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Chapter 1
Physical Geodesy and Its Boundary Value Problems

Physical geodesy is the science of determining the "figure" of the Earth and its external gravitational field.

The figure of the Earth is a rather fishy concept. For instance, one might believe that this should be identified with the external surface of the masses contained in a solid or liquid form in the body of the Earth. As we see in this definition we already exclude the Earth atmosphere, the external surface of which is not well defined and which bears a mass distribution rapidly and irregularly changing in time. Fortunately its gravimetric signal is rather faint and the largest part of it can be predicted by simple models, maybe nowadays defined by satellite observations (GRACE), and subtracted from the observable quantities we will define in the sequel. We shall make use again, later on, of this particular approach. Nevertheless the external surface of the solid and liquid Earth is still by far too irregular and rapidly changing. Be it enough to mention that on the solid Earth we have cities, vehicles, vegetation etc., while on the ocean we have meters high waves and tides and so forth.

Even in a purely geometrical discipline, like photogrammetry, we soon arrive at the concept of *Digital Terrain Model* where houses, vegetation (and moving objects) are purposely suppressed. Let us mention here once again that the gravity signal of such objects is mostly negligible, or it can be accounted for by more or less simple models, for instance ocean tides. So we arrive at a first, provisory definition of Earth surface "after some obvious smoothing, which is necessary to make it amenable to mathematical treatment, and also after some averaging with respect to time, since this surface undergoes temporal variations ... because of tidal effects, etc." ([1], Chap. 1, Sect. 1.1).

Despite the above simplifications, the surface so defined is still too complicated and subject to changes in time, for instance because of construction works or long-lasting geological phenomena.

Historically, the need of finding some smoother and more stable surface has led scientists to define it by means of the gravity field, which, as commented above, suffers minor variations due to the perturbations reported as examples. The con-

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cept, initially adopted by Gauss, was the geoid, namely a level surface of the gravity potential that is suitably close to the oceanic surface, although the name was adopted 50 years later, on suggestion of Listing. This has created the historical link between the determination of the figure of the Earth and of the Earth gravity field. The determination of the geoid from gravity observations is the problem that G. Stokes solved in 1849, after a linearization and in spherical approximation, by assuming that: (a) all the masses above the geoid have been removed; (b) the value of the gravity modulus is given directly on the geoid itself. This has urged one century of research by geodesists, trying to find good ways for approximating the effect of the topographic masses on the observations and downward continuing the observed gravity values from the actual topographic surface down to the geoid.

Unfortunately both these operations are either uncertain or mathematically non well-posed.

The first because our knowledge of the mass density inside topography (i.e., above the geoid) is quite poor, and insufficient for the target of the 1 cm accuracy which is expected these days. In fact, most of the methods use a mean constant density of 2.67 g/cm^{-3}, which certainly does not represent accurately the geological reality.

The second because, even in free space, the downward continuation of a harmonic function regular at infinity (or of its functionals) is always improperly posed, meaning that only under suitable regularization, i.e., at the expense of a loss of resolution, one can obtain reasonable results.

As such the Stokes approach to the main problem of physical geodesy, though genial, is not satisfactory and one century later, with the work of Molodensky [1, 2], we have found a formulation fully acceptable on a theoretical ground.

In fact the formulation of the so-called *Molodensky problem*, a free boundary, oblique derivative boundary value problem for the Laplace equation in an external domain, has primarily the function of setting up a theoretical frame which, once properly analyzed for its mathematical properties, constitutes a sound reference for the innumerable approximation methods used nowadays to find, globally or locally, the figure of the Earth and its gravity field.

At this point, following and developing the thought of Molodensky, Krarup, and others, we could arrive at our *definition* of figure of the Earth: this is any surface S on which we can assume (after a limit process) to know both the value of the gravity potential W and of its gradient modulus g to a pre-defined degree of accuracy; in addition at the points of this surface the horizontal coordinates $\sigma \equiv (\lambda, \varphi)$ are assumed to be known while the height of the surface, on a reference ellipsoid, $h(\sigma)$, is one of the unknowns of the problem. The potential W, after subtracting its centrifugal component that, when the z axis agrees with the rotation axis reads $C = \frac{1}{2}\omega^2(x^2 + y^2)$, has to be harmonic outside S and regular at infinity.

Here ω represents as usual the angular velocity of the Earth, supposed to be directed along the z axis and to be constant in modulus.

So S has to wander either strictly out of the masses or at most to be inside by a few meters, so that simple corrections can be applied to remove their effects.

In this respect a digital terrain model, i.e., the values $h(\sigma)$ roughly achieved for instance by space methods, is an essential ancillary data set to interpolate on S all the pointwise observed gravity values. Also we assume here that over the ocean we have already converted altimetric data into a gravity data set (see [3], Chap. 9), so that the above definition applies to the full surface S.

Formally the problem can be stated as follows: given $\sigma_P = (\lambda_P, \varphi_P)$ and $W_0(\sigma_P)$, $g_0(\sigma_P)$ for any $P \in S$, find $S \equiv \{h = h(\sigma_P)\}$ and $W(\sigma, h)$ such that

$$\begin{cases} \text{(a)}\ \Delta\{W - \frac{1}{2}\omega^2(x^2+y^2)\} = 0 \text{ in } \Omega \equiv \{h > h(\sigma\}) \\ \text{(b)}\ W|_{h=h(\sigma)} = W_0(\sigma) \qquad \text{on } S \\ \text{(c)}\ |\nabla W||_{h=h(\sigma)} = g_0(\sigma) \qquad \text{on } S \\ \text{(d)}\ W - \frac{1}{2}\omega^2(x^2+y^2) \to 0 \qquad \text{for } h \to \infty \end{cases} \tag{1.1}$$

This is called the *Scalar Non-Linear Molodensky Problem*. For technical reasons, due to the quasi-translation invariance of the problem [4], it is convenient to introduce a more stringent asymptotic condition instead of (d), namely

$$\text{(d')} \quad W - \frac{1}{2}\omega^2(x^2+y^2) \sim \frac{\mu}{r} + O\left(\frac{1}{r^3}\right) \qquad r \to \infty \tag{1.2}$$

introducing in parallel 3 more scalar unknowns in the boundary conditions $\{A_{-1}, A_0, A_1\}$ in (b), namely putting

$$\text{(b')} \quad W|_{h=h(\sigma)} = W_0(\sigma) + \sum_{k=-1}^{1} A_k \psi_{1k}(\sigma) \tag{1.3}$$

for suitable $\psi_{1k}(\sigma)$ (see [3, 5], Sect. 15.2).

In (1.3), and in the sequel, we use the short hand notation $\mu = GM$ where G is Newton's constant and M is the mass of the Earth.

The problem (a), (b'), (c), (d') has a unique unconditional solution in a band of data $W_0(\sigma)$, $g_0(\sigma)$, close enough to the spherical configuration, in suitable Hölder spaces [3, 5].

This problem has a linearized version, which is the one to which most of the literature refers as the Molodensky Problem. This is traditionally done by introducing the telluroid, $\widetilde{S}$, as the surface described by the ellipsoidal heights $\widetilde{h}(\sigma)$ obtained from the equation

$$U(\sigma, \widetilde{h}(\sigma)) = W_0(\sigma) = W(\sigma, h(\sigma)) , \tag{1.4}$$

where U is the normal gravity potential [3]. The solution $\widetilde{h}(\sigma_P)$ of (1.4) is usually called *normal height* of P. The unknown $h(\sigma)$ is then split according to

$$h(\sigma) = \widetilde{h}(\sigma) + \zeta(\sigma) , \tag{1.5}$$

with $\zeta(\sigma)$ the height anomaly. This geometrical quantity, though, is related to the anomalous potential,

$$T(\sigma, h) = W(\sigma, h) - U(\sigma, h) , \tag{1.6}$$

by the Bruns relation

$$\zeta(\sigma) = \frac{T(\sigma, h)}{\gamma(\sigma, h)} \tag{1.7}$$

with $\gamma(\sigma, h) = |\nabla U(\sigma, h)|$, the normal gravity modulus.

By further linearizing the observation equation of $g_0(\sigma) = |\nabla g(\sigma, h(\sigma))|$ one arrives at the relation (see (A.23))

$$\Delta g_0(\sigma) = g_0(\sigma) - \gamma(\sigma, \widetilde{h}(\sigma)) = -\frac{\partial T}{\partial h} + \frac{\frac{\partial \gamma}{\partial h}}{\gamma} T \Bigg|_{\widetilde{S}} . \tag{1.8}$$

So, considering that T has to be harmonic outside $\widetilde{S}$, because the non-harmonic part of W and U cancel in their difference (1.5), we arrive at the following, oblique derivative BVP:

$$\begin{cases} \nabla T = 0 & \text{in } \widetilde{\Omega} \equiv \{h \geq \widetilde{h}(\sigma)\} \\ -\frac{\partial T}{\partial h} + \frac{\frac{\partial \gamma}{\partial h}}{\gamma} T \Big|_{\widetilde{S}} = \Delta g_0(\sigma) . \end{cases} \tag{1.9}$$

This has generally to be complemented with an asymptotic condition of the form

$$T \sim \frac{\delta \mu}{r} + O\left(\frac{1}{r^3}\right) \tag{1.10}$$

with $\delta\mu = G(M - \widetilde{M})$, M the actual mass of the Earth and $\widetilde{M}$ the mass generating the normal potential. If we assume that the two are identical, the regularity condition (1.10) takes the form

$$T = O\left(\frac{1}{r^3}\right) . \tag{1.11}$$

We shall refer to (1.9) and (1.11) as the linearized Molodensky problem in classical form. Once solved, via Bruns relation (1.7) and the definition of height anomaly (1.5), one can retrieve the *Earth surface* S, namely that surface on which $W(\sigma)$ and $g(\sigma)$ were given.

We shall analyze in more detail in the next Chapter the linearization procedure trying to define precisely what is the band of linearization in which all linearized versions for the same non-linear problem are equivalent in the sense that the differences between them are to be considered *second order* effects.

In this sense, after recalling the definition of the Molodensky and of the Helmert approaches, we shall prove that, at the level of linearized BVP formulation, they are totally equivalent. The rest is in fact an attempt of solving the respective BVP by the so-called *downward continuation method*.

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Chapter 2
On the Linearization Band

In this chapter we will review the linearization process, following the general approach of T. Krarup, as presented in his famous three letters on Molodensky's Problem (see [1]), but adapted to the case of the Scalar Molodensky Problem, as introduced in Chap. 1 and discussed also in [2], Sect. 2.3.

There are several reasons to do that:

(i) First of all, from a general theoretical point of view, we aim at clarifying that in the linearization procedure, although the normal potential U and "some telluroid", $\{h = \widetilde{h}(\sigma)\}$, are introduced as approximations respectively of the gravity potential W and of the Earth surface $\{h = h(\sigma)\}$, so that the increments $T = W - U$ and $\zeta = h - \widetilde{h}$ can be considered as "first-order" infinitesimals, such a hypothesis cannot be considered as acceptable, unless a suitable compatibility condition is introduced relating the orders of magnitude of the two quantities.

(ii) Based on the above remark, once the order of magnitude of T and ζ are assessed, the Molodensky BVP can be expanded up to second-order terms with the purpose of verifying that they can be neglected for the level of accuracy we aim at. This defines the linearization band.

(iii) Finally, we shall establish the principle of equivalence stating that all the problems formulated by linearization of the same scalar non-linear Molodensky problem, with approximate reference potential and telluroid chosen in the linearization band, are essentially equivalent, up to second order of magnitude errors.

A similar problem has already been analyzed in literature (see [3]) with a strong numerical apparatus, taking into account also the spatial behaviour of the gravity field. However, in the quoted paper the purpose was more to compare different formulations of the GBVP, arriving at the conclusion that the scalar non-linear Molodensky problem was the most natural and useful formulation for geodetic purposes. So we

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build here in a sense on this conclusion. Our approach however is more elementary, though sufficient to achieve the most important result of the chapter, namely the definition of a "linearization band".

Before we get started, let us comment on the meaning of what we will use as "order of magnitude" for the different quantities, q, usually defined on the Earth sphere, namely the projection of the Earth surface on the unit sphere.

One first rigorous definition could be the mean square value of the distribution of $q(\sigma)$, namely

$$O(q) = \sigma(q) = \{\frac{1}{4\pi}\int q^2(\sigma)d\sigma\}^{1/2} \tag{2.1}$$
$$(\sigma = (\lambda, \varphi) \text{ spherical coordinates}) \; .$$

In (2.1) σ is used with two different meanings: to represent spherical coordinates, but also to mean the r.m.s. of some quantity on the sphere. The two concepts however should be clear by the context. The disadvantage of using this measure though, is that the extreme value $\max_\sigma |q(\sigma)|$ is not easily related to $\sigma(q)$, in particular considering that every quantity has generally a different spectral signature when expressed in terms of spherical harmonics ([2], Sect. 3.8). For instance the value of $3\sigma(q)$ is not always a good guess of the maximum value of q. So, since in the present reasoning we want to be on the safe side in evaluating the error we should try to find an index more related to the maximum of q. For this purpose we shall use a value $O_M(q)$, which is a very high value of q, only seldom met on the Earth globe and even more rarely exceeded. For instance a 90 % quantile. Generally we shall agree on a value that at least satisfies the following relation

$$O_M(q) \leq \max |q(\sigma)| \leq 2O_M(q) \; . \tag{2.2}$$

To avoid ambiguity, in the rest of the Chapter we shall use the following table of orders of magnitudes:

Table 2.1 Orders of magnitude of various geodetic quantities

Quantity q	$O_M(q)$
a, b, R	6×10^6 m
e^2	$150^{-1} = 6.7\ 10^{-3}$
W	6×10^9 Gal $\times$ m
γ, g	10^3 Gal
$\frac{\partial\gamma}{\partial r}, \frac{\partial g}{\partial r}$	0.3 Gal km^{-1}
$\frac{\partial T}{\partial r}, \delta g, \Delta g$	0.1 Gal
$\frac{\partial^2 T}{\partial r^2}$	6×10^{-4} Gal km^{-1}
δ	3×10^{-4} rad
H	6×10^3 m

where a, b are the semi-axes of the ellipsoid, R the mean Earth radius, e^2 the square of the first eccentricity, W, g are potential and gravity on the Earth surface, T is the anomalous potential, δg, Δg are gravity disturbance and anomaly, δ is the deflection of the vertical, H is the topographic height.

We shall use the symbol $\sim$ to express that $O_M(q)$ attains a certain numerical value, for instance $\gamma \sim 10^3$ Gal. Noteworthy, with the figures of Table 2.1, the following relations hold

$$T \sim 1.6 \times 10^{-5} W \tag{2.3}$$

$$\Delta g \sim 1 \times 10^{-4} \gamma \tag{2.4}$$

$$\frac{T}{\gamma} \sim 1.6 \times 10^{-5} \frac{W}{\gamma} \sim 10^2 \text{ m} \tag{2.5}$$

After these remarks let us go back to the linearization of the scalar Molodensky problem. We introduce the approximate potential U and some telluroid $\widetilde{S} = \{h = \widetilde{h}(\sigma)\}$, with $\sigma = (\lambda, \varphi)$ ellipsoidal coordinates, such that

$$W - U|_{\widetilde{S}} = T|_{\widetilde{S}} \tag{2.6}$$

$$\zeta(\sigma) = h(\sigma) - \widetilde{h}(\sigma) \tag{2.7}$$

should be considered as first-order infinitesimals. Note that by taking U, the normal potential, as an approximate solution for W, we will define a certain linearization band, that however could change with a different approximate potential, typically becoming narrower. Note also that with (2.6), $O_M(T)$ is *fixed* by Table 2.1 and (2.3).

The free-boundary relations to be linearized are

$$W_0(\sigma) = U(\sigma, h(\sigma)) + T(\sigma, h(\sigma)) \tag{2.8}$$

$$\begin{aligned} g_0(\sigma) &= |\nabla U(\sigma, h(\sigma)) + \nabla T(\sigma, h(\sigma))| = \\ &= |\boldsymbol{\gamma}(\sigma, h(\sigma)) + \nabla T(\sigma, h(\sigma))| \;. \end{aligned} \tag{2.9}$$

In order to appreciate the order of magnitude of the errors committed by substituting (2.8) and (2.9) with the linearized relations, we will push the Taylor development to the second order. For the sake of conciseness we shall use the symbol q' to express the vertical or radial derivative of q, according to the context.

Linearization of (2.8): we have

$$\begin{aligned} W_0 &= U(\widetilde{h} + \zeta) + T(\widetilde{h} + \zeta) = \\ &= U(\widetilde{h}) + U'(\widetilde{h})\zeta + \frac{1}{2} U''(\widetilde{h})\zeta^2 + \\ &+ T(\widetilde{h}) + T'(\widetilde{h})\zeta + O_3 \;. \end{aligned} \tag{2.10}$$

We call *geodetic anomaly of the potential W* the quantity

$$DW = W_0 - U(\widetilde{h}) \;. \tag{2.11}$$

We also note that

$$U'(\widetilde{h}) \sim -\gamma(\widetilde{h})$$
$$U''(\widetilde{h}) \sim -\gamma'(\widetilde{h}) \ ,$$

so that (2.10) can be reorganized as

$$DW = T(\widetilde{h}) - \gamma(\widetilde{h})\zeta - \frac{1}{2}\gamma'(\widetilde{h})\zeta^2 + T'(\widetilde{h})\zeta + O_3 \quad (2.12)$$

Now consider that in (2.12) we expect T, $\gamma\zeta$ to be the first-order terms, while $\frac{1}{2}\gamma'\zeta^2$, $T'\zeta$ should be the second-order terms, candidate to be neglected.

But this is true only if both T and $\gamma\zeta$ are of the *same* order of magnitude; since $O_M(T)$ is fixed, we must introduce then a *compatibility condition* stating that

$$\gamma\zeta \sim T \ , \quad (2.13)$$

which, on account of (2.5), implies

$$\zeta \sim 10^2 \text{ m}. \quad (2.14)$$

Notice that (2.13) is not the Bruns relation (1.7), because in general $\widetilde{h}$ doesn't need to be the Marussi telluroid defined by (1.4), i.e., by the condition $DW = 0$, yet $O_M(\zeta)$ has to be 100 m, i.e., the telluroid has to be in a band of 100–200 m from the Earth surface at most, if we want the linearization procedure to work. A larger height anomaly might bring us to false conclusions. The fact that the Marussi telluroid satisfies the compatibility condition is a lucky empirical fact that is verified a posteriori, once the solution T has been found and not an a priori statement.

Given the above, we can pass to evaluate the second-order terms and decide whether they are negligible or not. Before we do that, we must fix the order of magnitude of the negligible errors, ε_w, in potential. We state the rule that ε_w is negligible if

$$O_M(\varepsilon_w) = 1 \text{ cm} \cdot \gamma = 10 \text{ Gal} \times \text{ m}. \quad (2.15)$$

In fact, by using the value in Table 2.1, and (2.14), we have

$$\frac{1}{\gamma}(\frac{1}{2}\gamma'\zeta^2) \sim 1.5 \text{ mm} \quad (2.16)$$

Moreover,

$$\frac{T'\zeta}{\gamma} \sim 1 \text{ cm}. \quad (2.17)$$

As we see, this term is still in our acceptable error range.

Linearization of (2.9): before starting our computation, we recall the differential formula, valid up to the second order,

$$|\mathbf{v} + d\mathbf{v}| = |\mathbf{v}| + \mathbf{e} \cdot d\mathbf{v} + \frac{1}{2|\mathbf{v}|} d\mathbf{v} \cdot (I - P_e) d\mathbf{v} \tag{2.18}$$

where

$$\mathbf{e} = \frac{\mathbf{v}}{|\mathbf{v}|} \ , \ \ P_e d\mathbf{v} = \mathbf{e}(\mathbf{e} \cdot d\mathbf{v}) \ .$$

By applying (2.18) to (2.9), we get

$$g_0 = \gamma(h) + \mathbf{e} \cdot \nabla T(h) + \frac{1}{2} \frac{1}{\gamma(h)} \nabla T \cdot (I - P_e) \nabla T + O_3 \tag{2.19}$$

where all quantities are still evaluated at h. Note that in (2.19) one can write, with a very good order of approximation,

$$\mathbf{e} \cong -\boldsymbol{\nu} \ , \ \ \mathbf{e} \cdot \nabla T \cong -T' \ ;$$

this is because the tangent to the normal field lines is equal to $\boldsymbol{\nu}$ on the ellipsoid and it varies very slowly with altitude, at least at topographic heights.

