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Yves Coudène

Ergodic Theory and Dynamical Systems

Translated by Reinie Erné

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Contents

Introduction

Mais malheur à l'auteur qui veut toujours instruire !
Le secret d'ennuyer est celui de tout dire.

Voltaire (1694–1778)

These notes originated in a course for second-year graduate students given at the University of Rennes 1 in the period 2005–2008. It was an introductory course on ergodic theory and dynamical systems; the aim was to present a number of general ideas that form the basis of these two theories, before the students specialize by following more advanced courses.

The course consisted of 12 sessions of 2 h each; I chose to focus each session on one specific concept, and make the different sessions more or less independent of one another. Consequently, the material presented here lies at the junction of very diverse mathematical theories, and the audience interested in the subject often includes students and researchers from very different fields: probability theory, dynamical systems, geometry, physics, etc.

Each chapter begins with an informal presentation of the concepts and the problems we wish to resolve. These are followed by rigorous definitions and proofs, which we have attempted to illustrate with examples that are both simple and relevant. The figures we include shape the readers' intuition, while the exercises allow them to test their understanding of the subject. I found it interesting to add comments at the end of each chapter, in order to place the material into its historical context, present a number of actual problems, and orient the readers toward literature dealing with their own interests. These comments are meant rather for a second reading and assume a certain command of the concepts presented in this book.

As to the contents, I have chosen to put the emphasis on the ideas rather than the theoretical aspects, on the examples rather than the technique. There exist several books presenting the general theories in great detail, both in the domain of dynamical systems and in that of ergodic theory. These notes are not meant to replace them. For a number of classical results, I have given new or unusual proofs, in order to illustrate some lesser known aspects of the subject. These proofs have the potential to interest even the most hardened researcher. Readers are of course invited to consult reference material to learn the more classical approaches, which are summarized in the comments.

Themes

Ergodic theory and dynamical systems are two theories that go well together. The first gives the second its most remarkable quantitative results, while the second is a tireless supplier of examples quick to undermine the conjectures that are most dear to the first. Both originating at the beginning of the twentieth century, at least from the point of view of modern mathematics, under the leadership of one of the giants of the century, Henri Poincaré (1854–1912), they have known a sustained development up to today. Books that pretend to present a nonnegligible part of these theories are susceptible, by their volume and by their style, to frighten even the most motivated students.

This book was written with the aim to be accessible to a large audience, to arouse the interest in a very active field of mathematics, and to serve as an introduction into more advanced literature.

The great problems we wish to solve have not evolved much in one century. Take the example of a map that acts on some configuration space X. The points of X represent the different states the system can take on during its motion. Starting with an initial configuration given by a point x of X, the iterates of x correspond to the successive states the system visits during its evolution. This book is interested in the following questions:

- *Does the system return to its initial state during its evolution?*
  What do the concepts of recurrence and nonwandering try to formalize, both quantitatively (measure) and qualitatively (topology)?
- *It is possible to construct a representation of the system in which the evolution takes on a form that is particularly simple to describe?*
  The notions of local and global conjugation, isomorphism, coding, and symbolic model each try, in their own way, to put the system in a form where the evolution can be computed effectively.
- *Can the system evolve in such a way that it converges to a state given* a priori*, if we perturb it during its evolution?*
  This theme is dominant in hyperbolic dynamics, where the existence of local instabilities, modeled by the stable and unstable manifolds, leads to a uniform behavior of the system that is stable under perturbation.
- *In how far can the evolution of the system be predicted in the long term, or which probability quantity is the system prone to simulate?*
  The concept of entropy, introduced in 1958 by A.N. Kolmogorov in the theory of dynamical systems, has allowed us to make decisive progress on this question.

Organization of This Book

Chapters 1–4 deal with results from ergodic theory (recurrence, ergodicity, mixing), illustrated by examples from algebra, mechanics, or probability theory: Hamiltonian flows, Bernoulli shifts, toral automorphisms, flows on $\mathrm{SL}_2(\mathbf{R})$, etc. We have tried to highlight the role played by the weak topology in questions of ergodicity and mixing; the properties of this topology are recalled in Chaps. 16–18. Chapter 4 presents the Hopf argument, one of the well-known arguments of the hyperbolic theory of dynamical systems.

Chapters 5–9 deal with the dynamics of transformations, from a topological point of view. We introduce the concepts of nonwandering, transitivity, and conjugation, illustrated by the construction of a number of Morse–Smale transformations and by the study of the dynamics of certain polynomials (Schröder's examples). The theory of linearization of Hartman–Grobman allows us to analyze the behavior of the system in the neighborhood of its hyperbolic periodic points; we apply it to the study of a system obtained by perturbing a toral automorphism (called derived from Anosov).

Chapters 10–12 deal with entropy. We prove the Kolmogorov–Sinaï theorem on generating partitions. As applications, we compute the entropy of dilating maps (Rokhlin's formula) and of certain maps on the interval. One chapter deals with the interpretation of entropy in information theory.

The notions of Lebesgue space and ergodic decomposition are studied in Chapters 13–15. These important notions are rarely treated in detail in the literature. The object of these chapters is to present the theory of ergodic decomposition clearly, concisely, and completely. To do this, we have used the Hopf argument as inspiration, and construct the ergodic components "geometrically".

For Whom Is This Book

This book can be studied by a graduate student who has followed a course on measure theory and knows the vocabulary of the theory of Hilbert spaces. Some examples require a certain familiarity with the notions of flows and differential manifolds.

Researchers who wish to familiarize themselves with the issues at the intersection of dynamical systems and ergodic theory can also benefit from this text, through the comments appearing at the end of the chapters. These give a brief overview of the problems and methods that have characterized the theory, and mention a number of open questions in the field.

Chapters 16–18 summarize the results that are not necessarily part of undergraduate or graduate studies. From the first chapter on, we use the weak topology in the setting of Hilbert spaces. The properties of this topology are recalled in Chap. 16. Chapter 17 deals with the notion of conditional expectation.

Certain aspects of the theory of metric spaces and of measure theory are not studied in their most general form in undergraduate courses: separability, support, regularity, density of Lipschitz functions in the spaces L^p. These are presented in Chap. 18. If the reader is not familiar with these results, it is best to omit them during a first reading of the book. They become more or less evident when we work on open subsets of $\mathbf{R}^n$ with measures of the form $f(x)\,\mathrm{d}x$; it is with this type of space in mind that the reader is invited to begin reading.

Part I
Ergodic Theory

Chapter 1
The Mean Ergodic Theorem

The most useful piece of advice I would give to a mathematics student is always to suspect an impressive sounding theorem if it does not have a special case which is both simple and non-trivial.

M.F. Atiyah

1.1 Introduction

Ergodic theory is the study of the long-term behavior of systems preserving a certain form of energy.

From a mathematical point of view, a physical system can be modeled by the data of a space X, a transformation $T : X \to X$, and a measure μ defined on X and invariant under T: for every measurable set $A \subset X$, we have $\mu(T^{-1}(A)) = \mu(A)$. The quadruple consisting of the space X, the measure μ, the σ-algebra consisting of the measurable sets with respect to μ, and the measurable transformation T that preserves μ form what we call a *measure-preserving dynamical system*.

The space X consists of the set of all possible states of the system during its evolution. The transformation T describes its evolution in time; $T(x)$ is the state of the system at time 1 if it was in state x at time 0. The successive iterates $T^2(x)$, $T^3(x), \ldots$ give the state of the system at time $2, 3, \ldots$ Finally, the measure μ corresponds to an arbitrary extensive quantity defined on the space X and preserved during the motion.

The typical example comes from classical mechanics. It is given by a point mass that moves under the action of a time-independent potential. The set $X = \mathbf{R}^3 \times \mathbf{R}^3$ is the space (x, v) of positions and velocities, also called the phase space. The transformation T associates with the initial condition (x, v) the values of the position and velocity after a given time period, for example 1 s, 1 day, or 1 year, depending on the studied time scale. Finally, the measure μ is the standard volume $\mathrm{d}x\,\mathrm{d}v$ defined on the space X. Its invariance follows from the preservation of energy.

We want to determine the behavior of the sequence of iterates $T^n = T \circ T \circ \cdots \circ T$. The following remark, due to B. Koopman (1931), is crucial for what lies ahead. If we let the transformation T act by composition on the space $L^2(X, \mu)$ of square-integrable functions, the resulting map U is a linear isometry: if $f \in L^2$ and $Uf = f \circ T$, then $\|Uf\| = \|f\|$. This follows from the invariance of μ under T. We

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can therefore apply techniques from Hilbert analysis to study the "average" behavior of the sequence $f \circ T^n$, that is, its behavior in L^2 norm.

By focusing our attention on the L^2 action, we have replaced an a priori nonlinear problem in finite dimension with a linear problem in infinite dimension. Did we truly come out ahead? As it happens, Hilbert spaces have a certain number of properties reminiscent of finite dimension. The most useful is the weak compactness of the unit ball. Showing weak convergence therefore corresponds to identifying the limit through a property that characterizes it uniquely, a task that turns out to be simpler than that of showing the convergence.

These Hilbertian methods allow us to obtain the convergence of the averages $\frac{1}{n}\sum_{k=0}^{n-1} U^k$ for every linear map U satisfying the inequality $\|Uf\| \leqslant \|f\|$ for every $f \in L^2$. This result, initially obtained by J. Von Neumann (1932) in a slightly different context using functional calculus methods, illustrates a fact often used in analysis, namely that "taking averages tends to make things more regular".

A consequence of the ergodic theorem is that if the space X has finite measure, then almost every trajectory returns arbitrarily close to its initial state. This is one of the rare general conclusions we can draw on the character of motion in classical mechanics. Anterior to the ergodic theorem, this result, proved by H. Poincaré in 1899, is often considered as the first mathematical result of ergodic theory, and marks the birth of this discipline.

1.2 The Mean Ergodic Theorem

Theorem 1.1 *Let H be a Hilbert space, and let $U : H \to H$ be a linear map satisfying $\forall f \in H$, $\|Uf\| \leqslant \|f\|$. Set*

$$S_n(f) = \sum_{k=0}^{n-1} U^k f \quad \textit{and} \quad \mathrm{Inv} = \{f \in H \mid Uf = f\}.$$

Denote by $P : H \to H$ the orthogonal projection onto the subspace Inv *of U-invariant vectors. Then*

$$\frac{1}{n} S_n(f) \longrightarrow Pf \quad \textit{in norm.}$$

The proof we will present is based on an argument given by R. Mañé and uses the weak topology in Hilbert spaces. The properties of this topology are given in detail in Chap. 16. The proof also calls upon the adjoint U^* of the map U. Recall that this adjoint is a linear map from H to H defined by the equality

$$\langle U^* f, g\rangle = \langle f, Ug\rangle.$$

It satisfies the relations $(U^*)^* = U$ and $\|U^*\| = \|U\|$. We will prove that when this norm is bounded from above by 1, the map U^* has the same invariant vectors as U.

Lemma 1.1 *Under the assumptions of the theorem, every element $g \in H$ that is invariant under U is invariant under U^*. Likewise, every element $g \in H$ that is invariant under U^* is invariant under U.*

Proof Assume that g is invariant, that is, $Ug = g$. The equality $U^*g = g$ results from the following calculation:

$$\|g - U^*g\|^2 = \|g\|^2 + \|U^*g\|^2 - 2\langle g, U^*g\rangle \leqslant 2\|g\|^2 - 2\langle Ug, g\rangle = 2\langle g - Ug, g\rangle.$$

It suffices to replace U by U^* in this calculation to show that every element $g \in H$ that is invariant under U^* is invariant under U. □

Proof of Theorem 1.1 If f belongs to Inv, we have $\frac{1}{n}S_n(f) = f$ and the theorem holds. It suffices to prove that $\frac{1}{n}S_n(f)$ tends to 0 for $f \in \mathrm{Inv}^\perp$. Note that the spaces Inv and $\mathrm{Inv}^\perp$ are invariant under both U and U^* by virtue of the lemma.

We have the equality

$$\|\tfrac{1}{n}S_n(f)\|^2 = \langle f, \tfrac{1}{n}S_n^*\, \tfrac{1}{n}S_n(f)\rangle.$$

We must therefore verify, for every $f \in \mathrm{Inv}^\perp$, that the sequence $\frac{1}{n}S_n^*\, \frac{1}{n}S_n(f)$ converges weakly to 0, or equivalently that the accumulation points of this sequence are all 0. Because they are in $\mathrm{Inv}^\perp$, it suffices to prove that they are invariant under U or under U^*, by the lemma. To do this, we note that for every $h \in H$, we have the equality

$$(I - U^*)\,\tfrac{1}{n}S_n^*\,h = \frac{1}{n}(I - U^*)\sum_{k=0}^{n-1} U^{*k}\,h = \frac{1}{n}(I - U^{*n})\,h.$$

Let $h = \frac{1}{n}S_n(f)$; we have the following upper bound:

$$\|(I - U^*)\,\tfrac{1}{n}S_n^*\,\tfrac{1}{n}S_n(f)\| \leqslant \frac{1}{n}\,\|(I - U^{*n})\|\cdot\|\tfrac{1}{n}S_n(f)\| \leqslant \frac{2}{n}\,\|f\| \xrightarrow[n\to\infty]{} 0.$$

Convergence in norm implies weak convergence. Consequently, every weak accumulation point of the sequence $\frac{1}{n}S_n^*\, \frac{1}{n}S_n(f)$ is invariant under U, as desired. □

Let $(X, \mathcal{T}, \mu)$ be a measure space. Let H be the space $L^2(X)$ of square-integrable measurable functions with real values. From a measurable map $T : X \to X$ we define a linear map $U : H \to H$ by setting $Uf = f \circ T$. If T preserves the measure μ, the operator U satisfies $\|Uf\| = \|f\|$. We can apply the above to obtain the L^2 ergodic theorem (Theorem 1.2).

Theorem 1.2 (Von Neumann) *Let $(X, \mathcal{T}, \mu)$ be a measure space, let $T : X \to X$ be a measurable map that preserves μ, and let $f \in L^2(X)$. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{L^2} Pf,$$

where P is the orthogonal projection onto the subspace $\{f \in L^2 \mid f \circ T = f\}$.

A measurable function $f : X \to \mathbf{R}$ that satisfies $f \circ T = f$ is called *invariant* for the transformation T. Let A be a measurable set, and let $T^{-1}A$ be the set of points of X whose image belongs to A. The set A is called *invariant* under T if it satisfies the relation $T^{-1}A = A$. Its characteristic function is then invariant under T:

$$\mathbf{1}_A \circ T = \mathbf{1}_{T^{-1}A} = \mathbf{1}_A.$$

The measurable sets that are invariant under T form a σ-algebra that we will denote by $\mathcal{I}$. Let us show that the invariant functions are precisely the functions that are measurable with respect to this σ-algebra $\mathcal{I}$.

Proposition 1.1 *Let $(X, \mathcal{T}, \mu)$ be a measure space, let $T : X \to X$ be a measurable map that preserves μ, and let $f : X \to \mathbf{R}$ be a measurable function. Then f is invariant under T if and only if it is measurable with respect to the σ-algebra consisting of the invariant sets.*

Proof A function f is measurable with respect to $\mathcal{I}$ precisely when its level sets $f^{-1}(y)$ are invariant under T, that is, when we have the equality $T^{-1}f^{-1}(y) = f^{-1}(y)$ for every $y \in \mathbf{R}$. But this relation is equivalent to the equality $f(T(x)) = f(x)$ for every $x \in X$ satisfying $f(x) = y$. □

Let us now consider the properties of the projection P defined earlier. This projection is orthogonal, as illustrated in Fig. 1.1.

Properties of the projector P

- $\forall f \in L^2$, $\forall g \in L^2$ *such that* $g \circ T = g$, *we have* $\int Pf\, g \,\mathrm{d}\mu = \int f\, g \,\mathrm{d}\mu$;
- $\forall f \in L^2$, $\forall A \subset X$ *such that* $T^{-1}A = A$ *and* $\mu(A) < \infty$, *we have* $\int_A Pf \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu$;
- $\forall f \in L^2$ *such that* $f \geqslant 0$, *we have* $Pf \geqslant 0$.

Moreover, if $\mu(X) < \infty$, *then*

- $\forall f \in L^2$, *we have* $\int Pf \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu$;
- $\forall f \in L^2$ *such that* $f \geqslant 0$, *for almost all* $x \in X$, *the inequality* $f(x) > 0$ *implies* $Pf(x) > 0$.

Proof The projection P is orthogonal, it is equal to its adjoint: $P = P^*$. This implies

$$\int Pf\, g \,\mathrm{d}\mu = \langle Pf, g\rangle = \langle f, P^* g\rangle = \langle f, Pg\rangle = \int f\, g \,\mathrm{d}\mu.$$

This proves the first statement, the second follows from this equality by taking $g = \mathbf{1}_A$, and the case $A = X$ corresponds to the fourth statement.

Let us now prove the inequalities. For every $N > 0$, we have the upper bound

$$\mu(\{x \mid Pf(x) < -1/N\}) \leqslant N^2 \int |Pf|^2 \,\mathrm{d}\mu < \infty,$$

which implies

$$-\frac{1}{N}\,\mu(\{x \mid Pf(x) < -\tfrac{1}{N}\}) \geqslant \int_{\{Pf(x)<-\frac{1}{N}\}} Pf \,\mathrm{d}\mu = \int_{\{Pf(x)<-\frac{1}{N}\}} f \,\mathrm{d}\mu \geqslant 0.$$

It follows that $\mu(\{x \mid Pf(x) < -1/N\}) = 0$, and therefore that

$$\mu(\{x \mid Pf(x) < 0\}) = 0.$$

Finally, if the measure of the set $\{x \mid Pf(x) = 0\}$ is finite, we have the equality

$$\int_{\{x|Pf(x)=0\}} f \,\mathrm{d}\mu = \int_{\{x|Pf(x)=0\}} Pf \,\mathrm{d}\mu = 0.$$

The function f therefore vanishes on $\{x \mid Pf(x) = 0\}$ provided that it is nonnegative. □

As an application, we can now prove the Poincaré recurrence theorem, illustrated by Fig. 1.2.

Theorem 1.3 *Let $(X, \mathcal{T}, \mu)$ be a measure space, and let $T : X \to X$ be a measurable map that preserves μ. We assume $\mu(X) < +\infty$. Let $B \subset X$ be a measurable set. Then for almost all $x \in B$, there exist infinitely many $n \in \mathbf{N}$ such that $T^n(x) \in B$.*

Proof Recall that convergence in L^2 norm implies the strong convergence of a subsequence. This remark, combined with the ergodic theorem, shows that there exists a subsequence n_i such that for almost all $x \in X$, the sum $\frac{1}{n_i} S_{n_i} \mathbf{1}_B$ converges to $P\mathbf{1}_B(x)$. This quantity is strictly positive for almost all $x \in B$, by virtue of the last property of the projection P proved earlier.

If the trajectory of x passes through B only finitely many times, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_B(T^k(x)) \longrightarrow 0,$$

which gives a contradiction. □

We can prove the Poincaré recurrence theorem without using the ergodic theorem.

Another proof Set

$$\overline{\mathbf{1}}_B(x) = \overline{\lim}\, \frac{1}{n} S_n(\mathbf{1}_B)(x) = \overline{\lim}\, \frac{1}{n} \operatorname{Card}\{0 \leqslant k \leqslant n-1 \mid T^k(x) \in B\}.$$

This function is invariant under the transformation T: $\overline{\mathbf{1}}_B \circ T = \overline{\mathbf{1}}_B$. We have

$$\begin{aligned}
\mu\big(B \cap (\overline{\mathbf{1}}_B = 0)\big) &= \int \mathbf{1}_B\, \mathbf{1}_{(\overline{\mathbf{1}}_B=0)}\, \mathrm{d}\mu \\
&= \int \mathbf{1}_B \circ T^k\, \mathbf{1}_{(\overline{\mathbf{1}}_B=0)}\, \mathrm{d}\mu \qquad \text{for every } k, \text{ by invariance,} \\
&= \int \frac{1}{n} S_n(\mathbf{1}_B)\, \mathbf{1}_{(\overline{\mathbf{1}}_B=0)}\, \mathrm{d}\mu \qquad \text{by taking the average over } k, \\
&\leqslant \int \overline{\mathbf{1}}_B\, \mathbf{1}_{(\overline{\mathbf{1}}_B=0)}\, \mathrm{d}\mu \qquad \text{by Fatou's lemma,} \\
&= 0.
\end{aligned}$$

For almost all $x \in B$, the frequency $\overline{\mathbf{1}}_B(x)$ is nonzero, which proves the result. □

1.3 Application to Classical Mechanics

The main motivation of H. Poincaré comes from classical mechanics. Consider a point mass under the influence of a time-independent force field. We will show that if the space is closed and the energy is conserved during the motion, there exists an invariant finite measure in the phase space. We can then apply the recurrence theorem and conclude that the system certainly returns to a state close to its initial state.

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a C^2 function. We denote its gradient by ∇V. The energy associated with the potential V is given by

$$\forall (x, v) \in \mathbf{R}^n \times \mathbf{R}^n,\ E(x, v) = \tfrac{1}{2} m v^2 + V(x).$$

Assume that there exists $E_0 \in \mathbf{R}$ such that the energy surface $E^{-1}(E_0)$ is compact and $E^{-1}(E_0) \cap \{(x, 0) \mid \nabla V(x) = 0\} = \varnothing$. Then:

- *For every* $(x_0, v_0) \in E^{-1}(E_0)$*, the differential equation*

$$m \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} mv \\ -\nabla V(x) \end{pmatrix}$$

  admits a unique solution $\varphi_t(x_0, v_0)$ *satisfying* $\varphi_0(x_0, v_0) = (x_0, v_0)$ *and defined for every* t.

- *The energy E is constant along the trajectories of the flow φ_t.*
- *Denote by* vol_{2n-1} *the Riemannian volume form on the manifold* $E^{-1}(E_0)$. *The Borel measure* $\mathrm{d}\mu = \|\nabla E\|^{-1}\,\mathrm{d}\,\mathrm{vol}_{2n-1}$ *is a finite measure, invariant under the transformations* $(x, v) \mapsto \varphi_t(x, v)$ *for every* $t \in \mathbf{R}$. *Its support is equal to* $E^{-1}(E_0)$.

The first statement follows from general existence theorems for differential equations; here we can apply the Cauchy–Lipschitz theorem to deduce the existence of a solution on an open time-interval. We then obtain the invariance of the energy through an elementary calculation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi_t(x)) = \left\langle \nabla E, \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)\right\rangle &= mv\,\frac{\partial E}{\partial x} - \frac{\partial E}{\partial v}\,\nabla V(x)\\ &= mv\,\nabla V(x) - mv\,\nabla V(x) = 0.\end{aligned}$$

Once we know that the trajectory is restrained to a compact subset of the phase space, the usual existence theorems for differential equations allow us to assert that the solutions are defined for all t.

The invariance of the measure is due to J. Liouville. It is illustrated by Fig. 1.3 and can be deduced from the following result.

Lemma 1.2 *Let φ_t be a C^2 flow defined on an open subset U of $\mathbf{R}^d$, and let X be the associated vector field: $X(x) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)_{|t=0}$. Let $f : U \to \mathbf{R}$ be an integrable map, zero outside of a compact subset of U. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f(\varphi_t(x))\,\mathrm{d}x\,\Big|_{t=t_0} = \int f(\varphi_{t_0}(x))\,\mathrm{div}\,X(x)\,\mathrm{d}x.$$

Proof Using a partition of unity, we can restrict ourselves to the case where f is C^1 with compact support, localized in a box $R = \prod_i [a_i, b_i]$. After replacing f by $f \circ \varphi_{t_0}$, if necessary, we may moreover assume $t_0 = 0$. We differentiate under the summation sign:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R f(\varphi_t(x))\,\mathrm{d}x = \int_R \langle \nabla f(x), X(x)\rangle\,\mathrm{d}x = \int_R \sum_i \frac{\partial f}{\partial x_i}\cdot X_i\,\mathrm{d}x.$$

Next, we integrate by parts:

$$\int_R \frac{\partial f}{\partial x_i} X_i\,\mathrm{d}x = \int_R \frac{\partial (f X_i)}{\partial x_i}\,\mathrm{d}x - \int_R f\,\frac{\partial X_i}{\partial x_i}\,\mathrm{d}x,$$

$$\int\cdots\int\Big(\int \frac{\partial (fX_i)}{\partial x_i}\,\mathrm{d}x_i\Big)\,\mathrm{d}x_1\cdots\mathrm{d}x_n = \int\cdots\int [f X_i]_{a_i}^{b_i}\,\mathrm{d}x_1\cdots\mathrm{d}x_n = 0,$$

taking into account that the function f vanishes on the boundary of R. □

We can now show the invariance of the volume.

Proof of the invariance of the measure μ The divergence of the vector field $\binom{x}{v} \mapsto \binom{v}{-\nabla V(x)/m}$ is zero:

$$\frac{\partial v}{\partial x} - \frac{1}{m}\frac{\partial}{\partial v}\nabla V(x) = 0.$$

By the lemma, the volume form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_n$ is invariant under φ_t. Denote by ω the volume form on $E^{-1}(E_0)$ associated with the Riemannian volume form. It satisfies the relation

$$\|\nabla E\|^{-1}\omega \wedge dE = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_n.$$

To obtain this equality, it suffices to evaluate each of the two terms on a basis for $\mathbf{R}^n \times \mathbf{R}^n$ of the form $(\frac{\partial \psi}{\partial y_1}, \dots, \frac{\partial \psi}{\partial y_{2n-1}}, \nabla E)$, where $\psi(y_1, \dots, y_{2n-1})$ is a coordinate system on $E^{-1}(E_0)$. The invariance of $\|\nabla E\|^{-1}\omega$ then results from the following calculation:

$$\varphi_t^*\Big(\frac{\omega}{\|\nabla E\|}\Big) \wedge \mathrm{d}E = \varphi_t^*\Big(\frac{\omega}{\|\nabla E\|}\Big) \wedge \varphi_t^*\mathrm{d}E = \varphi_t^*\Big(\frac{\omega \wedge dE}{\|\nabla E\|}\Big) = \frac{\omega}{\|\nabla E\|} \wedge \mathrm{d}E.$$

□

It suffices to apply the recurrence theorem to the map $(x, v) \mapsto \varphi_1(x, v)$ to obtain the desired result.

Corollary 1.1 *Let $B \subset E^{-1}(E_0)$ be a measurable set with respect to* vol_{2n-1}. *Then for almost every point of B, the associated trajectory passes through B infinitely many times.*

The reader who wishes to know more about the link between classical mechanics and ergodic theory can consult the books of Arnold [1] and of Arnold and Avez [2].

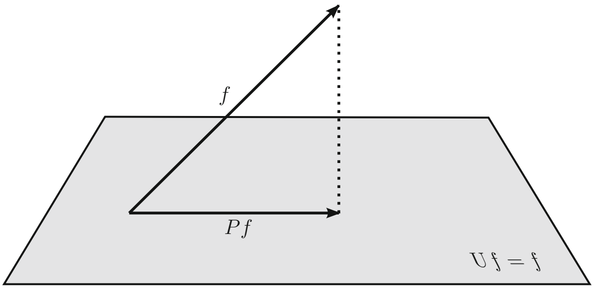


Fig. 1.1 Projection onto an invariant subspace

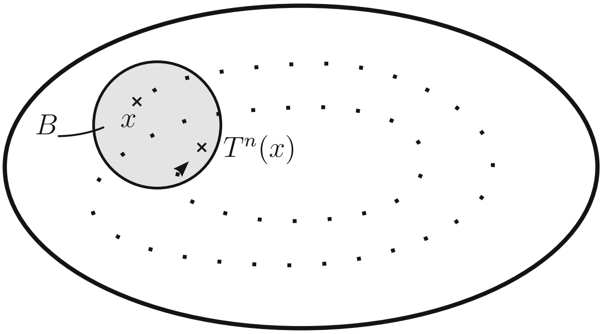


Fig. 1.2 Recurrence

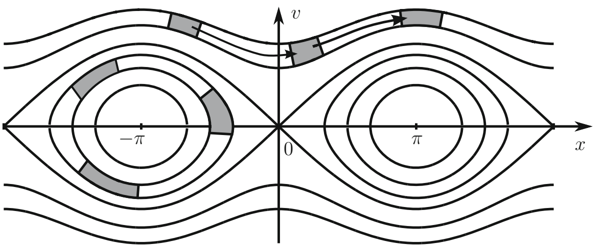


Fig. 1.3 observation of areas, physical pendulum: $\frac{1}{2}mv^2 + \cos x = \mathrm{C}$

1.4 Exercises

1.4.1 Basic Exercises

Exercise 1 Let H be a Hilbert space, let U be an invertible isometry of H, and let $f \in H$. Show that the sequence $\frac{1}{2n+1}\sum_{k=-n}^{n} U^k f$ converges in norm. What is its limit?

Exercise 2 Let H be a Hilbert space, let U be an isometry of H, and let $f \in H$. Prove the identity

$$f - \frac{1}{n} S_n(f) = g_n - U g_n, \quad \text{with } g_n = \frac{1}{n}\sum_{k=0}^{n-1} S_k(f).$$

A *coboundary* is an element of H of the form $g - Ug$, with $g \in H$. Show that the coboundaries are dense in the orthogonal complement of the U-invariant functions.

Exercise 3 Let H be a Hilbert space, let U be an isometry of H, let $\theta \in \mathbf{R}$, and let $P_\theta : H \to H$ be the orthogonal projection onto the subspace $\{f \in H \mid Uf = \mathrm{e}^{i\theta} f\}$. Show that

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathrm{e}^{-ik\theta}\, U^k f \xrightarrow[n\to\infty]{L^2} P_\theta f.$$

Exercise 4 Let $(X, \mathcal{T}, \mu)$ be a probability space, let $T : X \to X$ be a measurable map that preserves the measure μ, and let A, B be two measurable subsets of X. Determine the limit

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(A \cap T^{-k} B) \xrightarrow[n\to\infty]{} ?$$

Prove the equality $\langle P\mathbf{1}_A, P\mathbf{1}_B\rangle = \mu(A)\mu(B) + \langle P\mathbf{1}_A - \mu(A), P\mathbf{1}_B - \mu(B)\rangle$.

Exercise 5 Let $(X, \mathcal{T}, \mu)$ be a measure space such that $\mu(X) < \infty$, and let $T : X \to X$ be a measurable map that preserves the measure μ. Let $A \subset X$ be a measurable subset; show that $T^{-1}A \subset A$ implies $\mu(A \smallsetminus T^{-1}A) = 0$.

Exercise 6 Let $(X, \mathcal{T}, \mu)$ be a measure space such that $\mu(X) < \infty$, and let $T : X \to X$ be a measurable map that preserves the measure μ. Let $f : X \to (0, +\infty)$ be a measurable function. Show that $\sum_{k=0}^{\infty} f(T^k(x)) = \infty$ for almost all $x \in X$.

1.4.2 More Advanced Exercises

Exercise 7 Let H be a Hilbert space, let U be an isometry of H, and $f \in H$. Show that the sequence $U^n f$ converges in norm if and only if $Uf = f$.
Hint: What should the limit be equal to?

Exercise 8 A contraction U defined on a Hilbert space H is a continuous linear map of norm at most 1: $\|U\| \leqslant 1$. Let $\mathrm{Inv} = \{f \in H \mid Uf = f\}$, and let P be the orthogonal projection onto Inv.

- Show that $P = P^2 = P^* = PU = UP = PU^* = U^*P$.
- Set $L = \frac{1}{2}(\mathrm{Id} + U)$. Show that $L^n f \to Pf$ in norm.

  Hint: *We use the notation $C_n^k = n!/(k!(n-k)!)$.*
  Prove the inequality $\|L^n(1-U)\| \leqslant C_n^{n/2}/2^{n-1}$.

Exercise 9 Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map that preserves the measure μ. Let P be the orthogonal projection from $L^2(X)$ onto the linear subspace of T-invariant functions.

- Show that for every measurable $A \subset X$, we have $\|P\mathbf{1}_A\| \geqslant \mu(A)$.

  Hint: Use the Cauchy–Schwarz inequality.

- What is the limit of the sequence $\frac{1}{n}\sum_{k=0}^{n-1} \mu(A \cap T^{-k}A)$?
- Deduce the inequality $\overline{\lim}\, \mu(A \cap T^{-n}A) \geqslant \mu(A)^2$.

1.5 Comments

U. Krengel's book [13] contains a detailed presentation of the ergodic theorems.
The proof of the mean ergodic theorem that we have presented is due to à R. Mañé [15].
There exist other proofs:

- The most popular proof is due to F. Riesz [19] and dates back to the 1940s. It consists in verifying the theorem for the coboundaries $g - Ug$, and then showing the density of the coboundaries through a direct calculation. Assume that U is unitary; we must show that every vector $f \in H$ that is orthogonal to the coboundaries is U-invariant:
  if $\forall g \in H$, $\langle g - Ug, f\rangle = 0$, then $\langle g, f\rangle = \langle Ug, f\rangle = \langle g, U^{-1}f\rangle$; consequently, $U^{-1}f = f$.
- In his book [19], F. Riesz gives a proof of the ergodic theorem based on a convexity argument: *Let C be a convex subset in a Hilbert space, and let μ be the infimum of the norms of the elements of C. Then every sequence in C whose norm converges to μ is in fact convergent.* This can be proved using the parallelogram law.
- The original proof of J. Von Neumann (1931) used functional calculus for unitary operators. It can be found in the book by F. Riesz and B. Nagy [19]. This proof can be summarized as follows: *Let $U \in \mathcal{L}(H)$ be a unitary operator. We can construct an algebra homomorphism from the set of bounded Borel functions $f : S^1 \to \mathbf{C}$ to $\mathcal{L}(H)$ that sends $\mathbf{1}$ onto the identity and $z \mapsto z$ onto U. The image of $g : S^1 \to \mathbf{C}$ is denoted by $g(U)$. This morphism moreover satisfies that if g_n is a uniformly bounded sequence and*

$g_n \to g$ *pointwise, then for all* $f \in H$*, we have* $g_n(U)f \to g(U)f$ *in norm.* To obtain the ergodic theorem, it suffices to take $g_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} z^k$ and note that for all $z \in S^1$, we have $g_n(z) \to \mathbf{1}_{\{0\}}(z)$.

- Finally, we can give a proof that holds in an arbitrary reflexive Banach space (for example, in L^p for $1 < p < \infty$), using weak compactness and a convexity lemma in the style of Banach–Saks; we refer to the book by U. Krengel [13, Chap. 2]. The limit P is identified with the projection with image Inv and kernel $\overline{\mathrm{Im}(\mathrm{Id} - T)}$. Oddly enough, this generalization is nontrivial, even in finite dimension:
  Let Q *be a stochastic* $n \times n$ *matrix* ($\forall i, j,\ Q_{i,j} \geqslant 0$ *and* $\forall i,\ \sum_j Q_{i,j} = 1$)*, which we identify with the contraction of* $\mathbf{R}^n$ *endowed with the uniform norm. Then* $\frac{1}{n}\sum_1^n Q^k \to P$*, where* P *is the projection defined earlier.*
  Here is a direct proof of this result, in the style of F. Riesz: The space of stochastic matrices is compact and convex and contains the Q^k. It therefore suffices to show that P is the only possible accumulation point for $\frac{1}{n}\sum_1^n Q^k$. Let P_1 be such an accumulation point; a direct calculation shows that $P_1 x = x$ if $x \in \mathrm{Ker}(\mathrm{Id} - Q)$ and $P_1 x = 0$ if $x \in \mathrm{Im}(\mathrm{Id} - Q)$. The subspaces $\mathrm{Ker}(Q - \mathrm{Id})$ and $\mathrm{Im}(Q - \mathrm{Id})$ are in direct sum, and P_1 is the expected projection. The book by J. Kemeny, J. Snell, and A. Knapp (*Denumerable Markov Chains*, Chapter 6.1) gives a slightly different proof of this result.

There exists a process to generalize a result on unitary operators to arbitrary contractions. It relies on the following fact due to P. Halmos (1950) and presented in the book by F. Riesz and B. Nagy [19, App. §4]:
Let T *be a linear map defined on a Hilbert space* H *and satisfying* $\|T\| \leqslant 1$*. There exist a Hilbert space* H_1 *containing* H *and a unitary operator* $U : H_1 \to H_1$ *such that* $T^n f = PU^n f$ *and* ${T^*}^n f = PU^{-n} f$ *for every* $f \in H$*, where we have denoted by* P *the orthogonal projection from* H_1 *onto* H*.*
The ergodic theorem stated earlier is of little use when the measure μ is infinite and the transformation is ergodic, because in that case there is no nonzero L^2 invariant function. When the transformation is not ergodic, the ergodic components may be finite, in which case the limit may be nonzero; this is for example the case for a rotation defined on $\mathbf{R}^2$.
It is not necessary that U be linear to obtain the weak convergence in the mean ergodic theorem (Baillon's nonlinear ergodic theorem).
The L^2 convergence of $\frac{1}{n}\sum \mathrm{e}^{-ik\theta} f \circ T^k$ is uniform in θ (Wiener–Wintner theorem). Generalizations of this result can be found in the work of J. Bourgain (1990).
There exist topological versions of the Poincaré recurrence theorem. If X is a metric space, almost every point belonging to the support of the measure is topologically recurrent: not only does the trajectory emanating from the point return to B, it also admits a subsequence, in B, that converges to its initial point. This will be proved further on. In his book *Measure and Category*, in Chap. 17, J. Oxtoby gives an abstract version of the recurrence theorem that unites the topological and measurable aspects.
We can generalize the Poincaré recurrence theorem in several directions; for example, Von Neumann's ergodic theorem shows that almost every point of B returns to B with a positive frequency. Another generalization, which can be found in the books [18] and [13], is due to A. Khintchine:
For every $\varepsilon > 0$*, we can find* $L > 0$ *such that every interval of length* L *contains an integer* n *satisfying* $\mu(B \cap T^{-n}B) \geqslant \mu^2(B) - \varepsilon$.
This property was studied by T. Downarowicz and V. Bergelson (2008) in connection with the mixing of the transformation.

Chapter 2
The Pointwise Ergodic Theorem

Le second, de diviser chacune des difficultés que j'examinerois, en autant de parcelles qu'il se pourroit, et qu'il seroit requis pour les mieux résoudre.

R. Descartes (1596–1650)

2.1 Introduction

Consider a dynamical system, modeled by the data of a phase space X, a transformation $T : X \to X$ describing the evolution of the system over time, and a finite measure μ representing an extensive quantity conserved during the motion. We wish to study the sequence $\{T^n(x)\}_{n\in\mathbf{N}}$, which represents the succession of states the system takes on over time. This sequence makes up the *trajectory* of the point x, or its *orbit*.

Let us study the asymptotic behavior of this sequence. To do this, we consider an observable quantity $f : X \to \mathbf{R}$ and study its evolution in time. The quantities $S_n(f)(x) = \sum_{k=0}^{n-1} f(T^k(x))$ are called the Birkhoff sums of the function f and the averages

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x))$$

are the *Birkhoff averages* of f. In 1932, G.D. Birkhoff showed that the sequence of averages $\frac{1}{n}S_n(f)(x)$ converges for almost all $x \in X$, provided that the function f is integrable. When f is the characteristic function of a set $A \subset X$, these averages correspond to the frequency of passage of the iterates of x in the set A between the times 0 and $n-1$. These frequencies converge, and the limit is the average time x passes in A during its motion.

No doubt, the most natural idea for attacking a problem is to try to divide it up into several subproblems, which will, with any luck, be easier to treat than the initial problem. To study the dynamics of a transformation, we can try to "break up" the space X into several disjoint pieces, each of nonzero measure, so as to restrict the transformation to each of these pieces, as shown in Fig. 2.2. If this is not possible, the system is called *ergodic*.

Y. Coudène, *Ergodic Theory and Dynamical Systems*, Universitext,
DOI 10.1007/978-1-4471-7287-1_2

When a system is ergodic, it is possible to compute explicitly the limit of the Birkhoff averages $\frac{1}{n}S_n(f)$. This limit does not depend on x and is obtained by averaging f over X with respect to the studied measure. We can therefore say that for an ergodic system,

the time averages coincide with the space averages.

The Birkhoff ergodic theorem therefore allows us to pass from a qualitative property, namely *no nontrivial invariant sets*, to a quantitative statement, namely *the frequency of passage in an arbitrary set is proportional to the size of the set*. In particular, during the motion, the trajectories visit the whole space if the system is ergodic.

The simplest examples of ergodic systems come from probability theory. Consider a random experiment, such as rolling a die or drawing a ball from an urn. Let Ω be the set of possible outcomes, and let P be the probability measure on Ω associated with these outcomes. The repetition of this experiment, independently and an indefinite number of times, can be modeled by considering the space of sequences of outcomes $\Omega^{\mathbf{N}}$ endowed with the product probability $P^{\otimes\mathbf{N}}$ and the shift transformation, which consists in leaving out the first element of the sequence and shifting the other elements one to the left. In this context, the ergodic theorem, together with the ergodicity of the shift, gives the strong law of large numbers, whose first proof in this general setting is due to A.N. Kolmogorov (1933).

2.2 The Pointwise Ergodic Theorem

The following theorem is know as the Birkhoff (1932) ergodic theorem. It is a pointwise version of the ergodic theorem presented in Chap. 1. The statement invokes the notion of conditional expectation, presented in Chap. 17.

Theorem 2.1 (Birkhoff) *Let $(X, \mathcal{T}, \mu)$ be a measure space for which $\mu(X) < \infty$, let $T : X \to X$ be a measurable map that preserves the measure μ, and let $f : X \to \mathbf{R}$ be an integrable function. Set $\mathcal{I} = \{A \in \mathcal{T} \mid T^{-1}A = A\}$. Then, for μ-almost all $x \in X$,*

$$\frac{1}{n}\sum_{k=0}^{n-1} f\big(T^k(x)\big) \xrightarrow[n\to\infty]{} E(f \mid \mathcal{I})(x).$$

Proof Set

$$\overline{f}(x) = \overline{\lim}\,\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)), \quad \underline{f}(x) = \underline{\lim}\,\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)).$$

To obtain the almost everywhere convergence, it suffices to prove the inequalities $\int \overline{f}\mathrm{d}\mu \leqslant \int f\mathrm{d}\mu \leqslant \int \underline{f}\mathrm{d}\mu$, because if these hold, $\overline{f} - \underline{f}$ is a nonnegative function with zero integral, which implies that $\overline{f} - \underline{f} = 0$ for almost all x. Let us prove $\int \overline{f} \mathrm{d}\mu \leqslant \int f \mathrm{d}\mu$, the other inequality is obtained by replacing f by $-f$.

The function $\overline{f}$ is not a priori bounded from above and can take on the values $-\infty$ and $+\infty$. To bypass this difficulty, we fix a constant $M > 0$ and introduce the function $\overline{f}_M = \min(\overline{f}, M)$. At the end of the proof, the constant M will tend to infinity. The following calculation, which uses Fatou's lemma, shows that $\overline{f}_M(x)$ is unequal to $-\infty$ for almost all $x \in X$:

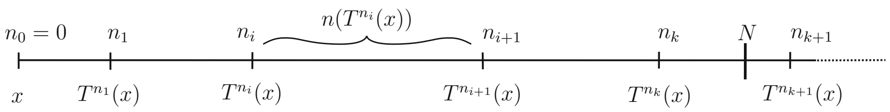
$$\int \overline{f}_M \geqslant \int \overline{\lim} \frac{1}{n} \sum -|f \circ T^k| \geqslant \overline{\lim} \int \frac{1}{n} \sum -|f \circ T^k| = -\int |f| > -\infty.$$

Let us now fix $\varepsilon > 0$. Recall that the upper limit of a sequence is the greatest accumulation point of the sequence. If $\overline{f}(x)$ is finite, we can find $n \in \mathbf{N}^*$ for which $\frac{1}{n} \sum f(T^k(x))$ is greater than $\overline{f}(x) - \varepsilon$. If $\overline{f}(x)$ is equal to $+\infty$, we can find $n \in \mathbf{N}^*$ such that $\frac{1}{n} \sum f(T^k(x))$ is greater than M. We can therefore consider the smallest integer $n(x) > 0$ for which the following equality is satisfied:

$$\overline{f}_M(x) \leqslant \frac{1}{n(x)} \sum_{k=0}^{n(x)-1} f(T^k(x)) \ + \varepsilon.$$

Set $A_R = \{x \in X \mid n(x) > R\}$. The intersection of all the A_R for $R \in \mathbf{N}$ is empty, and the integral $\int_{A_R} (|f| + M) \, \mathrm{d}\mu$ tends to 0 when R tends to infinity, by the dominated convergence theorem. Choose R such that this integral is less than ε, and write $A = A_R$ to lighten the notation.

We define by induction a sequence n_i depending on x, as follows:



- If $T^{n_i}(x) \notin A$, we set $n_{i+1} = n_i + n(T^{n_i}(x))$ and use the upper bound

$$n(T^{n_i}(x))\overline{f}_M(T^{n_i}x) \leqslant \sum_{k=0}^{n(T^{n_i}x)-1} f(T^k(T^{n_i}x)) \ + n(T^{n_i}(x))\,\varepsilon.$$

Since the function $\overline{f}_M$ is invariant under T, this implies

$$(n_{i+1} - n_i)\overline{f}_M(x) \leqslant \sum_{k=n_i}^{n_{i+1}-1} f(T^k(x)) \ + (n_{i+1} - n_i)\varepsilon.$$

- If $T^{n_i}(x) \in A$, we set $n_{i+1} = n_i + 1$ and bound from above: $\overline{f}_M(x) \leqslant M$.

The construction is illustrated by Fig. 2.1. In both cases, we have

$$(n_{i+1}-n_i)\overline{f}_M(x) \leqslant \sum_{k=n_i}^{n_{i+1}-1} \tilde{f}(T^k(x)) + (n_{i+1}-n_i)\varepsilon,$$

where $\tilde{f} = f + (|f| + M)\mathbf{1}_A$.

Let $N \in \mathbf{N}$, and let k be the integer (depending on x) satisfying $n_k \leqslant N < n_{k+1}$. We have constructed the n_k in such a way that the successive differences are bounded: $0 \leqslant N - n_k \leqslant n_{k+1} - n_k \leqslant R$. We take the sum of the previous inequalities:

$$\begin{aligned}
N\overline{f}_M(x) &= \sum_{i=0}^{k-1} (n_{i+1}-n_i)\overline{f}_M(x) + (N-n_k)\overline{f}_M(x) \\
&\leqslant \sum_{j=0}^{n_k-1} \tilde{f}(T^j(x)) + n_k\,\varepsilon + RM \\
&\leqslant \sum_{j=0}^{N-1} \tilde{f}(T^j(x)) - \sum_{j=n_k}^{N-1} \tilde{f}(T^j(x)) + N\,\varepsilon + RM \\
&\leqslant \sum_{j=0}^{N-1} \tilde{f}(T^j(x)) + \sum_{j=N-R}^{N-1} |f(T^j(x))| + N\,\varepsilon + RM.
\end{aligned}$$

This last inequality is obtained using the inequalities $n_k \geqslant N - R$ and $\tilde{f} \geqslant -|f|$. We now only need to integrate,

$$\forall N \in \mathbf{N}^*, \quad \int \overline{f}_M \,\mathrm{d}\mu \leqslant \int \tilde{f}\mathrm{d}\mu + R/N \int |f|\,\mathrm{d}\mu + \varepsilon + RM/N,$$

and let N tend to infinity, giving $\int \overline{f}_M \,\mathrm{d}\mu \leqslant \int \tilde{f}\mathrm{d}\mu + \varepsilon$. The integral of $\tilde{f}$ is bounded from above as follows:

$$\int \tilde{f}\mathrm{d}\mu = \int f\mathrm{d}\mu + \int_A (|f| + M)\,\mathrm{d}\mu \leqslant \int f\,\mathrm{d}\mu + \varepsilon.$$

Finally, we have obtained $\int \overline{f}_M \leqslant \int f + 2\varepsilon$ for every $\varepsilon > 0$, which gives $\int \overline{f}_M \leqslant \int f$. The sequence $\overline{f}_M$ is increasing and converges to $\overline{f}$, and we have seen that $\int \overline{f}_0 > -\infty$. We can now apply the monotone convergence theorem to obtain the desired inequality $\int \overline{f} \leqslant \int f$. □

Identifying the Limit Let us begin with the case where f is bounded and show that the almost everywhere limit, denoted by $\overline{f}$, satisfies the properties that characterize the conditional expectation. First, $\overline{f}$ is invariant under T, hence measurable with respect to $\mathcal{I}$. Next, by the dominated convergence theorem, $\int \overline{f}\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu$. Let A be a measurable invariant set. Since $\mathbf{1}_A = \mathbf{1}_A \circ T$, we have the equality $\mathbf{1}_A\overline{f} = \overline{\mathbf{1}_A f}$, which implies

$$\int_A \overline{f}\,\mathrm{d}\mu = \int \overline{\mathbf{1}_A f}\,\mathrm{d}\mu = \int \mathbf{1}_A f\,\mathrm{d}\mu = \int_A f\,\mathrm{d}\mu.$$

For unbounded f, we approximate f with a bounded function g and note that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} (f-g) \circ T^k \right\|_1 \leqslant \|f-g\|_1, \quad \|E(f-g \mid \mathcal{I})\|_1 \leqslant \|f-g\|_1.$$

Remarks

- The convergence also holds in L^1 norm. For bounded f, this is a consequence of the dominated convergence theorem; for integrable f, we reason as above.
- Recall that a function f is $\mathcal{I}$-measurable if and only if it is invariant under T; we saw this in Chap. 1, and it follows from the equivalence of the equalities $f(T(x)) = f(x)$ and $T^{-1}f^{-1}(\{f(x)\}) = f^{-1}(\{f(x)\})$.

Definition 2.1 Let $(X, \mathcal{T}, \mu)$ be a measure space, and let $T : X \to X$ be a measurable map that preserves the measure μ. The transformation T is called *ergodic* with respect to the measure μ if the only invariant measurable sets either are of measure 0 or have a complement of measure 0:

$$T^{-1}A = A \qquad \text{implies} \qquad \mu(A) = 0 \text{ or } \mu(A^c) = 0.$$

The limit in the ergodic theorem then takes on a particularly simple form.

Proposition 2.1 *A transformation T is ergodic if and only if the measurable functions that are invariant under T are constant almost everywhere. When $\mu(X)$ is finite and nonzero, this implies the equality*

$$\forall f \in L^1, \quad E(f \mid \mathcal{I}) = \frac{1}{\mu(X)} \int_X f\,\mathrm{d}\mu.$$

Proof Let g be an invariant function. Set

$$C = \sup\left\{t \mid \mu(g^{-1}((-\infty, t))) = 0\right\}.$$

By ergodicity, we have the equality $C = \inf\{t \mid \mu(g^{-1}([t,\infty))) = 0\}$. The constant C is finite provided $\mu(X) \neq 0$. The sets $g^{-1}((-\infty, C))$ and $g^{-1}((C,\infty))$ are then of measure 0, and the function g is constant, equal to C, almost everywhere. The function $E(f \mid \mathcal{I})$ is therefore constant and its integral is $\int f \,\mathrm{d}\mu$. This gives the desired equality. □

Suppose that T is ergodic and that $\mu(X)$ is finite, and let us apply the ergodic theorem to the characteristic function of a measurable set A. We obtain

$$\text{a.e. } x \in X, \qquad \frac{1}{n}\,\mathrm{Card}\{k \in \{0,1,\ldots,n-1\} \mid T^k(x) \in A\} \xrightarrow[n\to\infty]{} \frac{\mu(A)}{\mu(X)}.$$

The average time a trajectory spends in the set A is therefore the same for almost all trajectories; it is proportional to the measure of the set A. This is illustrated by Fig. 2.3.

2.3 Ergodicity of the Shift

The following example plays an important role in probability theory.

