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Yuming Qin

Analytic Inequalities and Their Applications in PDEs

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Analytic Inequalities and Their Applications in PDEs

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To my parents Zhenrong Qin, Xilan Xia

To my wife Yu Yin and my son Jia Qin

Contents

Preface xi

1 Integral Inequalities

1.1 The classical Bellman–Gronwall inequality 1

1.2 Linear generalizations of the Bellman–Gronwall inequalities 3

1.3 Simultaneous inequalities 17

1.4 The inequalities of Henry's type 20

1.4.1 The Henry inequalities 21

1.4.2 Henry–Gronwall–Bihari type integral inequalities 32

1.4.3 The Ou-Yang and Pachapatte type integral inequalities . . 44

1.4.4 Henry type inequalities with multiple integrals 48

1.5 Integral inequalities leading to upper bounds and decay rates . . . 51

2 Differential and Difference Inequalities

2.1 Differential inequalities leading to uniform bounds 79

2.2 Differential inequalities leading to asymptotic behavior 106

2.3 Differential and difference inequalities leading to decay rates . . . 130

2.4 Differential inequalities for non-existence of global solutions 169

3 Attractors for Evolutionary Differential Equations

3.1 Maximal attractors for nonlinear reaction-diffusion equations . . . 209

3.1.1 An initial boundary value problem 210

3.2 Attractors for autonomous differential equations with delay 213

3.3 H^2 estimates for the Cahn–Hilliard equation 215

3.4 Global existence, exponential stability and uniform attractors for a non-autonomous wave equation 218

3.4.1 Global existence and exponential stability 218

3.4.2 Uniform attractors 225

4 Global Existence and Uniqueness for Evolutionary PDEs
4.1 A weakly coupled Parabolic system 231
4.2 A convection equation with anomalous diffusion 234
4.3 Estimates on solutions for semilinear heat equations 237
4.4 Global existence decay estimates for a quasilinear parabolic system 240

5 Global Existence and Uniqueness for Abstract Evolutionary Differential Equations
5.1 Global existence of solutions to abstract evolutionary equations 245
5.1.1 Equivalent solutions 246
5.2 Uniqueness of solutions for differential equations in a Hilbert space 247
5.3 Dissipative estimates for PDEs 250

6 Global Existence and Asymptotic Behavior for Equations of Fluid Dynamics
6.1 Asymptotic behavior for the 2D homogeneous incompressible Navier–Stokes equations 257
6.1.1 Introduction 258
6.2 Large-time behavior for nonhomogeneous incompressible Navier–Stokes equations 262
6.3 A uniform lower Bound for density of a 1D viscous compressible barotropic fluid equation 268
6.4 Uniform bounds on specific volume for symmetrically quasilinear viscous barotropic fluid equations 273
6.5 Stabilization for the 1D compressible Navier–Stokes equations . . 281

7 Asymptotic Behavior of Solutions for Parabolic and Elliptic Equations
7.1 Decay estimates for flows in a semi-infinite straight channel 291
7.1.1 Uniform estimates 291
7.1.2 Exponential decay 294
7.2 Exact rates of convergence for nonlinear PDEs 296
7.2.1 Nonlinear diffusion equations and porous medium equations 296
7.3 Large-time behavior of solutions for parabolic equations 300
7.3.1 Large-time behavior for semilinear parabolic equations 300
7.3.2 Large-time behavior of solutions for parabolic equations 302

8 Asymptotic Behavior of Solutions to Hyperbolic Equations

8.1 Estimates on approximated solutions for 1D nonlinear wave equations . . . 307
8.2 Estimates on approximated solutions of wave equations . . . 312
8.3 Polynomial decay rate for nonlinear wave equations . . . 325
8.4 Decay rate estimates for dissipative wave equations . . . 330
8.5 Energy decay for a dissipative anisotropic elastic system . . . 335
8.6 Stabilization of weakly coupled wave equations . . . 344
8.7 Energy decay rates of nonlinear dissipative hyperbolic systems . . 348

9 Asymptotic Behavior of Solutions to Thermoviscoelastic, Thermoviscoelastoplastic and Thermomagnetoelastic Equations

9.1 Large-time behavior for thermoviscoelastic systems . . . 357
9.2 A thermoviscoelastoplastic system with hysteresis . . . 369
9.2.1 Thermoelastoplastic constitutive laws . . . 369
9.3 Asymptotic behavior for a linear thermomagnetoelastic system . . 377

10 Blow-up of Solutions to Nonlinear Hyperbolic Equations and Hyperbolic-Elliptic Inequalities

10.1 Blow-up of solutions for nonlinear wave equations . . . 393
10.2 Blow-up of solutions to semilinear wave equations . . . 403
10.3 Blow-up of solutions to nonlinear hyperbolic equations . . . 410
10.4 Breakdown of solutions to semilinear wave equations $\Box u + u_t = |u|^{1+\alpha}$ . . . 413
10.5 Blow-up of solutions to nonlinear wave equations with damping . . . 415
10.6 Blow-up of solutions to wave equations with a nonlinear dissipation . . . 417
10.7 Blow-up of solutions to the quasilinear hyperbolic-elliptic inequalities . . . 426

11 Blow-up of Solutions to Abstract Equations and Thermoelastic Equations

11.1 Blow-up of solutions to abstract nonlinear equations . . . 429
11.2 Blow-up of solutions to a class of abstract nonlinear equations . . 434
11.3 Blow-up of solutions to formally parabolic equations . . . 441
11.4 Blow-up of solutions to evolutionary PDEs . . . 448
11.5 Blow-up of solutions to initial boundary value problems . . . 452
11.6 Blow-up of solutions to the Cauchy problem in nonlinear one-dimensional thermoelasticity . . . 458

12 Appendix: Basic Inequalities
12.1 The Young inequalities . . . 467
12.2 The Hausdorff–Young inequalities and the Young inequalities . . . 471
12.3 The Hölder inequalities . . . 474
12.4 The Minkowski inequalities . . . 486
12.5 The Jensen inequalities . . . 490

Bibliography . . . 497

Index . . . 561

Preface

It is well known that inequalities are a very important tool in classical analysis. One application of these inequalities is the theory of PDEs (Partial Differential Equations). To our knowledge, some students such as Master or Diploma students do not have sufficient knowledge in this direction. Most of the basic courses in PDEs contains Bellman–Gronwall's inequality and nothing more. Advanced courses teach some classes of inequalities related to one topic of PDEs such as elliptic, parabolic and hyperbolic equations. On the other hand, there exist indeed some monographs about inequalities (cf. [7, 83, 58, 216, 724, 725, 726, 727, 942]), but all these monographs do not contain in detail explanations about applications to ODEs (Ordinary Differential Equations) and PDEs.

This book is aimed at presenting some analytic inequalities and their applications in (partial) differential equations. These inequalities include integral inequalities, differential inequalities and difference inequalities which play a crucial role in establishing (uniform) bounds, global existence, large-time behavior, decay rates and blow-up of solutions to various classes of evolutionary differential equations. The material of this book is selected by the author from a huge literature such as published papers, preprints and books categorized in terms of different properties that are consequences of those inequalities such as (uniform) bounds, global existence, large-time behavior, decay rates and blow-up of solutions for some partial differential equations.

There are essentially three parts in this book. The first part contains two chapters where we establish some important analytic inequalities. In Chapter 1, we carefully collect a number of integral inequalities including some famous integral inequalities such as the Bellman–Gronwall inequalities and the Henry inequalities. In Chapter 2, we consider differential and difference inequalities which are categorized in terms of implications such as uniform bounds, asymptotic behavior, decay rates and blow-up of solutions. The second part mainly discuss applications of some analytic inequalities introduced in the first part, which consists of nine chapters, Chapter 3–Chapter 11. In Chapter 3, we introduce some existence results on global and uniform attractors of some evolutionary differential equations. In Chapter 4, we introduce some results on global existence and uniqueness of solutions to evolutionary differential equations. Chapter 5 is concerned with the global existence and uniqueness for abstract evolutionary PDEs.

Chapter 6 investigates the global existence and asymptotic behavior of solutions to fluid equations. In Chapter 7, we establish the asymptotic behavior of solutions for parabolic and elliptic equations. Chapter 8 studies the asymptotic behavior of solutions to hyperbolic equations. In Chapter 9, we investigate the asymptotic behavior of solutions to thermoviscoelastic, thermoelastoplastic and thermomagnetoelastic systems. In Chapter 10, we study the blow-up of solutions to nonlinear hyperbolic equations and hyperbolic-elliptic inequalities. In Chapter 11, we are concerned with the blow-up of solutions to abstract equations and thermoelastic equations. The last part is Chapter 12, an appendix, which presents some basic inequalities including Young's inequalities, Hausdorff–Young's inequalities, Hölder's inequalities, Minkowski's inequalities and Jensen's inequalities.

One of the features of this book is that the reader may learn not only basic useful analytic inequalities from the first part of the book, but also many results and techniques in differential equations established in the second part. I believe that this book would be useful for Master, PHD students from mathematics, theoretic physics and other branches of science.

In the process of writing this book, many people have given me generous help. Among them, I appreciate greatly Professor Bert-Wolfgang Schulze for his spending a lot of time to improve earnestly the language word by word and giving me helpful suggestions, and also sincerely thank Professors Irena Lasiecka (University of Virgina, USA), Vilmos Komornik (Université de Strasbourg, France), Pavel Krejčí (Mathematical Institute, Academy of Sciences of the Czech Republic), Mokhtar Kirane (Université de La Rochelle, France), Patrick Martinez (Université Rennes I, France), Fatiha Alabau-Boussouira (Université de Metz et CNRS, France), Maurizio Grasselli (Politecnico di Milano, Italy), Hyeong-Ohk Bae (Ajou University, Republic of Korea) for providing me their recent results on analytic inequalities. Moreover, I would like to take this opportunity to thank all the authors of the references cited in this book for their excellent works which have increased the readability and highlights of the book. I also appreciate all the people who concern about me including my teachers, colleagues and collaborators. The book was initiated when I visited Germany and Brazil in 2008, so I particularly acknowledges the hospitality from Potsdam University in Germany and National Laboratory for Scientific Computing (LNCC) in Brazil. When I was visiting TU Bergakademie Freiberg, Germany, in August, 2012, Professor Michael Reissig proposed many good suggestions for the old book manuscript. Therefore I appreciate very much for his suggestions which indeed have improved the old book manuscript. Moreover, I would like to thank my students Lan Huang, Xinguang Yang, Shuxian Deng, Xin Liu, Zhiyong Ma, Taige Wang, Guili Hu, Xiaoke Su, Yaodong Yu, Lili Xu, Dongjie Ge, Xiaona Yu, Songtao Li, Tao Li, Xiaozhen Peng, Baowei Feng, Ming Zhang, Wei Wang, Haiyan Li, Jianlin Zhang, Xing Su, Yang Wang, Jie Cao, Tianhui Wei, Jia Ren, Jianpeng Zhang, Linlin Sun, Pengda Wang, Cheng Chen and Ying Wang for their hard work in typewriting and checking the gallery proof of the book manuscript.

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Chapter 1

Integral Inequalities

1.1 The classical Bellman–Gronwall inequality

It is well known that classical integral inequalities furnishing explicit bounds for an unknown function play a fundamental role in differential and integral equations. In this chapter, we shall first collect some basic integral inequalities.

Let us start with the famous Gronwall inequality (Gronwall [332]), which plays a crucial role in analysis, especially in studying existence, uniqueness and stability and estimates of solutions to differential equations (see, e.g., Bellman [94, 95, 96, 98], Gronwall [332]).

Theorem 1.1.1 (The Gronwall Inequality [332]). *Assume $u(t)$ is a continuous function on the interval $I = [\alpha, \alpha + h]$, and a and b are non-negative constants. Moreover, if for all $t \in I$, the following inequality holds*

$$0 \leq u(t) \leq \int_\alpha^t (bu(s) + a)\, ds. \tag{1.1.1}$$

Then, for all $t \in I$,

$$0 \leq u(t) \leq ahe^{bh}. \tag{1.1.2}$$

Proof. Set $u = z \exp[b(t - \alpha)]$. Let the maximum of z on I be attained at $t = t_1$. For this value of t, (1.1.1) implies

$$0 \leq z_{\max} \exp\left(b(t_1 - \alpha)\right) \leq \int_\alpha^{t_1} (bz(s) \exp[b(s - \alpha)] + a)\, ds$$

which, by the mean value theorem, gives us

$$\begin{aligned} 0 \leq z_{\max} \exp[b(t_1 - \alpha)] &\leq z_{\max} \int_\alpha^{t_1} b \exp\left(b(s - \alpha)\right) ds + \int_\alpha^{t_1} a ds \\ &\leq z_{\max} \left(\exp[b(t_1 - \alpha)] - 1\right) + a(t_1 - \alpha) \end{aligned}$$

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or

$$0 \le z_{\max} \le a(t_1 - \alpha) \le ah.$$

This immediately implies (1.1.2). □

Remark 1.1.1. Note that inequality (1.1.1) was already established by Peano (1885–1886), who explicitly considered the special case of the above theorem for $a = 0$, and obtained some general results on differential inequalities as well as maximal and minimal solutions of differential equations.

Theorem 1.1.2 (The Classical Bellman–Gronwall Inequality [96])**.** *Assume* $y(t)$ *and* $g(t)$ *are non-negative, continuous functions on* $0 \le t \le T$ *satisfying the inequality, for all* $t \in [0, T]$,

$$y(t) \le \eta + \int_0^t g(s)y(s)ds, \tag{1.1.3}$$

where η *is a non-negative constant. Then for all* $t \in [0, T]$,

$$y(t) \le \eta \exp\left(\int_0^t g(s)ds\right). \tag{1.1.4}$$

Proof. Let

$$v(t) = \eta + \int_0^t g(s)y(s)ds. \tag{1.1.5}$$

Thus from (1.1.3) it follows that for all $t \in [0, T]$,

$$v'(t) = g(t)y(t) \le g(t)v(t). \tag{1.1.6}$$

Multiplying (1.1.6) by $\exp\left(-\int_0^t g(s)ds\right)$, we can get

$$\frac{d}{dt}\left(v(t)\exp\left(-\int_0^t g(s)ds\right)\right) \le 0,$$

which thus gives us (1.1.4). □

Remark 1.1.2. In 1919, Gronwall [332] studied the case $g(t) = constant \ge 0$. Later on, in 1934, Bellman [95] extended this result to the form of Theorem 1.1.2. Since this type of inequalities is a very powerful and useful tool in analysis, more and more improvements and generalizations of the classical Bellman–Gronwall inequality have been established.

Remark 1.1.3. Clearly Bellman's inequality includes Gronwall's inequality because of $\int_\alpha^t ads \le ah$ for all $t \in I = [\alpha, \alpha + h]$. Since Bellman's inequality was found, its influence continued to increase, and it has been extended to various forms of inequalities.

We note that Theorem 1.1.2 indeed provides bounds on solutions of (1.1.3) in terms of the solution of a related linear integral equation

$$v(t) = \eta + \int_0^t g(s)v(s)ds \tag{1.1.7}$$

and belongs to the basic tools in the theory of differential equations. Motivated by various applications, it has been extended in different ways. For instance, on the one hand, in the Picard–Cauchy type iteration for establishing existence and uniqueness of solutions, this inequality and its various variants play a significant role, on the other hand, inequalities of the type (1.1.3) are also frequently used in the perturbation and stability theory of differential equations.

1.2 Linear generalizations of the Bellman–Gronwall inequalities

Since the above inequalities appeared, many different generalizations have been found, which include linear, nonlinear, singular, and uniform generalizations, and other generalizations, involving operators in partially ordered linear spaces, etc.

Reid [833] was one of the early users of the above inequalities in the theory of ordinary differential equations, who employed a slightly more general form than Theorem 1.1.2 to study the properties of solutions of infinite systems of linear ordinary differential equations. In this section, we shall gives some representative generalizations and emphasize their interconnections.

Bellman [97] established the following variant of Theorem 1.1.2 in order to study the asymptotic behavior of the solutions of linear differential-difference equations.

Theorem 1.2.1 (The Bellman Inequality [97]). *Let u and f be continuous and non-negative functions on $J = [\alpha, \beta]$, and let $n(t)$ be a continuous, positive and non-decreasing function on J. Then for all $t \in J$,*

$$u(t) \leq n(t) + \int_\alpha^t f(s)u(s)ds, \tag{1.2.1}$$

implies for all $t \in J$,

$$u(t) \leq n(t) \exp\left(\int_\alpha^t f(s)ds\right). \tag{1.2.2}$$

Proof. In fact, by virtue of (1.2.1), the function $w(t) = u(t)/n(t)$ satisfies

$$w(t) \leq 1 + \int_\alpha^t f(s)w(s)ds,$$

which, together with Theorem 1.1.2, yields

$$w(t) \le \exp\left(\int_\alpha^t f(s)ds\right).$$

Thus this gives us the required inequality (1.2.2). □

Theorem 1.2.2 (The Jones Inequality [407])**.** *Assume $y(t), f(t)$, and $g(t)$ are real-valued piecewise-continuous functions defined on $0 \le t \le T$, and g is non-negative on this interval. Then for all $t \in [0, T]$,*

$$y(t) \le f(t) + \int_0^t g(s)y(s)ds, \tag{1.2.3}$$

implies for all $t \in [0, T]$,

$$y(t) \le f(t) + \int_0^t g(s)f(s)\exp\left\{\int_s^t g(\theta)d\theta\right\} ds. \tag{1.2.4}$$

Proof. Let $h = \int_0^t g(s)y(s)ds$. Then (1.2.3) yields

$$h'(t) = g(t)y(t) \le g(t)f(t) + g(t)h(t),$$

whence

$$\frac{d}{dt}\left(h(t)\exp\left(-\int_0^t g(s)ds\right)\right) \le g(t)f(t)\exp\left(-\int_0^t g(s)ds\right). \tag{1.2.5}$$

Thus integrating (1.2.5) with respect to $t \in [0, T]$ yields

$$h(t) \le \int_0^t g(s)f(s)\exp\left\{\int_s^t g(\theta)d\theta\right\} ds$$

which, together with (1.2.3), implies (1.2.4). □

The above inequality was established by Jones [407] in 1964. Note that (1.1.4) provides the best possible result, in the sense that when we replace the inequality (1.1.3) by an equality, the same may be done in (1.1.4). Also, it is obvious that when $f(t) \equiv \eta$ (*a constant*), a direction integration in (1.2.5) yields

$$y(t) \le \eta \exp\left(\int_0^t g(s)ds\right)$$

which is precisely (1.1.4).

An alternative form of (1.2.4) can be stated as follows when $y(t), f(t)$ is more regular.

Theorem 1.2.3 (The Generalized Jones Inequality [407]). *Assume that $g(t)$ is a non-negative integrable function on $[0,T]$ $(0 < T)$, and that $f(t)$ and $y(t)$ are non-negative absolutely continuous functions on $[0,T]$ satisfying the following integral inequality for almost all $t \in [0,T]$,*

$$y(t) \le f(t) + \int_0^t g(s)y(s)ds. \tag{1.2.6}$$

Then, for almost all $t \in [0,T]$,

(1) $$y(t) \le f(0)\exp\left(\int_0^t g(s)ds\right) + \int_0^t \exp\left(\int_s^t g(\eta)d\eta\right) f'(s)ds. \tag{1.2.7}$$

(2) *If $f(t) \equiv A = constant > 0$, then for almost all $t \in [0,T]$,*

$$y(t) \le A\exp\left(\int_0^t g(s)ds\right). \tag{1.2.8}$$

If, further, $g(t) \equiv B = constant > 0$, then for almost all $t \in [0,T]$,

$$y(t) \le A\exp(Bt). \tag{1.2.9}$$

Proof. Since $f(t)$ and $y(t)$ are non-negative absolutely continuous functions on $[0,T]$, we know that $y'(t), f'(t)$ exist for almost all $t \in [0,T]$.

Then if we set

$$h(t) = f(t) + \int_0^t g(s)y(s)ds, \tag{1.2.10}$$

then it follows from (1.2.6) and (1.2.10) that for almost all $t \in [0,T]$,

$$h'(t) = f'(t) + g(t)y(t) \le f'(t) + g(t)h(t)$$

which implies

$$\frac{d}{dt}\left(h(t)\left(-\int_0^t g(s)ds\right)\right) \le f'(t)\left(-\int_0^t g(s)ds\right). \tag{1.2.11}$$

Integrating (1.2.11) with respect to t yields (1.2.7). Now (1.2.8) and (1.2.9) are direct consequences of (1.2.7). □

Theorem 1.2.4 (The Gollwitzer Inequality [322]). *Assume u, f, g and h are non-negative continuous functions on $J = [\alpha, \beta]$, and for all $t \in J$,*

$$u(t) \le f(t) + g(t)\int_\alpha^t h(s)u(s)ds. \tag{1.2.12}$$

Then for all $t \in J$,

$$u(t) \le f(t) + g(t)\int_\alpha^t h(s)f(s)\exp\left(\int_s^t h(\sigma)g(\sigma)d\sigma\right) ds. \tag{1.2.13}$$

Proof. Let

$$z(t) = \int_\alpha^t h(s)u(s)ds. \tag{1.2.14}$$

Then we have $z(\alpha) = 0, u(t) \leq f(t) + g(t)z(t)$, and

$$z'(t) = h(t)u(t) \leq h(t)f(t) + h(t)g(t)z(t). \tag{1.2.15}$$

Multiplying (1.2.15) by $\exp\left(-\int_\alpha^t h(\sigma)g(\sigma)d\sigma\right)$, we have

$$\frac{d}{dt}\left[z(t)\exp\left(-\int_\alpha^t h(\sigma)g(\sigma)d\sigma\right)\right] \leq h(t)f(t)\exp\left(-\int_\alpha^t h(\sigma)g(\sigma)d\sigma\right). \tag{1.2.16}$$

Setting $t = s$ in (1.2.16) and integrating the resulting equation over $[\alpha, t]$, we conclude that

$$z(t)\exp\left(-\int_\alpha^t h(\sigma)g(\sigma)d\sigma\right) \leq \int_\alpha^t h(s)f(s)\exp\left(-\int_\alpha^s h(\sigma)g(\sigma)d\sigma\right)ds. \tag{1.2.17}$$

Thus using (1.2.12), (1.2.17), we finally obtain (1.2.13). □

Remark 1.2.1. If $g(t) = 1$, then Theorem 1.2.4 reduces to Theorem 1.2.3 (Jones [407]). Moreover, some generalizations of Theorem 1.2.4 when $g(t) = 1$, including the subsequent extensions to discrete and discontinuous functional equations are also contained in Jones [407].

Remark 1.2.2. Note that in Theorem 1.2.4 equality in (1.2.13) holds for a subinterval $J_1 = [\alpha, \beta_1]$ of J if equality in (1.2.12) holds for $t \in J_1$. The results are still valid if "$\leq$" is replaced by "$\geq$" in (1.2.12). Both (1.2.12) and (1.2.13), with "$\leq$" replaced by "$\geq$", remain valid if $\int_\alpha^t$ is replaced by $\int_\alpha^\beta$ and $\int_s^t$ by $\int_t^s$ throughout.

Remark 1.2.3. In 1975, Beesack [86] pointed out that if the integrals in Theorem 1.2.4 are Lebesgue integrals, the hypotheses can be relaxed to: u, f, g and h are measurable functions such that hu, hf, $hg \in L(J)$. The equality and inequality conditions are then to be interpreted as almost everywhere, and the stated condition for equality is necessary as well as sufficient. Similar remarks apply to all subsequent theorems, which will be mostly stated for the continuous case.

Pachpatte [722] exploited the following variant of the inequality in Theorem 1.2.4 to obtain various generalizations of Bellman's inequality in Theorem 1.2.5.

Theorem 1.2.5 (The Pachpatte Inequality [722])**.** *Assume u, g and h are nonnegative continuous functions on $J = [\alpha, \beta]$ and $n(t)$ be a continuous, positive and non-decreasing function on J. Suppose that for all $t \in J$,*

$$u(t) \leq n(t) + g(t)\int_\alpha^t h(s)u(s)ds. \tag{1.2.18}$$

Then, for all $t \in J$,

$$u(t) \le n(t) \left[1 + g(t) \int_\alpha^t h(s) \exp\left(\int_s^t h(\sigma) g(\sigma) d\sigma \right) ds \right]. \tag{1.2.19}$$

Proof. Obviously, the proof follows in the same manner as that of Theorem 1.2.2, by using the inequality given in Theorem 1.2.4. □

Remark 1.2.4. We recall that this Gronwall inequality was given by Willett in [966], where an explicit bound for $u(t)$ was given under more general assumptions, e.g.,

$$u(t) \le n(t) + \sum_{i=1}^{n} g_i(t) \int_\alpha^t h_i(s) u(s) ds.$$

Theorem 1.2.6 (The Pachpatte Inequality [722])**.** *Assume that u, p, q, f and g are non-negative continuous functions on $J = [\alpha, \beta]$, and let for all $t \in J$,*

$$u(t) \le p(t) + q(t) \int_\alpha^t (f(s)u(s) + g(s))\, ds. \tag{1.2.20}$$

Then for all $t \in J$,

$$u(t) \le p(t) + q(t) \int_\alpha^t (f(s)p(s) + g(s)) \exp\left(\int_s^t f(\sigma) q(\sigma) d\sigma \right) ds. \tag{1.2.21}$$

Proof. Let

$$z(t) = \int_\alpha^t \Big(f(s)u(s) + g(s) \Big) ds.$$

Now we can follow the proof of Theorem 1.2.5 to get the desired inequality (1.2.21). □

Remark 1.2.5. Theorem 1.2.6 extends the result of Chandirov [137] where $q(t) = 1$. If we choose $g(t) = 0$ in Theorem 1.2.6, then Theorem 1.2.6 reduces to Theorem 1.2.4.

Theorem 1.2.7 (The Gollwitzer Inequality [322])**.** *Assume u, v, h and k are non-negative continuous functions on $J = [\alpha, \beta]$, and let for all $\alpha \le x \le t \le \beta$*

$$u(t) \ge v(x) - k(t) \int_x^t h(s) v(s) ds. \tag{1.2.22}$$

Then for all $\alpha \le x \le t \le \beta$,

$$u(t) \ge v(x) \exp\left(-k(t) \int_x^t h(s) ds \right). \tag{1.2.23}$$

Proof. Set

$$z(x) = u(t) + k(t)\int_x^t h(s)v(s)ds, \quad \alpha \le x \le t \le \beta. \tag{1.2.24}$$

Thus this, together with (1.2.22), gives for all $\alpha \le x \le t \le \beta$,

$$z'(x) = -h(x)v(x)k(t) \ge -h(x)z(x)k(t), \tag{1.2.25}$$

because $z(x) \ge v(x)$; here $z'(x)$ at the end points is defined to be the limit from the interior of $[\alpha, t]$. Then using the integral factor $r(x) = \exp\left(-k(t)\int_x^t h(s)ds\right)$, we have $(rz)'(x) \ge 0$ and hence $(rz)(t) \ge (rz)(x)$ on $[\alpha, t]$. This result is sharp in the sense that if equality in (1.2.22) holds on $[\alpha, t]$, then the equality in (1.2.23) holds on $[\alpha, t]$. □

Remark 1.2.6. The above result is similar to a special case of the Langenhop inequality (Langenhop [478]), and an estimate for u, independent of x, is obtained by taking $x = \alpha$.

The following generalization can be found in Qin [760, 761, 762, 763, 764, 765, 766].

Theorem 1.2.8 (The Generalized Bellman–Gronwall Inequality [763])**.** *Assume that $f(t), g(t)$ and $y(t)$ are non-negative integrable functions in $[0, T]$ $(0 < T)$ satisfying the integral inequality for all $t \in [0, T]$,*

$$y(t) \le g(t) + \int_0^t f(s)y(s)ds. \tag{1.2.26}$$

Then, for all $t \in [0, T]$,

$$y(t) \le g(t) + \int_0^t \exp\left(\int_s^t f(\theta)d\theta\right) f(s)g(s)ds. \tag{1.2.27}$$

In addition, if $g(t)$ is a non-decreasing function in $[0, T]$, then for all $t \in [0, T]$,

$$y(t) \le g(t)\left[1 + \int_0^t \exp\left(\int_s^t f(\theta)d\theta\right) f(s)ds\right] \tag{1.2.28}$$

$$\le g(t)\left[1 + \int_0^t f(s)ds \exp\left(\int_0^t f(\theta)d\theta\right)\right]. \tag{1.2.29}$$

Moreover, if $T = +\infty$ and $\int_0^{+\infty} f(s)ds < +\infty$, then for all $t \in [0, T]$,

$$y(t) \le Cg(t), \tag{1.2.30}$$

where $C = 1 + \int_0^{+\infty} f(s)ds \exp\left(\int_0^{+\infty} f(\theta)d\theta\right)$ is a positive constant.

Proof. (1) Let $h(t) = \int_0^t f(s)y(s)ds$. Then from (1.2.26), we obtain

$$h'(t) = f(t)y(t) \le f(t)g(t) + f(t)h(t) \tag{1.2.31}$$

with $h(0) = 0$. Multiplying (1.2.31) by $\exp\left(-\int_0^t f(s)ds\right)$ yields

$$\frac{d}{dt}\left(h(t)\exp\left(-\int_0^t f(s)ds\right)\right) \le f(t)g(t)\exp\left(-\int_0^t f(s)ds\right),$$

whence gives us

$$h(t) \le \int_0^t \exp\left(\int_s^t f(\theta)d\theta\right) f(s)g(s)ds. \tag{1.2.32}$$

Now inserting (1.2.32) into (1.2.26) yields (1.2.27).

(2) If $g(t)$ is a non-decreasing function in $[0,T]$, then (1.2.28) and (1.2.29) easily follow from (1.2.27).

(3) If, further, $T = +\infty$ and $\int_0^{+\infty} f(s)ds < +\infty$, then (1.2.30) easily follows from (1.2.29). □

The following result can be regarded as a corollary of Theorem 1.2.8, which can be found in Racke [822].

Corollary 1.2.1 ([822]). *Let $a > 0$, $\phi, h \in C([0,a]), h \ge 0$, and let $g : [0,a] \to \mathbb{R}$ be increasing. If for any $t \in [0,a]$,*

$$\phi(t) \le g(t) + \int_0^t h(s)\phi(s)ds, \tag{1.2.33}$$

then for all $t \in [0,a]$,

$$\phi(t) \le g(t)\exp\left(\int_0^t h(s)ds\right). \tag{1.2.34}$$

In 1973 and 1975, Pachpatte [717], [722] obtained the following two results, which can be regarded as generalizations of Theorem 1.2.2.

Theorem 1.2.9 (The Pachpatte Inequality [717, 722]). *Assume $u(t), f(t)$ and $g(t)$ are real-valued non-negative continuous functions on $I = [0,+\infty)$ satisfying the inequality for all $t \in I$,*

$$u(t) \le u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left[\int_0^s g(\tau)u(\tau)d\tau\right]ds, \tag{1.2.35}$$

where u_0 is a non-negative constant. Then, for all $t \in I$,

$$u(t) \le u_0\left\{1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau)+g(\tau))d\tau\right)ds\right\}. \tag{1.2.36}$$

Proof. If we define a function $v(t)$ by the right-hand side of (1.2.35), then we have

$$v'(t) = f(t)u(t) + f(t)\int_0^t g(\tau)u(\tau)d\tau, \quad v(0) = u_0$$

which in view of (1.2.35) implies

$$v'(t) \leq f(t)\left[v(t) + \int_0^t g(\tau)v(\tau)d\tau\right]. \tag{1.2.37}$$

Setting

$$m(t) = v(t) + \int_0^t g(\tau)v(\tau)d\tau, \quad m(0) = v(0) \equiv u_0,$$

it follows from (1.2.37) and the fact that $v(t) \leq m(t)$, that

$$m'(t) \leq (f(t) + g(t))\, m(t).$$

This implies due to $m(0) = u_0$,

$$m(t) \leq u_0 \exp\left[\int_0^t (f(s) + g(s))\, ds\right].$$

Then (1.2.37) yields

$$v'(t) \leq u_0 f(t) \exp\left[\int_0^t (f(s) + g(s))\, ds\right]. \tag{1.2.38}$$

Now integrating both sides of (1.2.38) over $[0, t]$ and substituting the value of $v(t)$ in (1.2.35), we can obtain the desired bound in (1.2.36). □

Theorem 1.2.10 (The Pachpatte Inequality [717, 722]). *Assume $u(t), f(t)$ and $g(t)$ are real-valued non-negative continuous functions on $I = [0, +\infty)$, and $n(t)$ is a positive, monotonic, non-decreasing continuous function on I, such that for all $t \in I$, the inequality holds,*

$$u(t) \leq n(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left[\int_0^s g(\tau)u(\tau)d\tau\right] ds. \tag{1.2.39}$$

Then for all $t \in I$,

$$u(t) \leq n(t)\left\{1 + \int_0^t f(s)\exp[\int_0^s (f(\tau) + g(\tau))d\tau]ds\right\}. \tag{1.2.40}$$

Proof. Thanks to the assumptions on $n(t)$ on I, it follows from (1.2.39) that

$$\begin{aligned} u(t)/n(t) &\leq 1 + \int_0^t f(s)u(s)/n(t)ds + \int_0^t f(s)\left[\int_0^s g(\tau)u(\tau)/n(t)d\tau\right]ds \\ &\leq 1 + \int_0^t f(s)u(s)/n(s)ds + \int_0^t f(s)\left[\int_0^s g(\tau)u(\tau)/n(\tau)d\tau\right]ds. \end{aligned} \tag{1.2.41}$$

Then by using Theorem 1.2.9, we can derive (1.2.40) from (1.2.41). □

Dhongade and Deo [206] established results similar to those in Theorems 1.2.9 and 1.2.10 when the second integral term on the right-hand side in (1.2.35) and (1.2.39) is absent. However, the bounds obtained in Theorems 1.2.9 and 1.2.10 are different from those given in [206].

Note that in Theorem 1.2.10, when $n(t)$ is not monotonic non-decreasing, estimate (1.2.40) is also obtained in Pachpatte [719], which will be stated in the following theorem.

In order to formulate this result, we introduce the definition of sub-multiplicative functions.

Definition 1.2.11. The non-negative function $W(u)$ is said to be sub-multiplicative if $W(uv) \leq W(u)W(v)$ for all $u, v \geq 0$.

Theorem 1.2.12 (The Pachpatte Inequality [719])**.** *Assume $u(t), f(t), g(t)$, and $h(t)$ are real-valued non-negative continuous functions on $I = [0, +\infty)$. Moreover, assume $W(u)$ is a continuous, positive, monotonic, non-decreasing and sub-multiplicative function for $u > 0, W(0) = 0$, and assume further that the inequality holds for all $t \in I$,*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right)ds + \int_0^t h(s)W(u(s))ds \tag{1.2.42}$$

where u_0 is a positive constant. Then for all $t \in [0, b]$,

$$\begin{aligned} u(t) \leq G^{-1}&\left[G(u_0) + \int_0^t f(s)W(1 + \int_0^s f(\tau)\exp(\int_0^\tau (f(k) + g(k))dk)d\tau)ds\right] \\ &\times\left[1 + \int_0^t f(s)\exp[\int_0^s (f(\tau) + g(\tau)d\tau)ds]\right], \end{aligned} \tag{1.2.43}$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{W(s)}, \quad r \geq r_0 > 0, \tag{1.2.44}$$

and G^{-1} is the inverse function of G, and t is in the subinterval $[0, b]$ of I such that

$$G(u_0) + \int_0^t h(s)W(1 + \int_0^s f(\tau)\exp[\int_0^\tau (f(k) + g(k))dk]d\tau)ds \in \mathrm{Dom}(G^{-1}). \tag{1.2.45}$$

Proof. Define

$$n(t) = u_0 + \int_0^t h(s)W(u(s))ds, \quad n(0) = u_0.$$

Then we infer from (1.2.42) that

$$u(t) \leq n(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right)ds. \tag{1.2.46}$$

Since $n(t)$ is positive, monotonic, non-decreasing on I, we derive from Theorem 1.2.10 and (1.2.46) that,

$$u(t) \leq n(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau) + g(\tau))d\tau\right)ds\right]. \tag{1.2.47}$$

Furthermore, we also have

$$W(u(t)) \leq W(n(t))W\left(1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau) + g(\tau))d\tau\right)ds\right), \tag{1.2.48}$$

since W is sub-multiplicative. Hence from (1.2.48) it follows that

$$\frac{h(t)W(u(t))}{W(n(t))} \leq h(t)W\left(1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau) + g(\tau))d\tau\right)ds\right). \tag{1.2.49}$$

Because of (1.2.44), this reduces to

$$\frac{d}{dt}G(n(t)) \leq h(t)W\left(1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau) + g(\tau))d\tau\right)ds\right). \tag{1.2.50}$$

Now integrating (1.2.50) over $[0, t]$, we obtain

$$G(n(t)) - G(n(0)) \leq \int_0^t h(s)W\left(1 + \int_0^s f(\tau)\exp\left(\int_0^\tau (f(k) + g(k))dk\right)d\tau\right)ds. \tag{1.2.51}$$

Thus (1.2.43) follows from (1.2.42) and (1.2.51), which completes the proof. □

The following result is a more general form of the above theorem which may be useful in certain situations. To this end, we introduce the definition of sub-additive functions.