So we can write, developing to the second order,

$$\begin{aligned} g_0 = {} & \gamma(\widetilde{h}) + \gamma'(\widetilde{h})\zeta + \frac{1}{2}\gamma''(\widetilde{h})\zeta^2 + \\ & -T'(\widetilde{h}) - T''(\widetilde{h})\zeta + \\ & + \frac{1}{2} \frac{1}{\gamma(\widetilde{h})} \nabla T(\widetilde{h}) \cdot (I - P_e) \nabla T(\widetilde{h}) + O_3 \ . \end{aligned} \tag{2.20}$$

Again we define the geodetic anomaly of g as

$$Dg = g_0 - \gamma(\widetilde{h}) \ , \tag{2.21}$$

observing that Dg will coincide with the usual free air gravity anomaly Δg as soon as $\widetilde{h}$ is chosen as the height of the Marussi telluroid. We note as well that the last term, being already a second-order term, can be directly evaluated at $\widetilde{h}$. So, reorganizing (2.20), we get

$$\begin{aligned} Dg = g_0 - \gamma(\widetilde{h}) = {} & -T'(\widetilde{h}) + \gamma'(\widetilde{h})\zeta + \\ & + \frac{1}{2}\gamma''(\widetilde{h})\zeta^2 - T''\zeta + \frac{1}{2\gamma(\widetilde{h})} |\nabla_h T|^2 + O_3 \end{aligned} \tag{2.22}$$

where $\nabla_h T$ is just the horizontal gradient of T. It is immediate to verify that T' and $\gamma' \ \zeta$ are of the same order of magnitude, so we need to analyze second order terms.

To verify whether the second-order terms are negligible we need to fix the order of a negligible error in g. We fix such error ε_g at the level

$$\varepsilon_g \sim 30\,\mu\text{Gal} \tag{2.23}$$

on the basis of the following simplistic reasoning. Since Δg, with an order of magnitude of 100 mGal, gives rise to a ζ of the order of 100 m, we could expect that, if we had an error with the same spectral shape as the signal and a mean square value of 10 μGal $= 10^{-4} \times 100$ mGal, the corresponding error in T/γ would be of the order of $10^{-4}100$ m $= 1$ cm, which is compatible with our previous choice. This however, as we shall see soon, is a severe restriction that we decide to relax at least by a factor of 3. The justification of this choice is that we expect the errors we are going to study (particularly the error related to T) to have more energy in the higher degrees, and since the operator that brings from Δg to T is a smoother, we would expect a more favorable error propagation. Based on that and on the direct experience, we will accept the threshold (2.23).

We now examine the three second-order terms in (2.22).

We have, in simple spherical approximation, i.e., with $\gamma = \frac{GM}{r}$

$$O_M\left(\frac{1}{2}\gamma''\zeta^2\right) \cong O_M\left(3\gamma\left(\frac{\zeta}{r}\right)^2\right) \cong 8 \cdot 10^{-4}\,\text{mGal}\,,$$

which is indeed totally irrelevant.

Let us consider then $O_M(T''\zeta)$. The value of $O_M(T'')$ in Table 2.1 is the 90 % quantile of T'' at zero level, according to a global model of the anomalous potential. With this value one has

$$O_M(T''\zeta) = 6 \times 10^{-5}\ \text{Gal} = 60\,\mu\text{Gal}.$$

With our definition (2.1) of O_M this is still compliant with (2.23), although it is clear that this term is mostly concerning us in the linearization procedure. As for the last term of (2.3), recalling that

$$\frac{|\nabla_h T|}{\gamma} \cong \delta\,,$$

we have

$$O_M\left(\frac{1}{2}\frac{|\nabla_h T|^2}{\gamma}\right) = \frac{1}{2}O_M(\gamma\delta^2) = 4.5 \times 10^{-4}\ \text{Gal} = 45\,\mu\text{Gal}.$$

Also for this term we are close to the maximum admissible value.

All in all one has the impression that by keeping only linear terms in (2.12) and (2.20) it is difficult to guarantee that the overall committed error is 1 cm as a maximum. More probably a few centimeters could be a more realistic figure. However, in some cases our estimates are really pessimistic. In this sense we want to elaborate a little more on the term $T''\zeta$, not only because it is the one that seems to have the largest impact if neglected, but also because its introduction into the BVP would change its nature because of the second-order oblique derivative of T. To reconduct such a term to a more favourable figure we will use the two well-known relations, valid in spherical approximation,

$$T' = -\frac{2}{r}T - \Delta g \, , \tag{2.24}$$

$$\Delta g' = -\frac{2}{r}\Delta g \, . \tag{2.25}$$

The relation (2.25) in particular gives an approximate vertical derivative of Δg in free air ([2], Sect. 2.4), as it is correct in the present case because we do not take into account the effects of the masses between S and $\widetilde{S}$.

Combining the above relations, one finds

$$\begin{aligned} T'' &= \frac{2}{r^2}T - \frac{2}{r}T' - \Delta g' = \\ &= \frac{2}{r^2}T + \left(\frac{4}{r^2}T + \frac{2}{r}\Delta g\right) + \frac{2}{r}\Delta g = \\ &= \frac{6}{r^2}T + \frac{4}{r}\Delta g \, . \end{aligned}$$

Accordingly, one can write

$$\begin{aligned} O_M(T''\zeta) &= 6O_M\left(\frac{T}{r}\frac{\zeta}{r}\right) + 4O_M\left(\Delta g\frac{\zeta}{r}\right) = \\ &= 1.6 \times 10^{-6}\,\text{Gal} + 6.6 \times 10^{-6}\,\text{Gal} = 8.2\,\mu\text{Gal} \, . \end{aligned}$$

As we see this estimate is almost one order of magnitude less than the one previously found.

With all the above discussions, we can finally say that, with an error of a few centimeters in geoid in the worst case, we can substitute the boundary relation of the non-linear Scalar Molodensky problem with the general linearized version

$$DW = T(\widetilde{h}) - \gamma(\widetilde{h})\zeta \tag{2.26}$$

$$Dg = -T'(\widetilde{h}) + \gamma'(\widetilde{h})\zeta \; ; \tag{2.27}$$

this estimate substantially agrees with the results of [3].

One has to recall that in the above boundary relations T' means $\frac{\partial T}{\partial h}$ and similarly γ'.

Solving (2.26) with respect to ζ, one gets the generalized Bruns relation

$$\zeta = \frac{T - DW}{\gamma} \tag{2.28}$$

and substituting into (2.27) one finds

$$-T' + \frac{\gamma'}{\gamma} T = Dg + \frac{\gamma'}{\gamma} DW\ , \tag{2.29}$$

which has to hold on the telluroid $\widetilde{S}$. All the above holds only if the compatibility condition (2.14) is verified.

After some reflections, the above discussion leads to conclude that the following *equivalence principle* holds:

> two linearized formulations of the Molodensky problem
>
> $$\begin{cases} \Delta T = O & \text{in } \widetilde{\Omega} \\ -T' + \frac{\gamma'}{\gamma} T = Dg + \frac{\gamma'}{\gamma} DW & \text{on } \widetilde{S} \\ T' = O\left(\frac{1}{r^3}\right) & r \to \infty \end{cases} \tag{2.30}$$
>
> are equivalent if they can be transformed one into the other, with the respective boundary relations given on telluroids that are in the same linearization band, in particular the two telluroids should be different from one another by no more that 100–200 m.

Notice that any linear problem

$$Ax = y$$

can be transformed into an equivalent one

$$A\xi = \eta$$

with $\xi = x - x_0$ and $y = y - Ax_0$. So, what gives rise to the equivalence of two BVPS of the type (2.30) is in particular that the two telluroids are in the same linearization band. It is interesting to note that the idea of using a "gravimetric" telluroid, i.e., one for which $Dg = 0$, already considered by Krarup [1] and later on by Sansò [4] for more theoretical reasons, is in fact at the boundary of the equivalence to the classical Molodensky problem (1.9). In fact the condition $Dg = 0$ would lead to pseudo-Bruns relation for ζ_G (see (2.22))

$$\zeta_G = \frac{T'}{\gamma'}\ ,$$

such that

$$O(\zeta_G) \cong \frac{100\,\text{mGal}}{0.3\ \text{mGal}\ \text{m}^{-1}} \cong 300\ \text{m}.$$

This is three times the order of magnitude of the Marussi height anomaly

$$\zeta_M = \frac{T}{\gamma}\ .$$

Even more absurd is the conclusions that one would get by putting directly

$$g_0 - \gamma(\widetilde{h}) + \frac{\gamma'(\widetilde{h})}{\gamma'(\widetilde{h})}[W_0 - U(\widetilde{h})] = 0\ .$$

On the other hand the reason why such a relation cannot be used as a definition of the telluroid is precisely that it takes us out of the linearization band.

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Chapter 3
On the Equivalent Linearized BVP's of Molodensky and Helmert

What is the Linearized Scalar Molodensky Problem has been clarified already in Chap. 1, so now it is time to look at the Helmert approach, but we will do it distinguishing sharply the BVP formulation, consequent upon the main idea of Helmert, from the downward continuation procedure, which is not necessarily a consequence of the above.

Helmert's idea [1, 2] is that "the Earth topography can be replaced by an infinitesimal thin layer of an areal density equal to the product of real mean topographical density and height. This condensation layer could be located anywhere on or beneath the geoid" [3].

So the Helmert procedure is primarily to change our unknown, the potential W, and the data, namely $W_0(\sigma)$, $g_0(\sigma)$, by consistently subtracting the effect of the attraction of topographic masses, with potential $V_t(P)$, and adding back the effect that we would have from a single layer on the condensation surface with layer density (see Fig. 3.1)

$$\nu(\sigma) = \int_{P_0(\sigma)}^{P(\sigma)} \rho(\sigma, z) dz \,, \tag{3.1}$$

where $\rho(\sigma, z)$ is the usual 3D mass density along the normal to the ellipsoid. We will call the potential of this single layer density $V_c(P)$.

Therefore, the Helmert potential correction amounts to change $W(P)$ by the, supposedly known, potential

$$\delta V^H(P) = V_t(P) - V_c(P) \,, \tag{3.2}$$

so that our new unknown is now

$$W^H(P) = W(P) - \delta V^H(P) \,. \tag{3.3}$$

One important evaluation we take from literature, namely that (cf. [4]).

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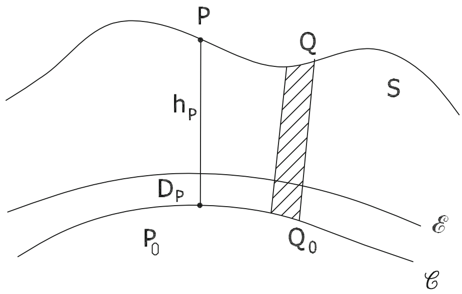

Fig. 3.1 Geometry of Helmert's correction

$$O_M\left(\frac{\delta V^H}{\gamma}\right) = 2\ \mathrm{m} \tag{3.4}$$

We consider this as our empirical truth, that will be basic in our equivalence reasoning. The literature has first of all considered the difficulty of using (3.1) because of the uncertainty in the knowledge of $\rho(\sigma, z)$. For instance in [4] it is considered that since the whole effect of δV^H is less than 2 m in geoid, it should be enough to know the averaged density along the column to an accuracy better than 0.5 %. It is recognized that this might be difficult in mountainous areas, while in flat areas a 5 % error in ρ could be ignored. Although a little simplistic, we accept for the moment the argument and in fact we assume that ρ is constant, so that (3.1) becomes simply

$$\nu(\sigma_P) = \rho[h_P + D_P] \tag{3.5}$$

where h_P is the height of P above the ellipsoid, and D_P is the depth of the compensation surface $\mathscr{C}$ with respect to $\mathscr{E}$: note that in principle D_P could be either positive or negative. Note also that (3.5) implicitly assumes the parallelism of vertical lines. A more rigorous formula can be found in Heck [3], accounting at least for the spherical convergence of the vertical.

Equation (3.5) however is worrying in that h_P is in fact unknown, since in h_P is hidden the geoid undulation, N_{P_0}, according to the well-known relation (e.g. see [5], Sect. 2.4).

$$h_P = H_P + N_P\ ,$$

with H_P the orthometric height of P. Many authors believe that H_P is a known quantity, because

$$H_P = \frac{W_0 - W_P}{\overline{g}}$$

with $\overline{g}$ the mean gravity between S and the geoid along the vertical through P.

That H_P might be known at the 1 cm accuracy level could be confuted again, specially because of the lack of knowledge of ρ inside the masses (see the discussion in [5], Sect. 2.4), however since we are working with a model of constant density, we don't dwell on this point. To bypass this difficulty Helmert invented the so-called *second condensation method*, by choosing

$$D_P = -N_P \ ,$$

i.e.,

$$\nu(P) = H_P \rho \ . \tag{3.6}$$

This means that the geoid itself is chosen as the condensation surface. Since the layer density is now known, we could think that the problem is solved, but this is not the case, because the geometry is now unknown, since the topographic layer $\{N_P \leq h \leq N_P + H_P\}$ is floating on the unknown geoid and taking the geometry into account to compute δV^H would change the nature of the linearized BVP.

Usually what is done is to flatten the geoid to a sphere, or even, as we will do here, to a plane (see [3]). We perform the computation in planar approximation to bound the order of magnitude of such a flattening error.

We choose an example of a plateau 1000 m high, with the underlying geoid and we map it to the flattened body as in Fig. 3.2.

For the second body the topographic potential can really be computed

$$V_t'(P) = G\rho \int dS_Q \int_0^{H_Q} \frac{dz}{\sqrt{s_{PQ}^2 + (z - H_P)^2}} \ , \tag{3.7}$$

where s_{PQ} is the horizontal distance between P and Q. While for the true body one should have

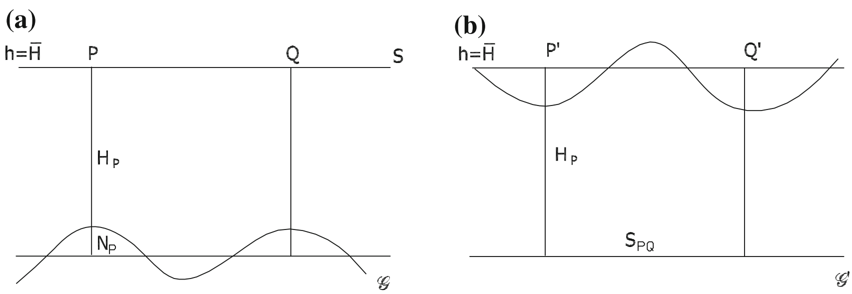

Fig. 3.2 **a** The topographic masses of the example, **b** the flattened body for which the potential can be computed

$$V_t(P) = G\rho \int dS_Q \int_{N_Q}^{H_Q+N_Q} \frac{dz}{\sqrt{s_{PQ}^2 + (z-H)^2}}, \tag{3.8}$$

The difference between (3.8) and (3.5) is the flattening error F, and we would like to find a bound for it. After some approximations one gets

$$\begin{aligned} F(P) &= V_t(P) - V_t'(P) \cong \\ &\cong G\rho \int dS_Q N_Q \left\{ \frac{1}{s_{PQ}} - \frac{1}{\sqrt{s_{PQ}^2 + \overline{H}^2}} \right\}. \end{aligned}$$

Recalling our definition of $O_M(\,)$, we can write

$$\begin{aligned} |F(P)| &\le G\rho O_M(N) \int dS_Q \left\{ \frac{1}{s_{PQ}} - \frac{1}{\sqrt{s_{PQ}^2 + \overline{H}^2}} \right\} = \\ &= 2\pi G\rho O_M(N) \int_0^{+\infty} ds \cdot s \left\{ \frac{1}{s} - \frac{1}{\sqrt{s^2 + \overline{H}^2}} \right\} = \\ &= 2\pi G\rho O_M(N) \cdot \overline{H}. \end{aligned} \tag{3.9}$$

With $O_M(N) = 10^2$ m, $\overline{H} = 10^3$ m we get

$$\frac{|F(P)|}{\gamma} \sim 1 \text{ cm}. \tag{3.10}$$

Since the majorization is really very crude, we can convince ourselves that, apart maybe from very high mountains, the flattening error is in fact negligible.

A very similar comment holds true for the other flattening error, due to a change of the single layer deposited on the geoid $\mathscr{G}$ with respect to a similar single layer deposited on the horizontal plane (Fig. 3.3).

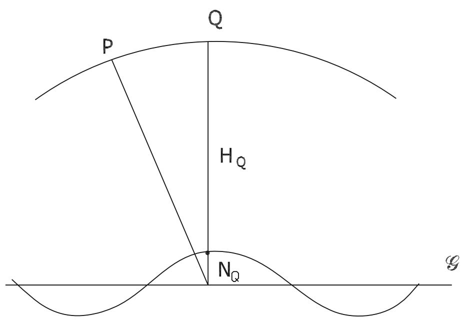

Fig. 3.3 The flattening of Helmert's single layer

The conclusion is that, with the above proviso, $\delta V^H(P)$ can be considered as computable and known.

Now we put the attention on how to modify consistently the data. Naturally the potential values $W_0(\sigma)$ are changed into

$$W_0^H(\sigma) = W_0(\sigma) - \delta V_0^H(\sigma) \; ; \tag{3.11}$$

here the index 0 stems from the fact that the quantities are known or computed on the surface S at the point P, with horizontal coordinates σ.