Proposition 2.2 *Let $(X, \mathcal{T}, \mu)$ be a probability space. On the product space $(X^{\mathbf{N}}, \mathcal{T}^{\otimes\mathbf{N}}, \mu^{\otimes\mathbf{N}})$, we define a transformation T by setting $T(\{x_i\}) = \{x_{i+1}\}$. Then T is ergodic.*

Proof Let f be an invariant integrable function. For every $\varepsilon > 0$, we can find $g \in L^1$ depending only on a finite number n of coordinates and such that $\|g - f\|_1 < \varepsilon/4$. This classical result is recalled in Chap. 17. The function g is not invariant, but it nevertheless satisfies the following estimate:

$$\|g - g\circ T^n\|_1 \leqslant \|g - f\|_1 + \|f - f\circ T^n\|_1 + \|f\circ T^n - g\circ T^n\|_1 \leqslant \varepsilon/2.$$

Let us calculate the norm of $g - g\circ T^n$ explicitly:

$$\begin{aligned}
\|g - g\circ T^n\|_1 &= \int |g(x_0,\ldots,x_{n-1}) - g(x_n,\ldots,x_{2n-1})|\mathrm{d}\mu(x_0)\cdots\mathrm{d}\mu(x_{2n-1})\\
&= \int |g(x_0,\ldots,x_{n-1}) - g(y_0,\ldots,y_{n-1})|\\
&\qquad\qquad \cdot\,\mathrm{d}\mu(x_0)\cdots\mathrm{d}\mu(x_{n-1})\mathrm{d}\mu(y_0)\cdots\mathrm{d}\mu(y_{n-1})\\
&= \int |g(x) - g(y)|\,\mathrm{d}\mu^{\otimes\mathbf{N}}(x)\,\mathrm{d}\mu^{\otimes\mathbf{N}}(y).
\end{aligned}$$

This last integral is therefore less than $\varepsilon/2$.

The integral $\int |f(x) - f(y)| \mathrm{d}\mu^{\otimes \mathbf{N}}(x) \mathrm{d}\mu^{\otimes \mathbf{N}}(y)$ is bounded from above by the sum of the terms

$$\int |f(x) - g(x)| \mathrm{d}\mu^{\otimes \mathbf{N}}(x) \mathrm{d}\mu^{\otimes \mathbf{N}}(y),$$

$$\int |g(x) - g(y)| \mathrm{d}\mu^{\otimes \mathbf{N}}(x) \mathrm{d}\mu^{\otimes \mathbf{N}}(y),$$

$$\int |g(y) - f(y)| \mathrm{d}\mu^{\otimes \mathbf{N}}(x) \mathrm{d}\mu^{\otimes \mathbf{N}}(y),$$

which gives, for every $\varepsilon > 0$,

$$\int |f(x) - f(y)| \mathrm{d}\mu^{\otimes \mathbf{N}}(x) \mathrm{d}\mu^{\otimes \mathbf{N}}(y) \leqslant \varepsilon/2 + 2\|f - g\| \leqslant \varepsilon.$$

This shows that $f(x) = f(y)$ for almost all $(x, y) \in \Omega^{\mathbf{N}} \times \Omega^{\mathbf{N}}$. By Fubini's theorem, we can find $y_0 \in \Omega^{\mathbf{N}}$ such that the set of $x \in \Omega^{\mathbf{N}}$ satisfying $f(x) = f(y_0)$ is of full measure, so that f is constant almost everywhere. □

As an application, let us show how the law of large numbers can be deduced from the ergodic theorem. Denote by $\mathbf{R}^{\mathbf{N}}$ the space of sequences of real numbers endowed with the product topology and the associated Borel σ-algebra. Let (X_i) be a sequence of random integrable variables defined on a probability space $(\Omega, \mathcal{T}, P)$. Let $\psi : \Omega \to \mathbf{R}^{\mathbf{N}}$ be the map defined by $\psi(\omega) = \{X_i(\omega)\}_{i \in \mathbf{N}}$, and let $\mu = \psi_* P$. Saying that the X_i are independent, identically distributed, random variables corresponds to saying that μ coincides with $(\psi_* P_{X_0})^{\otimes \mathbf{N}}$, where P_{X_0} is the probability distribution of X_0.

Define $f : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ by setting $f(\{x_i\}_{i \in \mathbf{N}}) = x_0$. Let T be the shift on the space $\mathbf{R}^{\mathbf{N}}$. Taking into account the equalities $f \circ T^k \circ \psi = X_k$, we see that the ergodic averages of f and the averages of the X_i are related through

$$\left(\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k\right) \circ \psi = \frac{1}{n} \sum_{k=0}^{n-1} X_k.$$

Let us apply the ergodic theorem to the function f, to the shift T defined on $\mathbf{R}^{\mathbf{N}}$, and to the probability measure μ. We obtain the strong convergence of the averages of f with respect to the measure μ, which is equivalent to the convergence of the averages of the X_i with respect to P. Set $E(X_0) = \int X_0 \, \mathrm{d}P = \int f \, \mathrm{d}\mu$. We have proved the following result.

Corollary 2.1 *Let $(\Omega, \mathcal{T}, P)$ be a probability space, and let $(X_i)_{i \in \mathbf{N}}$ be a sequence of independent, identically distributed, integrable random variables on this space. Then, for almost all $\omega \in \Omega$,*

$$\frac{1}{n} \sum_{k=0}^{n-1} X_i(\omega) \xrightarrow[n \to \infty]{} E(X_0).$$

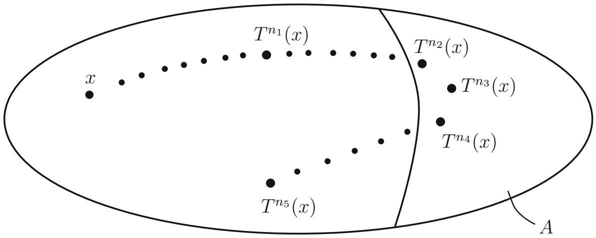


Fig. 2.1 Proof of the ergodic theorem

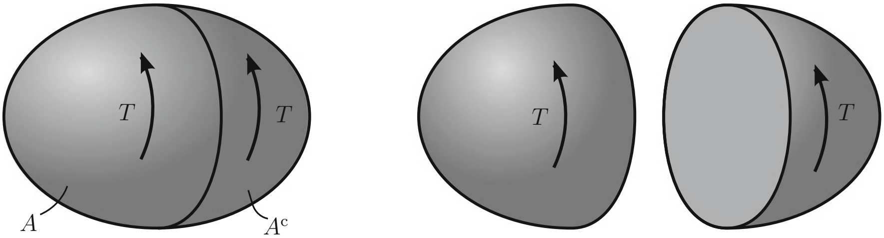


Fig. 2.2 A system that is not ergodic

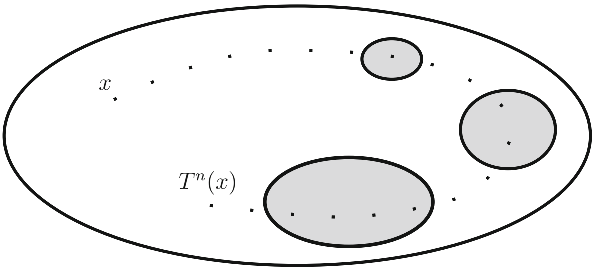


Fig. 2.3 Ergodicity: the time spent in a set is proportional to the measure of the set

2.4 Exercises

2.4.1 Basic Exercises

Exercise 1 Verify that the map $n(x)$ that is used in the proof of the ergodic theorem is measurable.

Exercise 2 Let $(X, \mathcal{T}, \mu)$ be a measure space with $\mu(X) < \infty$, let $T : X \to X$ be a measurable map that preserves the measure μ, and let $f : X \to \mathbf{R}$ be integrable. Show that for μ-almost all $x \in X$, we have $\frac{1}{n} f(T^n(x)) \to 0$.

Exercise 3 Let $(X, \mathcal{T}, \mu)$ be a measure space, and let $T : X \to X$ be a measurable map that preserves the measure μ. Show that T is ergodic if and only if for all measurable sets $A, B \subset X$ of positive measure, almost every point of A has an orbit that passes infinitely many times through B.

Exercise 4 Let T be an ergodic transformation of a measure space $(X, \mathcal{T}, \mu)$. We denote by $P : L^2(X) \to L^2(X)$ the orthogonal projection onto the T-invariant functions. Assume $\mu(X) = \infty$. Show that for all $f \in L^2(X)$, we have $Pf = 0$.

2.4.2 More Advanced Exercises

Exercise 5 Let α be an irrational real number. Show that the transformation T from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{R}/\mathbf{Z}$ given by $T(x) = x + \alpha \bmod 1$ preserves the Lebesgue measure and is ergodic.
Hint: Consider what the invariance of f under T means for its Fourier coefficients.

What condition on $\alpha \in \mathbf{R}^n$ is necessary for the transformation of $(\mathbf{R}/\mathbf{Z})^n$ given by $T(x) = x + \alpha \bmod 1$ to be ergodic with respect to the Lebesgue measure?

Exercise 6 Let α be an irrational real number. Show that the transformation T from $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ to $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ defined by $T : (x, y) \mapsto (x + \alpha, x + y)$ preserves the Lebesgue measure and is ergodic.
Hint: Again, consider what the invariance of $f \in L^2$ under T means for its Fourier coefficients.

Exercise 7 Give an example of an ergodic transformation T such that $T \circ T$ is not ergodic.

Exercise 8 Construct a transformation of the real line that preserves the Lebesgue measure and is ergodic.

2.5 Comments

The proof of the ergodic theorem that we have presented is due to Y. Katznelson and B. Weiss (1982). There exist other proofs:

- The original proof of the pointwise ergodic theorem, due to G. Birkhoff (1931), uses a *maximal inequality*. This equality was then generalized and simplified by N. Wiener (1939), K. Yosida and S. Kakutani (1939), H.R. Pitt (1942), F. Riesz (1945), E. Hopf (1954),... The following version, due to A. Garsia (1965), admits an elementary proof. *Let* $U : L^1 \to L^1$ *be a linear map such that* $\|Uf\| \leqslant \|f\|$. *Set*
$$E_n = \big\{x \mid \max_{0 \leqslant m \leqslant n} S_m f > 0\big\}.$$
Then $\int_{E_n} f \geqslant 0$.
- In 1966, E. Bishop gave a proof of the ergodic theorem inspired by the theory of martingales and based on upcrossing inequalities.
- In 1982, T. Kamae gave a proof based on nonstandard analysis. The proof by Y. Katznelson and B. Weiss presented earlier was inspired by the proof by T. Kamae.
- In 1987, P. Shields gave a new proof of the ergodic theorem, which was not based on a maximal inequality. In 1988, J. Bourgain proposed a proof based on variational inequalities. More recently, M. Keane and K. Petersen (2006) proposed "elementary" proofs of the ergodic theorem, in the style of Y. Katznelson and B. Weiss.

There exist versions of the ergodic theorem for the contractions of L^1. The most general statement is no doubt due to R. Chacon; it is included in Krengel's book [13, Chap. 4, Thm. 1.11] and is proved using a "filling scheme".
If $\mu(X) = \infty$, we still have convergence almost everywhere in the ergodic theorem, but the limit is no longer necessarily given by a conditional expectation. In particular, this limit is 0 if T is ergodic. When the measure is infinite but the transformation T is recurrent, E. Hopf gives a "ratio" version of the ergodic theorem: for nonnegative $f, g \in L^1$, the ratio $S_n f / S_n g$ converges to $\int f / \int g$. This statement was extended to positive contractions by R. Chacon and D. Ornstein (1960). Once again, it can be proved by passing through a maximal inequality. It can also be deduced from the finite measure theorem using induction (R. Zweimüller, 2004).
To prove the ergodic theorem, we can restrict ourselves to the case of a shift on $\mathbf{R}^{\mathbf{N}}$, with the projection onto the first coordinate as the observable quantity. The general case can be deduced from this by factoring the system through the morphism $\varphi : X \to \mathbf{R}^{\mathbf{N}}$ given by $x \mapsto \{f(T^i(x))\}_{i \in \mathbf{N}}$. This remark is used in the proof given by T. Kamae.
The ergodic theorem concerns the averages of the powers of an operator. There also exist results on the almost everywhere convergence of the powers themselves. The following theorem, due to G.-C. Rota (1962) and E. Stein (1961), can be applied, for example, to the auto-adjoint operator $Tf = 1/2(f \circ T + f \circ T^{-1})$ and gives a weighted version of the ergodic theorem.
Let $T : L^1 \to L^1$ *be such that* $\|T\|_1 \leqslant 1$ *and* $\|Tf\|_\infty \leqslant \|f\|_\infty$ *for bounded* f, $Tf \geqslant 0$ *for* $f \geqslant 0$, $T1 = 1$, *and* $T^*1 = 1$. *Then* $T^n T^{*n} f$ *converges almost everywhere if* $f \in L^p$ *for some* $1 < p < \infty$.
The convergence does not necessarily hold for every $f \in L^1$. A counterexample was constructed by D. Ornstein in 1968.
The behavior of ergodic sums from the topological point of view is different from its behavior from the point of view of measures. The set $\big\{(x_n) \in \{0,1\}^{\mathbf{N}} \mid \frac{1}{n}\sum x_i \text{ converges}\big\}$ has full measure for every probability measure defined on $\{0,1\}^{\mathbf{N}}$ that is invariant under the shift. Nevertheless, the set of sequences $(x_i)_{i \in \mathbf{N}}$ for which every real number in $[0,1]$ is an accumulation point for the averages $\frac{1}{n}\sum x_i$ is a G_δ-dense subset of $\{0,1\}^{\mathbf{N}}$.

Chapter 3
Mixing

Pour apprendre quelque chose aux gens, il faut mélanger ce qu'ils connaissent avec ce qu'ils ignorent.
P. Picasso (1881–1973)

3.1 Introduction

Consider a potential V defined on $\mathbf{R}^3$, and let us study the motion of a point mass under the action of the force field generated by this potential. Let $(x, v) \in \mathbf{R}^3 \times \mathbf{R}^3$ be the initial position and velocity of the point mass. We denote by $T(x, v)$ the position of the point at time 1. The initial energy of the system is given by the formula $E(x, v) = \frac{1}{2}mv^2 + V(x)$; it is preserved during the motion. When the energy surface $E(x, v) = E_0$ is bounded, we can restrict the Lebesgue measure $\mathrm{d}x\,\mathrm{d}v$ to this surface to obtain a probability measure, which we denote by μ.

Let us consider the propagation of a gas or liquid under the action of the potential V. The initial distribution of the gas can be represented by a probability measure of the form $\mathrm{d}\nu = h\,\mathrm{d}\mu$, where h is a nonnegative function on the energy surface in question. If A is a subset of this energy surface, $\nu(A)$ represents the quantity of gas or liquid present in A. We can also see it as the probability that a particle is in the region A at the initial time.

How should we model the evolution of the gas? A first, very naive, approach, consists in disregarding the interactions within the gas and supposing that each molecule moves in accordance with the classical laws of motion. The distribution of the gas at time 1 is then given by the measure $T_*\nu$ defined by $T_*\nu(A) = \nu(T^{-1}A)$ for every measurable $A \subset X$.

The sequence $T_*^n\nu$ represents the evolution of the gas over time. If this sequence converges to the measure μ, we say that the transformation is *mixing* with respect to μ: every initial distribution of gas of the form $h\mathrm{d}\mu$ ends up spreading uniformly on the energy surface, following the distribution μ.

The property of being mixing is stronger than ergodicity. It excludes periodic limit behavior (for example, $T^n = \mathrm{Id}$ for some $n \geqslant 2$), while such behavior is possible for an ergodic transformation. The ergodicity of the measure μ is in fact equivalent to the weak convergence of the averages $\frac{1}{n}\sum T_*^k\nu$ to μ for every probability measure ν of the form $h\mathrm{d}\mu$.

Y. Coudène, *Ergodic Theory and Dynamical Systems*, Universitext,
DOI 10.1007/978-1-4471-7287-1_3

Shifts on product spaces are mixing with respect to the product measures. For these systems, it is customary to deduce the ergodicity from the mixing property, because the proofs are of the same order of difficulty. A second family of mixing maps is given by the *hyperbolic automorphism* of tori. These maps are obtained by considering matrices with determinant 1, with integer coefficients and no eigenvalues of absolute value 1. The action of such a matrix on the quotient space $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ preserves the Lebesgue measure and gives a map that is mixing with respect to this measure.

3.2 Definition of Mixing

Definition 3.1 Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map that preserves the measure μ. The transformation T is mixing with respect to the measure μ if it satisfies

$$\forall \text{ measurable } A, B \subset X, \quad \mu(A \cap T^{-n}B) \xrightarrow[n\to\infty]{} \mu(A)\,\mu(B).$$

This definition is illustrated by Fig. 3.1. Mixing is a stronger property than ergodicity.

Proposition 3.1 *A mixing transformation is ergodic.*

Proof Let $A \subset X$ be an invariant set; since $T^{-n}A = A$, we must have $A \cap T^{-n}A = A$. The mixing property implies $\mu(A) = \mu(A)^2$, that is, $\mu(A) = 0$ or 1. □

To establish that a transformation is mixing, we will rely on the following criterion.

Criterion *Let $\mathcal{D}$ be a subset of L^2 that generates a dense linear subspace in L^2. The transformation T is mixing if and only if for all $f, g \in \mathcal{D}$,*

$$\int f \circ T^n\, g\, \mathrm{d}\mu \xrightarrow[n\to\infty]{} \int f\, \mathrm{d}\mu \int g\, \mathrm{d}\mu.$$

Proof We can rephrase the criterion in terms of the weak topology on the Hilbert space L^2: for all $f, g \in \mathcal{D}$, the composition $f \circ T^n$ converges weakly to $\int f\, \mathrm{d}\mu$. By the properties of weak convergence, stated in Chap. 16, Proposition 16.1, this implies the weak convergence of $f \circ T^n$ to $\int f\, \mathrm{d}\mu$ for every $f \in L^2$. We conclude by noting that $\int f \circ T^n\, g\, \mathrm{d}\mu = \mu(A \cap T^{-n}B)$ if $f = \mathbf{1}_B$ and $g = \mathbf{1}_A$. □

Note that if the sequence $f \circ T^n$ converges to a constant, this constant must equal $\lim\langle f \circ T^n, 1\rangle = \langle f, 1\rangle$. In terms of the weak topology, the map T is mixing if and only if the sequence $f \circ T^n$ *converges weakly* to a constant for every $f \in L^2$.

When X is a metric space and T and μ are Borel, mixing can be expressed using *weak convergence*. Recall that a sequence of probability measures μ_n converges weakly to μ if for every bounded continuous function f, the sequence $\int f \,\mathrm{d}\mu_n$ converges to $\int f \,\mathrm{d}\mu$. The transformation T is mixing if and only if

$$\text{for every } g \in L^2 \text{ such that } \int g \,\mathrm{d}\mu = 1, \quad T_*^n(g \,\mathrm{d}\mu) \xrightarrow[n\to\infty]{} \mathrm{d}\mu \quad \text{weakly.}$$

This is a consequence of the density of the bounded continuous functions in L^2, recalled in Chap. 18.

3.3 Example: Multiplication by 2

Consider the transformation from $[0, 1)$ to $[0, 1)$ given by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2})\,, \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1)\,. \end{cases}$$

Let us show that it preserves the Lebesgue measure and that it is mixing.

Proof The inverse image of an interval $[a, b]$ under T is a disjunct union of two intervals of length $(b - a)/2$, as illustrated by Fig. 3.2. The transformation preserves the Lebesgue measure.

To prove that the transformation is mixing, we can restrict ourselves to the case where A is of the form $[k/2^n, k + 1/2^n)$ for $n \in \mathbf{N}$ and $0 \leqslant k \leqslant 2^n - 1$, because these intervals generate the Borel σ-algebra. The set $T^{-N}[k/2^n, (k + 1)/2^n)$ is made up of the following 2^N intervals:

$$\left[\frac{(k + i\,2^n)}{2^{n+N}}, \frac{k + 1 + i\,2^n}{2^{n+N}}\right).$$

If $n + N > n'$, the intersection of these intervals with $B = [k'/2^{n'}, k' + 1/2^{n'})$ consists of $2^{N-n'}$ intervals of length 2^{-n-N}, which gives the desired relation, $\mu(B \cap T^{-n}A) = \mu(A)\,\mu(B)$. □

3.4 Example: The Bernoulli Shift

The following example comes from probability theory. The mixing property of the shift implies its ergodicity and allows us to recover the law of large numbers for independent random variables with the same probability distribution.

Proposition 3.2 *Let $(\Omega, \mathcal{T}, \mu)$ be a probability space. On the product space $(\Omega^{\mathbf{N}}, \mathcal{T}^{\otimes \mathbf{N}}, \mu^{\otimes \mathbf{N}})$, we define a transformation σ by setting $\sigma(\{x_i\}) = \{x_{i+1}\}$. Then σ is mixing with respect to $\mu^{\otimes \mathbf{N}}$.*

Proof The functions in $L^2(X)$ that depend only on a finite number of coordinates are dense in L^2. Hence, let f and g depend on j coordinates and assume $n > j$; then

$$\begin{aligned}
\int gf \circ \sigma^n \, \mathrm{d}\mu^{\otimes \mathbf{N}} &= \int g(x_0, \ldots, x_j) f(x_n, \ldots, x_{j+n}) \, \mathrm{d}\mu^{\otimes \mathbf{N}} \\
&= \int g(x_0, \ldots, x_j) f(x_n, \ldots, x_{j+n}) \, \mathrm{d}\mu(x_0) \cdots \mathrm{d}\mu(x_{j+n}) \\
&= \int g(x_0, \ldots, x_j) \, \mathrm{d}\mu(x_0) \cdots \mathrm{d}\mu(x_j) \\
&\qquad \cdot \int f(x_n, \ldots, x_{j+n}) \, \mathrm{d}\mu(x_n) \cdots \mathrm{d}\mu(x_{j+n}) \\
&= \int f \, \mathrm{d}\mu^{\otimes \mathbf{N}} \int g \, \mathrm{d}\mu^{\otimes \mathbf{N}}.
\end{aligned}$$

We have proved that the shift σ is mixing. □

When Ω is finite, the dynamical system consisting of the transformation σ on $\Omega^{\mathbf{N}}$ and the measure $\mu^{\otimes \mathbf{N}}$ is called a *Bernoulli shift*. We number the elements of $\Omega = \{x_1, \ldots, x_k\}$ and set $p_i = \mu(\{x_i\})$. The system is fully determined by the parameters $p_1, \ldots, p_k$.

3.5 Example: Toral Endomorphisms

Proposition 3.3 *Let A be an $n \times n$ matrix with integer coefficients and nonzero determinant. This matrix induces a map on the quotient space $\mathbf{T}^n = (\mathbf{R}/\mathbf{Z})^n$ that preserves the Lebesgue measure. This map is mixing if A does not have any eigenvalues that are roots of unity.*

The maps from $\mathbf{T}^n$ to itself obtained from matrices with determinant ± 1 without any eigenvalues of absolute value 1 are called *hyperbolic automorphisms of* $\mathbf{T}^n$.

Proof Let us show the invariance of the measure by using Fourier series. Let $k \in \mathbf{Z}^n$, and let $k \cdot x$ be the quantity $\sum k_i x_i$. Set $e_k(x) = \mathrm{e}^{2\pi i k \cdot x}$; these functions form a Hilbert basis for $L^2(\mathbf{T}^n)$. Let $f \in L^2$, and let c_k be its Fourier coefficients; then

$$\int f(Ax)\,\mathrm{d}x = \sum_{k\in\mathbf{Z}^n} c_k \int \mathrm{e}^{2\pi i\, k\cdot Ax}\,\mathrm{d}x = \sum_{k\in\mathbf{Z}^n} c_k \int \mathrm{e}^{2\pi i({}^t\!Ak)\cdot x}\,\mathrm{d}x$$
$$= c_0 = \int f(x)\,\mathrm{d}x.$$

Let $k, \ell \in \mathbf{Z}^n$. Let us now show that this transformation is mixing:

$$\int_{\mathbf{T}^n} e_k(x)\, e_\ell(A^n x)\,\mathrm{d}x = \int_{\mathbf{T}^n} \mathrm{e}^{2\pi i k\cdot x}\mathrm{e}^{2\pi i\ell\cdot A^n x}\mathrm{d}x = \int_{\mathbf{T}^n} \mathrm{e}^{2\pi i(k+{}^t\!A^n\ell)x}\mathrm{d}x.$$

This last quantity is 0 if ${}^t\!A^n\ell \neq -k$. If it does not tend to 0 when n tends to infinity, there exist distinct integers n_1, n_2 such that ${}^t\!A^{n_1}\ell = -k = {}^t\!A^{n_2}l$. We would therefore have ${}^t\!A^{n_2-n_1}\ell = \ell$. Since A does not have any eigenvalues that are roots of unity, it follows that $\ell = 0$ and $k = 0$. □

Remark The map induced by the matrix $\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$ on the torus $\mathbf{T}^2$ is sometimes called *Arnold's cat map*, in reference to an illustration in the book by Arnold and Avez [2] that shows the effect of this map on the image of a cat; see Fig. 3.3.

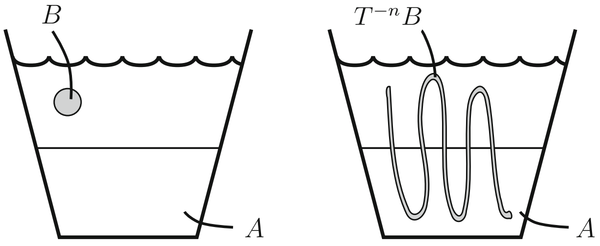


Fig. 3.1 Mixing property

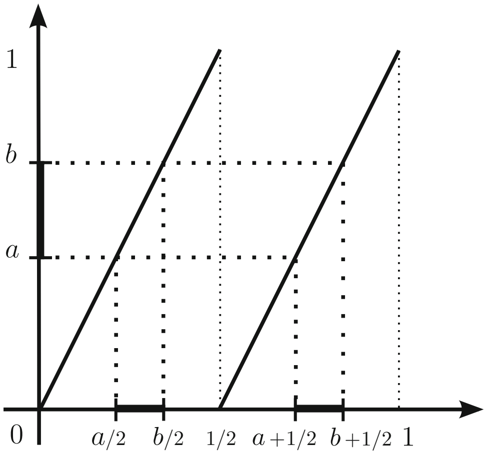


Fig. 3.2 Multiplication by 2

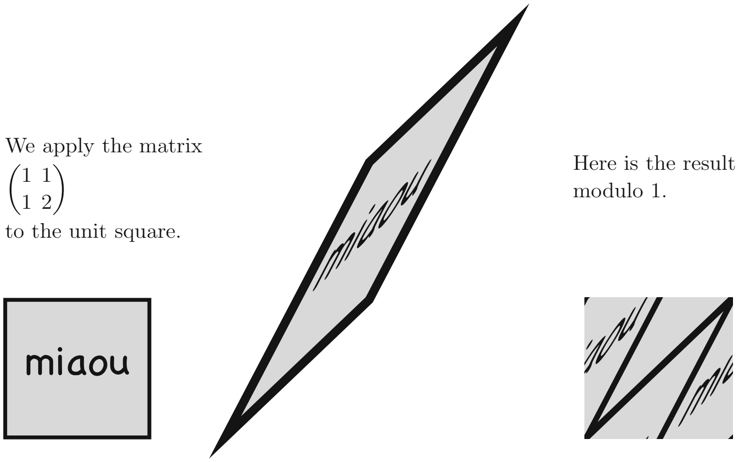


Fig. 3.3 Automorphism of the torus

3.6 Exercises

3.6.1 Basic Exercises

Exercise 1 Show that a rotation on the circle S^1 is not mixing with respect to the Lebesgue measure.
Hint: Use complex exponential functions.

Exercise 2 Show that if T is a mixing transformation, $T \circ T$ is also mixing.

Exercise 3 Show that the map $F : [0, 1] \to [0, 1]$ given by

$$F(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2], \\ 2 - 2x & \text{if } x \in (1/2, 1] \end{cases}$$

preserves the Lebesgue measure and is mixing.

Exercise 4 Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable transformation that preserves μ. Assume that T is mixing. Show that for every $A \subset X$ of nonzero measure and every sequence $n_i \to \infty$, we have $\bigcup_{i \in \mathbf{N}} T^{-n_i} A = X$ up to a negligible set.

Exercise 5 Does there exist a 3×3 matrix with integer coefficients and determinant 1 that is not hyperbolic but whose action on the torus $\mathbf{T}^3$ is mixing?

Exercise 6 Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable transformation that preserves μ. Show that if T is mixing, there do not exist any nonconstant measurable functions $f : X \to \mathbf{C}$ and complex numbers λ of absolute value 1 that satisfy

$$\text{for almost all } x \in X, \quad f\big(T(x)\big) = \lambda f(x).$$

3.6.2 More Advanced Exercises

Exercise 7 Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map that preserves the measure μ. Show that T is mixing if and only if for every measurable $A \subset X$, $\mu(A \cap T^{-n}A)$ converges to $\mu(A)^2$.
Hint: Consider the space generated by the functions $\mathbf{1}_A \circ T^n$ *for* $n \in \mathbf{N}$.

Exercise 8 Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map that preserves μ. Assume that T is mixing. Let k_i be a strictly increasing sequence of integers. Show that for every $f \in L^2$,

$$\frac{1}{n} \sum_{i=1}^{n} f \circ T^{k_i} \xrightarrow[n \to \infty]{L^2} \int f \, \mathrm{d}\mu.$$

Conversely, show that if this convergence holds for every $f \in L^2$ and every strictly increasing sequence k_i of integers, then T is mixing.
Hint: Use the characterization of mixing using the weak topology as well as the Banach–Saks theorem, included in Chap. 18.

Exercise 9 Give an example of a 4×4 matrix with integer coefficients and determinant 1 that is not hyperbolic and whose action on the torus $\mathbf{T}^4$ is mixing.

3.7 Comments

Given an arbitrary probability measure μ, it may happen that the sequence $\frac{1}{n}\sum T_*^k(h\mathrm{d}\mu)$ converges for all bounded nonnegative functions h, to a limit that is singular with respect to the measures $h\,\mathrm{d}\mu$. The simplest example is given by $T(x) = \frac{1}{2}x$ on $[0, 1]$ and $\mu = \lambda_{[0,1]}$. We then have $T_*^n(h\lambda_{[0,1]}) \to \delta_0$.

To what extent is mixing a typical property of dynamical systems? To answer this question, let us define a distance on the set of invertible Borel transformations of $[0, 1]^d$ or $\mathbf{T}^d$ preserving the Lebesgue measure. Let I_n be a sequence of boxes separating the points:

$$d(T, T') = \sum \frac{1}{2^n} \left(\lambda(T(I_n)\Delta T'(I_n)) + \lambda(T^{-1}(I_n)\Delta T'^{-1}(I_n))\right).$$

For this topology, the set of mixing transformations is meager (that is, a subset of a countable union of closed sets with empty interior). This also holds for the C^0 topology on the space of homeomorphisms, but does not hold if we consider the space of C^2 diffeomorphisms on $\mathbf{T}^d$ that preserve the Lebesgue measure. In the C^2 topology, every diffeomorphism that preserves the Lebesgue measure and is close to a hyperbolic automorphism is mixing.

The notion of mixing is more difficult to define when the measure is infinite. In 1969, U. Krengel and L. Sucheston proved that on an infinite σ-finite measure space, there do not exist any invertible transformations that preserve the measure and satisfy

$$\forall g \in L^\infty,\ \forall f \in L^1 \text{ such that } \int f\mathrm{d}\mu = 0, \qquad \int f \circ T^n g\,\mathrm{d}\mu \xrightarrow[n\to\infty]{} 0.$$

The mixing property of hyperbolic toral endomorphisms can be proved in several ways.

- The proof given earlier, using Fourier series, can be generalized to transitive automorphisms on Abelian compact groups. In general, the techniques from harmonic analysis work well in an algebraic setting.
- It is possible to encode these maps using a symbolic system; the proof of the mixing property then proceeds as with a shift. The simplest encoding is given by the decomposition in base 10. This decomposition gives a conjugation between the multiplication by 10 on $\mathbf{R}/\mathbf{Z}$ and the shift on the alphabet $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
- The mixing property can be deduced from the density, in the torus, of the projection of the stable subspaces of the matrix, a density that can be obtained using the ergodicity of the irrational translations on the torus. It then suffices to compute explicitly the images of the boxes whose faces are parallel to the eigenvectors of the matrix.
- Another method consists in showing that an accumulation point of the sequence $f \circ T^n$ is constant along the stable and unstable subspaces of the matrix. This argument can be generalized to geometric systems. It is the object of the next chapter.

A transformation preserving a probability measure is called mixing of order 3 if it satisfies the following property:

$$\forall \text{ measurable } A, B, C \subset X,$$
$$\mu(A \cap T^{-n_1} B \cap T^{-n_1-n_2} C) \xrightarrow[n_1,n_2\to\infty]{} \mu(A)\,\mu(B)\,\mu(C).$$

Do there exist mixing transformations that are not mixing of order 3? This question, asked by V. Rokhlin is 1949, is still open to this day. B. Host (1991) proved that a mixing transformation whose spectrum is singular is mixing of all orders.

Chapter 4
The Hopf Argument

The author has had complaints about too much detail missing in the presentation of the material in the latter paper. This has been rectified in the present paper.

E. Hopf (1902–1983)

4.1 Introduction

The ergodic theorem was proved by G.D. Birkhoff in 1932. At the time, there were already examples of ergodic systems. They came from probability theory and model random phenomena like throwing a die or drawing balls from an urn. It is therefore not surprising to see ergodicity appear in this context.

The next question was whether this ergodicity appears in classical mechanics. The focus was on geodesic flows in nonpositive curvature. J. Hadamard had in fact already proved in 1898 that these flows are unstable from the topological point of view.

Let us explain briefly how these dynamical systems are defined: Consider a surface whose points are locally saddle points. The manifold with equation

$$\{(x, y, z) \in \mathbf{T}^3 \mid \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi z) = 0\}$$

is an example of a surface embedded in the torus $\mathbf{T}^3$ whose curvature is negative outside of eight points; it is represented in Fig. 4.1. This surface is used in the study of a physical system consisting of three double pendula joined at their tips. The geodesic flow acts on the set of vectors of norm 1 tangent to the surface by translating these vectors along the geodesics. It preserves the canonical volume on the set of unit vectors.

G. Hedlund was the first to give an example of a surface with nonpositive curvature for which the geodesic flow is ergodic with respect to the volume. Carrying on the work of J. Hadamard, he showed, in 1934, that on some surfaces, the geodesic flow is semiconjugate to a symbolic system, which allowed him to reduce to a well-known situation.

Y. Coudène, *Ergodic Theory and Dynamical Systems*, Universitext,
DOI 10.1007/978-1-4471-7287-1_4

In 1936, E. Hopf proposed a geometric argument, which allowed him to prove the ergodicity of the geodesic flow on all surfaces of finite volume with nonpositive curvature. This argument turns out to be tricky to implement in higher dimension. However, it seems to apply to a larger class of dynamical systems: if there exist sufficiently many directions that are dilated and contracted by the transformation, then there is hope to prove the ergodicity of the system using the Hopf argument.

We will illustrate this argument by giving a new proof of the mixing property of hyperbolic toral automorphisms. Then, we will show the ergodicity of the geodesic flow on surfaces of finite volume with nonpositive constant curvature, using an algebraic model for this flow: it can be identified with the action of a diagonal subgroup of a quotient of the group $\mathrm{SL}_2(\mathbf{R})$ of 2×2 matrices with real coefficients and determinant 1.

4.2 Stable Foliation and Invariant Functions

Let X be a metric space, let $T : X \to X$ be a map, and let $x \in X$. The (strong) stable manifold of x for the map T is defined by

$$W^{\mathrm{ss}}(x) = \{y \in X \mid d\big(T^n(x), T^n(y)\big) \xrightarrow[n\to\infty]{} 0\}.$$

The stable manifolds partition the space X. If T is bijective, we can also define the (strong) unstable manifold $W^{\mathrm{su}}(x)$ of x: it is the stable manifold of x for T^{-1}.

Let μ be a measure that is invariant under T. A measurable function $f : X \to \mathbf{R}$ is called W^{ss}-*invariant* if, after restriction to a set X_0 satisfying $\mu(X_0^c) = 0$, it is constant on the stable manifolds: $\forall x, y \in X_0$, $y \in W^{\mathrm{ss}}(x)$ implies $f(x) = f(y)$.

The following result gives the link between the stable foliation and the weak accumulation points of the sequences of the form $f \circ T^n$.

Theorem 4.1 *Let X be a metric space, let μ be a finite Borel measure on X, and let $T : X \to X$ be a measurable map that preserves μ. Let $f \in L^2(X)$; then the accumulation points (for the weak topology) of the sequence $f \circ T^n$ are W^{ss}-invariant. If T is moreover invertible, then these accumulation points are also W^{su}-invariant.*

Proof In this proof, we will use several classical properties of the weak topology, which are set out in Chap. 16. Let n_i and g be such that $f \circ T^{n_i}$ converges to g for the weak topology.

Let us first assume that f is a bounded Lipschitz function. The Banach–Saks theorem, recalled in Chap. 16, provides subsequences $m_\ell, n_{i_k} \to \infty$ such that

$$\Psi_\ell(x) := \frac{1}{m_\ell}\sum_{k=1}^{m_\ell} f\circ T^{n_{i_k}}(x) \xrightarrow[\ell\to\infty]{} g(x) \quad \text{a.e.}$$

If $y \in W^{\mathrm{ss}}(x)$, then

$$|\Psi_\ell(x) - \Psi_\ell(y)| \leqslant C\frac{1}{m_\ell}\sum_{k=1}^{m_\ell} d\big(T^{n_{i_k}}(x), T^{n_{i_k}}(y)\big) \xrightarrow[\ell\to\infty]{} 0.$$

Consequently, the function g is W^{ss}-invariant.

Let us now proceed to the general case $f \in L^2$. For every $\varepsilon > 0$, we can find a Lipschitz function f' such that $\|f - f'\| < \varepsilon$. This classical density result is recalled in Chap. 18. The sequence $f' \circ T^n$ is bounded in L^2 norm. By weak compactness (Chap. 16), we can find a subsequence n_{i_j} such that $f' \circ T^{n_{i_j}}$ converges weakly to a function g'. We have just seen that this function g' is necessarily W^{ss}-invariant. The sequence $(f - f') \circ T^{n_{i_j}}$ therefore converges weakly to $g - g'$, which implies

$$\|g - g'\| \leqslant \underline{\lim}\, \|(f - f') \circ T^{n_{i_j}}\| \ \leqslant \|f - f'\| < \varepsilon.$$

We can therefore find a sequence of W^{ss}-invariant functions that converges to g in L^2 norm and, after taking a subsequence, almost everywhere. The function g is W^{ss}-invariant.

Let us proceed to the case where T is invertible. Let I be the subspace of W^{su}-invariant functions. We will show that if f belongs to $I^\perp$, then $f \circ T^n$ converges weakly to 0. Let g be a weak limit of $f \circ T^{n_i}$. We apply the above to T^{-1}; we can find a subsequence n_{i_k} and a function $g_0 \in I$ such that $g \circ T^{-n_{i_k}} \rightharpoonup g_0$. We obtain

$$\langle g, g\rangle = \lim_{k\to+\infty}\langle f\circ T^{n_{i_k}}, g\rangle = \lim_{k\to+\infty}\langle f, g\circ T^{-n_{i_k}}\rangle = \langle f, g_0\rangle = 0.$$

Every function $f \in L^2$ can be written as a sum $f = f_1 + f_2$ with $f_1 \in I$ and $f_2 \in I^\perp$. The sequence $f_2 \circ T^n$ tends weakly to 0. The accumulation points of $f \circ T^n$ are also accumulation points of the sequence $f_1 \circ T^n$, which belongs to I. □

If f is an invariant function, then $f \circ T^n = f$ and we obtain the following corollary.

Corollary 4.1 (Hopf) *Let X be a metric space, let μ be a finite Borel measure, and let $T : X \to X$ be a measurable map that preserves μ. Then every function $f \in L^2(X)$ that is invariant under T is W^{ss}-invariant. If T is moreover invertible, then f is also W^{su}-invariant.*

Set-theoretically, this corresponds to saying that every measurable set that is invariant under T coincides, up to a set of measure 0, with a union of stable manifolds.

The Hopf argument is only useful if the stable foliation is nontrivial. For an isometry, for example, we have $W^{ss}(x) = \{x\}$ for all x, and the argument does not give any information on the invariant sets.

4.3 Application to Toral Automorphisms

Let us apply this argument to hyperbolic automorphisms of the torus $\mathbf{T}^n$, in order to give a new proof of a proposition seen in Chap. 3.

Proposition 4.1 *Let A be an $n \times n$ matrix with integer coefficients and determinant 1, without any eigenvalues on the unit circle. This matrix induces a map on the quotient $\mathbf{T}^n = (\mathbf{R}/\mathbf{Z})^n$ that preserves the Lebesgue measure and is mixing.*

Proof Denote by E_s the projection onto the torus of the linear subspace associated with the eigenvalues of absolute value less than 1. Let E_u be the projection onto $\mathbf{T}^n$ of the subspace associated with the eigenvalues of absolute value greater than 1. The example of the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is illustrated by Fig. 4.2. These two subspaces are in direct sum and generate $\mathbf{R}^n$. The stable and unstable manifolds for the map induced by the matrix on the torus are given by $W^{ss}(x) = x + E_s$ and $W^{su}(x) = x + E_u$, respectively.

We choose a coordinate system in the directions of E_s and E_u, which gives a chart $(x, y) \in U$ defined in a neighborhood U of an arbitrary point of the torus. In this chart, the stable manifolds are horizontal, the unstable manifolds are vertical, and the Lebesgue measure takes on the form $\mathrm{d}x\,\mathrm{d}y$. Let $f \in L^2(\mathbf{T}^n)$ be a function defined on the torus, and let g be a weak accumulation point of the sequence $f \circ T^n$. We have shown that this function g is invariant under W^{ss} and W^{su}. In the coordinate system (x, y), it therefore does not depend on x or on y when restricted to a full-measure subset of U. The following lemma shows that it is constant almost everywhere on U.

Lemma 4.1 *Let $(X, \mathcal{T}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be two probability spaces, and let $f : X \times Y \to \mathbf{R}$ be an L^2 function. We assume that there exist two measurable functions $\varphi_1 : X \to \mathbf{R}$ and $\varphi_2 : Y \to \mathbf{R}$ and a subset $Z \subset X \times Y$ of full $\mu \otimes \nu$-measure such that*

$$\forall (x, y) \in Z, \quad f(x, y) = \varphi_1(x), \quad f(x, y) = \varphi_2(y).$$

Then f is constant almost everywhere.

Proof By Fubini's theorem, there exist $Y_0 \subset Y$ of full measure and $x_0 \in X$ such that $\{x_0\} \times Y_0 \subset Z$. For every $(x, y) \in Z \cap (X \times Y_0)$, the point (x_0, y) is in Z, which implies $\varphi_1(x_0) = \varphi_2(y) = f(x, y)$. This proves the lemma. $\square$

The function g is locally almost constant, by virtue of the lemma we just proved. We need to deduce from this that it is constant almost everywhere.

Lemma 4.2 *Let X be a metric space, let μ be a measure with connected support, and let g be a locally almost constant function. Then g is constant for almost all $x \in \operatorname{supp} \mu$.*

Proof For every $x_0 \in \operatorname{supp} \mu$, we can find $r_{x_0} > 0$ such that g is constant almost everywhere on $B(x_0, r_{x_0})$, equal to C_{x_0}. For $x \in \operatorname{supp} \mu$, we set

$$\overline{g}(x) = \overline{\lim_{r \to 0}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g \, \mathrm{d}\mu.$$

The function $\overline{g}$ is locally constant on $\operatorname{supp} \mu$; it has value C_{x_0} on the intersection $B(x_0, r_{x_0}) \cap \operatorname{supp} \mu$. By the connectedness of the support, we can deduce that $\overline{g}$ is constant and that the constant C_{x_0} does not depend on x_0; we denote it by C.

Recall that the support of a finite measure has a countable basis: we can find a countable family of open sets $\mathcal{D}$ such that every nonempty open set can be written as a union of elements of this family. This classical property is proved in Chap. 18. For every $x_0 \in \operatorname{supp} \mu$, we can therefore find $U \in \mathcal{D}$ such that $x_0 \in U \subset B(x_0, r_{x_0})$. The resulting family of open subsets $\{U_{x_0}\}_{x_0 \in X} \subset \mathcal{D}$ is countable and covers $\operatorname{supp} \mu$, and the function g is constant, equal to C almost everywhere on each of these open sets. It follows that g is constant almost everywhere on $\operatorname{supp} \mu$. □

We have shown that every accumulation point g of $f \circ T^n$ is constant almost everywhere. Since the integral of g is equal to that of f, we see that $\int f \mathrm{d}\mu$ is the only accumulation point of $f \circ T^n$ for the weak topology. By compactness, this implies the convergence of the sequence $f \circ T^n$ to the constant $\int f \, \mathrm{d}\mu$. The transformation is indeed mixing with respect to the Lebesgue measure. □

The key point in this proof is the existence of a coordinate system in which the stable and unstable manifolds can be identified with the horizontal and vertical vectors, respectively, and such that the invariant measure is equivalent to a product measure. This measure is called *absolutely continuous* along the stable and unstable foliations.

4.4 Flows on the Quotients of $\mathbf{PSL_2(R)}$

The geodesic flow on surfaces with constant nonpositive curvature can be described in algebraic terms. The space on which the system is defined can be identified with $\mathrm{PSL}_2(\mathbf{R})$, the quotient of the set of 2×2 matrices with real coefficients and determinant 1 by the subgroup $\{\mathrm{Id}, -\mathrm{Id}\}$. The flow is given by the following family of transformations:

$$\forall t \in \mathbf{R}, \quad \varphi_t\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

These transformations define a flow

$$\varphi_s \circ \varphi_t = \varphi_{s+t},$$

and preserve the measure $d\mu = da\,db\,dc/|a|$. This measure has infinite total mass. The group $\mathrm{PSL}_2(\mathbf{R})$ acts on itself by left multiplication and right multiplication. These two actions leave μ invariant, so that we obtain a measure on all quotients of $\mathrm{PSL}_2(\mathbf{R})$.

Consider the right action of a subgroup Γ of $\mathrm{PSL}_2(\mathbf{R})$. For $v \in \mathrm{PSL}_2(\mathbf{R})$, the orbit of v under the action of the group is defined by $\Gamma v = \{v\gamma \mid \gamma \in \Gamma\}$. From now on, we will assume that the projection onto the quotient $X = \mathrm{PSL}_2(\mathbf{R})/\Gamma$ is a local homeomorphism. This condition is, for example, guaranteed by the existence of a point $v \in \mathrm{PSL}_2(\mathbf{R})$ satisfying $d(v, \Gamma v \setminus \{v\}) > 0$. Let us, moreover, assume that X has finite μ-measure. A subgroup Γ satisfying these properties is called a *lattice*. The flow φ_t, which acts on the left, commutes with the action of Γ, which acts on the right. It therefore passes to the quotient and defines a new flow on X, which we also denote by φ_t. The measure μ on the quotient is preserved by this flow.

Definition 4.1 A flow $\{\varphi_t\}_{t\in\mathbf{R}}$ is *ergodic* with respect to an invariant measure μ if the only measurable sets invariant under all transformations φ_t for $t \in \mathbf{R}$ either are of measure 0 or have a complement of measure 0.

We will show that the flow $\{\varphi_t\}_{t\in\mathbf{R}}$ defined earlier is ergodic on X, using the Hopf argument.

Theorem 4.2 *Let Γ be a lattice in* $\mathrm{PSL}_2(\mathbf{R})$. *Then the flow φ_t is ergodic with respect to the measure induced by μ on the quotient* $\mathrm{PSL}_2(\mathbf{R})/\Gamma$.

Proof We endow $\mathrm{PSL}_2(\mathbf{R})$ with a distance d invariant under multiplication on the right. We can, for example, take $d(A, B) = \log(\|AB^{-1}\|\,\|BA^{-1}\|)$, for a well-chosen norm $\|\cdot\|$. The choice of this norm, which is not unique, is the object of Exercise 2. This distance passes to the quotient and defines a distance on $X = \mathrm{PSL}_2(\mathbf{R})/\Gamma$ that is compatible with the quotient topology.

We define three families of transformations by setting, for $t \in \mathbf{R}$ and $v \in X$,

$$\varphi_t(v) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} v, \quad h_t^{\mathrm{su}}(v) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} v, \quad h_t^{\mathrm{ss}}(v) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} v.$$

The equality $\varphi_t \circ h_s^{\mathrm{ss}} = h_{se^{-2t}}^{\mathrm{ss}} \circ \varphi_t$ shows that for every $s \in \mathbf{R}$, the points $h_s^{\mathrm{ss}}(M)$ belong to the stable manifold for φ_t passing through v. A similar calculation shows that $h_s^{\mathrm{su}}(v)$ belongs to the unstable manifold for φ_t.

Let $u' = e^{2t}u\,(1 - e^{2t}su)^{-1}$. We introduce a system of local coordinates in the neighborhood of an arbitrary point $v_0 \in X$ by setting $(t, s, u) \mapsto h_{u'}^{\mathrm{su}} h_s^{\mathrm{ss}} \varphi_t(v_0)$. In this coordinate system, the stable manifolds $W^{\mathrm{su}}(v)$ are vertical lines and the unstable manifolds $\bigcup_{t\in\mathbf{R}} \varphi_t(W^{\mathrm{ss}}(v))$ are horizontal planes. These observations follow from a calculation that is proposed in Exercise 3 and illustrated by Fig. 4.3.

We now consider a function $f \in L^2(X, \mu)$ that is invariant under the flow. In this coordinate system, it does not depend on t. We apply the Hopf argument. The function also does not depend on the coordinates s and u, up to a set of measure 0. Note that in our coordinate system, the measure μ is equivalent to a product measure because it can be calculated explicitly using the change of variables formula for C^1 diffeomorphisms. The two lemmas used to study the hyperbolic automorphims in the previous paragraph hold, and we conclude that f is constant almost everywhere. □

We have not proved that the maps φ_t for $t \neq 0$ are mixing. That is the object of Exercise 7.

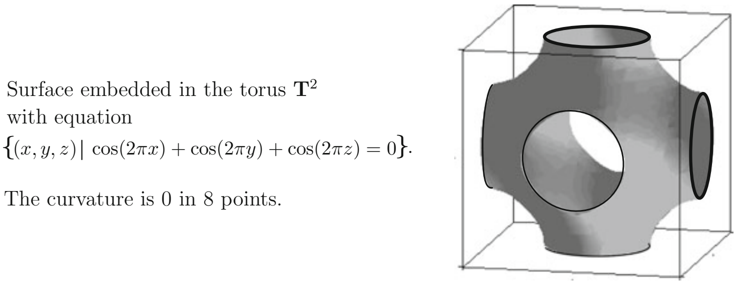


Fig. 4.1 A manifold with negative curvature

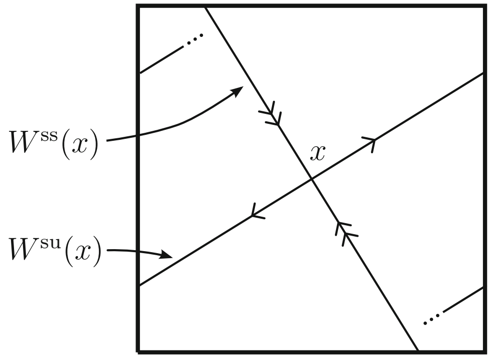


Fig. 4.2 Stable and unstable manifolds for the toral automorphism $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$

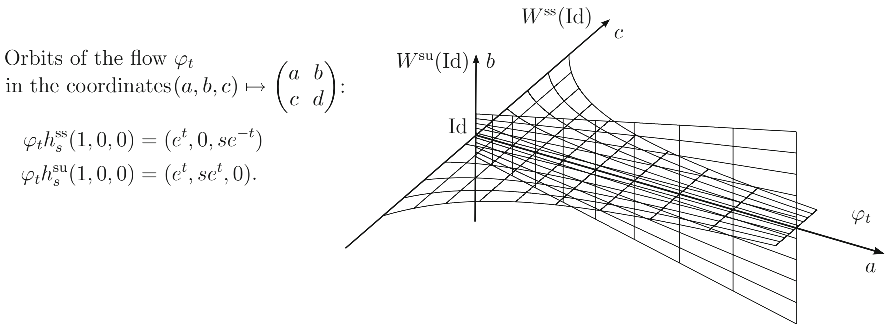


Fig. 4.3 Geodesic flow in constant negative curvature

4.5 Exercises

4.5.1 Basic Exercises

From here on forward, X is a metric space, μ is a finite Borel measure defined on X, and T is a measurable transformation that preserves μ.

