34, 37], Neel et al. designed games that adheres to bilateral symmetric interference assumption. This approach leads to BSI games. Babidi [4], Wu [52] and a few others further extended this approach.
- The formulation by Nie et al. [38] on interference minimization for distributed radios as well as our works on OFDMA systems [20–23] used an interference sum minimization objective. Symmetric observations are present for all pairs of players.

In subsequent chapters, we will review some of the applications listed above in details. For now, let us look at one particular formulation to demonstrate this principle.

Example 2.17 (Cognitive Radio Interference Minimization Game). Cognitive radios [29] are smart radio devices that can learn their environment and optimize their performance by adjusting their transmission parameters. In the distributed spectrum access problem among multiple radios, their interactions can be modeled as a game. Nie et al. [38] studied a channel allocation game among N cognitive radio pairs of transmitter and receiver. The N pairs of nodes constitute the set of players $\mathscr{N}$. There are K frequency bands ($K < N$) which represent the available resources; and each player must select one frequency to transmit its data. Thus, for player i, its strategy is the channel it selects, i.e., $S_i = k \in \{1, 2, \ldots, K\}$.

As the spectrum band is spatially reused by the distributed radios, co-channel interference is present which degrades the performance of the radios. We let p_i be the transmission power of pair i, and g_{ij} the channel gain between the transmitter of pair i and the receiver of pair j. We further define a variable δ_{ij}, which assumes a value 1 if players i and j are on the same channel, i.e., $S_i = S_j$; and 0 otherwise. As such, the possible co-channel interference that player i may experience from player j is given by $\delta_{ji} p_j g_{ji}$.

In this game, an interference minimization objective is adopted. The following utility function was firstly considered, i.e.,

$$U_i(S_i, S_{-i}) = -\sum_{j=1, j \neq i}^{N} \delta_{ji} p_j g_{ji} \tag{2.87}$$

in order for each player to minimize its received co-channel interference.

Naturally, the quantity $\delta_{ji}p_jg_{ji}$ perfectly represents the observation on player i's utility function due to the action of player j. On the other hand, player j's observation due to player i's action is given by $\delta_{ij}p_ig_{ij}$. In general, $p_j \neq p_i$ and $g_{ij} \neq g_{ji}$. Thus, the observation is not bilateral symmetric for this formulation.

However, between players i and j, the observations become symmetric if the total $\delta_{ji}p_jg_{ji} + \delta_{ij}p_ig_{ij}$ is considered instead. This quantity represents the total interference that a player generates to another player and also experiences from the same player. This leads to a second proposed utility function

$$V_i(S_i, S_{-i}) = -\sum_{j=1, j\neq i}^{N} \delta_{ji}p_jg_{ji} - \sum_{j=1, j\neq i}^{N} \delta_{ij}p_ig_{ij}. \tag{2.88}$$

We can see that the second utility function satisfies the BSI property. For any two players i and j, we have

$$w_{ij}(S_i, S_j) = w_{ji}(S_j, S_i) = -(\delta_{ji}p_jg_{ji} + \delta_{ij}p_ig_{ij}). \tag{2.89}$$

Thus, according to Theorem 2.21, the game $[\mathcal{N}, \mathbb{S}, \{V_i\}_{i\in\mathcal{N}}]$ is an exact potential game. Its potential function is given by

$$F(S) = -\sum_{i=1}^{N}\sum_{j=1, j<i}^{N} \left(\delta_{ji}p_jg_{ji} + \delta_{ij}p_ig_{ij}\right). \tag{2.90}$$

This can be alternatively written in the form

$$F(S) = \sum_{i=1}^{N}\left(-\frac{1}{2}\sum_{j=1, j\neq i}^{N} \delta_{ji}p_jg_{ji} - \frac{1}{2}\sum_{j=1, j\neq i}^{N} \delta_{ij}p_ig_{ij}\right) \tag{2.91}$$

as seen in [38].

In summary, this example demonstrates the forward method which works by identifying a utility function in accordance with certain desirable properties.

2.4.2.2 Backward Method

The forward method works by defining utility functions that ensure potential game properties first before obtaining the potential function. On the other hand, the backward method first defines a network objective as the potential function and works backward to obtain individual utility functions. The steps are as follows.

In the backward method, one may first define a network function F as a global objective to maximize, which will also serve as the potential function. Next, in order

Algorithm 2.3 Backward method

1: Define $F(S)$
2: Decompose $F(S)$, $\forall i$:

$$F(S) \rightarrow P_i(S_i, S_{-i}) + Q_i(S_{-i}) \tag{2.92}$$

3: Assign, $\forall i$:

$$U_i(S_i, S_{-i}) \leftarrow P_i(S_i, S_{-i}) \tag{2.93}$$

to define each player's utility function, a decomposition is applied to this global network function such that $F(S) = P_i(S_i, S_{-i}) + Q_i(S_{-i})$ where $Q_i(S_{-i})$ is a non-contributing term from the perspective of player i. Then, we can set $U_i = P_i(S_i, S_{-i})$.

Theorem 2.24. *Algorithm 2.3 results in an exact potential game.*

Proof. From (2.92),

$$U_i(S_i, S_{-i}) = F(S_i, S_{-i}) - Q_i(S_{-i}), \quad \forall i \tag{2.94}$$

By Definition 2.16, U_i is coordination-dummy separable. Therefore, the resulting game is an exact potential game. Its potential function is by default $F(S)$. □

Remark 2.9. Note that such a decomposition does not always give non-trivial utility functions. It is possible that $Q_i(S_{-i}) = 0$ for all players and the process results in an identical-interest game.

In the literature, examples of the backward method are:

- Menon et al. [27, 28] as well as Buzzi et al. [6] which defined a sum of inverse SINRs as the potential function and subsequently defined players' utility through a similar decomposition.
- Xu et al. [53] also defined two network objectives which are total network throughput and total network collisions. They serve as the potential functions for the potential games that follow.

These approaches will be discussed in more details in Chap. 5.

2.5 Further Readings

This chapter has covered the theory of potential games, including fundamental results with accompanied mathematical analysis. We make an attempt to generalize and present a collective summary on the results available from the literature. Some of the mathematical proofs are omitted in the text. However, interested readers may refer to the cited works. The authors are also aware that in the literature, more specialized topics are available for potential games. In what follows, we present a non-exhaustive list of related topics intended for further readings.

Although introduced, certain types of games such as generalized ordinal potential games and best-response potential games were not discussed in depth due to their rare appearances in communications applications. Readers who wish to explore their mathematical properties can refer to Monderer and Shapley [31] and Vorneveld [50].

There exist other notions of potential games in the literature. For example, in his paper [43], Schipper extended the class of pseudo-potential games to a broader class called *quasi-potential games*. However, the author did not elaborate further on the concept and their applications are not known. A few more generalizations from pseudo-potential games exist, such as *q-potential games* [30] and *nested potential games* [47]. One related concept to potential games is *near potential games* as proposed by Candogan et al. [8, 9]. The authors defined a notion of distance between games and those with a close distance to a potential game are called near-potential.

Extending potential properties of games outside the current static and complete information settings is a different line of literature. *Bayesian potential games* were studied by Facchini et al. [15] in which the games are set under incomplete information assumptions. Sandholm [41] presented an extension into potential games among *continuous populations*. On the other hand, Marden [24] defined *state-based potential games* in dynamic settings where there exists an underlying state space governing the system. González-Sánchez et al. [16] coupled potential games with dynamic stochastic control problems and characterized the conditions for the potential function in *dynamic stochastic potential games*. Another approach in the dynamic settings is to extend the potential game framework to continuous-time optimal control models, in which the concept of *Hamiltonian potential* and its use in characterization of open-loop equilibrium of differential games were proposed in Dragone et al. [11]. These recent interesting topics may attract further development.

There were also studies that linked potential games to the concept of Shapley value for coalitional games. Some fundamental results were presented by Monderer and Shapley [31], Ui [46], etc.

Our presented results on convergence of best/better-response dynamics give an elementary view of limiting behaviors of adaptive update rules for games. There is an extensive literature on this topic. Fictitious play and variants were discussed in Monderer and Shapley [31], Hofbauer and Sandholm [18], Marden et al. [26], and so on. Neel [32] defined some practical decision rules and demonstrated convergence properties. Logit-response dynamics are a different class considered in Alós-Ferrer et al. [2], Marden et al. [25], etc.

Regarding the identification of finite exact potential games, Cheng [10] gives some interesting results using the technique of semi-tensor product of matrices. His result may be of practical values, which due to the scope of this monograph we have omitted.

Appendix

Potential Function of Congestion Game

Example 2.15 introduces the congestion game. We have shown that it is a GSO game and thus is also an exact potential game. We now present and verify its potential function.

Rosenthal [40] introduced the following so-called Rosenthal's potential:

$$F(S) = \sum_{k=1}^{K} \sum_{j=1}^{x_j(S)} c_k(j) \tag{2.95}$$

and verified that it satisfies the definition of potential games.

Here, we present an alternative interpretation due to Vöcking [48]. The interpretation assumes that each strategy profile (joint selections of resources) is a result of individual selection taking place in sequence. At each individual selection, the corresponding player bears a "virtual" cost which depends on his/her choice and the selections of previous players. The sum of all virtual costs is the Rosenthal's potential.

To visualize this virtual cost calculation, look at an example for 4 players and 3 resources ($R1, R2, R3$) in Fig. 2.10. Imagine that the resources are represented by separate stacks. Each stack is comprised of several cells. Players are inserted into these cells one after another, according to their resource selections. In cell j of stack k, there is an associated cost which is equal to $c_k(j)$ if this cell is filled.

Without loss of generality, we can assume the order of players in making selection is $(1, 2, \ldots, N)$. Figure 2.10a shows the state of each resource after player 1's selection. At player 1's selection, all resources are unoccupied and his/her virtual cost is $c_1(1) + c_2(1)$.

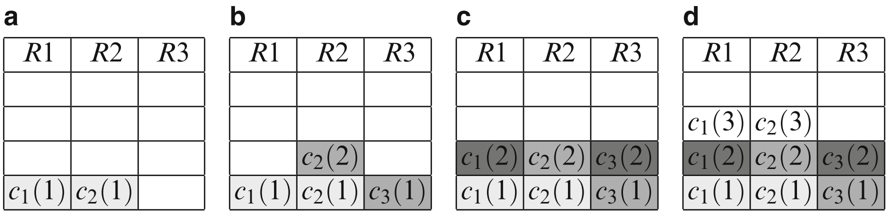

Fig. 2.10 Virtual costs in a 4-player 3-resource congestion game after the selections of (**a**) player 1, (**b**) player 2, (**c**) player 3 and (**d**) player 4, where each *color* indicates a different player. The Rosenthal's potential is equal to the sum of all values that fill up the cells

Figure 2.10b shows the state of each resource after player 2's selection. At player 2's selection, $R3$ is unoccupied while $R2$ already has one player. Hence, player 2's virtual cost is computed as $c_2(2) + c_3(1)$. The computation is similar for the next selecting players.

In general, at player i's selection, resources have been occupied by the previous $i-1$ players. For each of his/her selected resource $k \in S_i$, determining the current cost depends on the total number of users $\alpha_k(i)$ which includes him/herself and how many players have selected k previously. This number is estimated by

$$\alpha_k(i) = \left|\{j|k \in S_j, j \le i\}\right| \tag{2.96}$$

Player i's virtual cost is therefore given by

$$\gamma_i(S) = \sum_{k \in S_i} c_k(\alpha_k(i)). \tag{2.97}$$

The accumulated virtual costs for all players in this manner are given by

$$F'(S) = \sum_{i=1}^{N} \gamma_i(S) = \sum_{i=1}^{N} \sum_{k \in S_i} c_k(\alpha_k(i)). \tag{2.98}$$

This function is equal to the sum of values of cells that are filled in all the stacks (e.g., the sum of all values in Fig. 2.10d). By exchanging the order of summation, we can rewrite this into

$$F'(S) = \sum_{k=1}^{K} \sum_{j=1}^{x_j(S)} c_k(j) \tag{2.99}$$

which is exactly the Rosenthal's potential (2.95). That is, $F(S) = F'(S)$.

Now, for player N who is last to select resources, his/her virtual cost is exactly his/her real cost. That is, $\gamma_N(S) = U_N(S)$. We suppose that player N now wants to deviate to a new strategy unilaterally. By noting that

$$F(S) = \sum_{i=1}^{N-1} \gamma_i(S) + U_N(S) \tag{2.100}$$

where the first summation is unaffected by player N's strategy, it is apparent that $F(S)$ will be changed by exactly the same amount as $U_N(S)$.

This property should hold for every permutation of selection orders, and any player can be equivalently considered to be the last selector. In short, $F(S)$ is a potential function.

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Part II
Applications

Chapter 3
Frequency Assignment in Distributed OFDMA-Based Systems Using Potential Games

Abstract Wireless orthogonal frequency division multiple access (OFDMA) networks, where distributed users can access the spectrum dynamically, offer great flexibility and potentials for efficient utilization of the scarce spectrum resources. However, managing such large-scale networks while maintaining fair and efficient spectrum usage poses a great challenge, mainly due to co-channel interferences from the spatially reusable sub-carrier. This chapter considers the problem of interference minimization in such a scenario, from a game-theoretical viewpoint. A general framework that defines the distributed OFDMA-based spectrum access problem is presented. Next, our game-theoretical analysis shows that the proposed interference-minimizing utility function exhibits the properties of a potential game. Given a deterministic number of sub-carriers by each player, there exists a pure-strategy Nash equilibrium solution, and therefore convergence can be guaranteed via sequential best-response dynamics. Our simulation results verify the analysis, and at the same time the fairness and optimality of the solutions obtained through the proposed game are examined.

3.1 Overview

Orthogonal frequency division multiplexing (OFDM) [22] is a digital multi-carrier modulation technique, in which a large number of closely-spaced orthogonal subcarriers are used to transmit signals simultaneously, as depicted by Fig. 3.1. In OFDM systems, high-rate data which might otherwise suffer from severe frequency-selective fading, is first converted into a number of parallel orthogonal data streams occupying less than the coherence bandwidth before transmitting. The advantage of OFDM lies mainly in its robustness to multipath fadings and inter-symbol interferences due to the prolonged symbol duration on each subcarrier and the use of a cyclic prefix. Furthermore, its transceiver design and implementation can be simplified by the use of fast Fourier transform (FFT) and inverse FFT blocks, as shown in Fig. 3.2.

OFDMA [21] is the multiple-access version of OFDM. It has been adopted in various existing standards like IEEE 802.16e worldwide interoperability for microwave access (WiMAX), Third-Generation Partnership Project (3GPP)'s Long Term Evolution (LTE) standards and will also be a crucial part of future wireless

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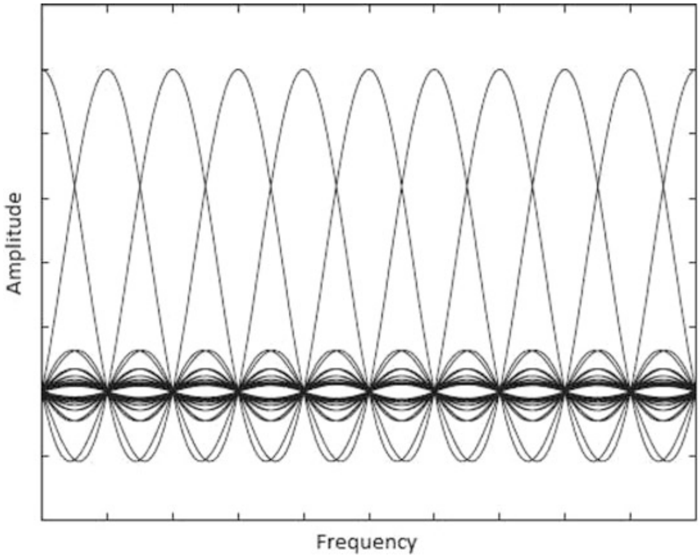

Fig. 3.1 The frequency spectrum of an OFDM signal. The subcarriers are orthogonal as at each subcarrier center frequency, as contributions from adjacent subcarriers are canceled

Fig. 3.2 Block diagram of a general OFDM transmitter

networks, such as 4G systems and the ongoing IEEE 802.22 standards. Multiple access in OFDMA can be achieved by partitioning separate subsets of subcarriers to users. OFDMA not only inherits all the advantages of OFDM but also provides multi-user diversity gain, because if a subcarrier is unfavorable to a user, it may still be favorable to another user and thus can be reassigned appropriately. Moreover, different number of subcarriers can be allocated for users with different QoS classes, which offers another degree of flexibility for radio resource management (RRM) in OFDMA systems.

In multi-user OFDMA networks, a key RRM issue is how to dynamically allocating the subcarriers and power to users in order to combat co-channel interference (CCI). Several subproblems exist for OFDMA resource allocation, such as downlink [9] vs. uplink [17], single-cell [6] vs. multi-cell [7], or distributed [11] vs. centralized [10] approaches. Pioneering works on RRM for centralized OFDMA systems were done by Yong et al. [23], who was among the first few to treat OFDMA resource allocation as mathematical optimization problems. Since then, enormous research efforts have been contributed, providing a wide variety of approaches and solutions to the problem. At the same time, early game-theoretical approaches for

OFDMA were proposed by [3, 6, 7, 11] and references therein. A good survey to these works can be found in [13]. These early works often focused on the traditional cellular systems involving uplink/downlink communications between the BS and several MSs. A common drawback of these approaches, as pointed out in [13], is that they do not always guarantee a stable solution, especially if no pure-strategy Nash equilibrium exists.

On the other hand, for the distributed, ad-hoc OFDMA system, Bazerque et al. [1] was one of the important works, where the authors relied on heavy information exchange in a fully connected network in order to obtain the decentralized algorithm. However, at the time of this research, there were still very few works directed towards the distributed OFDMA scenario. Nevertheless, it can be noticed that in the special case where each player only competes for a single frequency channel, the situation bears resemblance to the well-known distributed ad-hoc spectrum sharing games [4, 12, 15, 19], as well as cognitive radio spectrum access games [2, 18, 20]. Notably, Neel et al. [18] suggested the use of potential games for cognitive radio networks; and Nie et al. [19] formulated a potential game for the distributed, ad-hoc scenario. For such games, the existence of deterministic pure-strategy Nash equilibria is always guaranteed when assignment of a single channel is considered, which suggests the possibility of extending the same analytical platform to a multi-channel scenario. In summary, the lack of works in the literature on distributed, ad-hoc OFDMA systems and the stability issues of traditional OFDMA game-theoretical approaches provide the motivations to exploit the use of potential games to realize desirable solutions to resource allocation for OFDMA.

This chapter extends its investigation into the distributed, ad-hoc networks of multiple transmit-receive pairs. The system has an OFDMA interface with the bandwidth being divided into different orthogonal subcarriers, which has been a more practical approach for wireless networks recently. Each player has an OFDM front-end and can measure the channels in order to select a subset of the available subcarriers to transmit. As such, more complicated spatial reuse of spectrum occurs over the cell's area; and one will have to pay closer attention to the issue of CCI mitigation. By adopting the interference-sum minimization objective in the players' utility function, a potential game can be formulated similar to [19]. When this game is played, due to domination in the strategy, it is best for the players to be allocated with a given number of subcarriers in advance. The number of subcarriers for each player can be determined prior to the gameplay via a simple proposed method. This situation notwithstanding, the universal properties of potential games are not lost. Hence, Nash equilibrium convergence can always be obtained through sequential dynamics. Besides, the optimality of the Nash equilibria obtained through this method can also be characterized with the price of anarchy (PoA). Furthermore, extensive simulations are carried out to evaluate the proposed scheme as well as to compare the system and individual performance in terms of energy efficiency, fairness and optimality.

3.2 System Model

This section defines the mathematical model for the system and describes a framework for interference mitigation.

3.2.1 System Parameters

Our model considers a circular cell of radius r, where there are N players, each being a transmit-receive pair randomly distributed within the area. Once again, the terms *players* and *pairs* can be used interchangeably. Each pair individually competes with others for different combinations of OFDMA subcarriers among the shared K subcarriers available. It is assumed that $N > K$ so as to avoid trivial solutions, as well as the nodes' locations and channel conditions being static or changing slowly during the game. Moreover, the assumption of short-range distance between the transmitter and receiver of a pair still holds. Recall that the distance matrix is $\mathbf{D} \in \mathbb{R}^{N\times N}$, whose element d_{ij} is the distance between the ith transmitter and the jth receiver. Then, in general, it is assumed that $d_{ii} \ll d_{ji}$ for $j \neq i$.

The other parameters can be defined as follows. Let $\mathbf{A}$ denote the subcarrier assignment matrix, where $\mathbf{A} \in \{0, 1\}^{N\times K}$. Each of the elements in $\mathbf{A}$, $a_{ik} \in \{0, 1\}$, implies whether player i occupies subcarrier k for transmission; and $a_{ik} = 1$ if this is true and 0 otherwise. Hence, the choice of subcarriers for player i is reflected by $\mathbf{a}_i^T$, the $1 \times K$ ith row vector of $\mathbf{A}$. As the no-transmission strategy is not considered in this case, it can be understood that the all-zero vector $\mathbf{0}$ is not a valid choice.

Next, let us define the channel gain matrix $\mathbf{G} \in \mathbb{R}^{N\times N\times K}$, where g_{ij}^k gives the channel gain between transmitter i and receiver j through subcarrier k. Note that, in general, $g_{ij}^k \neq g_{ji}^k$. Similarly, the transmission power matrix is denoted by $\mathbf{P} \in \mathbb{R}^{N\times K}$, whose element p_{ik} is the transmitted power of player i over subcarrier k. The $1 \times K$ ith row vector of $\mathbf{P}$, denoted by $\mathbf{p}_i^T$, must be element-wise non-negative and the sum of all elements should be less than $P_{\max}$. That is,

$$\mathbf{p}_i^T(\mathbf{A}) \succeq \mathbf{0}, \quad \mathbf{p}_i^T\mathbf{1} \leq P_{\max}, \quad \forall i \tag{3.1}$$

where $\mathbf{1}$ is the $K \times 1$ all-one column vector.

Then, for player i with subcarrier k, its signal-to-interference-and-noise ratio (SINR) is expressed as

$$\gamma_{ik} = \frac{a_{ik}p_{ik}g_{ii}^k}{\sum_{j=1, j\neq i}^N a_{jk}p_{jk}g_{ji}^k + \sigma^2} \tag{3.2}$$

where σ^2 is the power of the receiver additive white Gaussian noise (AWGN) and is assumed to be identical between any two players.

Furthermore, the achievable rate for player i with subcarrier k is given by

$$R_{ik} = B \log_2 \left(1 + \frac{\gamma_{ik}}{\Gamma}\right) \tag{3.3}$$

where B is the bandwidth per subcarrier, $\Gamma = -\ln(5P_e)/1.5$ is a function of the required BER P_e, often known as the SINR gap [5].

The objective of any player i is to optimize its performance individually, which is reflected by a utility function U_i. Mathematically, the distributed optimization problem is stated as

$$\max_{\mathbf{a}_i^T} U_i, \ \forall i$$
$$\text{s.t.} \begin{cases} \mathbf{a}_i^T \in \{0,1\}^K \setminus \{\mathbf{0}\}, \\ \mathbf{p}_i^T(\mathbf{A}) \succeq 0, \quad \mathbf{p}_i^T \mathbf{1} \leq P_{\max}, \end{cases} \tag{3.4}$$

where the utility function $U_i(S)$ will be defined in Sect. 3.3.

3.2.2 An Interference Mitigation Framework

In this section, a framework for mitigation of CCI and the relevant parameters are discussed. Since the subcarriers are shared among the players, it is best that a subcarrier should not be reused by players which are within a certain vicinity of another player which is currently using it for transmission. As such, it is proposed that an area surrounding a player i with radius r_i be defined as the *circle of interference*, within which any device transmitting on the same subcarrier will cause significant CCI to the ith player. r_i is called the radius of interference. A similar concept was briefly mentioned in [20]. However, it is developed differently here due to the fact that the CCI affects mainly the receiver, so it makes sense to center the circle of interference at the receiver.

Using the previously defined distance matrix **D**, two rules governing the reuse of subcarriers are generalized:

1. **Rule 1:** For player i to successfully transmit using subcarrier k, it is required that $(\forall j \neq i, d_{ji} < r_i)$: $a_{jk} = 0$, i.e., all transmitters within the radius of interference should not use the same subcarrier as player i.
2. **Rule 2:** Two players i and j can reuse any subcarrier k if $d_{ij} > r_j$ and $d_{ji} > r_i$, i.e., the transmitter of one player falls outside the radius of interference of the other, and vice versa. A *conflict* is said to occur between players i and j if rule 2 is violated.

The definition of conflict allows us to construct the so-called *conflict matrix* $\mathbf{C} \in \{0,1\}^{N \times N}$, which maps the occurrence of conflicts in the entire network. Specifically, its elements are

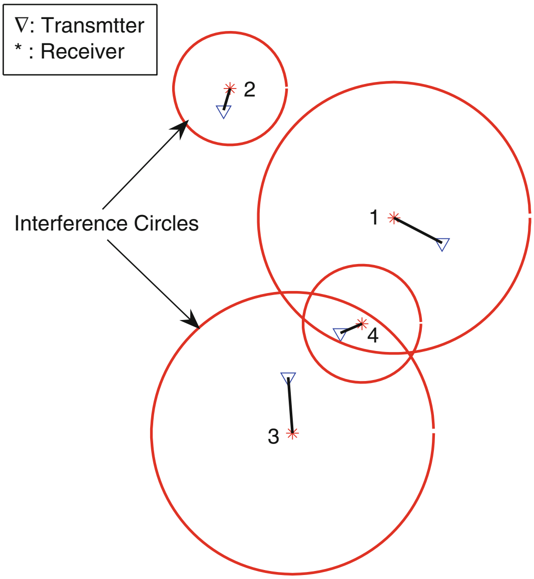

Fig. 3.3 Four transmit-receive pairs and their interference radii

$$C_{ij} = \begin{cases} 0 & i = j, \text{ or no conflict between } i \text{ and } j, \\ 1 & \text{otherwise.} \end{cases} \tag{3.5}$$

A simple example with $N = 4$ pairs of nodes is shown in Fig. 3.3. It is observed that player 2 does not have a conflict with any other player. On the other hand, player 4's transmitter is within both players 1 and 3's interference radii so it cannot use the subcarriers occupied by players 1 and 3. However, player 1 will not clash with player 3. Thus, the conflict matrix is given by

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}. \tag{3.6}$$

The symmetric conflict matrix is useful from a centralized point of view. In practice, an entry "1" in a row represents a one-hop neighbor of a particular player, and some available distributed MAC protocols can be used to obtain these one-hop neighbors. Hence, in the distributed context, it is reasonable to assume that each player knows only its corresponding row of the matrix. As will be seen in later parts of this chapter, this information can be useful for the players to autonomously compete for resources.

It has not been mentioned how the radius of interference is determined. This is a complex problem, as the value of r_i should depend on player i's as well as its neighbors' transmitted power and locations, its own SINR requirement, and channel gains. To be more accurate, for different subcarriers, player i may have different radii as a result of frequency-selective fading. However, a simple way to estimate r_i with only the path loss taken into consideration will be proposed, so that r_i will be identical for all subcarriers.