When we turn the attention to $g_0(\sigma)$, we find an obstacle in that g is a nonlinear functional of W, and so, we have to write

$$g(P) = |\nabla W(P)| = |\nabla W^H(P) + \nabla \delta V^H(P)| \; . \tag{3.12}$$

However, in view of the small size of δV^H, we are entitled to linearize (3.12), putting

$$g(P) = g^H(P) - \mathbf{n}^H \cdot \nabla \delta V^H(P) \tag{3.13}$$

where $g^H(P) = |\nabla W^H(P)|$ and $\mathbf{n}^H$ is the opposite of the direction of $\nabla W^H(P)$. Since $\mathbf{n}^H$ is very close to $\mathbf{n}$ and to $\boldsymbol{\nu}$ too, the normal of the ellipsoid, one can rewrite (3.13) as

$$g(P) = g^H(P) - \frac{\partial}{\partial h} \delta V^H(P) \; . \tag{3.14}$$

Taking P on S one can write, paralleling (3.11),

$$g_0^H(P) = g_0(\sigma) + \left(\frac{\partial}{\partial h} \delta V^H \right)(\sigma) \; . \tag{3.15}$$

Now, in order to repeat the same reasoning done in Chap. 2 and to linearize our problem with data $W_0^H(\sigma)$, $g_0^H(\sigma)$ and unknowns $W(P)$, S, we need to define our "Helmert" telluroid $\widetilde{S}^H$, what we do by establishing the mapping $P \in S \to \widetilde{P}^H \in \widetilde{S}^H$ according to

$$\begin{cases} W^H(h_P) = U(\widetilde{h}_P^H) \\ \sigma_P = \sigma_{\widetilde{P}^H} \; . \end{cases} \tag{3.16}$$

The second of (3.16) says that we use the ellipsoidal normal to create the mapping (Fig. 3.4). The first of (3.16) is a strict analogous of the Marussi telluroid mapping, only using $W^H(P)$ instead of $W(P)$

Let us recall that the height anomaly ζ_P is by definition

$$\zeta_P = h_P - \widetilde{h}_P \; , \tag{3.17}$$

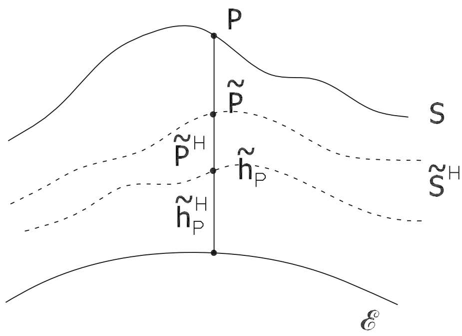

Fig. 3.4 Telluroid $\widetilde{S}$ and Helmert telluroid $\widetilde{S}^H$

so that

$$h_P = h_{\widetilde{P}} + \zeta_P = \widetilde{h}_P + \zeta_P \ . \tag{3.18}$$

Similarly we can define the Helmert height anomaly as

$$\zeta_P^H = h_P - \widetilde{h}_P^H \tag{3.19}$$

and note that, accordingly,

$$h_P = h_{\widetilde{P}^H} + \zeta_P^H = \widetilde{h}_P^H + \zeta_P^H \ . \tag{3.20}$$

Combining (3.18) and (3.20) we see that

$$\widetilde{h}^H - \widetilde{h} = -(\zeta^H - \zeta) \ . \tag{3.21}$$

Moreover, as by linearizing $W(h)$ starting from $U(\widetilde{h})$ we find the Bruns relation

$$\zeta_P = \frac{T(\widetilde{h})}{\gamma(\widetilde{h})} \ , \tag{3.22}$$

by linearizing $W^H(h)$ starting from $U(\widetilde{h}^H)$, we obtain

$$\zeta_P^H = \frac{T^H(\widetilde{h}^H)}{\gamma(\widetilde{h}^H)} \ , \tag{3.23}$$

where the Helmert anomalous potential T^H is defined by the relation

$$W^H(P) = U(P) + T^H(P) \ , \tag{3.24}$$

On the other hand, by using our basic definition (3.3), we see that

$$W^H(P) = W(P) - \delta V^H(P) = U(P) + T(P) - \delta V^H(P) \, . \tag{3.25}$$

Comparing (3.25) with (3.24) we conclude that

$$T^H(P) = T(P) - \delta V^H(P) \, . \tag{3.26}$$

Moreover, always from (3.25), we find

$$\begin{aligned} W^H(h) = U(\widetilde{h}^H) = W(h) - \delta V^H(h) = \\ = U(\widetilde{h}) - \delta V^H(h); \end{aligned}$$

by linearizing, this gives

$$U(\widetilde{h}^H) - U(\widetilde{h}) = -\gamma(\widetilde{h}^H - \widetilde{h}) = -\delta V^H$$

i.e.

$$\widetilde{h}^H - \widetilde{h} = \frac{\delta V^H}{\gamma} \, . \tag{3.27}$$

Recalling (3.4), we find that $\widetilde{S}$ and $\widetilde{S}^H$ are separated by a small height, at most 2 m, and therefore according to our compatibility condition (2.14) we see that *telluroid and Helmert telluroid are in the same linearization band*.

We note incidentally that (3.27), taking (3.21) into account, is consitent with what we get by subtracting (3.22) from (3.23) and neglecting higher order terms, namely

$$\delta\zeta = \zeta^H - \zeta = -\frac{\delta V^H}{\gamma} \, . \tag{3.28}$$

It is time now to linearize the observation equation for $g_0^H(\sigma)$. Following the same reasoning as in Chap. 2 and using (3.23) we find on the Helmert telluroid $\widetilde{S}^H$,

$$g_0^H - \gamma(\widetilde{h}^H) = \Delta g_0^H = -\frac{\partial T^H}{\partial h} + \frac{\gamma'}{\gamma} T^H \, . \tag{3.29}$$

This is indeed the Helmert analogous of Molodensky's linear problem (1.9). The quantity $\Delta g^H = g^H - \gamma(\widetilde{h}^H)$ is called the *Helmert gravity anomaly*.

It remains only to close the loop and show that solving the BVP (3.29) with a regular harmonic function T^H and computing $T = T^H + \delta V^H$ we get in fact a solution to (1.9).

To do that we first have to move (3.29), that holds on $\widetilde{S}^H$, to $\widetilde{S}$. Since this implies a vertical shift of at most 2 m, we note that the right hand side of (3.29), that is proportional to the first-order infinitesimals T^H, $\frac{\partial T^H}{\partial h}$, will not change significantly. To this purpose one can review the arguments of Chap. 2 on second-order terms, but with a much smaller variations in h, since $O_M(\zeta) = 100$ m, while $O_M(\delta\zeta) = 2$ m.

As for the left hand side, we note that g_0^H is observed/computed on S and as such it is invariant. On the contrary we have, recalling (3.27)

$$\begin{aligned}\gamma(\widetilde{h}^H) &= \gamma(\widetilde{h} + \widetilde{h}^H - \widetilde{h}) \cong \gamma(\widetilde{h}) + \gamma' \cdot (\widetilde{h}^h - \widetilde{h}) = \\ &= \gamma(\widetilde{h}) + \gamma' \frac{\delta V^H}{\gamma} \,.\end{aligned} \tag{3.30}$$

therefore the left hand side of (3.29), when we move from $\widetilde{S}^H$ to $\widetilde{S}$, changes according to the relation

$$\begin{aligned}\Delta g^H &= g_0^H - \gamma(\widetilde{h}^H) = \\ &= g_0^H - \gamma(\widetilde{h}) - \frac{\gamma'}{\gamma}\delta V^H \,.\end{aligned}$$

But, recalling (3.15), we can write

$$\begin{aligned}\Delta g^H(\widetilde{h}^H) &= g_0 - \gamma(\widetilde{h}) + \frac{\partial}{\partial h}\delta V^H - \frac{\gamma'}{\gamma}\delta V^H \\ &= \Delta g(\widetilde{h}) + \frac{\partial}{\partial h}\delta V^H - \frac{\gamma'}{\gamma}\delta V^H \,.\end{aligned} \tag{3.31}$$

Substituting in (3.29) we finally get

$$\Delta g(\widetilde{h}) = g_0 - \gamma(\widetilde{h}) = -\frac{\partial}{\partial h}(T^H + \delta V^H) + \frac{\gamma'}{\gamma}(T^H + \delta V^H) \,. \tag{3.32}$$

As we see (3.32) is exactly the linearized scalar Molodensky BVP, as defined in (1.9). As far as such a problem has a unique solution, as reported for instance in recent literature ([5–7]), in suitable Sobolev spaces, we can conclude that $T = T^H + \delta V^H$ and the equivalence between Molodensky's and Helmert's approaches is proved.

One last word of caution is useful on the asymptotic behaviour of our solutions when $r \to \infty$.

We have consistently defined T and fixed our reference system in such a way that

$$T = O\left(\frac{1}{r^3}\right) \,, \quad r \to \infty \,.$$

This means that in the asymptotic development of T we have (see [5])

$$T_{00} = 0 \,, \;\; T_{1,m} = 0 \;\; m = -1, 0, 1 \,. \tag{3.33}$$

The first part of (3.33) is related to the fact that the total mass generating T has to be 0. The second part of (3.33) says that the origin of the coordinate system is placed at the barycentre of those masses, or equivalently that the barycentre of the actual and of the normal fields coincide.

As for the first condition, this is guaranteed by the condensation mechanism which can conveniently be refined as in [3], when we want to take into account the convergence of vertical lines. As for the second condition, it can always be imposed, but then also Δg^H has to satisfy those conditions. Alternatively, one can introduce three scalar unknowns used to annihilate the first-degree coefficients of Δg^H. This is called the *Hörmander trick* also in recent Helmert literature [4, 8].

In any event, introducing a certain number of scalar unknowns to form a linear combination of functions to be added to Δg^H in the BVP is a condition to obtain existence and uniqueness of the solution of Molodensky's problem, under not too restrictive geometrical conditions on $\widetilde{S}$, as shown, e.g., in [6]. As mentioned this is important to prove that Molodensky's and Helmert's BVP's are equivalent. So we shall not dwell anymore on this point.

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Chapter 4
On the Equivalent BVPs of Stokes and Helmert, and Their Relations to the Molodensky BVP by Analytical Continuation

As stated in the abstract, we will closely analyze the two different BVPs most often presented in the literature, namely the HM and the HS problems, and then examine the equivalence of their solutions. For this reason, this chapter is articulated in two sections. To make the mathematical derivations easier to follow, especially because equations and parameters refer to different surfaces or points along the vertical, we introduce Fig. 4.1 which shows the boundary surfaces and heights for the Stokes and Molodensky BVPs in their original and in their 'Helmertized' versions.

4.1 The Helmert Stokes BVP

We have already seen in Chap. 3 that when the topography is condensed onto the geoid and we work with the Helmert gravity and disturbing potentials (see Eqs. 3.25 and 3.26)

$$W^H = U + T^H = W - \delta V^H, \tag{4.1}$$

$$T^H = T - \delta V^H, \tag{4.2}$$

where δV^H is given by Eq. (3.2), the telluroid is changed to the Helmert telluroid or co-telluroid $\tilde{S}^H$, which is used in the HM BVP (see also Eq. 3.29):

$$\nabla^2 T^H = 0 \text{ outside } \tilde{S}^H$$

$$-\frac{\partial T^H}{\partial h} + \frac{\gamma'}{\gamma} T^H = \Delta g^H = g^H(h) - \gamma(\tilde{h}^H) \text{ on } \tilde{S}^H \tag{4.3}$$

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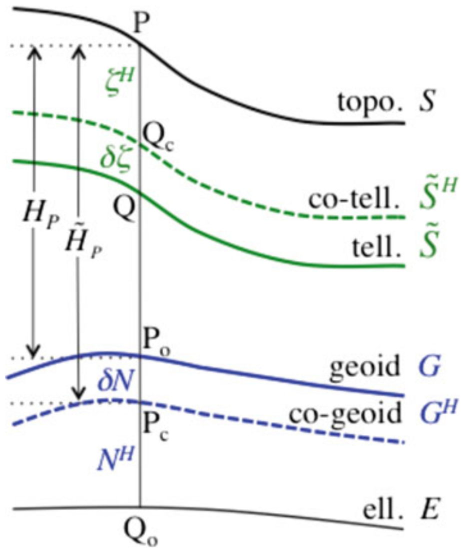

Fig. 4.1 Boundary surfaces for the Stokes and Molodensky BVPs

Analogously, the geoid G is changed to the co-geoid G^H, which will be the boundary surface for the Stokes Helmert BVP. The solution of this BVP with given Helmert gravity anomalies on the co-geoid will provide T^H on G^H or, by Bruns's equation, the co-geoidal undulation N^H

$$N^H = \frac{T^H(P_c)}{\gamma(Q_0)}. \tag{4.4}$$

The boundary condition on G^H can be derived as follows. Starting from Eq. (4.1), we obtain for point P_c

$$g^H(P_c) = -\frac{\partial W^H}{\partial h}\bigg|_{P_c} = -\frac{\partial U}{\partial h}\bigg|_{P_c} - \frac{\partial T^H}{\partial h}\bigg|_{P_c} = \gamma(P_c) - \frac{\partial T^H}{\partial h}\bigg|_{P_c}. \tag{4.5}$$

We have made here the acceptable assumption that the derivatives along the ellipsoidal normal and the plumb line are identical. Furthermore, just like in Chap. 3, we will also assume here that first-order Taylor series expansions are acceptable approximations when moving vertically between the various boundary surfaces.

Expressing the value of normal gravity at P_c by the linear term of a Taylor series around point Q_o, and taking Eq. (4.4) into account, yields

$$\gamma(P_c) \cong \gamma(Q_o) + \frac{\partial \gamma}{\partial h}\bigg|_{Q_o} N^H = \gamma(Q_o) + \frac{\gamma\prime}{\gamma}\bigg|_{Q_o} T^H(P_c). \tag{4.6}$$

We can now substitute Eq. (4.6) into Eq. (4.5) to obtain

$$g^H(P_c) = \gamma(Q_o) - \left.\frac{\partial T^H}{\partial h}\right|_{P_c} + \left.\frac{\gamma'}{\gamma}\right|_{Q_o} T^H(P_c). \tag{4.7}$$

The above provides us with the Helmert gravity anomalies on the co-geoid

$$\Delta g^H = g^H(P_c) - \gamma(Q_o). \tag{4.8}$$

Now the Helmert Stokes BVP can be defined analogously to the Helmert Molodensky BVP in Eqs. (4.3) as

$$\nabla^2 T^H = 0 \text{ outside } G^H$$

$$-\frac{\partial T^H}{\partial h} + \frac{\gamma'}{\gamma} T^H = \Delta g^H = g^H(P_c) - \gamma(Q_o) \text{ on } G^H \tag{4.9}$$

The Helmert gravity value on the co-geoid can be derived from the gravity values measured on the Earth's surface by a Taylor expansion (again, keeping the linear term only) around P_c as follows:

$$g^H(P_c) \cong g^H(P) - \left.\frac{\partial g^H}{\partial h}\right|_{P_c} (H_P + \delta N) = g^H(P) - \left.\frac{\partial g^H}{\partial h}\right|_{P_c} \tilde{H}_P \quad , \tag{4.10}$$

where H_P is the orthometric height of point P, $\tilde{H}_P$ is the "orthometric" height of P measured from the co-geoid, and δN is the indirect effect on the geoid due to the Helmert reduction, obtained by the Bruns equation as

$$\delta N = \frac{\delta V^H(P_c)}{\gamma(Q_o)}. \tag{4.11}$$

Writing the Helmert gravity in the gradient term of Eq. (4.10) as the sum of the Helmert gravity disturbance (see Eq. 4.5)

$$\delta g^H(P_c) = g^H(P_c) - \gamma(P_c) = -\left.\frac{\partial T^H}{\partial h}\right|_{P_c} \tag{4.12}$$

and normal gravity $\gamma(P_c)$, we obtain using Eq. (4.12)

$$\begin{aligned} g^H(P_c) &= g^H(P) - \left.\frac{\partial \delta g^H}{\partial h}\right|_{P_c} \tilde{H}_P - \left.\frac{\partial \gamma}{\partial h}\right|_{P_c} \tilde{H}_P \\ &= g^H(P) + \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P - \gamma'(H_P + \delta N) \end{aligned} \tag{4.13}$$

and therefore the equation to compute the Helmert gravity on the co-geoid from the measured gravity values is

$$g^H(P_c) = g(P) + \frac{\delta V^H}{\delta h}(P) - \gamma' H_P - \gamma' \delta N^H + \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P. \tag{4.14}$$

It should be noted here that, perhaps more appropriately, it is possible to derive at the same time the boundary condition in Eq. (4.9) with the Helmert gravity of Eq. (4.14) in its right-hand-side. To do this, we start by taking the vertical gradient of Eq. (4.2) and we obtain at point P

$$-\left.\frac{\partial T^H}{\partial h}\right|_P = -\left.\frac{\partial T}{\partial h}\right|_P + \left.\frac{\partial \delta V^H}{\partial h}\right|_P = g(P) - \gamma(P) + \left.\frac{\partial \delta V^H}{\partial h}\right|_P. \tag{4.15}$$

Expressing the values of normal gravity and the gradient of T by the linear term of Taylor series around points Q_o and P_c, respectively, yields

$$\gamma(P) \cong \gamma(Q_o) + \left.\frac{\partial \gamma}{\partial h}\right|_{Q_o} (H_P + N^H + \delta N) = \gamma(Q_o) + \gamma\prime(H_P + N^H + \delta N), \tag{4.16}$$

$$-\left.\frac{\partial T^H}{\partial h}\right|_P \cong -\left.\frac{\partial T^H}{\partial h}\right|_{P_c} - \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P. \tag{4.17}$$

Substituting the above expressions into Eq. (4.15) we get

$$-\left.\frac{\partial T^H}{\partial h}\right|_{P_c} = g(P) - \gamma(Q_o) - \gamma'(H_P + N^H + \delta N) + \left.\frac{\partial \delta V^H}{\partial h}\right|_P + \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P. \tag{4.18}$$

Finally, using N^H from Eq. (4.4), and Eq. (4.18) becomes

$$-\left.\frac{\partial T^H}{\partial h}\right|_{P_c} + \left.\frac{\partial \gamma}{\partial h}\right|_{Q_o} \frac{T^H(P_c)}{\gamma(Q_o)} = -\left.\frac{\partial T^H}{\partial h}\right|_{P_c} + \frac{\gamma'}{\gamma} T^H(P_c) = \Delta g^H = g^H(P_c) - \gamma(Q_o) =$$
$$= g(P) + \left.\frac{\partial \delta V^H}{\partial h}\right|_P - \gamma' H_P - \gamma' \delta N + \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P - \gamma(Q_o) \tag{4.19}$$

which is exactly the boundary condition on the co-geoid.

In Eqs. (4.14) and (4.19) we recognize, besides the measured gravity on the Earth's surface and the normal gravity on the ellipsoid, *the free-air correction*

$$F = -\left.\frac{\partial \gamma}{\partial h}\right|_{Q_o} H_P = -\gamma' H_P, \tag{4.20}$$

the topographic effect of Helmert's reduction

$$A_t - A_c = -\frac{\partial \delta V^H}{\partial h}\bigg|_P \tag{4.21}$$

computed at point P on the surface, *and the indirect effect on gravity*

$$\delta \Delta g = -\left.\frac{\partial \gamma}{\partial h}\right|_{Q_o} \delta N = -\gamma' \delta N = -\frac{\gamma'}{\gamma}\delta V^H(P_c) \tag{4.22}$$

computed from δV^H on the co-geoid. This term is typically at the sub-mGal level and, even though the co-geoid is not known before the solution of the BVP, it can be computed with sufficient accuracy using orthometric heights.

The computation of the term containing $\partial^2 T^H / \partial h^2$ is more problematic, as it requires the unknown Δg^H, T^H or, equivalently, δg^H on G^H for its computation:

$$\left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P = -\left.\frac{\partial \delta g^H}{\partial h}\right|_{P_c} \tilde{H}_P = \left.(-\frac{\partial \Delta g^H}{\partial h} + \frac{\gamma'}{\gamma}\frac{\partial T^H}{\partial h})\right|_{P_c} \tilde{H}_P. \tag{4.23}$$

Recalling the linear Taylor series expansions for δg^H, Δg^H and T^H, we see that (4.23) can be written as

$$\begin{aligned}\left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P &\cong \delta g^H(P_c) - \delta g^H(P) \\ &= \Delta g^H(P_c) - \Delta g^H(P) - \frac{\gamma'}{\gamma}[T^H(P_c) - T^H(P)]\end{aligned} \tag{4.24}$$

This development shows that this term represents the "analytical (downward) continuation" of the Helmert gravity disturbance from the Earth's surface to the co-geoid. In addition, it illustrates the need for an iterative-type of solution, starting from values on the surface as the initial approximations. This has been studied by Vanicek et al. [1] and is also discussed here in Chap. 5. However, a different and more rigorous mathematical interpretation will be given in Chap. 6.