Definition 1.2.13. The no-negative function $W(u)$ is said to be sub-additive if $W(u + v) \leq W(u) + W(v)$ for all $u, v \geq 0$.

Theorem 1.2.14 (The Pachpatte Inequality [721])**.** *Assume* $u(t), f(t), g(t)$, *and* $h(t)$ *are real-valued non-negative continuous functions on* $I = [0, +\infty)$, *and let* $W(u)$ *be a continuous, positive, monotonic, non-decreasing, subadditive and sub-multiplicative function for* $u > 0, W(0) = 0$. *Moreover, let* $p(t) > 0, M(t) \geq 0$ *be* *non-decreasing in t and continuous on I, $M(0) = 0$, and assume further that the inequality holds for all $t \in I$,*

$$u(t) \leq p(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right)ds + M\left(\int_0^t h(s)W(u(s))ds\right). \tag{1.2.52}$$

Then for all $t \in [0, b]$,

$$\begin{aligned} u(t) \leq \Bigg[p(t) + M\Bigg(G^{-1}\Bigg[G\Bigg(\int_0^t h(s)W(p(s)(1 \\ + \int_0^s f(\tau)\exp(f(\tau))\exp\Bigg(\int_0^\tau (f(k)+g(k))dk\Bigg)d\tau))\Bigg)ds\Bigg] \\ + \int_0^t h(s)W\left(1 + \int_0^s f(\tau)\exp\Bigg(\int_0^\tau (f(k)+g(k))dk\Bigg)d\tau\right)ds\Bigg)\Bigg] \\ \times\left[1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau)+g(\tau)d\tau)ds\right)\right], \end{aligned} \tag{1.2.53}$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{W(M(s))}, \qquad r \geq r_0 > 0, \tag{1.2.54}$$

and G^{-1} is the inverse function of G, and t is in the subinterval $[0, b]$ of I such that

$$G\Bigg(\int_0^t h(s)W\left(p(s)\left(1 + \int_0^s f(\tau)\exp(\int_0^\tau (f(k)+g(k))dk)d\tau\right)ds\right) \times \int_0^t h(s)W\left(1 + \int_0^s f(\tau)\exp\left(\int_0^\tau (f(k)+g(k))dk\right)d\tau\right)ds\Bigg) \in \mathrm{Dom}(G^{-1}).$$

Proof. The proof follows by using arguments similar to those in the proof of the above theorem, together with Theorem 1 in Deo and Murdeshwar [202]. We omit the details here. □

Theorem 1.2.15 (The Pachpatte–Pachpatte Inequality [728])**.** *Assume $u(t)$, $a(t)$, $b(t)$ are real-valued non-negative continuous functions defined for all $t \in \mathbb{R}^+$, such that $\int_0^{+\infty} b(s)ds < +\infty$ and assume that $a(t)$ is non-increasing for all $t \in \mathbb{R}^+$. If for all $t \in \mathbb{R}^+$,*

$$u(t) \leq a(t) + \int_t^{+\infty} b(s)u(s)ds, \tag{1.2.55}$$

then for all $t \in \mathbb{R}^+$,

$$u(t) \leq a(t)\exp\left(\int_t^{+\infty} b(s)ds\right). \tag{1.2.56}$$

Proof. First we assume that $a(t) > 0$ for all $t \in \mathbb{R}^+$. From (1.2.55) we easily obtain

$$\frac{u(t)}{a(t)} \leq 1 + \int_t^{+\infty} b(s) \frac{u(s)}{a(s)} ds. \tag{1.2.57}$$

Define a function $z(t)$ by the right-hand side of (1.2.57). Then $z(+\infty) = 1$, $\frac{u(t)}{a(t)} \leq z(t)$ and

$$z'(t) = -b(t) \frac{u(t)}{a(t)} \geq -b(t)z(t). \tag{1.2.58}$$

The inequality (1.2.58) implies that for all $t \in \mathbb{R}^+$,

$$z(t) \leq \exp\left(\int_t^{+\infty} b(s)ds\right). \tag{1.2.59}$$

Using (1.2.59) in $u(t)/a(t) \leq z(t)$, we get the desired estimate (1.2.56).

If $a(t)$ is non-negative, we carry out the above procedure for $a(t) + \varepsilon$ instead of $a(t)$, where $\varepsilon > 0$ is an arbitrary small constant, and then pass to the limit as $\varepsilon \to 0^+$ to obtain (1.2.56). □

A fairly general linear version of Theorem 1.1.2 may be stated as follows (see, e.g., Chu and Metcalf [164]).

Theorem 1.2.16 (The Chu–Metcalf Inequality [164]). *Let u and f be real continuous functions on $[0, T]$. Let K be continuous and non-negative on the triangle $\Delta : 0 \leq y \leq x \leq T$. If for all $x \in J = [0, T]$,*

$$u(x) \leq f(x) + \int_0^x K(x, y)u(y)dy, \tag{1.2.60}$$

then for all $x \in J$,

$$u(x) \leq f(x) + \int_0^x H(x, y)f(y)dy, \tag{1.2.61}$$

where for all $0 \leq y \leq x \leq T$,

$$H(x, y) = \sum_{i=1}^{+\infty} K_i(x, y)$$

is the resolvent kernel and the K_i $(i = 1, 2, \dots)$ are the iterated kernels of K.

Proof. In fact, from (1.2.60), we have for all $0 \leq x \leq T$,

$$\begin{aligned} u(x) &\leq f(x) + \int_0^x K(x, y)f(y)dy + \int_0^x K(x, y) \int_0^y K(y, z)u(z)dy \\ &= f(x) + \int_0^x K_1(x, y)f(y)dy + \int_0^x K_2(x, y)u(y)dy. \end{aligned}$$

The rest of the proof follows by induction and a standard estimate procedure shows the resulting series to be uniformly convergent. □

The previous results, in which an explicit upper bound for u was obtained, are the only cases for which the resolvent kernel H can be summed up in "closed form". For example, if $K(x,y) = g(x)h(y) \geq 0,\ 0 \leq y \leq x \leq T$, then

$$\begin{aligned} H(x,y) &= \sum_{i=1}^{+\infty} \frac{g(x)h(y)}{(i-1)!} \left[\int_y^x g(z)h(z)dz \right]^{i-1} \\ &= g(x)h(y) \exp\left(\int_y^x g(z)h(z)dz \right), \end{aligned}$$

since we can show by induction that each K_i $(i = 1, 2, \dots)$ is given by the appropriate term in the sum for H. □

As pointed out by Chu and Metcalf [164], the cases in which we obtain an explicit bound for u are precisely those in which the resolvent kernel (or a majorant of it) can be summed up in closed form. This is, in fact, the case when $K(x,y) = h(x)g(y) \geq 0$. Of particular interest is the case $h \equiv 1$.

Note that Beesack [85] extended Theorem 1.2.14 to the case where $u, f \in L^2(J)$ and $K \in L^2(\Delta)$ and the results are still valid if "$\leq$" is replaced by "$\geq$" in both (1.2.60) and (1.2.61). The inequality of Theorem 1.2.14 includes as a special case of the inequality given in Theorem 1.2.2.

Concerning Theorem 1.2.14, there is another interesting linear generalization due to Willett [966] under the assumption that either $K(x,y)$ or $\partial K(x,y)/\partial x$ is degenerate or directly separable in the following sense

$$K(x,y) \leq \sum_{i=1}^{n} h_i(x)k_i(y)$$

or a similar relation holds for $\partial K(x,y)/\partial x$.

The following theorem is a slight modification of the inequality given by Norbury and Stuart [700].

Theorem 1.2.17 (The Norbury–Stuart Inequality [700]). *Assume u and $K(t,s)$ are as in Theorem 1.2.14, and $K(t,s)$ is non-decreasing in t for each $s \in J$.*

(1) *If for all $t \in J = [0,T]$,*

$$u(t) \leq C + \int_0^t K(t,s)u(s)ds, \tag{1.2.62}$$

where $C \geq 0$ is a constant, then for all $t \in J$,

$$u(t) \leq C \exp\left(\int_0^t K(t,s)ds \right). \tag{1.2.63}$$

(2) *Let $n(t)$ be a positively continuous and non-decreasing function for all $t \in J$. If for all $t \in J$,*

$$u(t) \leq n(t) + \int_0^t K(t,s)u(s)ds, \tag{1.2.64}$$

then for all $t \in J$,

$$u(t) \leq n(t) \exp\left(\int_0^t K(t,s)ds\right). \tag{1.2.65}$$

The following result is due to Pata, Prouse and Vishik [732].

Theorem 1.2.18 (The Pata–Prouse–Vishik Inequality [732])**.** *Let $k_0, k_1 \geq 0, m \in L^1_{\text{loc}}(\mathbb{R}^+), m \geq 0$ almost everywhere, and $\phi \in C(\mathbb{R}^+), \phi \geq 0$. If for any $t > 0$ and some constant $\delta > 0$,*

$$\phi^2(t) \leq k_0 + k_1 e^{-\delta t} + \int_0^t m(s)\phi(s)e^{-\delta(t-s)}ds, \tag{1.2.66}$$

then for any $t > 0$,

$$\phi^2(t) \leq 2k_0 + 2k_1 e^{-\delta t} + \left(\int_0^t m(s)e^{-\delta(t-s)/2}ds\right)^2. \tag{1.2.67}$$

Proof. Let $\psi(t) = \phi^2(t)e^{\delta t}$. Then inequality (1.2.66) implies

$$\psi(t) \leq k_0 e^{\delta t} + k_1 + \int_0^t m(s)e^{\delta s/2}\psi^{1/2}(s)ds. \tag{1.2.68}$$

Fix a $t \in \mathbb{R}^+$, and let $t_0 \in [0,t]$ be such that $\psi(t_0) = \max_{s\in[0,t]} \psi(s)$. Then Young's inequality applied to (1.2.68) yields

$$\begin{aligned}\psi(t_0) &\leq k_0 e^{\delta t_0} + k_1 + \psi^{1/2}(t_0)\int_0^{t_0} m(s)e^{\delta s/2}ds \\ &\leq k_0 e^{\delta t_0} + k_1 + \frac{1}{2}\psi(t_0) + \frac{1}{2}\left(\int_0^{t_0} m(s)e^{\delta s/2}ds\right)^2\end{aligned}$$

which gives us (1.2.67). □

Remark 1.2.7. This result may have a general form of differential inequality (2.1.29), see, e.g., Theorem 2.1.6.

The following result can be used to prove a uniform bound for a non-negative function (see, e.g., Chi-Cheng Poon [157]).

Theorem 1.2.19 (The Chi-Cheng Poon Inequality [157])**.** *Let $\phi(s)$ be a non-negative function on $[0,1]$. Assume that there are constants γ and $\lambda, \gamma > \lambda > 0$, such that for any $1 \geq s_0 > 0$ and $0 < s < s_0/2$,*

$$\phi(s) \leq C\left(\left(\frac{s}{s_0}\right)^\gamma \phi(s_0) + s_0^\lambda\right). \tag{1.2.69}$$

Then for all $s \in (0, 1/2)$, we have

$$\phi(s) \leq Cs^\lambda < C(1/2)^\lambda. \tag{1.2.70}$$

Proof. Since $\gamma > \lambda > 0$, choose an $\alpha > 0$ such that $\gamma - \alpha > \lambda$. Let $Q > 1$ be a constant to be determined later. For any $s < s_0/Q$, we have

$$\phi(s) \leq C\left(\frac{s}{s_0}\right)^{\alpha}\left(\frac{s}{s_0}\right)^{\gamma-\alpha}\phi(s_0) + Cs_0^{\lambda}. \tag{1.2.71}$$

If Q is chosen large enough, then from (1.2.69), we infer that for all $s < s_0/Q$,

$$\phi(s) \leq \left(\frac{s}{s_0}\right)^{\gamma-\alpha}\phi(s_0) + Cs_0^{\lambda}. \tag{1.2.72}$$

Let $\eta = \gamma - \alpha$. Then from (1.2.72), fixing an $s < 1/Q$, we have

$$\phi(s) < s^{\eta}\phi(1) + C. \tag{1.2.73}$$

By induction, we obtain from (1.2.73) that for any positive integer k,

$$\phi(s^k) \leq s^{\eta k}\phi(1) + Cs^{k\lambda-\eta}\sum_{i=1}^{k} s^{i(\eta-\lambda)}. \tag{1.2.74}$$

Let $C_0 = \phi(1) + Cs^{-\eta}\sum_{i=1}^{+\infty} s^{i(\eta-\lambda)}$. Since $\eta > \lambda, C_0 > 0$ is finite, and then we have from (1.2.74)

$$\phi(s^k) \leq C_0 s^{k\lambda}. \tag{1.2.75}$$

For any $t > 0$, there is a positive integer k such that $s^{k+1} < t \leq s^k$. This, together with (1.2.75), implies (1.2.70). $\square$

1.3 Simultaneous inequalities

Greene [331] showed the following interesting inequality, which can be used in analysis of various problems in the theory of certain systems of simultaneous differential and integral equations.

Theorem 1.3.1 (The Greene Inequality [331]). *Let f, g, h_j $(1 \leq j \leq 4)$ be non-negative continuous on $[0, +\infty)$ and let h_j be bounded such that for any $t \geq 0$,*

$$f(t) \leq C_1 + \int_0^t h_1(s)f(s)ds + \int_0^t h_2(s)g(s)\exp(\mu s)ds, \tag{1.3.1}$$

$$g(t) \leq C_2 + \int_0^t h_3(s)f(s)\exp(-\mu s)ds + \int_0^t h_4(s)g(s)ds, \tag{1.3.2}$$

where C_1, C_2 and μ are non-negative constants. Then there exist positive constants β_k, M_k $(k = 1, 2)$ such that for any $t \geq 0$,

$$f(t) \leq M_1\exp(\beta_1 t), \quad g(t) \leq M_2\exp(\beta_2 t). \tag{1.3.3}$$

Proof. The proof due to Greene has been simplified by many authors, see, e.g., Wang [943] and Das [195]. Here we present the proof of Das [195].

First observe that (1.3.1) implies

$$e^{-\mu t}f(t) \le C_1 + \int_0^t e^{-\mu s}h_1(s)f(s)ds + \int_0^t h_2(s)g(s)ds.$$

Define

$$F(t) = e^{-\mu t}f(t) + g(t).$$

Then from (1.3.1)–(1.3.2) it follows that

$$F(t) \le C + \int_0^t h(s)F(s)ds \tag{1.3.4}$$

where $C = C_1 + C_2$ and h is defined by

$$h(t) = \max\Big\{|h_1(t) + h_3(t)|, |h_2(t) + h_4(t)|\Big\}.$$

Now applying Theorem 1.1.2 to (1.3.4) yields

$$F(t) \le C\exp\Big(\int_0^t h(s)ds\Big). \tag{1.3.5}$$

Inserting (1.3.5) into (1.3.4), we obtain

$$f(t) \le C\exp\Big(\mu t + \int_0^t h(s)ds\Big), \quad g(t) \le C\exp\Big(\int_0^t h(s)ds\Big)$$

which gives us the bounds in (1.3.3). □

The following system of inequalities can be considered as a simultaneously singular Bellman–Gronwall inequality (see, e.g., Dickstein and Loayza [210]).

Theorem 1.3.2 (The Dickstein–Loayza Inequality [210]). *Let* $A > 0, B > 0, k > 0, T > 0, 0 < \alpha_1, \alpha_2, \beta_1, \beta_2 < 1$. *Assume that continuous functions* $\phi, \psi : (0,T) \to \mathbb{R}^+$ *satisfy for any* $t \in (0,T)$,

$$\phi(t) \le A + kt^{\alpha_1}\int_0^t (t-s)^{-\alpha_1}s^{-\beta_1}\psi(s)ds, \tag{1.3.6}$$

$$\psi(t) \le B + kt^{\alpha_2}\int_0^t (t-s)^{-\alpha_2}s^{-\beta_2}\phi(s)ds. \tag{1.3.7}$$

Then there exists a constant $C = C(\alpha_1, \alpha_2, \beta_1, \beta_2, k, T) > 0$ *such that for all* $t \in (0,T)$,

$$\phi(t) \le C\left(A + Bt^{1-\beta_1}\right), \quad \psi(t) \le C\left(B + At^{1-\beta_2}\right). \tag{1.3.8}$$

Proof. Set $\tilde{\phi}(t) = \sup_{s\le t} \phi(s)$, $\tilde{\psi}(t) = \sup_{s\le t} \psi(s)$. Then (1.3.6)–(1.3.7) hold for $\tilde{\phi}$ and $\tilde{\psi}$. Indeed, if $\tau < t$, then

$$\begin{aligned}\tau^{\alpha_1}\int_0^\tau (t-s)^{-\alpha_1}s^{-\beta_1}\tilde{\psi}(s)ds &= \tau^{1-\beta_1}\int_0^1 (1-z)^{-\alpha_1}z^{-\beta_1}\tilde{\psi}(\tau z)dz\\ &\le t^{1-\beta}\int_0^1 (1-z)^{-\alpha_1}z^{-\beta_1}\tilde{\psi}(tz)dz\\ &= t^{\alpha_1}\int_0^t (t-s)^{-\alpha_1}s^{-\beta_1}\tilde{\psi}(s)ds,\end{aligned}$$

so that,

$$\phi(\tau) \le A + k\tau^{\alpha_1}\int_0^\tau (\tau-s)^{-\alpha_1}s^{-\beta_1}\tilde{\psi}(s)ds \le A + kt^{\alpha_1}\int_0^t (t-s)^{-\alpha_1}s^{-\beta_1}\tilde{\psi}(s)ds,$$

whence

$$\tilde{\phi}(t) \le A + kt^{\alpha_1}\int_0^t (t-s)^{-\alpha_1}s^{-\beta_1}\tilde{\psi}(s)ds.$$

A similar estimate holds for $\tilde{\psi}$. It then suffices to prove (1.3.8) for $\tilde{\phi}, \tilde{\psi}$. This is why we assume that ϕ and ψ are non-decreasing functions.

First we prove (1.3.8) for t small. To this end, define

$$M = k\max\left[\int_0^1 (1-s)^{-\alpha_1}s^{-\beta_1}ds, \int_0^1 (1-s)^{-\alpha_2}s^{-\beta_2}ds\right].$$

Then we deduce from (1.3.6)–(1.3.7),

$$\phi(t) \le A + Mt^{1-\beta_1}\psi(t), \quad \psi(t) \le B + Mt^{1-\beta_2}\phi(t) \tag{1.3.9}$$

which gives

$$\phi(t) \le A + BMt^{1-\beta_1} + M^2t^{2-\beta_1-\beta_2}\phi(t). \tag{1.3.10}$$

Fix $\tau > 0$ such that $M^2\tau^{2-\beta_1-\beta_2} < 1/2$. If $t \le \tau$, then we derive from (1.3.10)

$$\phi(t) \le 2(A + BMt^{1-\beta_1}). \tag{1.3.11}$$

Analogously, we have

$$\psi(t) \le 2(B + AMt^{1-\beta_2})$$

which, together with (1.3.11), proves (1.3.8) for all $t \le \tau$ with $C = 2$.

Consider now $t > \tau$ and choose a, b such that

$$\left\{\begin{aligned} &\left(\int_0^a + \int_b^1\right)(1-s)^{-\alpha_1}s^{-\beta_1}ds \le \frac{1}{2k}T^{-(1-\beta_1)}, &(1.3.12)\\ &\left(\int_0^a + \int_b^1\right)(1-s)^{-\alpha_2}s^{-\beta_2}ds \le \frac{1}{2k}T^{-(1-\beta_2)}. &(1.3.13)\end{aligned}\right.$$

Then by virtue of (1.3.12)–(1.3.13), we deduce from (1.3.6) that

$$\begin{aligned}\phi(t) &\le A + kt^{\alpha_1}\left(\int_0^{at} + \int_{at}^{bt} + \int_{bt}^{t}\right)(t-s)^{-\alpha_1}s^{-\beta_1}\psi(s)ds\\ &\le A + \frac{1}{2}(tT^{-1})^{1-\beta_1}\psi(t) + k(1-b)^{-\alpha_1}(a\tau)^{-\beta_1}\int_0^t \psi(s)ds\\ &\le A + \frac{1}{2}\psi(t) + k(1-b)^{-\alpha_1}(a\tau)^{-\beta_1}\int_0^t \psi(s)ds. \end{aligned} \tag{1.3.14}$$

Similarly, from (1.3.7), we obtain

$$\psi(t) \le B + \frac{1}{2}\phi(t) + k(1-b)^{-\alpha_2}(a\tau)^{-\beta_2}\int_0^t \phi(s)ds. \tag{1.3.15}$$

Set

$$J = \frac{4}{3}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \tag{1.3.16}$$

$$Q = \begin{pmatrix} 0 & k(1-b)^{-\alpha_1}(a\tau)^{-\beta_1} \\ k(1-b)^{-\alpha_2}(a\tau)^{-\beta_2} & 0 \end{pmatrix}, \tag{1.3.17}$$

$$P = JQ, \quad v = J\begin{pmatrix} A \\ B \end{pmatrix}, \tag{1.3.18}$$

and

$$f(t) = \begin{pmatrix} \int_0^t \phi(s)ds \\ \int_0^t \psi(s)ds \end{pmatrix}. \tag{1.3.19}$$

Then we derive from (1.3.14)–(1.3.19) that

$$f'(t) \le v + Pf(t),$$

whence

$$f'(t) \le e^{Pt}v.$$

This shows (1.3.8) for $C > 0$ and for all $t > \tau$, which completes the proof. □

1.4 The inequalities of Henry's type

In this section, we shall introduce the inequalities of Henry's type. Such kind of inequalities usually arises from the infinite-dimensional theory in a Banach space of evolutionary partial differential equalities which requires solving integral inequalities with singular kernels. Henry proposed a method in [355] to find solutions of such inequalities and proved some results concerning linear inequalities of Henry's type. More general linear inequalities of this type were established in Sano and Kunimatsu [845].

1.4.1 The Henry inequalities

In this subsection, we shall present a new method, due to [606], to solve nonlinear integral inequalities of Henry type and also their nonlinear Bihari version. These estimates are quite simple and the resulting formulas are similar to those in the classical Gronwall–Bihari inequalities. We also present results from [606] on integral inequalities containing multiple integrals which are modifications of the results recently published in [605] (see also [604]).

Theorem 1.4.1 (The Henry Inequality [355]). *Assume $b \geq 0, \beta > 0$ and let $a(t)$ be a non-negative function locally integrable on $[0, T)$ (for some $T \leq +\infty$), and assume that $u(t)$ is non-negative and locally integrable on $[0, T)$ such that*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1}u(s)ds \tag{1.4.1}$$

on this interval, then for all $t \in [0, T)$,

$$u(t) \leq a(t) + \int_0^t \left(\sum_{n=1}^{+\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right) ds. \tag{1.4.2}$$

Proof. Let $B\phi(t) = b\int_0^t (t-s)^{\beta-1}\phi(s)ds,\ t \geq 0$, for locally integrable functions ϕ. Then $u \leq a + Bu$ implies $u \leq \sum_{k=0}^{n-1} B^k a + B^n u$, and $B^n u(t) = \int_0^t (b\Gamma(\beta))^n (t-s)^{n\beta-1}u(s)ds/\Gamma(n\beta) \to 0$ as $n \to +\infty$ for each $t \in [0, T)$. Now from (1.4.1) it follows that

$$u(t) \leq a(t) + \int_0^t \sum_{n=1}^{+\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)ds. \tag{1.4.3}$$

□

Remark 1.4.1. In fact, the original form of (1.4.2) in Henry [355] should read

$$u(t) \leq a(t) + \theta\int_0^t E'_\beta(\theta(t-s))a(s)ds, \tag{1.4.4}$$

where

$$\theta = [b\Gamma(\beta)]^{1/\beta}, \quad E_\beta(z) = \sum_{n=0}^{+\infty} z^{n\beta}/\Gamma(n\beta+1), \quad E'_\beta(z) = \frac{d}{dz}E_\beta(z),$$

and $E'_\beta(z) \simeq z^{\beta-1}/\Gamma(\beta)$ as $z \to 0^+$, $E'_\beta(z) \simeq e^z/\beta$ as $z \to +\infty$, and $E_\beta(z) \simeq e^z/\beta$ as $z \to +\infty$. If $a(t) \equiv a$, a constant, then

$$u(t) \leq aE_\beta(\theta t). \tag{1.4.5}$$

Theorem 1.4.2 (The Henry Inequality [355]). *Assume $\beta > 0, \gamma > 0, \beta + \gamma > 1$ and $a \geq 0, b \geq 0$, and let u be non-negative and $t^{\gamma-1}u(t)$ locally integrable on $[0,T)$, and satisfy for a.e. in $[0,T)$,*

$$u(t) \leq a + b\int_0^t (t-s)^{\beta-1}s^{\gamma-1}u(s)ds. \tag{1.4.6}$$

Then

$$u(t) \leq aE_{\beta,\gamma}\left((b\Gamma(\beta))^{1/\nu}t\right), \tag{1.4.7}$$

where $\nu = \beta + \gamma - 1 > 0, E_{\beta,\gamma}(s) = \sum_{m=0}^{+\infty} C_m s^{m\nu}$ with $C_0 = 1, C_{m+1}/C_m = \Gamma(m\nu + \gamma)/\Gamma(m\nu + \gamma + \beta)$ for $m \geq 0$. As $s \to +\infty$, we have

$$E_{\beta,\gamma}(s) = O\left(s^{1/2(\nu/\beta-\gamma)}\exp\left(\frac{\beta}{\gamma}s^{\nu/\beta}\right)\right). \tag{1.4.8}$$

Proof. If

$$B\phi(t) = b\int_0^t (t-s)^{\beta-1}s^{\gamma-1}\phi(s)ds,$$

then an easy induction from (1.4.6) shows that

$$u(t) \leq a\sum_{m=0}^{n} C_m[b\Gamma(\beta)]^m t^{m\nu} + B^{n+1}u(t). \tag{1.4.9}$$

Also

$$B^n u(t) = \int_0^t K_n(t,s)s^{\gamma-1}u(s)ds, \tag{1.4.10}$$

where for $\gamma \geq 1$,

$$K_n(t,s) \leq Q_n t^{(n-1)(\gamma-1)}(t-s)^{n\beta-1},$$
$$Q_1 = b,\ Q_{n+1}/Q_n = b\Gamma(\beta)/\Gamma(n\beta)/\Gamma(n\beta+\beta).$$

If $\gamma \in (0,1)$, we have

$$K_n(t,s) \leq Q_n(t-s)^{n\nu-\gamma},\ \ Q_1 = b,$$
$$Q_{n+1}/Q_n = b\Gamma(\beta)/\Gamma(n\nu)/\Gamma(n\nu+\beta).$$

In either case, $Q_{n+1}/Q_n = O(n^{-\beta})$ as $n \to +\infty$, so $B^n u(t) \to 0$ as $n \to +\infty$, and

$$u(t) \leq aE_{\beta,\gamma}\left([b\Gamma(\beta)]^{1/\nu}t\right). \tag{1.4.11}$$

Now $\Gamma(z+p)/\Gamma(z+q) = z^{p-q}\{1 + (p-q)(p+q-1)/2z + O(z^{-2})\}$ as $z \to +\infty$, so if $\delta = (\beta\gamma + \nu)/2\nu$, then

$$\frac{\Gamma((n+1)\beta+\delta)C_{n+1}}{\Gamma(n\beta+\delta)C_n} = (\beta/\nu)^\beta[1 + O(n^{-2})]. \tag{1.4.12}$$

Thus $C_n\Gamma(n\beta+\delta)(\beta/\nu)^{-n\beta}$ converges as $n \to +\infty$ and has an upper bound K for all $n \geq 0$. Then for all $s > 0$,

$$E_{\beta,\gamma}(s^{\beta/\nu})s^{\delta-1} \leq K\sum_{n=0}^{+\infty}\frac{(\beta/\nu)^{n\beta}}{\Gamma(n\beta+\delta)}s^{n\beta+\delta-1}. \tag{1.4.13}$$

The Laplace transform of the right-hand side of (1.4.13) is

$$K\lambda^{-\delta}/\left[1-(\beta/\nu\lambda)^{\beta}\right],$$

therefore, the series is $O(\exp(\beta s/\nu))$ as $s \to +\infty$, which proves the result. □

The estimate of $E_\beta(z)$ and $E'_\beta(z)$ as $z \to +\infty$ follows from the fact that the Laplace transform

$$\int_0^{+\infty} e^{-\lambda z}E_\beta(z)dz = \lambda^{-1}/(1-\lambda^{-\beta})$$

has a simple pole at $\lambda = 1$ (see, e.g., Evgrafov [243]). For example, we can choose $\gamma \in (0,1)$ so $1-\lambda^{-\beta} \neq 0$ for $\Re\lambda \geq \gamma, \lambda \neq 1$, and then for $z > 0$,

$$E_\beta(z) = \frac{1}{\beta}e^z + \frac{1}{2\pi i}\lim_{N\to+\infty}\int_{\gamma-iN}^{\gamma+iN} e^{\lambda z}\lambda^{-1}/(1-\lambda^{-\beta})d\lambda, \tag{1.4.14}$$

where the shift in the line of integration is justified by $e^{\lambda z}\lambda^{-1}/(1-\lambda^{-\beta}) \to 0$ as $\Im\lambda \to \pm\infty$ for $\Re\lambda$ bounded. Integration by parts in the integral on the right-hand side of (1.4.14) shows that

$$\left|E_\beta(z) - \frac{1}{\beta}e^z\right| = O(e^z) \text{ as } z \to +\infty.$$

The following result was established by Nagumo [659], and can be viewed as a generalization of Theorem 1.4.2 for the special case when $a \equiv 0$, $\beta = 1$, $\gamma = 0$ $(\beta+\gamma = 1)$.

Corollary 1.4.1 (The Nagumo Inequality [659]). *Let $v(t) \in C[0,b]$ be a non-negative function such that $v(0) = 0$ and $\lim_{h\to0^+} v(h)/h = 0$. If $v(t)$ satisfies for all $t \in (0,b]$,*

$$v(t) \leq \int_0^t v(s)/sds, \tag{1.4.15}$$

then for all $t \in [0,b]$,

$$v(t) = 0.$$

Proof. For all $t > 0$, $\varepsilon > 0$, $t > \varepsilon$, let $F(t) = \int_0^t v(s)/sds$ for $s \in [\varepsilon,t]$. Then we have $F'(t) = v(t)/t$. If we add the condition $F'(0) = 0$, then since

$\lim_{h\to 0^+} v(h)/h = 0$, we have that $F'(t) \in C[0,b]$. By (1.4.15), we know that $F'(t) = v(t)/t \le F(t)/t$ for all $t > 0$, that is,

$$\big(\log F(t)\big)' \le 1/t, \quad t > 0. \tag{1.4.16}$$

Integrating (1.4.16) over $[\varepsilon, t]$ for any $\varepsilon > 0$ $(t \ge \varepsilon)$ yields

$$F(t) \le F(\varepsilon)t/\varepsilon. \tag{1.4.17}$$

By the l'Hospital Rule and noting that $\lim_{h\to 0^+} v(h)/h = 0$, we deduce that for all $t > 0$

$$F(t) \le t \lim_{\varepsilon\to 0^+} F(\varepsilon)/\varepsilon = t \lim_{\varepsilon\to 0^+} F'(\varepsilon) = 0$$

which together with (1.4.15) gives that for all $t \in (0,b]$,

$$v(t) = 0. \tag{1.4.18}$$

Combining $v(0) = 0$ and (1.4.18), we complete the proof. □

In the same manner, we may prove the following result (see, e.g., Henry [355], p. 190); the proof is left to the reader.

Theorem 1.4.3 (The Henry Inequality [355]). *If* α, β, γ *are positive with* $\beta+\gamma-1 = \nu > 0, \delta = \alpha+\gamma-1 > 0$, *and for all* $t > 0$,

$$u(t) \le at^{\alpha-1} + b\int_0^t (t-s)^{\beta-1}s^{\gamma-1}u(s)ds, \tag{1.4.19}$$

then

$$u(t) \le at^{\alpha-1}\sum_{m=0}^{+\infty} C'_m (b\Gamma(\beta))^m t^{m\nu} \tag{1.4.20}$$

where $C'_0 = 1, C'_{m+1} = C'_m = \Gamma(m\nu+\delta)/\Gamma(m\nu+\delta+\beta)$.

Corollary 1.4.2 (The Henry Inequality [355]). *Under the hypotheses of Theorem* 1.4.3*, let* $a(t)$ *be a non-decreasing function on* $[0, T)$. *Then*

$$u(t) \le a(t)E_\beta\left(g(t)\Gamma(\beta)t^\beta\right), \tag{1.4.21}$$

where E_β *is the Mittag-Leffler function defined by* $E_\beta(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\beta+1)}$.

In order to formulate the following singular Bellman–Gronwall inequality (see, e.g., Theorem 1.4.4) which can be viewed as a generalization of the above theorem (see, e.g., Amann [40]), we need to introduce first some basic concepts.

By a vector space, we always understand a vector space over K, where $K = \mathbb{R}$ or $K = \mathbb{C}$. If M is a subset of a vector space, we set

$$\dot{M} := M\backslash\{0\}.$$

If X is a topological space, by $BC(X,E)$ we denote the closed linear subspace of $B(X,E)$ consisting of all bounded and continuous functions.

Let J be a perfect subinterval of $\mathbb{R}$. Denote

$$J_\Delta := \{(t,s) \in J \times J;\ s \le t\}$$

and

$$J_\Delta^* := \{(t,s) \in J_\Delta;\ s < t\}.$$

Assume that J is a perfect subinterval of $\mathbb{R}^+$ containing 0 and let

$$J_T := J \cap [0,T], \quad T \in \mathbb{R}^+.$$

For any given $\alpha \in \mathbb{R}$, we denote by $\mathcal{K}(E,F,\alpha)$ the Fréchet space of all $k \in C(J_\Delta^*, \mathcal{L}(E,F))$ satisfying

$$||k||_{(\alpha),T} := ||k||_{(\alpha),T,\mathcal{L}(E,F)} := \sup_{0 \le s < t \le T} (t-s)^\alpha ||k(t,s)||_{\mathcal{L}(E,F)} < +\infty, \quad T \in \dot{J},$$

equipped with the topology induced by the seminorms $\{||\cdot||_{(\alpha),T};\ T \in \dot{J}\}$. We also put $\mathcal{K}(E,\alpha) := \mathcal{K}(E,E,\alpha)$. We see that

$$||\cdot||_{(\alpha),T} \le T^{\alpha-\beta} ||\cdot||_{(\beta),T}, \quad \alpha > \beta, \quad T \in \dot{J}, \tag{1.4.22}$$

so that

$$\mathcal{K}(E,F,\beta) \hookrightarrow \mathcal{K}(E,F,\alpha), \quad \alpha > \beta. \tag{1.4.23}$$

Let

$$||k||_{(\alpha)} := \sup_{(t,s) \in J_\Delta^*} (t-s)^\alpha ||k(t,s)||_{\mathcal{L}(E,F)}$$

and denote by $\mathcal{K}_\infty(E,F,\alpha)$ the Banach space consisting of all $k \in \mathcal{K}(E,F,\alpha)$ satisfying $||k||_{(\alpha)} < +\infty$, equipped with norm $||\cdot||_{(\alpha)}$. Note that

$$\mathcal{K}_\infty(E,F,\alpha) \hookrightarrow \mathcal{K}(E,F,\alpha) \tag{1.4.24}$$

and

$$\mathcal{K}_\infty(E,F,0) = BC(J_\Delta^*, \mathcal{L}(E,F)). \tag{1.4.25}$$

If $\alpha < 0$, then each $k \in \mathcal{K}(E,F,\alpha)$ can be continuously extended over J_Δ by putting $k(t,t) = 0$ for all $t \in J$ so that

$$\mathcal{K}(E,F,\alpha) \hookrightarrow C(J_\Delta, \mathcal{L}(E,F)), \quad \alpha < 0. \tag{1.4.26}$$

If $E = K$, we canonically identify $\mathcal{L}(K,F)$ with F via

$$\mathcal{L}(K,F) \ni B \leftrightarrow B \cdot 1 \in F.$$

Then $k \in \mathcal{K}(K,F,\alpha)$ if and only if $k \in C(J_\Delta^*, F)$ and

$$\sup_{0 \le s < t \le T} (t-s)^\alpha ||k(t,s)||_{\mathcal{L}(E,F)} < +\infty, \quad T \in \dot{J}.$$

In particular, we have an embedding

$$BC(\dot{J}, F) \hookrightarrow \mathcal{K}_\infty(K, F, 0) = BC(J^*_\Delta, F) \tag{1.4.27}$$

by the identification

$$C(\dot{J}, F) \ni u \leftrightarrow \Big[(t,s) \mapsto u(t)\Big] \in C(J^*_\Delta, F). \tag{1.4.28}$$

Let G be a Banach space. Assume that $k \in \mathcal{K}(E, F, \alpha)$ and $h \in \mathcal{K}(F, G, \beta)$ with $\alpha, \beta \in (-\infty, 1)$, and set

$$h * k(t,s) := \int_s^t h(t,\tau)k(\tau,s)d\tau, \quad (t,s) \in J_\Delta.$$

We easily verify that

$$h * k \in \mathcal{K}(E, G, \alpha + \beta - 1) \tag{1.4.29}$$

and

$$||h * k||_{(\alpha+\beta-1),T} \le B(1-\alpha, 1-\beta)||h||_{(\alpha),T}||k||_{(\beta),T}, \quad T \in \dot{J}, \tag{1.4.30}$$

where B is Euler's beta function. It follows from Fubini's theorem that the operation $*$ is associative.