Assuming that for pair i, its neighbor j who is using the same subcarrier k is the most significant interferer, and the effects from the remaining pairs are negligible. Furthermore, let us assume that they use roughly the same power $p_{ik} \approx p_{jk} = p$, and consider only the effect of path loss.

Proposition 3.1. *Under the above assumptions, to maintain its SINR above a target level γ_i^*, player i might set*

$$r_i = d_{ii}(\gamma_i^*)^{1/\lambda} \tag{3.7}$$

where λ is the path loss exponent.

Proof. Following the assumptions, the received power levels at receiver i from the ith and jth transmitters are proportional to p/d_{ii}^{λ} and p/d_{ji}^{λ}, respectively. By neglecting the background noise, the SINR perceived by the ith pair is roughly $(d_{ji}/d_{ii})^{\lambda}$. If we want this to be no less than γ_i^*, then $d_{ji} \geq d_{ii}(\gamma_i^*)^{1/\lambda}$. Thus, r_i can be estimated by $d_{ii}(\gamma_i^*)^{1/\lambda}$ according to rule 2 previously. □

In practice, if more than one interferers need to be accounted for, a safeguard margin (i.e., a multiplier greater than 1) might be introduced to this computed value. In the next section, the utility function is introduced in order to analyze the resulting game.

3.3 Analysis of Potential Game

3.3.1 Preliminaries

As investigated in Chap. 2, potential game formulations are desirable due to their attractive properties of pure-strategy Nash equilibrium existence and convergence. Recall that a game $\mathscr{G}$ is an exact potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists, such that

$$U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) = F(T_i, S_{-i}) - F(S_i, S_{-i}). \tag{3.8}$$

For the distributed system of transmit-receive radio pairs where each player selects only a single frequency channel, Nie et al. [19] formulated a potential game by adopting the following interference-minimizing utility function:

$$U_i = -(\text{Total CCI } \textit{generated} \text{ and } \textit{experienced} \text{ by } i). \tag{3.9}$$

We discussed the corresponding potential game analysis of this approach earlier in Example 2.17 of this book. From here on, we will label this game $\mathcal{G}_1$. Next, a generalization of $\mathcal{G}_1$ will be formulated which applies to the distributed OFDMA system under investigation.

3.3.2 *Potential Game Formulation*

Let us denote the distributed OFDMA subcarrier allocation game $\mathcal{G}_2$, where

$$\mathcal{G}_2 = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i\in\mathcal{N}}]. \tag{3.10}$$

The set of players is given by $\mathcal{N} = \{1, 2, \ldots, N\}$. For any player i, its strategies S_i are the subcarriers. Thus, effectively $S_i = \mathbf{a}_i^T \in \mathbf{S}_i$, where $\mathbf{S}_i = \{0, 1\}^K \setminus \{\mathbf{0}\}$ is the strategy set of player i. For each S_i, S_{-i} is the joint strategy by opponents of player i. Hence, $S = (S_i, S_{-i})$ is the joint strategy of all players, also known as a strategy profile. All strategy profiles belong to the strategy space, here defined by $\mathbb{S} = \mathbf{S}_1 \times \ldots \times \mathbf{S}_N$.

Every utility function $U_i(S)$ is then a mapping from $\mathbb{S}$ to $\mathbb{R}$, which is defined as a negative interference sum (3.9) over all the utilized subcarriers of player i, i.e., $U_i = \sum_{\{k|a_{ik}=1\}} U_{i,k}$. As the assignment of subcarriers $\{k|a_{ik} = 1\}$ is yet to be determined, another set of variables δ_{ij}^k needs to be introduced, such that

$$\delta_{ij}^k = \begin{cases} 1 & \text{Players } i, j \ (i \neq j) \text{ both transmit via subcarrier } k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Thus, δ_{ij}^k can be viewed as an indicator of whether two players i and j interfere each other through a common subcarrier k; and $\delta_{ii}^k = 0$ because a player does not interfere with itself. It is not difficult to verify that $\delta_{ij}^k = \delta_{ji}^k = a_{ik}a_{jk}$ $(i \neq j)$. With all these notations introduced, the utility function for player i can be expressed as

$$U_i(S) = -\sum_{k=1}^{K} \left(\sum_{j=1}^{N} \delta_{ji}^k p_{jk} g_{ji}^k + \sum_{j=1}^{N} \delta_{ij}^k p_{ik} g_{ij}^k \right). \tag{3.12}$$

Theorem 3.1. *Equation (3.8) will hold if the following exact potential function is considered:*

$$F(S) = \frac{1}{2} \sum_{i=1}^{N} U_i. \tag{3.13}$$

Proof. To simplify, let us define $\omega_{ij}^k = -\delta_{ij}^k p_{ik} g_{ij}^k$. Then, U_i in (3.12) can be written as

$$U_i(S_i, S_{-i}) = \sum_{k=1}^{K} \left(\sum_{j=1}^{N} \omega_{ij}^k + \sum_{j=1}^{N} \omega_{ji}^k \right) = \sum_{k=1}^{K} U_{i,k}(S_i, S_{-i}) \tag{3.14}$$

where $U_{i,k}(S_i, S_{-i}) \triangleq \sum_{j=1}^{N} \omega_{ij}^k + \sum_{j=1}^{N} \omega_{ji}^k$.

The potential function (3.13) can be simplified to

$$\begin{aligned} F(S_i, S_{-i}) &= \frac{1}{2} \sum_{i=1}^{N} \left(\sum_{k=1}^{K} \left(\sum_{j=1}^{N} \omega_{ij}^k + \sum_{j=1}^{N} \omega_{ji}^k \right) \right) \\ &= \sum_{k=1}^{K} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \omega_{ij}^k \right) \\ &= \sum_{k=1}^{K} F_k(S_i, S_{-i}) \end{aligned} \tag{3.15}$$

where $F_k(S_i, S_{-i}) \triangleq \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_{ij}^k$.

Suppose that player i switches its strategy from $S_i = \mathbf{a}_i^T = [a_{i1} \ a_{i2} \ \dots \ a_{iK}]$ to $T_i = \mathbf{a}_i'^T = \left[a'_{i1} \ a'_{i2} \ \dots \ a'_{iK}\right]$. Its power vector changes to $\mathbf{p}_i'^T = \left[p'_{i1} \ p'_{i2} \ \dots \ p'_{iK}\right]$. By comparing (3.14) and (3.15), it is seen that (3.8) holds if the following K separate identities hold simultaneously:

$$U_{i,k}(T_i, S_{-i}) - U_{i,k}(S_i, S_{-i}) = F_k(T_i, S_{-i}) - F_k(S_i, S_{-i}), \quad \forall k. \tag{3.16}$$

Noting that $\omega_{ii}^k = 0$, $F_k(S)$ can be rewritten into

$$\begin{aligned} F_k(S_i, S_{-i}) &= \sum_{l=1}^{N} \omega_{il}^k + \sum_{j=1}^{N} \omega_{ji}^k + \sum_{j=1, j\neq i}^{N} \sum_{l=1, l\neq i}^{N} \omega_{jl}^k \\ &= U_{i,k}(S_i, S_{-i}) + Q_k(S_{-i}) \end{aligned} \tag{3.17}$$

where $Q_k(S_{-i}) = \sum_{j=1,j\neq i}^{N} \sum_{l=1,l\neq i}^{N} \omega_{jl}^k$ is a constant value and independent of S_i, i.e., when player i changes its strategy to T_i, $Q_k(S_{-i})$ remains unchanged. Therefore,

$$\begin{aligned} F_k(T_i, S_{-1}) &= \sum_{l=1}^{N} \omega_{il}'^k + \sum_{j=1}^{N} \omega_{ji}'^k + \sum_{j=1, j\neq i}^{N} \sum_{l=1, l\neq i}^{N} \omega_{jl}^k \\ &= U_{i,k}(T_i, S_{-i}) + Q_k(S_{-i}). \end{aligned} \tag{3.18}$$

Subtracting (3.17) from (3.18) implies (3.16), and hence, (3.8) follows. □

As the proof made no assumption of the strategy space $\mathbb{S}$, Eq. (3.8) must be valid for any (S_i, S_{-i}) and $(T_i, S_{-i}) \in \mathbb{S}' \subset \mathbb{S}$. At the same time, the constraint in power plays no part in the proof and should not affect the analysis. The following corollaries are direct consequences.

Corollary 3.1. *The game $\mathscr{G}_2$ and every similar game whose strategy space is a subset of $\mathbb{S}$, including $\mathscr{G}_1$, are exact potential games.*

Corollary 3.2. *For such games, pure-strategy Nash equilibrium exists and can be obtained through sequential best/better-response dynamics.*

Remark 3.1. The previous analysis, as was presented in our original work [14], was based on a traditional technique of establishing potential games by first introducing a potential function and verifying it with the definition. On the other hand, one can alternatively make use of our framework in Chap. 2 to cross-examine such proofs.

To establish that the game $\mathscr{G}_2$ is an exact potential game, we note that its utility functions in (3.12) satisfy the bilateral symmetric information (BSI) property (2.55). The pairwise interactions between any two players i and j are represented by

$$w_{ij}(S_i, S_j) = w_{ji}(S_j, S_i) = -\sum_{k=1}^{K}(\omega_{ji}^k + \omega_{ij}^k) \tag{3.19}$$

which are symmetric between players i and j. Thus, Theorem 2.21 ensures that $\mathscr{G}_2$ is an exact potential game with the associated potential function

$$F(S) = -\sum_{i=1}^{N}\sum_{j=1, j<i}^{N} w_{ij}(S_i, S_j). \tag{3.20}$$

One can also easily verify that (3.20) is equivalent to (3.13).

Remark 3.2. In the formulation of $\mathscr{G}_2$, we have also followed the forward method documented in Sect. 2.4.2, similar to the formulation of $\mathscr{G}_1$. The utility functions were designed in a manner that exhibited symmetry of observations when formulating a potential game.

3.3.3 Domination in the Strategy Set

A careful examination reveals that the proposed game also exhibits dominated strategies, i.e., a strategy that always yields smaller utility than another for a player regardless of what its opponents' strategies are. This leads to the following theorem.

Theorem 3.2. *Given a strategy S_i of player i. Derive a new strategy T_i by replacing an element with value 1 in S_i by 0 (assuming T_i is valid). Provided that all other parameters remain the same, S_i is always (at least weakly) dominated by T_i for every player i, i.e.,*

$$U_i(T_i, S_{-i}) \geq U_i(S_i, S_{-i}), \quad \forall i \in \mathscr{N}. \tag{3.21}$$

Proof. Assume that $a_{im} = 1$ is replaced by $a'_{im} = 0$. Clearly, U_i can be rewritten into

$$\begin{aligned} U_i(S_i, S_{-i}) &= \left(\sum_{j=1}^{N} \omega_{ji}^m + \sum_{j=1}^{N} \omega_{ij}^m \right) + \sum_{k=1, k\neq m}^{K} \left(\sum_{j=1}^{N} \omega_{ji}^k + \sum_{j=1}^{N} \omega_{ij}^k \right) \\ &= U_{i,m}(S_i, S_{-i}) + W \end{aligned} \tag{3.22}$$

where $W = \sum_{k=1,k\neq m}^{K} \left(\sum_{j=1}^{N} \omega_{ji}^k + \sum_{j=1}^{N} \omega_{ij}^k \right)$ is a constant and $U_{i,m}(S_i, S_{-i}) \leq 0$.

With $a'_{im} = 0$, $\omega'^m_{ij} = \omega'^m_{ji} = 0$ for all i and j. Thus, $U_i(T_i, S_{-i}) = W$. Clearly, $U_i(S_i, S_{-i}) \leq U_i(T_i, S_{-i})$ and the theorem is proven. □

The weakly dominated case corresponds to when player i is the only one using the subcarrier. Under this circumstance, when it stops using the subcarrier, it will not affect the rest of the players and their utility functions remain unaltered.

However, for most of the time, the domination will be strict. Basically, it means that players will favor a strategy which yields less CCI to obtain a less negative utility value. That is, on subcarrier k, there are other co-channel players besides player i and by not transmitting, player i receives and generates less CCI. Following the same argument, players thus have a tendency to remove subcarriers. They will keep on doing so until each of them uses only one subcarrier, at which point the game $\mathscr{G}_2$ is reduced to $\mathscr{G}_1$. In that case, the following corollary is a direct implication.

Corollary 3.3. *In the strategy set of player i, the subset* $\Phi_i = \{\mathbf{a}_i^T \,|\, \mathbf{a}_i^T \mathbf{1} = 1\}$ *consists of dominant strategies. That is, transmitting on a single subcarrier is a dominant strategy.*

In defining the subset of dominant strategies above, it is understood that corresponding to each strategy S_i in Φ_i are a group of strategies dominated by S_i in the complement $\mathbf{S}_i \setminus \Phi_i$, but not all of them. For example, if $S_i = [1\ 0\ 0\ 0]$, then $[1\ 0\ 1\ 0]$, $[1\ 1\ 0\ 1]$ and so on are dominated by S_i, but $[0\ 1\ 1\ 0]$ is not dominated by S_i in the sense of Theorem 3.2. The domination in strategies is against the original objective for each player to have a sufficient number of subcarriers to transmit, so as to improve the overall spectrum usage. Hence, it is proposed that the domination in strategy for the game should be removed by allowing a player to request in advance how many subcarriers it is going to compete. The exact number of subcarriers for each player will be determined via a proposed method in Sect. 3.4. Under this condition, the original game can be modified accordingly.

3.3.4 The Modified Game

In the modified game, the previous parameters remain the same except that each player requests for a fixed number of subcarriers K_i, where $1 \le K_i \le K$. For player i, an eligible strategy is S_i, drawn from $\mathbf{S}'_i = \{\mathbf{a}_i^T \,|\, \mathbf{a}_i^T \mathbf{1} = K_i\}$, which is a subset of $\{0,1\}^K \setminus \{\mathbf{0}\}$. The overall strategy space is thus $\mathbb{S}' = \mathbf{S}'_1 \times \ldots \times \mathbf{S}'_N$. The modified game is denoted as

$$\mathscr{G}'_2 = [\,\mathscr{N}, \mathbb{S}', \{U_i\}_{i \in \mathscr{N}}\,]. \tag{3.23}$$

Mathematically, the previous optimization problem can also be restated as

$$\begin{aligned} &\max_{\mathbf{a}_i^T} U_i, \quad \forall i \in \mathscr{N} \\ &\text{s.t.} \begin{cases} \mathbf{a}_i^T \in \{0,1\}^K \setminus \{\mathbf{0}\}, & \mathbf{a}_i^T \mathbf{1} = K_i, \\ \mathbf{p}_i^T(\mathbf{A}) \ge 0, & \mathbf{p}_i^T \mathbf{1} \le P_{max}. \end{cases} \end{aligned} \tag{3.24}$$

We see that in $\mathscr{G}'_2$, strategy domination in the sense of Theorem 3.2 does not exist. Also, we have an immediate result.

Theorem 3.3. *The game $\mathscr{G}'_2$ is an exact potential game.*

Proof. This is a direct consequence of Corollary 3.1 since $\mathbb{S}' \subset \mathbb{S}$. □

Remark 3.3. The above result is also directly available from Theorem 2.23 in the previous chapter, where we established that imposing constraints on the strategy space of an exact potential game results in another exact potential game.

3.4 Allocation of the Number of Subcarriers

From the previous analysis, an important implication is that every player i will compete with a fixed number of subcarriers K_i to avoid the unnecessary strategy elimination. This fact gives rise to the need for a method to decide the number of subcarriers for the players before the actual gameplay. Thus, a distributed scheme is proposed where a player can use both its own QoS requirement, in terms of an expected data rate, and the locally available information about its neighbors in the network to autonomously decide the number of subcarriers. The method is referred to as the "Autonomous Number of Subcarrier Selection" (ANSS) scheme, which is described by Algorithm 3.1.

The ANSS scheme essentially defines a network etiquette by which at the beginning, players entering the network should go through some interactions prior to the actual gameplay. This allows them to locally explore the network, acquire and exchange information, thus enforcing certain degrees of "self-awareness" among

Algorithm 3.1 The ANSS scheme

I. ASSUMPTION:
Player i determines its interference radius r_i according to (3.7).
Player i's requested number of subcarriers is $\lceil R_{i,req}/R_{sub} \rceil$.
II. SELECTION STEP:
for player $i = 1 \to N$ **do**
    Upon entering the network, broadcast a signal of identical power on the control channel.
    The receiver listens and detects the number of transmitters within its radius of interference.
    Identify the number of conflicting neighbors $\tau_i = \sum_{j=1}^{N} C_{ij}$.
    Thus, determine the number of subcarriers according to (3.25).
end for

the players. This approach can also be easily utilized to monitor new players who are entering the networks. If the listening process lasts for an ample period of time, a player will gather sufficient information to determine how many conflicting neighbors within its radius of interference, which for player i is given by the sum of elements of the ith row in the conflict matrix $\mathbf{C}$, i.e., $\tau_i = \sum_{j=1}^{N} C_{ij}$.

The assumptions of ANSS and the selection rule given by Eq. (3.25) below are further explained as follows. It is assumed that every player i may request a certain data rate $R_{i,req}$ bps. Although failing to satisfy this requested rate is undesirable, the type of data service does not require the players to strictly fulfill the rate requirement. Depending on the OFDM system design parameters, one can assume that each subcarrier can support a rate of R_{sub} bps. Hence, it is natural that player i should request a total of $\lceil R_{i,req}/R_{sub} \rceil$ subcarriers. Hence, it is proposed that a player i can determine its number of subcarriers as follows:

$$K_i = \max\left[1, \min\left(\left\lfloor \frac{K}{1+\tau_i} \right\rfloor, \left\lceil \frac{R_{i,req}}{R_{sub}} \right\rceil\right)\right]. \tag{3.25}$$

Equation (3.25) considers the subcarrier allocation problem from both the player and the network perspective. On the one hand, based on the environment information obtained, it is likely that player i should be sharing the K available subcarriers with τ_i other conflicting neighbors. Apparently, among these τ_i neighbors there may exist non-conflicting pairs of players. However, due to incomplete information (player i not aware of the complete conflict matrix except its own row), and taking into account the fairness issue, the best action of that player is to take an equal proportion of the spectrum resource, i.e., $\lfloor K/(1+\tau_i) \rfloor$. This way, to some extent, it also helps minimizing its own CCI imposed on other players by not making excessive demand for subcarriers. On the one hand, it gives a player at least one subcarrier in case the pair is in a very crowded neighborhood, thus maintaining fairness. Furthermore, if the pair is in a very favorable situation with no significant co-channel interferers, it gives the player only a sufficient number of subcarriers rather than all the available subcarriers (i.e., when $\tau_i = 0$, although $\lfloor K/(1+\tau_i) \rfloor = K$, only $\lceil R_{i,req}/R_{sub} \rceil$ subcarriers will be allocated). This provides some flexibility for the player to select the good subcarriers and discard those with bad channel conditions, and hence achieve better energy efficiency while keeping the spectrum utilization high.

3.5 Game Algorithm

In this section, the algorithm for allocating transmission power and achieving a Nash equilibrium is described.

3.5.1 Power Mechanism

In OFDMA systems, waterfilling and its variations [7, 11] are often used for power allocation in order to optimize transmission capacity. However, when capacity-based power allocation is not the main concern and the key issue is CCI mitigation [12, 19], the common practice is to achieve a target transmission quality (e.g., target SINR) at minimal power. Similarly, in the proposed game where CCI minimization is the primary objective, a simpler power mechanism similar to Scheme 1 of the previous chapter can be employed which emphasizes more on fairness.

Consider player i and assume that the channel gain between its transmitter and receiver (g_{ii}^k) is known. During the pre-gameplay period (ANSS), the receiver has learned about its environment, measured the average amount of CCI from the surrounding players, and fed the information back to its transmitter. Player i who is aware of the expected amount of CCI previously would then set an interference margin μ_i to counter-measure this. The purpose of μ_i is to account for CCI. For player i to achieve a target SINR γ_i^* in an AWGN channel, it needs to transmit at a higher power to combat CCI in order to maintain the receiver performance. Real-time monitoring of CCI can be challenging and increases the implementation complexity. Therefore, the interference margin μ_i is used to account for the expected amount of CCI present at the receiver, the value of which can be estimated by the average amount of CCI seen at the receiver during the gameplay. The shortcoming of this is that the target SINR will not be strictly guaranteed after the gameplay, but it will be shown later via simulations that adequate SINR performance can still be obtained.

As such, if player i has a target SINR γ_i^* on each subcarrier, based on the channel gains (g_{ii}^k), the noise floor σ^2 and the margin μ_i it will be able to estimate the power level p_{ik} required to achieve this target. Furthermore, if the player uses multiple subcarriers, the same power budget will be distributed among K_i subcarriers and we can set maximum power per subcarrier at $P_{\max}/K_i$. The power level estimated in this way will be used during the gameplay. Such scheme reduces the complexity at the expense of slight derivation in SINR from the desired threshold. Specifically, assume player i uses subcarrier k then its transmitted power is estimated by

$$p_{ik(\mathrm{dBm})} = \min\left\{\sigma^2_{(\mathrm{dBm})} + \gamma^*_{i(\mathrm{dB})} + \mu_{i(\mathrm{dB})} - g^k_{ii(\mathrm{dB})}, (P_{\max}/K_i)_{(\mathrm{dBm})}\right\}. \tag{3.26}$$

Remark 3.4. Equation (3.26) ensures that there is a one-to-one correspondence from each subcarrier assignment matrix to a power matrix, i.e., $\mathbf{P} = \mathbf{P}(\mathbf{A})$. As such, subcarrier and power are two equivalent game strategies in this study.

3.5.2 Sequential Best-Response Play

As stated in Corollary 3.2, Nash equilibrium convergence for a potential game is ensured and can be obtained through a sequential best-response dynamic among the players. Let T_i be the new strategy of player i obtained via the best-response dynamic, defined by Eq. (1.16), which is restated here for the current game as follows

$$T_i = \underset{S_i \in \{\mathbf{a}_i^T \mid \mathbf{a}_i^T \mathbf{1} = K_i\}}{\arg\max} \; U_i(S_i, S_{-i}). \tag{3.27}$$

Algorithm 3.2 summarizes the details of the sequential iterative best-response scheme. The game is initialized with a random subcarrier assignment for each player. This is likely to be a nonequilibrium state and the first player will take action to try to improve its utility value by looking for its best-response strategy after observing the opponents' actions. After one active player has made the decision, the power matrix and assignment matrix are updated. The next player will become active and repeat the process to obtain its best response. The iteration continues until

Algorithm 3.2 Best-response algorithm for the distributed OFDMA game

I. INITIALIZATION STEP:
for player $i = 1 \rightarrow N$ **do**
  Assign a random strategy $S_i \in \{\mathbf{a}_i^T \mid \mathbf{a}_i^T \mathbf{1} = K_i\}$.
  if $a_{ik} = 1$ **then** set p_{ik} according to (3.26)
  else $p_{ik} \leftarrow 0$
  end if
end for
II. ITERATION STEP:
while Nash equilibrium has not been reached **do**
  for player i in a predetermined sequence **do**
    for subcarrier $k = 1 \rightarrow K$ **do**
      The receiver measures the noise and CCI, then feeds those values back to the transmitter.
    end for
    for subcarrier combination $S_i \in \{\mathbf{a}_i^T \mid \mathbf{a}_i^T \mathbf{1} = K_i\}$ **do**
      Use the power mechanism (3.26) to estimate the utility function (3.12).
    end for
    Decide the best-response strategy T_i according to (3.27).
    Update the new subcarrier and power vectors $\mathbf{a}_i^T$ and $\mathbf{p}_i^T$.
  end for
end while

a Nash equilibrium is reached. The sequential order by which players take turn can be determined by the order of network entry, during the ANSS stage.

The total search space for player i has a total of $\frac{K!}{K_i!(K-K_i)!}$ possibilities. The algorithm complexity is therefore $O(INL)$ where N is the number of players, I is the expected number of rounds until convergence (one round ends when all players have acted), and $L = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{K!}{K_i!(K-K_i)!}\right)$ is the mean number of strategy combinations over all players.

3.6 Optimality Studies via Price of Anarchy

In this OFDMA potential game, a pure-strategy Nash equilibrium can always be obtained via best response, which is computationally tractable. However, a typical characteristic of games with large, discrete strategy spaces is that multiple Nash equilibria can exist and there can be vast differences between two different Nash equilibria in terms of performance. At the same time, it is often that the Nash equilibrium is inefficient, compared to the social Pareto optimum point. This social optimum point can always be found using a centralized, exhaustive search method. However, it is known that such an approach is NP-hard. Therefore, it is important to assess the degree of optimality of the Nash equilibrium allocation point obtained via the best-response method with respect to the social optimum. The technique of PoA can provide us a tool to study this issue.

The concept of PoA is formally introduced in Sect. 1.3.4. Briefly, a performance measure (i.e., a real number) is assigned to each outcome of the game, including the Nash equilibria as well as the social optimum. A common measure is the utilitarian welfare function $\Theta(S)$, which is defined to be the sum of all individual utility functions in (1.18). Denote $\hat{S}$ the social optimum point. Then, the PoA of strategy S is defined in (1.19) to be the ratio between $\Theta(S)$ and $\Theta(\hat{S})$ or vice versa, depending on whether the game maximizes or minimizes players' utility functions.

Based on the definition and convention for PoA, when applied to the proposed game, it is noticed that each player maximizes the negative interference sum in the utility functions, which is equivalent to minimization of a positive interference sum. Thus, mathematically, the PoA for strategy S will be given by

$$PoA(S) = \frac{\Theta(S)}{\Theta(\hat{S})}. \tag{3.28}$$

Moreover, since the exact potential function $F(S)$ in (3.13) is exactly half the sum of all utility functions, it can be used equivalently as a scaled version of the utilitarian welfare function of the game. As such, one can now formally define

$$\Theta(S) \triangleq F(S) = \frac{1}{2}\sum_{i=1}^{N} U_i(S). \tag{3.29}$$

As $\Theta(S)$ refers to a negative interference sum, $\Theta(\hat{S})$ will have the least absolute value, so $PoA(S) \geq 1$, $\forall S$. A quantity of interest is the worst-case PoA corresponding to the worst Nash equilibrium $S_{wc} = \arg\min_{S\in\mathbb{E}} \Theta(S)$. Here, $\mathbb{E} \subset \mathbb{S}$ denotes the set of Nash equilibria. S_{wc} gives the performance lower bound for $\mathcal{G}_2$. In the next lemma, the performance upper bound is discussed.

Lemma 3.1. *The social optimum point of* $\Theta(S)$ *for the proposed potential game coincides with the best Nash equilibrium, i.e.,* $\hat{S} = \arg\max_{S\in\mathbb{E}} \Theta(S)$. *Thus, it is the upper bound for the achievable performance of the best-response algorithm.*

Proof. In a potential game, any Nash equilibrium corresponds exactly to a (local) maximum of the exact potential function $F(S)$ [16]. Therefore, the social optimum given by $\hat{S} = \arg\max_{S\in\mathbb{S}} \Theta(S)$, which is the global optimum of $\Theta(S)$ and $F(S)$, must be one Nash equilibrium (the best equilibrium) of $\mathcal{G}_3$, i.e., $\hat{S} \in \mathbb{E}$. As such, $\hat{S}$ is reachable via the best-response dynamics and is the upper performance bound of this algorithm. □

3.7 Simulation Results

Extensive computer simulation was carried out to study the system and the proposed approach using MATLAB. Firstly, the convergence property of the potential game will be verified. Secondly, in order to investigate the details of OFDMA resource allocation, the results of a particular one-shot are documented, as well as statistical results from long-run simulation in comparison with a few other schemes, which give more insights into the performance of the system. Numerical evaluation of PoA will also be carried out to estimate the optimality degree of the proposed algorithm. Furthermore, the effect of the number of allocated subcarrier K_i on the game's performance is also studied.