To close this section, we will now show that the boundary condition in Eq. (4.19) for the HS BVP is equivalent to the boundary condition on the co-geoid for the classical Stokes BVP. Using the following Taylor expansion around point P_c for the term containing δV^H

$$\left.\frac{\partial \delta V^H}{\partial h}\right|_P \cong \left.\frac{\partial \delta V^H}{\partial h}\right|_{P_c} + \left.\frac{\partial^2 \delta V^H}{\partial h^2}\right|_{P_c} \tilde{H}_P \tag{4.25}$$

and substituting into Eq. (4.19) using also Eq. (4.11), we obtain

$$
-\frac{\partial T^H}{\partial h}\bigg|_{P_c} + \gamma' \frac{T^H(P_c)}{\gamma(Q_o)} = g(P) + \frac{\partial \delta V^H}{\partial h}\bigg|_{P_c} + \frac{\partial^2 \delta V^H}{\partial h^2}\bigg|_{P_c} \tilde{H}_P - \gamma' H_P - \gamma' \frac{\delta V^H}{\gamma(Q_o)} + \frac{\partial^2 T^H}{\partial h^2}\bigg|_{P_c} \tilde{H}_P - \gamma(Q_o) \tag{4.26}
$$

or

$$
-\frac{\partial (T^H + \delta V^H)}{\partial h}\bigg|_{P_c} + \gamma' \frac{T^H(P_c) + \delta V^H(P_c)}{\gamma(Q_o)} = g(P) - \gamma(Q_o) - \gamma' H_P + \frac{\partial^2 (T^H + \delta V^H)}{\partial h^2}\bigg|_{P_c} \tilde{H}_P \tag{4.27}
$$

Since

$$
\frac{\partial^2 (T^H + \delta V^H)}{\partial h^2}\bigg|_{P_c} \tilde{H}_P = \frac{\partial^2 T}{\partial h^2}\bigg|_{P_c} \tilde{H}_P \cong \frac{\partial T}{\partial h}\bigg|_{P} - \frac{\partial T}{\partial h}\bigg|_{P_c} = -\delta g(P) + \delta g(P_c) = -g(P) + \gamma(P) + g(P_c) - \gamma(P_c) \tag{4.28}
$$

and

$$
\gamma' H_P \cong \gamma(P) - \gamma(P_c), \tag{4.29}
$$

we see that Eq. (4.27) reduces to

$$
-\frac{\partial (T^H + \delta V^H)}{\partial h}\bigg|_{P_c} + \frac{\gamma'}{\gamma}(T^H + \delta V^H)\bigg|_{P_c} = \Delta g = g(P_c) - \gamma(Q_o) \text{ on } G^H, \tag{4.30}
$$

which is exactly the boundary condition for the classical Stokes BVP on the co-geoid since $T^H + \delta V^H = T$.

Remark. It can be easily seen that if the Helmert Stokes BVP is set on the geoid G rather than the co-geoid G^H, the boundary condition on G given in the following Eq. (4.31) can easily be obtained by simply using in Eq. (4.19) P_o in place of P_c and H_P in place of $\tilde{H}_P$:

$$
-\frac{\partial T^H}{\partial h}\bigg|_{P_o} + \frac{\partial \gamma}{\partial h}\bigg|_{Q_o} \frac{T^H(P_o)}{\gamma(Q_o)} = -\frac{\partial T^H}{\partial h}\bigg|_{P_o} + \frac{\gamma'}{\gamma} T^H(P_o) = \Delta g^H = g^H(P_o) - \gamma(Q_o) = = g(P) + \frac{\partial \delta V^H}{\partial h}\bigg|_{P} - \gamma' H_P - \gamma' \delta N + \frac{\partial^2 T^H}{\partial h^2}\bigg|_{P_c} H_P - \gamma(Q_o) \tag{4.31}
$$

This is, for example, the same as Eq. (24) in [7], who derived the boundary condition of the Helmert Stokes problem directly on the geoid.

4.2 Equivalent Solutions of the Linear Helmert Molodensky and Helmert Stokes BVPs

As a first step, we will illustrate that the Helmert Stokes boundary values on the co-geoid can be obtained from the surface Helmert Molodensky ones. By using $\gamma'\delta N = \gamma'(N - N^H)$ in the right-hand side of Eq. (4.19), we can rewrite it as follows:

$$\begin{aligned}
&\Delta g^H(P_c) = g^H(P_c) - \gamma(Q_o) = \\
&= g(P) + \left.\frac{\partial \delta V^H}{\partial h}\right|_P - \gamma'(H_P + N) + \gamma' N^H + \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P - \gamma(Q_o) \\
&= g^H(P) - \gamma(P) + \gamma' N^H + \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P
\end{aligned} \tag{4.32}$$

Now with the Taylor approximation

$$\gamma(P) \cong \gamma(Q_c) + \gamma'\zeta^H \tag{4.33}$$

Eq. (4.32) becomes

$$\begin{aligned}
&\Delta g^H(P_c) = g^H(P_c) - \gamma(Q_o) = \\
&= g^H(P) - \gamma(Q_c) + \gamma'(N^H - \zeta^H) + \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P \\
&= g^H(P) - \gamma(Q_c) + \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P + \frac{\gamma'}{\gamma}[T^H(P_c) - T^H(P)] \\
&= g^H(P) - \gamma(Q_c) + \left.\frac{\partial^2 T^H}{\partial h^2}\right|_{P_c} \tilde{H}_P + \frac{\gamma'}{\gamma}\left.\frac{\partial T^H}{\partial h}\right|_{P_c} \tilde{H}_P \\
&= g^H(P) - \gamma(Q_c) + \left.\frac{\partial}{\partial h}\left(\frac{\partial T^H}{\partial h} - \frac{\gamma'}{\gamma} T^H\right)\right|_{P_c} \tilde{H}_P
\end{aligned} \tag{4.34}$$

Recognizing that $g^H(P) - \gamma(Q_c)$ is the Helmert Molodensky gravity anomaly, we can finally write

$$\Delta g^H(P_c) = \Delta g^H(P) - \left.\frac{\partial \Delta g^H}{\partial h}\right|_{P_c} \tilde{H}_P. \tag{4.35}$$

We have thus shown that Helmert Stokes anomalies can be obtained by Helmert Molodensky anomalies by "analytical (downward) continuation".

We now present briefly the solution to the HM BVP problem by analytical continuation to point level [2]. The surface Helmert anomalies in Eq. (4.3) are analytically 'reduced' from the Earth's surface to the level surface through point P. Using only the first Molodensky correction term g_1 [2] for the Helmert anomalies yields the anomalies at point level

$$\Delta g^{H^*} \cong \Delta g^H + g_1^H = \Delta g^H - \left.\frac{\partial \Delta g^H}{\partial h}\right|_P (\tilde{H} - \tilde{H}_P), \tag{4.36}$$

where we have used (for convenience in the developments that follow) $\tilde{H} - \tilde{H}_P$ instead of $H - H_P$ or $h - h_P$.

Then the solution for ζ^H can be obtained by evaluating at point P Stokes's integral operator, denoted here for brevity as S{.}, to obtain T^H and then ζ^H by applying Bruns's equation:

$$\zeta^H(P) = \frac{1}{\gamma} S\{\Delta g^{H^*}\} = \frac{1}{\gamma} S\{\Delta g^H - \left.\frac{\partial \Delta g^H}{\partial h}\right|_P (\tilde{H} - \tilde{H}_P)\}. \tag{4.37}$$

Adding the correction $\delta\zeta^H$ (indirect effect on the telluroid) yields the height anomaly at P:

$$\zeta(P) = \zeta^H(P) + \delta\zeta(P) = \frac{1}{\gamma} S\{\Delta g^H - \left.\frac{\partial \Delta g^H}{\partial h}\right|_P (\tilde{H} - \tilde{H}_P)\} + \frac{\delta V^H(P)}{\gamma} \tag{4.38}$$

Remark. It is easy to see that Eq. (4.37) consists of two components: the height anomaly at the co-geoid by the "downward continuation" of the surface anomalies to the co-geoid, and the upward continuation of the resulting height anomaly to point level (see also [3]):

$$\begin{aligned} \zeta^H(P) &= \frac{1}{\gamma} \mathrm{S}\{\Delta g^H - \left.\frac{\partial \Delta g^H}{\partial h}\right|_P \tilde{H}\} + \frac{1}{\gamma} \mathrm{S}\{\left.\frac{\partial \Delta g^H}{\partial h}\right|_P\} \tilde{H}_P) \\ &= \zeta^H(P_c) + \left.\frac{\partial \zeta^H}{\partial h}\right|_P \tilde{H}_P \end{aligned} \tag{4.39}$$

We recognize in the equation above that the first term in the right-hand side is

$$N^H(P_c) = \frac{1}{\gamma} \mathrm{S}\{\Delta g^H - \left.\frac{\partial \Delta g^H}{\partial h}\right|_P \tilde{H}\}. \tag{4.40}$$

Thus the geoid can be computed from "downward continued" Helmert Molodensky surface anomalies as follows:

$$N(P_o) = N^H(P_c) + \delta N(P_c) = \frac{1}{\gamma} \mathrm{S}\{\Delta g^H - \left.\frac{\partial \Delta g^H}{\partial h}\right|_P \tilde{H}\} + \frac{\delta V^H(P_c)}{\gamma}. \tag{4.41}$$

We now write the solution for the HS BVP using the gravity anomalies in Eq. (4.19) or, equivalently, in Eq. (4.35):

$$N(P_o) = N^H(P_c) + \delta N(P_c) = \frac{1}{\gamma}\mathrm{S}\{\Delta g^H - \left.\frac{\partial \Delta g^H}{\partial h}\right|_{P_c} \tilde{H}\} + \frac{\delta V^H(P_c)}{\gamma}. \quad (4.42)$$

We note that the "downward-continued" Helmert Molodensky solution of Eq. (4.41) is formally equivalent to the Helmert Stokes solution of Eq. (4.42) but the gradient of Δg^H needs to be computed at different surfaces, namely the Earth's surface and the co-geoid, respectively.

Remark. If the more usual case of continuation to the geoid instead of the co-geoid is used, then the solution of the two 'Helmertised' BVPs will still be given by Eqs. (4.38) and (4.41) but with P_o in place of P_c and H, H_P in place of $\tilde{H}$, $\tilde{H}_P$. In this case, we will also show below that the ζ and N we derived are related through the Bouguer anomaly of point P.

Using orthometric heights in Eqs. (4.38) and (4.41), and taking into account Eq. (4.40), we can write

$$\begin{aligned}\zeta(P) - N(P_o) &= \zeta^H(P) - N^H(P_o) + \delta\zeta(P) - \delta N(P_o)\\ &= \left.\frac{\partial \zeta^H}{\partial h}\right|_P H_P + \frac{\delta V^H(P) - \delta V^H(P_o)}{\gamma}\\ &\cong -\frac{\Delta g^H}{\gamma} H_P + \frac{\delta V^H(P) - \delta V^H(P_o)}{\gamma}\\ &= -\frac{\Delta g - (A_t - A_c)}{\gamma} H_P + \frac{\delta V^H(P) - \delta V^H(P_o)}{\gamma} \quad (4.43)\end{aligned}$$

For the attractions and potentials of the topography and the condensed topography, we will use the formulas given in [4] in planar approximation (see also [5]), specifically Eqs. (A.11) and (A.12) for $\delta V^H(P)$ and $\delta V^H(P_o)$, respectively, where we will keep only their first, most significant terms, and Eq. (A.14) for the difference in the attraction $A_t - A_c$ at P. Then the topographic effects are as follows:

$$\delta V^H(P) \cong \pi G\rho H_P^2, \quad (4.44)$$

$$\delta V^H(P_o) \cong -\pi G\rho H_P^2, \quad (4.45)$$

$$A_t(P) = 2\pi G\rho H_P - c(P), \quad (4.46)$$

$$A_c(P) = 2\pi G\rho H_P, \quad (4.47)$$

where G is Newton's gravitational constant, ρ is the density of the topographic masses (considered constant) and c is the classical terrain correction. Using Eqs. (4.44)–(4.48) into Eq. (4.43) and omitting the c term, we obtain

$$\zeta(P) - N(P_o) \cong -\frac{\Delta g + c}{\gamma} H_P + \frac{2\pi G\rho H_P^2}{\gamma}$$
$$\cong -\frac{\Delta g - 2\pi G\rho H_P}{\gamma} H_P = -\frac{\Delta g_B}{\gamma} H_P \quad (4.48)$$

We have thus recovered the well-known expression that related the height anomalies and the geoid undulations through the Bouguer anomaly Δg_B (see also [6], Sect. 2.4):

$$\zeta(P) - N(P_o) = -\frac{\Delta g_B}{\gamma} H_P. \quad (4.49)$$

Before closing this chapter, we summarize here the major findings:

- The HM BVP is equivalent to the Molodensky BVP, and the HS BVP is equivalent to the Stokes BVP.
- The boundary values used in the HS BVP can be obtained from the boundary values of the HM BVP by the so-called "analytical downward continuation" process.
- The solutions of the two BVPs are also related by the same process, i.e., the HM solution continued to the co-geoid is identical, in linear approximation, to the HS solution.
- The solutions of both the HM BVP and the HS BVP can be obtained by starting from the surface HM gravity anomalies Δg^H

$$\Delta g(P) = g(P) - \gamma(Q) + \frac{\partial \delta V^H}{\partial h}(P) + \frac{\gamma'}{\gamma}\delta V^H(P) \quad (4.50)$$

- They require the direct effect of the Helmert reduction on gravity, δA^H, on the Earth's surface, and the effect of the Helmert reduction on the potential, δV^H, either on the Earth's surface (HM BVP) or on the co-geoid (HS BVP).
- Both BVP solutions require the computation of the gradient of Δg^H. Although this gradient can be computed with surface HM anomalies as follows

$$\frac{\partial \Delta g^H}{\partial h} = -\frac{\Delta g_P^H}{R} + \frac{R^2}{2\pi}\int\int_\sigma \frac{\Delta g^H - \Delta g_P^H}{l_o^2} d\sigma, \quad l_o = 2R\sin\frac{\psi}{2}, \quad (4.51)$$

its computation on the co-geoid requires an iterative process (successive approximations) as Δg^H on the co-geoid are not known before the BVP is solved.

This is the main difficulty with the "analytical (downward) continuation" process, both from the theoretical and the computational point of view; see next chapter. The *change of boundary* approach in Chap. 7 will shed some new light into this problem.

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Chapter 5
The *Downward Continuation* Approach: A Long-Lasting Misunderstanding in Physical Geodesy

Up to now, we have proved the equivalence of Molodensky's and Helmert's approaches in terms of BVP formulation and of their classical solutions by what is termed the downward continuation method. Now we want to get a closer look at this method, showing that it is not and it cannot be just a downward continuation.

Let us observe that solving a BVP for the exterior domain $\widetilde{\Omega}$ means finding the anomalous potential T in $\widetilde{\Omega}$ and in particular on its boundary $\widetilde{S}$ because from $T(\sigma, \widetilde{h}(\sigma))$, through the Bruns relation (1.7), one can derive the height anomaly $\zeta(\sigma)$ and, subsequently, the shape of the Earth surface, namely $h(\sigma) = \widetilde{h}(\sigma) + \zeta(\sigma)$.

Naturally the knowledge of T in $\widetilde{\Omega}$ carries much more information, allowing us to compute all of its functionals like gravity anomalies at points in space, deflections of the vertical, gravity gradients, etc.; yet, obtaining $T|_{\widetilde{S}}$ is a primary goal of physical geodesy.

However, even if linearized, the BVP is still very hard to solve numerically. The reason lies in the extremely complicated structure of the boundary.

In fact, let us note that the oceanic surface is very smooth and its separation from the Earth ellipsoid is of the order of 100 m, i.e., the ellipsoid itself is in the linearization band and could be used as $\widetilde{S}$ in oceanic areas.

On the contrary, in continental areas, amounting to about 1.7×10^8 km^2, one has a totally irregular surface. Even though the significant effect of the topographic masses below the 1 km resolution can be directly computed and subtracted from the data, yet a 1 km resolution potential T, even restricted to S, requires 1.7×10^8 parameters as a minimum. Apart from some experimental studies [1–3] in the sense of a direct numerical solution of the BVP by the boundary elements method, the most widely used potential model at present, namely the EGM08 global model (see [4]), has a $\sim$10 km resolution and is described by some 4.6×10^6 parameters.

Even EGM08 could be computed only resorting to what is called the *downward continuation method*, which is believed to shift the data from $\widetilde{S}$ down to the ellipsoid so that subsequent simpler least squares or numerical quadrature techniques can be further applied (see [5]).

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As we have seen, both lines of thought, Molodensky's and Helmert's, in their evolution have resorted to the downward continuation concept as an intermediary step to *change the boundary* and then apply some simpler analytical solution based either on known systems of orthogonal functions (typically spherical harmonics) or on simple integral kernels (e.g., Poisson, Stokes, Hotine, etc.).

At this point, two statements have to be clearly made:

(a) mathematically the downward continuation is a completely (exponentially) unstable operation that cannot be performed numerically, unless we apply some kind of stabilization, typical for instance of Tikhonov methods or stochastic methods, like collocation;
(b) on the contrary, both approaches (Molodensky's and Helmert's) apply in some sense the downward continuation in a way that is capable of producing significant numerical results, verified by practical observations.

The conclusion can only be that when we apply the downward continuation in geodesy we automatically exploit some stabilization tool that protects the numerical results from instability. In other words, it means that we are in reality solving a different problem, which leads to a solution close to the correct one, or we solve a problem, leaving some errors that are reduced to zero by convenient iterations.

The formalization of the above "true" problems solved in geodesy will be the object of following chapters. Here we concentrate on understanding the properties of the DC either through the masses or in free air, namely a space free of masses. Indeed this item has been treated in geodetic and geophysical literature. See for instance [6–8] and the references therein, as well as [9, 10] as examples of the applied geophysical side. Here, however, we do not try to find a more clever way of performing the DC; we rather want to study its elementary mathematical properties to make clear that there is no way to avoid (i.e., eliminate) the instability of the problem. In fact one can at most control it.

The first clear perception of the intrinsic instability of DC can be achieved by looking at the simple formula representing a harmonic function in the complement of a sphere.

Let the support of the masses generating a certain potential u, be contained into the open sphere $B_R\{r < R\}$, with boundary S_R. Then (see [11], Sect. 3.4) it is well-known that the following representation holds:

$$u(r, \sigma) = \frac{\mu}{R}\sum_{n=0}^{+\infty} \sum_{m=-n}^{n} u_{nm} \left(\frac{R}{r}\right)^{n+1} Y_{nm}(\sigma) \tag{5.1}$$

where $\mu = GM$, M is the total mass generating u, $\sigma = (\lambda, \varphi)$ are the spherical longitude and latitude, $\{Y_{nm}(\sigma)\}$ is the set of fully normalized spherical harmonics, with respect to the (real) scalar product in L^2_σ, i.e.,

$$(u, v)_{L^2_\sigma} = \frac{1}{4\pi} \int u(\sigma)v(\sigma)d\sigma \tag{5.2}$$

and

$$d\sigma = \cos\varphi d\varphi d\lambda \ .$$

With the above choice of constants, the spherical coefficients $u_{nm} = u_{nm}(R)$ of u are purely non-dimensional.