Exercise 1 Show that every measurable function f invariant under T is W^{ss}-invariant. Show that a measurable function $f : X \to \mathbf{R}$ is W^{ss}-invariant if and only if it coincides almost everywhere with a function that is constant on the stable manifolds.

Exercise 2 Give an example of a norm on $\mathbf{R}^2$ for which the only linear orientation-preserving isometries are $\pm\,\mathrm{Id}$. Let $\|\cdot\|$ be the norm induced by this norm on the set of 2×2 matrices. Show that the expression

$$d(A, B) = \log(\|AB^{-1}\|) + \log(\|BA^{-1}\|)$$

defines a distance on $\mathrm{PSL}_2(\mathbf{R})$ that is invariant under right multiplication.
Hint: Take the L^1 norm in the first quadrant and the L^2 norm in the second.

Exercise 3 Set

$$u' = \mathrm{e}^{2t}u(1 - \mathrm{e}^{2t}su)^{-1}, \quad s' = s(1 - \mathrm{e}^{2t}su), \quad t' = t - \ln(1 - \mathrm{e}^{2t}su).$$

Show that

$$h^{\mathrm{su}}_{u'} \circ h^{\mathrm{ss}}_{\mathrm{s}} \circ \varphi_t = h^{\mathrm{ss}}_{s'} \circ \varphi_{t'} \circ h^{\mathrm{su}}_{\mathrm{u}}.$$

Show that the transformation $(s, t, u) \mapsto h^{\mathrm{su}}_{u'} \circ h^{\mathrm{ss}}_{\mathrm{s}} \circ \varphi_t(v)$ is a diffeomorphism from a neighborhood of the origin in $\mathbf{R}^3$ to a neighborhood of $v \in \mathrm{PSL}_2(\mathbf{R})$. Let v' be a point of this neighborhood; verify that in this coordinate system, the unstable manifolds $W^{\mathrm{su}}(v')$ correspond to the vertical lines and the stable manifolds $\bigcup_{t\in\mathbf{R}} W^{\mathrm{ss}}(\varphi_t(v'))$ correspond to the horizontal planes.

Exercise 4 The "stable foliation" W^{ss} is called *ergodic* with respect to the measure μ if every W^{ss}-invariant function is constant almost everywhere. Show that if this is the case, the transformation T is mixing with respect to the measure μ. Show the ergodicity of the stable foliation for the toral automorphism given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ with respect to the Lebesgue measure.

4.5.2 More Advanced Exercises

Exercise 5 The convex hull of a set $A \subset L^2(X)$ is denoted by $\mathrm{Conv}(A)$; it is the smallest convex set containing A. Let $f \in L^2(X)$. Show that all elements of the following convex set are W^{ss}-invariant:

$$\bigcap_{N \in \mathbf{N}} \overline{\mathrm{Conv}\big(\{f \circ T^n \mid n \geqslant N\}\big)}.$$

Exercise 6 Let $f_1, f_2 \in L^2(X)$. Assume that there exist two sequences of integers m_i, n_i that tend to infinity, such that the product $f_1 \circ T^{m_i} f_2 \circ T^{m_i+n_i}$ converges weakly. Show that the limit is W^{ss}-invariant. Generalize to a finite number of functions f_k.

Exercise 7 Prove the following relation for $\varepsilon, t \in \mathbf{R}$:

$$h^{\mathrm{ss}}_{(\mathrm{e}^{-t}-1)/\varepsilon} \circ h^{\mathrm{su}}_{\varepsilon} \circ h^{\mathrm{ss}}_{(\mathrm{e}^{t}-1)/\varepsilon} = h^{\mathrm{su}}_{\varepsilon \mathrm{e}^{t}} \circ \varphi_t.$$

Consider the quotient of $\mathrm{PSL}_2(\mathbf{R})$ by a lattice. Show that a W^{su}-invariant and W^{ss}-invariant measurable function f is invariant under the flow φ_t, in the sense that $f \circ \varphi_t = f$ almost everywhere. Deduce that the geodesic flow is mixing with respect to the measure μ:

$$\forall f \in L^2, \quad f \circ \varphi_t \xrightarrow[t \to \infty]{} \int f \, \mathrm{d}\mu \quad \text{weakly.}$$

Exercise 8 Let Γ be a subgroup of $\mathrm{PSL}_2(\mathbf{R})$. Set $\Gamma^* = \Gamma - \{\mathrm{Id}\}$. Recall that the group Γ acts properly on $\mathrm{PSL}_2(\mathbf{R})$ if for every compact subset $K \subset \mathrm{PSL}_2(\mathbf{R})$, the set $\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \varnothing\}$ is finite. Prove the equivalence of the following statements:

- The group Γ is discrete.
- The group Γ acts properly on $\mathrm{PSL}_2(\mathbf{R})$.
- The group Γ is countable, and Γv is closed for every $v \in \mathrm{PSL}_2(\mathbf{R})$.
- For every $v \in \mathrm{PSL}_2(\mathbf{R})$, we have $d(v, \Gamma^* v) > 0$.

4.6 Comments

The term "foliation" used for the partition of X given by the manifolds $W^{\mathrm{ss}}(x)$ for $x \in X$ is, of course, incorrect; in general, we should not expect this partition to form a foliation in the geometric sense of the word. The term "stable distributions" is sometimes used in the literature, but it clashes with the concept of distribution that comes from analysis. The term "lamination" is also used, but again, this term has a more restrictive meaning in differential geometry.

The Hopf argument also holds if we consider the "average" foliation

$$W_{\mathrm{moy}}^{\mathrm{ss}}(x) = \left\{ y \in X \mid \frac{1}{n} \sum_{k=1}^{n} d(T^k(x), T^k(y)) \xrightarrow[n \to \infty]{} 0 \right\}.$$

The functions that are invariant under T are $W_{\mathrm{moy}}^{\mathrm{ss}}$-invariant. More generally, the eigenfunctions of $f \mapsto f \circ T$ are $W_{\mathrm{moy}}^{\mathrm{ss}}$-invariant. On the other hand, the accumulation points of $f \circ T^n$ are in general not invariant under this foliation.

The previous theorems can in part be generalized to infinite measures. If we can find a countable family of open sets U_i of finite measure such that $\mu(X \setminus \cup U_i) = 0$, then the accumulation points of $f \circ T^n$ for $f \in L^2$ are W^{ss}-invariant. The proof is the same; the condition on the measure guarantees the density of the bounded L^2 Lipschitz functions in $L^2(X)$.

Meanwhile, in infinite measure, showing that the L^2 functions that are invariant under T are constant almost everywhere is not enough to obtain ergodicity. For example, for a translation on $\mathbf{R}$, there are no invariant sets of nonzero finite Lebesgue measure, and therefore no invariant L^2 functions; but there are many invariant sets of infinite measure whose complement has infinite measure, and many bounded invariant functions.

In infinite measure, the Hopf argument remains valid if we assume that the measure is conservative, that is, if every set of nonzero measure has an intersection of nonzero measure with one of its iterates. The proof is based on the "ratio" ergodic theorem, proved by E. Hopf in 1937 to extend the argument to infinite measures.

E. Hopf's original argument used the Birkhoff ergodic theorem, rather than the Banach–Saks theorem, and did not use the weak topology. From this point of view, it could not be used to tackle the question of whether the transformation is strongly mixing.

From the 1960s on, the mixing property was studied using entropic techniques. The σ-algebra of invariant sets is replaced by the Pinsker σ-algebra consisting of the sets belonging to a partition with zero entropy, and the Hopf argument is proved using increasing generalizations, in a series of articles that begin with D.V. Anosov and Y. Sinaï in 1967 and conclude with F. Ledrappier and L. S. Young in 1984. In their most general form, these results show the equivalence between the Pinsker σ-algebra and the σ-algebra of measurable sets that are the union of "fast" stable manifolds $W_{\mathrm{fast}}^{\mathrm{ss}}(x) = \{y \in X \mid \overline{\lim} \frac{1}{n} \log d(T^n(x), T^n(y)) < 0\}$ for every C^2 diffeomorphism on a compact manifold. The link with mixing is provided by the following remark: the accumulation points of the sequences of the form $f \circ T^n$ are measurable with respect to the Pinsker σ-algebra.

It is in general difficult to prove that the Lebesgue measure is absolutely continuous with respect to the stable foliation. For a geodesic flow on a compact manifold with negative curvature, the absolute continuity of the volume was proved by D. V. Anosov in 1963. The case of nonpositive curvature is not as well understood; the question of whether the volume is ergodic is still open to this day.

There exist a few examples of surfaces with nonpositive curvature for which we know how to prove the absolute continuity of the volume with respect to the stable and unstable foliations. For example, if there exists a point with nonpositive curvature on each geodesic, then the geodesic flow is Anosov (Eberlein, 1973), which implies absolute continuity, and therefore ergodicity. This holds for the surface with equation $\{(x, y, z) \in \mathbf{T}^3 \mid \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi z) = 0\}$ because its curvature vanishes at finitely many points.

The surface $\{(x, y, z) \in \mathbf{T}^3 \mid \cos(2\pi x) + \cos(2\pi y) + \cos(2\pi z) = 0\}$ appears in the work of T. J. Hunt and R. S. Mackay (2003), who give an example of a mechanical system that reduces to studying the geodesic flow on this surface. Maupertuis's principle, presented in Chap. 9, §45 of V. I. Arnold's book [1], provides another physical motivation for the study of geodesic flows. This principle states that at high energy, a Hamiltonian system behaves like a geodesic flow associated with a certain metric on the phase space. However, this metric rarely has nonpositive curvature.

The identification of connected complete orientable surfaces with constant nonpositive curvature with quotients of $\mathrm{PSL}_2(\mathbf{R})$ follows from Hadamard's theorem. This theorem states that the exponential map, defined on the tangent space to the surface at one of its points, is a covering map [9]; the metric can be calculated explicitly in polar coordinates: $\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{sh}^2(r)\,\mathrm{d}\theta^2$. This allows us to identify the universal cover of the surface with the upper half-plane $\mathbf{H} = \{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$ endowed with the metric $|\mathrm{d}z|/\mathrm{Re}(z)$. For this metric, the isometries that preserve the orientation are the homographies $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbf{R}$ with $ad - bc = 1$. Finally, two unit vectors of $T\mathbf{H}$ can be deduced from one another using a unique homography. From an algebraic point of view, the isomophism between $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbf{R})$ and $(x + iy, \theta) \in T^1\mathbf{H}$ is given by the Iwasawa decomposition of the matrix.

There exist algebraic methods for constructing quotients of $\mathrm{PSL}_2(\mathbf{R})$ of finite volume. We can, for example, take the quotient by $\mathrm{PSL}_2(K)$, where K is a quaternion algebra over a number field. These constructions are described by Katok [12]. Here is an example: Let $a, b \in \mathbf{N}$ be two prime numbers with a not a square modulo b. The quotient of $\mathrm{PSL}_2(\mathbf{R})$ by the following group is compact:

$$\left\{ \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a} \end{pmatrix} \,\middle|\, x_0, x_1, x_2, x_3 \in \mathbf{Z},\ x_0^2 - a\,x_1^2 - b\,x_2^2 + ab\,x_3^2 = 1 \right\}.$$

The most general constructions are geometric and use the identification of $\mathrm{PSL}_2(\mathbf{R})$ with the unit tangent bundle of the upper half-plane $\mathbf{H}$. The discrete subgroups of finite covolume are obtained from polygons in the plane that are symmetric with respect to the origin and have a finite number of sides. In fact, we associate with such a subspace the group generated by the elements of $\mathrm{PSL}_2(\mathbf{R})$ that identify opposite sides of the polygon. A well-known result of Poincaré states that we thus obtain the lattices of $\mathrm{PSL}_2(\mathbf{R})$. A reference for these questions is the book by A. Beardon [3].

Part II
Dynamical Systems

Chapter 5
Topological Dynamics

All truly wise thoughts have been thought already thousands of times; but to make them truly ours, we must think them over again honestly, until they take root in our personal experience.
J. W. von Goethe (1749–1832)

5.1 Introduction

A topological dynamical system is given by a topological space X and a transformation $T : X \to X$. We will study the case where X is a metric space and T satisfies certain compatibility conditions with the topology, for example that T is continuous or Borel. The sequence $x, T(x), T(T(x)), \ldots$ of iterates of a point $x \in X$ forms the *trajectory* or *orbit* of the point x.

Here are several examples of topological dynamical systems:

- *systems arising from physics*: the mechanics of a point mass can be described using the transformation that associates with initial conditions $(x_0, v_0) \in \mathbf{R}^3 \times \mathbf{R}^3$ the position and speed (x, v) of the point at time 1;
- *systems arising from algorithmics*: when an equation cannot be solved explicitly, we can try to approximate its solutions using a sequence $x_{n+1} = T(x_n)$ defined by induction;
- *systems arising from probability theory*: repeating an experiment independently can be described using a shift on a product space;
- *systems arising from geometry*: studying the evolution of an equation (geodesic flow, Ricci flow,. . .) gives information on the structure of the underlying space;
- *systems arising from arithmetic*: one of the first examples was given by Gauss, who noted that calculating the continued fraction representation can be done using a transformation that preserves an explicit measure;
- *systems arising from group theory*: we can study linear actions on quotients of groups of matrices, or let an element of the group act on the quotient by translation.

Let x be a point of X. What can be said about the behavior of the sequence $\{T^n(x)\}_{n\in\mathbf{N}}$ in general? This sequence may be convergent. That is the desired behavior when the system is meant for computing the solutions of an equation. On

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the other hand, the trajectory may be dense in the space X. This is almost always the case when the transformation is ergodic with respect to a finite measure with full support.

In general, these two behaviors can coexist in the same system, which leads us to study the notion of *nonwandering sets*. The points that do not belong to this set are those that have a neighborhood that is distinct from all of its iterates. Their trajectories cannot be dense; such a point also cannot return close to the initial point. The trajectory of an arbitrary point cannot converge toward a wandering point. It must therefore either go to infinity or end up in the nonwandering set.

5.2 Transitivity and Topological Mixing

Definition 5.1 Let X be a metric space, and let $T : X \to X$ be a map. This map is said to be *transitive* if for all nonempty open subsets U, V of X, there exists a sequence $n_i \to \infty$ such that $T^{-n_i}U \cap V$ is nonempty.

The map T is *topologically mixing* if for all nonempty open subsets U, V of X, there exists $N \in \mathbf{N}$ such that for every $n \geqslant N$, the set $T^{-n}U \cap V$ is nonempty.

The definition is illustrated by Fig. 5.1. A topologically mixing map is transitive. The following relationship between these notions and ergodicity and measure-theoretic mixing follows from the definitions.

Proposition 5.1 *Let X be a metric space. A Borel transformation T of X that preserves an ergodic finite Borel measure of full support is transitive. If the measure is mixing, T is topologically mixing.*

Let us introduce the notion of an ω-limit set.

Definition 5.2 Let X be a metric space, and let $T : X \to X$ be a map. The *ω-limit* set of x is the set of all accumulation points of the sequence $\{T^n(x)\}_{n\in\mathbf{N}}$:

$$\omega(x) = \{y \in X \mid \exists n_i \to \infty,\ T^{n_i}x \to y\} = \bigcap_{n\in\mathbf{N}} \overline{\{T^k(x) \mid k \geqslant n\}}.$$

We can show that when T is ergodic, there exists a point whose orbit is dense.

Proposition 5.2 *Let X be a metric space, let μ be a finite Borel measure, and let $T : X \to X$ be a Borel map that preserves μ. We assume that μ is ergodic. Then for almost every $x \in X$, we have* $\operatorname{supp}\mu \subset \omega(x)$.

Proof In Chap. 18, it is proved that the support of a finite measure is separable. Let $\{x_i\}_{i\in\mathbf{N}}$ be a countable subset of $\operatorname{supp}\mu$ that is dense in $\operatorname{supp}\mu$, and let $r > 0$ be rational. The sets $B(x_i, r)$ have positive measure. By ergodicity, there exists a set $\Omega_{i,r}$ with negligible complement such that for $x \in \Omega_{i,r}$, the point $T^n(x)$ belongs to $B(x_i, r)$ for infinitely many integers $n \in \mathbf{N}$.

Taking the intersection of the $\Omega_{i,r}$ for $i \in \mathbf{N}$ and $r > 0$ rational, we see that for almost every $x \in X$, for every $i \in \mathbf{N}$, and for every rational number $r > 0$, we can find infinitely many n such that $T^n(x) \in B(x_i, r)$. For these x, we have $x_i \in \omega(x)$ for every $i \in \mathbf{N}$. The support of μ is therefore contained in $\omega(x)$. $\square$

Corollary 5.1 *Let X be a metric space, and let T be a Borel transformation of X that preserves an ergodic finite Borel measure of full support. Then there exists a point $x \in X$ such that $\omega(x) = X$.*

If the measure has full support, the transformation is transitive. Figure 5.2 illustrates the transitivity of a toral automorphism.

In general, in a topological space, a set is dense if and only if it meets all nonempty open sets. The existence of a point x such that $\omega(x) = X$ therefore implies transitivity.

To establish the converse, we will use the Baire category theorem. This result plays an important role in topological dynamics. It holds in every *topologically complete* space, that is, in every topological space that is homeomorphic to a complete metric space, and reads as follows: in such a space, every countable intersection of dense open sets is dense. Complete or locally compact metric spaces are examples of topologically complete spaces, and the Baire category theorem is often stated in this setting.

Proposition 5.3 *Let X be a topologically complete separable space, and let $T : X \to X$ be a continuous map. Then T is transitive if and only if there exists $x \in X$ such that $\omega(x) = X$.*

Proof We have the following equality:

$$\{x \in X \mid \omega(x) = X\} = \bigcap_{\substack{U \text{ open} \\ \text{nonempty}}} \bigcap_{N \in \mathbf{N}} \bigcup_{n > N} T^{-n}U.$$

We can restrict the first intersection to the open sets belonging to a base of the topology; this notion is recalled in Chap. 18. If X has a countable base, the set $\{x \in X \mid \omega(x) = X\}$ can therefore be written as a countable intersection of open sets.

If T is transitive, the sets $\bigcup_{n>N} T^{-n}U$ are dense because they meet all open sets. The set $\{x \in X \mid \omega(x) = X\}$ is a countable intersection of dense open sets; it is therefore nonempty by the Baire category theorem.

Conversely, let $x \in X$ be such that $\omega(x) = X$, and let U and V be two nonempty open sets. The orbit of x admits accumulation points in U and V; it therefore passes infinitely many times through these open sets, which shows the transitivity. $\square$

5.3 Recurrent Points and the Nonwandering Set

Definition 5.3 A point $x \in X$ is *recurrent* if $x \in \omega(x)$, that is, if there exists a sequence $n_i \to \infty$ such that $T^{n_i}x \to x$. We will denote the set of recurrent points by $\mathcal{R}$.

We have just seen that if T is continuous, the set of $x \in X$ such that $\omega(x) = X$ is an intersection of open sets. The same holds for the set of recurrent points. This comes from the following equality:

$$\{x \in X \mid x \in \omega(x)\} = \bigcap_{k \in \mathbf{N}^*} \bigcap_{N \in \mathbf{N}} \bigcup_{n \geqslant N} \{x \in X \mid d(x, T^n x) < 1/k\}.$$

The following result is closely related to the Poincaré recurrence theorem.

Proposition 5.4 *Let X be a metric space, let μ be a finite Borel measure, and let $T : X \to X$ be a Borel map that preserves μ. Then almost every point of the support of μ is recurrent.*

Without the separability assumption on X, we cannot guarantee that the support has full measure. On the other hand, the support of a finite measure is always separable. The proof can be found in Chap. 18.

Proof Let $\{x_i\}_{i \in \mathbf{N}}$ be a countable subset of $\operatorname{supp} \mu$ that is dense in $\operatorname{supp} \mu$. Let $\mathcal{D}$ be the set of balls with center in one of the x_i and rational radius. For every $U \in \mathcal{D}$, the set

$$\{x \in X \mid x \notin U \text{ or } T^k x \in U \text{ for infinitely many } k\}$$

has full measure in X, by the Poincaré recurrence theorem. The following set therefore has full measure in X:

$$\{x \in X \mid \forall U \in \mathcal{D},\ x \in U \text{ implies } T^k x \in U \text{ for infinitely many } k\}.$$

We conclude by noting that for every $x \in \operatorname{supp} \mu$, the set of $U \in \mathcal{D}$ containing x forms a base of neighborhoods of x. □

Definition 5.4 A point $x \in X$ is *nonwandering* if for every open set U containing x, there exists a sequence $n_i \to \infty$ such that $T^{-n_i}(U) \cap U$ is nonempty. We denote the set of nonwandering points by Ω.

The nonwandering set contains all ω-limit sets: for all $x \in X$, we have $\omega(x) \subset \Omega$. The nonwandering set is closed, and recurrent points are nonwandering. We therefore have the following corollary.

Corollary 5.2 *Let X be a metric space, let μ be a finite Borel measure, and let $T : X \to X$ be a Borel map that preserves μ. Then* $\operatorname{supp} \mu \subset \overline{\mathcal{R}} \subset \Omega$.

For continuous transformations, we have the following criterion.

Proposition 5.5 *Let T be a continuous map on a metric space. A point x is wandering ($x \notin \Omega$) if and only if there exists an open set U containing x such that $T^{-n}U \cap U$ is empty for every $n \geqslant 1$.*

The proposition is illustrated by Fig. 5.3.

Proof First, suppose that x is periodic with period $p > 0$: $T^p x = x$. Then x is nonwandering because x is in $T^{-pk}U \cap U$ for every positive k.

Next, let $x \notin \Omega$. There exist an open set U containing x and $N \geqslant 1$ such that for every $n > N$, we have $T^{-n}U \cap U = \varnothing$. Set $r = \frac{1}{2}\min\{d(T^n x, x) \mid n \leqslant N\}$, and write

$$V = U \cap \bigcap_{0 \leqslant n \leqslant N} T^{-n} B(T^n x, r).$$

The set $T^{-i}V \cap V$ is included in $T^{-i}U \cap U$ if $i > N$, and included in $T^{-i}\big(B(x, r) \cap B(T^i x, r)\big)$ if $i \leqslant N$. The set $T^{-i}V \cap V$ is therefore empty for every $i \geqslant 1$. □

Finally, if T is an invertible transformation of X, the nonwandering set of T^{-1} coincides with that of T.

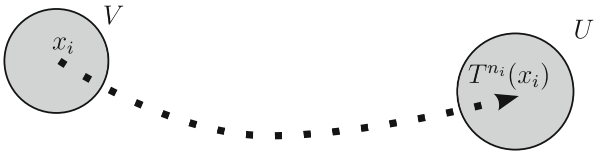


Fig. 5.1 Transitivity

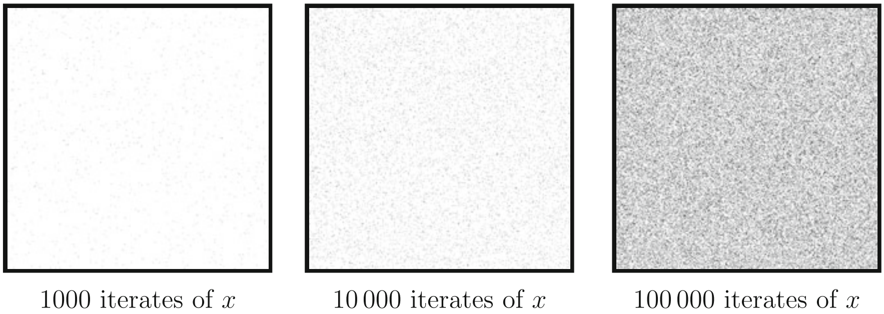


Fig. 5.2 Transitivity of hyperbolic toral automorphisms. We iterate an arbitrary point x of the torus $\mathbf{T}^2$ using the transformation $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$

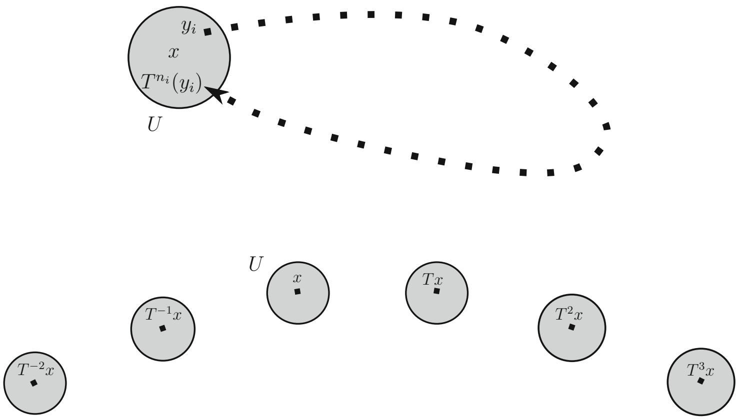


Fig. 5.3 Nonwandering set. Assume that T is invertible. A point x is wandering if and only if there exists a neighborhood U of x that is disjoint from all of its iterates

5.4 Exercises

5.4.1 Basic Exercises

Exercise 1 Let T be a homeomorphism on a metric space. Show that T^{-1} is transitive (resp. topologically mixing) if and only if T is transitive (resp. topologically mixing).

Exercise 2 Let X be a metric space, let $T : X \to X$ be a continuous map, and let $x \in X$. Show that $T\big(\omega(x)\big) \subset \omega(x)$, that $T(\mathcal{R}) \subset \mathcal{R}$, and that $T(\Omega) \subset \Omega$.

Assume that X is compact. Show that $T\big(\omega(x)\big) = \omega(x)$.

Exercise 3 Let X be a metric space without any isolated points, let $T : X \to X$ be a map, and let $x \in X$. Show that if $\{T^n(x) \mid n \in \mathbf{N}\}$ is dense in X, then $\omega(x) = X$ and T is transitive.

Exercise 4 Let X be a metric space, and let $T : X \to X$ be a map. Let $x, y \in X$. Show that if $d(T^n x, T^n y) \xrightarrow[n\to\infty]{} 0$, then $\omega(x) = \omega(y)$. Show that $\omega(Tx) = \omega(x)$.

Exercise 5 Consider the shift $\sigma(\{x_i\}_{i\in\mathbf{N}}) = \{x_{i+1}\}_{i\in\mathbf{N}}$ on the space $X = \{0,1\}^{\mathbf{N}}$; this space is endowed with the product topology. Consider the following set: $F = \{\{x_i\} \mid \forall k \in \mathbf{N},\, x_k = 1 \Rightarrow x_{k+1} = 0\}$. Find points $x \in X$ such that $\omega(x) = F$. Do the same for F reduced to the point $\{0\}_{i\in\mathbf{N}}$.

Exercise 6 Let X be a topologically complete metric space, and let $T : X \to X$ be a continuous map. Let x be a point whose orbit is a closed subset of X. Show that either x has an iterate that is periodic or $\omega(x)$ is empty.
Hint: Use the Baire category theorem on the orbit itself.

Exercise 7 Let X be a metric space without any isolated points, and let $T : X \to X$ be a homeomorphism. Assume that there exists $x \in X$ such that $\{T^n(x) \mid n \in \mathbf{Z}\}$ is dense in X. Show that T is transitive.

Exercise 8 Give an example of a continuous dynamical system that preserves a probability measure that is transitive but not ergodic.
Hint: Take a sum of ergodic measures.

5.4.2 More Advanced Exercises

Exercise 9 Give a homeomorphism of $\mathbf{R}^d$ whose nonwandering set is empty.

Exercise 10 Construct a bijective holomorphic map from the closed unit disk to itself that has a unique fixed point. Is it transitive, topologically mixing? Which Borel probability measures are invariant under this map? Which are ergodic?

Exercise 11 Let X be a metric space, let μ be a finite Borel measure, and let $T : X \to X$ be a Borel map. Assume that there exist $x \in X$ and $n_i \to \infty$ such that

$$\frac{1}{n_i}\sum_{k=0}^{n_i-1}\delta_{T^k x} \xrightarrow[i\to\infty]{} \mu \quad \text{weakly.}$$

Show that $\operatorname{supp}\mu \subset \omega(x)$.

Exercise 12 Let X be a compact metric space, and let $T : X \to X$ be a surjective continuous map. Let U be a proper open subset of X such that $T(U) \subset U$. Show that the set $\bigcup_{n\in\mathbf{N}} T^{-n}(U)$ is not equal to X.

5.5 Comments

We have just seen that for a transformation that preserves a finite measure of full support, almost all points are recurrent. Strictly speaking, this is the result that should be called the Poincaré recurrence theorem. Here is how it is stated in the 1890 paper *Sur le problème des trois corps et les équations de la dynamique*:

If we disregard certain exceptional trajectories whose realization is infinitely improbable, we can prove that the system will pass infinitely many times as close as we want to its initial position.

However, it is common practice to reserve the name of Poincaré recurrence theorem for the purely measurable statement proved earlier.

There exist other concepts of recurrence. A point x is said to be chain recurrent if for every $\varepsilon > 0$, there exist $n \in \mathbf{N}$ and $x_1, x_2, \ldots, x_n$ with $x_1 = x_n = x$ such that $d(T(x_i), x_{i+1}) < \varepsilon$ for every $i < n$. In other words, up to allowing a small error at every iteration, we can return arbitrarily close to the initial point. A nonwandering point is chain recurrent and the set of chain-recurrent points is closed and invariant under T. The following result, due to C. Conley (1978), is the first step in the decomposition of X in invariant sets: *Let X be a compact metric space, and let f be continuous. There exists a continuous function $\psi : X \to \mathbf{R}$ that is constant on the orbits of the chain-recurrent points and strictly decreasing on the other orbits; moreover, $\psi(\mathbf{R})$ has empty interior.*

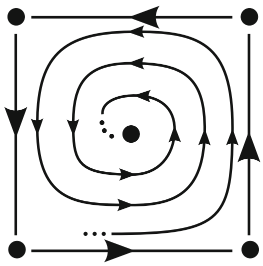
We can describe the transitivity in terms of invariant functions. The following result is due to H. Keynes and J. Robertson (1968):

Let $T : X \to X$ be a homeomorphism on a compact metric space with no isolated points. Then T is transitive if and only if every function that is invariant under T and whose set of points of continuity is dense, is constant on a G_δ-dense set.

We have used the Baire category theorem to show that if every nonempty invariant open set is dense, then the transformation admits a G_δ-dense set of points that all have dense orbits. Here, the Baire category theorem plays a role analogous to that of the ergodic theorem in the measurable setting: the absence of nontrivial invariant sets gives a "large" set of points whose orbits take up much space.

The Baire category theorem holds in all *topologically complete* spaces, that is, in every topological space homeomorphic to a complete metric space. A subset of a topologically complete space is topologically complete if and only if it is a countable intersection of open sets (Mazurkiewicz–Alexandrov theorem; this is Theorem 2.5.4 in the book by Dudley [6]). The locally complete metric spaces (i.e., those whose points each admit a complete neighborhood), *a fortiori* the locally compact metric spaces, are topologically complete. To prove this, it suffices to verify that such a space is the inter of all $1/k$-neighborhoods of itself in its completion. The open unit ball in a Hilbert space is an example of a topologically complete space that is neither locally compact nor complete for its natural metric.

Making the link with measure theory, it is interesting to work with separable spaces, that is, with spaces that admit a dense countable subset. A *Polish* space is a topological space homeomorphic to a complete separable metric space. Such a space admits a countable base of open sets, the Baire category theorem holds, and the finite Borel measures are automatically inner regular. This last point is proved in Chap. 18. Polish spaces therefore provide a natural setting for topological dynamics questions, if we wish to leave the setting of compact metric spaces.



Let T be a continuous map, and let Ω be its nonwandering set. The nonwandering set of the restriction of T to Ω is not necessarily equal to Ω. Unlike for recurrence, it does not suffice to study the trajectory of a point x to determine whether or not it is wandering. Indeed, the trajectories that return to the neighborhood of x can wander away from the orbit of x. We have drawn a counterexample on the right. The transformation is defined on $[0, \pi]^2$. It has five fixed points. The point at the center of the square is attracting and all points in the interior of the square converge to this point. The points on the boundary of the square converge to one of its four vertices. The nonwandering set of the transformation consists of all points on the boundary of the square. After restriction to the boundary of the square, the nonwandering set contains only the four vertices. Such a transformation can be constructed from the flow given by the equations

$$x' = \sin(x)\,(\cos(y) + 1/10\cos(x))\,,$$

$$y' = \sin(y)\,(1/10\cos(y) - \cos(x))\,.$$

Given a transitive transformation, it is in general difficult to determine whether a particular point has a dense orbit. For example, the transformation of the interval $[0, 1]$ given by $x \mapsto 4x(1-x)$ is transitive. Can we give an explicit rational number in $[0, 1]$ that admits a dense orbit under the action of this transformation? This question corresponds to finding a rational number r such that the sequence $\{(2^n/\pi)\arcsin(r)\}_{n\geqslant 0}$ is dense in $\mathbf{R}/\mathbf{Z}$. The question is open, even if, numerically, it seems that most rational numbers actually have a dense orbit.

Chapter 6
Nonwandering

With many a weary step, and many a groan,
Up the high hill he heaves a huge round stone;
The huge round stone, resulting with a bound,
Thunders impetuous down, and smokes along the ground.
Again the restless orb his toil renews,
Dust mounts in clouds, and sweat descends in dews.

Homer

6.1 Introduction

Consider a dynamical system given by a locally compact metric space X and a continuous map $T : X \to X$. A point in X is wandering if it admits a neighborhood that is disjoint from all of its iterates.

A point can be nonwandering and still have a trajectory that goes to infinity. Here is an example: consider a dynamical system on a compact set that preserves a probability measure and has a fixed point, for example an automorphism of a torus embedded in $\mathbf{R}^3$. We send the fixed point to infinity, making sure that the area of the obtained surface remains finite. All points are nonwandering because the system preserves a finite measure of full support, but the points that converged to the fixed point now go to infinity.

When all points are nonwandering, however, such behavior is exceptional. We will show that in this situation, the recurrent points form a G_δ-dense set of points on X: most of the trajectories return arbitrarily close to their initial positions.

On the other hand, what can we say if the nonwandering set is finite? In this case, the system has only finitely many periodic points, and all trajectories either go to infinity or converge to one of these periodic orbits. We come across this situation in mechanics when we take friction into account. The energy is not conserved, and the system ends up reaching a stable equilibrium. From a certain point of view, it is the simplest asymptotic behavior that we can observe for a dynamical system.

This type of dynamical system exists on all compact manifolds. Here is how to construct examples in dimension 2: every orientable compact surface is homeomorphic to a sphere, a torus, or a surface of genus greater than 1, in the shape of a "doughnut". It can therefore be cut up into "pants" and disks. Hence it suffices

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to construct transformations on each of the pieces, ensuring that they correspond through the gluings.

When the nonwandering set is finite, we can represent the dynamical system using a graph: the vertices of the graph are the nonwandering points of the set, and two vertices are connected if there exists a trajectory joining the two points. Such a representation is useful in classification problems.

6.2 Nonwandering

Let X be a metric space, and let $T : X \to X$ be a continuous map. We recall the definition of the nonwandering set:

$$\Omega = \{x \in X \mid \forall U \text{ open containing } x, \exists n_i \to \infty \text{ such that } T^{-n_i} U \cap U \neq \varnothing\}.$$

This set is closed, satisfies $T(\Omega) \subset \Omega$, and contains all limit sets

$$\omega(x) = \{y \mid \exists n_i \longrightarrow \infty \text{ such that } T^{n_i}(x) \longrightarrow y\}.$$

The stable manifold of a point $x \in X$ is defined by

$$W^{ss}(x) = \{y \in X \mid d\big(T^n(x), T^n(y)\big) \xrightarrow[n\to\infty]{} 0\}.$$

The results that follow describe the dynamics of the map T in the two extreme cases Ω finite and $\Omega = X$.

A point $x \in X$ is called *periodic* if there exists an integer $n > 0$ such that $T^n(x) = x$. The period of the point x is the smallest integer $n > 0$ satisfying this equality.

Proposition 6.1 *Let X be a locally compact metric space, and let $T : X \to X$ be a continuous map. We assume that the nonwandering set Ω of T is finite. Let $\mathcal{P}$ be the set of periodic points of T. Then the orbit of any point of X either is attracted by a periodic point or goes to infinity:*

$$X = \bigcup_{x\in\mathcal{P}} W^{ss}(x) \cup \{x \mid \omega(x) = \varnothing\}.$$

Proof Let $x_0 \in X$. Consider a point in $\omega(x_0) \subset \Omega$; since its orbit is contained in Ω, which is finite, one of its iterates p is periodic. Denote by ℓ the period of p. Let $\varepsilon_0 > 0$ be such that the set $\{x \in X \mid d(p, x) \leqslant \varepsilon_0\}$ is compact and disjoint from $\Omega \setminus \{p\}$. For every sufficiently small $\delta < \varepsilon_0$, we have the inclusion $T^\ell(B(p, \delta)) \subset B(p, \varepsilon_0)$. This is illustrated by Fig. 6.1.

Since the annulus $\{x \mid \delta \leqslant d(p,x) \leqslant \varepsilon_0\}$ is compact and disjoint from Ω, it contains only finitely many iterates of x_0. Consequently, there exists an integer $N \in \mathbf{N}$ such that for every $n \geqslant N$, the iterate $T^{n\ell}(x_0)$ is in $B(p,\delta) \cup B(p,\varepsilon_0)^c$. Let $n \geqslant N$; since the point $T^{n\ell}(x_0)$ is in $B(p,\delta)$, the point $T^{\ell}\big(T^{n\ell}(x_0)\big)$ belongs to $B(p,\varepsilon_0)$, and therefore $T^{(n+1)\ell}(x_0)$ is in $B(p,\delta)$. The sequence $T^{n\ell}(x_0)$ is in $B(p,\delta)$ for every $n \geqslant N$. This shows that $x_0 \in W^{ss}(p)$. ☐

Remark More generally, if $\omega(x)$ meets an invariant compact subset K of Ω that is isolated in Ω (i.e., $d(K, \Omega \smallsetminus K) > 0$), then $\omega(x)$ is contained in K.

Proposition 6.2 *Let X be a topologically complete metric space, and let $T : X \to X$ be a continuous map. We suppose that the nonwandering set of T equals X. Then the recurrent points are dense in X.*

Proof Set

$$U_{k,N} = \bigcup_{n \geqslant N} \{x \in X \mid d(x, T^n x) < 1/k\}.$$

The set of recurrent points coincides with the intersection of the $U_{k,N}$ for all $k, N \in \mathbf{N}^*$. If $\Omega = X$, then the open sets $U_{k,N}$ are dense in X. It suffices to apply the Baire category theorem to conclude. ☐

Remark In general, the recurrent points are dense in the interior of Ω (if this interior is nonempty), but not necessarily in Ω. Moreover, the nonwandering set of the transformation T restricted to Ω is not always equal to Ω.

6.3 Examples

In the following examples, we will construct transformations on manifolds of dimension d using local charts. Let X be a C^k manifold. A C^k *chart* on X is a homeomorphism ($k = 0$) or a (C^k) diffeomorphism φ from an open subset of X to an open subset of $\mathbf{R}^d$.

Let $f : X \to X$ be a continuous map, and let $x_0 \in X$. Choose a chart $\varphi : U \to V$ such that U contains x_0 and $f(x_0)$. The composition $\varphi \circ f \circ \varphi^{-1}$ is the expression of f in this chart. This map goes from the open subset $V \cap (\varphi f^{-1})(U) \subset \mathbf{R}^d$ to V. It allows us to represent f locally using a map between two open subsets of $\mathbf{R}^d$.

- *North-south dynamics*

  Let $x \in \mathbf{R}^d$ and let $\lambda \in \mathbf{R}$. We consider the map $x \mapsto \lambda x$. This map can be extended to the sphere $\mathbf{S}^d$ using the stereographic projection from the set $\{(x,t) \in \mathbf{R}^d \times \mathbf{R} \mid |x|^2 + t^2 = 1\}$ to $\mathbf{R}^d$ given by $(x,t) \mapsto 2x/(1-t)$. The resulting map is a homeomorphism of $\mathbf{S}^d$. If $|\lambda| \neq 1$, its nonwandering set consists of an attracting fixed point and a repelling fixed point. This example is illustrated by Fig. 6.2.

- *A saddle point on* $\mathbf{S}^2$

  We consider the map from $\mathbf{R}^2$ to $\mathbf{R}^2$ given by the formula

$$(x, y) \longmapsto \Big(\frac{1}{2}x + \frac{x}{1+x^2}, \frac{1}{2}y\Big).$$

  The nonwandering set consists of the points $(-1, 0)$ and $(1, 0)$, which attract, and the point $(0, 0)$, which is a "saddle point". Their stable manifolds are given by

$$W^{ss}(\pm 1, 0) = \{(x, y) \mid \pm x > 0\}, \quad W^{ss}(0, 0) = \{(x, y) \mid x = 0\}.$$

  This map can be lifted to the sphere $\mathbf{S}^2$ using the stereographic projection, in which case the north pole will become a repelling fixed point, as illustrated by Fig. 6.3. The map $(x, y) \mapsto \big(\frac{1}{2}(|x + \frac{1}{2}| - |x - \frac{1}{2}| + x), \frac{1}{2}y\big)$ has similar dynamics; it is linear in the neighborhood of the fixed points.
- *The sphere, again*

  Let A be a $d \times d$ matrix with real coefficients that has d distinct real eigenvalues, and let $f : \mathbf{S}^{d-1} \to \mathbf{S}^{d-1}$ be the map given by the formula $f(x) = Ax/\|Ax\|$. The nonwandering set consists of d points that can be either fixed or periodic with period 2, depending on the sign of the eigenvalues.
- *Gluing at the level of the attracting points*

  Consider two homeomorphisms f_1 and f_2 from the sphere $\mathbf{S}^2$ to itself. We suppose that f_1 has an attracting fixed point p_1 and that there exists a local chart U_1 in the neighborhood of which f_1 is of the form $x \mapsto \frac{1}{2}x$. We also suppose that f_2 has a repelling fixed point of the form $x \mapsto 2x$ in a well-chosen chart U_2.

  We can then construct a new surface by removing disks around p_1 and p_2 and gluing the two punctured spheres using a cylinder joining the holes. To do this, we take polar coordinates (r_1, θ_1) in the neighborhood of p_1 and (r_2, θ_2) in the neighborhood of p_2. We remove the disks $\{|r_1| \leqslant 1/4\}$ and $\{|r_2| \leqslant 1/4\}$ and join the annuli $\{1/4 < |r_1| < 1\}$ and $\{1/4 < |r_2| < 1\}$ using the map $(r_2, \theta_2) = (4r_1^{-1}, \theta_1)$. The maps f_1 and f_2 correspond to each other through this gluing and give a homeomorphism f of our new surface. The nonwandering set of f is the union of the nonwandering sets of f_1 and f_2 minus p_1 and p_2. This construction is illustrated by Fig. 6.4.

  We can also glue two fixed points p_1 and p_2 of a single homeomorphism f_1. If the basin of attraction of p_1 is disjoint from the basin of repulsion of p_2, the resulting nonwandering set equals that of f_1, minus the two points p_1 and p_2.

  On the other hand, if the two basins are not disjoint, the nonwandering set may be larger than that of the initial map. This is the case, for example, when we glue the north pole to the south pole in north-south dynamics; the resulting map is a translation on the torus, and all points are nonwandering.

6.4 The Graph Associated with the Dynamical System

We can construct an oriented graph to represent the dynamical system as follows.

The vertices are the points of Ω. We join two vertices by an oriented edge if there exists a point whose negative iterates tend to the first point and whose positive iterates tend to the second point. In other words, $x, y \in \Omega$ are joined if $W^{\mathrm{su}}(x) \cap W^{\mathrm{ss}}(y) \neq \varnothing$. The presence of cycles in this diagram can indicate the presence of recurrent points for the transformation. An example is shown in Fig. 6.5.

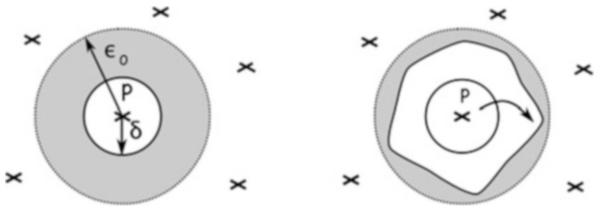


Fig. 6.1 Finite nonwandering set

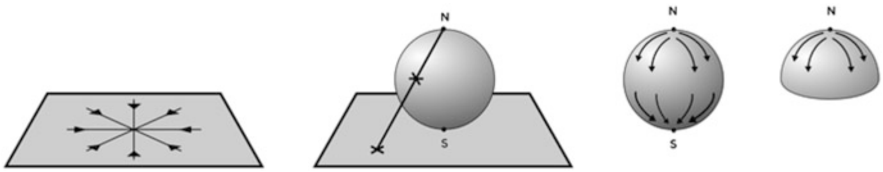


Fig. 6.2 North-south dynamics

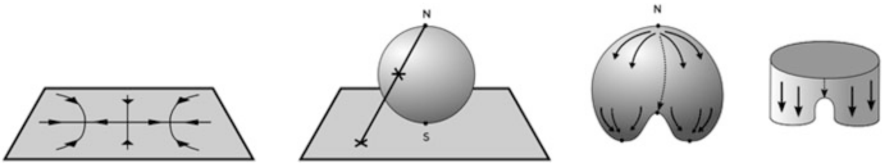


Fig. 6.3 A saddle point on the sphere

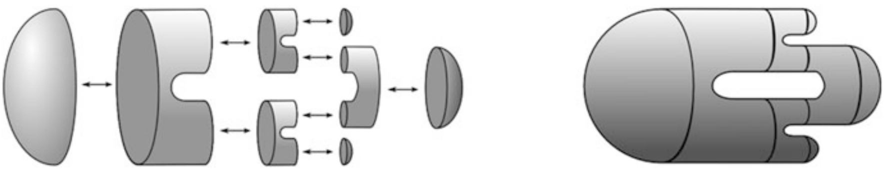


Fig. 6.4 Gluing

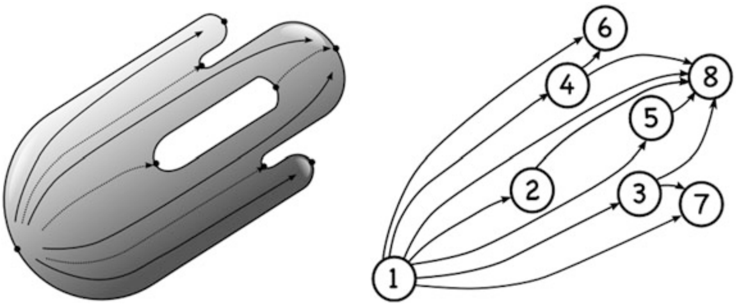


Fig. 6.5 Graph associated with the dynamical system

6.5 Exercises

6.5.1 Basic Exercises

Exercise 1 Let $v \in \mathbf{R}^d$. Consider the translation $T(x) = x + v$ on the torus $\mathbf{T}^d$. Determine the nonwandering set.

Exercise 2 Let X be a metric space, and let $T : X \to X$ be a homeomorphism whose nonwandering set is finite. Show that this set consists of periodic points.

Exercise 3 Let T be a homeomorphism on a compact space X. Let $x \in X$. Recall that the unstable manifold of x is defined by

$$W^{\mathrm{su}}(x) = \{y \in X \mid d\big(T^{-n}(x), T^{-n}(y)\big) \xrightarrow[n\to\infty]{} 0\}.$$

Assume that the nonwandering set Ω of T is finite. Prove the equality

$$X = \bigcup_{x\in\Omega} W^{\mathrm{su}}(x).$$

Exercise 4 Give an example of a continuous map on $\mathbf{S}^d$ whose nonwandering set is reduced to one point. What can be said about the dynamics of such a map?

Exercise 5 Study the dynamics of the map $T : S^1 \to S^1$ given by the formula

$$T(x) = x + \tfrac{1}{20}\big(1 - \cos(4\pi x)\big).$$

Next, study the dynamics of the map on $\mathbf{T}^2$ defined by $(x, y) \mapsto (Tx, Ty)$.

Exercise 6 Consider $a, b \in \mathbf{C}$ satisfying $|a|^2 - |b|^2 = 1$. Show that the linear fractional transformation $z \mapsto (az + b)/(\overline{b}z + \overline{a})$ sends the closed unit disk $\{z \in \mathbf{C} \mid |z| \leqslant 1\}$ to itself. Compute the fixed points of this transformation, and determine its nonwandering set.

Exercise 7 Give an example of a diffeomorphism on the surface of genus 2 (i.e., the "doughnut" with two holes) whose nonwandering set is finite.

6.5.2 More Advanced Exercises

Exercise 8 Construct homeomorphisms of the sphere $\mathbf{S}^2$ with a nonwandering set that has one, three, and seven elements, respectively.

Exercise 9 Let X be a locally compact space, let $T : X \to X$ be continuous, and let $K \subset \Omega$ be an isolated compact subset of Ω such that $T(K) \subset K$. Let $W^{\mathrm{ss}}(K) = \{y \in X \mid d(T^n(y), K) \xrightarrow[n\to\infty]{} 0\}$. Consider $x \in X$; show that if $\omega(x) \cap K \neq \varnothing$, then $x \in W^{\mathrm{ss}}(K)$.

Exercise 10 Let F be a closed subset of $[0, 1]$. Construct a homeomorphism of $[0, 1]$ whose set of fixed points equals F. Construct a homeomorphism of $[0, 1]$ whose nonwandering set equals F.

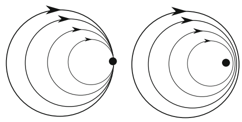
Exercise 11 Let T be a continuous map on a metric space X. Show that the nonwandering set of T^2 is included in that of T. Give an example where the converse inclusion does not hold.

Exercise 12 Let X be a compact metric space, and let $T : X \to X$ be a continuous map. Let $x \in X$ be such that $\omega(x)$ does not contain any fixed points of T. Show that

$$d\big(T^{n+1}(x), T^n(x)\big) \xrightarrow[n\to\infty]{} 0.$$

Deduce that $\omega(x)$ is connected. Assume, moreover, that the nonwandering set of T is countable. Show that the sequence $T^n(x)$ converges.

6.6 Comments



When we perturb a homeomorphism in the C^0 topology, the nonwandering set can remain close to its initial position or, on the contrary, "explode". Consider, for example, a linear fractional transformation $z \mapsto (az + b)/(\overline{b}z + \overline{a})$, $|a|^2 - |b|^2 = 1$, from the closed unit disk to itself. When $a + b = 2$, the nonwandering set is restricted to the fixed point that lies on the boundary of the disk. If $|a + b| < 2$, the map is the conjugate of a rotation, and all points of the disk are nonwandering. When $a + b = 2$, the graph associated with the dynamical system consists of a single point and a loop at this point. This is the simplest example of a graph containing a cycle. The presence of cycles often indicates a certain form of instability in the dynamical system. We can study these stability problems using filtrations; this is the object of Chaps. 2 and 3 of the book by Shub [21].
Given a compact manifold M, we can always construct, on M, a homeomorphism T whose nonwandering set is finite. On the other hand, it is not always possible to construct a homeomorphism T with finite nonwandering set and prescribed periodic points. Suppose, for example, that in the neighborhood of each periodic point, there exists a local chart in which T is linear, without any eigenvalues of absolute value 1. Let n_i be the number of periodic points for which the linearized map has an unstable subspace of dimension i. We have the following formula, due to S. Smale (1967):

$$\sum (-1)^i n_i = \chi,$$

where χ is the Euler characteristic of M. For an orientable surface M of genus g, this gives $n_0 - n_1 + n_2 = 2 - 2g$. We thus see, for example, that we cannot find a homeomorphism on M whose nonwandering set consists of an attracting point ($n_0 = 1$), a repelling point ($n_2 = 1$), and a saddle point ($n_1 = 1$).