In the sequel, we shall prove the following generalized Bellman–Gronwall inequality. First, we give the following two lemmas.

Assume that $k \in \mathcal{K}(E, \alpha)$ for some $\alpha \in [0, 1)$. By an easy induction argument, we see that for all $n \in \mathbb{N}$ and all $0 \le s < t \le T$,

$$||\underbrace{k * k * \cdots * k(t,s)}_{n}||_{\mathcal{L}(E)} \le \frac{[\Gamma(1-\alpha)||k||_{(\alpha),T}]^n}{\Gamma(n(1-\alpha))}(t-s)^{n(1-\alpha)-1}. \tag{1.4.31}$$

Set

$$\omega := \sum_{j=1}^{+\infty} \underbrace{k * \cdots * k}_{j}. \tag{1.4.32}$$

Then we have the following lemma.

Lemma 1.4.4 ([40]). *The function $w \in \mathcal{K}(E, \alpha)$ satisfies the estimate*

$$(t-s)^\alpha ||w(t,s)||_{\mathcal{L}(E)} \le c(\alpha, \varepsilon) m e^{(1+\varepsilon)m^{1/(1-\alpha)}(t-s)}, \tag{1.4.33}$$

for any given $\varepsilon > 0$ and $0 \le s < t \le T, T \in \dot{J}$, where

$$m := \Gamma(1-\alpha)||k||_{(\alpha),T}.$$

Proof. Let $\beta := 1-\alpha \in (0,1]$. Thanks to (1.4.31), it suffices to prove that for all $x>0$,

$$\sum_{j=1}^{+\infty} \frac{x^{j-1}}{\Gamma(\beta j)} \le c(\beta,\varepsilon) e^{(1+\varepsilon)x^{1/\beta}}. \tag{1.4.34}$$

Stirling's formula implies the existence of $\theta(t) \in (0,1)$ such that for all $t>0$,

$$\Gamma(t) = \sqrt{2\pi} t^{t-1/2} e^{-t+\theta(t)/(12t)},$$

which implies that for all $j \in \mathbb{N}$ that

$$\frac{\Gamma(j+1)^\beta}{\Gamma(\beta j)} = \frac{[j\Gamma(j)]^\beta}{\Gamma(\beta j)} \le (2\pi)^{(\beta-1)/2} e^{\beta/12} \beta^{1/2} \frac{j^{(1+\beta)/2}}{\beta^{\beta j}}.$$

Hence, by Hölder's inequality

$$\begin{aligned}
\sum_{j=1}^{+\infty} \frac{x^j}{\Gamma(\beta j)} &= \sum_{j=1}^{+\infty} \frac{x^j}{(j!)^\beta} \frac{\Gamma(j+1)^\beta}{\Gamma(\beta j)} \\
&\le c(\beta) \left[\sum_{j=1}^{+\infty} \frac{(\eta x^{1/\beta})^j}{j!} \right]^\beta \left[\sum_{j=1}^{+\infty} \frac{j^{(1+\beta)/(2(1-\beta))}}{(\eta\beta)^{j\beta/(1-\beta)}} \right]^{1-\beta},
\end{aligned}$$

where $\eta > 0$ is arbitrary. Since the last series converges for $\eta > 1/\beta$, it follows that for all $x>0$ and $\eta > 1/\beta$,

$$\sum_{j=1}^{+\infty} \frac{x^{j-1}}{\Gamma(\beta j)} \le c(\beta,\eta) \left(\frac{e^{\eta x^{1/\beta}} - 1}{\eta x^{1/\beta}} \right)^\beta \le c(\beta,\eta) e^{\beta \eta x^{1/\beta}}$$

which yields (1.4.33) with $\eta := (1+\varepsilon)/\beta$. □

Now it is easy to prove the following existence and uniqueness theorem for abstract linear Volterra equations.

Lemma 1.4.5 ([40])**.** *Assume that $\alpha, \beta \in [0,1)$ and $k \in \mathcal{K}(E,\alpha)$. Then the linear Volterra equations*

$$u = a + u * k, \quad v = b + k * v \tag{1.4.35}$$

possess for each $a \in \mathcal{K}(E,F,\beta)$ and $b \in \mathcal{K}(F,E,\beta)$ unique solutions

$$u \in \mathcal{K}(E,F,\beta), \quad \textit{and} \quad v \in \mathcal{K}(F,E,\beta),$$

respectively, which are given by

$$u = a + a * \omega, \quad v = b + \omega * b \tag{1.4.36}$$

respectively, where ω, the resolvent kernel of (1.4.35)*, belongs to $\mathcal{K}(E,\alpha)$ and is given by* (1.4.32)*.*

Proof. We consider the first equation in (1.4.35). The second one can be treated in a similar manner.

Define ω by (1.4.32) and u by (1.4.36), and observe that $\omega \in \mathcal{K}(E,\alpha)$ and $u \in \mathcal{K}(E,F,\beta)$ by Lemma 1.4.1 and by (1.4.23) and (1.4.29), respectively. It is obvious that u solves (1.4.35).

Let $T \in \dot{J}$ be fixed. Upon replacing J by J_T, it follows from (1.4.22), (1.4.23), (1.4.29), (1.4.30), and (1.4.31) that $*k \in \mathcal{L}(\mathcal{K}_\infty(E,F,\beta))$ and that the spectral radius of this operator equals zero. Hence (1.4.35) has at most one solution 'on J_T' for each $T \in \dot{J}$. This proves the lemma. □

Remark 1.4.2. **([40])** In the definition of $\mathcal{K}(E,F,\alpha)$, we can replace the assumption that $k \in C(J_\Delta^*, \mathcal{L}(E,F))$. Then everything remains true provided:

(1) $\sup_{0\le s<t\le T}$ is replaced by $\operatorname{ess\,sup}_{0\le s<t\le T}$ everywhere.
(2) (1.4.26) is replaced by

$$\mathcal{K}(E,F,\alpha) \cap C(J_\Delta^*, \mathcal{L}(E,F)) \hookrightarrow C(J_\Delta, \mathcal{L}(E,F)), \quad \alpha < 0.$$

(3) $C(J_\Delta^*, F)$ is replaced by $L_{\infty,\mathrm{loc}}(J_\Delta^*)$ in the interpretation of $\mathcal{K}(K,F,\alpha)$.
(4) BC is replaced by L_∞ in (1.4.25) and (1.4.27).

With this new definition of $\mathcal{K}(E,F,\alpha)$, and by using obvious notation, observe that

$$\mathcal{K}(F,G,\beta) * \mathcal{K}(E,F,\alpha) \hookrightarrow \mathcal{K}(E,G,\alpha+\beta-1) \cap C(J_\Delta, \mathcal{L}(E,G))$$

if $\alpha + \beta < 1$.

As a simple application of Lemma 1.4.2, we prove the following generalized Bellman–Gronwall inequality (see [40]).

Theorem 1.4.6. **([40])** *Given $\alpha, \beta \in [0,1)$ and $\varepsilon > 0$, there exists a positive constant $c := c(\alpha,\beta,\varepsilon)$ such that the following is true: If $u : J \to \mathbb{R}$ satisfies*

$$\left[t \mapsto t^\beta u(t) \right] \in L_{\infty,\mathrm{loc}}(J, \mathbb{R}) \tag{1.4.37}$$

and for a.e. $t \in \dot{J}$,

$$u(t) \le At^{-\beta} + B \int_0^t (t-\tau)^{-\alpha} u(\tau) d\tau, \tag{1.4.38}$$

where A and B are positive constants, then for a.e. $t \in \dot{J}$,

$$u(t) \le At^{-\beta} \left(1 + cBt^{1-\alpha} e^{(1+\varepsilon)\mu(\alpha,B)t} \right), \tag{1.4.39}$$

where $\mu(\alpha, B) := (\Gamma(1-\alpha)B)^{1/(1-\alpha)}$.

Proof. Let $E := F := \mathbb{R}$ and $k(t,s) := B(t-s)^{-\alpha}$ for $(t,s) \in J_\Delta^*$. Then that $k \in \mathcal{K}(E,\alpha)$ and $||k||_{(\alpha),T} = B$ for $T \in \dot{J}$. Let $a(t) := At^{-\beta}$ and observe that (1.4.28) implies $a \in \mathcal{K}(E,\beta)$ and $||a||_{(\beta),T} = A$ for $T \in \dot{J}$. Since $u \in \mathcal{K}(E,\beta)$ by (1.4.37) and Remark 1.4.2, it follows from (1.4.29) and (1.4.23) that

$$b := a + k * u - u \in \mathcal{K}(E,\beta).$$

Hence $u = a - b + k * u$, and now Lemma 1.4.2 implies that

$$u = (a-b) + \omega * (a-b).$$

Note that $b \geq 0$ by (1.4.38) and that $k \geq 0$ implies $\omega \geq 0$. Thus $u \leq a + \omega * a$, that is, for almost all $t \in \dot{J}$,

$$u(t) \leq At^{-\beta} + A\int_0^t \omega(t-\tau)\tau^{-\beta}d\tau.$$

Thus by Lemma 1.4.1, we have for all $t > 0$, $\varepsilon > 0$,

$$\omega(t) \leq c(\alpha,\varepsilon)Bt^{-\alpha}e^{(1+\varepsilon)\mu(\alpha,B)t}.$$

Since for all $t > 0$ and $\nu \geq 0$,

$$\begin{aligned}\int_0^t e^{\nu(t-\tau)}(t-\tau)^{-\alpha}\tau^{-\beta}d\tau &\leq e^{\nu t}\int_0^t (t-\tau)^{-\alpha}\tau^{-\beta}d\tau \\ &= B(1-\alpha,1-\beta)t^{1-\alpha-\beta}e^{\nu t},\end{aligned}$$

the assertion follows immediately. □

Corollary 1.4.3 ([40]). *Assume* (1.4.37) *and* (1.4.38) *hold. Then for any* $\varepsilon > 0$, *there exists a constant* $c := c(\varepsilon,\alpha,\beta,B) > 0$ *such that for almost all* $t \in \dot{J}$,

$$u(t) \leq Act^{-\beta}e^{(1+\varepsilon)\mu(\alpha,B)t}.$$

Remark 1.4.3.

(a) We note that, in general, the constant $c(\alpha,\varepsilon)$ in the estimate of Lemma 1.4.1 and, consequently, the constant c in Theorem 1.4.4 and Corollary 1.4.3, tend to infinity if $\varepsilon \to 0$. Moreover, if $\alpha = 0$, then $\varepsilon = 0$ is possible and $c(0,0) = 1$. In this case, the constant c of Theorem 1.4.4 equals $1/(1-\alpha)$ and (1.4.39) is then a consequence of the classical Gronwall inequality (e.g., [37], Corollary 6.2).

(b) The factor $e^{(1+\varepsilon)\mu(t-s)}$ in Lemma 1.4.1, Theorem 1.4.4 and Corollary 1.4.3, where $\mu := m^{1/(1-\alpha)}$ in Lemma 1.4.1, can be replaced by $e^{(\mu+\varepsilon)(t-s)}$.

It is well known that the generalized Bellman–Gronwall inequality has been proved, in a form somewhat less precise than the one of Theorem 1.4.4, by Amann in [28] and, independently, by means of Laplace transform techniques, by Henry [355]. The technique, used in the proof of Lemma 1.4.1 for estimating the majorant of the series (1.4.32) by means of Stirling's formula, is taken from [941].

The following result may be found in Ye and Li [980], and is a corollary of a special case of Theorem 1.4.4 with $\beta = 0$ and $\alpha \in (0, 1)$.

Corollary 1.4.4 ([980]). *Let $v(t) \geq 0$ be continuous on $[t_0, T]$. If there are positive constants a, b and $\alpha < 1$ such that for all $t \in [t_0, T]$,*

$$v(t) \leq a + b \int_{t_0}^{t} (t-s)^{\alpha-1} v(s) ds, \tag{1.4.40}$$

then there is a constant $M > 0$, independent of a, such that

$$v(t) \leq Ma. \tag{1.4.41}$$

Proof. By iterating (1.4.40) and exploiting the identity

$$\int_0^t (t-s)^{-\alpha-1}(s-\tau)^{\beta-1} ds = (t-\tau)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

we obtain

$$\begin{aligned} v(t) &\leq a + b \int_{t_0}^{t} (t-s)^{\alpha-1} \Big\{ a + b \int_{t_0}^{s} (s-\tau)^{\alpha-1} v(\tau) d\tau \Big\} ds \\ &\leq a \Big\{ 1 + b \frac{(T-t_0)^{\alpha}}{a} \Big\} + b^2 \int_{t_0}^{t} \Big\{ \int_{\tau}^{t} (t-s)^{\alpha-1}(s-\tau)^{\alpha-1} ds \Big\} v(\tau) d\tau \\ &= a(1+b) \Big\{ 1 + b \frac{(T-t_0)^{\alpha}}{a} \Big\} + b^2 \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} \int_{t_0}^{t} (t-\tau)^{2\alpha-1} v(\tau) d\tau, \end{aligned}$$

which implies that

$$v(t) \leq a \sum_{j=0}^{n-1} \left[\frac{b(T-t_0)^{\alpha}}{a} \right]^j + \frac{[b\Gamma(\alpha)]^n}{\Gamma(n\alpha)} \int_{t_0}^{t} (t-\tau)^{n\alpha-1} v(\tau) d\tau. \tag{1.4.42}$$

Choosing n so large that $n\alpha - 1 > 0$, we conclude that

$$v(t) \leq C_1 a + C_2 \int_{t_0}^{t} v(\tau) d\tau \tag{1.4.43}$$

where C_1, C_2 are positive constants depending only on $T - t_0$ and b, but not on α and a. Thus (1.4.41) follows from (1.4.43), by the Bellman–Gronwall inequality. □

Recently, Ye et al. [979] gave the following inequality to prove the continuous dependence on parameters of fractional differential equations, which can be viewed as a general form of the above theorem.

Theorem 1.4.7 ([979]). *Let $\beta > 0$, and assume that $a(t)$ is a non-negative function locally integrable on $[0, T)$ for some $T \leq +\infty$ and let $g(t)$ be a non-negative, non-decreasing continuous function on $[0, T), g(t) \leq M$ (a constant). Assume that $u(t)$ is non-negative and locally integrable on $0 \leq t < T$ and satisfies*

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) ds \tag{1.4.44}$$

on this interval. Then for all $0 \leq t < T$,

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{+\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds. \tag{1.4.45}$$

Proof. Let $B\varphi(t) = g(t) \int_0^t (t-s)^{\beta-1} \varphi(s) ds, t \geq 0$, for locally integrable functions φ. Then

$$u(t) \leq a(t) + Bu(t)$$

implies

$$u(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n u(t).$$

Let us prove by induction that

$$B^n u(t) \leq \int_0^t \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} u(s) ds \tag{1.4.46}$$

and $B^n u(t) \to 0$ as $n \to +\infty$ for each $t \in [0, T)$.

We know that (1.4.46) holds for $n = 1$. Assume that it holds for some $n = k$. If $n = k+1$, then

$$\begin{aligned} B^{k+1}u(t) &= B(B^k u(t)) \\ &\leq g(t) \int_0^t (t-s)^{\beta-1} \left[\int_0^s \frac{(g(t)\Gamma(\beta))^k}{\Gamma(k\beta)} (s-\tau)^{k\beta-1} u(\tau) d\tau \right] ds. \end{aligned} \tag{1.4.47}$$

Since $g(t)$ is non-decreasing, it follows that

$$B^{k+1}u(t) \leq (g(t))^{k+1} \int_0^t (t-s)^{\beta-1} \left[\int_0^s \frac{(\Gamma(\beta))^k}{\Gamma(k\beta)} (s-\tau)^{k\beta-1} u(\tau) d\tau \right] ds. \tag{1.4.48}$$

Interchanging the order of integration, we obtain

$$B^{k+1}u(t) \leq \int_0^t \frac{(g(t)\Gamma(\beta))^{k+1}}{\Gamma((k+1)\beta)} (t-s)^{(k+1)\beta-1} u(s) ds, \tag{1.4.49}$$

where the integral

$$\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{k\beta-1}ds = (t-\tau)^{k\beta+\beta-1}\int_0^1 (1-z)^{\beta-1}z^{k\beta-1}dz$$
$$= (t-\tau)^{k\beta+\beta-1}B(k\beta,\beta) = \frac{\Gamma(\beta)\Gamma(k\beta)}{\Gamma((k+1)\beta)}(t-\tau)^{k\beta+\beta-1}$$

is evaluated with the help of the substitution $s=\tau+z(t-\tau)$ and the definition of the beta function (cf. [747], pp. 6–7). This gives us the relation (1.4.46). Moreover, $B^n u(t) \le \int_0^t \frac{(M\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}u(s)ds \to 0$ as $n\to+\infty$ for all $t\in[0,T)$, hence the theorem is proved. □

For $g(t)\equiv$ constant $=b$ in the theorem, we obtain the following inequality, which can be found in Henry ([355], p. 188).

Corollary 1.4.5 ([355]). *Let $b\ge 0$, $\beta>0$, and let $a(t)$ be a non-negative function locally integrable on $[0,T)$ for some $T\le+\infty$. Assume that $u(t)$ is non-negative and locally integrable on $[0,T)$ and that*

$$u(t) \le a(t) + b\int_0^t (t-s)^{\beta-1}u(s)ds \tag{1.4.50}$$

on this interval. Then for all $0\le t<T$,

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{+\infty}\frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right]ds. \tag{1.4.51}$$

Corollary 1.4.6 ([355]). *Under the hypotheses of Theorem 1.4.5, let $a(t)$ be a non-decreasing function on $[0,T)$. Then*

$$u(t) \le a(t)E_\beta(g(t)\Gamma(\beta)t^\beta), \tag{1.4.52}$$

where E_β is the Mittag-Leffler function, defined by $E_\beta(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k\beta+1)}$.

1.4.2 Henry–Gronwall–Bihari type integral inequalities

A new approach due to Medveď [606] to the analysis of nonlinear integral inequalities with weakly singular kernels will be used to prove Theorem 1.4.6, which concerns a nonlinear integral inequality. Linear inequalities investigated by Henry [355] (see also Section 1.4.1) are special cases of this nonlinear one.

First let us define a special class of nonlinear functions.

Definition 1.4.1. Let $q>0$ be a real number and $0<T\le+\infty$. We say that a function $\omega:\overline{\mathbb{R}^+}=[0,+\infty)\to\mathbb{R}$ satisfies condition (q), if for all $u\in\overline{\mathbb{R}^+}$, $t\in[0,T)$,

$$e^{-qt}[\omega(u)]^q \le R(t)\omega(e^{-qt}u^q), \tag{1.4.53}$$

where $R(t)$ is a continuous non-negative function.

Remark 1.4.4. If $\omega(u) = u^m, m > 0$, then for any $q > 1$,

$$e^{-qt}[\omega(u)]^q = e^{(m-1)qt}\omega(e^{-qt}u^q), \tag{1.4.54}$$

i.e., condition (q) (i.e., (1.4.53)) is satisfied with $R(t) = e^{(m-1)qt}$.

Let $\omega(u) = u + au^m$, where $0 \le a \le 1, \quad m \ge 1$. We shall show that ω satisfies condition (q). We also need the following well-known inequality, which is a consequence of Jensen's inequality,

$$(A_1 + A_2 + \cdots + A_n)^r \le n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r) \tag{1.4.55}$$

for any non-negative real numbers $A_1, A_2, \ldots, A_n$, where $r > 1$ is a real number and n is a natural number. Using (1.4.55) with $r = q$ and $n = 2$, we have

$$e^{-qt}[\omega(u)]^q = e^{-qt}(u + au^m)^q \le 2^{q-1}e^{-qt}(u^q + a^q u^{qm}), \tag{1.4.56}$$

$$\begin{aligned} 2^{q-1}e^{qmt}\omega(e^{-qt}u^q) &= 2^{q-1}e^{qmt}[e^{-qt}u^q + ae^{-qmt}u^{qm}] \\ &= 2^{q-1}e^{-qt}[e^{qmt}u^q + au^{qm}] \ge 2^{q-1}e^{-qt}[u^q + a^q u^{qm}] \end{aligned} \tag{1.4.57}$$

and thus (1.4.56) yields condition (q), i.e., (1.4.53), with $R(t) = 2^{q-1}e^{qmt}$.

The following three theorems are due to Medveď [606].

Theorem 1.4.8 (The Medveď Inequality [606]). *Let $a(t)$ be a non-decreasing, non-negative C^1-function on $[0,T)$, $F(t)$ a continuous, non-negative function on $[0,T), \omega : \overline{\mathbb{R}^+} \to \mathbb{R}$ a continuous, non-decreasing function, $\omega(0) = 0, \omega(u) > 0$ on $(0,T)$, and $u(t)$ a continuous, non-negative function on $(0,T)$ satisfying for all $t \in [0,T)$,*

$$u(t) \le a(t) + \int_0^t (t-s)^{\beta-1}F(s)\omega(u(s))ds, \tag{1.4.58}$$

for a constant $\beta > 0$. Then the following assertions hold:

(1) *Assume $\beta > \frac{1}{2}$, and let ω satisfy condition (q) with $q = 2$. Then for all $t \in [0,T_1]$,*

$$u(t) \le e^t \left\{\Omega^{-1}\left[\Omega\left(2a^2(t)\right) + g_1(t)\right]\right\}^{1/2}, \tag{1.4.59}$$

where

$$g_1(t) = \frac{\Gamma(2\beta-1)}{4^{\beta-1}}\int_0^t R(s)F^2(s)ds,$$

and Γ is the gamma function, $\Omega(v) = \int_{v_0}^{v}\left(\frac{dy}{\omega(y)}\right), v_0 > 0, \Omega^{-1}$ the inverse of Ω, and $T_1 \in \mathbb{R}^+$ such that $\Omega\left(2a^2(t)\right) + g_1(t) \in \mathrm{Dom}(\Omega^{-1})$ for all $t \in [0,T_1]$.

(2) *Let $\beta \in (0, \frac{1}{2}]$, and let ω satisfy condition (q) with $q = z + 2$, where $z = \frac{1-\beta}{\beta}$ (i.e., $\beta = \frac{1}{z+1}$). Then for all $t \in [0,T_1]$,*

$$u(t) \le e^t \left\{\Omega^{-1}\left[\Omega\left(2^{q-1}a^q(t)\right) + g_2(t)\right]\right\}^{1/q}, \tag{1.4.60}$$

where

$$g_2(t) = 2^{q-1} K_z^q \int_0^t F^q(s) R(s) ds,$$

$$K_z = \left[\frac{\Gamma(1-ap)}{p^{1-ap}}\right]^{1/p}, \quad \alpha = \frac{z}{z+1}, \ p = \frac{z+2}{z+1}, \tag{1.4.61}$$

where $T_1 \in \mathbb{R}^+$ *is such that* $\Omega\left(2^{q-1} a^q(t)\right) + g_2(t) \in \mathrm{Dom}(\Omega^{-1})$ *for all* $t \in [0, T_1]$.

Proof. First we shall prove the assertion (1). Using the Cauchy–Schwarz inequality, we obtain from (1.4.58)

$$\begin{aligned} u(t) &\le a(t) + \int_0^t (t-s)^{\beta-1} e^s F(s) e^{-s} \omega(u(s)) ds \\ &\le a(t) + \left[\int_0^t (t-s)^{2\beta-2} e^{2s} ds\right]^{1/2} \left[\int_0^t F^2(s) e^{-2s} \omega^2(u(s)) ds\right]^{1/2}. \end{aligned} \tag{1.4.62}$$

For the first integral in (1.4.62), we have

$$\begin{aligned} \int_0^t (t-s)^{2\beta-2} e^{2s} ds &= \int_0^t \tau^{2\beta-2} e^{2(t-\tau)} d\tau \\ &= e^{2t} \int_0^t \tau^{2\beta-2} e^{-2\tau} d\tau = \frac{2e^{2t}}{4^\beta} \int_0^t \sigma^{2\beta-2} e^{-\sigma} d\sigma \\ &\le \frac{2e^{2t}}{4^\beta} \Gamma(2\beta - 1). \end{aligned} \tag{1.4.63}$$

Therefore, (1.4.62) yields

$$u(t) \le a(t) + \left[\frac{2e^{2t}}{4^\beta} \Gamma(2\beta-1)\right]^{1/2} \left[\int_0^t F^2(s) e^{-2s} \omega^2(u(s)) ds\right]^{1/2}. \tag{1.4.64}$$

Using the inequality (1.4.55) with $n = 2, r = 2$, we obtain

$$u^2(t) \le 2a^2(t) + \frac{e^{2t}\Gamma(2\beta-1)}{4^{\beta-1}} \int_0^t F^2(s) e^{-2s} \omega^2(u(s)) ds \tag{1.4.65}$$

and applying condition (q) for $q = 2$, we have

$$v(t) \le \alpha(t) + K \int_0^t F^2(s) R(s) \omega(u(s)) ds, \tag{1.4.66}$$

where

$$v(t) = \left(e^{-t} u(t)\right)^2, \qquad \alpha(t) = 2a^2(t), \qquad K = \frac{\Gamma(2\beta-1)}{4^{\beta-1}}. \tag{1.4.67}$$

Now we shall proceed in a standard way. Let $V(t)$ be the right-hand side of (1.4.66). Then

$$\omega(v(t))[\omega(V(t))]^{-1} \leq 1,$$

which yields

$$\left[\alpha'(t) + KF^2(t)R(t)\omega(v(t))\right][\omega(V(t))]^{-1} \leq \alpha'(t)\left[\omega(\alpha(t))\right]^{-1} + KF^2(t)R(t), \tag{1.4.68}$$

i.e.,

$$\frac{V'(t)}{\omega(V(t))} \leq \frac{\alpha'(t)}{\omega(\alpha(t))} + KF^2(t)R(t) \tag{1.4.69}$$

or

$$\frac{d}{dt}\Omega(V(t)) \leq \frac{d}{dt}\Omega(a(t)) + KF^2(t)R(t). \tag{1.4.70}$$

Integrating this inequality from 0 to t, we obtain

$$\Omega(V(t)) \leq \Omega(\alpha(t)) + g_1(t), \tag{1.4.71}$$

where

$$g_1(t) = K\int_0^t F^2(s)R(s)ds$$

whence

$$v(t) \leq V(t) \leq \Omega^{-1}\left[\Omega(\alpha(t)) + g_1(t)\right].$$

Using (1.4.67), we obtain (1.4.59).

Now let us prove assertion (2). Obviously, $\beta - 1 = -\alpha = \frac{-z}{(z+1)}$. Let p, q be as in the theorem. Since $1/p + 1/q = 1$, Hölder's inequality yields

$$\begin{aligned}
u(t) &\leq a(t) + \int_0^t (t-s)^{\beta-1}F(s)\omega(u(s))ds \\
&= a(t) + \int_0^t (t-s)^{-\alpha}e^sF(s)e^{-s}\omega(u(s))ds \\
&\leq a(t) + \left[\int_0^t (t-s)^{-ap}e^{ps}ds\right]^{1/p}\left[\int_0^t F^q(s)e^{-qs}\omega^q(u(s))ds\right]^{1/q}.
\end{aligned} \tag{1.4.72}$$

For the first integral in (1.4.72), we have

$$\begin{aligned}
\int_0^t (t-s)^{-\alpha p}e^{ps}ds &= e^{pt}\int_0^t \tau^{-\alpha p}e^{-p\tau}d\tau = \frac{e^{pt}}{p^{1-\alpha p}}\int_0^{pt}\sigma^{-\alpha p}e^{-\sigma}d\sigma \\
&\leq \frac{e^{pt}}{p^{1-\alpha p}}\Gamma(1-\alpha p).
\end{aligned} \tag{1.4.73}$$

Obviously, $1-\alpha p = \frac{1}{(z+1)^2} > 0$ and so $\Gamma(1-\alpha p) \in \mathbb{R}$. Thus by (1.4.62) and condition (q), we get

$$u(t) \le a(t) + e^t K_z \left[\int_0^t F^q(s)R(s)\omega(e^{-qs}u^q(s))ds\right]^{1/q}, \tag{1.4.74}$$

where K_z is defined by (1.4.61). Now using the inequality (1.4.55) for $n=2, r=q$, we obtain

$$u^q(t) \le 2^{q-1}a^q(t) + 2^{q-1}e^{qt}K_z^q \int_0^t F^q(s)R(s)\omega(e^{-qs}u(s)^q)ds, \tag{1.4.75}$$

which yields

$$v(t) \le \phi(t) + 2^{q-1}K_z^q \int_0^t F^q(s)R(s)\omega(v(s))ds, \tag{1.4.76}$$

where

$$v(t) = \left(e^{-t}u(t)\right)^q, \ \ \phi(t) = 2^{q-1}a^q(t). \tag{1.4.77}$$

Let $V(t)$ be the right-hand side of (1.4.76). Then $\omega(V(t))[\omega(V(t))]^{-1} \le 1$ which yields

$$\begin{aligned}&\left[\phi'(t) + 2^{q-1}K_z^q F^q(t)R(t)\omega(v(t))\right][\omega(V(t))]^{-1}\\ &\quad \le \phi'(t)\left[\omega(\phi(t))\right]^{-1} + 2^{q-1}K_z^q F^q(t)R(t),\end{aligned} \tag{1.4.78}$$

i.e.,

$$\frac{V'(t)}{\omega(V(t))} \le \frac{\phi'(t)}{\omega(\phi(t))} + 2^{q-1}K_z^q F^q(t)R(t), \tag{1.4.79}$$

or

$$\frac{d}{dt}\Omega(V(t)) \le \frac{d}{dt}\Omega(\phi(t)) + 2^{q-1}K_z^q F^q(t)R(t). \tag{1.4.80}$$

Integrating (1.4.80) from 0 to t, we conclude

$$\Omega(V(t)) \le \Omega(\phi(t)) + g_2(t), \tag{1.4.81}$$

where

$$g_2(t) = 2^{q-1}K_z^q \int_0^t F^q(s)R(s)ds.$$

Consequently,

$$v(t) \le V(t) \le \Omega^{-1}[\Omega(\phi(t)) + g_2(t)]. \tag{1.4.82}$$

Using (1.4.79), we can obtain (1.4.60). □

As a consequence of Theorem 1.4.6, we have the following corollary.

Theorem 1.4.9 (The Medveď Inequality [606]). *Let $0 < T \leq +\infty, a(t), F(t)$ be as in Theorem 1.4.6, and let $u(t)$ be a continuous, non-negative function on $[0,T)$ such that*

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} F(s) u(s) ds, \tag{1.4.83}$$

for a constant $\beta > 0$. Then the following assertions hold:

(1) *If $\beta > \frac{1}{2}$, then for all $t \in [0,T)$,*

$$u(t) \leq \sqrt{2} a(t) \exp\left[\frac{2\Gamma(2\beta-1)}{4^\beta} \int_0^t F^2(s) ds + t\right]. \tag{1.4.84}$$

(2) *If $\beta = \frac{1}{z+1}$ for some $z \geq 1$, then for all $t \in [0,T)$,*

$$u(t) \leq (2^{q-1})^{1/q} a(t) \exp\left[\frac{2^{q-1}}{q} K_z^q \int_0^t F^q(s) ds + t\right], \tag{1.4.85}$$

where K_z is defined by (1.4.61), $q = z + 2$.

The method used in the proof of Theorem 1.4.6 enables us to prove the following theorem concerning the inequality (1.4.83), where $a(t), F(t)$, and $u(t)$ are integrable on $[0,T)$.

Theorem 1.4.10 (The Medveď Inequality [606]). *Let $a(t), b(t)$ be non-negative, integrable functions on $[0,T)$ for $0 < T \leq +\infty$, and let $F(t), u(t)$ be integrable, non-negative functions on $[0,T)$ such that for a.e. on $[0,T)$,*

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} F(s) u(s) ds. \tag{1.4.86}$$

Then the following assertions hold:

(1) *If $\beta > 1/2$, then for a.e. on $[0,T)$,*

$$u(t) \leq e^t \Phi^{1/2}(t), \tag{1.4.87}$$

where

$$\Phi(t) = 2a^2(t) + 2Kb^2(t) \int_0^t a^2(s) F^2(s) \exp\left[K \int_s^t b^2(r) F^2(r) dr\right] ds,$$

$$K = \frac{\Gamma(2\beta-1)}{4^{\beta-1}}.$$

(2) *If $\beta = 1/(z+1)$ for some $z \geq 1$, then for a.e. on $[0,T)$,*

$$u(t) \leq e^t \Psi^{1/q}(t), \tag{1.4.88}$$

where

$$\Psi(t) = 2^{q-1}a^q(t) + 2^{q-1}K_z^q b^q(t) \times \int_0^t a^q(s)F^q(s)\exp\left[2^{q-1}K_z^q \int_s^t b^q(r)F^q(r)dr\right] ds,$$

$q = z + 2$, *and* K_z *is defined by* (1.4.61).

Proof. First we shall prove assertion (1). Using the same procedure as in the proof of assertion (1) of Theorem 1.4.6, we can show that

$$v(t) \le 2a^2(t) + \frac{\Gamma(2\beta - 1)}{4^{\beta-1}} b^2(t) \int_0^t F^2(s)v(s)ds, \tag{1.4.89}$$

where $v(t) = (e^{-t}u(t))^2$. From Theorem 1.4 in [591], we obtain inequality (1.4.87).

Using the procedure from the proof of the assertion (2) of Theorem 1.4.6, we can show that

$$v(t) \le 2^{q-1}a^q(t) + 2^{q-1}K_z^q B^q(t) \int_0^t F^q(s)v(s)ds, \tag{1.4.90}$$

where $v(t) = (e^{-t}u(t))^q$, and the inequality (1.4.88) is a direct consequence of Theorem 1.4 in [591]. □

The following result is an analogue of Theorem 1.4.7 (see, e.g., [115]).

Theorem 1.4.11 (The Brandolese Inequality [115]). *Assume that a non-negative and locally bounded function* $h = h(t)$ *satisfies the inequality for all* $t \ge 0$,

$$h(t) \le C_1(1 + t) + C_2 \int_0^t (t - \tau)^{-a}(1 + \tau)^{-b}h(\tau)d\tau \tag{1.4.91}$$

for some $a \in (0, 1)$, $b > 0$, *positive constants* C_1 *and* C_2.

If $a + b > 1$, *then for all* $t \ge 0$,

$$h(t) \le C(1 + t) \tag{1.4.92}$$

for a constant $C > 0$ *independent of* t. *The same conclusion* (1.4.92) *holds true in the limit case* $a + b = 1$ *under the weaker assumption*

$$h(t) \le C_1(1 + t) + C_2 \int_0^t (t - \tau)^{-a}\tau^{-b}h(\tau)d\tau, \tag{1.4.93}$$

provided that $C_2 > 0$ *is sufficiently small.*

Proof. If $a+b=1$, we deduce from (1.4.93) the inequality

$$h(t) \leq C_1(1+t) + C_2 K(a,b) \sup_{0\leq\tau\leq t} h(\tau),$$

where

$$K(a,b) = \int_0^t (t-\tau)^{-a}\tau^{-b}d\tau = \int_0^1 (1-s)^{-a}s^{-b}ds.$$

Consequently,

$$\sup_{0\leq\tau\leq t} h(\tau) \leq \frac{C_1}{1-C_2K(a,b)}(1+t),$$

provided that $C_2 < 1/K(a,b)$. This gives us (1.4.92).

In the case $a+b>1$, using (1.4.91), we write $b = b_1+\eta$ for $a+b_1=1$ and $\eta>0$, and we fix $t_1>0$ such that

$$C_2(1+t_1)^{-\eta} < \frac{1}{K(a,b_1)}.$$

Now splitting the integral in (1.4.92) at t_1 yields

$$h(t) \leq C(1+t) + C_2K(a,b_1)(1+t_1)^{-\eta} \sup_{0\leq\tau\leq t} h(\tau)$$

for some constant $C>0$ independent of t. Hence the conclusion follows. □

Now we shall prove a result which is a modification of Lemma 7.1.2 in Henry [355] (see, e.g., Theorem 1.4.3) and is due to Medveď [606].

