SPRINGER BRIEFS IN ELECTRICAL AND COMPUTER ENGINEERING · CONTROL, AUTOMATION AND ROBOTICS

Xu-Guang Li
Silviu-Iulian Niculescu
Arben Çela

Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays

Springer

SpringerBriefs in Electrical and Computer Engineering

Control, Automation and Robotics

More information about this series at http://www.springer.com/series/10198

Xu-Guang Li · Silviu-Iulian Niculescu
Arben Çela

Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays

Springer

Xu-Guang Li
School of Information Science
and Engineering
Northeastern University
Shenyang, Liaoning
China

Arben Çela
Department of Computer Science
and Telecommunication
UPE, ESIEE Paris
Noisy-le-Grand
France

Silviu-Iulian Niculescu
Laboratoire des Signaux et Systèmes
(L2S, UMR CNRS 8506)
CNRS-CentraleSupélec-Université
Paris-Sud
Gif-Sur-Yvette
France

ISSN 2191-8112 ISSN 2191-8120 (electronic)
SpringerBriefs in Electrical and Computer Engineering
ISSN 2192-6786 ISSN 2192-6794 (electronic)
Control, Automation and Robotics
ISBN 978-3-319-15716-0 ISBN 978-3-319-15717-7 (eBook)
DOI 10.1007/978-3-319-15717-7

Library of Congress Control Number: 2015933150

Mathematical Subject Classifications: 34K06, 93B60, 93C23, 93D20

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media
(www.springer.com)

To our families

Preface

Time-delay systems have been intensively studied in various disciplines. On one hand, delay phenomena exist in many dynamical systems encountered in engineering, physics, chemistry, biology, and economics. For instance, examples can be found in population dynamics [63], biological systems [76], as well as engineering systems [92]. On the other hand, sometimes, complex dynamics can be well approximated by a time-delay system (see [36] and the references therein) and a high-order linear model can be approximately "reduced" to a low-order one with delays (see [47]). In other words, delays may induce some simplifications in modeling a dynamical system (less parameters to be taken into account) although the corresponding time-delay system is infinite dimensional.

It is well known that the delay considerably affects the system stability and related performances. One may tend to have some intuition that the delay always has a *negative* effect (i.e., increasing the value of delay in a system must deteriorate the system dynamics and even brings instability). This intuition does not always hold. As pointed out in the control literature, the delay may have a *positive* effect on the system dynamics. For instance, several examples in this monograph illustrate that increasing the delay properly may stabilize some unstable systems.

In this book, we will mainly focus on the analysis of the delay's effect on the stability and our objective is to find the whole stability domain with respect to the delay parameter in the case of linear systems with commensurate delays. Both retarded and neutral systems will be addressed. This problem, referred to as the *complete stability problem* for time-delay systems, has attracted a lot of attention since the 1950s, but it has not received full characterization. For an overview of the stability study of time-delay systems, one may refer to [102].

Actually, the complete stability problem is generally much more complicated than we can expect, due to the intricate spectral characteristics. *First*, a time-delay system has infinitely many characteristic roots. For this reason, time-delay systems represent a class of infinite-dimensional systems. In this context, it is important to point out that, given a delay, the unstable roots (if any) are always in finite number for a retarded time-delay system as well as a neutral time-delay system whose neutral operator is stable. By existing mathematical tools, it is impossible to accurately detect all the infinitely many characteristic roots. *Second*, a critical imaginary root of a time-delay system has infinitely many critical delays. Thus, a thorough asymptotic behavior analysis for a general time-delay system is very difficult to achieve.

In our opinion, the current bottleneck mainly lies in the involved singularities associated with the spectra (the case without singularities can be studied by the existing methods). As a matter of fact, the complexity of the singular case (as illustrated by various examples proposed in this book) was underestimated until recently.

In order to systematically address the singularities and eventually solve the complete stability problem, a new methodology will be proposed in this book. Roughly speaking, a singular point of a time-delay system can be reformulated (from a new analytical curve perspective) such that its asymptotic behavior can be studied from the corresponding singular point of the frequency-sweeping curves associated to this time-delay system. Since such an approach covers the regular case, it is quite general. The methodology proposed in this book is called a *new frequency-sweeping framework*. It is worth mentioning that the origin of the classical frequency-sweeping method for studying the stability of time-delay systems goes back to Tsypkin in 1946 (see [114]). Further insights into such approaches and techniques will be addressed throughout this volume.

Outline of the Book

First, a new analytic curve perspective will be introduced, making the line of this book distinct from the existing ones in the literature. From this new perspective, the asymptotic behavior of the critical imaginary roots with respect to the critical delays can be systematically investigated. One of the most important results is that the asymptotic behavior of a critical imaginary root can be accurately described and studied by means of the Puiseux series. Next, we will propose to prove the *general invariance property*, to overcome the peculiarity that a critical imaginary root has infinitely many critical delays. In order to determine whether the general invariance property holds, we will improve the classical frequency-sweeping method by adopting the analytic curve perspective. We will show that the asymptotic behavior of the frequency-sweeping curves can be reflected by the dual Puiseux series. With the help of such dual Puiseux series, the frequency-sweeping curves will play an important role in studying the complete stability problem. The general invariance property will be confirmed by studying carefully the (equivalence) relation between the Puiseux series and the dual Puiseux series. As a consequence, the complete stability problem can be fully solved with the following important results.

- An explicit expression of the number of the unstable roots with respect to the delay parameter will be found. By using such a function, the analysis and design of time-delay systems may be significantly simplified.
- The ultimate stability problem (the system stability property when the delay approaches infinity) can be thoroughly studied. Moreover, all time-delay systems, according to their ultimate stability properties, may be classified into three types: Type 1: A time-delay system has infinitely many unstable roots as delay tends to infinity, Type 2: A time-delay system has a fixed number of unstable roots for all delay values (delay-independently hyperbolic system), and Type 3: A time-delay system has a fixed number of unstable roots except at the critical delays.
- A simple frequency-sweeping criterion will be presented. Using this graphical test, the asymptotic behavior analysis for the critical imaginary roots at all the positive critical delays can be fulfilled by simply observing the frequency-sweeping curves (without any calculation).

In most part of the book, the time-delay systems under consideration are of retarded type with commensurate delays. The proposed methodology will be extended to the time-delay systems of neutral type with commensurate delays, by paying attention to the additional features of the corresponding neutral operators.

The book is mainly based on the contributions of the authors in the last five years, namely [66–74]. Further extensions could be made in the future, based on the methodology proposed in this book.

Further Extensions

In our opinion, the ideas proposed here may be applied to some other problems not covered in this book. In the sequel, we list two possible directions:

From τ-decomposition to D-decomposition: If one differentiates the system parameters in two categories: delays and others, appropriate methods have been developed to handle the stability problems in frequency-domain. More precisely, the τ-decomposition [64] corresponds to the case when the delay is "free" and the other parameters are "fixed". Similarly, the D-decomposition [89] corresponds to the counterpart case: "free" parameters for all system parameters, except the delay which is assumed to be "fixed". This book addresses the τ-decomposition problem. Despite the differences between the two classes of problems, the methodology proposed here may bring some useful insights into the D-decomposition problem as it also relies on the asymptotic behavior analysis of the critical characteristic roots.

From single delay parameter to multiple delay parameters: As mentioned earlier, the time-delay systems considered in this book are supposed to have commensurate delays. This means that the problem involves in fact only one (delay) parameter and we will see that the results to be presented are mathematically elegant. However, if the delays are not commensurate, the problem has multiple (delay) parameters and will become much more complicated (the problem is generally nondeterministic polynomial (NP)-hard [113]). Extending the results of this book to the problem with multiple incommensurate delays may not be straightforward as, to the best of the authors' knowledge, we still lack an effective mathematical tool for multiple-parameter asymptotic behavior analysis.

Acknowledgments

The idea of writing this monograph appeared when the first author was a postdoctoral fellow at ESIEE, Paris and Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), Supélec, France, in 2008 and 2009.

We greatly acknowledge the financial support of ESIEE, Paris (Noisy-Le-Grand, France), Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), Supélec (Gif-Sur-Yvette, France), French CNRS (National Center for Scientific Research), Northeastern University (Shenyang, China) (Grant number N120404017), as well as the National Natural Science Foundation of China (Grant numbers 61104100 and 61473065) that helped us to finish this book project.

We are grateful to Tamer Başar, Antonio Bicchi, and Miroslav Krstic (SpringerBriefs Series Editors) and anonymous reviewers for their constructive feedbacks and insightful suggestions which greatly helped us in improving the overall quality of the monograph. Next, we wish to thank Oliver Jackson, Karin de Bie, as well as Shalini Kumari from Springer for their patience and help during the whole process.

We would like to thank our friends, past and present collaborators, who implicitly or explicitly made a significant contribution to the research results presented in the book. Among them, we mention (alphabetically): Chaouki T. Abdallah (University of New Mexico at Albuquerque, USA), Catherine Bonnet (Inria Saclay and L2S at Gif-Sur-Yvette, France), Islam Boussaada (IPSA and L2S at Gif-Sur-Yvette, France), Tiao-Yang Cai (Northeastern University at Shenyang, China), Jie Chen (City University of Hong Kong, China), André R. Fioravanti (UNICAMP at Campinas, Brazil), Emilia Fridman (Tel Aviv University, Israel), Peilin Fu (National University at San Diego, USA), José Geromel (UNICAMP at Campinas, Brazil), Keqin Gu (Southern Illinois University at Edwardsville, USA), Vladimir L. Kharitonov (Saint Petersburg University, Russia), Xu Li (Northeastern University at Shenyang, China), Wim Michiels (KU Leuven, Belgium), Mihai Putinar (University of California at Santa Barbara, USA), Rifat Sipahi (Northeastern University at Boston, USA), Hong-Hai Wang (Northeastern University at Shenyang, China), and Lu Zhang (Northeastern University at Shenyang, China).

Last but not least, X.-G. Li would like to thank his parents, Ya-Wen Ding and Ji-Xue Li, and his future wife, Jing Zhang, for their love and support. Concerning S.-I. Niculescu, there is a special person in his life, Laura, to whom he owes the exceptional support that she gave to Silviu to overcome all the difficulties both professional and extra-professional in the last 20 years. A. Çela cannot thank enough his wife, Anila, and their children, Deborah, Alexander, and Adrianna, for their patience and support during all phases of this book. We dedicate this monograph to all of them, in love and gratitude.

Shenyang, People's Republic of China, December 2014 Xu-Guang Li
Gif-Sur-Yvette, France, December 2014 Silviu-Iulian Niculescu
Noisy-Le-Grand, France, December 2014 Arben Çela

Contents

Symbols

$\mathbb{N}$ ($\mathbb{N}_+$)	Set of non-negative (positive) integers
$\mathbb{R}$ ($\mathbb{R}_+$)	Set of (positive) real numbers
$\mathbb{Z}$	Set of integers
$\mathbb{C}$	Set of complex numbers
$\mathbb{C}_0$	Imaginary axis
$\partial\mathbb{D}$	Unit circle
$\mathrm{Re}(\lambda)$ ($\mathrm{Im}(\lambda)$)	Real (imaginary) part of λ ($\lambda \in \mathbb{C}$)
$\mathbb{C}_-$ ($\mathbb{C}_+$)	Set $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$ ($\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$)
$\mathbb{C}_L$ ($\mathbb{C}_U$)	Set $\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) < 0\}$ ($\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) > 0\}$)
$\mathfrak{I}_1$	Line parallel to the abscissa axis with ordinate equal to 1
$\mathrm{Arg}(\lambda) \in (-\pi, \pi]$	Principal argument of λ ($\lambda \in \mathbb{C}$)
$\det(A)$	Determinant of matrix A
$\rho(C)$	Spectral radius of matrix C
$\lVert\cdot\rVert$	Vector norm
$\lvert z\rvert$	Norm of z ($z \in \mathbb{C}$)
$\lceil \gamma \rceil$	Smallest integer greater than or equal to γ ($\gamma \in \mathbb{R}$)
mod	For real numbers $a, b > 0$, and c, $a \bmod b = c$ ($a = c(\mathrm{mod}(b))$) denotes that $a = kb + c$, where $k \in \mathbb{Z}$ and $\lvert c\rvert < b$
$\deg(\cdot)$	Degree of a polynomial
$\mathrm{ord}(\varphi(x)) = \kappa$	For a function $\varphi(x)$, $\mathrm{ord}(\varphi(x)) = \kappa$ for $x = x^*$ denotes that $\frac{d^i\varphi(x)}{dx^i} = 0$ ($i = 0, \ldots, \kappa - 1$) and $\frac{d^\kappa\varphi(x)}{dx^\kappa} \neq 0$ when $x = x^*$
$\phi_{x^\alpha y^\beta}$	Partial derivative $\frac{\partial^{\alpha+\beta}\phi(x,y)}{\partial x^\alpha \partial y^\beta}$ ($\alpha \in \mathbb{N}, \beta \in \mathbb{N}$)
$\dot{x}$	Derivative of x with respect to time t, $\frac{dx}{dt}$
I	Identity matrix of appropriate dimensions
j	Imaginary unit, i.e., $j^2 = -1$
ε	A sufficiently small positive real number (generic notation)

τ	Time delay (generic notation)
λ	Characteristic root (generic notation)
ω	Frequency (generic notation)

Chapter 1
Introduction to Complete Stability of Time-Delay Systems

Time delays are widely encountered in various types of control systems and usually affect the stability and related performances considerably. Here, we only mention a few examples: data transfer in high-speed networks [93], sampled-data control systems [12, 31], networked control systems [16, 126], design of PID controllers [99, 106], consensus for multi-agents [87, 96], supply chain systems [103], traffic flow [84], cell dynamics [100], switched systems [50], fuzzy systems [35, 42], fractional-order systems [28, 81], and neural networks [78, 125].

In this introductory chapter, some preliminaries and prerequisites regarding the complete stability problem and the sketch of the technical line of the book are given.

1.1 Preliminaries and Prerequisites

Recall now some fundamentals concerning the stability of linear time-delay systems in the frequency domain framework.

1.1.1 Basic Concepts

In most part of this book, we consider linear systems with commensurate delays:

$$\dot{x}(t) = \sum_{\ell=0}^{m} A_\ell x(t - \ell\tau), \tag{1.1}$$

under appropriate initial conditions, where $x(t) \in \mathbb{R}^r$ ($r \in \mathbb{N}_+$) is the system state at time t, $A_\ell \in \mathbb{R}^{r\times r}$ ($\ell = 0, \ldots, m, m \in \mathbb{N}_+$) are constant matrices, and $\tau \in \mathbb{R}_+ \cup \{0\}$ is the delay parameter.

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_1

The time-delay system (1.1) is of *retarded* type. If the system dynamics also depend on the derivatives of the past states, such a time-delay system, subject to some appropriate assumptions (associated with the neutral operator), may enter the class of time-delay systems of *neutral* type and will be studied specifically in Chap. 10.

We start by recalling some fundamentals needed for our development of the stability analysis, which can be found in, e.g., [6, 39, 45, 85, 92, 112].

First, the characteristic function of time-delay system (1.1) is given by[1]

$$f(\lambda, \tau) = \det\left(\lambda I - \sum_{\ell=0}^{m} A_\ell e^{-\ell\tau\lambda}\right), \tag{1.2}$$

which is a *quasipolynomial* of the form

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}, \tag{1.3}$$

where $a_0(\lambda), \ldots, a_q(\lambda)$ ($q \in \mathbb{N}_+$) are polynomials in λ with real coefficients. A complex number λ such that $f(\lambda, \tau) = 0$ is called a characteristic root for the time-delay system (1.1).

A distinctive feature of the time-delay system (1.1) is that it has *an infinite number of characteristic roots* for a $\tau > 0$, representing a class of *infinite-dimensional* systems. It is important to point out that the structure of the spectrum changes when τ increases from 0 to a sufficiently small positive number $+\varepsilon$. When $\tau = 0$, the system (1.1) is *finite dimensional* with r characteristic roots (i.e., the eigenvalues of $\sum_{\ell=0}^{m} A_\ell$). During the transition as τ increases from 0 to $+\varepsilon$, infinitely many new characteristic roots appear (see [6, 85]). A natural question may arise about the initial locations of these infinitely many new roots.

As the time-delay system (1.1) is of retarded type, it follows that

$$\deg(a_0(\lambda)) > \max\{\deg(a_1(\lambda)), \ldots, \deg(a_q(\lambda))\}.$$

Such a condition ensures that as τ increases from 0 to $+\varepsilon$, all the new roots appear at far left of the complex plane. However, this property does not necessarily hold for a neutral time-delay system, as will be discussed in Chap. 10. More precisely, in the neutral case, an infinitesimal change in the delays may sharply change the roots distribution in the right half-plane $\mathbb{C}_+$. Such discontinuities are strongly related to the essential spectrum of the system (see [6, 85]).

We now introduce the asymptotic stability definition and a necessary and sufficient condition characterizing such a notion.

Definition 1.1 The trivial solution $x(t) = 0$ of time-delay system (1.1) is said to be stable if for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\max_{t_0 - m\tau \le t \le t_0} \|x(t)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for $t \ge t_0$. Furthermore, it is said to be

[1] The characteristic function can be equivalently obtained in two ways: (1) substituting a sample solution into (1.1) or (2) applying the Laplace transformation to (1.1) under the assumption of zero initial conditions.

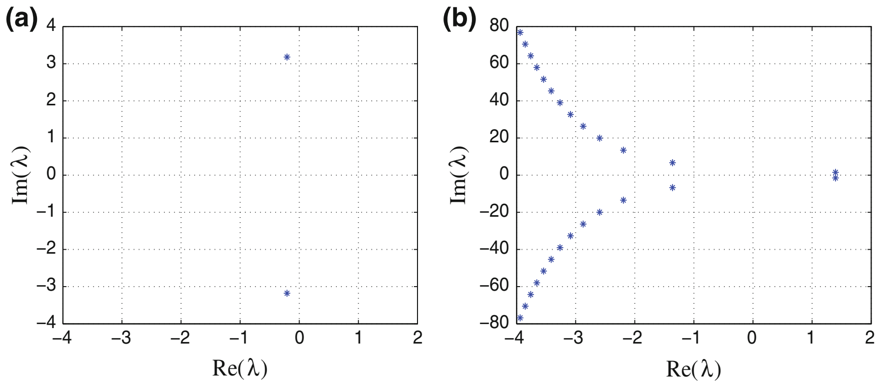

Fig. 1.1 Characteristic roots for Example 1.1. **a** $\tau = 0.01$. **b** $\tau = 1$

asymptotically stable if it is stable, and for any $t_0 \in \mathbb{R}$, there exists a $\delta_a = \delta_a(t_0) > 0$ such that $\max_{t_0 - m\tau \le t \le t_0} \|x(t)\| < \delta_a$ implies $\lim_{t\to\infty} x(t) = 0$.

For convenience, throughout this book, we will simply adopt the expression "time-delay system (1.1) is asymptotically stable" instead of "the trivial solution $x(t) = 0$ of time-delay system (1.1) is asymptotically stable".

As the system (1.1) is linear and of retarded type, the asymptotic stability is equivalent to the exponential stability (the definition of exponential stability can be found in [85]).

Theorem 1.1 *The time-delay system (1.1) is asymptotically stable if and only if all the characteristic roots are located in the open left half-plane* $\mathbb{C}_-$.

However, a direct application of Theorem 1.1 is generally *impossible* since the system has an infinite number of characteristic roots. The following simple example illustrates the root locations intuitively.

Example 1.1 Consider the following time-delay system:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ -9 & -1.5 \end{pmatrix} x(t - \tau),$$

with the characteristic function $f(\lambda, \tau) = (1.5\lambda + 9)e^{-\tau\lambda} + \lambda^2 - \lambda + 1$.

When $\tau = 0$, the system has only two characteristic roots $-0.2500 \pm 3.1524j$. As τ increases from 0 to $+\varepsilon$, infinitely many new characteristic roots appear at far left of the complex plane. Figure 1.1a shows the case when $\tau = 0.01$, where the two points denote the locations of the original roots. In the domain of Fig. 1.1a, the other (infinitely many) roots do not appear as they are still at far left of the complex plane. Next, as τ increases, some roots move to the selected domain. For instance, when $\tau = 1$, some roots enter the selected domain and the two original roots enter

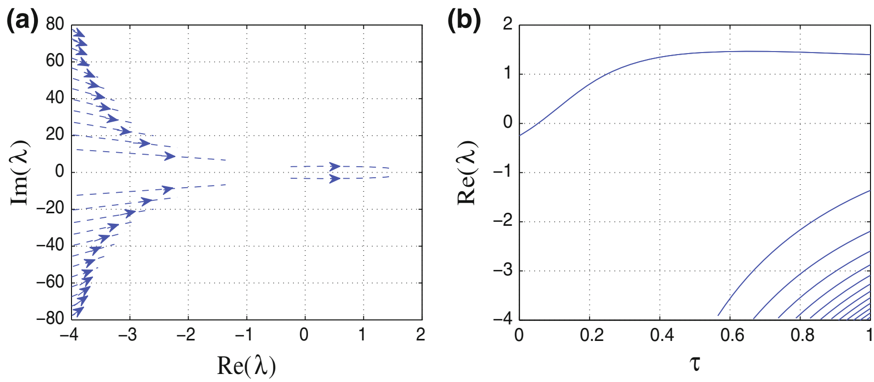

Fig. 1.2 Root loci for Example 1.1. **a** Re(λ) versus Im(λ). **b** Re(λ) versus τ

the right-half plane as shown in Fig. 1.1b. For further illustration, Fig. 1.2 gives the corresponding root loci with respect to the increasing delay. □

The root loci in Fig. 1.2 are *numerically* generated by using the DDE-BIFTOOL [24]. To verify the theoretical results, we will often provide the corresponding root loci using this MATLAB®-based package. Additional numerical methods or algorithms for estimating the spectra of time-delay systems include [10, 11, 43, 44, 118].

1.1.2 Complete Stability Problem

Following the notation widely used in the literature (see [64, 97, 109]), we use $NU(\tau) \in \mathbb{N}$ to denote the number of unstable roots (i.e., the characteristic roots located in $\mathbb{C}_+$) in the presence of delay τ. According to Theorem 1.1, a time-delay system is asymptotically stable for a given τ, if and only if it does not have roots located on the imaginary axis and $NU(\tau) = 0$.

For a time-delay system with commensurate delays, the objective of this book is to obtain its exhaustive stability domain for the delay parameter τ (i.e., the whole domain for $\tau \geq 0$ such that $NU(\tau) = 0$ excluding the possible critical points), which is known as the *complete stability problem*.

For a finite-dimensional system, e.g., an autonomous system $\dot{x}(t) = Ax(t)$ or a linear system $\dot{x}(t) = Ax(t) + Bu(t)$ with the feedback control $u(t) = Kx(t)$, the characteristic roots are the eigenvalues of A or $A + BK$, respectively. Thus, the number of the unstable roots can be easily known. However, the case of time-delay systems (monitoring $NU(\tau)$ as τ varies) becomes much more involved and, to the best of the authors' knowledge, there is no straightforward way to compute $NU(\tau)$.

Only some partial results were reported on the explicit computation of $NU(\tau)$ (related references, e.g., [21, 48, 52, 97, 112, 122], will be discussed in Sect. 9.2). To fix better the ideas, some representative examples from the literature are presented below.

Example 1.2 Consider the time-delay system (Example 5.11 in [39])

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -2 & 0.1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(t-\tau).$$

This system is unstable when $\tau = 0$. While, as τ increases, the system may become asymptotically stable. More precisely, this system is asymptotically stable if and only if $\tau \in (0.1002, 1.7178)$. The root loci are shown in Fig. 1.3a. □

It is commonly accepted that increasing the delay tends to destabilize a system. However, this assertion does not necessarily hold. Example 1.2 and the subsequent Example 1.3 are counterexamples, where *increasing the delay may induce stability*.

Example 1.3 The following time-delay system (Example of Sect. 3 in [97])

$$\dot{x}(t) = \begin{pmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{pmatrix} x(t) + \begin{pmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{pmatrix} x(t-\tau)$$

is asymptotically stable if and only if $\tau \in [0, 0.1624) \cup (0.1859, 0.2219)$. The root loci are shown in Fig. 1.3b. □

We underline here an interesting phenomenon, as shown in Example 1.3, that a time-delay system may have multiple stability intervals. Especially, the system to be considered in Example 6.3 has 12 stability intervals. For such an interesting phenomenon, one may also refer to [1, 75, 92, 108].

Solving the complete stability problem requires to analyze $NU(\tau)$ along the whole positive τ-axis. Some reported approaches can be used to approximately estimate the delay margin (i.e., the first stability interval, including $\tau = 0$). The approach based on simple Lyapunov-Krasovskii functionals (see [30, 124]) requires a low computational load and has the advantage of treating non-nominal models. The complete

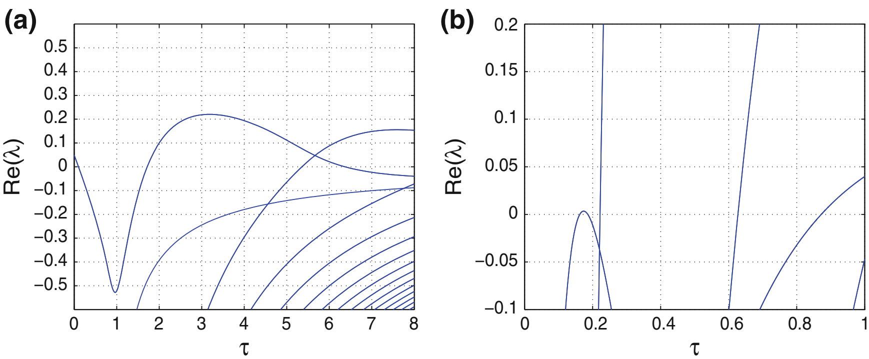

Fig. 1.3 Root loci for Examples 1.2 and 1.3. **a** Re(λ) versus τ for Example 1.2. **b** Re(λ) versus τ for Example 1.3

Lyapunov-Krasovskii functionals together with the discretization technique (called the discretized Lyapunov-Krasovskii approach [37]) may be used in the case of multiple stability intervals. However, both the aforementioned approaches inevitably involve conservatism (i.e., some stability interval may be missing). For the above time-domain approaches, one may refer to [29, 39, 57, 92].

In order to compute $NU(\tau)$ for a given delay value, one of the classical ways is to study the argument of the characteristic function (by complex analysis, see [3]) along some particular contour in the complex plane. Such an idea was explicitly used in [112] for both retarded and neutral systems and applied to construct the so-called *stability chart*, which is a region in the parameter plane (defined by two appropriate parameters other than delay) such that the corresponding system is asymptotically stable for parameters belonging to this region. For some extensions of the work of [112], one may refer to [48], where the derived criterion provides further information about $NU(\tau)$ even in the case when some characteristic roots are located on the imaginary axis. Similar algebraic ideas can also be found in [51, 119].

Although it appears quite complicated to directly compute $NU(\tau)$, the root continuity property offers an indirect way. We now recall this important property.

First, if $\tau = 0$, the system (1.1) is a delay-free system with r roots. When τ increases from 0 to $+\varepsilon$, as discussed in Sect. 1.1.1, infinitely many new roots appear in $\mathbb{C}_-$. In parallel, the original r roots vary continuously with respect to τ (Rouché's theorem [3] offers a simple way to prove such a property). Next, as τ increases from $+\varepsilon$, all the (infinitely many) roots vary continuously with respect to τ. Therefore, $NU(\tau)$ changes as τ increases from $+\varepsilon$ toward $+\infty$ only if for some τ the system has characteristic roots located on the imaginary axis $\mathbb{C}_0$, called the critical imaginary roots. These delays are called the critical delays. For further continuity properties of time-delay systems, one may refer to [85].

Remark 1.1 It is worth mentioning that we keep track of $NU(\tau)$ from $\tau = +\varepsilon$ (not from $\tau = 0$) because some systems involve critical imaginary roots when $\tau = 0$ (for instance, the systems considered in Example a1 of [109] and Example 2 of [122] have simple critical imaginary roots when $\tau = 0$). This case has to be treated carefully and the related result will be given and explained later.

Based on the root continuity argument, the τ-decomposition idea was proposed and largely used in the literature. Roughly speaking, this corresponds to a two-step method requiring to solve two problems, which will be reviewed in the next subsection. The approach we are proposing is in line of such an idea.

1.1.3 τ-Decomposition Idea

The τ-decomposition idea may be traced back to at least the 1960s (see [64] and the references therein), along which a great number of stability results have been reported, see [7, 19, 21, 85, 97, 122]. In our opinion, the τ-decomposition method can be applied in two steps by solving the two problems (Problems 1 and 2) described below.

According to the root continuity argument, if $NU(\tau)$ changes as τ increases, there must be a critical delay at which the system has a (simple or multiple) critical imaginary root. This gives rise to the first problem along the τ-decomposition idea.

Problem 1 The exhaustive detection (if any!) of the critical imaginary roots and the corresponding critical delays.

Various effective methods for solving Problem 1 have been proposed in the literature (one may refer to a survey paper [110]) and the frequency-sweeping approach proposed in this book also covers Problem 1. In our opinion, Problem 1 has been well studied and understood.

Unlike for the critical imaginary roots, the analytic computation for the other characteristic roots is generally very difficult. For instance, one may compute the characteristic roots located in a vertical line parallel to the imaginary axis. But the procedure is much more complicated. In order to solve the complete stability problem, we have to further analyze the variation of a critical imaginary root as τ increases near the corresponding critical delay (called the asymptotic behavior of a critical imaginary root).

So, we are now led to the second problem of the τ-decomposition method.

Problem 2 The asymptotic behavior analysis of the critical imaginary roots with respect to the corresponding critical delays.

A pair (λ, τ), where $\tau \in \mathbb{R}_+ \cup \{0\}$ and $\lambda \in \mathbb{C}_0$, such that $f(\lambda, \tau) = 0$ is called a *critical pair*. Most of the work in this book will focus on the algebraic properties of the critical pairs, from a new *analytic curve perspective* to be introduced in the forthcoming Chaps. 2 and 3.

Remark 1.2 Owing to the conjugate symmetry of the spectrum, it suffices to consider only the critical imaginary roots with nonnegative imaginary parts. More precisely, if a critical imaginary root $j\omega$ at a critical delay is detected, the system necessarily has a critical imaginary root $-j\omega$ for the same critical delay. Furthermore, the asymptotic behavior of $j\omega$ and $-j\omega$ are symmetric with respect to the real axis.

Remark 1.3 Under the assumption that Problem 1 is solved, the critical delays divide the positive τ-axis into infinitely many subintervals and within each subinterval $NU(\tau)$ is constant. We will next monitor $NU(\tau)$ as τ increases by means of solving Problem 2. For instance, consider a subinterval $\tau \in (\tau', \tau'')$ where τ' and τ'' are two positive critical delays such that there are no other critical delays inside this subinterval. If the value of $NU(\tau' - \varepsilon)$ is known and the asymptotic behavior of the critical imaginary roots at $\tau = \tau'$ is properly studied, we may precisely know the value of $NU(\tau' + \varepsilon)$. According to the root continuity argument mentioned previously, for any $\tau \in (\tau', \tau'')$, $NU(\tau) = NU(\tau' + \varepsilon)$. It is worth noting that this result is accurate, without any conservatism.

However, Problem 2 is rather involved and we need to further divide it into two sub-problems (see Sect. 1.3 for details).

1.1.4 Frequency-Sweeping Framework of this Book

We will propose a frequency-sweeping framework along the τ-decomposition idea in this book. In this subsection, we explain the distinction of the frequency-sweeping approach used in this book with respect to the existing ones in the literature.

The frequency-sweeping idea is not new and has been largely used in studying the stability of time-delay systems. In [20], some necessary and sufficient delay-independent stability conditions based on the frequency-sweeping technique were proposed. However, the interest of the authors focused more on the delay-independent stability instead of characterizing the delay intervals guaranteeing the asymptotic stability. Next, some frequency-sweeping criteria were introduced and discussed in [39] for detecting the stability interval including $\tau = 0$ for time-delay systems. In [41], the frequency-sweeping test was used to determine the stability crossing set for a class of quasipolynomials with two delays by an appropriate geometric interpretation (triangle geometry) of the characteristic equation. An extension of this idea to the case of three delays can be found in [40]. In [107], a frequency-sweeping approach within the cluster treatment of characteristic roots (CTCR) framework was given, by which the potential stability switching hyperspace can be obtained for multiple-delay systems. Such a result makes use of the so-called Rekasius substitution in order to detect critical imaginary roots.

The above frequency-sweeping methods were mainly used for solving Problem 1 ([20, 39] consider commensurate delays case while [40, 107] treat incommensurate delays case), without considering Problem 2. In [21, 64], frequency-sweeping tests were used for studying Problem 2. However, the characteristic functions considered therein are confined to a class of simple quasipolynomials.

Unlike the existing approaches, we will study in-depth the asymptotic behavior of the frequency-sweeping curves (the procedure for plotting them will be given in Sect. 1.2.3) from a new analytic curve perspective. As a consequence, the frequency-sweeping approach will cover both Problems 1 and 2 for general time-delay systems with commensurate delays. In this sense, the frequency-sweeping approach in this book refers not only to the frequency-sweeping curves but also to the new mathematical framework to be established.

1.2 Detecting Critical Pairs and Frequency-Sweeping Curves

In this section, we will show that Problem 1 can be easily solved from the frequency-sweeping curves (the graphical tool adopted in this book) with the aid of an immediate yet important transformation of the quasipolynomial. In addition, two useful indices will be introduced and discussed.

1.2.1 Rewriting Characteristic Function

The characteristic function $f(\lambda, \tau)$ (1.3) is often transformed by letting $z = e^{-\tau\lambda}$ into a more tractable form, including two variables $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}$ (see [55]):

$$p(\lambda, z) = \sum_{i=0}^{q} a_i(\lambda) z^i. \tag{1.4}$$

The function $p(\lambda, z)$ (1.4) is a two-variate polynomial, which may help to facilitate our study. The detection of the critical pairs (λ, τ) for $f(\lambda, \tau) = 0$ amounts to detecting the critical pairs (λ, z) ($\lambda \in \mathbb{C}_0$ and $z \in \partial\mathbb{D}$, i.e., λ and z are located on the imaginary axis and the unit circle, respectively) such that $p(\lambda, z) = 0$.

Remark 1.4 The time-delay system (1.1) is said to be *hyperbolic* at a given τ, if it has no critical imaginary roots at this τ [46]. Furthermore, the time-delay system (1.1) is said to be *hyperbolic independently of delay*, if it has no critical imaginary roots for any $\tau \geq 0$. According to the root continuity argument, a time-delay system with delay-independent hyperbolicity is either asymptotically stable or unstable for all $\tau \geq 0$. The former case corresponds to the well-known *delay-independent stability*, see [20, 22, 23, 49, 55]. The latter case representing the so-called *delay-independent instability*, to the best of the authors' knowledge, is seldom encountered in the literature. Such an example will be presented in Sect. 9.1.3.

Consider now that the system is not delay-independently hyperbolic. Such a condition will be automatically assumed in the forthcoming chapters. In this context, without any loss of generality, suppose there are u critical pairs for $p(\lambda, z) = 0$ throughout this book: $(\lambda_0 = j\omega_0, z_0)$, $(\lambda_1 = j\omega_1, z_1)$, $\ldots$, $(\lambda_{u-1} = j\omega_{u-1}, z_{u-1})$ where $\omega_0 \leq \omega_1 \leq \cdots \leq \omega_{u-1}$. Notice that two critical pairs may share the same critical imaginary root.

Once all the critical pairs $(\lambda_\alpha, z_\alpha)$, $\alpha = 0, \ldots, u-1$, are found, all the critical pairs (λ, τ) can be obtained. More precisely, for each critical imaginary root λ_α, the corresponding critical delays are given by $\tau_{\alpha,k} \overset{\Delta}{=} \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}$, $k \in \mathbb{N}$, where $\tau_{\alpha,0} \overset{\Delta}{=} \min\{\tau \geq 0 : e^{-\tau\lambda_\alpha} = z_\alpha\}$. The pairs $(\lambda_\alpha, \tau_{\alpha,k})$, $k = 0, 1, \ldots$, define a set of critical pairs associated with $(\lambda_\alpha, z_\alpha)$.

We illustrate the notations introduced above through the following example.

Example 1.4 Consider a time-delay system with the characteristic function $f(\lambda, \tau) = e^{-2\tau\lambda} + (\lambda^2 + 1)e^{-\tau\lambda} + \lambda^4 - 2$. The system has five sets of critical pairs: $(\lambda_\alpha, \tau_{\alpha,k})$, $\alpha = 0, \ldots, 4$. In particular, $\lambda_0 = 0$ and hence this system cannot be asymptotically stable for any $\tau \geq 0$. To further characterize the system property (marginally stable or unstable), we need to know $NU(\tau)$ and the distribution of the critical imaginary roots. The second and the third sets of critical pairs have the same critical imaginary root λ: $(\lambda_1 = j, \tau_{1,k} = 2k\pi)$ and $(\lambda_2 = j, \tau_{2,k} = (2k+1)\pi)$. The fourth and the fifth sets of critical pairs also involve the same critical imaginary root λ: $(\lambda_3 = 1.3161j, \tau_{3,k} = \frac{1.1961+2k\pi}{1.3161})$ and $(\lambda_4 = 1.3161j, \tau_{4,k} = \frac{5.0871+2k\pi}{1.3161})$. □

Remark 1.5 For a critical imaginary root $\lambda_\alpha = j\omega_\alpha$, regardless of its multiplicity, ω_α must be bounded (see Proposition 1.8 in [85]). In other words, all the characteristic roots located on the imaginary axis are *bounded*. Further remarks on the estimation of the frequency interval including all the critical imaginary roots can be found in [27, 95].

Remark 1.6 A critical imaginary root λ_α is *invariant* with respect to the *delay shift* $\frac{2\pi}{\omega_\alpha}$ (i.e., if λ_α is a critical imaginary root for $\tau = \tau_{\alpha,0}$, then the system has a critical imaginary root λ_α for all $\tau = \tau_{\alpha,0} + k\frac{2\pi}{\omega_\alpha}$, $k \in \mathbb{N}$). However, the multiplicity of a critical imaginary root is not necessarily fixed with the delay shift (see the next subsection). This phenomenon is a source of the complexity of the problem.

1.2.2 Two Useful Indices and Related Remarks

Introduce now two indices n and g associated with a critical pair. Such indices play an important role in the asymptotic behavior analysis.

For a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$, denote by $n \in \mathbb{N}_+$ the multiplicity of λ_α at $\tau_{\alpha,k}$. Clearly, a critical imaginary root is called a *simple critical imaginary root* (a *multiple critical imaginary root*) if the corresponding index $n = 1$ ($n > 1$). In other words, the index n simply implies that for $\lambda = \lambda_\alpha$ and $\tau = \tau_{\alpha,k}$,

$$f_{\lambda^0} = \cdots = f_{\lambda^{n-1}} = 0,\ f_{\lambda^n} \neq 0. \tag{1.5}$$

Next, introduce the index $g \in \mathbb{N}_+$ at $(\lambda_\alpha, \tau_{\alpha,k})$, by which we may *artificially* treat $\tau_{\alpha,k}$ as a g-multiple root for $f(\lambda, \tau) = 0$ when $\lambda = \lambda_\alpha$, having the property that when $\lambda = \lambda_\alpha$ and $\tau = \tau_{\alpha,k}$,

$$f_{\tau^0} = \cdots = f_{\tau^{g-1}} = 0,\ f_{\tau^g} \neq 0. \tag{1.6}$$

Both indices n and g are bounded for critical imaginary roots distinct from the origin, as given in the following property, with q defined in (1.3).

Property 1.1 *For a critical imaginary root λ_α of the time-delay system* (1.1)*, it follows that $n < \infty$ and that $g \leq q$ ($g = \infty$) if $\lambda_\alpha \neq 0$ ($\lambda_\alpha = 0$).*

Proof In any vertical strip of the complex plane, there are only a finite number of (multiplicity taken into account) roots for the system (1.1) (Corollary 1.9 in [85]). With this remark, the first part of the proof is finished. Next, we have $f_\tau = -p_z z\lambda$, $f_{\tau\tau} = (-1)^2 p_{zz}(z\lambda)^2 + (-1)^2 p_z z\lambda^2, \ldots$ Thus, for $\lambda = 0$, $f_{\tau^\kappa} = 0, \forall\kappa \in \mathbb{N}_+$. For $\lambda \neq 0$, the index g implies that $p_{z^0} = \cdots = p_{z^{g-1}} = 0$. Given a λ, z can be treated as a g-multiple root for polynomial equation $p(\lambda, z) = 0$ with degree q. As a q-degree polynomial has at most q roots, the second part of the proof is complete. □

We next recall two important properties reported in [66]:

Property 1.2 *For a critical imaginary root $\lambda_\alpha \neq 0$, the index g is a constant for all the critical delays $\tau_{\alpha,k}$.*

Proof Following the proof of Property 1.1, for a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ with $\lambda_\alpha \neq 0$, the condition (1.6) is equivalent to that at the critical pair $(\lambda_\alpha, z_\alpha)$

$$p_{z^0} = \cdots = p_{z^{g-1}} = 0,\ p_{z^g} \neq 0. \tag{1.7}$$

The proof is now complete as the condition (1.7) does not explicitly depend on τ. □

Property 1.3 *For a critical imaginary root λ_α, the index n may vary with respect to different critical delays $\tau_{\alpha,k}$.*

Examples 6.1 and 6.2, to be seen in this book (Chap. 6), illustrate Property 1.3.

Remark 1.7 It would be more formal to express indices n and g as functions of the critical imaginary root and the corresponding critical delay. For brevity, we simply denote them by "n" and "g" throughout this book with a slight abuse of notations.

1.2.3 Frequency-Sweeping Curves

Problem 1 can be effectively solved from the frequency-sweeping curves generated by the method recently proposed by the authors in [70], which is easy to implement with a low computational load.

Frequency-Sweeping Curves Sweep $\omega \geq 0$ and for each $\lambda = j\omega$ we have q solutions of z such that $p(j\omega, z) = 0$ (denoted by $z_1(j\omega), \ldots, z_q(j\omega)$). In this way, we obtain q frequency-sweeping curves $\Gamma_i(\omega)$: $|z_i(j\omega)|$ versus ω, $i = 1, \ldots, q$.

If $(\lambda_\alpha, \tau_{\alpha,k})$ is a critical pair with index g, g frequency-sweeping curves collide with $\Im_1$ (throughout this book we denote by $\Im_1$ the line parallel to the abscissa axis with ordinate 1) at $\omega = \omega_\alpha$ and the frequency ω_α is called a *critical frequency*.

Remark 1.8 For each given ω, $p(j\omega, z) = 0$ is a polynomial equation of z, which can be easily solved by using the MATLAB command `roots`.

Remark 1.9 If some $\Gamma_i(\omega)$ collide with $\Im_1$ at $\omega = 0$ with the associated $z = 1$ for $p(\lambda, z) = 0$ (i.e., $\lambda = 0$ is a critical imaginary root), as earlier discussed, the system cannot be asymptotically stable for any $\tau \geq 0$. However, if the corresponding $z \neq 1$, $\lambda = 0$ is evidently not a critical imaginary root (since $e^{0\tau} = 1$ for all $\tau \geq 0$) and this point should be ignored. For instance, a system with $f(\lambda, \tau) = \lambda + 1 + e^{-\tau\lambda}$ exhibits the delay-independent stability (sometimes called *weak delay-independent stability* [92]). Obviously, $\lambda = 0$ is not a critical imaginary root, though $p(0, -1) = 0$.

Remark 1.10 If no critical imaginary roots are detected from the frequency-sweeping curves, the system is, clearly, hyperbolic independently of delay (Remark 1.4).

Remark 1.11 As previously mentioned, for a critical imaginary root (say, $\lambda = j\omega'$) there may exist multiple critical pairs (say, two pairs $(j\omega', z_1)$ and $(j\omega', z_2)$). In this case, $\lambda = j\omega'$ corresponds to two sets of critical delays, $\tau_{1,k} \overset{\Delta}{=} \tau_{1,0} + \frac{2k\pi}{\omega'}$ with $\tau_{1,0} \overset{\Delta}{=} \min\{\tau \geq 0 : e^{-\tau\omega' j} = z_1\}$ and $\tau_{2,k} \overset{\Delta}{=} \tau_{2,0} + \frac{2k\pi}{\omega'}$ with $\tau_{2,0} \overset{\Delta}{=} \min\{\tau \geq 0 : e^{-\tau\omega' j} = z_2\}$. Such a case will be presented and discussed in Example 7.4.

Consider now a simple example to demonstrate how all the critical imaginary roots and the corresponding critical delays for time-delay system (1.1) can be detected according to the frequency-sweeping curves.

Example 1.5 Consider the system in Example 1.2, where $f(\lambda, \tau) = -e^{-\tau\lambda} + \lambda^2 - 0.1\lambda + 2$ and $p(\lambda, z) = -z + \lambda^2 - 0.1\lambda + 2$. This system has only one frequency-sweeping curve and it can be easily generated using MATLAB (or other software for scientific computation). For instance, in the MATLAB environment, for each given ω, we assign its value to a variable `w`. The solution of $z_1(j\omega)$ for $p(j\omega, z) = 0$ can be obtained by using the command `roots([-1,(w*j)^2-0.1*w*j+2])`. The frequency-sweeping curve is shown in Fig. 1.4.

Two critical pairs (λ, z) for $p(\lambda, z) = 0$ are found from the frequency-sweeping curve: $(\lambda_0 = 1.0025j, z_0 = 0.9950-0.1003j)$ and $(\lambda_1 = 1.7277j, z_1 = -0.9850-0.1728j)$. For the first critical pair, we calculate the corresponding critical delays such that $e^{-\tau\lambda_0} = z_0 = e^{-(0.1004+2k\pi)j}$. We get: $\tau_{0,k} = 0.1002 + \frac{2k\pi}{1.0025}$, $k \in \mathbb{N}$. Similarly, for the second critical pair, the associated critical delays can be computed by the condition: $e^{-\tau\lambda_1} = z_1 = e^{-(2.9680+2k\pi)j}$. We have that $\tau_{1,k} = 1.7178 + \frac{2k\pi}{1.7277}$, $k \in \mathbb{N}$. In addition, $n = 1$ (i.e., the critical imaginary roots are simple) and $g = 1$ for both sets of critical pairs, as $f_\lambda \neq 0$ and $f_\tau \neq 0$ at these critical pairs.

Following the τ-decomposition idea, the positive τ-axis can be divided into subintervals: $[0, 0.1002)$, $(0.1002, 1.7178)$, $(1.7178, 5.3546)$, $(5.3546, 6.3676)$, ... When τ lies in each such subinterval, $NU(\tau)$ is a constant. □

Later in this book, some algebraic properties of the frequency-sweeping curves will be derived from an analytic curve perspective and, consequently, a new frequency-

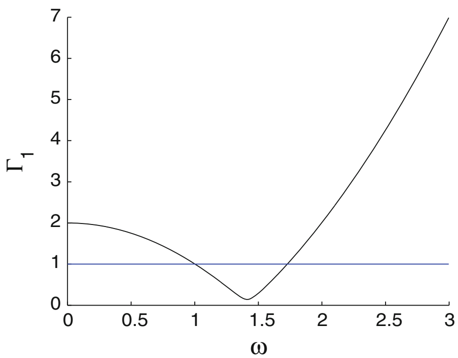

Fig. 1.4 Frequency-sweeping result for Example 1.5

sweeping framework will be established. Such a framework will be helpful in order to characterize and to explicitly compute $NU(\tau)$ in each detected interval.

1.3 Asymptotic Behavior of Critical Imaginary Roots

Unlike Problem 1, Problem 2 has not been fully investigated (it received some partial characterizations). Due to its complexity, we further divide it into two sub-problems (Problems 2.1 and 2.2).

1.3.1 Critical Imaginary Root at a Critical Delay

The first sub-problem of Problem 2 is as follows:

Problem 2.1 The analysis of the asymptotic behavior of a critical imaginary root at a critical delay.