Here is how we prove the formula given above for surfaces: Note that the unstable manifolds W^{su} of the saddle points form the edges of a "triangulation" of the surface whose vertices are the saddle points and the repelling points of T, and whose faces are the basins of attraction W^{ss} of the attracting points. The Morse–Smale formula then follows from the

Euler relation, which links the number of vertices, edges, and faces of a "triangulation" of the surface to its Euler characteristic.

In the examples given earlier by gluing half-spheres and "pants", the Euler characteristic of the surface equals the difference between the number of half-spheres and the number of pants. The number of half-spheres equals $n_0 + n_2$, while the number of pants equals n_1. We indeed obtain the desired formula.

A Morse–Smale diffeomorphism is a diffeomorphism on a connected compact manifold, whose nonwandering set is finite and whose periodic points are hyperbolic; we moreover require that the stable and unstable manifolds of the periodic points intersect transversally whenever they intersect. The examples given earlier are examples of Morse–Smale diffeomorphisms. These transformations are stable, in the sense that every diffeomorphism close to such a map for the C^1 topology is C^0-conjugate to it.

Determining the nonwandering set can prove to be difficult in practice: for example, we still do not know whether all points are nonwandering for an Anosov diffeomorphism on a connected manifold. Likewise, studying the dynamics of a quadratic polynomial $z \mapsto z^2 + c$ involves passing through a sophisticated proof of a nonwandering theorem.

Chapter 7
Conjugation

The source of all great mathematics is the special case, the concrete example. It is frequent in mathematics that every instance of a concept of seemingly great generality is in essence the same as a small and concrete special case.

P. R. Halmos (1916–2006)

7.1 Introduction

Let Y be a metric space, and let $T : Y \to Y$ be a continuous map. It is not in general possible to explicitly compute the iterates $T^n(x)$ in order to decide whether the individual orbits converge or diverge.

We can, however, look for a new coordinate system in which the transformation has a simpler expression, allowing the effective computation of its iterates. Let X be another metric space, and let $S : X \to X$ be a continuous map. We say that T is *conjugate* to S, from a topological point of view, if there exists a homeomorphism $\varphi : X \to Y$ that satisfies $\varphi \circ S = T \circ \varphi$. The map φ represents the change of coordinates, and S is the expression of T in this new coordinate system.

From the relation $\varphi \circ S = T \circ \varphi$, we deduce the equality $T^n = \varphi \circ S^n \circ \varphi^{-1}$ for every integer n. The conjugation φ therefore gives a bijection between the orbits of S and those of T. If a trajectory of S is convergent, then its image by φ is also convergent; if it is dense, its image is dense. If a set or measure is invariant under S, its image is also invariant under T, etc. The dynamics of S and T correspond to each other and studying the transformation T reduces to studying S.

Can we, using conjugation, reduce the study of a general dynamical system to that of a small number of well-chosen examples? Here are two families of dynamical systems for which this has been done:

- completely integrable mechanical systems: the two-body problem, geodesic motion on an ellipsoid, harmonic oscillators, the displacement of a rigid body in an ideal fluid, a symmetric top fixed at its extremity. These can be conjugated to rotations on tori.
- transformations of compact Riemann manifolds that dilate a metric. Again, we can reduce to studying an algebraic system on a homogeneous space. These transformations have a clearly more disorganized behavior than the integrable

Y. Coudène, *Ergodic Theory and Dynamical Systems*, Universitext,
DOI 10.1007/978-1-4471-7287-1_7

systems; for example, they are topologically mixing, which is never the case for a completely integrable system.

Further on, we will study three examples: the first is given by a map on the interval, which turns out to be dilating after a suitable change of coordinates. The second comes from physics; it is a simple pendulum under the influence of a constant force field. The third is obtained by iterating a rational function on the extended complex plane $\mathbf{C} \cup \{\infty\}$. In each of these examples, the conjugation depends on a peculiar identity satisfied by a well-chosen function. From this point of view, these systems may seem rather exceptional.

7.2 Conjugation and Semiconjugation

Definition 7.1 Let X and Y be two metric spaces, and let $S : X \to X$ and $T : Y \to Y$ be two (continuous, C^k) Borel maps. The map T is semiconjugate to S via a map $\varphi : X \to Y$ if we have the relation $\varphi \circ S = T \circ \varphi$. In other words, the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "T"] & Y \end{tikzcd}$$

We speak of conjugation when φ is Borel (continuous, C^k) and bijective, with Borel (continuous, C^k) inverse.

Examples

- The map $x \to 2x \bmod 1$ is semiconjugate to the shift on a two-symbol alphabet via the continuous map

$$\varphi : \{0, 1\}^{\mathbf{N}} \longrightarrow [0, 1)$$
$$\{a_i\}_{i \in \mathbf{N}} \longmapsto \sum \frac{a_i}{2^{i+1}}.$$

- Consider the map $T : [0, 1) \to [0, 1)$ defined by

$$T(y) = 4\,y\,(1 - y).$$

Set $\varphi(x) = \sin^2(\frac{\pi}{2}x)$. We take the square of the following trigonometric identity:

$$\sin(2\,\tfrac{\pi}{2}x) = 2\,\sin(\tfrac{\pi}{2}x)\,\cos(\tfrac{\pi}{2}x).$$

This gives the equality $\varphi(2x) = 4\varphi(x)(1 - \varphi(x))$, that is, $\varphi(2x) = T(\varphi(x))$. The map T is therefore semiconjugate to the map $S(x) = 2x$ from $\mathbf{R}$ to $\mathbf{R}$.

Proposition 7.1 *Let $S : X \to X$ and $T : Y \to Y$ be Borel maps that are semiconjugate via a Borel map φ. Let μ be an S-invariant Borel measure. Then its image $\varphi_*\mu$ is a Borel measure that is invariant under T. If S is ergodic with respect to μ (resp. mixing), then T is ergodic (resp. mixing) with respect to $\varphi_*\mu$.*

Proof

- Let us show the invariance of $\varphi_*\mu$:

$$T_*(\varphi_*\mu) = (T \circ \varphi)_*\mu = (\varphi \circ S)_*\mu = \varphi_*(S_*\mu) = \varphi_*\mu.$$

- Let $A \subset Y$ be a T-invariant set. Then $\varphi^{-1}(A)$ is S-invariant:

$$S^{-1}\left(\varphi^{-1}(A)\right) = (\varphi \circ S)^{-1}(A) = (T \circ \varphi)^{-1}(A) = \varphi^{-1}T^{-1}(A) = \varphi^{-1}(A).$$

  Since μ is ergodic, $\varphi^{-1}(A)$ either is negligible or has negligible complement, with respect to μ. The same therefore holds for A with respect to $\varphi_*\mu$.
- For the mixing property, let $A, B \subset Y$ be two Borel subsets; then

$$\varphi_*\mu(T^{-n}A \cap B) = \mu(\varphi^{-1}T^{-n}A \cap \varphi^{-1}B) = \mu(S^{-n}\varphi^{-1}A \cap \varphi^{-1}B).$$

  This quantity converges to $\mu(\varphi^{-1}A)\,\mu(\varphi^{-1}B)$, which is the desired value.

□

Example Set

$$S(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2)\,, \\ 2 - 2x & \text{if } x \in [1/2, 1)\,, \end{cases}$$

that is, $S(x) = 1 - |2x - 1|$ for $x \in [0, 1)$. A direct calculation shows that S is conjugate to $T(y) = 4y(1 - y)$ on $[0, 1)$ via the homeomorphism

$$y = \varphi(x) = \sin^2(\tfrac{\pi}{2}x).$$

The following diagram is commutative:

$$\begin{tikzcd}
[0,1) \arrow[r, "x \mapsto 1-|2x-1|"] \arrow[d, "\varphi"'] & [0,1) \arrow[d, "\varphi"] \\
[0,1) \arrow[r, "y \mapsto 4y(1-y)"'] & [0,1)
\end{tikzcd}$$

This diagram is illustrated by Fig. 7.1. Since S is mixing with respect to the measure $\mathrm{d}x$, the map T is mixing with respect to the measure $\mathrm{d}y/(\pi\sqrt{y(1-y)})$. Indeed, this measure is the image of the Lebesgue measure by φ:

$$\mathrm{d}y = \mathrm{d}\left(\sin^2(\tfrac{\pi}{2}x)\right) = 2\frac{\pi}{2}\sin(\tfrac{\pi}{2}x)\cos(\tfrac{\pi}{2}x)\,\mathrm{d}x = \pi\sqrt{y(1-y)}\,\mathrm{d}x.$$

The density of the T-invariant measure is clearly visible in Fig. 7.2, which was obtained using numerical experiments.

7.3 Elliptic Functions

A priori, every peculiar identity may lead to an interesting conjugation. We will study elliptic functions, which generalize the usual trigonometric functions. We fix a parameter $k \in (0, 1)$ and define the *Jacobi elliptic function* sn via its inverse:

$$\mathrm{sn}^{-1}(z) = \int_0^z \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Set $K = \mathrm{sn}^{-1}(1)$. The function sn depends on the parameter k and defines a continuous bijection from $[-1, 1]$ to $[-K, K]$. This bijection is holomorphic in the neighborhood of 0 in the complex plane. The derivative of sn can be calculated using the formula for the derivative of an inverse function; we obtain the equality $\mathrm{sn}'(x)^2 = (1-\mathrm{sn}^2(x))(1-k^2\,\mathrm{sn}^2(x))$. The function sn satisfies the identity

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}(u)\ \mathrm{sn}'(v) + \mathrm{sn}'(u)\ \mathrm{sn}(v)}{1-k^2\ \mathrm{sn}^2(u)\ \mathrm{sn}^2(v)}.$$

To establish this identity, it suffices to differentiate the right-hand side, first with respect to u, and then with respect to v, and to note that the two derivatives are equal. This shows that the right-hand side can be written as a function of the sum $u+v$. We determine this function by taking $v = 0$. It follows that

$$\mathrm{sn}(2u) = \frac{2\,\mathrm{sn}(u)\,\mathrm{sn}'(u)}{1-k^2\,\mathrm{sn}^4(u)}.$$

This equality allows us to extend the map sn to a meromorphic function on the complex plane: for every z, there exists an integer n such that $\mathrm{sn}(z/2^n)$ is well defined; we deduce the value of $\mathrm{sn}(z)$ by applying the identity.

Now, let us calculate a few values of sn using the earlier formulas:

$$\mathrm{sn}(K) = 1,\ \mathrm{sn}'(K) = 0,\ \mathrm{sn}(2K) = 0,\ |\,\mathrm{sn}'(2K)| = 1,\ |\,\mathrm{sn}(u+2K)| = |\,\mathrm{sn}(u)|.$$

It follows that

$$\mathrm{sn}(u + 4K) = \mathrm{sn}(u).$$

The function sn is periodic, with period $4K$. Set

$$K' = \int_1^{1/k} \frac{\mathrm{d}t}{\sqrt{(t^2 - 1)(1 - k^2 t^2)}}.$$

By the formula given earlier, $\mathrm{sn}^{-1}(\frac{1}{k}) = K + iK'$, which implies

$$\mathrm{sn}(K + iK') = \frac{1}{k},\ \mathrm{sn}'(K + iK') = 0,\ \mathrm{sn}(2K + 2iK') = 0.$$

As above, it follows that

$$\mathrm{sn}(u + 4iK') = \mathrm{sn}(u + 4K + 4iK') = \mathrm{sn}(u).$$

Surprisingly, the function sn has a second period $4iK'$. Let us give two applications of the peculiar identities we have just proved.

7.4 The Simple Pendulum

Elliptic functions allow us to explicitly integrate certain equations from classical mechanics. Consider a simple pendulum under the influence of the Earth's gravitational field. Denote by θ the angle the pendulum makes with the vertical axis. It is represented on the left in Fig. 7.3. The energy is preserved during the motion, which gives the equation

$$\theta'^2 - 2\omega^2 \cos\theta = E = C$$

for some constant C. Set $\psi = \dfrac{2\omega}{\sqrt{E + 2\omega^2}} \sin(\theta/2)$. This function satisfies the relation

$$\psi'^2 = \omega^2 (1 - \psi^2)\Big(1 - \Big(\frac{1}{2} + \frac{E}{4\omega^2}\Big)\psi^2\Big).$$

When $|E| < 2\omega^2$, this equation admits the solutions $\psi(t) = \mathrm{sn}(\omega t + C_0)$ with $k^2 = 1/2 + E/4\omega^2$ and C_0 some constant determined by the initial conditions. In the phase space (ψ, ψ'), the motion is conjugate to a rotation.

7.5 Schröder's Examples (1871)

The function $z \mapsto \mathrm{sn}^2(z)$ has two periods. We pass to the quotient in order to obtain a map on the torus $\mathbf{T}^2 = \mathbf{C}/(2K\mathbf{Z} + 2iK'\mathbf{Z})$ with values in $\mathbf{C} \cup \{\infty\}$. Let us verify that this holomorphic map is surjective. The image of an open set by a nonconstant holomorphic map is also open, and the image of a compact set by a continuous map is compact. The set $\mathrm{sn}^2(\mathbf{T}^2)$ is a subset that is both open and closed in the connected space $\mathbf{C} \cup \{\infty\}$; it is therefore equal to $\mathbf{C} \cup \{\infty\}$.

The function sn^2 gives a semiconjugation between the mixing transformation of the torus $\mathbf{T}^2$ given by $x \mapsto 2x$ and the rational function

$$z \longmapsto \frac{4z\,(1-z)\,(1-k^2z)}{(1-k^2z^2)^2}.$$

This rational function is therefore transitive on $\mathbf{C} \cup \{\infty\}$, and ergodic and mixing with respect to a measure that is absolutely continuous with respect to the Lebesgue measure.

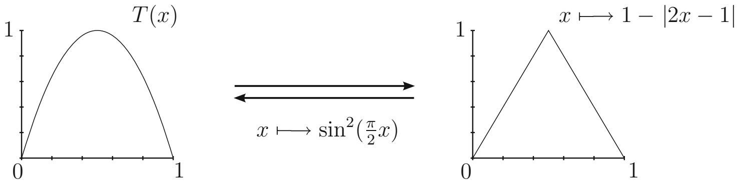


Fig. 7.1 Conjugation of $T(x) = 4x(1-x)$ to the "tent" map

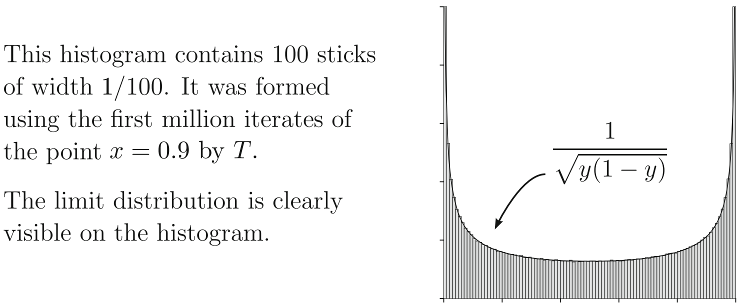


Fig. 7.2 Statistical behavior of the orbits of $T(x) = 4x(1-x)$

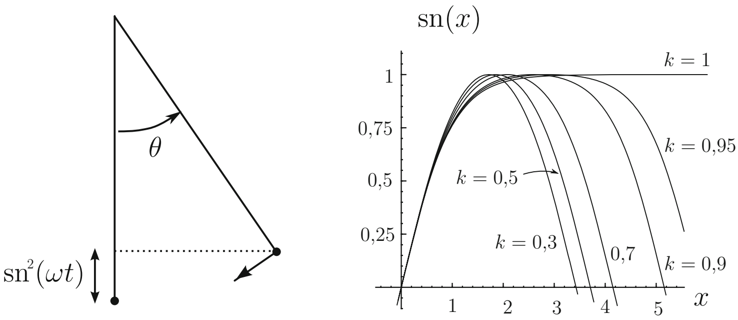


Fig. 7.3 Simple pendulum

7.6 Exercises

7.6.1 Basic Exercises

Exercise 1 Give an explicit C^0 conjugation between the maps $f_1, f_2 : \mathbf{R} \to \mathbf{R}$ defined by $f_1(x) = 2x$ and $f_2(x) = 3x$. Show that there exists a C^1 conjugation on $\mathbf{R} - \{0\}$. Does there exist a C^1 conjugation on all of $\mathbf{R}$?

Exercise 2 Let A be an $n \times n$ matrix with integer coefficients, and let v be a vector in $\mathbf{R}^n$. Assume that 1 is not an eigenvalue of A. Show that the maps on $\mathbf{T}^n$ induced by $x \mapsto Ax$ and $x \mapsto Ax + v$ are conjugate via a translation.

Deduce that these two transformations have the same properties from a topological point of view and from the point of view of the Lebesgue measure.

Exercise 3 Show that the transformation of the interval $(0, 1/2)$ given by the formula $x \mapsto 2x(1-x)$ is conjugate to multiplication by 2 on $(-\infty, 0)$ via the conjugation $x \mapsto \frac{1}{2}(1 - \mathrm{e}^x)$.

Exercise 4 Study the dynamics of the map $T : [-1, 1] \to [-1, 1]$ given by the formula $T(x) = 4x^3 - 3x$.
Hint: Use a trigonometric identity for the cosine.

Exercise 5 Let $a, b, c, d \in \mathbf{R}$ with $ad - bc \neq 0$. Show that a linear fractional transformation $z \mapsto (az + b)/(cz + d)$ on $\mathbf{C} \cup \{\infty\}$ is conjugate, via a linear fractional transformation, to a translation $z \mapsto z + \alpha$ with $\alpha \in \mathbf{R}$, to a homothety $z \mapsto \lambda z$ with $\lambda \in \mathbf{R}$, or to a rotation $z \mapsto \mathrm{e}^{i\theta} z$ with $\theta \in \mathbf{R}$.

Exercise 6 The motion of the pendulum is periodic; determine its period. Assume that the pendulum hangs vertically downward at time $t = 0$. Show that its height at time t is proportional to $\mathrm{sn}^2(\omega t)$.

Exercise 7 Show that Schröder's examples admit infinitely many periodic points, and that these periodic points are dense in $\mathbf{C} \cup \{\infty\}$. Explicitly determine the invariant measure that is absolutely continuous with respect to the Lebesgue measure.

7.6.2 More Advanced Exercises

Exercise 8 Classify all linear maps of the plane up to C^0 conjugation. Do the same up to C^1 conjugation.

Exercise 9 Consider a norm on the algebra of $n \times n$ matrices with real coefficients. Let A be such a matrix, with norm less than 1 and positive determinant. Assume that the eigenvalues of A are distinct. Show that the map from $\mathbf{R}^n$ to $\mathbf{R}^n$ given by $x \mapsto Ax$ is C^0-conjugate to the map $x \mapsto \frac{1}{2}x$.
Hint: Begin with the case where all eigenvalues are real.

Exercise 10 Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a 1-periodic continuous map; set $f(x) = 2x + \varphi(x)$.

- Show that the sequence $\sum_{n=0}^{\infty} 2^{-n-1}\varphi(f^n(x))$ converges; denote its limit by ψ.
- Show that ψ is continuous and periodic with period 1.
- Show the relation $2\psi(x) = \varphi(x) + \psi(f(x))$.
- Let $h(x) = x + \psi(x)$. Show that h is surjective and that $h(f(x)) = 2h(x)$.
- Show that if f is C^1 and $f' > 1$, then the map h is injective.
- Let $\overline{f} : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ be the map obtained from f by passing to the quotient. Show that $\overline{f}$ is transitive and that it admits an invariant finite measure of full support that is mixing.

7.7 Comments

The first duplication formulas for elliptic integrals appeared in the eighteenth century. The following formula can be found in the treatise *Produzioni Mathematische* (1750) by G. Fagnano:

$$\int_0^r \frac{\mathrm{d}t}{\sqrt{1-t^4}} = 2\int_0^\rho \frac{\mathrm{d}t}{\sqrt{1-t^4}}, \qquad r^2 = \frac{4\rho^2(1-\rho^4)}{(1+\rho^4)^2}.$$

These integrals play a role in the problem of cutting up an ellipse into arcs of equal length [14].

The functions sn can be constructed from the Jacobi θ functions and are related to the Weierstraß $\wp$ functions through the formula

$$\wp(u) = -\frac{1}{3}(1+k^2) + \frac{1}{\mathrm{sn}^2(u)}.$$

Here are some physical systems whose motion equations can be integrated using elliptic functions, the calculations are given in the book by D. F. Lawden, *Elliptic functions and applications* [14]: an object under the influence of a force proportional to $1/r^4$ or $1/r^5$, relativistic effects in the two-body problem, a whirling chain, a rigid body rotating freely about a fixed point, electric current in a rectangular conducting plate.

Rational functions that act transitively on $\mathbf{C} \cup \{\infty\}$ are called Lattès functions, even though the work of S. Lattès (1918) was carried out after that of E. Schröder. J. Milnor (2004) has given a historical overview of this problem.

We can obtain other families of transitive rational functions on $\mathbf{C} \cup \{\infty\}$ using peculiar identities for $\mathrm{sn}(mt)$ with $m \in \mathbf{N}$. Schröder's example is the only one-parameter family of degree 4. There exist families of degree n^2 for every $n > 1$. We can also construct isolated examples that can be obtained using the transformation formulas for elliptic functions with respect to the parameter k. For example, the function $\mathrm{sn}(u; k)$ associated with the parameter k can be expressed as a function of $\mathrm{sn}(u; \ell)$ with $\ell = 2\sqrt{k}/(1+k)$ using the formula

$$\mathrm{sn}((1+k)u; k) = (1+k)\,\frac{\mathrm{sn}(u;\ell)}{1+k\,\mathrm{sn}^2(u;\ell)}.$$

The equation $\ell = k$ admits the solutions $k = \pm\frac{1}{2}(1 \pm i\sqrt{7})$. For these values, the map $z \mapsto (1+k)z/(1+kz^2)$ is therefore transitive.
In many classification results, the system is conjugate to an algebraic model. We must then classify these models. For example, J.-P. Conze and J. Marcuard (1970) have shown that two automorphisms of a torus or of a nilmanifold are topologically conjugate if and only if they are algebraically conjugate.
We know how to classify up to topological conjugation the C^1 maps of connected, compact, differentiable manifolds that dilate a Riemann metric, thanks to the work of M. Shub (1969) and M. Gromov (1981). The manifold must be diffeomorphic to the quotient of a simply connected nilpotent Lie group G by a group of affine transformations acting properly discontinuously without any fixed points; such a manifold is called an "infra-nilmanifold". The transformation is conjugate to the quotient of an automorphism of G whose eigenvalues are greater than 1.
In 1978, A. Manning showed that Anosov diffeomorphisms on infra-nilmanifolds are topologically conjugate to automorphisms without any eigenvalues of absolute value 1. The infra-nilmanifolds endowed with an Anosov diffeomorphism have been classified in dimension less than 9 by J. Lauret and C. E. Will (2004). Those associated with a nilpotent Lie algebra $\mathcal{G}$ of class 2 ($[\mathcal{G}, [\mathcal{G}, \mathcal{G}]] = 0$) have been classified by T. L. Payne (2009).
There exist examples of manifolds homeomorphic to tori or to infra-nilmanifolds, but not diffeomorphic to such manifolds, which admit Anosov diffeomorphisms. These examples were constructed by F. Farrell and L. Jones (1978) and F. Farrell and A. Gogolev (2010). We do not know, however, whether a manifold endowed with an Anosov diffeomorphism is necessarily homeomorphic to an infra-nilmanifold.

Chapter 8
Linearization

I specifically remember discussions among ourselves and with visitors about what is now known as nonlinear mathematics—truly a strange expression, for it is like saying "I will discuss nonelephant animals".

S. Ulam (1909–1984)

8.1 Introduction

To study the dynamics of a transformation f on a space X, we can try to conjugate f to a simpler model $A : Y \to Y$ via a homeomorphism $\varphi : X \to Y$ satisfying $A \circ \varphi = \varphi \circ f$. When Y is an open subset of $\mathbf{R}^d$ and the model A is linear, we speak of *linearization*.

The dynamics of a linear map is sufficiently simple that we can completely describe the asymptotic behavior of its orbits and deduce that of its conjugates. Figure 8.1 shows the phase portraits of the linear maps on $\mathbf{R}^2$ that are diagonalizable on $\mathbf{C}$.

Which maps can be linearized? This question can be tackled using perturbative methods: is a transformation close to a linear map A conjugate to it? This is the case whenever A does not have any eigenvalues of absolute value 1, as illustrated in Fig. 8.2. Such a map is called *hyperbolic*.

This result can be generalized to the infinite dimensional setting, provided that the space can be decomposed into two subspaces that are respectively contracted and dilated by the linear map. To construct the conjugation, we must solve the equation $A \circ \varphi = \varphi \circ f$, that is, construct a fixed point for the map $\varphi \mapsto A \circ \varphi \circ f^{-1}$. To do this, we apply the hyperbolic fixed point theorem which follows from the fixed point theorem for a contractive Lipschitz map.

We can also tackle the conjugation problem locally. Suppose that the transformation f is differentiable and has a fixed point p. In the neighborhood of this fixed point, the transformation f is close to its differential $D_p f$. P. Hartman and D. M. Grobman (1960) have shown that we can conjugate f to its differential in a small neighborhood of the fixed point if this differential is hyperbolic.

When f is invertible and all eigenvalues of $D_p f$ have absolute value less than 1, the conjugation can even be extended to the set of points whose orbits tend to p,

Y. Coudène, *Ergodic Theory and Dynamical Systems*, Universitext,
DOI 10.1007/978-1-4471-7287-1_8

which shows, *a posteriori*, that this set is open and homeomorphic to $\mathbf{R}^d$. This set is called the *basin of attraction* of the fixed point p.

8.2 The Hyperbolic Fixed Point Theorem

Recall that a map on a metric space is *contractive* if it is Lipschitz with Lipschitz constant less than 1. The contraction mapping theorem, due to S. Banach, states that every contractive map on a complete metric spaces admits a unique fixed point. We will give a hyperbolic version of this result.

Definition 8.1 Let E be a Banach space. A continuous linear map $T : E \to E$ is called hyperbolic if there exist two closed subspaces E_{s} and E_{u} of E such that $E = E_{\mathrm{s}} \oplus E_{\mathrm{u}}$ and

$$T(E_{\mathrm{s}}) \subset E_{\mathrm{s}}, \quad \|T_{|E_{\mathrm{s}}}\| < 1,$$
$$T(E_{\mathrm{u}}) = E_{\mathrm{u}}, \quad T_{|E_{\mathrm{u}}} \text{ is invertible}, \quad \|(T_{|E_{\mathrm{u}}})^{-1}\| < 1.$$

Theorem 8.1 *Let E be a Banach space, let $T : E \to E$ be a hyperbolic linear map, and let $F : E \to E$ be a continuous map. We assume that $F - T$ is Lipschitz, with sufficiently small Lipschitz constant. Then F has a unique fixed point.*

Proof Consider the two identities

$$(\mathrm{Id} - T)(\mathrm{Id} + T + \cdots + T^n) = \mathrm{Id} - T^{n+1},$$
$$(T - \mathrm{Id})(T^{-1} + T^{-2} + \cdots + T^{-n}) = \mathrm{Id} - T^{-n}.$$

The first identity shows that $\mathrm{Id} - T : E_{\mathrm{s}} \to E_{\mathrm{s}}$ is invertible, the second that $\mathrm{Id} - T : E_{\mathrm{u}} \to E_{\mathrm{u}}$ is invertible. The linear map $\mathrm{Id} - T : E \to E$ is therefore invertible.

A point x is a fixed point of F if and only if $(F - T)(x) = x - T(x)$, that is, if and only if it is a fixed point of the map $(\mathrm{Id} - T)^{-1}(F - T)$. This map is contractive if the Lipschitz constant of $F - T$ is less than $\|(\mathrm{Id} - T)^{-1}\|^{-1}$. It therefore has a unique fixed point. □

8.3 The Linearization Theorem, Lipschitz Case

The following result shows that a continuous map close to a hyperbolic linear map is necessarily conjugate to this map.

Proposition 8.1 *Let B be a Banach space, let $A : B \to B$ be an invertible, hyperbolic, continuous linear map, and let $f : B \to B$ be an invertible continuous map. We assume that $f - A$ is bounded and Lipschitz with sufficiently small Lipschitz constant. Then f and A are conjugate to each other.*

Proof Let $g : B \to B$ be a map satisfying the same assumptions as f. Let us show that there exists a continuous map φ such that $g \circ \varphi = \varphi \circ f$. Such a map φ must be a fixed point of the transformation given by $\varphi \mapsto g \circ \varphi \circ f^{-1}$. We are looking for a φ of the form $\varphi = \mathrm{Id} + \psi$, with $\psi : B \to B$ bounded and continuous. We have

$$\begin{aligned} g \circ (\mathrm{Id} + \psi) \circ f^{-1} &= (g - A) \circ (\mathrm{Id} + \psi) \circ f^{-1} + A \circ (\mathrm{Id} + \psi) \circ f^{-1} \\ &= (g - A) \circ (\mathrm{Id} + \psi) \circ f^{-1} + A \circ f^{-1} + A \circ \psi \circ f^{-1}. \end{aligned}$$

Set

$$\begin{aligned} T(\psi) &= A \circ \psi \circ f^{-1}, \\ F(\psi) &= T(\psi) + (g - A) \circ (\mathrm{Id} + \psi) \circ f^{-1} + A \circ f^{-1} - \mathrm{Id}. \end{aligned}$$

These two maps go from E to E, where $E = C_b(B, B)$ is the Banach space of the bounded continuous functions from B to B, endowed with the uniform norm. The function F has been chosen such that we have the equality

$$g \circ (\mathrm{Id} + \psi) \circ f^{-1} = \mathrm{Id} + F(\psi),$$

so that we wish to show that F has a fixed point. To do this, let us show that T is hyperbolic and that the Lipschitz constant of $F - T$ is small.

Let B_{s} and B_{u} be the stable and unstable manifolds of A, respectively. Set $E_{\mathrm{s}} = C_b(B, B_{\mathrm{s}})$ and $E_{\mathrm{u}} = C_b(B, B_{\mathrm{u}})$. These two spaces are the stable and unstable manifolds of T, respectively.

$$\text{If } \psi \in E_{\mathrm{s}}, \text{ then } \|T(\psi)\| = \|A \circ \psi \circ f^{-1}\|_\infty = \|A \circ \psi\|_\infty \leqslant \|A_{|B_{\mathrm{s}}}\| \cdot \|\psi\|_\infty.$$

If $\psi \in E_{\mathrm{u}}$, then

$$(T_{|E_{\mathrm{u}}})^{-1} = (A_{|B_{\mathrm{u}}})^{-1} \circ \psi \circ f \quad \text{and} \quad \|(T_{|E_{\mathrm{u}}})^{-1}(\psi)\|_\infty \leqslant \|(A_{|B_{\mathrm{u}}})^{-1}\| \cdot \|\psi\|_\infty.$$

The Lipschitz constant of $F - T$ is less than that of $g - A$, which we denote by K:

$$\begin{aligned} \|(F - T)(\psi) - (F - T)(\psi')\|_\infty &= \|(g - A) \circ (\mathrm{Id} + \psi) - (g - A) \circ (\mathrm{Id} + \psi')\|_\infty \\ &\leqslant K \|\psi - \psi'\|_\infty. \end{aligned}$$

The hyperbolic fixed point theorem holds and gives a unique map $\psi \in C_b(B, B)$ satisfying $F(\psi) = \psi$, as desired. It remains to show that the map $\varphi = \mathrm{Id} + \psi$ is invertible. To do this, we reverse the roles of g and f. We obtain a function φ' that satisfies $f \circ \varphi' = \varphi' \circ g$. This gives $g \circ \varphi \circ \varphi' = \varphi \circ \varphi' \circ g$ and $f \circ \varphi' \circ \varphi = \varphi' \circ \varphi \circ f$. By the uniqueness of the conjugation, we have $\varphi \circ \varphi' = \varphi' \circ \varphi = \mathrm{Id}$. □

Remark In the theorem above, the invertibility of f follows from the other assumptions. Let K be the Lipschitz constant of $f - A$. Let us show that if this constant is less than $\|A^{-1}\|^{-1}$, the function f is invertible.

Set $A_y(x) = A(x) - y$; this map is Lipschitz, with inverse $A_y^{-1}x = A^{-1}x + A^{-1}y$. The map $A_y^{-1}(A - f)$ is contractive, so has a fixed point, which satisfies $y = f(x)$. The relation

$$x = A^{-1}(A - f)(x) + A^{-1}y$$

gives

$$\|x - x'\| \leqslant \|A^{-1}\| \, K\|x - x'\| + \|A^{-1}\| \cdot \|y - y'\|.$$

The map f^{-1} is therefore Lipschitz.

8.4 The Linearization Theorem, Differentiable Case

We want to understand the dynamics of a diffeomorphism in the neighborhood of a periodic orbit. The Hartman–Grobman theorem concerns the case where the orbit is hyperbolic.

Definition 8.2 Let M be a differentiable manifold, and let $f : M \to M$ be a C^1 map. A periodic point p of f with period n is called *hyperbolic* if $D_p f^n$ is invertible and does not have any eigenvalues of absolute value 1. We speak of an *attracting* (resp. repelling) point when all eigenvalues have absolute values less than 1 (resp. greater than 1).

Let us show that this notion of hyperbolicity agrees with the one introduced earlier. Consider a $d \times d$ matrix A without any eigenvalues of absolute value 1. The origin is a hyperbolic fixed point in the sense of the earlier definition. We must verify that the matrix is hyperbolic according to the definition given at the beginning of the chapter, for a well-chosen norm on $\mathbf{R}^d$.

Let Λ be the set of eigenvalues of A of absolute value less than 1. Set

$$E_{\mathrm{s}} = \oplus_{\lambda \in \Lambda} \operatorname{Ker}(A - \lambda \operatorname{Id})^d.$$

On the invariant subspace $\operatorname{Ker}(A - \lambda \operatorname{Id})^d$, we have

$$A^m = \big(\lambda \operatorname{Id} + (A - \lambda \operatorname{Id})\big)^m = \sum_{k=0}^{d-1} C_m^k \, \lambda^{m-k} (A - \lambda \operatorname{Id})^k.$$

The norm of the matrix A^m is bounded from above by $m^{d-1}|\lambda|^m$ times a constant that is independent of m. We choose $\lambda_0, \lambda_1 \in \mathbf{R}$ such that

$$\max_{\lambda \in \Lambda} \lambda < \lambda_0 < 1 \quad \text{and} \quad \lambda_0 < \lambda_1 < 1.$$

Since $\|A^m\| \leqslant \lambda_0^m$ for m sufficiently large, the sequence

$$\|x\|_{\mathrm{s}} = \sum_{m=0}^{\infty} \lambda_1^{-m} \|A^m x\|$$

converges for every $x \in E_{\mathrm{s}}$. We thus obtain a norm on E_{s} that satisfies the relation

$$\|Ax\|_{\mathrm{s}} = \lambda_1(\|x\|_{\mathrm{s}} - \|x\|) \leqslant \lambda_1 \|x\|_{\mathrm{s}}.$$

For the eigenvalues of absolute value greater than 1, we proceed likewise by considering A^{-1}, which give another norm $\|\cdot\|_{\mathrm{u}}$ on the sum of the generalized eigenspaces associated with the eigenvalues of A of absolute value greater than 1. This norm is contracted by A^{-1}. The matrix A is then hyperbolic with respect to the maximum of the two constructed norms. Note that if A does not have any eigenvalues of absolute value greater than 1, the ball $B(0, r)$ for the norm $\|\cdot\|_{\mathrm{s}}$ satisfies $A\big(B(0, r)\big) \subset B(0, \lambda_1 r) \subset B(0, r)$.

Let us now state the Hartman–Grobman linearization theorem.

Theorem 8.2 (Hartman–Grobman) *Let M be a differential manifold of dimension d, and let $f : M \to M$ be a C^1 differentiable map with a hyperbolic fixed point p. We can then find neighborhoods $U_1, U_2 \subset M$ of p and $V_1, V_2 \subset \mathbf{R}^d$ of 0, as well as a homeomorphism $\varphi : U_1 \cup U_2 \to V_1 \cup V_2$ that locally conjugates f and $D_p f$:*

$$\forall x \in U_1, \quad \varphi\big(f(x)\big) = D_p f\big(\varphi(x)\big)$$

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\varphi"'] & U_2 \arrow[d, "\varphi"] \\ V_1 \arrow[r, "D_p f"'] & V_2 \end{tikzcd}$$

Proof Consider a local chart centered at p. Let us show that $f - D_p f$ extends to a map on $\mathbf{R}^d$ with small Lipschitz constant.

Let $g : \mathbf{R}^d \to [0, 1]$ be a C^1 map equal to 0 for $|x| > 2$ and equal to 1 for $|x| < 1$. Set $g_r(x) = g(x/r)$. The extension of f is given by $\tilde{f} = g_r f + (1 - g_r) D_p f$ for a well-chosen r. In order to get an upper bound for the Lipschitz constant of $f - D_p f$, we bound the norm of the differential of $g_r(f - D_p f)$ for $|x| < 2r$:

$$\begin{aligned} \|D_x\big(g_r(f - D_p f)\big)\| &\leqslant \frac{1}{r}\|Dg\|_\infty \, \|f(x) - D_p f(x)\| + \|g_r\|_\infty \|D_x f - D_p f\| \\ &\leqslant \frac{1}{r}\|Dg\|_\infty \Big(\sup_{|x|<2r} \|D_x f - D_p f\|\Big)|x| + \|D_x f - D_p f\|. \end{aligned}$$

It suffices to take r such that $\sup_{|x|<2r} \|D_x f - D_p f\| < \varepsilon/(1+2\|Dg\|_\infty)$ with ε small, to be able to apply the linear conjugation theorem. We obtain a global conjugation $\varphi : \mathbf{R}^d \to \mathbf{R}^d$ between $\tilde{f}$ and $D_p f$. On $B(0,r)$, we have the equality $\tilde{f} = f$. For U_2 we take the open subset of M corresponding to $B(0,r)$, and we set $U_1 = f^{-1}(U_2) \cap U_2$. The open sets V_1 and V_2 are the image of U_1 and U_2, respectively, by φ. □

When p is an attracting fixed point, we can choose the sets such that we have the inclusions $\overline{U_2} \subset U_1$ and $\overline{V_2} \subset V_1$. If f is moreover invertible, the conjugation extends to a homeomorphism

$$\varphi : \bigcup_{k \in \mathbf{N}} f^{-k}(U_1) \longrightarrow \mathbf{R}^d$$

by setting $\varphi = (D_p f)^{-k} \circ \varphi \circ f^k$ on $f^{-k}(U_1)$. This extension is illustrated by Fig. 8.3. The set $\bigcup_{k \in \mathbf{N}} f^{-k}(U_1)$ consists of the points whose orbits converge to p. This is the *basin of attraction* of p. With the notation of the previous chapters, this basin coincides with $W^{ss}(p)$, which is therefore an open set homeomorphic to $\mathbf{R}^d$.

The Hartman–Grobman theorem can also be used to study the dynamics in the neighborhood of a hyperbolic periodic point p. Let n be the period of the point p. This point is fixed by f^n, and we can apply the linearization theorem to this iterate of f. In the case of an attracting periodic point, we obtain n disjoint open sets $W^{ss}(p)$, $W^{ss}(f(p)), \ldots, W^{ss}(f^{n-1}(p))$, each sent to the next by f. These open sets, which are homeomorphic to $\mathbf{R}^d$, are permuted cyclically by f. The orbit of a point belonging to any of these open sets ends up identifying with that of p.

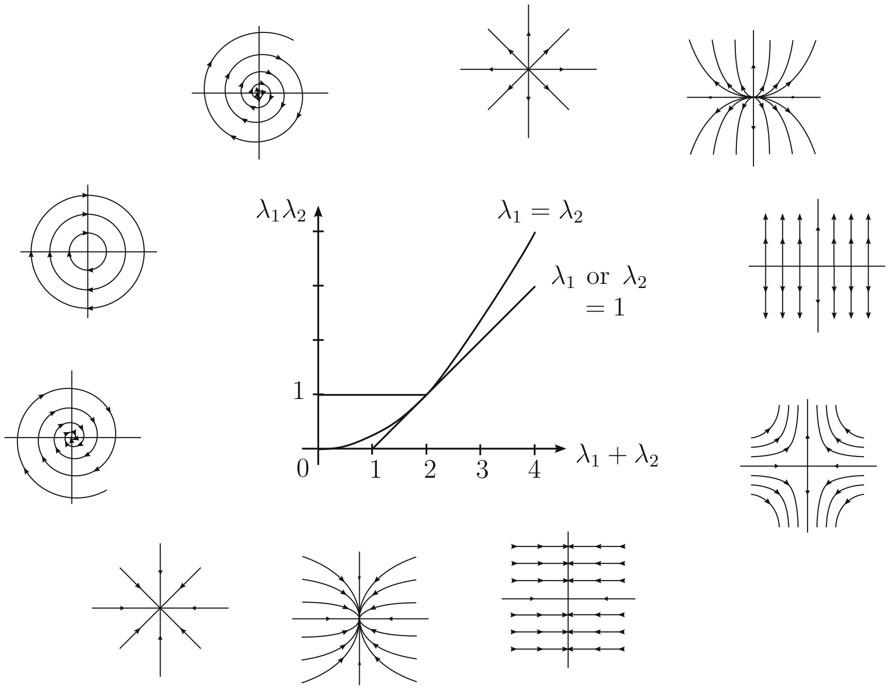


Fig. 8.1 Dynamical systems of the linear maps of $\mathbf{R}^2$ with eigenvalues λ_1 and λ_2

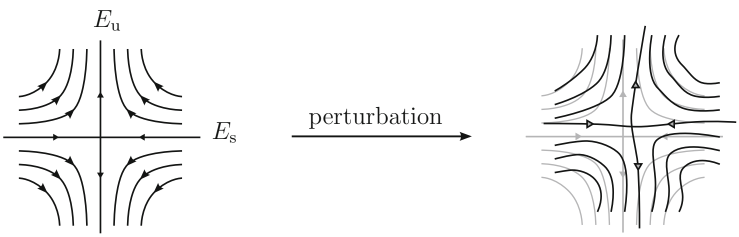


Fig. 8.2 Perturbation of a hyperbolic linear map

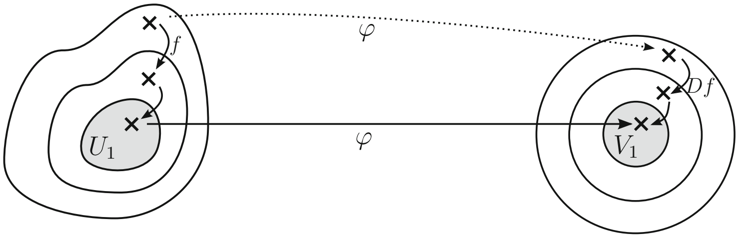


Fig. 8.3 Extension of the conjugation in the contractive case

8.5 Exercises

8.5.1 Basic Exercises

Exercise 1 Let X be a compact metric space, and let $T : X \to X$ be a map that for every pair of distinct points $x, y \in X$ satisfies the inequality $d(T(x), T(y)) < d(x, y)$.

Show that T has a unique fixed point x_0 and that for all $x \in X$, we have $T^n(x) \xrightarrow[n\to\infty]{} x_0$.

Hint: Consider $\max\{d(Tx, Ty)/d(x, y) \mid d(x, y) \geqslant \varepsilon\}$.

Exercise 2 Let M be a manifold, and let $T : M \to M$ be a differentiable map with a hyperbolic fixed point p that is not attracting. Show that the set of points attracted by p has empty interior.

Hint: Use the Baire category theorem.

Exercise 3 Consider a diffeomorphism T on a compact manifold M. Assume that its nonwandering set is finite and made up of hyperbolic periodic points. Show that the union of the basins of the attracting periodic points is a dense open set.

Exercise 4 Let f be a differentiable map on a compact manifold M. Let p be a hyperbolic periodic point. Show that there exists a neighborhood of p such that all periodic points in this neighborhood have periods greater than that of p.

Suppose that all periodic points of T are hyperbolic. Let $n \in \mathbf{N}^*$; show that the set of periodic points of T with period n is finite.

Exercise 5 Let T be a diffeomorphism on a compact manifold M. Let p be a hyperbolic fixed point. Assume that there exists a point $x \neq p$ such that $x \in W^{\mathrm{ss}}(p) \cap W^{\mathrm{su}}(p)$. Show that x is not recurrent. Show that it belongs to the nonwandering set. Deduce that the nonwandering set of T is infinite.

Exercise 6 Consider a diffeomorphism T on a manifold M and a hyperbolic fixed point p. Let E_{s} be the stable manifold of $D_p f$. Show that we have a bijective continuous semiconjugation $\varphi : E_{\mathrm{s}} \to W^{\mathrm{ss}}(p)$ that satisfies

$$\forall x \in E_{\mathrm{s}}, \quad f\big(\varphi(x)\big) = \varphi\big(D_p f(x)\big)$$

$$\begin{tikzcd}
E_{\mathrm{s}} \arrow[r, "D_p f"] \arrow[d, "\varphi"'] & E_{\mathrm{s}} \arrow[d, "\varphi"] \\
W^{\mathrm{ss}}(p) \arrow[r, "f"] & W^{\mathrm{ss}}(p)
\end{tikzcd}$$

8.5.2 More Advanced Exercises

Exercise 7 Let M be a connected compact manifold, and let $f : M \to M$ be a diffeomorphism that admits an attracting fixed point. Show that the complement of the basin of attraction is a nonempty, connected, compact set. Deduce that this complement is not countable if it contains more than one point. Is there a counterexample when M is not compact?

Exercise 8 Let T be a diffeomorphism of a connected compact manifold M of dimension greater than 1. Assume that the nonwandering set Ω is finite and made up of hyperbolic periodic points. Show that if $\mathrm{Card}(\Omega) = 2$, then M is homeomorphic to a sphere. If $\mathrm{Card}(\Omega) > 2$, then Ω contains a periodic orbit that is neither attracting nor repelling.

Exercise 9 Show that there does not exist a diffeomorphism T on a connected compact manifold, whose nonwandering set consists of three hyperbolic fixed points.

Exercise 10 Under the assumptions of the Lipschitz linearization theorem, let $p_\mathrm{u}, p_\mathrm{s} : B \to B$ be the projections onto B_u and B_s, respectively. Show that the conjugation φ between A and f is given by the formulas

$$\begin{aligned}\varphi &= \mathrm{Id} + (\mathrm{Id} - T)^{-1}\big((A-f)f^{-1}\big),\\ &= \mathrm{Id} + \sum_{k\geqslant 0} p_\mathrm{u}A^{-k-1}(A-f)f^k + p_\mathrm{s}A^k(A-f)f^{-k-1}\\ &= \lim_{n\to\infty} p_\mathrm{u}A^{-n}f^n + p_\mathrm{s}A^n f^{-n}.\end{aligned}$$

8.6 Comments

There exist different methods for obtaining a conjugation between two dynamical systems:

- Apply a fixed point theorem.
- Apply a local inversion theorem.
- Try to obtain the conjugation in the form of a series.
- Construct the conjugation geometrically.

The proof of the linearization theorem given earlier uses a fixed point theorem. We could also have called upon the following local inversion theorem:

> *Let U and V be two open subsets in Banach spaces E and F, respectively, and let $f : U \to V$ be a homeomorphism whose inverse is Lipschitz. Let $g : U \to F$ be such that the Lipschitz constant of $f - g$ is less than the inverse of that of f^{-1}. Then g is a homeomorphism from U to $g(U)$ and its inverse is Lipschitz.*

The proof of this result, which uses the contraction mapping theorem, can be found in the book by Shub [21, App. 5.1]. For a proof that uses the geometry of the stable and unstable manifolds, we refer to the book by Palis and de Melo [16]. If we wish to study the regularity of φ, it is better to use a series expansion, for example the one presented in Exercise 206. When f is C^∞ and there are no resonances between the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of D_pf, that is,

$$\forall i,\ \forall k_1, \ldots, k_n \in \mathbf{N} \text{ such that } \sum k_i > 1, \quad \lambda_i \neq \lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_n^{k_n},$$

S. Sternberg (1959) has shown that the conjugation is in fact C^∞. The proof can be found in the book by Katok and Hasselblatt [11, Theorem 6.6.6]. An example where there is resonance is given by a diffeomorphism of $\mathbf{R}^n$ or $\mathbf{T}^n$ preserving the Lebesgue measure ($\Pi\lambda_i = 1$). In this case, we can show that two such C^∞ maps that coincide at all orders at p are conjugate via a conjugation preserving the volume; we refer to the work of A. Banyaga, R. de la Llave, and C. E. Wayne (1996).

On the other hand, if we do not make any assumptions on the Lipschitz constant of $f - A$, we still have a continuous semiconjugation φ such that $A \circ \varphi = \varphi \circ f$ (take $g = A$ in the proof). If the Lipschitz constant of $f - A$ is small, φ satisfies the following regularity condition: there exist $C, D \in \mathbf{R}$ with $0 < C < 1$, such that for all $n \in \mathbf{N}^*$, we have $\|x - y\| \leqslant D^n \|\varphi(x) - \varphi(y)\| + C^n$.

A Lipschitz map on a subspace of a metric space, with real values, always admits a Lipschitz extension with the same Lipschitz constant; this result can be found in the book by Dudley [6, Theorem 6.1.1]. For every Lipschitz map f from a subset of $\mathbf{R}^n$ to $\mathbf{R}^n$, where the set $\mathbf{R}^n$ is endowed with the Euclidean metric, there exists a Lipschitz extension with the same Lipschitz constant. This is the Kirszbraun theorem; a proof can be found in the book by Federer [7].

Here is a global conjugation theorem that can be deduced from the Lipschitz case of the Hartman–Grobman theorem: every C^1 map on the torus $\mathbf{T}^n$ that is C^1-close to a hyperbolic automorphism is in fact topologically conjugate to this automorphism. Consequently, such a map is transitive, topologically mixing, etc.

The situation is very different when we perturb an automorphism whose eigenvalues have absolute value 1. The standard example is the family $f_\lambda : \mathbf{T}^2 \to \mathbf{T}^2$ defined by $f_\lambda(x, y) = (x + y, y + \lambda \sin(2\pi(x + y)))$. The maps f_λ preserve the Lebesgue measure, and all points are nonwandering. For $\lambda = 0$, the system is integrable; the trajectories of f_0 are contained in the circles $\{y = C\}$, and there do not exist any orbits asymptotic to periodic points: $W^{ss}(p) = \{p\}$ for all periodic p. On the other hand, for $\lambda > 0$ there exists a periodic point that is hyperbolic; its stable manifold is therefore nontrivial. We do not know whether there can exist an orbit whose closure has nonempty interior. Likewise, we do not know whether there exists an invariant open set such that the Lebesgue measure restricted to it is ergodic. Sinaï [22] has conjectured that the entropy of f_λ with respect to the volume is nonzero for every $\lambda > 0$.

A diffeomorphism is not in general conjugate to its differential in the neighborhood of a fixed point if the differential at this point has eigenvalues of absolute value 1. Studying the dynamics in the neighborhood of such a fixed point involves Birkhoff normal forms and KAM theory [2].

Consider the case of C^∞ diffeomorphisms of the unit disk that preserve the area and have the origin as a fixed point, with differential at this point equal to a rotation over the angle $2\pi\alpha$. For every number α that is neither Diophantine nor rational, B. Fayad and M. Saprykina (2005) have constructed examples that are ergodic with respect to the area measure. Such diffeomorphisms cannot be conjugate to a rotation in the neighborhood of the origin, because if they were, we would obtain invariant open sets by taking the images of concentric circles by the conjugation. On the other hand, for α Diophantine, M. Herman has shown that we can always find invariant closed simple curves in the neighborhood of the fixed point on which the transformation is conjugate to a rotation. Moreover, these curves form a set of positive area. We refer to an article by B. Fayad and R. Krikorian (2009) for a presentation of M. Herman's results.

Chapter 9
A Strange Attractor

The perfect square has no corners.
Great talents ripen late.
The highest notes are hard to hear.
The greatest form has no shape.

Lao Tzu

9.1 Introduction

Using the Hartman–Grobman theorem, we can show that a small perturbation f of a hyperbolic toral automorphism is conjugate to this automorphism. For such a transformation, all points are therefore nonwandering, and there exists a dense set of recurrent points.