Now if we compute $u(r, \sigma)$ on an outer sphere $S_{R+\delta R}$ we see that

$$(u(R + \delta R, \sigma), Y_{nm}(\sigma)) = u_{nm}(R + \delta R) = u_{nm}(R) \left(\frac{R}{R + \delta R} \right)^{n+1} \ . \tag{5.3}$$

This means that if we identify functions in $L^2(S_R)$ with functions in L^2_σ, i.e., we pull back $u(R, \sigma)$ as

$$u(\sigma) = u(R, \sigma) \ , \tag{5.4}$$

and we do the same with functions in $L^2(S_{R+\delta R})$, i.e.

$$v(\sigma) = v(R + \delta R, \sigma) \ , \tag{5.5}$$

we can consider the correspondence between $u(R, \sigma)$ and $u(R + \delta R, \sigma)$, namely the upward continuation operator U defined by

$$Uu = U \sum_{n,m} u_{nm} Y_{n,m}(\sigma) = \sum_{n,m} \left(\frac{R}{R + \delta R} \right)^{n+1} u_{nm} Y_{nm}(\sigma) \ , \tag{5.6}$$

as an operator acting from L^2_σ into L^2_σ. This operator is indeed symmetric (we could also say self-adjoint, but we are working only with spaces of real functions) and its eigenvalues and eigenfunctions are

$$U \sim \left(\frac{R}{R + \delta R} \right)^{n+1} \ ; \ \{Y_{nm}(\sigma); \ m = -n, \ldots, 0, \ldots n\} \ . \tag{5.7}$$

This shows at once that, since $\left(\frac{R}{R+\delta R}\right)^{n+1} \to 0$ for $n \to \infty$, U is a compact operator. Notice that no eigenvalue is equal to zero and so U is invertible: by definition the inverse U^{-1} is the downward continuation operator

$$D = U^{-1} \ . \tag{5.8}$$

But then D is an unbounded operator, meaning that:

(a) the domain of D is not the whole L^2_σ. This is because $\{UL^2_\sigma\}$ is a strict subset of L^2_σ, densely contained in such a space;

(b) there are sequences in L^2_σ that tend to zero while their counterimage through D tends to infinity: for example

$$u_n(\sigma) = \left(\frac{R}{R+\delta R}\right)^{(n+1)/2} Y_{n,0}(\sigma) \ ,$$

such that

$$\| \ u_n \ \|^2_{L^2_\sigma} = \left(\frac{R}{R+\delta R}\right)^{n+1} \to 0 \ ,$$

while

$$v_n(\sigma) = Du_n = \left(\frac{R+\delta R}{R}\right)^{(n+1)/2} Y_{n,0}(\sigma)$$

so that

$$\| \ v_n \ \|^2_{L^2_\sigma} = \left(\frac{R+\delta R}{R}\right)^{n+1} \to \infty \ .$$

The geodetic relevance of these elementary statements is the following: given any L^2_σ function f on $S_{R+\delta R}$ it is not true that we can find its DC to S_R, i.e. it is not true that in general we can find a function harmonic down to S_R of which the given function is the trace on $S_{R+\delta R}$. Indeed, we can find a sequence $\{f_n\}$ in L^2_σ of functions which on the contrary can be continued down to S_R (which is basically the Runge–Krarup theorem, see [11], Sect. 3.5) but while the convergence of f_n to f on $S_{R+\delta R}$ is granted, we expect that their image on S_R will be not convergent and possibly even divergent.

To be specific, we will examine the following example.

Let

$$f(r,\sigma) = \sum_{n,m} \frac{1}{(1+n^2)} \left(\frac{R+\delta R}{r}\right)^{n+1} Y_{nm}(\sigma) \ ;$$

as it is clear $f(R+\delta R, \sigma) \in L^2_\sigma$, since

$$\| \ f(R+\delta R,\sigma) \ \|^2_{L^2_\sigma} = \sum_{n=0}^{+\infty} \frac{2n+1}{(1+n^2)^2} < +\infty \ .$$

Now $f(r,\sigma)$ can be approximated by the finite sum function

$$f_N(r,\sigma) = \sum_{n,m=0}^{N} \frac{1}{(1+n^2)} \left(\frac{R+\delta R}{r}\right)^{n+1} Y_{nm}(\sigma)$$

in such a way that

$$\lim_{N\to\infty} \| f(R+\delta R,\sigma) - f_N(R+\delta R,\sigma) \|^2_{L^2_\sigma} = 0 \, .$$

Note that each f_N, for N fixed, as a finite sum of spherical harmonics, is regular on any sphere outside the origin.

On the other hand, one has obviously

$$\lim_{N\to\infty} \| f_N(R,\sigma) \|^2_{L^2_\sigma} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{2n+1}{(1+n^2)^2} \left(\frac{R+\delta R}{R} \right)^{2n+2} = \infty \, .$$

Such elementary example is well present and formally understood in geodetic literature; however, its relevance is sometimes overlooked for several reasons that we try to examine:

(i) The example is deemed maybe too elementary because it involves only a spherical geometry and not realistic surfaces. The observation is inconsistent since, as proved in Appendix A, defining in general the upward continuation operator U between two surfaces S_e and S_i (one external, the other internal) it turns out that, though non self-adjoint, U is a compact invertible operator, with dense image in $L^2(S_e)$, so that its inverse D is unbounded, and the statements made above hold unaltered.

(ii) Sometimes the DC procedure is believed to be a pointwise operation, performed by a Taylor development from the upper point down to the internal surface. We shall dwell in Chap. 7 on such a procedure and its correct interpretation in terms of pseudo boundary value problems. However, we can comment already here that if the Taylor point P is in the harmonicity domain, then $u(P)$ is in fact a real analytic function in a neighborhood of P, and therefore the Taylor series is convergent. However, its use implies that we do know all its derivatives in P and this is indeed an awfully improperly posed problem. Naturally truncating the Taylor series is a powerful regularizer of this improperly posed problem, however this leaves residuals between data and model and only iterations can reduce them. The item of convergence of such iteration will be discussed later on.

(iii) One could also claim that the successful numerical experience of global models, in particular of the EGM08 model with its high resolution of 10 km on ground, computed by reducing data to the ellipsoid, shows that DC can in any way produce good results. The point, however, is that the DC for EGM08 is performed by a local least-squares collocation procedure which is in fact a regularized approach based on a statistical technique. Furthermore, at a very high degree the model is a truncated series, namely an approximate solution in a finite dimensional space, so that compact operators and their inverse can become indeed manageable in a finite dimensional space. This is in fact one of the most popular regularization procedures, offering good results when the truncation degree is wisely chosen on the basis of the resolution and accuracy of the data.

(iv) Finally, one could claim that after all one can penetrate down the surface S into the masses by using Bruns equation ([11], Sect. 2.3) or better its version for the gravity anomaly

$$\frac{\partial \Delta g}{\partial h} = -2C(h)\Delta g(h) + 4\pi G\rho \ , \tag{5.9}$$

coupled with the other well-known equation

$$\frac{\partial}{\partial h}\left(\frac{T}{\gamma}\right) = -\frac{\Delta g}{\gamma} \ , \tag{5.10}$$

as discussed in ([11], Sect. 2.4). Namely this seems to imply only the integration of ordinary differential equations. However this is first of all false, because the mean curvature $C(h)$ of equipotential surfaces hides as a matter of fact the horizontal Laplacian of W ([12], Sect. 2.3). But even assuming that there is a constant $C_0 \cong \frac{1}{R}$, such that $O(C - C_0) \sim 10^{-3}C_0$ (see [11], formula (2.128)), namely transforming (5.9) into

$$\frac{\partial \Delta g}{\delta h} = -2C_0\Delta g + 4\pi G\rho \tag{5.11}$$

and assuming that (5.10) could be simplified to

$$\frac{\partial T}{\partial h} = -\Delta g \ , \tag{5.12}$$

one can see that the instability of DC is clearly displayed in (5.11). In fact the negative sign of the constant coefficient $2C_0$, says that Δg decays exponentially going upward ($dh > 0$) and therefore it increases exponentially going downward ($dh < 0$).

Considering that, with P the point on the surface S,

$$\left|\frac{h_P - h}{R}\right| \leq 10^{-3}$$

for topographic heights, one can express the solution of the two equations as (see [11], Sect. 2.3)

$$\Delta g(h) = \Delta g_P[1 + 2C_0(h_P - h)] - 4\pi G\rho(h_p - h) \tag{5.13}$$

$$T(h) = T(P) + \Delta g_P[(h_P - h) + C_0(h_P - h)^2] + \\ -2\pi G\rho(h_P - h)^2 \ . \tag{5.14}$$

Again, in such equations the polynomial functional shape still keeps the instability character, although it is clear that each term is rather small in the topographic layer, apart form high mountains where $\frac{T(h)-T(P)}{\gamma}$ can amount to some decimeters.

Before going to the conclusion, one remark is in order: (5.14) shows that if we abandon the strict hypothesis that ρ is constant, but we assume for instance that, specially in mountainous areas, it can have fluctuations up to 10 %, we find that the corresponding error in $\frac{T(0)-T(P)}{\gamma}$ due to the constant density model can amount up to 1 cm at 1 km height, 4 cm at 2 km height and so forth. In other words, this errors is small but not negligible in such areas.

Concluding the discussion of this Chapter we arrive at the following statement: strictly speaking the DC of the anomalous potential $T(P)$ below the Earth surface S is mathematically impossible, being plagued by an exponential divergence with depth. Yet the introduction of some a-priori information or the use of some suitable mathematical trick (Tikhonov theory, collocation, etc.) can help in stabilizing the solution, although the so derived potential is uncertain and unreliable at cm-level between S and the geoid. Yet there is hope that the numerical solutions derived up to now can be taken as consistent estimates in the sense that increasing the surface information, with a continuous limit in mind, we might expect that our estimates behave exactly as in the example presented at the beginning of the Chapter, namely they converge to the correct solution from the surface S upward, while they have an increasingly fuzzy behavior between S and the reference sphere S_0.

The next two chapters are devoted to the study of possible interpretations of the actual practice, showing that this can have a more sound mathematical basis.

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Chapter 6
The Change of Boundary Approach

In this chapter, we will try to show that what is called downward continuation is in reality something different, namely an iterative solution based on the change of boundary (CB) approach.

The concept has been illustrated first by Sansò [1] for the simple example of two concentric spheres, for which the theory is rather clear. This will be presented and further developed here, as well.

The general case with an internal sphere and an external, general surface (telluroid) is not yet completely proved, although on the basis of the analysis performed in the Appendix A we could say, by a perturbative argument with a first order theory in the topographic heights compared to the Earth radius, that the CB approach should be valid too, at least for global models of finite maximum degree.

One further comment of historical nature is that what we shall do here is a formalization of a method that has been in fact used numerically in building the first high resolution global model, namely the model of Wentzel [2], with a maximum degree up to 1800.

Since our problem is essentially of Dirichlet type, as defined in the previous Chapter, we can focus the discussion of the CB approach for such a problem.

We do it first in the simplest form, which is basically illustrated by the scheme in Fig. 6.2. We assume to have a star-shaped surface $\widetilde{S} \equiv \{r = R(\sigma)\}$, where $R(\sigma)$ is for instance a Lipschitz function. Moreover, we define a sphere S_0 totally internal to $\widetilde{S}$; $S_0 \equiv \{r = R_0\}$ with $R_0 < \min_\sigma R(\sigma)$ (see Fig. 5.1).

Note that a shift along a radius creates a bi-univocal correspondence between three points $(1, \sigma)$, on the unit sphere, (R_0, σ) on S_0, and (R_σ, σ) on $\widetilde{S}$, so that any function on $\widetilde{S}$ can be considered too as a function of σ only, i.e. it can be pulled back to the unit sphere (Fig. 6.1).

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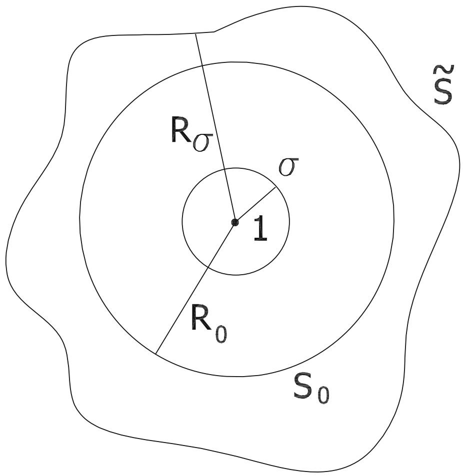

Fig. 6.1 Geometry of the CB

The iterative procedure envisaged as Change of Boundary approach is as follows:

1. Start from $f(\sigma)$ on $\widetilde{S}$.
2. Pull back f to S_0, i.e., consider the same $f(\sigma)$ on S_0, so the pull back operator in this case is just the identity, I.
3. Solve the Dirichlet problem for S_0 and $f(\sigma)$, namely apply the Poisson integral to $f(\sigma)$

$$u_1(r, \sigma) = \Pi f \; .$$

4. Take the trace of u_1 on $\widetilde{S}$,

$$u_1(\sigma) \equiv \Gamma u_1(r, \sigma) \equiv u_1(R_\sigma, \sigma) \; .$$

5. Compute the residual, or error, $r_1(\sigma)$

$$r_1(\sigma) = f(\sigma) - u_1(\sigma)$$

then iterate.

The procedure is represented in Fig. 6.2 where the sets $S_0 \equiv \{r = R_0\}$, $\Omega_0 \equiv \{r > R_0\}$, $\widetilde{S} \equiv \{r = R_0\}$, tell us where the functions in the columns are defined.

We shall call V the operator leading to $u_1(\sigma)$ from f, as well as to $u_2(\sigma)$ from r_1 and so forth

$$V = \Gamma \Pi P B \; , \tag{6.1}$$

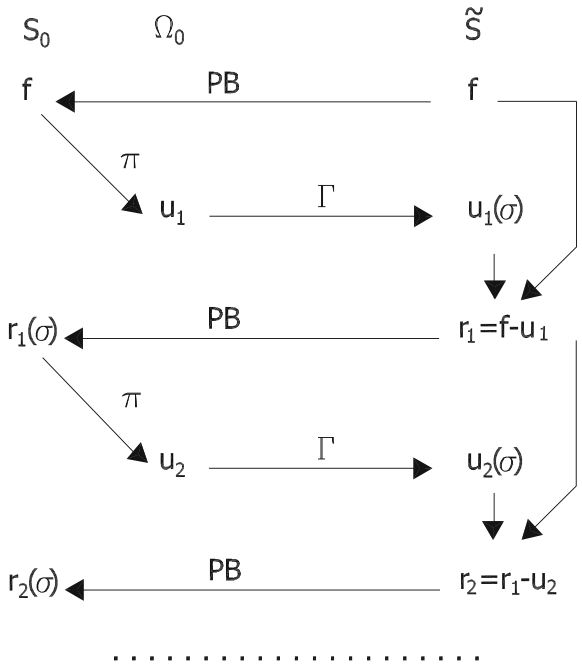

Fig. 6.2 The iterative scheme of the CB

when $PB = I$ as in the present case $V = U$, the upward continuation operator already discussed in Chap. 5 and in the Appendix.

As we can see, the following iterative relations hold:

$$\begin{aligned} r_\ell &= (I - U)^\ell f = \\ &= r_{\ell-1} - u_\ell = \\ &\quad \cdots\cdots\cdots\cdots \\ &= f - \left(\sum_{k=1}^{\ell} u_k \right) . \end{aligned} \tag{6.2}$$

From (6.2) it follows that if we can prove that

$$\lim_{r \to \infty} r_\ell = \lim_{\ell \to \infty} (I - U)^\ell f = 0 \tag{6.3}$$

in L^2_σ, then we also have

$$f(\sigma) = \sum_{k=1}^{+\infty} u_k(\sigma) \ , \tag{6.4}$$

the series being convergent in L^2_σ, so that the corresponding harmonic series of the $u_k(r,\sigma)$ will also converge to the solution of the Dirichlet problem with $f(\sigma)$ as boundary data, according to well-known theorems [3–5]. This is illustrated by the following example.

Example 6.1 Assume $\widetilde{S}$ is a sphere of radius $\widetilde{R}$. In this case, the upward continuation operator U is just

$$\widetilde{U}f = \sum \left(\frac{R_0}{\widetilde{R}}\right)^{n+1} f_{nm} Y_{nm}(\sigma) \ ,$$

so that $I - \widetilde{U}$ is given by

$$(I - \widetilde{U})f = \sum \left[I - \left(\frac{R_0}{\widetilde{R}}\right)^{n+1} \right] f_{nm} Y_{nm}(\sigma) \ .$$

We note that $(I - \widetilde{U})$ is selfadjoint with eigenvalues $\left[I - \left(\frac{R_0}{\widetilde{R}}\right)^{n+1} \right]$ so that its spectral radius is just 1 and therefore

$$\| I - \widetilde{U} \|_{L^2_\sigma} = 1 \ .$$

On the other hand, if we restrict $I - \widetilde{U}$ to the finite space L^2_N, spanned by $\{Y_{nm} \ ; \ n \leq N\}$, we have

$$\| I - \widetilde{U} \|_{L^2_N} = q_{ON} = \left[1 - \left(\frac{R_0}{\widetilde{R}}\right)^{N+1} \right] . \tag{6.5}$$

Moreover, if we put, with obvious notation,

$$I - \widetilde{U} = (I - \widetilde{U})_N + \mathscr{R}_N \tag{6.6}$$

we see that L^2_N is invariant under $(I - \widetilde{U})_N$ and the orthogonal complement of L^2_N, i.e., $(L^2_N)^\perp$ is invariant under $\mathscr{R}_N$. Furthermore, $(I - \widetilde{U})_N$ and $\mathscr{R}_N$ commute and their product is always zero:

$$(I - \widetilde{U})_N \mathscr{R}_N = \mathscr{R}_N (I - \widetilde{U})_N = 0 \ . \tag{6.7}$$

We also observe that

$$\| \mathscr{R}_N \| = \sup_{N<n} \left[1 - \left(\frac{R_0}{\widetilde{R}}\right)^{n+1} \right] = 1 \ . \tag{6.8}$$

Consequently we have

$$(I - U)^\ell = (I - \widetilde{U})^\ell_N + \mathscr{R}^\ell_N \tag{6.9}$$

with

$$\| (I - \widetilde{U})^\ell_N \| \leq q^\ell_{ON} \tag{6.10}$$

and

$$\| \mathscr{R}^\ell_N \| \leq \| \mathscr{R}_N \|^\ell = 1 \ . \tag{6.11}$$

Now let f be any function in L^2_σ, then, after fixing an arbitrary $\varepsilon > 0$, we can find N such that

$$f = f_N + r_N \ , \ f_N \in L^2_N \ , \ r_N \in \mathscr{R}_N$$

and

$$\| f_N \| \leq \| f \| \ , \ \| r_N \| \leq \varepsilon \ .$$

Therefore, recalling (6.9) and (6.10)

$$(I - \widetilde{U})^\ell f = (I - \widetilde{U})^\ell_N f_N + \mathscr{R}^\ell_N r_N \tag{6.12}$$

and we have

$$\| (I - \widetilde{U})^\ell f \| \leq q^\ell_{ON} \| f \| + \varepsilon \ . \tag{6.13}$$

Taking the upper limit of (6.13) we get

$$\overline{\lim_{\ell \to \infty}} \| (I - \widetilde{U})^\ell f \| \leq \varepsilon \ ,$$

and, given the arbitrariness of ε, it has been proven that

$$\lim_{\ell \to \infty} (I - \widetilde{U})^\ell f = 0 \ .$$

Unfortunately, the general case where $\widetilde{S}$ is not a sphere, does not allow us to draw such clear conclusions. However, the analysis reported in the Appendix A shows the following interesting result. Let R_0, S_0 be as before and let $S \equiv \{r = R_\sigma\}$ be our surface such that (see Fig. 6.3)

$$R_0 < R_- = \min R_\sigma < Max R_\sigma = R_+ \ ;$$

call $\Delta = R_+ - R_-$, then, at least to the first order in $\frac{\Delta}{R_0}$, one has (according to what we have called the *Sandwich conjecture* (see (A.65))

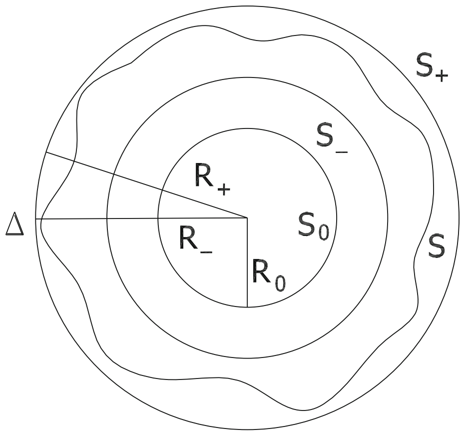

Fig. 6.3 The general geometry of CB

$$\| I - U \|_{L^2_N} \leq \left[1 - \left(\frac{R_0}{R_-} \right)^{N+1} \right] < 1 \, . \tag{6.14}$$

Naturally in this case the operator U

$$Uf = \sum \left(\frac{R_0}{R_\sigma} \right)^{n+1} f_{nm} Y_{nm}(\sigma) \tag{6.15}$$

is not anymore selfadjoint, nor $\{Y_{nm}(\sigma)\}$ are its eigenfunctions. Consequently the nice orthogonal decomposition (6.6) does not hold, nor the powers $(I - U)^\ell$ can be expressed in an elementary way as in (6.12).