In the entire simulation, the parameters were set according to Table 3.1. In particular, the target SINR per subcarrier was $\gamma_i^* = 13\,\text{dB}$ for every player, which is necessary for 16-Quadrature amplitude modulation (QAM) to meet the BER requirement of 10^{-5}. In addition, all the links were assumed to undergo independently and identically distributed Rayleigh fading, as well as path loss. The OFDM subcarrier was designed to support a rate of 10 kSymbols/s; thus, with 16-QAM, the bit rate supported is $R_{sub} = 40$ kbps. Moreover, the players' requested rates $R_{i,req}$ were assumed to be within 30 and 150 kbps, following a truncated exponential distribution with mean at 60 kbps.

Table 3.1 Simulation settings for the distributed OFDMA system

System parameters	Values
Number of players, N	20
Number of subcarriers, K	5
Cell radius, r	1 km
Path loss exponent, λ	3
Noise power level, σ^2	10^{-13} W
Bandwidth per subcarrier, B	10 kHz
Maximum power per user, $P_{\max}$	20 mW
Modulation scheme	16-QAM
Supported OFDM symbol rate	10 kSymbols/s
BER requirement, P_e	10^{-5}
Target SINRs per subcarrier, γ_i^*, $\forall i$	13 dB
Interference margins, μ_i, $\forall i$	3 dB

3.7.1 Nash Equilibrium Convergence

To verify the convergence to a Nash equilibrium of the best-response algorithm, a special case when all players occupied two subcarriers is demonstrated in Fig. 3.4. The following example is meant only for the purpose of graphical illustration. In an actual game, due to the ANSS scheme, the number of subcarriers varied from player to player, but Nash equilibria would be obtained similarly and convergence properties would be maintained.

It is observed that the game quickly settled down to a Nash equilibrium solution. With $K_i = 2$, there were $L = 10$ possible valid strategies for each player to select from, each of which is labeled with a binary sequence on the vertical axis. At the same time, Fig. 3.5 shows the evolution of the exact potential function during the course of the example game. Each step on the horizontal axis refers to a decision made by a single player. Thus, N (= 20) steps constitutes one round. As expected, a monotonically increasing behavior until saturation can be noticed. Furthermore, in Fig. 3.4, it is seen that the number of rounds I was 4. It was observed that most of the actual simulations also quickly converged, where the expected value of the number of rounds I was only 4 or 5. Essentially, this illustrates three important facts: (1) existence of a pure-strategy Nash equilibrium; (2) convergence of the sequential best-response dynamics; and (3) finite improvement paths with no improvement cycles. All of these exhibit the properties of a potential game.

3.7.2 Performance of a One-Shot Game

The performance of the system will be evaluated to assess the efficiency and fairness of the proposed game. The criteria for evaluation include power, SINR and capacity. Firstly, a one-shot is demonstrated. In the example, the nodes' distribution in the

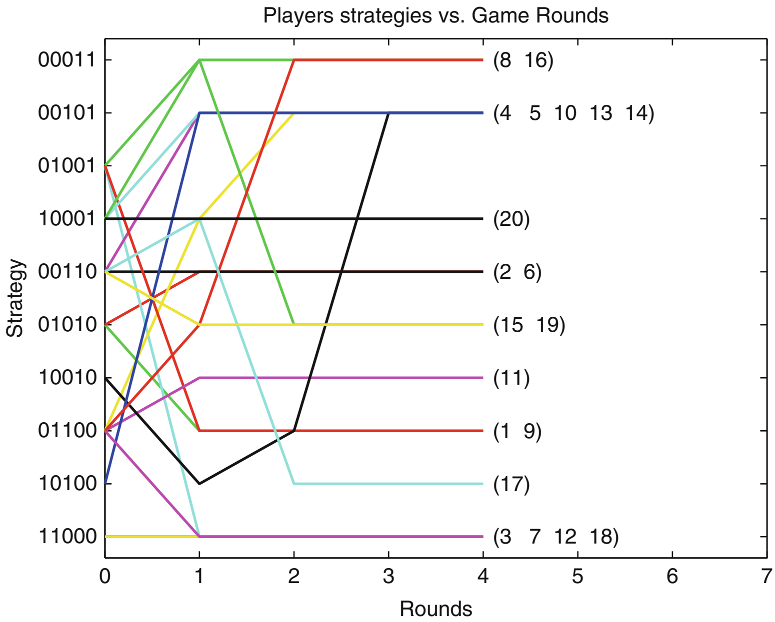

Fig. 3.4 Convergence of the game $\mathscr{G}_2'$ towards Nash equilibrium

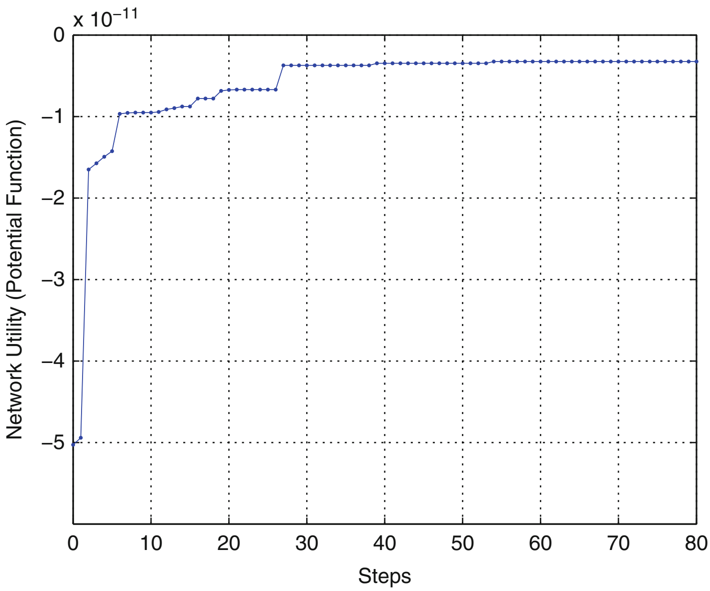

Fig. 3.5 Evolution of the exact potential function of $\mathscr{G}_2'$

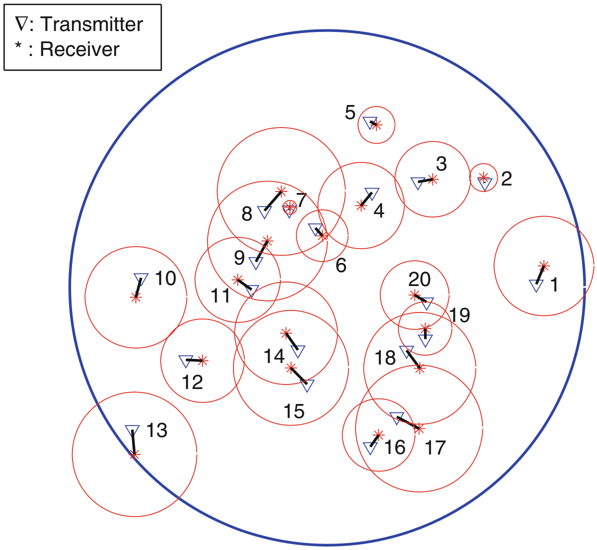

Fig. 3.6 Locations of the transmit-receive pairs for the one-shot

Table 3.2 Allocation of subcarriers at equilibrium for the one-shot

Subcarrier	Player(s)
1	1, 3, 9, 12, 15, 20
2	1, 2, 3, 5, 6, 12, 13, 14, 17
3	2, 5, 7, 10, 11, 12, 16, 20
4	2, 3, 5, 8, 10, 13, 14, 19
5	1, 4, 5, 11, 12, 13, 15, 16, 18

cell is depicted in Fig. 3.6. For viewing purpose, the players' indices are all labeled. Moreover, their radii of interferences which were determined according to (3.7) are also drawn. The players then followed the ANSS scheme and the best-response play to successfully reach a Nash equilibrium. Table 3.2 shows which players were occupying a certain subcarrier at Nash equilibrium.

Closer investigation shows that the algorithm yielded reasonable and desirable results. In Fig. 3.6, conflicts were observed among pairs 6, 7, 8, 9 and 11. They would likely cause heavy CCI if they transmitted on the same subcarrier. Nevertheless, the ANSS and iterative sequential play have allowed those players to intelligently avoid CCI by selecting non-overlapping subcarriers. This fact is evident in Table 3.2, which shows that none of the subcarriers was shared by more than one in the cluster. Notice that players 7 and 11 were an exception as they did not have conflict and could share subcarrier 3. For other conflicting clusters, heavy CCI was also avoided.

Next, in Figs. 3.7 and 3.8, both initial and final SINR values are displayed. The SINR values in dB are grouped according to the subcarrier, and plotted against the indices of the players occupying that subcarrier. It is clear that the final allocation

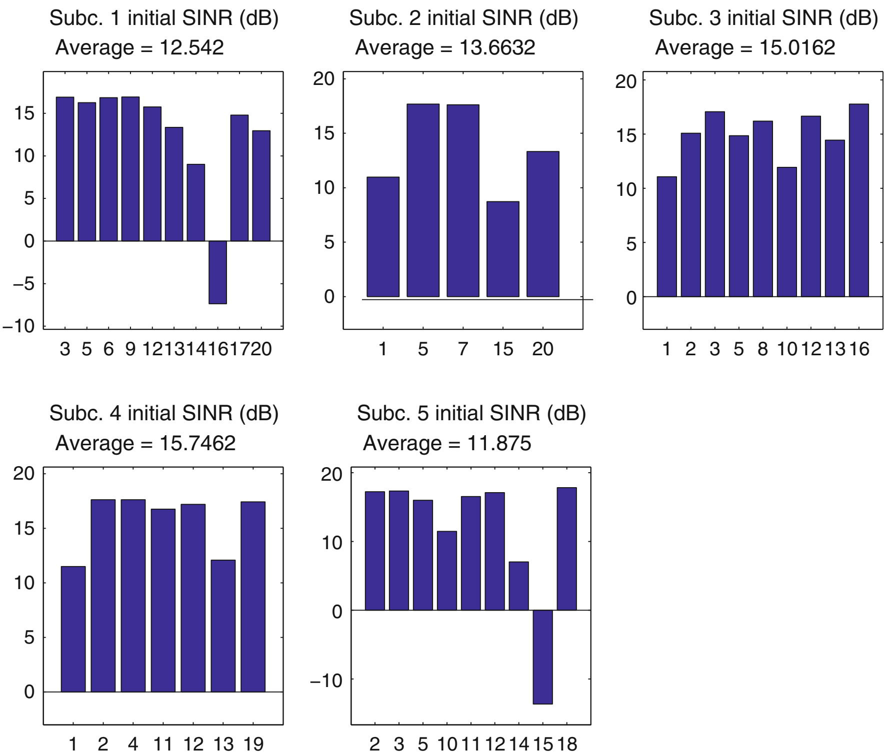

Fig. 3.7 Initial SINR seen from each subcarrier for the one-shot

point showed improvement in both spectrum efficiency and fairness. Spectrum efficiency was enhanced as most players achieved better SINR in all the subcarriers. Most significantly, in subcarrier 5, the average SINR was increased from 11.88 dB (see Fig. 3.7) to 15.50 dB (see Fig. 3.8). Furthermore, at the Nash equilibrium, the variation in SINR of different players in the same subcarriers as well as the deviation from the SINR target of 13 dB had been decreased. Besides, no players had to suffer from severely low SINR (see Fig. 3.8).

The power and capacity performance obtained in Figs. 3.9 and 3.10 give more insights into the system performance. Compared to the initial power levels which were relatively high, the power consumption at the Nash equilibrium was reduced considerably for several players and the whole system (see Fig. 3.9). Besides, no players had to use up their entire power budget $P_{\max}$. Hence, the higher efficiency in SINR performance could be achieved with lower power consumption, which reaffirms the efficiency of the algorithm. For the capacity performance shown in Fig. 3.10, the achievable capacities were compared with the desired (requested) capacities. For every player i, its achievable capacity is estimated by $\frac{1}{B}\sum_{k=1}^{K} a_{ik}R_{ik}$ with R_{ik} given in (3.3), while its requested capacity is $R_{i,req}/B$. It is observed that commonly the achievable capacity would satisfy the requested rate or was only

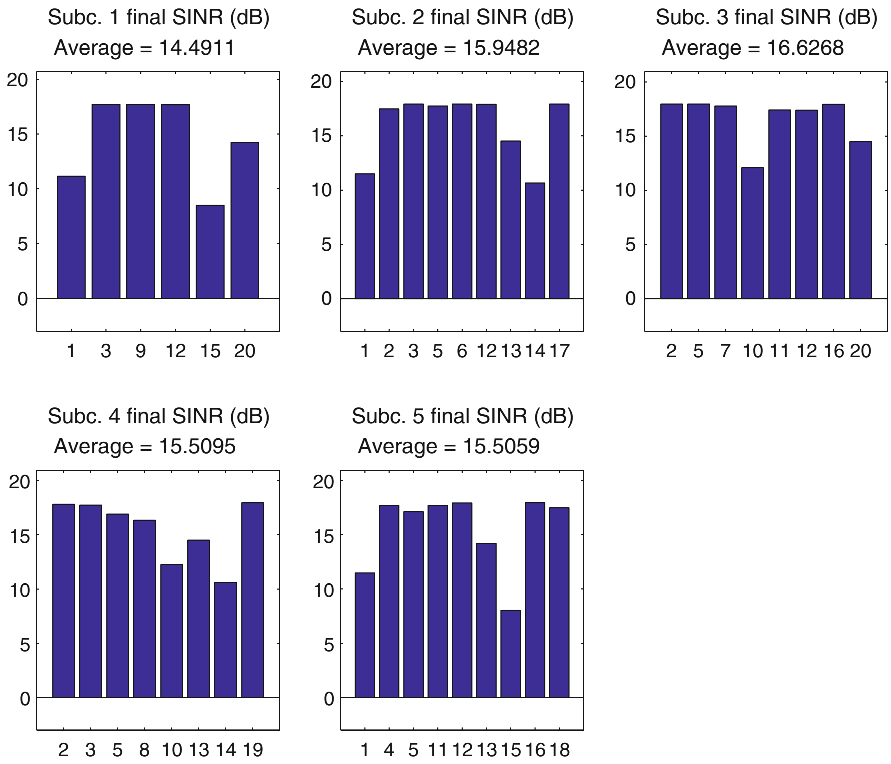

Fig. 3.8 Final SINR seen from each subcarrier for the one-shot

below by a small margin. Exceptions occurred when the players suffered from excessive conflicts with other neighbors and could only use a smaller number of subcarriers than the requested one, as obtained by (3.25). Examples of these are players 8, 9, 19 and 20 which were in crowded neighborhoods (see Fig. 3.6) and thus had to give up their required rates. As CCI reduction is the primary objective, some players may need to sacrifice for the welfare of the entire network.

3.7.3 Effects of the Number of Subcarriers

In this part of the simulation, the effects of K_i on the achievable capacity of the network were studied. To make the comparison compatible, it was assumed that every player had an identical number of subcarrier, i.e. $K_i = K_j, \forall i, j$ and the ANSS scheme was not in effect. For the same players' locations and channel conditions, the average achievable capacities per player of the game $\mathscr{G}_2'$ at the Nash equilibrium were compared for the different number of subcarriers used by each player K_i. Note

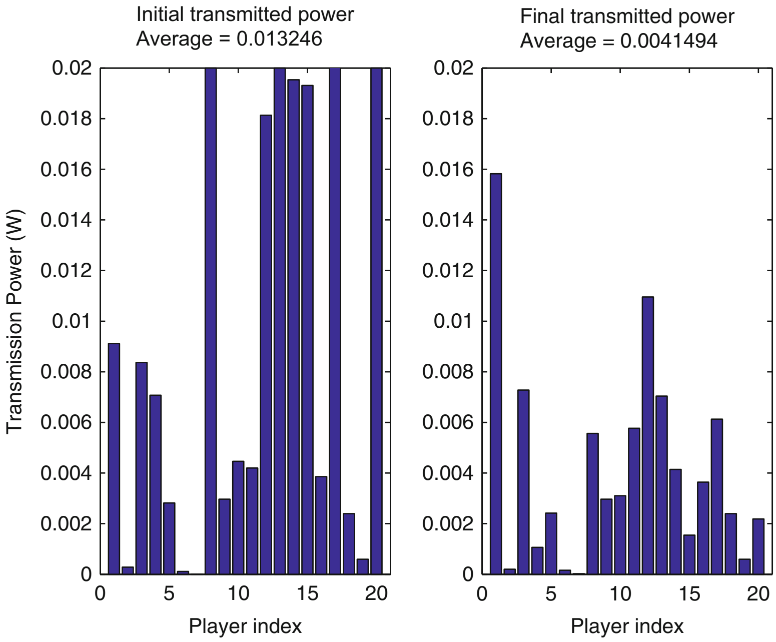

Fig. 3.9 Initial and final transmitted power for the one-shot

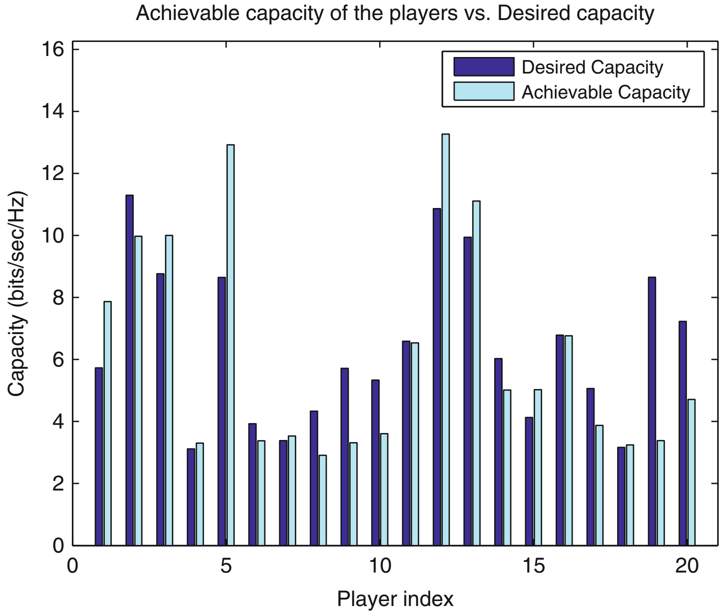

Fig. 3.10 Players' achievable and desired capacities for the one-shot

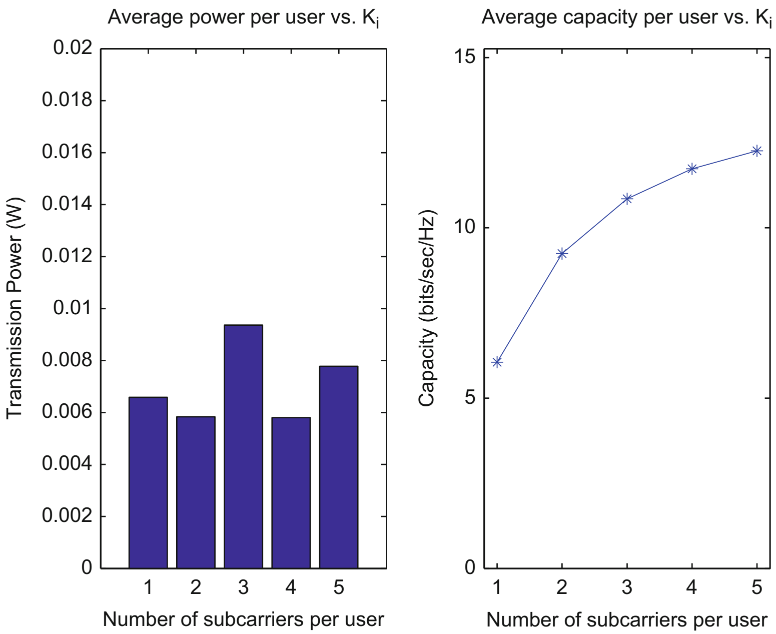

Fig. 3.11 Average power and capacity per user vs. number of subcarriers K_i

that if $K_i = 1$, the game $\mathcal{G}_2'$ reduces to $\mathcal{G}_1$. For fair comparison, the power mechanism was regulated so that for each K_i, all players consumed nearly the same amount of power. If one takes the single-subcarrier case as reference, then for K_i subcarriers, the expected SINR target per subcarrier should be reduced by K_i times. Specifically, $\gamma_{(dB)}^{*(K_i)} = \gamma_{(dB)}^{*} - 10\log(K_i)$. All other parameters were the same as in Table 3.1. K_i took values from 1 to 5. Although only a small number of subcarriers is used in the graphical illustration, the algorithm is scalable to the size of network nodes and number of subcarriers.

Figure 3.11 plots the power usage as well as capacity per player at Nash equilibrium against K_i. First, it should be pointed out that with the previous regulation in transmission power, the variation in average power per player for different values of K_i was not significant. Despite that, the average capacity for per player in the system became better for higher K_i, implying that better efficiency can be achieved with more sharing of spectrum. The capacity curve became saturated since the higher K_i, the higher number of players occupying the same subcarrier and as a result, more CCI was introduced. This is also reflected in the change in the equilibrium values of the exact potential function. Figure 3.12 illustrates that the potential function F became more negative when K_i increased, since more interference terms were added to F. This is in accordance with Theorem 3.2, which predicts a lower potential value for more channel sharing. Nevertheless, in the proposed algorithm, the increase in CCI to a certain extent is still acceptable as long as improvement in average players' capacity can still be achieved.

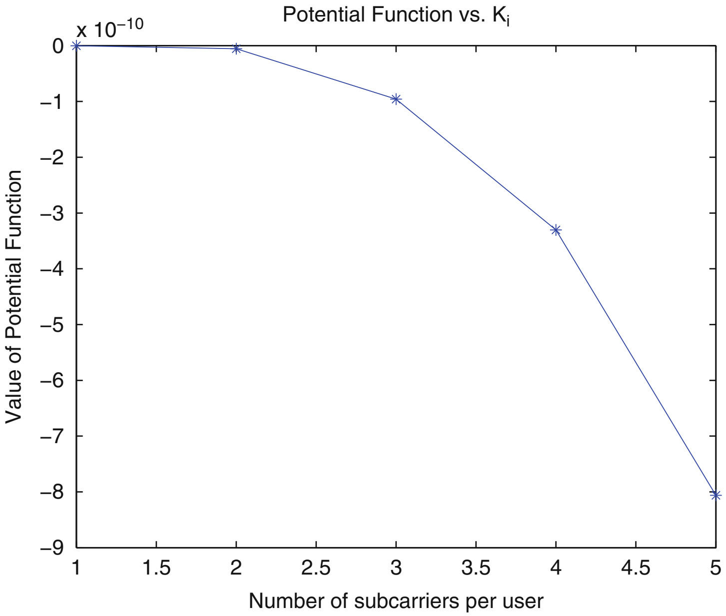

Fig. 3.12 Value of $F(S)$ at Nash equilibrium vs. number of subcarriers K_i

Table 3.3 Comparison of distributed OFDMA schemes

Schemes	Convergence	Fairness index
Random allocation	N.A.	0.693
Proposed game	100 %	0.840
SM/PP	94.06 %	0.768
OFDMA/WF	0 %	N.A.

3.7.4 Performance Evaluation in the Long Run

In addition to looking at one particular instance of the game, the system performance was also statistically evaluated over a long time horizon. This test case considered 5000 replications of the one-shot game, during which most of the global parameter settings were the same as before, except that the players' locations and channel conditions were randomly and independently generated for each replication. Furthermore, to benchmark the performance of the proposed algorithm, it was compared with a few existing RRM schemes, as shown in Table 3.3, including the *SINR Maximization with Power Pricing* (SM/PP) scheme [12], the *OFDMA Iterative Waterfilling* (OFDMA/WF) scheme (a commonly used method in OFDMA networks, partially adopted in [7, 11]); as well as the *Random Allocation* scheme. In SM/PP [12], the utility function consists of a SINR reward and a power pricing term but does not possess the property of a potential game. In OFDMA/WF, the players also select subcarriers in a best-response manner and allocate power through the waterfilling algorithm to maximize their rates. Random allocation is an intuitive method where player i was randomly preassigned with K_i subcarriers to transmit regardless of the other players' actions.

The most important consideration for an allocation scheme is its ability to converge to a stable solution. From Table 3.3, the convergence probabilities for the schemes in comparison over the 5000 one-shot games can be seen. As expected, the proposed scheme converged for every one-shot game, as predicted by the property of potential games. The SM/PP scheme, despite not having 100 % convergence, seemed to perform reasonably well as 94 % of the one-shots converged. However, the OFDMA/WF scheme, which was employed in cellular OFDMA networks (e.g., [7, 11]), did not stabilize in this distributed system. For the random allocation, testing for convergence is not necessary as the subcarrier assignment was preassigned.

Next, for the stable schemes (the proposed game, SM/PP and random allocation), the degree of fairness of the final spectrum allocation was compared. In order to obtain a measure of fairness, the test used the *fairness index* proposed by R. Jain et al. [8]. Considering an N-user system for which user i achieves a quantity of x_i under a particular resource allocation strategy, the fairness index for this allocation is defined as

$$\varphi = \frac{\left(\sum_{i=1}^{N} x_i\right)^2}{N\left(\sum_{i=1}^{N} x_i^2\right)}. \tag{3.30}$$

In the proposed game, this quantity is chosen to be the normalized rate/capacity achieved by the players. Normalized values are necessary as each player has a different requested rate $R_{i,req}$. At the final Nash equilibrium, the achievable rate of player i is given by $R_i = \sum_{k=1}^{K} R_{ik}$, and hence, the normalized rate is taken as $x_i = R_i/R_{i,req}$. Thus, for any one-shot game simulated, various fairness indices could be computed for different schemes. The mean values of the fairness indices over the 5000 replications are displayed in Table 3.3. It can now be seen that the proposed game obtained the highest degree of fairness over random allocation and SM/PP, with an index of 0.840. In practice, this implies that statistically, the proposed scheme let the players meet their requested rates better than the competing schemes. As the quantity x_i used in computing the Jain's index is a player's achievable rate normalized by its requested rate, if the index is closer to 1, then all players receive more even proportions of the resources and hence the values of x_i are closer to each other and to 1. Thus, players transmitting at higher than their requested rates can give up part of their capacities so that those transmitting at lower than their requested rates can improve their capacities. The reason for the proposed game's fairly high index despite a decentralized scheme can be attributed to the utility function which allows for less selfish behaviors and more cooperation among competing players.