Consider the hyperbolic automorphism on $\mathbf{T}^2$ given by the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. What happens if we carry out a local perturbation in the neighborhood of the origin $(0,0)$ that transforms this fixed point into an attracting point? By the Hartman–Grobman theorem, there exists an open set U of points that will be attracted by $(0,0)$. In this chapter, we consider an explicit example of perturbation. The open set U in this example is depicted in Fig. 9.1. What can be said about this set?

We will see that there exists a hyperbolic fixed point p on the boundary of this open set. Its stable and unstable manifolds

$$\begin{aligned} W^{\mathrm{ss}}(p) &= \{x \in \mathbf{T}^2 \mid d\big(f^n(x), f^n(p)\big) \xrightarrow[n\to\infty]{} 0\} \\ &= \{x \mid d\big(f^n(x), p\big) \xrightarrow[n\to\infty]{} 0\}, \end{aligned}$$

$$\begin{aligned} W^{\mathrm{su}}(p) &= \{x \in \mathbf{T}^2 \mid d\big(f^{-n}(x), f^{-n}(p)\big) \xrightarrow[n\to\infty]{} 0\} \\ &= \{x \mid d\big(f^{-n}(x), p\big) \xrightarrow[n\to\infty]{} 0\} \end{aligned}$$

form two immersed submanifolds of dimension 1. The stable manifold of p cannot, of course, belong to the open set U of points attracted by $(0,0)$. We will show that its closure K has empty interior and coincides with the complement of U. Consequently, most orbits converge to the origin.

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What can we be said about the dynamics of f^{-1}? The origin is now a repelling point, and all other points converge to the invariant compact set K. Moreover, the transformation restricted to K is transitive. The structure of K is interesting: it has empty interior in $\mathbf{T}^2$, but contains an immersed submanifold of dimension 1 that is both dense in K and has empty interior in K. This is therefore a geometric object that is halfway between a line and a plane. It is indicated in black on Fig. 9.1, while its complement, in white, corresponds to the open set U.

The proofs are based on the Hartman–Grobman linearization theorem and on the existence of an invariant direction on K that is dilated by the differential of f. In fact, the transformation A has an eigenvalue that is greater than 1 and the associated dilation is undisturbed by the perturbation when we are far from the origin.

Historically, the compact set K is the first example of a uniformly hyperbolic attractor that is not a submanifold. It was constructed by S. Smale in 1972. Since the transformation f^{-1} comes from a toral automorphism, which is the simplest example of an Anosov diffeomorphism, it is called a *diffeomorphism derived from Anosov* (DA diffeomorphism for short). We can carry out this type of construction on any transformation with a hyperbolic fixed point.

9.2 Perturbation of a Toral Automorphism

We begin with the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. We denote the golden mean by $\lambda = \frac{1+\sqrt{5}}{2} \simeq 1.618$. The matrix A admits two eigenvalues λ^2 and λ^{-2}; the associated eigenvectors $\mathbf{e}_\mathrm{u} = \frac{1}{\sqrt{1+\lambda^2}}\left(\begin{smallmatrix} \lambda \\ 1 \end{smallmatrix}\right)$ and $\mathbf{e}_\mathrm{s} = \frac{1}{\sqrt{1+\lambda^2}}\left(\begin{smallmatrix} -1 \\ \lambda \end{smallmatrix}\right)$ form an orthonormal basis for $\mathbf{R}^2$. We have

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix}.$$

Let us perturb A in such a manner that the point 0 becomes attracting. For $(x, y) \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2$, set

$$\begin{aligned} f\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^2 + p_1 k(r/a) & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{p_1}{1+\lambda^2} k(r/a) \begin{pmatrix} \lambda^2 & \lambda \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

with $r = \sqrt{x^2 + y^2}$ and $k(r) = (1 - r^2)^2 \mathbf{1}_{[-1,1]}(r)$ used as a C^1 "bump". The parameter a controls the extent of the perturbation, while the parameter p_1 controls its amplitude. When $a \in [0, 1/2]$, the map f passes to the quotient and defines a transformation from $\mathbf{T}^2$ to $\mathbf{T}^2$, also denoted by f. Let us establish some of its properties.

Properties

- *For every* $(x, y) \in \mathbf{T}^2$*, we have* $f((x, y) + \mathbf{R}\mathbf{e}_u) \subset f(x, y) + \mathbf{R}\mathbf{e}_u$.
- *For* $p_1 \in (-\lambda^2, 0]$ *and* $a \in [0, 1/2]$*, the map* f *is a diffeomorphism of the torus* $\mathbf{T}^2$.
- *For* $p_1 \in (-\lambda^2, 1 - \lambda^2]$*, the point* 0 *is an attracting fixed point. We denote its basin of attraction by* U.
- *For* $p_1 \in (-\lambda^2, 1 - \lambda^2]$*, the map* f *has a fixed point* $p \in (0, a)\mathbf{e}_u$ *such that* $[0, p) \subset U$.
- *The open ball* $B(0, |p|)$ *is included in the basin of attraction* U *of* 0.
- *For every* $(x, y) \in U^c$*, we have* $|d_{(x,y)}f \cdot \mathbf{e}_u| > 1$.

Proof

- For every $(x, y) \in \mathbf{T}^2$, the point $f(x, y) - A(x, y)$ belongs to $\mathbf{R}\mathbf{e}_u$. Consequently, the point $f\big((x, y) + t\mathbf{e}_u\big) - A\big((x, y) + t\mathbf{e}_u\big)$ is also in $\mathbf{R}\mathbf{e}_u$, and therefore

$$f\big((x, y) + t\mathbf{e}_u\big) - f(x, y) \in \mathbf{R}\mathbf{e}_u.$$

- Let us determine the Jacobian of f in the orthonormal basis $(\mathbf{e}_u, \mathbf{e}_s)$:

$$\begin{aligned}\det(df) &= \frac{\partial}{\partial x}\big(x + \lambda^{-2} p_1 x k(r/a)\big) \\ &= 1 + \lambda^{-2} p_1 k(r/a) + \lambda^{-2} p_1 \frac{x^2}{ra} k'(r/a) \\ &\geqslant 1 + \lambda^{-2} p_1.\end{aligned}$$

  The map f is therefore a local diffeomorphism.

  Let us show that it is bijective. Let S_r be the circle with radius r and center 0. The transformation f restricted to S_r is linear, and $f(S_r)$ is an ellipse with minor axis $\lambda^{-2}\, r\, \mathbf{e}_s$ and major axis $(\lambda^2 + p_1 k(r/a))\, r\, \mathbf{e}_u$. The lengths of these two axes are strictly increasing functions for $r \in [0, a]$. The sets $f(S_r)$ for $r \in [0, a]$ are therefore disjoint, and the transformation f is bijective from the ball $B(0, a)$ onto the set $f(B(0, a))$. This set coincides with the interior of the ellipse $f(S_a)$; it therefore equals the image of $B(0, a)$ by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Outside of $B(0, a)$, this matrix coincides with f, which is bijective.
- The fixed point 0 is attracting for the values given above because the differential $D_0 f$ admits λ^{-2} and $\lambda^2 + p_1$ as eigenvalues.
- The map $h(t) = \lambda^2 t + p_1 t\, k(t/a)$ admits a fixed point in the interval $(0, a)$ because $h(0) = 0$, $h'(0) \in (0, 1)$, and $h(a) > a$. Let t_0 be the smallest fixed point of h in $(0, a)$. We set $p = t_0\, \mathbf{e}_u$ and note that $\lambda^2 + p_1 k(|p|/a) = 1$.
- Let us show that $|f(x, y)| < |(x, y)|$ if $|(x, y)| < |p|$. In the basis $(\mathbf{e}_u, \mathbf{e}_s)$, we have

$$\begin{aligned}|f(x, y)|^2 &= \lambda^{-4} y^2 + \big(\lambda^2 x + p_1 x k(r/a)\big)^2 \\ &< \lambda^{-4} y^2 + \big(\lambda^2 x + p_1 x k(|p|/a)\big)^2 \\ &= \lambda^{-4} y^2 + x^2.\end{aligned}$$

We have used the fact that k is strictly decreasing on $(0, a)$ and the equality $p_1 k(|p|/a) = 1 - \lambda^2$. The function $(x, y) \mapsto f(x, y)/|(x, y)|$ reaches its maximum on the annulus $\{\varepsilon \leqslant |m| \leqslant |p| - \varepsilon\}$. It is therefore contractive on this annulus. Every point of $B(0, |p|)$ has an orbit that ends up entering $B(0, \varepsilon)$. The orbit of the point therefore converges to 0.

- Let us determine the differential in the direction $\mathbf{e}_\mathrm{u}$. In the basis $(\mathbf{e}_\mathrm{u}, \mathbf{e}_\mathrm{s})$, we have

$$\begin{aligned} d_{(x,y)}f.\mathbf{e}_\mathrm{u} &= \lambda^2 + p_1 k(r/a) + p_1 \frac{x^2}{ra} k'(r/a) \\ &= 1 - p_1 \big(k(|p|/a) - k(r/a)\big) + p_1 \frac{x^2}{ra} k'(r/a). \end{aligned}$$

  This is greater than 1 if $r \geqslant |p|$ and equal to 1 if $(x, y) = (0, |p|)$. This point is in U.

$\square$

From here on, we take $p_1 = -2.236$ and $a = 0.5$. We denote the basin of attraction of 0 by U and the complement of U by K. Finally, we fix a linearization φ from a neighborhood V of p to $(0, 1)^2$.

9.3 Perturbed Dynamics

We wish to show that the map f restricted to K is transitive. For the proof, we study the stable and unstable manifolds of the fixed point p on the boundary of U. We will need to verify that $W^{\mathrm{su}}(p)$ is dense in $\mathbf{T}^2$, and then that $W^{\mathrm{ss}}(p)$ is dense in K.

Lemma 9.1 *Let $x \in K$. Then for every $\varepsilon > 0$, the segment $x - [0, \varepsilon]\, \mathbf{e}_\mathrm{u}$ meets U. The open set U is therefore dense in $\mathbf{T}^2$, and K has empty interior.*

Proof The open set U consists of the points whose iterates converge to 0; it is invariant under f. If the segment $x - [0, \varepsilon]\, \mathbf{e}_\mathrm{u}$ does not meet U, then the same holds for all of its iterates. Since $D_y f \cdot \mathbf{e}_\mathrm{u} > 1$ if $y \in K$, these iterates are of the form $f^n(x) - [0, c_n \varepsilon]\, \mathbf{e}_\mathrm{u}$, with $c_n \geqslant C^n$ for some constant $C > 1$.

Since the set $\mathbf{R}_+ \mathbf{e}_\mathrm{u}$ is dense in $\mathbf{T}^2$, we can find $n \in \mathbf{N}$ such that every point of $\mathbf{T}^2$ is at a distance less than $|p|$ from $[0, C^n \varepsilon]\, \mathbf{e}_\mathrm{u}$. In particular, the point $f^n(x)$ is at a distance less than $|p|$ from $[0, C^n \varepsilon]\, \mathbf{e}_\mathrm{u}$; in other words, the point 0 is at a distance less than $|p|$ from the subset $f^n(x) - [0, C^n \varepsilon]\, \mathbf{e}_\mathrm{u} \subset K$. This gives a contradiction. $\square$

Proposition 9.1 *The set $p + \mathbf{R}_+ \mathbf{e}_\mathrm{u}$ is included in $W^{\mathrm{su}}(p)$. The unstable manifold $W^{\mathrm{su}}(p)$ is therefore dense in $\mathbf{T}^2$.*

Proof Suppose, to the contrary, that the set is not included in $W^{\mathrm{su}}(p)$. We can then set

$$t_1 = \inf\{t \in \mathbf{R}_+ \mid p + t\mathbf{e}_\mathrm{u} \not\subseteq W^{\mathrm{su}}(p)\}.$$

Because of the form of the map $t \mapsto f(p + t\mathbf{e}_u)$, the real number t_1 is positive. The image of $p + [0, t_1)\,\mathbf{e}_u$ by f is of the form $p + [0, s)\,\mathbf{e}_u$. Since $W^{su}(p)$ is invariant under f, we have $s = t_1$, and the point $p' = p + t_1\,\mathbf{e}_u$ is a fixed point of f.

The point p' is distinct from the origin. Indeed, since p is in the set $\mathbf{R}_+\mathbf{e}_u$, we would otherwise have $0 \in \mathbf{R}^*\mathbf{e}_u$, which contradicts the irrationality of λ. The set U is the basin of attraction of the origin. It follows that the fixed point p' is not in U; the slope of the curve $t \mapsto f(p + t\mathbf{e}_u)$ at t_1 is therefore greater than 1. Consequently, the points on $p + [0, t_1)\,\mathbf{e}_u$ close to $p' = p + t_1\mathbf{e}_u$ have negative iterates that approach both p and p', which is absurd. □

Proposition 9.2 *Let $m \in U$ and $t > 0$ be such that $m + [0, t)\,\mathbf{e}_u \subset U$ and $m + t\mathbf{e}_u \notin U$. Then $m + t\mathbf{e}_u$ belongs to $W^{ss}(p)$. Moreover, the set $W^{ss}(p) \cap W^{su}(p)$ is dense in K.*

Proof Let $\varphi : V \to (-1, 1)^2$ be a linearization on an open neighborhood of p. Since $W^{su}(p) \cap U$ contains $(0, p)$, there exists $x' \in (0, p) \cap V$ such that $[x', f(x')]$ is in $U \cap V$. Hence, there exists in $\varphi(V)$ a rectangle $[-\delta, \delta] \times [x', f(x')]$ contained in $\varphi(U)$. Its positive iterates under the action of $D_p f^{-1}$ are also in $\varphi(U)$ and cover $[-\delta, \delta] \times [x', p)$. This reasoning is illustrated by Fig. 9.2. The open set U comes to lean against the stable manifold of p.

Consider a curve in the open set $[-\delta, \delta] \times [x', -x')$ originating in the lower half-plane, and not entirely contained in $\varphi(U)$. The first point of the curve that is not in $\varphi(U)$ must lie on the x-axis, that is, on $\varphi(W^{ss}(p))$.

For large n, the iterate $f^n(m + [0, t]\,\mathbf{e}_u)$ is a line from a small neighborhood of 0 in the direction of $\mathbf{e}_u$. The first point of the curve that belongs to K must therefore be in V, and belongs to $W^{ss}(p)$.

Let us now show the density of $W^{ss}(p) \cap W^{su}(p)$ in K. Let $m' \in K$ and $\varepsilon > 0$ be such that $m = m' - \varepsilon\,\mathbf{e}_u$ is in U. Since $p + \mathbf{R}_+\mathbf{e}_u$ is dense in $\mathbf{T}^2$, there exists $C > 0$ such that $p + C\,\mathbf{e}_u$ is arbitrarily close to m. Taking up the previous reasoning, we see that the iterate $f^n(p + [C, C + 2t]\,\mathbf{e}_u)$ is close to $f^n(m + [0, 2t]\,\mathbf{e}_u)$. It therefore meets $W^{ss}(p)$ at a point x such that $f^{-n}(x)$ is as close to $m + t\,\mathbf{e}_u$ as we want. □

9.4 Transitivity and the Mixing Property

Corollary 9.1 *The map f restricted to K is transitive and topologically mixing.*

Proof Let U_1 be an open set intersecting K, let $x_1 \in W^{ss}(p) \cap W^{su}(p) \cap U_1$, and let n_1 be such that $f^{-n}(x_1)$ is in V for every $n \geqslant n_1$. Set $x_1' = f^{-n_1}(x_1)$. We begin by showing that every segment in the direction of $\mathbf{e}_u$, passing close to x_1', meets $W^{ss}(x_1')$ in the neighborhood of the point x_1'.

Let $R \subset V$ be a small rectangle with center x_1' and oriented in the directions of $\mathbf{e}_s$ and $\mathbf{e}_u$. There exists N such that for every $n \geqslant N$, the iterate $f^n(x_1')$ is in V. The set $f^N(R)$ contains a small rectangle R' with center $f^N(x_1')$ and, after increasing N if necessary, we may assume that R' crosses the open set V from top to bottom.

Since $f^N(x_1')$ is in $\varphi((-1,1)\times\{0\})$, the vertical lines in R' meet $\varphi((-1,1)\times\{0\}) \subset W^{\mathrm{ss}}\big(f^N(x_1')\big)$ in the neighborhood of $f^N(x_1')$. The vertical lines of R therefore meet $W^{\mathrm{ss}}(x_1')$ in the desired manner. Figure 9.3 summarizes the situation.

Let U_2 be another open set intersecting K. To prove the transitivity, it suffices to construct a point $x' \in K$ with a negative iterate in U_2 and a positive iterate in U_1. Let $x_2 \in U_2 \cap W^{\mathrm{ss}}(p)$ and $\varepsilon > 0$ be such that $x_2 + [-\varepsilon, \varepsilon]\,\mathbf{e}_{\mathrm{u}}$ is in U_2. For large n, the image $f^n(x_2 + [-\varepsilon, \varepsilon]\,\mathbf{e}_{\mathrm{u}})$ is a segment in the direction of $\mathbf{e}_{\mathrm{u}}$, close to p, which crosses V from top to bottom. It therefore meets $W^{\mathrm{ss}}(x_1') \cap f^{n_1}(U_1)$ at a point x' that is in K.

The previous reasoning shows that for every sufficiently large n, the set $f^n(U_2) \cap K$ meets $f^{n_1}(U_1)$. This implies that the restriction of f to K is topologically mixing. $\square$

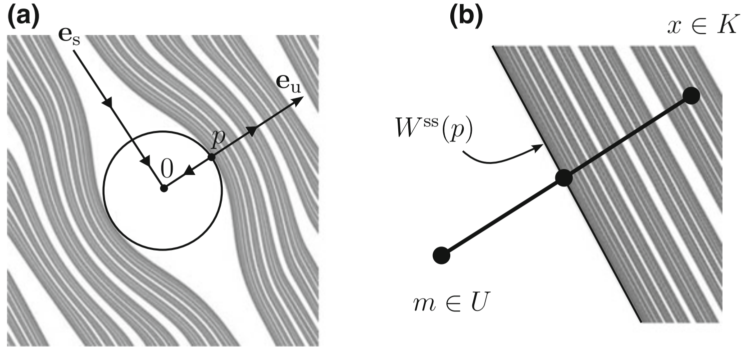

Fig. 9.1 Basin of attraction of the origin. (**a**) 0 is an attracting fixed point. *p* is a hyperbolic fixed point. (**b**) A segment in the direction $\mathbf{e}_u$ that joins *U* to *K* must meet the stable manifold of *p*

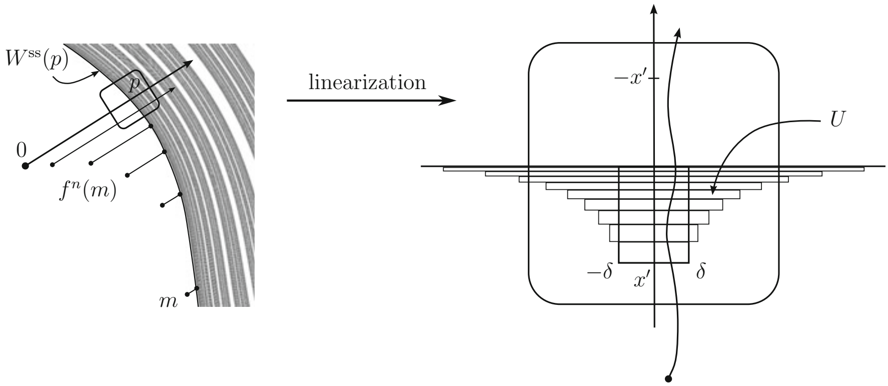

Fig. 9.2 The stable manifold of *p*

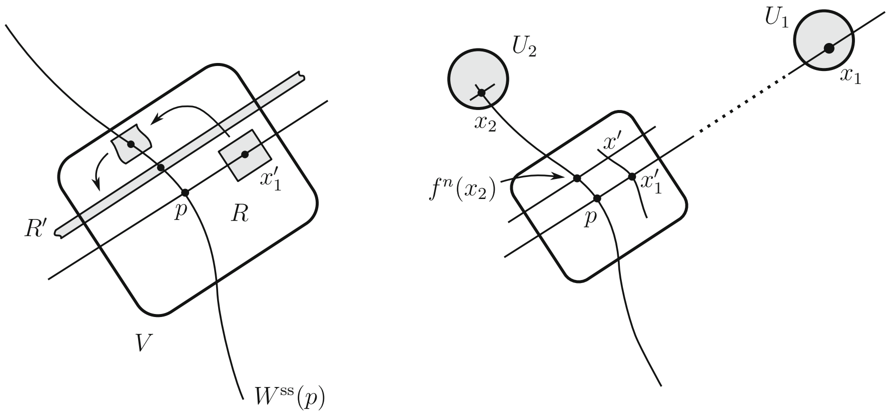

Fig. 9.3 Proof of the transitivity

9.5 Exercises

9.5.1 Basic Exercises

Exercises 1–5 and 8–11 concern the diffeomorphism f we have just studied.

Exercise 1 Show that K is compact, connected, and uncountable.

Exercise 2 Show that for every $m \in \mathbf{T}^2$, we have $f(-m) = -f(m)$. Deduce that there exists a hyperbolic fixed point $p' \in [-a, 0]\,\mathbf{e}_\mathrm{u}$ and that $W^{\mathrm{ss}}(p')$ is dense in K.

Exercise 3 Show that $W^{\mathrm{ss}}(p)$ has empty interior in K.
Hint: Note that no point of $W^{\mathrm{ss}}(p)$ has dense orbit.

Exercise 4 Let $\varepsilon > 0$. Show that $(p+[0,\varepsilon]\,\mathbf{e}_\mathrm{u}) \cap K$ is compact, without any isolated points, and with empty interior in $p + [0,\varepsilon]\,\mathbf{e}_\mathrm{u}$. Deduce that it is uncountable.

Exercise 5 Show that the points of $W^{\mathrm{ss}}(p)$ can be reached from U in the following sense: for every $x \in W^{\mathrm{ss}}(p)$ there exists a continuous map $\gamma : [0,1] \to \mathbf{T}^2$ such that $\gamma([0,1)) \subset U$ and $\gamma(1) = x$.

Exercise 6 Show that we can glue two systems derived from Anosov in such a way that we obtain a diffeomorphism f on a surface of genus 2 whose nonwandering set is the union of two uncountable connected compact sets K_1 and K_2 restricted to which f is transitive.

Exercise 7 Let M be a differential manifold, and let $f : M \to M$ be a C^1 diffeomorphism with a hyperbolic fixed point p. Show that if $W^{\mathrm{ss}}(p)$ and $W^{\mathrm{su}}(p)$ are dense in M, then f is topologically mixing.

9.5.2 More Advanced Exercises

Exercise 8 Show that K is not locally connected.
Hint: Note that every neighborhood of p contains a point of U that belongs to $W^{\mathrm{su}}(p)$ and iterate a neighborhood of this point.

Exercise 9 Let $\gamma : [0,1] \to K$ be a continuous map starting at p: $\gamma(0) = p$. On an open neighborhood V of p on which we have a linearization, we consider a partial path $\gamma([0,\delta])$ contained in V. Show that $\gamma([0,\delta]) \subset W^{\mathrm{ss}}(p)$.

Does it follow that $\gamma([0,1]) \subset W^{\mathrm{ss}}(p)$?

Exercise 10 Let $\gamma : [0,1] \to \mathbf{T}^2$ be a continuous map that satisfies $\gamma([0,1)) \subset W^{\mathrm{ss}}(p)$. Show that $\gamma(1) \in W^{\mathrm{ss}}(p)$.
Hint: Use contradiction and show that $\gamma(1)$ is a hyperbolic fixed point.

Exercise 11 Show that K is not path-connected.
Hint: Show that p and $-p$ cannot be connected by a path that remains in K.

Exercise 12 Let M be a differential manifold, and let $f : M \to M$ be a C^1 diffeomorphism with a hyperbolic fixed point p. Suppose that D_pf has a unique eigenvalue of absolute value less than 1 and that this eigenvalue is real and positive. Show that $W^{ss}(p) \smallsetminus \{p\}$ has two connected components.

9.6 Comments

The perturbation f studied in this chapter is C^1 and has Lipschitz derivative. We could have constructed a C^∞ map by taking a "bump" function of the type

$$k(r) = \exp\Big(-\frac{1}{1-r^2}\Big)\mathbf{1}_{[-1,1]}(r).$$

From a numerical point of view, a polynomial "bump" seems preferable.
Here are three algorithms that allow us to visualize the compact set K.

- We choose a point x arbitrarily and iterate it a million times using the map f^{-1}. If the point is not the origin, its trajectory will converge to the attractor K. The transformation f^{-1} restricted to K is transitive. For most x, the trajectory should therefore converge to all points of the attractor. This is what is seen in practice. This method is the fastest one from a numerical point of view.
- The origin is an attracting fixed point for the map f and K is the complement of its basin of attraction. To visualize this basin, we fix a small disk with center the origin, and then color the points of the plane as a function of the number of iterations needed to reach this disk. In practice, most points reach the disk in less than 70 steps. We could, for example, color all points needing more than twenty iterations in black, which would allow us to represent a small neighborhood of K.
- We can show that the periodic points of f are dense in K. The set of periodic points with period less than n, for n sufficiently large, therefore gives a good approximation of K. Calculating the periodic points turns out to be very costly numerically, so this method is seldom recommended.

We can describe the dynamics of f restricted to K in a more precise way. On K, the transformation f is semiconjugate to a topologically mixing shift of finite type. R. F. Williams (1974) has shown that it is conjugate to a shift on a generalized solenoid. The behavior of f is therefore highly unpredictable.
The compact set K is locally homeomorphic to the product of a segment and a Cantor set. We can verify this in the neighborhood of p by showing that the intersection of K and the local unstable manifold of p has empty interior in K. To prove it in the neighborhood of every point x of K, we must study in detail the structure of the stable manifolds $W^{ss}(x)$ and show that they are all immersed submanifolds of dimension 1.
The points of $W^{ss}(p)$ and of $W^{ss}(-p)$ make up the accessible boundary of U. They are the only points of ∂U that are the endpoints of a curve $\gamma : [0,1] \to \mathbf{T}^2$ contained in U for $t \in [0,1)$. This notion of accessible boundary no doubt corresponds better to the intuitive idea one can have of the boundary of a set.
A DA diffeomorphism is an example of an Axiom A diffeomorphism: its periodic points form a dense subset of the nonwandering set $\{0\} \cup K$; when restricted to the latter, the tangent space can be decomposed into the sum of two invariant subbundles, respectively contracted and dilated by the differential of the map.

The nonwandering set of an Axiom A diffeomorphism decomposes into a finite number of invariant compact sets, restricted to which the transformation is transitive. This can be proved by studying the stable and unstable manifolds of the periodic points, as was done for the DA diffeomorphism.

Part III
Entropy Theory

Chapter 10
Entropy

The sun comes up just about as often as it goes down, in the long run, but this doesn't make its motion random.

D. Knuth

10.1 Introduction

Let us study the problem of conjugation from the measure-theoretic viewpoint. Consider two measure-preserving dynamical systems given by a map $T_1 : X_1 \to X_1$ preserving a measure μ_1 and a map $T_2 : X_2 \to X_2$ preserving a measure μ_2. These two systems are *isomorphic* if there exist two subsets of X_1 and X_2, each with negligible complement, as well as a measurable bijection φ with measurable inverse between these two sets that satisfies $\varphi \circ T_1 = T_2 \circ \varphi$ and sends μ_1 to μ_2, that is, $\varphi_* \mu_1 = \mu_2$.

Which properties are preserved by isomorphisms? If T_1 is ergodic or mixing, the same holds for T_2. The measure of the full space and the average number of preimages are quantities that are invariant under isomorphisms. We can construct other invariants by letting the maps T_1 and T_2 act on the space L^2 by composition: for every $f \in L^2$, we set $U_1 f = f \circ T_1$ and $U_2 f = f \circ T_2$. If T_1 and T_2 are isomorphic, the maps U_1 and U_2 are conjugate via a unitary operator. They therefore have the same eigenvalues, which gives new numerical invariants.

Using these invariants, we can classify the rotations of the circle or the ergodic toral translations up to isomorphism. Two such systems are isomorphic if the induced maps on L^2 are conjugate, and this happens if and only if they have the same eigenvalues. The situation is more complex for hyperbolic toral automorphisms or Bernoulli shifts on a finite alphabet. We can show that the induced linear maps on L^2 are all conjugate to one another, but this conjugation is not obtained via an isomorphism. The classification of these systems up to isomorphism, which was started in the 1930s, was resolved only in 1970, by D. Ornstein.

An important step in this resolution was the construction in 1958 by A.N. Kolmogorov of a new isomorphism invariant called *entropy*. This numerical invariant measures the "randomness" in the system. Here is how it is defined. We begin by partitioning the space X into a finite number of pieces, and then take their iterates by the transformation T, so that we cut up X into increasingly small pieces. If the initial

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partition is well chosen, the measure of the pieces tends to 0 and the entropy can be obtained from the rate at which the measure decreases; Figs. 10.1 and 10.2 give several examples of partitions for affine toral maps. The average measure decreases polynomially for the rotations but exponentially for the Bernoulli shifts, and it is the exponent that makes it possible to differentiate the latter.

A priori, the rate at which the measure decreases depends on the initial partition. This leads to a number of technical complications and makes entropy a difficult concept to define and to compute.

10.2 Definition of Entropy

Let $(X, \mathcal{T}, \mu)$ be a probability space. A (finite, countable) *partition* ξ of X consists of a (finite, countable) set of measurable subsets of X that are mutually disjoint and whose union covers almost all of X. We denote by $\xi(x)$ the element of the partition containing the point $x \in X$.

Let $T : X \to X$ be a measurable map, and let $\xi = \{A_i\}_{i=1,\dots,n}$ be a partition. We set $T^{-1}\xi = \{T^{-1}(A_i)\}_{i=1,\dots,n}$. Let $\xi_1, \xi_2, \dots, \xi_n$ be partitions of X. The *partition generated* by the ξ_i is defined by the relation $\bigvee_{i=1}^{n} \xi_i(x) = \bigcap_{i=1}^{n} \xi_i(x)$. Figure 10.3 illustrates this notion.

Definition 10.1 Let $(X, \mathcal{T}, \mu)$ be a probability space, and let ξ be a countable partition of X. The *information function* and *entropy* of ξ are defined by

$$I(\xi)(x) = -\log\big(\mu\big(\xi(x)\big)\big) = \sum_{A \in \xi} -\log\big(\mu(A)\big)\,\mathbf{1}_A(x),$$

$$H(\xi) = \int_X I(\xi)\,\mathrm{d}\mu = \sum_{A \in \xi} -\mu(A)\,\log\big(\mu(A)\big).$$

The information function $I(\xi)$ is a nonnegative measurable function; the entropy $H(\xi)$ is a nonnegative real number, which can be infinite if the partition is not finite. To compute these quantities, we introduce a conditional version of entropy.

Definition 10.2 Let $\mathcal{A} \subset \mathcal{T}$ be a σ-algebra. The conditional entropy of ξ given $\mathcal{A}$ is defined by

$$H(\xi \mid \mathcal{A}) = -\sum_{A \in \xi} \int E_\mu(\mathbf{1}_A \mid \mathcal{A})\,\log\big(E_\mu(\mathbf{1}_A \mid \mathcal{A})\big)\,\mathrm{d}\mu.$$

Let η be a countable partition; we denote by $H(\xi \mid \eta)$ the conditional entropy of ξ given the σ-algebra generated by the elements of η.

Properties *Let ξ, ξ_1, ξ_2, η be countable partitions, let $T : X \to X$ be a measurable map that preserves the measure μ, and let $\mathcal{A}$ and $\mathcal{B}$ be sub-σ-algebras of $\mathcal{T}$.*

- $H(\xi \mid \mathcal{T}) = 0$, $H(\xi \mid \{X\}) = H(\xi)$.
- *If* $\mathcal{A} \subset \mathcal{B}$, *then* $H(\xi) \geqslant H(\xi \mid \mathcal{A}) \geqslant H(\xi \mid \mathcal{B}) \geqslant 0$.
- $H(\xi_1 \vee \xi_2 \mid \eta) = H(\xi_1 \mid \xi_2 \vee \eta) + H(\xi_2 \mid \eta)$; *in particular,* $H(\xi_1 \vee \xi_2) \leqslant H(\xi_1) + H(\xi_2)$.

Proof

- This follows from the equalities

$$E(\mathbf{1}_A \mid \{X\}) = \mu(A) \quad \text{and} \quad E(\mathbf{1}_A \mid \mathcal{T}) = \mathbf{1}_A.$$

- The function $f(x) = x \log x$ is convex. Jensen's inequality gives

$$f\big(E(\mathbf{1}_A \mid \mathcal{A})\big) \leqslant E\big(f(E(\mathbf{1}_A \mid \mathcal{B})) \mid \mathcal{A}\big).$$

  Consequently, we have

$$E\left(f\big(E(\mathbf{1}_A \mid \mathcal{A})\big)\right) \leqslant E\left(E\big(f(E(\mathbf{1}_A \mid \mathcal{B})) \mid \mathcal{A}\big)\right) = E\big(f(E(\mathbf{1}_A \mid \mathcal{B})\big).$$

  It suffices to take the sum over all elements A of ξ to obtain the desired inequality.
- Set $\xi_1 = \{A_i\}$, $\xi_2 = \{B_j\}$, and $\eta = \{C_k\}$, in which case $\xi_1 \vee \xi_2 = \{A_i \cap B_j\}$ and $\xi_2 \vee \eta = \{B_j \cap C_k\}$. Set $\mu(A \mid B) = \mu(A \cap B)/\mu(B)$. We have

$$\begin{aligned}
H(\xi_2 \mid \eta) &= -\sum_{j,k} \mu(B_j \cap C_k) \, \log \mu(B_j \mid C_k) \\
&= -\sum_{i,j,k} \mu(A_i \cap B_j \cap C_k) \, \log \mu(B_j \mid C_k) \\
H(\xi_1 \mid \xi_2 \vee \eta) &= -\sum_{i,j,k} \mu(A_i \cap B_j \cap C_k) \, \log \mu(A_i \mid B_j \cap C_k) \\
H(\xi_1 \vee \xi_2 \mid \eta) &= -\sum_{i,j,k} \mu(A_i \cap B_j \cap C_k) \, \log \mu(A_i \cap B_j \mid C_k).
\end{aligned}$$

  It suffices to take the sum of the first two equalities to obtain the third. □

Definition 10.3 Let $(X, \mathcal{T}, \mu)$ be a probability space, let $T : X \to X$ be a measurable map that preserves the measure μ, and let ξ be a countable partition of X. The entropy of T relative to the partition ξ and the entropy of T are defined by

$$h_\mu(T, \xi) := \lim_{N \to \infty} \frac{1}{N} H\Big(\bigvee_{i=0}^{N-1} T^{-i}\xi\Big),$$

$$h_\mu(T) = \sup\{h_\mu(T, \xi) \mid \xi \text{ is a finite partition of } X\}.$$

The sequence $\frac{1}{n}H\left(\bigvee_{k=0}^{n-1} T^{-k}\xi\right)$ converges because it is nonincreasing. We prove this below.

10.3 Properties of Entropy

Here are several formulas concerning the entropy of a transformation.

Properties *Let ξ and η be two countable partitions, let $T : X \to X$ be a measurable map that preserves the measure μ, and let $\mathcal{A}$ and $\mathcal{B}$ be sub-σ-algebras of $\mathcal{T}$.*

- $H(\xi \mid \mathcal{A}) = H(T^{-1}\xi \mid T^{-1}\mathcal{A})$.
- $h_\mu(T, \eta) \leqslant h_\mu(T, \xi) + H(\eta \mid \xi)$.
- $h_\mu(T, \xi) = \lim_{n\to\infty} H(\xi \mid \bigvee_1^n T^{-i}\xi)$.
- *For every $n \in \mathbf{N}$, we have $h_\mu(T, \xi) = h_\mu(T, \bigvee_{i=0}^n T^{-i}\xi)$.*

Proof

- This follows from the relation $E_\mu(T^{-1}A \mid T^{-1}\mathcal{A}) = E_\mu(A \mid \mathcal{A}) \circ T$.
- We begin with the relation

$$H(\bigvee_0^n T^{-i}\eta) \leqslant H(\bigvee_0^n T^{-i}\eta \ \bigvee_0^n T^{-i}\xi) = H(\bigvee_0^n T^{-i}\xi) + H(\bigvee_0^n T^{-i}\eta \mid \bigvee_0^n T^{-i}\xi)$$

and then bound the last term as follows:

$$\begin{aligned} H(\bigvee_0^n T^{-i}\eta \mid \bigvee_0^n T^{-i}\xi) &\leqslant \sum_i H(T^{-i}\eta \mid \bigvee_0^n T^{-i}\xi) \\ &\leqslant \sum_i H(T^{-i}\eta \mid T^{-i}\xi) \\ &= (n+1)H(\eta \mid \xi). \end{aligned}$$

The desired inequality follows by dividing by $n+1$ and taking the limit.
- We also have

$$H(\bigvee_0^n T^{-i}\xi) = H(\bigvee_1^n T^{-i}\xi) + H(\xi \mid \bigvee_1^n T^{-i}\xi) = H(\bigvee_0^{n-1} T^{-i}\xi) + H(\xi \mid \bigvee_1^n T^{-i}\xi),$$

hence, by applying this relation recursively,

$$H(\bigvee_{k=0}^n T^{-k}\xi) = H(\xi) + \sum_{j=1}^n H(\xi \mid \bigvee_{k=1}^j T^{-k}\xi).$$

Since the sequence $H(\xi \mid \bigvee_{k=1}^n T^{-k}\xi)$ is nonincreasing, the same holds for the average $\frac{1}{n+1}H(\bigvee_{k=0}^n T^{-k}\xi)$, and these two sequences have the same limit.

- The last point is a consequence of the following equalities:

$$\begin{aligned}
h\big(T, \bigvee_{i=0}^{n} T^{-i}\xi\big) &= \lim \frac{1}{N} H\Big(\bigvee_{j=0}^{N-1} \bigvee_{i=0}^{n} T^{-i-j}\xi\Big) \\
&= \lim \frac{1}{N+n} H\Big(\bigvee_{k=0}^{N+n-1} T^{-k}\xi\Big) \\
&= h(T, \xi).
\end{aligned}$$

□

10.4 Generating Partitions

To compute the entropy of a transformation, we first construct partitions that are well suited to the transformation.

Definition 10.4 Let $(X, \mathcal{T}, \mu)$ be a probability space, let $T : X \to X$ be a measurable map that preserves the measure μ, and let ξ be a finite partition of X. The partition ξ is called *one-sided generating* or *a one-sided generator* if the σ-algebra generated by the elements of the partitions $T^{-i}\xi$ for $i \in \mathbf{N}$ and the negligible sets equals $\mathcal{T}$.

When the transformation is invertible, we speak of a generating partition or generator if the σ-algebra generated by the elements of the partitions $T^{-i}\xi$ for $i \in \mathbf{Z}$ and the negligible sets equals $\mathcal{T}$.

The explicit computation of the entropy is based on the following theorem, due to A. Kolmogorov and Y. Sinaï (1958).

Theorem 10.1 (Kolmogorov, Sinaï) *Let $(X, \mathcal{T}, \mu)$ be a probability space, let $T : X \to X$ be a measurable map that preserves the measure μ, and let ξ be a finite one-sided generator of X. Then*

$$h_\mu(T) = h_\mu(T, \xi) = H(\xi \mid T^{-1}\mathcal{T}).$$

Proof Let ξ_n be a sequence of countable partitions satisfying $\xi_{n+1}(x) \subset \xi_n(x)$ for every $n \in \mathbf{N}$ and every $x \in X$. Let $\mathcal{A}$ be the σ-algebra generated by the elements of all the ξ_n. Finally, let η be a finite partition of X. Let us show the convergence $H(\eta \mid \xi_n) \to H(\eta \mid \mathcal{A})$.

Let A be an element of η. By the martingale convergence theorem in L^2, recalled in Chap. 17, we have $E_\mu(\mathbf{1}_A \mid \xi_n) \to E_\mu(\mathbf{1}_A \mid \mathcal{A})$ in L^2 norm. We can find a subsequence n_i for which the convergence occurs almost everywhere. This implies

$$E_\mu(\mathbf{1}_A \mid \xi_{n_i}) \log\big(E_\mu(\mathbf{1}_A \mid \xi_{n_i})\big) \longrightarrow E_\mu(\mathbf{1}_A \mid \mathcal{A}) \log\big(E_\mu(\mathbf{1}_A \mid \mathcal{A})\big) \quad \text{a.e.}$$

The left-hand side is bounded by e^{-1}. By the dominated convergence theorem, we have

$$\int E_\mu(\mathbf{1}_A \mid \xi_{n_i}) \log\bigl(E_\mu(\mathbf{1}_A \mid \xi_{n_i})\bigr)\,\mathrm{d}\mu \longrightarrow \int E_\mu(\mathbf{1}_A \mid \mathcal{A}) \log\bigl(E_\mu(\mathbf{1}_A \mid \mathcal{A})\bigr)\,\mathrm{d}\mu.$$

Taking the sum over all elements A of η, we obtain $H(\eta \mid \xi_{n_i}) \to H(\eta \mid \mathcal{A})$. Since the sequence $H(\eta \mid \xi_n)$ is nonincreasing, we have the desired relation.

Consider a one-sided generating partition ξ.

- If we set $\xi_n = \bigvee_{i=1}^{n} T^{-i}\xi$, then

$$h_\mu(T,\xi) = \lim H(\xi \mid \xi_n) = H(\xi \mid T^{-1}\mathcal{T}).$$

- If we set $\xi_n = \bigvee_{i=0}^{n} T^{-i}\xi$, then $\lim H(\eta \mid \xi_n) = H(\eta \mid \mathcal{T}) = 0$.

By the properties established earlier, we also have

$$h_\mu(T,\eta) \leqslant h_\mu(T,\xi_n) + H(\eta \mid \xi_n) = h_\mu(T,\xi) + H(\eta \mid \xi_n).$$

We have just seen that the last term tends to 0 when n tends to infinity, so we have the inequality $h_\mu(T,\eta) \leqslant h_\mu(T,\xi)$ for every finite partition η. The desired equality $h_\mu(T) = h_\mu(T,\xi)$ follows. □

In the invertible case, we can use generating partitions to compute the entropy.

Theorem 10.2 *Let $(X,\mathcal{T},\mu)$ be a probability space, let $T : X \to X$ be a measurable map that preserves the measure μ, and let ξ be a generating finite partition of X. Then*

$$h_\mu(T) = h_\mu(T,\xi).$$

Proof The proof proceeds as in the one-sided case. Set

$$\xi_n = \bigvee_{i=-n}^{n} T^{-i}\xi.$$

By the definition of a generating partition, the ξ_n generate the σ-algebra $\mathcal{T}$. Consequently, we have

$$H(\eta \mid \xi_n) \longrightarrow H(\eta \mid \mathcal{T}) = 0.$$

By the properties of entropy, we also have the inequality

$$h_\mu(T,\eta) \leqslant h_\mu(T,\xi_n) + H(\eta \mid \xi_n).$$

It remains to verify the inequality $h_\mu(T,\xi_n) \leqslant h_\mu(T,\xi)$:

$$\begin{aligned}
h(T,\xi_n) &= \lim_{N\to\infty} \frac{1}{N} H\Big(\bigvee_{j=0}^{N-1} \bigvee_{i=-n}^{n} T^{-i-j}\xi\Big)\\
&= \lim \frac{1}{N} H\Big(\bigvee_{j=-n}^{N+n-1} T^{-j}\xi\Big)\\
&\leqslant \lim \frac{1}{N}\Big[H\Big(\bigvee_{j=0}^{N-1} T^{-j}\xi\Big)+H\Big(\bigvee_{j=-n}^{-1} T^{-j}\xi\Big)+H\Big(\bigvee_{j=N}^{N+n-1} T^{-j}\xi\Big)\Big]\\
&\leqslant h_\mu(T,\xi) + 0 + \lim \frac{1}{N} H\Big(T^{-N}(\bigvee_{j=0}^{n-1} T^{-j}\xi)\Big)\\
&\leqslant h_\mu(T,\xi) + \lim_{N\to\infty} \frac{1}{N} H\Big(\bigvee_{j=0}^{n-1} T^{-j}\xi\Big)\\
&\leqslant h_\mu(T,\xi).
\end{aligned}$$

We therefore have $h_\mu(T,\eta) \leqslant h_\mu(T,\xi)$ for every finite partition η. The result has been proved. □

10.5 Entropy and Isomorphisms

Definition 10.5 Let $(X_1,\mathcal{T}_1,\mu_1)$ and $(X_2,\mathcal{T}_2,\mu_2)$ be two probability spaces, let $T_1 : X_1 \to X_1$ be a measurable map that preserves μ_1, and let $T_2 : X_2 \to X_2$ be a measurable map that preserves μ_2. These two measure-preserving dynamical systems are called *isomorphic* if there exist two subsets $X_1' \subset X_1$ and $X_2' \subset X_2$, each with negligible complement, as well as a measurable bijection $\varphi : X_1' \to X_2'$, with measurable inverse, that conjugates T_1 and T_2 and sends μ_1 to μ_2:

$$\varphi \circ T_1 = T_2 \circ \varphi, \quad \varphi_* \mu_1 = \mu_2.$$

We will show that entropy is invariant under isomorphisms.

Proposition 10.1 *Two isomorphic measure-preserving dynamical systems have the same entropy.*

Proof Let $(X_1,\mathcal{T}_1,\mu_1)$ and $(X_2,\mathcal{T}_2,\mu_2)$ be two isomorphic systems; we use the notation of Definition 10.5. Let ξ be a finite partition of X_2; we restrict this partition to X_2' and consider its inverse image under φ. We obtain a partition of X_1' to which we add the element $X_1'^{\,c}$ in order to get a partition of X_1, which we will abusively denote by $\varphi^{-1}\xi$. This partition has the same entropy as ξ:

$$\begin{aligned}H_{\mu_1}(\varphi^{-1}\xi) &= \sum_{A\in\xi} \mu_1(\varphi^{-1}A)\log(\mu_1(\varphi^{-1}A))\\ &= \sum_{A\in\xi} \mu_2(A)\log(\mu_2(A))\\ &= H_{\mu_2}(\xi).\end{aligned}$$

Let us now use the commutativity of φ, T_1, and T_2:

$$\begin{aligned}h_{\mu_1}(T_1,\varphi^{-1}\xi) &= \lim \frac{1}{n} H_{\mu_1}\Big(\bigvee_{k=0}^{n-1} T_1^{-k}\varphi^{-1}\xi\Big) = \lim \frac{1}{n} H_{\mu_1}\Big(\bigvee_{k=0}^{n-1} \varphi^{-1}T_2^{-k}\xi\Big)\\ &= \lim \frac{1}{n} H_{\mu_1}\Big(\varphi^{-1}(\bigvee_{k=0}^{n-1} T_2^{-k}\xi)\Big) = \lim \frac{1}{n} H_{\mu_2}\Big(\bigvee_{k=0}^{n-1} T_2^{-k}\xi\Big)\\ &= h_{\mu_2}(T_2,\xi).\end{aligned}$$

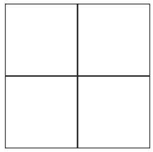
Next, we take the upper limit over all finite partitions ξ:

$$h_{\mu_2}(T_2) = \sup_{\xi}\{h_{\mu_2}(T_2,\xi)\} \leqslant \sup_{\eta}\{h_{\mu_1}(T_1,\eta)\} = h_{\mu_1}(T_1).$$

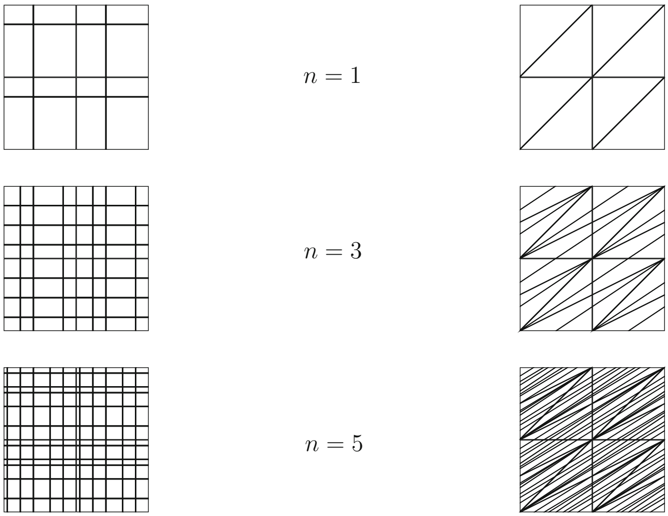
The inverse inequality can be obtained by exchanging the two dynamical systems.

$\square$

We consider the partition
$\xi = \{[0,1/2[^2, [0,1/2[\times [1/2,1[,$
$[1/2,1[\times [0,1/2[, [1/2,1[^2\}$
and its iterates $\bigvee_{k=0}^{n} T^{-k}\xi$
by the following maps:



$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+\sqrt{2} \\ y+\sqrt{3} \end{pmatrix} \qquad\qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



At the nth step, we obtain $(2n+2)^2$ pieces that are approximately of the same size for translation, while the toral automorphism gives 2^{n+2} pieces.

Fig. 10.1 Iteration of a partition of the torus

$\bigvee_{k=-7}^{7} T^{-k}\xi$

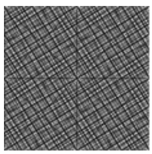


Fig. 10.2 Iteration of the partition ξ by the automorphism $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

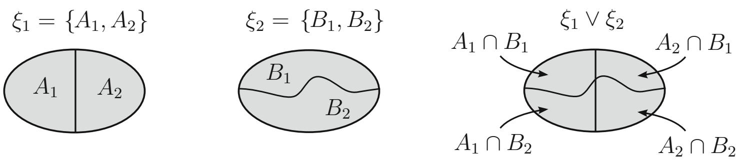


Fig. 10.3 Generating partitions

10.6 Exercises

10.6.1 Basic Exercises

From this point on, $(X, \mathcal{T}, \mu)$ is a probability space.

***Exercise* 1** Consider the map $T : [0,1) \to [0,1)$ given by $T(x) = 2x \bmod 1$. This map preserves the Lebesgue measure. Let $\xi = \{[0,1/2), [1/2,1)\}$.

- Compute $T^{-1}\xi$, and then $\xi \vee T^{-1}\xi$.
- Compute $H(\xi)$, and then $H(\xi \vee T^{-1}\xi)$.
- Prove the following formula by induction:

$$\bigvee_0^{n-1} T^{-i}\xi = \left\{ \left[\frac{\ell - 1}{2^n}, \frac{\ell}{2^n} \right) \mid \ell \in \{1, \dots, 2^n\} \right\}.$$

- Compute $H(\bigvee_0^{n-1} T^{-i}\xi)$ and $h(T, \xi)$.
- Show that the partition ξ is generating. Deduce $h(T)$.

***Exercise* 2** Let ξ be a finite partition of X. Prove the inequality $H(\xi) \leqslant \log\big(\mathrm{Card}(\xi)\big)$. Show that we have equality if and only if all elements of ξ have the same measure.

***Exercise* 3** Let $(X, \mathcal{T}, \mu)$ be a probability space, let $T : X \to X$ be a measurable map that preserves μ, and let ξ be a countable partition of X. Let k be a nonnegative integer. Prove the following equalities: $h_\mu(T^k, \xi) = k h_\mu(T, \xi)$ and $h_\mu(T^k) = k h_\mu(T)$.

***Exercise* 4** Let ξ and η be two finite partitions of X. Set $d(\xi, \eta) = H(\xi \mid \eta) + H(\eta \mid \xi)$. Let ξ_1, ξ_2, and ξ_3 be three finite partitions of X. Prove the triangle inequality

$$d(\xi_1, \xi_3) \leqslant d(\xi_1, \xi_2) + d(\xi_2, \xi_3).$$

***Exercise* 5** Let $(X, \mathcal{T}, \mu)$ be a nonatomic probability space: $\mu(\{x\}) = 0$ for all $x \in X$. Show that the identity map does not admit a generating partition. Generalize to the case of a transformation for which all points are periodic.