However, as observed at the end of Appendix A, if we restrict our analysis to L^2_N, namely we restrict $I - U$ changing it into $P_N(I - U)P_N = P_N - P_N U P_N$, where P_N is the orthogonal projection onto L^2_N, then (6.14) tells us exactly that $P_N(I - U)P_N$ is a contraction for any fixed N. This means that if we start with a global model at maximum degree N and at each step we project the result on L^2_N, then the sequence of the downward computed potentials $u_k(r, \sigma)$ provides a convergent series (see (6.2) and (6.3)). The result is not sharp and we cannot call it a theorem, however it seems to provide a reasonable justification of the CB approach and of its convergency for global models.

One might object however that in practice, before iterating, it is common to pull back the values of $f(\sigma)$ from $\widetilde{S}$ to S_0, not just shifting them along the radius, but rather by using a Taylor development up to some order.

We shall see, by continuing the Example 5.1 of the two spheres, that in fact such operation is a booster of the convergency, in the sense that with the above pull back operator the rate of convergence is higher.

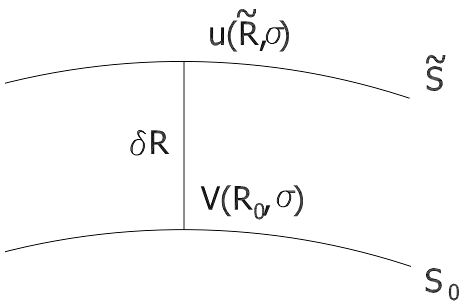

Fig. 6.4 The pull back from $(\widetilde{R}, \sigma)$ to (R, σ)

Example 6.2 We assume to be in the same situation as in Example 6.1.

However we define in this case a pull back operator *PB*, through the relation (see also Fig. 6.4)

$$v(R_0, \sigma) = u(\widetilde{R}, \sigma) - \delta R \frac{\partial}{\partial r} u(\widetilde{R}, \sigma) + \frac{1}{2} \delta R^2 u''(\widetilde{R}, \sigma) = PBu \tag{6.16}$$

Naturally *PB* could be defined by a higher order Taylor formula, yet (6.16) is sufficient to illustrate our point. In fact, note that if

$$u(\widetilde{R}, \sigma) = \sum \left(\frac{\widetilde{R}}{r} \right)^{n+1} u_{nm} Y_{nm}$$

then

$$\begin{aligned} PBu &= \sum \left[1 + (n+1) \frac{\delta R}{\widetilde{R}} + \frac{1}{2} (n+1)(n+2) \left(\frac{\delta R}{\widetilde{R}} \right)^2 \right] u_{nm} Y_{nm}(\sigma) \\ &= \sum b_n u_{nm} Y_{nm}(\sigma) \end{aligned}$$

where $b_n > 1$.

Therefore the spectrum of the operator

$$V = U \cdot PB \,,$$

is now

$$\lambda_n(V) \sim \left(\frac{R_0}{\widetilde{R}} \right)^{n+1} b_n$$

and so

$$\lambda_n(I - V) \sim 1 - \left(\frac{R_0}{\widetilde{R}} \right)^{n+1} b_n \,.$$

Let us explicitly note here that we always have

$$b_n < \frac{1}{\left(1 - \frac{\delta R}{\widetilde{R}}\right)} = \left(\frac{\widetilde{R}}{R_0}\right)^{n+1}$$

so that it is also

$$\left(\frac{R_0}{\widetilde{R}}\right)^{n+1} b_n < 1\ ;$$

therefore $\lambda_n(I - V)$ are always positive.

Accordingly, if we restrict $I - U$ to L^2_N, we have

$$\| I - V \|_{L^2_N} = 1 - \left(\frac{R_0}{\widetilde{R}}\right)^{N+1} b_N < 1 - \left(\frac{R_0}{\widetilde{R}}\right)^{N+1} = \| I - U \|_{L^2_N}$$

and the acceleration of $(I - V)^{\ell}_{L^2_n}$ to zero is proved.

The conclusion of the Example is that the application of a *PB* operator of Taylor type is useful to improve the performance of the CB approach. The same problem should be studied for a general geometry.

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Chapter 7
The Pseudo-Boundary Value Problem (ψ-BVP) Interpretation

There is a first interpretation of the method of "stabilized" downward continuation that, at least up to the linear term, justifies its application. This has been proposed in [1], and consists basically in solving a different problem that approximates the given BVP with another one where boundary values of the unknown potential u are combined with the vertical derivative of u but at points inside the harmonicity domain, i.e., out of the boundary. For this reason such an interpretation could be called an *out of boundary* or *pseudoboundary* value problem.

As explained hereafter we can limit ourselves to examine the Dirichlet BVP.

As a starting point, let us note that the actual geodetic BVP can be successfully analyzed as a perturbation of the so-called simple Molodensky problem, which is nothing but the spherical approximation of the above [2].

In such an approximation, one has

$$\frac{\partial T}{\partial h} \sim \frac{\partial T}{\partial r} = T' \; ; \; \frac{\gamma'}{\gamma} \sim \frac{-2\mu/r^3}{\mu/r^2} = -\frac{2}{r} \, , \tag{7.1}$$

so that the problem becomes

$$\begin{cases} \Delta T = 0 & \text{outside } \widetilde{S} \\ -T' - \frac{2}{r} T = \Delta g + \frac{a}{r^2} + \frac{\mathbf{b} \cdot \mathbf{r}}{r^4} & \text{on } \widetilde{S} \\ T = 0\left(\frac{1}{r^3}\right) & r \to \infty \end{cases} \tag{7.2}$$

Note that the parameters $(a, \mathbf{b})$ are introduced to compensate the loss of zero and first order harmonics for T, implicit in the third equation of (7.2).

A significant step forward in the study of (7.2) has been accomplished by the so-called Prague method of Krarup [3], in which it is observed that the potential

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$$v = r\frac{\partial T}{\partial r} + T \tag{7.3}$$

has to be harmonic too, in the same domain where T is.

In this case in fact we can transform (7.2) into a Dirichlet problem, i.e.,

$$\begin{cases} \Delta v = 0 & \text{outside } \widetilde{S} \\ v = -r\Delta g + \frac{a}{r} + \frac{\mathbf{b}\cdot\mathbf{r}}{r^3} & \text{on } \widetilde{S} \\ v = O\left(\frac{1}{r^3}\right) & r \to \infty \,. \end{cases} \tag{7.4}$$

Furthermore, it is enough to call

$$u(r,\sigma) = v(r,\sigma) - \frac{a}{r} - \frac{\mathbf{b}\cdot\mathbf{r}}{r^3}$$

to realize that u is just the regular solution of a Dirichlet problem, without any further unknown parameters and special asymptotic conditions, i.e.,

$$\begin{cases} \Delta u = 0 & \text{outside } \widetilde{S} \\ u = f(\sigma) & \text{on } \widetilde{S} \end{cases} \tag{7.5}$$

$$(f = -r\Delta g) \,. \tag{7.6}$$

In essence, the idea of the ψ-BVP is that, instead of claiming that a known function $f(P)$, with $P \in S$ is the value attained by u on S, we just say that at P_0 the projection of P on $S_0 \equiv \{r = R_0\}$, the value of u is approximately

$$u(P_0) \cong u(P) - h\frac{\partial}{\partial r}u(P) = f(P) - h\frac{\partial u(P)}{\partial r} \,, \tag{7.7}$$

with $h = r - R_0$ (see Fig. 7.1).

Indeed in doing so we implicitly assume that $u(P)$ is harmonic down to S_0, but let us underline once more that this does not mean that we downward continue the true potential, only that we approximate the true potential by another one harmonic outside the Bjerhammer sphere S_0, which is the solution of an approximate problem.

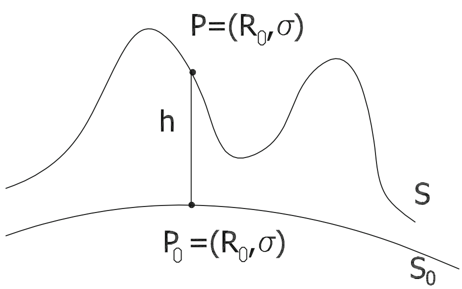

Fig. 7.1 Geometry of the ψ-BVP

This ψ-BVP then reads: find u, regular at infinity, such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_0 \equiv (r > R_0) \\ u(R_0, \sigma) + h(\sigma)\frac{\partial}{\partial r}u(R_0 + h(\sigma), \sigma) = f(\sigma) & \text{on } S_0 \end{cases} \tag{7.8}$$

We seek a solution of (7.8) in the topology of harmonic functions with boundary values in $C(S_0)$, i.e., continuous with the sup norm.

In fact (7.8) can be reduced to a simple integral equation for the boundary trace

$$u_0(\sigma) = u(R_0, \sigma) \ . \tag{7.9}$$

Noting that one can write, with the help of the Poisson function (see Appendix A, (A.31))

$$U(\sigma, \sigma') = \frac{1}{4\pi} \frac{R_0[(R_0 + h_\sigma)^2 - R_0^2]}{[(R_0 + h_\sigma)^2 + R_0^2 - 2R_0(R_0 + h_\sigma)\cos\psi_{\sigma\sigma'}]^{3/2}} \ , \tag{7.10}$$

the operator

$$\begin{aligned} h_\sigma \frac{\partial}{\partial h} u(R_0 + h_\sigma, \sigma) = K[u_0] = \\ = \int K(\sigma, \sigma')u_0(\sigma')d\sigma' \ , \end{aligned} \tag{7.11}$$

has kernel K (see [4], Sect. 1.18),

$$\begin{aligned} &K(\sigma, \sigma') = \\ &= \frac{h_\sigma R_0}{4\pi} \frac{[5R_0^2(R_0 + h_\sigma) - (R_0 + h_\sigma)^3 - R_0(R_0 + h_\sigma)^2\cos\psi - 3R_0^3\cos\psi]}{[(R_0 + h_\sigma)^2 + R_0^2 - 2R_0(R_0 + h_\sigma)\cos\psi]^{5/2}} \ . \end{aligned} \tag{7.12}$$

By using the operator K we can re-write the second equation of (7.8), simply as

$$u_0 + K[u_0] = f \ . \tag{7.13}$$

If (7.13) has a solution, then u is just the solution of the Dirichlet problem with boundary values $u_0(\sigma)$.

The key point is that it is easy to show that (7.13) has one and only one solution $u_0 \in C(S_0)$, when $f \in C(S_0)$. This fact has determined the present unusual norm choice. In fact, in this case one has, introducing polar coordinates (α, ψ) on the sphere,

$$\begin{aligned} \| K \| &\le \sup_\sigma \int d\sigma' |K(\sigma, \sigma')| = \\ &= \sup_\sigma \int_0^{2\pi} d\alpha \int_0^\pi |K(\sigma, \sigma')| \sin\psi d\psi \end{aligned} \tag{7.14}$$

Since for fixed σ, $K(\sigma, \sigma')$ does not depend on α but only on ψ, namely

$$K(\sigma, \sigma') = K(\sigma, \psi_{\sigma\sigma'}) \ ,$$

we can write (7.14) as

$$\| \ K \ \| \leq \sup_{\sigma} 2\pi \int_0^{\pi} |K(\sigma, \psi)| d\psi \equiv q_\sigma \ . \tag{7.15}$$

Now it is just matter of a patient exercise of integration to prove that $\| \ K \ \| < 1$, so that K is a contraction operator and, as claimed before, (7.13) has one and only one solution in $C(S_0)$.

To work out (7.15) it is convenient to introduce the notation

$$\eta_\sigma = 1 + \frac{h_\sigma}{R_0} \ , \tag{7.16}$$

as well as the change of variable $\cos\psi = t$, to the effect that we have now to compute

$$\begin{aligned} q &= \frac{\eta - 1}{2} \int_{-1}^{1} \frac{|5\eta - \eta^3 - (\eta^2 + 3)t|}{[1 + \eta^2 - 2\eta t]^{5/2}} dt = \\ &= \frac{\eta - 1}{2} \frac{\eta^2 + 3}{(2\eta)^{5/2}} \int_{-1}^{1} \frac{|a - t|}{(b - t)^{5/2}} dt \end{aligned} \tag{7.17}$$

where

$$a = \frac{5\eta - \eta^3}{\eta^2 + 3}, \ b = \frac{\eta^2 + 1}{2\eta} \ . \tag{7.18}$$

We note that physical values of η_σ are confined to the layer

$$1 \leq \eta \leq 1.0015 \ , \tag{7.19}$$

but, as we shall see, $\sup_\sigma q_\sigma < 1$, at least for $\eta \leq 9$. In any event, we will find our inequality valid even for $\eta = 1$, meaning that for the present problem S can be partly coinciding with S_0. Without reproducing all the cumbersome calculations, let us report that the final result is

$$q_\sigma = \frac{(\eta^2 + 3)^{3/2}}{3\sqrt{3}\eta^2(\eta + 1)} \ . \tag{7.20}$$

It is not difficult to verify that q, as a function of η, has a negative derivative into the interval

$$1 \leq \eta \leq 9.6 \ ,$$

meaning that, for the above values,

$$q[\eta_\sigma] \leq q[1] = 0.7689 < 1 \,. \tag{7.21}$$

Therefore $\sup_\sigma q[\eta_\sigma] \leq 0.7689$ and the operator K is a contraction, as claimed before. We can summarize the result by the following theorem.

Theorem 7.1 *For every $f(\sigma)$ in $C(S_0)$, the problem (7.8) has one and only one solution with trace in $C(S_0)$.*

At this point several comments are in order on the above result.

Remark 7.1 An identical approach can be applied in a Cartesian approximation, according to which the sphere S_0 goes into the plane $h = 0$. In this case, the relevant kernel becomes

$$K(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \frac{h\rho^2 - 2h^3}{[\rho^2 + h^2]^{5/2}} \,, \tag{7.22}$$
$$(\rho = |\mathbf{x} - \mathbf{x}'|), \;\; h = h(\mathbf{x}),$$

and one can compute, with $\rho^2 = sh^2$,

$$q = \frac{1}{2\pi} \int d\alpha \int_0^{+\infty} \frac{|h\rho^2 - 2h^3|}{[\rho^2 + h^2]^{5/2}} \rho d\rho = \tag{7.23}$$
$$= \frac{1}{2} \int_0^{+\infty} \frac{|s-2|}{[s+1]^{5/2}} ds = \frac{4}{3\sqrt{3}} = 0.7689 \,.$$

As we can see, the result is in fact identical to that in spherical geometry when $\eta \equiv 1$.

This might not be surprising because when we let $\eta \to 1$, keeping the point $\mathbf{x}$ fixed on S_0, we are looking at the limit sphere S_0 from very close with a kernel having surface integral equal to 1 and tending to a $\delta_{S_0}(\mathbf{x})$ distribution, exactly as (7.22).

Remark 7.2 The result could be very stimulating if, continuing the Taylor development to e.g. the second order, and defining the ψ-BVP

$$u(R_0, \sigma) + h_\sigma \frac{\partial u}{\partial r}(R_0 + h_\sigma, \sigma) - \frac{1}{2} h_\sigma^2 \frac{\partial^2 u}{\partial r^2}(R_0 + h_\sigma, \sigma) = f(\sigma)$$

one would obtain a norm estimate q better than that of the first order. This unfortunately is not the case. Repeating the calculus of q for the Cartesian case, one gets $q = 1.24$ and the contraction property of the pseudoboundary operator is lost.

This is probably due to the rough approximations implied by (7.15), (7.23), where the oscillations of the kernel K are killed by the modulus.

Maybe another topology would do better, yet for the moment we can only claim that the ψ-BVP interpretation is capable of explaining only the linear term of the

Taylor development, which by the way has the largest impact on the solution, but we cannot claim that the present approach has a general significance.

Remark 7.3 It is interesting to observe that if, instead of taking the linearization point P inside the domain of harmonicity of the sought solution, one would take the same point P_0 (see Fig. 7.1), or in other words if one tries to approximate the BVP by the boundary relation

$$u(R_0, \sigma) + h_\sigma \frac{\partial}{\partial r} u(R_0, \sigma) = f(\sigma) \tag{7.24}$$

one would be faced immediately with a contradiction, as explained for instance in [5]. In fact, limiting oneself to the 2-sphere example so as to have $h_\sigma = \overline{h} =$ const, one can express (7.24) in terms of spherical harmonics as

$$(1 - \frac{n+1}{R_0}\overline{h})u_{nm} = f_{nm} \; . \tag{7.25}$$

Indeed, the solution of (7.25) for $\overline{h}$ equal to a fraction of R_0 becomes unstable for a degree corresponding to the inverse of such a fraction.

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Chapter 8
One Further Example, Some Remarks and Conclusions

First we return to the elementary case of Example 6.1, putting some realistic numbers into that rather abstract frame, in order to further appreciate the difference/analogy between the Molodensky and the Helmert approaches.

We will interpret the first as a solution by the CB approach and the second basically the same, but after we have subtracted from the data the effects of condensed masses coming form the density distribution between the two spheres.

Example 8.1 We assume to have a potential T on a sphere $\widetilde{S}$ of radius $\widetilde{R}$

$$T|_{\widetilde{S}} = \sum_{n>2} T_{nm} Y_{nm}(\sigma) \ . \tag{8.1}$$

The potential T is generated by a mass distribution inside a sphere S_0 of radius $R_0 < \widetilde{R}$ plus a "topographic" mass distribution between S_0 and $\widetilde{S}$. We shall take

$$R_0 = \ 6.371 \ \text{ km} \quad \delta R = \widetilde{R} - R_0 = 1 \text{ km} \ . \tag{8.2}$$

We note that for a realistic model of topographic heights (cfr. [1]) we have the spectrum of Fig. 8.1 in terms of full power degree variances $\sigma_n^2(H)$

The mean and the standard deviation of such heights are respectively

$$\mu(H) \sim 230 \text{ m} \ , \ \sigma(H) \sim 630 \text{ m} \tag{8.3}$$

The topographic layer, modelled with a constant density of $\rho_c = 2.67$ g/ cm^{-3}, generates an external potential $t(r, \sigma')$.

We shall mimic this potential by introducing a mass distribution, depending only on the horizontal coordinates,

$$R_0 \leq r \leq \widetilde{R} \ , \ \rho(\sigma) = \sum \rho_{nm} Y_{nm}(\sigma) \tag{8.4}$$

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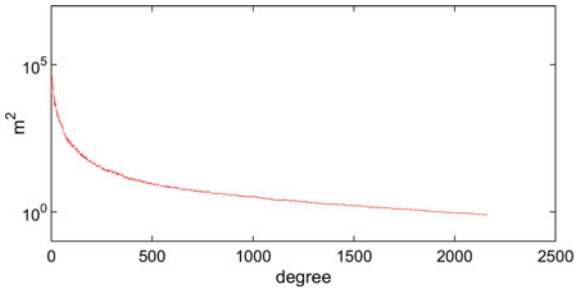

Fig. 8.1 Spectrum of topographic heights (continental heights only). Units in m^2

in such a way that its potential on $\widetilde{S}$ is similar to what we expect to be $t(\widetilde{R}, \sigma)$.

To accomplish this, we first put

$$\begin{cases} t(\widetilde{R}, \sigma) = \sum t_{nm} Y_{nm}(\sigma) \\ t_{nm} = 4\pi G \rho_c R_0 q_0^{n+1} \frac{H_{nm}}{2n+1} \end{cases}, \tag{8.5}$$

$$(q_0 = \frac{R_0}{\widetilde{R}} = 1 - \ 1.57 \ \times 10^{-4}) ,$$

corresponding to condensing the topographic masses to a single layer on S_0 (see [2], Sect. 4.3), and then we consider $t(r, \sigma)$ as our true model, from $r = \widetilde{R}$ onwards.