Introduce now some notations describing the asymptotic behavior of a critical imaginary root from the stability perspective. Suppose (α, β) (with $\beta > 0$) is a critical pair with the index n. Near this critical pair, there exist n roots $\lambda_i(\tau)$ continuous with respect to τ satisfying $\alpha = \lambda_i(\beta)$, $i = 1, \ldots, n$. Under some perturbation ε $(-\varepsilon)$ on β, the n roots are expressed by $\lambda_i(\beta + \varepsilon)$ $(\lambda_i(\beta - \varepsilon))$, $i = 1, \ldots, n$. Denote the number of unstable roots among $\lambda_1(\beta+\varepsilon), \ldots, \lambda_n(\beta+\varepsilon)$ $(\lambda_1(\beta-\varepsilon), \ldots, \lambda_n(\beta-\varepsilon))$ by $NU_\alpha(\beta^+)$ $(NU_\alpha(\beta^-))$. With these notations, we define:

$$\Delta NU_\alpha(\beta) \triangleq NU_\alpha(\beta^+) - NU_\alpha(\beta^-). \tag{1.8}$$

In other words, the notation $\Delta NU_\alpha(\beta)$ simply stands for the change in $NU(\tau)$ caused by the variation in the critical imaginary root $\lambda = \alpha$ in some neighborhood of the root as τ increases from $\beta - \varepsilon$ to $\beta + \varepsilon$. For a critical pair $(\lambda_\alpha, \tau_{\alpha,k} > 0)$, we need to compute the value of $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ in order to solve Problem 2.1.

Remark 1.12 If $\tau_{\alpha,0} = 0$ for a critical imaginary root λ_α, such a situation corresponds to the case where the system (1.1) free of delays has original critical imaginary roots. In this case, the asymptotic behavior of the critical pair $(\lambda_\alpha, \tau_{\alpha,0})$ refers to how the original critical imaginary root λ_α varies as τ increases from 0. This information is necessary for computing $NU(+\varepsilon)$ (see Sect. 5.1 for details).

Remark 1.13 The notation $\Delta NU(\tau)$ has been largely used in the literature. However, at a critical delay, the system may happen to have more than one critical imaginary root. That is the reason for taking explicitly into account the critical imaginary root as a subscript in the notation (1.8).

Problem 2.1 will be studied in detail in Chap. 4. Once Problems 1 and 2.1 are solved, we may accurately know the stability property for system (1.1) in the presence of any finitely large τ (see Sect. 5.3). It is worth mentioning that such a stability result is *accurate without any conservatism*. For general linear time-delay systems with commensurate delays, this result represents a new contribution. However, it is still not sufficient for solving the complete stability problem. First, if τ is very large, the procedure may be computationally prohibitive. As we will see in Chap. 4, solving Problem 2.1 is not a trivial task. Second, it is impossible to "calculate" $NU(\tau)$ as $\tau \to \infty$ by using only the solution of Problem 2.1. Since a critical imaginary root has an infinite number of critical delays, we are unable to analyze the asymptotic behavior at all the critical delays one-by-one.

Hence, we need to explore deeply Problem 2.2 discussed below.

1.3.2 Critical Imaginary Root with Infinitely Many Positive Critical Delays

The second sub-problem of Problem 2 can be described as follows:

Problem 2.2 The analysis of the asymptotic behavior of a critical imaginary root with respect to all the infinitely many positive critical delays.

Problem 2.2 (though has not been explicitly proposed in the literature) has been noticed and solved for some specific time-delay systems.

If a critical imaginary root is simple for all the critical delays, it was proved in [97] together with [109] that the way the critical imaginary root moves as τ increases near each positive critical delay always has the same effect on $NU(\tau)$. If a multiple critical imaginary root appears, the case will become much more complicated and computationally involved. To the best of the authors' knowledge, only one paper [54] explicitly discusses such a case. More precisely, in [54], it was proved that if λ_α is a double critical imaginary root for one of the critical delays, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant for all $k \in \mathbb{N}$ with $\tau_{\alpha,k} > 0$. However, in order to perform the analysis, additional assumptions are needed in [54] (details are given in Chap. 6).

The above intriguing property ($\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a *constant* for all $k \in \mathbb{N}$ with $\tau_{\alpha,k} > 0$) is called the *invariance property*. Though this property was only found for some very specific time-delay systems, we are inspired to consider if the following more general result is valid:

The invariance property holds for any time-delay system with commensurate delays.

If the above result (named the *general invariance property*) holds, Problem 2.2 will be tractable and the complete stability problem can be solved. In Chaps. 6–8 of this book, we will study in detail the general invariance property. It will be exciting to see in Chap. 8 that the general invariance property really holds!

Consider once again the time-delay system in Example 1.5 in order to better illustrate the objective of Problem 2.

Example 1.6 In the sequel, we will continue the analysis of Example 1.5, for which all the critical imaginary roots and the critical delays have been found. These critical delays divide the positive τ-axis into infinitely many subintervals.

First, the system has two unstable roots as $\tau = 0$. Therefore, $NU(\tau) = 2$ for τ lies in the first subinterval $[0, 0.1002)$. In order to compute $NU(\tau)$ when τ lies in the second subinterval $(0.1002, 1.7178)$, we need to know the asymptotic behavior of the critical imaginary root $1.0025j$ at the boundary point $\tau = 0.1002$. As this is a simple critical imaginary root case, we have at least two effective ways to accomplish it. We may either compute the derivative of λ with respect to τ by using the implicit function theorem (see [97]) or observe the frequency-sweeping curve (see [21]). It is not hard to include that, as τ increases near $\tau = 0.1002$, a root crosses $\mathbb{C}_0$ at $1.0025j$ from right to left. Due to the conjugate symmetry, a root crosses $\mathbb{C}_0$ at $-1.0025j$ from right to left as well when τ increases near $\tau = 0.1002$. Therefore, $NU(\tau) = 0$ for $\tau \in (0.1002, 1.7178)$. Continuing the above procedure, we have: $NU(\tau) = 2$ for $\tau \in (1.7178, 5.3546)$; $NU(\tau) = 4$ for $\tau \in (5.3546, 6.3676)$; $NU(\tau) = 2$ for $\tau \in (6.3676, 8.9913)$; ....

As the invariance property has been proved for the simple critical imaginary root case, we have stronger results: Each time τ increases near some $\tau_{0,k}$ ($\tau_{1,k}$), two roots cross $\mathbb{C}_0$ from right to left (from left to right). Moreover, such results can be directly known from the frequency-sweeping curve. More precisely, for this time-delay system we have two fascinating properties [70]: (1) The frequency-sweeping curve crosses $\mathfrak{I}_1$ from below to above (from above to below) if and only if the corresponding critical imaginary root crosses $\mathbb{C}_0$ from left to right (from right to left). (2) The frequency-sweeping curve touches without crossing $\mathfrak{I}_1$ if and only if the corresponding critical imaginary root touches without crossing $\mathbb{C}_0$.

Consequently, we may easily keep track of $NU(\tau)$ and analyze the complete stability: In this case study, the system is asymptotically stable if and only if $\tau \in (0.1002, 1.7178)$. □

Example 1.6 shows that the frequency-sweeping curves can be used for solving Problem 2 and considerably simplify the analysis. However, the time-delay system in Example 1.6 is specific ($n = g = 1$ for all critical pairs) and extremely simple to handle by using the existing methods in the literature. For general time-delay systems with commensurate delays, a new frequency-sweeping methodology will be established step-by-step and next illustrated through some examples.

1.4 Book Structure

Characterizing the general invariance property as well as the (complete) stability of linear time-delay systems need a better understanding of the asymptotic behavior of the critical imaginary roots. In this book, we will introduce a new analytic curve perspective to address the problem. From the analytic curve perspective, a series of new mathematical properties will be obtained regarding the asymptotic behavior of the critical imaginary roots and the frequency-sweeping curves, which allow us to thoroughly solve Problems 1 and 2 in the commensurate delays case and to open interesting perspectives in handling the incommensurate delays case.

The remainder of the book is organized as follows:

In Chap. 2, we introduce some preliminary results concerning analytic curves, including the notions of analytic curves and the Puiseux series, the classical Newton diagram, as well as some useful properties.

In Chap. 3, we show that the analytic curve perspective may be appropriately adopted for studying the asymptotic behavior of the critical imaginary roots and the frequency-sweeping curves of time-delay systems.

In Chap. 4, we show how to obtain the Puiseux series in order to solve Problem 2.1. Moreover, some important mathematical properties are given.

In Chap. 5, we first present a procedure to compute $NU(\tau)$ for a finitely large τ. Next, we explain in detail why it is not sufficient for the complete stability problem as well as the necessity of studying Problem 2.2. In order to find a solution to Problem 2.2, we propose to prove the general invariance property.

In Chap. 6, we prove the invariance property for a specific case where the index g is always "1", using the frequency-sweeping curves. An important feature of the frequency-sweeping curves is that they are independent of the critical delays. It will play a key role in addressing the general invariance property in Chaps. 6–8.

In Chap. 7, the invariance property for simple critical imaginary roots is revisited. A contribution of this chapter is that an embryonic form of the frequency-sweeping framework is presented.

In Chap. 8, we study whether the invariance property holds for general time-delay systems. First, a new frequency-sweeping mathematical framework is established based on the embryonic form given in Chap. 7. Next, a series of new mathematical properties regarding the asymptotic behavior of the critical imaginary roots and the frequency-sweeping curves are found. Finally, the general invariance property is confirmed in virtue of these new properties.

In Chap. 9, we give a systematic approach, the *frequency-sweeping approach*, to study the complete stability of time-delay systems with commensurate delays. We obtain the explicit expression of $NU(\tau)$ and solve the complete stability problem.

In Chap. 10, we extend the proposed frequency-sweeping framework to the time-delay systems of neutral type by taking care of an additional necessary condition for the stability of such time-delay systems.

Finally, some concluding remarks and future perspectives are given in Chap. 11.

Chapter 2
Introduction to Analytic Curves

The study of *analytic curves*, which at first sight appears to be unrelated to the stability analysis of time-delay systems, will be extremely helpful for addressing the stability problem.

In this book, we will see that the mathematical properties concerning the singularities of analytic curves provide us with a new angle (called the analytic curve perspective or point of view in this book) to study the stability of time-delay systems. New insights for the complete stability problem will be developed based on this analytic curve perspective. To be more precise, two aspects are essential. First, it will be used for studying the asymptotic behavior of the critical pairs. Second, the analytic curve perspective will be used to improve the classical frequency-sweeping approach. Moreover, as we will discuss later, the analytic curve perspective may be applied to many other important problems.

In this chapter, we start by presenting some fundamentals concerning *analytic curves*. Especially, as an important tool for studying analytic curves, the Puiseux series will be introduced and discussed in detail.

In Sect. 2.1, we will first present the related concepts on analytic curves and show that an analytic curve can be understood in an intuitive manner. In Sect. 2.2, the *Puiseux series* will be introduced for describing and analyzing an analytic curve. The convergence of the Puiseux series will be discussed in Sect. 2.3. In Sect. 2.4, we will briefly review a classical method, the *Newton diagram*, for computing the Puiseux series. In Sect. 2.5, we will explain how to analyze the asymptotic behavior of an analytic curve by means of the Puiseux series. Finally, some notes and comments will be given in Sect. 2.6.

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_2

2.1 Introductory Remarks to Singularities of Analytic Curves

Consider a power series $\Phi(y, x)$ in two variables $x \in \mathbb{C}$ and $y \in \mathbb{C}$:

$$\Phi(y, x) = \sum_{\alpha,\beta \geq 0} \phi_{\alpha,\beta} y^{\alpha} x^{\beta}, \tag{2.1}$$

where $\phi_{\alpha,\beta}$ ($\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$) are complex coefficients.

We suppose that $\Phi(0, 0) = 0$ (that is, the constant term $\phi_{0,0} = 0$) and that the power series $\Phi(y, x)$ is convergent in a small neighborhood of the point $(x = 0, y = 0)$.

Remark 2.1 If there exists a point (y^*, x^*) other than $(0, 0)$ such that $\Phi(y^*, x^*) = 0$, we may obtain a new power series with a zero constant term. More precisely, we may define two new variables $\widetilde{x} = x - x^*$ and $\widetilde{y} = y - y^*$. As a result, we obtain a new power series $\widetilde{\Phi}(\widetilde{y}, \widetilde{x})$ satisfying that $\widetilde{\Phi}(0, 0) = 0$ from the original power series equation $\Phi(y^*, x^*) = 0$ and the local behavior of the original equation $\Phi(y, x) = 0$ as $y \to y^*$ and $x \to x^*$ is reflected by that of the new one $\widetilde{\Phi}(\widetilde{y}, \widetilde{x}) = 0$ as $\widetilde{y} \to 0$ and $\widetilde{x} \to 0$.

Remark 2.2 One may have a question why we are now considering a power series. The reason is related to the fact that for many stability problems in the control area, we need to study characteristic functions of the form $\rho(\lambda, \xi)$, where λ and ξ denote, respectively, the characteristic root and the system parameter under consideration, and $\rho(\lambda, \xi)$ is usually analytic. One may notice that in the case of time-delay system (1.1), the corresponding characteristic function $f(\lambda, \tau)$ falls in this class. Next, near a critical pair (λ^*, ξ^*) such that $\rho(\lambda^*, \xi^*) = 0$, we may expand $\rho(\lambda, \xi)$ as a two-variable Taylor series, which is exactly a power series of the $\Phi(y, x)$ type.

From the algebraic geometry point of view, in e.g., [15, 121], the equation $\Phi(y, x) = 0$ defines an *analytic curve* in the $\mathbb{C}^2$ plane.[1] Instead of studying the whole curve, we are interested in a small neighborhood of the origin O (i.e., the point $(x = 0, y = 0)$) in the $\mathbb{C}^2$ plane. In other words, we study how y varies near "0" with respect to an infinitesimal variation of x near "0". Such a local study will be extremely useful in the subsequent study of the asymptotic behavior of time-delay systems.

Throughout this book, we define the notation $\operatorname{ord}(\cdot)$ as follows.

Definition 2.1 For a function $\varphi(x)$, $\operatorname{ord}(\varphi(x)) = \kappa$ for $x = x^*$ denotes that $\frac{d^i \varphi(x)}{dx^i} = 0$ $(i = 0, \ldots, \kappa - 1)$ and that $\frac{d^{\kappa} \varphi(x)}{dx^{\kappa}} \neq 0$ when $x = x^*$.

Furthermore, for simplicity, we denote by ord_y and ord_x, respectively, the values of $\operatorname{ord}(\Phi(y, 0))$ when $y = 0$ and $\operatorname{ord}(\Phi(0, x))$ when $x = 0$. If $\operatorname{ord}_x = 1$ and/or

[1] Note that we cannot explicitly draw such a curve since there are two complex variables.

$\text{ord}_y = 1$, the curve defined by $\Phi(y, x) = 0$ is called *non-singular* at the origin O and the origin O is called a *non-singular point* of the curve. If both ord_x and ord_y are larger than 1, the curve defined by $\Phi(y, x) = 0$ is called *singular* at the origin O and the origin O is called a *singular point* of the curve.

In order to have a better understanding of the above notions and notations, consider now two simple examples.

Example 2.1 Consider $\Phi(y, x) = y^3 + yx + x$ (polynomials represent a specific type of power series). At the point $(0, 0)$, it follows that $\Phi(0, 0) = 0$, $\text{ord}_y = 3$ ($\frac{d\Phi(y,0)}{dy} = \frac{d^2\Phi(y,0)}{dy^2} = 0$, $\frac{d^3\Phi(y,0)}{dy^3} \neq 0$), and $\text{ord}_x = 1$ ($\frac{d\Phi(0,x)}{dx} \neq 0$). The curve defined by $\Phi(y, x) = y^3 + yx + x = 0$ is non-singular at the origin O ($\text{ord}_x = 1$). □

Example 2.2 Consider $\Phi(y, x) = y^3 + yx + x^2$. At the point $(0, 0)$, it follows that $\Phi(0, 0) = 0$, $\text{ord}_y = 3$ ($\frac{d\Phi(y,0)}{dy} = \frac{d^2\Phi(y,0)}{dy^2} = 0$, $\frac{d^3\Phi(y,0)}{dy^3} \neq 0$), and $\text{ord}_x = 2$ ($\frac{d\Phi(0,x)}{dx} = 0$, $\frac{d^2\Phi(0,x)}{dx^2} \neq 0$). The curve defined by $\Phi(y, x) = y^3 + yx + x^2 = 0$ is singular at the origin O (both ord_y and ord_x are larger than 1). □

As we will show later in the book, a critical pair for the time-delay system (1.1) can be viewed as a non-singular (singular) point if $n = 1$ and/or $g = 1$ (both n and g are greater than 1). As expected, the singular case is much more complicated than the non-singular case.

For simplicity, we will only study the case where both ord_y and ord_x are bounded. In fact, we will see that this case corresponds to the complete stability problem under consideration in this book.

The study of singularities of analytic curves is a meeting point for various mathematical fields such as algebra, geometry, topology, and function theory. The first systematic contribution on curve singularities is due to Isaac Newton. Later on, some theoretical framework (for analysis and classification of curve singularities) was established by many geometers such as Puiseux, Smith, Noether, Halphen, Enriques, and Zariski. A detailed introduction to this subject can be found in e.g., [2, 15, 121]. It is worth mentioning that the analytic curve perspective to be introduced in this book is at an *elementary* level at present.

Intuitively speaking, we may view $y = 0$ as a root for $\Phi(y, x) = 0$ when $x = 0$, whose multiplicity is ord_y. Clearly, the equation $\Phi(y, x) = 0$ determines the corresponding ord_y root loci near the origin O. Such an angle (we interpret the relation between y and x as local root loci in the $\mathbb{C}^2$ plane) is easy to follow and will be frequently used in the sequel.

We now recall the classical *Weierstrass preparation theorem* (see, e.g., [15, 60, 91, 121]). It states that in a small neighborhood of O, $\Phi(y, x)$ can be decomposed as

$$\Phi(y, x) = G(y, x)Q(y, x), \tag{2.2}$$

where $G(y, x)$ is a convergent power series with $G(0, 0) \neq 0$ and $Q(y, x)$ is a polynomial in y

$$Q(y,x) = y^{\mathrm{ord}_y} + \sum_{i=0}^{\mathrm{ord}_y - 1} q_i(x) y^i,$$

where for $i = 0, \ldots, \mathrm{ord}_y - 1$, $q_i(x)$ are convergent power series at $x = 0$ such that $q_i(0) = 0$. This polynomial $Q(y,x)$ is called a *Weierstrass polynomial*.

In other words, in a small neighborhood of O, the root loci of y with respect to x governed by the equation $\Phi(y,x) = 0$ coincide with those for the equation $Q(y,x) = 0$.

Now we know that in a small neighborhood of O, for each x there are ord_y continuous solutions for y, denoted by $y(x)$, such that $\Phi(y(x),x) = 0$ (since a polynomial equation with degree ord_y always has ord_y solutions in $\mathbb{C}$).

In addition, it is not hard to anticipate that the solutions of $y(x)$ can be expressed by some appropriate convergent series.

Two questions arise here. First, which class of series do the solutions of $y(x)$ belong to? Second, how to obtain the corresponding series? In the following two sections, we will give some answers. It should be pointed out that the factorization (2.2) is in general difficult to find since all $q_i(x)$ are power series.

2.2 Puiseux Series

In this section we will introduce an effective tool, the Puiseux series, to describe the local behavior of power series $\Phi(y,x)$ (i.e., the solutions $y(x)$ in a small neighborhood of O). We start with a specific case. If $\frac{\partial \Phi(y,x)}{\partial y} \neq 0$ at O (i.e., $\mathrm{ord}_y = 1$), we may apply the well-known implicit function theorem (see Appendix A). In this particular case (corresponding to the case where the linear time-delay system with commensurate delays has a simple critical imaginary root), $y(x)$ corresponds to a Taylor series, and we can calculate the derivatives of y with respect to x (based on the implicit function theorem) to determine the coefficients of the Taylor series.

However, in the general case, i.e., ord_y is allowed to be greater than 1 (corresponding to the general case where the time-delay system is allowed to have a critical imaginary root with any multiplicity), the implicit function theorem does not allow to conclude. For this reason, the analysis of $y(x)$ calls for a different mathematical tool.

In mathematics, the local variation of $y(x)$ can be well studied by using the *Puiseux' theorem*, see, e.g., [91, 121]. Actually, this theorem has multiple versions. In the sequel, we briefly recall some results closely related to the objective of our study.

According to the Puiseux' theorem, the general solutions of $y(x)$ such that $\Phi(y(x),x) = 0$ are some series "s" of the form

$$s = \sum_{i=1}^{\infty} C_i x^{\frac{i}{\mathrm{N}}}, \qquad (2.3)$$

where C_i are complex coefficients and N is a positive integer.

The fractional power series of the form (2.3) are called the *Puiseux series*. The concept of Puiseux series is not new in mathematics. It was first introduced by Issac Newton in his correspondence with Leibniz and Oldenburg in 1676 [90] and further developed by Victor Puiseux in 1850 [101]. The naming of the series after Puiseux rather than Newton is based upon the fact that Puiseux investigated this series expansion more thoroughly. The above information can be found in [13].

Remark 2.3 Unlike the well-known Taylor series, the exponents of a Puiseux series are allowed to be positive fractional numbers.

A Puiseux series s is called a y-root for $\Phi(y, x) = 0$ if $\Phi(s, x) = 0$. In Sect. 2.4, we will introduce an effective tool for obtaining such y-roots.

Remark 2.4 It should be stressed that a Puiseux series has an infinite number of terms and, hence, we are unable to entirely obtain a Puiseux series by calculation. Fortunately, for the stability problem, we only need to invoke finitely many terms of a Puiseux series (see Chap. 4). In particular, we only need to obtain the first-order term of a Puiseux series in the nondegenerate case. Of course, the more terms we obtain, a more elaborate picture of the root loci we have.

At the end of this section, we borrow two examples from the literature on solving polynomial[2] equations.

Example 2.3 Consider a polynomial equation $y^3 - 3xy + x^3 = 0$, where $y = 0$ is a root when $x = 0$. Following the discussions in Sect. 2.1, there exist three $y(x)$ solutions near the origin O as $\mathrm{ord}_y = 3$. The solutions, which can be found in [115], are the Puiseux series $y = \frac{1}{3}x^2 + o\left(x^2\right)$ and $y = \pm\sqrt{3}x^{\frac{1}{2}} + o(x^{\frac{1}{2}})$. □

Remark 2.5 It shall be noticed that solving a polynomial equation generally cannot be accomplished by radicals (for a power series equation, it is obviously more difficult). It has been proved that the general equation of the fifth degree is not solvable by radicals [53].

Example 2.4 Consider a polynomial equation $y^5 + 2xy^4 - xy^2 - 2x^2y - x^3 + x^4 = 0$, for which $y = 0$ is a root when $x = 0$. As $\mathrm{ord}_y = 5$, the equation has five $y(x)$ solutions near the origin O. The solutions, reported in [120], are as follows: Two solutions are of the form $y = -x + o\,(x)$ and the other three ones are $y = x^{\frac{1}{3}} + o(x^{\frac{1}{3}})$, $y = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)x^{\frac{1}{3}} + o(x^{\frac{1}{3}})$, and $y = (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)x^{\frac{1}{3}} + o(x^{\frac{1}{3}})$. □

In Sect. 2.4, we will provide some details on how to acquire the above Puiseux series solutions.

---

[2] For simplicity, we here give two examples where $\Phi(y, x)$ are polynomials, which represent a specific form of power series. The approach applies to the general power series equations. Historically, the study of the singularities of analytic curves stemmed from solving the polynomial equations.

2.3 Convergence of Puiseux Series

Before discussing deeper the way to derive the Puiseux series, it is necessary to pay attention to the corresponding convergence property. Needless to say, a divergent series will not be useful for the problem studied in this book. A property regarding the convergence of a Puiseux series is given as follows, see [15].

Property 2.1 *A Puiseux series* $\sum_{i=1}^{\infty} C_i x^{\frac{i}{\mathrm{N}}}$ *is a convergent series if and only if the power series* $\sum_{i=1}^{\infty} C_i \chi^i$ *is convergent.*

We see from Property 2.1 that the convergence of a Puiseux series $\sum_{i=1}^{\infty} C_i x^{\frac{i}{\mathrm{N}}}$ depends only on the coefficients C_i, $i = 1, \ldots, \infty$ (it does not depend on the integer N).

As the Puiseux series considered in this chapter are derived from the power series $\Phi(y, x)$, a nice result for the convergence property is available from [15] and given below.

Property 2.2 *If the power series* $\Phi(y, x)$ *are convergent, all the* y*-roots for* $\Phi(y, x) = 0$ *are convergent series.*

In light of property 2.2, the convergence of all the Puiseux series used in this book associated with the complete stability problem for time-delay systems with commensurate delays (including the Puiseux series for studying the asymptotic behavior of the critical imaginary roots as well as the dual Puiseux series, to be proposed later in this book, for studying the asymptotic behavior of the frequency-sweeping curves) can be guaranteed.

2.4 Newton Diagram

The Newton diagram (or Newton polygon) is a geometrical approach proposed by Newton in order to obtain the y-roots for the equation $\Phi(y, x) = 0$ in terms of the Puiseux series. In this section, we briefly review this approach.

Consider power series $\Phi(y, x)$ described by (2.1), where both ord_y and ord_x are bounded. As we just mentioned, according to the Puiseux' Theorem, all the y-root solutions are in the form of Puiseux series.

In the sequel, we demonstrate how to find the initial terms of the corresponding Puiseux series by using the classical Newton diagram. More precisely, we will determine the solutions of the form

$$y = Cx^{\mu} + o(x^{\mu}), \tag{2.4}$$

where C is the complex coefficient and μ is a rational number. Obviously, C and μ may have multiple values.

We mark the point (α, β) by a "dot" in a coordinate plane if there is a nonzero coefficient $\phi_{\alpha,\beta}$ in (2.1). In this way, we obtain a discrete set of points with non-negative integral coordinates in the coordinate plane, called the *Newton diagram* of $\Phi(y, x)$.

We draw a line through the point $(0, \mathrm{ord}_x)$ (this point belongs to the Newton diagram) coinciding with the ordinate axis and we rotate this line counterclockwise around the point $(0, \mathrm{ord}_x)$ until it touches other points from the Newton diagram. Among the touched points from the Newton diagram, we select the one with the greatest abscissa, say $(\mathscr{M}_1, \mathscr{N}_1)$. We now have a segment linking the two points $(0, \mathrm{ord}_x)$ and $(\mathscr{M}_1, \mathscr{N}_1)$. We next rotate the line counterclockwise around the point $(\mathscr{M}_1, \mathscr{N}_1)$ until it touches new points from the Newton diagram. We also select the one with the greatest abscissa, say $(\mathscr{M}_2, \mathscr{N}_2)$, among the touched points. We have a new segment linking two points $(\mathscr{M}_1, \mathscr{N}_1)$ and $(\mathscr{M}_2, \mathscr{N}_2)$. We continue this procedure till the segment ending at the point $(\mathrm{ord}_y, 0)$ (this point belongs to the Newton diagram) is found.

As a result, we obtain the so-called *Newton polygon* which consists of all the segments found by the above procedure (referred to as the *rotating ruler method*). Without any loss of generality, suppose that the Newton polygon of $\Phi(y, x)$ consists of $p \in \mathbb{N}_+$ segments. The starting point and the ending point of the ith segment are denoted by $(\mathscr{M}_{i-1}, \mathscr{N}_{i-1})$ and $(\mathscr{M}_i, \mathscr{N}_i)$ (it is easy to see that $\mathscr{M}_0 = 0, \mathscr{N}_0 = \mathrm{ord}_x, \mathscr{M}_p = \mathrm{ord}_y, \mathscr{N}_p = 0$), respectively. The Newton polygon is depicted in Fig. 2.1.

Note that on a segment of the Newton polygon, say, the ith segment with the endpoints $(\mathscr{M}_{i-1}, \mathscr{N}_{i-1})$ and $(\mathscr{M}_i, \mathscr{N}_i)$, there may exist other points from the Newton diagram. Without loss of generality, suppose there are q points other than the endpoints lying on the ith segment: $(\widetilde{\mathscr{M}_{i1}}, \widetilde{\mathscr{N}_{i1}}), \ldots, (\widetilde{\mathscr{M}_{iq}}, \widetilde{\mathscr{N}_{iq}})$, with $\mathscr{M}_i > \widetilde{\mathscr{M}_{i1}} > \cdots > \widetilde{\mathscr{M}_{iq}} > \mathscr{M}_{i-1}$.

Each segment of the Newton polygon determines a set of solutions of C and μ. More precisely, from the ith segment linking points $(\mathscr{M}_{i-1}, \mathscr{N}_{i-1})$ and $(\mathscr{M}_i, \mathscr{N}_i)$, we have $\mathscr{M}_i - \mathscr{M}_{i-1}$ roots in the form (2.4) with $\mu = \frac{\mathscr{N}_{i-1} - \mathscr{N}_i}{\mathscr{M}_i - \mathscr{M}_{i-1}}$ (note that $-\mu$ is the slope of the segment). The coefficient C associated with this exponent μ has

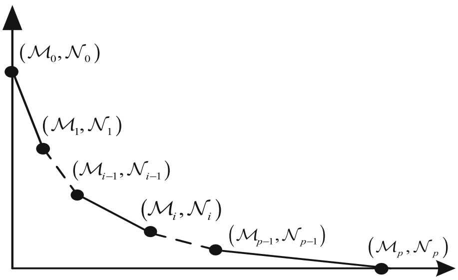

Fig. 2.1 Newton polygon

$\mathcal{M}_i - \mathcal{M}_{i-1}$ (note that this value is equal to the length of the ith segment's projection on the abscissa axis) solutions, which are given by the solutions of the polynomial equation.

$$\phi_{\mathcal{M}_i,\mathcal{N}_i} C^{\mathcal{M}_i - \mathcal{M}_{i-1}} + \phi_{\widetilde{\mathcal{M}}_{i1},\widetilde{\mathcal{N}}_{i1}} C^{\widetilde{\mathcal{M}}_{i1} - \mathcal{M}_{i-1}} + \cdots + \phi_{\mathcal{M}_{i-1},\mathcal{N}_{i-1}} = 0. \quad (2.5)$$

A rigorous proof of the above results can be found in e.g., [115]. In summary, a segment of the Newton polygon gives rise to some initial terms of the Puiseux series with the same exponent. To be more precise, the number of the obtained Puiseux series equals to the length of the projection of this segment on the abscissa and the exponent is the negative slope of this segment.

One can see that the p sets of Puiseux series derived from the p segments of the Newton polygon include all the ord_y y-roots (expressed by the first-order terms of the Puiseux series) for $\Phi(y, x) = 0$.

We now give the Newton polygons, for Examples 2.3 and 2.4, respectively, in Fig. 2.2a, b, from which one may obtain the Puiseux series solutions by employing the Newton diagram introduced above.

2.5 A Direct Application of Puiseux Series

It should be pointed out that, to the best of the authors' knowledge, there are at least two ways to express the Puiseux series solutions. The expression given in the sequel is relatively simple to understand.[3]

Without any loss of generality, the Newton polygon for the power series $\Phi(y, x)$ is supposed to have p segments.

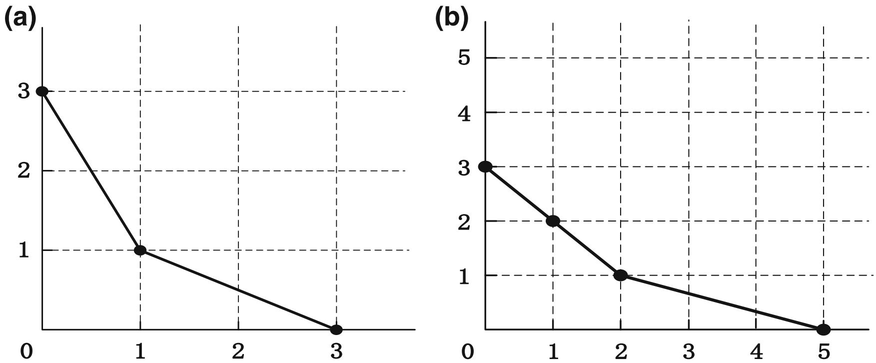

Fig. 2.2 Newton polygons for Examples 2.3 and 2.4. **a** Example 2.3. **b** Example 2.4

[3] In Chap. 4, the expression of the Puiseux series will be simplified. However, some additional algebraic properties (mainly concerning the concept of the *conjugacy class*) will be required.

Following Sect. 2.4, the ith segment determines a set of Puiseux series

$$y = \widetilde{C}_{\mu_i,l} x^{\mu_i} + o(x^{\mu_i}), l = 1, \ldots, \mathscr{M}_i - \mathscr{M}_{i-1}, \tag{2.6}$$

where μ_i is the negative slope of the ith segment, $\widetilde{C}_{\mu_i,l}$ are the corresponding coefficients calculated according to (2.5), and $\mathscr{M}_i - \mathscr{M}_{i-1}$ equals to the length of the segment's projection on the abscissa axis.

Totally, the p segments give rise to the following Puiseux series

$$\begin{cases} y = \widetilde{C}_{\mu_1,l} x^{\mu_1} + o(x^{\mu_1}), l = 1, \ldots, \mathscr{M}_1 - \mathscr{M}_0, \\ \vdots \\ y = \widetilde{C}_{\mu_p,l} x^{\mu_p} + o(x^{\mu_p}), l = 1, \ldots, \mathscr{M}_p - \mathscr{M}_{p-1}. \end{cases} \tag{2.7}$$

With the expression (2.7), we may consider each x^{μ_i} as a single-valued number[4] in $\mathbb{C}$. As a result, the $\mathscr{M}_i - \mathscr{M}_{i-1}$ Puiseux series corresponding to the ith segment as described by (2.6) have $\mathscr{M}_i - \mathscr{M}_{i-1}$ values for $y(x)$. The total p sets of Puiseux series corresponding to all the p segments (i.e., the total ord_y Puiseux series) as described by (2.7) present all the ord_y solutions $y(x)$.

Remark 2.6 In (2.7), we only present the first-order terms (also called the initial terms) of the Puiseux series. As we will see later in this book, the first-order terms are sufficient for the stability analysis in the nondegenerate case. However, in the degenerate case, we need to obtain higher order terms. We will see in Sect. 4.3 that the Newton diagram can be used in an iterative manner such that higher order terms of the Puiseux series can be obtained.

Remark 2.7 On may notice that invoking the Puiseux series (by using the Newton diagram) is not a trivial work, even if only the first-order terms are required. Some representative examples will be given in Chap. 4. Fortunately, the calculation of the Puiseux series may be bypassed. It will be interesting to see that we can accomplish the complete stability analysis for time-delay systems with commensurate delays (by adopting the frequency-sweeping approach to be proposed in this book) without explicitly employing the Newton diagram.

2.6 Notes and Comments

In this chapter, we introduced some useful results for analytic curves including the basic concepts, the Puiseux series, and the Newton diagram. More precisely, we followed the ideas proposed by [15, 60, 91, 121] in order to introduce some of the

[4] In fact, each x^{μ_i} may have multiple values. We may choose any one among them. As will be illustrated by the examples in Chap. 4, the value set of all the Puiseux series is identical for any choice.

notions and properties needed in the forthcoming chapters. From the next chapter, we will apply these results to study the complete stability problem of time-delay systems.

In our opinion, the analytic curve idea in fact may be used for a broader range of stability and stabilization problems in the area of control, as it is applicable to both continuous-time and discrete-time systems.

For continuous-time systems (including the time-delay systems considered in the forthcoming chapters), we are concerned with the variation of the critical roots with respect to the imaginary axis $\mathbb{C}_0$ as some system parameters vary. Recall that for a continuous-time system a critical root refers to a characteristic root located on the imaginary axis $\mathbb{C}_0$. We may perform a qualitative stability analysis through the real parts of the corresponding Puiseux series.

For discrete-time systems (e.g., the state transition expression of a networked control system is a discrete-time model [80]), we are concerned with the variation of the critical roots (note that for discrete-time systems a critical root refers to a characteristic root located on the unit circle $\partial\mathbb{D}$) with respect to the unit circle $\partial\mathbb{D}$, as some system parameters vary. In this case, the stability analysis requires to know the variation directions of the critical roots with respect to the unit circle $\partial\mathbb{D}$, based on the Puiseux series. For instance, if for a critical root its variation direction points to the outside (inside) of the unit circle $\partial\mathbb{D}$, it implies that the critical root becomes an unstable (stable) root.

It was already pointed out that we only adopt some preliminary results on the singularities of analytic curves and one will find that they are not hard to follow. The studies from a decidedly geometrical viewpoint (e.g., resolution of singularities and classification of singularities) are generally much more complicated and can be found in [2, 15, 121].

Chapter 3
Analytic Curve Perspective for Time-Delay Systems

In this chapter, we will apply the *analytic curve point of view* to the stability problem of time-delay systems with commensurate delays, using the prerequisites introduced in Chap. 2.

In Sect. 3.1, we will explain in detail why the analytic curve standpoint helps us to understand the asymptotic behavior of the critical imaginary roots more deeply. We will first present a motivating example in Sect. 3.1.1 to show that some key information may be hidden behind the characteristic function. Then, in Sect. 3.1.2, we will propose to obtain the series expansion expression of the characteristic function, instead of using it directly. Based on this series expansion expression, we will see in Sect. 3.1.3 that the analytic curve perspective fits well with the asymptotic behavior analysis of the critical imaginary roots for time-delay systems. In Sect. 3.2, we will roughly demonstrate that the analytic curve perspective is also applicable to the asymptotic behavior analysis for the frequency-sweeping curves. A motivating example will be given to show that such a new idea is important as the classical frequency-sweeping method fails to handle the general case.

3.1 Further Focus on Asymptotic Behavior Analysis of Critical Imaginary Roots

Consider the time-delay system (1.1)

$$\dot{x}(t) = \sum_{\ell=0}^{m} A_\ell x(t - \ell\tau),$$

with the characteristic function (1.3)

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}.$$

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_3

As discussed in Chap. 1, in order to analyze the stability of system (1.1), for a critical imaginary root we need to know its asymptotic behavior at a critical delay. This is the objective of Problem 2.1.

We now explain through a motivating example why the existing methods (based on a direct study of $f(\lambda, \tau)$) do not allow solving Problem 2.1.

3.1.1 A Motivating Example

It is true that the asymptotic behavior of a critical imaginary root with respect to a critical delay is fully determined by the characteristic function $f(\lambda, \tau)$. In fact, most of the existing results are based on a direct study of $f(\lambda, \tau)$.

However, we here point out that some key information may be hidden behind the characteristic function $f(\lambda, \tau)$ (or, its explicit form), as demonstrated below.

Example 3.1 Consider the following three different characteristic functions:

$$f^{(1)}(\lambda, \tau) = e^{-3\tau\lambda} - (\lambda^6 - \lambda^4 + \lambda^2)e^{-2\tau\lambda} - (\lambda^{10} - \lambda^8 + \lambda^6)e^{-\tau\lambda} + \lambda^{12}, \quad (3.1)$$

$$f^{(2)}(\lambda, \tau) = e^{-2\tau\lambda} + \left(\frac{\pi}{2}\lambda^3 - \lambda^2 + \frac{\pi}{2}\lambda + 1\right) e^{-\tau\lambda} - \frac{\pi}{2}\lambda^5 - \frac{\pi}{2}\lambda^3 - \lambda^2, \quad (3.2)$$

$$f^{(3)}(\lambda, \tau) = e^{-\tau\lambda} + \frac{3\pi}{8}\lambda^5 - \frac{\pi^2}{8}\lambda^4 + \frac{5\pi}{4}\lambda^3 - \frac{\pi^2}{4}\lambda^2 + \frac{7\pi}{8}\lambda - \frac{\pi^2}{8} + 1. \quad (3.3)$$

For all the above three quasipolynomials (3.1), (3.2), and (3.3), $\lambda = j$ is a triple critical imaginary root at $\tau = \pi$. However, the asymptotic behavior of the critical pair (j, π) is quite different for the quasipolynomials above, which can be observed from the root loci near (j, π) shown in Figs. 3.1, 3.2, and 3.3.

The root loci of $f^{(3)}(\lambda, \tau)$ given in (3.3), as shown in Fig. 3.3, exhibit a nearly symmetric structure with respect to the critical imaginary root. By contrast, the root loci of $f^{(1)}(\lambda, \tau)$ given in (3.1), as shown in Fig. 3.1, are asymmetric (in fact, they are independent of each other, as will be further illustrated in Chap. 4). Finally, the root loci of $f^{(2)}(\lambda, \tau)$ given in (3.2), as shown in Fig. 3.2, are mixed with the nearly symmetric and the asymmetric structures (among the three root loci, two are nearly symmetric while the other one is independent).

What is the reason causing different structures of the root loci? The answer is related to their respective dominating partial derivatives of the quasipolynomials. Some nonzero partial derivatives induce the leading effect on the root loci.[1]

Leaving the detailed analysis in Chap. 4, we are listing the respective dominant factors for the three characteristic functions: For $f^{(1)}(\lambda, \tau)$ given in (3.1), $f^{(1)}_{\lambda^3} \neq 0$ and $f^{(1)}_{\tau^3} \neq 0$ and these two quantities are the dominant factors. For $f^{(2)}(\lambda, \tau)$ given

[1] This argument will be verified later in Chap. 4.

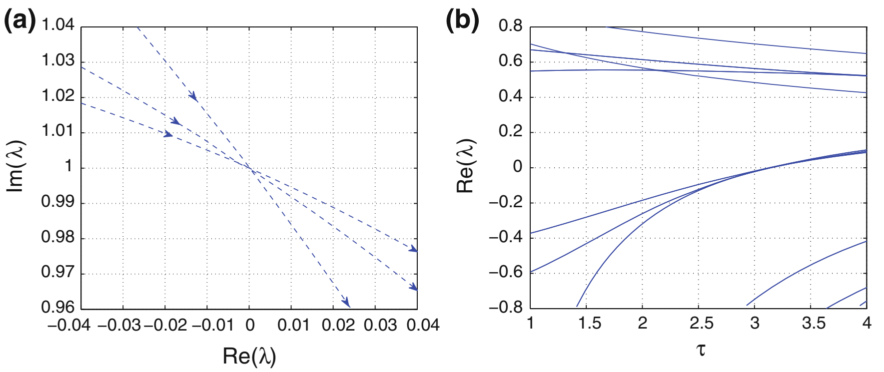

Fig. 3.1 Root loci for the characteristic function given in (3.1). **a** Re(λ) versus Im(λ). **b** Re(λ) versus τ

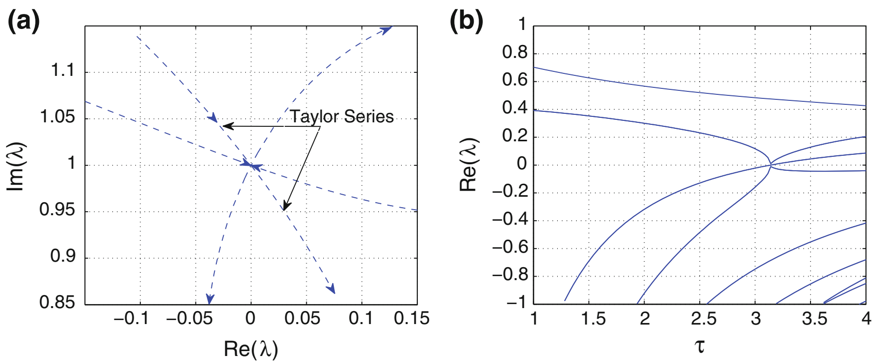

Fig. 3.2 Root loci for the characteristic function given in (3.2). **a** Re(λ) versus Im(λ). **b** Re(λ) versus τ

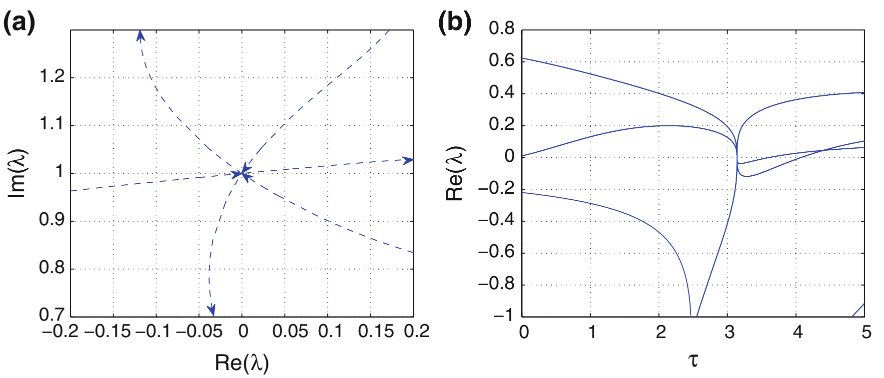

Fig. 3.3 Root loci for the characteristic function given in (3.3). **a** Re(λ) versus Im(λ). **b** Re(λ) versus τ

in (3.2), $f^{(2)}_{\lambda^3} \neq 0$, $f^{(2)}_{\lambda\tau} \neq 0$, and $f^{(2)}_{\tau^2} \neq 0$ are the dominant factors. Finally, for $f^{(3)}(\lambda, \tau)$ given in (3.3), $f^{(3)}_{\lambda^3} \neq 0$ and $f^{(3)}_{\tau} \neq 0$ are the dominant factors. □

Remark 3.1 It should be pointed out that most of the existing methods (e.g., [21, 97, 122]) cannot be used to analyze the asymptotic behavior of the quasipolynomials in Example 3.1. Generally, these methods are based on the *implicit function theorem* and not applicable to Problem 2.1 in the case of multiple critical imaginary roots (the reason will be given in detail in Sect. 4.1).

3.1.2 Two-Variable Taylor Expansion

Inspired by Example 3.1, we realized that the asymptotic behavior of a critical pair is determined by the information concerning the partial derivatives of the characteristic function $f(\lambda, \tau)$ with respect to λ and τ.

So, instead of treating the characteristic function $f(\lambda, \tau)$ directly, we propose to transform it into a power series. As $f(\lambda, \tau)$ (1.3) is a quasipolynomial, it is analytical with respect to variables λ and τ. Thus, in a small neighborhood of a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$, the characteristic function $f(\lambda, \tau)$ can be expanded as a convergent power series of the form:

$$f(\lambda, \tau) = f(\lambda_\alpha, \tau_{\alpha,k}) + (f_\lambda \Delta\lambda + f_\tau \Delta\tau) + \frac{f_{\lambda\lambda}(\Delta\lambda)^2 + 2 f_{\lambda\tau}\Delta\lambda\Delta\tau + f_{\tau\tau}(\Delta\tau)^2}{2!} + \frac{f_{\lambda^3}(\Delta\lambda)^3 + 3 f_{\lambda^2\tau}(\Delta\lambda)^2\Delta\tau + 3 f_{\lambda\tau^2}\Delta\lambda(\Delta\tau)^2 + f_{\tau^3}(\Delta\tau)^3}{3!} + \cdots, \tag{3.4}$$

where $\lambda = \lambda_\alpha + \Delta\lambda$ and $\tau = \tau_{\alpha,k} + \Delta\tau$.

The expression (3.4) is a standard two-variable Taylor expansion of $f(\lambda, \tau)$. All the partial derivatives of $f(\lambda, \tau)$ with respect to λ and τ are included in this formula.

Remark 3.2 We can see that the relation between $\Delta\lambda$ and $\Delta\tau$ is completely determined by the two-variable expansion given in (3.4). In fact, such a relation implicitly describes the asymptotic behavior we are interested in.

Remark 3.3 We will focus mainly on the case with no characteristic roots at the origin for time-delay system (1.1). If $\lambda = 0$ is a characteristic root, this root is invariant with respect to the delay parameter and hence the system cannot be asymptotically stable for any $\tau \geq 0$. It is worth mentioning that an example (Example 4.7) will be presented later to explain the particularity of such a case and the corresponding asymptotic behavior.