Theorem 1.4.12 (The Medveď Inequality [606])**.** *Let $a(t)$ be a non-negative, non-decreasing C^1 function on $[0,T)$ for $0<T\leq+\infty$, and $F(t)$ a continuous, non-negative function on $[0,T)$. Let $u(t)$ be a non-negative, continuous function on $[0,T)$ such that for constant $\beta>0, \gamma>0$, for all $t\in[0,T)$,*

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)u(s)ds. \tag{1.4.94}$$

Then the following assertions hold:

(1) *If $\beta>\frac{1}{2}$ and $\gamma>1-\frac{1}{2p}$, where $p>1$, then for all $t\in[0,T)$,*

$$u(t) \leq 2^{1-\frac{1}{2q}}a(t)\exp\left[\frac{4^q}{2q}K^qL^q\int_0^t F^{2q}(s)e^{qs}ds + t\right], \tag{1.4.95}$$

where

$$K = \frac{\Gamma(2\beta-1)}{4^{\beta-1}}, \quad L = \left[\frac{\Gamma((2\gamma-2)p+1)}{p^{(2\gamma-2)p}}\right]^{1/p}, \quad \frac{1}{p}+\frac{1}{q}=1.$$

(2) *Let* $\beta = \frac{1}{m+1}$ *for some real number* $m \geq 1, \gamma > 1 - \frac{1}{\kappa q}$, *where* $\kappa > 1$, $p = \frac{m+2}{m+1}, q = m+2$. *Then for all* $t \in [0, T)$,

$$u(t) \leq 2^{(\kappa r-1)/qr} a(t) \exp\left[\frac{p}{\kappa r}\int_0^t e^{rs}F^{rq}(s)ds + t\right], \tag{1.4.96}$$

where $r > 1$ *is such that* $1/\kappa + 1/r = 1$,

$$P = \left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}\right]^{rq/p}\left[\frac{\Gamma(\kappa q(\gamma-1)+1)}{\kappa^{\kappa q(\gamma-1)}}\right]^{r/\kappa}$$

and $-\alpha = \beta - 1 = \frac{-m}{m+1}$.

Proof. Let us prove assertion (1). From the inequality (1.4.94), we have

$$\begin{aligned} u(t) &\leq a(t) + \left[\int_0^t (t-s)^{2\beta-2}e^{2s}ds\right]^{1/2}\left[\int_0^t s^{2\gamma-2}F^2(s)e^{-2s}u^2(s)ds\right]^{1/2} \\ &\leq a(t) + e^t K^{1/2}\left[\int_0^t s^{2\gamma-2}F^2(s)(e^{-s}u(s))^2 ds\right]^{1/2}, \end{aligned} \tag{1.4.97}$$

which yields

$$u^2(t) \leq 2a^2(t) + 2e^{2t}K\int_0^t s^{2\gamma-2}F^2(s)(e^{-s}u(s))^2 ds, \tag{1.4.98}$$

whence

$$v(t) \leq c(t) + 2K\int_0^t s^{2\gamma-2}F^2(s)v(s)ds, \tag{1.4.99}$$

where

$$c(t) = 2a^2(t), \quad v(t) = (e^{-t}u(t))^2. \tag{1.4.100}$$

Thus from (1.4.99) it follows that

$$\begin{aligned} v(t) &\leq c(t) + 2K\int_0^t s^{2\gamma-2}e^{-s}F^2(s)e^s v(s)ds \\ &\leq c(t) + 2K\left[\int_0^t s^{(2\gamma-2)p}e^{-ps}ds\right]^{1/p}\left[\int_0^t F^{2q}(s)e^q(v(s))^q ds\right]^{1/q}, \end{aligned} \tag{1.4.101}$$

where $q > 1, \frac{1}{q} + \frac{1}{p} = 1$. For the first integral in (1.4.101), we have

$$\int_0^t s^{(2\gamma-2)p}e^{-ps}ds \leq \frac{e^{pt}}{p^{(2\gamma-2)p}}\Gamma((2\gamma-2)p+1).$$

Obviously, the assumption yields

$$(2\gamma-2)p + 1 > \left[2\left(1 - \frac{1}{2p}\right) - 2\right]p + 1 = 0.$$

Hence,

$$\Gamma((2\gamma - 2)p + 1) \in \mathbb{R}.$$

Let L be as in Theorem 1.4.10. From (1.4.101), we have

$$v^q(t) \leq 2^{q-1}c^q(t) + \frac{4^q}{2}K^qL^q \int_0^t F^{2q}(s)e^{qs}v^q(s)ds, \tag{1.4.102}$$

which yields

$$v^q(t) \leq 2^{q-1}c^q(t) \exp\left[\frac{4^q}{2}K^qL^q \int_0^t F^{2q}(s)e^{qs}ds\right]. \tag{1.4.103}$$

From this inequality and (1.4.100), we obtain (1.4.95). Now let us prove assertion (2). From the inequality (1.4.96), we obtain

$$\begin{aligned} u(t) &\leq a(t) + \left[\int_0^t (t-s)^{-p\alpha}e^{ps}ds\right]^{1/p} \left[\int_0^t s^{q(\gamma-1)}e^{-qs}F^q(s)u^q(s)ds\right]^{1/q} \\ &\leq a(t) + e^t \left[\frac{\Gamma(1-\alpha p)}{p^{(1-\alpha p)}}\right]^{1/p} \left[\int_0^t s^{\kappa q(\gamma-1)}e^{-\kappa s}ds\right]^{1/(\kappa q)} \\ &\quad \times \left[\int_0^t e^{rs}F^{rq}(s)(e^{-s}u(s))^{rq}ds\right]^{1/(rq)} \\ &\leq a(t) + e^t \left[\frac{\Gamma(1-\alpha p)}{p^{(1-\alpha p)}}\right]^{1/p} \frac{\Gamma(\kappa q(\gamma-1)+1)^{1/(\kappa q)}}{\kappa^{\kappa q(\gamma-1)-1}} \\ &\quad \times \left[\int_0^t e^{rs}F^{rq}(s)(e^{-s}u(s))^{rq}ds\right]^{1/(rq)}, \end{aligned} \tag{1.4.104}$$

where r is as in the theorem. We assume that $\gamma > 1 - \frac{1}{\kappa q}$ and thus we have $\kappa q(\gamma-1)+1 > \kappa q\left(\frac{-1}{\kappa q}\right) + 1 = 0$, i.e., $\Gamma(\kappa q(\gamma-1)+1) \in \mathbb{R}$. Therefore, (1.4.104) yields

$$v(t) \leq 2^{rq-1}\left[a^{qr}(t) + P\int_0^t e^{rs}F^{rq}(s)v(s)ds\right], \tag{1.4.105}$$

where $v(t) = (e^{-t}u(t))^{rq}$ and P is defined as in the theorem. Therefore, we obtain

$$v(t) \leq 2^{rq-1}a^{rq}(t) \exp\left[P\int_0^t e^{rs}F^{rq}(s)ds\right] \tag{1.4.106}$$

which yields the inequality (1.4.96). □

For the special case when $a(t) = t^{-\alpha}$ ($\alpha > 0,\ a\,\text{constant}$), $\beta = 1/2$, $\gamma = 1/2$, $F = \text{constant} > 0$, we have Theorem 1.4.11 below, whose proof needs the following lemma, due to Bae and Jin [57].

Lemma 1.4.13 ([57]). *Let $a < 1,\ b > 0,\ d < 1$. If $b + d < 1$, then for all $t > 0$,*

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds \le Ct^{1-a-d}(1+t)^{-b}. \tag{1.4.107}$$

If $b + d = 1$, then for all $t > 0$,

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds \le Ct^{-a}\ln(1+t). \tag{1.4.108}$$

If $b + d > 1$, then for all $t > 0$,

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds \le Ct^{-a}. \tag{1.4.109}$$

Proof. Set

$$I := \int_{t/2}^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds, \quad II := \int_0^{t/2} (t-s)^{-a}(s+1)^{-b}s^{-d}ds.$$

Then we have

$$\begin{aligned} I &\le C(1+t)^{-b}t^{-d}\int_{t/2}^t (t-s)^{-a}ds = C(1+t)^{-b}t^{-d}t^{1-a} \\ &= C(1+t)^{-b}t^{1-a-d}, \end{aligned} \tag{1.4.110}$$

$$II \le Ct^{-a}\int_0^{t/2} (s+1)^{-b}s^{-d}ds. \tag{1.4.111}$$

If $t \ge 2$, then

$$\begin{aligned} \int_0^{t/2} (s+1)^{-b}s^{-d}ds &= \int_1^{t/2} (s+1)^{-b}s^{-d}ds + \int_0^1 (s+1)^{-b}s^{-d}ds \\ &\le C\int_1^{t/2} (s+1)^{-b-d}ds + C\int_0^1 s^{-d}ds \\ &\le \begin{cases} C(t+1)^{1-b-d}, & \text{if } \ b+d<1, \\ C\ln(t+1), & \text{if } \ b+d=1, \\ C, & \text{if } \ b+d>1. \end{cases} \end{aligned} \tag{1.4.112}$$

If $t \le 2$, then

$$\int_0^{t/2} (s+1)^{-b}s^{-d}ds \le C\int_0^1 (s+1)^{-b}s^{-d}ds \le C. \tag{1.4.113}$$

Hence, we derive from (1.4.111)–(1.4.112)

$$II \le \begin{cases} Ct^{-a}(t+1)^{1-b-d}, & \text{if } \ b+d<1, \\ Ct^{-a}\ln(t+1), & \text{if } \ b+d=1, \\ Ct^{-a}, & \text{if } \ b+d>1. \end{cases} \tag{1.4.114}$$

Thus it follows from (1.4.110)–(1.4.114) that

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds$$
$$\leq \begin{cases} Ct^{1-a-d}(t+1)^{-b} + Ct^{-a}(t+1)^{1-b-d}, & \text{if } \ b+d<1, \\ Ct^{1-a-d}(t+1)^{-b} + Ct^{-a}\ln(t+1), & \text{if } \ b+d=1, \\ Ct^{1-a-d}(t+1)^{-b} + Ct^{-a}, & \text{if } \ b+d>1. \end{cases} \tag{1.4.115}$$

The proof is thus complete. □

Theorem 1.4.14 (The Bae–Jin Inequality [57]). *Assume that $x(t) \geq 0$ satisfies that the inequality for all $t > 0$,*

$$x(t) \leq Ct^{-\alpha} + \varepsilon \int_0^t (t-s)^{-1/2}s^{-1/2}x(s)ds. \tag{1.4.116}$$

Then for all $t > 0$,

$$x(t) \leq Ct^{-\alpha} + C\varepsilon t^{-1/2}\int_0^t s^{-1/2}x(s)ds, \tag{1.4.117}$$

where $\alpha > 0, C > 0$ are constants independent of $t > 0$ and $\varepsilon > 0$.

Proof. Let

$$I := \int_{t/2}^t (t-s)^{-1/2}s^{-1/2}x(s)ds, \quad II := \int_0^{t/2} (t-s)^{-1/2}s^{-1/2}x(s)ds.$$

Then we infer that

$$I \leq Ct^{-1/2}\int_{t/2}^t (t-s)^{-1/2}x(s)ds, \qquad II \leq Ct^{-1/2}\int_0^{t/2} s^{-1/2}x(s)ds. \tag{1.4.118}$$

If we insert (1.4.116) in I, we obtain

$$\begin{aligned} t^{1/2}I &\leq C\int_{t/2}^t (t-s)^{-1/2}x(s)ds \\ &\leq C\varepsilon \int_{t/2}^t (t-s)^{-1/2}\Big\{s^{-\beta} + \varepsilon\int_0^s (s-\tau)^{-1/2}\tau^{-1/2}x(\tau)d\tau\Big\}ds \\ &= C\int_{t/2}^t (t-s)^{-1/2}s^{-\beta}ds \\ &\quad + C\varepsilon\int_{t/2}^t\int_0^s (t-s)^{-1/2}(s-\tau)^{-1/2}\tau^{-1/2}x(\tau)d\tau ds \\ &= I_1 + I_2. \end{aligned} \tag{1.4.119}$$

A straightforward computation yields

$$I_1 \le C^2 t^{-\beta+1/2}. \tag{1.4.120}$$

Next, Fubini's theorem gives

$$\begin{aligned}
I_2 &= C\varepsilon \int_0^{t/2} \int_{t/2}^t (t-s)^{-1/2}(s-\tau)^{-1/2}\tau^{-1/2}x(\tau)dsd\tau \\
&\quad + C\varepsilon \int_{t/2}^t \int_\tau^t (t-s)^{-1/2}(s-\tau)^{-1/2}\tau^{-1/2}x(\tau)dsd\tau \\
&= C\varepsilon \int_0^{t/2} \tau^{-1/2}x(\tau)\Big\{ \int_{t/2}^t (t-s)^{-1/2}(s-\tau)^{-1/2}ds\Big\}d\tau \\
&\quad + C\varepsilon \int_{t/2}^t \tau^{-1/2}x(\tau)\Big\{ \int_\tau^t (t-s)^{-1/2}(s-\tau)^{-1/2}ds\Big\}d\tau \\
&\le C\varepsilon \int_0^t \tau^{-1/2}x(\tau)d\tau,
\end{aligned} \tag{1.4.121}$$

where we have used the following estimates:

$$\begin{cases}
\displaystyle\int_\tau^t (t-s)^{-1/2}(s-\tau)^{-1/2}ds = C\int_0^{t-\tau} z^{-1/2}(t-\tau-z)^{-1/2}dz \\
\qquad\qquad\qquad\qquad \le C \ \text{ for } \ t/2 < \tau < t, \\
\displaystyle\int_{t/2}^t (t-s)^{-1/2}(s-\tau)^{-1/2}ds \le C\int_\tau^t (t-s)^{-1/2}(s-\tau)^{-1/2}ds \\
\qquad\qquad\qquad\qquad \le C \ \text{ for } \ 0 < \tau < t/2,
\end{cases}$$

with some constant $C > 0$ independent of t owing to Lemma 1.4.3. Hence from (1.4.119)–(1.4.121), we conclude that

$$I \le Ct^{-\beta} + C\varepsilon t^{-1/2}\int_0^t \tau^{-1/2}x(\tau)d\tau. \tag{1.4.122}$$

Combining (1.4.118) and the above estimates on I and II, we complete the proof. □

1.4.3 The Ou-Yang and Pachapatte type integral inequalities

In this subsection, we begin to study the integral inequality

$$u^r(t) \le a(t) + \int_0^t (t-s)^{\beta-1}F(s)\omega(u(s))ds, \ \ \beta > 0. \tag{1.4.123}$$

The following result concerns the case $r = 2$, which was studied in Medveď [606].

Theorem 1.4.15 (The Medveď Inequality [606]). *Assume that $r = 2$. Let $a(t)$ be a non-decreasing, non-negative C^1 function on $[0,T)$ for $0 < T \leq +\infty$, $F(t)$ be a continuous, non-negative function, and $\omega : \mathbb{R}^+ \to \mathbb{R}$, $d\omega(u)/du$ be continuous, non-decreasing functions such that $\omega(0) = 0, \omega(u) > 0$ on $(0,T)$. Let $u(t)$ be a continuous, non-negative function on $[0,T)$ satisfying the inequality* (1.4.123)*. Then the following assertions hold:*

(i) *Assume that $\beta > \frac{1}{2}$ and that ω satisfying condition (q) (i.e., (1.4.53)) for $q = 2$. Then*

$$u(t) \leq e^t \left\{ \Lambda^{-1} \left[\Lambda(2a^2(t)) + K \int_0^t F^2(s)R(s)ds \right] \right\}^{1/4}, \tag{1.4.124}$$

for all $t \in [0,T_1)$, where

$$K = \frac{\Gamma(2\beta - 1)}{4^{\beta-1}}, \qquad \Lambda(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sqrt{\sigma})}, \quad v_0 > 0 \tag{1.4.125}$$

and $T_1 \in \mathbb{R}^+$ is such that $\Lambda(2a^2(t)) + K \int_0^t F^2(s)R(s)ds \in \mathrm{Dom}(\Lambda^{-1})$ for all $t \in [0,T_1]$.

(ii) *Let $\beta \in (0, \frac{1}{2}]$ and let ω satisfy condition (q) (i.e., (1.4.53)) for $q = z+2$, where $z = \frac{1-\beta}{\beta}$, i.e., $\beta = \frac{1}{z+1}$. Then for all $t \in [0,T_1]$,*

$$u(t) \leq e^t \left\{ \Lambda^{-1} \left[\Lambda(2^{q-1}a^q(t)) + 2^{q-1}K_z^q \int_0^t F^q(s)R(s)ds \right] \right\}^{1/(2q)}, \tag{1.4.126}$$

where

$$K_z = \left[\frac{\Gamma(1-\beta p)}{p^{1-\beta p}} \right]^{1/p}, \quad \beta = \frac{1}{z+1}, \quad p = \frac{z+2}{z+1}, \tag{1.4.127}$$

and $T_1 \in \mathbb{R}^+$ is such that $\Lambda(2^{q-1}a^q(t)) + 2^{q-1}K_z^q \int_0^t F^q(s)R(s)ds \in \mathrm{Dom}(\Lambda^{-1})$ for all $t \in [0,T_1]$.

Proof. First let us prove the assertion (i). Following the proof of Theorem 1.4.6, we can show that

$$v^2(t) \leq \alpha(t) + K \int_0^t F^2(s)R(s)\omega(v(s))ds, \tag{1.4.128}$$

with

$$v(t) = (e^{-t}u(t))^2, \quad \alpha(t) = 2a^2(t), \quad K = \frac{\Gamma(2\beta-1)}{4^{\beta-1}}. \tag{1.4.129}$$

Let $V(t)$ denote the right-hand side of (1.4.128). Then $v(t) \le \sqrt{V(t)}$. This yields $\omega(v(t)) \le \omega(\sqrt{V(t)})$ and thus

$$\begin{aligned} \frac{V'(t)}{\omega(\sqrt{V(t)})} &= \frac{\alpha'(t) + KF^2(t)R(t)\omega(v(t))}{\omega(\sqrt{V(t)})} \\ &\le \frac{\alpha'(t)}{\omega(\sqrt{\alpha(t)})} + KF^2(t)R(t). \end{aligned} \tag{1.4.130}$$

This yields

$$\frac{d}{dt}\int_0^{V(t)} \frac{d\sigma}{\omega(\sqrt{V(\sigma)})} \le \frac{d}{dt}\int_0^{\alpha(t)} \frac{d\sigma}{\omega(\sqrt{\alpha(\sigma)})} + KF^2(t)R(t). \tag{1.4.131}$$

Thus we have

$$\frac{d}{dt}\Lambda(V(t)) \le \frac{d}{dt}\Lambda(\alpha(t)) + KF^2(t)R(t), \tag{1.4.132}$$

where Λ is defined by (1.4.125). This yields

$$V(t) \le \Lambda^{-1}\left[\Lambda(\alpha(t)) + K\int_0^t F^2(s)R(s)ds\right], \tag{1.4.133}$$

whence

$$v(t) \le \sqrt{V(t)} \le \left\{\Lambda^{-1}\left[\Lambda(\alpha(t)) + K\int_0^t F^2(s)R(s)ds\right]\right\}^{1/2}. \tag{1.4.134}$$

Using (1.4.129), we can obtain (1.4.128). Now we shall prove assertion (ii). Following the proof of assertion (2) of Theorem 1.4.6, we can show that

$$v^2(t) \le \phi(t) + 2^{q-1}K_z^q\int_0^t F^q(s)R(s)\omega(v(s))ds, \tag{1.4.135}$$

where

$$v(t) = (e^{-t}u(t))^q, \qquad \phi(t) = 2^{q-1}a^q(t). \tag{1.4.136}$$

Following the procedure from the proof of assertion (i), we obtain

$$v(t) \le \left\{\Lambda^{-1}(\Lambda(\phi(t))) + 2^{q-1}K_z^q\int_0^t F^q(s)R(s)ds\right\}^{1/2} \tag{1.4.137}$$

and using (1.4.136), we can obtain (1.4.126). □

Remark 1.4.5. We can prove a result similar to Theorem 1.4.12 for an analogue of the inequality (1.4.123) involving multiple integrals. We do not give the details here.

The nonsingular version of (1.4.123) for $r = 2$, $\beta = 1$ was studied by Pachpatte in [723], where a result obtained by Ou-Yang [715] is generalized. Applying the method developed in Medveď [606], the following theorem can be proved in Medveď [606].

Theorem 1.4.16 (The Medveď Inequality [606]). *Let $a(t)$ be a non-negative, non-decreasing C^1-function on the interval $[0,T]$ $(0 < T < +\infty)$, let $F(t)$ be a non-negative, continuous function on $[0,T]$, $0 < \beta < 1, r \geq 1$, and let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, non-decreasing, positive function. Assume that $u(t)$ is a continuous, non-negative function on $[0,T]$ satisfying the inequality (1.4.123). Then for all $t \in [0,T]$,*

$$\Lambda_{qr}(u^{qr}(t)) \leq \Lambda_{qr}(2^{q-1}a^q) + K_q \int_0^t e^{-qs} F^q(s) ds, \tag{1.4.138}$$

or

$$u(t) \leq \left\{ \Lambda_{qr}^{-1} \left[\Lambda_{qr}(2^{q-1}a^q) + K_q \int_0^t e^{-qs} F^q(s) ds \right] \right\}^{1/qr}, \tag{1.4.139}$$

where $\beta = \frac{1}{1+z}$, $z > 0$, $q = \frac{1}{\beta} + \varepsilon = 1 + z + \varepsilon$, $p = \frac{1+z+\varepsilon}{z+\varepsilon}$, $\varepsilon > 0$,

$$\Lambda_{qr}(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega^q(\sigma^{1/rq})}, \tag{1.4.140}$$

with $2^{q-1}a^q(0) \geq v_0 > 0$, and Λ_{qr}^{-1} is the inverse of Λ_{qr}, $a = a(t)$,

$$K_q = \frac{2^{q-1}e^{pT}}{p^{1-\alpha p}} \Gamma(1 - \alpha p),$$

with $\alpha = 1 - \beta = \frac{z}{1+z}$. Also, Γ is Euler's Gamma function, and $T_1 > 0$ is such that for all $t \in [0,T_1]$,

$$\Lambda_{qr}(2^{q-1}a^q) + K_q \int_0^t e^{-qs} F^q(s) ds \in \text{Dom } (\Lambda_{qr}^{-1}).$$

Proof. Obviously, $\frac{1}{p} + \frac{1}{q} = 1$. Using the Hölder inequality, we obtain from (1.4.123)

$$\begin{aligned} u^r(t) &\leq a(t) + \int_0^t (t-s)^{-\alpha} e^s e^{-s} F(s) \omega(u(s)) ds \\ &\leq a(t) + \left[\int_0^t (t-s)^{-\alpha p} e^{ps} ds \right]^{1/p} \left[\int_0^t e^{-qs} F^q(s) \omega^q(u(s)) ds \right]^{1/q}. \end{aligned} \tag{1.4.141}$$

Since $(A+B)^q \leq 2^{q-1}(A^q + B^q)$ holds for any $A \geq 0, B \geq 0$ and

$$\int_0^t (t-s)^{-\alpha p} e^{ps} ds = e^{pt} \int_0^t \tau^{-\alpha p} e^{-p\tau} d\tau \leq \frac{e^{pt}}{p^{1-\alpha p}} \Gamma(1 - \alpha p), \tag{1.4.142}$$

for $1-\alpha p = \frac{\varepsilon}{(1+z)(z+\varepsilon)} > 0$, we obtain from (1.4.141) that for all $t \in [0,T]$,

$$u^{rq}(t) \le 2^{q-1}a^q + K_q \int_0^t e^{-qs}F(s)\omega^q(u(s))ds. \tag{1.4.143}$$

Let $W(t)$ denote the right-hand side of the inequality (1.4.143). Then $u(t) \le W^{1/rq}(t)$ which yields $\omega^q(u(t)) \le \omega^q(W^{1/rq}(t))$.

From (1.4.143), we obtain

$$\frac{W'(t)}{\omega^q(W^{1/rq}(t))} \le \frac{K_q e^{-qt}F^q(t)\omega^q(u(t))}{\omega^q(W^{1/rq}(t))} + \frac{\alpha'(t)}{\omega^q(\alpha^{1/rq}(t))},$$

i.e.,

$$\frac{d}{dt}\int_0^{W(t)} \frac{d\sigma}{\omega^q(\sigma^{1/rq})} \le K_q e^{-qt}F^q(t) + \frac{d}{dt}\int_0^{\alpha(t)} \frac{d\sigma}{\omega^q(\sigma^{1/rq})}, \tag{1.4.144}$$

or

$$\frac{d}{dt}\Lambda_{qr}(W(t)) \le K_q e^{-qt}F^q(t) + \frac{d}{dt}\Lambda_{qr}(d(t)), \tag{1.4.145}$$

where Λ_{qr} is defined by (1.4.140) and $\alpha(t) = 2^{q-1}a^q(t)$. Integrating inequality (1.4.145) from 0 to t, we can obtain the inequality (1.4.138). □

1.4.4 Henry type inequalities with multiple integrals

The following theorem, due to Medveď [606], is a modification of Theorem 1.4.6.

Theorem 1.4.17 (The Medveď Inequality [606]). *Let $a(t)$, $a'(t)$, $\ldots$, $a^{(m-1)}(t)$ ($a^{(i)} = \frac{d^i a}{dt^i}$) be non-negative, continuous functions on $[0,T)$ ($0 < T \le +\infty$), $F_i(t)$ ($i = 1,2,\ldots,m$) be non-negative, continuous functions on $[0,T)$, ω as in Theorem 1.4.13, and let $u(t)$ be a continuous, non-negative function on $[0,T)$ such that for all $t \in [0,T]$,*

$$\begin{aligned} u(t) \le a(t) &+ \int_0^t (t-s)^{\beta_1 - 1}F_1(s)\omega(u(s))ds \\ &+ \int_0^t\int_0^{t_1} (t_1 - s)^{\beta_2 - 1}F_2(s)\omega(u(s))dsdt_1 + \cdots \\ &+ \int_0^t\int_0^{t_1}\cdots\int_0^{t_{m-1}} (t_{m-1} - s)^{\beta_m - 1}F_m(s)\omega(u(s))ds\cdots dt_1, \end{aligned} \tag{1.4.146}$$

where $\beta_i > 1/2$ $(i = 1,2,\ldots,m)$ and ω satisfies condition (q) (i.e., (1.4.53)) for $q = 2$.

Then for all $t \in [0,T_1]$,

$$u(t) \le e^t\chi^{1/2}(t), \tag{1.4.147}$$

where $\chi(t) = \Omega^{-1}[\Omega\{(m+1)a^2(t)\} + G(t)]$,

$$G(t) = h_1(t) + \int_0^t h_2(s)ds + \cdots + \int_0^t \int_0^{t_1} \int_0^{t_{m-1}} h_m(s)ds \cdots dt_1,$$

$$h_i(t) = \eta_i(m+1)F_i^2(t)R(t), \qquad \eta_i = \frac{\Gamma(2\beta_i - 1)}{2^{2\beta_i+m-1}},\ i = 1,2,\ldots,m \tag{1.4.148}$$

and $T_1 \in \mathbb{R}^+$ *is such that* $\Omega\{(m+1)a(t)^2\} + G(t) \in \text{Dom}\ (\Omega^{-1})$ *for all* $t \in [0, T_1]$.

Proof. In fact, the inequality (1.4.146) yields

$$\begin{aligned} u(t) \le a(t) &+ \left[\int_0^t (t-s)^{2\beta_1-2} e^{2s} ds\right]^{1/2} \left[\int_0^t F_1(s)^2 e^{-2s}\omega^2(u(s))ds\right]^{1/2} \\ &+ \cdots + \left[\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} (t_{m-1}-s)^{2\beta_m-2} e^{2s} ds \cdots dt_1\right]^{1/2} \\ &\times \left[\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} F_1^2(s)e^{-2s}\omega^2(u(s))ds \cdots dt_1\right]^{1/2} \\ &\le a(t) + e^t \eta_1^{1/2} \left[\int_0^t F_1^2(s)e^{-2s}\omega^2(u(s))ds\right]^{1/2} \\ &+ \cdots + e^t \eta_m^{1/2} \left[\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} F_m^2(s)e^{-2s}\omega^2(u(s))ds \cdots dt_1\right]^{1/2}, \end{aligned} \tag{1.4.149}$$

where η_i $(i = 1,2,\ldots,m)$ are defined by (1.4.148). Here we have used the following estimate

$$\begin{aligned} &\int_0^t \int_0^{t_1} \cdots \int_0^{t_{i-1}} (t_{i-1}-s)^{2\beta_i-1} e^{2s} ds \cdots dt_1 \\ &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{i-2}} e^{2t_{i-1}} \int_0^{t_{i-1}} \sigma^{2\beta_i-1} e^{-2\sigma} d\sigma \cdots dt_1 \\ &\le \frac{e^{2t}}{2^{2\beta_i}} \Gamma(2\beta_i - 1) \int_0^t \int_0^{t_1} \cdots \int_0^{t_{i-2}} e^{2t_{i-2}} dt_{i-1} \cdots dt_1 \\ &\le \frac{e^{2t}\Gamma(2\beta_i-1)}{2^{2\beta_i+i-1}}, \quad i = 1,2,\ldots,m. \end{aligned} \tag{1.4.150}$$

The inequalities (1.4.149) and (1.4.55) yield

$$\begin{aligned} u^2(t) \le (m+1)\Big[a^2(t) &+ e^{2t}\eta_1 \int_0^t F_1^2(s)e^{-2s}\omega^2(u(s))ds \\ &+ \cdots + e^{2t}\eta_m \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} F_m^2(s)e^{-2s}\omega^2(u(s))ds \cdots dt_1\Big] \end{aligned} \tag{1.4.151}$$

and using the property (q) (i.e., (1.4.53)) for $q = 2$, we obtain

$$v(t) \le (m+1)\Big[a^2(t) + \eta_1 \int_0^t F_1^2(s)R(s)\omega(u(s))ds + \cdots + \eta_m \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} F_m(s)R(s)\omega(u(s))ds \cdots dt_1\Big], \tag{1.4.152}$$

where

$$v(t) = (e^{-t}u(t))^2. \tag{1.4.153}$$

Let $V(t)$ be the right-hand side of (1.4.150) and

$$\alpha(t) = (m+1)a^2(t), \quad h_i(t) = c_i^2\eta_i(m+1)F_i^2(t)R(t). \tag{1.4.154}$$

Then for all $t \in [0, T)$,

$$V'(t) - \alpha'(t) - h_1(t)\omega(v(t)) = V_1(t), \tag{1.4.155}$$
$$V_1'(t) - h_2(t)\omega(v(t)) = V_2(t), \ldots, \tag{1.4.156}$$
$$V_{m-2}'(t) - h_{m-1}\omega(v(t)) = V_{m-1}(t), \tag{1.4.157}$$
$$V_{m-1}'(t) = h_m(t)\omega(v(t)) \le h_m(t)\omega(v(t)). \tag{1.4.158}$$

□

We need the following lemma, due to Medveď [606].

Lemma 1.4.18. ([606]) *If $H(t)$ is a C^1-function on $[0, T)$, $H(t) \ge 0$ for all $t \in [0, T)$, and $H(0) = 0$, then for all $t \in [0, T)$,*

$$\int_0^t \frac{H'(s)}{\omega(V(s))}ds \ge \frac{H(t)}{\omega(V(t))}. \tag{1.4.159}$$

Proof. Integrating by parts on the left-hand side of (1.4.156), we obtain

$$\int_0^t \frac{H'(s)}{\omega(V(s))}ds = \frac{H(t)}{\omega(V(t))} + \int_0^t H(s)\frac{\omega'(V(s))}{[\omega(V(s))]^2}V'(s)ds \ge \frac{H(t)}{\omega(V(t))}.$$ □

Now let us continue the proof of the theorem. Using (1.4.155) and (1.4.156), we have

$$\frac{V_{m-1}(t)}{\omega(V(t))} \le \int_0^t \frac{V_{m-1}'(s)}{\omega(V(t))}ds \le \int_0^t h_m(s)ds. \tag{1.4.160}$$

Therefore, for the equality (1.4.154) and the inequalities (1.4.159), (1.4.160), it follows

$$\begin{aligned}\frac{V_{m-2}(t)}{\omega(V(t))} &\le \int_0^t \frac{V_{m-2}'(s)}{\omega(V(t))}ds \le \int_0^t h_{m-1}(s)ds + \int_0^t \frac{V_{m-1}(s)}{\omega(V(s))}ds \\ &\le \int_0^t h_{m-1}(s)ds + \int_0^t \int_0^{t_1} h_m(s)dsdt_1.\end{aligned} \tag{1.4.161}$$

Continuing in this way, we can prove that

$$\frac{V_1(t)}{\omega(V(t))} \leq \int_0^t h_2(s)ds + \int_0^t \int_0^{t_1} h_3 ds dt_1 + \cdots + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} h_m(s) ds dt_{m-1} \cdots dt_1. \tag{1.4.162}$$

Using this inequality, we obtain

$$\begin{aligned} \frac{V'(t)}{\omega(V(t))} - \frac{\alpha'(t)}{\omega(\alpha(t))} \leq \frac{V'(t) - \alpha'(t)}{\omega(V(t))} &\leq h_1(t) + \frac{V_1(t)}{\omega(V(t))} \\ &\leq h_1(t) + \int_0^t h_2(s)ds + \int_0^t \int_0^{t_1} h_3 ds dt_1 \\ &\quad + \cdots + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} h_m(s) ds dt_{m-1} \cdots dt_1 \\ &:= G(t), \end{aligned} \tag{1.4.163}$$

whence for all $t \in [0, T_1)$,

$$v(t) \leq \Omega^{-1}[\omega(\alpha(t)) + G(t)] \tag{1.4.164}$$

where $T_1 \in \mathbb{R}^+$ is as in Theorem 1.4.14. Using (1.4.153), we can obtain (1.4.159). □

Remark 1.4.6. We note that the assertion for the case $\beta_j = \frac{1}{z+1}, z \geq 1$ for all j and its proof are similar to the assertion (2) of Theorem 1.4.6. We omit the details here. The case $\beta_i > \frac{1}{z+1}$ for a real number $z \geq 1$ is more complicated and we also omit it here.

1.5 Integral inequalities leading to upper bounds and decay rates

In this section, we shall introduce some integral inequalities leading to upper bounds and decay rates. Bae and Jin [57] proved the following theorem.

Theorem 1.5.1 (The Bae–Jin Inequality). *Let $\alpha < 1/2$ and $\varepsilon > 0$ be given constants. Assume that $x(t) \geq 0$ is a function satisfying the integral inequality for all $t > 0$,*

$$x(t) \leq Ct^{-\alpha} + \varepsilon t^{-1/2} \int_0^t s^{-1/2} x(s) ds, \tag{1.5.1}$$

with a constant $C > 0$. If

$$\lim_{t \to 0^+} t^{-\varepsilon} \int_0^t s^{-1/2} x(s) ds = 0, \tag{1.5.2}$$

then there exists a constant $\varepsilon_0 > 0$ such that if $\varepsilon \le \varepsilon_0$, then for all $t > 0$,

$$x(t) \le C_1 t^{-\alpha}, \tag{1.5.3}$$

for some constant $C_1 > 0$.

Proof. Let $J(t) = \int_0^t s^{-1/2} x(s) ds$. Then $J(t)$ satisfies

$$J'(t) \le C t^{-\alpha} t^{-1/2} + \varepsilon t^{-1} J(t)$$

which implies

$$\frac{d}{dt}\Big(t^{-\varepsilon} J(t)\Big) \le C t^{-\alpha-1/2-\varepsilon}. \tag{1.5.4}$$

Integrating (1.5.4) over $(0, t)$ gives us

$$t^{-\varepsilon} J(t) \le C \int_0^t s^{-\alpha-1/2-\varepsilon} ds. \tag{1.5.5}$$

Here we used the assumption that $\lim\limits_{t\to 0^+} t^{-\varepsilon} J(t) = 0$.

Assume that $\alpha + \varepsilon < 1/2$. Then Lemma 1.4.3 yields

$$\int_0^t s^{-\alpha-1/2-\varepsilon} ds \le C t^{-\alpha+1/2-\varepsilon}. \tag{1.5.6}$$

Thus we infer from (1.5.3) that

$$J(t) \le C t^{-\alpha+1/2}$$

which, inserted in (1.5.1), gives us (1.5.3). □

Remark 1.5.1. If $\alpha + \varepsilon < 1/2$, then (1.5.3) still holds.

The next result, obtained by Kawashima, Nakao and Ono [423] in 1995, is an analogue of Theorem 1.5.1.

Theorem 1.5.2 (The Kawashima–Nakao–Ono Inequality [423])**.** *Let $y(t)$ be a non-negative function on $[0, T)$, $0 < T \le +\infty$, which satisfies the integral inequality for all $t \in [0, T]$,*

$$y(t) \le k_0(1+t)^{-\alpha} + k_1 \int_0^t (1+t-s)^{-\beta}(1+s)^{-\gamma} y^{\mu}(s) ds \tag{1.5.7}$$

with some constants $k_0, k_1 > 0, \alpha, \beta, \gamma \ge 0$ and $0 \le \mu < 1$. Then for all $t \in [0, T]$,

$$y(t) \le c(1+t)^{-\theta} \tag{1.5.8}$$

with some constant $c > 0$ and

$$\theta = \min\left\{\alpha, \beta, \frac{\gamma}{1-\mu}, \frac{\beta+\gamma-1}{1-\mu}\right\}, \tag{1.5.9}$$

with the following exceptional case: If $\alpha \geq \hat{\theta}$ *and* $(\beta+\gamma-1)/(1-\mu) = \hat{\theta} \leq 1$, *where*

$$\hat{\theta} = \min\left\{\beta, \frac{\gamma}{1-\mu}\right\}, \tag{1.5.10}$$

then for all $t \in [0,T]$,

$$y(t) \leq c(1+t)^{-\hat{\theta}}\Big(\log(2+t)\Big)^{1/(1-\mu)}. \tag{1.5.11}$$

Remark 1.5.2. Once we have known that $y(t)$ is a bounded function, we can also apply Theorem 1.5.2 to the case $\mu = 1$. In particular, if $\gamma > 0$ and $\beta + \gamma - 1 > 0$, we can obtain (1.5.8) for

$$\theta = \min\{\alpha, \beta\}. \tag{1.5.12}$$

Note that even for the exceptional case, (1.5.8) is valid if θ is replaced by $\theta - \varepsilon$, $0 < \varepsilon \ll 1$.