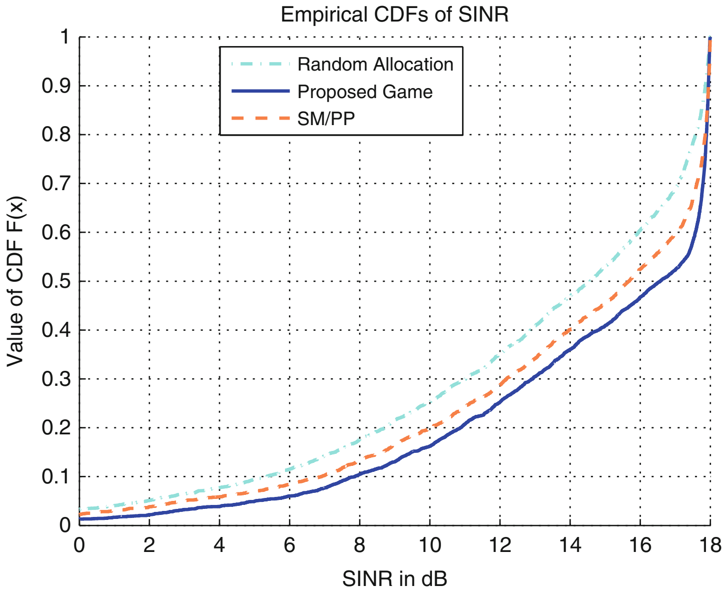

Fig. 3.13 Empirical CDFs of SINR for multiple distributed OFDMA schemes

The empirical cumulative distribution functions (CDF) for the resulting players' SINR and capacities are also presented in Figs. 3.13 and 3.14 for the proposed, SM/PP and random allocation schemes. For SINR, the samples are the SINRs γ_{ik} for any player i over its used subcarrier k; and for capacity, the samples are the individual total capacities $C_i = \sum_{k=1}^{K} C_{ik}$. The results show that the proposed game consistently maintained an advantage over the SM/PP scheme and outperformed the random allocation scheme in achieving both better SINR and capacities. From the CDF for SINR of the proposed game, it is noticed that $Prob[\gamma_{ik} < 13\,\text{dB}] \approx 0.3$. In other words, about 70 % of the time, the achieved SINR for the players can satisfy the target SINR γ_i^* of 13 dB. In addition, based on this numerical studies, 70–75 % of the time, the requested rate can be adequately met by the achievable rate.

An interesting phenomenon could be noticed, which is the piecewise continuous shapes of the CDFs for capacity. This is likely due to the differences in the number of subcarriers that the players were allocated, which is a discrete random variable. That is, if a player gets one subcarrier (i.e., its capacity consists of a single term C_{ik} only), then it is likely that its achievable capacity may fall into a certain region distinct from that of another player who gets two subcarriers (i.e., its capacity includes two terms $C_{jm} + C_{jn}$). In this sense, the first "segment" of the CDF curve corresponds to the capacities of those with one subcarrier, the second "segment" to those with two subcarriers, and so on. Therefore, the aggregate CDF might be seen as the sum of these individual curves.

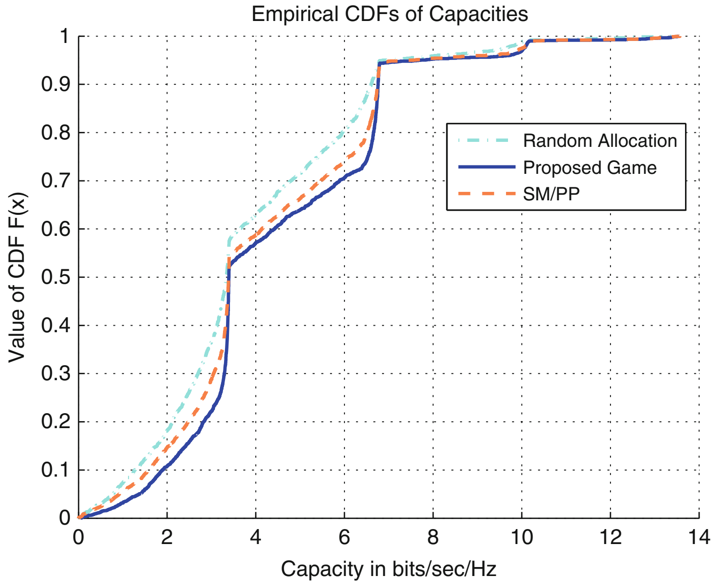

Fig. 3.14 Empirical CDFs of capacities for multiple distributed OFDMA schemes

3.7.5 PoA Evaluation

In order to examine the optimality degree of the proposed method, the PoA of the game was numerically computed during the simulations, which were done over 100 different randomized one-shot games using the global settings in Table 3.1. For each one-shot game, the test searched for the set of Nash equilibria $\mathbb{E}$. The Nash equilibrium which gives the social optimum was then identified. On the other hand, an independent run of the proposed best-response sequential algorithm was performed, and the algorithm would converge to one of the Nash equilibria. For comparison purpose, a random allocation (as described in the previous section) of each one-shot game in question was also considered. Hence, for any one-shot game, different PoA metrics could be computed, including the PoA_{game} for the proposed game, PoA_{rand} for random allocation, and the PoA_{wc} which represents the price of anarchy for the *worst-case* Nash equilibrium. PoA_{wc} is an important performance metric which specifies a lower bound for the efficiency of a game-theoretical method.

In Fig. 3.15, the different PoAs and the social optimum PoA bound, which is always 1, are plotted for various one-shot games, sorted in an increasing order of PoA_{wc} to clearly indicate the performance bounds. As predicted by Lemma 3.1, the PoA_{game} graph is maximum bounded by the PoA_{wc} graph, and minimum bounded by 1. Moreover, it can also be observed that the optimality degree of the proposed game was fairly acceptable as it was consistently close to the optimum bound, within the

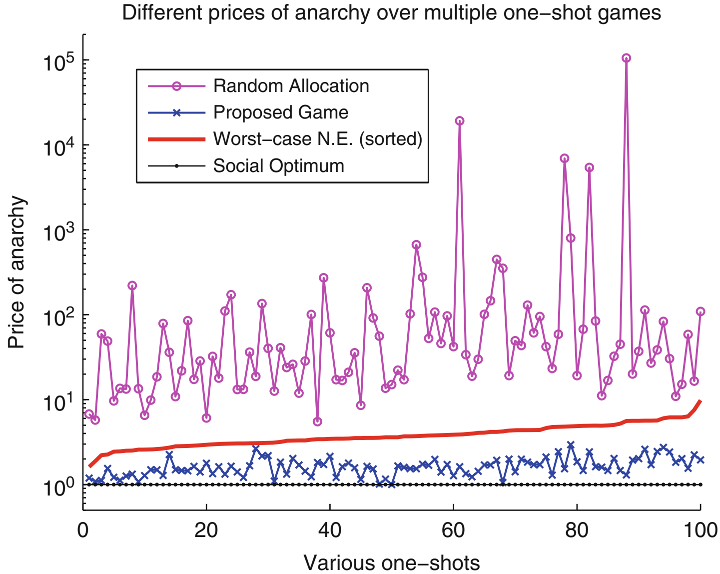

Fig. 3.15 Comparison of PoA for the distributed OFDMA game

10^0–10^1 range, whereas the random allocation was generally worse, occasionally resulting in PoA of order 10^4. In fact, numerical results showed that the mean value of PoA_{game} was about 1.65, and that of the worst case scenario was only 3.92, while on average the random allocation method had a mean PoA of around 10^3. The results imply that the proposed game could reduce the interference power generated to the network by 10–20 dB over the random allocation and was only 2–4 dB behind the optimal solution in the worst case.

3.8 Concluding Remarks

This chapter studies the problem of designing game-theoretical approaches to RRM algorithms to achieve fair and efficient spectrum access for the distributed, ad-hoc OFDMA network of transmit-receive pairs with spatial frequency reuse. The main contributions of this chapter include introducing a framework for distributed OFDMA spectrum sharing and CCI mitigation, formulating the problem into a potential game and analyzing its properties, implementing the iterative best-response algorithm to obtain Nash equilibrium solutions, and evaluating the system performance with extensive simulations. It is worth noting that the game exhibits strategy domination, which motivates the use of the ANSS scheme to deal with the allocation of number of subcarriers to players prior to the gameplay. Numerical results suggest that the scheme not only provides a stable solution to the OFDMA spectrum sharing system, but also improves the spectrum efficiency, and at the same time maintaining a reasonable degree of fairness and optimality.

Although the game-theoretical solution proposed in this chapter is valid for a distributed OFDMA system, its methodology is general enough to be applied to several other systems, one of which is the infrastructure-based OFDMA networks, i.e., ones with existing infrastructures like BSs and access points. It is not difficult to visualize that if one BS serves only one MS, then this link is identical to a transmit-receive pair and the two problems are equivalent. However, for a BS serving multiple MSs, reformulation of the model in order to maintain the potential game properties is a nontrivial problem. In the next chapter, the RRM issues for the multi-cell OFDMA system will be addressed.

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Chapter 4
Potential Game Approach to Downlink Multi-Cell OFDMA Networks

Abstract This chapter investigates the subcarrier allocation problem for a downlink multi-cell multiuser OFDMA network using potential game theory. Each player is considered to be a central base station together with all the mobiles distributed within its coverage area. In such a system, co-channel interferences (CCI), if left uncontrolled, could hinder the transmissions and limit the throughputs of the users, especially those near the cell-edge area. Certain remedies, including power control with pricing, did not seem to solve the problem completely. We specifically address this issue from an interference-minimizing approach, where the utility function adopted is meant to minimize the total CCI among players. Under such a formulation, we show that the formulated game can be mathematically described by a potential game. Hence, a Nash equilibrium will be guaranteed for the proposed game and stable solutions can be achieved via myopic gameplays such as the best/better responses. We propose our iterative algorithm for obtaining the Nash equilibria and address several performance issues such as fairness for edge-users and the price of anarchy. Numerical results show the improvement in efficiency and fairness using this approach.

4.1 Overview

Unlike wireless ad-hoc networks, in an infrastructure-based wireless network, there exist infrastructures such as BSs, access points, switches, routers, etc. that provide wireless connectivities and spectrum access to mobile users. Over the past decades, cellular networks have emerged as a prominent type of infrastructure-based wireless network topologies where a large service area is divided into several smaller regions. Each region is called a cell, with a BS serving the MSs within the cell. OFDMA is one of the leading modulation and multiple access strategies for current and future cellular networks. The same set of frequency channels can be spatially reused by two cells if the resulting CCI levels are tolerable. In order to efficiently utilize the spectrum resources, the frequency channels and transmission power should be dynamically and adaptively allocated in such a way that minimizes CCI and fulfills the QoS performance of users in a single cell or in the entire system. A centralized RRM scheme for cellular OFDMA (e.g., [3, 9, 10, 28]) must take into account the parameters of the entire system, where decisions are made at a central network controller, which is connected to each BS by wired, backhaul transmission. On the

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other hand, in distributed RRM schemes (e.g., [2, 7, 12, 19, 20, 23]), resources can be allocated on a per-BS basis, based on the performance of a particular cell. Distributed schemes might be less optimal compared to centralized schemes, but are less computationally complex. Centralized and distributed schemes can be also used together in multi-layered approaches [18, 21], where the network controller mitigates inter-cell interferences at a longer "super-frame" level while the BSs assist in decision-making by distributively assigning channels to MSs on a frame-by-frame basis. RRM schemes for cellular OFDMA networks have been studied to a considerable extent and comprise a rich body of works in the literature. Papers [9, 24, 27, 29] were among the seminal works on OFDMA in the late 1990s and early 2000s, which formulated mathematical optimization problems in order to maximize the total rates or minimize the total power of the OFDMA system. Thereafter, a large number of research works have been proposed, employing analytical tools from various fields of sciences, including auction [28], graph-theoretical techniques [3], evolutionary algorithms [1], machine learning [2] and especially game-theoretical methods (e.g., [13, 15–17] and references therein). Comprehensive surveys of OFDMA RRM techniques for the interested readers can be found in [22] for earlier schemes, and [25] for more recent approaches.

Since the last decade, various approaches to cellular OFDMA resource allocation based on game theory have also been proposed by several authors. Some of the important works prior to 2010, which were extensively surveyed by the author in [14], can be loosely divided into *cooperative games* and *non-cooperative games*. For cooperative games, a notable work is [6], where the optimal rate allocation for multiple users was associated with the Nash bargaining solution; however, the drawback of the algorithm, which involves the Hungarian assignment method [11], was its high complexity degree. In Chee et al. [4], the Kalai-Smorodinsky bargaining solution was considered and an algorithm of reduced complexity order was suggested. Ibing et al. [8] later made a comparison for four different fairness and bargaining schemes, i.e., the utilitarian, egalitarian, Nash and Kalai-Smorodinsky solutions. On the other hand, other authors also proposed non-cooperative game-theoretical approaches. For example, Han et al. [7] devised a power-minimization game in which power was allocated via a dual waterfilling method, with further rate adaptation via a virtual referee game. Kwon et al. [12] considered rate maximization objectives with linear power pricing in their game, with a utility function comprised of weighted total throughputs and negative total power consumption. The same concept of power pricing was adopted by a few others, for example, by Wang et al. [26] for multi-cell OFDMA with a sigmoid-shaped reward function, or by Yu et al. [30] for the OFDMA-relay network. Capacity maximization was also studied by Liang et al. [19], where the authors introduced another dimension of integer bit-loading to the strategy set of the players.

One common strategy that the previous game-theoretical approaches for cellular OFDMA systems relied upon is the use of iterative, myopic gameplay in order to obtain convergence to a Nash equilibrium. As previously mentioned, such gameplay

Table 4.1 Descriptions of $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{G}_3$

Game	$\mathscr{G}_1$	$\mathscr{G}_2$	$\mathscr{G}_3$
Player	Transmit-receive pairs (ad hoc)	Transmit-receive pairs (ad hoc)	Multiple BSs in downlink cellular OFDMA network
Strategy	Each pair selects a single frequency channel	Each pair selects a combination of subcarriers	Each BS allocates to each of its MSs one subcarrier
Utility function	Interference sum (3.9)	Generalization of (3.9) for all used subcarriers	Generalization of (3.9), to be formulated
Potential game	Yes	Yes	To be determined

can often result in unstable, oscillatory behaviors [14], which is, at first glance, because of the non-existence of pure-strategy Nash equilibria. The underlying cause is due to the prevalence of CCI, which affects not only the stability, but also the fairness of the allocation, where the edge-users are likely to suffer more than the center-users. Meanwhile, in Chap. 3, a potential game formulation for the distributed, ad-hoc OFDMA scenario has been studied, which guarantees that stable Nash equilibrium solutions can be achieved via the use of best-response iterative dynamics. At the same time, the CCI minimization objectives exhibit a reasonable fairness index, implying the feasibility of such a direct approach.

Following up on the previous chapter, the current chapter further develops the potential game formulation for the multi-cell OFDMA systems, based on the earlier generalized interference-sum minimization objective (3.9). The resulting game will be labeled $\mathscr{G}_3$. Table 4.1 provides a description and comparison of the key potential game candidates $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{G}_3$. It will be proven in this chapter that $\mathscr{G}_3$ is indeed a potential game. The key contributions of this chapter are as follows.

- *Potential game formulation* for the downlink multi-cell OFDMA system. The mathematical analysis and proof of the potential game are provided.
- *Optimality analysis*. The performance bounds are again investigated using PoA.
- *Distributed algorithm to solve for Nash equilibria*. Various aspects of the algorithm design are discussed, such as the classification of edge-users and center-users, the best-response and better-response dynamics, power mechanism and signaling issues.
- *Numerical results*. Simulations show that the proposed method could be a good alternative for cellular OFDMA systems due to its stability and fairness advantages.

In the following section, a system model for the multi-cell OFDMA networks under investigation will be described.

4.2 System Model

Consider the downlink of a multi-cell OFDMA network with N cells. Each cell has a BS at the center and several MSs randomly distributed around the cell area. For cell i, define Ω_i as the set of MSs belonging to this cell. The total number of MSs in the entire system is given by $M = \sum_{i=1}^{N} |\Omega_i|$, where $|\Omega_i|$ is the number of MSs of the set Ω_i. There are K available orthogonal subcarriers and each subcarrier has the same bandwidth, which is assumed smaller than the coherent bandwidth so that the links are subject to only flat fading. Interferences from adjacent subcarriers or adjacent symbols are assumed to be negligible. The channel conditions and locations of the MSs are further assumed to be static throughout the duration of a gameplay. Figure 4.1 is an example of such an OFDMA network with $N = 3$, $M = 6$ and $K = 3$.

In the OFDMA downlink, each BS makes use of a subset of the available subcarrier pool and assigns one subcarrier to each of its MSs in a distributed manner. The cellular network has a reuse factor of 1. A subcarrier can be used by more than one BSs, if the resulting CCI is tolerable. Let us denote $\mathbf{A}$ the subcarrier assignment matrix, where $\mathbf{A} \in \{0, 1\}^{N \times M \times K}$, whose element a_{im}^{k} takes a value of 1 if BS i transmits to MS m ($m \in \Omega_i$) via subcarrier k, and 0 otherwise. Consequently, the following constraints apply to any BS i

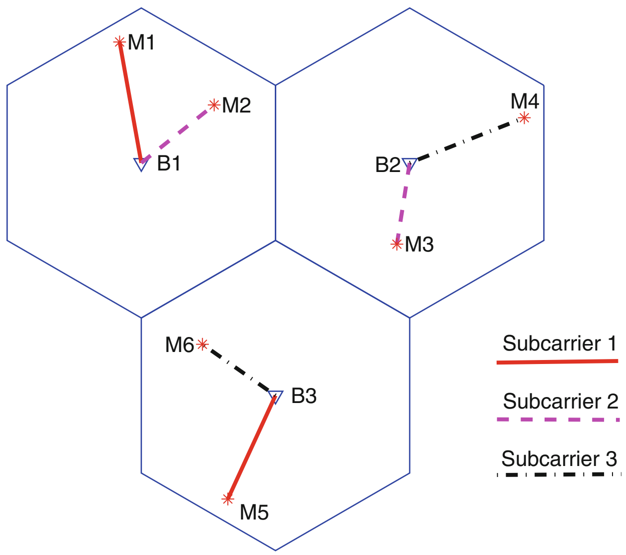

Fig. 4.1 Example of a downlink cellular OFDMA network

$$\sum_{m \in \Omega_i} a_{im}^k \leq 1, \quad \forall k \tag{4.1a}$$

$$\sum_{k=1}^{K} a_{im}^k = \begin{cases} 1, & m \in \Omega_i, \\ 0. & m \notin \Omega_i. \end{cases} \tag{4.1b}$$

Equation (4.1a) suggests that no subcarrier can be assigned to two MSs in the same cell, while (4.1b) means that each BS gives one subcarrier to a MS in its cell. From (4.1b), the overload of MSs in a cell occurs if $\exists i : |\Omega_i| > K$. In well-designed practical systems, this MS overload can be avoided by admission control mechanisms which limit the number of ongoing MSs in each cell. In this study, this specific scenario will be taken into account in Sect. 4.3.3. Otherwise, normally it is understood that $|\Omega_i| \leq K, \forall i$. As such, the system will always admit feasible solutions.

Next, let us denote the channel gain matrix $\mathbf{G} \in \mathbb{R}^{N \times M \times K}$, where g_{im}^k is the channel gain between BS i and MS m through subcarrier k. If $m \notin \Omega_i$, g_{im}^k is in fact the channel gain of the interference path from external BS i to MS m. On the other hand, the transmission power matrix is defined by $\mathbf{P} \in \mathbb{R}^{N \times K}$. Each element p_{ik} represents the transmitted power of BS i on subcarrier k. If a subcarrier is unused, the BS simply transmits 0. The $1 \times K$ ith row vector $\mathbf{p}_i^T$ of $\mathbf{P}$ belongs entirely to BS i and has to satisfy the non-negative and the maximum constraints, given by

$$\mathbf{p}_i^T(\mathbf{A}) \succeq 0, \quad \mathbf{p}_i^T \mathbf{1} \leq P_{\max}, \quad \forall i \tag{4.2}$$

where $\mathbf{1}$ is the all-one $K \times 1$ column vector.

For the link between BS i and MS m on subcarrier k, its SINR is expressed as

$$\gamma_{im}^k = \frac{a_{im}^k p_{ik} g_{im}^k}{\sum_{j=1, j \neq i}^{N} a_{jm}^k p_{jk} g_{jm}^k + \sigma^2} \tag{4.3}$$

where σ^2 is the power of the receiver AWGN and is assumed to be identical for all links. Consequently, the achievable capacity for this link in bps/Hz is given by

$$C_{im}^k = \log_2 \left(1 + \frac{\gamma_{im}^k}{\Gamma} \right). \tag{4.4}$$

Again, $\Gamma = -\ln(5P_e)/1.5$ is a function of the required BER P_e, often known as the SINR gap [5].

4.3 Potential Game Formulation and Analysis

4.3.1 *Game Formulation*

In the multi-cell OFDMA system, each of the N BSs will independently distribute subcarriers to its MSs in order to optimize the performance of the individual cell, which can be viewed as an N-player strategic-form game. One can denote the set of players $\mathcal{N} = \{1, 2, \ldots, N\}$, which represents the N BSs. For BS i, its available strategy S_i is a feasible assignment of subcarriers to its MSs. The action of choosing subcarrier k can be symbolized by its index k, $1 \leq k \leq K$. Then, an eligible strategy S_i of BS i may be represented by a $1 \times |\Omega_i|$ vector

$$S_i = \left[k_1 \; k_2 \;\; \ldots \;\; k_{|\Omega_i|}\right] \tag{4.5}$$

in which BS i has assigned subcarrier k_m to the mth MS of cell i (not the mth MS of the system). Referring to Fig. 4.1, the strategies taken by B1, B2 and B3 are $S_1 = [1\ 2]$, $S_2 = [3\ 2]$ and $S_3 = [1\ 3]$, respectively. Let the strategy set $\mathbf{S}_i$ denote the set of all possible combinations that S_i can take. Moreover, a strategy profile S can be understood as the joint strategy of all players, i.e., $S = (S_i, S_{-i})$. The domain of S is called the strategy space, defined by $\mathbb{S} = \mathbf{S}_1 \times \ldots \times \mathbf{S}_N$.

Now, every player's objective in the game is to maximize its own utility function $U_i(S) : \mathbb{S} \mapsto \mathbb{R}$. The original interference-sum utility function (3.9) for player i can now be redefined, which incorporates the total CCI which BS i imposes on other MSs $m \notin \Omega_i$, as well as those which all MSs $m \in \Omega_i$ suffer as a result of external BSs. This sum can be written as

$$U_i(S) = -\left[\sum_{m \notin \Omega_i} I[i, m] + \sum_{m \in \Omega_i} \sum_{j=1, j \neq i}^{N} I[j, m]\right] \tag{4.6}$$

where $I[i, m]$ is the CCI caused by BS i on MS m. If the CCI is present, it will equal $p_{i\kappa(m)} g_{im}^{\kappa(m)}$, where $\kappa(m)$ is the subcarrier assigned to MS m. Since the assigned subcarrier is yet to be determined, the indicator variable δ_{im}^{k} is introduced such that

$$\delta_{im}^{k} = \begin{cases} 1 & \text{If BS } i \text{ interferes with MS } m \ (m \notin \Omega_i) \text{ via subcarrier } k, \\ 0 & \text{Otherwise.} \end{cases} \tag{4.7}$$

Basically, δ_{im}^{k} indicates whether there is an interference path between BS i and MS m via subcarrier k. In particular, note that $\delta_{im}^{k} = 0$ if $m \in \Omega_i$ because the BS does not interfere its own MSs. The utility function can now be expressed as

$$U_i(S) = -\sum_{k=1}^{K}\left[\sum_{m\notin\Omega_i}\delta_{im}^{k}p_{ik}g_{im}^{k} + \sum_{m\in\Omega_i}\sum_{j=1}^{N}\delta_{jm}^{k}p_{jk}g_{jm}^{k}\right]. \tag{4.8}$$

The formulated game will therefore be denoted by

$$\mathscr{G}_3 = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i\in\mathscr{N}}]. \tag{4.9}$$

Mathematically, the multi-cell OFDMA game can be described by the following distributed optimization problem applied to any player i, i.e.,

$$\max\ U_i \quad \text{s.t. (4.1a), (4.1b) and (4.2)}. \tag{4.10}$$

In the next section, the proof that the multi-cell OFDMA game $\mathscr{G}_3$ given above is an exact potential game will be derived.

4.3.2 Existence of the Exact Potential Function

To verify the potential game property of $\mathscr{G}_3$, the exact potential function $F(S)$ must be established such that the defining Eq. (3.8) for exact potential games is satisfied. In order to derive this function, a novel approach is employed where a fictional game $\mathscr{G}_4$ which is a dual to $\mathscr{G}_3$ is built, whose utility function will be defined so that it becomes a potential game. First, the concept of a "pseudo-player" is introduced as follows.

Definition 4.1. In the multi-cell OFDMA system, a pseudo-player refers to any pairing of a BS with one of its MSs.

In fact, a pseudo-player is a transmit-receive pair where the BS is the transmitter and the MS is the receiver. The number of pseudo-players is exactly M, the number of MSs. Hence, in Fig. 4.1, six pseudo-players are identified: B1-M1, B1-M2, B2-M3, B2-M4, B3-M5 and B3-M6.

The fictional game $\mathscr{G}_4$ is the one played by the M pseudo-players, i.e., the subcarrier allocation game among multiple transmit-receiver pairs. The set of players will be $\mathscr{M} = \{1, 2, \ldots, M\}$. Note that even though a transmitter can be common to a few pseudo-players, they will be treated as different entities in this fictional game. The grouping of the MSs into N cells corresponds exactly to a disjoint partition of the player set $\mathscr{M} = \bigcup_{i=1}^{N}\Omega_i$. For each pseudo-player m, an eligible strategy is $S_m = \mathbf{a}_m^T$ which is a binary $1\times K$ row vector serving as the subcarrier assignment indicator (when subcarrier k is used, its kth element is 1). The strategy set of a pseudo-player is given by $\mathbf{S}'_m = \{0, 1\}^K \setminus \{\mathbf{0}\}$; and the strategy space is $\mathbb{S}' = \mathbf{S}'_1 \times \ldots \times \mathbf{S}'_M$. The counterpart of the subcarrier assignment matrix in $\mathscr{G}_4$ is given by $\mathbf{A}' = [\mathbf{a}_1\ \mathbf{a}_2\ \ \ldots\ \ \mathbf{a}_M]^T$. The alternative channel gain matrix $\mathbf{H} \in \mathbb{R}^{M\times M\times K}$

and power matrix $\mathbf{Q} \in \mathbb{R}^{M \times K}$ can be defined in the same manner as $\mathbf{G}$ and $\mathbf{P}$ of $\mathscr{G}_3$. Furthermore, the cross-interference indicators ϵ_{im}^k for pseudo-players are defined in the same manner as (4.7).

The utility function V_m of the pseudo-player m is modeled after the game $\mathscr{G}_2$ of Chap. 3, i.e.,

$$V_m = -\sum_{k=1}^{K}\left[\sum_{j=1}^{M} \epsilon_{mj}^k q_{mk} h_{mj}^k + \sum_{j=1}^{M} \epsilon_{jm}^k q_{jk} h_{jm}^k\right]. \tag{4.11}$$

Finally, the game $\mathscr{G}_4$ is fully defined as follows.

$$\mathscr{G}_4 = [\mathscr{M}, \mathbb{S}', \{V_m\}_{m\in\mathscr{M}}]. \tag{4.12}$$

Regarding the non-overlapping assignment of a single subcarrier to the MSs in the same cell, a joint strategy such that those two MSs get the same subcarriers, which is valid in $\mathscr{G}_4$ but not in $\mathscr{G}_3$, should be deemed infeasible for $\mathscr{G}_4$. Hence, the feasible strategy space $\tilde{\mathbb{S}}$ is defined as

$$\tilde{\mathbb{S}} = \{S \mid \forall i : \forall m, j \in \Omega_i,\ \mathbf{a}_m^T \text{ and } \mathbf{a}_j^T \text{ have a single 1 at different locations }\}. \tag{4.13}$$

Thus, $\mathscr{G}_4$ may be considered over the feasible strategy space $\tilde{\mathbb{S}}$.