Furthermore, we may reformulate (3.4) in a convenient form. Since $f(\lambda, \tau) = f(\lambda_\alpha, \tau_{\alpha,k}) = 0$, we have:

$$0 = (f_\lambda \Delta\lambda + f_\tau \Delta\tau) + \frac{f_{\lambda\lambda}(\Delta\lambda)^2 + 2 f_{\lambda\tau} \Delta\lambda\Delta\tau + f_{\tau\tau}(\Delta\tau)^2}{2!}$$
$$+ \frac{f_{\lambda^3}(\Delta\lambda)^3 + 3 f_{\lambda^2\tau}(\Delta\lambda)^2\Delta\tau + 3 f_{\lambda\tau^2}\Delta\lambda(\Delta\tau)^2 + f_{\tau^3}(\Delta\tau)^3}{3!} + \cdots \quad (3.5)$$

Recall that, in light of the index n, $f_\lambda = \cdots = f_{\lambda^{n-1}} = 0$ and $f_{\lambda^n} \neq 0$. As a result, for a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ with indices n and g, we now obtain a series expression $F_{(\lambda_\alpha,\tau_{\alpha,k})}(\Delta\lambda, \Delta\tau)$ (from the right-hand side of (3.5)) describing the relation between $\Delta\lambda$ and $\Delta\tau$ as follows:

$$F_{(\lambda_\alpha,\tau_{\alpha,k})}(\Delta\lambda, \Delta\tau) = \sum_{i=n}^{\infty} L_{i0}(\Delta\lambda)^i + \sum_{i=0}^{\infty} (\Delta\lambda)^i \sum_{l=1}^{\infty} L_{il}(\Delta\tau)^l = 0, \quad (3.6)$$

where $L_{il} = \frac{f_{\lambda^i \tau^l}}{(i+l)!}\binom{i+l}{i}$ ($\binom{i+l}{i}$ denotes the number of i-combinations from a set of $i+l$ elements). In addition, in view of the index g, we have that $L_{01} = \cdots = L_{0(g-1)} = 0$ and $L_{0g} \neq 0$.

From the root-locus point of view, for a $\Delta\tau$, $\Delta\lambda$ must have n solutions (multiplicity taken into account) satisfying that $F_{(\lambda_\alpha,\tau_{\alpha,k})}(\Delta\lambda, \Delta\tau) = 0$ and that $\Delta\lambda \to 0$ as $\Delta\tau \to 0$. The n solutions of $\Delta\lambda$ represent the local root loci near the critical pair for the time-delay system.

Remark 3.4 For the sake of simplicity, in the remaining part of the book, we usually use a more concise expression $F(\Delta\lambda, \Delta\tau)$ (i.e., we omit the subscript "$(\lambda_\alpha, \tau_{\alpha,k})$") when no confusion occurs.

3.1.3 Analytic Curves and Asymptotic Behavior of Critical Imaginary Roots

Now, we can see that the asymptotic behavior of a critical pair for the time-delay system (1.1) is fully determined by the corresponding power series $F(\Delta\lambda, \Delta\tau)$. Moreover, $F(\Delta\lambda, \Delta\tau)$ belongs to the class of the power series $\Phi(y, x)$ discussed in Chap. 2 since $F(\Delta\lambda, \Delta\tau)$ is convergent near the point $(0, 0)$ with $F(0, 0) = 0$. As a consequence, we can use the existing mathematical results for analytic curves to study the time-delay system (1.1).

The first important result concerning the variation of $\Delta\lambda$ with respect to $\Delta\tau$, based on the expression (3.6), can be summarized as follows:

Theorem 3.1 *Consider the time-delay system (1.1) and assume $\lambda_\alpha \neq 0$ is an n-multiple imaginary root for $\tau = \tau_{\alpha,k}$. If τ is perturbed at $\tau_{\alpha,k}$ by $\Delta\tau$, the variation $\Delta\lambda$ of λ at λ_α corresponds to n Puiseux series solutions with respect to $\Delta\tau$. Any Puiseux series solution converges in a neighborhood of* $(\Delta\lambda = 0, \Delta\tau = 0)$.

The proof of Theorem 3.1 can be found in [71]. In order to make the book self-contained, a modified version of the proof is given as follows:

Proof The solutions of $\Delta\lambda$ in terms of $\Delta\tau$ are determined by $F(\Delta\lambda, \Delta\tau) = 0$, where $F(\Delta\lambda, \Delta\tau)$ is a power series given by (3.6). In $F(\Delta\lambda, \Delta\tau)$, the term $(\Delta\lambda)^n$ appears with a nonzero coefficient, in view of the multiplicity n. According to the discussions in Sects. 2.1 and 2.2, the solutions are in the form of Puiseux series.

As $F(\Delta\lambda, \Delta\tau)$ is a convergent power series, any Puiseux series solution converges in some neighborhood of the origin (Property 2.2). □

Unlike the well-known Taylor series, the Puiseux series are generally with fractional exponents. Thus, it is much more complicated to determine a Puiseux series (by means of calculating the exponents as well as the associated coefficients). The approach for computing the Puiseux series (based on the Newton diagram) will be detailed in Chap. 4. Furthermore, in Chap. 4, some useful mathematical properties of the Puiseux series will be presented. With the aid of these properties we may express the Puiseux series in a more convenient form.

At the end of this subsection, we give some useful remarks concerning the power series $F(\Delta\lambda, \Delta\tau)$.

In order to explicitly compute the Puiseux series solutions $\Delta\lambda(\Delta\tau)$, we need to invoke the power series $F(\Delta\lambda, \Delta\tau)$. Generally, it is neither necessary nor possible to obtain all the infinitely many terms of $F(\Delta\lambda, \Delta\tau)$. We only need to calculate a finite number of partial derivatives of $f(\lambda, \tau)$ with respect to λ and τ, as will be seen in Chap. 4.

Without any loss of generality, $F(\Delta\lambda, \Delta\tau)$ may be decomposed as a product of some power series:

$$F(\Delta\lambda, \Delta\tau) = U(\Delta\lambda, \Delta\tau)F_1(\Delta\lambda, \Delta\tau)\cdots F_v(\Delta\lambda, \Delta\tau), \tag{3.7}$$

where $U(0,0) \neq 0$, $F_1(0,0)=\cdots=F_v(0,0)=0$, and each $F_l(\Delta\lambda, \Delta\tau)$ ($l = 1, \ldots, v$) is irreducible (i.e., it cannot be further decomposed into a product of some power series which equal to 0 at the point $(0, 0)$).

Property 3.1 *For a critical pair* $(\lambda_\alpha \neq 0, \tau_{\alpha,k})$*, neither* $\Delta\lambda$ *factor nor* $\Delta\tau$ *factor appears in the right-hand side of (3.7).*

Proof According to Property 1.1 (Chap. 1), for a critical pair $(\lambda_\alpha \neq 0, \tau_{\alpha,k})$, the indices n and g must be bounded. Thus, $F(\Delta\lambda, \Delta\tau)$ cannot be decomposed into the form $U(\Delta\lambda, \Delta\tau)(\Delta\lambda)^\chi(\Delta\tau)^\kappa F_1(\Delta\lambda, \Delta\tau)\cdots F_v(\Delta\lambda, \Delta\tau)$ ($\chi \in \mathbb{N}$, $\kappa \in \mathbb{N}$, $\chi + \kappa \geq 1$), as both the $(\Delta\lambda)^n$ term and the $(\Delta\tau)^g$ term exist in $F(\Delta\lambda, \Delta\tau)$. □

The decomposition form (3.7) indicates that each equation $F_l(\Delta\lambda, \Delta\tau) = 0$ independently determines some Puiseux series solutions (equivalently, some local root loci). That is, according to (3.7), the Puiseux series can be divided into groups.

It is important to point out that the decomposition form (3.7) is generally very difficult to find in practice (since $F_1(\Delta\lambda, \Delta\tau), \ldots, F_v(\Delta\lambda, \Delta\tau)$ are all power series) and that we do not need to really decompose $F(\Delta\lambda, \Delta\tau)$.

Instead, the decomposition form (3.7) will be *implicitly* used along with the concept of the *conjugacy class*, to be introduced later in this book. We will see in Chap. 8 that such an idea is very useful for a macroscopic study of the asymptotic behavior of the critical pairs.

3.2 Asymptotic Behavior Analysis of Frequency-Sweeping Curves

The frequency-sweeping approach is by now a classical tool in the field of control theory and has been extensively used for the stability analysis of time-delay systems. One of its first versions is the so-called *Tsypkin's criterion* [114] (see, e.g., [92] and the discussions therein) used for studying delay-independent stability of some closed-loop systems. However, to the best of the authors' knowledge, in most of the existing applications, the frequency-sweeping curves are only used to detect the critical pairs (if any!) of time-delay systems, see, e.g., [39]. Only a few attempts have been made to employ the frequency-sweeping curves to the asymptotic behavior analysis. For instance, in [64], the (single) frequency-sweeping curve is used for analyzing the asymptotic behavior of the critical imaginary roots. However, the scenario considered therein is specific and it is not easy to extend the approach to the general case.

In the sequel, we give a motivating example to show that the frequency-sweeping curves of a time-delay system may possess some involved characteristics, even if the time-delay system under consideration has only simple critical imaginary roots.

Example 3.2 Consider a time-delay system with the quasipolynomial $f(\lambda, \tau) = e^{-3\tau\lambda} + 3e^{-2\tau\lambda} + 3e^{-\tau\lambda} + \lambda^3 - \lambda^2 + \lambda$, where $\lambda = j$ (for $\tau = (2k+1)\pi, k \in \mathbb{N}$) is a *simple critical imaginary root* with $g = 3$. At the critical frequency $\omega = 1$, the frequency-sweeping curves (see Fig. 3.4) have a multiple point. To the best of the authors' knowledge, such a case has not been reported and investigated so far. □

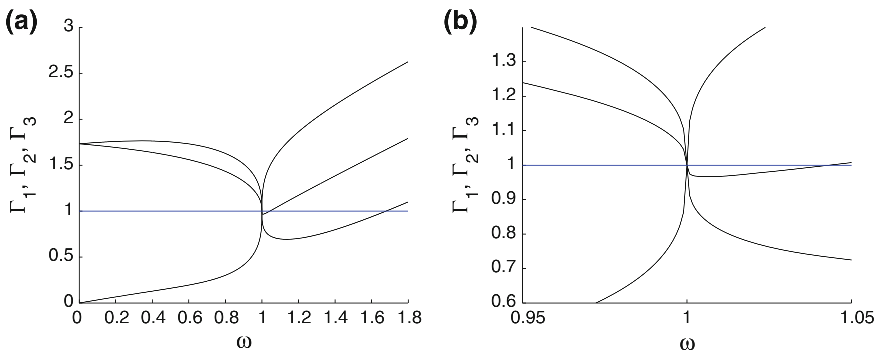

Fig. 3.4 Frequency-sweeping result for Example 3.2. **a** Frequency-sweeping curves for $0 \le \omega \le 1.8$. **b** Zoomed-in figure near $\omega = 1$

Similar examples will also be encountered in Chaps. 7 and 8, for which the existing frequency-sweeping approaches do not allow to conclude.

As mentioned in Chap. 1, we will adopt the analytic curve perspective to study the asymptotic behavior of the frequency-sweeping curves and gradually we will establish a new frequency-sweeping framework (detailed development will be given from Chap. 6). Roughly speaking, the following new results will be obtained.

First, we will introduce a new concept: the *dual Puiseux series*. For the characteristic equation $f(\lambda, \tau) = 0$, we propose to consider the variation of τ in $\mathbb{C}$ with respect to λ. This is equivalent to analyzing the way $\Delta\tau$ varies in $\mathbb{C}$ with respect to $\Delta\lambda$ for the equation $F(\Delta\lambda, \Delta\tau) = 0$. The resultant series are the so-called dual Puiseux series $\Delta\tau(\Delta\lambda)$.

Second, we will prove that the frequency-sweeping curves have a close connection with the dual Puiseux series. Consequently, the asymptotic behavior for general frequency-sweeping curves can be fully studied by means of the dual Puiseux series.

With the above mentioned novelties, the classical frequency-sweeping approach will be improved and the frequency-sweeping approach in this book is also referred to as the the frequency-sweeping framework. Using this frequency-sweeping framework, the complete stability problem will be systematically solved for linear time-delay systems with commensurate delays. Moreover, computing the Puiseux series (through first transforming $f(\lambda, \tau)$ into $F(\Delta\lambda, \Delta\tau)$ and then employing the Newton diagram), which is clearly not a trivial work, will not be necessary. The complete stability may be graphically studied from the frequency-sweeping curves. Such an idea will be proposed and discussed in Chap. 9.

3.3 Notes and Comments

In this chapter, we briefly explained how the mathematical tool for studying the analytic curves (introduced in Chap. 2) can be used for the stability analysis of time-delay systems. Such an analytic curve perspective presents a novelty with respect to the existing studies in the literature (see, e.g., [6, 39, 45, 85]) and most of the results to be proposed in this book arise from this idea.

Although the study of analytic curves appears to be complex and computationally involved, most of the relevant results used in this book can be appropriately interpreted from an intuitive root-locus angle, making the contents of this book not difficult to follow.

Chapter 4
Computing Puiseux Series for a Critical Pair

As pointed out in Chap. 3, the Puiseux series is an effective tool for analyzing the asymptotic behavior of the critical imaginary roots for general time-delay systems. Thus, computing the Puiseux series is important in solving the stability problem and the Puiseux series will be extensively used throughout this volume.

In Sect. 4.1 we will explain why the existing methods for describing the asymptotic behavior do not work in the general case. Next, in Sect. 4.2, an algorithm for calculating all the Puiseux series will be proposed. Furthermore, we will show (Sect. 4.3) that the proposed approach can be used in an iterative manner to obtain higher order terms of the Puiseux series such that the degenerate case can be studied appropriately. In Sect. 4.4, some useful properties on the Puiseux series will be presented. Finally, concluding remarks will be given in Sect. 4.5.

4.1 Why Puiseux Series Are a Necessary Tool

To the best of the authors' knowledge, the Puiseux series was first introduced in the stability analysis of time-delay systems in a recent paper [19] (see Sect. 4.5 for more details). In this section, we will explain the necessity for adopting this tool. It is worth mentioning that the idea may be extended for dealing with systems with incommensurate delays.

4.1.1 Introductory Remarks

Consider the time-delay system (1.1)

$$\dot{x}(t) = \sum_{\ell=0}^{m} A_\ell x(t - \ell\tau),$$

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_4

with the characteristic function (1.3)

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}.$$

As discussed in Chap. 1, we need to know how a critical imaginary root behaves at a critical delay, which is nothing else than the objective of Problem 2.1.

If a critical imaginary root λ_α is *simple* at a critical delay $\tau_{\alpha,k}$ (i.e., $f = 0$ and $f_\lambda \neq 0$ at $(\lambda_\alpha, \tau_{\alpha,k})$), we may compute the derivative of λ with respect to τ based on the *implicit function theorem* (see Appendix A):

$$\frac{d\lambda}{d\tau} = -\frac{f_\tau}{f_\lambda}. \tag{4.1}$$

As a result, we have a straightforward asymptotic behavior characterization for a critical pair $(\lambda_\alpha, \tau_{\alpha,k} > 0)$ as: As τ increases from $\tau_{\alpha,k} - \varepsilon$ to $\tau_{\alpha,k} + \varepsilon$, a characteristic root λ such that $f(\lambda_\alpha, \tau_{\alpha,k}) = 0$ crosses the imaginary axis $\mathbb{C}_0$ from the left half-plane $\mathbb{C}_-$ to the right half-plane $\mathbb{C}_+$ if $\mathrm{Re}(\frac{d\lambda}{d\tau}) > 0$, while it crosses the imaginary axis $\mathbb{C}_0$ from the right half-plane $\mathbb{C}_+$ to the left half-plane $\mathbb{C}_-$ if $\mathrm{Re}(\frac{d\lambda}{d\tau}) < 0$.

Remark 4.1 As mentioned in Remark 1.12, the case when $\tau = 0$ is a critical delay will be specifically discussed in Sect. 5.1 (for computing $NU(+\varepsilon)$).

Throughout this book, we usually adopt a more concise (but less precise) expression "a critical imaginary root λ_α crosses (or touches) $\mathbb{C}_0$ near a positive critical delay $\tau_{\alpha,k}$" instead of "a characteristic root λ such that $f(\lambda_\alpha, \tau_{\alpha,k}) = 0$ crosses (or touches) $\mathbb{C}_0$ *as* τ increases from $\tau_{\alpha,k} - \varepsilon$ to $\tau_{\alpha,k} + \varepsilon$".

If $\mathrm{Re}(\frac{d\lambda}{d\tau}) = 0$, the first-order derivative is not sufficient for concluding on the variation of the critical imaginary root with respect to $\mathbb{C}_0$. This is the so-called *degenerate case*, for which higher order derivatives are needed. To intuitively illustrate such degenerate cases, we present two examples:

Example 4.1 Consider a time-delay system with $f(\lambda, \tau) = e^{-2\tau\lambda} + (-\lambda^6 - 3\lambda^4 - 3\lambda^2 + \lambda + 2)e^{-\tau\lambda} - \lambda^7 - 2\lambda^6 - 3\lambda^5 - 6\lambda^4 - 3\lambda^3 - 6\lambda^2$, where $\lambda = j$ is a simple critical imaginary root at $\tau = \pi$. For the critical pair (j, π), $\mathrm{Re}(\frac{d\lambda}{d\tau}) = \mathrm{Re}(\frac{d^2\lambda}{d\tau^2}) = 0$, $\mathrm{Re}(\frac{d^3\lambda}{d\tau^3}) > 0$. Thus, the critical imaginary root j crosses $\mathbb{C}_0$ from $\mathbb{C}_-$ to $\mathbb{C}_+$ near the critical delay π. The corresponding root locus is given in Fig. 4.1a. □

Example 4.2 Consider a time-delay system with $f(\lambda, \tau) = e^{-2\tau\lambda} - (\lambda^2 - 1)e^{-\tau\lambda} + \lambda^6 - \lambda^5 + \lambda + 2$, where $\lambda = j$ is a simple critical imaginary root for $\tau = \pi$. For the critical pair (j, π), $\mathrm{Re}(\frac{d\lambda}{d\tau}) = 0$, $\mathrm{Re}(\frac{d^2\lambda}{d\tau^2}) \neq 0$. Therefore, the critical imaginary root j touches $\mathbb{C}_0$ near the critical delay π. The corresponding root locus is given in Fig. 4.1b. □

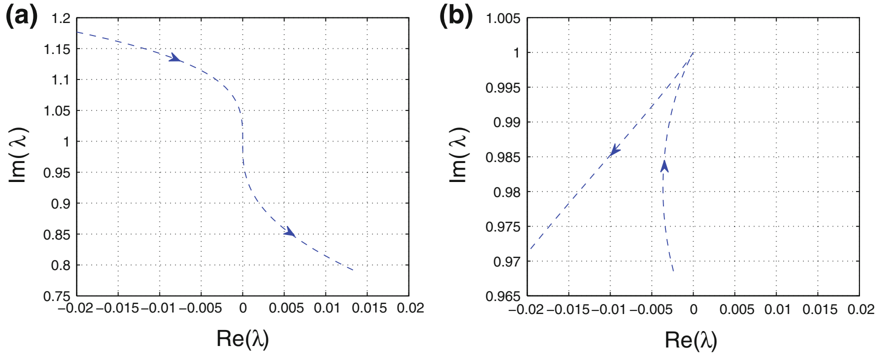

Fig. 4.1 Root loci for Examples 4.1 and 4.2. **a** Example 4.1. **b** Example 4.2

A more formal description on the asymptotic behavior of a simple critical imaginary root is given in Theorem 7.1[1] stating that:

The variation of a simple critical imaginary root, $\Delta\lambda$, *with respect to an infinitesimal perturbation* $\Delta\tau$ *at the critical delay, subjects to the Taylor series* $\Delta\lambda = \sum_{i=g}^{\infty} C_i(\Delta\tau)^i$, *where* $C_g \neq 0, C_{g+1}, C_{g+2}, \ldots$ *are complex coefficients.*

Recall that g denotes the index defined in (1.6) for a critical pair. The coefficients $C_1 = \frac{d\lambda}{d\tau}, C_2 = \frac{1}{2!}\frac{d^2\lambda}{d\tau^2}, \ldots$ can be obtained using the implicit function theorem.

However, if the critical imaginary root is multiple, i.e., $f = f_\lambda = 0$, the implicit function theorem is no longer valid (the denominator of the right-hand side of (4.1) is 0). In such a situation, we have to seek a new mathematical tool describing the asymptotic behavior.

The asymptotic behavior analysis plays a critical role in describing the qualitative behavior of the dynamics of physical systems with respect to the change of the system parameters (see [104, 116], and the references therein). If such a methodology was largely applied to various classes of dynamical systems, to the best of the authors' knowledge, its application to time-delay systems needs further developments. The research in this book as well as some earlier contributions (e.g., [64, 122]) related to Problem 2.1 follows this line.

4.1.2 Asymptotic Behavior Must Correspond to Puiseux Series

As pointed out by Theorem 3.1 (Chap. 3), the asymptotic behavior of a critical imaginary root must correspond to some Puiseux series $\Delta\lambda(\Delta\tau)$. For a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ with indices n and g, the critical imaginary root λ_α is an n-multiple root for the equation $f(\lambda, \tau) = 0$. Equivalently, for a $\Delta\tau$, there must be n solutions for $\Delta\lambda$ such that $F(\Delta\lambda, \Delta\tau) = 0$, where $F(\Delta\lambda, \Delta\tau)$ is defined in (3.6).

[1] The theorem as well as the proof will be given in Chap. 7.

Obviously, the Taylor series is only valid for the simple critical imaginary root case since it is a single-valued function with respect to $\Delta\tau$. Unlike the Taylor series, the Puiseux series are generally with fractional exponents and hence they are multiple-valued for each $\Delta\tau$. As a result, we may use the Puiseux series to appropriately describe the local root loci of a multiple critical imaginary root if the exponents and the associated coefficients are available.

Remark 4.2 Notice that a Puiseux series has *infinitely many coefficients* since it is a power series. Fortunately, in practice, calculating finitely many terms of a Puiseux series will be sufficient for the stability analysis.

4.2 How to Obtain Puiseux Series

In Sect. 4.2.1, we will present an algorithm, which is an application of the Newton diagram for computing the Puiseux series. Some illustrative examples will be given in Sect. 4.2.2.

4.2.1 An Algorithm for General Case

For some specific time-delay systems, the Puiseux series were calculated in [71] (Propositions 1 and 2 of [71]). In the sequel, we present an algorithm (Algorithm 4.1) for general time-delay systems with commensurate delays.

Algorithm 4.1 Algorithm for calculating the Puiseux series

Step 0: Let $\alpha_0 = 0$ and $\beta_0 = g$.
Step 1: Define $\mu = \max\{\frac{\beta_0-\beta}{\alpha-\alpha_0} > 0 : L_{\alpha\beta} \neq 0, \alpha > \alpha_0, \beta < \beta_0\}$, where the coefficients $L_{\alpha\beta}$ are defined in (3.6).
Step 2: If there exists a μ, go to Step 3. Otherwise, skip to Step 5.
Step 3: Collect all the nonzero $L_{\alpha\beta}$ satisfying $\frac{\beta_0-\beta}{\alpha-\alpha_0} = \mu$ to form a set

$$\left\{ L_{\alpha_1\beta_1}(\Delta\lambda)^{\alpha_1}(\Delta\tau)^{\beta_1},\ L_{\alpha_2\beta_2}(\Delta\lambda)^{\alpha_2}(\Delta\tau)^{\beta_2},\ \dots \right\}, \tag{4.2}$$

with the order $\alpha_1 > \alpha_2 > \dots$ We find a set of Puiseux series

$$\Delta\lambda = \widetilde{C}_{\mu,l}(\Delta\tau)^{\mu} + o((\Delta\tau)^{\mu}), l = 1, \dots, \alpha_1 - \alpha_0, \tag{4.3}$$

where the coefficients $\widetilde{C}_{\mu,l}$ are the solutions of the polynomial equation

$$L_{\alpha_1\beta_1}C^{\alpha_1-\alpha_0} + L_{\alpha_2\beta_2}C^{\alpha_2-\alpha_0} + \dots + L_{\alpha_0\beta_0} = 0. \tag{4.4}$$

Step 4: Let $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$ and return to Step 1.
Step 5: The algorithm stops.

Theorem 4.1 *For an n-multiple nonzero critical imaginary root of the time-delay system* (1.1)*, all the Puiseux series can be obtained by Algorithm* 4.1.

Proof The relation between $\Delta\lambda$ and $\Delta\tau$ is determined by the equation $F(\Delta\lambda, \Delta\tau) = 0$, where $F(\Delta\lambda, \Delta\tau)$ is the power series described by (3.6). We apply the Newton diagram to obtain the Puiseux series, as introduced in Sect. 2.4. The Newton polygon may be constructed via Steps 0, 1, and 4. Each segment of the Newton polygon determines a set of Puiseux series: The exponent is given by the negative slope of the segment (Step 1) and the associated coefficients are determined by all the points from the Newton diagram lying on the segment (Step 3).

According to (1.5) and (1.6), $\text{ord}(F(\Delta\lambda, 0))|_{\Delta\lambda=0} = n$ and $\text{ord}(F(0, \Delta\tau))|_{\Delta\tau=0} = g$. In light of Property 1.1, n and g are bounded. Thus, the Newton polygon starts at the point $(0, g)$ (Step 0) and must terminate at the point $(n, 0)$. The first segment can be obtained by performing Step 1 for the first time, the endpoint found of which is denoted by $(\mathscr{M}_1, \mathscr{N}_1)$. Step 1 acts as the rotating ruler method to construct the segments of the Newton polygon. Then, we set $(\mathscr{M}_1, \mathscr{N}_1)$ as the initial point for the next segment, which is realized by Step 4. Repeating Steps 1, 2 and 4, we can construct the remaining segments. Without any loss of generality, we assume that the Newton polygon is composed of p ($p \in \mathbb{N}_+$) segments linking the points $(\mathscr{M}_0, \mathscr{N}_0)$, $(\mathscr{M}_1, \mathscr{N}_1), \ldots, (\mathscr{M}_p, \mathscr{N}_p)$ ($\mathscr{M}_0 = 0, \mathscr{N}_0 = g, \mathscr{M}_p = n, \mathscr{N}_p = 0$), as shown in Fig. 2.1.

During the ith ($i = 1, \ldots, p$) implementation of Step 1, there must exist a μ with $0 < \mu < \infty$. The repeating of Steps 1–4 must stop when the pth segment is found. After that, no new μ can be found and Algorithm 4.1 stops. The Puiseux series obtained from the p segments correspond to totally $\sum_{i=1}^{p} \mathscr{M}_i - \mathscr{M}_{i-1} = n$ roots. In other words, all the Puiseux series can be found. □

Remark 4.3 The boundedness of n and g ensures that Algorithm 4.1 necessarily terminates. However, if $\lambda = 0$ is a characteristic root, $g = \infty$ and consequently Algorithm 4.1 does not stop. Such an example will be studied specifically in Example 4.7 and show that this case can be treated in light of the proposed approach.

4.2.2 Illustrative Examples

We first revisit the three quasipolynomials discussed in Example 3.1. All the three quasipolynomials have a triple critical imaginary root. However, they exhibit different types of Puiseux series. Next, we present an example with $g > n$. To the best of the authors' knowledge, such cases have not been sufficiently discussed in the literature. Finally, we consider an example with a critical imaginary root at the origin.

Example 4.3 Consider a time-delay system with $f(\lambda, \tau) = e^{-3\tau\lambda} - (\lambda^6 - \lambda^4 + \lambda^2)e^{-2\tau\lambda} - (\lambda^{10} - \lambda^8 + \lambda^6)e^{-\tau\lambda} + \lambda^{12}$ (i.e., the quasipolynomial (3.1)). For $\tau = \pi, \lambda = j$ is a triple critical imaginary root ($f_\lambda(j, \pi) = f_{\lambda\lambda}(j, \pi) = 0$ and $f_{\lambda^3}(j, \pi) \neq 0$). As

$f_\tau(j,\pi) = f_{\tau\tau}(j,\pi) = 0$ and $f_{\tau^3}(j,\pi) \neq 0$, $g = 3$. By calculation, $f_{\lambda\tau}(j,\pi) = 0$. Now, we invoke Algorithm 4.1. In Step 0, we let $\alpha_0 = 0$ and $\beta_0 = 3$. In Step 1, we find $\mu = 1$. Next, in Step 2, as there exists a μ, go to Step 3. The set satisfying $\frac{\beta_0-\beta}{\alpha-\alpha_0} = 1$ is $\left\{\frac{1}{6}f_{\lambda^3}(j,\pi)(\Delta\lambda)^3, \frac{1}{2}f_{\lambda^2\tau}(j,\pi)(\Delta\lambda)^2\Delta\tau, \frac{1}{2}f_{\lambda\tau^2}(j,\pi)\Delta\lambda(\Delta\tau)^2\right\}$. We obtain three series $\Delta\lambda = \widetilde{C}_{1,l}\Delta\tau + o(\Delta\tau)$, $l = 1, 2, 3$, where $\widetilde{C}_{1,l}$ are the solutions of the equation $f_{\lambda^3}(j,\pi)C^3 + 3f_{\lambda^2\tau}(j,\pi)C^2 + 3f_{\lambda\tau^2}(j,\pi)C + f_{\tau^3}(j,\pi) = 0$, with $f_{\lambda^3}(j,\pi) = -643.34 - 422.61j$, $f_{\lambda^2\tau}(j,\pi) = 150.80 - 28.78j$, $f_{\lambda\tau^2}(j,\pi) = -18.85 + 24j$, $f_{\tau^3}(j,\pi) = -6.00j$. Thus, we have $\Delta\lambda = (0.14 - 0.23j)\Delta\tau + o(\Delta\tau)$, $\Delta\lambda = (0.15 - 0.12j)\Delta\tau + o(\Delta\tau)$, and $\Delta\lambda = (0.13 - 0.07j)\Delta\tau + o(\Delta\tau)$. After Step 3, in Step 4, we let $\alpha_0 = 3$, $\beta_0 = 0$ and return to Step 1. In Step 1, there does not exist a $\mu > 0$ and hence we skip to Step 5. All the Puiseux series have been explicitly found and, in particular, they are three Taylor series.

We now study how the asymptotic behavior of the critical imaginary root affects the stability (more precisely, the number change of the unstable roots caused by the splitting of the critical imaginary root) based on the Taylor series. For the three Taylor series, we compare the real parts of $\Delta\lambda$ when $\Delta\tau = +\varepsilon$ and $\Delta\tau = -\varepsilon$, respectively. It is easy to conclude that as τ increases from $\pi - \varepsilon$ to $\pi + \varepsilon$, three roots cross the imaginary axis $\mathbb{C}_0$ at j from $\mathbb{C}_-$ to $\mathbb{C}_+$, i.e., the asymptotic behavior of the triple critical imaginary root leads to three more unstable roots. The above analysis is consistent with the root loci given in Fig. 3.1. □

Example 4.4 Consider a time-delay system with $f(\lambda,\tau) = e^{-2\tau\lambda} + (\frac{\pi}{2}\lambda^3 - \lambda^2 + \frac{\pi}{2}\lambda + 1)e^{-\tau\lambda} - \frac{\pi}{2}\lambda^5 - \frac{\pi}{2}\lambda^3 - \lambda^2$ (i.e., the quasipolynomial (3.2)). For $\tau = \pi$, $\lambda = j$ is a triple critical imaginary root with $g = 2$. We now invoke Algorithm 4.1. In Step 0, we let $\alpha_0 = 0$ and $\beta_0 = 2$. In Step 1, we find $\mu = 1$ ($f_{\lambda\tau}(j,\pi) \neq 0$). Next, in Step 2, as there exists a μ, go to Step 3. The set satisfying $\frac{\beta_0-\beta}{\alpha-\alpha_0} = 1$ is $\{f_{\lambda\tau}(j,\pi)\Delta\lambda\Delta\tau\}$. We have a Taylor series $\Delta\lambda = \widetilde{C}_{1,1}\Delta\tau + o(\Delta\tau)$, where $\widetilde{C}_{1,1}$ is the solution of $\frac{1}{2}(2f_{\lambda\tau}(j,\pi)C + f_{\tau\tau}(j,\pi)) = 0$, with $f_{\lambda\tau}(j,\pi) = 2 + \pi j$, $f_{\tau\tau}(j,\pi) = -2$. Thus we have $\Delta\lambda = (0.14 - 0.23j)\Delta\tau + o(\Delta\tau)$. After Step 3, in Step 4, we let $\alpha_0 = 1$, $\beta_0 = 1$, and return to Step 1. In Step 1, we find $\mu = \frac{1}{2}$. In Step 2, as there exists a μ, go to Step 3. The set satisfying $\frac{\beta_0-\beta}{\alpha-\alpha_0} = \frac{1}{2}$ is $\left\{\frac{1}{6}f_{\lambda^3}(j,\pi)(\Delta\lambda)^3\right\}$. We have the Puiseux series $\Delta\lambda = \widetilde{C}_{\frac{1}{2},l}(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$, $l = 1, 2$, where $\widetilde{C}_{\frac{1}{2},l}$ are the solutions of $\frac{1}{6}f_{\lambda^3}(j,\pi)C^2 + f_{\lambda\tau}(j,\pi) = 0$, with $f_{\lambda\tau}(j,\pi) = 2 + \pi j$, $f_{\lambda^3}(j,\pi) = -36.47 + 148.04j$. We have $\Delta\lambda = (0.15 + 0.35j)(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$ and $\Delta\lambda = -(0.15 + 0.35j)(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$. After Step 3, in Step 4, we let $\alpha_0 = 3$, $\beta_0 = 0$, and return to Step 1. In Step 1, there is no $\mu > 0$. In Step 2, we skip to Step 5. All the Puiseux series have been explicitly found.

We now study the effect of the asymptotic behavior on the stability. We first consider the Taylor series. We conclude that one root crosses $\mathbb{C}_0$ at j from left to right as delay increases. We next consider the Puiseux series. Unlike the Taylor series case, for a Puiseux series, the fractional power of a $\Delta\tau$ has multiple values. In fact, we may choose any one among them. For this example, two values of $(\Delta\tau)^{\frac{1}{2}}$ corresponding to $\Delta\tau = +\varepsilon$ are $\pm\sqrt{\Delta\tau}$. For either value, the two Puiseux series take the values $\Delta\lambda = \pm(0.15 + 0.35j)\sqrt{\Delta\tau} + o(\sqrt{\Delta\tau})$. Similarly, for $\Delta\tau = -\varepsilon$, two values of $(\Delta\tau)^{\frac{1}{2}}$ are $\pm j\sqrt{-\Delta\tau}$. For either value, the two Puiseux series take the values $\Delta\lambda = \pm(0.15 + 0.35j)j\sqrt{-\Delta\tau} + o(\sqrt{-\Delta\tau})$. We conclude that two roots collide at j as τ increases from $\pi - \varepsilon$ to π, one from $\mathbb{C}_-$ while the other from $\mathbb{C}_+$. As τ increases from π to $\pi + \varepsilon$, the root j splits into two branches (one heads for $\mathbb{C}_-$ while the other heads for $\mathbb{C}_+$). Thus, from the stability point of view, the asymptotic behavior of the triple root leads to one more unstable root. The root loci (Fig. 3.2) are consistent with the above analysis. In particular, the root locus corresponding to the Taylor series is marked in the figure. In this case, unlike the ones corresponding to the Puiseux series, the root locus is smooth near the critical imaginary root. □

Example 4.5 Consider a time-delay system with $f(\lambda, \tau) = e^{-\tau\lambda} + \frac{3\pi}{8}\lambda^5 - \frac{\pi^2}{8}\lambda^4 + \frac{5\pi}{4}\lambda^3 - \frac{\pi^2}{4}\lambda^2 + \frac{7\pi}{8}\lambda - \frac{\pi^2}{8} + 1$ (i.e., the quasipolynomial (3.3)). For $\tau = \pi$, $\lambda = j$ is a triple critical imaginary root with $g = 1$. We invoke Algorithm 4.1. In Step 0, we let $\alpha_0 = 0$ and $\beta_0 = 1$. In Step 1, $\mu = \frac{1}{3}$. In Step 2, as there exists a μ, go to Step 3. The set satisfying $\frac{\beta_0-\beta}{\alpha-\alpha_0} = \frac{1}{3}$ is $\left\{\frac{1}{6}f_{\lambda^3}(j, \pi)(\Delta\lambda)^3\right\}$. We have the Puiseux series $\Delta\lambda = \widetilde{C}_{\frac{1}{3},l}(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$, $l = 1, 2, 3$, where $\widetilde{C}_{\frac{1}{3},l}$ are the solutions of $\frac{1}{6}f_{\lambda^3}(j, \pi)C^3 + f_\tau(j, \pi) = 0$, with $f_{\lambda^3}(j, \pi) = -16.12 - 29.61j$, $f_\tau(j, \pi) = j$. We have $\Delta\lambda = (0.55+0.09j)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$, $\Delta\lambda = (-0.36+0.43j)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$, and $\Delta\lambda = (-0.20 - 0.53j)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$. After Step 3, in Step 4, we let $\alpha_0 = 3$, $\beta_0 = 0$, and return to Step 1. In Step 1, there does not exist a $\mu > 0$. In Step 2, we skip to Step 5. Algorithm 4.1 terminates. Similar to the analysis presented in Example 4.4, the number of the unstable roots decreases by 1 due to the asymptotic behavior of the triple critical imaginary root (see the root loci in Fig. 3.3). □

Example 4.6 Consider $f(\lambda, \tau) = e^{-3\tau\lambda} - 3e^{-2\tau\lambda} + 3e^{-\tau\lambda} + \lambda^4 + 2\lambda^2$. For $\tau = 0$, $\lambda = j$ is a double critical imaginary root with $g = 3$ ($f_\lambda(j, 2\pi) = 0$, $f_{\lambda\lambda}(j, 2\pi) = -8$, $f_\tau(j, 2\pi) = f_{\tau\tau}(j, 2\pi) = 0$, $f_{\lambda\tau}(j, 2\pi) = 0$, $f_{\tau^3}(j, 2\pi) = 6j$). This is a case with $g > n$. By Algorithm 4.1, the Puiseux series are $\Delta\lambda = (0.35+0.35j)(\Delta\tau)^{\frac{3}{2}} + o((\Delta\tau)^{\frac{3}{2}})$ and $\Delta\lambda = -(0.35 + 0.35j)(\Delta\tau)^{\frac{3}{2}} + o((\Delta\tau)^{\frac{3}{2}})$. Unlike the previous examples, the numerators of the first-order exponents in this example are not 1. □

For a more intuitive illustration, we explicitly give the Newton polygons for the above four examples in Fig. 4.2.

In the previous part of this chapter, we have not considered the case where the origin is a critical imaginary root. In this case, there is always a root at the origin (invariant root) for all delay values and thus the system can never be asymptotically stable. When studying the stability, it appears as a common assumption that the origin is not a critical imaginary root. However, one may still wonder how the critical imaginary roots at the origin vary with respect to delay (Algorithm 4.1 is not valid in this case). In the sequel, we give such an example and show that we can also explicitly compute the Puiseux series following the idea of the Newton diagram.

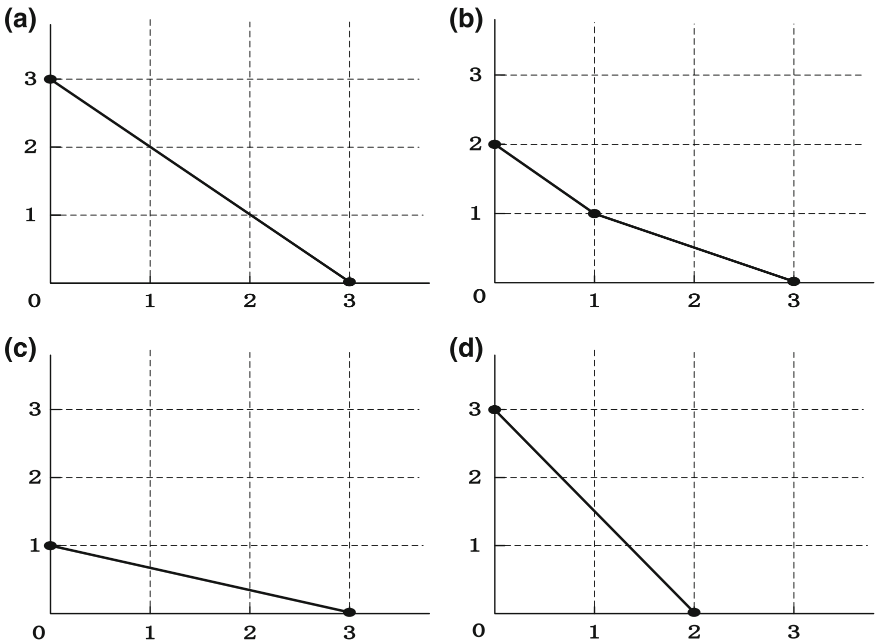

Fig. 4.2 Newton polygons for Examples 4.3–4.6. **a** Example 4.3. **b** Example 4.4. **c** Example 4.5. **d** Example 4.6

Example 4.7 Consider the example of an inverted pendulum controlled by a delayed controller as discussed in [105]. The characteristic function of the linearized system in closed-loop writes as (by choosing, e.g., $\varepsilon = 2/3, a = b = 1$ in [105]): $f(\lambda, \tau) = \lambda^2 + 2(e^{-\tau\lambda}(1+\lambda) - 1)$. It is easy to see that $f(0, \tau) = 0$ for all τ. We study the asymptotic behavior near $\tau = 1$. For $\lambda = 0$ and $\tau = 1$, $f_\lambda = f_{\lambda\lambda} = 0$, $f_{\lambda^3} = 4$. Thus, $\lambda = 0$ is a triple root at $\tau = 1$. For any $\gamma \in \mathbb{N}_+$, $f_{\tau^\gamma} = 2(1+\lambda)(-\lambda)^\gamma e^{-\tau\lambda} = 0$. Algorithm 4.1 can not be applied since $g = \infty$. However, it is not hard to treat the problem. We have that $f_{\lambda\tau} = -2 \neq 0$. As discussed in [71], to make the right-hand side of (3.5) equal to 0, the only possibility is to let $\frac{f_{\lambda^3}}{6}(\Delta\lambda)^3 + f_{\lambda\tau}\Delta\lambda\Delta\tau = o(\Delta\lambda\Delta\tau)$. It is easy to have $\Delta\lambda = (3\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$ (it is a degenerate case, which will be discussed in the next section). The Puiseux series provide the information on two roots passing through the origin. Since there are three root loci (the multiplicity is 3), we need to find the expression of $\Delta\lambda$ for the remaining one. Obviously, it is $\Delta\lambda = 0$, which is easy to understand: $\lambda = 0$ is an invariant root for all delays and as τ increases to 1 two new roots collide at the origin. The root loci are shown in Fig. 4.3. The fixed root $\lambda = 0$ is seen in Fig. 4.3b. □

Remark 4.4 It is worth mentioning that, in Example 4.7, the multiplicity of the root at the origin is larger than the degree of the polynomial corresponding to the delay-free system. For a deeper discussion on tracking the root multiplicity at the origin

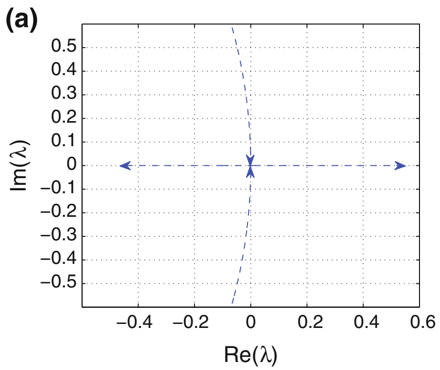

Fig. 4.3 Root loci for Example 4.7. **a** Re(λ) versus Im(λ). **b** Re(λ) versus τ

of time-delay systems as well as related upper bounds and properties, one may refer to [8, 9].

4.3 Studying Some Degenerate Cases

For Examples 4.3–4.6,[2] all the first-order terms of the Puiseux series do not contain purely imaginary numbers when $\Delta\tau = \pm\varepsilon$. Thus, the first-order terms of the Puiseux series are sufficient for the asymptotic behavior analysis. However, the first-order terms of the Puiseux series are not sufficient for our problem if they involve purely imaginary numbers when $\Delta\tau = \pm\varepsilon$ (this is the so-called *degenerate case*). The following example will show that we may invoke Algorithm 4.1 in some iterative manner to obtain the higher order terms of the Puiseux series. As a consequence, the asymptotic behavior in the degenerate case may also be analyzed.

Example 4.8 Consider the time-delay system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2\pi - \pi^2 & 2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\pi^3 & -2\pi & 2-\pi \end{pmatrix} x(t-\tau),$$

with $f(\lambda, \tau) = \lambda^3 - 2\lambda^2 + (2\pi + \pi^2)\lambda + ((\pi - 2)\lambda^2 + 2\pi\lambda + \pi^3)e^{-\tau\lambda}$. As $\tau = 1$, $\lambda = j\pi$ is a double critical imaginary root with $g = 1$ ($f_\lambda = 0$, $f_{\lambda\lambda} = \pi(2j-2)(\pi - 1) \neq 0$). Near the critical pair $(j\pi, 1)$, the power series $F(\Delta\lambda, \Delta\tau)$ (3.6) is

$$F(\Delta\lambda, \Delta\tau) = f_\tau \Delta\tau + \frac{1}{2} f_{\lambda\lambda} (\Delta\lambda)^2 + \cdots$$

[2] Example 4.7 is specific as there is an invariant characteristic root at the origin.

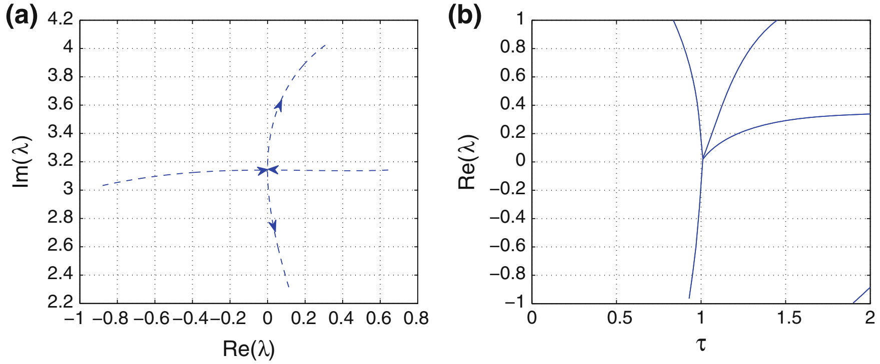

Fig. 4.4 Root loci for Example 4.8. **a** Re(λ) versus Im(λ). **b** Re(λ) versus τ

(the terms denoted by "$\cdots$" are not needed for computing the first-order terms of the Puiseux series). Using Algorithm 4.1, we have the Puiseux series $\Delta\lambda = 3.04j(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$ and $\Delta\lambda = -3.04j(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$. This is a degenerate case and we need to calculate the higher order terms. For $\Delta\lambda = 3.04j(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$, we let $\Delta\tau_1 = (\Delta\tau)^{\frac{1}{2}}$. By substituting $\Delta\lambda = \Delta\tau_1(3.04j + \Delta\lambda_1)$ into $F(\Delta\lambda, \Delta\tau)$, we have $F(\Delta\lambda, \Delta\tau) = (\Delta\tau_1)^2 \widetilde{F}_1(\Delta\lambda_1, \Delta\tau_1)$ with

$$\widetilde{F}_1(\Delta\lambda_1, \Delta\tau_1) = 3.04j f_{\lambda\lambda}\Delta\lambda_1 + (3.04j f_{\lambda\tau} + \frac{1}{3!}(3.04j)^3 f_{\lambda^3})\Delta\tau_1 + \cdots$$

Applying Algorithm 4.1 to $\widetilde{F}_1(\Delta\lambda_1, \Delta\tau_1) = 0$ yields $\Delta\lambda_1 = (2.15+0.68j)\Delta\tau_1 + o(\Delta\tau_1)$. Hence, we obtain a Puiseux series sufficient for the asymptotic behavior analysis $\Delta\lambda = 3.04j(\Delta\tau)^{\frac{1}{2}} + (2.15 + 0.68j)\Delta\tau + o(\Delta\tau)$. Similarly, for $\Delta\lambda = -3.04j(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$, we have $\Delta\lambda = -3.04j(\Delta\tau)^{\frac{1}{2}} + (2.15 + 0.68j)\Delta\tau + o(\Delta\tau)$. Root loci near $(j\pi, 1)$ (Fig. 4.4) illustrate the results. □

4.4 Useful Properties for Puiseux Series

In this section, we will introduce some algebraic properties of analytic curves (see [2, 15, 121]). These properties will be very useful for invoking and analyzing the Puiseux series.