Proof. The case $\mu = 0$ is well known. We define $M(t)$ by

$$M(t) := \sup_{0\leq s\leq t}\left\{(1+s)^{\theta} y(s)\right\}. \tag{1.5.13}$$

Then we infer from (1.5.7) and (1.5.13) that

$$\begin{aligned} y(t) &\leq k_0(1+t)^{-\alpha} + k_1\int_0^t (1+t-s)^{-\beta}(1+s)^{-\gamma-\mu\theta} ds\, M^{\mu}(t) \\ &\leq k_0(1+t)^{-\alpha} + c(1+t)^{-\theta^*} M^{\mu}(t) \end{aligned}$$

with a constant $c > 0$ and $\theta^* := \min\{\beta, \gamma+\mu\beta, \beta+\gamma+\mu\theta-1\}$, where we have assumed that $\beta \neq 1$ and $\gamma + \mu\theta \neq 1$. Now it is easy to see that $\min\{\alpha, \theta^*\} = \theta$, and hence

$$(1+t)^{\theta} y(t) \leq k_0 + cM^{\mu}(t). \tag{1.5.14}$$

Since $0 < \mu < 1$, (1.5.14) implies $M(t) \leq C < +\infty$, which is equivalent to (1.5.8). The exceptional case where $\beta = 1$ or $\gamma + \mu\theta = 1$ can be treated in a similar way. □

The following result was proved by Vaigant [931] (see also Kaliev and Podkuiko [410]).

Theorem 1.5.3 (The Vaigant Inequality [931])**.** *Assume that* $y(t)$ *is a continuous and non-negative function satisfying for all* $t > 0$,

$$y^n(t) \leq a + b\int_0^t c(\tau) y^{n-1}(\tau) d\tau \tag{1.5.15}$$

where $a, b \geq 0$ *and* $n \geq 1$ *are constants,* $c(t) \in L^1[0,T]$.

Then we have for all $t > 0$,

$$y(t) \le a^{1/n} + \frac{b}{n}\int_0^t c(\tau)d\tau \tag{1.5.16}$$
$$\le C(T) \tag{1.5.17}$$

with a constant $C(T) > 0$.

Proof. Without loss of generality, we assume $c(t)$ is a continuous and non-negative function. For general $c(t)$, we may use continuous functions from $L^1[0,T]$ to approximate $c(t)$.

Let $z(t) = \int_0^t c(\tau)y^{n-1}(\tau)d\tau$. Then it is easy to verify $z(t)$ is a continuous function satisfying for all $t > 0$,

$$z(0) = 0, \quad z'(t) = c(t)y^{n-1}(t). \tag{1.5.18}$$

From (1.5.15), we derive

$$y^n(t) \le a + bz(t). \tag{1.5.19}$$

Then (1.5.18) gives

$$\left(\frac{z'(t)}{c(t)}\right)^{n/(n-1)} \le a + bz(t), \quad z'(t) \le c(t)(a + bz(t))^{(n-1)/n}$$

or

$$\frac{d}{dt}\left(a + bz(t)\right)^{1/n} \le \frac{b}{n}c(t). \tag{1.5.20}$$

Integrating (1.5.20) with respect to t, we arrive at

$$(a + bz(t))^{1/n} \le a^{1/n} + \frac{b}{n}\int_0^t c(\tau)d\tau$$

which together with (1.5.19) proves (1.5.16); (1.5.17) is a direct result of (1.5.16). □

Inequalities involving sequences of real numbers, which may be considered as discrete analogues of the Bellman–Gronwall inequality, have been used in the analysis of finite difference equations. These discrete analogues have been also proved to be very useful for partial differential equations. Now we shall review some of the typical results of this type.

Theorem 1.5.4 (The Hull and Luxemburg Inequality [383])**.** *Let m be a positive integer, $u_0, u_1, \dots, u_m$ be a sequence of $m+1$ non-negative numbers, and $z_0, z_1, \dots, z_m$ a non-decreasing sequence of $m+1$ real numbers. Furthermore, let $\{f_m\}$ be a non-decreasing sequence of non-negative real numbers and $L \ge 0$. Assume that*

$$u_l \le f_l + L\sum_{j=0}^{l-1} u_j(z_{j+1} - z_j) = \{f_l + Lu_0(z_1 - z_0)\} + \sum_{j=0}^{l-1} u_j(z_{j+1} - z_j) \tag{1.5.21}$$

for $l = 1, 2, \ldots, m$. *Then the inequality*

$$u_l \le \{f_l + Lu_0(z_1 - z_0)\} \prod_{j=1}^{l} [1 + L(z_j - z_{j-1})] \tag{1.5.22}$$

holds for $l = 1, 2, \ldots, m$.

Proof. Set $h_j = z_{j+1} - z_j, \;\; j = 0, 1, \ldots, m-1$. By hypothesis,

$$u_l \le f_l + Lu_0h_0 + L\sum_{j=1}^{l-1} u_jh_j. \tag{1.5.23}$$

Since $1 + Lh_0 \ge 1$, the inequality (1.5.22) certainly holds for $l = 1$. Assume that it holds for $l \le n-1$. We show that it holds for $l = n$.

Now

$$\begin{aligned}
u_n &\le (f_n + Lu_0h_0) + L\sum_{j=1}^{n-1} u_jh_j \\
&\le (f_n + Lu_0h_0) + L\sum_{j=1}^{n-1} h_j(f_j + Lu_0h_0)\prod_{i=1}^{j}(1 + Lh_{i-1}) \\
&\le (f_n + Lu_0h_0)\left\{1 + L\sum_{j=1}^{n-1} h_j\prod_{i=1}^{j}(1 + Lh_{i-1})\right\} \\
&\le (f_n + Lu_0h_0)\prod_{j=1}^{n}(1 + Lh_{j-1})
\end{aligned} \tag{1.5.24}$$

since $\{f_n\}$ is non-decreasing, and

$$\begin{aligned}
&\left\{1 + L\sum_{j=1}^{n-1} h_j\prod_{i=1}^{j}(1 + Lh_{i-1})\right\} \\
&\quad = 1 + Lh_1(1 + Lh_0) + Lh_2(1 + Lh_0)(1 + Lh_1) \\
&\qquad + \cdots + Lh_{n-1}(1 + Lh_0)\cdots(1 + Lh_{n-2}) \\
&\quad \le (1 + Lh_0)\Big\{1 + Lh_1 + Lh_2(1 + Lh_1) \\
&\qquad + \cdots + Lh_{n-1}(1 + Lh_1)\cdots(1 + Lh_{n-2})\Big\} \\
&\quad = (1 + Lh_0)(1 + Lh_1)\cdots(1 + Lh_{n-1}) \\
&\quad = \prod_{j=1}^{n}(1 + Lh_{j-1}),
\end{aligned}$$

which completes the proof. □

In the investigation of convergence properties of several finite difference schemes for nonlinear parabolic equations, Lees [495] has used the following result.

Theorem 1.5.5 (The Lees Inequality [495]). *Let u and f be non-negative functions defined on the integers $1, 2, \dots, m$. Let f be non-decreasing. If*

$$u_l \le f_l + Lk \sum_{i=1}^{l-1} u_i, \quad l = 1, 2, \dots, m \tag{1.5.25}$$

where L and k are positive constants, then

$$u_l \le f_l \exp(Lkl), \quad l = 1, 2, \dots, m. \tag{1.5.26}$$

Proof. This theorem easily follows by setting $u_0 = 0$ and $u_j - u_{j-1} \equiv k,\ k > 0$, for $j = 1, 2, \dots, m$. For, under these assumptions, (1.5.25) yields the estimate

$$u_l \le f_l \prod_{j=1}^{l} (1 + Lk) \le f_l \exp(Lkl),$$

which proves (1.5.26). □

For other useful discrete analogue, we refer to Hull and Luxemburg [383], Jones [407], Li [542], Willett and Wong [967].

Ladyzhenskaya, Solonnikov and Ural'ceva [472] established the following two discrete forms of the Bellman–Gronwall inequalities.

Theorem 1.5.6 (The Ladyzhenskaya–Solonnikov–Ural'ceva Inequality [472]). *Let a sequence y_i $(i = 0, 1, \dots)$ of non-negative numbers satisfy the recursion relation*

$$y_{i+1} \le C b^i y_i^{1+\varepsilon}, \quad i = 0, 1, \dots \tag{1.5.27}$$

for some positive constants C, ε and $b \ge 1$. Then

$$y_i \le C^{[(1+\varepsilon)^i - 1]/\varepsilon} b^{[(1+\varepsilon)^i - 1]/\varepsilon^2 - i/\varepsilon} y_0^{(1+\varepsilon)^i}. \tag{1.5.28}$$

In particular, if $y_0 \le \theta = C^{-1/\varepsilon} b^{-1/\varepsilon^2}$ and $b > 1$, then

$$y_i \le \theta b^{-i/\varepsilon} \tag{1.5.29}$$

and consequently

$$y_i \to 0 \tag{1.5.30}$$

when $i \to +\infty$.

Proof. This result is proved directly by induction. We leave the details to the reader. □

Theorem 1.5.7 (The Ladyzhenskaya–Solonnikov–Ural'ceva Inequality [472]). *Assume that the non-negative numbers y_i and z_i $(i = 0, 1, \dots)$ are connected by the system of recursion inequalities*

$$\begin{cases} y_{i+1} \le Cb^i \left(y_i^{1+\delta} + z_i^{1+\varepsilon} y_i^{\delta} \right), & (1.5.31) \\ z_{i+1} \le Cb^i \left(y_i + z_i^{1+\varepsilon} \right) & (1.5.32) \end{cases}$$

where C, b, ε and δ are certain fixed positive numbers and $b \ge 1$. Then

$$y_i \le \lambda b^{-i/d}, \quad z_i \le (\lambda b^{-i/d})^{1/(1+\varepsilon)} \tag{1.5.33}$$

where

$$d = \min\left(\delta, \varepsilon/(1+\varepsilon)\right), \quad \lambda = \min\left((2C)^{-1/\delta} b^{-1/(\delta d)}, (2C)^{-(1+\varepsilon)/\varepsilon} b^{-1/(\varepsilon b)} \right),$$

when $y_0 \le \lambda$ and $z_0 \le \lambda^{1/(1+\varepsilon)}$.

Proof. Indeed, inequalities (1.5.33) hold by assumption for $i = 0$. Assume that they hold for y_i and z_i. Then from (1.5.31)–(1.5.32) it follows that

$$\begin{cases} y_{i+1} \le Cb^i 2(\lambda b^{-i/d})^{1+\delta} = 2C\lambda^{1+\delta} b^{i(1-(1+\delta)/d)}, & (1.5.34) \\ z_{i+1} \le 2C\lambda b^{i(1-1/d)}. & (1.5.35) \end{cases}$$

But, as is easily verified, the right-hand sides of these inequalities (1.5.34)–(1.5.35) do not exceed $\lambda b^{-(i+1)/d}$ and $(\lambda b^{-(i+1)/d})^{1/(1+\varepsilon)}$, respectively, and hence the inequalities in (1.5.33) also hold for y_{i+1} and z_{i+1}. □

Ammari and Tucsnak [42] recently established the following discrete inequality with a uniform bound (see, e.g., Rauch, Zhang and Zuazua [830]).

Theorem 1.5.8 (The Ammari–Tucsnak Inequality [42]). *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers satisfying for $k = 0, 1, 2, \dots,$*

$$a_{k+1} \le a_k - Ca_{k+1}^{2+\alpha}, \tag{1.5.36}$$

for some constants $C > 0$ and $\alpha > -1$. Then there is a constant $M = M(C, \alpha) > 0$ such that for $k = 0, 1, 2, \dots,$

$$a_k \le \frac{M}{(k+1)^{1/(1+\alpha)}}. \tag{1.5.37}$$

Proof. Let $F_k = \frac{M}{(k+1)^{1/(1+\alpha)}}$, where $M > 0$ is to be determined later on. After a simple calculation, we obtain

$$\frac{1}{M} \lim_{k \to +\infty} \left[(F_k - F_{k+1}) k (k+2)^{1/(1+\alpha)} \right] = \frac{1}{1+\alpha}, \tag{1.5.38}$$

so there is an integer $k_0 > 0$ such that for all $k \geq k_0$,

$$F_k - F_{k+1} \leq \frac{2M}{(1+\alpha)k(k+2)^{1/(1+\alpha)}}, \tag{1.5.39}$$

which implies that for all $k \geq k_1 = \max\big[k_0, 2\big]$,

$$F_k - F_{k+1} \leq \frac{4}{(1+\alpha)M^{1+\alpha}} F_{k+1}^{2+\alpha}. \tag{1.5.40}$$

If we now assume that

$$\frac{4}{(1+\alpha)M^{1+\alpha}} < C, \quad \frac{M}{(k_1+1)^{1/(1+\alpha)}} \geq a_{k_1}, \tag{1.5.41}$$

then we infer from (1.5.40), that for all $k \geq k_1$,

$$F_k - F_{k+1} \leq C F_{k+1}^{2+\alpha}. \tag{1.5.42}$$

It obviously suffices to show that for all $k \geq k_1$,

$$a_k \leq F_k. \tag{1.5.43}$$

We shall do this by induction over k. In fact, if $k = k_1$, (1.5.43) follows directly from (1.5.41). If we assume that (1.5.43) is fulfilled for $k \leq m$, by combining (1.5.36) and (1.5.42), we obtain

$$a_{m+1} + C a_{m+1}^{2+\alpha} \leq F_{m+1} + C F_{m+1}^{2+\alpha}$$

which obviously implies that $a_{m+1} \leq F_{m+1}$. □

The following two theorems concern integral inequalities which yield polynomial or exponential decay of solutions (see, e.g., Haraux [347] and Lagnese [474], and Komornik [451]).

Theorem 1.5.9 (The Haraux–Lagnese Inequality [347, 474])**.** *Let $E : [0, +\infty) \to [0, +\infty)$ be a non-increasing function and assume that there exists a constant $T > 0$ such that for all $t \in \mathbb{R}^+$,*

$$\int_t^{+\infty} E(s)ds \leq TE(t). \tag{1.5.44}$$

Then for all $t \geq T$,

$$E(t) \leq E(0)e^{1-t/T}. \tag{1.5.45}$$

Proof. Define for all $x \in [0, +\infty)$,

$$f(x) = e^{x/T} \int_x^{+\infty} E(s)ds.$$

Then f is locally absolutely continuous and it is also non-increasing by (1.5.44):

$$f'(x) = T^{-1}e^{x/T}\left(\int_x^{+\infty} E(s)ds - TE(x)\right) \le 0$$

almost everywhere in $[0,+\infty)$. Hence, using again (1.5.44), for all $x \in [0,+\infty)$,

$$f(x) \le f(0) = \int_0^{+\infty} E(s)ds \le TE(0),$$

i.e., for all $x \in [0,+\infty)$,

$$\int_x^{+\infty} E(s)ds \le TE(0)e^{-x/T}. \tag{1.5.46}$$

Since E is non-negative and non-increasing, we have

$$\int_x^{+\infty} E(s)ds \ge \int_x^{x+T} E(s)ds \ge TE(x+T). \tag{1.5.47}$$

Inserting (1.5.47) in (1.5.46), we obtain for all $x \in [0,+\infty)$,

$$E(x+T) \le E(0)e^{-x/T},$$

which, by setting $t = x + T$, gives us (1.5.44). The proof is complete. □

Remark 1.5.3. Note that the inequality (1.5.45) also holds for $0 \le t < T$; indeed, it is weaker than the trivial inequality $E(t) \le E(0)$.

Remark 1.5.4. Theorem 1.5.9 is optimal in the following sense: given $T > 0$ and $t' \ge T$ arbitrarily, there exists a non-increasing function $E : [0,+\infty) \to [0,+\infty)$, non-identically zero, satisfying (1.5.44) and such that

$$E(t') = E(0)e^{1-t'/T}.$$

Remark 1.5.5. If the function E is also continuous, then the inequality (1.5.45) is strict; in particular, $E(T) < E(0)$. This result is also optimal, see Komornik [451].

The following theorem, due to Komornik [451], is a nonlinear generalization of Theorem 1.5.9, which also improves some earlier results of Haraux [347] and Lagnese [474] (see also Theorem 1.5.9).

Theorem 1.5.10 (The Komornik Inequality [451])**.** *Let $E : [0,+\infty) \to [0,+\infty)$ be a non-increasing function and assume that there are two constants $\alpha > 0$ and $T > 0$ such that for all $x \in [0,+\infty)$,*

$$\int_t^{+\infty} E^{\alpha+1}(s)ds \le TE^{\alpha}(0)E(t). \tag{1.5.48}$$

Then, for all $t \ge T$,

$$E(t) \le E(0)\left(\frac{T+\alpha t}{T+\alpha T}\right)^{-1/\alpha}. \tag{1.5.49}$$

Proof. First, if $E(0)=0$, then $E(t)=0$ and there is nothing to prove. Otherwise, replacing the function E by $E/E(0)$, we may assume that $E(0)=1$, and we have to prove the estimate for all $t \ge T$,

$$E(t) \le \left(\frac{T+\alpha t}{T+\alpha T}\right)^{-1/\alpha}. \tag{1.5.50}$$

Define the function

$$F:[0,+\infty)\to[0,+\infty),\quad F(t)=\int_t^{+\infty} E^{\alpha+1}(s)ds.$$

Obviously, F is non-increasing and locally absolutely continuous. Differentiating and using (1.5.48), we get

$$-F''(t) \ge T^{-\alpha-1}F^{\alpha+1}(t) \text{ a.e. in } (0,+\infty),$$

whence

$$(F^{-\alpha}(t))' \ge \alpha T^{-\alpha-1} \text{ a.e. in } (0,B),\quad B=\sup\{t : E(t)>0\}. \tag{1.5.51}$$

(Observe that $F^{-\alpha}(t)$ is defined for $0<t<B$). Integrating (1.5.51) over $[0,s]$ gives for every $s\in[0,B)$,

$$F^{-\alpha}(s)-F^{-\alpha}(0) \ge \alpha T^{-\alpha-1}s,$$

whence for every $s\in[0,B)$,

$$F(s) \le \left(F^{-\alpha}(0)+\alpha T^{-\alpha-1}s\right)^{-1/\alpha}. \tag{1.5.52}$$

Since $F(s)=0$ if $s\ge B$, this inequality holds, in fact, for every $s\in[0,+\infty)$. Since $F(0)\le TE^{\alpha+1}(0)=T$ by (1.5.48), the right-hand side of (1.5.52) is less than or equal to

$$\left(T^{-\alpha}+\alpha T^{-\alpha-1}s\right)^{-1/\alpha} = T^{(\alpha+1)/\alpha}(T+\alpha s)^{-1/\alpha}. \tag{1.5.53}$$

On the other hand, due to E being non-negative and non-increasing, the left-hand side of (1.5.52) can be estimated as follows:

$$F(s)=\int_s^{+\infty} E^{\alpha+1}(t)dt \ge \int_s^{T+(\alpha+1)s} E^{\alpha+1}(t)dt \ge (T+\alpha s)E^{\alpha+1}(T+(\alpha+1)s). \tag{1.5.54}$$

Therefore, we can deduce from (1.5.53)–(1.5.54) that

$$(T+\alpha s)E^{\alpha+1}(T+(\alpha+1)s) \le T^{(\alpha+1)/\alpha}(T+\alpha s)^{-1/\alpha},$$

whence for all $s\ge 0$,

$$E(T+(\alpha+1)s) \le (1+\alpha s/T)^{-1/\alpha}. \tag{1.5.55}$$

Choosing here $t=T+(\alpha+1)s$ we obtain (1.5.49). □

Remark 1.5.6 ([451])**.** Note that the inequality (1.5.49) also holds for $0 \le t < T$; indeed, it follows from the inequality $E(t) \le E(0)$.

Remark 1.5.7 ([451])**.** If we let $\alpha \to 0$ in (1.5.49), then Theorem 1.5.10 reduces to Theorem 1.5.9.

Remark 1.5.8 ([451])**.** In fact, Theorem 1.5.10 is optimal in the following case: given $\alpha > 0, T > 0, C > 0$ and $t' \ge T$ arbitrarily, there exists a non-increasing function $E : [0, +\infty) \to [0, +\infty)$ satisfying (1.5.48) and such that

$$E(0) = C, \quad E(t') = E(0)\left(\frac{T + \alpha t'}{T + \alpha T}\right)^{-1/\alpha}.$$

We leave to the reader to verify that the following example has these properties:

$$E(t) = \begin{cases} C(1 + \alpha C^{-\alpha} t/T)^{-1/\alpha}, & \text{if } 0 \le t \le t'', \\ C(1+\alpha)^{1/\alpha}(1 + \alpha C^{-\alpha} t'/T)^{-1/\alpha}, & \text{if } t'' \le t \le t', \\ 0, & \text{if } t > t' \end{cases} \tag{1.5.56}$$

where $t'' = (t' - TC^\alpha)/(\alpha + 1)$. Note that for $0 \le t < T$, we cannot state more than the trivial estimate $E(t) \le E(0)$. Indeed, for any given $\alpha > 0, T > 0, C > 0$ and $t' < T$, the function

$$E(t) = \begin{cases} C, & \text{if } 0 \le t \le T, \\ 0, & \text{if } t > T \end{cases} \tag{1.5.57}$$

satisfies (1.5.48) and $E(t') = E(0) = C$.

Remark 1.5.9. ([451]) Assume that E is also continuous. Then the inequality (1.5.50) is strict; in particular, $E(T) < E(0)$. See Komornik [451] for this result, for a detailed study of integral inequalities of type (1.5.48) (also for $\alpha < 0$), and for the study of closely related differential inequalities.

Obviously, the above two theorems can be stated as the following form, which was used in [451].

Corollary 1.5.1 (The Komornik Inequality [451])**.** *Let* $E : [0, +\infty) \to [0, +\infty)$ *be a non-increasing function and assume that there exist two constants* $p \ge 1$ *and* $A > 0$*, such that for all* $0 \le s < +\infty$*,*

$$\int_s^{+\infty} E^{(p+1)/2}(t)dt \le AE(s). \tag{1.5.58}$$

Then we have for all $t \ge 0$*,*

$$E(t) \le \begin{cases} E(0)e^{1-t/A}, & \textit{if} \quad p = 1, \\ (A(1 + 2/(p-1)))^{2/(p-1)} \frac{1}{(1+t)^{2/(p-1)}}, & \textit{if} \quad p > 1. \end{cases} \tag{1.5.59}$$

Now we shall study integral inequalities of type (1.5.48) in detail. This also includes the case $\alpha < 0$ (see, e.g., Komornik [451]). Such integral inequalities are related with the Lyapunov methods. For details, we may refer the reader to Komornik [451].

We next introduce some Alabau inequalities, which were established in [19].

Assume H is a separable real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Let $A : D(A) \subset H \to H$ be the infinitesimal generator of a strongly continuous semigroup $\exp(tA)$ of bounded operators on H with its domain $D(A)$. It is well known that the H-valued function $U(t) = e^{tA}U_0$ is continuous for any $U_0 \in H$, of class $C^1([0, +\infty), H)$ for any $U_0 \in D(A)$, and that, in the latter case, $U(t) = e^{tA}U_0$ solves the Cauchy problem for all $t \geq 0$,

$$U'(t) = AU(t), U(0) = U_0. \tag{1.5.60}$$

Moreover, for any $k \in \mathbb{N}$,

$$U_0 \in D(A^k) \Rightarrow U^{(k)} = e^{tA}A^kU_0 = A^ke^{tA}U_0.$$

We recall that a semigroup e^{tA} is said to be strongly stable if for all $U_0 \in H$,

$$\lim_{t\to+\infty} e^{tA}U_0 = 0.$$

If there exist two constants $M, \omega > 0$ such that for all $t \geq 0$,

$$\|e^{tA}\| \equiv \sup\{\|e^{tA}x\|_H : x \in H, \|x\|_H \leq 1\} \leq Me^{-\omega t},$$

then e^{tA} is called exponentially stable.

The following result, given by Alabau [17] in 1999, is a generalization of the integral inequalities due to Haraux [347] and Komornik [451], i.e., Theorems 1.5.9–1.5.10.

Theorem 1.5.11 (The Alabau Inequality [17])**.** *Assume that there exists a functional E on $C([0, +\infty), H) \times [0, +\infty)$ such that for every $U_0 \in H$, $E(U(\cdot), \cdot)$ is a non-increasing, locally absolutely continuous function from $[0, +\infty)$ on $[0, +\infty)$. Assume, moreover, that there exist a positive integer k and a non-negative constant c such that for all $0 \leq s \leq T$, for all $U_0 \in D(A^k)$,*

$$\int_s^T E(U(t), t)dt \leq c \sum_{p=0}^{k} E(U^{(p)}(s), s).$$

Then for every positive integer n, we have for all $t > 0$, for all $U_0 \in D(A^{kn})$,

$$E(U(t), t) \leq c_n \left(\sum_{p=0}^{kn} E(U^{(p)}(0), 0) \right) t^{-n},$$

for a certain constant c_n depending on n.

Since the above theorem is a slight modification of the next result, we omit its proof and only give the proof of the following theorem, obtained in 2002 by Alabau, Cannarsa and Komornik [19] in a slightly different form, which is a polynomial decay criterion.

Theorem 1.5.12 (The Alabau–Cannarsa–Komornik Inequality [19]). *Let H, A be same as the above statement. Let $L : H \to [0, +\infty)$ be a continuous function such that, for some integer $K \geq 0$ and some constant $c \geq 0$, for all $T \geq 0$, for all $x \in D(A^K)$,*

$$\int_0^T L(e^{tA}x)dt \leq c\sum_{k=0}^{K} L(A^k x). \tag{1.5.61}$$

Then, for any integer $n \geq 1$ and any $x \in D(A^{nK})$, for all $0 \leq s \leq T$,

$$\int_s^T L(e^{tA})\frac{(t-s)^{n-1}}{(n-1)!}dt \leq c^n(1+K)^{n-1}\sum_{k=0}^{nK} L(e^{sA}A^k x). \tag{1.5.62}$$

If, in addition, $L(e^{tA}x) \leq L(e^{sA}x)$ for all $x \in H$ and any $0 \leq s \leq t$, then for all $t > 0$,

$$L(e^{tA}x) \leq c^n(1+K)^{n-1}\frac{n!}{t^n}\sum_{k=0}^{nK} L(A^k x), \tag{1.5.63}$$

for any integer $n \geq 1$, and any $x \in D(A^{nK})$.

Proof. To prove (1.5.62), we proceed by induction on n. Let us first show (1.5.62) for $n = 1$: for any $x \in D(A^K)$ and any $0 \leq s \leq T$, the assumption (1.5.61) yields

$$\int_0^{T-s} L(e^{tA}e^{sA}x)dt \leq c\sum_{k=0}^{K} L(A^k e^{sA}x),$$

which, since A commutes with e^{sA}, yields

$$\int_s^T L(e^{tA}x)dt \leq c\sum_{k=0}^{K} L(e^{sA}A^k x). \tag{1.5.64}$$

Now assume that the conclusion holds for $n \geq 1$ and let $x \in D(A^{(n+1)K})$. Integrating (1.5.62) over $[S, T]$, for $0 \leq S \leq T$, we have

$$\int_S^T ds \int_s^T L(e^{tA}x)\frac{(t-s)^{n-1}}{(n-1)!}dt \leq c^n(1+K)^{n-1}\sum_{k=0}^{nK}\int_S^T L(e^{sA}A^k x)ds. \tag{1.5.65}$$

Next, apply Fubini's Theorem and (1.5.62) for $n = 1$ to find

$$\int_S^T L(e^{tA}x)dt \int_S^t \frac{(t-s)^{n-1}}{(n-1)!}ds \leq c^{n+1}(1+K)^{n-1}\sum_{k=0}^{nK}\sum_{h=0}^{K} L(A^h e^{SA}A^k x). \tag{1.5.66}$$

Since

$$\sum_{k=0}^{nK}\sum_{h=0}^{K} L(e^{SA}A^{k+h}x) = \sum_{h=0}^{K}\sum_{k=0}^{nK} L(e^{SA}A^{h+k}x) \le (1+K)\sum_{k=0}^{(n+1)K} L(e^{SA}A^{k}x),$$

(1.5.66) implies

$$\int_S^T L(e^{tA}x)\frac{(t-S)^n}{n!}dt \le c^{n+1}(1+K)^n \sum_{k=0}^{(n+1)K} L(e^{SA}A^k x) \tag{1.5.67}$$

for any $0 \le S \le T$, as desired.

Finally, to prove (1.5.63), it suffices to observe that

$$L(e^{tA}x)\frac{T^n}{n!} \le \int_0^T L(e^{tA}x)\frac{t^{n-1}}{(n-1)!}dt \le c^n(1+K)^{n-1}\sum_{k=0}^{nK} L(A^k x), \tag{1.5.68}$$

as L is non-increasing along $e^{tA}x$. Thus the proof is complete. □

In 1999, Martinez [586] extended the results of Haraux [347], [349] and Komornik [449] (i.e., Theorems1.5.9–1.5.10) with a weighted function.

Theorem 1.5.13 (The Martinez Inequality [586]). *Let $E : [0,+\infty) \to [0,+\infty)$ be a non-increasing function and $\phi : [0,+\infty) \to [0,+\infty)$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0, \quad \phi(t) \to +\infty \ \text{ as } \ t \to +\infty. \tag{1.5.69}$$

Assume that there exist $\sigma \ge 0$ and $\omega > 0$ such that for all $S \ge 0$,

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)dt \le \frac{1}{\omega}E^\sigma(0)E(S). \tag{1.5.70}$$

Then $E(t)$ has the following decay property: for all $t \ge 0$,

$$\begin{cases} \text{if } \ \sigma = 0, \ \ \text{then } \ E(t) \le E(0)e^{1-\omega\phi(t)}, & (1.5.71)\\[2mm] \text{if } \ \sigma > 0, \ \ \text{then } \ E(t) \le E(0)\left(\dfrac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{1/\sigma}. & (1.5.72)\end{cases}$$

Proof. Define now a function $f : [0,+\infty) \to [0,+\infty)$ by

$$f(\tau) = E(\phi^{-1}(\tau)).$$

Then it is easy to verify that f is non-increasing and satisfies that for any $0 \le S < T < +\infty$,

$$\begin{aligned}\int_{\phi(S)}^{\phi(T)} f^{1+\sigma}(\tau)d\tau &= \int_{\phi(S)}^{\phi(T)} E^{1+\sigma}(\phi^{-1}(\tau))d\tau \qquad (1.5.73)\\ &= \int_S^T E^{1+\sigma}(t)\phi'(t)dt \le \frac{1}{\omega}E^\sigma(0)E(S) = \frac{1}{\omega}f^\sigma(0)f(\phi(S)).\end{aligned}$$

Setting $s = \phi(S)$, then as $\lim_{T\to+\infty} \phi(T) = +\infty$, we derive that f satisfies that for all $s \geq 0$,

$$\int_s^{+\infty} f^{1+\sigma}(\tau)d\tau \leq \frac{1}{\omega} f(s). \tag{1.5.74}$$

Hence now applying Theorem 1.5.10 (or Corollary 1.4.5) to (1.5.74) we obtain for all $s \geq 0$,

$$\begin{cases} \text{if} \quad \sigma = 0, \ f(s) \leq f(0)e^{1-\omega s}, & (1.5.75) \\ \text{if} \quad \sigma > 0, \ f(s) \leq f(0)\left(\dfrac{1+\sigma}{1+\omega\sigma s}\right)^{1/\sigma}. & (1.5.76) \end{cases}$$

Since $E(t) = f(\phi(t))$, (1.5.71) and (1.5.72) follow from (1.5.75) and (1.5.76). □

Remark 1.5.10. We know that the new feature of Theorem 1.5.13 is concerned with the weighted function $\phi'(t)$, which allows us to consider functions $E(t)$ that can decay slowly to zero. For example, if E satisfies (1.5.70) for

$$\phi(t) = \ln(\ln(3+t)) - \ln(\ln 3), \quad \sigma = 0,$$

or

$$\phi(t) = \ln(3+t) - \ln 3, \quad \sigma > 0,$$

then from Theorem 1.5.13 it follows that for all $t \geq 0$

$$E(t) \leq \frac{CE(0)}{(\ln(3+t))^{\gamma}},$$

with $\gamma = \omega$ if $\sigma = 0$, and $\gamma = 1/\sigma$ if $\sigma > 0$.

Remark 1.5.11. In fact, if $\phi(0) \neq 0$, then it suffices to replace $\phi(t)$ by $\phi(t) - \phi(0)$ in (1.5.71) and (1.5.72).

Corollary 1.5.2 (The Martinez Inequality [586]). *Let $f : [0, +\infty) \to [0, +\infty)$ be a non-increasing continuous function. Assume that there exist constants $\sigma > 0$, $\sigma' \geq 0$, and $C > 0$, such that for all $t \geq 0$,*

$$\int_t^{+\infty} f^{1+\sigma}(\tau)d\tau \leq \frac{Cf^{\sigma}(0)f(t)}{(1+t)^{\sigma'}}. \tag{1.5.77}$$

Then there exists a constant $\widetilde{C} > 0$, such that for all $t \geq 0$,

$$f(t) \leq \frac{\widetilde{C}f(0)}{(1+t)^{(1+\sigma')/\sigma}}. \tag{1.5.78}$$

Proof. Without loss of generality, we may assume that $f(0) = 1$. Define for all $t \geq 0$,

$$g(t) = \frac{f(t)}{(1+t)^{\sigma'}}.$$

Then g is non-increasing and satisfies for all $t \geq 0$,

$$\int_t^{+\infty} g^{1+\sigma}(\tau)(1+\tau)^{\sigma'(1+\sigma)} d\tau \leq Cg(t). \tag{1.5.79}$$

Thus we can apply Theorem 1.5.13 with $\phi(t) = (1+t)^{\sigma'(1+\sigma)} - 1$ to (1.5.79) to deduce that g decays as

$$g(t) \leq C/(1+t)^{[\sigma'(1+\sigma)+1]/\sigma} = C/(1+t)^{\sigma'}(1+t)^{(1+\sigma')/\sigma}$$

which gives (1.5.78). □

We can use Corollary 1.5.2 to show the following integral inequality (see, e.g., Martinez [586]).

Corollary 1.5.3 (The Martinez Inequality [586]). *Let $f : [0, +\infty) \to [0, +\infty)$ be a non-increasing continuous function. Assume that there exist constants $\sigma > 0$, $\sigma' \geq 0$ and $c > 0$, such that for all $t \geq 0$,*

$$\int_t^{+\infty} f^{1+\sigma}(\tau) d\tau \leq c f^{1+\sigma}(t) + \frac{c}{(1+t)^{\sigma'}} f^{\sigma}(0) f(t). \tag{1.5.80}$$

Then there exists a constant $C > 0$, such that for all $t \geq 0$,

$$f(t) \leq \frac{Cf(0)}{(1+t)^{(1+\sigma')/\sigma}}. \tag{1.5.81}$$

Proof. Without loss of generality, we may assume that $f(0) = 1$. Note that we may neglect the influence of the term $f^{1+\sigma}(t)$ on the decay estimate of f. It is known that if there is only the term $f^{1+\sigma}(t)$ on the right-hand side of (1.5.80), then f decays exponentially to zero, but if there is only the second term, then f decays at least polynomially to zero. Now we prove (1.5.81) by an induction argument. In the following, by C we shall denote all universal different positive constants.