If we think of t as generated by "topographic" masses with a lateral variation only in the layer $R_0 \leq r \leq \widetilde{R}$ i.e.,

$$t(\widetilde{R}, \sigma_p) = G \int d\sigma_Q \rho(\sigma_Q) \int_{R_0}^{\widetilde{R}} \frac{r^2 dr}{\ell_{PQ}} , \tag{8.6}$$

we find the spectral relation

$$t_{nm} = 4\pi G \widetilde{R}^2 \frac{(1 - q_0^{n+3})}{n+3} \frac{\rho_{nm}}{2n+1} . \tag{8.7}$$

By comparing (8.7) with (8.5), we find the direct relation between our model ρ_{nm} and H_{nm}, namely

$$\rho_{nm} = \rho_c \frac{n+3}{(1 - q_0^{n+3})} q_0^{n+2} \frac{H_{nm}}{\widetilde{R}} . \tag{8.8}$$

At this point it is important to make a spectral delimitation. In fact, consider that we could write

$$T = T_I + t_M + t + t_r \tag{8.9}$$

where T_I is the potential generated by masses internal to S_0, t_M is the topographic potential up to a low degree M, e.g. $M = 360$, t_r is the residual "topographic" potential from a high degree L onward, e.g. $L = 1440$, and finally t is the topographic potential with spectrum between degrees M and L.

We want to determine T by CB or by first subtracting a condensed potential and then applying CB. In this way the two approaches have indeed the same effect on T_I so we can disregard this component in the comparison. Furthermore, before we try to compute any solution, we usually subtract a global model up to some degree M; this is the same in both approaches.

We assume that such a model has totally absorbed the t_M component in (8.9). Finally, considering t_r we assume that this is basically due to the residual terrain effect, which for $L = 1440$ is just the effect of the terrain between the actual surface, which might be known with a 100 m resolution, and an averaged topographic surface, with a resolution of 14 km, corresponding to degree 1440. Also, this component can be computed and removed before processing. So our comparison will concentrate on t, with its spectrum between degrees 360 and 1440.

In this spectral interval, the (true) degree variances of H_{nm} are well approximated by the formula

$$\begin{aligned} &\sigma_n^2(H) = Ke^{-\alpha n}\ ,\ 360 \le n \le 1440 \\ &(K = \ 17.78^2\ \text{m}^2\ ,\ \alpha = \ 1.56\ \times 10^{-3})\ , \end{aligned} \tag{8.10}$$

corresponding to a total power $\sigma(H) \sim 73$ m. In any event (8.10) is considered to be exact for our model. This spectrum can then be propagated to t through (8.5), obtaining

$$\sigma_n^2(t) = A\frac{e^{-\varepsilon n}}{(2n+1)}\ ,\ 360 \le n \le 1440\ , \tag{8.11}$$

with

$$\begin{aligned} &\sqrt{A} = 4\pi G\rho_c R_0\sqrt{K} \cong \\ &\cong 25 \times 10^3\ \text{Gal} \times\ \text{m} \\ &\varepsilon = 2\log\frac{1}{q_0} + \alpha \cong\ 1.87\ \times 10^{-3}\ . \end{aligned}$$

This corresponds roughly to a signal of ~ 15 cm in geoid and ~ 7 mGal in δg. The application of 1 cycle of CB to t (see Appendix A) then corresponds to a spectral filter of the form

$$(I - U)t \sim (1 - q_0^{n+1})t_{nm}\ . \tag{8.12}$$

In our range of degrees, this filter is almost linear in n, with values going from 5,5 % for $n = 360$ to 20 % for $n = 1440$.

Further cycles reduce residuals, at the level of $\widetilde{S}$, by the same rate, i.e., by powers of the filter (8.12).

As commented in Appendix A, the application of a Taylor development (see Example 6.2) only speeds up this process.

Let us see now what happens with the Helmert approach. Here the signal t has to be substituted by $t - t^c$, where the condensed potential t^c is given by

$$t^c(\widetilde{R}, \sigma_P) = G \int R_0^2 d\sigma_Q \rho(\sigma_Q) \frac{\delta R}{\ell_{PQ}} \tag{8.13}$$
$$(\delta R = \widetilde{R} - R_0)$$

or, spectrally

$$t_{nm}^c = 4\pi G R_0 \delta R \frac{\rho_{nm}}{2n+1} q_0^{n+1} \; . \tag{8.14}$$

Substituting (8.7) in (8.14) we find

$$t_{nm}^c = t_{nm} \frac{\delta R}{R_0} \frac{(n+3) q_0^{n+3}}{(1 - q_0^{n+3})} \; . \tag{8.15}$$

On account of the relation

$$\frac{\delta R}{R_0} = \frac{1}{q_0} - 1 = \frac{1 - q_0}{q_0} \; ,$$

(8.15) can be written as

$$t_{nm}^c = t_{nm} q_0^{n+1} \frac{q_0 (1 - q_0)(n+3)}{(1 - q_0^{n+3})} \; , \tag{8.16}$$

implying also that

$$t_{nm} - t_{nm}^c = \left[1 - \frac{q_0 (1 - q_0)(n+3)}{(1 - q_0^{n+3})} q_0^{n+1} \right] t_{nm} \; . \tag{8.17}$$

It is enough now to evaluate the factor

$$F_n = \frac{q_0 (1 - q_0)(n+3)}{(1 - q_0^{n+3})}$$

at the extremes of the spectrum of t, i.e.,

$$F_{360} = \; 1.029 \qquad F_{1440} = \; 1.117$$

to realize that (8.17) is quite close to $(1 - q_0^{n+1})t_{nm}$ that we already encountered in (8.12) as the first step of CB. From here on, further iterations of CB correspond to a multiplication by the filter $(1 - q_0^{n+1})$ and so have the same rate of convergence for all degrees.

The conclusion is that by starting from the Helmert potential and then applying the *downward continuation*, namely an iterated CB, we basically obtain the same sequence as for the direct application of CB, only starting already from the first iteration.

Let us note that in the example we have assumed that $\rho(\sigma)$ is perfectly known before Helmert's condensation, so that the advantage of skipping the first iteration by applying Helmert's approach could be attenuated by the imperfect knowledge of ρ. Yet, despite the roughness of the arguments and the elementary set up of this example, we came again to the same conclusion of a substantial equivalence of the two methods.

Let us add here some comments on the original results obtained and highlighted in the text:

(1) The concept of physical surface of the Earth has been made more precise, as the basis of the so-called scalar GBVP.
(2) The linearization process has been reviewed along the lines of Krarup's lesson, although here a quantitative definition of the linearization band has been pursued, providing an operative definition of the equivalent linearized GBVP's. In addition, a compatibility condition (namely (2.14)), has been introduced as necessary for the definition of the linearization band, which was missing in Krarup's papers.
(3) The equivalence of Helmert's and Molodensky's approaches has been strictly proved at the level of BVP definitions, in the sense commented at point 2 above.
(4) The use of the DC to solve a GBVP has been proven to be inconsistent, thus showing that the good results obtained by numerical practice need to be explained by a different theory. The iterated CB approach has been proposed as a sounder interpretation. This is justified only by a first order perturbation theory and a conjecture, without a rigorous proof.

Lastly, based on the material presented in the text, we can draw three main conclusions.

Conclusion 1: on a theoretical ground it is not allowed to say that either Helmert's or Molodensky's BVP is correct and the other wrong. Any further dispute on this point is non-scientific.

Conclusion 2: to avoid wrong statements, the Downward Continuation should never be invoked as a solution method without adding an adjective like "regularized".

Conclusion 3: the iterated Change of Boundary method needs further mathematical investigations for two reasons: to make rigorous the basis of our numerical geodetic solutions, but also because it is interesting in itself since, to the knowledge of the authors, it provides a novel approach to classical BVPs, at least for star-shaped domains, not yet discussed in mathematical literature.

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Appendix A
On the Analysis of the Upward Continuation Operator

The purpose of this Appendix is to study the properties of the upward continuation operator U, particularly its spectral behaviour.

In the most general case, at least in this context, we define U as follows: let S_i and S_e be two star-shaped surfaces satisfying some regularity condition, e.g., the cone condition (cf. [1]), and internal one to the other, as in Fig. A.1. Let us further define a space of functions harmonic down to S_i and square integrable on such a surface.

In this space, to every function $u(P) = u(r, \sigma)$, we can associate by taking traces and pulling back to the unit sphere

$$u_i(\sigma) = u(R_{i\sigma}, \sigma) \tag{A.1}$$

$$u_e(\sigma) = u(R_{e\sigma}, \sigma) \ . \tag{A.2}$$

Since the pull back operators $S_i \to \{\sigma\}$, $S_e \to \{\sigma\}$ are bounded and invertible, as far as

$$O < \text{ const } \leq R_{i\sigma} \leq R_{e\sigma} \leq \text{ const } < +\infty \ ,$$

and $R_{i\sigma}$, $R_{e\sigma}$ are Lipschitz functions, as implied by the cone condition, in a slightly restricted form, the norms in $L^2(S_e)$, $L^2(S_i)$, L^2_σ are all equivalent. So we expect $\{u(R_i, \sigma)\}$ to span the whole L^2_σ.

Therefore we can define the operator U, with domain L^2_σ, putting

$$u_e(\sigma) = U u_i(\sigma) \ ; \tag{A.3}$$

so basically $u_e(\sigma)$ is the trace on S_e of the solution to the Dirichlet boundary value problem, with $u_i(\sigma)$ as boundary value on S_i.

Remark A.1 It is important here and in what follows that we assume S_i to be totally internal to S_e, namely

$$\sup_\sigma R_{i\sigma} < \inf_\sigma R_{e\sigma} \ . \tag{A.4}$$

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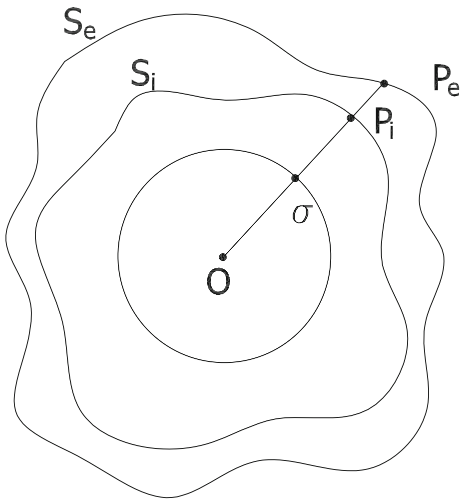

Fig. A.1 The general set up of the upward continuation

The case that S_i and S_e are partially coinciding could be relevant to physical geodesy, but is out of the scope of the present work.

Given the above definition and calling $G(P, Q)$ the Green function relative to the surface S_i and its exterior domain, it is obvious that outside S_i we can write

$$u(P) = \frac{-1}{4\pi} \int_{S_i} \frac{\partial}{\partial n_Q} G(P, Q) u(Q) dS_{iQ} \,. \tag{A.5}$$

Therefore if we take $Q \sim (R_{i\sigma'}, \sigma') \in S_i$, $P \sim (R_{e\sigma}, \sigma) \in S_e$ and we observe that

$$\begin{aligned} dS_{iQ} &= R^2_{i\sigma'} J_{\sigma'} d\sigma' \\ J_{\sigma'} &= \frac{1}{\mathbf{n}_Q \cdot \mathbf{e}_{rQ}} \leq \text{ const } , \end{aligned} \tag{A.6}$$

we find that the operator U is just an integral operator with kernel

$$U(\sigma, \sigma') = -\frac{1}{4\pi} G_{n_Q}[(R_\sigma, \sigma), (R_{\sigma'}, \sigma')] R^2_{i\sigma'} J_{\sigma'} \,, \tag{A.7}$$

such that

$$u_e(\sigma) = \int U(\sigma, \sigma') u_i(\sigma') d\sigma' \,. \tag{A.8}$$

Thanks to the well-known estimates on the Green function (see [2, 3]) and keeping in mind the Remark A.1, we see that $U(\sigma, \sigma')$ is a continuous, and then bounded function, implying that U is a compact integral operator $L^2_\sigma \to L^2_\sigma$.

At this point we could already apply a classical result by Schmidt (see [4], n. 142) to get an explicit spectral representation of $U(\sigma, \sigma')$, also attributed to Picard in geodetic literature (see [5]). However, we prefer here to derive this result from the well-known spectral theorem for selfadjoint compact operators [6].

Proposition A.1 *Let K be a selfadjoint compact operator on a Hilbert space H; then there is a sequence of real numbers $\{\lambda_n\}$, such that*

$$\lim_{n\to\infty} \lambda_n = 0 \, , \tag{A.9}$$

and a sequence of functions $\{\varphi_n\}$, which is orthonormal and complete in H, such that

$$K\varphi_n = \lambda_n \varphi_n \; ; \tag{A.10}$$

each eigenvalue $\lambda_n \neq 0$ can have at most a finite multiplicity. In addition, one has

$$K = \sum_{n=1}^{+\infty} \lambda_n \varphi_n \otimes \varphi_n \; , \tag{A.11}$$

namely if K is an integral operator with kernel $K(x, y)$, then

$$K(x, y) = \sum_{n=1}^{+\infty} \lambda_n \varphi_n(x) \varphi_n(y) \; . \tag{A.12}$$

Remark A.2 The value $\lambda_n = 0$ can in general be an eigenvalue of K and its eigenspace can be even infinite-dimensional, however this is not the case when we know that K is an invertible operator.

We can apply the above proposition to the operator on $L^2_\sigma \otimes L^2_\sigma$

$$A = \begin{bmatrix} 0 & U^T \\ U & 0 \end{bmatrix} . \tag{A.13}$$

A few remark are in order here: if U is bounded, then U^T defined on the whole space is bounded too and has the same norm as U.

- If U is compact, U^T is compact too. This is true in general, but it is obvious when U^T is an integral operator too, with kernel $U(\sigma', \sigma)$.
- If U is invertible, U^T is invertible too.

Accordingly, we see that

- A is a bounded compact operator defined on the whole $L^2_\sigma \otimes L^2_\sigma$.
- A is selfadjoint, since

$$< \begin{vmatrix} u \\ v \end{vmatrix}, A \begin{vmatrix} f \\ g \end{vmatrix} >=< u, U^T g > + < v, Uf >=$$
$$=< U^T v, f > + < Uu, g >=< A \begin{vmatrix} u \\ v \end{vmatrix}, \begin{vmatrix} f \\ g \end{vmatrix} > \ .$$

- A is invertible if U is invertible, on account of the explicit formula

$$A^{-1} = \begin{vmatrix} O & U^{-1} \\ U^{T-1} & O \end{vmatrix} \ . \tag{A.14}$$

It is time now to establish the invertibility of U.

Proposition A.2 *The operator U is invertible in $L^2(\sigma)$.*

Proof We have to show that

$$u_0 \in L^2_\sigma, \ Uu_0 = 0 \quad \Rightarrow \quad u_0 = 0 \ . \tag{A.15}$$

But this is trivial, because if we call $u(r, \sigma)$ the solution of the Dirichlet problem with boundary values $u(R_{i\sigma}, \sigma) = u_0(\sigma)$, we see that, by (A.15),

$$u|_{S_e} = u(R_{e\sigma}, \sigma) \equiv 0 \ .$$

Then $u \equiv 0$ outside S_e and, by the unique continuation property of harmonic functions (cfr. Miranda, [7]), $u \equiv 0$ in all of its harmonicity domain and so $u_0 = 0$.

We can now apply Proposition A.1 and Remark A.2 to claim the existence of eigenvalues $\{\wedge_n\}$, with $\wedge_n \to 0$, and eigenfunctions $\left\{ \begin{vmatrix} \varphi_n \\ \psi_n \end{vmatrix} \right\}$ which form an ONCS (Ortho-Normal Complete System) in $L^2_\sigma \otimes L^2_\sigma$, such that

$$U^T \psi_n = \wedge_n \varphi_n \ , \tag{A.16}$$
$$U \varphi_n = \wedge_n \psi_n \ . \tag{A.17}$$

From (A.17), multiplying the first equation by U, the second by U^T and then substituting the resulting formula, one gets

$$\begin{aligned} UU^T \psi_n &= \wedge_n^2 \psi_n \\ U^T U \varphi_n &= \wedge_n^2 \varphi_n \ . \end{aligned} \tag{A.18}$$

As eigenfunctions of two positive definite selfadjoint compact operators, we can state that

- $\{\varphi_n\}$, $\{\psi_n\}$ are both orthogonal systems in L^2_σ;
- since $\left\{\left|\begin{matrix}\varphi_n\\ \psi_n\end{matrix}\right|\right\}$ is complete $L^2_\sigma \otimes L^2_\sigma$, $\{\varphi_n\}$, $\{\psi_n\}$ have to be each separately complete in L^2_σ.
- We see from (A.18) that $\{\wedge_n^2\}$, $\{\varphi_n\}$ are the set of eigenvalues and eigenfunctions of $U^T U$ while $\{\wedge_n^2\}$, $\{\psi_n\}$ are the set of eigenvalues and eigenfunctions of UU^T and as such could be defined from the very beginning, imposing to both φ_n and ψ_n to be normalized.
- Naturally, with the above normalization choice,

$$\wedge_n^2 = \| U^T \psi_n \|^2 \Rightarrow \| U^T \|^2 = \sup_n \wedge_n^2 \tag{A.19}$$

$$\wedge_n^2 = \| U \varphi_n \|^2 \Rightarrow \| U \|^2 = \sup_n \wedge_n^2 \tag{A.20}$$

and the two norms, as already stated, are equal to each other.

Another property of (A.17) is of interest for us. In fact, one immediately verifies by substitution that the eigenvalues $\wedge_n$ come in couples of opposite sign. Namely, if $\left(\wedge_n, \left|\begin{matrix}\varphi_n\\ \psi_n\end{matrix}\right|\right)$ satisfy (A.17), then $\left(-\wedge_n, \left|\begin{matrix}\varphi_n\\ -\psi_n\end{matrix}\right|\right)$ satisfy too such equations.

Accordingly, we can organize the spectral representation by considering first the $\wedge_n$ all positive and then adding the corresponding negative image.

We can therefore apply (A.12) to get

$$\begin{aligned} A &= \frac{1}{2}\left\{\sum_{n=1}^{+\infty} \wedge_n \left|\begin{matrix}\varphi_n\\ \psi_n\end{matrix}\right| \otimes |\varphi_n \psi_n| + \sum_{n=1}^{+\infty}(-\wedge_n)\left|\begin{matrix}\varphi_n\\ (-\psi_n)\end{matrix}\right| \otimes |\varphi_n(-\psi_n)|\right\} \\ &= \frac{1}{2}\left|\begin{matrix} O & 2\sum_{n=1}^{+\infty} \wedge_n \varphi_n \otimes \psi_n \\ 2\sum_{n=1}^{+\infty} \wedge_n \psi_n \otimes \varphi_n & O \end{matrix}\right| , \end{aligned} \tag{A.21}$$

where the factor $\frac{1}{2}$ is to re-normalize $\left|\begin{matrix}\varphi_n\\ \psi_n\end{matrix}\right|$ which has a squared norm equal to 2.

The relation (A.21) proves the following theorem.

Theorem A.1 *The operator U has the spectral representation*

$$U = \sum_{n=1}^{+\infty} \wedge_n \psi_n \otimes \varphi_n , \tag{A.22}$$

namely U is an integral operator in L^2_σ with kernel

$$U(\sigma, \sigma') = \sum_{n=1}^{+\infty} \wedge_n \; \psi_n(\sigma)\varphi_n(\sigma') \; , \tag{A.23}$$

where

$$\lim_{h \to \infty} \wedge_n = 0 \; , \tag{A.24}$$

and $\{\varphi_n(\sigma)\}$, $\{\psi_n(\sigma)\}$ are two complete orthonormal systems in L^2_σ.

This Theorem has an obvious Corollary which is relevant to our discussion.