4.4.1 Conjugacy Class

In the examples of this chapter, for an n-multiple critical imaginary root, we invoke n *independent* Puiseux series. Such expressions are in fact a little clumsy. The Puiseux series can be expressed in a more compact form if we introduce the concept of *conjugacy class*, see, e.g., [15].

For a Puiseux series $s = \sum_{i=1}^{\infty} C_i x^{\frac{i}{N}}$ (2.3), N is called the *polydromy order* if N and all i with $C_i \neq 0$ have no common factor greater than 1. Without any loss of generality, if a Puiseux series is given in the form (2.3), N is the polydromy order.

For a Puiseux series s described in (2.3), the Puiseux series $\sigma_\xi(s)$ ($\xi^N = 1$)

$$\sigma_\xi(s) = \sum_{i=1}^{\infty} \xi^i C_i x^{\frac{i}{N}} \tag{4.5}$$

will be called the conjugates of s. The set of all the N conjugates of s will be called the *conjugacy class* of s. Recall that for a power series $\Phi(y, x)$, the Puiseux series s is called a y-root for $\Phi(y, x) = 0$ if $\Phi(s, x) = 0$ (Sect. 2.2). We have:

Property 4.1 *If s is a y-root for $\Phi(y, x) = 0$, then all conjugates of s are y-roots too.*

According to Property 4.1, for the Puiseux series belonging to one conjugacy class, one expression will be sufficient to describe all of them. We demonstrate the advantage through the following example.

Example 4.9 Revisit now Example 4.5, where (j, π) is a critical pair with indices $n = 3$ and $g = 1$. For the critical pair, three expressions of the Puiseux series are: $\Delta\lambda = (0.55+0.09j)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$, $\Delta\lambda = (-0.36+0.43j)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$, and $\Delta\lambda = (-0.20-0.53j)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$. These three expressions correspond to a same conjugacy class. Therefore, any one among the above three expressions is enough to fully express the asymptotic behavior of the triple critical imaginary root. For instance, we choose the expression $\Delta\lambda = (0.55 + 0.09j)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$. The variation of the triple critical imaginary root as delay increases from π to $\pi + \varepsilon$ ($\pi - \varepsilon$ to π) can be deduced by substituting the three values of $(+\varepsilon)^{\frac{1}{3}}$ ($(-\varepsilon)^{\frac{1}{3}}$) into $(\Delta\tau)^{\frac{1}{3}}$ for this expression. One may notice that the value sets of the Puiseux series by the substitution of the values of $(+\varepsilon)^{\frac{1}{3}}$ and $(-\varepsilon)^{\frac{1}{3}}$ do not change if we choose the other two expressions of the Puiseux series. □

Remark 4.5 It should be pointed out that a critical pair may correspond to multiple conjugacy classes of Puiseux series. In this book, without any loss of generality, we suppose that a critical pair corresponds to $v \in \mathbb{N}_+$ conjugacy classes of Puiseux series. For the sake of simplicity, we adopt the short expression "v Puiseux series" instead of "v conjugacy classes of Puiseux series".

4.4.2 Structure of Puiseux Series

Algorithm 4.1 only allows us to obtain finitely many terms of the Puiseux series. In the remaining part of the book, we will often need to know the general expression (also called the structure) of the Puiseux series. The general expression is determined by the number of conjugacy classes, polydromy orders, and first-order coefficients. We now explain it through revisiting Examples 4.4 and 4.6.

For Example 4.4, (j, π) is a critical pair with $n = 3$ and $g = 2$. The asymptotic behavior corresponds to two conjugacy classes of Puiseux series. The general expression of the Puiseux series is

$$\begin{cases} \Delta\lambda = \sum_{i=1}^{\infty} C_{1i}(\Delta\tau)^{i}, \\ \Delta\lambda = \sum_{i=1}^{\infty} C_{2i}(\Delta\tau)^{\frac{i}{2}}, \end{cases}$$

where C_{1i} ($C_{11} \neq 0$) and C_{2i} ($C_{21} \neq 0$) are complex coefficients.

For Example 4.6, $(j, 2\pi)$ is a critical pair with $n = 2$ and $g = 3$. The asymptotic behavior corresponds to one conjugacy class of Puiseux series. The general expression of the Puiseux series is

$$\Delta\lambda = \sum_{i=3}^{\infty} C_{1i}(\Delta\tau)^{\frac{i}{2}},$$

where C_{1i} ($C_{11} = C_{12} = 0$, $C_{13} \neq 0$) are complex coefficients.

The structures of Puiseux series for some specific classes of time-delay systems will be given in Chaps. 6 and 7. The structure for general time-delay systems with commensurate delays will be presented in Chap. 8.

Remark 4.6 For a critical imaginary root λ_α associated with infinitely many critical delays $\tau_{\alpha,k}$, $k \in \mathbb{N}$, the structure of the Puiseux series may vary considerably with respect to different k, which makes the analysis of the complete stability involved.

4.5 Notes and Comments

To the best of the authors' knowledge, the Puiseux series was first introduced in studying the stability of time-delay systems by a cornerstone work [18, 19],[3] impacting significantly our research. The perturbation analysis for analytic matrix functions (see [5, 56]) was adopted therein. However, in our opinion, the method of [18, 19] cannot be applied directly to the general case. As discussed in Chap. 3 and this chapter, the analytic curve perspective allows us to fully study the asymptotic behavior of a critical pair. Moreover, some useful properties for analytic curves will be introduced later for the complete stability problem.

Most of the results in the first three sections of this chapter are taken from [71]. The materials in Sect. 4.4 are new.

---

[3] The earlier conference versions of the work were reported in [17, 33].

Chapter 5
Invariance Property: A Unique Idea for Complete Stability Analysis

In Chap. 4, we have pointed out that the asymptotic behavior of a critical imaginary root must correspond to some Puiseux series. Furthermore, an algorithm was presented to obtain all the Puiseux series of a critical pair. Therefore, we can now solve Problem 2.1.

In the first three sections of this chapter, we will show that the number of unstable roots ($NU(\tau)$) can be precisely calculated for any finitely large τ, based on the solution of Problem 2.1. More precisely, in Sect. 5.1, we will give the procedure to compute $NU(+\varepsilon)$ and in Sect. 5.2, we will explain how to calculate $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ for a critical pair $(\lambda_\alpha, \tau_{\alpha,k} > 0)$. A method for the calculation of $NU(\tau)$ will be presented in Sect. 5.3. However, it is still far from being able to solve the complete stability problem for time-delay systems with commensurate delays. The obstacles will be explained in Sect. 5.4, giving rise to Problem 2.2. Since there exist no appropriate "routine" ways to address Problem 2.2, we need to seek for a new analysis line. The existing literature is helping us, and a very useful property will be recalled in Sect. 5.5: The invariance property has been found for some specific types of time-delay systems. If such an invariance property holds for any time-delay system with commensurate delays, then Problem 2.2 can be fully investigated. However, it appears rather uncertain if this hypothesis is true. In Sect. 5.6, we will formally formulate the problem of proving the *general invariance property*.

A main purpose of this chapter is to propose a potential solution (by means of proving the general invariance property) for Problem 2.2. Rigorous proof will be given in subsequent chapters.

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_5

5.1 Infinitesimal Delay Case and Spectral Properties

As mentioned earlier, it is always required to know $NU(+\varepsilon)$ for precisely studying the stability of a time-delay system. If the delay-free system contains no critical imaginary roots, then, according to the root continuity argument, $NU(+\varepsilon) = NU(0)$. If the delay-free system has simple critical imaginary roots (without multiple ones), the method based on the implicit function theorem for computing $NU(+\varepsilon)$ was proposed in e.g., [109, 122]. Such a situation appears naturally when oscillatory systems are controlled by using delay (block) controllers (see [58, 85, 92]). However, to the best of the authors' knowledge, the scenario that the delay-free system involves multiple critical imaginary roots has not been studied.

In this section, we give a procedure to compute $NU(+\varepsilon)$, covering the general case. The procedure is stated in the following straightforward result.

Theorem 5.1 *If the system* (1.1) *has no critical imaginary roots when* $\tau = 0$, $NU(+\varepsilon) = NU(0)$. *Otherwise,* $NU(+\varepsilon) - NU(0)$ *equals to the number of the values in* $\mathbb{C}_+$ *of the Puiseux series for all the corresponding critical imaginary roots when* $\tau = 0$ *with* $\Delta\tau = +\varepsilon$.

Example 5.1 Consider the time-delay system in Example 4.6 with $f(\lambda, \tau) = e^{-3\tau\lambda} - 3e^{-2\tau\lambda} + 3e^{-\tau\lambda} + \lambda^4 + 2\lambda^2$. When $\tau = 0$, this system has four characteristic roots. More precisely, $\lambda = j$ as well as $\lambda = -j$ is a double critical imaginary root. Using Algorithm 4.1, we have the following Puiseux series for the critical pair $(j, 0)$[1]:

$$\Delta\lambda = (0.3536 + 0.3536j)(\Delta\tau)^{\frac{3}{2}} + o\left((\Delta\tau)^{\frac{3}{2}}\right). \tag{5.1}$$

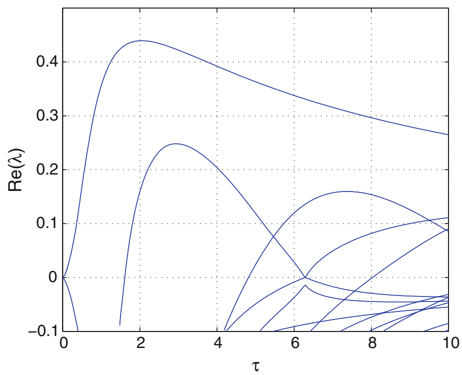

Fig. 5.1 Re(λ) versus τ for Example 5.1

[1] Notice that the two independent Puiseux series calculated in Example 4.6 can be expressed by a conjugacy class of Puiseux series (5.1) following the discussions in Sect. 4.4.1.

Substituting $\Delta\tau = +\varepsilon$ into (5.1) indicates that as τ increases from 0, the double root j splits into two branches toward $\mathbb{C}_-$ and $\mathbb{C}_+$ respectively, which is verified by the root loci given in Fig. 5.1. Thus, $NU(+\varepsilon) = +2$ in light of the conjugate symmetry. □

5.2 Quantifying Asymptotic Behavior of a Critical Imaginary Root

In the sequel, we show that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ can be accurately calculated by means of the Puiseux series for the critical pair $(\lambda_\alpha, \tau_{\alpha,k})$. More precisely, for a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ with $\tau_{\alpha,k} > 0$, we substitute $\Delta\tau = +\varepsilon$ ($\Delta\tau = -\varepsilon$) into the corresponding Puiseux series. Then, the value of $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ can be obtained by comparing the numbers of the values of the Puiseux series in $\mathbb{C}_+$ when $\Delta\tau = +\varepsilon$ and $\Delta\tau = -\varepsilon$, respectively.

Example 5.2 Revisit the time-delay system in Example 4.5 with $f(\lambda,\tau) = e^{-\tau\lambda} + \frac{3\pi}{8}\lambda^5 - \frac{\pi^2}{8}\lambda^4 + \frac{5\pi}{4}\lambda^3 - \frac{\pi^2}{4}\lambda^2 + \frac{7\pi}{8}\lambda - \frac{\pi^2}{8} + 1$, where $\lambda = j$ is a triple critical imaginary root at $\tau = \pi$. For the critical pair (j, π), the Puiseux series have one (two) value(s) in $\mathbb{C}_+$ when $\Delta\tau = +\varepsilon$ ($\Delta\tau = -\varepsilon$). Thus, $\Delta NU_j(\pi) = 1 - 2 = -1$. □

5.3 Stability Test for Bounded Delay

Now we can effectively solve Problems 1 and 2.1. As a consequence, we are able to compute $NU(\tau)$ for any finitely large delay value τ.

Theorem 5.2 *For a finitely large τ which is not a critical delay, $NU(\tau)$ for the time-delay system* (1.1) *can be computed as*

$$NU(\tau) = NU(+\varepsilon) + 2 \sum_{0<\tau_{\alpha,k}<\tau} \Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}). \tag{5.2}$$

Proof Due to the root continuity argument, as τ increases from $+\varepsilon$, only at critical delays $NU(\tau)$ may change. At a critical delay $\tau_{\alpha,k}$, the variation of $NU(\tau)$ caused by the asymptotic behavior of λ_α is $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$. In view of the conjugate symmetry of the spectrum, at the critical delay $\tau_{\alpha,k}$, $-\lambda_\alpha$ is also a critical imaginary root and the effect of its asymptotic behavior on $NU(\tau)$ can also be quantified as $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$. Thus, the proof is completed. □

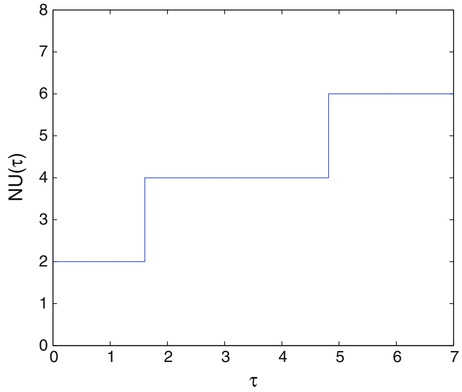

Fig. 5.2 $NU(\tau)$ for Example 5.3

An example is given below to show that we may plot $NU(\tau)$ for any bounded interval of τ.

Example 5.3 Consider the time-delay system in Example 5.1. Our task is to compute $NU(7)$ and plot $NU(\tau)$ for $\tau \in [0, 7]$. This system has two sets of critical pairs $(j, 2k\pi)$ and $(1.9566j, \frac{(2k+1)\pi}{1.9566})$. According to Theorem 5.2, it follows that

$$\begin{aligned} NU(7) = NU(+\varepsilon) + 2\Delta NU_j(2\pi) + 2\Delta NU_{1.9566j}(1.6056) \\ + 2\Delta NU_{1.9566j}(4.8169), \end{aligned}$$

where $NU(+\varepsilon) = 2$ (see Example 5.1). Using the approach in Chap. 4, we have that the Puiseux series for the critical pairs $(j, 2\pi)$, $(1.9566j, 1.6056)$, and $(1.9566j, 4.8169)$ are respectively: $\Delta\lambda = (0.3536 + 0.3536j)(\Delta\tau)^{\frac{3}{2}} + o((\Delta\tau)^{\frac{3}{2}})$, $\Delta\lambda = (0.6036 - 0.5253j)\Delta\tau + o(\Delta\tau)$, and $\Delta\lambda = (0.1357 - 0.3543j)\Delta\tau + o(\Delta\tau)$.

Next, $\Delta NU_j(2\pi) = 0$, $\Delta NU_{1.9566j}(1.6056) = +1$, and $\Delta NU_{1.9566j}(4.8169) = +1$, following the discussions in Sect. 5.2. We have $NU(7) = 6$ and the variation of $NU(\tau)$ for $\tau \in [0, 7]$ can be obtained, see Fig. 5.2. □

In Example 5.3, we invoke four Puiseux series for four critical pairs (including the critical pair $(j, 0)$ in order to compute $NU(+\varepsilon)$). For each Puiseux series, we only need to compute the first-order terms as no degenerate case occurs. However, the computational load may significantly increase for a more involved case (e.g., when τ is large and when the degenerate case occurs).

5.4 Limitations

Although we can now compute $NU(\tau)$ for any bounded τ, the existing results are still not enough for the complete stability problem. In this section, we list the main limitations of the currently available results.

5.4.1 Computational Complexity Issues

The calculation of $NU(\tau)$ for a given τ requires a series of tedious algebraic manipulations. We have to *manually* obtain the Puiseux series for all the critical pairs $(\lambda_\alpha, \tau_{\alpha,k})$ with $\tau_{\alpha,k} < \tau$. Unfortunately, for the moment, this task cannot be automatically fulfilled by existing software. Note that, even for a same critical imaginary root, the Puiseux series may exhibit remarkable differences (not only in the coefficients but also in the structure) for different critical delays. If the number of critical pairs to be analyzed is large, the computational load increases accordingly. Thus, it is very difficult to explicitly calculate $NU(\tau)$ by Theorem 5.2 when τ is large.

5.4.2 Large Delays and Ultimate Stability Problem

In order to thoroughly solve the complete stability problem, we need to understand the way the spectrum changes when $\tau \to +\infty$. This is the so-called *ultimate stability problem*, which has not been fully investigated so far.

As seen in Example 1.2 (i.e., Example 5.11 in [39]), Example 1.3 (i.e., the case study of [97]), and Example 2 in [122], increasing the delay in a certain interval may bring a stabilizing effect. One may naturally wonder the effect of delay (stabilizing or destabilizing) if it keeps increasing. To answer this question, we need to know the limit behavior as $\tau \to \infty$, i.e., $\lim\limits_{\tau\to\infty} NU(\tau)$.

To the best of the authors' knowledge, it is impossible to "compute" $NU(\infty)$ by using the existing methods. Without a general understanding of $NU(\infty)$, we will always suspect if some stability intervals distant in the τ-axis are missing since, as already discussed, we can only compute $NU(\tau)$ by Theorem 5.2 for a finitely large τ in practice.

5.5 Invariance Property for Some Specific Delay Systems

A very useful property of the form "$\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant for all $k \in \mathbb{N}$ with $\tau_{\alpha,k} > 0$" (called the *invariance property*) has been found for some specific time-delay systems. Most of the studies reported in the literature were devoted to case with only simple critical imaginary roots, see [21, 122], and the references therein. One may think that the issue of proving the invariance property in the case without multiple critical imaginary roots should be easy. In fact, it was in two recent papers [97, 109] that this issue was systematically solved. However, the above approaches cannot be extended to general time-delay systems as the mathematical tools used are only valid for the specific scenario without multiple critical imaginary roots.

To the best of the authors' knowledge, only a few results have been reported for the invariance property related to time-delay systems with multiple critical imaginary roots. A pioneering work is [64], although the invariance issue was not explicitly discussed therein. An interesting geometric criterion was proposed, which offers a simple way to analyze the asymptotic behavior: An any-multiple critical imaginary root's asymptotic behavior can be graphically determined by the criterion. However, the time-delay system considered therein is restricted to have a simple form of characteristic functions. The result in [64] is very enlightening as the critical imaginary root is allowed to be with any multiplicity. However, in our opinion, the mathematical argument proposed in [64] (based on the *Lagrange's inversion formula*) cannot be easily extended to the general case. Another work motivating considerably our study is the recent paper [54]. Its novelty consists in characterizing explicitly an invariance property for multiple critical imaginary roots of time-delay systems, though some strong constraints (it is required that $n \leq 2$ and $g = 1$ and that the degenerate case does not occur) were imposed.

5.6 General Invariance Property Statement

As discussed earlier, Problem 2.2 cannot be addressed by a "routine" method. Inspired by the invariance property for some specific time-delay systems recalled in Sect. 5.5, we wonder if this useful property holds for any time-delay system with commensurate delays. If such a property is true, Problem 2.2 then can be fully investigated and new insights for time-delay systems may be derived.

However, this appears very uncertain as the Puiseux series for a critical imaginary root may vary considerably with respect to different critical delays. We now formally present the concept of *general invariance property* in this study, as follows:

General Invariance Property For a critical imaginary root λ_α, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant for all $k \in \mathbb{N}$ with $\tau_{\alpha,k} > 0$.

In the subsequent three chapters, we will concentrate on proving this general invariance property.

5.7 Notes and Comments

As we will see it is not a trivial work to verify whether or not the general invariance property holds. The difficulty lies in that we do not have an effective tool at hand for this issue. Furthermore, no existing mathematical results in the literature can be directly employed to prove it. The frequency-sweeping framework to be proposed in this volume will allow addressing such a problem.

Chapter 6
Invariance Property for Critical Imaginary Roots with Index $g = 1$

We start confirming the invariance property with the specific case where $g = 1$ for all the critical pairs, which is the central task of this chapter. For this specific case, we will find in this chapter a useful *equivalence relation* between the critical imaginary roots' asymptotic behavior and the frequency-sweeping curves, based on which the invariance property can be proved in the case $g = 1$. In the following two chapters, such an equivalence relation will be further studied and play a pivotal role in confirming the general invariance property.

6.1 Preliminaries

Consider the time-delay system (1.1)

$$\dot{x}(t) = \sum_{\ell=0}^{m} A_\ell x(t - \ell\tau),$$

with the characteristic function $f(\lambda, \tau)$ described by (1.3)

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}.$$

The aim of this chapter is to confirm the invariance property, under the following assumption:

Assumption 6.1 Assume that $g = 1$ for all critical pairs.

If Assumption 6.1 is violated, the problem will be deferred to the latter chapters. Furthermore, Assumption 6.1 ensures that $\lambda = 0$ is not a characteristic root (otherwise, the system cannot be asymptotically stable for any $\tau \geq 0$).

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_6

Property 6.1 *Under Assumption 6.1, $\lambda = 0$ is not a characteristic root of the system* (1.1).

Proof If $\lambda = 0$ is a characteristic root, $f_\tau = 0$ at all the associated critical pairs, which contradicts Assumption 6.1. □

We now recall the results of [54], which considerably motivate the current study.

Property 6.2 ([54]) *If λ_α is a double root at $\tau_{\alpha,k'}$ ($k' \in \mathbb{N}$) of the system* (1.1) *and Assumption 6.1 holds, then λ_α is a simple root for the system* (1.1) *at any $\tau_{\alpha,k}$ ($k \neq k'$).*

Sketch of the proof: For a double critical imaginary root, $f_\lambda = 0$. Thus, at $(\lambda_\alpha, \tau_{\alpha,k'})$, $f_\lambda = p_\lambda - p_z z \tau_{\alpha,k'} = 0$ ($p(\lambda, z)$ is defined in (1.4)). It is natural that at any $(\lambda_\alpha, \tau_{\alpha,k})$ with $k \neq k'$, $f_\lambda = -p_z z(\tau_{\alpha,k} - \tau_{\alpha,k'}) \neq 0$, as $p_z \neq 0$ under Assumption 6.1.

In fact, the condition "$f_\lambda = 0$" will be used for any critical imaginary root with multiplicity larger than 1 (to be seen in the proof of Property 6.5).

Theorem 6.1 ([54]) *Suppose Assumption 6.1 holds and λ_α is a double root at $\tau_{\alpha,k'} > 0$ of the system* (1.1). *Near $(\lambda_\alpha, \tau_{\alpha,k'})$, the variation of λ with respect to τ can be expressed as the Puiseux series*

$$\Delta\lambda = C_1(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}}), \tag{6.1}$$

where C_1 is the complex coefficient with $C_1^2 = -2\frac{f_\tau}{f_{\lambda\lambda}}$. If C_1^2 is not purely real, it follows that for any $\tau_{\alpha,k} > 0$, $k \neq k'$,

$$\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NU_{\lambda_\alpha}(\tau_{\alpha,k'}). \tag{6.2}$$

However, if C_1^2 is purely real, Theorem 6.1 does not help in concluding on the asymptotic behavior and this is called a *degenerate case*. In this chapter, we will extend the results of Theorem 6.1 in order to handle the cases of any multiplicity as well as the degenerate cases. For a general time-delay system (1.1), to the best of the authors' knowledge, the Puiseux series expansion represents the only effective tool for studying the asymptotic behavior of critical imaginary roots. In Chap. 4, a systematic approach for computing the Puiseux series was proposed. However, the closed form of the Puiseux series for a general time-delay system has not yet been explicitly reported in the literature. As discussed in Sect. 4.4.2, a critical imaginary root may exhibit various types of Puiseux series. In the sequel, we will give the general expression of the Puiseux series for the systems under consideration in this chapter. Next, we will propose a more general invariance property and an easily implemented *frequency-sweeping criterion*.

6.2 General Expression of Puiseux Series When $g = 1$

For a critical imaginary root satisfying Assumption 6.1, according to Property 1.1 (Chap. 1), the multiplicity n must be finite. We have the following result:

Theorem 6.2 *For an n-multiple critical imaginary root λ_α of the system (1.1) at a critical delay $\tau_{\alpha,k'}$, if Assumption 6.1 holds, all its root loci are subject to the Puiseux series*

$$\Delta\lambda = C_1(\Delta\tau)^{\frac{1}{n}} + C_2(\Delta\tau)^{\frac{2}{n}} + C_3(\Delta\tau)^{\frac{3}{n}} + \cdots, \tag{6.3}$$

where $C_1 \neq 0$, C_2, C_3, ... are the complex coefficients.

The proof of Theorem 6.2 can be found in [72], following a similar line of [123]. In the sequel, we give a more concise proof.

Proof First, using Algorithm 4.1, we have that the exponent of the first (nonzero) term of the Puiseux series is $\frac{1}{n}$. We may next finish the proof according to the fact that the Puiseux series has n values for a $\Delta\tau$. As the term $(\Delta\tau)^{\frac{1}{n}}$ has n values for a $\Delta\tau$, the general form of the Puiseux series must have only one conjugacy class and the polydromy order must be n. The general form (6.3) is now obtained. □

Remark 6.1 When $n = 1$, the Puiseux series (6.3) reduces to the well-known Taylor series. We can study the stability accordingly, see [70].

For an n-multiple critical imaginary root λ_α at $\tau_{\alpha,k'}$, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k'})$ can be obtained according to the Puiseux series (6.3). If it is the non-degenerate case, the first-order term $C_1(\Delta\tau)^{\frac{1}{n}}$ ($C_1 = (-n!\frac{f_\tau}{f_{\lambda^n}})^{\frac{1}{n}}$ by Algorithm 4.1) is sufficient. However, in the degenerate case, the higher order terms are required. In the sequel, we will show that we can estimate $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k'})$ *without invoking the Puiseux series*.

6.3 Invariance Property When $g = 1$

Introducing $z = e^{-\tau\lambda}$, we can rewrite $f(\lambda, \tau)$ as (1.4)

$$p(\lambda, z) = \sum_{i=0}^{q} a_i(\lambda)z^i.$$

We assume that $a_q(j\omega) \neq 0$ for any $\omega \in \mathbb{R}_+$. As discussed in [70], this assumption is introduced here only for brevity and does not affect the result of the work.

Define

$$\widetilde{p}(\lambda, z) = \sum_{i=0}^{q} \widetilde{a}_i(\lambda)z^i = 0, \tag{6.4}$$

where $\widetilde{a}_i(\lambda) = \frac{a_i(\lambda)}{a_q(\lambda)}$. It is easy to see that $\widetilde{a}_q(\lambda) = 1$. Given $a_i(\lambda)$ ($i = 0, \ldots, q$), there exist continuous functions $L_l(\lambda)$ ($l = 1, \ldots, q$) such that

$$\widetilde{p}(\lambda, z) = \prod_{l=1}^{q} f_l = \prod_{l=1}^{q} (z + L_l(\lambda)). \tag{6.5}$$

We do not need the explicit expressions of $L_l(\lambda)$. The following properties will be sufficient for confirming the invariance property.

Property 6.3 *The continuous mappings $L_l(\lambda) : \mathbb{C} \mapsto \mathbb{C}$ $(l = 1, \ldots, q)$ are unique.*

Proof For each given λ, $p(\lambda, z) = 0$ can be viewed as a polynomial equation in z, with the solutions $-L_1(\lambda), \ldots, -L_q(\lambda)$. □

The following property will reveal that $f(\lambda, \tau) = 0$ at $(\lambda_\alpha, \tau_{\alpha,k'})$ if and only if one factor $(z + L_l(\lambda))$ is 0 at $(\lambda_\alpha, \tau_{\alpha,k'})$. In addition, the corresponding function $L_l(\lambda)$ is analytic with respect to λ near λ_α.

Property 6.4 *Assume that $(\lambda_\alpha, \tau_{\alpha,k'})$ is a critical pair for the system* (1.1)*. Under Assumption* 6.1*, there is only one factor $f_l = z + L_l(\lambda) = 0$ at $(\lambda_\alpha, \tau_{\alpha,k'})$. The function $L_l(\lambda)$ with $z + L_l(\lambda) = 0$ at $(\lambda_\alpha, \tau_{\alpha,k'})$ is analytic with respect to λ at $\lambda = \lambda_\alpha$.*

Proof As $f_\tau = -p_z z \lambda$, $p_z \neq 0$ at any critical pair under Assumption 6.1. Thus, $z = e^{-\tau_{\alpha,k'}\lambda_\alpha}$ is a simple root of $p(\lambda, z) = 0$ when $\lambda = \lambda_\alpha$. By the implicit function theorem (see Appendix A), there exists a unique solution $z(\lambda)$ satisfying $z(\lambda_\alpha) = e^{-\tau_{\alpha,k'}\lambda_\alpha}$, which is analytic at λ_α. □

Without any loss of generality, we let $f_1 = 0$ and $f_l \neq 0, l = 2, \ldots, q$ at $(\lambda_\alpha, \tau_{\alpha,k'})$. In fact, by this setting, $f_1 = 0$ and $f_l \neq 0, l = 2, \ldots, n$, at all $(\lambda_\alpha, \tau_{\alpha,k})$, $k \in \mathbb{N}$.

Property 6.5 *Suppose λ_α is an n-multiple root at $\tau_{\alpha,k'}$ for the system* (1.1)*. If Assumption* 6.1 *holds, it follows that:*

(i) *λ_α is an n-multiple (a simple) root at $\tau_{\alpha,k'}$ (any $\tau_{\alpha,k}, k \neq k'$) for the system* (1.1).
(ii) *Only factor $f_1 = 0$ when $\lambda = \lambda_\alpha$ and $\tau = \tau_{\alpha,k}, k \in \mathbb{N}$.*

Proof First, λ_α is a κ-multiple ($\kappa \in \mathbb{N}_+$) root of the system (1.1) at $\tau_{\alpha,k'}$ if and only if λ_α is a κ-multiple root for $f_1 = 0$ at $\tau_{\alpha,k'}$, due to Property 6.4 and our setting. Next, if λ_α is a multiple root at $\tau_{\alpha,k'}$, the condition $f_\lambda = 0$ holds at $(\lambda_\alpha, \tau_{\alpha,k'})$. As $f_1 = 0$ and $f_l \neq 0, l = 2, \ldots, q$, at $(\lambda_\alpha, \tau_{\alpha,k'})$, $f_\lambda = (-\tau e^{-\tau\lambda} + \frac{dL_1(\lambda)}{d\lambda}) f_2 \cdots f_q a_q(\lambda)$ at $(\lambda_\alpha, \tau_{\alpha,k'})$. According to the context, $-\tau e^{-\tau\lambda} + \frac{dL_1(\lambda)}{d\lambda} = 0$ and $f_2 \cdots f_q a_q(\lambda) \neq 0$ at $(\lambda_\alpha, \tau_{\alpha,k'})$. Then, at any $(\lambda_\alpha, \tau_{\alpha,k})$, $k \neq k'$, $f_\lambda \neq 0$ since $-\tau e^{-\tau\lambda} + \frac{dL_1(\lambda)}{d\lambda} = -(\tau_{\alpha,k} - \tau_{\alpha,k'}) e^{-\tau_{\alpha,k'}\lambda_\alpha} \neq 0$ (the value of $f_2 \cdots f_q a_q(\lambda)$ is independent of k). Thus, for any $\tau_{\alpha,k}$, $k \neq k'$, λ_α is a simple root for the system (1.1). □

To summarize, all these properties lead to the following result:

Theorem 6.3 *Under Assumption* 6.1*, if $\lambda_\alpha = j\omega_\alpha$ is an n-multiple root at $\tau_{\alpha,k'} > 0$, it follows that, for any $\tau_{\alpha,k} > 0$ $(k \neq k')$, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NU_{\lambda_\alpha}(\tau_{\alpha,k'})$.*

Furthermore, $\lambda_\alpha = j\omega_\alpha$ is an n-multiple root for $f_1 = 0$ at $\tau_{\alpha,k'}$.

Finally, for all positive critical delays $\tau_{\alpha,k}$, $k \in \mathbb{N}$, the following properties hold:

(i) $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = +1(-1)$ *if and only if as* ω *increases,* $|L_1(j\omega)| - 1$ *changes its sign near* ω_α *from negative to positive (from positive to negative).*
(ii) $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = 0$ *if and only if as* ω *increases,* $|L_1(j\omega)| - 1$ *does not change its sign near* ω_α.

The proof relies on the following lemma:

Lemma 6.1 *Assume that* $\lambda_\alpha = j\omega_\alpha$ *is an* n*-multiple critical imaginary root at* $\tau_{\alpha,k'}$ *for the characteristic equation*

$$e^{-\tau\lambda} + \phi(\lambda) = 0, \tag{6.6}$$

where $\phi(\lambda)$ *is an analytic function at* λ_α. *At all positive critical delays* $\tau_{\alpha,k}$, $k \in \mathbb{N}$, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ *is mirrored by the frequency-sweeping property of* $\phi(\lambda)$:

(i) $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = +1(-1)$ *if and only if as* ω *increases,* $|\phi(j\omega)| - 1$ *changes its sign near* ω_α *from negative to positive (from positive to negative).*
(ii) $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = 0$ *if and only if as* ω *increases,* $|\phi(j\omega)| - 1$ *does not change its sign near* ω_α.

Lemma 6.1 is slightly extended from Theorem 15 in [64], where $\phi(\lambda)$ is of the form $\frac{h(\lambda)}{g(\lambda)}$ ($h(\lambda)$ and $g(\lambda)$ are polynomials). The proof of Lemma 6.1 is given below.

Proof The characteristic equation (6.6) can be rewritten as $\tau = \frac{\zeta}{\lambda}$ with $\zeta = -\ln(-\phi(\lambda)) + 2k\pi j$ ($\ln(\cdot)$ denotes the principal value of the logarithmic function). The geometric criterion used in [64] is obtained by analyzing the derivatives of ζ with respect to λ, which is equivalent to the frequency-sweeping test here. More precisely, if λ_α is an n-multiple root at $\tau_{\alpha,k'}$, we can define: $\tau_{\alpha,k'} = \frac{\zeta_{\alpha,k'}}{\lambda}$, $\zeta_{\alpha,k'} = -\ln(-\phi(\lambda)) + 2k_0\pi j$, $\tau_{\alpha,k} = \frac{\zeta_{\alpha,k}}{\lambda}$, $\zeta_{\alpha,k} = \zeta_{\alpha,k'} + 2(k-k')\pi j$, where $k_0 \in \mathbb{Z}$. By Property 6.5, at all $\tau_{\alpha,k}$, $k \neq k'$, λ_α is a simple root. It is easy to see that $\zeta_{\alpha,k'}$ and $\zeta_{\alpha,k}$ have the same derivatives (of all orders) with respect to λ. Thus, according to the approach in [64], $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant for all positive $\tau_{\alpha,k}$ and it can be examined by the frequency-sweeping test as stated in the lemma. It can be seen that the result in [64] is applicable to any $\phi(\lambda)$ only if $\phi(\lambda)$ is analytic at λ_α. □

According to Property 6.4, $L_1(\lambda)$ is analytic at λ_α. We can now prove Theorem 6.3 based on Lemma 6.1.

Remark 6.2 The explanation of Theorem 6.3 is twofold. First, it proves the invariance property of any-multiple critical imaginary roots, including the degenerate case. Second, it provides a simple method to compute $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$. We may simply observe the frequency-sweeping curves, without invoking the Puiseux series.

We recall here the procedure for generating the frequency-sweeping curves given in Sect. 1.2.3: Sweep ω and for each $\lambda = j\omega$ we have q solutions of z such that $p(\lambda, z) = 0$ (denoted by $z_1(j\omega), \ldots, z_q(j\omega)$). In this way, we obtain q frequency-sweeping curves $\Gamma_i(\omega)$: $|z_i(j\omega)|$ versus ω (i.e., in the context of this chapter, $|L_i(j\omega)|$ versus ω since $z_i(j\omega) = -L_i(j\omega)$), $i = 1, \ldots, q$.

6.4 Simple Class of Quasipolynomials

In this section, we will consider specifically a simple class of quasipolynomial:

$$Q(\lambda) + P(\lambda)e^{-\tau\lambda}, \tag{6.7}$$

where $Q(\lambda)$ and $P(\lambda)$ are co-prime polynomials in λ with $\deg(Q(\lambda)) > \deg(P(\lambda))$.

This class of quasipolynomials (6.7) have been largely studied in the literature, see [7, 21, 64, 77, 82]. However, the complete stability has not been solved so far. We now make use of the approach proposed in this chapter and we will see that the complete stability problem can be fully investigated.

Property 6.6 *For any nonzero critical imaginary root of the system* (6.7)*, the index* g *equals to* 1.

Property 6.6 follows straightforwardly from Property 1.1 (Chap. 1).

Thus, all the results derived in this chapter apply to the time-delay systems whose characteristic functions are in the form (6.7). We have the following results:

Corollary 6.1 *For an* n*-multiple nonzero critical imaginary root of the system* (6.7)*, the Puiseux series is in the form* (6.3)*.*

Corollary 6.2 *The invariance property holds for a time-delay system whose characteristic function is of the form* (6.7)*.*

6.5 Illustrative Examples

We first revisit some of the examples proposed and discussed in Chap. 4.

Example 6.1 Consider the time-delay system of Example 4.5 with $f(\lambda, \tau) = e^{-\tau\lambda} + \frac{3\pi}{8}\lambda^5 - \frac{\pi^2}{8}\lambda^4 + \frac{5\pi}{4}\lambda^3 - \frac{\pi^2}{4}\lambda^2 + \frac{7\pi}{8}\lambda - \frac{\pi^2}{8} + 1$. As $\tau = \pi$, $\lambda = j$ is a triple critical imaginary root with $f_\tau \neq 0$. Using the approach proposed in Chap. 4, we have the Puiseux series $\Delta\lambda = (0.55 + 0.09j)(\Delta\tau)^{\frac{1}{3}} + o((\Delta\tau)^{\frac{1}{3}})$. Thus, $\Delta NU_j(\pi) = -1$.

This result can be directly obtained from the frequency-sweeping curve as shown in Fig. 6.1a. Moreover, the invariance property holds (Theorem 6.3). At the critical frequency $\omega = 1$, the frequency-sweeping curve crosses the line $\Im_1$ from above to below. Therefore, according to Theorem 6.3, $\Delta NU_j((2k+1)\pi) = -1$. Both the two cases are included: (1) As τ increases near π, the number of unstable roots decreases by 1 due to the splitting of the triple critical imaginary root. (2) As τ increases near $3\pi, 5\pi, \ldots$, each simple critical imaginary root $\lambda = j$ enters in $\mathbb{C}_-$. That is to say, at the point $\lambda = j$, despite simple or multiple critical imaginary root, the influence of the asymptotic behavior on $NU(\tau)$ is equivalent. To further argue the analysis, we may observe the root loci in Fig. 6.1b. □

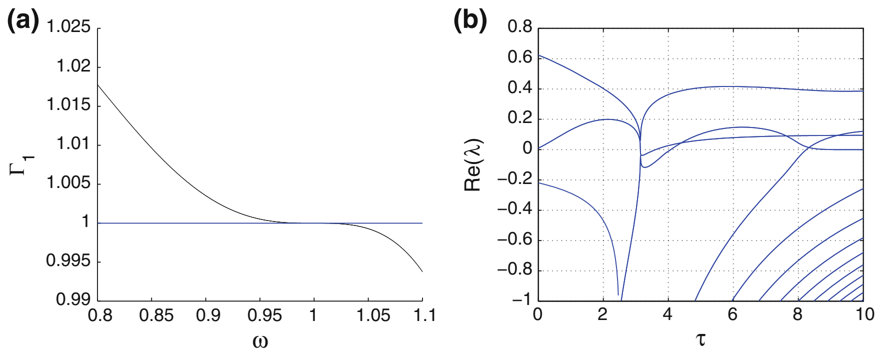

Fig. 6.1 Frequency-sweeping curve and root loci for Example 6.1. **a** Frequency-sweeping result. **b** Re(λ) versus τ

Example 6.2 Consider the time-delay system of Example 4.8 with $f(\lambda, \tau) = \lambda^3 - 2\lambda^2 + (2\pi + \pi^2)\lambda + ((\pi - 2)\lambda^2 + 2\pi\lambda + \pi^3)e^{-\tau\lambda}$. As $\tau = 1$ and $\lambda = j\pi$, $f_\lambda = 0$, $f_{\lambda\lambda} = \pi(2j - 2)(\pi - 1) \neq 0$. Thus, at $\tau = 1$, $\lambda = j\pi$ is a double root. In addition, Assumption 6.1 holds as $f_\tau = \pi^3(2j - 2) \neq 0$. This is a degenerate case and hence one cannot use the method proposed by [54], i.e., Theorem 6.1.

According to Theorem 6.2, the Puiseux series is in the form $\Delta\lambda = C_1(\Delta\tau)^{\frac{1}{2}} + C_2(\Delta\tau)^{\frac{2}{2}} + \cdots$ and, as discussed in Example 4.8, the Puiseux series is $\Delta\lambda = 3.04j(\Delta\tau)^{\frac{1}{2}} + (2.15 + 0.68j)\Delta\tau + o(\Delta\tau)$. Therefore, $\Delta NU_{j\pi}(1) = +1$.

The above result can be easily derived without invoking the Puiseux series by observing the corresponding frequency-sweeping curve as shown in Fig. 6.2a. At the critical frequency $\omega = \pi$, the frequency-sweeping curve crosses the line $\Im_1$ from below to above. Therefore, according to Theorem 6.3, $\Delta NU_{j\pi}(1) = +1$. Moreover, according to Theorem 6.3, we may have a stronger result, the invariance property. No matter the critical imaginary root $\lambda = j\pi$ is simple or double (it is double at $\tau = 1$, while it is simple at the critical delays other than $\tau = 1$), its asymptotic behavior always makes the number of unstable roots increase by 1. The above analysis is verified by the root loci as shown in Fig. 6.2b. For further illustration, we also list two Taylor series, which are also degenerate, corresponding to the critical delays $\tau = 3$ and $\tau = 5$: $\Delta\lambda = -1.5708j\Delta\tau + 2.4118j(\Delta\tau)^2 + (1.8506 - 8.1679j)(\Delta\tau)^3 + o((\Delta\tau)^3)$ for $\tau = 3$ and $\Delta\lambda = -0.7854j\Delta\tau + 0.4978j(\Delta\tau)^2 + (0.1157 - 0.6155j)(\Delta\tau)^3 + o((\Delta\tau)^3)$ for $\tau = 5$. We may notice that the proposed approach significantly simplifies the analysis since we have to invoke two terms of the Puiseux series and three terms of the Taylor series if the series expansion analysis is used. □

Finally, we give two interesting examples to illustrate the results in Sect. 6.4.

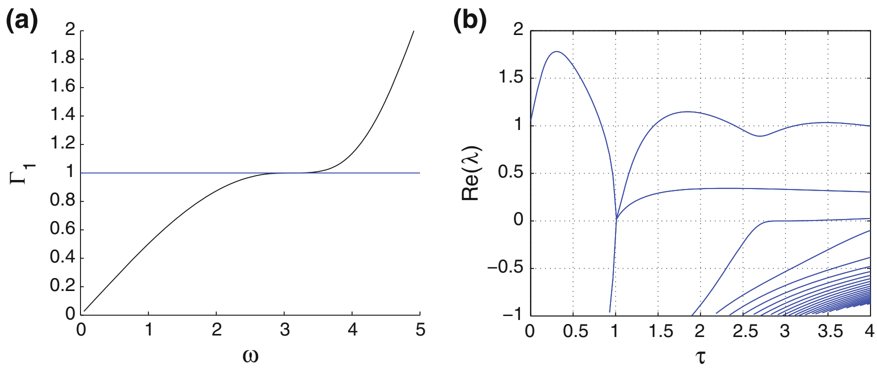

Fig. 6.2 Frequency-sweeping curve and root loci for Example 6.2. **a** Frequency-sweeping result. **b** Re(λ) versus τ

Example 6.3 Consider the following oscillator system (see, e.g., [58])

$$\left[\frac{d^4}{dt^4} + 5\frac{d^2}{dt^2} + 4\right] : x(t) = u(t), \tag{6.8}$$

where $x(t)$ is a scalar. It has been pointed out in the literature that the system (6.8) cannot be stabilized by a state feedback $u(t) = Kx(t)$ (K is the scalar gain). Interestingly, the system can be stabilized by a delayed controller $u(t) = Kx(t - \tau)$.

The characteristic function of the closed-loop system is $f(\lambda, \tau) = \lambda^4 + 5\lambda^2 + 4 - Ke^{-\tau\lambda}$. As the characteristic function falls in the class discussed in Sect. 6.4, we may straightforwardly study the complete stability (here, we let $K = -0.1$) from the frequency-sweeping curve. Note that, the system has critical imaginary roots for $\tau = 0$. As mentioned earlier, we may calculate $NU(+\varepsilon)$ using Theorem 5.1.

The frequency-sweeping curve is given in Fig. 6.3a, from which we find four sets of critical pairs: ($\lambda_0 = 0.9834j$, $\tau_{0,k} = 3.1947 + \frac{2k\pi}{0.9834}$), ($\lambda_1 = 1.0167j$, $\tau_{1,k} = \frac{2k\pi}{1.0167}$), ($\lambda_2 = 1.9916j$, $\tau_{2,k} = \frac{2k\pi}{1.9916}$), and ($\lambda_3 = 2.0082j$, $\tau_{3,k} = 1.5643 + \frac{2k\pi}{2.0082}$).

The "$NU(\tau)$ versus τ" plot can be obtained (see Fig. 6.3b). This system has multiple stability intervals and $NU(\tau) \to \infty$ as $\tau \to \infty$. □

Example 6.4 Consider a chain of three integrators (see [94])

$$\frac{d^3}{dt^3}x(t) = u(t), \tag{6.9}$$

($x(t)$ is a scalar) controlled by a proportional + delay controller $u(t) = K_2x(t) + K_1x(t - \tau)$, where K_1 and K_2 are scalar gains. The characteristic function of the closed-loop system is $f(\lambda, \tau) = \lambda^3 - K_2 - K_1e^{-\tau\lambda}$. It is easy to see that the system cannot be asymptotically stable if $K_1 = 0$. We now try to find the appropriate parameters (K_1, K_2, and τ) such that the closed-loop system is asymptotically stable.