Lemma 4.1. *$\mathscr{G}_4$ is an exact potential game with the exact potential function $F(S) = \frac{1}{2}\sum_{m=1}^{M} V_m$.*

Proof. $\mathscr{G}_4$ is the OFDMA game among transmit-receive pairs similar to $\mathscr{G}_2$ (See Table 4.1). Its strategy space, either $\mathbb{S}'$ or $\tilde{\mathbb{S}}$, is a nonempty subset of that of $\mathscr{G}_2$, so it is an exact potential game and admits the exact potential function $F(S)$, according to Corollary 3.1 from the previous chapter. □

The next lemma features an important result which connects the two games $\mathscr{G}_3$ and $\mathscr{G}_4$.

Lemma 4.2. *Within $\tilde{\mathbb{S}}$, the same subcarrier and power allocation profile for both $\mathscr{G}_3$ and $\mathscr{G}_4$ yields $\sum_{m\in\Omega_i} V_m = U_i$.*

Proof. The two expressions will be equal as they refer to the same physical quantity, i.e., the total CCI. Mathematically, it can indeed be shown that they are equivalent.

In (4.8), let $U_i = -\sum_{k=1}^{K}(X_k + Y_k)$ where $X_k = \sum_{j\notin\Omega_i} \delta_{ij}^k p_{ik} g_{ij}^k$ (index m has been changed to j without affecting the formula) and $Y_k = \sum_{m\in\Omega_i}\sum_{j=1}^{N} \delta_{jm}^k p_{jk} g_{jm}^k$.

For the term X_k, under the conditions stated in the lemma, for all k and j, it follows that $g_{ij}^k = h_{mj}^k$ and $p_{ik} = q_{mk}$, $\forall m \in \Omega_i$. Moreover, due to the restriction of non-overlapping subcarrier assignment in the same cell, the relationships between the δ-parameters and the ϵ-parameters are established as follows.

- If $\delta_{ij}^k = 1$, there exists only one $m \in \Omega_i$ that $\epsilon_{mj}^k = 1$.
- If $\delta_{ij}^k = 0$, then $\epsilon_{mj}^k = 0, \forall m \in \Omega_i$.

For any of the two cases above, the following identity holds

$$\delta_{ij}^k p_{ik} g_{ij}^k = \sum_{m\in\Omega_i} \epsilon_{mj}^k q_{mk} h_{mj}^k. \tag{4.14}$$

Thus,

$$X_k = \sum_{j\notin\Omega_i} \sum_{m\in\Omega_i} \epsilon_{mj}^k q_{mk} h_{mj}^k. \tag{4.15}$$

Besides, if both $m, j \in \Omega_i$, then $\epsilon_{mj}^k = 0, \forall k$. It allows us to add the dummy term $\sum_{j\in\Omega_i} \sum_{m\in\Omega_i} \epsilon_{mj}^k q_{mk} h_{mj}^k = 0$ to X_k, which yields

$$\begin{aligned} X_k &= \sum_{m\in\Omega_i} \sum_{j\in\Omega_i \cup (\mathscr{M}\setminus\Omega_i)} \epsilon_{mj}^k q_{mk} h_{mj}^k \\ &= \sum_{m\in\Omega_i} \sum_{j=1}^{M} \epsilon_{mj}^k q_{mk} h_{mj}^k. \end{aligned} \tag{4.16}$$

For the second term Y_k, similar to (4.14), it can be derived that

$$\delta_{jm}^k p_{jk} g_{jm}^k = \sum_{l\in\Omega_j} \epsilon_{lm}^k q_{lk} h_{lm}^k. \tag{4.17}$$

Hence,

$$\begin{aligned} Y_k &= \sum_{m\in\Omega_i} \sum_{j=1}^{N} \sum_{l\in\Omega_j} \epsilon_{lm}^k q_{lk} h_{lm}^k = \sum_{m\in\Omega_i} \sum_{l\in\bigcup_{j=1}^{N}\Omega_j} \epsilon_{lm}^k q_{lk} h_{lm}^k \\ &= \sum_{m\in\Omega_i} \sum_{l=1}^{M} \epsilon_{lm}^k q_{lk} h_{lm}^k \\ &= \sum_{m\in\Omega_i} \sum_{j=1}^{M} \epsilon_{jm}^k q_{jk} h_{jm}^k \quad \text{(index } j \text{ is used to replace } l\text{).} \end{aligned} \tag{4.18}$$

From (4.11), (4.16) and (4.18), it is clear that $U_i = \sum_{k=1}^{K}(X_k + Y_k) = \sum_{m\in\Omega_i} V_m$. □

We are ready to prove the main result in the next theorem.

Theorem 4.1. *$\mathcal{G}_3$ is an exact potential game with the exact potential function $F(S) = \frac{1}{2}\sum_{i=1}^{N} U_i$.*

Proof. First, due to Lemma 4.2,

$$\frac{1}{2}\sum_{i=1}^{N} U_i = \frac{1}{2}\sum_{i=1}^{N}\sum_{m\in\Omega_i} V_m = \frac{1}{2}\sum_{m=1}^{M} V_m = F(S)$$

which is the same potential function given in Lemma 4.1. Now, suppose that an arbitrary BS i changes its strategy from $S_i = [k_1\ k_2\ \ \ldots\ \ k_{|\Omega_i|}]$ to $T_i = [k_1'\ k_2'\ \ \ldots\ \ k_{|\Omega_i|}']$ where both (S_i, S_{-i}) and (T_i, S_{-i}) are in $\tilde{\mathbb{S}}$. One can now prove that (3.8) is true by introducing a strategy sequence $\{S_i^{(n)}\}$ as follows.

$$\begin{cases} S_i^{(0)} &= S_i, \\ S_i^{(1)} &= \left[k_1'\ k_2\ \ \ldots\ \ k_{|\Omega_i|}\right], \\ & \vdots \\ S_i^{(n)} &= \left[k_1'\ \ \ldots\ \ k_n'\ k_{n+1}\ \ \ldots\ \ k_{|\Omega_i|}\right], \\ & \vdots \\ S_i^{(|\Omega_i|)} &= T_i. \end{cases} \tag{4.19}$$

An intermediate strategy $S_i^{(n)}$ may correspond to an infeasible allocation in $\mathcal{G}_3$ where BS i assigns a subcarrier to more than one MSs (e.g., in multi-casting), as a new subcarrier index k_j' may overlap with an existing subcarrier index k_l. These are allowed as transitional strategies in the context of the proof but are not treated as the feasible actions in the game.

Nevertheless, $(S_i^{(n)}, S_{-i})$ corresponds to a valid strategy profile of the fictional game $\mathcal{G}_4$ in the general strategy space $\mathbb{S}'$. Each step from $S_i^{(n)}$ to $S_i^{(n+1)}$ corresponds to a change in strategy due to a *single* pseudo-player $m_n \in \Omega_i$, e.g., from (S_{m_n}, S_{-m_n}) to (T_{m_n}, S_{-m_n}). As $\mathcal{G}_4$ is a potential game with potential function F, it follows that

$$\begin{aligned} F(S_i^{(n+1)}, S_{-i}) - F(S_i^{(n)}, S_{-i}) &= F(T_{m_n}, S_{-m_n}) - F(S_{m_n}, S_{-m_n}) \\ &= V_{m_n}' - V_{m_n} \end{aligned} \tag{4.20}$$

where V_{m_n}' is the utility function of pseudo-player m_n because of the strategy change. As a result,

$$\begin{aligned} F(T_i, S_{-i}) - F(S_i, S_{-i}) &= (F(T_i, S_{-i}) - F(S_i^{(|\Omega_i|-1)}, S_{-i})) + \ldots + (F(S_i^{(2)}, S_{-i}) \\ &\quad - F(S_i^{(1)}, S_{-i})) + (F(S_i^{(1)}, S_{-i}) - F(S_i, S_{-i})) \\ &= (V_{m_{|\Omega_i|}}' - V_{m_{|\Omega_i|}}) + \ldots + (V_{m_2}' - V_{m_2}) + (V_{m_1}' - V_{m_1}) \\ &= \sum_{m\in\Omega_i} V_m' - \sum_{m\in\Omega_i} V_m = U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}). \end{aligned} \tag{4.21}$$

The final equality of (4.21) is a consequence of Lemma 4.2 and it requires only S_i and T_i to be feasible strategies. □

4.3.3 A Special Case: Overload of MSs

The singular scenario where there can be more MSs than the total number of available subcarriers in one cell is addressed. Suppose for cell i, $|\Omega_i| > K$. It is then proposed that BS i continues to allocate one subcarrier to K MSs. For the other $|\Omega_i| - K$, the BS employs the "no-transmission" strategy on them, leaving them unoccupied temporarily until the resource is released by other MSs. The no-transmission strategy allows the users to cease or delay transmissions if the conditions are unfavorable.

If no-transmission is represented by the index 0, then in (4.5), each k_j, $1 \leq j \leq |\Omega_i|$, can take values from 0 to K. This results in a new strategy space $\mathbb{T}$ and a modified game

$$\mathscr{G}_3' = [\mathscr{N}, \mathbb{T}, \{U_i\}_{i \in \mathscr{N}}]. \tag{4.22}$$

Theorem 4.2. *$\mathscr{G}_3'$ is also an exact potential game with the potential function $F(S)$.*

Proof. It is observed that the only difference introduced by the addition of no-transmission is that the all-zero vector $\mathbf{a}_m^T = \mathbf{0}$ is now valid in the strategy set of a pseudo-player m in $\mathscr{G}_4$. However, such an inclusion does not affect the proof of Theorem 3.1 that $\mathscr{G}_2$ (and $\mathscr{G}_4$) are potential games, as the addition of terms of zero values does not affect the exact potential function. □

4.4 Price of Anarchy Analysis

In this section, the PoA of the proposed game $\mathscr{G}_3$ is studied. Similar to the previous chapter, due to the interference minimization of the game and the definition of the exact potential function, the PoA for a particular strategy profile $PoA(S)$ can be defined according to (3.28), with the same utilitarian welfare function $\Theta(S)$ given by (3.29). The fact that the utilitarian welfare can be represented equivalently by the exact potential function $F(S)$ illustrates a convenient property of potential games, where each strategy profile has a latent "potential" equal to the value of the exact potential function taken at this profile, indicative of its relative strength.

Analogous to the previous chapter, similar performance bounds can be made regarding the PoA of the game $\mathscr{G}_3$.

Lemma 4.3. *The performance of the game $\mathscr{G}_3$ under sequential dynamics such as best/better responses are upper bounded by the social optimum point of $\Theta(S)$ which is also the best Nash equilibrium, i.e., $\hat{S} = \arg\max_{S \in \mathbb{E}} \Theta(S)$; and lower bounded* *by the worst Nash equilibrium* $S_{wc} = \arg\min_{S\in\mathbb{E}} \Theta(S)$*, where* $\mathbb{E}$ *is the set of all pure-strategy Nash equilibria of* $\mathcal{G}_3$.

Proof. The proof is obtained by using the same argument as in Lemma 3.1. □

The best/better-response dynamics can always converge to a Nash equilibrium for potential games, which is another convenient property of this class of games. In the next section, details of these Nash equilibrium convergence dynamics will be discussed, together with other design issues.

4.5 Distributed Algorithm for Nash Equilibrium

4.5.1 Edge-Users Versus Center-Users

For cellular systems, the BS requires higher power to communicate with MSs at the cell edge than to those nearer to the center. Since all BSs tend to do so, CCI increases significantly and the throughputs of edge-MSs hardly improve, sometimes even worsen. A simple greedy throughput maximization approach will end up assigning all resources to the center-MSs, leaving very little to the edge-MSs, consequently resulting in a poor system fairness index. To overcome this issue, besides adopting interference avoidance, it is proposed that all the MSs be separated into two groups and subsequently allocated power based on such grouping, which has not been considered in some previous approaches in the literature (e.g., [6, 7, 12, 20]). Specifically, different weighing factors w_m are given to MSs belonging to different groups. Denote r and d_{im} the cell radius and the distance from MS m to its BS i, respectively. Then, a threshold distance $r_{th} < r$ can be defined, such that MS m is a center-user if $d_{im} \le r_{th}$, and an edge-user if $r_{th} < d_{im} \le R$.

4.5.2 Power Mechanism

In Chap. 3, when the main objective of the system is CCI mitigation, the previously used power mechanism which aims at achieving a target SINR at a minimal power consumption level and maintaining fairness has shown its feasibility without any significant loss in users' performance. Therefore, it will again be adapted in this current game with necessary adjustments to account for the weighing factors.

Consider the link between BS i and its own MS m. The BS computes the receiver sensitivity at the mth MS based on a target SINR γ_m^*, an interference margin μ_m, and the weighing factors w_m, as follows

$$p_{m(dBm)}^{\min} = \sigma_{(dBm)}^2 + \mu_{m(dB)} + \gamma_{m(dB)}^* + w_{m(dB)}. \tag{4.23}$$

Then, the estimated power that BS i should use on subcarrier k assigned to MS m is the function of the MS index m, i.e.,

$$p_{ik}(m)_{(dBm)} = \max\left\{p^{\min}_{m(dBm)} - g^{k}_{im(dB)}, P_{\max}/|\Omega_i|_{(dBm)}\right\} \tag{4.24}$$

where $P_{\max}/|\Omega_i|$ is present to ensure the power constraint. Note that in (4.24), in the special case when $|\Omega_i| > K$ for some i, $P_{\max}/|\Omega_i|$ is replaced by $P_{\max}/K$. Finally, Eq. (4.24) ensures that there is a one-to-one correspondence from each strategy profile to a power matrix, i.e., $\mathbf{P} = \mathbf{P}(S)$. As such, subcarrier and power are two equivalent game strategies in this study.

4.5.3 Signaling Issues

It is worth mentioning that the feasibility of the utility function (4.8) requires complete channel state information. Therefore, the number of channel gains to be estimated, the amount of signaling traffic and information exchange are expected to be very high. It is also necessary to modify the existing protocol to support these estimation and information exchange.

There is a need for each BS to know the amount of CCI it imposes on all the MSs in neighboring cells. For BS i and neighboring MS $m \in \Omega_j$, the MS needs to estimate the gains g^k_{im}. Assuming that channel conditions do not change significantly over the period of gameplay, the following method is suggested. That is, an out-of-band common pilot channel is employed using sufficiently high power to form a fully connected network to perform channel estimations and information exchange. Thus, a network protocol can be enforced such that, prior to the actual gameplay, BS i can send probing messages to external MSs in order for them to estimate their channel gains g^k_{im} to BS i. Each MS then takes turn to broadcast its estimate to BS i. Once BS i completes, it signals another BS and the process is repeated for all other BSs.

4.5.4 Iterative Convergence Dynamics

This section covers the iterative schemes that are used to obtain Nash equilibria. For potential games, there exists an improvement path of the potential function to reach a pure-strategy Nash equilibrium. In order to establish such a path, the game can be played sequentially, where each player employs actions such as best/better responses.

The best response is already employed for the games proposed in the previous chapters. For the current game $\mathcal{G}_3$, player i is to select its best-response strategy T_i if and only if

Algorithm 4.1 Iterative gameplay for the multi-cell OFDMA game

I. INITIALIZATION STEP:
for player $i = 1 \rightarrow N$ **do**
  Select a random strategy S_i that satisfies (4.1a) and (4.1b).
  if $a_{im}^{k} = 1$ **then** $p_{ik} \leftarrow p_{ik}(m)$ according to (4.24).
  else if $\sum_{m \in \Omega_i} a_{im}^{k} = 0$ **then** $p_{ik} \leftarrow 0$.
  end if
end for
II. ITERATION STEP:
while Nash equilibrium has not been reached **do**
  for player i in a predetermined sequence **do**
    for MS $m = 1 \rightarrow |\Omega_i|$ **do**
      for subcarrier $k = 1 \rightarrow K$ **do**
        Measure the noise and CCI, then feed those values back to the BS.
      end for
    end for
    for subcarrier combination $S_i \in \mathbf{S}_i$ **do**
      Use the power mechanism (4.24) to estimate the utility function (4.8).
    end for
    Decide the new strategy T_i according to (4.25) or (4.26).
    Update the new subcarrier and power matrices $\mathbf{A}$ and $\mathbf{P}$.
  end for
end while

$$T_i = \arg\max_{S_i \in \mathbf{S}_i} U_i(S_i, S_{-i}). \tag{4.25}$$

The best-response dynamic requires an exhaustive search over the strategy set $\mathbf{S}_i$ for the best option. Since BS i serves $|\Omega_i|$ MSs, where each MS is allocated with one out of K subcarriers, the total number of possible strategies that belong to player i amounts to ${}_K P_{|\Omega_i|} = \frac{K!}{(K-|\Omega_i|)!}$ options, which limits the scalability of the system. Thus, in this game, the algorithm also considers a method that required less searching, i.e., the better-response dynamic (1.17), whereby player i's better-response strategy T_i is any random strategy such that

$$U_i(T_i, S_{-i}) > U_i(S_i, S_{-i}). \tag{4.26}$$

Algorithm 4.1 shows the detailed steps. The multiple BSs should act in a sequential order, which can easily be determined at the system level during the network planning stage. This is different and easier to handle than in the ad-hoc context where players need to self-organize before each gameplay.

4.6 Simulation Results

Simulations were done to study the performance of the proposed solutions. The system parameters were specified in Table 4.2. The network consisted of $N = 7$ hexagonal cells with omnidirectional BSs and the layout is similar to Fig. 4.2. There were $M = 21$ MSs randomly distributed around the area competing for $K = 5$ subcarriers. On average, there were 3 MSs per cell. The threshold distance to classify MS in the cell edge and cell center was given by $r_{th} = 2r/3$. There was a 50 % probability that a MS fell within the edge area. This study assigned the edge-MSs

Table 4.2 Simulation settings for the multi-cell OFDMA system

System parameters	Values
Number of BSs, N	7
Total number of MSs, M	21
Number of subcarriers, K	5
Cell radius, r	100 m
Threshold radius, r_{th}	$2r/3$
Path loss exponent, λ	3
Noise power level, σ^2	10^{-13} W
Maximum power, P_{max}	40 mW
BER requirement, P_e	10^{-5}
Target SINRs per user per subcarrier, γ_m^*, $\forall m$	13 dB
Interference margins, μ_m, $\forall m$	3 dB
Weighing factor for edge-users, w_e	1
Weighing factor for center-users, w_c	2
Probability of a MS being an edge-user, P_{edge}	0.5

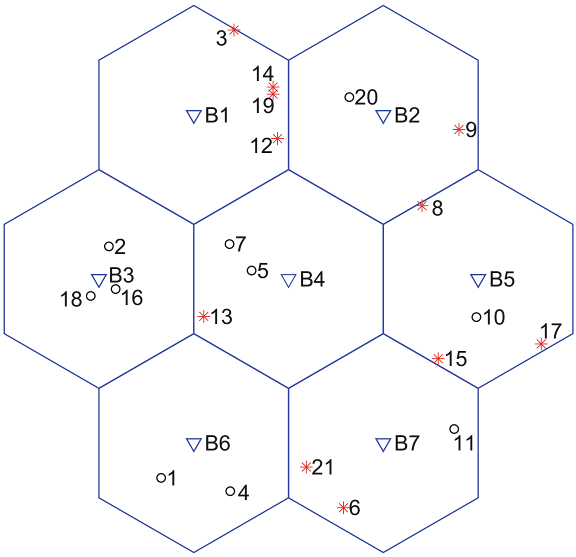

Fig. 4.2 The OFDMA cellular network in the one-shot

a smaller weight $w_e = 1$ than the center-MSs' $w_c = 2$ with the aim to limit CCI. Moreover, in the simulation, the system was not overloaded. The channels were subject to identically and independently distributed Rayleigh fading as well as path loss. The path loss exponent was 3.

4.6.1 Convergence of the Game

The convergence of the proposed algorithm was demonstrated via a one-shot simulation in Fig. 4.2. The indices of the BSs and MSs are indicated, with the edge-MSs depicted by stars and the center-MSs by circles. The better-response dynamic (4.26) was employed in the iterative algorithm. The game was played until a Nash equilibrium was found as the convergent point of the dynamics. In each step which represents a player's decision, the values of the potential function as well as the values of the $N = 7$ utility functions were plotted in Fig. 4.3. As expected, the potential function increased monotonically and converged to a Nash equilibrium.

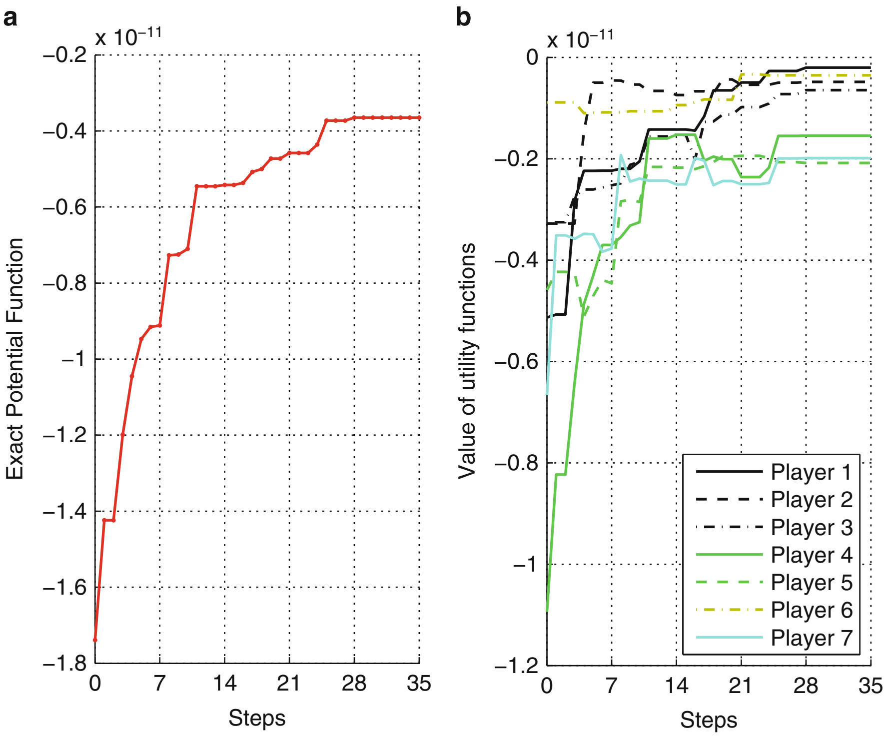

Fig. 4.3 Convergence of the game $\mathscr{G}_3$ in (**a**) The potential function, and (**b**) Separate utility functions

Note that one's utility function improved within its turn by the same amount as the potential function according to the game's property (e.g., in Fig. 4.3b, player 7 in steps 1, 8, and so on). It is also seen that for each player, only 5 rounds (i.e., 35 steps) were needed before convergence occurred.

4.6.2 Performance Evaluation of the One-Shot Game

The achieved SINR, power and capacity allocation for this particular one-shot were examined in this section. In Fig. 4.4, initial SINRs and final SINRs at the Nash equilibrium in dB were displayed for the edge-MSs and center-MSs separately. Note that an improvement in SINR could be seen within both groups. In fact, the average SINR increased from 8.84 to 13.63 dB for the edge-MSs, and from 10.78 to 16.06 dB for the center-MSs. Furthermore, it is observed that at the Nash equilibrium, all the center-MSs and almost all the edge-MSs achieved SINR near their target SINR of 13 dB, except for example, MS 15 which was in a more crowded cell. The overall Jain fairness index of the system, as given by (3.30), with the quantity x_i represented by the individual SINR was estimated to be 0.937, which suggests a high degree of fairness in this approach.

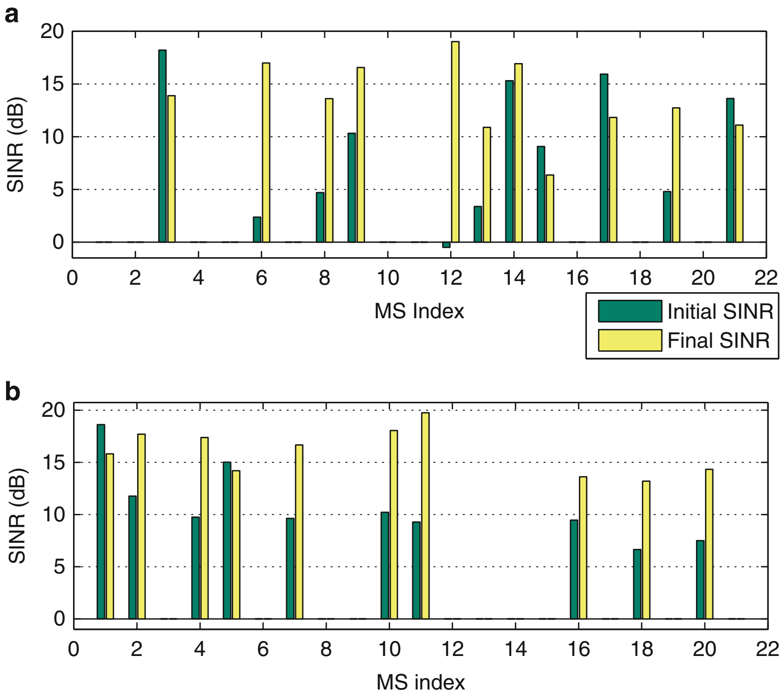

Fig. 4.4 Comparison of SINR for (**a**) Edge-MSs, and (**b**) Center-MSs

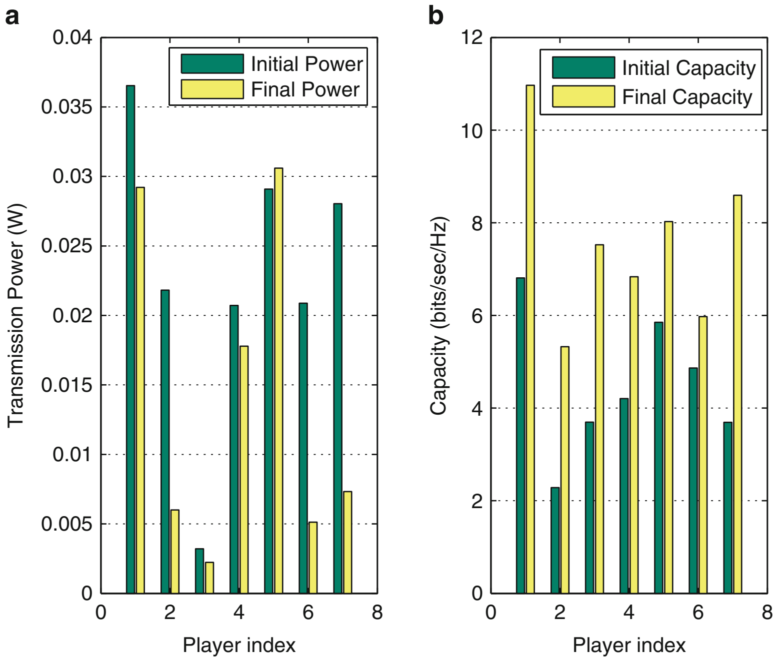

Fig. 4.5 Comparison of (**a**) BS's total power, and (**b**) BS' total capacity

In Fig. 4.5, the achievable power and capacity were compared from the BSs' perspective. Both showed a significant improvement at the Nash equilibrium over the initial point. Due to the interference minimization nature of the game, players showed the tendency to avoid overlapping subcarriers and preferred subcarriers that cost less power. Consequently, the total power and CCI were reduced, which led to improved throughputs. Statistically, the average power consumption of a BS was reduced from 22.9 mW initially to 14.0 mW at the Nash equilibrium. Moreover, the average capacity per BS (i.e., $\frac{1}{N}\sum_{i=1}^{N} C_i$ where $C_i = \sum_{m\in\Omega_i} C_{im}^{\kappa(m)}$) increased from 4.49 to 7.61 bps/Hz.