***Exercise* 6** Consider a measurable map $T : X \to X$ that preserves the measure μ. Let ξ be a finite one-sided generating partition of X. Suppose that T is invertible. Show that $h_\mu(T) = 0$.
Hint: What is the σ-algebra $T^{-1}\mathcal{T}$? What is its relation to the entropy?

10.6.2 More Advanced Exercises

***Exercise* 7** Let $(X, \mathcal{T}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be two probability spaces, and let $T : X \to X$ and $S : Y \to Y$ be two measurable maps that preserve the measure. Suppose that these two dynamical systems are isomorphic. Show that they have the same entropy.

***Exercise* 8** Let ξ and η be two countable partitions of X. Show that $H(\xi \mid \eta) = 0$ if and only if for almost all $x \in X$, we have $\eta(x) \subset \xi(x)$.

Suppose $H(\xi) < \infty$ and $H(\eta) < \infty$. Show that the equality $H(\xi \vee \eta) = H(\xi) + H(\eta)$ is satisfied if and only if the partitions ξ and η are independent:

$$\forall A \in \xi,\ \forall B \in \eta, \quad \mu(A \cap B) = \mu(A)\mu(B).$$

***Exercise* 9** In the proof of the Kolmogorov–Sinaï theorem, we saw that if an increasing sequence of partitions ξ_n generates $\mathcal{T}$, then $H(\eta \mid \xi_n)$ converges to 0 for every finite partition η. Show the converse.
Hint: Begin by showing that the entropy of a partition of the form $\{A, A^c\}$ given the σ-algebra generated by all the ξ_n is 0.

10.7 Comments

The sequence $a_n = H(\vee_{k=0}^{n} T^{-k}\xi)$ satisfies the inequality $a_{n+m} \leqslant a_m + a_n$. We call it subadditive. The convergence of the sequence a_n/n could therefore also have been deduced from the following elementary lemma: *Let $a_n \in \mathbf{R}$ be a sequence satisfying $a_{n+m} \leqslant a_m + a_n$. Then the sequence a_n/n admits a limit in $[-\infty, \infty)$, and this limit coincides with* $\inf\{a_n/n\}$. We could have defined the entropy by taking the upper limit over all countable partitions with finite entropy. We would have obtained the same value for $h_\mu(T)$. The Kolmogorov–Sinaï theorem also holds for all countable generating partitions, but we must then justify the use of the dominated convergence theorem. To do this, we can use the following lemma, due to K.L. Chung (1961), and proved again in the book by Parry [18, Chap. 2]: *Let ξ be a countable partition, and let $\{\xi_n\}_{n\in\mathbf{N}}$ be an increasing sequence of countable partitions. Then*

$$\int \sup_n I(\xi|\xi_n)\,\mathrm{d}\mu \leqslant H(\xi) + 1.$$

When can a system with positive entropy be called random? This certainly applies for Bernoulli shifts, which model rolling a die or flipping a coin. In 1964, Y.G. Sinaï showed that every ergodic measure-preserving dynamical system with positive entropy admits a quotient isomorphic to a Bernoulli shift with the same entropy. This result allows us to interpret entropy in probabilistic terms: a system has entropy greater than or equal to $-\sum p_i \log p_i$ if and only if there exists a partition $\{A_1, \ldots, A_n\}$ of the space satisfying $\mu(A_i) = p_i$ and such that the $X_n(x) = \sum i\,\mathbf{1}_{A_i}(T^n(x))$ form a sequence of independent, identically distributed random variables. For example, a system has entropy greater than $\log(6)$ if and only if it allows us to simulate rolling a six-sided die.

The first regular systems to have been classified using entropy are the ergodic automorphisms of the torus $\mathbf{T}^2$, by R.L. Adler and B. Weiss (1967). Two such systems are isomorphic if and only if they have the same entropy. In particular, there exist automorphisms of $\mathbf{T}^2$ that are isomorphic without being algebraic conjugates, for example $\left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 5 & 4 \\ 1 & 1 \end{smallmatrix}\right)$. The isomorphism is constructed using a Markov partition.
In 1970, D. Ornstein showed that two two-sided Bernoulli shifts are isomorphic if and only if they have the same entropy. This result extends to mixing shifts of finite type. Y.G. Sinaï, R. Bowen, and M. Ratner then showed, using Markov partitions, that most uniformly hyperbolic systems are isomorphic to Bernoulli shifts; examples are C^2 diffeomorphisms derived from Anosov preserving the Lebesgue measure and C^1 Axiom A diffeomorphisms preserving a mixing Gibbs measure. Pesin theory allows us to extend these results to certain classes of nonuniformly hyperbolic systems. Measure-theoretically, the behavior of these systems is highly unpredictable.
Classifying noninvertible transformations up to isomorphism is more delicate. The only isomorphisms between two two-sided Bernoulli shifts are obtained by permuting the symbols. In 2001, B. Marcus and S. Tuncel gave a classification of the two-sided shifts of finite type. A rational transformation of the Riemann sphere has a unique probability measure with maximal entropy: in 2002, D. Heicklen and C. Hoffman showed that this system is isomorphic to a Bernoulli shift.
A measure-preserving dynamical system can have infinite entropy: Consider a sequence p_i of real numbers in $[0, 1]$ satisfying $\sum p_i = 1$ and $-\sum p_i \log p_i = \infty$. The sequence of the p_i defines a probability measure μ on the set of natural numbers $\mathbf{N}$, and the shift on the product space $\mathbf{N}^{\mathbf{Z}}$ has infinite entropy relative to the product measure $\mu^{\otimes \mathbf{Z}}$. We can, however, show that a Lipschitz homeomorphism with Lipschitz constant K, defined on a finite-dimensional compact metric space and preserving a Borel probability measure, has finite entropy, bounded from above by $\dim(X) \log^+ K$ (A.G. Kouchnirenko, 1965). In particular, a C^1 diffeomorphism on a compact manifold preserving a Borel probability measure has finite entropy.
In 1970, W. Krieger proved that every ergodic invertible transformation of a Lebesgue space that preserves the measure and has finite entropy admits a one-sided generating partition with at most $e^{h_\mu(T)} + 1$ elements. The ergodic invertible transformations with strictly positive entropy do not have one-sided generating partitions.
In 1977, D.A. Lind and J.P. Thouvenot showed that every ergodic transformation with finite entropy of a Lebesgue space is isomorphic to a toral homeomorphism preserving the Lebesgue measure. The existence of C^1 realizations for transformations with finite entropy remains an open question.

Chapter 11
Entropy and Information Theory

Von Neumann told me, "You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one really knows what entropy really is, so in a debate you will always have the advantage."

C.E. Shannon (1916–2001)

11.1 Introduction

The term entropy was first used by R. Clausius in 1865, in the setting of his research on heat. The underlying concept would play a crucial role in the development of thermodynamics and statistical mechanics with the work of J.W. Gibbs and L. Boltzmann at the end of the nineteenth century. It was, however, not these two theories that inspired A.N. Kolmogorov when he introduced a new invariant called "entropy" to study dynamical systems, but rather the work of C.E. Shannon on information theory.

In a well-known paper published in 1948, marking the birth of information theory, C.E. Shannon introduced a quantity meant to quantify the information lost in telephone transmissions when there is static on the line.

The following experiment shows how we can understand entropy from the point of view of information theory. Consider an information source that produces a value or result belonging to a set of n symbols $x_1, \ldots, x_n$ with respective probabilities $p_1, \ldots, p_n$. We wish to determine the result produced by the information source by asking yes-no questions, of the type "Is the result equal to x_1?" or "Does the result belong to such and such a subset?". Set $H = -\sum p_i \log p_i$, where the logarithm is in base 2. C.E. Shannon showed that the average number of questions necessary lies between H and $H + 1$ provided that the choice of the questions is optimal.

In Chap. 10, we introduced the concept of partition and associated information function. What is the connection with the question we just stated? First, note that

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a question partitions the sample space into two subsets. A sequence of questions therefore gives a sequence of partitions of our space. If this sequence of questions is able to distinguish between all possible results, this means that the generated partition is the partition into singletons, where $\xi(x) = \{x\}$ for all x. The entropy of this partition is exactly equal to H.

The value of H manifests as the average amount of information needed to determine the result produced by the source. Later on, we will explain how to quantify this concept of information and treat a number of concrete examples. We will compute explicitly the entropy H when the source produces a sequence of mutually independent symbols or when the probability of a symbol depends only on the symbol that precedes it (Markov case).

11.2 The Notion of Information

Alice is informed of the result of a random experiment. Bob wants to determine this result and asks Alice to give him information. Alice allows Bob to ask one question, which she will answer with "yes" or "no".

Bob asks his question, and Alice gives a positive answer. How much information has Bob received?

Let us try to understand the value of this information. We first note that Bob's question partitions the sample space Ω into two subsets: on the one hand, the results that lead to a positive answer from Alice and on the other hand, those that lead to a negative answer.

The amount of information received by Bob depends on the probability p of obtaining a positive answer to his question; we denote this amount by $I(p)$. Next, suppose that Alice carries out the experiment twice and that the answer to the first question is positive in both cases, and let us state the following postulate.

The value of the information provided by the two results obtained in an independent manner is equal to the sum of the amount of information associated with each of the results.

The amount of information obtained by Bob is therefore equal to twice the amount that would have resulted from a positive answer to a single execution of the experiment. On the other hand, the probability of obtaining a positive answer twice is equal to p^2. So we have $I(p^2) = 2I(p)$. It follows that $I(p^m) = mI(p)$ when we repeat the experiment m times. If the function $p \mapsto I(p)$ is continuous on $(0, 1)$, this leads to the equality $I(p^x) = xI(p)$ for every real number x. By convention, we set $I(\frac{1}{2}) = 1$, which gives $I(y) = -\log_2(y)$.

Let us compute the average amount of information given by Alice's answer: the answer is positive with probability p, in which case the amount of information received is equal to $\log_2 p$; the answer is negative with probability $1-p$, in which case the amount of information received is equal to $\log_2(1-p)$. We therefore have

$$H(p) = -p\log_2 p - (1-p)\log_2(1-p).$$

Denote by $\xi = \{yes, no\}$ the partition of Ω associated with Alice's question, by x the result of the random experiment, and by μ the probability measure defined on Ω. By the definition of p, we have $p = P(yes) = \mu(\xi(x))$, and we recover the usual formula for the entropy of a partition with two elements. Let us now treat a concrete example Sect. 11.3.

11.3 The Game of Questions and Answers

Alice rolls two six-sided dice and takes the sum of the outcomes (Table 11.1). Bob tries to guess the result by asking questions to which Alice replies with "yes" or "no".

Bob asks, for example, whether the result is greater than or equal to 7, then whether it is even. The respective answers to these questions are "no" and "yes". He then asks whether the result is equal to 6, and after receiving a negative answer, whether it is equal to 2. This is summarized in Table 11.2. We can calculate explicitly the information given by Alice's replies. Denote Alice's result by x, and by $\xi_1,\ldots, \xi_n$ the partitions associated with Bob's successive questions. The information Bob obtains from the answers to questions 1 through n is equal to $I(\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n)(x)$; it can be found in the penultimate column of the table.

The information gain given by answer n is $I(\xi_n \mid \xi_1 \vee \cdots \vee \xi_{n-1})(x)$, that is, $-\log_2 \mu(\xi_n(x) \mid \xi_1 \vee \cdots \vee \xi_{n-1}(x))$. This is the difference between $I(\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n)(x)$ and $I(\xi_1 \vee \xi_2 \vee \cdots \vee \xi_{n-1})(x)$; it can be found in the last column of the table.

To guess the result x, Bob must obtain a total gain of information equal to $-\log_2\big(P(\{x\})\big)$. We can follow his progress in the table. For example, the reply to question 3, "*Is the result equal to* 6*?*", is rather favorable (information greater than 1), even if on average such a question brings little information in this context (the average relative information is equal to 0.25). After having asked question 4, "*Is the result equal to* 2*?*", Bob has enough information to guess the number. Indeed, 4 is the only result that induces the series of answers "no-yes-no-no" to his questions, and in fact, Bob has reached the necessary amount of information: $\log_2(12) \simeq 3.58$.

11.4 Information and Markov Chains

Here is another example from probability theory. Let $X_0, X_1, \ldots, X_n, \ldots$ be a sequence of stationary random variables on a probability space $(\Omega, \mathcal{T}, P)$, with

Table 11.1 Probability distribution of the sum of the outcomes of rolling two six-sided dice

Result x	2	3	4	5	6	7	8	9	10	11	12
Probability p	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
Information	5.17	4.17	3.58	3.17	2.85	2.58	2.85	3.17	3.58	4.17	5.17

Table 11.2 List of questions and answers ($x = 4$)

i	Question	An element of ξ_i	Cumulative entropy	Gain	Answer	Cumulative information	Gain
1	$\geqslant 7$?	$\{7, 8, 9, 10, 11, 12\}$	0.98	0.98	No	1.26	1.26
2	Even?	$\{2, 4, 6, 8, 10, 12\}$	1.96	0.98	Yes	2	0.74
3	6?	$\{6\}$	2.21	0.25	No	3.17	1.17
4	2?	$\{2\}$	2.3	0.09	No	3.58	0.41

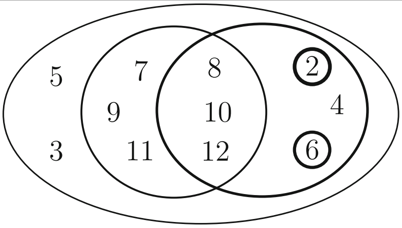
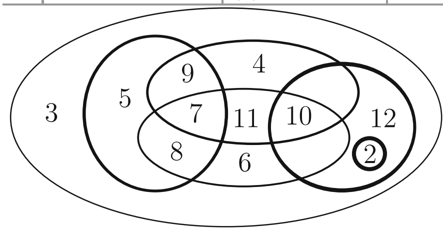


Table 11.3 List of questions and answers ($x = 12$)

i	Question	An element of ξ_i	Cumulative entropy	Gain	Answer	Cumulative information	Gain
1	6, 7, 8, 10, or 11?	$\{6, 7, 8, 10, 11\}$	0.98	0.98	No	1.26	1.26
2	4, 7, 9, 10, or 11?	$\{4, 7, 9, 10, 11\}$	1.98	1	No	2.16	0.9
3	5, 7, 8, or 9?	$\{5, 7, 8, 9\}$	2.97	0.99	No	3.18	1.02
4	2, 10, or 12?	$\{2, 10, 12\}$	3.22	0.25	Yes	4.17	0.99
5	2?	$\{2\}$	3.28	0.06	No	5.17	1



values in the finite set $\mathcal{A} = \{1, \ldots, N\}$. Suppose that we know the values of the X_i for $i \geqslant 1$. What amount of information do we need, on average, to know the value of X_0?

We may assume that the space Ω on which the X_i are defined is equal to $\mathcal{A}^{\mathbf{N}}$, where the random variable X_i corresponds to the projection onto the coordinate i. We denote by $\sigma : \Omega \to \Omega$ the shift to the left. Let us consider the partitions ξ_n of Ω defined by $\xi_n = \{(X_n = a) \mid a \in \mathcal{A}\}$; note that $\xi_n = \sigma^{-n}\xi_0$. Let $H(X_0 \mid X_1, \ldots, X_n)$ be the average amount of information needed to know the value of X_0 if we know that of the X_i for $i = 1, \ldots, n$. We have the equality

$$H(X_0 \mid X_1, \ldots, X_n) = H(\xi_0 \mid \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n) = H\big(\xi_0 \mid \bigvee_{i=1}^{n} \sigma^{-i}\xi_0\big).$$

This amount converges to the entropy of the shift $h(\sigma, \xi_0)$. When $\Omega = \mathcal{A}^{\mathbf{N}}$, the partition ξ_0 is generating because the elements of the partition $\bigvee_0^n \sigma^{-i}\xi_0$ are the cylinder sets of length $n + 1$. The entropy of the shift can therefore be seen as the average amount of information needed to know the "current" value X_0 if we know the "past" values X_i of the process for $i \geqslant 1$. Let us carry out the calculation when the X_i form a Markov chain.

Proposition 11.1 *Let $\mathcal{A}$ be a finite set; the shift $\sigma : \mathcal{A}^{\mathbf{N}} \to \mathcal{A}^{\mathbf{N}}$ is defined by the formula $\sigma(\{x_i\}_{i\in\mathbf{N}}) = \{x_{i+1}\}_{i\in\mathbf{N}}$. For $i, j \in \mathcal{A}$, we consider elements p_i, $p_{i,j}$ of $[0, 1]$ satisfying $\sum_i p_i = 1$, $\sum_j p_{i,j} = 1$, and $\sum_i p_i p_{i,j} = p_j$. Denote by P the probability satisfying*

$$P\big(\{\{x_i\}_{i\in\mathbf{N}} \mid x_0 = i_0, \ldots, x_n = i_n\}\big) = p_{i_0} p_{i_0,i_1} p_{i_1,i_2} \cdots p_{i_{n-1},i_n}.$$

The entropy of σ with respect to P is given by

$$h_P(\sigma) = -\sum_{i,j} p_i p_{i,j} \log_2 p_{i,j}.$$

Proof By virtue of the equalities $P(X_k = j \mid X_{k+1} = i) = p_{i,j}$ and $p_i = P(X_1 = i)$ and the Markov property, we have

$$\begin{aligned} H(X_0 \mid X_1, \ldots, X_n) &= H(X_0 \mid X_1) \\ &= -\sum_{i,j} P(X_0 = j, X_1 = i) \log_2 P(X_0 = j \mid X_1 = i) \\ &= -\sum_{i,j} p_i p_{i,j} \log_2 p_{i,j}. \end{aligned}$$

□

As a corollary, we deduce that a Bernoulli shift on an alphabet with n symbols with respective probabilities $p_1, \ldots, p_n$ has entropy $-\sum p_i \log_2 p_i$. For example, the Bernoulli shift corresponding to flipping a fair coin ($p_1 = p_2 = \frac{1}{2}$) has entropy equal to $\log_2 2 = 1$. The Bernoulli shift corresponding to rolling a six-sided die ($p_1 = \cdots = p_6 = \frac{1}{6}$) has entropy $\log_2 6$.

In Chap. 10, we saw that two measure-preserving dynamical systems with different entropies cannot be isomorphic.

Corollary 11.1 *Two Bernoulli shifts with different entropies are not isomorphic. In particular, the Bernoulli shift with probability vector* $(\frac{1}{2}, \frac{1}{2})$ *is not isomorphic to the Bernoulli shift with probability vector* $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$.

11.5 Interpretation in the Dynamical Setting

Consider a transformation T that admits a one-sided generating partition ξ, in the sense that the elements of the $T^{-i}\xi$ for $i \in \mathbf{N}$ generate the σ-algebra of measurable sets. Let us try to interpret the entropy $h(T)$ of T in terms of information. We saw in Chap. 10 that this entropy is given by the formula $h(T) = h(T, \xi) = H(\xi \mid T^{-1}\mathcal{T})$; it corresponds to the average amount of information needed to determine to which element of ξ the point x belongs if we know the positions of the iterates $T^i(x)$ in the partition ξ for $i \geqslant 1$.

Under very general hypotheses, we can show that ξ is generating if and only if the sequence $\xi(T^i(x))$ for $i \geqslant 0$ determines x uniquely if x belongs to a certain well-chosen set of full measure. To formalize this result, we need preliminaries on measure theory that are the object of the last part of this book.

Proposition 11.2 *Let* $(X, \mathcal{T}, \mu)$ *be a Lebesgue space, and let* $T : X \to X$ *be a measurable map that preserves the measure* μ. *A finite partition* ξ *is one-sided generating if and only if there exists a set* $X_0 \subset X$ *of full measure such that the map*

$$\begin{aligned} X &\longrightarrow \xi^{\mathbf{N}} \\ x &\longmapsto \{\xi(T^i(x))\}_{i \in \mathbf{N}} \end{aligned}$$

restricted to X_0 *is injective.*

In other words, once the points of X_0^c have been dismissed, the position of the point x is fixed if we know the list of elements of ξ to which x and its iterates belong. This result will be proved in Chap. 15. We use the notion of Lebesgue space, a notion that will be explained in Part IV of this book. All probability spaces one comes across in practice are Lebesgue spaces. For example, every Borel space of a complete metric space, endowed with a Borel probability measure, is a Lebesgue space.

Using this result, we can interpret the entropy of a transformation in terms of information. Knowing the position of all the iterates $T^i(x)$ for $i > 0$ with respect to the generating partition ξ corresponds to knowing the point $T(x)$, and the only thing missing to determine the point x completely is the position of x with respect to the partition itself. The entropy $h(T)$ can therefore be seen as the average amount of information needed to know x if we know $T(x)$.

If the transformation T is invertible, the point x is completely determined by the knowledge of $T(x)$, and the amount of information needed to know x if we know $T(x)$ is 0. The entropy of an invertible transformation with a one-sided generating partition is 0. This does not mean that all invertible transformations have entropy 0, rather that in general, these transformations do not have one-sided generating partitions, whence the need to turn to generating partitions to calculate their entropy.

11.6 Exercises

11.6.1 Basic Exercises

***Exercise* 1** Alice rolls two six-sided dice and takes the sum of the outcomes. She agrees to answer Bob's questions about the value of the sum with "yes" or "no".

Can Bob be certain to guess the correct value using only three questions? What is the minimal number of questions he must ask to be certain to conclude regardless of the result? Repeat this exercise for three and then four dice.

***Exercise* 2** Alice rolls two six-sided dice and Bob tries to guess the sum of the outcomes. He is allowed to ask four questions.

Alice's first three answers have given him an amount of information equal to 3.17, and he only has one question left.

- Can the result be 6?
- Can the result be 4?
- If so, which question should Bob ask?

Recall that $\log_2(3) = 1.58$ *and* $\log_2(5) = 2.32$.

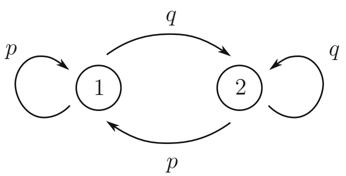
***Exercise* 3** Let $X_0, X_1, \ldots, X_n, \ldots$ be a stationary Markov sequence of random variables. Show the inequality $H(X_n \mid X_0) \leqslant H(X_{n+1} \mid X_0)$.

***Exercise* 4** We roll two n-sided dice and take the sum of the outcomes. Calculate the entropy of the system obtained by repeating this experiment independently. Compare this with the entropy associated with the independent repetition of a uniformly distributed experiment on a set with $2n - 1$ elements.

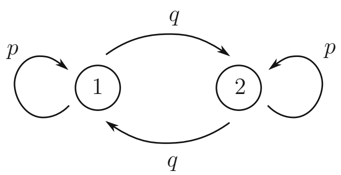
***Exercise* 5** Show that the two-sided Bernoulli shift (where "two-sided" means indexed by $\mathbf{Z}$) on an alphabet with three symbols with respective probabilities $\frac{1}{3}$,

$\frac{1}{3}$, and $\frac{1}{3}$ is not isomorphic to the Bernoulli shift on an alphabet with two symbols with respective probabilities $\frac{1}{2}$ and $\frac{1}{2}$.

***Exercise* 6** Let $p, q \in [0, 1]$ satisfy $p+q = 1$. Compute the entropy of the following Markov chains:



Transition matrix: $\begin{pmatrix} p & q \\ p & q \end{pmatrix}$



Transition matrix: $\begin{pmatrix} p & q \\ q & p \end{pmatrix}$

11.6.2 More Advanced Exercise

***Exercise* 7** We consider a random experiment with n possible outcomes with respective probabilities $p_1, \dots, p_n$. Bob tries to guess the outcome of the experiment using only yes-no questions. Show that the minimal number of questions Bob needs to ask to be certain to conclude regardless of the result is always greater than the entropy $-\sum p_i \log_2 p_i$. Show that equality is possible only if all outcomes of the experiment have the same probability. The number of possible outcomes must then be a power of 2.

11.7 Comments

Let us return to the case of a six-sided die. Consider a set of questions $\xi_1, \dots, \xi_n$ that allows us to conclude regardless of the outcome: $\xi_1 \vee \cdots \vee \xi_n(x) = \{x\}$, that is, $H(\xi_1 \vee \cdots \vee \xi_n) = -\sum p_i \log p_i = 3.27$. For some outcomes, it is not necessary to ask n questions to conclude. For example, for the questions in Table 11.3, it suffices to ask the first three to find the result if it is $4, 5, 6, 7, 8$, or 9.

On average, what is the number of questions truly asked to find the outcome? In his fundamental paper of 1948, C.E. Shannon showed that this average number is always greater than the entropy. In 1952, D. Huffman proposed an algorithm to construct a sequence of questions that minimizes the average number of questions that need to be asked. For the rolling of two dice, Table 11.3 was obtained using this algorithm. The average number of questions truly asked is 3.306; this is optimal. The compression methods jpeg, mp3, and pkzip use this algorithm by D. Huffman.

Note that at least four questions need to be asked to distinguish between all outcomes. Indeed, three questions partition the set of outcomes into at most $2^3 = 8$ parts, whereas there are 11 different outcomes.

We can try to determine a set of four questions such that the average number of questions that need to be asked out of this set of four is minimal. This can be obtained using a "numismatic" algorithm, which gives the questions "6, 7, 8, or 9?"; "4, 5, 7, 8, or 10?"; "3, 5, 6, 7, or 11?"; and "2, 3, or 4?". The average number of questions needed is 3.333.
The entropy associated with rolling two n-sided dice is given by

$$H = -\sum p_i \log p_i = 2\log n - \frac{1}{n^2}\Big(\sum_{i=1}^{n-1} 2i \log i + n\log n\Big) \sim \log(n) + \frac{1}{2} + o(1/n).$$

The reader may want to compare this with the entropy of the uniform distribution on a set with $2n-1$ elements: $H = \log(2n-1) \sim \log(n) + \log(2) + o(1)$.

Chapter 12
Computing Entropy

L'entropie est une loi générale de l'univers: la tendance naturelle des choses à passer de l'ordre au désordre sous l'effet d'un hasard calculable.

F. Jacob

12.1 Introduction

In general, computing the entropy of a measure-preserving transformation is a delicate problem. We will study a class of noninvertible maps for which the computation does not present too many problems.

A map T on a metric space X is called *uniformly dilating* if there exists a finite partition $\{X_i\}$ of X such that the restriction of T to each of the X_i dilates the metric in the following sense: there exists a constant $K > 1$ such that for every i,

$$\forall x,\ y \in X_i, \quad d\big(T(x), T(y)\big) \geqslant K\, d(x, y).$$

We have already come across several examples of dilating maps: the toral automorphisms whose eigenvalues all have absolute value greater than 1 and the piecewise C^1 maps on the interval whose derivatives are greater than some constant $K > 1$ satisfy this property. A few dilating transformations on the interval are shown in Fig. 12.1.

For these maps, we can give an explicit expression for the entropy of invariant probability measures. This expression uses the dilation factor of the measure under the action of the transformation. From an informal point of view, we could say that the inherent randomness of the system is proportional to this dilation factor.

This factor is easily computed when the transformation is regular and preserves an invariant measure that is absolutely continuous with respect to the Lebesgue measure. It can be obtained through a simple change of variables. The entropy then equals the integral of the logarithm of the Jacobian of the transformation. We thus link a measurable quantity, defined globally and measuring the uncertainty in the evolution of the system over time, to a quantity obtained by averaging the infinitesimal dilation observed in the neighborhood of each point of the space.

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The Bernoulli shifts defined on a finite alphabet I can be interpreted in terms of a dilating map. More generally, we can introduce a distance on the set of admissible sequences of a Markov chain with finite state space, for which the shift is dilating. It then becomes possible to compute the shift's entropy by evaluating its Jacobian. This computation method is not the most elementary one, but it illustrates well the concept of a dilating map.

12.2 The Rokhlin Formula

Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map that preserves the measure μ. Suppose that T restricted to a measurable set $A \subset X$ is injective. We define the inverse of the Jacobian of T on A by the formula

$$\frac{1}{|T'_\mu|} = \frac{\mathrm{d}}{\mathrm{d}\mu}\big(T_*(\mu_{|A})\big) \circ T,$$

so that we have the usual change of variables:

$$\int_A g \circ T \,\mathrm{d}\mu = \int_{TA} \frac{g}{|T'_\mu| \circ T^{-1}_{|A}} \,\mathrm{d}\mu.$$

Proposition 12.1 *Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map that preserves the measure μ. Suppose that there exists a finite partition X_i, for $1 \leqslant i \leqslant k$, such that the TX_i are measurable and $T : X_i \to TX_i$ is bijective with measurable inverse. Then*

$$\textit{for almost all } x_0 \in X, \quad E(f \mid T^{-1}\mathcal{T})(x_0) = \sum_{Tx = Tx_0} \frac{f(x)}{|T'_\mu(x)|},$$

$$H(\{X_i\}_{i=1,\dots,k} \mid T^{-1}\mathcal{T}) = \int_X \log |T'_\mu| \,\mathrm{d}\mu.$$

Proof Denote by $T_i^{-1} : TX_i \to X_i$ the inverse of T on the domains in question; we extend this function to X arbitrarily. Note that for all $x_0 \in X_i$, we have

$$\{x \mid Tx = Tx_0\} = \{T_i^{-1}(Tx_0) \mid i \text{ such that } T(x_0) \in T(X_i)\}.$$

To compute the conditional expectation, we write $f = \sum_i \mathbf{1}_{X_i} f \circ T_i^{-1} \circ T$, which gives

$$E(f \mid T^{-1}\mathcal{T}) = \sum_i E(\mathbf{1}_{X_i} \mid T^{-1}\mathcal{T}) f \circ T_i^{-1} \circ T.$$

It therefore suffices to prove the equality $E(\mathbf{1}_{X_i} \mid T^{-1}\mathcal{T}) = \dfrac{\mathbf{1}_{TX_i} \circ T}{|T'_\mu| \circ T_i^{-1} \circ T}$:

$$\begin{aligned}\int g \circ T\, E(\mathbf{1}_{X_i} \mid T^{-1}\mathcal{T})\, \mathrm{d}\mu &= \int_{X_i} g \circ T\, \mathrm{d}\mu \\ &= \int \frac{\mathbf{1}_{TX_i}\, g}{|T'_\mu| \circ T_i^{-1}}\, \mathrm{d}\mu \\ &= \int \frac{\mathbf{1}_{TX_i} \circ T\, g \circ T}{|T'_\mu| \circ T_i^{-1} \circ T}\, \mathrm{d}\mu.\end{aligned}$$

It remains to compute the entropy of the partition $\{X_i\}$:

$$\begin{aligned}H(\{X_i\} \mid T^{-1}\mathcal{T}) &= \sum_i \int -\mathbf{1}_{X_i} \log E(\mathbf{1}_{X_i} \mid T^{-1}\mathcal{T})\, \mathrm{d}\mu \\ &= \sum_i \int_{X_i} \log |T'_\mu| \circ T_i^{-1} \circ T\, \mathrm{d}\mu \\ &= \sum_i \int_{X_i} \log |T'_\mu|\, \mathrm{d}\mu \\ &= \int_X \log |T'_\mu|\, \mathrm{d}\mu. \qquad \square\end{aligned}$$

The following formula, due to V. Rokhlin, will allow us to calculate the entropy of maps that are piecewise dilating.

Corollary 12.1 *The entropy is bounded from below by the integral of the Jacobian:*

$$h_\mu(T) \geqslant \int_X \log |T'_\mu|\, \mathrm{d}\mu,$$

with equality if the partition $\{X_i\}$ is a one-sided generator.

Remarks

- Suppose that we have $X \subset \mathbf{R}^n$, that μ is the Lebesgue measure, that T is C^1 on the interior of X_i (Lipschitz suffices), and that $\mu(\partial X_i) = 0$. Then $|T'_\mu| = |\det(DT)|$ almost everywhere.
- The quantity $E(\mathbf{1}_{X_i} \mid T^{-1}\mathcal{T})(x)$ can be seen as the probability that x is in X_i, given the value of $T(x)$.
- If T has a one-sided generator, then $h(T)$ can be seen as the average amount of information needed to know x, given that Tx is known.
- The operator $L_\mu f(y) = \sum_{Tx=y} f(x)/|T'_\mu(x)|$ is the *transfer operator* associated with T; it is the adjoint of the isometry $f \mapsto f \circ T$ defined on $L^2(X, \mu)$.

12.3 Entropy of Shifts

Let us compute the entropy of the shift on a Markov chain using the observations above. Let I be a finite alphabet. Let $X = I^{\mathbf{N}}$, and consider the shift on X given by $T(\{x_i\}_{i\in\mathbf{N}}) = \{x_{i+1}\}_{i\in\mathbf{N}}$.

We define a measure on X using a transition matrix $\{p_{i,j}\}$. For every $i, j \in I$, we take real numbers $p_{ij} \in [0, 1]$ and $p_i \in [0, 1]$ satisfying

$$\sum_i p_i = 1, \quad \sum_j p_{ij} = 1, \quad \text{and} \quad \sum_i p_i p_{ij} = p_j.$$

Let $a_1, \ldots, a_n$ be elements of I. Recall that the cylinder set $[a_1, \ldots, a_n] \subset X$ consists of the elements of X that begin with the sequence $a_1, \ldots, a_n$. By the Kolmogorov extension theorem, there exists a probability measure μ on X that satisfies

$$\mu([a_1, a_2, \ldots, a_n]) = p_{a_1} p_{a_1 a_2} \cdots p_{a_{n-1} a_n}.$$

This measure is T-invariant. The entropy of the transformation T relative to this measure is given by the following proposition.

Proposition 12.2 $h_\mu(T) = -\sum_{i,j} p_i\, p_{ij}\, \log p_{ij}$.

Proof Let N be the cardinality of I. Each point of X has exactly N preimages and T is bijective from $[i]$ to X. We therefore take $X_i = [i]$. The partition $\{[i] \mid i \in I\}$ is a generator because the partition generated by its first k inverse images under T consists of all cylinder sets of length k. Let us calculate the Jacobian of $T : [i] \to X$ restricted to the cylinder set $[i, j]$, when it has nonzero measure:

$$\mu(T^{-1}[j, a_1, \ldots, a_n] \cap [i]) = \mu([i, j, a_1, \ldots, a_n]) = \frac{p_i p_{ij}}{p_j}\, \mu([j, a_1, \ldots, a_n]).$$

This shows that $|T'_\mu|$ is constant on the cylinder set $[i, j]$, with value $p_j / p_i p_{ij}$. Finally,

$$h_\mu(T) = -\sum_{i,j} p_i\, p_{ij} \log(p_{ij}) - \sum_i p_i \log(p_i) \Big(\sum_j p_{ij} \Big) + \sum_j \log(p_j) \Big(\sum_i p_i p_{ij} \Big).$$

The last two terms cancel each other out. □

The space X can be endowed with the following distance:

$$d\big(\{x_i\}, \{y_i\}\big) = 2^{-\min\{j\in\mathbf{N} \mid x_j \neq y_j\}}.$$

The shift is dilating with respect to this distance when restricted to the cylinder sets $[a] = \{\, \{x_i\} \in I^{\mathbf{N}} \mid x_0 = a\}$, for every $a \in I$. This follows from the relation

$$\forall x, y \in [a], \quad d\big(Tx, Ty\big) \geqslant 2\, d(x, y).$$

The dynamical system we have just studied is therefore an example of a piecewise dilating map.

12.4 Entropy of Dilating Transformations

The trajectories of a dilating map T tend to separate over time, as illustrated by Fig. 12.2. This allows us to show that a partition $\{X_i\}$ is generating if the restriction of T to each piece of the partition is dilating.

Proposition 12.3 *Let X be a metric space, let μ be a Borel probability measure, and let $T : X \to X$ be a Borel map that preserves the measure μ. Let ξ be a finite partition whose elements have finite diameters; we suppose that for some $K > 1$ and for every $A \in \xi$,*

$$\forall x,\, y \in A, \quad d(Tx, Ty) \geqslant K\, d(x, y).$$

Then ξ is a one-sided generator.

Proof Let us show that the diameters of the sets $\big(\bigvee_0^n T^{-i}\xi\big)(x)$ tend to 0 when n tends to ∞.

If this is not the case, we can find $\delta > 0$ and, for every integer $n \in \mathbf{N}$, points x_n in $\bigvee_0^n T^{-i}\xi\,(x)$ such that $d(x_n, x) > \delta$. For every $i \in \{0, \dots, n\}$, the point $T^i x_n$ belongs to $\xi(T^i x)$, which gives

$$d(T^{i+1}x_n, T^{i+1}x) \geqslant K\, d(T^i x_n, T^i x).$$

Consequently, we have

$$\operatorname{diam} \xi(T^n x) \geqslant d(T^n x_n, T^n x) \geqslant K^n d(x_n, x) \geqslant K^n\, \delta.$$

The diameter of the $\xi(T^n x)$ is therefore not bounded. We conclude using the following lemma. □

Lemma 12.1 *Let X be a metric space, and let ξ_n be a sequence of countable partitions that satisfies, for every $x \in X$,* $\operatorname{diam} \xi_n(x) \to 0$. *Then the elements of the ξ_n for $n \in \mathbf{N}$ generate the Borel σ-algebra of X.*

Proof Let U be an open subspace of X. For every $x \in U$, there exists $n \in \mathbf{N}$ such that $\xi_n(x) \subset U$. We therefore have

$$U = \bigcup_{n \in \mathbf{N}} \bigcup_{\substack{A \in \xi_n \\ \text{and } A \subset U}} A.$$

The open subspace U is in the σ-algebra generated by the ξ_n. The proof is illustrated by Fig. 12.3. □

We have shown that the entropy of a piecewise dilating map can be obtained by integrating the Jacobian.

Theorem 12.1 *Let X be a metric space, and let $T : X \to X$ be a uniformly piecewise dilating Borel map: there exist a finite partition $\{X_i\}$ of X by bounded Borel sets and a constant $K > 1$ such that*

$$\forall i,\ \forall x,\ y \in X_i, \quad d\big(T(x), T(y)\big) \geqslant K\, d(x, y).$$

Let μ be a Borel probability measure that is invariant under T. Then the entropy of T relative to μ is given by

$$h_\mu(T) = \int_X \log |T'_\mu| \,\mathrm{d}\mu.$$

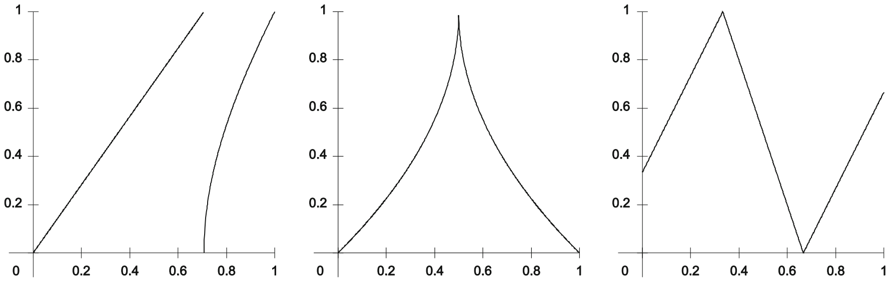

Fig. 12.1 A few dilating maps on the interval [0, 1]

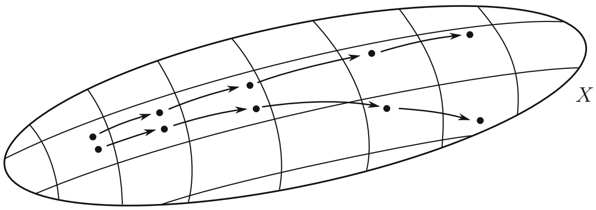

Fig. 12.2 Dilation and generating partition

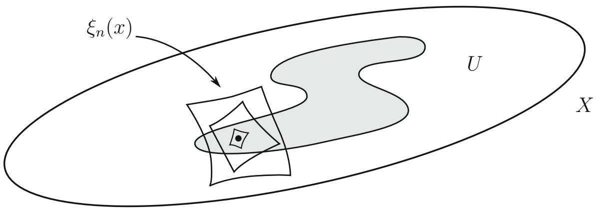

Fig. 12.3 Sequence of partitions with arbitrarily small diameter

12.5 Exercises

12.5.1 Basic Exercises

***Exercise* 1** Consider the map $T : [0, 1] \to [0, 1]$ given by

$$T(x) = \begin{cases} \sqrt{2}\,x & \text{if } x \in [0, 1/\sqrt{2}], \\ \sqrt{2x^2 - 1} & \text{if } x \in [1/\sqrt{2}, 1]. \end{cases}$$

Show that T preserves the measure $2x\mathrm{d}x$ and determine its entropy.

***Exercise* 2** Let A be an $n \times n$ matrix with integral coefficients and nonzero determinant, whose eigenvalues all have absolute value greater than 1. This matrix induces a map on the torus $\mathbf{T}^n$ by passing to the quotient. Show that this map preserves the Lebesgue measure, and then that it is injective and dilating on every set with sufficiently small diameter. Determine its entropy relative to the Lebesgue measure on the torus.

***Exercise* 3** Let U be an open subset of $\mathbf{R}^n$, and let $T : U \to \mathbf{R}^n$ be an injective C^1 map preserving a finite measure of the form $\mathrm{d}\mu = h\,\mathrm{d}x$, with h measurable. Compute $|T'_\mu|$.

Suppose that the function $\log(h)$ is μ-integrable. Prove the following formula:

$$\int_U \log|T'_\mu|\,\mathrm{d}\mu = \int_U \log|\det D_x T|\,\mathrm{d}\mu.$$

***Exercise* 4** Let X be a compact metric space, let $T : X \to X$ be a continuous map, and let μ be a Borel probability measure that is invariant under T. Let ξ be a finite partition satisfying the following property:

$$\forall A \in \xi,\ \forall x, y \in \overline{A} \text{ distinct}, \quad d(Tx, Ty) > d(x, y).$$

Show that the partition ξ is generating.

***Exercise* 5** Show that the following map $T : [0, 1] \to [0, 1]$ preserves the Lebesgue measure:

$$T(x) = \begin{cases} \sqrt{|2x - 1|} & \text{if } x \in [0, \frac{1}{2}], \\ 1 - \sqrt{|2x - 1|} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Compute its entropy.

12.5.2 More Advanced Exercises

***Exercise* 6** Let $T : [0,1] \to [0,1]$ be the map defined by

$$T(x) = \begin{cases} 2x + 1/3 & \text{if } x \in [0, \frac{1}{3}], \\ -3x + 2 & \text{if } x \in [\frac{1}{3}, \frac{2}{3}], \\ 2x - 4/3 & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

Show that T preserves a probability measure that is absolutely continuous with respect to the Lebesgue measure. Compute the entropy of T with respect to this measure. Does there exist a generating partition with two elements?

***Exercise* 7** Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map that preserves μ. We suppose that there exists a finite partition $\{X_i\}$ such that the TX_i are measurable and that the restriction of T to each of the X_i is a bijective, bimeasurable map from X_i to TX_i. Prove the equality

$$\int_X |T'_\mu| \, \mathrm{d}\mu = \int_X \mathrm{Card}\big(T^{-1}(y)\big) \, \mathrm{d}\mu(y).$$

***Exercise* 8** Let $p, q \in [0,1]$ satisfy $p + q = 1$. Compute the Jacobian of the unilateral shift associated with the Markov chain with transition matrix $\left(\begin{smallmatrix} p & q \\ p & q \end{smallmatrix}\right)$. Then, do the same for the transition matrix $\left(\begin{smallmatrix} p & q \\ q & p \end{smallmatrix}\right)$. Deduce that the associated measure-preserving dynamical systems are not isomorphic.

12.6 Comments

For a measurable map $T : X \to X$ that is injective when restricted to the elements of a finite partition $\{X_i\}$, the Jacobian $|T'_\mu|$ as defined earlier a priori does not depend on the chosen partition. However, if $\{Y_j\}$ is another partition such that T is injective when restricted to its elements, then the Jacobians $T'_{\mu|X_i}$ and $T'_{\mu|Y_j}$ coincide almost everywhere when restricted to $X_i \cap Y_j$. The function $|T'_\mu|$ can therefore be considered well defined up to a set of measure 0, without needing to refer to a specific partition.

The inverse of the Jacobian admits a probabilistic interpretation. The quantity $1/|T'_\mu(x)|$ corresponds to the probability of obtaining the value x among all values of $T^{-1}(T(x))$ if we know the value of $T(x)$.

We have restricted ourselves to the case of a partition $\{X_i\}$ with finite cardinality because it is in this context that the concept of entropy was defined. The results proved earlier also hold if the partition $\{X_i\}$ is countable; the proofs are the same.

A Lebesgue space X on which there is a measurable map $T : X \to X$ that preserves the measure, and for which the cardinality of the fibers $T^{-1}(x)$ is countable for every $x \in X$, automatically admits a countable partition $\{X_i\}_{i \in \mathbf{N}}$ such that T restricted to each of the X_i is injective. This result is proved in the book by Parry [17].

We can relax the assumption of uniform dilation in the proposition showing that the partition $\{X_i\}$ is generating (Proposition 12.3). Indeed, it suffices that a power of the transformation be dilating. When the space X is compact, it suffices to have a bound of the form $d(Tx, Ty) > d(x, y)$ on the elements of the partition $\{X_i\}$. When the transformation is ergodic, we can settle for a dilation that is uniform on only one element X_i, and a bound of the form $d(Tx, Ty) \geqslant d(x, y)$ on all other elements. Almost all points in X have an orbit that passes infinitely many times through the element X_i on which we have the dilation, and this suffices to conclude.

Let X be a compact manifold, and let T be a C^1 differentiable map on X. The condition that $\det D_x T > 1$ at every point ensures that the transformation restricted to any set of sufficiently small diameter is uniformly dilating. Not all differential manifolds admit such a transformation. Only the manifolds that can be written as the quotient of a nilpotent Lie group by a discrete subgroup are likely to admit such maps.

The Jacobian is invariant by measurable conjugation. It can be used to distinguish between the unilateral shifts. For example, the unilateral shift associated with the Markov chain with transition matrix $\left(\begin{smallmatrix} p & q \\ p & q \end{smallmatrix}\right)$ and the one associated with the Markov chain with transition matrix $\left(\begin{smallmatrix} p & q \\ q & p \end{smallmatrix}\right)$ have the same entropy. However, the partitions given by the level sets of the Jacobian cannot be isomorphic: in the first case, this partition generates the Borel sets under the action of the shift, whereas in the second case, the generated partition is invariant under the permutation of the two symbols.

Part IV
Ergodic Decomposition

Chapter 13
Lebesgue Spaces and Isomorphisms

Nous avons la chance unique d'avoir à notre disposition une langue universelle, la numérotation décimale écrite, utilisons-la.

H. Lebesgue (1875–1941)

13.1 Introduction

Let us study the spaces on which the measure-preserving dynamical systems are defined. We will say that two probability spaces are isomorphic if, after having dismissed a negligible set of points in both spaces, we can find a measure-preserving measurable bijection whose inverse is also measurable. If two dynamical systems are isomorphic, so are their underlying spaces.

Can we classify the probability spaces up to isomorphism? First, note that an isomorphism sends the points with positive measure (the atoms of the measure) onto points with the same measure. The number and mass of these atoms are therefore isomorphism invariants.

Surprisingly, most probability spaces without atoms are isomorphic to $[0, 1]$ endowed with the Lebesgue measure. Only when the space is not measurable or separable does such an isomorphism not exist.

Let us explain why this is so, by considering the case of a Borel space X in a complete measurable metric space. By the separability of X, we can encode the points of X by sequences of 0's and 1's, in the same way that we can represent a real number by the sequence of its digits in base 2. Hence there exists a measurable injection from X into $\{0, 1\}^{\mathbf{N}}$. The only problem left is to show that this map sends measurable sets onto measurable sets.

This question appears very early in the history of measure theory. H. Lebesgue already wondered under which conditions a Borel map transforms a Borel set into a Borel set. Injectivity of the transformation turns out to be sufficient. Proving this required the introduction of a new class of sets, namely Suslin (or analytic) sets, and the structure of these sets needed to be studied extensively.

It is possible to bypass the difficulties presented by these notions by making do with only showing that the image of a Borel set by an injective Borel map is

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measurable. This result is relatively easy to obtain, because the measurability refers to a measure. We can therefore obtain it through a direct computation. This is the approach we will follow in this chapter.

13.2 Measurable Isomorphism

Let us begin by recalling the difference between a Borel set and a measurable set. Let X be a topological space, let $\mathcal{B}(X)$ be the Borel σ-algebra of X, and let μ be a measure on this σ-algebra. We will say that a subset of X is *μ-measurable* if it belongs to the completion $\overline{\mathcal{B}(X)}$ of the Borel σ-algebra with respect to μ:

$$\overline{\mathcal{B}(X)} = \{A \subset X \mid \exists B, C \in \mathcal{B}(X) \text{ such that } B \subset A \subset C \text{ and } \mu(C \setminus B) = 0\}.$$

The σ-algebra $\overline{\mathcal{B}(X)}$ is *complete* in the following sense: *the subsets of the negligible sets all belong to this σ-algebra*.

Recall that a *map* f between two spaces $(X, \mathcal{T}, \mu)$ and $(Y, \mathcal{S}, \nu)$ is measurable if the inverse images of the elements of $\mathcal{S}$ are in $\mathcal{T}$. For a real-valued function, the measurability refers to a different notion: a *function* $f : X \to \mathbf{R}$ on a measure space $(X, \mathcal{T}, \mu)$ is called measurable if the inverse image of every Borel subset of $\mathbf{R}$ is in $\mathcal{T}$. If the σ-algebra $\mathcal{T}$ is complete, these two notions coincide: every measurable function $f : X \to \mathbf{R}$ is a measurable map from $(X, \mathcal{T}, \mu)$ to $(\mathbf{R}, \overline{\mathcal{B}(\mathbf{R})}, f_*\mu)$; that is, the inverse images of the $f_*\mu$-measurable sets are in $\mathcal{T}$. For this reason, it is preferable to work with complete σ-algebras.

Definition 13.1 Two measure spaces $(X, \mathcal{T}, \mu)$ and $(Y, \mathcal{S}, \nu)$ are called *isomorphic* if there exist two measurable sets $X_0 \subset X$ and $Y_0 \subset Y$ such that $\mu(X_0^c) = 0$ and $\nu(Y_0^c) = 0$, as well as a bijective measurable map $\varphi : X_0 \to Y_0$ that has a measurable inverse and satisfies $\varphi_*\mu = \nu$.

Recall that a measure μ is called *nonatomic* if the singletons are all negligible: $\mu(\{x\}) = 0$ for every x.

Example Let μ be a nonatomic Borel probability measure on $[0, 1]$. Let us show that $([0, 1], \overline{\mathcal{B}([0, 1])}, \mu)$ is isomorphic to $([0, 1], \overline{\mathcal{B}([0, 1])}, \lambda)$, where λ denotes the Lebesgue measure.

The isomorphism φ is given by the distribution function of μ:

$$\varphi(x) = \mu([0, x)).$$

Since μ does not have any atoms, the map φ is continuous, nondecreasing, and surjective. An example is given in Fig. 13.1.

- Let us show that $\varphi_* \mu = \lambda$. Let $y \in [0, 1]$, and let

$$x = \min\{x' \in [0, 1] \mid \varphi(x') = y\}.$$

We have the equality $\varphi(x) = y$ and the relation $\varphi_* \mu = \lambda$ is a consequence of the following calculation:

$$\varphi_* \mu([0, y)) = \mu(\varphi^{-1}([0, \varphi(x)))) = \mu([0, x)) = \varphi(x) = y = \lambda([0, y)).$$

- Let us construct a Borel subset $X_0 \subset [0, 1]$ restricted to which the map φ is injective. To do this, note that every open subset of $(0, 1)$ is a disjoint countable union of open intervals. Hence there exist $a_i, b_i \in (0, 1)$ with $a_i < b_i$, such that

$$(0, 1) \smallsetminus \operatorname{supp} \mu = \coprod_{i \in \mathbf{N}} (a_i, b_i)\,.$$

Set

$$X_0 = (0, 1) \smallsetminus (\bigcup [a_i, b_i]) = \operatorname{supp} \mu \smallsetminus \bigcup \{a_i, b_i\} \cup \{0, 1\},$$
$$Y_0 = \varphi(X_0) = (0, 1) \smallsetminus \bigcup \{\varphi(a_i)\}.$$

The function φ is then an increasing bijection between X_0 and Y_0. Since $\varphi : X_0 \to Y_0$ and $\varphi^{-1} : Y_0 \to X_0$ are increasing, they are Borel; since $\varphi_* \mu = \lambda$, they are also measurable. The map φ is indeed an isomorphism.