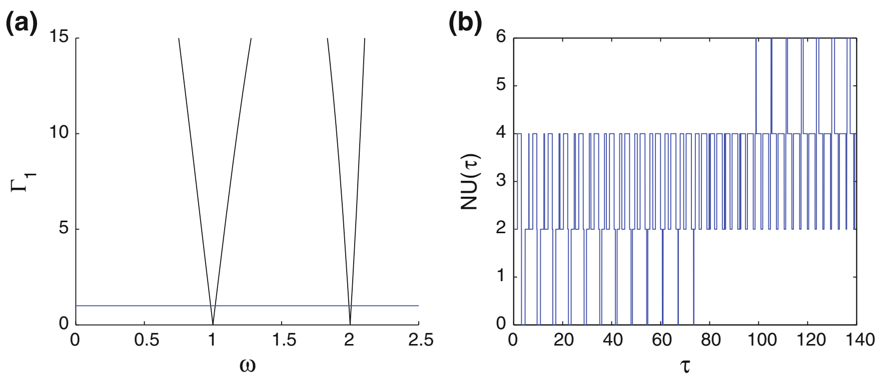

Fig. 6.3 Frequency-sweeping curve and $NU(\tau)$ for Example 6.3. **a** Frequency-sweeping result. **b** $NU(\tau)$

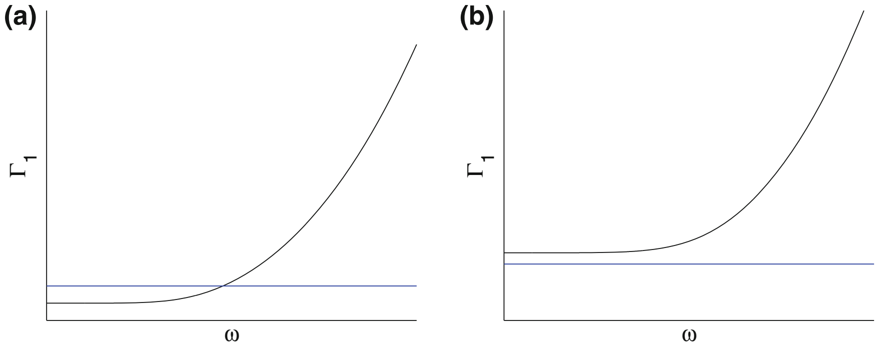

Fig. 6.4 Frequency-sweeping curves for Example 6.4. **a** $\left|\frac{K_2}{K_1}\right| < 1$. **b** $\left|\frac{K_2}{K_1}\right| > 1$

We first consider the cases when $\left|\frac{K_2}{K_1}\right| < 1$ and $\left|\frac{K_2}{K_1}\right| > 1$. For both cases, $NU(0) > 0$. If the system can be stable for some τ, there must exist critical imaginary roots whose asymptotic behavior causes a decrease in $NU(\tau)$. When $\left|\frac{K_2}{K_1}\right| < 1$, the frequency-sweeping curve is depicted in Fig. 6.4a. We see from Fig. 6.4a that the system has one and only one positive critical imaginary root, whose asymptotic behavior always causes an increase in $NU(\tau)$ (by Corollary 6.2). If $\left|\frac{K_2}{K_1}\right| > 1$, the system has no critical imaginary roots (see the frequency-sweeping curve depicted in Fig. 6.4b). Next, we consider the case $\left|\frac{K_2}{K_1}\right| = 1$. If $\frac{K_2}{K_1} = -1$, $\lambda = 0$ is a characteristic root. If $\frac{K_2}{K_1} = 1$, the system has no critical imaginary roots (see Remark 1.9).

By the above discussions, the system (6.9) cannot be stabilized by a proportional + delay controller. This conclusion is consistent with Proposition 1 in [94]. □

6.6 On Some Limitations (Lack of Analyticity)

The development of the results in this chapter is relatively simple as we may follow the idea of [64] through introducing a factorization (6.5). However, there are two limitations for extending the approach in this chapter to general time-delay systems.

First, if Assumption 6.1 does not hold, the factor f_1 is not necessarily analytic at the critical imaginary root and hence the method of this chapter will be invalid (having in mind that the analyticity of f_1 is the prerequisite of Theorem 6.3). Such examples will be seen in Chaps. 7 and 8. Especially, we will see in Chap. 7 that the index g may be larger than 1 even for a simple critical imaginary root.

Second, the mathematical tool lying behind Theorem 6.3, the Lagrange's inversion formula (which plays a critical role in the study of [64]), is not easy to apply to general time-delay systems. As mentioned earlier, we need a mathematical machinery compatible with the Puiseux series.

6.7 Notes and Comments

The work of this chapter, though only applied to a specific case, opens some interesting perspectives in analyzing the general case. More precisely, as the frequency-sweeping curves are independent of different critical delays, we may choose their asymptotic behavior as a "reference object". If for each positive critical delay $\tau_{\alpha,k}$, an equivalence between $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ and the asymptotic behavior of the frequency-sweeping curves at $\omega = \omega_\alpha$ is proved, the general invariance property can be proved.

Particularly, for the time-delay system considered in this chapter, the asymptotic behavior of the frequency-sweeping curves refers to their intersection with respect to $\Im_1$ at $\omega = \omega_\alpha$. More specifically, we know from Theorem 6.3 that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = +1$ (-1) if and only if the corresponding frequency-sweeping curve crosses $\Im_1$ from below to above (from above to below) and that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = 0$ if and only if the corresponding frequency-sweeping curve "touches" without crossing $\Im_1$. It is worth mentioning that the key lemma (Lemma 6.1) in developing the above results is an extension of Theorem 15 in [64].

However, the above result will be invalid when the frequency-sweeping curves may have multiple points at some critical frequencies. Such a situation will be encountered and studied in the next chapter. A more general description and discussion on the asymptotic behavior of the frequency-sweeping curves will be proposed, providing an embryonic form of the new mathematical framework of this book.

Most of the results of this chapter were reported in [72]. Some new materials are added in Sects. 6.4–6.6.

Chapter 7
Invariance Property for Critical Imaginary Roots with Index $n = 1$

First of all, it should be emphasized that the invariance property for simple critical imaginary roots has been proved in [97, 109]. However, as mentioned earlier, the mathematical tool used therein is not applicable to general time-delay systems. In this chapter, we will "revisit" the invariance issue for simple critical imaginary roots by adopting the frequency-sweeping approach. In this context, some new perspectives will be introduced and they will be crucial for confirming the general invariance property in Chap. 8.

More precisely, we will equip the classical frequency-sweeping approach with a new mathematical tool. First, we will point out that the frequency-sweeping curves may involve multiple points if $g > 1$ and the existing results will be no longer valid in this case. To cover the general case, we will introduce a new notation $\Delta NF_{z_\alpha}(\omega_\alpha)$ to describe the asymptotic behavior of the frequency-sweeping curves. Next, we will propose a new mathematical tool, the *dual Puiseux series*, and we will prove that the value of $\Delta NF_{z_\alpha}(\omega_\alpha)$ is fully determined by the dual Puiseux series. Finally, a useful equivalence relation between the Puiseux series and the dual Puiseux series will be found, based on which the invariance property can be proved.

Based on the new ideas proposed in Chap. 6 and this chapter, we will establish a new frequency-sweeping mathematical framework in Chap. 8.

7.1 Preliminaries

Consider the time-delay system (1.1)

$$\dot{x}(t) = \sum_{\ell=0}^{m} A_\ell x(t - \ell\tau),$$

with the characteristic function $f(\lambda, \tau)$ described by (1.3)

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_7

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}.$$

Without any loss of generality, suppose that the time-delay system (1.1) has u sets of critical pairs $(\lambda_\alpha, \tau_{\alpha,k})$, $\alpha = 0, \ldots, u-1$, $k \in \mathbb{N}$. As usual, we adopt the common assumption that $\lambda = 0$ is not a characteristic root. Otherwise, the system (1.1) cannot be asymptotically stable for any $\tau \geq 0$. As discussed in Chap. 1, the asymptotic behavior of a critical imaginary root λ_α near a positive critical delay $\tau_{\alpha,k}$ can be simply described by using $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$. The aim of this chapter is to confirm the invariance property, under the following assumption:

Assumption 7.1 Assume that all the critical imaginary roots are simple.

In particular, the asymptotic behavior of a simple critical imaginary root may be simply described by its crossing direction with respect to the imaginary axis $\mathbb{C}_0$. Due to the conjugate symmetry discussed in Remark 1.2, it suffices to consider only the critical imaginary roots with nonnegative imaginary parts.

7.2 Embryo of New Frequency-Sweeping Framework

Concerning the general invariance property, we have two important observations from the previous chapters:

(1) The technical line should be *compatible* with the Puiseux series.
(2) The frequency-sweeping curves are very useful as they are *invariant* with respect to different critical delays.

We will seriously take the above two points into account in the subsequent study.

Although the time-delay systems (under Assumption 7.1) considered in this chapter are still specific, we will find some new insights. Moreover, an embryonic form of the new mathematical framework will be explicitly built.

7.2.1 Asymptotic Behavior of Simple Critical Imaginary Roots

The asymptotic behavior of a simple critical imaginary root corresponds to a Taylor series, which can be regarded as a specific type of the Puiseux series.

Theorem 7.1 *Consider a simple critical imaginary root λ_α at a critical delay $\tau_{\alpha,k}$ with the index g of system* (1.1)*. For an infinitesimal perturbation $\Delta\tau$ of τ, the variation of λ, $\Delta\lambda$, subjects to the Taylor series in the form*

$$\Delta\lambda = \sum_{i=g}^{\infty} C_i (\Delta\tau)^i, \tag{7.1}$$

where $C_g \neq 0, C_{g+1}, C_{g+2}, \ldots$ *are complex coefficients.*

Proof As $f_\lambda \neq 0$ (condition automatically satisfied for a simple root), by the implicit function theorem (see Appendix A), there exists a unique solution $\lambda(\tau)$, which is analytic at $\tau_{\alpha,k}$ with $\lambda(\tau_{\alpha,k}) = \lambda_\alpha$. Thus, near $(\lambda_\alpha, \tau_{\alpha,k})$, the asymptotic behavior subjects to a Taylor series $\Delta\lambda = \sum_{i=1}^{\infty} C_i (\Delta\tau)^i$. Furthermore, by the method given in Chap. 4, $C_1 = \cdots = C_{g-1} = 0$. □

7.2.2 Some New Angles for Frequency-Sweeping Curves

As seen in Chap. 6, the asymptotic behavior of the frequency-sweeping curves may act as a *reference object* for addressing the invariance issue. In this chapter, we will further study the related properties. Though the case studied in this chapter is also specific (under Assumption 7.1), the frequency-sweeping curves may exhibit some complicated characteristics (see Example 3.2).

It is worth mentioning that, in the case $g > 1$, the asymptotic behavior of the frequency-sweeping curves cannot be described by the existing results. For this reason, we now introduce a new notation.

Suppose $(\lambda_\alpha, \tau_{\alpha,k})$, $k \in \mathbb{N}$, is a set of critical pairs (as usually assumed, $\lambda_\alpha \neq 0$) with the index g (having in mind that g is a constant with respect to different k, see Property 1.2). There must exist g frequency-sweeping curves such that $z_i(j\omega_\alpha) = z_\alpha = e^{-\tau_{\alpha,0}\lambda_\alpha}$ colliding with $\Im_1$ when $\omega = \omega_\alpha$. Among such g frequency-sweeping curves, we denote the number of the frequency-sweeping curves when $\omega = \omega_\alpha + \varepsilon$ ($\omega = \omega_\alpha - \varepsilon$) above $\Im_1$ by $NF_{z_\alpha}(\omega_\alpha + \varepsilon)$ ($NF_{z_\alpha}(\omega_\alpha - \varepsilon)$). Introduce now a new notation $\Delta NF_{z_\alpha}(\omega_\alpha)$ as

$$\Delta NF_{z_\alpha}(\omega_\alpha) \triangleq NF_{z_\alpha}(\omega_\alpha + \varepsilon) - NF_{z_\alpha}(\omega_\alpha - \varepsilon). \tag{7.2}$$

Remark 7.1 It is a very useful property that $\Delta NF_{z_\alpha}(\omega_\alpha)$ is invariant with respect to different critical delays. In addition, we may straightforwardly know $\Delta NF_{z_\alpha}(\omega_\alpha)$ from the frequency-sweeping curves, simplifying thus the implementation of the method.

A single frequency-sweeping curve may either cross $\Im_1$ (from below to above or the other way) or simply touch $\Im_1$ (e.g., the frequency-sweeping curve is tangent to $\Im_1$), at the critical frequency. Thus, if $g = 1$, $\Delta NF_{z_\alpha}(\omega_\alpha)$ must be ± 1 or 0. However, in the case where $g > 1$, g frequency-sweeping curves collide with $\Im_1$ at $\omega = \omega_\alpha$ and $\Delta NF_{z_\alpha}(\omega_\alpha)$ generally takes more possible values.

As mentioned in Sect. 3.2, we will address the frequency-sweeping curves from a new analytic curve angle. If we let $\tau \in \mathbb{C}$, we may treat $\tau_{\alpha,k}$ as a g-multiple root of $f(\lambda, \tau) = 0$ when $\lambda = \lambda_\alpha$.

Theorem 7.2 *For a simple critical imaginary root* $\lambda = \lambda_\alpha$ *for* $\tau = \tau_{\alpha,k}$ *with the index* g *of system* (1.1)*, if* λ *is imposed by an infinitesimal* $\Delta\lambda$ *at* λ_α*, the variation of* τ *subject to* $f(\lambda, \tau) = 0$*,* $\Delta\tau$*, corresponds to the Puiseux series in the form*

$$\Delta\tau = \sum_{i=1}^{\infty} D_i (\Delta\lambda)^{\frac{i}{g}}, \tag{7.3}$$

where $D_1 \neq 0, D_2, D_3, \ldots$ *are complex coefficients.*

The proof is in the same line of the proof of Theorem 6.2 in Chap. 6 and, for the sake of brevity, it is omitted. The Puiseux series of the form (7.3) will be called the *dual Puiseux series*.

Remark 7.2 In the literature, the asymptotic behavior of the frequency-sweeping curves has only been analyzed in the case $g = 1$, due to the limitations of the adopted mathematical tools (e.g., computing the derivatives of the module of the frequency-sweeping curves with respect to ω, see, e.g., [21]). We will show that the frequency-sweeping curves of general time-delay systems can be appropriately treated by means of the dual Puiseux series.

The following property brings a new angle to study the asymptotic behavior of the frequency-sweeping curves.

Property 7.1 *For a simple critical imaginary root* $\lambda_\alpha = j\omega_\alpha$*,* $\Delta NF_{z_\alpha}(\omega_\alpha)$ *can be determined by the dual Puiseux series* (7.3)*:*

$$\Delta NF_{z_\alpha}(\omega_\alpha) = ND_{(\lambda_\alpha,\tau_{\alpha,k})}(+\varepsilon j) - ND_{(\lambda_\alpha,\tau_{\alpha,k})}(-\varepsilon j),$$

where $ND_{(\lambda_\alpha,\tau_{\alpha,k})}(+\varepsilon j)$ $(ND_{(\lambda_\alpha,\tau_{\alpha,k})}(-\varepsilon j))$ *denotes the number of the values in* $\mathbb{C}_U$ *of the dual Puiseux series* (7.3) *evaluated when* $\Delta\lambda = +\varepsilon j$ $(\Delta\lambda = -\varepsilon j)$.

Proof To study the frequency-sweeping curves, we consider (7.3) with $\Delta\lambda = \pm\varepsilon j$. For a $\Delta\lambda$, we have g values of $\Delta\tau$ from (7.3). By (1.4), near $\omega = \omega_\alpha$, a corresponding frequency-sweeping curve reflects the variation of $\left|e^{-\tau\lambda}\right|$ with $\left|e^{-\tau_{\alpha,k}\lambda_\alpha}\right| = 1$. As $\lambda_\alpha + \Delta\lambda$ is a positive imaginary number, $\left|e^{-(\tau_{\alpha,k}+\Delta\tau)(\lambda_\alpha+\Delta\lambda)}\right| - \left|e^{-\tau_{\alpha,k}\lambda_\alpha}\right| > 0\ (< 0)$ if and only if the corresponding $\mathrm{Im}(\Delta\tau) > 0\ (< 0)$. □

Remark 7.3 Using Property 7.1, we can now study the frequency-sweeping curves in an algebraic way (by means of the dual Puiseux series), such that the graphical frequency-sweeping criterion will be applicable to general time-delay systems with commensurate delays.

In Chap. 6, an equivalence relation between the asymptotic behavior of the critical imaginary roots and the frequency-sweeping curves was found. In the sequel, we will study if such an equivalence relation also holds when $g > 1$.

7.3 Equivalence Relation Between Two Types of Asymptotic Behavior

It is exciting that the equivalence relation between $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ and $\Delta NF_{z_\alpha}(\omega_\alpha)$ exists in the context of this chapter, as stated in the following theorem:

Theorem 7.3 *For a simple critical imaginary root λ_α at a positive critical delay $\tau_{\alpha,k}$, it follows that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha)$.*

Theorem 7.3 is based on analyzing the connections between the Puiseux series (7.1) and the dual Puiseux series (7.3). This new idea will be used in the next chapter. Detailed development of Theorem 7.3 is given in the following two subsections:

A useful property, extracted from Lemmas 3 and 4 in [64], is given below.

Property 7.2 *For a complex number $x \neq 0$, the following results hold.*

(1) If $(p+1)/2$ is even and $x \in \mathbb{C}_+$, then among the p values of $x^{\frac{1}{p}}$, $(p-1)/2$ ones lie in $\mathbb{C}_+$ and $(p+1)/2$ ones lie in $\mathbb{C}_-$. If $x \in \mathbb{C}_-$, then the reverse is true.

(2) If $(p+1)/2$ is odd and $x \in \mathbb{C}_+$, then among the p values of $x^{\frac{1}{p}}$, $(p+1)/2$ ones lie in $\mathbb{C}_+$ and $(p-1)/2$ ones lie in $\mathbb{C}_-$. If $x \in \mathbb{C}_-$, then the reverse is true.

(3) If p is even and $x \notin \mathbb{R}_+$, then among the p values of $x^{\frac{1}{p}}$, $p/2$ ones lie in $\mathbb{C}_U$ and $p/2$ ones lie in $\mathbb{C}_L$.

One may prove the property by simply analyzing the arguments of $x^{\frac{1}{p}}$.

7.3.1 *Nondegenerate Case*

We first consider the nondegenerate case with $\mathrm{Re}(C_g) \neq 0$ in (7.1). Substituting (7.3) into (7.1), we have that $D_1 = (\frac{1}{C_g})^{\frac{1}{g}}$.

If g is odd, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = +1(-1)$ if and only if $\mathrm{Re}(C_g) > 0\ (< 0)$ in view of (7.1). For $\Delta\lambda = \pm\varepsilon j$, $D_1(\Delta\lambda)^{\frac{1}{g}} = \left(\frac{\Delta\lambda}{C_g}\right)^{\frac{1}{g}} = \left(\frac{\pm\varepsilon j(\mathrm{Re}(C_g)-\mathrm{Im}(C_g)j)}{|C_g|^2}\right)^{\frac{1}{g}}$. According to Properties 7.1 and 7.2, $\Delta NF_{z_\alpha}(\omega_\alpha) = +1\ (-1)$ if and only if $\mathrm{Re}(C_g) > 0\ (< 0)$.

Remark 7.4 We may express $\left(\frac{\Delta\lambda}{C_g}\right)^{\frac{1}{g}} = \left(\frac{\Delta\lambda j^g}{C_g}\right)^{\frac{1}{g}}(-j)$ and then analyze $\left(\frac{\Delta\lambda j^g}{C_g}\right)^{\frac{1}{g}}$ using Property 7.2. Such a simple manipulation will also be used later.

If g is even, it is easy to see from (7.1) that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = 0$. We now consider the term $D_1(\Delta\lambda)^{\frac{1}{g}}$. For $\Delta\lambda = \pm\varepsilon j$, $D_1(\Delta\lambda)^{\frac{1}{g}} = (\frac{\Delta\lambda}{C_g})^{\frac{1}{g}}$ has $\frac{g}{2}$ values in $\mathbb{C}_U$ and $\frac{g}{2}$ values in $\mathbb{C}_L$ by Property 7.2. Thus, according to Property 7.1, $\Delta NF_{z_\alpha}(\omega_\alpha) = 0$.

From the above analysis, we have:

Lemma 7.1 *If $\mathrm{Re}(C_g) \neq 0$, Theorem 7.3 holds.*

7.3.2 Degenerate Case

We now consider the degenerate case, i.e., $\mathrm{Re}(C_g) = 0$.[1] Without any loss of generality, suppose $\mathrm{Re}(C_g) = \cdots = \mathrm{Re}(C_{g+M-1}) = 0, \mathrm{Re}(C_{g+M}) \neq 0$.

By substituting (7.3) into (7.1), we know that $D_1 = (\frac{1}{C_g})^{\frac{1}{g}}$, $D_i = r_{i-1}D_1^i, 2 \leq i \leq M$, and $D_{1+M} = \frac{D_1^{1+M}}{g}(\frac{-C_{g+M}}{C_g} + r_M)$, where $r_1, \ldots, r_M$ are real numbers. We have the following two properties:

Property 7.3 *If g is odd, for $\Delta\lambda = \pm\varepsilon j$, $\sum_{i=1}^{M} D_i(\Delta\lambda)^{\frac{i}{g}}$ has $\frac{g-1}{2}$ values in $\mathbb{C}_U$, $\frac{g-1}{2}$ values in $\mathbb{C}_L$, and one real value.*

Proof The property can be proved by the equation

$$\sum_{i=1}^{M} D_i(\Delta\lambda)^{\frac{i}{g}} = \left(\frac{\Delta\lambda}{C_g}\right)^{\frac{1}{g}} + \sum_{i=2}^{M} r_{i-1}\left(\frac{\Delta\lambda}{C_g}\right)^{\frac{i}{g}}. \tag{7.4}$$

The proof ends by noting that C_g is an imaginary number. □

Property 7.4 *If g is even, one of the following two cases must happen*:

(1) *For $\Delta\lambda = -\varepsilon j$, $\sum_{i=1}^{M} D_i(\Delta\lambda)^{\frac{i}{g}}$ has $\frac{g}{2}$ values in $\mathbb{C}_L$ and $\frac{g}{2}$ values in $\mathbb{C}_U$. For $\Delta\lambda = +\varepsilon j$, $\sum_{i=1}^{M} D_i(\Delta\lambda)^{\frac{i}{g}}$ has $\frac{g}{2} - 1$ values in $\mathbb{C}_L$, $\frac{g}{2} - 1$ values in $\mathbb{C}_U$, and two* (*one negative and one positive*) *real values.*

(2) *For $\Delta\lambda = -\varepsilon j$, $\sum_{i=1}^{M} D_i(\Delta\lambda)^{\frac{i}{g}}$ has $\frac{g}{2} - 1$ values in $\mathbb{C}_L$, $\frac{g}{2} - 1$ values in $\mathbb{C}_U$, and two (one negative and one positive) real values. For $\Delta\lambda = +\varepsilon j$, $\sum_{i=1}^{M} D_i(\Delta\lambda)^{\frac{i}{g}}$ has $\frac{g}{2}$ values in $\mathbb{C}_L$ and $\frac{g}{2}$ values in $\mathbb{C}_U$.*

Proof In light of (7.4), case (1) (case (2)) happens if and only if C_g is a positive imaginary number (negative imaginary number). □

According to Properties 7.3 and 7.4, we need to explicitly take into account the term $D_{1+M}(\Delta\lambda)^{\frac{1+M}{g}}$ when analyzing the dual Puiseux series (7.3):

$$\Delta\tau = \sum_{i=1}^{M} D_i(\Delta\lambda)^{\frac{i}{g}} + \frac{1}{g}\left(-\frac{C_{g+M}}{C_g} + r_M\right)\left(\frac{\Delta\lambda}{C_g}\right)^{\frac{1+M}{g}} + o\left(\left(\frac{\Delta\lambda}{C_g}\right)^{\frac{1+M}{g}}\right). \tag{7.5}$$

To summarize, we have the following result:

[1] Notice that "$\mathrm{Re}(C_g) = 0$" is the degeneracy condition for the specific case where $n = 1$. The general degeneracy condition will be presented in Appendix B.

Lemma 7.2 *If* $\mathrm{Re}(C_g) = 0$*, Theorem* 7.3 *holds.*

Proof We need to consider the four possible cases ($g+M$ is odd and g is odd, $g+M$ is odd and g is even, $g+M$ is even and g is odd, and $g+M$ is even and g is even) separately. For the sake of brevity, we suppose that $g+M$ is odd and g is odd in the sequel. The other three cases can be studied by similarity.

According to (7.1), $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = +1\ (-1)$ if and only if $\mathrm{Re}(C_{g+M}) > 0$ (< 0). We now study $\Delta\tau$ via the dual Puiseux series (7.3). We first assume that C_g is a positive imaginary number. It follows from (7.4) that for $\Delta\lambda = +\varepsilon j\ (-\varepsilon j)$, $\sum_{i=1}^{M} D_i(\Delta\lambda)^{\frac{i}{g}}$ has $\frac{g-1}{2}$ values in $\mathbb{C}_U$, $\frac{g-1}{2}$ values in $\mathbb{C}_L$, and a positive real value (negative real value). We need to further consider the term $D_{1+M}(\Delta\lambda)^{\frac{1+M}{g}}$. Observe that $1+M$ is odd if $g+M$ is odd and g is odd. In light of (7.5), for $\Delta\lambda = +\varepsilon j\ (-\varepsilon j)$, if $\mathrm{Re}(C_{g+M}) > 0$, $\Delta\tau$ has $1+\frac{g-1}{2}$ $(\frac{g-1}{2})$ values in $\mathbb{C}_U$. Similarly, for $\Delta\lambda = +\varepsilon j$ $(-\varepsilon j)$, if $\mathrm{Re}(C_{g+M}) < 0$, $\Delta\tau$ has $\frac{g-1}{2}$ $(1+\frac{g-1}{2})$ values in $\mathbb{C}_U$. Thus, by Property 7.1, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha)$ when C_g is a positive imaginary number. Analogously, we can prove the result when C_g is a negative imaginary number. Now the proof is complete. □

7.4 Invariance Property for Simple Critical Imaginary Roots

With the methodology described previously in this chapter, the invariance property for the time-delay systems without multiple critical imaginary roots can be proved in a new way.

Theorem 7.4 *For a simple critical imaginary root* $\lambda_\alpha = j\omega_\alpha$ *of the time-delay system* (1.1)*, it follows that for all the corresponding positive critical delays* $\tau_{\alpha,k}$ *(*$k \in \mathbb{N}$*),*

$$\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha).$$

One may easily prove the result according to Theorem 7.3 and Remark 7.1.

In this chapter, we proposed a simple frequency-sweeping criterion. As we will show in Sect. 7.5 that the asymptotic behavior of all the critical pairs with positive critical delays can be directly known from the frequency-sweeping curves. No calculation related to the analysis of the asymptotic behavior is required.

The most important contribution of this chapter lies in the introduced technical line (description and analysis of the asymptotic behavior of the frequency-sweeping curves, the dual Puiseux series, and the equivalence relationship $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha)$). Along this new technical line, we will establish a new frequency-sweeping mathematical framework for general time-delay systems with commensurate delays in the next chapter.

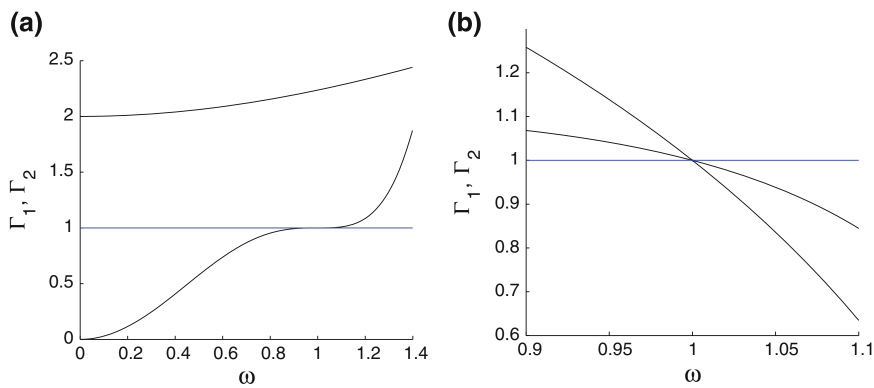

Fig. 7.1 Frequency-sweeping results for **a** Example 7.1 and **b** Example 7.4

7.5 Illustrative Examples

We now give some illustrative examples. As the nondegenerate case with $g = 1$ is relatively simple (see, e.g., [68]), it will not be discussed here.

Example 7.1 Consider the system in Example 4.1 with $f(\lambda, \tau) = e^{-2\tau\lambda} + (-\lambda^6 - 3\lambda^4 - 3\lambda^2 + \lambda + 2)e^{-\tau\lambda} - \lambda^7 - 2\lambda^6 - 3\lambda^5 - 6\lambda^4 - 3\lambda^3 - 6\lambda^2$. For this system, $\lambda = j$ is a simple critical imaginary root with $g = 1$ for $\tau = \pi, 3\pi, 5\pi, \ldots$ According to Theorem 7.1, near $\lambda = j$ and $\tau = \pi, 3\pi, 5\pi, \ldots$, the asymptotic behavior can be expressed by the Taylor series $\Delta\lambda = C_1\Delta\tau + C_2(\Delta\tau)^2 + C_3(\Delta\tau)^3 + \cdots$. This is a degenerate case as $\mathrm{Re}(C_1) = \mathrm{Re}(C_2) = 0$, $\mathrm{Re}(C_3) > 0$.

According to Theorem 7.4, we directly know from the frequency-sweeping curves shown in Fig. 7.1a that $\Delta NU_j(\tau) = \Delta NF_{-1}(1) = +1$ for all the critical delays. This frequency-sweeping criterion considerably simplifies the analysis, since we do not need to invoke the Puiseux series. □

Example 7.2 Consider again the time-delay system treated in Example 3.2 with $f(\lambda, \tau) = e^{-3\tau\lambda} + 3e^{-2\tau\lambda} + 3e^{-\tau\lambda} + \lambda^3 - \lambda^2 + \lambda$, for which the frequency-sweeping curves have been given in Fig. 3.4. This system has three simple critical imaginary roots $\lambda = j$ $(\tau = (2k+1)\pi)$ with $g = 3$, $\lambda = 1.0433j$ $(\tau = 2.5228 + 2k\pi/1.0433)$ with $g = 1$, and $\lambda = 1.6791j$ $(\tau = 2.9051 + 2k\pi/1.6791)$ with $g = 1$. For the critical imaginary roots $\lambda = 1.0433j$ and $\lambda = 1.6791j$, it is easy to know the corresponding crossing directions (both from $\mathbb{C}_-$ to $\mathbb{C}_+$) by using Theorem 8.5 (the method in [70] also applies as $g = 1$).

Consider now the critical imaginary root $\lambda = j$. The existing frequency-sweeping methods are not applicable as $g = 3$. According to Theorem 7.4, we know the corresponding crossing direction (from $\mathbb{C}_+$ to $\mathbb{C}_-$) as $\Delta NF_{-1}(1) = -1$.

For $\tau = 0$, the system has two (unstable) roots $1.2442 \pm 1.7764j$. We can now precisely know $NU(\tau)$, see Fig. 7.2. □

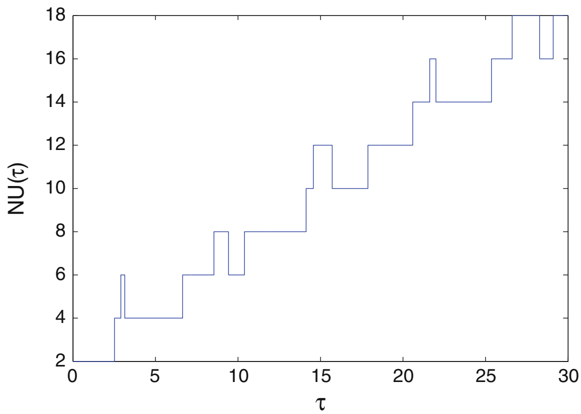

Fig. 7.2 $NU(\tau)$ for Example 7.2

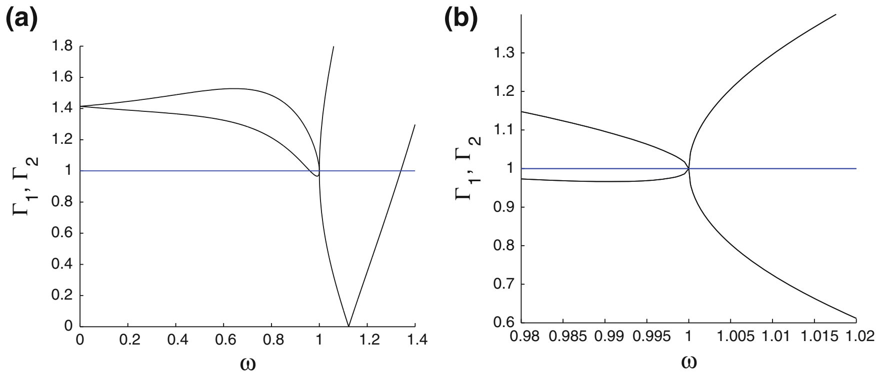

Fig. 7.3 Frequency-sweeping result for Example 7.3. **a** Frequency-sweeping curves for $0 \le \omega \le 1.4$. **b** Zoomed-in figure near $\omega = 1$

Example 7.3 Consider the system in Example 4.2 with $f(\lambda, \tau) = e^{-2\tau\lambda} - (\lambda^2 - 1)e^{-\tau\lambda} + \lambda^6 - \lambda^5 + \lambda + 2$, where $\lambda = j$ is a simple critical imaginary root with $g = 2$ for $\tau = (2k+1)\pi$. The frequency-sweeping curves are shown in Fig. 7.3. According to Theorem 7.4, the critical imaginary root j touches without crossing $\mathbb{C}_0$ as τ increases near each corresponding critical delay. To verify the result, we now invoke the Puiseux series. All the critical pairs $(j, (2k+1)\pi)$ correspond to a same Puiseux series $\Delta\lambda = \frac{-1-2j}{20}(\Delta\tau)^2 + o((\Delta\tau)^2)$ (this is a special case as $f_\lambda = -4 + 8j$ for all the critical pairs $(j, (2k+1)\pi)$), which is consistent with our analysis. □

Finally, we present an interesting example, where *a simple critical imaginary root has multiple sets of critical delays*. This case may lead to a confusion with the case of multiple critical imaginary roots. We will show that such a confusion can be easily avoided and the results proposed in this chapter are still applicable.

Example 7.4 Consider a time-delay system with $f(\lambda, \tau) = e^{-2\tau\lambda} + (\lambda^2 + 1)e^{-\tau\lambda} + \lambda^4 - 2$. For this system, $\lambda = j$ is a simple critical imaginary root with $g = 1$ for both $\tau = 0, 2\pi, 4\pi, \ldots$ and $\tau = \pi, 3\pi, 5\pi, \ldots$

Observe the frequency-sweeping curves (Fig. 7.1b). Although two frequency-sweeping curves simultaneously collide with $\Im_1$ at $\omega = 1$, they correspond to two sets of critical delays as they correspond to two sets of critical pairs $(j, \tau = 0, 2\pi, 4\pi, \ldots)$ and $(j, \tau = \pi, 3\pi, 5\pi, \ldots)$.

For the former critical pair $e^{-\tau\lambda} = 1$, while for the later critical pair $e^{-\tau\lambda} = -1$. Thus, the two frequency-sweeping curves corresponding to two sets of critical pairs can be easily distinguished, as depicted in Fig. 7.1b. In addition, using the notation defined earlier, we have that $\Delta NF_1(1)$ ($\Delta NF_{-1}(1)$) corresponds to the critical pairs $(j, \tau = 0, 2\pi, 4\pi, \ldots)$ $((j, \tau = \pi, 3\pi, 5\pi, \ldots))$. □

7.6 Notes and Comments

In this chapter, we studied the invariance property for time-delay systems with only simple critical imaginary roots. Though this property has already been proved, we introduced some new ideas that will play a crucial role in confirming the general invariance property.

First, we introduced a new notation $\Delta NF_{z_\alpha}(\omega_\alpha)$ in order to describe the asymptotic behavior of the frequency-sweeping curves. Then, we proved that the value of $\Delta NF_{z_\alpha}(\omega_\alpha)$ can be fully determined by the *dual Puiseux series*. Furthermore, we found the equivalence relation that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha)$ through analyzing the Puiseux series and the dual Puiseux series. Finally, the invariance property was proved based on the fact that $\Delta NF_{z_\alpha}(\omega_\alpha)$ is a constant with respect to different critical delays.

The above ideas provide a preliminary scheme for confirming the invariance property in the general case where both the indices n and g for a critical pair are allowed to be greater than 1. Motivated by the results in Chap. 6 and this chapter, we will study if the equivalence relation $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha)$ is satisfied for general time-delay systems with commensurate delays. If so, the general invariance property will be naturally confirmed. To this end, the current mathematical framework will be further improved in the next chapter.

Some of the ideas proposed in this chapter can be found in [67]. However, in that paper the proof was not given and the notation $\Delta NU_{\lambda_\alpha}(\tau_\alpha)$ was not explicitly adopted.

Chapter 8
A New Frequency-Sweeping Framework and Invariance Property in General Case

In Chaps. 6 and 7, we proved the invariance property for two specific types of time-delay systems and proposed an embryonic form of the new frequency-sweeping mathematical framework.

In this chapter, we will further discuss the results of Chaps. 6 and 7 and a more sophisticated frequency-sweeping methodology will be established. With this new frequency-sweeping framework, the invariance property for general time-delay systems with commensurate delays will be eventually confirmed.

8.1 Preliminaries

Consider the time-delay system (1.1)

$$\dot{x}(t) = \sum_{\ell=0}^{m} A_\ell x(t - \ell\tau),$$

with the characteristic function $f(\lambda, \tau)$ given by (1.3)

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}.$$

As usual, we adopt the trivial assumption that $\lambda = 0$ is not a characteristic root.

Inspired by Chaps. 6 and 7, we find that the frequency-sweeping curves have a close relationship with the asymptotic behavior of the critical imaginary roots. In addition, in Chap. 7, a new idea for studying the frequency-sweeping curves (studying the asymptotic behavior of the frequency-sweeping curves by means of the dual Puiseux series) was adopted, from which some useful algebraic properties regarding the frequency-sweeping curves can be obtained. In the sequel, we will build a new frequency-sweeping framework as our mathematical tool to address the invariance

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_8

issue for general time-delay systems with commensurate delays, based on its embryonic form proposed in Chap. 7.

8.2 Constructing a New Frequency-Sweeping Framework

As pointed out in Chap. 3, the asymptotic behavior for a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ is determined by the equation $F_{(\lambda_\alpha,\tau_{\alpha,k})}(\Delta\lambda, \Delta\tau) = 0$, where $F_{(\lambda_\alpha,\tau_{\alpha,k})}(\Delta\lambda, \Delta\tau)$ is a convergent power series given in (3.6). For simplicity, when no confusion occurs, we usually omit the subscript "$(\lambda_\alpha, \tau_{\alpha,k})$". It follows that $\mathrm{ord}(F(\Delta\lambda, 0)) = n$ at $\Delta\lambda = 0$ and $\mathrm{ord}(F(0, \Delta\tau)) = g$ at $\Delta\tau = 0$. As discussed in Sect. 3.1.3, without any loss of generality, $F(\Delta\lambda, \Delta\tau)$ can be decomposed as (3.7):

$$F(\Delta\lambda, \Delta\tau) = U(\Delta\lambda, \Delta\tau)\prod_{l=1}^{v} F_l(\Delta\lambda, \Delta\tau),$$

where, in a sufficiently small neighborhood of $(0, 0) \in \mathbb{C}^2$, $F_l(\Delta\lambda, \Delta\tau)$ $(l = 1, \ldots, v)$ are irreducible and $U(0, 0) \neq 0$ in the ring of convergent power series. For each $F_l(\Delta\lambda, \Delta\tau)$, $\mathrm{ord}(F_l(\Delta\lambda, 0))$ at $\Delta\lambda = 0$ and $\mathrm{ord}(F_l(0, \Delta\tau))$ at $\Delta\tau = 0$ are denoted by $n_l \in \mathbb{N}_+$ and $g_l \in \mathbb{N}_+$, respectively. It is true that $n = \sum_{l=1}^{v} n_l$ and $g = \sum_{l=1}^{v} g_l$. Note that, in the right-hand side of (3.7), repeated $F_l(\Delta\lambda, \Delta\tau)$ are allowed and, according to Property 3.1, neither $(\Delta\lambda)^\alpha$ $(\alpha \in \mathbb{N}_+)$ factor nor $(\Delta\tau)^\beta$ $(\beta \in \mathbb{N}_+)$ factor appears.

We now present a useful property concerning the Puiseux series solutions for an irreducible power series equation.

Property 8.1 *Let $\Phi_I(y, x)$ is an irreducible power series in $x \in \mathbb{C}$ and $y \in \mathbb{C}$, which is convergent in a small neighborhood of the point $(x = 0, y = 0)$ with $\Phi_I(0, 0) = 0$. Denote by* $\mathrm{ord}_{\mathrm{I}y}$ *(*$\mathrm{ord}_{\mathrm{I}x}$*) the value of* $\mathrm{ord}(\Phi_I(y, 0))$ *at $y = 0$ (*$\mathrm{ord}(\Phi_I(0, x))$ *at $x = 0$). All the y-roots for $\Phi_I(y, x) = 0$ can be expressed by a conjugacy class of Puiseux series "s" with the general form*

$$s = \sum_{i=\mathrm{ord}_{\mathrm{I}x}}^{\infty} C_i x^{\frac{i}{\mathrm{ord}_{\mathrm{I}y}}},$$

where C_i are complex coefficients.

Proof As $\Phi_I(y, x)$ is irreducible, the equation $\Phi_I(y, x) = 0$ determines a conjugacy class of Puiseux series (Proposition 2.2.1 in [15]). From $\Phi_I(y, x)$, the polydromy order and the initial term of this Puiseux series can be derived. First, by Corollary 1.8.5 in [15], the polydromy order is $\mathrm{ord}_{\mathrm{I}y}$. Next, the first exponent must be $\frac{\mathrm{ord}_{\mathrm{I}x}}{\mathrm{ord}_{\mathrm{I}y}}$ (see Exercise 11.3.1 in [91]). The general expression of the Puiseux series is hence obtained. □

We are now in a position to give the *general form of the Puiseux series* for a critical pair for the time-delay system (1.1).

Theorem 8.1 *For a critical pair* $(\lambda_\alpha, \tau_{\alpha,k})$ *with the indices n and g, the asymptotic behavior corresponds to v (counted with multiplicities) Puiseux series*

$$\begin{cases} \mathrm{PS}_1 : \Delta\lambda = \sum_{i=g_1}^{\infty} C_{1i}(\Delta\tau)^{\frac{i}{n_1}}, \\ \vdots \\ \mathrm{PS}_v : \Delta\lambda = \sum_{i=g_v}^{\infty} C_{vi}(\Delta\tau)^{\frac{i}{n_v}}, \end{cases} \tag{8.1}$$

where $C_{1i}, \dots, C_{vi}$ *are complex coefficients with* $C_{lg_l} \neq 0$ *(*$l = 1, \dots, v$*),* $n_l \in \mathbb{N}_+$ *and* $g_l \in \mathbb{N}_+$ *satisfy that* $n_1 + \cdots + n_v = n$ *and* $g_1 + \cdots + g_v = g$*.*

Proof First, in view of the factorization form (3.7) and Property 8.1, there are totally v (counted with multiplicities) Puiseux series PS_l ($l = 1, \dots, v$) and each PS_l is determined by the equation $F_l(\Delta\lambda, \Delta\tau) = 0$. Next, for each PS_l the general form is known according to Property 8.1. The general expression (8.1) is hence obtained. □

The *frequency-sweeping approach* has been largely applied in studying the stability of time-delay systems, see [20, 39, 64, 114], and the references therein. However, its application to the complete stability analysis of general time-delay systems has not been reported. In Chaps. 6 and 7, some new ideas were introduced concerning the frequency-sweeping curves. In the sequel, we will extend these ideas to the general case and establish a new frequency-sweeping framework.

As proposed in Chap. 7, we consider how τ varies in $\mathbb{C}$ with respect to λ ($\tau_{\alpha,k}$ is viewed as a g-multiple root for $f(\lambda, \tau) = 0$) and, consequently, the dual Puiseux series is introduced for describing such asymptotic behavior. Similar to the Puiseux series, we have v (counted with multiplicities) dual Puiseux series as well, determined by $F_1(\Delta\lambda, \Delta\tau), \dots, F_v(\Delta\lambda, \Delta\tau)$, respectively. Although the idea of dual Puiseux series seems to be a little abstract since $\Delta\tau \in \mathbb{R}$ for a practical system ($\Delta\tau \in \mathbb{C}$ when considering the dual Puiseux series), it will give rise to a series of important properties for addressing the general invariance property. A fundamental feature of the new frequency-sweeping framework is given below.

Theorem 8.2 *For a critical pair* $(\lambda_\alpha, \tau_{\alpha,k})$ *with the indices n and g, each Puiseux series* $\mathrm{PS}_l : \Delta\lambda = \sum_{i=g_l}^{\infty} C_{li}(\Delta\tau)^{\frac{i}{n_l}}$ *(*$C_{lg_l} \neq 0$*,* $1 \leq l \leq v$*) corresponds to a dual Puiseux series* $\mathrm{DPS}_l : \Delta\tau = \sum_{i=n_l}^{\infty} D_{li}(\Delta\lambda)^{\frac{i}{g_l}}$*, where* D_{li} *are complex coefficients with* $D_{ln_l} \neq 0$*.*

Proof By using the same idea as for the proof of Theorem 8.1, we may obtain the general expression of the dual Puiseux series. □

By Theorems 8.1 and 8.2, for a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$, we have a group of Puiseux series (8.1) as well as a group of *dual Puiseux series*:

$$\begin{cases} \text{DPS}_1 : \Delta\tau = \sum_{i=n_1}^{\infty} D_{1i}(\Delta\lambda)^{\frac{i}{g_1}}, \\ \vdots \\ \text{DPS}_\nu : \Delta\tau = \sum_{i=n_\nu}^{\infty} D_{\nu i}(\Delta\lambda)^{\frac{i}{g_\nu}}. \end{cases} \tag{8.2}$$

The above dual Puiseux series group (8.2) has an important connection with $\Delta NF_{z_\alpha}(\omega_\alpha)$ ($\Delta NF_{z_\alpha}(\omega_\alpha)$ was defined in (7.2) for describing the asymptotic behavior of the general frequency-sweeping curves):

Property 8.2 *For a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ with any indices n and g, it follows that*

$$\Delta NF_{z_\alpha}(\omega_\alpha) = ND_{(\lambda_\alpha, \tau_{\alpha,k})}(+\varepsilon j) - ND_{(\lambda_\alpha, \tau_{\alpha,k})}(-\varepsilon j),$$

where $ND_{(\lambda_\alpha, \tau_{\alpha,k})}(+\varepsilon j)$ ($ND_{(\lambda_\alpha, \tau_{\alpha,k})}(-\varepsilon j)$) denotes the number of the values in $\mathbb{C}_U$ of the dual Puiseux series (8.2), evaluated when $\Delta\lambda = +\varepsilon j$ ($\Delta\lambda = -\varepsilon j$).

Property 7.1 (with the constraint $n = 1$) can be straightforwardly extended to Property 8.2 (with any index n), since the information on n is not explicitly used in the development of Property 7.1.

Now we have equipped the "classical" frequency-sweeping approach with a new mathematical tool (we study the asymptotic behavior of the frequency-sweeping curves in terms of the dual Puiseux series) and, consequently, we establish a *new frequency-sweeping framework*, as depicted in Fig. 8.1.

A very useful property of the frequency-sweeping framework is that $\Delta NF_{z_\alpha}(\omega_\alpha)$ is independent of different critical delays in light of (1.4). Therefore, the remaining

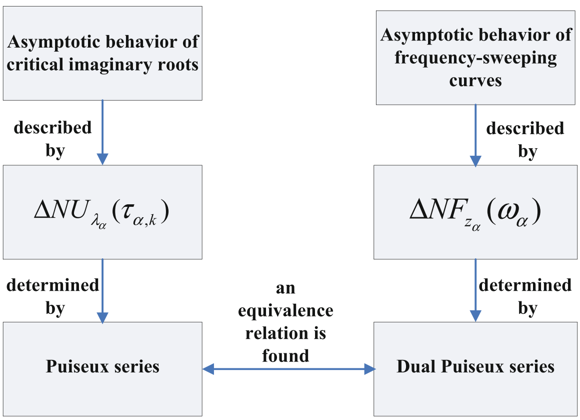

Fig. 8.1 Scheme of the new frequency-sweeping mathematical framework

task in verifying the general invariance property is to see if the equivalence relation $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha)$ holds for general time-delay systems with commensurate delays.

8.3 Proving General Invariance Property

In the sequel, we will prove the general invariance property. First, we will prove the invariance property for the case where the critical imaginary root involves only one conjugacy class of Puiseux series. Next, we will generalize the result to the case of any number of conjugacy classes of Puiseux series.

8.3.1 Critical Imaginary Roots with One Puiseux Series

In this subsection, we adopt the following assumption:

Assumption 8.1 Assume that a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ has only one Puiseux series.