First, it is obvious from (1.5.80) that for all $t \geq 0$,

$$\int_t^{+\infty} f^{1+\sigma}(\tau) d\tau \leq Cf(t). \tag{1.5.82}$$

Therefore from Corollary 1.5.2 and (1.5.82) it follows that for all $t \geq 0$,

$$f(t) \leq \frac{C}{(1+t)^{1/\sigma}}. \tag{1.5.83}$$

Then inserting the inequality (1.5.83) in (1.5.80), we can obtain that for all $t \geq 0$,

$$\int_t^{+\infty} f^{1+\sigma}(\tau) d\tau \leq \frac{Cf(t)}{(1+t)} + \frac{Cf(t)}{(1+t)^{\sigma'}}. \tag{1.5.84}$$

Now if we set $\sigma_1 = \min(1, \sigma')$, then we can derive from (1.5.84) that for all $t \geq 0$,

$$\int_t^{+\infty} f^{1+\sigma}(\tau) d\tau \leq \frac{Cf(t)}{(1+t)^{\sigma_1}}, \tag{1.5.85}$$

which, together with (1.5.78), implies that for all $t \geq 0$,

$$f(t) \leq \frac{C}{(1+t)^{(1+\sigma_1)/\sigma}}. \tag{1.5.86}$$

If $\sigma' \leq 1$, then we can conclude (1.5.81). Otherwise, for all $t \geq 0$,

$$f(t) \leq \frac{C}{(1+t)^{2/\sigma}}. \tag{1.5.87}$$

Now if we take $n \in \mathbb{N}$ such that $\sigma \in [n, n+1]$, then we can prove by induction that for all $k \in \mathbb{N}, k \leq n$, f satisfies for all $t \geq 0$,

$$f(t) \leq \frac{C_k}{(1+t)^{(1+k)/\sigma}}. \tag{1.5.88}$$

We hence have proved (1.5.88) for $k = 0$ and for $k = 1$ if $n \geq 1$. Assume that $n \geq 2$ and that (1.5.88) is true for some $k < n$. Then we may use (1.5.88) to derive from (1.5.80) that f satisfies for all $t \geq 0$,

$$\int_t^{+\infty} f^{1+\sigma}(\tau) d\tau \leq \frac{Cf(t)}{(1+t)^{1+k}} + \frac{Cf(t)}{(1+t)^{\sigma'}}. \tag{1.5.89}$$

Since $1 + k \leq n \leq \sigma'$, we obtain for all $t \geq 0$,

$$\int_t^{+\infty} f^{1+\sigma}(\tau) d\tau \leq \frac{Cf(t)}{(1+t)^{1+k}}. \tag{1.5.90}$$

Therefore from Corollary 1.5.2 it follows that for all $t \geq 0$,

$$f(t) \leq \frac{C_k}{(1+t)^{(2+k)/\sigma}}.$$

This shows that for all $t \geq 0$,

$$f(t) \leq \frac{C_k}{(1+t)^{(n+1)/\sigma}}, \tag{1.5.91}$$

and for all $t \geq 0$,

$$f(t) \leq \frac{C_k}{(1+t)^{(\sigma'+1)/\sigma}}. \tag{1.5.92}$$

□

Now we may use Corollary 1.5.3 to prove the following integral inequality (see Martinez [586]).

Theorem 1.5.14 (The Martinez Inequality [586]). *Assume that $E : [0,+\infty) \to [0,+\infty)$ is a non-increasing function and $\phi : [0,+\infty) \to [0,+\infty)$ a strictly increasing function of class C^1, such that*

$$\phi(0) = 0, \quad \phi(t) \to +\infty \ \text{ as } \ t \to +\infty. \tag{1.5.93}$$

Assume that there exist constants $\sigma > 0$, $\sigma' \geq 0$, and $c > 0$ such that for all $S \geq 0$,

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)dt \leq cE^{1+\sigma}(S) + \frac{c}{(1+\phi(S))^{\sigma'}} E^{\sigma}(0)E(S). \tag{1.5.94}$$

Then there exists a constant $C > 0$ such that for all $t \geq 0$,

$$E(t) \leq E(0)\frac{C}{(1+\phi(t))^{(1+\sigma')/\sigma}}. \tag{1.5.95}$$

Proof. In fact, if we introduce $f(\tau) = E(\phi^{-1}(\tau))$ and use Corollary 1.5.3, then the desired conclusion easily follows. □

Next let us establish the following lemmas.

Lemma 1.5.15 (The Martinez Inequality [586]). *Let $\Phi(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing C^1 function, such that*

$$\sigma(0) = 0 \quad \text{and} \quad \sigma(t) \to +\infty \quad \text{as} \quad t \to +\infty. \tag{1.5.96}$$

Assume that there exist constants $m \geq 0$ and $\omega > 0$, such that for all $S \geq 0$,

$$\int_S^{+\infty} \Phi^{1+m}(t)\sigma'(t)dt \leq \frac{1}{\omega}\Phi^m(0)\Phi(S). \tag{1.5.97}$$

Then Φ has the following decay property: if $m = 0$, then for all $t \geq 0$,

$$\Phi(t) \leq \Phi(0)e^{1-\omega\sigma(t)}, \tag{1.5.98}$$

and if $m > 0$, then for all $t \geq 0$,

$$\Phi(t) \leq \Phi(0)\left(\frac{1+m}{1+\omega m\sigma(t)}\right)^{1/m}. \tag{1.5.99}$$

Proof. Define a function $f : [0,+\infty) \to \mathbb{R}^+$ by

$$f(\tau) = \Phi(\sigma^{-1}(\tau)).$$

Then it is easy to verify that f is non-increasing for all $0 \leq S < T < +\infty$

$$\int_{\sigma(S)}^{\sigma(T)} f^{1+m}(\tau)d\tau = \int_{\sigma(S)}^{\sigma(T)} \Phi^{1+m}(\sigma^{-1}(\tau))d\tau = \int_S^T \Phi^{1+m}(t)\sigma'(t)dt$$
$$\leq \frac{1}{\omega}\Phi^m(0)\Phi(S) = \frac{1}{\omega}f^m(0)f(\sigma(S)).$$

Setting $s = \sigma(S)$, then since $\lim_{T\to+\infty} \sigma(T) = +\infty$, we have that f satisfies for all $s \geq 0$,

$$\int_s^{+\infty} f^{1+m}(\tau)d\tau \leq \frac{1}{\omega}f(s).$$

Then applying Theorem 1.5.10 to the above inequality shows that if $m = 0$, then for all $s \geq 0$,

$$f(s) \leq f(0)e^{1-\omega s},$$

and if $m > 0$, then for all $t \geq 0$,

$$f(s) \leq f(0)\left(\frac{1+m}{1+\omega ms}\right)^{1/m}.$$

Noting that $\Phi(t) = f(\sigma(t))$, we can obtain (1.5.98) and (1.5.99). □

In turn, Lemma 1.5.1 implies the following result.

Lemma 1.5.16 (The Martinez Inequality [586])**.** *Assume that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing continuous function, and there exist constants $m > 0$, $n \geq 0$, and c, such that for all $t \geq 0$,*

$$\int_t^{+\infty} f^{1+m}(\tau)d\tau \leq c\frac{f^m(0)f(t)}{(1+t)^n}. \tag{1.5.100}$$

Then there exists a constant $C > 0$, such that for all $t \geq 0$,

$$f(t) \leq f(0)\frac{C}{(1+t)^{(1+n)/m}}. \tag{1.5.101}$$

Proof. Without loss of generality, we may assume that $f(0) = 1$. Define, for all $t \geq 0$,

$$g(t) = \frac{f(t)}{(1+t)^n}.$$

Then the function g is non-increasing and satisfies for all $t \geq 0$,

$$\int_t^{+\infty} g^{1+m}(\tau)(1+\tau)^{n(1+m)}d\tau \leq Cg(t). \tag{1.5.102}$$

Now we can apply Lemma 1.5.1 with

$$\sigma(t) = (1+t)^{n(1+m)+1} - 1$$

to derive that g decays as

$$g(t) \leq \frac{C}{(1+t)^{(n(1+m)+1)/m}} = \frac{C}{(1+t)^n(1+t)^{(1+n)/m}}.$$

Therefore (1.5.101) follows. □

The following lemma, similar to Theorem 1.5.14, from Martinez [586], will be of essential use in establishing the next result.

Lemma 1.5.17 (The Martinez Inequality [586]). *Assume that $\Phi(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing function and $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing C^1 function, with $\sigma(t) \to +\infty$ as $t \to +\infty$, and that there exist constants $p, q \geq 0$, and $c > 0$, such that for all $1 \leq s < +\infty$.*

$$\int_s^{+\infty} \sigma'(t)\Phi^{1+p}(t)dt \leq c\Phi^{1+p}(s) + \frac{c\Phi(s)}{\sigma^q(s)}. \tag{1.5.103}$$

Then there exist positive constants k and δ, such that for all $t \geq 1$,

$$\begin{cases} \Phi(t) \leq ke^{-\delta\sigma(t)}, & \text{if } p = q = 0, \quad (1.5.104) \\ \Phi(t) \leq \dfrac{k}{\sigma(t)^{(1+q)/p}}, & \text{if } p > 0. \quad (1.5.105) \end{cases}$$

Proof. The desired conclusion follows readily if we employ Corollary 1.5.3 for $f(t) = \Phi(\sigma^{-1}(t))$. □

Now we establish decay properties, due to Alabau [18]. We first give a result for a general weight function F^{-1}. We consider here the special choice of a weight function which will become clear in an application (see Chapter 8, Section 7), where the link between dissipation and the linear and nonlinear kinetic energy of the solution will be made, through convexity arguments).

Assume $\eta > 0$ and $T_0 > 0$ are fixed given real numbers and F is a strictly increasing function from $[0, +\infty)$ on $[0, \eta)$, with $F(0) = 0$ and $\lim_{y \to +\infty} F(y) = \eta$.

For any $r \in (0, \eta)$, we define a function $K_r : (0, r] \to [0, +\infty)$ by

$$K_r(\tau) = \int_\tau^r \frac{dy}{yF^{-1}(y)} \tag{1.5.106}$$

and a strictly increasing onto function ψ_r defined from

$$[1/F^{-1}(r), +\infty) \quad \text{to} \quad [1/F^{-1}(r), +\infty)$$

by for all $z \geq \frac{1}{F^{-1}(r)}$,

$$\psi_r(z) = z + K_r\left(F\left(\frac{1}{z}\right)\right) \geq z. \tag{1.5.107}$$

Now we give the first weighted integral inequality due to Alabau [18].

Theorem 1.5.18 (The Alabau Inequality [18]). *Assume that E is a non-increasing, absolutely continuous function from $[0, +\infty)$ on $[0, +\infty)$, satisfying $0 < E(0) < \eta$ and the inequality for all $0 \leq S \leq T$,*

$$\int_S^T E(t)F^{-1}(E(t))dt \leq T_0 E(S). \tag{1.5.108}$$

Then $E(t)$ satisfies the estimate: for all $t \geq \frac{T_0}{F^{-1}(r)}$,

$$E(t) \leq F\left(\frac{1}{\psi_r^{-1}(t/T_0)}\right), \tag{1.5.109}$$

where r is any real number such that

$$\frac{1}{T_0}\int_0^{+\infty} E(\tau)F^{-1}(E(\tau))d\tau \leq r \leq \eta. \tag{1.5.110}$$

Moreover, we have

$$\lim_{t\to+\infty} E(t) = 0, \tag{1.5.111}$$

the decay rate being given by the estimate (1.5.109).

Proof. Indeed, if we define functions k and M respectively by, for all $t \geq 0$,

$$k(t) = \int_t^{+\infty} M(E(\tau))d\tau, \tag{1.5.112}$$

and for all $y \geq 0$,

$$M(y) = yF^{-1}(y), \tag{1.5.113}$$

then, thanks to (1.5.108), we have for all $t \geq 0$,

$$k(t) \leq T_0 E(t). \tag{1.5.114}$$

Moreover, since F^{-1} is a strictly non-negative function, M is an increasing non-negative function. Thus, differentiating (1.5.112) and using (1.5.114), we deduce that for all $s \geq 0$,

$$-k'(s) = M(E(s)) \geq M(k(s)/T_0),$$

which, when integrated between 0 and t, yields for all $t \geq 0$,

$$\int_t^0 \frac{k'(s)}{T_0 M(k(s)/T_0)} ds \geq t/T_0. \tag{1.5.115}$$

The change of variable $y = k(t)/T_0$ in (1.5.115) gives for all $t \geq 0$,

$$\int_{k(t)/T_0}^B \frac{dy}{M(y)} \geq t/T_0, \tag{1.5.116}$$

where B is defined by

$$0 < B = \frac{1}{T_0} \int_0^{+\infty} E(\tau) F^{-1}(E(\tau)) d\tau \leq E(0) < \eta.$$

Hence, since M is positive on $(0, \eta]$, it follows for all $r \in [B, \eta]$ that for all $t \geq 0$,

$$\int_{k(t)/T_0}^r \frac{dy}{M(y)} \geq t/T_0. \tag{1.5.117}$$

We define K_r by (1.5.106). On the other hand, since F^{-1} is strictly increasing on $[0, \eta)$, for all $r \in [B, \eta)$ and all $\tau \in (0, r]$, it holds that for all $y \in [\tau, r]$,

$$\frac{1}{yF^{-1}(r)} \leq \frac{1}{M(y)}. \tag{1.5.118}$$

Thus we have for all $0 < \tau \leq r$,

$$\frac{1}{F^{-1}(r)} (\ln r - \ln \tau) \leq K_r(\tau), \tag{1.5.119}$$

whence, $\lim_{\tau \to 0^+} K_r(\tau) = +\infty$. Thus, K_r is a strictly decreasing function from $(0, r]$ onto $[0, +\infty)$. This, together with (1.5.117), gives us for all $t \geq 0$,

$$k(t) \leq T_0 K_r^{-1}(t/T_0). \tag{1.5.120}$$

In particular, since M is increasing and non-negative on $[0, \eta)$, while E is non-increasing, we infer that for all $t \geq 0$, $\theta > 0$,

$$\theta M(E(t+\theta)) \leq \int_t^{t+\theta} M(E(\tau)) d\tau \leq k(t) \leq T_0 K_r^{-1}(t/T_0). \tag{1.5.121}$$

Hence, we have for all $t \geq 0$,

$$E(t) \leq M^{-1}\Big(\min_{\theta \in (0,t]} (T_0 \gamma_t(\theta)) \Big), \tag{1.5.122}$$

where for all $\theta \in (0, t]$,

$$\gamma_t(\theta) = \frac{1}{\theta} K_r^{-1}((t-\theta)/T_0). \tag{1.5.123}$$

Let now $t > 0$ be fixed for the moment. Thus θ^* is a critical point of γ_t if and only if it satisfies the relation

$$K_r^{-1}\left(\frac{t-\theta^*}{T_0}\right) + \frac{\theta^*}{T_0 K_r'(K_r^{-1}((t-\theta^*)/T_0))} = 0,$$

or, equivalently, if and only if it satisfies

$$K_r^{-1}\left(\frac{t-\theta^*}{T_0}\right) = \frac{\theta^*}{T_0} M\left(K_r^{-1}\left(\frac{t-\theta^*}{T_0}\right)\right).$$

Using the definition of M, we deduce that θ^* is a critical point of γ_t if and only if it satisfies

$$\frac{T_0}{\theta^*} = F^{-1}\left(K_r^{-1}\left(\frac{t-\theta^*}{T_0}\right)\right)$$

which implies that θ^* is a critical point of γ_t if and only if

$$\psi_r\left(\frac{\theta^*}{T_0}\right) = \frac{t}{T_0}, \tag{1.5.124}$$

where ψ_r is defined by (1.5.107) from $[1/F^{-1}(r), +\infty)$ to $[1/F^{-1}(r), +\infty)$. Since F is strictly increasing and K_r is strictly decreasing, we deduce that ψ_r is strictly increasing and onto from $[1/F^{-1}(r), +\infty)$ to $[1/F^{-1}(r), +\infty)$. Hence, for all $t \geq T_0(1/F^{-1}(r))$, γ_t has a unique critical point $\theta(t)$ at which it attains a minimum, which is given by

$$\theta(t) = T_0 \psi_r^{-1}(t/T_0). \tag{1.5.125}$$

Moreover, by the definition of $\theta(t)$, we may write

$$M^{-1}(T_0\phi(\theta(t))) = K_r^{-1}\left(\frac{t-\theta(t)}{T_0}\right) = F\left(\frac{T_0}{\theta(t)}\right).$$

Thus, using these identities in (1.5.122), and (1.5.125), we can obtain (1.5.109). Noting now that $\psi_r^{-1}(\tau) \to +\infty$ as $\tau \to +\infty$ and since F is continuous at zero with $F(0) = 0$, we deduce that

$$\lim_{t\to+\infty} F(1/\psi_r^{-1}(t/T_0)) = 0.$$

So (1.5.109) indeed gives the decay rate of energy as time goes to infinity. □

Recall that if ϕ is a proper convex function from $\mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, then its convex conjugate ϕ^* is defined as

$$\phi^*(y) = \sup_{x\in\mathbb{R}} \{xy - \phi(x)\}.$$

In order to prove a weighted integral inequality, we need the following lemma.

Lemma 1.5.19 ([18]). *Assume that g is an odd, strictly increasing C^1 function from $\mathbb{R}$ to $\mathbb{R}$ such that $g'(0) = 0$, and that there exists an $r_0 > 0$ such that the function H defined by*

$$H(x) = \sqrt{x}g(\sqrt{x}) \tag{1.5.126}$$

is strictly convex on $[0, r_0^2]$. Setting

$$\hat{H}(x) = \begin{cases} H(x), & \text{if } x \in [0, r_0^2], \\ +\infty, & \text{if } x \in \mathbb{R} - [0, r_0^2], \end{cases} \tag{1.5.127}$$

and defining a function F by

$$F(y) = \begin{cases} \frac{\hat{H}^*(y)}{y}, & \text{if } y \in (0, +\infty), \\ 0, & \text{if } y = 0, \end{cases} \tag{1.5.128}$$

where $\hat{H}^*$ stands for the convex conjugate function of $\hat{H}$, then F is a strictly increasing continuous onto function from $[0, +\infty)$ to $[0, r_0^2)$ given by

$$F(y) = \begin{cases} (H')^{-1}(y) - \frac{H((H')^{-1}(y))}{y}, & \text{if } y \in [0, H'(r_0^2)], \\ r_0^2 - \frac{H(r_0^2)}{y}, & \text{if } y \in [H'(r_0^2), +\infty). \end{cases} \tag{1.5.129}$$

Proof. Noting that $g \in C^1(\mathbb{R})$, we have $H \in C^1([0, r_0^2])$. Moreover, for all $x \in (0, r_0^2]$,

$$H'(x) = \frac{g'(\sqrt{x})}{2} + \frac{g(\sqrt{x})}{2\sqrt{x}} > 0,$$

and $H(0) = 0$. On the other hand, since H on $[0, r_0^2]$ is strict convex, the function H' is strictly increasing from $[0, r_0^2]$ onto $[0, H'(r_0^2)]$ and satisfies that for all $0 \le x < z \le r_0^2$,

$$H'(x) < \frac{H(z) - H(x)}{z - x} < H'(z). \tag{1.5.130}$$

Thus, the convex conjugate of $\hat{H}$ defined as

$$\hat{H}^*(y) = \sup_{x \in \mathbb{R}} \{xy - \hat{H}(x)\}$$

is a continuous function on $\mathbb{R}$ given by

$$\hat{H}^*(y) = \begin{cases} y(H')^{-1}(y) - H((H')^{-1}(y)), & \text{if } y \in [0, H'(r_0^2)], \\ r_0^2 y - H(r_0^2), & \text{if } y \in [H'(r_0^2), +\infty), \\ 0, & \text{if } y \le 0. \end{cases} \tag{1.5.131}$$

Thus now using (1.5.130) for $x = 0$, we can obtain for all $z \in (0, r_0^2]$,

$$0 < \frac{H(z)}{z} < H'(z). \tag{1.5.132}$$

Next, choosing $z = (H')^{-1}(y)$ in the above inequality, we obtain for all $y \in (0, H'(r_0^2)]$,

$$0 < \frac{H((H')^{-1}(y))}{y} < (H')^{-1}(y). \tag{1.5.133}$$

Thus,

$$\lim_{y \to 0} \frac{H((H')^{-1}(y))}{y} = 0. \tag{1.5.134}$$

Hence noting that $F(0) = 0$ and using (1.5.134), we can define F as in (1.5.129).

Moreover, F is continuous on $[0, +\infty)$. Since (1.5.133) holds, we deduce that $F > 0$ on $(0, H'(r_0^2)]$. On the other hand, using (1.5.132) for $z = r_0^2$, we deduce that $F > 0$ on $[H'(r_0^2), +\infty)$. Now we need to prove that F is strictly increasing from $[0, +\infty)$ onto $[0, r_0^2)$. Indeed we first consider the interval $[0, H'(r_0^2)]$. Assume y_1, y_2 is given in $[0, H'(r_0^2)]$ such that $y_1 < y_2$ and set $x_i = (H')^{-1}(y_i)$ for $i = 1, 2$. Then $0 \le x_1 < x_2 \le r_0^2$. Thus we can use (1.5.130) for $x = x_1$ and $z = x_2$. Thus this, together with $H'(x_2) > 0$, implies

$$\frac{H(x_2)}{H'(x_2)} - \frac{H(x_1)}{H'(x_2)} < x_2 - x_1. \tag{1.5.135}$$

However, since H' is non-negative and increasing on $[0, r_0^2]$,

$$\frac{H(x_2)}{H'(x_2)} - \frac{H(x_1)}{H'(x_1)} \le \frac{H(x_2)}{H'(x_2)} - \frac{H(x_1)}{H'(x_2)} < x_2 - x_1. \tag{1.5.136}$$

Now in the above inequalities if we replace x_i by $(H')^{-1}(y_i)$ for $i = 1, 2$, then we can obtain $F(y_1) < F(y_2)$. Thus F is increasing on $[0, H'(r_0^2)]$. On the other hand, since F is clearly strictly increasing on $[H'(r_0^2), +\infty)$, now it remains to consider the case $y_1 \in [0, H'(r_0^2)]$, $y_2 \in [H'(r_0^2), +\infty)$, with $y_1 < y_2$. Noting that $F(H'(r_0^2)) < F(y_2)$ if $y_2 > H'(r_0^2)$, we easily conclude that we also have $F(y_1) < F(y_2)$ in this case. □

We are ready to state the second weighted integral inequality due to Alabau [18].

Theorem 1.5.20 (The Alabau Inequality [18])**.** *Let g be a given odd, strictly increasing C^1 function from $\mathbb{R}$ to $\mathbb{R}$ such that $g'(0) = 0$. We assume that there exists an $r_0 > 0$ such that g is of class of C^2 on $[0, r_0]$ and H defined by* (1.5.126) *is strictly convex on $[0, r_0^2]$. We define $\hat{H}$ and F as in Lemma* 1.5.4*. Moreover, let $T_0 > 0$ be a fixed real number, let E be a given non-increasing, absolutely continuous, non-negative real function on $[0, +\infty)$, and let $\beta > 0$ be a given real number such that*

$$0 < \frac{E(0)}{2F(H'(r_0^2))} \le \beta \tag{1.5.137}$$

and for all $0 \le S \le T$,

$$\int_S^T E(t) F^{-1}\left(\frac{E(t)}{2\beta}\right) dt \le T_0 E(S). \tag{1.5.138}$$

Then $E(t)$ *decays at* $+\infty$ *for all* $t \geq \frac{T_0}{H'(r_0^2)}$,

$$E(t) \leq 2\beta z^2(t)\frac{z(t)g'(z(t)) - g(z(t))}{z(t)g'(z(t)) + g(z(t))} \tag{1.5.139}$$

where

$$z(t) = \phi^{-1}(t/T_0). \tag{1.5.140}$$

Here, ϕ *is a strictly decreasing and onto function defined from* $(0, r_0]$ *to* $[1/H'(r_0^2), +\infty)$ *by*

$$\phi(v) = \frac{2v}{vg'(v) + g(v)} + 4\alpha(v), \tag{1.5.141}$$

where α *is defined on* $(0, r_0]$ *by the integral expression*

$$\alpha(\tau) = \int_\tau^{r_0} \frac{g(u)\left(u^2 g''(u) + ug'(u) - g(u)\right)}{(ug'(u) + g(u))^2(ug'(u) - g(u))} du. \tag{1.5.142}$$

Proof. Set $\hat{E}(t) = \frac{E(t)}{2\beta}$. Then by virtue of (1.5.138), the function $\hat{E}(t)$ satisfies (1.5.108). Further, since $E(t)$ is non-increasing, and thanks to Lemma 1.5.4 applied to $\hat{E}$, we deduce that

$$\hat{E}(t) \leq \hat{E}(0) \leq F(H'(r_0^2)) = r_0^2 - \frac{H(r_0^2)}{H'(r_0^2)} < r_0^2. \tag{1.5.143}$$

Set $\eta = r_0^2$. Then, again by Lemma 1.5.4, F is a strictly increasing onto function from $[0, +\infty)$ to $[0, \eta)$. Now define B by

$$0 < B = \frac{1}{T_0}\int_0^{+\infty} \hat{E}(\tau)F^{-1}(\hat{E}(\tau))d\tau \leq \hat{E}(0) < \eta. \tag{1.5.144}$$

We also set $r = F(H'(r_0^2))$. Then (1.5.143) and (1.5.144) show that $r \in [B, \eta)$, so we can apply Theorem 1.5.15 to $\hat{E}(t)$ with r and B defined as above. This gives us the estimate for all $t \geq \frac{T_0}{F^{-1}(r)}$,

$$\hat{E}(t) \leq F\left(\frac{1}{\psi_r^{-1}(t/T_0)}\right), \tag{1.5.145}$$

where ψ_r is the strictly increasing onto function defined from $[1/F^{-1}(r), +\infty)$ to $[1/F^{-1}(r), +\infty)$ by (1.5.107) and K_r is defined by (1.5.106). On the other hand, since F is increasing and (1.5.144) and (1.5.137) hold, we have $F^{-1}(r) = H'(r_0^2)$. Hence, $F(v)$ from Lemma 1.5.4 is given by, for all $v \in [0, F^{-1}(r)]$,

$$F(v) = (H')^{-1}(v) - \frac{H((H')^{-1}(v))}{v}.$$

We can easily check that F is differentiable. Thus, making the change of variable $v = F^{-1}(y)$ in (1.5.106),

$$K_r\left(F(1/s)\right) = \int_{1/s}^{F^{-1}(r)} \frac{F'(v)}{vF(v)}dv. \tag{1.5.146}$$

On the other hand, a straightforward computation shows that $F'(v) = \frac{H(H')^{-1}(v)}{v^2}$. Therefore, using the expression of $F(v)$ and $F'(v)$ in (1.5.146), and making the change of variable $\tau = (H')^{-1}(v)$, in the resulting equality we conclude

$$K_r\left(F(1/s)\right) = \int_{(H')^{-1}(1/s)}^{r_0^2} \frac{H(\tau)H''(\tau)}{(H'(\tau))^2(\tau H'(\tau) - H(\tau))}d\tau.$$

Replacing $H(\tau) = \sqrt{\tau}g(\sqrt{\tau})$ in the above expression, and making the change of variable $u = \sqrt{\tau}$, we obtain

$$K\left(F(1/s)\right) = 4\alpha\left(\sqrt{(H')^{-1}(1/s)}\right)$$

where

$$\alpha(\tau) = \int_{\tau}^{r_0} \frac{g(u)(u^2g''(u) + ug'(u) - g(u))}{(ug'(u) + g(u))^2(ug'(u) - g(u))}du. \tag{1.5.147}$$

Now we set for all $v \in (0, r_0]$,

$$\phi(v) = \psi_r\left(\frac{1}{H'(v^2)}\right). \tag{1.5.148}$$

Recall that ψ_r is strictly increasing and onto from

$$[1/F^{-1}(r), +\infty) \quad \text{to} \quad [1/F^{-1}(r), +\infty),$$

and $H'(v^2)$ is strictly increasing and onto from $(0, r_0]$ to $(0, H'(r_0^2)]$. Hence, ϕ is strictly decreasing and onto from $(0, r_0]$ to $[1/H'(r_0^2), +\infty)$ and ϕ is given by (1.5.141). For all $t \geq T_0/H'(r_0^2)$, we set

$$s(t) = \psi_r^{-1}(t/T_0), \quad z(t) = \sqrt{(H')^{-1}(1/s(t))}.$$

Then for all $t \geq T_0/H'(r_0^2)$,

$$\phi(z(t)) = \psi_r(s(t)) = t/T_0. \tag{1.5.149}$$

Hence, $z(t)$ satisfies (1.5.140) and $z(t) \in [0, r_0]$ for all $t \geq T_0/H'(r_0^2)$. We now need to rewrite the decay rate given in (1.5.109).

On the other hand, $H'(z^2(t)) \leq H'(r_0^2)$, for all $t \geq T_0/H'(r_0^2)$. Hence, by noting that

$$F\left(\frac{1}{\psi_r^{-1}(t/T_0)}\right) = F(H'(z^2(t))),$$

and by Lemma 1.5.4, we have

$$F(H'(z^2(t))) = z^2(t) - \frac{H(z^2(t))}{H'(z^2(t))}.$$

Using these last two relations in (1.5.145), together with the expression of H in terms of g, we can obtain

$$E(t) \leq 2\beta z^2(t)\frac{z(t)g'(z(t)) - g(z(t))}{z(t)g'(z(t)) + g(z(t))} \tag{1.5.150}$$

which thus concludes the proof. □

The following result is a generalization of a power form of a non-negative continuous function $y(t)$ (see, e.g., Caraballo, Rubin, and Valero [131]).

Theorem 1.5.21 (The Caraballo–Rubin–Valero Inequality [131]). *Let $g(t) \geq 0$ belong to $L^1(0,T)$ and $M \geq 0, 0 < \alpha \leq 2$. Moreover, let $y(t)$ be a non-negative continuous function on $[0,T]$ such that for all $t \in [0,T]$,*

$$y^2(t) \leq M^2 + 2\int_0^t g(\tau)y^\alpha(\tau)d\tau. \tag{1.5.151}$$

Then for all $t \in [0,T]$, we have

$$\begin{cases} y(t) \leq \left(M^{2-\alpha} + (2-\alpha)\displaystyle\int_0^t g(s)ds\right)^{1/(2-\alpha)}, & \text{if } \alpha < 2, \quad (1.5.152)\\ y(t) \leq M\exp\left(\displaystyle\int_0^t g(s)ds\right), & \text{if } \alpha = 2. \quad (1.5.153)\end{cases}$$

Proof. Set $U(s) = \sqrt{M^2 + 2\int_0^s g(\tau)y^\alpha(\tau)d\tau}$, which is a non-decreasing function. Differentiating $U^2(t)$, we get

$$2U(s)\frac{dU(s)}{ds} = 2g(s)y^\alpha(s) \leq 2g(s)U^\alpha(s). \tag{1.5.154}$$

Since $U(t)$ is non-decreasing, there exists a constant $0 \leq \beta \leq T$, such that $U(t) = M$ for all $t \in [0,\beta]$, and $U(t) > M$ for all $\beta \in [0,T]$. Clearly, (1.5.152)–(1.5.153) are satisfied for all $t \in [0,\beta]$. If $t > \beta$, then integrating over (β, t), we can obtain

$$\begin{cases} \dfrac{U^{2-\alpha}(t)}{2-\alpha} \leq \dfrac{M^{2-\alpha}}{2-\alpha} + \displaystyle\int_0^t g(s)ds, & \text{if } \alpha < 2, \quad (1.5.155)\\ U(t) \leq M\exp\left(\displaystyle\int_0^t g(s)ds\right), & \text{if } \alpha = 2. \quad (1.5.156)\end{cases}$$

It therefore follows that

$$\begin{cases} y(t) \leq U(t) \leq \left(M^{2-\alpha} + (2-\alpha)\displaystyle\int_0^t g(s)ds\right)^{1/(2-\alpha)} & \text{if } \alpha < 2, \quad (1.5.157)\\ y(t) \leq U(t) \leq M\exp\left(\displaystyle\int_0^t g(s)ds\right) & \text{if } \alpha = 2. \quad (1.5.158)\end{cases}$$

□

Chapter 2

Differential and Difference Inequalities

In this chapter, we establish differential and difference inequalities in analysis that play a role in applications in the subsequent chapters.

2.1 Differential inequalities leading to uniform bounds

In this section, we introduce certain differential inequalities that provide uniform bounds and play a crucial role in studying the global well-posedness of solutions, especially for the existence of a global (uniform) attractor for a semigroup or a semiflow (a semigroup or a process).

The following two results may be regarded as generalizations of the Bellman–Gronwall inequality. Since they can be derived by Theorems 1.1.1 and 1.1.2.

Theorem 2.1.1 (The Generalized Bellman–Gronwall Inequality [95, 331])**.** *Assume that $T > 0$, $f(t) \in L^1(0,T)$, $\phi(t) \in W^{1,1}(0,T)$, $f(t) \geq 0$ a.e. on $[0,T]$, $\phi(t) \geq 0$ on $[0,T]$, and for a.e. $t \in [0,T]$,*

$$\phi'(t) \leq 2f(t)\sqrt{\phi(t)}. \tag{2.1.1}$$

Then for a.e. $t \in [0,T]$, we have

$$\sqrt{\phi(t)} \leq \sqrt{\phi(0)} + \int_0^t f(s)ds \leq C(T), \tag{2.1.2}$$

where $C(T)$ is a positive constant depending on $T > 0$.

Proof. Let $h(t) = \sqrt{\phi(t)}$. It follows from (2.1.1) that

$$2h(t)h'(t) \leq 2f(t)h(t),$$

i.e.,

$$h'(t) \leq f(t). \tag{2.1.3}$$

Thus integrating (2.1.3) with respect to t yields (2.1.2). □

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The next theorem can be found in Renardy, Hrusa and Nohel [835]; it is also called Ou-Yang inequality [715].

Theorem 2.1.2 (The Ou-Yang Inequality [715]). *Let $f(t) \in L^1(0,T)$ be such that $f(t) \geq 0$ a.e. on $[0,T]$, and let $a \geq 0$ be a constant. Assume that $w(t) \in C([0,T])$ satisfies for any $t \in [0,T]$,*

$$\frac{1}{2}w^2(t) \leq \frac{a^2}{2} + \int_0^t f(s)w(s)ds. \tag{2.1.4}$$

Then for any $t \in [0,T]$, we have

$$|w(t)| \leq a + \int_0^t f(s)ds \leq C(T) \tag{2.1.5}$$

where $C(T)$ is a positive constant depending only on $T > 0$.

Proof. By (2.1.4),

$$|w(t)|^2 \leq a^2 + 2\int_0^t f(s)|w(s)|ds \equiv F(t). \tag{2.1.6}$$

Thus it follows from (2.1.5)–(2.1.6) that

$$F'(t) \leq 2f(t)|w(t)| \leq 2f(t)\sqrt{F(t)},$$

which together with Theorem 2.1.1 yields

$$\sqrt{F(t)} \leq \sqrt{F(0)} + \int_0^t f(s)ds. \tag{2.1.7}$$

Thus it follows from (2.1.6)–(2.1.7) that

$$|w(t)| \leq \sqrt{F(t)} \leq a + \int_0^t f(s)ds \leq C(T). \qquad \square$$

The following theorem can be found in Temam [915].

Theorem 2.1.3 (The Uniform Bellman–Gronwall Inequality [915]). *Assume that $g(t)$, $h(t)$ and $y(t)$ are three positive locally integrable functions on $(t_0, +\infty)$ such that $y'(t)$ is locally integrable on $(t_0, +\infty)$ and the following inequalities are satisfied for all $t \geq t_0$,*

$$\begin{cases} y'(t) \leq g(t)y(t) + h(t), & (2.1.8) \\ \displaystyle\int_t^{t+r} g(s)ds \leq a_1, \quad \int_t^{t+r} h(s)ds \leq a_2, \quad \int_t^{t+r} y(s)ds \leq a_3, & (2.1.9) \end{cases}$$

where r, a_i $(i = 1, 2, 3)$ are positive constants. Then for all $t \geq t_0$,

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right) e^{a_1}. \tag{2.1.10}$$

Proof. Assume that $t_0 \le t \le s \le t+r$. We may rewrite (2.1.8) with t replaced by s, multiply (2.1.8) by $\exp\{-\int_t^s g(\tau)d\tau\}$, and obtain the relation

$$\frac{d}{dt}\left\{y(s)\exp\left(-\int_t^s g(\tau)d\tau\right)\right\} \le h(s)\exp\left\{-\int_t^s g(\tau)d\tau\right\} \le h(s). \tag{2.1.11}$$

Then integrating (2.1.11) between $t_1 \in [t, t+r]$ and $t+r$ yields

$$\begin{aligned} y(t+r) &\le y(t_1)\exp\left(\int_{t_1}^{t+r} g(\tau)d\tau\right) + \left(\int_{t_1}^{t+r} h(s)ds\right)\exp\left(\int_{t_1}^{t+r} g(\tau)d\tau\right) \\ &\le (y(t_1)+a_2)\exp(a_1). \end{aligned} \tag{2.1.12}$$

Therefore integrating (2.1.12) with respect to t_1 between t and $t+r$, gives us precisely (2.1.10). □

The following inequality is a variant of the generalized Bellman–Gronwall inequality which is due to Ladyzhenskaya, Solonnikov and Ural'ceva [472].

Theorem 2.1.4 (The Ladyzhenskaya–Solonnikov–Ural'ceva Inequality [472])**.** *Assume that the non-negative function $y(t)$ is absolutely continuous on $[0,T]$, is equal to zero for $t=0$, and satisfies the inequality for almost all $t \in [0,T]$,*

$$y'(t) \le c(t)y(t) + F(t) \tag{2.1.13}$$

with non-negative functions $c(t)$ and $F(t)$ that are integrable over $[0,T]$.