We now state the measurability lemma, which will allow us to establish the isomorphism theorem. The notions from measure theory that play a part in this lemma (inner regularity, Lusin's theorem) are recalled in Chap. 18.

Lemma 13.1 (Measurability Lemma) *Let X be a Hausdorff topological space, and let μ be an inner regular finite Borel measure on X. Let Y be a separable metric space, and let $\varphi : X \to Y$ be a Borel map. Let $A \subset X$ be a Borel or μ-measurable set satisfying*

$$\varphi(A) \cap \varphi(A^c) = \varnothing.$$

Then $\varphi(A)$ is a measurable set with respect to the measure $\varphi_* \mu$.

In particular, if φ is injective, then it is a measurable isomorphism:

$$\varphi : (X, \overline{\mathcal{B}(X)}, \mu) \xrightarrow{\sim} (Y, \overline{\mathcal{B}(Y)}, \varphi_* \mu).$$

Proof By the inner regularity, there exists, for every $\varepsilon > 0$, a compact set $K \subset A$ such that $\mu(A \smallsetminus K) < \varepsilon$. By Lusin's theorem, there exists a compact subset $K' \subset K$ such that $\mu(K \smallsetminus K') < \varepsilon$ and $\varphi_{|K}$ is continuous. In particular, $\varphi(K)$ is compact, hence Borel and $\varphi_* \mu$-measurable.

We then construct by induction mutually disjoint compact sets K_i, restricted to which φ is continuous and whose union gives almost all of A: there exists $N \subset A$ with $\mu(N) = 0$ such that

$$A = \coprod K_i \amalg N.$$

We can apply the same reasoning to A^c and write $A^c = \coprod K_i' \amalg N'$. Figure 13.2 illustrates this decomposition.

Let us give a lower bound for the measure of the set $\varphi(\cup K_i)$, which is Borel:

$$\varphi_*\mu\big(\varphi(\cup K_i)\big) = \mu\big(\varphi^{-1}(\varphi(\cup K_i))\big) \geqslant \mu(\cup K_i) = \mu(A).$$

The same reasoning gives the lower bound $\varphi_*\mu\big(\varphi(\cup K_i')\big) \geqslant \mu(A^c)$.

The sets $\varphi(\cup K_i)$ and $\varphi(\cup K_i')$ are disjoint by virtue of the two inclusions $\varphi(\cup K_i) \subset \varphi(A)$ and $\varphi(\cup K_i') \subset \varphi(A^c)$. We therefore have

$$\begin{aligned}\varphi_*\mu\big(\varphi(\cup K_i) \amalg \varphi(\cup K_i')\big) &= \varphi_*\mu(\varphi\big(\cup K_i)\big) + \varphi_*\mu(\varphi\big(\cup K_i')\big)\\ &\geqslant \mu(A) + \mu(A^c)\\ &= \varphi_*\mu(Y).\end{aligned}$$

Consequently, the set $\varphi(\cup K_i) \cup \varphi(\cup K_i')$ has full measure. Its complement is negligible; it contains the set $\varphi(A) \smallsetminus \varphi(\cup K_i)$, which is therefore measurable. It follows that $\varphi(A)$ is measurable. □

Remarks

- The result also holds if φ is μ-measurable, in the following sense: the inverse image of every Borel set in $\mathbf{R}$ is μ-measurable.
- We of course have $\varphi(X) \cap \varphi(X^c) = \varnothing$. This shows that φ is almost surjective: $\varphi(X)$ is measurable and $\varphi_*\mu\big(\varphi(X)^c\big) = 0$.
- The condition $\varphi(A) \cap \varphi(A^c) = \varnothing$ is equivalent to the equality $\varphi^{-1}(\varphi(A)) = A$, and also to $A = \bigcup_{y\in\varphi(A)} \varphi^{-1}(\{y\})$. The set A is the union of level sets of φ.

13.3 Lebesgue Spaces

Definition 13.2 A Lebesgue space $(X, \mathcal{T}, \mu)$ is a measure space, endowed with a probability measure μ defined on a complete σ-algebra $\mathcal{T}$, that is isomorphic to $([0, 1], \overline{\mathcal{B}([0, 1])}, \lambda)$.

Here, we are interested in nonatomic probability spaces: $\mu(\{x\}) = 0$ for every $x \in X$. There also exists a notion of Lebesgue space "with atoms": such a space is isomorphic to the union of an interval endowed with the Lebesgue measure and a countable measure space whose points have positive measure.

The following theorem shows that most nonatomic probability spaces are Lebesgue spaces.

Theorem 13.1 (Isomorphism Theorem) *Let X be a Borel subset of a complete separable metric space, and let μ be a nonatomic Borel probability measure on X. Then $(X, \overline{\mathcal{B}(X)}, \mu)$ is a Lebesgue space.*

Proof By the Oxtoby–Ulam theorem, of which a proof is included in Chap. 18, the measure μ, seen as a measure on the complete separable metric space, is inner regular, and this property also holds after restriction to any measurable subset of the space. The measure μ is therefore inner regular on X.

Since every subset of a separable metric space is a separable metric space, there exists a countable base of open sets $\{U_i\}_{i\in\mathbf{N}}$ for the topology on X. Consider the injections

$$\begin{aligned} \varphi_1 : X &\hookrightarrow \{0,1\}^{\mathbf{N}} & \qquad \varphi_2 : \{0,1\}^{\mathbf{N}} &\hookrightarrow [0,1] \\ x &\longmapsto \{\mathbf{1}_{U_i}(x)\}_{i\in\mathbf{N}} & a_i &\longmapsto \sum \frac{a_i}{3^i}\,. \end{aligned}$$

The Borel injection $\varphi_2 \circ \varphi_1$ establishes an isomorphism between $(X, \overline{\mathcal{B}(X)}, \mu)$ and $\big([0,1], \overline{\mathcal{B}([0,1])}, \varphi_2 \circ \varphi_{1*}(\mu)\big)$; it is illustrated by Fig. 13.3. This reduces the problem to the example following Definition 13.1. $\square$

The measurability lemma can be generalized to Lebesgue spaces.

Proposition 13.1 *Let φ be a measure-preserving measurable map between two Lebesgue spaces, and let A be a measurable subset satisfying $\varphi(A) \cap \varphi(A^c) = \varnothing$. Then $\varphi(A)$ is measurable. In particular, if φ is injective, φ is an isomorphism.*

Proof We reduce to the space $([0,1], \overline{\mathcal{B}([0,1])}, \lambda)$ by isomorphism. A small check is necessary to ensure that the negligible sets that appear in the definition of an isomorphism do not pose any problems. We denote the two Lebesgue spaces by X and Y, where $\varphi : X \to Y$.

Let $X_0 \subset X$ be a subset with negligible complement, and let $B \subset [0,1]$ be a λ-measurable subset that has negligible complement and is in bimeasurable bijection with X_0. After passing to a subset if necessary, we may assume that B is Borel. Let $Y_0 \subset Y$ be a subset with negligible complement, and let $B' \subset [0,1]$ be a Borel subset that has negligible complement and is in bimeasurable bijection with Y_0. Set $X' = X_0 \cap \varphi^{-1}(Y_0)$; we identify this with a Borel subset of $[0,1]$ of full measure, using earlier bijections. The set $\varphi(A \cap X')$ is measurable, because we can identify $A \cap X'$ with a λ-measurable subset of $[0,1]$ and apply the measurability lemma (Lemma 13.1) to $\varphi : X' \to Y_0$. Likewise, the set $\varphi(A^c \cap X')$ is measurable.

The map φ preserves the measure; the set $\varphi(A \cap X') \amalg \varphi(A^c \cap X')$ has full measure in Y. Since $\varphi(A)$ and $\varphi(A^c)$ are disjoint subsets of Y, we have the inclusion

$$\varphi(A) \smallsetminus \varphi(A \cap X') \subset \Big(\varphi(A \cap X') \amalg \varphi(A^c \cap X')\Big)^c .$$

The set $\varphi(A) \smallsetminus \varphi(A \cap X')$ is therefore negligible, and $\varphi(A)$ is measurable. $\square$

In the proof above, we have shown that the set $\varphi(X')$ is measurable (take $A = X$ in the proof); on the other hand, the set $\varphi(X'^c)$ need not be measurable. An example is constructed in the exercises. In general, the image of a negligible set by a measure-preserving measurable map is not necessarily measurable.

13.4 The Measurable Stone–Weierstraß Theorem

The following is an application to density problems in the L^p spaces.

Theorem 13.2 *Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, let $p \in [1, \infty)$, and let $f_n : X \to \mathbf{R}$ be a sequence of bounded measurable functions that separates the points:*

$$\forall x, y \in X,\ x \neq y,\ \exists n \in \mathbf{N} \text{ such that } f_n(x) \neq f_n(y).$$

Then the algebra generated by the functions f_n and the constants is dense in $L^p(X, \mathcal{T}, \mu)$.

Proof The separation hypothesis implies that the following map in injective:

$$\begin{aligned} \varphi : X &\hookrightarrow \mathbf{R}^{\mathbf{N}} \\ x &\longmapsto \{f_i(x)\}_{i \in \mathbf{N}}. \end{aligned}$$

It therefore gives an isomorphism from $(X, \mathcal{T}, \mu)$ to $(\mathbf{R}^{\mathbf{N}}, \overline{\mathcal{B}(\mathbf{R}^{\mathbf{N}})}, \varphi_* \mu)$.

The set $\varphi(X)$ is contained in the compact set $K = \prod_{i \in \mathbf{N}} [-\|f_i\|, \|f_i\|]$, and the measure $\varphi_* \mu$ is supported on K. The map φ therefore induces an invertible isometry from $L^p(X, \mathcal{T}, \mu)$ to $L^p(K, \overline{\mathcal{B}(K)}, \varphi_* \mu)$, which sends the functions f_i to the projections onto the coordinates.

These projections are continuous functions on K that separate its points. By the usual Stone–Weierstraß theorem, the algebra generated by these projections is uniformly dense in $C(K)$ and $C(K)$ is dense in $L^p(K, \overline{\mathcal{B}(K)}, \varphi_* \mu)$. The latter is a classical result elaborated in Chap. 18. □

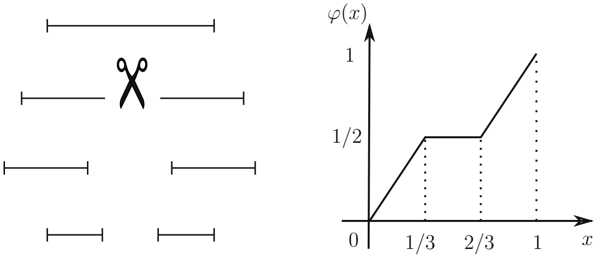


Fig. 13.1 Isomorphism between $[0, 1]$ and $[0, 1/3] \cup [2/3, 1]$, endowed with the renormalized Lebesgue measure

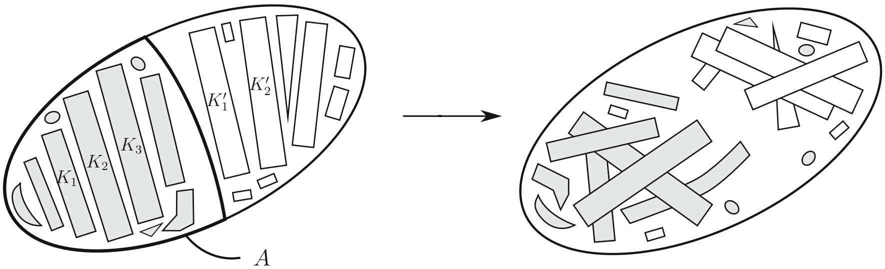


Fig. 13.2 Image of a Borel set

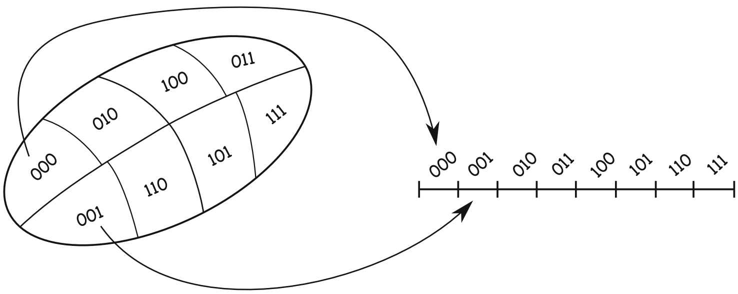


Fig. 13.3 Construction of the isomorphism

13.5 Exercises

13.5.1 Basic Exercises

***Exercise* 1** Let A and B be two Borel subsets of $[0,1]$, and let $\varphi : A \to B$ be a nondecreasing map from A to B. Show that φ is Borel.

***Exercise* 2** Let $(X, \mathcal{T}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be two Lebesgue spaces, let $\varphi : X \to Y$ be a measurable map, and let $A \in \mathcal{T}$.

- Show that there exists a set $A' \subset A$ such that $\mu(A \setminus A') = 0$ and $\varphi(A') \in \mathcal{S}$.
- Is the set $\varphi(A \setminus A')$ necessarily measurable?

***Exercise* 3** Let X be a Hausdorff topological space with a countable base of open sets, and let μ be an inner regular nonatomic Borel probability measure. Show that the space $(X, \overline{\mathcal{B}(X)}, \mu)$ is a Lebesgue space.

***Exercise* 4** Let $f : [0,1] \to \mathbf{R}$, and let λ be the Lebesgue measure on $[0,1]$. Compute $f_*\lambda$ when

- the map f is constant;
- the map f is C^1 and injective;
- the map f is piecewise affine.

***Exercise* 5** Let B be a Borel set in a complete metric space, endowed with a nonatomic finite Borel measure. Show that for every $t \in [0, \mu(B)]$, there exists a Borel set $A \subset B$ such that $\mu(A) = t$.

***Exercise* 6** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let $\{B_i\}_{i\in\mathbf{N}}$ be a countable family of elements of $\mathcal{T}$. Suppose that for every x, y in X with $x \neq y$, there exists $i \in \mathbf{N}$ such that $x \in B_i$ and $y \notin B_i$. Show that together, the B_i and the negligible sets generate the σ-algebra $\mathcal{T}$.

13.5.2 More Advanced Exercises

***Exercise* 7** Let μ be a finite Borel measure on a separable metric space X. Let A be a Borel subset of X satisfying the following property: for every measurable $B \subset A$, we have either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. Show that there exists $x \in A$ such that $\mu(A \setminus \{x\}) = 0$.

***Exercise* 8** Give a set $N \subset [0,1]$ that is negligible for the Lebesgue measure and admits a bijection to $\mathbf{R}$. Construct a subset N' of $[0,1]$ that is not measurable for the Lebesgue measure and that admits a bijection to $\mathbf{R}$.

- We define a function φ as follows: φ is equal to the identity on $[0,1] \setminus N$ and φ is a bijection from N to N'. Show that $\varphi : [0,1] \to [0,1]$ is a measurable map that preserves the Lebesgue measure.

- Show that the image of a negligible set by a measure-preserving measurable map between two Lebesgue spaces is not necessarily measurable.

***Exercise* 9** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let $\{f_n\}_{n\in\mathbf{N}}$ be a family of bounded functions that generates an algebra that is dense in $L^2(X)$. Show that there exists a set $X_0 \subset X$ of full measure such that the restrictions of the f_n to this set X_0 separate the points of X_0.
Hint: Approximate the characteristic functions of a countable base of open sets.

***Exercise* 10** The Lebesgue spaces that we have considered in this chapter are *nonatomic* probability spaces. We can also define Lebesgue spaces of finite measure with atoms: these are measure spaces isomorphic to $[0, a] \cup \mathbf{N}$ endowed with the measure $\lambda_{[0,a]} + \sum_{i\in\mathbf{N}} a_i\delta_i$, with $a, a_i \in \mathbf{R}_+$ and $a + \sum a_i < +\infty$.

- Classify these spaces up to isomorphism.
- Show that the measurability lemma is also true for these spaces.
- Show that the Stone–Weierstraß theorem is also true for these spaces.
- Show that every Borel set in a complete separable metric space, endowed with a finite Borel measure, is isomorphic to such a space.

13.6 Comments

A separable metric space can always be embedded into a complete separable metric space, for example its completion. But nothing guarantees that it is a Borel subspace of its completion. If that is the case, we say that this topological space is a *standard Borel space*. Such a space becomes a Lebesgue space if we endow it with a Borel probability measure.
We can give a stronger version of the measurability lemma: *the image of a Borel subspace by an injective Borel map between two standard Borel spaces is a Borel subspace.* This result is more difficult to establish than the lemmas given earlier; the proof usually passes through the theory of analytic sets. We refer the reader to the book by Cohn [4, Chap. 7]. Without the injectivity hypothesis, we have the following result: for a Borel map between two standard Borel spaces, the image of a Borel set is *universally measurable*, that is, measurable with respect to every finite Borel measure on the target space.
Lusin's theorem admits a purely topological version: Let X be a separable metric space, and let $f : X \to \mathbf{R}$ be a Borel function. Then there exists a dense subset G_δ restricted to which f is continuous.
The compact subsets given by Lusin's theorem are not connected in general, contrary to what might be implied by Fig. 13.2. We can show that for a dense set G_δ of Borel maps $f : [0, 1] \to \mathbf{R}$ (for example in L^p with $p \in [0, \infty)$), there does not exist an open subset on which f is continuous.
Let $f : [0, 1] \to [0, 1]$ be a Borel map. In general, there is no reason for the inverse image of a measurable Lebesgue space to be a measurable Lebesgue space. We have seen in the exercises that this does hold if the measure $f_*\lambda$ is equal to λ. More generally, it is the case if f satisfies the following property: a Borel subset of $[0, 1]$ is negligible if and only if its inverse image under f is negligible. We then say that f is *nonsingular*.
Here are two types of probability spaces that are not Lebesgue spaces:

- spaces that are too large, for example an uncountable product of circles, endowed with its Haar measure;

- spaces that are too small, for example a nonmeasurable subset of $[0, 1]$, endowed with the restriction of the Lebesgue measure.

There exist other notions of "sympathetic" probability spaces: standard spaces, the compact classes of Marczewski, the perfect spaces of Gnedenko and Kolmogorov, Lusin spaces, etc. All these notions amount to supposing more or less that every separable sub-σ-algebra is a Borel σ-algebra, for an ad hoc topology with respect to which X is almost σ-compact. These notions are equivalent to one another; their aim is to recover the disintegration theorem stated further on. The terminology "Lebesgue space" was introduced by Rokhlin [20]. The measurability lemma can be found in a paper by J. Haezendonck (1973).
There exist versions of the measurable Stone–Weierstraß theorem for Baire measures on locally compact spaces; we refer to the work of R.H. Farrell, S. Cater, or R.J. Nagel.
Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space. Denote by $L^0(X)$ the set of real-valued measurable functions, endowed with the topology of convergence in probability. This topology corresponds to the distance

$$d(f, g) = \inf_{\varepsilon > 0} \{\varepsilon + \mu(\omega \mid d(f(\omega), g(\omega)) > \varepsilon)\}.$$

The measurable Stone–Weierstraß theorem also holds in this space, and it is not necessary to assume that the functions are bounded in the statement of the theorem. The proof remains the same.
Lebesgue spaces enjoy a number of properties that are not true for all probability spaces. The Stone–Weierstraß theorem is an example. Another example is given by the following result: Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space. Every morphism of σ-algebras from $\mathcal{T}$ to $\mathcal{T}$ is of the form $A \mapsto F^{-1}A$, where $F : X \to X$ is a measurable map.
We can look for topological equivalents to the isomorphism theorem. It is, for example, possible to associate a symbolic representation with each of the points of a compact metric space K: we can always construct a continuous surjection $\varphi : \{0, 1\}^{\mathbf{N}} \to K$, so that the space K is homeomorphic to the quotient of $\{0, 1\}^{\mathbf{N}}$ by the equivalence relation $x \sim y \leftrightarrow \varphi(x) = \varphi(y)$.
In certain specific cases, we can construct isomorphisms between measure spaces that are continuous. For example, J. Oxtoby and S. Ulam (1941) have shown that two nonatomic probability measures of full support on the cube $[0, 1]^n$ that do not charge the boundary of the cube can be deduced from each other through a homeomorphism. This result does not hold for $\{0, 1\}^{\mathbf{N}}$; on this space, the subset of $\mathbf{R}$ given by the measures associated with the closed open subsets (that is, with the cylinders) is a nontrivial homeomorphism invariant. The problem of the classification of measures on $\{0, 1\}^{\mathbf{N}}$ up to homeomorphism remains open to this day, even in the case of product measures. We refer the reader to the paper by R. Daniel Mauldin and A. Yingst (2009) on this subject.

Chapter 14
Ergodic Decomposition

I was very concretely minded (...). Yet I felt a little bit that I ought to do these abstract things, and Steinhaus, whom I met a little later, said, "You shouldn't; you must earn the right to generalize."

M. Kac (1914–1984)

14.1 Introduction

When a system is not ergodic, it is possible to decompose the underlying space into several pieces, so that the transformation is ergodic on each of these pieces. We call this a *partition into ergodic components*. The number of components may be uncountable, but the resulting partition still satisfies a certain regularity property: it is possible to approximate it with partitions having finitely many pieces.

Strictly speaking, constructing the partition into ergodic components is not difficult. In general, we can associate with each σ-algebra a partition such that the functions that are measurable with respect to this σ-algebra are exactly the functions that are constant on the elements of the partition. We will present this construction in the next chapter. It therefore suffices to consider the partition associated with the σ-algebra of functions invariant under the transformation; a measurable set is invariant mod 0 if and only if it is a union of elements of this partition, up to a negligible set.

But in order to speak of the ergodicity of the transformation when restricted to each of its ergodic components, it is not enough to partition the space into these components; we also need to deduce from the measure defined on the whole space a measure that is invariant on each of the components. These measures are obtained by *disintegration*. Such a disintegration is possible whenever we work with a *measurable* partition ξ, that is, with a partition that can be approximated with partitions that are finite in the following sense: there exists a sequence of partitions ξ_n whose elements are measurable, finite in number, and satisfy $\xi(x) = \bigcap_{n\in\mathbf{N}} \xi_n(x)$ for almost all x.

Y. Coudène, *Ergodic Theory and Dynamical Systems*, Universitext,
DOI 10.1007/978-1-4471-7287-1_14

The notion of ergodic decomposition is a purely measurable notion. When we work with measures on topological spaces or on metric spaces, it is useful to have a geometric definition of the ergodic components. To obtain this, we can use the Hopf argument as inspiration. Rather than pairing the points whose trajectories are asymptotic, we can pair the points whose Birkhoff sums, associated with an arbitrary bounded continuous (or Lipschitz) function, are asymptotic. This way, we obtain a Borel realization of the ergodic components that depends only on the topology of the space. The measure then selects the components of the points that are typical, from the point of view of the Birkhoff ergodic theorem. And when we define the components using Lipschitz functions, the Hopf argument becomes an immediate consequence of the ergodic decomposition theorem.

14.2 Disintegration

A partition $\{\xi(x)\}$ is called measurable if there exist measurable sets B_n, for $n \in \mathbf{N}$, each the union of elements of the partition, such that for almost all $x \in X$, we have the equality $\xi(x) = \bigcap_{B_n \ni x} B_n \ \bigcap_{B_n^c \ni x} B_n^c$.

These elements B_n allow us to approximate the partition ξ with finite partitions. If we define the partitions by setting

$$\eta_k = \{B_k, B_k^c\}, \qquad \xi_n = \bigvee_{k=0}^{n} \eta_k,$$

then we have the relation $\xi(x) = \bigcap_{n \in \mathbf{N}} \xi_n(x)$ for almost all $x \in X$.

We will study the notion of measurable partition in detail in the next chapter. Its significance comes from the following disintegration theorem (Figs. 14.1 and 14.2).

Theorem 14.1 *Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let ξ be a measurable partition. Then there exists a unique family of probability measures $\mu_{\xi(x)}$ supported on the elements of the partition such that, for every $A \in \mathcal{T}$,*

- *the set A is $\mu_{\xi(x)}$-measurable for μ-almost all x;*
- *the function $x \mapsto \mu_{\xi(x)}(A)$ is μ-integrable and its integral is $\mu(A)$:*

$$\mu(A) = \int \mu_{\xi(x)}(A)\, \mathrm{d}\mu(x).$$

Proof We may assume that X is of the form $\{0,1\}^{\mathbf{N}}$. We denote by $\mathcal{C}$ the algebra of finite unions of cylinders sets in $\{0,1\}^{\mathbf{N}}$ and by $\mathcal{B}$ the σ-algebra generated by $\mathcal{C}$. The algebra $\mathcal{C}$ satisfies the Kolmogorov criterion: every nonincreasing sequence of elements of this family whose intersection is empty, eventually takes on the value $\varnothing$; it therefore has measure 0 with respect to every finitely additive measure on $\mathcal{C}$. Consequently, every finitely additive measure on $\mathcal{C}$ admits a unique σ-additive extension to $\mathcal{B}$. It therefore suffices to define the $\mu_{\xi(x)}$ on the algebra generated by the cylinder sets.

Let $\widehat{\xi} \subset \mathcal{T}$ be the σ-algebra of sets that are unions of elements of the partition. Suppose that the integral relation in the theorem is satisfied. Consider $C \in \mathcal{C}$ and $A \in \widehat{\xi}$. Since A is constant on the elements of ξ, we must have $\mu(A \cap C) = \int_A \mu_{\xi(x)}(C)\,\mathrm{d}\mu$. By the properties of the conditional expectation, this is equivalent to the equality

$$\forall C \in \mathcal{C},\ \text{for } \mu\text{-almost all } x, \quad \mu_{\xi(x)}(C) = E(\mathbf{1}_C|\,\widehat{\xi})(x).$$

The function $E(\mathbf{1}_C|\,\widehat{\xi})$ is a priori defined only almost everywhere. We choose a representative that is constant on the elements of the partition. Since the number of elements of $\mathcal{C}$ is countable, there exists a set Ω, with $\mu(\Omega^c) = 0$, for which the quantities $E(\mathbf{1}_C|\,\widehat{\xi})(x)$ are well defined for all $C \in \mathcal{C}$ and all $x \in \Omega$. After restricting Ω again if necessary, we may assume that the function $C \mapsto E(\mathbf{1}_C|\,\widehat{\xi})(x)$ is finitely additive, which gives the desired measure $\mu_{\xi(x)}$.

The integral relation is satisfied for $A \in \mathcal{C}$. Let us show that it also holds if $A \in \mathcal{T}$. Since every open set is a nondecreasing union of elements of $\mathcal{C}$, the equation is also true for the open sets, and by taking the complement, for the closed sets. Let $A \in \mathcal{B}$; for every $\varepsilon > 0$, there exist a closed set F and an open set U such that $\mu(U \setminus F) < \varepsilon$ and $F \subset A \subset U$. We therefore have

$$\mu(A) - \varepsilon \leqslant \mu(F) = \int \mu_{\xi(x)}(F)\,\mathrm{d}\mu(x) \leqslant \int \mu_{\xi(x)}(U)\,\mathrm{d}\mu(x) = \mu(U) \leqslant \mu(A) + \varepsilon.$$

This proves the desired relation for $A \in \mathcal{B}$. It remains to verify it for μ-negligible $A \in \mathcal{T}$. Such a set is contained in a negligible set $B \in \mathcal{B}$. The relation shows that B is μ-negligible for μ-almost all x. The same therefore holds for A. This implies $\int \mu_{\xi(x)}(A)\,\mathrm{d}\mu = 0$.

It remains to prove that $\mu_{\xi(x)}$ is supported on $\xi(x)$. Let B_n be the sequence of sets from the definition of ξ. For μ-almost all x and all $n \in \mathbf{N}$, we have $\mu_{\xi(x)}(B_n) = E(\mathbf{1}_{B_n}|\,\widehat{\xi})(x) = \mathbf{1}_{B_n}(x)$. Consequently, the B_n and B_n^c that contain x have full measure for $\mu_{\xi(x)}$, and the same holds for $\xi(x)$, which is a countable intersection of such sets. □

Remark The measures $\mu_{C(x)}$ are not a priori defined on the whole σ-algebra $\mathcal{T}$. However, if we consider a σ-algebra $\mathcal{B}' \subset \mathcal{T}$ generated by a countable number of elements, then for μ-almost all $x \in X$, all elements of $\mathcal{B}'$ are $\mu_{C(x)}$-measurable. And every family of measures defined on $\mathcal{B}'$ and satisfying the conclusions of the theorem coincides with $\mu_{C(x)}$ on $\mathcal{B}'$ for μ-almost all x.

14.3 Ergodic Decomposition

To define the ergodic components topologically, we consider *standard Borel spaces*: we call a topological space X a standard Borel space if it is homeomorphic to a Borel subset of a complete metric space. A standard Borel space endowed with a Borel probability measure is of course a Lebesgue space, and every Lebesgue space is isomorphic to a standard Borel space endowed with a probability measure (after completion).

Let X be a metric space, and let $x \in X$. The *ergodic component* of x is defined by

$$C(x) = \{y \in X \mid \forall f \in C_b(X),\ \tfrac{1}{n}S_n(f)(x) - \tfrac{1}{n}S_n(f)(y) \to 0\}.$$

The sets $C(x)$ are T-invariant and partition the space X. Figure 14.3 shows examples where T is a rotation.

Proposition 14.1 *Let X be a metric space, let $T : X \to X$ be a Borel map, and let $x \in X$. There exists at most one invariant Borel probability measure supported on $C(x)$, that is, such that $C(x)$ is measurable and has negligible complement. When it exists, this measure is denoted by ν_x. In that case, ν_x is ergodic and we have*

$$\forall y \in C(x),\ \forall f \in C_b(X), \quad \frac{1}{n}S_n(f)(y) \longrightarrow \int f d\nu_x.$$

Proof Let $f \in C_b(X)$, and let $n_i \in \mathbf{N}$ be a sequence such that $\frac{1}{n_i}S_{n_i}(f)(x)$ converges; we denote its limit by ℓ. Since $\frac{1}{n_i}S_{n_i}(f)(y)$ also converges to ℓ for every $y \in C(x)$, for every invariant probability measure μ with support $C(x)$, we have

$$\int f\,\mathrm{d}\mu = \int_{C(x)} f\,\mathrm{d}\mu = \int_{C(x)} \frac{1}{n_i}S_{n_i}(f)(y)\,\mathrm{d}\mu(y) = \ell\,\mu(C(x)) = \ell.$$

Two invariant probability measures with support $C(x)$ give the same value for the integral of f, and are therefore equal. If, moreover, such a probability measure ν_x exists, then $\int f d\nu_x$ is the only possible accumulation point for the sequence $\frac{1}{n}S_n(f)(y)$, which is therefore convergent. □

We can now state and prove the ergodic decomposition theorem.

Theorem 14.2 *Let X be a standard Borel space, let $T : X \to X$ be a Borel map, and let μ be a probability measure that is invariant under T. Then the partition $\{C(x)\}_{x\in X}$ is measurable. The set of x for which $C(x)$ supports an invariant probability measure has full μ-measure, and for every positive Borel function f and positive invariant μ-integrable function g, we have*

$$\int_X f\,g\,\mathrm{d}\mu = \int_X \Big(\int_{C(x)} f\,\mathrm{d}\nu_x\Big) g(x)\,\mathrm{d}\mu(x).$$

Proof We begin by constructing a countable family $\{f_k\}_{k\in\mathbf{N}}$ of bounded Lipschitz functions, which allows us to approximate from below the characteristic functions of the open sets. We begin with a countable base of open sets $\mathcal{D}$ that is invariant under taking finite unions: $U_1, \dots, U_n \in \mathcal{D}$ implies $\bigcup U_i \in \mathcal{D}$. Then, for every element U of this base, we approximate $\mathbf{1}_U$ from below by the sequence of continuous functions $f_{j,U}(x) = \min(1, j\,d(x, U^c))$. Every nonempty open subset U of X can be written as a nondecreasing union of elements U_j of $\mathcal{D}$, and the sequence f_{j,U_j} is nondecreasing, converging to $\mathbf{1}_U$.

Let μ_n and μ be probability measures. If the sequence $\int f_k\,\mathrm{d}\mu_n$ converges to $\int f_k\,\mathrm{d}\mu$ for every $k \in \mathbf{N}$, then we have the inequalities $\underline{\lim}\,\mu_n(U) \geqslant \mu(U)$ for every open subset $U \subset X$, simply by the monotone convergence theorem. These inequalities imply the weak convergence of the sequence μ_n to μ: for every $f \in C_b(X)$, we have $\int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu$. This follows from the portmanteau theorem [6, Theorem 11.1.1]. Exercise 5 proposes a proof of that theorem.

Let P be the projection from $L^2(X)$ onto the subspace of T-invariant functions. The Birkhoff ergodic theorem provides a set $\widetilde{\Omega}$ of full measure for which $\frac{1}{n}S_n(f_k)(x)$ converges to $Pf_k(x)$ for every $k \in \mathbf{N}$. For these x, we set

$$\begin{aligned}\widetilde{C}(x) &= \{y \in \widetilde{\Omega} \mid \forall k \in \mathbf{N},\ \tfrac{1}{n}S_n(f_k)(y) - \tfrac{1}{n}S_n(f_k)(x) \longrightarrow 0\}\\ &= \{y \in \widetilde{\Omega} \mid \forall k \in \mathbf{N},\ Pf_k(x) = Pf_k(y)\}.\end{aligned}$$

The partition $\widetilde{C}$ is measurable because its elements are intersections of sets of the form $\{x \mid Pf_k(x) \in [r_1, r_2]\}$ for $m \in \mathbf{N}$ and $r_1, r_2 \in \mathbf{Q}$. It is T-invariant. Let $\mu_{\widetilde{C}(x)}$ be the disintegrations of μ along $\widetilde{C}$. The sets $\widetilde{C}(x)$ and the measure μ are T-invariant; the measures $T_*\mu_{\widetilde{C}(x)}$ therefore form a new family of measures disintegrating μ. By the uniqueness of the disintegration, we must have $T_*\mu_{\widetilde{C}(x)} = \mu_{\widetilde{C}(x)}$ for almost all x. For these x, we have found an invariant measure with support $\widetilde{C}(x)$.

For every $y \in \widetilde{C}(x)$, the sequence $\frac{1}{n}S_n(f_k)(y)$ converges to $Pf_k(x)$. We apply the dominated convergence theorem and use the invariance of $\mu_{\widetilde{C}(x)}$ with respect to T:

$$\int f_k(y)\,\mathrm{d}\mu_{\widetilde{C}(x)}(y) = \int \frac{1}{n}S_n(f_k)(y)\,\mathrm{d}\mu_{\widetilde{C}(x)}(y) \xrightarrow[n\to\infty]{} Pf_k(x).$$

It follows that $\int f_k\,\mathrm{d}\mu_{\widetilde{C}(x)} = Pf_k(x)$ for almost all $x \in X$.

We wish to show that the partitions C and $\widetilde{C}$ are equal almost everywhere. We set $\Omega = \{x \in \widetilde{\Omega} \mid \forall k,\ \frac{1}{n}S_n(f_k)(x) \to \int f_k\,\mathrm{d}\mu_{\widetilde{C}(x)}\}$ and use the disintegration formula:

$$\int_X \mu_{\widetilde{C}(x)}(\Omega^c)\,\mathrm{d}\mu(x) = \mu(\Omega^c) = 0.$$

The set Ω^c is therefore $\mu_{\widetilde{C}(x)}$-negligible for μ-almost all x.

Let $y \in \widetilde{C}(x) \cap \Omega$. Set $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{T^k(y)}$. We have the following convergence:

$$\forall k \in \mathbf{N}, \qquad \int f_k \,\mathrm{d}\mu_n = \frac{1}{n}S_n(f_k)(y) \xrightarrow[n\to\infty]{} \int f_k \,\mathrm{d}\mu_{\widetilde{C}(x)}.$$

This implies the weak convergence of the sequence μ_n to the measure $\mu_{\widetilde{C}(x)}$. The point y is therefore in $C(x)$, which shows the equality

$$\widetilde{C}(x) \cap \Omega = C(x) \cap \Omega.$$

The partition C is measurable and for μ-almost all x, the set $C(x)$ supports an invariant probability measure: $\nu_x = \mu_{\widetilde{C}(x)} = \mu_{C(x)}$.

To obtain the integral equation that figures in the statement of the theorem, we consider a positive integrable function g and integrate the relation $\int f_k \,\mathrm{d}\mu_{C(x)} = Pf_k(x)$, which holds for almost all $x \in X$, with respect to the measure $g\,\mathrm{d}\mu$. From this, we deduce the desired integral relation for the f_k, and then for the characteristic functions f of open sets by the monotone convergence theorem. We pass to the characteristic functions of measurable sets using outer regularity and monotone convergence, and then to positive functions by linearity and, again, monotone convergence. □

Let us conclude with two remarks.

- If A is a μ-measurable set, then it is $\mu_{C(x)}$-measurable for μ-almost all x. Indeed, we can find two Borel sets $B \subset A \subset C$ such that $\mu(C \smallsetminus B) = 0$, and the disintegration formula gives $\mu_{C(x)}(C \smallsetminus B) = 0$ for μ-almost all $x \in X$.
- The integral equation generalizes to μ-integrable f and T-invariant μ-integrable g, simply by decomposing f and g into positive and negative parts and noting that f coincides μ-almost everywhere with a Borel function.

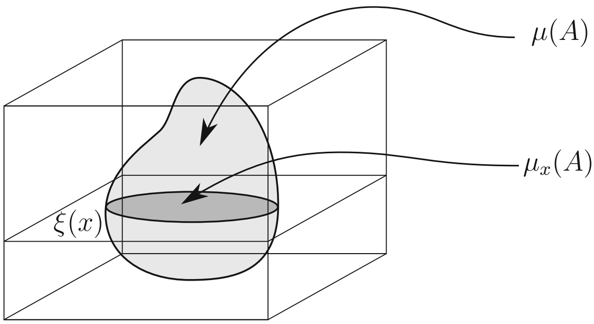


Fig. 14.1 Disintegration

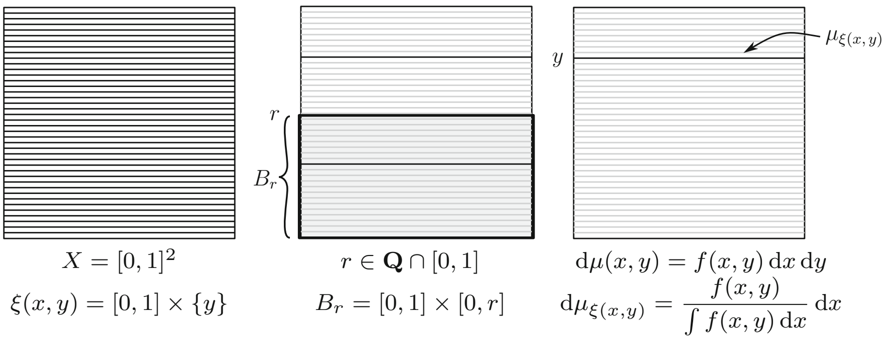


Fig. 14.2 Disintegration of $f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ along the *horizontal lines*

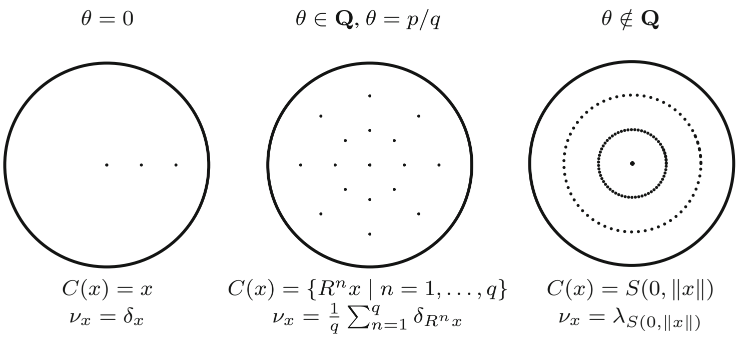


Fig. 14.3 Ergodic disintegration of the measure associated with a rotation R of angle $2\pi\theta$ on the disk

14.4 Exercises

14.4.1 Basic Exercises

***Exercise* 1** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let ξ be a measurable partition. Verify that for every measurable function f and almost all $x \in X$, we have $E(f \mid \widehat{\xi})(x) = \int f \, \mathrm{d}\mu_{\xi(x)}$. Express the usual properties of the conditional expectation using the $\mu_{\xi(x)}$.

***Exercise* 2** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, let $T : X \to X$ be a measurable map that preserves μ, and let ξ be a measurable partition that satisfies $\xi(T(x)) = T(\xi(x))$ for almost all x. Show that for almost all x, we have $T_* \mu_{\xi(x)} = \mu_{\xi(T(x))}$.

***Exercise* 3** Give a Lebesgue space $(X, \mathcal{T}, \mu)$ and a measurable partition ξ such that μ is not atomic, while the $\mu_{\xi(x)}$ are all atomic.

***Exercise* 4** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let ξ be a measurable partition. Verify that for almost all x, the spaces X and $\xi(x)$, endowed with the measure $\mu_{\xi(x)}$ and the σ-algebra of $\mu_{\xi(x)}$-measurable sets, are Lebesgue spaces (with atoms).

***Exercise* 5** Let X be a metric space, and let μ_n and μ be Borel probability measures. We assume that for every open subset $U \subset X$, we have $\underline{\lim}\, \mu_n(U) \geqslant \mu(U)$.

- Show that for every closed subset $F \subset X$, we have $\overline{\lim}\, \mu_n(F) \leqslant \mu(F)$.
  Hint: Consider the complement.
- Show that for every Borel set A such that $\mu(\partial A) = 0$, we have $\lim \mu_n(A) = \mu(A)$.
  Hint: Take $F = \overline{A}$ and $U = \mathring{A}$.
- Show that for every bounded continuous function f, we have $\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$.
  Hint: Approximate f with $\sum k\varepsilon \, \mathbf{1}_{f^{-1}([k\varepsilon,(k+1)\varepsilon))}$, where ε has been chosen such that $\mu(f^{-1}(\mathbf{Z}\varepsilon)) = 0$.

***Exercise* 6** Let X be a separable metric space, and let $T : X \to X$ be a Borel map. Show that every ergodic invariant finite measure is supported on an ergodic component.
Hint: Note that $x \mapsto \int f \, \mathrm{d}\mu_{C(x)}$ is invariant under T.

***Exercise* 7** Let X be a metric space, let $T : X \to X$ be a Borel map, and let μ be a T-invariant finite measure. Show that every T-invariant measurable set coincides, almost everywhere, with a set that is a union of ergodic components.
Hint: Show that $\mathbf{1}_I(x) - \mu_x(I) = 0$ for μ-almost all x if I is an invariant set.

14.4.2 More Advanced Exercises

***Exercise* 8** Let X be a standard Borel space, let $T : X \to X$ be a Borel map, and let μ be a T-invariant finite Borel measure. Show that for μ-almost all x, the component $C(x)$ is a standard Borel space.

***Exercise* 9** Let X be a standard Borel space, and let d be a metric on X that generates the topology. Let $BL(X)$ be the set of bounded Lipschitz functions. Show that the ergodic components coincide, up to a negligible set, with the partition

$$C'(x) = \{y \in X \mid \forall f \in BL(X),\ \tfrac{1}{n}S_n(f)(x) - \tfrac{1}{n}S_n(f)(y) \to 0\}.$$

***Exercise* 10** Let X be a standard Borel space, and let $T : X \to X$ be a Borel map. Show that the set

$$\left\{x \in X \mid \tfrac{1}{n}\sum_{k=1}^{n} \delta_{T^k x} \text{ converges weakly to a measure supported on } C(x)\right\}$$

has full measure for every invariant finite Borel measure. Does it have full measure for every invariant Borel measure? Can this set be empty?

Remark The points in this set are called generic points.

***Exercise* 11** Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $F : X \to \mathbf{R}^n$ be an integrable map. Show that $\int F \, \mathrm{d}\mu$ is in the closure of the convex hull of $F(X)$.

Suppose that $(X, \mathcal{T}, \mu)$ is a Lebesgue space, and let $T : X \to X$ be a measurable map that preserves μ. Show that every invariant finite measure can be approximated with linear combinations of ergodic finite measures (*Choquet theorem*).

14.5 Comments

The construction of conditional measures $\mu_{\xi(x)}$ is based on the Kolmogorov criterion. If X is locally compact, we can also use the Riesz representation theorem and give a proof that does not use a symbolic model; this is the approach followed in the book by H. Furstenberg [8]. When μ is a product measure on a product space and ξ is the partition in horizontal or vertical subspaces, the disintegration theorem reduces to the Fubini theorem. We therefore recover the pathology associated with this type of situation. For example, the measurability of a set A with respect to a probability measure μ cannot be deduced from the measurability of A with respect to all disintegrations $\mu_{\xi(x)}$. A well-known counterexample was given by W. Sierpinski in 1920: modulo the continuum hypothesis, Sierpinski constructed a subset of $[0, 1] \times [0, 1]$ that intersects every line in at most two points, but whose outer measure, in the sense of Lebesgue, is equal to 1.

The sets $C(x)$ are not in general Borel. If X is a compact metric space, or if X is a totally bounded metric space and we require that the convergence in the definition of the $C(x)$ take place only for the uniformly continuous functions, then the $C(x)$ are Borel sets. The partition $C(x)$ is even Borel when restricted to the set $\Omega = \{x \mid \forall f,\ \tfrac{1}{n}S_n(f)(x) \text{ converge}\}$: There exist Borel sets B_n such that for every $x \in \Omega$, the set $C(x) \cap \Omega$ coincides with the

intersection of all B_n and B_n^c that contain x. This follows from the existence of a countable subset, dense for the uniform topology, in the set of uniformly continuous functions.
Is it customary to use the term "ergodic components" to denote the class mod 0 of the partition $\{C(x)\}$ constructed earlier, which means that the components depend on the chosen measure μ. In applications, it is sometimes preferable to have a Borel version of these components that is independent of the measure. The version stated earlier was inspired by the Hopf argument. We could also have taken

$$C(x) = \{y \in X \mid \forall f \in C_b, \exists n_i \text{ such that } \tfrac{1}{n_i} S_{n_i}(f)(x) - \tfrac{1}{n_i} S_{n_i}(f)(y) \to 0\},$$

which allows the use of the L^2 ergodic theorem rather than the almost everywhere ergodic theorem. We could also have taken

$$C(x) = \{y \in X \mid \forall f \in BL(X),\ \tfrac{1}{n} S_n(f)(x) - \tfrac{1}{n} S_n(f)(y) \to 0\},$$

where $BL(X)$ is the space of Lipschitz bounded functions, with respect to a distance compatible with the topology on X. In this case, the components depend on the chosen distance function. Another possibility is to take

$$C(x) = \{y \in X \mid \rho\,(\tfrac{1}{n} \textstyle\sum_{k=1}^{n} \delta_{T^k x}, \tfrac{1}{n} \sum_{k=1}^{n} \delta_{T^k y}) \to 0\},$$

where ρ is the Prokhorov distance function. In this case, all ergodic components are standard Borel spaces.
There exist several approaches to the ergodic decomposition theorem. We can use the abstract Choquet theorem: *every point in a compact convex space contained in a locally convex topological linear space is the barycenter of a measure supported on the extreme points of the compact space.* The ergodic decomposition is obtained by applying this result to $\mathcal{M}^1(X)$. This does not provide an explicit description of the ergodic components.
Another approach, due to N. Krylov and N. Bogolyoubov, is based on the concept of generic points. These points $x \in X$ are those for which $\frac{1}{n} \sum_{k=1}^{n} f(T^k x)$ converges for every $f \in C_b(X)$. When X is locally compact, the Riesz theorem allows us to associate with each of these points a conditional measure ν_x, which in turn allows us to define the ergodic component as the set of y such that $\nu_x = \nu_y$. This point of view is presented in the book by M. Denker, C. Grillenberger, and K. Sigmund [5].
The ergodic decomposition theorem also holds for the actions of countable groups on standard Borel spaces. This extension is due to V. S. Varadarajan (1963); a proof can be found in the book by E. Glasner [10]. There also exist ergodic decomposition theorems for quasi-invariant actions, or in the setting of infinite measures. For this, we refer the reader to the book *Cocycles on ergodic transformation groups* by K. Schmidt.

Chapter 15
Measurable Partitions and σ-Algebras

> *The title of Rokhlin's paper* [On the fundamental ideas of measure theory] *seems to suggest that measurable partitions are the main object of measure theory. But probably this would seem very doubtful to most analysts ("all my life I have worked with the Lebesgue integral, and this is the first time I have heard about measurable partitions").*
>
> D.V. Anosov

15.1 Introduction

This chapter studies the notion of measurable partition in detail. As far as we know, there exists no elementary treatise of this notion, and yet it plays an important role in ergodic theory. We have seen it play a role in Chap. 14 when we studied the decomposition of a transformation into ergodic components.

The notion of partition is intimately linked with the notions of an algebra of measurable functions and of a σ-algebra. In fact, with every partition we can associate the algebra of measurable functions that are constant on each of the elements of the partition. We can also associate with a partition the σ-algebra generated by the elements of the partition.

Consider a Lebesgue space $(X, \mathcal{T}, \mu)$. In this chapter, we will show that there is a bijection between the sub-σ-algebras of $\mathcal{T}$ and a certain type of partition of X that was already used in the last chapter, namely measurable partitions. The terminology is somewhat misleading. By a measurable partition, we do not mean just a partition into measurable sets, but rather a partition that can be approximated by finite partitions into measurable sets. The first task of this chapter will therefore be to formalize this notion.

The elements of the functional space $L^0(X)$ are equivalence classes of measurable functions. In order to establish a correspondence between certain subspaces of L^p and measurable partitions, we will need to identify the partitions that coincide modulo a negligible set. Once again, we will need to explain what we mean by this.

Modulo these identifications, we will be able to associate a unique measurable partition with each σ-algebra of $\mathcal{T}$ that contains all negligible sets. The σ-algebra will then be generated by the elements of this partition and the negligible sets.

Y. Coudène, *Ergodic Theory and Dynamical Systems*, Universitext,
DOI 10.1007/978-1-4471-7287-1_15

If we apply this correspondence to the σ-algebra of sets invariant under a transformation of X, we obtain the partition into ergodic components that was studied in the last chapter. It is worth noting that we can define these ergodic components without using the disintegration theorem.

In summary, the notion of measurable partition, introduced by V.A. Rokhlin in the 1940s, allows us to give a concrete realization of the notion of a σ-algebra. One of its most salient applications can be found in entropy theory.

15.2 Measurable Partitions

Definition 15.1 Let $(X, \mathcal{T}, \mu)$ be a probability space. A partition ξ of X consists of a set of mutually disjoint subsets of X that cover X. The element of the partition that contains the point $x \in X$ is denoted by $\xi(x)$.

The partition is called *measurable* if there exists a countable family of measurable sets $\{B_n\}$, where each B_n is the union of elements of the partition, such that for all distinct $C_1, C_2 \in \xi$, there exists $n \in \mathbf{N}$ such that

$$C_1 \subset B_n \text{ and } C_2 \subset B_n^c \qquad \text{or} \qquad C_1 \subset B_n^c \text{ and } C_2 \subset B_n.$$

We say that the sets B_n separate the elements of the partition ξ. The elements of a measurable partition are sometimes called the *atoms* of the partition. They are measurable sets; it suffices to remark that the sets B_n from the definition of the partition determine this partition fully: $\xi(x) = \bigcap_{B_n \ni x} B_n \ \bigcap_{B_n^c \ni x} B_n^c$.

Two partitions ξ and η coincide up to a negligible set, or *modulo* 0, if there exists a measurable set $\Omega \subset X$ that has negligible complement and satisfies $\xi(x) \cap \Omega = \eta(x) \cap \Omega$ for almost all $x \in X$. From here on, we identify any two partitions that are equal mod 0, and we also use the term measurable partition for a partition that coincides mod 0 with a partition that is measurable in the sense of the definition given above.