The following property follows straightforwardly from Theorems 8.1 and 8.2.

Property 8.3 *For a critical pair* $(\lambda_\alpha, \tau_{\alpha,k})$ *satisfying Assumption 8.1, we have the following Puiseux series*

$$\Delta\lambda = \sum_{i=g}^{\infty} C_i (\Delta\tau)^{\frac{i}{n}}, \tag{8.3}$$

as well as the following dual Puiseux series

$$\Delta\tau = \sum_{i=n}^{\infty} D_i (\Delta\lambda)^{\frac{i}{g}}, \tag{8.4}$$

where C_i *and* D_i *are complex coefficients with* $C_g \neq 0$ *and* $D_n \neq 0$.

Theorem 8.3 *For a critical pair* $(\lambda_\alpha, \tau_{\alpha,k} > 0)$ *with any indices* n *and* g, *if Assumption 8.1 holds, it follows that:*

$$\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{z_\alpha}(\omega_\alpha). \tag{8.5}$$

The proof is given in Appendix B.

To summarize, under Assumption 8.1, the invariance property holds.

Theorem 8.4 *For a critical imaginary root* λ_α, *if Assumption 8.1 holds for all the critical pairs* $(\lambda_\alpha, \tau_{\alpha,k} > 0)$, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ *is a constant* $\Delta NF_{z_\alpha}(\omega_\alpha)$ *for all* $\tau_{\alpha,k} > 0$.

Proof By using Theorem 8.3, the proof can be completed as $\Delta NF_{z_\alpha}(\omega_\alpha)$ is independent of different $\tau_{\alpha,k}$. □

8.3.2 Critical Imaginary Roots with Multiple Puiseux Series

Without any loss of generality, when Assumption 8.1 is removed, we let a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$ have v pairs of Puiseux series and dual Puiseux series:

$$\begin{cases} \mathrm{PS}_l : \Delta\lambda = \sum_{i=g_l}^{\infty} C_{li}(\Delta\tau)^{\frac{i}{n_l}}, \\ \mathrm{DPS}_l : \Delta\tau = \sum_{i=n_l}^{\infty} D_{li}(\Delta\lambda)^{\frac{i}{g_l}}, \end{cases} \quad l = 1, \ldots, v. \tag{8.6}$$

We call the Puiseux series PS_l together with the dual Puiseux series DPS_l as expressed in (8.6) the *lth dual Puiseux series pair*.

Based on the decomposition (3.7), we can extend Theorem 8.4 to the general case as follows:

Theorem 8.5 *For a critical imaginary root λ_α of the time-delay system* (1.1)*, it always holds that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant $\Delta NF_{z_\alpha}(\omega_\alpha)$ for all $\tau_{\alpha,k} > 0$.*

Proof For a critical pair $(\lambda_\alpha, \tau_{\alpha,k})$, all the v dual Puiseux series pairs are determined by $F(\Delta\lambda, \Delta\tau)$ (3.6). Furthermore, in light of (3.7), the lth dual Puiseux series pair is determined by $F_l(\Delta\lambda, \Delta\tau)$. Now, denote by $\Delta NU_{l,\lambda_\alpha}(\tau_{\alpha,k})$ the number change of the values of the lth Puiseux series in $\mathbb{C}_+$ as τ increases from $\tau_{\alpha,k} - \varepsilon$ to $\tau_{\alpha,k} + \varepsilon$. Similarly, denote by $\Delta NF_{l,z_\alpha}(\omega_\alpha)$ the number change of the values of the lth dual Puiseux series in $\mathbb{C}_U$ as λ varies from $(\omega_\alpha - \varepsilon)j$ to $(\omega_\alpha + \varepsilon)j$. As each $F_l(\Delta\lambda, \Delta\tau)$ corresponds to one conjugacy class of Puiseux series, in the same spirit of Theorem 8.3, $\Delta NU_{l,\lambda_\alpha}(\tau_{\alpha,k}) = \Delta NF_{l,z_\alpha}(\omega_\alpha), l = 1, \ldots, v$. Since $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \sum_{l=1}^{v} \Delta NU_{l,\lambda_\alpha}(\tau_{\alpha,k})$ and $\Delta NF_{z_\alpha}(\omega_\alpha) = \sum_{l=1}^{v} \Delta NF_{l,z_\alpha}(\omega_\alpha)$, the result (8.5) follows. As a consequence, the proof is completed by noting that $\Delta NF_{z_\alpha}(\omega_\alpha)$ is invariant with respect to k. □

We have proved the invariance property for general time-delay system with commensurate delays (1.1), using the new frequency-sweeping framework proposed in this book. Now, we are able to systematically solve the complete stability problem (see the next chapter).

8.4 Illustrative Examples

We have presented some illustrative examples on the invariance property. In Example 1.6, all critical imaginary roots are with $n = g = 1$. In Chap. 6, the critical imaginary root considered in Example 6.1 is with $n = 3$ and $g = 1$ and the critical imaginary root considered in Example 6.2 with $n = 2$ and $g = 1$ corresponds to a degenerate case. In Chap. 7, some examples for the degenerate cases with $n = 1$ and $g > 1$ are presented. In the sequel, we illustrate the cases where $n > 1$ and $g > 1$ by some numerical examples.

Example 8.1 Consider the time-delay system studied in Example 5.1 with $f(\lambda,\tau) = e^{-3\tau\lambda} - 3e^{-2\tau\lambda} + 3e^{-\tau\lambda} + \lambda^4 + 2\lambda^2$. The frequency-sweeping result is shown in Fig. 8.2a (there are three frequency-sweeping curves where two of them above $\Im_1$ coincide), from which we detect two critical imaginary roots $\lambda = j$ (with the critical delays $2k\pi$) and $\lambda = 1.9566j$ (with the critical delays $\frac{(2k+1)\pi}{1.9566}$).

The root $\lambda = 1.9566j$ is simple for $\tau = \frac{(2k+1)\pi}{1.9566}$. From Fig. 8.2a, we see that $\Delta NF_{-1}(1.9566) = +1$. Thus, according to the invariance property (Theorem 8.5), $\Delta NU_{1.9566j}(\frac{(2k+1)\pi}{1.9566}) = +1$ for all $k \in \mathbb{N}$. To verify such a result, we may choose a critical delay and invoke the Puiseux series. For instance, near $(1.9566j, 1.6056)$, the Puiseux series is $\Delta\lambda = (0.6036 - 0.5253j)\Delta\tau + o(\Delta\tau)$, indicating that $\Delta NU_{1.9566j}(1.6056) = +1$. The analysis is also consistent with the root loci, see Fig. 5.1.

The root $\lambda = j$ is double for $\tau = 2k\pi$, with $f_{\lambda\lambda} = -8.00$, $f_\tau = f_{\tau\tau} = 0$, $f_{\lambda\tau} = 0$, and $f_{\tau^3} = 6.00j$ (i.e., $n = 2$ and $g = 3$) for all $k \in \mathbb{N}$. Therefore, for any critical pair $(j, 2k\pi)$ ($k \in \mathbb{N}$), the Puiseux series is (5.1). It is seen from the frequency-sweeping curves that $\Delta NF_1(1) = 0$. Thus, by the invariance property (Theorem 8.5), $\Delta NU_j(2k\pi) = 0$ for all $k \in \mathbb{N}_+$, which is consistent with the Puiseux series analysis as well as the root loci near $(j, 2\pi)$ shown in Fig. 8.2b. □

Example 8.2 Consider a time-delay system with the quasipolynomial $f(\lambda,\tau) = \sum_{i=0}^{5} a_i(\lambda)e^{-i\tau\lambda}$, where $a_0(\lambda) = \lambda^4 + 2\lambda^3 + 5\lambda^2 + 4\lambda + 4$, $a_1(\lambda) = 5\lambda^3 + 10\lambda^2 + 15\lambda + 10$, $a_2(\lambda) = 4\lambda^3 + 14\lambda^2 + 24\lambda + 14$, $a_3(\lambda) = \lambda^3 + 11\lambda^2 + 21\lambda + 11$, $a_4(\lambda) = 5\lambda^2 + 10\lambda + 5$, and $a_5(\lambda) = \lambda^2 + 2\lambda + 1$.

The frequency-sweeping result is shown in Fig. 8.3a. Five sets of critical pairs are found[1]: $(0.7266j, 5.1884 + \frac{2k\pi}{0.7266})$, $(0.8753j, 2.9402 + \frac{2k\pi}{0.8753})$, $(j, \pi + 2k\pi)$, $(3.0777j, 0.4083 + \frac{2k\pi}{3.0777})$, and $(5.9358j, 0.1577 + \frac{2k\pi}{5.9358})$.

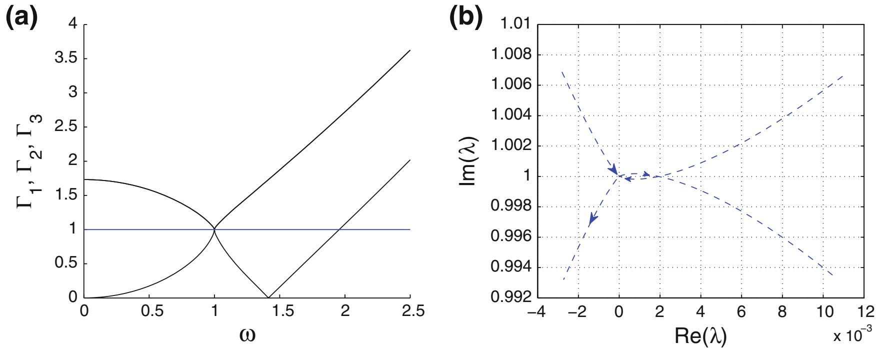

Fig. 8.2 Frequency-sweeping curves and root loci for Example 8.1. **a** Frequency-sweeping result. **b** Re(λ) versus Im(λ)

[1] Although $p(\lambda = 0, z = -0.5 \pm 0.866j) = 0$ with $|z| = 1$, according to Remark 1.9, $\lambda = 0$ is, however, not a critical imaginary root.

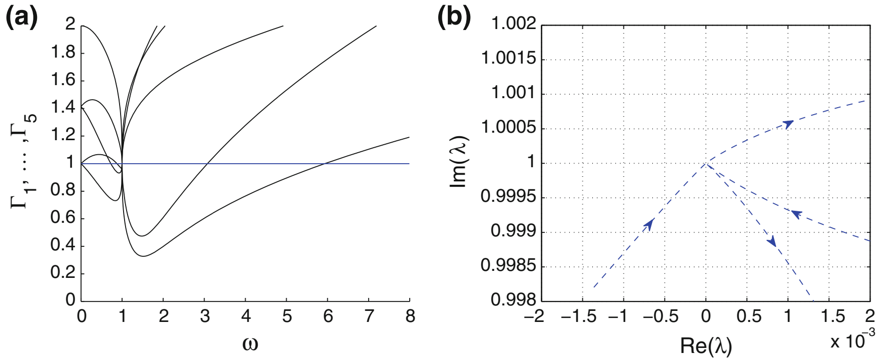

Fig. 8.3 Frequency-sweeping curves and root loci for Example 8.2. **a** Frequency-sweeping result. **b** Re(λ) versus Im(λ)

We study the critical pairs $(j, \pi + 2k\pi)$ (the asymptotic behavior of the others is relatively simple). For all $k \in \mathbb{N}$, the indices are $n = 2$ and $g = 5$. By Theorem 8.5, from the frequency-sweeping curves, $\Delta NU_j((2k+1)\pi) = +1$ for all $k \in \mathbb{N}$. We next verify this result. The Puiseux series for $(j, (2k+1)\pi)$ consists of two Taylor series

$$\begin{cases} \Delta\lambda = (0.5 - 0.5j)(\Delta\tau)^2 + o((\Delta\tau)^2), \\ \Delta\lambda = (0.5 + 0.5j)(\Delta\tau)^3 + o((\Delta\tau)^3). \end{cases} \tag{8.7}$$

We also give the root loci near (j, π) in Fig. 8.3b. □

Remark 8.1 It is worth mentioning that, for both Examples 8.1 and 8.2, the invariance property can be *analytically proved* from the Puiseux series owing to a nice feature that for both examples the first-order terms of the Puiseux series (5.1) and (8.7) are *invariant* with respect to different k (this feature is not always satisfied as generally a Puiseux series varies with respect to k).

8.5 Notes and Comments

In this chapter, we established a new frequency-sweeping mathematical framework, based on its embryonic form proposed in Chaps. 6 and 7. Using this frequency-sweeping framework, we confirmed the invariance property for general time-delay systems with commensurate delays. The skeleton of the work presented in this chapter can be found in [66, 74] (see also [73]).

With the aid of the general invariance property, some exciting results (the explicit expression of $NU(\tau)$, the ultimate stability property, and a unified approach for the complete stability problem) will be derived in the next chapter.

Chapter 9
Complete Stability for Time-Delay Systems: A Unified Approach

With the aid of the general invariance property, proved in Chap. 8, we focus now on the complete stability problem for time-delay system (1.1) with commensurate delays.

In order to thoroughly solve the complete stability problem, we will first study the so-called *ultimate stability problem*, which has not been fully investigated so far. Next, we will present the explicit expression of $NU(\tau)$ for general time-delay systems. We will see that the complete stability problem (which consists in solving both Problems 1 and 2) can be systematically solved by the frequency-sweeping approach proposed in this book.

9.1 Ultimate Stability Property

As introduced in Chap. 5, it is necessary to understand the way the spectrum of time-delay system (1.1) behaves as $\tau \to \infty$ (equivalently, to know $\lim_{\tau \to \infty} NU(\tau)$). Such a problem, called the *ultimate stability problem*, has only been studied for some specific time-delay systems (see Theorem 1 in [21] for a simple form of quasipolynomials). In this section, we will further characterize the ultimate stability problem. First, a core result will be presented in Sect. 9.1.1. Then, we can classify time-delay systems from the viewpoint of the ultimate stability property (see Sect. 9.1.2). Finally, in Sect. 9.1.3, the delay-independent cases will be discussed.

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_9

9.1.1 Characterizing Some Limit Cases

As the time-delay system (1.1) is of retarded type, the characteristic function $f(\lambda,\tau)$ (1.3) satisfies that the degree of $a_0(\lambda)$ is greater than the degrees of $a_1(\lambda),\ldots,a_q(\lambda)$. Therefore, we have the following property:

Property 9.1 *The frequency-sweeping curves of time-delay system (1.1) satisfy*

$$\lim_{\omega\to+\infty}|z_i(j\omega)|=+\infty,\, i=1,\ldots,q.$$

Notice that we do not need to distinguish the q frequency-sweeping curves though we have multiple choices to label them. It is important to point out that, due to the invariance property, different choices do not affect the stability analysis.

A critical frequency ω_α is called a *crossing (touching) frequency* for a $\Gamma_i(\omega)$, if $\Gamma_i(\omega)$ crosses (touches without crossing) $\Im_1$ as ω increases near ω_α. If a $\Gamma_i(\omega)$ crosses $\Im_1$, we denote the crossing frequencies by $\omega_{i,1},\omega_{i,2},\ldots$ with $\omega_{i,1}>\omega_{i,2}>\cdots$. By Property 9.1, at the crossing frequencies $\omega_{i,\rho}$ where ρ are odd, $\Gamma_i(\omega)$ crosses $\Im_1$ from below to above (the number of such intersections is denoted by $N_{odd,i}$), while at the crossing frequencies $\omega_{i,\rho}$ where ρ are even, $\Gamma_i(\omega)$ crosses $\Im_1$ from above to below (the number of such intersections is denoted by $N_{even,i}$). It must be true that either $N_{odd,i}=N_{even,i}$ or $N_{odd,i}=N_{even,i}+1$.

In the case $q=1$ (the system has only one frequency-sweeping curve), if the system has crossing frequencies, then $NU(\tau)$ increases more frequently on average than it decreases as τ increases from $+\varepsilon$, following the discussions in [21].

Lemma 9.1 *If $q=1$ and the frequency-sweeping curve has a crossing frequency, there exists some delay value τ^* such that the time-delay system (1.1) is unstable for all $\tau>\tau^*$ and $\lim\limits_{\tau\to\infty}NU(\tau)=\infty$.*

Next, we extend the result of Lemma 9.1 to the case with any q.

Theorem 9.1 *If the frequency-sweeping curves have a crossing frequency, there exists some delay value τ^* such that the time-delay system (1.1) is unstable for all $\tau>\tau^*$ and $\lim\limits_{\tau\to\infty}NU(\tau)=\infty$.*

Proof Due to Theorem 8.5, we may equivalently consider a time-delay system with the characteristic function $\widetilde{f}(\lambda,\tau)=\prod_{i=1}^{q}\widetilde{f}_i(\lambda,\tau)=\prod_{i=1}^{q}(e^{-\tau\lambda}+\phi_i(\lambda))$ satisfying: (a) each $\phi_i(\lambda)$ is an analytical function and (b) the frequency-sweeping curve for each $\widetilde{f}_i(\lambda,\tau)$, denoted by $\widetilde{\Gamma}_i(\omega)$, has the same crossing frequencies of $\Gamma_i(\omega)$. By Lemma 6.1, the asymptotic behavior of the critical imaginary roots for $\widetilde{f}_i(\lambda,\tau)=0$ satisfies the invariance property and can be fully mirrored by $\widetilde{\Gamma}_i(\omega)$. Therefore, $\widetilde{f}(\lambda,\tau)$ and $f(\lambda,\tau)$ have the same number change of unstable roots as τ increases from $+\varepsilon$. If $\widetilde{\Gamma}_i(\omega)$ has a crossing frequency, similar to Lemma 9.1, $\widetilde{f}_i(\lambda,\tau)$ has infinitely many unstable roots as $\tau\to+\infty$. Since the union of the characteristic roots for $\widetilde{f}_i(\lambda, \tau)$ $(i = 1, \ldots, q)$ constitutes the spectrum of $\widetilde{f}(\lambda, \tau)$, the result holds for $\widetilde{f}(\lambda, \tau)$ as well as $f(\lambda, \tau)$. □

From Theorem 9.1 we have an important conclusion that once the time-delay system (1.1) has crossing frequencies, the system will become "more and more unstable" (the value $NU(\tau) > 0$ denotes the instability degree [85]) as $\tau \to \infty$. Although this conclusion seems natural, it has not been rigorously proved in the literature.

Theorem 9.1 is the core result for the *ultimate stability problem*, with which we will obtain a macroscopic understanding of time-delay systems, from the stability point of view.

9.1.2 Classification

With the remarks and results above, we can now categorize all time-delay systems according to the ultimate stability property, as follows:

Theorem 9.2 *A time-delay system (1.1) must fall in the following three types:*

Type 1: The system has crossing frequencies and $\lim_{\tau\to\infty} NU(\tau) = \infty$.

Type 2: The system has neither crossing frequencies nor touching frequencies and $NU(\tau) = NU(0)$ *for all* $\tau > 0$.

Type 3: The system has touching frequencies but no crossing frequencies and $NU(\tau)$ *is a constant for all* $\tau \geq 0$ *except for the critical delays.*

One may easily prove Theorem 9.2 according to Theorem 9.1 and the root continuity argument for time-delay systems. Time-delay systems of Type 1 are often encountered in the literature and will be seen in Sect. 9.3. Time-delay systems of Type 3 can be found in Example 4 in [70] (simple critical imaginary root case) and Sect. 3 in [54] (a double critical imaginary root case). A time-delay system of Type 2 must be either asymptotically stable or unstable independently of delay. We will discuss this type of time-delay systems specifically in the next subsection.

9.1.3 Delay-Independent Stability (Instability)

Time-delay systems of Type 2 are called *hyperbolic* independently of delay [45]. For a time-delay system of Type 2, it exhibits the well-known *delay-independent stability* if $NU(0) = 0$. It is easy to have that $NU(0) = 0$ for the time-delay system (1.1) if and only if all the eigenvalues of $\sum_{\ell=0}^{m} A_\ell$ are located in $\mathbb{C}_-$. The delay-independent stability has been extensively studied in the literature, see [20, 22, 23, 49, 55]. If $NU(0) > 0$ for a time-delay system of Type 2, this system is unstable independently

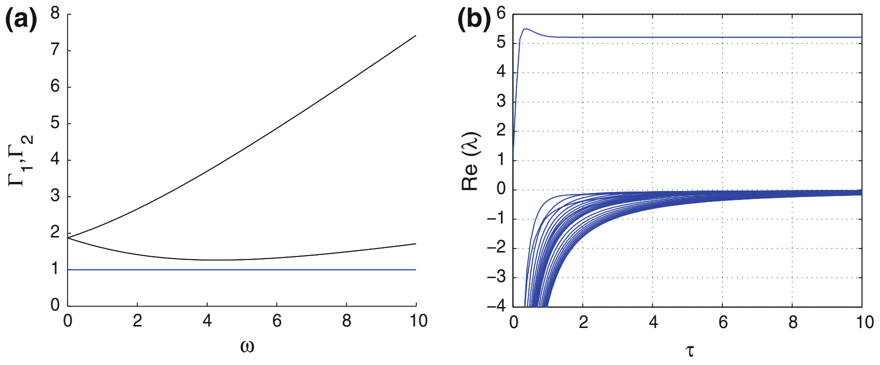

Fig. 9.1 Frequency-sweeping curves and root loci for Example 9.1. **a** Frequency-sweeping result. **b** Re(λ) versus τ

of delay[1] (i.e., this system is unstable for all $\tau \geq 0$). We present now such a time-delay system.

Example 9.1 Consider the time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau),$$

with

$$A_0 = \begin{pmatrix} 5.6035 & -3.2483 \\ 2.1573 & 4.8251 \end{pmatrix}, \; A_1 = \begin{pmatrix} -7.9037 & -7.4422 \\ 0.9908 & -0.2954 \end{pmatrix}.$$

When $\tau = 0$, the characteristic roots of this system are $1.1148 \pm 4.6897j$. Thus, $NU(0) = 2$. This system has no critical imaginary roots, which can be verified by the frequency-sweeping result given in Fig. 9.1a. Therefore, for this system, $NU(\tau) = NU(0) = 2$ for all $\tau \in [0, \infty)$. The root loci for this system are given in Fig. 9.1b, illustrating thus the analysis. □

In the sequel, we give some complementary discussions regarding the effects of the system matrices. For simplicity, we consider a time-delay system $\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau)$ as studied in Example 9.1. A straightforward property is already mentioned in this book that the system is asymptotically stable when $\tau = 0$ if and only if the matrix $A_0 + A_1$ is asymptotically stable (i.e., all the eigenvalues of the matrix $A_0 + A_1$ are in $\mathbb{C}_-$). Another necessary condition for the delay-independent stability is the asymptotic stability of the matrix A_0. We now discuss it through referring to a delay-independent stability theorem reported in [20].

[1] It is worth mentioning that a delay-independently unstable system is not necessarily delay-independently hyperbolic (see Example 9.2).

Theorem 9.3 *A time-delay system $\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau)$ is asymptotically stable independently of delay if and only if*

(i) A_0 *is asymptotically stable,*
(ii) $\rho((j\omega I - A_0)^{-1} A_1) < 1, \forall \omega > 0$, *and*
(iii) either

(1) $\rho(A_0^{-1} A_1) < 1$ *or*
(2) $\rho(A_0^{-1} A_1) = 1$ *and* $\det(A_0 + A_1) \neq 0$.

For the system in Example 9.1, conditions (ii) and (iii) of Theorem 9.3 are met. However, condition (i) is violated (the eigenvalues of A_0 are $5.2143 \pm 2.6184j$, i.e., A_0 is unstable). One may notice that the conditions (ii) and (iii) of Theorem 9.3 are equivalent to the frequency-sweeping test used in this book. For the time-delay system in Example 9.1, the matrix A_0 is the key factor affecting the stability property (delay-independent stable or unstable). In addition, some discussions on the spectrum of the matrix $A_0 - A_1$ and its link with the delay-independent stability can be found in [92].

9.2 A Unified Approach for Complete Stability

We now present the steps of the new frequency-sweeping approach, a unified approach for studying the complete stability problem.

Step 1: Generate the frequency-sweeping curves, through which we can detect all the critical imaginary roots and the corresponding critical delays.
Step 2: For each critical imaginary root λ_α, we may choose any positive critical delay $\tau_{\alpha,k}$ to compute $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ (the value is denoted by U_{λ_α}). Alternatively, we may directly have from the frequency-sweeping curves that $U_{\lambda_\alpha} = \Delta NF_{z_\alpha}(\omega_\alpha)$, according to Theorem 8.5.
Step 3: Compute $NU(+\varepsilon)$ (by Theorem 5.1).

With the steps above, we obtain the explicit expression of $NU(\tau)$ for the time-delay system (1.1), as stated in the following theorem.

Theorem 9.4 *For any $\tau > 0$ which is not a critical delay, $NU(\tau)$ for the time-delay system (1.1) can be explicitly expressed as*

$$NU(\tau) = NU(+\varepsilon) + \sum_{\alpha=0}^{u-1} NU_\alpha(\tau), \tag{9.1}$$

where

$$NU_\alpha(\tau) = \begin{cases} 0, \ \tau < \tau_{\alpha,0}, \\ 2U_{\lambda_\alpha} \left\lceil \frac{\tau - \tau_{\alpha,0}}{2\pi/\omega_\alpha} \right\rceil, \ \tau > \tau_{\alpha,0}, \end{cases} \text{if } \tau_{\alpha,0} \neq 0,$$

$$NU_\alpha(\tau) = \begin{cases} 0, \tau < \tau_{\alpha,1}, \\ 2U_{\lambda_\alpha}\left\lceil \frac{\tau-\tau_{\alpha,1}}{2\pi/\omega_\alpha} \right\rceil, \tau > \tau_{\alpha,1}, \end{cases} \text{if } \tau_{\alpha,0} = 0.$$

Proof Since $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant U_{λ_α} for all $k \in \mathbb{N}$ with $\tau_{\alpha,k} > 0$ due to the invariance property as stated in Theorem 8.5, $NU(\tau)$ can be expressed by the closed form (9.1). □

Now we have proposed a systematic approach (a new frequency-sweeping framework) to solve the complete stability problem. The time-delay system (1.1) is asymptotically stable for the domain of τ with $NU(\tau) = 0$ excluding the critical delays. In addition, according to Theorem 9.2, the ultimate stability property is known.

Remark 9.1 In the literature, similar results have only been obtained for some specific time-delay systems, see [21, 97, 122]. The analysis and design of a general time-delay system have long been considered rather involved. In our opinion, the explicit form of $NU(\tau)$ (9.1) may help to simplify the existing analysis and design procedures for time-delay systems and open some new perspectives in this domain.

Remark 9.2 For fixed delay parameters, some interesting formulas for counting the number of unstable roots have been reported in the literature, see [48, 52, 112], which are in a substantially different line (argument principle-based methods) compared with the τ-decomposition one adopted in [21, 97, 122], and this book. However, in our opinion, it is not easy to apply the formulas in [48, 52, 112] to the complete stability problem discussed in this book (specifically, it is difficult to apply them to achieve the "$NU(\tau)$ versus τ" plot) since one explicitly needs to know the critical pairs and the corresponding asymptotic behavior.

9.3 Illustrative Examples

In the sequel, the examples considered in Chap. 8 are completely solved in the context of the complete stability.

Example 9.2 Go on with the analysis in Example 8.1, for which the characteristic function is $f(\lambda, \tau) = e^{-3\tau\lambda} - 3e^{-2\tau\lambda} + 3e^{-\tau\lambda} + \lambda^4 + 2\lambda^2$. The frequency-sweeping result has been given in Fig. 8.2a (there are three frequency-sweeping curves where two of them above $\Im_1$ coincide). From the frequency-sweeping result, we detect two critical imaginary roots $\lambda_0 = j$ (with the critical delays $2k\pi$) and $\lambda_1 = 1.9566j$ (with the critical delays $\frac{(2k+1)\pi}{1.9566}$). The root $\lambda = 1.9566j$ is simple for $\tau = \frac{(2k+1)\pi}{1.9566}$ and the root $\lambda = j$ is double for $\tau = 2k\pi$. In fact, we do not need to know the multiplicities of the critical imaginary roots and further information. According to Step 2 we may directly know that $U_j = 0$ and $U_{1.9566j} = +1$. When $\tau = 0$ this system has multiple critical imaginary roots. We need to first compute $NU(+\varepsilon)$ using Theorem 5.1. This task has been finished in Example 5.1 with the result $NU(+\varepsilon) = +2$.

By Theorem 9.4, we have the explicit expression of $NU(\tau)$ for any τ other than critical delays:

$$NU(\tau) = 2 + NU_1(\tau),$$

where

$$NU_1(\tau) = \begin{cases} 0, \tau < 1.6056, \\ 2U_{1.9566j}\left\lceil \frac{\tau - 1.6056}{3.2113} \right\rceil, \tau > 1.6056. \end{cases}$$

The plot of $NU(\tau)$ is shown in Fig. 9.2a. Finally, we have that $NU(\tau) \to \infty$ as $\tau \to \infty$ by Theorem 9.1, which is illustrated by Fig. 9.2a. This system is unstable for all $\tau \in [0, \infty)$ without being delay-independently hyperbolic. □

Example 9.3 Consider the system of Example 8.2 with the characteristic function $f(\lambda, \tau) = \sum_{i=0}^{5} a_i(\lambda)e^{-i\tau\lambda}$, where $a_0(\lambda) = \lambda^4 + 2\lambda^3 + 5\lambda^2 + 4\lambda + 4$, $a_1(\lambda) = 5\lambda^3 + 10\lambda^2 + 15\lambda + 10$, $a_2(\lambda) = 4\lambda^3 + 14\lambda^2 + 24\lambda + 14$, $a_3(\lambda) = \lambda^3 + 11\lambda^2 + 21\lambda + 11$, $a_4(\lambda) = 5\lambda^2 + 10\lambda + 5$, and $a_5(\lambda) = \lambda^2 + 2\lambda + 1$. The frequency-sweeping curves are given in Fig. 8.3a. Five sets of critical pairs are found: $(\lambda_0 = 0.7266j, \tau_{0,k} = 5.1884 + \frac{2k\pi}{0.7266})$, $(\lambda_1 = 0.8753j, \tau_{1,k} = 2.9402 + \frac{2k\pi}{0.8753})$, $(\lambda_2 = j, \tau_{2,k} = \pi + 2k\pi)$, $(\lambda_3 = 3.0777j, \tau_{3,k} = 0.4083 + \frac{2k\pi}{3.0777})$, and $(\lambda_4 = 5.9358j, \tau_{4,k} = 0.1577 + \frac{2k\pi}{5.9358})$. We may directly have from the frequency-sweeping curves that $U_{0.7266j} = -1$, $U_{0.8753j} = -1$, $U_j = +1$, $U_{3.0777j} = +1$, and $U_{5.9358j} = +1$. All the characteristic roots when $\tau = 0$ lie in $\mathbb{C}_-$. Thus, $NU(+\varepsilon) = 0$, according to Theorem 5.1.

According to Theorem 9.4, for a τ which is not a critical delay, we have

$$NU(\tau) = \sum_{\alpha=0}^{4} NU_\alpha(\tau),$$

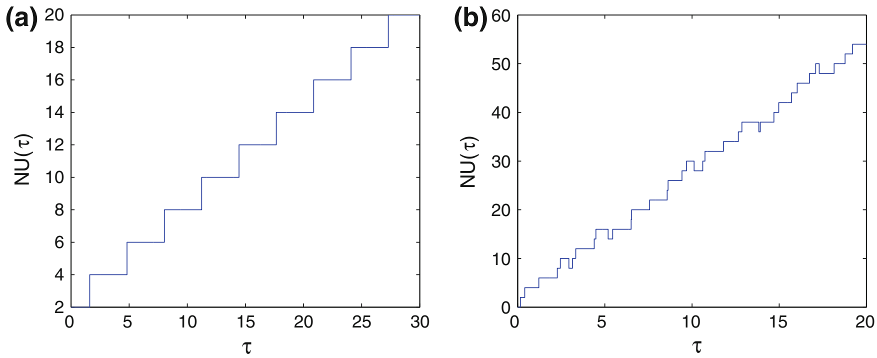

Fig. 9.2 $NU(\tau)$ for Examples 9.2 and 9.3. **a** Example 9.2. **b** Example 9.3

where

$$NU_0(\tau) = \begin{cases} 0, \tau < 5.1884, \\ 2U_{0.7266j} \left\lceil \frac{\tau - 5.1884}{8.6474} \right\rceil, \tau > 5.1884, \end{cases}$$

$$NU_1(\tau) = \begin{cases} 0, \tau < 2.9402, \\ 2U_{0.8753j} \left\lceil \frac{\tau - 2.9402}{7.1783} \right\rceil, \tau > 2.9402, \end{cases}$$

$$NU_2(\tau) = \begin{cases} 0, \tau < \pi, \\ 2U_{j} \left\lceil \frac{\tau - \pi}{2\pi} \right\rceil, \tau > \pi, \end{cases}$$

$$NU_3(\tau) = \begin{cases} 0, \tau < 0.4803, \\ 2U_{3.0777j} \left\lceil \frac{\tau - 0.4803}{2.0415} \right\rceil, \tau > 0.4803, \end{cases}$$

$$NU_4(\tau) = \begin{cases} 0, \tau < 0.1577, \\ 2U_{5.9358j} \left\lceil \frac{\tau - 0.1577}{1.0585} \right\rceil, \tau > 0.1577. \end{cases}$$

The plot of $NU(\tau)$ is given in Fig. 9.2b. The ultimate stability property ($\lim_{\tau\to\infty} NU(\tau) = \infty$ by Theorem 9.1) is illustrated by Fig. 9.2b. This system has only one stability interval: $\tau \in [0, 0.1577)$. That is to say, this system is asymptotically stable if and only if $\tau \in [0, 0.1577)$. □

Through the above examples, we see that the complete stability of time-delay systems with commensurate delays can be systematically studied. Both Problems 1 (detecting all the critical imaginary roots and the critical delays) and 2 (analyzing the asymptotic behavior of the critical imaginary roots) can be solved by the frequency-sweeping approach proposed in this book. The asymptotic behavior at all the (infinitely many) positive critical delays can be studied by a graphical test of the frequency-sweeping curves. Thus, the frequency-sweeping approach appears to be simple to implement in practice.

9.4 Notes and Comments

Now, we have systematically solved the complete stability problem for a general time-delay system with commensurate delays (1.1) in the retarded case. We will see in the next chapter that the proposed approach can be extended to another class of time-delay systems, the neutral time-delay systems.

As the delays appearing in time-delay system (1.1) are commensurate, the problem considered in this book involves in fact only one parameter τ. If the delays are incommensurate (e.g., consider a time-delay system $\dot{x}(t) = A_0 x(t) + \sum_{\ell=1}^{m} A_\ell x(t - \tau_\ell)$

where $\tau_1, \ldots, \tau_m$ are incommensurate delays), the problem will contain multiple parameters and be much more involved. Such a problem is far beyond the scope of this book. For solving Problem 1 in the case of multiple incommensurate delays, some effective methods have been proposed, see [25, 40, 41, 88, 107]. However, to the best of the authors' knowledge, no result has been reported on Problem 2 for such a time-delay system so far. In the small-gain analysis framework, some necessary and sufficient delay-independent and sufficient delay-dependent stability conditions can be obtained for a system with incommensurate delays (one may refer to Part III of [39] for a detailed introduction). Apart from the above results, some sufficient criteria have been reported, see [59, 86]. Overall, there is much room for the improvement concerning the stability research of systems with multiple incommensurate delays.

The skeleton of the work in this chapter was reported in [66, 74] (see also [73]).

Chapter 10
Extension to Neutral Time-Delay Systems

In the preceding chapters, we solved the complete stability problem for time-delay systems of retarded type (shortly called retarded systems) by proposing a frequency-sweeping framework. In this chapter, we will see that this new framework is applicable to time-delay systems of neutral type (shortly called neutral systems) as well.

In the stability analysis framework (in the case of commensurate delays), compared to retarded systems, the major distinction of neutral systems lies in that infinitely many new characteristic roots may appear in the right-half plane $\mathbb{C}_+$ when delay increases from 0 to $+\varepsilon$. For this reason, we have to first check the stability of the neutral operator (sometimes called delay-difference operator), which ensures that all the infinitely many new characteristic roots appear in the left-half plane $\mathbb{C}_-$. In other words, the stability of the neutral operator is a necessary condition for the stability of the neutral time-delay system. We will show in this chapter that this necessary condition can be *embedded* in the frequency-sweeping approach, i.e., we may directly verify this condition from the frequency-sweeping curves.

If the stability of the neutral operator is guaranteed, we will proceed to address the invariance and ultimate stability issues. It will turn out that the general invariance property holds for neutral systems. Thus, the results derived in the retarded case can also be applied to neutral systems. Next, we will analyze the ultimate stability problem for neutral systems. Although the frequency-sweeping curves of neutral systems exhibit different limit characteristics from the ones of retarded systems, two types of time-delay systems possess the same ultimate stability property essentially.

Combining the aforementioned results, we will demonstrate that the frequency-sweeping framework also represents a unified approach for the stability analysis of neutral time-delay systems with commensurate delays. The complete stability of general neutral time-delay systems has not been solved in the literature, though a great number of results have been reported. To the best of the authors' knowledge, the newest results were reported in [98, 109]. However, it was assumed therein that the neutral system under consideration has only simple critical imaginary roots.

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_10

Finally, some complementary discussions concerning the cases with multiple delays will be given. In particular, we will show that, in the commensurate delays case, the complete stability is covered by the approach proposed in this chapter.

10.1 Preliminaries

10.1.1 Basic Concepts

Consider the following time-delay system of neutral type

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + C\dot{x}(t-\tau), \tag{10.1}$$

under appropriate initial conditions, where $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times r}$, and $C \in \mathbb{R}^{r\times r}$ are constant matrices. The characteristic function of system (10.1) is given by [45]

$$f_N(\lambda, \tau) = \det(\lambda I - A - Be^{-\tau\lambda} - \lambda Ce^{-\tau\lambda}), \tag{10.2}$$

which is a quasipolynomial of the form

$$f_N(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}, \tag{10.3}$$

where $a_0(\lambda), \ldots, a_q(\lambda)$ are polynomials in λ with real coefficients.

For a $\tau > 0$, the neutral time-delay system (10.1) has *infinitely many* characteristic roots (i.e., the roots for $f_N(\lambda, \tau) = 0$). We have the following theorem [85]:

Theorem 10.1 *The trivial solution $x(t) = 0$ of neutral time-delay system* (10.1) *is exponentially stable if and only if all the characteristic roots lie in the open left-half plane $\mathbb{C}_-$ and are bounded away from the imaginary axis $\mathbb{C}_0$.*

For the sake of brevity, in the sequel we simply say "*neutral system* (10.1) *is exponentially stable*" instead of "*the trivial solution $x(t) = 0$ of neutral system* (10.1) *is exponentially stable*".

Remark 10.1 For the retarded time-delay system (1.1), it is exponentially stable if and only if it is asymptotically stable. However, for the neutral time-delay system (10.1), the asymptotic stability does not in general imply the exponential stability (the converse holds), see the example given in Sect. 4.2 of [117].

Remark 10.2 Different from Theorem 1.1 concerning the stability of retarded systems, it is additionally required in Theorem 10.1 that *all the characteristic roots must be bounded away from the imaginary axis* $\mathbb{C}_0$. In fact, such an additional condition can be guaranteed by the stability of the neutral operator [34], which will be discussed later. In other words, if the neutral operator is stable, we only need to verify if all the characteristic roots lie in $\mathbb{C}_-$.

The objective of this chapter is to find the whole domain for τ (delay intervals) where the neutral system (10.1) is exponentially stable. As usual, we denote the number of unstable roots (i.e., the characteristic roots in $\mathbb{C}_+$) by $NU(\tau)$ in the presence of delay τ. We need to find the whole domain for $\tau \in [0, \infty)$ such that $NU(\tau) = 0$.

To rule out a trivial case, we assume that $a_0(\lambda), \ldots, a_q(\lambda)$ have no common zeros in $\mathbb{C}_+ \cup \mathbb{C}_0$ (otherwise, the system (10.1) is not exponentially stable for any $\tau \geq 0$). Another straightforward remark is that if $\lambda = 0$ is a characteristic root, the system (10.1) has one invariant root at the origin for all $\tau \geq 0$ and hence we will not compute $NU(\tau)$ for such systems.

With the notation $z = e^{-\tau\lambda}$, $f_N(\lambda, \tau)$ can be rewritten as the following form:

$$p_N(\lambda, z) = \sum_{i=0}^{q} a_i(\lambda) z^i. \tag{10.4}$$

The detection of the critical imaginary roots and the corresponding critical delays for $f_N(\lambda, \tau) = 0$ amounts to detecting the critical pairs (λ, z) ($\lambda \in \mathbb{C}_0$ and $z \in \partial\mathbb{D}$) such that $p_N(\lambda, z) = 0$. Without any loss of generality, assume that there are u critical pairs denoted by $(\lambda_0 = j\omega_0, z_0)$, $(\lambda_1 = j\omega_1, z_1)$, ..., $(\lambda_{u-1} = j\omega_{u-1}, z_{u-1})$ where $\omega_0 \leq \omega_1 \leq \cdots \leq \omega_{u-1}$. Once all the critical pairs $(\lambda_\alpha, z_\alpha)$, $\alpha = 0, \ldots, u-1$, are found, all the critical pairs $(\lambda_\alpha, \tau_{\alpha,k})$, $\alpha = 0, \ldots, u-1$, $k \in \mathbb{N}$, can be obtained: For each critical imaginary root λ_α, the corresponding critical delays are given by $\tau_{\alpha,k} \stackrel{\Delta}{=} \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}$ with $\tau_{\alpha,0} \stackrel{\Delta}{=} \min\{\tau \geq 0 : e^{-\tau\lambda_\alpha} = z_\alpha\}$.

In light of (10.4), we may employ the frequency-sweeping test to obtain the frequency-sweeping curves.

Frequency-Sweeping Curves Sweep $\omega \geq 0$ and for each $\lambda = j\omega$ we have q solutions of z such that $p_N(\lambda, z) = 0$ (denoted by $z_1(j\omega), \ldots, z_q(j\omega)$). In this way, we obtain q frequency-sweeping curves $\Gamma_i(\omega)$: $|z_i(j\omega)|$ versus ω, $i = 1, \ldots, q$.

In addition, we define "$\Delta NF_{z_\alpha}(\omega_\alpha)$" as in (7.2) to describe the asymptotic behavior of the frequency-sweeping curves. It is easy to see that all the critical pairs can be detected from the frequency-sweeping curves. In the sequel, we will show that the other issues for the complete stability problem can also be solved by using the frequency-sweeping approach.

10.1.2 Subtleties of Neutral Time-Delay Systems

Though retarded systems and neutral systems share the same type of characteristic functions (their characteristic functions are both quasipolynomials of the form (10.3)), they exhibit the following "subtle" difference:

(i) For the system (10.1) with $C = 0$ (i.e., a retarded system), it follows that

$$\deg(a_0(\lambda)) > \max\{\deg(a_1(\lambda)), \ldots, \deg(a_q(\lambda))\}.$$

(ii) For the system (10.1) with $C \neq 0$ (i.e., a neutral system), it happens that

$$\deg(a_0(\lambda)) = \max\{\deg(a_1(\lambda)), \ldots, \deg(a_q(\lambda))\}.$$

The above difference results in a distinction between two types of time-delay systems. For a retarded time-delay system, when τ increases from 0 to $+\varepsilon$, all the infinitely many new roots appear at far left of the complex plane (i.e., all with $-\infty$ real parts). Therefore, all these infinitely many new characteristic roots are "dormant" from the stability point of view. However, when τ increases from 0 to $+\varepsilon$, infinitely many new roots may appear in the right-half plane for a neutral time-delay system and, hence, the neutral system may be unstable for all $\tau > 0$. That is, the spectrum of a neutral system may exhibit some discontinuity properties (see [4, 83, 85]). This gives rise to the stability issue of the neutral operator.

For the system (10.1), the *neutral operator* refers to the following difference equation:

$$x(t) = Cx(t - \tau). \tag{10.5}$$

The exponential stability of the neutral operator (10.5) is a necessary condition for the exponential stability of system (10.1). Recall the following well-known condition, see [85].

Lemma 10.1 *The neutral operator* (10.5) *is exponentially stable for any positive* τ *if and only if*

$$\rho(C) < 1. \tag{10.6}$$

Therefore, in order to analyze the complete stability for neutral system (10.1), we have to first check the necessary condition (10.6). For a comprehensive introduction to the spectral properties of linear neutral time-delay systems, we recommend a recent review article [38]. See also [14] for further discussions.

10.2 Complete Stability Characterization

In this section, the technical issues (the stability of the neutral operator, the invariance property, and the ultimate stability property) required by the complete stability problem of neutral time-delay systems will be studied. Finally, the unified approach will be presented.

10.2.1 Embedding Stability Condition of Neutral Operator

In this chapter, all eigenvalues of C are denoted by $\lambda_1(C), \ldots, \lambda_r(C)$ and naturally the spectrum of C corresponds to the set $\{\lambda_1(C), \ldots, \lambda_r(C)\}$. We have the following properties connecting the spectrum of C with the frequency-sweeping curves:

Lemma 10.2 *If* $q = r$, *as* $\omega \to \infty$, $\{\frac{1}{z_1(j\omega)}, \ldots, \frac{1}{z_r(j\omega)}\} \to \{\lambda_1(C), \ldots, \lambda_r(C)\}$.

Proof In the case $q = r$, we have r values of z (i.e., r frequency-sweeping curves) from $p_N(j\omega, z) = (j\omega)^r \det(M(\omega) - N(\omega)z) = 0$ where $M(\omega) = I - \frac{A}{j\omega}$ and $N(\omega) = \frac{B}{j\omega} + C$. It is easy to see that $M(\omega) \to I$ and $N(\omega) \to C$ as $\omega \to \infty$. For each $\lambda_i(C) \neq 0$, we may find a $z(j\omega) \to \frac{1}{\lambda_i(C)}$ such that $p_N(j\omega, z(j\omega)) = 0$ as $\omega \to \infty$. For each $\lambda_i(C) = 0$ (if any!), we may find a $z(j\omega) \to \infty$ such that $p_N(j\omega, z(j\omega)) = 0$ as $\omega \to \infty$. □

If $q < r$, $r - q$ eigenvalues of C cannot be reflected by the q frequency-sweeping curves. Without any loss of generality, we denote the spectrum of C by the set $\{\lambda_1(C), \ldots, \lambda_q(C)\} \cup \{\lambda_{q+1}(C), \ldots, \lambda_r(C)\}$, where the q eigenvalues in $\{\lambda_1(C), \ldots, \lambda_q(C)\}$ connect with the frequency-sweeping curves while the $r - q$ eigenvalues in $\{\lambda_{q+1}(C), \ldots, \lambda_r(C)\}$ do not. We have the following lemma concerning the eigenvalues $\lambda_{q+1}(C), \ldots, \lambda_r(C)$.

Lemma 10.3 *If $q < r$, $\lambda_{q+1}(C) = \cdots = \lambda_r(C) = 0$.*

Proof Following the proof of Lemma 10.2, $p_N(j\omega, z) = (j\omega)^r \det(M(\omega) - N(\omega)z)$ and we can express $\det(M(\omega) - N(\omega)z)$ as a polynomial: $b_q(\omega)z^q + b_{q-1}(\omega)z^{q-1} + \cdots + b_0(\omega)$ where $b_q(\omega) \not\equiv 0, b_{q-1}(\omega), \ldots, b_0(\omega)$ are continuous functions of ω. As $M(\omega) \to I$ and $N(\omega) \to C$ as $\omega \to \infty$, $\det(I - Cz)$ is a polynomial in z whose degree is not larger than q. More precisely, $\det(I - Cz) = c_q z^q + c_{q-1} z^{q-1} + \cdots + c_0$ such that $b_q(\omega) \to c_q, \cdots, b_0(\omega) \to c_0 = 1$ as $\omega \to \infty$. Let $s = z^{-1}$. Then, it follows straightforwardly that $\det(sI - C) = s^r \det(I - Cz) = s^r + c_1 s^{r-1} + \cdots + c_q s^{r-q} = s^{r-q}(s^q + c_1 s^{q-1} + \cdots + c_q)$, which is the characteristic function of C. That is to say, $\lambda_{q+1}(C) = \cdots = \lambda_r(C) = 0$. □

The ideas of Lemmas 10.2 and 10.3 lead to the following lemma:

Lemma 10.4 *If $q < r$, as $\omega \to \infty$, $\{\frac{1}{z_1(j\omega)}, \ldots, \frac{1}{z_q(j\omega)}\} \to \{\lambda_1(C), \ldots, \lambda_q(C)\}$.*

Combining Lemmas 10.2–10.4, we have the following result, which allows embedding the necessary condition (10.6) in the frequency-sweeping approach.