Corollary A.1 *The downward continuation operator*

$$D = U^{-1}$$

is an unbounded operator in L^2_σ with spectral representation

$$D = U^{-1} = \sum_{n=1}^{+\infty} \wedge_n^{-1} \; \varphi_n \otimes \psi_n \; . \tag{A.25}$$

Proof The proof of (Corollary A.1) relies just on the fact that $\{\varphi_n\}$ and $\{\psi_n\}$ are complete orthonormal systems, so that

$$\begin{aligned} DU &= \sum_{j=1}^{+\infty} \wedge_j^{-1} \; \varphi_j \otimes < \psi_j, \sum_{n=1}^{+\infty} \wedge_n \; \psi_n \otimes \varphi_n >= \\ &= \sum_{j,n=1}^{+\infty} \wedge_j^{-1} \; \wedge_n \varphi_j \otimes < \psi_j, \psi_n > \varphi_n = \\ &= \sum_{n=1}^{+\infty} \varphi_n \otimes \varphi_n = I \; . \end{aligned}$$

That D is unbounded is seen from the fact that the sequence $\{\psi_n\}$, which is bounded, is transformed by D to

$$D\psi_n = \frac{1}{\wedge_n} \varphi_n$$

which is clearly unbounded, because of (A.24).

The problem whether the inverse $D = U^{-1}$ could be computed and used in approximate form, by first approximating U with a discrete matrix operation, U_α, and then inverting U_α, has been nicely analyzed in the book ([5], Sect. 8.6.1).

Both numerical tests, over a limited area in the Rocky Mountains, and a theoretical assessment with an example with 2 spheres, similar to what we discuss below, have been conducted. The conclusion is that "the posedness of the discrete downward continuation problem should be treated separately for each specific case. Making a grid of topographical heights denser and denser there is a step size (depending on the condition number of the discrete operator) which breaks down the stable behaviour of the discrete downward continuation."

Such a conclusion (a) describes appropriately a typical non well-posed problem, (b) proves that the discretization is in fact one of a family of stabilization methods, including the spectral truncation of series of spherical harmonics.

What is not discussed in the mentioned book is that by broadening the mesh of the grid one gets indeed a more stable problem but at the same time one has a worse approximation of the discrete to the continuous solution. The tradeoff between the two effects is exactly the critical point of any regularization theory.

In any case, particularly simple and clarifying is the example, treated also in this Appendix, where

$$\begin{aligned} S_i &= S_0 = \text{ sphere with radius } R_0 \\ S_e &= S = \text{ sphere with radius } R > R_0 \ . \end{aligned}$$

In such a case, U_0 has the explicit representation

$$U_0(\sigma) = \frac{1}{4\pi} \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \left(\frac{R_0}{R} \right)^{n+1} Y_{nm}(\sigma) Y_{nm}(\sigma') \ , \tag{A.26}$$

which shows at once that in this example the eigenvalues are $(2n + 1)$-multiple

$$\wedge_{0nm} = \left(\frac{R_0}{R} \right)^{n+1} = \wedge_{0n}, \ m = -n, \ldots, n \ , \tag{A.27}$$

and the eigenfunctions ψ_n, φ_n are given by

$$\psi_n(\sigma) = \varphi_n(\sigma) = \frac{Y_{nm}(\sigma)}{\sqrt{4\pi}} \ . \tag{A.28}$$

Equation (A.26) is in fact symmetric, meaning that the operator U is selfadjoint. Moreover, one has

$$\| U \| = \sup \wedge_{0n} = \wedge_{00} = \frac{R_0}{R} < 1 \ , \tag{A.29}$$

namely U is a contraction operator.

Finally, since $\wedge_{0n} > 0$, the operator U_0 is positive definite.

From now on, we shall restrict the analysis of the operator U to the case that

$$S_i = S_0 = \text{ sphere of radius } R_0 \ , \tag{A.30}$$

while the external surface S_e is still general, i.e., $S_e \equiv \{r = R_\sigma;\ \inf R_\sigma > R_0\}$.

Under this hypothesis, the shape of the kernel $U(\sigma, \sigma')$ is still perfectly known to be equal to the Poisson kernel, namely

$$U(\sigma, \sigma') = \frac{1}{4\pi} \frac{R_0(R_\sigma^2 - R_0^2)}{\ell_{\sigma\sigma'}^3} \ , \tag{A.31}$$

$$\ell_{\sigma\sigma'} = [R_\sigma^2 + R_0^2 - 2R_\sigma R_0 \cos\psi_{\sigma\sigma'}]^{\frac{1}{2}} \ . \tag{A.32}$$

Moreover, we shall take advantage of the well-known series representation of (A.31), namely [7, 8]

$$U(\sigma, \sigma') = \frac{1}{4\pi} \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \left(\frac{R_0}{R_\sigma}\right)^{n+1} Y_{nm}(\sigma) Y_{nm}(\sigma') \ ; \tag{A.33}$$

here it has to be strongly stressed that (A.33) is not the spectral representation (see (A.22)) of the operator U, because the functions

$$\left\{ \left(\frac{R_0}{R_\sigma}\right)^{n+1} Y_{nm}(\sigma) \right\}$$

do not represent an orthogonal system in L_σ^2.

As a matter of fact we could say that the rest of this Appendix is devoted to inferring from (A.31), (A.33) the form of the spectrum $\{\wedge_{nm}\}$ of our operator U at least to the first order in a perturbative form, where the perturbation parameter is just the oscillation of the "topographic heights"

$$\begin{cases} \Delta = R_+ - R_- \\ R_+ = \sup_\sigma R_\sigma \ , \ R_- = \inf_\sigma R_\sigma \ . \end{cases} \tag{A.34}$$

To be specific, we shall put (see Fig. A.2)

$$H_\sigma = R_\sigma - R_0 \tag{A.35}$$

$$H_- = R_- - R_0 > 0 \ , \ H_+ = R_+ - R_0 = H_- + \Delta \ , \tag{A.36}$$

and we shall assume that Δ, as well as H, are small with respect to R_0. Here small means of a relative order between 10^{-3} and 10^{-2}.

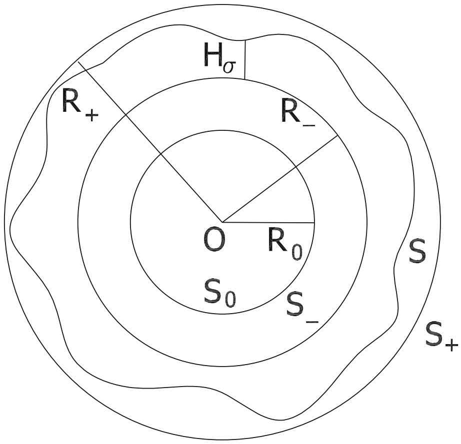

Fig. A.2 The geometry of the surface S and of the concentric spheres (S_0, S_-, S_+) with radius $R_0 < R_- < R_+$

Corresponding to Fig. A.2 we shall use the following notation:

$$\begin{cases} U_\pm \sim U_\pm(\sigma,\sigma') = \frac{1}{4\pi}\frac{R_0(R_\pm^2 - R_0^2)}{\ell_\pm(\sigma,\sigma')^3} \\ \ell_\pm(\sigma,\sigma') = [R_0 + R_\pm^2 - 2R_0R_\pm \cos\psi_{\sigma\sigma'}]^{\frac{1}{2}} \ , \end{cases} \tag{A.37}$$

while we will call U the upward continuation to the surface $S \equiv \{r = R_\sigma\}$.

Now it is clear that if we fix S_- and we squeeze the distance Δ between S_+ and S_-, we will have

$$\lim_{\Delta\to 0} u(R_- + \Delta H_\sigma, \sigma) = u(R_-, \sigma) \ . \tag{A.38}$$

In fact the limit (A.38) holds true even uniformly in σ, since we are inside the harmonicity domain, i.e., $H_- > 0$, (see [9] Proposition 6.23).

However, since we want to be quantitative, we shall provide our own estimate of

$$\| \delta U_\pm \| = \| U - U_\pm \| \ .$$

Before we go to the general case, however, it is convenient to develop the example of two concentric spheres, namely putting $R_\sigma = R_+$, because in this event we are able to compute exactly the $\| U_+ - U_- \|$, and this will suggest to us how to tune the various parameters, particularly the value of H_-, given that $\Delta \sim 10^{-3}R$ is fixed by the actual physical reality of the Earth.

Example A.1 Assume that $S = S_+$ (see Fig. A.2), i.e., that $R_\sigma = R_+$. Then, for a u harmonic outside S_0,

$$U_\pm u = \sum \left(\frac{R_0}{R_\pm}\right)^{n+1} u_{nm} Y_{nm}(\sigma) \tag{A.39}$$

so that

$$(U_- - U_+)u = \sum \left[\left(\frac{R_0}{R_-} \right)^{n+1} - \left(\frac{R_0}{R_+} \right)^{n+1} \right] u_{nm} Y_{nm}(\sigma) \ . \tag{A.40}$$

Accordingly

$$\| u_- - u_+ \|^2 = 4\pi \sum \left[\left(\frac{R_0}{R_-} \right)^{n+1} - \left(\frac{R_0}{R_+} \right)^{n+1} \right]^2 u_{nm}^2 \tag{A.41}$$

and one has indeed, noting that $\int u^2 d\sigma = 4\pi \sum u_{nm}^2$,

$$\| U_- - U_+ \|^2 = \sup_n \left[\left(\frac{R_0}{R_-} \right)^{n+1} - \left(\frac{R_0}{R_+} \right)^{n+1} \right]^2 . \tag{A.42}$$

Since the term in the brackets is always positive, one has directly

$$\| U_- - U_+ \| = \sup_n \left[\left(\frac{R_0}{R_-} \right)^{n+1} - \left(\frac{R_0}{R_+} \right)^{n+1} \right] . \tag{A.43}$$

The sup is achieved at

$$\overline{n} + 1 = \frac{\log \log \frac{R_+}{R_0} - \log \log \frac{R_-}{R_0}}{\log \frac{R_+}{R_0} - \log \frac{R_-}{R_0}} \ . \tag{A.44}$$

Since

$$\frac{R_-}{R_0} = 1 + \frac{H_-}{R_0} \ , \ \frac{R_+}{R_0} = 1 + \frac{H_-}{R_0} + \frac{\Delta}{R_0}$$

and $\varepsilon = \frac{\Delta}{R_0}$ is fixed at 10^{-3}, we find that $\overline{n} + 1$ depends basically on the ration $\frac{H_-}{R_0}$.

As an example one can compute the values of the following small table (Table A.1): From this we see that by taking a deeper R_0 (i.e., a larger H_-) and leaving Δ fixed, one gets a smaller norm of $U_- - U_+$.

Table A.1 $\| U_- – U_+ \|$ as function of H_-/R_0

H_-/R_0	5ε	10ε	20ε
$\overline{n} + 1$	183	95	49
$\| U_- – U_+ \|$	0.066	0.035	0.017

However, already at the level of

$$H_- = 10\varDelta \cong 60 \ \text{km (i.e., } R_0 = 6311 \text{ km)}$$

we find a not very deep surface, in the sense that $\frac{H_-}{R_0} \sim 10^{-2}$, and at the same time $\frac{\varDelta}{H_-} \sim 10^{-1}$. Moreover, in this case one has obviously

$$\| U_- \| = \frac{R_0}{R_-} = 0.99$$

while

$$\| U_- - U_+ \| \cong 3.505 \cdot 10^{-2} \| U_- \| \tag{A.45}$$

so that $U_- - U_+$ can be reasonably considered as a perturbation of U_-.

On the basis of the above example, we will favour the choice $H_- = 10\varDelta \cong 60$ km with the warning that this has no particular physical meaning but it is just a numerical value fitting our needs for further analysis.

We are ready now for a slightly more general result.

Proposition A.3 *Let us fix*

$$H_- = 10\varDelta \tag{A.46}$$

as in the Example A.1, then

$$\| \delta U_- \| \le 3.5 \cdot 10^{-2} \tag{A.47}$$

and δU_- can be considered as a first order perturbation of U_-, which has norm close to 1.

Proof We note that

$$|u_\sigma - u_-|^2 = \left| \int_{R_-}^{R_\sigma} u' dr \right|^2 \le \varDelta_\sigma \int_{R_-}^{R_\sigma} (u')^2 dr \le$$
$$\le \varDelta \int_{R_-}^{R_+} (u')^2 dr$$

and therefore, when $u = \sum \left(\frac{R_0}{r}\right)^{n+1} u_{nm} Y_{nm}(\sigma)$,

$$
\begin{aligned}
&\int |u_\sigma - u_-|^2 d\sigma = \| u_\sigma - u_- \|^2 \leq &\text{(A.48)}\\
&\leq \Delta \int_{R_-}^{R_+} dr \int d\sigma (u')^2 = \\
&= \frac{\Delta}{R_0} \cdot 4\pi \sum_{n,m} u_{nm}^2 \frac{(n+1)^2}{2n+1} \left[\left(\frac{R_0}{R_-} \right)^{2n+1} - \left(\frac{R_0}{R_+} \right)^{2n+1} \right] \\
&\leq q 4\pi \sum u_{nm}^2 = q \parallel u \parallel^2 u_{nm}^2
\end{aligned}
$$

with

$$
q = \| \delta U_- \|^2 = \sup_n \frac{\Delta}{R_0} \frac{(n+1)^2}{2n+1} \left[\left(\frac{R_0}{R_-} \right)^{2n+1} - \left(\frac{R_0}{R_+} \right)^{2n+1} \right] . \tag{A.49}
$$

If one sets

$$
x = 2n+1, \quad \varepsilon = \frac{\Delta}{R_0} = 10^{-3}, \quad \alpha = \frac{H_-}{R_0} = 10^{-2}, \quad \beta = \frac{H_+}{R_0} = \alpha + \varepsilon \, ,
$$

one finds the sup of the function above again at $\overline{n} = 94$, namely

$$
q = 0.0012
$$

and therefore

$$
\| \delta U_- \| \leq 3.464 \times 10^{-2} \, . \tag{A.50}
$$

Closing the proof, one can remark that the above bound should be valid also for $\| U_+ - U_- \|$. This is in fact the case, as we see by comparing (A.50) with (A.45), although the fact that the two numbers are so close indicates that we have not lost much in our inequality.

Remark A.3 An identical proof holds also for $U_+ - U$, so that we can claim that $\delta U_\pm$ can be both considered as perturbations of $U_\pm$, respectively, and the inequality

$$
\| \delta U_+ \| \leq 0.035 \| U_+ \| \tag{A.51}
$$

holds for chosen values of Δ and H_-, also considering that $\| U_+ \| \cong 1$.

Given the above statement, we can proceed now to study the norm of the operator $I - U$, which is of primary interest for the discussion in Chap. 6.

We note first that $I - U$ is not selfadjoint and therefore to study its norm we could look at the spectral structure of the operator

$$
K = (I - U^T)(I - U) \, , \tag{A.52}
$$

knowing that, if k_n are the eigenvalues of K, then

$$\| I - U \| = \sqrt{\sup k_n} \ . \tag{A.53}$$

Unfortunately we are not able to find k_n directly. However, if we put

$$K_\pm = (I - U_\pm^T)(I - U_\pm) \ ,$$

for which we already know (see (A.39)) that

$$k_{n\pm} = \left[1 - \left(\frac{R_0}{R_\pm} \right)^{n+1} \right]^2 \tag{A.54}$$

we can consider K as a perturbation of $K_\pm$ and at least compute the perturbation $\delta k_{n\pm}$ at the first order in Δ (i.e., in δU) by applying the well-known perturbative theory of Kato (see also [10], Ch. 3). More precisely, we can put

$$K = K_\pm + \delta K_\pm + O(\Delta^2) \ , \tag{A.55}$$

where, recalling that $U_\pm^T = U_\pm$,

$$\delta K_\pm = -[\delta U_\pm^T (I - U_\pm) + (I - U_\pm)\delta U] \tag{A.56}$$

is an infinitesimal of the first order in Δ.

Remark A.4 We are here in the more complicated situation in which to any $k_{n\pm}$ corresponds a $2n + 1$ dimensional eigenspace of spherical harmonics of degree n. We know however that when we perturb $K_\pm$ by $\delta U_\pm$ we get exactly $2n+1$ eigenvalues

$$k_{nm} = k_{n\pm} + \delta k_{nm\pm} \tag{A.57}$$

close to $k_{n\pm}$, because the spherical symmetry that causes the multiplicity of $k_{n\pm}$ is broken. Therefore the eigenfunctions $W_{nm}(\sigma)$ corresponding to k_{nm} might not be close to any of the classical $Y_{nm}(\sigma)$ to which we are used. Nevertheless, they are close, to the first order in Δ, to some spherical harmonic of degree n, that we will call again $Y_{nm}(\sigma)$ since we will not need its specific form, but only that it belongs to the eigenspace corresponding to $k_{n\pm}$.

We can now prove the following proposition.

Proposition A.4 *Let* $k_{n\pm}$, $\delta k_{nm\pm}$, $W_{nm}(\sigma)$, $Y_{nm}(\sigma)$ *be as in Remark A.4, then one has*

$$\delta k_{nm+} \le 0 \ , \ \delta k_{nm-} \ge 0 \tag{A.58}$$

respectively, implying that

$$k_{n+} + O(\Delta^2) < k_{nm} < k_{n_-} + O(\Delta^2) \ . \tag{A.59}$$

Proof According to the well-known perturbation theory (see [11]), we find

$$\begin{aligned}\delta k_{nm\pm} &= < Y_{nm}, \delta K_\pm, Y_{nm} >= \\ &= - < Y_{nm}, \delta U_\pm^T (I - U_\pm) Y_{nm} > - < Y_{nm}, (I - U_\pm)\delta U Y_{nm} >= \\ &= -2k_{n\pm} < Y_{nm}, \delta U_\pm Y_{nm} > \ .\end{aligned} \tag{A.60}$$

On the other hand, one has

$$< Y_{nm}, \delta U_\pm Y_{nm} >= \sum_{mn} \int \left[\left(\frac{R_0}{R_\sigma} \right)^{n+1} - \left(\frac{R_0}{R_\pm} \right)^{n+1} \right] Y_{nm}^2(\sigma) d\sigma \ . \tag{A.61}$$

But since

$$\left(\frac{R_0}{R_\sigma} \right)^{n+1} - \left(\frac{R_0}{R_-} \right)^{n+1} < 0 \ , \ \left(\frac{R_0}{R_\sigma} \right)^{n+1} - \left(\frac{R_0}{R_+} \right)^{n+1} > 0$$

and $k_{n\pm} > 0$, (A.61) and (A.60) prove (A.59).

Up to here our reasoning has been rigorous. However, in order to derive useful conclusions, we will make a conjecture, namely a statement that seems likely to be true, although we do not have a strict proof.

The "Sandwich" conjecture: given that the terms $O(\Delta^2)$ in (A.59) should be small, we shall stipulate that the "Sandwich inequality"

$$\left[1 - \left(\frac{R_0}{R_+} \right)^{n+1} \right]^2 = k_{n+} < k_{nm} < k_{n-} = \left[1 - \left(\frac{R_0}{R_-} \right)^{n+1} \right]^2 \tag{A.62}$$

holds true.

This conjecture has two consequences: the first is that

$$\sup_{n,m} k_{nm} = 1 \tag{A.63}$$

but then, thanks to (A.53),

$$\| I - U \| = 1 \ , \tag{A.64}$$

which is still not a useful result for the change of boundary theory.

On the other hand, if we restrict all our analysis to the subspace L^2_N of $L^2(\sigma)$, namely the linear space including all spherical harmonics up to a finite maximum degree N, we get

$$\| I - U \|_{L^2_N} \leq \left[1 - \left(\frac{R_0}{R_-} \right)^{N+1} \right] < 1 \tag{A.65}$$

and, at least in this restricted case, the operator $I - U$ is a contraction. As a last comment, let us underline that the above statement does not mean at all that $(I - U)^\ell P_N$, with P_N the orthogonal projection on L^2_N, tends to zero when $\ell \to \infty$. Rather, it means that $[P_N(I - U)P_N]^\ell \to 0$ when $\ell \to \infty$. This means that, starting from a function $f_N \in L^2_N$, when we compute the residuals $(I - U) f_N$, we have further to re-project them on L^2_N and then iterate. Fortunately, this is exactly what is done in practice in the estimation of a global gravity model (see [12]).

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