We will say that a sub-σ-algebra $\mathcal{A}$ of $\mathcal{T}$ is *complete* if it contains the μ-measurable sets that are negligible with respect to the measure μ. This is a relative completion. Every sub-σ-algebra $\mathcal{A} \subset \mathcal{T}$ admits a unique completion $\overline{\mathcal{A}}$: this is the σ-algebra generated by $\mathcal{A}$ and the μ-negligible sets. Equivalently, the completion consists of the subsets $\widetilde{A} \in \mathcal{T}$ for which there exist a set $\Omega \in \mathcal{T}$ of full measure and a set $A \in \mathcal{A}$ satisfying $\widetilde{A} \cap \Omega = A \cap \Omega$. It also consists of the subsets $\widetilde{A} \in \mathcal{T}$ for which there exists $A \in \mathcal{A}$ such that the symmetric difference of $\widetilde{A}$ and A is negligible, that is, $\mu(A \Delta \widetilde{A}) = 0$.

We will associate a σ-algebra with every partition, and vice versa.

15.3 The σ-Algebra Associated with a Partition

With the partition ξ, we associate the completion of the σ-algebra generated by the measurable sets saturated by ξ:

$$\widehat{\xi} = \overline{\{A \in \mathcal{T} \mid A = \cup_{x \in A} \xi(x)\}}.$$

Note that a measurable function is $\widehat{\xi}$-measurable if and only if it coincides almost everywhere with a function that is constant on the atoms of the partition; it suffices to approximate the function with linear combinations of characteristic functions to see this.

The following lemma will allow us to define a bijection between the partitions and σ-algebras.

Lemma 15.1 *Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, let ξ be a measurable partition of X, and let B_n be the sets from its definition. Then the σ-algebra $\widehat{\xi}$ is generated by the B_n and the negligible sets.*

Proof Let $\varphi : X \to \{0, 1\}^{\mathbf{N}}$ be the function defined by $\varphi(x) = \{\mathbf{1}_{B_n}(x)\}$. Note that the inverse image of a Borel set under φ is measurable. Moreover, the inverse image of the σ-algebra of $\varphi_*\mu$-measurable sets is contained in the σ-algebra generated by the B_n and the negligible sets. We must therefore show that it contains the σ-algebra $\{A \in \mathcal{T} \mid A = \bigcup_{x \in A} \xi(x)\}$.

Consider a set $A \in \mathcal{T}$; we have the equality $\varphi^{-1}\varphi(A) = \cup_{x \in A} \xi(x)$. Consequently, if $A = \cup_{x \in A} \xi(x)$, then $\varphi^{-1}\varphi(A) = A$. In the chapter on Lebesgue spaces, we saw a measurability lemma that allows us to assert that $\varphi(A)$ is $\varphi_*\mu$-measurable, which concludes the proof. □

15.4 The Partition Associated with a σ-Algebra

We will say that a complete σ-algebra $\mathcal{A} \subset \mathcal{T}$ is *separable* if it is the completion of a σ-algebra generated by a countable family of sets.

Lemma 15.2 *Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space. Every complete σ-algebra $\mathcal{A} \subset \mathcal{T}$ is separable.*

Proof The space $L^1(X, \mathcal{T}, \mu)$ is separable. Moreover, every subset of a separable metric space is separable. We refer to Chap. 18 for the proof of these classical results. It follows that the set $\{\mathbf{1}_A \mid A \in \mathcal{A}\}$ is separable for the L^1 norm. Let $\{\mathbf{1}_{A_n}\}$ be a dense countable subset; let us show that the A_n generate $\mathcal{A}$.

For every $A \in \mathcal{A}$, there exists a sequence n_k such that the characteristic functions $\mathbf{1}_{A_{n_k}}$ converge almost everywhere to $\mathbf{1}_A$. The symmetric difference of A and $\overline{\lim} A_{n_k}$ therefore has measure 0. The set A is in the completion of the σ-algebra generated by the A_n. □

Let $\mathcal{A}$ be a complete sub-σ-algebra of $\mathcal{T}$, and let $\{B_n\}_{n\in\mathbf{N}}$ be a countable set of subsets such that $\mathcal{A}$ is generated by the B_n and the negligible sets. We associate with $\mathcal{A}$ the partition $\xi_{\mathcal{A}}$ whose atoms are given by the formula

$$\xi_{\mathcal{A}}(x) = \bigcap_{\substack{n\in\mathbf{N}\\ B_n\ni x}} B_n \bigcap_{\substack{n\in\mathbf{N}\\ B_n^c\ni x}} B_n^c.$$

This partition is measurable; the B_n indeed separate the elements of the partition.

Lemma 15.3 *The definition of the partition $\xi_{\mathcal{A}}$ does not depend on the choice of the B_n.*

Proof Let $\{B_n\}$ be a family of subsets, and let $\langle B_n\rangle$ be the σ-algebra they generated. We have the equality

$$\bigcap_{B_n\ni x} B_n \bigcap_{B_n^c\ni x} B_n^c = \bigcap_{\substack{A\ni x\\ A\in\langle B_n\rangle}} A.$$

To see this, note that $\langle B_n\rangle_{x,y} = \{A \in \langle B_n\rangle \mid x \in A \leftrightarrow y \in A\}$ is a σ-algebra that contains the sets B_n if $x \in B_n \leftrightarrow y \in B_n$ for all n.

Let B_n and B'_n be two countable families with completion $\mathcal{A}$. For each n, there exist sets $A_n^1, A_n^2 \in \langle B_n\rangle$ such that $A_n^1 \subset B'_n \subset A_n^2$ and $\mu(A_n^2 - A_n^1) = 0$. Likewise, there exist sets $A_n'^1, A_n'^2 \in \langle B'_n\rangle$ such that $A_n'^1 \subset B_n \subset A_n'^2$ and $\mu(A_n'^2 - A_n'^1) = 0$. The partitions associated with the B_n and B'_n coincide on $\Omega = (\cup A_n^2 - A_n^1)^c \cap (\cup A_n'^2 - A_n'^1)^c$. □

Proposition 15.1 *Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space. We have a bijection between the measurable partitions of X and the complete sub-σ-algebras of $\mathcal{T}$. This bijection is given by*

$$\xi \longrightarrow \widehat{\xi}$$
$$\xi_{\mathcal{A}} \longleftarrow \mathcal{A}.$$

Proof We take a partition ξ, denote by B_n the sets from its definition, and associate with it the σ-algebra $\widehat{\xi}$. By Lemma 15.1, this σ-algebra is generated by the B_n and the negligible sets. We can use these sets B_n to define the partition $\xi_{\widehat{\xi}}$, which shows the equality $\xi_{\widehat{\xi}} = \xi$.

Let $\mathcal{A}$ be a complete σ-algebra, and let $\{B_n\}$ be a countable family of subsets that generate $\mathcal{A}$ after completion. These sets B_n can be used to define the partition $\xi_{\mathcal{A}}$. By Lemma 15.1, the partition $\widehat{\xi_{\mathcal{A}}}$ is generated by the B_n and the negligible sets, which shows the equality $\widehat{\xi_{\mathcal{A}}} = \mathcal{A}$. □

15.5 Factors and Partitions

We introduce the notion of a factor and show that there exists a bijection between the factors of a Lebesgue space and the measurable partitions of the space.

Definition 15.2 Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space. A factor of $(X, \mathcal{T}, \mu)$ consists of a Lebesgue space $(Y, \mathcal{S}, \nu)$ and a measurable map $\varphi : X \to Y$ satisfying $\varphi_* \mu = \nu$.

In Chap. 13, we saw that the map φ is almost surjective. We will call φ the projection associated with the factor.

With every factor of $(X, \mathcal{T}, \mu)$, we can associate a partition of X using the formula

$$\xi = \{\varphi^{-1}(\{y\}) \mid y \in Y\}.$$

This partition is measurable; it suffices to take the B_n to be the inverse images of a countable family of subsets of Y that separates the points. Conversely, with every measurable partition of $(X, \mathcal{T}, \mu)$, we can associate a factor by taking the quotient of X by the equivalence relation

$$x \sim y \text{ if and only if } y \in \xi(x).$$

The quotient space is denoted by X/ξ and the canonical projection by $\pi : X \to X/\xi$. This space is endowed with the σ-algebra $\pi_* \mathcal{T} = \{A \subset X/\xi \mid \pi^{-1} A \in \mathcal{T}\}$ and the measure $\pi_* \mu$.

Lemma 15.4 *The space* $(X/\xi, \pi_* \mathcal{T}, \pi_* \mu)$ *is a Lebesgue space.*

Proof It is immediate that $\pi_* \mathcal{T}$ is complete with respect to $\pi_* \mu$. We denote by B_n the sets from the definition of the partition ξ. The map $\varphi : X \to \{0, 1\}^{\mathbf{N}}$ defined by $\varphi(x) = \{\mathbf{1}_{B_n}(x)\}$ passes to the quotient and gives an injective measurable map $\overline{\varphi} : (X/\xi, \pi_* \mathcal{T}, \pi_* \mu) \to \{0, 1\}^{\mathbf{N}}$. The measurability lemma shows that the images of the elements of $\pi_* \mathcal{T}$ are $\varphi_* \mu$-measurable sets. The map $\overline{\varphi}$ is therefore an isomorphism:

$$\overline{\varphi} : (X/\xi, \pi_* \mathcal{T}, \pi_* \mu) \xrightarrow{\sim} \left(\{0, 1\}^{\mathbf{N}}, \overline{\mathcal{B}(\{0, 1\}^{\mathbf{N}})}, \varphi_* \mu\right).$$

□

We will call two factors *isomorphic* if there exists an isomorphism between the factors that commutes with the two projections.

We consider a factor $\varphi : (X, \mathcal{T}, \mu) \to (Y, \mathcal{S}, \nu)$ and denote by ξ the associated partition. Note that the map φ passes to the quotient and gives a measurable injection from $(X/\xi, \pi_* \mathcal{T}, \pi_* \mu)$ to $(Y, \mathcal{S}, \nu)$. Since these two spaces are Lebesgue spaces, this injection is an isomorphism.

It follows that there is a bijection between the factors and the measurable partitions of Lebesgue spaces.

15.6 σ-Algebras and Algebras of Functions

Let $(X, \mathcal{T}, \mu)$ be a probability space; we denote by $L^0(X, \mathcal{T}, \mu)$ the algebra of real-valued measurable functions, and we endow it with the distance

$$d(f, g) = \inf_{\varepsilon > 0} \big\{\varepsilon + \mu\big(\{x \in X \mid d(f(x), g(x)) > \varepsilon\}\big)\big\}.$$

This makes L^0 into a complete (but not locally convex) metric space. The convergence associated with this distance is the convergence in probability.

Proposition 15.2 *There exists a bijection between the complete sub-σ-algebras of $\mathcal{T}$ and the closed subalgebras of $L^0(X, \mathcal{T}, \mu)$. This bijection is given by*

$$\begin{aligned} \mathcal{A} &\longrightarrow L^0(X, \mathcal{A}, \mu) \\ \{A \in \mathcal{T} \mid \mathbf{1}_A \in \mathbf{A}^0\} &\longleftarrow \mathbf{A}^0. \end{aligned}$$

Proof The equality $\mathcal{A} = \{A \in \mathcal{T} \mid \mathbf{1}_A \in L^0(X, \mathcal{A}, \mu)\}$ is proved by approximating the $\mathcal{A}$-measurable functions with linear combinations of characteristic functions of elements of $\mathcal{A}$.

The equality $\mathbf{A}^0 = L^0(X, \{A \mid \mathbf{1}_A \in \mathbf{A}^0\}, \mu)$ corresponds to showing that if $f \in \mathbf{A}^0$, then $\mathbf{1}_{f^{-1}(I)} \in \mathbf{A}^0$ for every open interval $I \subset R$. This is a consequence of the following fact, left as an exercise: there exists a sequence of P_n defined on $\mathbf{R}$ that converges simply to $\mathbf{1}_I$. $\square$

15.7 The Rokhlin Correspondence

From the earlier lemmas, we deduce the following theorem.

Theorem 15.1 *Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space. There exist bijections between*

- *the measurable partitions of X;*
- *the complete sub-σ-algebras of $\mathcal{T}$;*
- *the closed subalgebras of $L^0(X, \mathcal{T}, \mu)$;*
- *the factors of $(X, \mathcal{T}, \mu)$, up to isomorphism.*

Let us give two examples to illustrate this correspondence. The first concerns the decomposition into ergodic components, while the second refers to the notion of a generating partition.

We consider a Lebesgue space $(X, \mathcal{T}, \mu)$ and a measurable map $T : X \to X$ that preserves the measure μ. We can consider the completion of the σ-algebra of sets invariant under the transformation T. With this σ-algebra is associated a measurable partition C of X into invariant sets. Its elements are the ergodic components of T.

By the above, the partition C into ergodic components is uniquely determined by the following two properties:

- All elements of the partition are invariant under T.
- Every invariant set is the union of elements of the partition.

These two properties hold up to negligible sets: two sets are considered to be equal if their symmetric difference is negligible.

The space X/C is the set of ergodic components of T, while the associated subalgebra of $L^0(X, \mathcal{T}, \mu)$ consists of the measurable functions that are invariant under the transformation T. The partition C of course coincides with the one constructed in the last chapter. It is worth noting that we can define it without using the disintegration theorem for measures. Nonetheless, this theorem gives ergodic measures on each of the components, and this is what turns out to be useful in practice.

Let us give a second illustration of the correspondence seen earlier by proving the following proposition, which was used in Chap. 11.

Proposition 15.3 *Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let $T : X \to X$ be a measurable map that preserves the measure μ. A finite partition ξ is a one-sided generator if and only if there exists a set $X_0 \subset X$ of full measure such that the map*

$$X \longrightarrow \xi^{\mathbf{N}}$$

$$x \longmapsto \{\xi(T^i(x))\}_{i\in\mathbf{N}}$$

restricted to X_0 is injective.

Proof We denote this map by $\varphi : X \to \xi^{\mathbf{N}}$, and view it as a factor, where the set $\xi^{\mathbf{N}}$ is endowed with the measure $\varphi_*\mu$. With this factor is associated a measurable partition ξ^*, given by

$$\xi^*(x) = \varphi^{-1}(\{\varphi(x)\}) = \bigcap_{k\in\mathbf{N}} (T^{-k}\xi)(x).$$

We order this family $\{T^{-k}A \mid k \in \mathbf{N}, A \in \xi\}$ so as to obtain a sequence of measurable sets $\{B_n\}_{n\in\mathbf{N}}$, and note that the complement of an element of this family can be written as a finite union of elements of this family. The partition ξ^* is associated with the B_n through the relation

$$\xi^*(x) = \bigcap_{\substack{k\in\mathbf{N}, A\in\xi \\ x\in T^{-k}A}} T^{-k}A = \bigcap_{B_n \ni x} B_n = \bigcap_{B_n \ni x} B_n \bigcap_{B_n^c \ni x} B_n^c.$$

The σ-algebra associated with the partition ξ^* is the σ-algebra generated by the B_n:

$$\widehat{\xi^*} = \langle B_n \mid n \in \mathbf{N}\rangle = \langle T^{-k}A \mid k \in \mathbf{N}, A \in \xi\rangle.$$

The partition ξ is generating if this σ-algebra equals $\mathcal{T}$, or equivalently if and only if the partition ξ^* coincides, up to a negligible set, with the partition into singletons $\xi_{\mathcal{T}}$. In other words, the partition ξ is generating if and only if there exists $X_0 \subset X$ with $\mu(X_0^c) = 0$ such that

$$\forall x \in X_0, \quad \xi^*(x) \cap X_0 = \xi_{\mathcal{T}}(x) \cap X_0 = \{x\}.$$

This condition is equivalent to the injectivity of the restriction of φ to X_0:

$$\forall x \in X_0, \quad \varphi^{-1}(\{\varphi(x)\}) \cap X_0 = \xi^*(x) \cap X_0 = \{x\}.$$

□

15.8 Exercises

15.8.1 Basic Exercises

***Exercise* 1** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let $\mathcal{A}$ be a closed subalgebra of $L^0(X, \mathcal{T}, \mu)$. Show that there exists a partition of X such that the functions of $\mathcal{A}$ are exactly those that coincide almost everywhere with the functions that are constant on each of the elements of the partition.

***Exercise* 2** Determine the measurable partitions associated with the closure of the following subalgebras of $L^0([-1, 1], \mathcal{B}([-1, 1]), \lambda)$:

- the algebra of even functions;
- the algebra generated by the polynomials of odd degree;
- the algebra generated by the function $x \mapsto x(x-1)(x+1)$.

***Exercise* 3** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let $f : X \to \mathbf{R}$ be a measurable function. Let $\mathcal{A}$ be the closed subalgebra of L^0 generated by f.

- Show that the partition of X given by the level sets of f, that is, $\xi(x) = f^{-1}(\{f(x)\})$, is measurable.
- Show that this partition is the one associated with the subalgebra $\mathcal{A}$.
- This result is well known to probabilists. In what type of problems is it used?

***Exercise* 4** Consider the map on the torus $(\mathbf{R}/\mathbf{Z})^2$ defined by $T(x, y) = (x + y, y)$. This map preserves the Lebesgue measure. Explicitly give the partition of T into ergodic components with respect to this measure.

The map T also preserves the Dirac measure at the origin $(0, 0)$. Explicitly give the ergodic components of T with respect to this measure.

15.8.2 More Advanced Exercises

***Exercise* 5** Let α be an irrational real number. We take $\mathbf{R}/\mathbf{Z}$ endowed with the Lebesgue measure, and consider the map $T : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ given by $T(x) = x + \alpha$. Show that the partition whose elements are the orbits of the map T is not measurable.

***Exercise* 6** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space. We say that a measurable partition ξ_1 of X is finer than a partition ξ_2 if there exists a set $\Omega \subset X$ with negligible complement such that $\xi_1(x) \cap \Omega \subset \xi_2(x) \cap \Omega$ for almost all $x \in X$.

Let ξ be an arbitrary partition of X. Show that there exists a measurable partition $\overline{\xi}$ of X such that every measurable function that is constant on the elements of ξ coincides almost everywhere with a function that is constant on the elements of $\overline{\xi}$. Show that among these measurable partitions $\overline{\xi}$, there exists one that is finer than the others.

***Exercise* 7** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let Γ be a countable group of measurable transformations of X, all invertible with measurable inverse and preserving μ. A set is called invariant under Γ if it is invariant under all transformations in the group. Show that there exists a measurable partition of X consisting of invariant sets, such that every invariant measurable set is the union of elements of this partition.

***Exercise* 8** Let $(X, \mathcal{T}, \mu)$ be a Lebesgue space, and let $T : X \to X$ be a measurable map that preserves the measure μ. Denote by $\mu_{C(x)}$ the measures obtained by the disintegration of μ into ergodic components along the partition C, and by $\pi : X \to X/C$ the canonical projection. Show that the following formula holds:

$$h_\mu(T) = \int_X h_{\mu_{C(x)}}(T)\, \mathrm{d}\mu(x) = \int_{X/C} h_{\mu_C}(T)\, \mathrm{d}\pi_*\mu(C).$$

Part V
Appendices

Chapter 16
Weak Convergence

The study of various topologies and the relations among them is, despite its current popularity in the theory of topological linear spaces, a pretty dull business.

P.R. Halmos (1916–2006)

16.1 Convergence in a Hilbert Space

A Hilbert space H is a vector space endowed with an inner product such that the norm associated with the inner product is complete. We use the notation $\langle\ ,\ \rangle$ for the inner product, and $\|f\| = \sqrt{\langle f, f\rangle}$ for the norm.

We can endow H with two topologies:

- the *strong topology*: a sequence f_n of elements of H converges strongly to $f \in H$ if $\|f_n - f\| \xrightarrow[n\to\infty]{} 0$;
- the *weak topology*: a sequence f_n of elements of H converges weakly to $f \in H$ if $\langle f_n, g\rangle \xrightarrow[n\to\infty]{} \langle f, g\rangle$ for all $g \in H$.

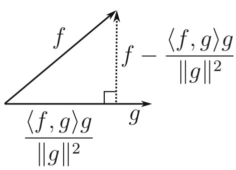
Theorem 16.1 *Every strongly convergent sequence is weakly convergent.*

This is a consequence of the following inequality.

Cauchy-Schwarz Inequality *For all $f, g \in H$, we have $|\langle f, g\rangle| \leqslant \|f\|\cdot\|g\|$.*

Proof This is a consequence of the Pythagorean theorem:

$$\|f\|^2 = \left\|\left\langle f, \frac{g}{\|g\|}\right\rangle \frac{g}{\|g\|}\right\|^2 + \left\|f - \left\langle f, \frac{g}{\|g\|}\right\rangle \frac{g}{\|g\|}\right\|^2.$$



We have equality if and only if f is proportional to g. $\square$

Y. Coudène, *Ergodic Theory and Dynamical Systems*, Universitext,
DOI 10.1007/978-1-4471-7287-1_16

Theorem 16.2 *Every weakly convergent sequence is bounded.*

Proof It suffices to show that there exist $g \in H$ and $\varepsilon > 0$ such that

$$\sup\{\langle f_n, h\rangle \mid n \in \mathbf{N},\ h \in B(g, \varepsilon)\} < \infty,$$

because then the following equality shows that $\|f_n\|$ is bounded independently of n:

$$\|f_n\| = \frac{1}{\varepsilon}\big(\langle f_n, g + \varepsilon f_n/\|f_n\|\rangle - \langle f_n, g\rangle\big).$$

Suppose that we cannot find g and ε as above. We construct, by induction, an increasing sequence of integers n_k and a nested decreasing sequence of closed balls B_k whose diameters tend to 0, such that for all $h \in B_k$, we have $\langle f_{n_k}, h\rangle \geqslant 2^k$.

Let g_∞ be the unique common point of all B_k. We then have the inequality $\langle f_{n_k}, g_\infty\rangle \geqslant 2^k$ that contradicts the weak convergence of the sequence f_n. □

Weak convergence corresponds to "coordinatewise" convergence.

Proposition 16.1 *Let $\mathcal{D}$ be a set of elements of H that generates a dense linear subspace of H; to show that f_n converges weakly to f, it suffices to verify the convergence $\langle f_n, g\rangle \xrightarrow[n\to\infty]{} \langle f, g\rangle$ for $g \in \mathcal{D}$.*

Proof Let $\varepsilon > 0$, and let $g \in H$ and $g_k \in \mathrm{Vect}(\mathcal{D})$ be such that $\|g_k - g\| \xrightarrow[n\to\infty]{} 0$. We have the inequality

$$|\langle f_n - f, g\rangle| \leqslant |\langle f_n - f, g_k\rangle| + \|f_n - f\|\cdot\|g_k - g\|.$$

Since the sequence $\|f_n - f\|$ is bounded, we can find $k \in \mathbf{N}$ such that the last term of this inequality is less than ε for all n. For this value of k, the sequence $\langle f_n - f, g_k\rangle$ tends to 0; we can therefore bound $|\langle f_n - f, g\rangle|$ from above by $2\,\varepsilon$ for n sufficiently large. □

Here are two other results on the weak topology.

16.2 Weak Sequential Compactness

Theorem 16.3 (Banach–Alaoglu) *Let H be a Hilbert space, and consider a bounded sequence $\{f_n\}_{n\in\mathbf{N}}$ of elements of H. Then we can find $f \in H$ and a subsequence $f_{n_1}, \ldots, f_{n_k}, \ldots$ such that*

$$f_{n_k} \longrightarrow f \quad \textit{weakly when} \quad k \longrightarrow +\infty.$$

Proof Let F be the closed linear subspace generated by the f_n, and let $\{v_m\}$ be a Hilbert basis for F. Such a basis can be obtained by applying the Gram–Schmidt orthonormalization process to the f_i. Since $\langle v_1, f_n\rangle$ is a bounded sequence, we can find a subsequence $f_{n'_i}$ such that $\langle v_1, f_{n'_i}\rangle$ converges. From this subsequence, we can extract a subsequence such that $\langle v_2, f_{n''_i}\rangle$ converges, and so on. By taking the kth term of the kth subsequence, we construct a sequence f_{n_k} such that for all $m \in \mathbf{N}$, the sequence $\langle v_m, f_{n_k}\rangle$ converges. We denote its limit by c_m. Let $N \in \mathbf{N}$; we have

$$\|f_{n_k}\|^2 \geqslant \sum_{m=0}^{N} |\langle v_m, f_{n_k}\rangle|^2 \xrightarrow[k\to\infty]{} \sum_{m=0}^{N} |c_m|^2.$$

Since the sequence $\|f_{n_k}\|$ is bounded, this last quantity converges, and we obtain a well-defined element of H by setting $f = \sum c_i v_i$. For all $m \in \mathbf{N}$, this element satisfies $\langle v_m, f_{n_k}\rangle \xrightarrow[k\to\infty]{} \langle v_m, f\rangle$. We therefore have $\langle h, f_{n_k}\rangle \to \langle h, f\rangle$ if $h \in F$. If $h \in F^{\perp}$, then $\langle h, f_{n_k}\rangle = 0$ and $\langle h, f\rangle = 0$. This proves the theorem. □

16.3 Convex Closed Subsets

Theorem 16.4 (Banach–Saks) *Let H be a Hilbert space. Consider elements f and $\{f_n\}_{n\in\mathbf{N}}$ of H such that the sequence f_n converges weakly to f. Then there exists a subsequence $f_{n_1}, \ldots, f_{n_k}, \ldots$ such that*

$$\frac{1}{m}\sum_{k=1}^{m} f_{n_k} \xrightarrow[m\to\infty]{} f \quad \textit{strongly.}$$

Proof We may assume $f = 0$. Let us construct a subsequence f_{n_k} satisfying

$$\sum_{i<j} |\langle f_{n_i}, f_{n_j}\rangle| < +\infty.$$

We suppose that we have constructed $f_{n_1}, \ldots, f_{n_k}$ such that if $i < j$, we have $|\langle f_{n_i}, f_{n_j}\rangle| < 1/2^j$.

For all $i \leqslant k$, we know that $\langle f_{n_i}, f_n\rangle \xrightarrow[n\to\infty]{} 0$. We can therefore find n_{k+1} such that for all $i \leqslant k$, we have $|\langle f_{n_i}, f_{n_{k+1}}\rangle| < 1/2^{k+1}$, which gives

$$\sum_{i<j} |\langle f_{n_i}, f_{n_j}\rangle| < \sum \frac{j}{2^j} \leqslant 2.$$

We can now give an upper bound for the average along the sequence n_k:

$$\begin{aligned}\left\|\frac{1}{m}\sum_{k=1}^{m} f_{n_k}\right\|^2 &= \frac{1}{m^2}\sum_{k=1}^{m}\|f_{n_k}\|^2 + \frac{2}{m^2}\sum_{1\leqslant i<j\leqslant m}\langle f_{n_i}, f_{n_j}\rangle \\ &\leqslant \frac{1}{m}\sup\|f_{n_k}\|^2 + \frac{4}{m^2}.\end{aligned}$$

This quantity indeed tends to 0. □

Remarks

- It follows that every strongly closed convex set is weakly closed.
- Let $(X, \mathcal{T}, \mu)$ be a measure space. Every L^2 sequence that converges in L^2 norm admits a subsequence that converges almost everywhere. If H is of the form $L^2(X)$, we can therefore find m_i such that

$$\frac{1}{m_i}\sum_{k=1}^{m_i} f_{n_k} \xrightarrow[i\to\infty]{} f \quad \text{almost everywhere.}$$

Chapter 17
Conditional Expectation

Probability theory is measure theory with a soul.

M. Kac (1914–1984)

17.1 Definition of the Conditional Expectation

This appendix recalls a number of results on the notion of conditional expectation. Let $(X, \mathcal{T}, \mu)$ be a measure space, with μ a finite measure, and let $\mathcal{A}$ be a σ-algebra contained in $\mathcal{T}$. With every function $f \in L^1(X, \mathcal{T}, \mu)$, we can associate a signed measure on the σ-algebra $\mathcal{A}$ using the formula

$$\mu_f(A) = \int_A f \, \mathrm{d}\mu.$$

The *conditional expectation* of f with respect to the σ-algebra $\mathcal{A}$ is equal to the Radon–Nikodym derivative of $\mu_{f|\mathcal{A}}$ with respect to $\mu_{|\mathcal{A}}$. This derivative is denoted by $E(f \mid \mathcal{A})$; it is the unique function in $L^1(X, \mathcal{A}, \mu)$ that satisfies, for all $A \in \mathcal{A}$,

$$\mu_f(A) = \int_A E(f \mid \mathcal{A}) \, \mathrm{d}\mu.$$

In other words, the function $E(f \mid \mathcal{A})$ is uniquely determined by the following two relations:

$$E(f \mid \mathcal{A}) \in L^1(X, \mathcal{A}, \mu),$$
$$\forall A \in \mathcal{A}, \quad \int_A E(f \mid \mathcal{A}) \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu.$$

When the function f is square integrable, its conditional expectation is also square integrable. The operator $f \mapsto E(f \mid \mathcal{A})$ then identifies with the orthogonal projection of $f \in L^2(X, \mathcal{T}, \mu)$ onto the subspace $L^2(X, \mathcal{A}, \mu)$. We could have defined the conditional expectation of L^2 functions using this projection, and then extended the definition to all integrable functions using a density argument.

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17.2 Properties of the Conditional Expectation

Let us give a few properties of the conditional expectation. They follow from the previous description, and are left as an exercise.

- For all $f_1, f_2 \in L^1(X, \mathcal{T}, \mu)$ such that $f_1 \leqslant f_2$,

$$E(f_1 \mid \mathcal{A}) \leqslant E(f_2 \mid \mathcal{A}).$$

- For all $f \in L^1(X, \mathcal{T}, \mu)$,

$$|E(f \mid \mathcal{A})| \leqslant E(|f| \mid \mathcal{A}).$$

- Let $f \in L^1(X, \mathcal{T}, \mu)$, and let $g \in L^\infty(X, \mathcal{A}, \mu)$; we have the equality

$$E(fg \mid \mathcal{A}) = g\, E(f \mid \mathcal{A}).$$

- Let $\mathcal{A}' \subset \mathcal{A}$ be a sub-σ-algebra; we have the equality

$$E(E(f \mid \mathcal{A}) \mid \mathcal{A}') = E(f \mid \mathcal{A}').$$

- Let $T : X \to X$ be a measurable map that preserves the measure μ; we have

$$E(f \mid \mathcal{A}) \circ T = E(f \circ T \mid T^{-1}\mathcal{A}).$$

17.3 The Martingale Convergence Theorem in L^2

The conditional expectation is a continuous operator with respect to the first argument: if f_n is a sequence of integrable functions that converges in L^1 norm to a function f, then the conditional expectations of the f_n converge to that of f. We will prove a continuity property with respect to the second argument: if $\mathcal{A}_n$ is an increasing sequence of σ-algebras, then the expectation of a function f with respect to these σ-algebras converges to the expectation of f with respect to the σ-algebra generated by all the $\mathcal{A}_n$.

We begin by stating an abstract convergence result in Hilbert spaces.

Proposition 17.1 *Let H be a Hilbert space, and let E_i be a nested increasing sequence of closed linear subspaces of H: $E_i \subset E_j$ if $i < j$. We denote by E_∞ the closure of the union of the E_i for $i \in \mathbf{N}$. The orthogonal projections onto the E_i are denoted by π_i. Then, for all $v \in H$, we have the convergence in norm*

$$\pi_i v \xrightarrow[i \to \infty]{} \pi_\infty v.$$

Proof Our proof is based on the weak compactness of the unit ball in a Hilbert space. The sequence $\pi_n v$ is bounded by $\|v\|$; we will show that it converges weakly. Consider an accumulation point v_∞ of $\pi_n v$; let us show that it equals the orthogonal projection of v on E_∞.

All $\pi_n v$ are in E_∞; the same holds for their limit v_∞. Next, the terms of the sequence $v - \pi_n v$ are orthogonal to E_i for n greater than i. The limit $v - v_\infty$ is orthogonal to E_i for all i; it therefore belongs to the orthogonal complement of E_∞:

$$v_\infty \in E_\infty, \quad v - v_\infty \in E_\infty^\perp.$$

These properties determine the projection of v on E_∞. We have shown that $\pi_\infty v$ is the only accumulation point of the sequence $\pi_n v$; the sequence $\pi_n v - \pi_\infty v$ converges weakly to 0. In particular, this implies

$$\langle \pi_\infty v - \pi_n v, v \rangle \xrightarrow[n\to\infty]{} 0.$$

This quantity equals the square of the norm that we wish to bound from above:

$$\begin{aligned} \|\pi_\infty v - \pi_n v\|^2 &= \langle (\pi_\infty - \pi_n)^* (\pi_\infty - \pi_n)\, v, v \rangle \\ &= \langle (\pi_\infty - \pi_n)^2\, v, v \rangle \\ &= \langle (\pi_\infty - \pi_n)\, v, v \rangle. \end{aligned}$$

This norm converges to 0, and the theorem is proved. □

Let us now turn to the martingale convergence theorem in L^2.

Theorem 17.1 *Let $(X, \mathcal{T}, \mu)$ be a probability space, and let $\mathcal{A}_i$ be a nested increasing sequence of σ-algebras contained in $\mathcal{T}$: $\mathcal{A}_i \subset \mathcal{A}_j$ if $i < j$. We denote by $\mathcal{A}$ the σ-algebra generated by the $\mathcal{A}_i$. Then for all $f \in L^2(X, \mathcal{T}, \mu)$, we have the convergence in L^2 norm*

$$E(f \mid \mathcal{A}_n) \xrightarrow[n\to\infty]{} E(f \mid \mathcal{A}).$$

Proof To alleviate the notation, we suppose $\mathcal{A} = \mathcal{T}$; the general case can be deduced from this one by replacing f by $E(f \mid \mathcal{A})$. The conditional expectation of f with respect to the σ-algebra $\mathcal{A}_n$ is the orthogonal projection of f on $L^2(X, \mathcal{A}_n)$. We denote by π_∞ the limit of these projections. We need to show that this limit equals the identity. Since measurable simple functions are dense in the L^2 functions, it suffices to verify that the following collection of sets coincides with $\mathcal{A}$:

$$\mathcal{A}' = \{A \in \mathcal{A} \mid \pi_\infty \mathbf{1}_A = \mathbf{1}_A\}.$$

This class contains the union of the $\mathcal{A}_i$, which is an algebra of subsets of X. Let us verify that $\mathcal{A}'$ is a monotone class. It is indeed invariant under taking the complement; let us show that it is invariant under taking an increasing countable union. Let A_n be an increasing sequence of elements of $\mathcal{A}'$; the sequence $\mathbf{1}_{A_n}$ converges to $\mathbf{1}_{\cup A_n}$ in L^2 norm, by the dominated convergence theorem. It follows that

$$\pi_\infty\big(\mathbf{1}_{\cup A_n}\big) = \lim_{n\to\infty} \pi_\infty\big(\mathbf{1}_{A_n}\big) = \lim_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_{\cup A_n}.$$

By the monotone class theorem, $\mathcal{A}'$ contains the σ-algebra generated by the $\mathcal{A}_n$. It coincides with $\mathcal{A}$, and the projection π_∞ equals the identity, as desired. □

To illustrate this theorem, we take a probability space $(X, \mathcal{T}, \mu)$ and consider the product $(X^{\mathbf{N}}, \mathcal{T}^{\otimes \mathbf{N}}, \mu^{\otimes \mathbf{N}})$. Let $p_n : X^{\mathbf{N}} \to X^n$ be the projection onto the first n coordinates, and set $\mathcal{A}_n = p_n^{-1}(\mathcal{T}^{\otimes n})$. The σ-algebras $\mathcal{A}_n$ generate $\mathcal{T}^{\otimes \mathbf{N}}$, by definition of the tensor product. Therefore, for all $f \in L^2(X^{\mathbf{N}})$, we have

$$E(f \mid \mathcal{A}_n) \longrightarrow f.$$

The function f is thus approximated explicitly in L^2 norm by functions that depend on only a finite number of coordinates. This result also holds for any integrable function f, because the space $L^2 \cap L^1$ is dense in L^1.

Finally, note that the convergence in the martingale convergence theorem also holds almost everywhere. A proof of this can be found in the second chapter of Parry's book [18].

Chapter 18
Topology and Measures

We think in generalities, but we live in details.

A.N. Whitehead (1861–1947)

18.1 Separability

Definition 18.1 A metric space X is *separable* if it contains a dense countable subset. A topological space is said to have a *countable base* if there exists a countable collection of open sets such that every open set can be written as a union of elements of this collection.

Note that in this case, every open set can be written as a countable union of elements of the base, more specifically, as the union of all elements of the base that it contains.

Proposition 18.1 *A metric space is separable if and only if it has a countable base.*

This implies that every subset of a separable metric space X is separable. Indeed, such a subset admits a countable base of open sets, which is obtained by restricting the base of X to the subset.

Proof If X has a countable base, it suffices to take a point in each element of the base. The resulting set meets all open sets and is therefore dense.

Suppose that X is separable; let $\mathcal{D}$ be a dense countable subset. Let us show that the set $\{B(y,r) \mid y \in \mathcal{D},\ r \in \mathbf{Q}\}$ is a base of open sets. Let U be an open set, and let $x \in U$. We can find a point $y \in \mathcal{D}$ such that $d(y,x) < d(y,U^c)$. Let $r \in \mathbf{Q}$ be such that $d(y,x) < r < d(y,U^c)$. The relation $x \in B(y,r) \subset U$ shows that U is indeed a union of open balls with rational radius and center in $\mathcal{D}$. □

18.2 The Support of a Measure

Definition 18.2 Let X be a metric space, and let μ be a Borel measure. A point $x \in X$ belongs to the support of the measure μ if every neighborhood of x has positive measure. The *support* is denoted by $\operatorname{supp}\mu$.

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Let us give a few properties of the support.

Proposition 18.2

- *The support is a closed set.*
- *If X is separable, then* $\mu\big((\operatorname{supp}\mu)^c\big) = 0$.
- *Let $A \subset X$ be such that $\mu(A^c) = 0$. Then* $\operatorname{supp}\mu \subset \overline{A}$.
- *If the measure μ is finite, then its support is separable: there exists a countable subset of the support that is dense in the support.*

Proof We have the equality

$$(\operatorname{supp}\mu)^c = \bigcup_{\substack{U \text{ open} \\ \mu(U)=0}} U.$$

The union can be restricted to a base of open sets; this proves the first two points. Let $x \in \operatorname{supp}\mu$, and let U be an open set containing x. Since $\mu(U) > 0$ and $\mu(A^c) = 0$, we have $\mu(U \cap A) > 0$. The open set U meets A, which shows that $x \in \overline{A}$; that is, $\operatorname{supp}\mu \subset \overline{A}$.

For every $k, n \in \mathbf{N}^*$, we consider the sets A satisfying the following two conditions:

- For all $x, y \in A$, the inequality $x \neq y$ implies $d(x, y) > 2/k$.
- For all $x \in A$, we have $\mu\big(B(x, 1/k)\big) > 1/n$.

These sets have finite cardinal, bounded from above by $n\mu(X)$. For each k and n, we choose such a set $A_{k,n}$ of maximal cardinality. The union of these sets, for $k, n \in \mathbf{N}^*$, forms a dense countable subset of the support of μ.

Indeed, if $x \in \operatorname{supp}\mu$ and $k > 0$, there exists $n \in \mathbf{N}^*$ such that $\mu\big(B(x, 1/k)\big) > 1/n$. Because $A_{k,n}$ is maximal, we have $d(x, y) < 2/k$ for some $y \in A_{k,n}$; hence $x \in A_{k,n}$. □

18.3 Density in the L^p Spaces

Let X be a topological space. We denote its Borel σ-algebra by $\mathcal{B}(X)$. Let μ be a Borel measure defined on X. The completion of $\mathcal{B}(X)$ with respect to μ will be denoted by $\overline{\mathcal{B}(X)}$; its elements are the *$\mu$-measurable* sets:

$$\overline{\mathcal{B}(X)} = \{A \subset X \mid \exists B, C \in \mathcal{B}(X) \text{ such that } B \subset A \subset C \text{ and } \mu(C \setminus B) = 0\}.$$

We can show that the Lipschitz functions are dense in the L^p spaces. The proof uses the regularity properties of the measure.

Proposition 18.3 (Outer Regularity) *Let X be a metric space, let μ be a finite Borel measure, and let $A \subset X$ be a μ-measurable set. Then for every $\varepsilon > 0$, there exists an open set U containing A such that $\mu(U \smallsetminus A) < \varepsilon$.*

We say that the measure is *outer regular*. This implies that every μ-measurable set can be written as $(\cap U_i) \smallsetminus N$, where U_i is a sequence of open subsets of X and N is a negligible set.

Proof Set

$$\mathcal{T} = \left\{ A \text{ measurable} \;\middle|\; \begin{array}{l} \forall \varepsilon > 0,\ \exists U \text{ open such that} A \subset U \text{ and } \mu(U \smallsetminus A) < \varepsilon \\ \forall \varepsilon > 0,\ \exists V \text{ open such that } A^c \subset V \text{ and } \mu(V \smallsetminus A^c) < \varepsilon \end{array} \right\}.$$

The open sets are in $\mathcal{T}$: the complement of any open set U can be written as an intersection of open sets $V_n = \{x \in X \mid d(x, U^c) < 1/n\}$, so the measure of the V_n tends to that of U^c. The negligible sets are also in $\mathcal{T}$. It now suffices to show that $\mathcal{T}$ is a σ-algebra. The invariance under taking the complement is clear. Let A_i be measurable sets in $\mathcal{T}$, and let U_i be open sets such that $A_i \subset U_i$ and $\mu(U_i \smallsetminus A_i) < \varepsilon/2^i$. The union of the U_i is indeed an approximation of the union of the A_i:

$$\bigcup A_i \subset \bigcup U_i, \quad \mu(\bigcup U_i \smallsetminus \bigcup A_i) \leqslant \sum \mu(U_i \smallsetminus A_i) < \varepsilon.$$

For the complements, there exists n such that $\mu\big(\bigcap_1^n A_i \smallsetminus \bigcap_1^\infty A_i\big) < \varepsilon/2$, and we use the inclusion

$$\bigcap_1^n V_i \smallsetminus \bigcap_1^n A_i^c \subset \bigcup_1^n (V_i \smallsetminus A_i^c).$$

□

Theorem 18.1 (Density in the L^p Spaces) *Let X be a metric space, let μ be a finite Borel measure, and let $1 \leqslant p < \infty$. Then every function $f \in L^p(X, \mu)$ can be approximated, in L^p norm, by a sequence of bounded Lipschitz functions belonging to L^p.*

Proof Since the functions that are linear combinations of characteristic functions are dense in the L^p spaces, we can restrict ourselves to the case where f is a characteristic function: $f = \mathbf{1}_A$. Let U be an open set containing A such that $\mu(U \smallsetminus A) < \varepsilon$. We have

$$\|\mathbf{1}_U - \mathbf{1}_A\|_p = \mu(U \smallsetminus A)^{1/p} < \varepsilon^{1/p}.$$

It now suffices to approximate $\mathbf{1}_U$ by a sequence of bounded Lipschitz functions. Set $f_k = \min(1, kd(x, U^c))$. These functions are Lipschitz:

$$\forall x, y \in X, \quad |f_k(x) - f_k(y)| \leqslant k\, d(x, y).$$

The sequence f_k converges pointwise to $\mathbf{1}_U$, and the functions $|\mathbf{1}_U - f_k|$ are uniformly bounded from above by 1. This ensures the convergence of the f_k to $\mathbf{1}_U$ in L^p norm. □

Remark This theorem also holds if X is a separable metric space and μ is locally finite.

18.4 Inner Regularity

Let us now study a property that is stronger than outer regularity.

Definition 18.3 A Borel measure μ defined on X is said to be *inner regular* if for every Borel subset $A \subset X$ of finite measure and every $\varepsilon > 0$, there exists a compact subset $K \subset A$ such that $\mu(A \smallsetminus K) < \varepsilon$.

This property is satisfied whenever the underlying space is a complete separable metric space.

Theorem 18.2 (Oxtoby–Ulam) *A finite Borel measure defined on a complete separable metric space is inner regular.*

Proof Let $\{x_i\}$ be a dense sequence in X, and let $n \in \mathbf{N}^*$. The family of closed balls $\overline{B}(x_i, 1/n)$ for $i \in \mathbf{N}$ covers X, so that there exists $N \in \mathbf{N}$, depending on n, such that

$$\mu\Big(X \smallsetminus \bigcup_{i=0}^{N} \overline{B}(x_i, 1/n)\Big) < \varepsilon/2^n.$$

Set

$$K' = \bigcap_{n\in\mathbf{N}^*} \bigcup_{i=0}^{N} \overline{B}(x_i, 1/n).$$

This set is closed and totally bounded; it is therefore compact. We moreover have the inequality $\mu(K'^c) < \varepsilon$. Let $A \subset X$ be a Borel set. By the outer regularity, there exists a closed subset $F \subset A$ such that $\mu(A \smallsetminus F) < \varepsilon$. Set $K = K' \cap F \subset A$. This set is compact, and $\mu(A \smallsetminus K) < 2\varepsilon$. The theorem has been proved. □

We conclude with a result due to Lusin, which shows that a measurable function is continuous on a set with small complement. When the measure is inner regular, this set can be taken to be compact.

Theorem 18.3 (Lusin) *Let X be a Hausdorff topological space, let μ be an inner regular finite measure, and let Y be a separable metric space. Let $f : X \to Y$ be a measurable function (that is, the inverse image of any Borel set is μ-measurable). Then for every μ-measurable $A \subset X$ and every $\varepsilon > 0$, there exists a compact subset $K \subset A$ such that $\mu(A \smallsetminus K) < \varepsilon$ and $f_{|K}$ is continuous.*

Proof Fix $n \in \mathbf{N}^*$. Let E_i be a disjoint countable family of Borel sets with diameter less than $2/n$ that covers Y. We can construct the E_i by induction from a dense sequence $\{y_i\}_{i \in \mathbf{N}^*}$ in Y by setting $E_i = B(y_i, 1/n) \setminus \bigcap_{j<i} E_j$.

By regularity, for every i, there exist a Borel set A'_i and a compact set K_i such that

$$K_i \subset A'_i \subset A \cap f^{-1}(E_i), \quad \mu\big(A \cap f^{-1}(E_i) \setminus K_i\big) < \varepsilon/2^{i+n}.$$

We then have $\mu(A \setminus \cup K_i) < \varepsilon/2^n$, so that there exists $N \in \mathbf{N}$, depending on n, such that

$$\mu\Big(A \setminus \bigcup_{i=0}^{N} K_i\Big) < \varepsilon/2^n.$$

We define a function f_n on $K'_n = \cup_0^N K_i$ by setting $f_n(x) = y_i$ for every $x \in K_i$. The compact sets K_i are disjoint; hence the function f_n is continuous. It moreover satisfies $d(f_n(x), f(x)) < 1/n$ on K'_n. The sequence f_n converges uniformly to f on $\cap_n K'_n$, which shows that f is continuous on this compact set. This concludes the proof. □

18.5 Exercises

***Exercise* 1** Give an example of a locally compact metric space that is not separable.

***Exercise* 2** Let X be a separable metric space. Show that there exists a countable base of open sets $\mathcal{D}$ that is invariant under taking finite unions:

$$\forall n \in \mathbf{N},\ \forall U_1, \dots, U_n \in \mathcal{D}, \quad \bigcup U_i \in \mathcal{D}.$$

Hint: Take the family of open sets that are finite unions of sets belonging to a given countable base of open sets.

***Exercise* 3** Let X be a separable metric space. Show that there exists a sequence of continuous functions $f_n : X \to \mathbf{R}$ that separates points: for all distinct $x, y \in X$, there exists n such that $f_n(x) \neq f_n(y)$. Deduce that there exists a continuous injection from X to $\mathbf{R}^{\mathbf{N}}$.

***Exercise* 4** Give an explicit sequence of compact sets $K_n \subset [0, 1] \setminus \mathbf{Q}$ such that $\lambda(K_n)$ tends to 1 when n tends to $+\infty$.

***Exercise* 5** Let X be a Hausdorff topological space, endowed with a finite Borel measure. We say that the measure μ is *tight* if there exists a sequence of compact subsets K_i such that $\mu\big(X \setminus \bigcup_{i \in \mathbf{N}} K_i\big) = 0$. Show the equivalence of the following properties:

- The measure μ is tight.
- The measure μ is inner regular.
- For every Borel subset $A \subset X$, the measure μ restricted to A is tight.

***Exercise* 6** Show that every metric space endowed with an inner regular Borel probability measure is *almost separable*, that is, it admits a separable subset of full measure.
Hint: A totally bounded metric space is separable.

***Exercise* 7** Show that $\mathbf{R}^{\mathbf{N}}$ is not σ-compact, but is tight, with respect to every finite Borel measure.
Hint: Use the inner regularity of the measure.

***Exercise* 8** Show that the Oxtoby–Ulam theorem also holds if instead of assuming that the measure is finite, we assume that it is σ-finite, that is, that there exists a sequence of Borel sets B_i of finite measure such that $X = \bigcup_{i\in\mathbf{N}} B_i$.

***Exercise* 9** A σ-algebra $\mathcal{T}$ is said to be *complete* if it has the following property: for every $B \in \mathcal{T}$ such that $\mu(B) = 0$ and every $A \subset B$, the set A belongs to $\mathcal{T}$. Let X be a topological space. Show that $\overline{\mathcal{B}(X)}$ is complete.

***Exercise* 10** Let $(X, \mathcal{T}, \mu)$ be a measure space with $\mathcal{T}$ complete, let Y be a topological space, and let $f : X \to Y$ be a function such that the inverse image of every Borel set is in $\mathcal{T}$. Show that the inverse image of every $f_*\mu$-measurable set is in $\mathcal{T}$.

Let X and Y be two topological spaces, let μ and ν be Borel measures on X and Y, respectively, and let $\varphi : X \to Y$ be a Borel map such that $\varphi_*\mu = \nu$. Show that φ is measurable: the inverse image of a ν-measurable set is μ-measurable.

***Exercise* 11** Let $\varphi : [0, 1] \to [0, 1]$ be a measurable map in the following sense: the inverse image of a Borel set is Lebesgue-measurable. Show that there exists a Borel subset $X_0 \subset [0, 1]$ with Lebesgue-negligible complement such that $\varphi(X_0)$ is Borel and $\varphi : X_0 \to \varphi(X_0)$ is Borel.
Hint: Use Lusin's theorem.

Notation

The sets of natural numbers (including 0), integers, rational numbers, real numbers, and complex numbers are denoted respectively by **N**, **Z**, **Q**, **R**, **C**.

$\mathbf{1}_A$	Characteristic function of A
$B \ni x$	B contains x
$B(x, r)$	Open ball with center x and radius r
$BL(X)$	Space of bounded Lipschitz functions
$C_b(B, B)$	Space of bounded continuous maps
$C_b(X)$	Space of bounded continuous functions
Δ	Symmetric difference
∇V	Gradient of the function V
h_μ	Entropy of the measure μ
$\mathbf{H}$	Upper half-plane
$H(\xi)$	Entropy of the partition ξ
K	Compact set or Lipschitz constant
L^p	Space of classes of pth-power summable functions
$\overline{\lim}$	Upper limit
M	Riemann manifold
$\mathcal{M}^1(X)$	Space of invariant probability measures
μ	Measure
$\mathbf{N}^*$	Positive integers: $\{1, 2, 3, 4, \ldots\}$
$\omega(x)$	ω-limit set
Ω	Nonwandering set
$\varnothing$	Empty set
φ_t	Flow
a.e.	Almost everywhere
$\square$	Symbol indicating the end of a proof
$\mathrm{PSL}_2(\mathbf{R})$	Matrices with determinant ± 1

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supp μ	Support of μ
S^1	Unit circle
$S_n(f)$	Birkhoff sum of f
$\mathcal{T}$	σ-algebra
$\mathbf{T}^n$	Torus of dimension n
U^*	Adjoint of U
$\vee$	Generated partition
W^{ss}	Stable manifold
W^{su}	Unstable manifold
$\wedge$	Exterior product
X	Set
X_i	Random variable
ξ	Partition

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