4.6.3 Performance Evaluation in the Long Run

In this section, the performance of the proposed game with the previous settings was studied over 500 independent one-shots, each with different MS locations and channel conditions. Four RRM methods were compared, namely, Fixed Allocation (FA), Potential Game with better response (PG-better), Potential Game with best response (PG-best), and Waterfilling with Power Pricing game [12] (WF/PP). In the FA scheme, subcarriers and power were randomly assigned to the MSs at the

Table 4.3 Comparison of multi-cell OFDMA schemes

Schemes	Convergence (%)	Iterations	Fairness	Energy efficiency
FA	N.A.	N.A.	0.7030	260.27
PG-better	100	7.892	0.9072	583.63
PG-best	100	3.604	0.9032	561.39
WF/PP	87.20	27.89	0.2808	644.42

beginning without further adaptation. The power allocation given in (4.24) was also used for fair comparison. WF/PP was subject to the same power constraint as the PG schemes, and the subcarriers were assigned greedily to the MSs with highest throughputs and reallocated iteratively until convergence. Its utility function consists of the total transmission rate and a negative cost related to the power consumed [12]. The key performance indicators for comparison were shown in Table 4.3.

The convergence probability is a critical measure for any game theoretic approaches. Both PG schemes could guarantee pure-strategy Nash equilibrium convergence with a 100 % convergence rate. WF/PP achieved a reasonably high convergence rate of 87.2 %. However, the number of iterations (rounds) required for WF/PP before convergence was considerably higher than for PG. PG-best needed less rounds than PG-better; however, within a round, the searching required for PG-better was less due to the non-optimal selection of new strategy. Convergence and iteration counts did not apply to FA.

The other two indicators are the Jain fairness index (3.30), and the energy efficiency defined as the achieved capacity per unit power, i.e.,

$$\eta = \frac{\sum_{i=1}^{N} C_i}{\sum_{i=1}^{N} \mathbf{p}_i^T \mathbf{1}} \text{ (bps/Hz/W)}. \tag{4.27}$$

In terms of fairness, PG-better and PG-best were both superior in the ranking, with fairness indices around 0.9. WF/PP returned a low fairness score, perhaps due to the greedy allocation which allowed some favorable MSs to hog the resources. Nevertheless, WF/PP obtained the best energy efficiency (644.42 bps/Hz/W), due to its throughput maximization nature. PG-better gave roughly the same energy efficiency as PG-best (583.63 vs. 561.39 bps/Hz/W); and both of them were significantly superior to FA (260.27 bps/Hz/W).

For this simulation studies, the empirical CDFs of the achievable capacity of BSs for the four schemes were also obtained. Figure 4.6 shows that FA was consistently worst than the rest. WF/PP, however, obtained occasionally high capacity (over 15 bps/Hz, for about 6 % of the time) for some BSs hogging the resources, while the rest of the BSs had much lower throughputs (50 % having capacity less than 5 bps/Hz) compared to PGs (only 15 %). This partially explains its low fairness score. There was no significant performance difference in the long run between PG-best and PG-better.

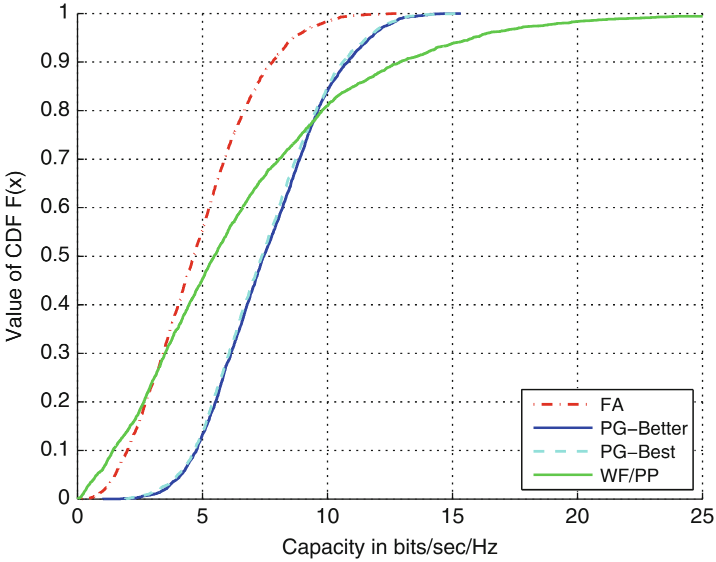

Fig. 4.6 Empirical CDFs of BS's achievable capacity

4.6.4 Price of Anarchy Evaluation

In order to study the degree of optimality of the game, the PoA was numerically evaluated. In each of the 50 simulated one-shots, the set of Nash equilibria $\mathbb{E}$ was obtained. Independent runs of PG-best and PG-better will converge to one of the Nash equilibria. Hence, PoAs can be computed for the Nash equilibria achieved by both dynamics, i.e., PoA_{best} and PoA_{better}, for the worst-case Nash equilibrium, PoA_{wc}, and for the initial allocation point, PoA_{init}. The various PoAs (sorted in increasing order of PoA_{wc}) are plotted in Fig. 4.7. The social optimum bound (which is always 1) is also included. As explained before, PoA_{best} and PoA_{better} are bounded between PoA_{wc} and the optimum bound. Both PG-best and PG-better were able to reach the best Nash equilibrium once or twice, and none of them was consistently better than the other. The mean values of PoA_{best}, PoA_{better} and PoA_{wc} were 1.385, 1.344 and 2.611, respectively. PoA_{init} was far behind at 30.898 on average. A relatively low PoA value suggests the proposed game has a fair degree of optimality.

4.6.5 Impacts of Weighing Factors

In this section, the effects of assigning different weights to the edge-MSs (w_e) and center-MSs (w_c) were examined. Three scenarios were considered: (a) Equal

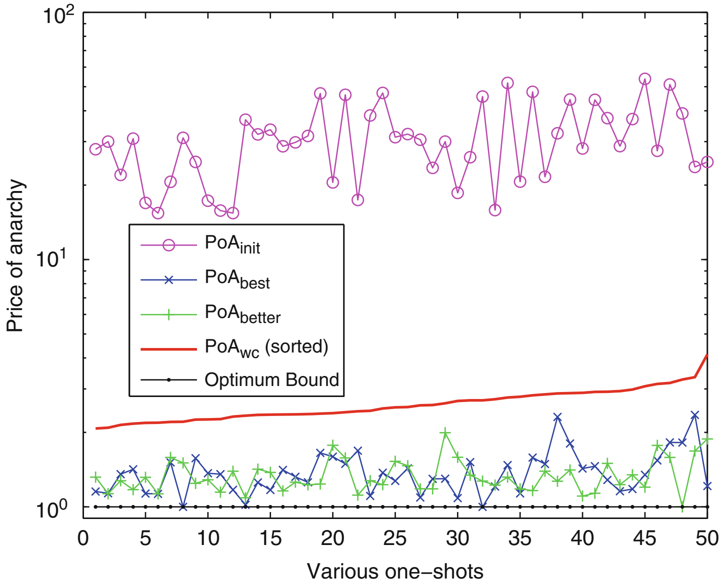

Fig. 4.7 Comparison of PoA for the multi-cell OFDMA game

weights $(w_e, w_c) = (1, 1)$, (b) Center-MSs had higher weights $(w_e, w_c) = (1, 2)$ and (c) Center-MSs had lower weights $(w_e, w_c) = (1, 0.5)$. Note that more power is consumed by the group with a larger weight. The performance was compared in terms of players' capacity and system fairness over 500 independent one-shots with the previous settings. In Fig. 4.8, the CDFs of BS achievable capacity for the three cases are shown. The obtained fairness indices for case (a), (b) and (c) were 0.9250, 0.9043 and 0.9531, respectively. As expected, there was a trade-off between fairness and throughput for the two opposite weighing schemes (b) and (c). The higher weight assigned to the center-MSs, the better the player's throughput, as shown by the gain in the CDF of case (b) over the other two. Case (c) resulted in the highest fairness index, although the differences were relatively small. Hence, different weight combinations can be flexibly assigned depending on whether the concerned objective is fairness or throughput.

4.6.6 System Performance with Increasing Loads

The impact of the number of MSs on the system performance was investigated for $N = 7$ cells and $K = 16$ subcarriers. A maximum power $P_{\max} = 0.1$ W was imposed on each BS, while the other parameters in Table 4.2 was maintained. For the PG scheme, best-response dynamic was used and the performance was compared with FA and WF/PP, in terms of the fairness index (Fig. 4.9a) and the

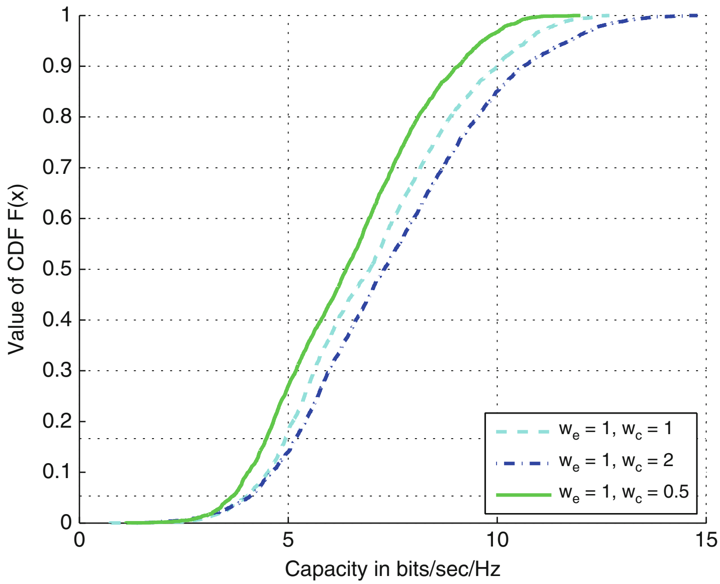

Fig. 4.8 Comparison of various weighing factor combinations

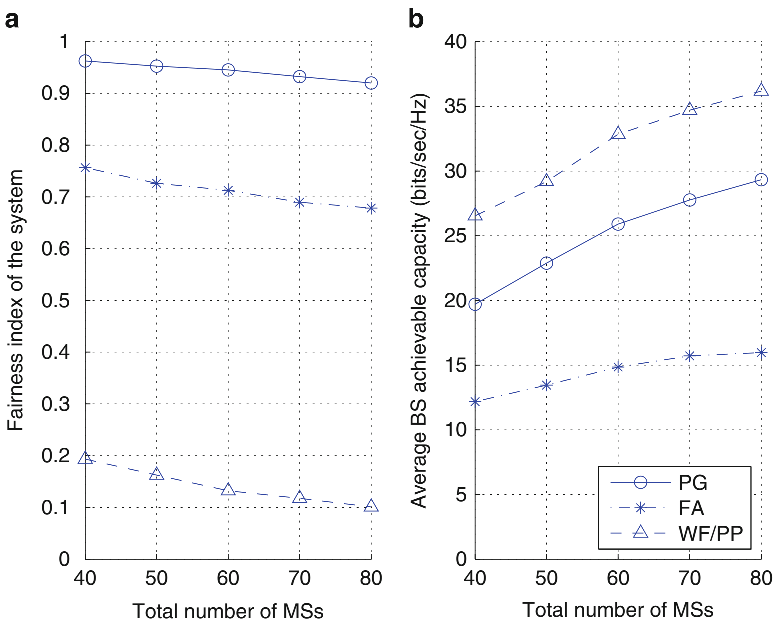

Fig. 4.9 Impacts of the number of MSs on (**a**) Fairness and (**b**) Capacity

average BS achievable capacity (Fig. 4.9b), for different loads. It was seen that the higher the system load was, the overall system fairness deteriorated for all schemes, probably due to the increasing CCI and MS density which further widened the performance gaps among different users. For PG, however, the decrease in fairness had a smaller impact as its score was consistently in the range of 0.9. WF/PP was still worst in terms of fairness due to reasons previously discussed, i.e., the performance gap between the few users hogging the subcarriers and the rest in WF/PP was emphasized and reduced its fairness index. However, because of the users with very high throughputs, WF/PP showed a gain in average BS achievable capacity over PG. The trade-off between fairness and throughput was expected of PG, whose throughput was still significantly better than FA.

4.7 Concluding Remarks

In this chapter, an OFDMA subcarrier assignment method is proposed using game theory. The derivation of the potential game is achieved via the concept of a fictional game involving the pseudo-players. This property also holds for an overloaded system if the no-transmission strategy is included for those MSs who are not able to get a free subcarrier. The system always admits a stable Nash equilibrium solution and can be played using best/better responses, both of which were studied and whose performances were compared. Simulation results indicate the validity of the interference minimization approach in combating CCI, achieving a convergent solution for the RRM problem and enhancing fairness among users, especially when the system load becomes high. Although throughput maximization is not the main objective, the proposed game is still able to maintain a fair degree of energy efficiency and optimality.

Due to the generality of the potential game formulation, although the proposed method is considered for a traditional cellular network, the results discussed in this chapter can also be valid for a much broader range of systems and for other different contexts. One of the important examples is the new femtocell network, which are small-range low-power BSs serving clusters of users. Nowadays, due to the growing wireless traffics, there can be increasingly many wireless hotspots in public areas like shopping centers, airports and train stations, within which there are many densely-placed femtocell configurations. Similarly, in the unlicensed bands, there are also situations where multiple clusters of users try to access the shared spectrum through their own network gateways; and the level of CCI needs to be controlled among these clusters. Each of the coexisting systems can be thought of as an independent player and the protocols described in the chapter can be used as a spectrum access mechanism.

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Chapter 5
Other Applications of Potential Games in Communications and Networking

Abstract This chapter provides summaries of other notable potential game formulations in the area of communications and networking. To align with the theme of this monograph, the selected materials are divided according to the types of potential games involved. Readers can find applications of exact potential games in Sect. 5.1, and applications of pseudo-potential games in Sect. 5.2.

5.1 Applications of Exact Potential Games

Exact potential games can be used to solve a wide range of wireless communications problems, especially in the distributed allocation of radio resources. In earlier chapters of this book, some of these applications have been introduced, e.g., in Examples 2.12–2.14, 2.16, 2.17, as well as in Chaps. 3 and 4. In this section, we discuss a few more notable problem formulations using the potential game framework established in Chap. 2.

5.1.1 Potential Games with Sum of Inverse SINRs as the Potential Functions

Firstly, we examine a number of studies in the literature which define the sum of inverse SINRs as the network objective function and subsequently derive the utility functions in order to define potential games. It is notable that this method of formulating potential games follow the backward method which we have generalized in Sect. 2.4.2. This section covers two applications: the waveform adaptation game (Sect. 5.1.1.1) and the subcarrier assignment game for uplink multi-cell OFDMA systems (Sect. 5.1.1.2).

5.1.1.1 Waveform Adaptation Game in Distributed Networks

Menon et al. [13] proposed a waveform adaptation framework for networks consisting of co-located receivers at a base station (e.g., uplink of cellular networks) which enabled the implementation of distributed, convergent algorithms such as

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best-response dynamics. In [12], the authors expanded the framework to generic networks of distributed receivers with spatial frequency reuse, whose topology is akin to the distributed networks of transmit-receive pairs introduced in Chap. 3 of the book.

Waveform Adaptation Problem

In the waveform adaptation problem, a player which is a transmit-receive pair must adjust their transmitted waveforms, which are often characterized by a *signal-space representation* [20]. That is, the transmitted waveform of a player is assumed to be represented using a set of *M orthonormal basis functions*. By projecting this waveform onto those orthonormal dimensions, one obtains an M-dimensional column vector which is the *signature sequence* associated with this player. Assume that there are N players and for player i, its signature sequence is denoted by $\mathbf{s}_i \in \mathbb{R}^M$. In [12], it is further assumed that signature sequences have unit Euclidean norm, i.e., $||\mathbf{s}_i||^2 = 1, \forall \mathbf{s}_i$.

We denote p_i the transmit power level of player i which is predetermined, and g_{ij} the known channel gain between transmitter of player i and receiver of player j which remains constant over all signal dimensions and waveform adaptation duration. Using the signal-space representation, the SINR experienced by the receiver of player i is derived in [12] as

$$\gamma_i = \frac{p_i g_{ii}}{\mathbf{s}_i^H \left(\sum_{j=1, j \neq i}^{N} p_j g_{ji} \mathbf{s}_j \mathbf{s}_j^H + \mathbf{R}_{zz} \right) \mathbf{s}_i} \tag{5.1}$$

where again $\mathbf{s}_i$ refers to the transmitted signature sequence for waveform of player i; and $\mathbf{R}_{zz} = \mathbb{E}[\mathbf{z}\mathbf{z}^H]$ is the noise covariance matrix for the zero-mean additive Gaussian noise vector $\mathbf{z} \in \mathbb{R}^N$.

Potential Game Formulation

The distributed waveform adaptation problem above was formulated as an exact potential game in [12]. Here, the N transmit-receive pairs form the player set $\mathcal{N}$. The strategy set of each player is defined as the set of available waveforms, or signature sequences, which can be used by the player for transmission. That is,

$$\mathbf{S}_i = \{\mathbf{s}_i |\ \mathbf{s}_i \in \mathbb{R}^M, ||\mathbf{s}_i||^2 = 1\}, \quad \forall i \in \mathcal{N}. \tag{5.2}$$

The strategy space is $\mathbb{S} = \mathbf{S}_1 \times \ldots \times \mathbf{S}_N$.

First, [12] defined a network objective function which is chosen to be the sum of inverse SINRs, given by

$$\Phi(\mathbf{s}_i, \mathbf{s}_{-i}) = \sum_{i=1}^{N} \frac{1}{\gamma_i(\mathbf{s}_i, \mathbf{s}_{-i})}$$
$$= \sum_{i=1}^{N} \frac{\mathbf{s}_i^H \left(\sum_{j=1, j\neq i}^{N} p_j g_{ji} \mathbf{s}_j \mathbf{s}_j^H + \mathbf{R}_{zz}\right) \mathbf{s}_i}{p_i g_{ii}} \tag{5.3}$$

where we use $\mathbf{s}_{-i}$ to denote the joint strategy of player i's opponents.

This function Φ can also be interpreted as a weighted sum of interferences and noise, where the weight for player i is its inverse received power level. Since the sum of inverse SINRs decreases as individual SINRs increase, minimizing this sum $\Phi(\mathbf{s}_i, \mathbf{s}_{-i})$ is a feasible network objective as the network is expected to perform better with all players operate at higher SINRs.

Thus, a candidate potential function for the game is $F(\mathbf{s}) = -\Phi(\mathbf{s})$, where $\mathbf{s}$ denotes the joint strategy profile. Next, $F(\mathbf{s})$ can be decomposed as follows:

$$F(\mathbf{s}_i, \mathbf{s}_{-i}) = -\frac{\mathbf{s}_i^H \left(\sum_{j=1, j\neq i}^{N} p_j g_{ji} \mathbf{s}_j \mathbf{s}_j^H + \mathbf{R}_{zz}\right) \mathbf{s}_i}{p_i g_{ii}} - \sum_{j=1, j\neq i} \frac{\mathbf{s}_j^H \left(\sum_{l=1, l\neq j}^{N} p_l g_{lj} \mathbf{s}_l \mathbf{s}_l^H + \mathbf{R}_{zz}\right) \mathbf{s}_j}{p_j g_{jj}}$$
$$= -\frac{\mathbf{s}_i^H \left(\sum_{j=1, j\neq i}^{N} p_j g_{ji} \mathbf{s}_j \mathbf{s}_j^H + \mathbf{R}_{zz}\right) \mathbf{s}_i}{p_i g_{ii}} - \sum_{j=1, j\neq i} \frac{p_i g_{ij}}{p_j g_{jj}} \mathbf{s}_j^H \mathbf{s}_i \mathbf{s}_i^H \mathbf{s}_j$$
$$\underbrace{- \sum_{j=1, j\neq i} \frac{\mathbf{s}_j^H \left(\sum_{l=1, l\neq i,j}^{N} p_l g_{lj} \mathbf{s}_l \mathbf{s}_l^H + \mathbf{R}_{zz}\right) \mathbf{s}_j}{p_j g_{jj}}}_{\text{Non-contributing term } Q_i(\mathbf{s}_{-i})}. \tag{5.4}$$

Since the candidate potential function can be decomposed into the form in (5.4), the utility function of player i can subsequently be defined in accordance with (2.92) as

$$U_i(\mathbf{s}_i, \mathbf{s}_{-i}) = -\mathbf{s}_i^H \mathbf{X}_i \mathbf{s}_i, \tag{5.5}$$

where $\mathbf{X}_i$ is a symmetric M-by-M matrix and can be written as

$$\mathbf{X}_i = \frac{\sum_{j=1, j\neq i}^{N} p_j g_{ji} \mathbf{s}_j \mathbf{s}_j^H + \mathbf{R}_{zz}}{p_i g_{ii}} + \sum_{j=1, j\neq i} \frac{p_i g_{ij}}{p_j g_{jj}} \mathbf{s}_j \mathbf{s}_j^H. \tag{5.6}$$

The resulting game is therefore defined as $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i\in\mathcal{N}}]$.

Theorem 5.1. *The aforementioned game $\mathcal{G}$ is an exact potential game with potential function $F(\mathbf{s})$.*

Proof. This result is obvious due to the application of backward method, as established in Sect. 2.4.2. □

Nash Equilibrium Characterization

As $\mathcal{G}$ is an exact potential game, a Nash equilibrium can be obtained using best-response dynamics. Furthermore, the Nash equilibria of this game can be further characterized [12]. Recall our original assumption that $||\mathbf{s}_i||^2 = \mathbf{s}_i^H \mathbf{s}_i = 1$. As such, (5.5) can be rewritten as

$$U_i(\mathbf{s}_i, \mathbf{s}_{-i}) = -\frac{\mathbf{s}_i^H \mathbf{X}_i \mathbf{s}_i}{\mathbf{s}_i^H \mathbf{s}_i}, \tag{5.7}$$

where the expression on the right-hand side is the negative Rayleigh quotient of matrix $\mathbf{X}_i$ [9]. Thus, from matrix theory, $U_i(\mathbf{s}_i, \mathbf{s}_{-i})$ can be maximized by the eigenvector $\mathbf{s}_i^*$ corresponding to the minimum eigenvalue $\lambda_{k,\min}$ of $\mathbf{X}_i$. Note that $\mathbf{X}_i$ is real and symmetric so $\lambda_{i,\min}$ is real. That is, the Nash equilibrium $(\mathbf{s}_i^*, \mathbf{s}_{-i}^*)$ satisfies

$$\mathbf{X}_i \mathbf{s}_i^* = \lambda_{i,\min} \mathbf{s}_i^*, \quad \forall i \in \mathcal{N}. \tag{5.8}$$

5.1.1.2 Subcarrier Assignment Game in Uplink Multi-Cell OFDMA Systems

The cellular OFDMA system can be modeled as a potential game via the interference minimization approach, as shown in Chap. 4. Alternatively, the sum of inverse SINRs can also be used as a potential function in formulating a potential game. This approach was investigated by Buzzi et al. [3], as well as Cai et al. [4].

Problem Formulation

The uplink of a multi-cell OFDMA network with M cells is considered. Each cell has a BS and several associated MSs. There are K available orthogonal subcarriers to be allocated. For cell m, Ω_m is the set of MSs assigned to this cell. The total number of MSs in the cellular network is $N = \sum_{m=1}^{M} |\Omega_m|$, where $|\Omega_m|$ is the number of MSs of the set Ω_m.

For the distributed uplink scenario, the N MSs instead of the BSs act as players. Here, we can denote $\mathbf{A} \in \{0, 1\}^{N \times K}$ the subcarrier assignment matrix, whose element a_{ik} takes a value of 1 if MS i selects subcarrier k to transmit, and 0 otherwise. Hence, the subcarrier assignment vector for MS i is given by $\mathbf{a}_i^T$, the $1 \times K$ ith row vector extracted from $\mathbf{A}$.

The channel gain matrix is $\mathbf{G} \in \mathbb{R}^{N \times M \times K}$, where $g_{i,m}^k$ is the channel gain from MS i to BS m via subcarrier k. The transmit power matrix is $\mathbf{P} \in \mathbb{R}^{M \times K}$. Each element $p_{ik} : 0 \le p_{ik} \le P_{\max}$ represents the transmit power of MS i on subcarrier k. It is assumed that power allocation is decoupled from the subcarrier assignment process, i.e., p_{ik} is assumed to be known, or is set to 0 if MS i does not transmit on subcarrier k.

The SINR of the link from MS i to its associated BS m_i on subcarrier k is given by

$$\gamma_{ik} = \frac{p_{ik} g_{i,m_i}^k}{\sum_{j=1, j \neq i}^{N} a_{jk} p_{jk} g_{j,m_i}^k + \sigma^2} \tag{5.9}$$

Potential Game Analysis

A strategic form game can be formulated for the uplink multi-cell OFDMA subcarrier assignment problem above. The set of players $\mathcal{N}$ consists of the N MSs. The strategy for player i can be expressed by $S_i = \mathbf{a}_i^T$, similar to Chap. 3. It was further assumed in [3] that each MS may select L subcarriers out of the K available subcarriers to transmit, where L is a fixed, predetermined number. Therefore, the strategy sets are given by

$$\mathbf{S}_i = \{\mathbf{a}_i^T \,|\, \mathbf{a}_i^T \in \{0, 1\}^{1 \times K}, \mathbf{a}_i^T \mathbf{1} = L\}, \forall i \in \mathcal{N}. \tag{5.10}$$

Similar to Sect. 5.1.1.1, an exact potential game can be formulated via the backward method by adopting the (negative) sum of inverse SINRs as the candidate potential function. This function can be expressed as

$$\begin{aligned} F(S) &= -\sum_{k=1}^{K} \sum_{i=1}^{N} \frac{a_{ik}}{\gamma_{ik}(S_i, S_{-i})} \\ &= -\sum_{k=1}^{K} \sum_{i=1}^{N} \frac{a_{ik} \left(\sum_{j=1, j \neq i}^{N} a_{jk} p_{jk} g_{j,m_i}^k + \sigma^2 \right)}{p_{ik} g_{i,m_i}^k} \end{aligned} \tag{5.11}$$

where $\gamma_{ik}(S_i, S_{-i})$ indicates the dependency of γ_{ik} on the game's strategy profile.