Then for all $t \in [0,T]$,

$$\begin{cases} y(t) \le \exp\left\{\int_0^t c(\tau)d\tau\right\}\int_0^t F(\tau)d\tau \le C(T), & (2.1.14) \\ y'(t) \le c(t)\exp\left\{\int_0^t c(\tau)d\tau\right\}\int_0^t F(\tau)d\tau + F(t) & (2.1.15) \end{cases}$$

where $C(T) > 0$ is a constant depending on $T > 0$. In particular, for $c(t) = C =$ constant *and non-decreasing $F(t)$, we have for all $t > 0$,*

$$\begin{cases} y(t) \le C^{-1}F(t)(e^{Ct}-1), & (2.1.16) \\ y'(t) \le F(t)e^{Ct}. & (2.1.17) \end{cases}$$

Proof. If we rewrite (2.1.13) with t replaced by s, and multiply (2.1.13) by $\exp\left\{-\int_0^t c(s)ds\right\}$, we obtain

$$\begin{aligned} y(t) &\le \int_0^t F(\tau)\exp\left\{\int_\tau^t c(\eta)d\eta\right\}d\tau \\ &\le \int_0^t F(\tau)d\tau\exp\left\{\int_0^t c(\tau)d\tau\right\} \le C(T) \end{aligned} \tag{2.1.18}$$

which, when inserted in (2.1.13), yields (2.1.14). If $c(t) = C =$ constant and $F(t)$ is non-decreasing, then we infer from (2.1.18) that

$$\begin{aligned} y(t) &\le F(t) \int_0^t \exp\left(C(t-\tau)\right) d\tau \\ &\le C^{-1} F(t)(e^{Ct} - 1) \end{aligned}$$

which, when inserted in (2.1.13), yields (2.1.17). □

If in Theorem 1.2.2 we take $\tau = 0$ and $f(t) = w(0) + \int_0^t h(s)ds, g(t) = \alpha(t), y(t) = w(t)$, then we get the following corollary, which can be also regarded as a generalization of Theorem 2.1.4.
The next corollary can be found in [472].

Corollary 2.1.1 (The Ladyzhenskaya–Solonnikov–Ural'ceva Inequality [472])**.** *Assume that $T > 0, \alpha(t) \in L^1(0,T), h(t) \in L^1(0,T), \alpha(t), h(t), w(t) \ge 0$ a.e. on $[0,T]$. If $w(t) \in W^{1,1}(0,T)$ satisfies for a.e. $t \in [0,T]$,*

$$w'(t) \le \alpha(t)w(t) + h(t), \tag{2.1.19}$$

then for a.e. $t \in [0,T]$,

$$w(t) \le w(0) \exp\left(\int_0^t \alpha(s)ds\right) + \int_0^t \exp\left(\int_s^t \alpha(\sigma)d\sigma\right) h(s)ds. \tag{2.1.20}$$

The following theorem can be found in Babin and Vishik [54] (see also, e.g., Chepyzhov and Vishik [154]), and can be viewed as a generalization of Theorem 2.1.4.

Theorem 2.1.5 (The Babin–Vishik Inequality [54, 154])**.** *Let $y(t) \in C^1[t_0, t_1]$, $y(t) \ge 0$ for all $t \in [t_0, t_1]$, and the following inequality holds for all $t \in [t_0, t_1]$,*

$$y'(t) \le a(t)y(t) + h(t) \tag{2.1.21}$$

where $a(t), h(t) \in C^1[t_0, t_1], a(t) \ge 0, h(t) \ge 0$. Then for all $t \in [t_0, t_1]$,

$$y(t) \le \left(y(t_0) + \int_{t_0}^t h(\tau)d\tau\right) \exp\left(\int_{t_0}^t a(\tau)d\tau\right). \tag{2.1.22}$$

If for all $t \in [t_0, t_1]$, the following inequality

$$y'(t) + \gamma y(t) \le h(t), \tag{2.1.23}$$

holds with a constant $\gamma \ge 0$, then for all $t \in [t_0, t_1]$,

$$y(t) \le \int_0^t e^{-\gamma(t-\tau)} h(\tau)d\tau + y(0)e^{-\gamma t}. \tag{2.1.24}$$

In particular, if $h(t) =$ constant $= C, \gamma > 0$, then for all $t \in [t_0, t_1]$,

$$y(t) \le C\gamma^{-1}\left(1 - e^{-\gamma t}\right) + y(0)e^{-\gamma t}. \tag{2.1.25}$$

Proof. Multiplying (2.1.21) by $\exp\left\{-\int_{t_0}^t a(\tau)d\tau\right\}$, we arrive at

$$\frac{d}{dt}\left(y(t)\exp\left\{-\int_{t_0}^t a(\tau)d\tau\right\}\right) \le h(t)\exp\left\{-\int_{t_0}^t a(\tau)d\tau\right\}. \tag{2.1.26}$$

Integrating (2.1.26) over $[0,t]$ yields

$$y(t) \le \left(y(t_0) + \int_{t_0}^t h(\tau)\exp\left\{-\int_{t_0}^\tau a(s)ds\right\}d\tau\right)\exp\left\{\int_{t_0}^t a(\tau)d\tau\right\}$$

which in turn gives us (2.1.22). Similarly, multiplying (2.1.23) by $\exp(\gamma t)$ and integrating the resulting inequality, we can derive (2.1.24). Estimate (2.1.25) is a direct result of (2.1.24). □

The following Theorems 2.1.6–2.1.7 are due to Chepyzhov, Pata and Vishik [156].

Theorem 2.1.6 (The Chepyzhov–Pata–Vishik Inequality [156]).

(i) *For every $\tau \in \mathbb{R}$, every non-negative locally summable function ϕ on $\mathbb{R}_\tau \equiv [\tau, +\infty)$, and every $\beta > 0$, we have for a.e. $t \ge \tau$,*

$$\sup_{t\ge\tau}\int_\tau^t \phi(s)e^{-\beta(t-s)}ds \le \frac{1}{1-e^{-\beta}}\sup_{t\ge\tau}\int_t^{t+1}\phi(s)ds. \tag{2.1.27}$$

(ii) *Let $\xi(t), \phi_1(t)$ and $\phi_2(t)$ be non-negative locally summable functions on $\mathbb{R}_\tau$ satisfying the differential inequality, for a.e. $t \in \mathbb{R}_\tau$,*

$$\frac{d}{dt}\xi(t) + 2\beta\xi(t) \le \phi_1(t) + \phi_2(t)\xi^{1/2}(t), \tag{2.1.28}$$

with some constant $\beta > 0$. Then for a.e. $t \in \mathbb{R}_\tau$,

$$\xi(t) \le 2\xi(\tau)e^{-2\beta(t-\tau)} + 2\int_\tau^t \phi_1(s)e^{-2\beta(t-s)}ds + \left(\int_\tau^t \phi_2(s)e^{-\beta(t-s)}ds\right)^2. \tag{2.1.29}$$

Proof. (i) Writing $t - \tau = N + w$ for some non-negative integer N and some $w \in [0,1)$, we obtain

$$\int_\tau^t \phi(s)e^{-\beta(t-s)}ds \le \sum_{n=0}^{N-1} e^{-\beta n}\int_{t-n-1}^{t-n}\phi(s)ds + e^{-\beta N}\int_\tau^{\tau+w}\phi(s)ds, \tag{2.1.30}$$

where the sum vanishes for $N = 0$. Therefore,

$$\int_\tau^t \phi(s)e^{-\beta(t-s)}ds \le \sum_{n=0}^{N} e^{-\beta n}\sup_{t\ge\tau}\int_\theta^{\theta+1}\phi(s)ds \le \frac{1}{1-e^{-\beta}}\sup_{t\ge\tau}\int_\theta^{\theta+1}\phi(s)ds,$$

which gives (2.1.27).

(ii) In order to prove (2.1.29), we multiply (2.1.28) by $e^{2\beta t}$ and integrate the resulting inequality. This yields

$$\xi(t)e^{2\beta t} \le \xi(\tau)e^{2\beta\tau} + \int_\tau^t \phi_1(s)e^{2\beta s} + \int_\tau^t \phi_2(s)\xi^{1/2}(s)e^{2\beta s}ds. \tag{2.1.31}$$

Let $\psi(t) = \xi(t)e^{2\beta t}$ for any $t \in \mathbb{R}_\tau$, and let $t_0 \in [\tau, t]$ be such that $\psi(t_0) = \max_{s\in[\tau,t]} \psi(s)$. Then it follows from (2.1.31) that

$$\begin{aligned}
\psi(t_0) &\le \xi(\tau)e^{2\tau} + \int_\tau^{t_0} \phi_1(s)e^{2\beta s}ds + \int_\tau^{t_0} \phi_2(s)\xi^{1/2}(s)e^{2\beta s}ds \\
&\le \xi(\tau)e^{2\tau} + \int_\tau^{t_0} \phi_1(s)e^{2\beta s}ds + \psi^{1/2}(t_0)\int_\tau^{t_0} \phi_2(s)e^{\beta s}ds \\
&\le \xi(\tau)e^{2\tau} + \int_\tau^{t_0} \phi_1(s)e^{2\beta s}ds + \frac{1}{2}\psi(t_0) + \frac{1}{2}\left(\int_\tau^{t_0} \phi_2(s)e^{2\beta s}ds\right)^2,
\end{aligned}$$

which immediately yields (2.1.29). □

The next result can be viewed as a corollary of Theorem 2.1.6 (see, e.g., Nagasawa [657]).

Corollary 2.1.2 (The Nagasawa Inequality [657]). *Let* $\lambda(t)$ (≥ 0) *and* $\omega(t)$ *be continuous functions for which there exist positive constants* C_i $(i = 1, 2, 3, 4)$, *such that for all* $0 \le \tau \le t$,

$$C_1 e^{C_2(t-\tau)} \le \exp\left(\int_\tau^t \omega(s)ds\right) \le C_3 e^{C_4(t-\tau)}. \tag{2.1.32}$$

Define $\Lambda(t) := \int_t^{t+1} \lambda(\tau)d\tau$. *Then*

$$\begin{aligned}
C^{-1} \lim_{t\to+\infty} \inf \Lambda(t) &\le \lim_{t\to+\infty} \inf \int_0^t \exp\left(-\int_\tau^t \omega(s)ds\right)\lambda(\tau)d\tau \\
&\le \lim_{t\to+\infty} \sup \int_0^t \exp\left(-\int_\tau^t \omega(s)ds\right)\lambda(\tau)d\tau \\
&\le C \lim_{t\to+\infty} \sup \Lambda(t).
\end{aligned} \tag{2.1.33}$$

Proof. We will prove the estimate from above in (2.1.33), the estimate from below can be derived in a similar manner. In fact, using (2.1.32); we may obtain

$$\begin{aligned}
&\int_0^t \exp\left(-\int_\tau^t \omega(s)ds\right)\lambda(\tau)d\tau \\
&\quad \le \exp\left(-\int_0^t \omega(s)ds\right)\int_0^{T+1} \exp\left(\int_0^\tau \omega(s)ds\right)\lambda(\tau)d\tau
\end{aligned}$$

$$+\sum_{j=0}^{\max(0,[t-T]-1)} \int_{t-j-1}^{t-j} \exp\left(-\int_\tau^t \omega(s)ds\right) \lambda(\tau)d\tau$$
$$\leq C(T)\exp\left(-\int_0^t \omega(s)ds\right) + C^{-1}\left(\sup_{t\geq T}\Lambda(t)\right) \sum_{j=0}^{\max(0,[t-T]-1)} e^{-C_2 j}.$$

Here $[\cdot]$ is the integer part symbol. The desired estimate follows with no difficulty. □

Recently, Qin and Ren [803] generalized the result (2.1.27) in (i) of Theorem 2.1.6.

Theorem 2.1.7 (The Qin–Ren Inequality [803]). *For every $\tau \in \mathbb{R}$, assume that ϕ_0 and η are non-negative locally summable function on $\mathbb{R}_\tau \equiv [\tau, +\infty)$, η is non-increasing on $\mathbb{R}_\tau$. Then for every $\nu > 0$ we have for a.e. $t \geq \tau$,*

$$\sup_{t\geq\tau} \int_\tau^t \phi_0(s)e^{-\nu\int_s^t \eta(w)dw} ds \leq \frac{1}{1-e^{-\nu\eta(0)}} \sup_{t\geq\tau}\int_t^{t+1} \phi_0(s)ds. \tag{2.1.34}$$

Proof. Writing $t-\tau = N+\varpi$, for some non-negative integer N and some $\varpi \in [0,1)$, we have

$$\int_\tau^t \phi_0(s)e^{-\nu\int_s^t \eta(w)dw} ds$$
$$\leq e^{-\nu\int_s^{s+N}\eta(w)dw}\int_\tau^{\tau+\varpi}\phi_0(s)ds + \sum_{n=0}^{N-1} e^{-\nu\int_s^{s+n}\eta(w)dw}\int_{t-n-1}^{t-n}\phi_0(s)ds$$
$$\leq \sum_{n=0}^{N} e^{-\nu\int_s^{s+n}\eta(w)dw}\int_t^{t+1}\sup_{t\geq\tau}\phi_0(s)ds \leq \frac{1}{1-e^{-\nu\int_s^{s+1}\eta(w)dw}}\sup_{t\geq\tau}\int_t^{t+1}\phi_0(s)ds$$
$$\leq \frac{1}{1-e^{-\nu\eta(0)}}\sup_{t\geq\tau}\int_t^{t+1}\phi_0(s)ds$$

as claimed. □

The next result is thus due to Chepyzhov, Pata and Vishik [156].

Theorem 2.1.8 (The Chepyzhov–Pata–Vishik Inequality [156]). *Let $\xi(t) : \mathbb{R}_\tau \to \mathbb{R}^+$ satisfy for almost every $t \geq \tau$ the differential inequality*

$$\frac{d}{dt}\xi(t) + \phi_1(t)\xi(t) \leq \phi_2(t), \tag{2.1.35}$$

where for almost every $t \geq \tau$, the scalar functions ϕ_1 and ϕ_2 satisfy

$$\int_\tau^t \phi_1(s)ds \geq \beta(t-\tau)-\gamma, \quad \int_t^{t+1}\phi_2(s)ds \leq M, \tag{2.1.36}$$

for some constants $\beta > 0, \gamma \geq 0$ *and* $M \geq 0$. *Then for almost all* $t \geq \tau$,

$$\xi(t) \leq e^{\gamma}\xi(\tau)e^{-\beta(t-\tau)} + \frac{Me^{\gamma}}{1-e^{-\beta}}. \tag{2.1.37}$$

Proof. Indeed, fix $t > \tau$ and define for all $s \in [\tau, t]$,

$$w(s) := \int_s^t \phi_1(y)dy \geq \beta(t-s) - \gamma.$$

Multiplying (2.1.35) by $\exp\left(\int_\tau^t \phi_1(s)ds\right)$ and integrating over $[\tau, t]$, we obtain

$$\begin{aligned}\xi(t) &\leq \xi(\tau)e^{-w(\tau)} + \int_\tau^t e^{-w(s)}\phi_2(s)ds \\ &\leq e^{\gamma}\xi(\tau)e^{-\beta(t-\tau)} + e^{\gamma}\int_\tau^t e^{-\beta(t-s)}\phi_2(s)ds.\end{aligned} \tag{2.1.38}$$

From (2.1.36) and (2.1.38) it follows that

$$\int_\tau^t e^{-\beta(t-s)}\phi_2(s)ds \leq \frac{M}{1-e^{-\beta}},$$

which combined with (2.1.38) completes the proof. □

The following lemma can be found in Belleri and Pata [92].

Theorem 2.1.9 (The Belleri–Pata Inequality [92]). *Let* X *be a Banach space, and let* $\mathcal{C} \subset C([0,1); X)$, *the space of* X*-valued functions on* $[0,1)$. *Let* $\Phi : X \to [0, +\infty)$ *be a mapping such that* $\Phi(v(0)) \leq c$, *for some* $c \geq 0$ *and every* $v \in \mathcal{C}$. *In addition, assume that for every* $v \in \mathcal{C}$ *the function* $t \mapsto \Phi(v(t))$ *is continuously differentiable and satisfies the differential inequality*

$$\frac{d}{dt}\Phi(v(t)) + k||v(t)||_X^2 \leq \omega \tag{2.1.39}$$

for some constants $\omega > 0$ *and* $k \geq 0$ *independent of* $v \in \mathcal{C}$. *Then for every* $\delta > 0$ *there is a* $t_\delta > 0$, *such that for all* $t \geq t_\delta$,

$$\Phi(v(t)) \leq \sup_{v \in X}\left\{\Phi(v) : \ k||v||_X^2 \leq \omega + \delta\right\}.$$

Proof. First, we can verify that the following inequality (for a fixed t),

$$\frac{d}{dt}\Phi(v(t)) \geq -\delta \tag{2.1.40}$$

yields

$$\Phi(v(t)) \leq \sup_{v \in X}\left\{\Phi(v) : \ k||v||^2 \leq \omega + \delta\right\}. \tag{2.1.41}$$

Indeed, if (2.1.40) holds, then from (2.1.39) it follows readily that $k||v||^2 \le \omega + \delta$. Thus setting $t_\delta = c/\delta$ and choosing $v \in \mathcal{C}$, we know that there is a $t_0 \in [0, t_\delta]$ depending on v, such that (2.1.40) holds for $t = t_0$. If not, we would have

$$\Phi(v(t_\delta)) \le -\delta t_\delta + \Phi(v(0)) \le -\delta t_\delta + c = 0,$$

contradicting the positivity of Φ. Now define

$$t^* = \sup\left\{\tau > t_0 : \text{ (2.1.41) holds for all } t \in [t_0, \tau]\right\}.$$

We shall show that $t^* = +\infty$ and that (2.1.41) holds for every $t \ge t_\delta$, independently of $v \in \mathcal{C}$. Indeed, if $t^* < +\infty$, then there is a sequence $t_n \downarrow t^*$ such that

$$\Phi(v(t_n)) - \Phi(v(t^*)) > 0,$$

whence

$$\frac{d}{dt}\Phi(v(t^*)) \ge 0.$$

By the continuity of the derivative, it follows that there exists a right neighborhood J of t^* such that (2.1.40) holds for every $t \in J$. Hence (2.1.41) holds for every $t \in J$, which contradicts to the maximality of t^*. □

Pata and Zelik (see, e.g., [735, 78]) used the following result, which can be regarded as a corollary of Theorem 2.1.9, to show the existence and regularity of solutions of 2D wave equations with a nonlinear damping.

Corollary 2.1.3 (The Pata–Zelik Inequality [735, 78]). *Let $E : \mathcal{H}_0 \to \mathbb{R}$ satisfy, for all $\zeta \in \mathcal{H}_0$,*

$$\beta\|\zeta\|_{\mathcal{H}_0} - m \le E(\zeta) \le Q(\|\zeta\|_{\mathcal{H}_0}) + m,$$

for some constants $\beta > 0$ and $m \ge 0$, where Q is a generic positive increasing function and $\mathcal{H}_0$ is a Banach space. Let $\xi \in C([0, +\infty), \mathcal{H}_0)$ be given. Assume that the mapping $t \mapsto E(\xi(t))$ is continuously differentiable and fulfills the differential inequality

$$\frac{d}{dt}E(\xi(t)) + \lambda\|\xi(t)\|_{\mathcal{H}_0} \le k,$$

for some constants $\lambda > 0$ and $k > 0$. Then for all $t \ge t_0$,

$$\|\xi(t)\|_{\mathcal{H}_0} \le Q(k + m + \beta^{-1}),$$

where $t_0 = Q(\|\xi(0)\|_{\mathcal{H}_0}) + Q(k)$.

From Corollary 2.1.3 and the usual Bellman–Gronwall inequality we easily derive the following corollary.

Corollary 2.1.4 (The Pata–Zelik Inequality [735, 78]). *Let $\Lambda : [0, +\infty) \to [0, +\infty)$ be an absolutely continuous function satisfying*

$$\frac{d}{dt}\Lambda(t) + 2\lambda\Lambda(t) \le h(t)\Lambda(t) + k$$

where $\lambda > 0$, $k \ge 0$ are constants and $\int_s^t h(\tau)d\tau \le \lambda(t-s) + m$, for all $t \ge s \ge 0$ and some constant $m \ge 0$. Then we have for all $t \ge t_0$,

$$\Lambda(t) \le \Lambda(0)e^m e^{-\lambda t} + \frac{\lambda e^m}{\lambda}.$$

The next result is a Bellman–Gronwall type lemma (see, e.g., [327]).

Theorem 2.1.10 (The Grasselli–Pata Inequality [327]). *Let Φ be an absolutely continuous positive function on $[0, +\infty)$, which satisfies for some constant $\varepsilon > 0$, the differential inequality for almost every $t \in [0, +\infty)$,*

$$\frac{d}{dt}\Phi(t) + 2\varepsilon\Phi(t) \le f(t)\Phi(t) + h(t) \tag{2.1.42}$$

where f and h are functions on $[0, +\infty)$, such that for almost all $t \ge \tau$,

$$\int_\tau^t |f(y)|dy \le \alpha\Big(1 + (t-\tau)^\omega\Big), \quad \sup_{t\ge 0}\int_t^{t+1} |h(y)|dy \le \beta \tag{2.1.43}$$

with some constants α, $\beta > 0$ and $\omega \in [0, 1)$. Then for every $t \in [0, +\infty)$,

$$\Phi(t) \le \gamma\Phi(0)e^{-\varepsilon t} + K \tag{2.1.44}$$

with some constants $\gamma = \gamma(f) \ge 1$ and $K = K(\varepsilon, f, h) \ge 0$.

Proof. Let us first recall the following general fact (see, e.g., [732], or recall (2.1.27)). For any function h on $[0, +\infty)$ satisfying

$$\sup_{t\ge 0}\int_t^{t+1} |h(y)|dy \le \beta,$$

it follows from (2.1.43) that for any $\varepsilon > 0$,

$$\int_0^{+\infty} e^{-\varepsilon(t-y)}|h(y)|dy \le \beta C(\varepsilon),$$

where $C(\varepsilon) = \frac{e^\varepsilon}{1-e^{-\varepsilon}}$. Now set $m(t) = -2\varepsilon + f(t)$, then for $0 \le \tau < t$, we have

$$\int_\tau^t m(y)dy = -2\varepsilon(t-\tau) + \int_\tau^t f(y)dy \le -\varepsilon(t-\tau) + \ln\gamma$$

with some constant $\gamma = \gamma(\omega, \alpha) \geq 1$. Hence, by the generalized Jones inequality (i.e., Theorem 1.2.3), we conclude from (2.1.42) that

$$\Phi(t) \leq \Phi(0)e^{\int_0^t m(s)ds} + \int_0^t e^{\int_s^t m(y)dy}|h(s)|ds \leq \gamma\Phi(0)e^{-\varepsilon t} + \gamma\beta C(\varepsilon),$$

which yields (2.1.44). □

Note that $h \equiv 0$ implies exponential decay.

Remark 2.1.1. In fact, Theorem 2.1.10 can be generalized (see Lemma A.1, [732]) allowing the right-hand side of the above differential inequality to have an extra term of the form $\tilde{h}(t)\Phi^\sigma(t)$, with $\sigma \in [0, 1)$ and $\sup_{t\geq 0} \int_t^{t+1} |\tilde{h}(y)|dy \leq \tilde{\beta}$, for some constant $\tilde{\beta} \geq 0$. Then the conclusion of Theorem 2.1.10 still holds for $\gamma = \gamma(f) \geq \frac{1}{1-\sigma}$ and $K = K(\varepsilon, f, h, \tilde{h}) \geq 0$.

Remark 2.1.2. Notice that the condition $\int_\tau^t |h(y)|dy \leq \alpha(1 + (t - \tau)^\omega)$, for some constants $\alpha \geq 0$ and $\omega \in [0, 1)$, implies that $\sup_{t\geq 0} \int_t^{t+1} |h(y)|dy < +\infty$.

The next result summarizes Lemma A.5 in [117] and some results in [732], whose proof will be left to the reader.

Theorem 2.1.11 (The Brézis–Pata–Prouse–Vishik Inequality [117, 732])**.** *Let* Φ, r_1, r_2 *be non-negative, locally summable functions on* $[\tau, +\infty), \tau \in \mathbb{R}$, *which satisfy, for some* $\varepsilon > 0$ *and* $0 < \sigma < 1$, *the differential inequality for a.e.* $t \in [\tau, +\infty)$,

$$\frac{d}{dt}\Phi(t) + \varepsilon\Phi(t) \leq r_2(t)\Phi^{1-\sigma}(t) + r_1(t). \tag{2.1.45}$$

Assume also that, for $j = 1, 2$,

$$m_j = \sup_{t\geq\tau} \int_r^{r+1} r_j(y)dy < +\infty.$$

Then, setting $C(\nu) = e^\nu/(1 - e^\nu)$, *for any* $t \in [\tau, +\infty)$,

$$\Phi(t) \leq \frac{1}{\sigma}\Phi(\tau)e^{-\varepsilon(t-\tau)} + \frac{1}{\sigma}m_1C(\varepsilon) + [m_2C(\varepsilon\sigma)]^{1/\sigma}. \tag{2.1.46}$$

Finally, we introduce the uniform Bellman–Gronwall inequality (cf. Theorem 2.1.3) for $r = 1$, which can be regarded as a corollary of Theorem 2.1.11 with $\sigma = 0$.

Corollary 2.1.5 (The Pata–Prouse–Vishik Inequality [732])**.** *Let* $\Phi(t)$ *be an absolutely continuous positive function on* $[0, +\infty)$ *satisfying the differential inequality for almost every* $t \in [0, +\infty)$,

$$\frac{d}{dt}\Phi(t) \leq f_1(t)\Phi(t) + f_2(t),$$

where f_1 *and* f_2 *satisfy* $\sup_{t\geq 0}\int_t^{t+1}|f_j(y)|dy \leq \alpha_j$, *for some constants* $\alpha_j \geq 0$ $(j=1,2)$. *Assume, in addition, that* $\sup_{t\geq 0}\int_t^{t+1}\Phi(y)dy \leq \alpha_3$, *for some constant* $\alpha_3 \geq 0$. *Then for almost all* $t \in [0,+\infty)$,

$$\Phi(t+1) \leq (\alpha_2+\alpha_3)e^{\alpha_1}.$$

The following result may be found in Temam [915]. It is due to Ghidaglia and furnishes a uniform bound for large time.

Theorem 2.1.12 (The Ghidaglia Inequality [915])**.** *Let* $y(t)$ *be a positive absolutely continuous function on* $(0,+\infty)$ *satisfying*

$$y'(t) + \gamma y^p(t) \leq \delta \tag{2.1.47}$$

for constants $p > 1, \gamma > 0, \delta > 0$. *Then for any* $t > 0$,

$$y(t) \leq (\delta/\gamma)^{1/p} + (\gamma(p-1)t)^{-1/(p-1)}. \tag{2.1.48}$$

Proof. If $y(0) \leq \left(\delta/\gamma\right)^{1/p}$, then for all $t \geq 0$,

$$y(t) \leq (\delta/\gamma)^{1/p}. \tag{2.1.49}$$

If $y(0) > (\delta/\gamma)^{1/p}$, then there is a $t_0 \in (0,+\infty)$ such that

$$\begin{cases} y(t) \geq (\delta/\gamma)^{1/p}, \quad \text{for all} \quad 0 \leq t < t_0, & (2.1.50)\\ y(t) \leq (\delta/\gamma)^{1/p}, \quad \text{for all} \quad t \geq t_0. & (2.1.51)\end{cases}$$

For all $t \in [0,t_0]$, we write $z(t) = y(t) - (\delta/\gamma)^{1/p} \geq 0$. Since $a^p + b^p \leq (a+b)^p$ for all $a, b \geq 0, p > 1$, we have

$$y^p(t) = [z(t) + (\delta/\gamma)^{1/p}]^p \geq z^p(t) + \delta/\gamma. \tag{2.1.52}$$

From (2.1.47) and (2.1.52) it follows that

$$z'(t) + \gamma z^p(t) \leq y'(t) + \gamma\Big(y^p(t) - \delta/\gamma\Big) \leq 0. \tag{2.1.53}$$

Integrating (2.1.53) over $[0,t]$ yields

$$z^{p-1}(t) \leq \frac{1}{z(0)^{1-p} + \gamma(p-1)t} \leq \frac{1}{\gamma(p-1)t},$$

which implies (2.1.48) for all $t \in [0,t_0)$ and, since this inequality is obvious for all $t \geq t_0$ from (2.1.51), the proof is complete. □

The following result is due to Zlotnik [1024].

Theorem 2.1.13 (The Zlotnik Inequality [1024]). *Let $N_0 \geq 0, N_1 \geq 0$, and $\epsilon_0 > 0$ be three parameters. Let $f \in C(\mathbb{R})$ and $y, b \in W^{1,1}(0,T)$, for any $T > 0$. Then we have the following estimates:*

(1) *if for all $t > 0$,*

$$y'(t) \geq f(y(t)) + b'(t), \tag{2.1.54}$$

where $f(-\infty) = +\infty$ and $b(t_2) - b(t_1) \geq -N_0 - N_1(t_2 - t_1)$ for any $0 \leq t_1 \leq t_2$, then the uniform lower bound for all $t \geq 0$,

$$\min\{y(0), \tilde{z}\} - N_0 \leq y(t), \tag{2.1.55}$$

holds, where $\tilde{z} = \tilde{z}(N_1)$ is such that $f(z) \geq N_1$ for all $z \leq \tilde{z}$;

(2) *if for all $t > 0$,*

$$y'(t) \leq f(y(t)) + b'(t), \tag{2.1.56}$$

where $\limsup_{z \to +\infty} f(z) \leq 0$ and $b(t_2) - b(t_1) \leq N_0 - \epsilon_0(t_2 - t_1)$ for any $0 \leq t_1 \leq t_2$, then the uniform upper bound for all $t \geq 0$,

$$y(t) \leq \max\left\{y(0), \hat{z}\right\} + N_0 \tag{2.1.57}$$

holds, where $\hat{z} = \hat{z}(\epsilon_0)$ is such that $f(z) \leq \epsilon_0$ for all $z \geq \hat{z}$.

(3) *if $f(+\infty) = -\infty$ and $b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)$ for all $0 \leq t_1 \leq t_2$, then we have an estimate of the form* (above formula) *with a quantity $\tilde{z} = \tilde{z}(N_1)$ such that $f(z) \leq -N_1$ for all $z \geq \tilde{z}$.*

Proof. (1) It obviously suffices to verify inequality (2.1.55) for points $t = t_2$ such that $y(t_2) < \tilde{y}^0 = \min\{y(0), \tilde{z}\}$. By the continuity of y on $\overline{\mathbb{R}^+}$ and the initial condition $y(0) = y^0 \geq \tilde{y}^0$, for any such point (if any), there exists a point $t_1 \in [0, t_2)$ such that $y(t) < \tilde{y}^0$ for $t_1 < t \leq t_2$ and $y(t_1) = \tilde{y}^0$. Integrating the differential inequality (2.1.54) over (t_1, t_2) and taking into account the choice of the points t_1 and t_2 and the assumption on b, we obtain

$$y(t_2) \geq y(t_1) + \int_{t_1}^{t_2} f(y(t))dt + b(t_2) - b(t_1) \geq \tilde{y}^0 - N_0, \tag{2.1.58}$$

because $f(y(t)) \geq N_1$ on $[t_1, t_2]$. This completes the proof of (2.1.55).

(2) Similarly as in (1), it suffices to verify (2.1.57) for points $t = t_2$ such that $y(t_2) > \hat{y}^0 = \max\{y(0), \hat{z}\}$. For any such point, there exists a $t_1 \in [0, t_2)$ such that $y(t) > \hat{y}^0$ for $t_1 < t \leq t_2$ and $y(t_1) = \hat{y}^0$. Therefore, we have

$$y(t_2) \leq y(t_1) + \int_{t_1}^{t_2} f(y(t))dt + b(t_2) - b(t_1) \leq \hat{y}^0 + N_0,$$

because $f(y(t)) \leq \varepsilon_0$ on $[t_1, t_2]$, which completes the proof of (2.1.57).

(3) This case can be reduced to case (1), since $z'(t) = -f(-z) + (-b)'$ for $z = -y$. □

Remark 2.1.3. In Theorem 2.1.13, we may drop the conditions $f(-\infty) = +\infty$ and $\limsup_{z\to+\infty} f(z) \le 0$, take $f \in C(\mathbb{R} \times \overline{\mathbb{R}^+})$, and replace $f(y)$ by $f(y,t)$. The claim (1) remains valid provided that, for a fixed N_1, there exists a point $\tilde{z}$ such that $f(z,t) \ge N_1$, for all $z \ge \tilde{z}$ and all $t \ge 0$. Similarly, claim (2) remains valid provided that, for a fixed $\varepsilon_0 \in \mathbb{R}$, there exists a point $\hat{z}$ such that $f(z,t) \le \varepsilon_0$, for all $z \ge \hat{z}$ and all $t \ge 0$.

Remark 2.1.4. Theorem 2.1.13 is chosen from Zlotnik [1024], where both claims are concerned with differential equations are used, but the proof remains valid for inequalities, and so is Remark 2.1.3.

The next result is due to Galdi [295].

Theorem 2.1.14 (The Galdi Inequality [295]). *Let $y(t) \in C^1(\overline{\mathbb{R}^+})$ be a non-negative function satisfying the inequality for all $t \ge 0$,*

$$ay(t) \le b + y'(t), \tag{2.1.59}$$

where $a > 0, b \ge 0$. If

$$\liminf_{t\to+\infty} y(t)e^{-at} = 0, \tag{2.1.60}$$

then $y(t)$ is uniformly bounded and

$$\sup_{t\ge0} y(t) \le \frac{b}{a}. \tag{2.1.61}$$

Proof. From (2.1.59) it follows that

$$-\frac{d}{dt}\left(y(t)e^{-at}\right) \le be^{-at},$$

which, once integrated from t to t_1 ($t_1 > t$), gives

$$-y(t_1)e^{-at_1} + y(t)e^{-at} \le \frac{b}{a}\left(e^{-at} - e^{-at_1}\right). \tag{2.1.62}$$

If we take the inferior limit of both sides of (2.1.62) as $t_1 \to +\infty$ and use (2.1.60), then we obtain (2.1.61). The proof is complete. □

We shall next give a result due to Dlotko [213] which is based on a system of differential inequalities.

Theorem 2.1.15 (The Dlotko Inequality [213]). *Let $y(t), z(t) : [0, +\infty) \longrightarrow [0, +\infty)$, y of class C^1, z of class C^0, satisfy the following system of inequalities*

$$\begin{cases} y'(t) \le -\alpha z(t) + \beta\big(1 + z^\sigma(t)\big), \quad \sigma \in [0,1), & (2.1.63)\\ y(t) \le Cz(t). & (2.1.64)\end{cases}$$

Then for all $t \geq 0$,

$$\begin{cases} y(t) \leq \max\{y(0), Cz_1\}, & (2.1.65) \\ \overline{\lim\sup}_{t\to+\infty} y(t) \leq Cz_1 & (2.1.66) \end{cases}$$

where $C, \alpha, \beta > 0$ *are constants, and* z_1 *is a positive root of the equation* $-\alpha r + \beta(1 + r^\sigma) = 0$.

Proof. Let $\phi(s) = -\alpha s + \beta(1 + s^\sigma)$ and for all $s \geq 0$ define a new function $\phi_1(s)$ as

$$\phi_1(s) = \begin{cases} \phi(s_0), & \text{for } 0 \leq s \leq s_0, \\ \phi(s), & \text{for } s_0 < s \end{cases}$$

where $s_0 > 0$ is the point of maximum of ϕ. Evidently $\phi(s) \leq \phi_1(s), s \geq 0$, and $\phi_1(s)$ is non-increasing. Then by (2.1.63), it follows

$$y'(t) \leq \phi_1(z(t)) \leq \phi_1(C^{-1}y(t)), \tag{2.1.67}$$

and so $y'(t) \leq 0$ whenever $C^{-1}y(t) \geq z_1^{-1}$. This proves both estimates (2.1.65) and (2.1.66). □

A differential form of Corollary 1.4.1 can be stated as follows:

Theorem 2.1.16. *Assume that* $v \in C^1([0, +\infty)), v(0) = 0$, *and there is a positive constant* C *such that for all* $x \in [0, +\infty)$,

$$|v'(x)| \leq C|v(x)|. \tag{2.1.68}$$

Then for all $x \in [0, +\infty)$,

$$v(x) \equiv 0. \tag{2.1.69}$$

Proof. Indeed, (2.1.68) implies

$$|v(x)| \leq C \int_0^x |v(y)|dy, \tag{2.1.70}$$

because $v(0) = 0$. Thus (2.1.69) follows from (2.1.70) by the Bellman–Gronwall inequality, i.e., Theorem 1.1.2. □

In 1992, Ohara [703] proved the following result.

Theorem 2.1.17 (The Ohara Inequality [703])**.** *Let* $y(t)$ *be a non-negative* C^1 *function on* $(0, T]$ *satisfying for all* $t \in (0, T]$,

$$y'(t) \leq Ct^{-1-\delta} + By(t) - At^{\lambda\theta-1}y^{\theta+1}(t), \tag{2.1.71}$$

with some constants $A > 0, B \geq 0, C \geq 0, \lambda > 0, \theta > 0$ *and* $\delta \in \mathbb{R}$. *Assume that* $\lambda\theta \geq 1$ *and* $\lambda > \delta$. *Then for all* $t \in (0, T]$,

$$y(t) \leq \left\{ \left(\frac{2\lambda + 2BT}{A} \right)^{1/\theta} + \frac{2Ct^{\lambda-\delta}}{\lambda + BT} \right\} t^{-\lambda}. \tag{2.1.72}$$

Proof. Without loss of generality, we may assume that $y(t) > 0$. Firstly, we consider the case $C = 0$.