Theorem 10.2 *The neutral operator (10.5) is exponentially stable if and only if all the frequency-sweeping curves are above $\Im_1$ as $\omega \to \infty$.*

To illustrate Theorem 10.2, two numerical examples are proposed.

Example 10.1 Consider the neutral time-delay system of Example case 2 in [98], i.e., the system (10.1) with matrices:

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -3.346 & -2.715 & 2.075 & -2.007 \\ -4 & 0 & 2 & 0 \\ -3 & 0 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & 2 & -1 \\ 3 & 3 & -2 & 0 \\ 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & -3 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.2 & -0.1 & 0.5 & -0.1 \\ -0.3 & 0.09 & -0.15 & -0.027 \\ -3.333 & 0.1 & 0.2 & 1 \\ -1 & 2 & 0.5 & 1 \end{pmatrix}.$$

The four eigenvalues of C are $0.0881 \pm 0.8494j$ and $0.6569 \pm 0.5284j$. Hence, $\rho(C) < 1$ (i.e., the neutral operator (10.5) is exponentially stable). This result can be directly obtained from the frequency-sweeping curves shown in Fig. 10.1a. We see that as $\omega \to \infty$, $|z_i(j\omega)| > 1$, $i = 1, \ldots, 4$. By Theorem 10.2, the neutral operator (10.5) is exponentially stable. □

Example 10.2 Consider the neutral time-delay system of Example b2 in [109], i.e., the system (10.1) with matrices:

$$A = \begin{pmatrix} 12 & 10 & -6 & 14 \\ 7 & 8 & 11 & 9 \\ -5 & 7 & 3 & 3 \\ 6 & 2 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -169 & -276.85 & -445.76 & -675.75 \\ -11 & -46 & -61 & -83 \\ 249 & 360.05 & 1070.43 & 1431.02 \\ 81.65 & 158.32 & 127.61 & 230.85 \end{pmatrix},$$

$$C = \begin{pmatrix} -4 & 12 & 3 & 1 \\ 0 & 1 & -2 & 6 \\ 12 & -8 & 4 & 2 \\ 1.47 & -10.09 & -4.33 & 0.03 \end{pmatrix}.$$

The four eigenvalues of C are $0.2816 \pm 1.3641j$, –1.3469, and 1.8138. As $\rho(C) > 1$, the neutral operator (10.5) is unstable. That is, $NU(\tau) = +\infty$ for any $\tau > 0$. We now apply the frequency-sweeping test. It is seen from the frequency-sweeping curves, Fig. 10.1b, that as $\omega \to \infty$, $|z_i(j\omega)| < 1$, $i = 1, \ldots, 4$. By Theorem 10.2, we directly know that the neutral operator (10.5) is not exponentially stable. □

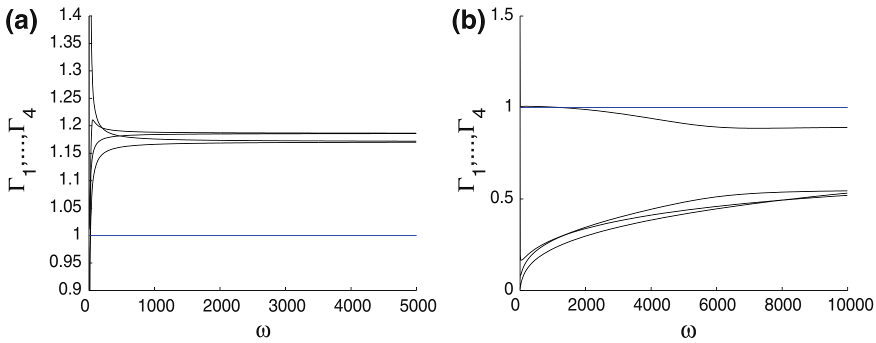

Fig. 10.1 Frequency-sweeping results for **a** Example 10.1 and **b** Example 10.2

10.2.2 Infinitesimal Delay Case

As a straightforward extension of Theorem 5.1, we present the method for computing $NU(+\varepsilon)$ of neutral systems in the case when the neutral operator is stable.

Theorem 10.3 *If the system (10.1) with $\rho(C) < 1$ has no critical imaginary roots when $\tau = 0$, then $NU(+\varepsilon) = NU(0)$. If the system (10.1) with $\rho(C) < 1$ has critical imaginary roots when $\tau = 0$, then $NU(+\varepsilon) - NU(0)$ equals to the number of the values in $\mathbb{C}_+$ of the Puiseux series for all the corresponding critical imaginary roots when $\tau = 0$ with $\Delta\tau = +\varepsilon$.*

10.2.3 General Invariance Property for Neutral Time-Delay Systems

It is not hard to prove that the invariance property, which was confirmed for general retarded systems in Chap. 8, also holds for general neutral systems, as two types of time-delay systems share the same type of characteristic functions.

Theorem 10.4 *For a critical imaginary root λ_α of the system (10.1), $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant $\Delta NF_{z_\alpha}(\omega_\alpha)$ for all $\tau_{\alpha,k} > 0$.*

Recall that "$\Delta NF_{z_\alpha}(\omega_\alpha) \in \mathbb{Z}$" is a constant number reflecting the asymptotic behavior of the frequency-sweeping curves near $\omega = \omega_\alpha$.

10.2.4 Ultimate Stability Property

For retarded time-delay systems, we studied the ultimate stability problem in Chap. 9 relying on the invariance property and an important property on the frequency-sweeping curves (Property 9.1): $\lim_{\omega\to+\infty} |z_i(j\omega)| = +\infty, i = 1, \ldots, q$, for the system (1.1). Property 9.1 ensures that the largest crossing frequency for each frequecy-sweeping curve must correspond to the intersection of $\mathfrak{I}_1$ from below to above. Next, based on the invariance property we have that as τ increases $NU(\tau)$ increases more frequently than it decreases, on average.

We now consider the frequency-sweeping curves for neutral system (10.1) as $\omega \to \infty$. In light of Theorem 10.2, we have:

Property 10.1 *If $\rho(C) < 1$ for the neutral time-delay system (10.1), the frequency-sweeping curves satisfy that*

$$\lim_{\omega\to+\infty} |z_i(j\omega)| > 1, \quad i = 1, \ldots, q. \tag{10.7}$$

Property 10.1 also ensures that at the largest crossing frequency each frequency-sweeping curve must intersect $\Im_1$ from below to above. As a result, we may now have the same results on the ultimate stability for neutral time-delay system (10.1) as for retarded time-delay system (1.1).

Theorem 10.5 *If the frequency-sweeping curves have a crossing frequency, there exists some delay value* τ^* *such that the neutral time-delay system* (10.1) *with* $\rho(C) < 1$ *is unstable for all* $\tau > \tau^*$ *and* $\lim_{\tau\to\infty} NU(\tau) = \infty$.

Theorem 10.6 *A neutral time-delay system* (10.1) *with* $\rho(C) < 1$ *must belong to the following three types:*

Type 1: The system has a crossing frequency and $\lim_{\tau\to\infty} NU(\tau) = \infty$.
Type 2: The system has neither crossing frequencies nor touching frequencies and $NU(\tau) = NU(0)$ *for all* $\tau > 0$.
Type 3: The system has touching frequencies but no crossing frequencies and $NU(\tau)$ *is a constant for all* $\tau \geq 0$ *except for the critical delays.*

10.2.5 Frequency-Sweeping Framework: A Unified Approach

We can now solve the complete stability of neutral time-delay systems within the frequency-sweeping framework. We summarize the steps as Algorithm 10.1.

Algorithm 10.1 Algorithm for analyzing the complete stability of a neutral system

Step 1: Perform the frequency-sweeping test to obtain the frequency-sweeping curves.
Step 2: Verify condition (10.6) by Theorem 10.2. If the condition is violated, the system (10.1) is not exponentially stable for any $\tau > 0$. Otherwise, go to Step 3.
Step 3: Calculate all the critical pairs $(\lambda_\alpha, \tau_{\alpha,k})$ from the frequency-sweeping curves.
Step 4: Compute $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ for all $\alpha = 0, \ldots, u-1$, and $\tau_{\alpha,k} > 0$ by the invariance property (Theorem 10.4).
Step 5: The explicit expression of $NU(\tau)$ is of the form (9.1), where the value of $NU(+\varepsilon)$ is computed according to Theorem 10.3.

The neutral time-delay system (10.1) is exponentially stable if and only if τ lies in the domain where $NU(\tau) = 0$ excluding critical delays. The ultimate stability property is obtained by Theorem 10.6.

10.3 Discussions on Neutral Systems with Multiple Delays

In the previous sections of this chapter, the considered neutral system (10.1) has a single delay parameter. In this section, we will briefly discuss the cases with multiple delays. If the multiple delays are commensurate, we will see in Sect. 10.3.1 that this case can be well studied. However, for the case where the multiple delays are incommensurate, the problem is rather involved (see Sect. 10.3.2).

10.3.1 Multiple Commensurate Delays

Consider a neutral system with commensurate delays

$$\dot{x}(t) = Ax(t) + \sum_{\ell=1}^{m} B_\ell x(t - \ell\tau) + \sum_{\ell=1}^{m} C_\ell \dot{x}(t - \ell\tau), \tag{10.8}$$

where $A \in \mathbb{R}^{r\times r}$, $B_\ell \in \mathbb{R}^{r\times r}$, and $C_\ell \in \mathbb{R}^{r\times r}$ ($\ell = 1, \ldots, m$) are constant matrices. The characteristic function of system (10.8) is

$$f_{N_C}(\lambda, \tau) = \det\left(\lambda I - A - \sum_{\ell=1}^{m} B_\ell e^{-\ell\tau\lambda} - \sum_{\ell=1}^{m} \lambda C_\ell e^{-\ell\tau\lambda}\right). \tag{10.9}$$

Since the characteristic function (10.9) involves only one parameter τ, the asymptotic behavior of the critical imaginary roots can be analyzed by the frequency-sweeping approach. We will show that the examination of the stability of the neutral operator can also be properly embedded in the frequency-sweeping approach as for the neutral system (10.1) with a single delay.

Remark 10.3 Unlike for the characteristic function $f_N(\lambda, \tau)$ (10.2) (we may generate the frequency-sweeping curves via computing the generalized eigenvalues of the matrix pencil $((j\omega I - A), (B + j\omega C))$), in general, we cannot directly obtain the frequency-sweeping curves for the characteristic function f_{N_C} in the form (10.9). An efficient way is to acquire the scalar form of $f_{N_C}(\lambda, \tau)$ (i.e., the quasipolynomial form (10.3)) and then employ the procedure introduced in Sect. 10.1.1.

The neutral operator of system (10.8) is

$$x(t) = \sum_{\ell=1}^{m} C_\ell x(t - \ell\tau). \tag{10.10}$$

A necessary condition for the exponential stability of system (10.8) is the exponential stability of the neutral operator (10.10), with the criterion given below.

Theorem 10.7 ([34]) *The neutral operator* (10.10) *is exponentially stable for any positive* τ *if and only if*

$$\rho(\widehat{C}) < 1,$$

where

$$\widehat{C} = \begin{pmatrix} C_1 & \cdots & C_{m-1} & C_m \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix}.$$

Property 10.2 *The characteristic function of* $\widehat{C}$, $\det(\lambda I - \widehat{C})$, *is equivalent to* $\det(\Delta\widehat{N})$ *where*

$$\Delta\widehat{N} = \lambda^m I - \sum_{\ell=1}^{m} \lambda^{m-\ell} C_\ell.$$

Proof For simplicity, we here consider the case $m = 3$. For the characteristic function $\det(\lambda I - \widehat{C})$, we may transform the characteristic matrix $\lambda I - \widehat{C}$ by a series of elementary transformations into the form

$$\Delta\widetilde{N} = \begin{pmatrix} \Delta\widehat{N} & -\lambda C_2 - C_3 & -C3 \\ 0 & \lambda^2 I & 0 \\ 0 & 0 & \lambda I \end{pmatrix}$$

The elementary transformations can be described by $(\lambda I - \widehat{C})S_1 S_2 = \Delta\widetilde{N}$, with

$$S_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & \lambda I & 0 \\ 0 & I & I \end{pmatrix}, \quad S_2 = \begin{pmatrix} \lambda^2 I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

It is easy to have that

$$\det((\lambda I - \widehat{C})S_1 S_2) = \det(\lambda I - \widehat{C})\det(\lambda I)\det(\lambda^2 I),$$

$$\det(\Delta\widetilde{N}) = \det(\Delta\widehat{N})\det(\lambda^2 I)\det(\lambda I).$$

Therefore, $\det(\lambda I - \widehat{C}) = \det(\Delta\widehat{N})$ as $\det((\lambda I - \widehat{C})S_1 S_2) = \det(\Delta\widetilde{N})$. One may easily extend the above analysis to the case with any $m \in \mathbb{N}_+$. □

Based on Property 10.2, we may have the following stability criterion for the neutral operator (10.10), following the idea of Theorem 10.2.

Theorem 10.8 *The neutral operator* (10.10) *is exponentially stable if and only if all the frequency-sweeping curves are above* $\Im_1$ *as* $\omega \to \infty$.

10.3.2 Multiple Incommensurate Delays

In this subsection, we will briefly discuss the case where a neutral system has multiple incommensurate delays.

Consider such a neutral system

$$\dot{x}(t) = Ax(t) + \sum_{\ell=1}^{m} B_\ell x(t - \tau_\ell) + \sum_{\ell=1}^{m} C_\ell \dot{x}(t - \tau_\ell), \tag{10.11}$$

where $A \in \mathbb{R}^{r\times r}$, $B_\ell \in \mathbb{R}^{r\times r}$, and $C_\ell \in \mathbb{R}^{r\times r}$ are constant matrices and $\tau_\ell \in \mathbb{R}_+ \cup \{0\}$ are independent delays ($\ell = 1, \ldots, m$).

Generally speaking, the complete stability for this type of time-delay systems is very complicated. One may refer to, e.g., [111] and the references therein.

First, it is impossible to accurately verify the stability of the neutral operator by existing methods in the general case.[1]

The corresponding neutral operator for system (10.11) is

$$x(t) = \sum_{\ell=1}^{m} C_\ell x(t - \tau_\ell), \tag{10.12}$$

It should be emphasized that, even if the neutral operator (10.12) is exponentially stable for some commensurate delays τ_ℓ, the stability may be fragile (the stability may be lost when sufficiently small perturbations are imposed on the delays τ_ℓ). Thus, we need to ensure the *strong stability* of the neutral operator (10.12). We have the following theorem from the literature (more details can be found in [4, 38]).

Theorem 10.9 *The neutral operator* (10.12) *is exponentially stable for arbitrary positive delays if and only if*

$$\max_{0\le\theta_\ell\le 2\pi, \ell=1,\ldots,m} \rho\left(\sum_{\ell=1}^{m} e^{j\theta_\ell} C_\ell\right) < 1 \tag{10.13}$$

However, it is difficult to check condition (10.13) in practice. To the best of the authors' knowledge, no method has been reported for precisely checking this condition when $m > 2$ (in the case $m = 2$, see [32]). An easily testable criterion was proposed in [14], which is, however, sufficient but not necessary.

If the neutral operator (10.12) is exponentially stable, we next need to study the asymptotic behavior of the critical imaginary roots with respect to multiple delay parameters τ_ℓ ($\ell = 1, \ldots, m$). Such an issue will be much more complicated than the single delay parameter case addressed in this book.

---

[1] In the specific case where the system includes two independent delays, a matrix pencil-based approach allows to check the stability of the neutral operator (see [32]).

10.4 Illustrative Examples

We now give two illustrative examples.

Example 10.3 Consider the neutral time-delay system (10.1) with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{\pi} & 1 \\ 0 & \frac{1}{3\pi} \end{pmatrix},$$

$$a_{11} = \frac{14\pi - 150\pi^2 - 1182\pi^3 + 2}{\pi^2(3\pi+1)^2} - 120,$$

$$a_{12} = \frac{5961}{5} - \frac{2422\pi/15 - 13931\pi^2/10 - 63003\pi^3/5 + 691/30}{\pi^2(3\pi+1)^2},$$

$$a_{21} = -10, a_{22} = 100,$$

$$b_{11} = \frac{14\pi - 370\pi^2 - 1842\pi^3 + 2}{\pi^2(3\pi+1)^2} - 245,$$

$$b_{12} = \frac{13149}{5} - \frac{2417\pi/15 - 19621\pi^2/5 - 100959\pi^3/5 + 23}{\pi^2(3\pi+1)^2},$$

$$b_{21} = -20, b_{22} = 215.$$

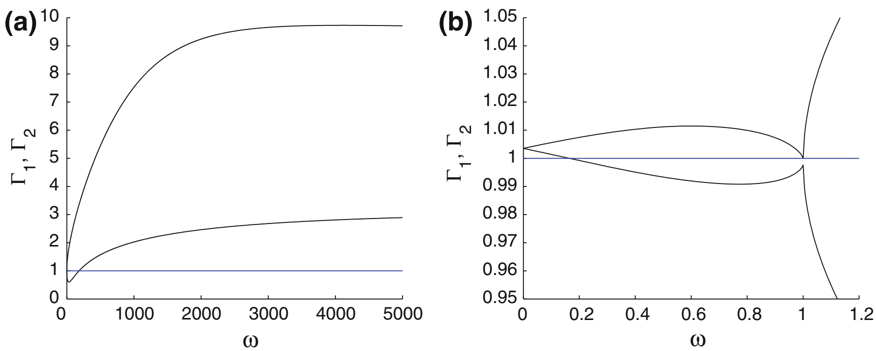

Fig. 10.2 Frequency-sweeping result for Example 10.3. **a** Frequency-sweeping curves for $0 \le \omega \le 5000$, **b** Frequency-sweeping curves for $0 \le \omega \le 1.2$

First, from the frequency-sweeping result given in Fig. 10.2a, we know that according to Theorem 10.2 the necessary condition (10.6) is satisfied. In light of the frequency-sweeping curves, we have that the system has three sets of critical pairs: ($\lambda_0 = 0.1638j, \tau_{0,k} = 19.8105 + \frac{2k\pi}{0.1638}$), ($\lambda_1 = j, \tau_{1,k} = (2k+1)\pi$), and ($\lambda_2 = 182.6684j, \tau_{2,k} = 0.0062 + \frac{2k\pi}{182.6684}$) ($\lambda_0$ and λ_1 can be observed from Fig. 10.2b and λ_2 can be observed from Fig. 10.2a).

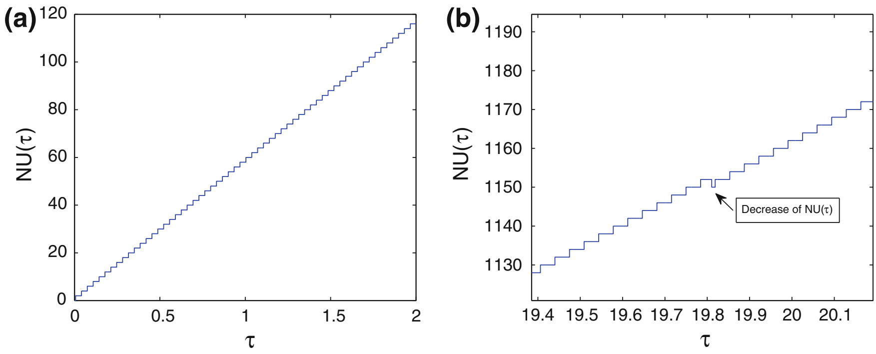

Fig. 10.3 $NU(\tau)$ versus τ for Example 10.3. **a** $NU(\tau)$ for $0 \leq \tau \leq 2$, **b** First decrease of $NU(\tau)$

Then, by Theorem 10.4, $\Delta NU_{\lambda_0}(\tau_{0,k}) = -1$, $\Delta NU_{\lambda_1}(\tau_{1,k}) = 0$, and $\Delta NU_{\lambda_2}(\tau_{2,k}) = +1$, for all $k \in \mathbb{N}$ (the asymptotic behavior for critical pairs $(\lambda_0, \tau_{0,k})$ and $(\lambda_1, \tau_{1,k})$ can be studied from the zoomed-in figure given in Fig. 10.2b).

We have the explicit expression of $NU(\tau)$ for any τ other than critical delays:

$$NU(\tau) = NU_0(\tau) + NU_2(\tau),$$

with

$$NU_0(\tau) = \begin{cases} 0, \tau < 19.8105, \\ -2\left\lceil \frac{\tau - 19.8105}{2\pi/0.1638} \right\rceil, & \tau > 19.8105, \end{cases}$$

$$NU_2(\tau) = \begin{cases} 0, \tau < 0.0062, \\ +2\left\lceil \frac{\tau - 0.0062}{2\pi/182.6684} \right\rceil, & \tau > 0.0062. \end{cases}$$

This system is exponentially stable if and only if $\tau \in [0, 0.0062)$.

We now verify the above result by invoking the Puiseux series. Here, we only analyze the critical pair (j, π) (the critical pairs $(\lambda_0, \tau_{0,k})$ and $(\lambda_2, \tau_{2,k})$ correspond to simple critical imaginary roots and can be studied by the existing methods). The critical imaginary root $\lambda = j$ is double at $\tau = \pi$. Using the approach in Chap. 4, the asymptotic behavior of (j, π) corresponds to the Puiseux series $\Delta\lambda = (0.0049 + 0.0248j)(\Delta\tau)^{\frac{1}{2}} + o((\Delta\tau)^{\frac{1}{2}})$, which is consistent with the analysis based on Theorem 10.4. As $\Delta NU_{\lambda_1}(\tau_{1,k}) = 0$ for all $k \in \mathbb{N}$, we need not explicitly take "$NU_1(\tau)$" into account in the expression of $NU(\tau)$.

Furthermore, by Theorem 10.5, $NU(\tau) \to \infty$ as $\tau \to \infty$, which is verified by the plot of $NU(\tau)$, see Fig. 10.3a. Compared to the increase of $NU(\tau)$, the decrease of $NU(\tau)$ takes place much less frequently, as shown in Fig. 10.3b, where we see that when the first decrease of $NU(\tau)$ occurs $NU(\tau)$ is already 1152. □

Example 10.4 We now consider the neutral system of Example 1.17 in [85]:

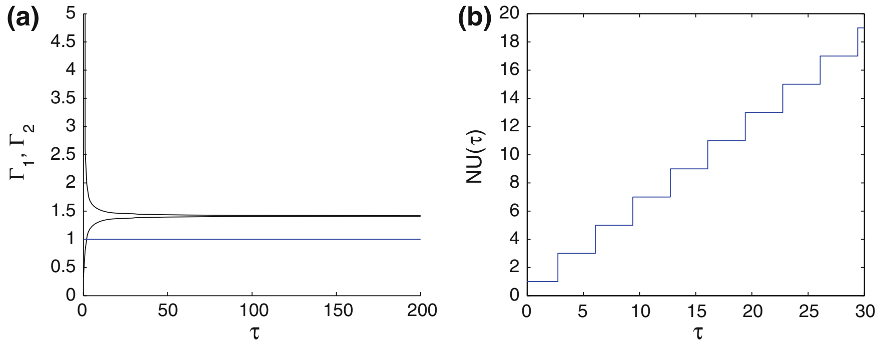

Fig. 10.4 Frequency-sweeping curves and $NU(\tau)$ for Example 10.4. **a** Frequency-sweeping result, **b** $NU(\tau)$

$$\dot{x}(t) = \frac{1}{4}x(t) + \frac{3}{4}x(t-\tau_1) + \frac{3}{4}\dot{x}(t-\tau_1) - \frac{1}{2}\dot{x}(t-\tau_2).$$

This system has multiple delays. We first consider the commensurate delays case where $\tau_1 = \tau$ and $\tau_2 = 2\tau$. According to Theorem 10.7, the neutral operator is exponentially stable for any positive delay τ, as $\rho(\widehat{C}) = 0.7071 < 1$ with

$$\widehat{C} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

This condition can be directly obtained from the frequency-sweeping curve (Fig. 10.4a) according to Theorem 10.8. Then, using the frequency-sweeping approach we may obtain the "$NU(\tau)$ versus τ" plot as shown in Fig. 10.4b.

Next, we analyze the incommensurate delays case, i.e., the case where τ_1 and τ_2 are two independent delay parameters. It is worth mentioning that, although the neutral operator $x(t) = \frac{3}{4}x(t-\tau_1) - \frac{1}{2}x(t-\tau_2)$ is exponentially stable for any positive τ in the commensurate delays case discussed above, the condition of the strong stability (Theorem 10.9) does not hold. One may easily find a counterexample, e.g., $\rho(e^{j\theta_1}\frac{3}{4} - e^{j\theta_2}\frac{1}{2}) = \frac{5}{4} > 1$ when $\theta_1 = 0$ and $\theta_2 = \pi$. □

10.5 Notes and Comments

In this chapter, we studied the complete stability of neutral systems. Compared with the retarded systems studied in Chaps. 1–9, an additional necessary condition, the stability of the neutral operator, is required. It was shown that this necessary condition can be effectively embedded in the frequency-sweeping framework proposed in this book. Thus, the complete stability for neutral systems can also be thoroughly studied by using the frequency-sweeping framework.

Main results of this chapter have been reported in [68] (see also [69]). Some complementary discussions were added in Sect. 10.3.

Chapter 11
Concluding Remarks and Further Perspectives

11.1 Concluding Remarks

It is generally difficult to study the spectrum of a linear time-delay system (infinite-dimensional system) mainly due to two reasons: (1) a time-delay has infinitely many characteristic roots and (2) a critical imaginary root of a time-delay system corresponds to infinitely many critical delays. Owing to the involved spectral features, the complete stability of linear time-delay systems has remained an open problem.

In this book, we introduced a new analytic curve perspective for addressing the spectrum of a time-delay system in both retarded and neutral cases. We may systematically study the asymptotic behavior of the critical imaginary roots as well as the frequency-sweeping curves from this new perspective. It turned out that the asymptotic behavior of the critical imaginary roots (frequency-sweeping curves) can be fully investigated through the Puiseux series (dual Puiseux series). In addition, some useful properties in the area of the singularities of analytic curves can be adopted for the stability analysis of time-delay systems.

Consequently, a new frequency-sweeping mathematical framework was gradually established allowing a deeper study of the asymptotic behavior related to time-delay systems. One of the most important results, derived within this new framework, is that we can now confirm the invariance property for general time-delay systems with commensurate delays (i.e., the general invariance property termed in this book). With the aid of the general invariance property, we have been able to solve the complete stability problem for linear time-delay systems. It is worth mentioning two interesting results: First, we find the explicit expression of the number of unstable roots with respect to the delay parameter. Second, the ultimate stability property of time-delay systems can be fully understood and all time-delay systems can be appropriately categorized. In addition, the proposed approach is applicable to both retarded and neutral time-delay systems.

It is natural that this book cannot cover all the aspects of time-delay systems. We do not claim to give the best approach and tool for the stability analysis of

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7_11

time-delay systems. Some problems still remain open due to the lack of a more in-depth understanding and more powerful mathematical tools for time-delay systems. For instance, as mentioned earlier, if a time-delay system involves multiple incommensurate delay parameters, the asymptotic behavior issue will become much more complicated and the ideas proposed here are helpful but do not allow handling directly such cases. However, combining some of the ideas proposed here with the geometric method discussed in [41] seems to open interesting perspectives in fully characterizing the stability regions in the delay-parameter space for the case of quasipolynomials including two delays.

Most of the results in this book stemmed from the analytic curve perspective. However, as we mentioned, such an analytic curve perspective is only at an elementary level in algebraic geometry. We believe that this book just opens some new perspectives for (rather than closes) the stability study of time-delay systems.

11.2 Future Perspectives

In the future, we may consider extending the approach proposed in this book to other problems, e.g., the *D*-decomposition and the multiple-delay problems mentioned in the Preface of the book.

Recently, the authors applied the methodology of this book to analyze the stability of linear systems with multiple incommensurate delays and some new results were obtained. Though a direct asymptotic behavior analysis with respect to multiple delay parameters is quite difficult, we may analyze the asymptotic behavior of a critical imaginary root with respect to one delay parameter at a time (fixing the other delay parameters). The invariance property in this case can be proved and then the number of unstable roots for any given combination of multiple delays can be calculated by applying the frequency-sweeping test in an iterative way. If for the given multiple delays the system has a critical imaginary root, we may analyze the asymptotic behavior with respect to each delay parameter such that we may find a stabilizing combination of multiple delays (if any!) near the given one.

In the sequel, we list some additional interesting options for the future work. In general, they are much more involved than the problem discussed in this book.

11.2.1 Extra Requirements on Spectra of Time-Delay Systems

Sometimes, we may have more stringent requirements on the system dynamics than the asymptotic stability. For instance, in some situation, we are concerned not only with the convergence of the system states but also with how fast they converge. The decay rate[1] is usually used as an index for measuring the convergence speed. In order

[1] For the time-delay system (1.1), if there exist two positive numbers α and β such that $\|x(t)\| \leq \beta e^{-\alpha t}(\max_{t-m\tau \leq t \leq 0} \|x(t)\|)$, α is called the decay rate.

to guarantee a desired decay rate $\alpha \in \mathbb{R}_+$, we need to make sure that all the real parts of the characteristic roots are less than $-\alpha$, see Theorem 6.2 in [45].

Such a problem appears similar to the one studied in this book as the only difference lies in the "boundaries" we choose: In this book, the boundary is the imaginary axis in the complex plane, while for the decay rate analysis the boundary is the vertical line with abscissa $-\alpha$ in the complex plane. However, for the latter, many nice spectral properties will be lost due to this difference (for instance, the periodicity discussed in Remark 1.6 is not satisfied), making the analysis in fact much more involved.

11.2.2 Design Problem

In this book, we have mainly considered the "analysis" problem for time-delay systems with commensurate delays. However, for some simple cases (see Examples 6.3 and 6.4: controlling an oscillator and a chain of integrators by using delay blocks), we have pointed out that the proposed methodology allows handling the stability analysis of the closed-loop systems. In this context, it is natural to consider if the results proposed in this book can be extended to the "design" problem. For instance, consider a linear system $\dot{x}(t) = Ax(t) + Bu(t - \tau)$ with an input delay τ, where the control signal $u(t)$ is a widely-used state feedback $u(t) = Kx(t)$ (K is the controller matrix to be designed). The task is to design K such that the stability domain of τ for the closed-loop system is as large as possible.

Such a design problem is much more complicated than the analysis problem discussed in this book. For given K and τ, the system has infinitely many roots and there does not exist an explicit relation between K and the corresponding stability domain of τ. Thus, the adjustment of K is generally computationally demanding [85]. In our opinion, this is a very challenging topic.

11.2.3 Other Types of Time-Delay Systems

The time-delay systems considered in this book belong to the classical type of functional differential equations. The reported results may be applicable to other types of time-delay systems, e.g., fractional-order time-delay systems (one may refer to [26] for a detailed introduction). In this case, it appears clearly that the Puiseux series still works and allows concluding on the asymptotic behavior.

As other types of time-delay systems are generally infinite dimensional as well and the corresponding stability problems need to be recast into the asymptotic behavior analysis of the characteristic roots located on the stability boundaries. We believe that the ideas proposed in this book would also be helpful.

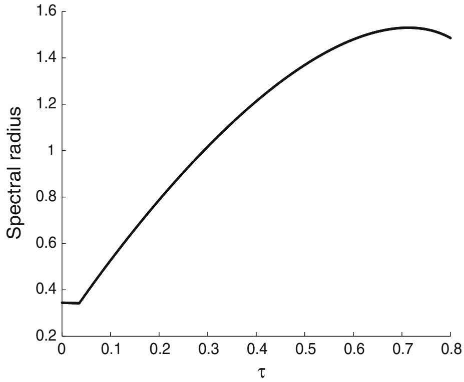

Fig. 11.1 Delay-sweeping result

11.2.4 Applying Parameter-Sweeping Techniques to Other Problems

A further effort is also suggested here concerning the graphical tool used in this book. The "frequency-sweeping" test is a class of *parameter-sweeping techniques*. Parameter-sweeping techniques have been widely used in the control area, owing to the more and more powerful computer-processing power. For instance, parameter-sweeping criteria have been used for studying the stability and related problems for networked-control systems (NCSs), see [16, 65], where the authors sweep the sampling period and/or the network-induced delay parameters. Here, we borrow a *delay-sweeping* test for Example 10.2 of [16], as shown in Fig. 11.1 (sweeping the networked-induced delay τ to monitor the spectral radius of the transition matrix of the NCS). For that example, the NCS is asymptotically stable if and only if the spectral radius is less than 1. In this way, the stability domain of the network-induced delay can be precisely obtained.

However, such an application is relatively elementary as no deeper analysis concerning the parameter-sweeping curve has been reported. In our opinion, there is still much room for further improvement regarding the parameter-sweeping techniques.

Appendix A
Implicit Function Theorem

In the sequel we briefly recall an important (classical) theorem, the implicit function theorem. Although this theorem has some limitations, it is very helpful for understanding the spectral characteristics related to the stability problem considered in this book, especially when one variable can be regarded as a simple root.

A.1 Implicit Function Theorem and Related Remarks

The implicit function theorem has multiple versions. In this section, we recall a simple one.

Theorem A.1 ([79]) *Let $F(z, w)$ be a function of two complex variables which is analytic in a neighborhood of the point (z_0, w_0), and suppose that*

$$F(z_0, w_0) = 0 \text{ and } F_w \neq 0 \text{ at } (z_0, w_0).$$

Then there are neighborhoods $N(z_0)$ and $N(w_0)$ such that the equation $F(z, w) = 0$ has a unique root $w = w(z)$ in $N(w_0)$ for any given $z \in N(z_0)$. Moreover, the function $w(z)$ is single-valued and analytic in $N(z_0)$, and satisfies the condition $w(z_0) = w_0$.

Furthermore, according to Theorem A.1, the derivatives of w with respect to z are well-defined. For instance, $\frac{dw}{dz}$ and $\frac{d^2w}{dz^2}$ can be calculated as follows:

$$\frac{dw}{dz} = -\frac{F_z}{F_w},$$

$$\frac{d^2w}{dz^2} = \frac{\partial}{\partial z}\left(-\frac{F_z}{F_w}\right) + \frac{\partial}{\partial w}\left(-\frac{F_z}{F_w}\right)\frac{dw}{dz} = -\frac{F_{zz}F_w^2 - 2F_{zw}F_zF_w + F_{ww}F_z^2}{F_w^3}.$$

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7

As mentioned, Theorem A.1 is a simple version of the implicit function theorem. This version fits well with the methodology in the current book.

When the number of the variables increases (besides, the number of the equations may also increase), one may adopt a more advanced version of the implicit function theorem, see, e.g., Theorem 1.4.11 (the holomorphic implicit function theorem) in [61].

For a more comprehensive introduction to the implicit function theorem, we recommend the book [62]. As pointed out in [62], the implicit function theorem should be better understood as an *ansatz*, which is a way of looking at various problems.

A.2 Application of Implicit Function Theorem

As discussed above, if $F_w \neq 0$ at (z_0, w_0), $w(z)$ represents a (single) root locus near (z_0, w_0). Furthermore, the local behavior of $w(z)$ can be reflected by the derivatives of w with respect to z.

Remark A.1 Conversely, we may also consider how z varies with respect to w. Namely, if $F_z \neq 0$ at (z_0, w_0), $z(w)$ denotes the corresponding (single) root locus near (z_0, w_0).

In summary, based on the implicit function theorem, we may analyze the local behavior of a simple root-path. As often mentioned, the implicit function theorem can not be used to address a multiple root.

In the case with multiple roots, we need to employ the Puiseux' theorem as discussed in Chap. 2, to study the asymptotic behavior from the analytic curve perspective. In our opinion, in analyzing the asymptotic behavior for the equation $F(z, w) = 0$ ($F(z, w)$ is an analytic function), Puiseux' theorem covers the implicit function theorem.

Appendix B
Proof of Theorem 8.3 (One Conjugacy Class)

When a critical imaginary root has only one Puiseux series (i.e., $v = 1$), the dual Puiseux series pair is given by (8.3) and (8.4). For the first-order coefficients, it is true that $D_n = (\frac{1}{C_g})^{\frac{n}{g}}$ [66]. For higher-order coefficients, we have the following relation.

Property B.1 *For any integer $h \geq 1$, it follows that*

$$D_{n+h} = \left(\sum_{i_1+2i_2+\cdots+hi_h=h} \alpha_{i_1,\ldots,i_h} \prod_{\mathrm{w}=1}^{h} \gamma_{h,\mathrm{w}}^{i_\mathrm{w}} \right) \left(\frac{1}{C_g} \right)^{\frac{n+h}{g}},$$

where $\alpha_{i_1,\ldots,i_h}$ ($i_1, \ldots, i_h$ are non-negative integers) are real coefficients, $\gamma_{h,\mathrm{w}} = 1$ if $C_{g+\mathrm{w}} = 0$ and $i_\mathrm{w} = 0$, $\gamma_{h,\mathrm{w}} = \frac{C_{g+\mathrm{w}}}{C_g}$ otherwise.

Proof Substituting (8.3) into (8.4), we have that

$$\Delta\tau = \sum_{h=0}^{\infty} D_{n+h} C_g^{\frac{n+h}{g}} (\Delta\tau)^{\frac{n+h}{n}} \left(\sum_{i=0}^{\infty} \frac{C_{g+i}}{C_g} (\Delta\tau)^{\frac{i}{n}} \right)^{\frac{n+h}{g}}. \tag{B.1}$$

By the binomial theorem (pp. 90 in [2]),

$$\left(\sum_{i=0}^{\infty} \frac{C_{g+i}}{C_g} (\Delta\tau)^{\frac{i}{n}} \right)^{\frac{n+h}{g}} = 1 + \sum_{k=1}^{\infty} \left(\frac{(\frac{n+h}{g}) \cdots (\frac{n+h}{g} - k + 1)}{k!} \left(\sum_{i=1}^{\infty} \frac{C_{g+i}}{C_g} (\Delta\tau)^{\frac{i}{n}} \right)^k \right).$$

Thus, the right-hand side of (B.1) is of the form $\sum_{h=0}^{\infty} \chi_h (\Delta\tau)^{\frac{n+h}{n}}$, where $\chi_0 = 1$ and $\chi_h = 0$ for all $h \geq 1$. We have that, for any $h \geq 1$, χ_h is of the form $\sum_{\mathrm{v}=0}^{h-1} D_{n+\mathrm{v}} C_g^{\frac{n+\mathrm{v}}{g}} \left(\sum_{i_1+\cdots+(h-\mathrm{v})i_{h-\mathrm{v}}=h-\mathrm{v}} \beta_{i_1,\ldots,i_{h-\mathrm{v}}}^{(\mathrm{v})} \prod_{\mathrm{w}=1}^{h-\mathrm{v}} \gamma_{h-\mathrm{v},\mathrm{w}}^{i_\mathrm{w}} \right) + D_{n+h} C_g^{\frac{n+h}{g}}$, where

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7

$\beta^{(\mathrm{v})}_{i_1,\ldots,i_{h-\mathrm{v}}}$ are real coefficients. First, D_n is known from $\chi_0 = 1$. It can be seen that after determining $D_n, \ldots, D_{n+h-1}$, we can proceed to determine D_{n+h} from $\chi_h = 0$, which can be viewed as a linear equation with the variable D_{n+h} ($D_n, \ldots, D_{n+h-1}$ are already known). In this way, we obtain the general forms of D_{n+h} by induction as stated by the property. □

For $\Delta\tau = +\varepsilon$ $(-\varepsilon)$, the n principal arguments of $(\Delta\tau)^{\frac{1}{n}}$ are denoted by a set $\Theta^{(+)}$ $(\Theta^{(-)})$ with elements $\theta_i^{(+)}$ $(\theta_i^{(-)})$, $i = 1, \ldots, n$. For $\Delta\lambda = +\varepsilon j$ $(-\varepsilon j)$, the g principal arguments of $(\frac{\Delta\lambda}{C_g})^{\frac{1}{g}}$ are denoted by a set $\Psi^{(+)}$ $(\Psi^{(-)})$ with elements $\psi_i^{(+)}$ $(\psi_i^{(-)})$, $i = 1, \ldots, g$.

Without any loss of generality, suppose that the greatest common factor of n and g is $\eta \in \mathbb{N}_+$, i.e., $\frac{g}{n} = \frac{\eta\widetilde{g}}{\eta\widetilde{n}} = \frac{\widetilde{g}}{\widetilde{n}}$, where $\widetilde{n} \in \mathbb{N}_+$ and $\widetilde{g} \in \mathbb{N}_+$ are co-prime. Hence, there are three possible cases (Case 1: $\widetilde{n}$ is odd and $\widetilde{g}$ is odd, Case 2: $\widetilde{n}$ is odd and $\widetilde{g}$ is even, and Case 3: $\widetilde{n}$ is even and $\widetilde{g}$ is odd).

If the first term $C_g(\Delta\tau)^{\frac{g}{n}}$ when $\Delta\tau = \pm\varepsilon$ of the Puiseux series (8.3) contains purely imaginary values, we need to further consider higher-order terms until we can conclude on the value of $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$. This corresponds to the so-called *degenerate case*. The higher-order terms $C_{g+1}(\Delta\tau)^{\frac{g+1}{n}}, \ldots$ are also called degenerate if the corresponding branches of these terms still contain purely imaginary values plus 0. Similarly, if the first term $D_n(\Delta\lambda)^{\frac{n}{g}}$ when $\Delta\lambda = \pm\varepsilon j$ of the dual Puiseux series (8.4) contains purely real values, we need to further consider the higher-order terms until we can conclude on the value of $\Delta NF_{z_\alpha}(\omega_\alpha)$ according to Property 8.2. This also corresponds to the so-called *degenerate case*. The higher-order terms $D_{n+1}(\Delta\lambda)^{\frac{n+1}{g}}, \ldots$ are also called degenerate if the corresponding branches of these terms still contain purely real values.

It is necessary to understand the condition causing a degenerate case (i.e., the degeneracy condition). We will see (by Lemmas B.2, B.3, B.5, B.6, B.8, and B.9 given later) that a degenerate Puiseux series (8.3) must be concurrent with a degenerate dual Puiseux series (8.4) under the degeneracy condition: When $\widetilde{n}$ is odd (even), the degenerate case occurs if and only if $\mathrm{Re}(C_g^{\widetilde{n}}) = 0$ $(\mathrm{Im}(C_g^{\widetilde{n}}) = 0)$.

We first consider the non-degenerate case, for which $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is as follows. For Case 1: (1) when $\widetilde{n} \bmod 4 = 1$, if $\mathrm{Re}(C_g^{\widetilde{n}}) > 0$ (< 0), $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = \eta$ $(-\eta)$; (2) when $\widetilde{n} \bmod 4 = 3$, if $\mathrm{Re}(C_g^{\widetilde{n}}) > 0$ (< 0), $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = -\eta$ (η). For Case 2 and Case 3: $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = 0$.

Regarding the value of $\Delta NF_{z_\alpha}(\omega_\alpha)$, we can precisely calculate it via studying the dual Puiseux series (8.4), according to Property 8.2. As $D_n(\Delta\lambda)^{\frac{n}{g}} = (\frac{\Delta\lambda}{C_g})^{\frac{n}{g}} = (\frac{\Delta\lambda}{C_g})^{\frac{\widetilde{n}}{\widetilde{g}}}$ and we are interested in the case where $\Delta\lambda = \pm\varepsilon j$, we have that

$$\left(\frac{\Delta\lambda}{C_g}\right)^{\frac{\widetilde{n}}{\widetilde{g}}} = \left(\frac{(\pm\varepsilon)^{\widetilde{n}} j^{\widetilde{n}} (\mathrm{Re}(C_g^{\widetilde{n}}) - \mathrm{Im}(C_g^{\widetilde{n}}) j)}{\left|C_g^{\widetilde{n}}\right|^2}\right)^{\frac{1}{\widetilde{g}}}. \tag{B.2}$$

By Property 7.2, we have the results for the non-degenerate case. For Case 1: (1) when $\widetilde{n}$ mod $4 = 1$, if $\mathrm{Re}(C_g^{\widetilde{n}}) > 0$ (< 0), $\Delta NF_{z_\alpha}(\omega_\alpha) = \eta$ ($-\eta$); (2) when $\widetilde{n}$ mod $4 = 3$, if $\mathrm{Re}(C_g^{\widetilde{n}}) > 0$ (< 0), $\Delta NF_{z_\alpha}(\omega_\alpha) = -\eta$ (η). For Case 2 and Case 3: $\Delta NF_{z_\alpha}(\omega_\alpha) = 0$.

From the above analysis, we obtain the following result.

Lemma B.1 *Theorem 8.3 holds for the non-degenerate case.*

In the forthcoming sections, we suppose that the degenerate cases occur and we will explicitly address Cases 1, 2, and 3. The following notations will be adopted in the sequel. For the elements in $\Theta^{(+)}$ ($\Theta^{(-)}$) causing a degenerate case, we denote them by a set $\Theta_{\mathbb{D}}^{(+)}$ ($\Theta_{\mathbb{D}}^{(-)}$). For the elements in $\Psi^{(+)}$ ($\Psi^{(-)}$) causing a degenerate case, we denote them by a set $\Psi_{\mathbb{D}}^{(+)}$ ($\Psi_{\mathbb{D}}^{(-)}$).

B.1 Case 1 ($\widetilde{n}$ Odd and $\widetilde{g}$ Odd)

According to the degeneracy condition, $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$ or $\mathrm{Arg}(C_g) = -\frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$, $k_1 \in \mathbb{Z}$. In this section, we let $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$ (the proof when $\mathrm{Arg}(C_g) = -\frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$ can be completed in the same manner). For an odd $\widetilde{n}$, it has two possibilities: $\widetilde{n}$ mod $4 = 1$ and $\widetilde{n}$ mod $4 = 3$. In this section, we let $\widetilde{n}$ mod $4 = 1$ (the proof when $\widetilde{n}$ mod $4 = 3$ can be completed in the same spirit). To avoid confusion, we emphasize that the results to be given below are developed in the case where $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_1\pi}{\widetilde{n}}$ and $\widetilde{n}$ mod $4 = 1$. For the other possibilities, we may prove the result analogously.

By analyzing the arguments of $C_g(\Delta\tau)^{\frac{g}{n}}$, we have that $\Theta_{\mathbb{D}}^{(+)}$ has η elements such that

$$g\theta_i^{(+)} + \mathrm{Arg}(C_g) = \frac{\pi}{2}(\mathrm{mod}(2\pi)), \tag{B.3}$$

and that $\Theta_{\mathbb{D}}^{(-)}$ has η elements such that

$$g\theta_i^{(-)} + \mathrm{Arg}(C_g) = -\frac{\pi}{2}(\mathrm{\,mod\,}(2\pi)). \tag{B.4}$$

In light of (B.2), $\Psi_{\mathbb{D}}^{(+)}$ contains η elements satisfying that

$$n\psi_i^{(+)} = 0(\mathrm{mod}(2\pi)), \tag{B.5}$$

and $\Psi_{\mathbb{D}}^{(-)}$ contains η elements satisfying that

$$n\psi_i^{(-)} = \pi(\mathrm{mod}(2\pi)). \tag{B.6}$$

With proper relabeling, we let $\Theta_{\mathbb{D}}^{(+)} = \{\theta_1^{(+)}, \ldots, \theta_\eta^{(+)}\}$, $\Theta_{\mathbb{D}}^{(-)} = \{\theta_1^{(-)}, \ldots, \theta_\eta^{(-)}\}$, $\Psi_{\mathbb{D}}^{(+)} = \{\psi_1^{(+)}, \ldots, \psi_\eta^{(+)}\}$, and $\Psi_{\mathbb{D}}^{(-)} = \{\psi_1^{(-)}, \ldots, \psi_\eta^{(-)}\}$.