Subsequently, the decomposition is done as follows:

$$F(S_i, S_{-i}) = -\sum_{k=1}^{K} \frac{a_{ik}\left(\sum_{j=1,j\neq i}^{N} a_{jk} p_{jk} g_{j,m_i}^{k} + \sigma^2\right)}{p_{ik} g_{i,m_i}^{k}} - \sum_{k=1}^{K} \sum_{j=1,j\neq i}^{N} \frac{a_{jk} a_{ik} p_{ik} g_{i,m_j}^{k}}{p_{jk} g_{j,m_j}^{k}} \underbrace{- \sum_{k=1}^{K} \sum_{j=1,j\neq i}^{N} \frac{a_{jk}\left(\sum_{l=1,l\neq i,j}^{N} a_{lk} p_{lk} g_{l,m_j}^{k} + \sigma^2\right)}{p_{jk} g_{j,m_j}^{k}}}_{\text{Non-contributing term } Q_i(S_{-i})} \quad (5.12)$$

Thus, the utility functions are defined as

$$U_i(S_i, S_{-i}) = -\sum_{k=1}^{K} \frac{a_{ik}}{\gamma_{ik}} - \sum_{k=1}^{K} \sum_{j=1,j\neq i}^{N} \frac{a_{jk} a_{ik} p_{ik} g_{i,m_j}^{k}}{p_{jk} g_{j,m_j}^{k}}, \forall i \in \mathcal{N}. \quad (5.13)$$

The following result is apparent from application of backward method:

Theorem 5.2. *The corresponding game $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i\in\mathcal{N}}]$ is an exact potential game with potential function $F(S)$.*

Remark 5.1. The utility function $U_i(S_i, S_{-i})$ in (5.13) consists of the negative sum of player i's own inverse SINRs, minus the sum of "signal-to-individual-interference" ratios $\frac{a_{jk} a_{ik} p_{ik} g_{i,m_j}^{k}}{p_{jk} g_{j,m_j}^{k}}$ over all subcarriers, as seen at each of player i's opponents. Note that in this formulation, to calculate $U_i(S_i, S_{-i})$, a large amount of channel information exchange is required.

5.1.2 Potential Games Under Synthetic Symmetric Interference

We next look at games where the constructions of utility functions can be grouped under the technique of synthetic symmetric interference. Synthetic symmetry of interference refers to man-made adjustment of observations (in this case, the observed signal interferences) in order to create bilateral symmetric interactions (i.e., the BSI property which was introduced in Sect. 2.4.1). Consequently, associated games enjoy the BSI property and are exact potential games.

5.1.2.1 Preliminary: Synthetic Symmetric Interference

Motivated by the convergence properties associated with BSI potential games, Neel et al. [15–17] discussed certain conditions and adjustments that could result in pairwise symmetric interference in cognitive radio networks. First, assume that each player's objective is to adapt its waveform ω_i (which is represented by a signature sequence as discussed in Sect. 5.1.1.1) in order to avoid interferences from other radios. A player can be either a single radio user, a link, or a cluster of radios depending on the context. Then, in [15], the *unaltered* observation (i.e., interference) seen by player i due to player j is given by

$$I_{ji} = p_j g_{ji} \rho(\omega_j, \omega_i) \tag{5.14}$$

where p_j is the transmit power of player j, g_{ji} is the channel gain from j to i and $\rho(\omega_j, \omega_i)$ is the correlation between the basis functions of ω_j and ω_i. Recall that the BSI condition holds if $I_{ji} = I_{ij}$, $\forall i, j$.

The simplest case of BSI [15] is in a scenario of two cognitive radio users, each functioning as a transceiver (with co-located transmitting and receiving antennae). They transmit at the same time with identical power level $p_i = p_j = p$, using a set of orthogonal waveforms which satisfy $\rho(\omega_j, \omega_i) = 1$ if $\omega_i = \omega_j$ and 0 otherwise. Moreover, channel reciprocity is assumed, i.e., $g_{ij} = g_{ji}$. Under these conditions, $I_{ji} = I_{ij}$ can be readily obtained.

For more general case, BSI does not hold. However, [15] showed that alterations or approximations can be introduced to design synthetic symmetric interference. One notable alteration is the procedure of scaling the observations by the player's transmit power. The assumption $p_i = p_j$ is usually infeasible as autonomous radios often use adaptive power control. Nevertheless, multiplying the observation by the player's own transmit power (i.e., $I'_{ji} \triangleq p_i I_{ji}$) can overcome this issue, by recognizing that $p_i p_j = p_j p_i$. Assuming orthogonal waveforms and that channel reciprocity $g_{ij} = g_{ji}$ still holds, we can easily see that for the altered observations,

$$I'_{ji} = p_i p_j g_{ji} \rho(\omega_j, \omega_i) = p_j p_i g_{ij} \rho(\omega_i, \omega_j) = I'_{ij}, \forall i, j \tag{5.15}$$

We also see that multiplication by one's own power is a cost-effective procedure because players are assumed to know their own power and no additional information is required [15].

5.1.2.2 Interference Avoidance Game in Canonical Networks

The aforementioned synthetic symmetric interference approach was adopted in order to study performance of greedy asynchronous distributed interference avoidance (also known as GADIA) algorithms by Babadi et al. [2]. Their formulation was subsequently extended in [24, 26, 29, 31].

Problem Formulation

Canonical networks are general network structures proposed in [2]. The network is comprised of multiple spatially distributed, autonomous *nodes*. Each node is not a single communications device, but instead a collection of multiple entities. These entities are closely located and there are intra-node communications among them. There is also one representative entity in each node. A good example of such networks is wireless body-area networks where each person (e.g., medical patients or soldiers) wearing body sensors can be considered a node, within which there is a coordinator acting as the cluster head.

The problem considered in [2] is of channel selection in canonical networks. Each node as an autonomous decision maker is equivalent to a player. As usual, we denote the set of nodes by $\mathscr{N} = \{1, 2, \ldots, N\}$. The spectrum consists of K non-overlapping bands. It is also assumed that each node uses one channel (i.e., channel $c_i, i = 1, \ldots, K$) for intra-node communications and spatial reuse of spectrum is allowed amongst nodes. As such there exists inter-node interference. The strategy of node i is thus c_i and its opponents' strategies is c_{-i}. The interference indicator is defined [2, 24, 26] by a Kronecker delta function

$$\delta(c_i, c_j) = \begin{cases} 1 & c_i = c_j, \\ 0 & c_i \neq c_j. \end{cases} \tag{5.16}$$

We see that this is equivalent to the waveform correlation function $\rho(\omega_i, \omega_j)$ in (5.14) in the one-dimension case of orthogonal channel selection.

The total experienced interference for node i is given [2, 24, 26] by

$$I_i = \sum_{j \neq i} p_j w_{ji} \delta(c_j, c_i) \tag{5.17}$$

where w_{ji} is the mutual interference coefficient between node j and node i. Subsequently, the utility function of player i is defined as the negative weighted interference sum as

$$U_i(c_i, c_{-i}) = -p_i I_i = -\sum_{j \neq i} p_i p_j w_{ji} \delta(c_j, c_i) \tag{5.18}$$

For BSI, we need pairwise symmetric interference, i.e., $w_{ji} = w_{ij}$. In Babadi et al. [2], the authors simply assumed that interference between any two nodes is reciprocal and symmetric. In Wang et al. [24], it was argued that as entities of a node are closely located such that the distance between two nodes is significantly greater than intra-node distances, channel gains between two nodes are approximated by path loss as follows:

$$w_{ij} = w_{ji} \approx \begin{cases} \left(\frac{1}{d_{ij}}\right)^{\alpha} & d_{ij} < D_I, \\ 0 & d_{ij} \geq D_I \end{cases} \tag{5.19}$$

where d_{ij} is the distance between nodes i and j, and D_I is an interference range. The strategic game associated with the above formulation is denoted by $\mathscr{G}$.

Alternatively, Wu et al. [26] considered block fading channels, where identically and independently distributed (i.i.d.) random fadings are experienced across different channels, but are constant during each decision window. Moreover, the interference coefficients among two nodes i and j are taken to be their expected values, i.e.,

$$\bar{w}_{ij}^{k} = (d_{ij})^{-\alpha}\mathbb{E}[\epsilon_{ij}^{k}] \tag{5.20}$$

where ϵ_{ij}^{k} is an instantaneous fading component for the link between i and j across channel k. From i to j and from j to i, the expected values of these random variables are assumed to be the same for channel k, i.e., $\mathbb{E}[\epsilon_{ij}^{k}] = \mathbb{E}[\epsilon_{ji}^{k}], \forall i, j$. Thus, $\bar{w}_{ij}^{k} = \bar{w}_{ji}^{k}, \forall i, j$ and $\forall k$.

The alternative utility function is then given by

$$\bar{U}_i(c_i, c_{-i}) = -p_i\bar{I}_i = -\sum_{j\neq i} p_i p_j \bar{w}_{ji}^{c_i}\delta(c_j, c_i). \tag{5.21}$$

The resulting associated game is denoted by $\bar{\mathscr{G}}$.

Potential Game Analysis

Due to the symmetric interference considerations, the above formulated games are expected to exhibit the BSI property.

Theorem 5.3. *The aforementioned games $\mathscr{G}$ and $\bar{\mathscr{G}}$ are BSI games. Thus, they are exact potential games with potential functions*

$$F(c_i, c_{-i}) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} p_i p_j w_{ij}\delta(c_j, c_i), \tag{5.22}$$

and

$$\bar{F}(c_i, c_{-i}) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} p_i p_j \bar{w}_{ji}^{c_i}\delta(c_j, c_i), \tag{5.23}$$

respectively.

Proof. It is straightforward to see that in $\mathcal{G}$, the pairwise symmetric observation between players i and j is $p_i p_j w_{ij} \delta(c_j, c_i)$, $\forall i, j \in \mathcal{N}$. Similarly, in $\bar{\mathcal{G}}$, this term is equal to $p_i p_j \bar{w}_{ji}^{c_i} \delta(c_j, c_i)$, $\forall i, j \in \mathcal{N}$.

Thus, both games are BSI games according to Theorem 2.21; and their corresponding potential functions are given by (5.22) and (5.23), respectively. □

Remark 5.2. An alternative proof of the game $\mathcal{G}$ being an exact potential game by verifying its potential function with the definition can be found in Wu et al. [25].

Remark 5.3. The GADIA algorithm in [2] is equivalent to the best-response dynamics. Thus, it inherits the Nash equilibrium convergence property as the associated game is an exact potential game.

Remark 5.4. A stochastic, dynamic formulation of games with synthetic symmetric interference for canonical networks was studied in Zheng et al. [31], in which the dynamic variation of the set of active nodes was modeled by a state space. Potential games cast in dynamic stochastic settings are not in the scope of this book. Interested readers can refer to [31].

Finally, we comment on the issue of power allocation in this game-theoretic framework. Suppose that power can be allocated via a joint channel and power allocation scheme using the aforementioned formulation. As these games are exact potential games, the network can achieve equilibria by maximizing the potential functions, i.e.,

$$\max_{\mathbf{c},\mathbf{p}} \left(-\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} p_i p_j w_{ij} \delta(c_j, c_i) \right) \tag{5.24}$$

where $\mathbf{c}$ and $\mathbf{p}$ are the channel and power allocation vectors, respectively. However, it is easily seen that the all-zero power vector $\mathbf{p} = \mathbf{0}$ maximizes this network objective. This observation was made by Neel et al. [15], where they argued that power allocation should not be used jointly with channel/waveform selection. Similarly, [2] also decoupled power allocation from the channel allocation process.

5.1.3 Potential Games with Channel Capacity-Based Utility Functions

For some wireless communications problems, optimizing the Shannon capacity, which is given by $\log_2(1 + \mathsf{SINR})$, is a natural choice for the utility functions. Often, games with this utility function are not potential games. However, there are special scenarios where such an approach does lead to a potential game formulation. A particular example is the uplink power control problem in single-cell single-frequency networks considered by Fattahi et al. [6] and Scutari et al. [22]. This scenario can be generalized to the case of *multiple receivers, each of which operates* *on one of the orthogonal channels*. Examples include the BS selection problem by Perlaza et al. [18], or the channel selection problem in Mertikopoulos et al. [14] and Perlaza et al. [19].

Problem Formulation

The system model is based on [14, 18, 19]. The players in the model belong to the set $\mathscr{N} = \{1, 2, \ldots, N\}$ of N single-antenna transmitters. Each player wishes to transmit their data to one or several of the K single-antenna receivers. Each receiver occupies one of the K non-overlapping (i.e., orthogonal) channels. Denote the receiver set or equivalently, channel set, as $\mathscr{K} = \{1, 2, \ldots, K\}$. Index k can represent both a receiver or its corresponding channel. We remark that this model can describe either the receiver (i.e., BS) selection problem of Perlaza et al. [18] or the channel selection problem of Mertikopoulos et al. [14], Perlaza et al. [19].

The channel/receiver assignment matrix is $\mathbf{A} \in \{0, 1\}^{N\times K}$. Its element $a_{ik} = 1$ if player i selects channel/receiver k, and 0 otherwise. Again, the selection of player i is represented by $\mathbf{a}_i^T$, the $1 \times K$ ith row vector of $\mathbf{A}$. Since players are allowed to use more than one channel, multiple entries of $\mathbf{a}_i^T$ can be 1 simultaneously.

The channel gain matrix is $\mathbf{G} \in \mathbb{R}^{N\times K}$, where g_{ik} is the channel gain from transmitter i to receiver k. The power matrix is denoted by $\mathbf{P} \in \mathbb{R}^{N\times K}$, whose element p_{ik} is the power used by transmitter i on channel k. The row vector $\mathbf{p}_i^T$, extracted from the i-th vector of $\mathbf{P}$, refers to the power allocation vector of player i. Note that $p_{ik} = 0$ whenever $a_{ik} = 0$. In [14], a maximum power constraint $P_{\max}$ is imposed on every player, i.e., $\sum_{k=1}^{K} a_{ik}p_{ik} \le P_{\max}, \forall i$. The feasible power allocation vectors of player i thus belong to the following set

$$\Delta_i = \{\mathbf{p}_i^T \,|\, \mathbf{p}_i^T \in \mathbb{R}^{1\times K}, p_{ik} \ge 0 \;\forall k, \sum_{k=1}^{K} a_{ik}p_{ik} \le P_{\max}\}. \tag{5.25}$$

Let $\mathbb{P} = \times_{i=1}^{N} \Delta_i \subset \mathbb{R}^{N\times K}$ be the set of all feasible power matrix $\mathbf{P}$. In this game, we consider a player's strategy to be a joint channel and power allocation $S_i = (\mathbf{a}_i^T, \mathbf{p}_i^T)$, thus the strategy space is written as the following Cartesian product

$$\mathbb{S} = \{0, 1\}^{N\times K} \times \mathbb{P}. \tag{5.26}$$

The total achievable rate of player i is chosen to be its utility function, given by

$$U_i(S_i, S_{-i}) = \sum_{k=1}^{K} B_k \log_2\left(1 + \frac{a_{ik}p_{ik}g_{ik}}{\sum_{j=1, j\ne i}^{N} a_{jk}p_{jk}g_{jk} + \sigma^2}\right) \tag{5.27}$$

where B_k is the bandwidth of channel k and σ^2 is the noise power. The resulting strategic game is denoted by $\mathscr{G} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i\in\mathscr{N}}]$.

Potential Game Analysis

Using the framework presented in Chap. 2, $\mathcal{G}$ can be identified to be an exact potential game due to the separability property.

Theorem 5.4. *The game $\mathcal{G}$ is coordination-dummy separable.*

Proof. For an arbitrary player i and channel k, we have

$$\log_2\left(1+\frac{a_{ik}p_{ik}g_{ik}}{\sum_{j=1,j\neq i}^{N}a_{jk}p_{jk}g_{jk}+\sigma^2}\right)=\log_2\left(\frac{a_{ik}p_{ik}g_{ik}+\sum_{j=1,j\neq i}^{N}a_{jk}p_{jk}g_{jk}+\sigma^2}{\sum_{j=1,j\neq i}^{N}a_{jk}p_{jk}g_{jk}+\sigma^2}\right)$$
$$=\log_2\left(\sigma^2+\sum_{j=1}^{N}a_{jk}p_{jk}g_{jk}\right)-\log_2\left(\sigma^2+\sum_{j=1,j\neq i}^{N}a_{jk}p_{jk}g_{jk}\right),\forall i\in\mathcal{N},k\in\mathcal{K}. \tag{5.28}$$

Thus, $\forall i:$

$$U_i(S_i,S_{-i})=\underbrace{\sum_{k=1}^{K}B_k\log_2\left(\sigma^2+\sum_{j=1}^{N}a_{jk}p_{jk}g_{jk}\right)}_{\text{coordination term}}-\underbrace{\sum_{k=1}^{K}B_k\log_2\left(\sigma^2+\sum_{j=1,j\neq i}^{N}a_{jk}p_{jk}g_{jk}\right)}_{\text{dummy term}}. \tag{5.29}$$

Clearly, the first term is common among all players while the second term does not involve any decision variables from player i. Hence, according to Definition 2.16, this form of $U_i(S_i,S_{-i})$ satisfies coordination-dummy separability. □

The next corollary is a direct result of Theorem 2.20.

Corollary 5.1. *The game $\mathcal{G}$ is an exact potential game with the potential function*

$$F(S)=\sum_{k=1}^{K}B_k\log_2\left(\sigma^2+\sum_{i=1}^{N}a_{ik}p_{ik}g_{ik}\right). \tag{5.30}$$

Remark 5.5. In [14, 18, 19], the authors did not explain how the potential function $F(S)$ in (5.30) was derived. However, by applying our generalized framework, the derivation of $F(S)$ can be intuitively understood. Unlike the approach in Sect. 5.1.1, this game is formulated via the forward method. In practice, the function $F(S)$ of this example does not seem to represent a meaningful network objective, but is nevertheless the potential function.

Remark 5.6 (Power Allocation Problem). The uplink power allocation problem in single-cell single-frequency networks considered in [6, 22] were claimed to be potential games. In fact, we remark that this problem is a special case of the currently investigated game with a single channel/receiver, i.e., $K=1$ and $a_{iK}=1,\forall i$.

Without the dimension of channel selection, the actions of players are reduced to solely adaptation of power. As such, the strategy space of the power allocation game is a subset of the current strategy space $\mathbb{S}$. Theorem 2.23 guarantees that the power allocation game is also an exact potential game.

Remark 5.7 (Inclusion of Power Pricing). In (5.27), we can modify U_i so as to include power pricing as follows

$$U_i(S_i, S_{-i}) = \sum_{k=1}^{K} B_k \log_2 \left(1 + \frac{a_{ik} p_{ik} g_{ik}}{\sum_{j=1, j \neq i}^{N} a_{jk} p_{jk} g_{jk} + \sigma^2}\right) - c_i(\mathbf{p}_i^T) \tag{5.31}$$

where $c_i : \Delta_i \mapsto \mathbb{R}$ is a generic pricing function, usually assumed to be convex, non-decreasing and continuously differentiable in $\mathbf{p}_i^T$, $\forall i \in \mathcal{N}$ [22]. Since the introduction of pricing terms is equivalent to linearly combining the original potential game with a no-conflict game, Theorem 2.18 assures us that the resulting game is also an exact potential game.

Discussion

A word of caution is that capacity maximization games are potential games only if for every channel, a common receiver is assumed (where this receiver can be randomly located). In a more general scenario where multiple randomly located, distributed receivers are allowed to reuse the same channels, such as the case considered in Chap. 3, a similar formulation unfortunately does not automatically lead to an exact potential game.

A simple mathematical counterargument is given as follows. Using the settings and notations in Chap. 3 for the problem of transmit-receive pairs, the total achievable rate (per unit bandwidth) for player i is given by

$$\begin{aligned} C_i &= \sum_{k=1}^{K} \log_2 \left(1 + \frac{a_{ik} p_{ik} g_{ii}^k}{\sum_{j=1, j \neq i}^{N} a_{jk} p_{jk} g_{ji}^k + \sigma^2}\right) \\ &= \sum_{k=1}^{K} \log_2 \left(\sigma^2 + \sum_{j=1}^{N} a_{jk} p_{jk} g_{ji}^k\right) - \sum_{k=1}^{K} \log_2 \left(\sigma^2 + \sum_{j=1, j \neq i}^{N} a_{jk} p_{jk} g_{ji}^k\right). \end{aligned} \tag{5.32}$$

While the same decomposition method yields a dummy second term to all players, the first term in (5.32) is not identical among all players [compared to the first term in (5.29)] and cannot function as the coordination term and hence the potential function. This is because the summation $\sum_{j=1}^{N} a_{jk} p_{jk} g_{ji}^k$, which refers to the total power received by player i for channel k, differs across two different referenced players i_1 and i_2, since $g_{ji_1}^k \neq g_{ji_2}^k$. Thus, this formulation for distributed transmit-receiver pairs does not guarantee an exact potential game. In fact, simulation studies [10, 11] revealed that the Nash equilibrium convergence rate for such a formulation is not 100 %.

5.1.4 Potential Games for Players with Local Interactions in Cognitive Radio Networks

In opportunistic spectrum access, cognitive radios (i.e., secondary users) are allowed to transmit on spectrum bands unoccupied by the licensed (i.e., primary) users at certain times and locations [1]. How the distributed cognitive radio users select the best available channels opportunistically for transmission is often formulated as a game.

Xu et al. [28] proposed potential game formulations under the assumption that the communication ranges of cognitive radio users are such that each user will only interfere a few of its local neighboring users. The authors of [28] referred to this context as *local interaction games*.

Problem Formulation

In the system, it is assumed that there are N pairs of transmitters and receivers acting as the N players of the game. They compete for M licensed channels where $M < N$. Channels are only available to a player if no primary user occupies them. The set of players is denoted by $\mathscr{N}$ and the set of channels by $\mathscr{M}$. Channel availability is characterized by the matrix $\mathbf{C} \in \{0, 1\}^{N \times M}$, whose element $c_{nm} = 1$ if channel m is available to player n, and 0 otherwise. Then, assuming each player can select a single channel to transmit, the set of available channels for player i constitutes its strategy set, i.e.,

$$\mathbf{S}_i = \{m \mid m \in \mathscr{M}, c_{im} = 1\} \subseteq \mathscr{M}. \tag{5.33}$$

It is possible that no channel is available for player i; in this case, $\mathbf{S}_i = \emptyset$.

Next, the local interaction settings can be characterized by an interference graph. Let $\Gamma = (\mathscr{N}, \mathscr{E})$ be the corresponding graph where its set of nodes coincides with the set of players $\mathscr{N}$ and the set of edges $\mathscr{E}$ is such that two neighboring nodes are connected by an edge. Players i and j are neighbors if they are separated by a physical distance less than a predefined value D_I and can interfere each other during transmission. We denote $J_i = \{j \mid j \in \mathscr{N}, (i, j) \in \mathscr{E}\}$ the set of neighbors for player i. As the system is distributed, each player i only knows its locally available information, such as its own neighbors in J_i and its own available channels (i.e., row i of $\mathbf{C}$). Thus, the local interaction problem can be represented by $< \mathscr{G}, \Gamma >$ where $\mathscr{G}$ is any strategic game that is formulated for players within an underlying interference graph Γ.

The authors further assumed a slotted ALOHA transmission protocol for the system. In each timeslot, all players may transmit on an available channel with probability p and stay silent with probability $1 - p$. When two players transmit simultaneously on the same channel (i.e., $S_i = S_j$), collision will occur. We define the collision indicator variables δ_{ij} as follows.

$$\delta_{ij} = \begin{cases} 1 & \text{if } S_i = S_j, \\ 0 & \text{otherwise.} \end{cases} \tag{5.34}$$

Then, the individual throughput of player i is expressed as

$$f_i(S_i, S_{J_i}) = p \prod_{k \in J_i} (1-p)^{\delta_{ij}} \tag{5.35}$$

when $\mathbf{S}_i \neq \emptyset$. In case $\mathbf{S}_i = \emptyset, f_i = 0$.

In our notations, $f_i(S_i, S_{J_i})$ suggests that it depends only on the strategies (S_i, S_{J_i}) selected by player i as well as its **neighbors** in J_i. This is the difference between local interaction games and traditional strategic games.

Next, two utility functions, namely the local altruistic function and the local collision function, were considered in [28].

Potential Games for Local Altruistic Objectives

Under the local altruistic consideration, cooperative behaviors among neighboring players are encouraged. As such, [28] defined the following utility function

$$U_i(S_i, S_{J_i}) = f_i(S_i, S_{J_i}) + \sum_{j \in J_i} f_j(S_j, S_{J_j}), \forall i \in \mathscr{N} \tag{5.36}$$

which maximizes not only the individual throughput of player i but also the aggregate throughput of its neighbors. A player that maximizes this objective exhibits altruistic behaviors. This results in a local interaction problem $< \mathscr{G}^{(U)}, \Gamma >$ where

$$\mathscr{G}^{(U)} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i \in \mathscr{N}}], \tag{5.37}$$

with $\mathbb{S} = \mathbf{S}_1 \times \ldots \times \mathbf{S}_N$.

We establish the following result.

Theorem 5.5. *The game $\mathscr{G}^{(U)}$ is cooperation-dummy separable.*

Proof. We define the network throughput as the sum of individual throughputs, i.e.,

$$F(S) = \sum_{i \in \mathscr{N}} f_i(S_i, S_{J_i}), \tag{5.38}$$

which can be rewritten with respect to player i as

$$F(S_i, S_{-i}) = f_i(S_i, S_{J_i}) + \sum_{j \in J_i} f_j(S_j, S_{J_j}) + \sum_{j \in \mathscr{N} \setminus (J_i \cup \{i\})} f_j(S_j, S_{J_j}). \tag{5.39}$$

To player i, $Q_i(S_{-i}) = \sum_{j \in \mathcal{N} \setminus (J_i \cup \{i\})} f_j(S_j, S_{J_j})$ is a dummy term since it involves the throughputs of players which are not player i's neighbors and is independent from player i's strategy.

Thus, from (5.36) and (5.39), one can write

$$U_i(S_i, S_{J_i}) = F(S_i, S_{-i}) - Q_i(S_{-i}), \forall i \tag{5.40}$$

which shows that $\mathscr{G}^{(U)}$ is cooperation-dummy separable, according to Definition 2.16. □

The following result is directly from Theorem 2.20.

Corollary 5.2. *The game $\mathscr{G}^{(U)}$ is an exact potential game with potential function $F(S)$ in (5.38).*

Remark 5.8. Although we present the formulation of $\mathscr{G}^{(U)}$ in the forward method, i.e., defining utility functions satisfying the separability property, it is possible that U_i in (5.36) might have been conceptualized via the **backward method**. Here, the network objective which is the sum of individual throughputs $F(S)$ is first set. Subsequently, by discounting from $F(S)$ the non-contributing terms for player i, the remaining terms are player i's utility, which represents an altruistic objective.

Potential Games for Local Collision Objectives

Previously, $\mathscr{G}^{(U)}$ is formulated as an exact potential game in which ultimately, the potential function to be maximized corresponds to the network objective of total throughput maximization. Alternatively, [28] proposed a second network objective, which is the network collision level $\Phi(S)$, defined as

$$\Phi(S) = \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in J_i} \delta_{ij} \tag{5.41}$$

which represents the total number of collision occurrences in the networks. The local collision objective is to minimize the overall number of collisions.