Property B.2 *For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}}^{(+)}$, $1 \le a \le \eta$, there exists a unique $\psi_{c(a)}^{(+)} \in \Psi_{\mathbb{D}}^{(+)}$, $1 \le c(a) \le \eta$, such that $\psi_{c(a)}^{(+)} = \theta_a^{(+)}$. Furthermore, for different $\theta_a^{(+)}$, the corresponding $c(a)$ are different.*

Proof From (B.3) and (B.5), $\eta\theta_1^{(+)}, \ldots, \eta\theta_\eta^{(+)}$ correspond to a same principal argument (denoted by $\overline{\theta}^{(+)}$) and $\eta\psi_1^{(+)}, \ldots, \eta\psi_\eta^{(+)}$ correspond to a same principal argument (denoted by $\overline{\psi}^{(+)}$). The proof will be complete if $\overline{\psi}^{(+)} = \overline{\theta}^{(+)}$.

Since $\overline{\psi}^{(+)}$ corresponds to a principal argument of $(\frac{+\varepsilon j}{C_g})^{\frac{1}{\widetilde{g}}}$, it follows that

$$\overline{\psi}^{(+)} = \frac{1}{\widetilde{g}}(\frac{\pi}{2} + 2s_1\pi - \mathrm{Arg}(C_g)), \tag{B.7}$$

where $s_1 \in \mathbb{Z}$. From (B.3), we have that $\mathrm{Arg}(C_g) = -\widetilde{g}\overline{\theta}^{(+)} + \frac{\pi}{2} + 2s_2\pi$, $s_2 \in \mathbb{Z}$. Then, from (B.7),

$$\overline{\psi}^{(+)} = \overline{\theta}^{(+)} + \frac{(s_1 - s_2)2\pi}{\widetilde{g}}. \tag{B.8}$$

Note that $\overline{\theta}^{(+)} = \frac{2s_3\pi}{\widetilde{n}}$ with $s_3 \in \mathbb{Z}$ (corresponding to a principal argument of $(+\varepsilon)^{\frac{1}{\widetilde{n}}}$) and that $\overline{\psi}^{(+)} = \frac{2s_4\pi}{\widetilde{n}}$ with $s_4 \in \mathbb{Z}$ (by (B.5)). According to (B.8), it must be true that $s_1 - s_2 = 0$, as $\widetilde{n}$ and $\widetilde{g}$ are co-prime. Thus, $\overline{\psi}^{(+)} = \overline{\theta}^{(+)}$. □

Similarly, from (B.4) and (B.6), we have:

Property B.3 *For each $\theta_b^{(-)} \in \Theta_{\mathbb{D}}^{(-)}$, $1 \le b \le \eta$, there exists a unique $\psi_{d(b)}^{(-)} \in \Psi_{\mathbb{D}}^{(-)}$, $1 \le d(b) \le \eta$, such that $\psi_{d(b)}^{(-)} = \theta_b^{(-)}$. Furthermore, for different $\theta_b^{(-)}$, the corresponding $d(b)$ are different.*

Lemma B.2 *For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}}^{(+)}$, $1 \le a \le \eta$, satisfying that the corresponding branches of $C_g(+\varepsilon)^{\frac{g}{n}}, \ldots, C_{g+M(a)-1}(+\varepsilon)^{\frac{g+M(a)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ is not degenerate, there exists a unique $c(a)$, $1 \le c(a) \le \eta$, satisfying the following properties.*

(1) For $\psi_{c(a)}^{(+)}$, the corresponding branches of $D_n(+\varepsilon j)^{\frac{n}{g}}, \ldots, D_{n+M(a)-1}(+\varepsilon j)^{\frac{n+M(a)-1}{g}}$ are all degenerate.

(2) For $\theta_a^{(+)}$ and $\psi_{c(a)}^{(+)}$, the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) if and only if the corresponding branch of $D_{n+M(a)}(+\varepsilon j)^{\frac{n+M(a)}{g}}$ lies in $\mathbb{C}_U$ ($\mathbb{C}_L$).

Proof We start by proving property (1) for $h = 1, \ldots, M(a) - 1$ (the degeneracy of $D_n(+\varepsilon j)^{\frac{n}{g}}$ is apparent from (B.2)). First, assume that $C_{g+h} \neq 0$ for $h = 1, \ldots, M(a) - 1$. According to the conditions of the lemma, $((g+h)\theta_a^{(+)} + \mathrm{Arg}(C_{g+h})) \bmod 2\pi = \pm\frac{\pi}{2}$, $h = 0, \ldots, M(a) - 1$. It follows that for $h = 1, \ldots, M(a) - 1$,

$$(\mathrm{Arg}(C_{g+h}) - \mathrm{Arg}(C_g) + h\theta_a^{(+)}) \bmod 2\pi = 0 \text{ or } \pi. \tag{B.9}$$

In light of Property B.1, for $h = 1, \ldots, M(a) - 1$, $D_{n+h}(+\varepsilon j)^{\frac{n+h}{g}}$ is the sum of a finite number of terms subject to the form $\alpha_{i_1,\ldots,i_h} \prod_{w=1}^{h} \gamma_{h,w}^{i_w} (\frac{+\varepsilon j}{C_g})^{\frac{n+h}{g}}$. For each such term, the argument corresponding to $\psi_{c(a)}^{(+)}$ is (by (B.9)) $(n+h)\psi_{c(a)}^{(+)} - h\theta_a^{(+)} + \kappa\pi$, $\kappa \in \mathbb{Z}$, having in mind that $i_1 + \cdots + hi_h = h$ and $\alpha_{i_1,\ldots,i_h} \in \mathbb{R}$. Then, according to Property B.2 and (B.5), for $h = 1, \ldots, M(a) - 1$, the value of $D_{n+h}(+\varepsilon j)^{\frac{n+h}{g}}$ associated with $\psi_{c(a)}^{(+)}$ is purely real. We may also prove property (1) in the same spirit if some $C_{g+h} = 0$ $(1 \le h \le M(a) - 1)$ by noting that 0 corresponds to a degenerate term for both the Puiseux series and the dual Puiseux series.

We next consider property (2). Clearly, $C_{g+M(a)} \neq 0$ by the conditions of the lemma. It follows from Property B.1 that $D_{n+M(a)} = (D' + D'')(\frac{1}{C_g})^{\frac{n+M(a)}{g}}$ where $D' = \sum_{i_1+\cdots+M(a)i_{M(a)}=M(a)} \alpha_{i_1,\ldots,i_{M(a)}} \prod_{w=1}^{M(a)} \gamma_{M(a),w}^{i_w}$ $(i_{M(a)} = 0)$ and $D'' = \alpha_{0,\ldots,0,1}\frac{C_{g+M(a)}}{C_g}$ $(\alpha_{0,\ldots,0,1} = -\frac{n}{g})$. By the same idea of the proof for property (1), we have that the value of $D'(\frac{+\varepsilon j}{C_g})^{\frac{n+M(a)}{g}}$ associated with $\psi_{c(a)}^{(+)}$ is purely real. Consequently, property (2) can be proved if the following condition holds

$$\mathrm{Arg}\left(D''(\frac{+\varepsilon j}{C_g})^{\frac{n+M(a)}{g}}\right) - \mathrm{Arg}\left(C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}\right) = \frac{\pi}{2}. \tag{B.10}$$

Taking into account (B.3), (B.5), and Property B.2, we see that (B.10) is satisfied and hence the proof is complete. □

Lemma B.3 *For each $\theta_b^{(-)} \in \Theta_{\mathbb{D}}^{(-)}$, $1 \le b \le \eta$, satisfying that the corresponding branches of $C_g(-\varepsilon)^{\frac{g}{n}}, \ldots, C_{g+M(b)-1}(-\varepsilon)^{\frac{g+M(b)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(b)}(-\varepsilon)^{\frac{g+M(b)}{n}}$ is not degenerate, there exists a unique $d(b)$, $1 \le d(b) \le \eta$, satisfying the following properties.*

(1) For $\psi_{d(b)}^{(-)}$, the corresponding branches of $D_n(-\varepsilon j)^{\frac{n}{g}}, \ldots, D_{n+M(b)-1}(-\varepsilon j)^{\frac{n+M(b)-1}{g}}$ are all degenerate.

(2) For $\theta_b^{(-)}$ and $\psi_{d(b)}^{(-)}$, the corresponding branch of $C_{g+M(b)}(-\varepsilon)^{\frac{g+M(b)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) if and only if the corresponding branch of $D_{n+M(b)}(-\varepsilon j)^{\frac{n+M(b)}{g}}$ lies in $\mathbb{C}_U$ ($\mathbb{C}_L$).

The proof is in the same spirit of that of Lemma B.2.

Combining the above discussions, we have:

Lemma B.4 *Theorem 8.3 holds if $\widetilde{n}$ is odd and $\widetilde{g}$ is odd.*

Proof For the elements in $\Theta^{(+)} - \Theta_{\mathbb{D}}^{(+)}$ ($\Theta^{(-)} - \Theta_{\mathbb{D}}^{(-)}$), the number of the corresponding values of the Puiseux series (8.3) in $\mathbb{C}_+$ is denoted by $NU_{\Theta^{(+)}-\Theta_{\mathbb{D}}^{(+)}}(+\varepsilon)$ ($NU_{\Theta^{(-)}-\Theta_{\mathbb{D}}^{(-)}}(-\varepsilon)$). Similarly, for the elements in $\Psi^{(+)} - \Psi_{\mathbb{D}}^{(+)}$ ($\Psi^{(-)} - \Psi_{\mathbb{D}}^{(-)}$), the number of the corresponding values of the dual Puiseux series (8.4) in $\mathbb{C}_U$ is denoted by $ND_{\Psi^{(+)}-\Psi_{\mathbb{D}}^{(+)}}(+\varepsilon j)$ ($ND_{\Psi^{(-)}-\Psi_{\mathbb{D}}^{(-)}}(-\varepsilon j)$). It follows that $NU_{\Theta^{(+)}-\Theta_{\mathbb{D}}^{(+)}}(+\varepsilon) = NU_{\Theta^{(-)}-\Theta_{\mathbb{D}}^{(-)}}(-\varepsilon)$ and $ND_{\Psi^{(+)}-\Psi_{\mathbb{D}}^{(+)}}(+\varepsilon j) = ND_{\Psi^{(-)}-\Psi_{\mathbb{D}}^{(-)}}(-\varepsilon j)$ for Case 1. Thus, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is determined by the effect of the elements in $\Theta_{\mathbb{D}}^{(+)}$ and $\Theta_{\mathbb{D}}^{(-)}$ on the Puiseux series (8.3) and $\Delta NF_{z_\alpha}(\omega_\alpha)$ is determined by the effect of the elements in $\Psi_{\mathbb{D}}^{(+)}$ and $\Psi_{\mathbb{D}}^{(-)}$ on the dual Puiseux series (8.4). According to Lemmas B.2 and B.3, the above two effects are equivalent. □

B.2 Case 2 ($\widetilde{n}$ Odd and $\widetilde{g}$ Even)

According to the degeneracy condition, $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_2\pi}{\widetilde{n}}$ or $\mathrm{Arg}(C_g) = -\frac{\pi}{2\widetilde{n}} + \frac{2k_2\pi}{\widetilde{n}}$, $k_2 \in \mathbb{Z}$. For an odd $\widetilde{n}$, there are two possibilities: $\widetilde{n} \bmod 4 = 1$ and $\widetilde{n} \bmod 4 = 3$. In this section, we suppose $\mathrm{Arg}(C_g) = \frac{\pi}{2\widetilde{n}} + \frac{2k_2\pi}{\widetilde{n}}$ and $\widetilde{n} \bmod 4 = 1$.

By analyzing the arguments of $C_g(\Delta\tau)^{\frac{g}{n}}$, we have that $\Theta_{\mathbb{D}}^{(+)}$ has η elements satisfying (B.3) and that $\Theta_{\mathbb{D}}^{(-)}$ has η elements satisfying that

$$g\theta_i^{(-)} + \mathrm{Arg}(C_g) = \frac{\pi}{2}(\bmod\,(2\pi)). \tag{B.11}$$

By (B.2), $\Psi_{\mathbb{D}}^{(+)}$ contains two subsets ($\Psi_{\mathbb{D}_1}^{(+)}$ and $\Psi_{\mathbb{D}_2}^{(+)}$) and $\Psi_{\mathbb{D}}^{(-)}$ is empty. More precisely, $\Psi_{\mathbb{D}_1}^{(+)}$ has η elements satisfying (B.5) and $\Psi_{\mathbb{D}_2}^{(+)}$ has η elements satisfying that

$$n\psi_i^{(+)} = \pi(\mathrm{mod}(2\pi)). \tag{B.12}$$

After proper relabeling, we let $\Theta_{\mathbb{D}}^{(+)} = \{\theta_1^{(+)}, \ldots, \theta_\eta^{(+)}\}$, $\Theta_{\mathbb{D}}^{(-)} = \{\theta_1^{(-)}, \ldots, \theta_\eta^{(-)}\}$, and $\Psi_{\mathbb{D}}^{(+)} = \{\psi_1^{(+)}, \ldots, \psi_{2\eta}^{(+)}\} = \Psi_{\mathbb{D}_1}^{(+)} \cup \Psi_{\mathbb{D}_2}^{(+)}$ where $\Psi_{\mathbb{D}_1}^{(+)} = \{\psi_1^{(+)}, \ldots, \psi_\eta^{(+)}\}$ and $\Psi_{\mathbb{D}_2}^{(+)} = \{\psi_{\eta+1}^{(+)}, \ldots, \psi_{2\eta}^{(+)}\}$.

Property B.4 *For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}}^{(+)}$, $1 \le a \le \eta$, there exists a unique $\psi_{c(a)}^{(+)} \in \Psi_{\mathbb{D}_1}^{(+)}$, $1 \le c(a) \le \eta$, such that $\psi_{c(a)}^{(+)} = \theta_a^{(+)}$. Furthermore, for different $\theta_a^{(+)}$, the corresponding $c(a)$ are different.*

Property B.5 *For each $\theta_b^{(-)} \in \Theta_{\mathbb{D}}^{(-)}$, $1 \leq b \leq \eta$, there exists a unique $\psi_{d(b)}^{(+)} \in \Psi_{\mathbb{D}_2}^{(+)}$, $\eta + 1 \leq d(b) \leq 2\eta$, such that $\psi_{d(b)}^{(+)} = \theta_b^{(-)}$. Furthermore, for different $\theta_b^{(-)}$, the corresponding $d(b)$ are different.*

Lemma B.5 *For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}}^{(+)}$, $1 \leq a \leq \eta$, satisfying that the corresponding branches of $C_g(+\varepsilon)^{\frac{g}{n}}, \ldots, C_{g+M(a)-1}(+\varepsilon)^{\frac{g+M(a)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ is not degenerate, there exists a unique $c(a)$, $1 \leq c(a) \leq \eta$, satisfying the following properties.*
(1) For $\psi_{c(a)}^{(+)}$, the corresponding branches of $D_n(+\varepsilon j)^{\frac{n}{g}}, \ldots, D_{n+M(a)-1}(+\varepsilon j)^{\frac{n+M(a)-1}{g}}$ are all degenerate.
(2) For $\theta_a^{(+)}$ and $\psi_{c(a)}^{(+)}$, the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) if and only if the corresponding branch of $D_{n+M(a)}(+\varepsilon j)^{\frac{n+M(a)}{g}}$ lies in $\mathbb{C}_U$ ($\mathbb{C}_L$).

Lemma B.6 *For each $\theta_b^{(-)} \in \Theta_{\mathbb{D}}^{(-)}$, $1 \leq b \leq \eta$, satisfying that the corresponding branches of $C_g(-\varepsilon)^{\frac{g}{n}}, \ldots, C_{g+M(b)-1}(-\varepsilon)^{\frac{g+M(b)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(b)}(-\varepsilon)^{\frac{g+M(b)}{n}}$ is not degenerate, there exists a unique $d(b)$, $\eta + 1 \leq d(b) \leq 2\eta$, satisfying the following properties.*
(1) For $\psi_{d(b)}^{(+)}$, the corresponding branches of $D_n(+\varepsilon j)^{\frac{n}{g}}, \ldots, D_{n+M(b)-1}(+\varepsilon j)^{\frac{n+M(b)-1}{g}}$ are all degenerate.
(2) For $\theta_b^{(-)}$ and $\psi_{d(b)}^{(+)}$, the corresponding branch of $C_{g+M(b)}(-\varepsilon)^{\frac{g+M(b)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) if and only if the corresponding branch of $D_{n+M(b)}(+\varepsilon j)^{\frac{n+M(b)}{g}}$ lies in $\mathbb{C}_L$ ($\mathbb{C}_U$).

Combining Properties B.4 and B.5 and Lemmas B.5 and B.6, we have:

Lemma B.7 *Theorem 8.3 holds if $\widetilde{n}$ is odd and $\widetilde{g}$ is even.*

B.3 Case 3 ($\widetilde{n}$ Even and $\widetilde{g}$ Odd)

According to the degeneracy condition, $\text{Arg}(C_g) = \frac{2k_3\pi}{\tilde{n}}$ or $\text{Arg}(C_g) = \frac{\pi}{\tilde{n}} + \frac{2k_3\pi}{\tilde{n}}$, $k_3 \in \mathbb{Z}$. For an even $\widetilde{n}$, there are two possibilities: $\widetilde{n} \bmod 4 = 0$ and $\widetilde{n} \bmod 4 = 2$. In this section, we assume $\text{Arg}(C_g) = \frac{2k_3\pi}{\tilde{n}}$ and $\widetilde{n} \bmod 4 = 0$.

In light of the arguments of $C_g(\Delta\tau)^{\frac{g}{n}}$, $\Theta_{\mathbb{D}}^{(+)}$ has two subsets ($\Theta_{\mathbb{D}_1}^{(+)}$ and $\Theta_{\mathbb{D}_2}^{(+)}$), while $\Theta_{\mathbb{D}}^{(-)}$ is empty. More precisely, $\Theta_{\mathbb{D}_1}^{(+)}$ contains η elements satisfying (B.3), and $\Theta_{\mathbb{D}_2}^{(+)}$ contains η elements satisfying that

$$g\theta_i^{(+)} + \text{Arg}(C_g) = -\frac{\pi}{2}(\text{mod}(2\pi)). \quad \text{(B.13)}$$

According to (B.2), $\Psi_{\mathbb{D}}^{(+)}$ has η elements satisfying (B.5), and $\Psi_{\mathbb{D}}^{(-)}$ has η elements satisfying that

$$n\psi_i^{(-)} = 0(\mathrm{mod}(2\pi)). \tag{B.14}$$

With suitable relabeling, we let $\Theta_{\mathbb{D}}^{(+)} = \{\theta_1^{(+)}, \ldots, \theta_{2\eta}^{(+)}\} = \Theta_{\mathbb{D}_1}^{(+)} \cup \Theta_{\mathbb{D}_2}^{(+)}$ (where $\Theta_{\mathbb{D}_1}^{(+)} = \{\theta_1^{(+)}, \ldots, \theta_\eta^{(+)}\}$ and $\Theta_{\mathbb{D}_2}^{(+)} = \{\theta_{\eta+1}^{(+)}, \ldots, \theta_{2\eta}^{(+)}\}$), $\Psi_{\mathbb{D}}^{(+)} = \{\psi_1^{(+)}, \ldots, \psi_\eta^{(+)}\}$, and $\Psi_{\mathbb{D}}^{(-)} = \{\psi_1^{(-)}, \ldots, \psi_\eta^{(-)}\}$.

Property B.6 *For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}_1}^{(+)}$, $1 \le a \le \eta$, there exists a unique $\psi_{c(a)}^{(+)} \in \Psi_{\mathbb{D}}^{(+)}$, $1 \le c(a) \le \eta$, such that $\psi_{c(a)}^{(+)} = \theta_a^{(+)}$. Furthermore, for different $\theta_a^{(+)}$, the corresponding $c(a)$ are different.*

Property B.7 *For each $\theta_b^{(+)} \in \Theta_{\mathbb{D}_2}^{(+)}$, $\eta + 1 \le b \le 2\eta$, there exists a unique $\psi_{d(b)}^{(-)} \in \Psi_{\mathbb{D}}^{(-)}$, $1 \le d(b) \le \eta$, such that $\psi_{d(b)}^{(-)} = \theta_b^{(+)}$. Furthermore, for different $\theta_b^{(+)}$, the corresponding $d(b)$ are different.*

Lemma B.8 *For each $\theta_a^{(+)} \in \Theta_{\mathbb{D}_1}^{(+)}$, $1 \le a \le \eta$, satisfying that the corresponding branches of $C_g(+\varepsilon)^{\frac{g}{n}}, \ldots, C_{g+M(a)-1}(+\varepsilon)^{\frac{g+M(a)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ is not degenerate, there exists a unique $c(a)$, $1 \le c(a) \le \eta$, satisfying the following properties.*
(1) For $\psi_{c(a)}^{(+)}$, the corresponding branches of $D_n(+\varepsilon j)^{\frac{n}{g}}, \ldots, D_{n+M(a)-1}(+\varepsilon j)^{\frac{n+M(a)-1}{g}}$ are all degenerate.
(2) For $\theta_a^{(+)}$ and $\psi_{c(a)}^{(+)}$, the corresponding branch of $C_{g+M(a)}(+\varepsilon)^{\frac{g+M(a)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) if and only if the corresponding branch of $D_{n+M(a)}(+\varepsilon j)^{\frac{n+M(a)}{g}}$ lies in $\mathbb{C}_U$ ($\mathbb{C}_L$).

Lemma B.9 *For each $\theta_b^{(+)} \in \Theta_{\mathbb{D}_2}^{(+)}$, $\eta+1 \le b \le 2\eta$, satisfying that the corresponding branches of $C_g(+\varepsilon)^{\frac{g}{n}}, \ldots, C_{g+M(b)-1}(+\varepsilon)^{\frac{g+M(b)-1}{n}}$ are all degenerate and that the corresponding branch of $C_{g+M(b)}(+\varepsilon)^{\frac{g+M(b)}{n}}$ is not degenerate, there exists a unique $d(b)$, $1 \le d(b) \le \eta$, satisfying the following properties.*
(1) For $\psi_{d(b)}^{(-)}$, the corresponding branches of $D_n(-\varepsilon j)^{\frac{n}{g}}, \ldots, D_{n+M(b)-1}(-\varepsilon j)^{\frac{n+M(b)-1}{g}}$ are all degenerate.
(2) For $\theta_b^{(+)}$ and $\psi_{d(b)}^{(-)}$, the corresponding branch of $C_{g+M(b)}(+\varepsilon)^{\frac{g+M(b)}{n}}$ lies in $\mathbb{C}_+$ ($\mathbb{C}_-$) if and only if the corresponding branch of $D_{n+M(b)}(-\varepsilon j)^{\frac{n+M(b)}{g}}$ lies in $\mathbb{C}_L$ ($\mathbb{C}_U$).

We have the following result for Case 3:

Lemma B.10 *Theorem 8.3 holds if $\widetilde{n}$ is even and $\widetilde{g}$ is odd.*

The proofs of the properties and lemmas for Cases 2 and 3 are omitted as they follow the same ideas as for Case 1.

The proof of Theorem 8.3 is now complete according to Lemmas B.1, B.4, B.7, and B.10.

References

1. C.T. Abdallah, P. Dorato, J. Benitez-Read, R. Byrne, Delayed positive feedback can stabilize oscillatory systems. In: *American Control Conference* (San Francisco, CA, USA 1993)
2. S.S. Abhyankar, *Algebraic Geometry for Scientists and Engineers* (American Mathematical Society, Providence, 1990)
3. L.V. Ahlfors, *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*, 3rd edn. (McGraw-Hill, New York, 1979)
4. C.E. Avellar, J.K. Hale, On the zeros of exponential polynomials. J. Math. Anal. Appl. **73**(2), 434–452 (1980)
5. H. Baumgärtel, *Analytic Perturbation Theory for Matrices and Operators* (Birkhäuser, Basel, 1985)
6. R. Bellman, K.L. Cooke, *Differential-Diffrence Equations* (Acamemic Press, New York, 1963)
7. F.G. Boese, Stability with respect to the delay: on a paper of K. L. Cooke and P. van den Driessche. J. Math. Anal. Appl. **228**(2), 293–321 (1998)
8. I. Boussaada, D.-A. Irofti, S.-I. Niculescu, Computing the codimension of the singularity at the origin for delay systems in the regular case: a vandermonde-based approach. In: *13th European Control Conference* (Strasbourg, France, 2014)
9. I. Boussaada, S.-I. Niculescu, Computing the codimension of the singularity at the origin for delay systems: the missing link with birkhoff incidence matrices. In: *21st International Symposium on Mathematical Theory of Networks and Systems* (Groningen, The Netherlands, 2014)
10. D. Breda, S. Maset, R. Vermiglio, Pseudospectral differencing methods for characteristic roots of delay differential equations. SIAM J. Sci. Comput. **27**(2), 482–495 (2005)
11. D. Breda, S. Maset, R. Vermiglio, *Stability of Linear Delay Differential Equations: A Numerical Approach with MATLAB* (Springer, New York, 2015)
12. C. Briat, Convergence and equivalence results for the Jensen's inequality-application to time-delay and sampled-data systems. IEEE Trans. Autom. Control **56**(7), 1660–1665 (2011)
13. E. Brieskorn, H. Knörrer, *Plane Algebraic Curves* (Birkhäuser Verlag, Basel, 1986)
14. L.A.V. Carvalho, On quadratic Liapunov functionals for linear difference equations. Linear Algebra Appl. **240**, 41–64 (1996)
15. E. Casas-Alvero, *Singularities of Plane Curves* (Cambridge University Press, Cambridge, 2000)
16. A. Çela, M. Ben Gaid, X.-G. Li, S.-I. Niculescu, *Optimal Design of Distributed Control and Embedded Systems* (Springer, London, 2014)

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7

17. J. Chen, P. Fu, S.-I. Niculescu, Asymptotic behavior of imaginary zeros of linear systems with commensurate delays. In: *45th IEEE Conference on Decision and Control* (San Diego, CA, USA, 2006)
18. J. Chen, P. Fu, S.-I. Niculescu, Z. Guan, An eigenvalue perturbation approach to stability analysis, part I: eigenvalue series of matrix operators. SIAM J. Control Optim. **48**(8), 5564–5582 (2010)
19. J. Chen, P. Fu, S.-I. Niculescu, Z. Guan, An eigenvalue perturbation approach to stability analysis, part II: when will zeros of time-delay systems cross imaginary axis? SIAM J. Control Optim. **48**(8), 5583–5605 (2010)
20. J. Chen, H.A. Latchman, Frequency sweeping tests for stability independent of delay. IEEE Trans. Autom. Control **40**(9), 1640–1645 (1995)
21. K.L. Cooke, P. van den Driessche, On zeros of some transcendental equations. Funkcialaj Ekvacioj **29**(1), 77–90 (1986)
22. R. Datko, Remarks concerning the asymptotic stability and stabilization of linear delay differential equations. J. Math. Anal. Appl. **111**(2), 571–584 (1985)
23. I.I. Delice, R. Sipahi, Delay-independent stability test for systems with multiple time-delays. IEEE Trans. Autom. Control **57**(4), 963–972 (2012)
24. K. Engelborghs, T. Luzyanina, D. Roose, Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL. ACM Trans. Math. Softw. **28**(1), 1–21 (2002)
25. H. Fazelinia, R. Sipahi, N. Olgac, Stability robustness analysis of multiple time-delayed systems using "building block" concept. IEEE Trans. Autom. Control **52**(5), 799–810 (2007)
26. A.R. Fioravanti, *$\mathscr{H}_\infty$ Analysis and Control of Time-Delay Systems by Methods in Frequency-Domain*. PhD thesis, Université Paris-Sud XI, France (2011)
27. A.R. Fioravanti, C. Bonnet, S.-I. Niculescu, J.C. Geromel, Matrix norm approach for control of linear time-delay systems. In: *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)* (Orlando, USA, 2011)
28. A.R. Fioravanti, C. Bonnet, H. Özbay, S.-I. Niculescu, A numerical method for stability windows and unstable root-locus calculation for linear fractional time-delay systems. Automatica **48**(11), 2824–2830 (2012)
29. E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control* (Birkhäuser, Basel, 2014)
30. E. Fridman, M. Dambrine, Control under quantization, saturation and delay: an LMI approach. Automatica **45**(10), 2258–2264 (2009)
31. E. Fridman, A. Seuret, J.-P. Richard, Robust sampled-data stabilization of linear systems: an input delay approach. Automatica **40**(8), 1441–1446 (2004)
32. P. Fu, J. Chen, S.-I. Niculescu, Generalized eigenvalue-based stability tests for 2-D linear systems: necessary and sufficient conditions. Automatica **42**(9), 1569–1576 (2006)
33. P. Fu, J. Chen, S.-I. Niculescu, High-order analysis of critical stability properties of linear time-delay systems. In: *2007 American Control Conference* (New York, USA, 2007)
34. P. Fu, S.-I. Niculescu, J. Chen, Stability of linear neutral time-delay systems: exact conditions via matrix pencil solutions. IEEE Trans. Autom. Control **51**(6), 1063–1069 (2006)
35. H. Gao, X. Liu, J. Lam, Stability analysis and stabilization for discrete-time fuzzy systems with time-varying delay. IEEE Trans. Syst. Man Cybern. Part B Cybern. **39**(2), 306–317 (2009)
36. H. Górecki, S. Fuksa, P. Gabrowski, A. Korytowski, *Analysis and Synthesis of Time Delay Systems* (Wiley, New York, 1989)
37. K. Gu, Discretized LMI set in the stability problem of linear uncertain time-delay systems. Int. J. Control **68**(4), 923–934 (1997)
38. K. Gu, A review of some subtleties of practical relevance for time-delay systems of neutral type. ISRN Appl. Math. **2012**, 1–46 (2012)
39. K. Gu, V.L. Kharitonov, J. Chen, *Stability of Time-Delay Systems* (Birkhäuser, Boston, 2003)
40. K. Gu, M. Naghnaeian, Stability crossing set for systems with three delays. IEEE Trans. Autom. Control **56**(1), 11–26 (2011)

41. K. Gu, S.-I. Niculescu, J. Chen, On stability crossing curves for general systems with two delays. J. Math. Anal. Appl. **311**(1), 231–253 (2005)
42. X.-P. Guan, C.-L. Chen, Delay-dependent guaranteed cost control for T-S fuzzy systems with time delays. IEEE Trans. Fuzzy Syst. **12**(2), 236–249 (2004)
43. S. Gumussoy, W. Michiels, A predictor-corrector type algorithm for the pseudospectral abscissa computation of time-delay systems. Automatica **46**(4), 657–664 (2010)
44. S. Gumussoy, W. Michiels, Root locus for SISO dead-time systems: a continuation based approach. Automatica **48**(3), 480–489 (2012)
45. J.K. Hale, *Theory of Functional Differential Equations* (Springer, New York, 1977)
46. J.K. Hale, E.F. Infante, F.-S.P. Tsen, Stability in linear delay equations. J. Math. Anal. Appl. **105**(2), 533–555 (1985)
47. Y. Halevi, Reduced-order models with delay. Int. J. Control **64**(4), 733–744 (1996)
48. B.D. Hassard, Counting roots of the characteristic equation for linear delay-differential systems. J. Differ. Equ. **136**(2), 222–235 (1997)
49. D. Hertz, E.I. Jury, E. Zeheb, Simplified analytic stability test for systems with commensurate time delays. IEE Proc. D (Control Theory Appl.) **131**(1), 52–56 (1984)
50. L. Hetel, J. Daafouz, C. Iung, Stabilization of arbitrary switched linear systems with unknown time-varying delays. IEEE Trans. Autom. Control **51**(10), 1668–1674 (2006)
51. G.-D. Hu, M. Liu, Stability criteria of linear neutral systems with multiple delays. IEEE Trans. Autom. Control **52**(4), 720–724 (2007)
52. T. Insperger, G. Stépán, *Semi-Discretization for Time-Delay Systems: Stability and Engineering Applications* (Springer, New York, 2011)
53. N. Jacobson, *Lectures in Abstract Algebra III-Theory of Fields and Galois Theory* (Springer, New York, 1964)
54. E. Jarlebring, W. Michiels, Invariance properties in the root sensitivity of time-delay systems with double imaginary roots. Automatica **46**(6), 1112–1115 (2010)
55. E.W. Kamen, On the relationship between zero criteria for two-variable polynomials and asymptotic stability of delay differential equations. IEEE Trans. Autom. Control **25**(5), 983–984 (1980)
56. T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1995)
57. V.L. Kharitonov, Time-Delay Systems: Lyapunov Functionals and Matrices (Birkhäuser, Basel, 2013)
58. V.L. Kharitonov, S.-I. Niculescu, J. Moreno, M. Michiels, Static output feedback stabilization: necessary conditions for multiple delay controllers. IEEE Trans. Autom. Control **50**(1), 82–86 (2005)
59. C.R. Knospe, M. Roozbehani, Stability of linear systems with interval time delays excluding zero. IEEE Trans. Autom. Control **51**(8), 1271–1288 (2006)
60. J. Kollár, *Lectures on Resolution of Singularities* (Princeton University Press, Princeton, 2007)
61. S.G. Krantz, *Function Theory of Several Complex Variables*, 2nd edn. (Wadsworth & Brooks/Cole, Pacific Grove, 1992)
62. S.G. Krantz, H.R. Parks, *The Implicit Function Theorem, History, Theory, and Applications* (Birkhäuser, Boston, 2003)
63. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics* (Academic Press, Boston, 1993)
64. M.S. Lee, C.S. Hsu, On the τ-decomposition method of stability analysis for retarded dynamical systems. SIAM J. Control **7**(2), 242–259 (1969)
65. X.-G. Li, A. Çela, S.-I. Niculescu, A. Reama, Some problems in the stability of networked-control systems with periodic scheduling. Int. J. Control **83**(5), 996–1008 (2010)
66. X.-G. Li, S.-I. Niculescu, A. Çela. Complete stability of linear time-delay systems: A new frequency-sweeping approach. In: *10th IEEE International Conference on Control and Automation* (Hangzhou, China, 2013)
67. X.-G. Li, S.-I. Niculescu, A. Çela, Critical simple imaginary characteristic roots and invariance properties for quasipolynomials: a missing link. In: *11th IFAC Workshop on Time-Delay Systems* (Grenoble, France, 2013)

68. X.-G. Li, S.-I. Niculescu, A. Çela, Complete stability for neutral time-delay systems: a unified frequency-sweeping approach. In: *33rd Chinese Control Conference* (Nanjing, China, 2014)
69. X.-G. Li, S.-I. Niculescu, A. Çela, X. Li, L. Zhang, Complete stability for neutral time-delay systems: a frequency-sweeping framework. Submitted
70. X.-G. Li, S.-I. Niculescu, A. Çela, H.-H. Wang, T.-Y. Cai, On τ-decomposition frequency-sweeping test for a class of time-delay systems. Part I: simple imaginary roots case. In: *10th IFAC Workshop on Time-Delay Systems* (Boston, USA, 2012)
71. X.-G. Li, S.-I. Niculescu, A. Çela, H.-H. Wang, T.-Y. Cai, On computing Puiseux series for multiple imaginary characteristic roots of LTI systems with commensurate delays. IEEE Trans. Autom. Control **58**(5), 1338–1343 (2013)
72. X.-G. Li, S.-I. Niculescu, A. Çela, H.-H. Wang, T.-Y. Cai, Invariance properties for a class of quasipolynomials. Automatica **50**(3), 890–895 (2014)
73. X.-G. Li, S.-I. Niculescu, A. Çela, L. Zhang, X. Li, A new frequency-sweeping framework from an analytic curve perspective for complete stability of time-delay systems. Submitted
74. X.-G. Li, S.-I. Niculescu, A. Çela, L. Zhang, X. Li, Characterizing invariance properties of linear time-delay systems: the general case. In: *52nd IEEE Conference on Decision and Control* (Florence, Italy, 2013)
75. J. Louisell, A matrix method for determining the imaginary axis eigenvalues of a delay system. IEEE Trans. Autom. Control **46**(12), 2008–2012 (2001)
76. N. MacDonald, *Biological Delay Systems: Linear Stability Theory* (Cambridge University Press, Cambridge, 1989)
77. E. Malakhovski, L. Mirkin, On stability of second-order quasi-polynomials with a single delay. Automatica **42**(6), 1041–1047 (2006)
78. C.M. Marcus, R.M. Westervelt, Stability of analog neural networks with delay. Phys. Rev. A **39**(1), 347–359 (1989)
79. A.I. Markushecvich, *Theory of Functions of a Complex Variable*, vol. II (Prentice-Hall, Englewood Cliffs, 1965)
80. C.-F. Méndez-Barrios, S.-I. Niculescu, J. Chen, M. Maya-Méndez, Output feedback stabilisation of single-input single-output linear systems with I/O network-induced delays: an eigenvalue-based approach. Int. J. Control **87**(2), 346–362 (2014)
81. A. Mesbahi, M. Haeri, Stability of linear time invariant fractional delay systems of retarded type in the space of delay parameters. Automatica **49**(5), 1287–1294 (2013)
82. W. Michiels, Stability analysis of oscillatory systems subject to large delays: a synchronization point of view. J. Vib. Control **16**(7–8), 1087–1111 (2010)
83. W. Michiels, K. Engelborghs, D. Roose, D. Dochain, Sensitivity to infinitesimal delays in neutral equations. SIAM J. Control Optim. **40**(4), 1134–1158 (2002)
84. W. Michiels, C.-I. Morărescu, S.-I. Niculescu, Consensus problems with distributed delays, with application to traffic flow models. SIAM J. Control Optim. **48**(1), 77–101 (2009)
85. W. Michiels, S.-I. Niculescu, *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach* (SIAM, Philadelphia, 2007)
86. U. Münz, C. Ebenbauer, T. Haag, F. Allgöwer, Stability analysis of time-delay systems with incommensurate delays using positive polynomials. IEEE Trans. Autom. Control **54**(5), 1019–1024 (2009)
87. U. Münz, A. Papachristodoulou, T. Haag, F. Allgöwer, Delay robustness in consensus problems. Automatica **46**(8), 1252–1265 (2010)
88. M. Naghnaeian, K. Gu, Stability crossing set for systems with two scalar-delay channels. Automatica **49**(7), 2098–2106 (2013)
89. J. Neimark, D-subdivisions and spaces of quasi-polynomials. Prikl. Math. Mech. **13**, 349–380 (1949)
90. I. Newton, The correspondence of Sir Isaac Newton, vol. II (1676–1687). Cambridge University Press, Cambridge, pp. 20–42 and 110–163 (1960)
91. L.I. Nicolaescu, *Notes on the Topology of Complex Singularities* (2001)
92. S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach* (Springer, Heidelberg, 2001)

93. S.-I. Niculescu, On delay robustness analysis of a siple control algorithm in high-speed networks. Automatica **38**(5), 885–889 (2002)
94. S.-I. Niculescu, W. Michiels, Stabilizing a chain of integrators using multiple delays. IEEE Trans. Autom. Control **49**(5), 802–807 (2004)
95. S.-I. Niculescu, M. Putinar, A toric Positivstellensatz with applications to time-delay systems. C R Math **349**(5-6), 327–329 (2011)
96. R. Olfati-Saber, R.M. Murray, Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Autom. Control **49**(9), 1520–1533 (2004)
97. N. Olgac, R. Sipahi, An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems. IEEE Trans. Autom. Control **47**(5), 793–797 (2002)
98. N. Olgac, R. Sipahi, A practical method for analyzing the stability of neutral type LTI-time delayed systems. Automatica **40**(5), 847–853 (2004)
99. L.-L. Ou, W.-D. Zhang, L. Yu, Low-order stabilization of LTI systems with time delay. IEEE Trans. Autom. Control **54**(4), 774–787 (2009)
100. H. Özbay, C. Bonnet, H. Benjelloun, J. Clairambault, Stability analysis of cell dynamics in leukemia. Math. Model. Nat. Phenom. **7**(01), 203–234 (2012)
101. V. Puiseux, Recherches sur les fonctions algébriques. Journal de Mathématiques pures et Appliquées **15**, 365–480 (1850)
102. J.-P. Richard, Time-delay systems: an overview of some recent advances and open problems. Automatica **39**(10), 1667–1694 (2003)
103. C.E. Riddalls, S. Bennett, The stability of supply chains. Int. J. Prod. Res. **40**(2), 459–475 (2002)
104. A.P. Seyranian, A.A. Mailybaev, *Multiparameter Stability Theory with Mechanical Applications* (World Scientific, Singapore, 2003)
105. J. Sieber, B. Krauskopf, Bifurcation analysis of an inverted pendulum with delayed feedback control near a triple-zero eingevalue singularity. Nonlinearity **17**(1), 85–103 (2004)
106. G.J. Silva, A. Datta, S.P. Bhattacharyya, *PID Controllers for Time-Delay Systems* (Birkhäuser, Boston, 2005)
107. R. Sipahi, I.I. Delice, Advanced clustering with frequency sweeping methodology for the stability analysis of multiple time-delay systems. IEEE Trans. Autom. Control **56**(2), 467–472 (2011)
108. R. Sipahi, S.-I. Niculescu, C.T. Abdallah, W. Michiels, K. Gu, Stability and stabilization of systems with time delay, limitations and opportunities. IEEE Control Syst. Mag. **31**(1), 38–65 (2011)
109. R. Sipahi, N. Olgac, Degenerate cases in using the direct method. ASME J. Dyn. Syst. Measur. Control **125**(2), 194–201 (2003)
110. R. Sipahi, N. Olgac, A comparative survey in determining the imaginary characteristic roots of LTI time delayed systems. In: *16th IFAC World Congress* (Prague, Czech Republic, 2005)
111. R. Sipahi, N. Olgac, D. Breda, A stability study on first-order neutral systems with three rationally independent time delays. Int. J. Syst. Sci. **41**(12), 1445–1455 (2010)
112. G. Stépán, *Retarded Dynamical Systems: Stability and Characteristic Functions* (Longman Scientific & Technical, Harlow, 1989)
113. O. Toker, H. Özbay, Complexity issues in robust stability of linear delay-differential systems. Math. Control Signals Syst. **9**(4), 386–400 (1996)
114. Ya.Z. Tsypkin, The systems with delayed feedback. Avtomathika i Telemech. **7**, 107–129 (1946)
115. M.M. Vainberg, V.A. Trenogin, *Theory of Branching of Solutions of Non-Linear Equations* (Noordhoff International Publishing, Leyden, 1974)
116. F. Verhulst, *Methods and Applications of Singular Perturbations: Boundary Layers and Multiple Timescale Dynamics* (Springer, New York, 2005)
117. E.I. Verriest, W. Michiels, Inverse routh table construction and stability of delay equations. Syst. Control Lett. **55**(9), 711–718 (2006)
118. T. Vyhlídal, P. Zítek, Mapping based algorithm for large-scale computation of quasi-polynomial zeros. IEEE Trans. Autom. Control **54**(1), 171–177 (2009)

119. T. Vyhlídal, P. Zítek, Modification of mikhaylov criterion for neutral time-delay systems. IEEE Trans. Autom. Control **54**(10), 2430–2435 (2009)
120. R.J. Walker, *Algebraic Curves* (Princeton University Press, Princeton, 1950)
121. C.T.C. Wall, *Singular Points of Plane Curves* (Cambridge University Press, Cambridge, 2004)
122. K. Walton, J.E. Marshall, Direct method for TDS stability analysis. IEE Proc. (Control Theory Appl.) **134**(2), 101–107 (1987)
123. A. Welters, On explicit recursive formulas in the spectral perturbation analysis of a Jordan block. SIAM J. Matrix Anal. Appl. **32**(1), 1–22 (2011)
124. M. Wu, Y. He, J.-H. She, *Stability Analysis and Robust Control of Time-Delay Systems* (Springer, Berlin, 2010)
125. H. Zhang, Z. Liu, G.-B. Huang, Z. Wang, Novel weighting-delay-based stability criteria for recurrent neural networks with time-varying delay. IEEE Trans. Neural Networks **21**(1), 91–106 (2010)
126. W. Zhang, M.S. Branicky, S.M. Phillips, Stability of networked control systems. IEEE Control Syst. Mag. **21**(1), 84–99 (2001)

Series Editors' Biography

Tamer Başar is with the University of Illinois at Urbana-Champaign, where he holds the academic positions of Swanlund Endowed Chair, Center for Advanced Study Professor of Electrical and Computer Engineering, Research Professor at the Coordinated Science Laboratory, and Research Professor at the Information Trust Institute. He received the B.S.E.E. degree from Robert College, Istanbul, and the M.S., M.Phil, and Ph.D. degrees from Yale University. He has published extensively in systems, control, communications, and dynamic games, and has current research interests that address fundamental issues in these areas along with applications such as formation in adversarial environments, network security, resilience in cyber-physical systems, and pricing in networks.

In addition to his editorial involvement with these *Briefs*, Tamer Başar is also the Editor-in-Chief of *Automatica*, Editor of two Birkhäuser Series on *Systems & Control and Static & Dynamic Game Theory*, the Managing Editor of the *Annals of the International Society of Dynamic Games* (ISDG), and member of editorial and advisory boards of several international journals in control, wireless networks, and applied mathematics. He has received several awards and recognitions over the years, among which are the Medal of Science of Turkey (1993); Bode Lecture Prize (2004) of IEEE CSS; Quazza Medal (2005) of IFAC; Bellman Control Heritage Award (2006) of AACC; and Isaacs Award (2010) of ISDG. He is a member of the US National Academy of Engineering, Fellow of IEEE and IFAC, Council Member of IFAC (2011–14), a past president of CSS, the founding president of ISDG, and president of AACC (2010–11).

Antonio Bicchi is Professor of Automatic Control and Robotics at the University of Pisa. He graduated at the University of Bologna in 1988 and was a postdoc scholar at M.I.T. A.I. Lab between 1988 and 1990. His main research interests are in:

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7

- dynamics, kinematics, and control of complex mechanical systems, including robots, autonomous vehicles, and automotive systems;
- haptics and dextrous manipulation; and
- theory and control of nonlinear systems, in particular hybrid (logic/dynamic, symbol/signal) systems.

He has published more than 300 papers on international journals, books, and refereed conferences.

Professor Bicchi currently serves as the Director of the Interdepartmental Research Center "E. Piaggio" of the University of Pisa, and President of the Italian Association or Researchers in Automatic Control. He has served as Editor in Chief of the Conference Editorial Board for the IEEE Robotics and Automation Society (RAS), and as Vice President of IEEE RAS, Distinguished Lecturer, and Editor for several scientific journals including the *International Journal of Robotics Research*, the *IEEE Transactions on Robotics and Automation*, and *IEEE RAS Magazine*. He has organized and co-chaired the first WorldHaptics Conference (2005), and Hybrid Systems: Computation and Control (2007). He is the recipient of several best paper awards at various conferences, and of an Advanced Grant from the European Research Council. Antonio Bicchi has been an IEEE Fellow since 2005.

Miroslav Krstic holds the Daniel L. Alspach chair and is the founding director of the Cymer Center for Control Systems and Dynamics at University of California, San Diego. He is a recipient of the PECASE, NSF Career, and ONR Young Investigator Awards, as well as the Axelby and Schuck Paper Prizes. Professor Krstic was the first recipient of the UCSD Research Award in the area of engineering and has held the Russell Severance Springer Distinguished Visiting Professorship at UC Berkeley and the Harold W. Sorenson Distinguished Professorship at UCSD. He is a Fellow of IEEE and IFAC. Professor Krstic serves as Senior Editor for *Automatica and IEEE Transactions on Automatic Control* and as Editor for the Springer series *Communications and Control Engineering*. He has served as Vice President for Technical Activities of the IEEE Control Systems Society. Krstic has co-authored eight books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.

Index

X.-G. Li et al., *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays*, SpringerBriefs in Control, Automation and Robotics,
DOI 10.1007/978-3-319-15717-7