We will see how the backward method can help us obtain an exact potential game. In doing so, we will take our network function as $-\Phi(S)$ where the minus sign is to convert the minimization problem to maximization. Subsequently,

$$-\Phi(S_i, S_{-i}) = -\frac{1}{2} \left(\sum_{j \in J_i} \delta_{ij} + \sum_{j: i \in J_j} \delta_{ji} + \sum_{k \neq i} \sum_{j \in J_k \setminus \{i\}} \delta_{kj} \right) \tag{5.42}$$

Then, $-\frac{1}{2} \sum_{k \neq i} \sum_{j \in J_k \setminus \{i\}} \delta_{kj}$ is a non-contributing term. According to the backward method, the utility function V_i of player i can be set to $-\frac{1}{2}(\sum_{j \in J_i} \delta_{ij} + \sum_{j: i \in J_j} \delta_{ji})$. Because $\delta_{ji} = \delta_{ij}$, this equals to

$$V_i(S_i, S_{J_i}) = -\sum_{j \in J_i} \delta_{ij}. \tag{5.43}$$

which corresponds to the total number of neighbors selecting the same channel as player i. We have ended up with exactly the utility function proposed in [28].

This results in the local interaction problem $< \mathscr{G}^{(V)}, \Gamma >$ where

$$\mathscr{G}^{(V)} = [\mathscr{N}, \mathbb{S}, \{V_i\}_{i \in \mathscr{N}}] . \tag{5.44}$$

Theorem 5.6. *The game $\mathscr{G}^{(V)}$ is an exact potential game with potential function $-\Phi(S)$ in (5.41).*

Proof. This theorem follows directly from application of backward method. □

5.2 Applications of Pseudo-Potential Games

In this section, we touch on the applications of pseudo-potential games, or specifically, subclasses of pseudo-potential games known as games of *weak strategic complements with aggregation* (WSC-A) and games of *weak strategic substitutes with aggregation* (WSS-A). These games were investigated in the work of Dubey et al. [5]. Subsequently, Heikkinen [8] applied these results to study the convergence of distributed power control algorithms in wireless networks. We will present a brief theoretical introduction to these concepts in Sect. 5.2.1, followed by a couple of applications in Sects. 5.2.2 and 5.2.3.

5.2.1 Strategic Complements and Substitutes

The notions of strategic complements/substitutes [5] rest on the assumption that the strategy space can be "ordered", i.e., a strategy is "preferred" over another via the relation $\succ$. In this discussion, the standard strategic form notation of games $\mathscr{G} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i \in \mathscr{N}}]$ is used, unless otherwise stated.

Definition 5.1 (Strategic Complements and Substitutes). Consider arbitrary strategies $S_i, T_i \in \mathbf{S}_i$ where $S_i \succ T_i$. We denote $\Delta_{S_i,T_i} U_i(S_{-i}) = U_i(S_i, S_{-i}) - U_i(T_i, S_{-i})$. Then, *strategic complement* refers to the condition that

$$\forall x, y \in \mathbf{S}_{-i}: \quad x \succ y \Rightarrow \Delta_{S_i,T_i} U_i(x) > \Delta_{S_i,T_i} U_i(y), \forall i. \tag{5.45}$$

That is, player i's and opponents' strategies are complements to each other.

On the other hand, if $x \succ y \Rightarrow \Delta_{S_i,T_i} U_i(x) < \Delta_{S_i,T_i} U_i(y)$, we have *strategic substitutes*.

Weak strategic complements/substitutes are relaxations of previous notions to some *best response* of player i, similar to how best-response potential games are generalized from ordinal potential games.

Definition 5.2 (Weak Strategic Complements and Substitutes). For player i, let $b_i(x)$ be some best-response function selected out of its best-response correspondence $\mathscr{B}_i(x)$. If,

$$\forall i, \forall x, y \in \mathbf{S}_{-i}: \quad x \succ y \Rightarrow b_i(x) > b_i(y), \tag{5.46}$$

then the game is of *weak strategic complements*.

If in the above, we replace the > sign with the < sign, we have *weak strategic substitutes*.

Furthermore, the preference order among the opponents' strategies can be restricted to only a class of *additive aggregation* [5]. That is, we associate the opponents' joint strategies S_{-i} with their additive aggregate, $\sum_{j\in\mathscr{N}\setminus\{i\}} S_j$; and the previous notation $x = S_{-i} \succ y = T_{-i}$ is now equivalent to $\sum_{j\in\mathscr{N}\setminus\{i\}} S_j > \sum_{j\in\mathscr{N}\setminus\{i\}} T_j$. We see that additive aggregation can be frequently seen in practice, where the accumulation of other players' actions affects one player's utility. For example, signal interferences are in the form of summation of received signals from interfering channels.

We are ready to formally define games of WSC-A and WSS-A as follows.

Definition 5.3. A game $\mathscr{G}$ is of weak strategic complements with aggregation (WSC-A) if for any player i, there exists a particular best-response function $b_i : \mathbf{S}_{-i} \mapsto \mathbf{S}_i$ such that $b_i(x) \in \mathscr{B}_i(x)\ \forall x \in \mathbf{S}_{-i}$, b_i is continuous on $\mathbf{S}_{-i}$ and

$$b_i(x) > b_i(y) \text{ whenever } x \succ y, \forall x, y \in \mathbf{S}_{-i}. \tag{5.47}$$

Note that the condition that b_i is continuous on $\mathbf{S}_{-i}$ does not need to hold for finite games [5]. In (5.47) above, one could also define WSS-A games by replacing the $\succ$ sign with the $\prec$ sign.

A key result for WSC-A and WSS-A games is as follows.

Theorem 5.7 (Dubey). *WSC-A and WSS-A games are pseudo-potential games. As such, they have pure-strategy Nash equilibria, reachable via sequential best-response dynamics. Furthermore, for WSC-A and WSS-A games with convex strategy sets and single-valued best responses, simultaneous best-response dynamics also converge to a Nash equilibrium.*[1]

[1] Here, it is not specified whether the convergence can be obtained in finite steps. However, in Chap. 2, our survey of the finite and approximate finite improvement path properties suggested that in finite games, convergence can be finitely achieved. Meanwhile for continuous games, such dynamics ultimately approach a Nash equilibrium but may not necessarily be finite.

Proof. The proof that WSC-A and WSS-A games are pseudo-potential games is given in Theorem 1 of [5], which we omit here.

As a result, both have pure-strategy Nash equilibria which are obtainable from sequential best-response dynamics.

Convergence of simultaneous best-response dynamics in WSC-A and WSS-A games with convex strategy sets and single-valued best responses is shown in Remark 2 of [5].[2] □

Also in [5], the results for additive aggregation can still hold if we consider broader classes of aggregation, among which includes the weighted linear aggregator $\alpha : \mathbf{S}_{-i} \mapsto \mathbb{R}$, defined by

$$\alpha(S_{-i}) = \sum_{j \neq i} a_j S_j \tag{5.49}$$

where $a_j, j \neq i$ are real scalars, such that $\alpha(S_{-i}) > 0, \forall i, \forall S_{-i}$.

Proposition 5.1 (Dubey). *The results of Theorem 5.7 still hold if in the definitions of WSC-A and WSS-A games, we replace the additive aggregation* $\sum_{j \in \mathscr{N} \setminus \{i\}} S_j$ *by the weighted aggregator* $\alpha(S_{-i})$ *above.*

Proof. A proof can be obtained from [5]. □

It can be seen that in power control problems, a player's total experienced interferences, which is a summation of the other nodes' transmit power levels, scaled by the channel gains, can be treated directly as the weighted aggregation of opponents' strategies. In the next sections, we see how these results can be applied in some relevant practical problems.

5.2.2 *Convergence of Some Power Control Games with Interference Aggregation*

Power Control Game

We revisit the problem of uplink power control in wireless networks. Some power control games have been previously introduced in Examples 2.12 and 2.16. Distributed power control problems are widely studied in the literature [8, 21, 27];

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[2]In addition, a generic potential function for such games is given in [5] as

$$F(S) = -\sum_{i \in \mathscr{N}} S_i - \sum_{i \in \mathscr{N}} \sum_{j<i} S_i S_j + \sum_{i \in \mathscr{N}} P_i(S_i) \tag{5.48}$$

where $P_i(S_i) = \int_{-1}^{\max(\Sigma_{-i})+1} \min(\tau_i(x), S_i) \mathrm{d}x$. Here, $\tau_i(.)$ is the linearly extended function of best response $b_i(.)$ to the convex hull Σ_{-i} of the set $\mathbf{S}_{-i}$.

and convergence of iterative algorithms for distributed power control is well-understood via the theory of *standard function* [30]. However, not all algorithms can be verified under this technique. Meanwhile, in some cases, one can make use of strategic complements to identify the convergence conditions.

The players are N mobile nodes, each of which transmits to a common destination, e.g., a base station, access point or cluster head. Player i's strategy is to transmit at power level p_i which is bounded in a certain range, e.g., $[0, P_{\max}]$. Subsequently, we use $\mathbf{p} = (p_i, p_{-i})$ to refer to strategy profiles in this game.

The SINR γ_i of player i is calculated as

$$\gamma_i = \frac{p_i g_i}{\sum_{j=1, j\neq i}^{N} p_j g_j + \sigma^2} \tag{5.50}$$

where g_i is the channel gain between player i and the destination node; and σ^2 is the noise power.

We can formulate a power control game (with general utility function U_i) as

$$\mathscr{G} = [\mathscr{N}, \mathbb{S}, \{U_i\}_{i\in\mathscr{N}}] \tag{5.51}$$

whose strategy space is

$$\mathbb{S} = \{\mathbf{p} | \ \mathbf{p} \in [0, P_{\max}]^N\}. \tag{5.52}$$

Often, utility function $U_i(p_i, p_{-i})$ is directly related to player i's SINR, i.e., $U_i = U_i(\gamma_i(p_i, p_{-i}))$. Its SINR γ_i is in turn a function of player i's accumulated interference. Thus, the theory of WSC-A and WSS-A games can be applied to this scenario.

For player i, we denote its *interference aggregator* by

$$\mathsf{I}_{-i} = \sum_{j\neq i} p_j g_j \tag{5.53}$$

which is the total interference power experienced by player i. Clearly, I_{-i} belongs to the class of weighted aggregators in (5.49).

We will focus on the following class of utility functions:

$$U_i(p_i, p_{-i}) = f_i(\gamma_i) - c_i p_i, \forall i \tag{5.54}$$

where $c_i p_i$ is a linear power pricing term and $f_i(.)$ is a continuous, differentiable function of γ_i.

In general, if utility function U_i is continuous, differentiable and concave, a best response for player i can be derived based on the first-order derivative condition $\frac{\partial U_i}{\partial p_i} = 0$ which gives

$$f_i'(\gamma_i)\frac{\partial \gamma_i}{\partial p_i} - c_i = 0, \quad \text{or } f_i'(\gamma_i) = \frac{c_i(\mathsf{I}_{-i} + \sigma^2)}{g_i} \tag{5.55}$$

Hence,

$$\hat{\gamma}_i = \frac{\hat{p}_i g_i}{\mathsf{I}_{-i} + \sigma^2} = (f_i')^{-1}\left(\frac{c_i(\mathsf{I}_{-i} + \sigma^2)}{g_i}\right) \tag{5.56}$$

where $(f_i')^{-1}(.)$ is defined in the region where $f_i(.)$ is concave. Consequently, the best-response function for player i can be written as

$$\hat{p}_i = b_i(\mathsf{I}_{-i}) = \frac{\mathsf{I}_{-i} + \sigma^2}{g_i}(f_i')^{-1}\left(\frac{c_i(\mathsf{I}_{-i} + \sigma^2)}{g_i}\right). \tag{5.57}$$

Then, from (5.47), the condition for WSC-A/WSS-A (under weighted aggregation) in our game $\mathcal{G}$ is that *the best response* $b_i(\mathsf{I}_{-i})$ *in (5.57) is an increasing/decreasing function in* I_{-i}.

We will next check this condition for some commonly adopted function $f_i(.)$.

Shannon Capacity Utility

Firstly, we consider $f_i(.)$ to be the Shannon capacity function, i.e.,

$$f_i(\gamma_i) = \log_2(1 + \gamma_i), \forall i \tag{5.58}$$

whose first derivative is

$$f_i'(\gamma_i) = \frac{1}{(1 + \gamma_i)\ln 2}, \forall i. \tag{5.59}$$

We observe the following result.

Proposition 5.2. *In* $\mathcal{G}_1$*, best-response* $b_i(\mathsf{I}_{-i})$ *is decreasing in* I_{-i}*.*

Proof. We notice that $f_i(\gamma_i)$ above is concave for all $\gamma_i \geq 0$. Thus, $(f_i')^{-1}(.)$ is well-defined and is given by

$$(f_i')^{-1}(u) = \frac{1}{u \ln 2} - 1. \tag{5.60}$$

As such, from (5.57), we have the best-response function

$$b_i(\mathsf{I}_{-i}) = \frac{1}{c_i \ln 2} - \frac{\mathsf{I}_{-i} + \sigma^2}{g_i}. \tag{5.61}$$

Clearly, $b_i(\mathsf{I}_{-i})$ is linearly proportional to $-\mathsf{I}_{-i}$. This proves our proposition. □

As such, games with the Shannon capacity as the utilities are of *weak strategic substitutes* and best-response dynamics can converge to a Nash equilibrium.

Remark 5.9. Existence of a Nash equilibrium is also established by an alternative argument as we notice that $\mathbb{S} = [0, P_{\max}]^N$ is a nonempty, convex and compact subset of Euclidean space, and the utility function U_i is continuous and quasi-concave in each p_i. This is a direct result from Theorem 1.2.

Sigmoid Utility

The second function $f_i(.)$ we consider is the sigmoid function, which has been used in wireless power control by several authors [7, 21, 27].

A sigmoid function $f_i(\gamma_i)$ is meant to reflect the level of satisfaction of player i with respect to its obtained SINR level γ_i. In general, such a function yields the following desirable properties

- $f_i(0) = 0$,
- $\lim_{x\to\infty} f_i(x) = 1$, i.e., satisfaction level saturates to 1 at infinity.
- $f_i'(x) > 0$ for $x > 0$, i.e., $f_i(x)$ increases with respect to x.

A good candidate for $f_i(\gamma_i)$ is the logistic sigmoid function

$$f_i(\gamma_i) \triangleq \frac{1}{1 + e^{-\alpha_i(\gamma_i - \beta_i)}} - \delta_i \tag{5.62}$$

where α_i, β_i are constant. Figure 5.1 shows the sigmoid function and its shapes with respect to various α_i and β_i. Furthermore, $\delta_i = 1/(1 + e^{\alpha_i\beta_i})$ is present to offset it to 0. Often this value is negligibly small, so we can set $\delta_i \approx 0$. This function appears in mathematical studies of population growth [23] and has a wide application in sciences and engineering.

We should note that the sigmoid function $f_i(\gamma_i)$ is concave only where $\gamma_i \geq \beta_i$, i.e., to the right of its inflexion point $(\beta_i, 0.5)$. Thus, the inverse function of its derivative, $(f_i')^{-1}(.)$, should be defined over $[\beta_i, \infty)$.

Figure 5.2 depicts the derivative $f_i'(.)$ and also presents us a good interpretation of finding the best response for player i (which was originally suggested by Xiao et al. [27]). From (5.57), $b_i(\mathsf{I}_{-i})$ is a product of two factors. The first factor $(f_i')^{-1}\left(\frac{c_i(\mathsf{I}_{-i}+\sigma^2)}{g_i}\right)$ is in fact the abscissa of the second intersection between $f_i'(.)$ and the horizontal line $y = \frac{c_i(\mathsf{I}_{-i}+\sigma^2)}{g_i}$. The second factor $\frac{\mathsf{I}_{-i}+\sigma^2}{g_i}$ is equal to the height of the previous line, scaled by $1/c_i$. Thus, $b_i(\mathsf{I}_{-i})$ is directly proportional to the area of the rectangle indicated in Fig. 5.2. We denote this area by $\mathscr{A}$.

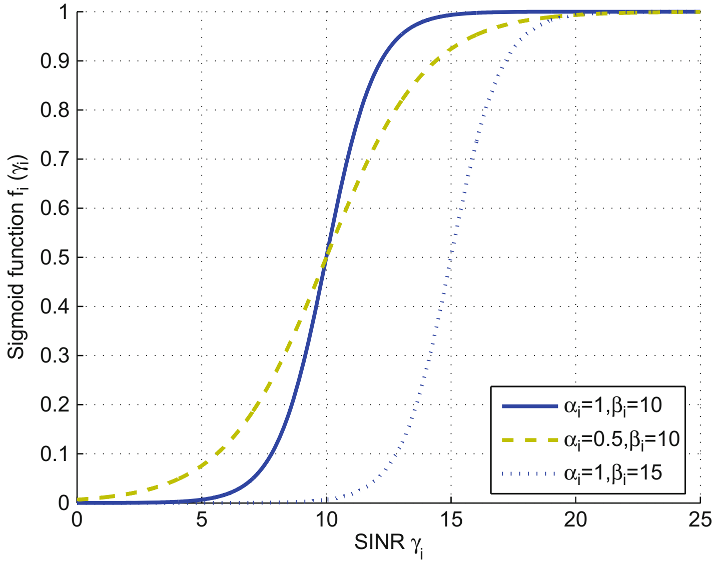

Fig. 5.1 The sigmoid function $f_i(\gamma_i)$ with various α_i and β_i

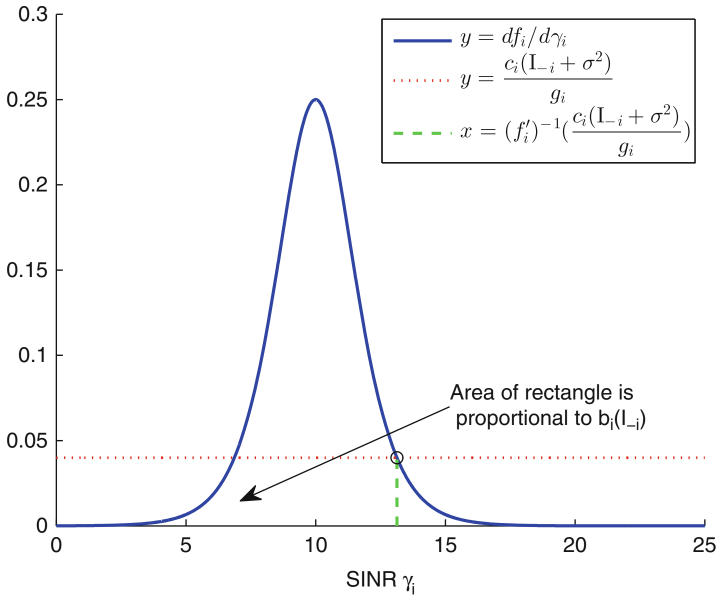

Fig. 5.2 Derivative of sigmoid function $f_i'(\gamma_i)$

In closed form, $b_i(\mathsf{I}_{-i})$ can be computed as

$$b_i(\mathsf{I}_{-i}) = \frac{\mathsf{I}_{-i} + \sigma^2}{g_i} \left[\beta_i - \frac{1}{\alpha_i} \ln \left(\frac{1}{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{c_i(\mathsf{I}_{-i} + \sigma^2)}{\alpha_i g_i}}} - 1 \right) \right]. \tag{5.63}$$

In [27], a necessary condition for convergence of iterative best-response dynamics was stated as follows.

Proposition 5.3. *In the power control game with sigmoid utility functions, iterative best-response dynamics converge if $\mathscr{A}$ increases with its height.*

Proof. Clearly, that $\mathscr{A}$ increases with its height $\frac{c_i(\mathsf{I}_{-i}+\sigma^2)}{g_i}$ means $b_i(\mathsf{I}_{-i})$ is an increasing function of I_{-i}, which in turn means the game is of *weak strategic complements*. Convergence follows due to WSC-A games being pseudo-potential games. □

Regarding the validity of the condition on $\mathscr{A}$ in Proposition 5.3, [27] also stated that this holds for "almost all practical situations" due to the large value of α_i and β_i in practice. However, neither did the authors of [27] give an analysis nor elaborate further on this statement. Here, we formally validate this claim as follows.

Proposition 5.4. *Suppose that player i chooses suitable values of α_i, β_i and c_i, such that the following condition is satisfied:*

$$\alpha_i \beta_i > \ln\left(\frac{2}{1 + \sqrt{1 - 4X_i}} - 1\right) + \frac{1}{\sqrt{1 - 4X_i}} \tag{5.64}$$

where $X_i = \frac{c_i(\mathsf{I}_{-i}+\sigma^2)}{\alpha_i g_i}$ is the height of $\mathscr{A}$ (scaled by $\frac{1}{\alpha_i}$), then $b_i(\mathsf{I}_{-i})$ is an increasing function of X_i (and hence, in I_{-i}).

Proof. This condition is directly derived from the requirement that $\frac{\partial b_i(\mathsf{I}_{-i})}{\partial X_i} > 0$ for $b_i(\mathsf{I}_{-i})$ to be increasing in X_i. Equation (5.63) can be written as

$$b_i(\mathsf{I}_{-i}) = \frac{X_i}{c_i} \left[\alpha_i \beta_i - \ln \left(\frac{2}{1 + \sqrt{1 - 4X_i}} - 1 \right) \right],$$

whose derivative can be shown to be

$$\frac{\partial b_i}{\partial \mathsf{I}_{-i}} = \frac{1}{c_i} \left[\alpha_i \beta_i - \ln \left(\frac{2}{1 + \sqrt{1 - 4X_i}} - 1 \right) - \frac{1}{\sqrt{1 - 4X_i}} \right]. \tag{5.65}$$

Clearly, (5.64) follows. Intuitively, we see that this requires $\alpha_i \beta_i$ to be sufficiently large, which to some extent explains the claim in [27] for large α_i and β_i. □

We further remark that firstly, in order for the best response $b_i(\mathsf{I}_{-i})$ to exist, the range of I_{-i} should be in the region such that $0 < X_i < 0.25$. We next illustrate how condition (5.64) could be satisfied across almost this entire feasible region, given a large value of $\alpha_i \beta_i$. In fact, (5.64) reduces to $X_i < \tilde{X}_i$ for given α_i and β_i; and $\tilde{X}_i$ approaches 0.25 quickly as α_i and β_i become larger. For instance, $\alpha_i = 1$ and $\beta_i = 10$ (as chosen in [27]), then (5.64) becomes $X_i < \tilde{X}_i \approx 0.2476$. With $\alpha_i = 1$ and $\beta_i = 20$, $\tilde{X}_i \approx 0.2493$; and with $\alpha_i = 1$ and $\beta_i = 50$, $\tilde{X}_i \approx 0.2499$.

5.2.3 A Pseudo-Potential Game Analysis for the Power Minimization Problem

A power minimization game with coupled constraints was introduced in Example 2.16, where we showed that it is an exact potential game. In this section, we provide an alternative analysis using pseudo-potential games, which was proposed by Heikkinen [8].

We will keep the same notations of Sect. 5.2.2, in particular, the interference aggregator I_{-i} in (5.53). Now, let us investigate a power minimization game with the following utility functions

$$U_i \triangleq -p_i, \forall i \in \mathcal{N}, \tag{5.66}$$

and the minimum SINR constraints

$$\gamma_i \geq \bar{\gamma}_i, \forall i \in \mathcal{N}, \tag{5.67}$$

where $\bar{\gamma}_i > 0$ is a target SINR for player i, specified by the player's QoS requirements.

The power minimization game, which we call $\mathcal{G}$, is therefore the following distributed optimization problem

$$\begin{aligned} \forall i : \quad & \max_{p_i} \, (-p_i) \\ & \text{s.t. } p_i \in [0, P_{\max}], \quad \gamma_i \geq \bar{\gamma}_i. \end{aligned} \tag{5.68}$$

Proposition 5.5. *The game $\mathcal{G}$ is a WSC-A game, considering the weighted aggregator I_{-i}.*

Proof. First, we need to find the best response for player i. From the SINR constraint, we have

$$\frac{p_i g_i}{\sum_{j \neq i} p_j g_j + \sigma^2} \geq \bar{\gamma}_i \tag{5.69}$$

which leads to

$$p_i \geq \frac{\bar{\gamma}_i \left(\mathsf{I}_{-i} + \sigma^2 \right)}{g_i}. \tag{5.70}$$

Clearly, $b_i(\mathsf{I}_{-i}) = \frac{\bar{\gamma}_i(\mathsf{I}_{-i}+\sigma^2)}{g_i}$. This is an increasing function in terms of I_{-i}. Hence, $\mathscr{G}$ has *weak strategic complements*. □

Thus, $\mathscr{G}$ is a pseudo-potential game, which is not surprising since by the argument of Example 2.16, it must also be an exact potential game. In [8], a potential function associated with the pseudo-potential game was also provided.

Proposition 5.6. *The following function serves as a (pseudo) potential function for* $\mathscr{G}$*:*

$$F(\mathbf{p}) \triangleq \left(\sigma^2 + \sum_{i=1}^{N} g_i p_i \right)^2 - \sum_{i=1}^{N} \frac{1+\bar{\gamma}_i}{\bar{\gamma}_i} (g_i p_i)^2. \tag{5.71}$$

Proof. To validate $F(\mathbf{p})$, we verify it against (2.10).

Firstly, for any player i, its best-response correspondence is single-valued, i.e., $\mathscr{B}_i(S_{-i}) = \{b_i(\mathsf{I}_{-i})\}$.

Next, we find p_i that maximizes $F(p_i, p_{-i})$, which can be found via the first-order derivative condition, i.e.,

$$\begin{aligned} \frac{\partial F}{\partial p_i} &= 2\left(\sigma^2 + \sum_{j=1}^{N} g_j p_j \right) g_i - \frac{1+\bar{\gamma}_i}{\bar{\gamma}_i}(2 g_i^2 p_i) \\ &= 2 g_i \left((\sigma^2 + \mathsf{I}_{-i}) + g_i p_i - g_i p_i - \frac{g_i p_i}{\bar{\gamma}_i} \right) \\ &= 2 g_i \left((\sigma^2 + \mathsf{I}_{-i}) - \frac{g_i p_i}{\bar{\gamma}_i} \right) = 0 \end{aligned} \tag{5.72}$$

This equation is solved by $\hat{p}_i = \frac{\bar{\gamma}_i(\mathsf{I}_{-i}+\sigma^2)}{g_i}$, which is exactly $b_i(\mathsf{I}_{-i})$. It remains to verify that $\hat{p}_i$ is indeed a maximum by checking the Hessian matrix of $F(\mathbf{p})$. We omit the details.

In summary, $b_i(\mathsf{I}_{-i}) = \arg\max_{p_i'} F(p_i', p_{-i})$ which guarantees that $F(\mathbf{p})$ is indeed a potential function for the game $\mathscr{G}$. □

5.3 Concluding Remarks

In this chapter, we have looked at a number of potential game formulations in wireless communications and networking. Among various types of potential games, exact potential games have been the most popular; and the major formulations using this approach include games with the sum of inverse SINRs as the potential function, games under synthetic symmetric interference, games with Shannon capacity-based utility functions, as well as games of local interactions among players in a graphical network. In addition, we also examine the use of pseudo-potential games in understanding the convergence of various power control games. In summarizing these major existing works, we present the results according to the framework laid out in Chap. 2. The list of problems described above is non-exhaustive, and represents, at the time of writing, a collection of potential game-related applications that are gaining popularity in the literature. Finally, the authors wish you a fruitful journey for this topic of research for your future work.

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