(i) Setting $\mu(t) = y(t)^{-\theta}$, we infer from (2.1.71) that

$$\mu'(t) \geq -B\theta\mu(t) + A\theta t^{\lambda\theta-1}. \tag{2.1.73}$$

Solving the differential inequality (2.1.73), we obtain for all $t \in (0, T]$,

$$\begin{aligned} \mu(t)\exp\{B\theta t\} &\geq A\theta \int_0^t s^{\lambda\theta-1}\exp(B\theta s)ds \\ &\geq \frac{A}{\lambda+BT} t^{\lambda\theta}\exp(B\theta t), \end{aligned} \tag{2.1.74}$$

which gives us (2.1.72).

(ii) Next, we assume $C > 0$. Note that for any $a \geq 0, b \geq 0$ and $k > 1$,

$$(a+b)^{\theta+1} \leq k^\theta \left(a^{\theta+1} + (k-1)^{-\theta} b^{\theta+1}\right). \tag{2.1.75}$$

Then, setting $\phi(t) = y(t) + \nu t^{-\lambda}$, we obtain from (2.1.71) and (2.1.75)

$$\phi'(t) - B\phi(t) + Ak^{-\theta}t^{\lambda\theta-1}\phi^{\theta+1}(t) \leq t^{-\lambda-1}[Ct^{\lambda-\delta} - \nu(\lambda + Bt - A(k-1)^{-\theta}\nu^\theta)]. \tag{2.1.76}$$

Setting $k = 2^{1/\theta}$ and fixing a sufficiently small positive constant ν, we conclude from (2.1.76) for all $t \in (0, T_1]$ that

$$\phi'(t) \leq B\phi(t) - (A/2)t^{\lambda\theta-1}\phi^{\theta+1}(t), \tag{2.1.77}$$

for some positive constant T_1 ($\leq T$).

In view of (i), (2.1.77) implies that for all $t \in (0, T_1]$,

$$y(t) \leq \phi(t) \leq \left(\frac{2\lambda+2BT}{A}\right)^{1/\theta} t^{-\lambda}. \tag{2.1.78}$$

(iii) Assume that there exists an $s \in (T_1, T)$ (under the assumption $T_1 < T$, otherwise for $T_1 = T$, our assertion is valid) such that

$$y(s) > \left\{\left(\frac{2\lambda+2BT}{A}\right)^{1/\theta} + \frac{2Cs^{\lambda-\delta}}{\lambda+BT}\right\} s^{-\lambda}. \tag{2.1.79}$$

Set

$$t_0 = \inf\left\{t \in (T_1, T] : y(t)t^\lambda > \left(\frac{2\lambda+2BT}{A}\right)^{1/\theta} + \frac{2Ct^{\lambda-\delta}}{\lambda+BT}\right\}.$$

Then we have

$$y(t_0) = \left\{\left(\frac{2\lambda+2BT}{A}\right)^{1/\theta} + \frac{2Ct_0^{\lambda-\delta}}{\lambda+BT}\right\} t_0^{-\lambda},$$

whence

$$y(t)t^{\lambda} > \left(\frac{3\lambda + 3BT}{2A}\right)^{1/\theta} + \frac{2Ct^{\lambda-\delta}}{\lambda + BT} \tag{2.1.80}$$

for any $t \in [t_0, T_2]$, with a constant T_2 $(> t_0)$.

From (2.1.80) and (2.1.71), we obtain that for any $t \in [t_0, T_2]$,

$$y'(t) < \frac{1}{2}(\lambda + BT)t^{-1}y(t) + By(t) - \frac{3}{2}(\lambda + BT)t^{-1}y(t) < -\lambda t^{-1}y(t). \tag{2.1.81}$$

Solving the inequality (2.1.81), we obtain for all $t \in (t_0, T_2]$,

$$\begin{aligned} y(t)t^{\lambda} < y(t_0)t_0^{\lambda} &= \left(\frac{2\lambda + 2BT}{A}\right)^{1/\theta} + \frac{2Ct_0^{\lambda-\delta}}{\lambda + BT} \\ &< \left(\frac{2\lambda + 2BT}{A}\right)^{1/\theta} + \frac{2Ct^{\lambda-\delta}}{\lambda + BT}, \end{aligned}$$

which contradicts the definition of t_0. From (ii) and (iii), we arrive at the desired result (2.1.72). □

Chen [140] extended the above result (the proof is left to the reader as an exercise).

Theorem 2.1.18 (The Chen Inequality [140])**.** *Let $y(t)$ be a non-negative C^1 function on $(0, T]$ satisfying for all $t \in (0, T]$,*

$$y'(t) + At^{\lambda\theta-1}y^{\theta+1}(t) \le Ct^{\delta} + Bt^{-k}y(t) \tag{2.1.82}$$

with some constants $A, \theta > 0, \lambda\theta \ge 1, B, C \ge 0, k \le 1$ and $\delta \in \mathbb{R}$. Then for all $t \in (0, T]$,

$$y(t) \le \left(\frac{2\lambda + 2BT^{1-k}}{A}\right)^{1/\theta} t^{-\lambda} + 2C(\lambda + BT^{1-k})^{-1}t^{-1-\delta}. \tag{2.1.83}$$

Giga and Kohn [312] proved the following result.

Theorem 2.1.19 (The Giga–Kohn Inequality [312])**.** *Let $g(s)$ be a non-negative H^1 function defined for $s \ge s_0$. Assume that $g'(s)$ satisfies*

$$\begin{cases} C_1 g^{p+1}(s) \le g(s)g'(s) + C_2 A, & (2.1.84) \\ \displaystyle\int_{s_0}^{+\infty} (g'(s))^2 ds \le A, & (2.1.85) \end{cases}$$

for some positive constants C_1, C_2 and $A \le 1$. Then there is a constant $C > 0$ only depending on C_1, C_2 and p such that for all $s \ge s_0$,

$$g(s) \le CA^{1/(2p)}. \tag{2.1.86}$$

Proof. From (2.1.84) we see that, for almost all s, either

$$g(s) \le A^{1/(2p)} \quad \text{or} \quad C_1 g^p(s) \le g'(s) + C_2 A^{1-/(2p)}.$$

Since $A \le 1$ and $p > 1$, it follows from (2.1.85) that

$$\int_s^{s+1} g^{2p}(\tau)d\tau \le A + 2C_1^{-2} \int_s^{s+1} \Big((g'(\tau))^2 + C_2^2 A^{2-1/p} \Big) d\tau \le C_3 A, \quad (2.1.87)$$

with $C_3 > 0$ only depending on C_1 and C_2. Applying the interpolation inequality

$$\sup_{s \le \tau \le s+1} g(\tau) \le C(p) \left[\int_s^{s+1} \left[(g'(\tau))^2 + g^2(\tau) \right] d\tau \right]^{\theta/2} \left[\int_s^{s+1} g^{2p}(\tau) d\tau \right]^{(1-\theta)/(2p)} \quad (2.1.88)$$

for $\theta = 1/(p+1)$, we conclude from (2.1.87)–(2.1.88), and Hölder's inequality that for all $s \ge s_0$,

$$g(s) \le C\Big(A^{1/2} + A^{1/(2p)}\Big)^{\theta} A^{(1-\theta)/(2p)} \le C A^{1/(2p)},$$

which is the estimate (2.1.86). □

In 2009, Yang and Jin [976] proved the following result, which leads to a uniform bound, e.g., the existence of an absorbing set.

Theorem 2.1.20 (The Yang–Jin Inequality [976])**.** *Let $z(t)$ be a non-negative absolutely continuous function on $[0, +\infty)$ satisfying for a.e. $t > 0$ the differential inequality for all $\varepsilon \in (0, \varepsilon_0]$,*

$$z'(t) + \varepsilon \rho z(t) \le k + C\varepsilon^{2q} z^q(t), \quad (2.1.89)$$

where $\varepsilon_0 = (2\rho/3k)(k/4C)^{1/q}, k, C, \rho > 0$, and $q > 1$ are constants. Then for $t \ge T(z_0)$,

$$z(t) \le R_2 := \frac{k}{2\varepsilon_0 \rho}, \quad (2.1.90)$$

where $T(z_0) > 0$ is a constant depending on $z_0 = z(0)$.

Proof. By our assumptions, we have for all $\varepsilon \in (0, \varepsilon_0]$,

$$\frac{3\varepsilon k}{2\rho} \le \frac{3\varepsilon_0 k}{2\rho} = \left(\frac{k}{4C} \right)^{1/q}. \quad (2.1.91)$$

We distinguish the following three cases:

(1) If

$$z_0 \le \frac{1}{\varepsilon_0^2} \left(\frac{k}{4C} \right)^{1/q} := R_1, \quad (2.1.92)$$

then there exists a constant $T > 0$ such that, for any $t \in [0, T]$

$$z(t) \leq \frac{1}{\varepsilon_0^2} \left(\frac{k}{2C} \right)^{1/q}. \tag{2.1.93}$$

It can be either

$$T < +\infty, \quad C\varepsilon_0^{2q} z^q(T) = k/2 \tag{2.1.94}$$

or

$$T = +\infty. \tag{2.1.95}$$

Now we claim that (2.1.94) cannot happen; this means that (2.1.95) holds. Suppose (2.1.94) holds true. Then it follows from (2.1.89) and (2.1.93) that for all $t \in [0, T]$,

$$\begin{cases} z'(t) + \varepsilon_0 \rho z(t) \leq 3k/2, \ \ 0 \leq t \leq T, & (2.1.96) \\ z^q(t) \leq \left(z_0 e^{-\varepsilon_0 \rho t} + \dfrac{3k}{2\varepsilon\rho}(1 - e^{-\varepsilon_0 \rho t}) \right)^q & \\ \qquad \leq z_0^q e^{-\varepsilon_0 \rho t} + \left(\dfrac{3k}{2\varepsilon_0 \rho} \right)^q (1 - e^{-\varepsilon_0 \rho t}), & (2.1.97) \end{cases}$$

where we have used the fact that $f(x) = x^q$ $(x > 0, q > 1)$ is a convex function, i.e.,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta) f(y)$$

for all $x > 0, y > 0, 0 \leq \theta \leq 1$. By (2.1.91) and (2.1.92), we have for all $t \in [0, T]$,

$$C\varepsilon_0^{2q} z^q(t) \leq C\varepsilon_0^{2q} z_0^q e^{-\varepsilon_0 \rho t} + C \left(\frac{3\varepsilon_0 k}{2\rho} \right)^q (1 - e^{-\varepsilon_0 \rho t}) \leq \frac{k}{4} e^{-\varepsilon_0 \rho t} + \frac{k}{4}(1 - e^{-\varepsilon_0 \rho t}) = \frac{k}{4}, \tag{2.1.98}$$

which contradicts (2.1.94). Hence $T = +\infty$, and for all $0 \leq t < +\infty$, $0 < \varepsilon \leq \varepsilon_0$,

$$z(t) \leq z_0 e^{-\varepsilon \rho t} + \frac{3k}{2\varepsilon\rho}(1 - e^{-\varepsilon \rho t}). \tag{2.1.99}$$

(2) If $z_0 > R_1$, we set $\varepsilon_1 = \sqrt{\frac{R_1}{z_0}} \varepsilon_0$ $(< \varepsilon_0)$, that is,

$$z_0 = \frac{1}{\varepsilon_1^2} \left(\frac{k}{4C} \right)^{1/q}. \tag{2.1.100}$$

Repeating the same argument as in case (1), we have for all $0 \leq t < +\infty$,

$$z(t) \leq z_0 e^{-\varepsilon_1 \rho t} + \frac{3k}{2\varepsilon_1 \rho} \left(1 - e^{-\varepsilon_1 \rho t} \right) \leq z_0, \tag{2.1.101}$$

where we have used the fact

$$\frac{3k}{2\varepsilon_1 \rho} = \frac{\varepsilon_0}{\varepsilon} \frac{3\varepsilon_0 k}{2\rho} \frac{1}{\varepsilon_0^2} = \sqrt{\frac{z_0}{R_1}} \frac{1}{\varepsilon_0^2} \left(\frac{k}{4C} \right)^{1/q} = \sqrt{z_0 R_1} < z_0.$$

Substituting (2.1.101) into (2.1.89) for $\varepsilon_1 = \varepsilon$, we get for all $t \geq 0$,

$$\begin{cases} z'(t) + \varepsilon_1 \rho z(t) \leq k + C\varepsilon_1^{2q} z_0^q = \dfrac{5k}{4}, & (2.1.102) \\ z(t) \leq z_0 e^{-\varepsilon_1 \rho t} + \dfrac{5k}{4\varepsilon_1 \rho}(1 - e^{-\varepsilon_1 \rho t}) & \\ \qquad \leq z_0 e^{-\varepsilon_1 \rho t} + \dfrac{5}{6} z_0 (1 - e^{-\varepsilon_1 \rho t}). & (2.1.103) \end{cases}$$

Obviously, (2.1.103) implies for all $t \geq T_1(z_0)$,

$$z(t) \leq \frac{5}{6} z_0. \tag{2.1.104}$$

(3) Taking $\bar{z}_0 = z(t_0)$ $(\leq \frac{5}{6} z_0)$, for $t_0 \geq T_1(z_0)$, as an initial datum, we have the following cases:

(i) If $\bar{z}_0 \leq R_1$, repeating the same argument as in case (1), we obtain for all $t > 0, 0 < \varepsilon \leq \varepsilon_0$,

$$z(t) \leq \bar{z}_0 e^{-\varepsilon \rho t} + \frac{3k}{2\varepsilon \rho}(1 - e^{-\varepsilon \rho t}), \tag{2.1.105}$$

which implies $z(t) \leq R_2$ as $t \geq T_2(z_0)$.

(ii) If $\bar{z}_0 > R_1$, taking $\bar{\varepsilon}_1 = \sqrt{R_1/\bar{z}_0}\varepsilon_0$ $(< \varepsilon_0)$ and using the same arguments as in case (2), we get for all $t \geq T_2(z_0)$,

$$z(t) \leq \frac{5}{6} \bar{z}_0 \leq \left(\frac{5}{6}\right)^2 z_0. \tag{2.1.106}$$

Repeating the above process for finite times, it follows that for all $t \geq T_n(z_0)$,

$$z(t) \leq \left(\frac{5}{6}\right)^n z_0 \leq R_1. \tag{2.1.107}$$

Taking $\tilde{z}_0 = z(t^*)$ $(\leq R_1)$ with $t^* \geq T_n(z_0)$ as an initial datum, and repeating the same arguments as in case (1), we can obtain (2.1.90), which completes the proof. □

Remark 2.1.5. Note that the proof used here is direct and transparent, and quite different from that of the following theorem, also obtained in 2009 by Gatti et al. [297], which implies a uniform bound, e.g., the existence of an absorbing set.

Theorem 2.1.21 (The Gatti–Pata–Zelik Inequality [297]). *Let $\alpha > \beta \geq 1$ and $\gamma \geq 0$ be such that*

$$\frac{\beta - 1}{\alpha - 1} < \frac{1}{1 + \gamma}. \tag{2.1.108}$$

Let ψ be a non-negative absolutely continuous function on $[0, +\infty)$ which satisfies, for some constants $K \geq 0, Q \geq 0, \varepsilon_0 > 0$ and every $\varepsilon \in (0, \varepsilon_0]$, the differential inequality

$$\psi'(t) + \varepsilon\psi(t) \leq K\varepsilon^\alpha[\psi(t)]^\beta + \varepsilon^{-\gamma}q(t), \tag{2.1.109}$$

where $q(t)$ is any non-negative function for which

$$\sup_{t\geq 0} \int_t^{t+1} q(y)dy \leq Q. \tag{2.1.110}$$

Then there exists a constant $R_0 > 0$ with the following property: for every $R \geq 0$, there is some time $t_R \geq 0$ such that for all $t \geq t_R$,

$$\psi(t) \leq R_0, \tag{2.1.111}$$

whenever $\psi(0) \leq R$. Both R_0 and t_R can be computed explicitly.

Proof. Estimate (2.1.110) on $q(t)$ implies that for any $t \geq 0$ and for all $\tau > 0$,

$$\int_t^{t+\tau} q(y)dy \leq Q(1 + \tau). \tag{2.1.112}$$

Under the assumptions on α, β and γ, we choose $\theta \in (0, 1)$ such that

$$1 - \theta > \max(\beta - \alpha\theta, \gamma\theta).$$

For $\omega := 1 - \gamma\theta > \theta$, we consider the function

$$J(r) = -\omega r^{\omega-\theta} + \omega K r^{\beta-\alpha\theta-\gamma\theta}. \tag{2.1.113}$$

Since $\lim_{r\to+\infty} J(r) = -\infty$, we can find a $\rho \geq \omega Q$ such that $\rho^{-\theta/\omega} \leq \varepsilon_0$ and for all $r \geq \rho^{1/\omega}$,

$$J(r) \leq -1 - 2\omega Q. \tag{2.1.114}$$

Now let us introduce the auxiliary function

$$\phi(t) = (\psi(t))^\omega.$$

Note that, for a.e. t such that $\phi(t) \geq \rho$, we have

$$\phi'(t) \leq -1 - 2\omega Q + \omega q(t). \tag{2.1.115}$$

Indeed, for (almost) any fixed t, setting $\varepsilon := (\phi(t))^{-\theta/\omega}$ (where we observe that $\varepsilon \leq \varepsilon_0$ when $\phi(t) \geq \rho$), the differential inequality (2.1.115) reads

$$\phi'(t) \leq J([\phi(t)]^{1/\omega}) + \omega q(t). \tag{2.1.116}$$

Now we distinguish two cases:

(1) If $\phi(t) \le \rho$ for some $t \ge 0$, then $\phi(t+\tau) \le 2\rho$, for every $\tau \ge 0$. If not, let $\tau_1 > 0$ be such that $\phi(t+\tau_1) > 2\rho$, and set $\tau_0 = \sup\{\tau \in [0,\tau_1] : \phi(t+\tau) \le \rho\}$. Integrating (2.1.115) over $[t+\tau_0, t+\tau_1]$, we eventually obtain a contradiction:

$$2\rho < \phi(t+\tau_1) \le \rho - (\tau_1 - \tau_0) - 2\omega Q(\tau_1 - \tau_0) + \omega Q(1+\tau_1-\tau_0) < 2\rho. \quad (2.1.117)$$

(2) If $\phi(0) > \rho$, then $\phi(t_*) \le \rho$ for some $t_* \le \phi(0)\,(1+\omega Q)^{-1}$. Indeed, let $t > 0$ be such that $\phi(\tau) > \rho$ for all $\tau \in [0,t]$. Integrating (2.1.115) on $[0,t]$, we have

$$\rho < \phi(t) \le \phi(0) - t - 2\omega Q t + \omega Q(1+t) \le \phi(0) - (1+\omega Q)t + \rho, \quad (2.1.118)$$

which gives us $t < \phi(0)(1+\omega Q)^{-1}$.

In order to come back to the original $\psi(t)$, we just define

$$R_0 = (2\rho)^{1/\omega}, \quad t_R = R^{1/\omega}(1+\omega Q)^{-1}.$$

By applying (1) and (2), we hence obtain the assertion. □

We immediately get the following corollary of Theorem 2.1.21.

Corollary 2.1.6 (The Gatti–Pata–Zelik Inequality [297]). *Under the assumptions of Theorem* 2.1.21, *we have*

$$\psi(t) \le \mathcal{Q}(\psi(0))e^{-\nu t} + C_* \quad (2.1.119)$$

for some constants $\nu > 0, C_* \ge 0$ *and some non-negative increasing function* $\mathcal{Q}$, *which along with constants* ν *and* C_* *can be explicitly computed in terms of the parameters* α, β *and* C.

Theorem 2.1.22 (The Gatti–Pata–Zelik Inequality [297]). *Assume that* $z : \mathbb{R}^+ \to \mathbb{R}^+$ *is a positive integrable function and* $C, t^* \ge 0$ *are two constants such that for all* $t \ge t^*$, *the following inequality holds,*

$$z(t) \le C. \quad (2.1.120)$$

Moreover, let $y : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfy the estimate*

$$y'(t) + \varepsilon y(t) \le z(t). \quad (2.1.121)$$

Then for every $\eta > 0$, *there exists some time* $t_\eta \ge 0$ *such that for all* $t \ge t_\eta$,

$$y(t) \le \frac{C}{\varepsilon} + \eta. \quad (2.1.122)$$

Proof. Let $t \ge t^*$ and integrate (2.1.121). Then it is easy to see that

$$y(t) \le \left(y(0) + \int_0^{t^*} z(s)e^{\varepsilon s} ds \right) e^{-\varepsilon t} + e^{-\varepsilon t} \int_{t^*}^{t} z(s)e^{\varepsilon s} ds$$

whence, by (2.1.120),

$$y(t) \leq \left(y(0) + \int_0^{t^*} z(s) e^{\varepsilon s} ds \right) e^{-\varepsilon t} + \frac{C}{\varepsilon} e^{-\varepsilon t^*}.$$

Now since the quantity in parentheses in the last inequality is bounded and independent of t, we obtain (2.1.222). □

Next we turn to an abstract result, due to Agmon and Nirenberg [14], which implies lower bounds and uniqueness of solutions to an abstract differential inequality in a Hilbert space of the form

$$\left\| \frac{du}{dt} - Bu(t) \right\| \leq \Phi(t) \|u(t)\| \tag{2.1.123}$$

where $u(t)$ is a function with values in a (complex) Hilbert space, B is an unbounded linear operator, and Φ is a scalar function.

Let H be a Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|$. We shall consider functions $u(t)$ defined on an interval $[0, T)$, taking their values in H, and satisfying a differential inequality of the form:

$$\left\| \frac{du}{dt} - B(t)u(t) \right\| \leq \Phi(t) \left\{ \|u(t)\|^2 + \int_t^T \omega(\tau) \|u(\tau)\|^2 d\tau \right\}^{1/2}. \tag{2.1.124}$$

Here $B(t)$ (for each t) is a linear operator in H with domain $D_{B(t)}$. We shall assume that $u(t) \in D_{B(t)}$, that $u \in C^1([0,T); H)$, and that $B(t)u(t) \in C([0,T); H)$. $\Phi(t)$ denotes a non-negative measurable function which is bounded in every finite interval $[0, T']$ with $T' < T$, $\omega(t)$ is a non-negative continuous function on $[0, T)$ and we assume that

$$\int_0^T \omega(\tau) \|u(\tau)\|^2 d\tau < +\infty. \tag{2.1.125}$$

Our main assumptions are as follows: $B(t)$ admits a decomposition of the form

$$B(t) = B_+(t) + B_-(t) = B_+(t) + B'_-(t) + B''_-(t) \tag{2.1.126}$$

where $B_+(t)$ is a linear symmetric operator on $D_{B(t)}$, while $B'_-(t)$ and $B''_-(t)$ are linear skew-symmetric on $D_{B(t)}$. (Thus, for all $x \in D_{B(t)}$, $(B_+(t)x, x)$ is real and $\Re(B'_-(t)x, x) = \Re(B''_-(t)x, x) = 0$). Moreover, we now give the following hypotheses.

(1) For each $t \in [0, T)$ and all $x \in D_{B(t)}$, there holds that

$$\Re(B_+(t)x, B'_-(t)x) \geq -\gamma_1(t) \|B_+(t)x\| \|x\| - \beta_1(t) \|x\|^2. \tag{2.1.127}$$

(2) For each $t \in [0, T)$ and all $x \in D_{B(t)}$, there holds that

$$\|B''_-(t)x\|^2 \leq \gamma_2(t)\|B_+(t)x\|\|x\| + \beta_2(t)\|x\|^2. \tag{2.1.128}$$

(3) For the solution u of (2.1.123), the function $(B_+(t)u(t), u(t))$ is differentiable for $0 \leq t < \tau$ and there holds that

$$\begin{aligned} &\frac{d}{dt}(B_+(t)u(t), u(t)) - 2\Re\left(B_+(t)u(t), \frac{du}{dt}\right) \\ &\quad \geq -\gamma_3(t)\|B_+(t)u(t)\|\|u(t)\| - \beta_3(t)\|u(t)\|^2. \end{aligned} \tag{2.1.129}$$

In the sequel, we denote by γ_i, β_i $(i = 1, 2, 3)$, non-negative measurable functions that are bounded on every closed finite subinterval of $[0, T)$.

Concerning the last hypothesis (3), we make the following remark.

Remark 2.1.6. **([14])** Assume that $D_{B(t)} = D$ is independent of t and that for every $x \in D$ and $t \in [0, T)$, there exists the strong limit

$$\lim_{h\to 0} \frac{B(t+h)x - B(t)x}{h} = \dot{B}(t)x$$

hereinafter we use a dot to represent differentiation with respect to t. Moreover, assume that

$$\|\dot{B}(t)x\| \leq \gamma_3(t)\|B_+(t)x\| + \beta_3(t)\|x\|. \tag{2.1.130}$$

Then hypothesis (3) is valid for a solution u of (2.1.123) provided that $B_+(t)u(t)$ is strongly continuous. Indeed it is easily seen in this case (taking the difference quotient and passing to the limit) that the function $(B_+(t)u(t), u(t))$ is differentiable there holds that

$$\frac{d}{dt}\Big(B_+(t)u(t), u(t)\Big) = 2\Re\left(B_+(t)u(t), \frac{du}{dt}\right) + (\dot{B}_+(t)u(t), u(t)). \tag{2.1.131}$$

Combining (2.1.131) and (2.1.130), we can obtain (2.1.129), so that hypothesis (3) holds.

If we set $\gamma(t) = \max_{i=1,2,3}\gamma_i(t)$, $\beta = \max_{i=1,2,3}\beta_i(t)$, then we can state the following main result.

Theorem 2.1.23 (The Agmon–Nirenberg Inequality [14])**.** *Let $u(t)$ be a solution of* (2.1.123) *in $[0, T)$ and assume that all the hypotheses introduced above hold. Set*

$$q(t) = \|u(t)\|^2 + \int_t^T \omega(\tau)\|u(\tau)\|^2 d\tau \tag{2.1.132}$$

and let $[t_0, t_1)$ be a sub-interval of $[0, T)$ such that $q(t) > 0$ for all $t_0 \leq t < t_1$. Let

$$f(t) = u'(t) - B(t)u(t) \tag{2.1.133}$$

and define for all $t_0 \leq t < t_1$,

$$\psi(t) := \frac{2\Re(f(t), u(t))}{q(t)} - \omega(t)\frac{\|u(t)\|^2}{q(t)} \tag{2.1.134}$$

and

$$l(t) := \log q(t) - \int_{t_0}^{t} \psi(\tau)d\tau. \tag{2.1.135}$$

Then $l(t)$ is twice differentiable and satisfies the following second-order differential inequality in the interval $[t_0, t_1)$:

$$l''(t) + a(t)|l'(t)| + b(t) \geq 0, \tag{2.1.136}$$

where

$$a(t) = 4\gamma(t) + \omega(t), \quad b(t) = 16\gamma^2(t) + 8\beta(t) + 4\Phi^2(t). \tag{2.1.137}$$

Before proving the above theorem, note that we can write inequality (2.1.123) in the form

$$\|f(t)\| \leq \Phi(t)q^{1/2}(t). \tag{2.1.138}$$

Therefore, from (2.1.134), (2.1.138), and the estimate $\|u(t)\|^2 \leq q(t)$, it follows that

$$|\psi(t)| \leq 2\Phi(t) + \omega(t) \tag{2.1.139}$$

whence from (2.1.135) and (2.1.139), we derive

$$\log q(t) \geq l(t) - \int_{t_0}^{t} (2\Phi(\tau) + \omega(\tau))\, d\tau. \tag{2.1.140}$$

It will be seen later on that the differential inequality (2.1.136) yields a lower bound for $l(t)$ and thus we shall derive a lower bound for $\log q(t)$, and this is our required result.

Proof of Theorem 2.1.23. Differentiating $q(t)$, using (2.1.133), (2.1.126), (2.1.134), the symmetry of B_+ and the skew-symmetry of B'_- and B''_-, we find for all $t \in [t_0, t_1)$:

$$\begin{aligned} q(t) &= 2\Re(u'(t), u(t)) - \omega(t)\|u(t)\|^2 \\ &= 2\Re(B(t)u(t), u(t)) + 2\Re(f(t), u(t)) - \omega(t)\|u(t)\|^2, \\ q'(t) &= 2(B_+(t)u(t), u(t)) + \psi(t)q(t). \end{aligned} \tag{2.1.141}$$

Let

$$s(t) = \exp\Big(- \int_{t_0}^{t} \psi(\tau)d\tau\Big), \tag{2.1.142}$$

which clearly satisfies

$$s'(t) + \psi(t)s(t) = 0. \tag{2.1.143}$$

Now setting

$$p(t) = s(t)q(t), \tag{2.1.144}$$

thus from (2.1.141) and (2.1.143)–(2.1.144), it follows that

$$l'(t) = \frac{p'(t)}{q(t)} = \frac{s(t)q'(t) + s'(t)q(t)}{p(t)} = \frac{2}{q(t)}(B_+(t)u(t), u(t)). \tag{2.1.145}$$

Next, it follows from hypothesis (3) and from (2.1.145) that $p(t)$ is twice differentiable, and using (2.1.141) and (2.1.140), we conclude readily

$$\begin{aligned} l''(t) &= \frac{2}{q}\frac{d}{dt}(B_+u, u) - \frac{2}{q^2}(B_+u, u)[2(B_+u, u) + \psi q] \\ &\geq \frac{4}{q}\Re(B_+u, u') - \frac{2\gamma}{q}\|B_+u\|\|u\| - \frac{2\beta}{q}\|u\|^2 \\ &\quad - \frac{4}{q^2}(B_+u, u)^2 - \frac{2\psi}{q}(B_+u, u) \\ &= \frac{4}{q}(B_+u, B_+u) - \frac{4}{q^2}(B_+u, u)^2 + \frac{4}{q}\Re(B_+u, B_-u) \\ &\quad + \frac{4}{q}Re(B_+u, f) - \frac{2\gamma}{q}\|B_+u\|\|u\| - \frac{2\beta}{q}\|u\|^2 - \frac{2\psi}{q^2}(B_+u, u). \end{aligned} \tag{2.1.146}$$

Now using $\|u\|^2 \leq q$ and noting the following inequality

$$\frac{4}{q}\left(\|B_+u\|^2 - q^{-1}(B_+u, u)^2\right) \geq \frac{4}{q}\left\|B_+u - \frac{(B_+u, u)}{q}u\right\|^2,$$

we easily arrive at

$$\begin{aligned} l''(t) \geq \frac{4}{q}\left\|B_+u - \frac{(B_+u, u)}{q}u\right\|^2 + \frac{4}{q}\Re(B_+u, B_-u) \\ + \frac{4}{q}\Re(B_+u, f) - \frac{2\gamma}{q}\|B_+u\|\|u\| - \frac{2\beta}{q}\|u\|^2 - \frac{2\psi}{q^2}(B_+u, u). \end{aligned}$$

Using (2.1.126) and hypothesis (1), we derive that

$$\begin{aligned} l''(t) \geq \frac{4}{q}\left\|B_+u - \frac{(B_+u, u)}{q}u\right\|^2 + \frac{4}{q}\Re(B_+u, B''_-u) \\ - \frac{6\gamma}{q}\|B_+u\|\|u\| - \frac{6\beta}{q}\|u\|^2 + \frac{4}{q}\Re(B_+u, f) - \frac{2\psi}{q^2}(B_+u, u). \end{aligned} \tag{2.1.147}$$

Since $\Re(B''_- u, u) = 0$,

$$|\Re(B_+ u, B''_- u)| = \left|\Re\left(B_+ u - \frac{(B_+ u, u)}{q} u, B''_- u\right)\right| \\ \leq \frac{1}{2}\left\|B_+ u - \frac{(B_+ u, u)}{q} u\right\|^2 + \frac{1}{2}\|B''_- u\|^2.$$

Inserting this in (2.1.147), and using hypothesis (2) and the fact that $\|u\|^2 \leq q$, we find

$$l''(t) \geq \frac{2}{q}\left\|B_+ u - \frac{(B_+ u, u)}{q} u\right\|^2 - \frac{8\gamma}{q}\|B_+ u\|\|u\| \\ - 8\beta + \frac{4}{q}\Re(B_+ u, f) - \frac{2\psi}{q^2}(B_+ u, u).$$

By the definition (2.1.134),

$$\begin{aligned} &\frac{4}{q}\Re(B_+ u, f) - \frac{2\psi}{q}(B_+ u, u) \\ &\quad = \frac{4}{q}\Re\left(B_+ u - \frac{(B_+ u, u)}{q} u, f\right) + \frac{2\omega}{q^2}\|u\|^2(B_+ u, u) \\ &\quad \geq -\frac{1}{q}\left\|B_+ u - \frac{(B_+ u, u)}{q} u\right\|^2 - \frac{4}{q}\|f\|^2 + \frac{2\omega}{q^2}\|u\|^2(B_+ u, u), \end{aligned} \tag{2.1.148}$$

which, inserted in the preceding expression, yields, with the help of (2.1.138),

$$\begin{aligned} l''(t) \geq \frac{1}{q}\left\|B_+ u - \frac{(B_+ u, u)}{q} u\right\|^2 - \frac{8\gamma}{q}\|B_+ u\|\|u\| - 8\beta \\ - 4\Phi^2(t) - \frac{2\omega}{q}|(B_+ u, u)|. \end{aligned} \tag{2.1.149}$$

Now if θ denotes the angle between the vectors $B_+ u$ and u in the Hilbert space, we have

$$\left\|B_+ u - \frac{(B_+ u, u)}{q} u\right\|^2 \geq \|B_+ u\|^2 \sin^2\theta, \tag{2.1.150}$$

and

$$|l'(t)| = \frac{2}{q}|(B_+ u, u)| = \frac{2}{q}\|B_+ u\|\|u\||\cos\theta|. \tag{2.1.151}$$

Hence,

$$\begin{aligned} \frac{8\gamma}{q}\|B_+ u\|\|u\| &= \frac{8\gamma}{q}\|B_+ u\|\|u\|(\sin^2\theta + \cos^2\theta) \\ &\leq \frac{1}{q}\|B_+ u\|^2\sin^2\theta + \frac{16\gamma^2}{q}\|u\|^2\sin^2\theta + 4\gamma|l'(t)|. \end{aligned} \tag{2.1.152}$$

Inserting (2.1.152) into (2.1.149), using (2.1.150)–(2.1.151) and the inequality $\|u\|^2 \leq q$, we conclude

$$l''(t) \geq -16\gamma^2 - 8\beta - 4\Phi^2(t) - (4\gamma + \omega)|l'(t)|$$

which thus gives us the desired inequality (2.1.136). □

2.2 Differential inequalities leading to asymptotic behavior

In this section, we shall collect some differential inequalities which may be used to determine the large time behavior of functions. This class of inequalities plays a very significant role in the study of the asymptotic behavior of global solutions to some evolutionary differential equations, and is a very convenient and powerful tool in establishing large-time behavior of global solutions when we use energy methods.

We begin with some familiar results of the classical calculus for the single real variable analysis.

Lemma 2.2.1.

(1) *Let $y(t) \in L^1([0, +\infty))$ with $y(t) \geq 0$ for a.e. $t \geq 0$, and assume that $\lim_{t\to+\infty} y(t)$ exists. Then*

$$\lim_{t\to+\infty} y(t) = 0. \tag{2.2.1}$$

(2) *Let $y(t) \in L^1([0, +\infty))$ with $y(t) \geq 0$ for a.e. $t \geq 0$, $y'(t) \in L^1([0, +\infty))$. Then*

$$\lim_{t\to+\infty} y(t) = 0. \tag{2.2.2}$$

(3) *Let $y(t)$ be uniformly continuous on $[0, +\infty)$, $y(t) \in L^1([0, +\infty))$. Then*

$$\lim_{t\to+\infty} y(t) = 0. \tag{2.2.3}$$

(4) *Let $y(t)$ be a monotonic function on $[0, +\infty)$ and $y(t) \in L^1([0, +\infty))$. Then*

$$\lim_{t\to+\infty} y(t) = 0 \tag{2.2.4}$$

and as $t \to +\infty$,

$$y(t) = o(1/t). \tag{2.2.5}$$

Proof. (1) Let $\lim_{t\to+\infty} y(t) = A$. Since $y(t) \in L^1([0, +\infty))$, by the Cauchy convergence criterion, we have

$$\lim_{n\to+\infty} \int_n^{n+1} y(t)dt = 0. \tag{2.2.6}$$

Let $\alpha_n = \inf f([n, n+1]), \beta_n = \sup f([n, n+1])$. Then it follows from the integration mean value theorem that there exists a sequence $\lambda_n \in [\alpha_n, \beta_n]$ such that

$$\int_n^{n+1} y(t)dt = \lambda_n \tag{2.2.7}$$

which, together with (2.2.6), yields

$$\lim_{n\to+\infty} \lambda_n = 0. \tag{2.2.8}$$

Noting that $\lim_{t\to+\infty} y(t) = A$, we infer that for any $\varepsilon > 0$, there exists a constant $M_1 > 0$ such that for $t > M_1$,

$$A - \varepsilon < y(t) < A + \varepsilon. \tag{2.2.9}$$

Hence, if $N > M_1$, then for $n > N$ and for any $t \in [n, n+1]$, we have

$$A - \varepsilon < y(t) < A + \varepsilon. \tag{2.2.10}$$

Then from (2.2.7) we conclude that

$$A - \varepsilon \le \alpha_n \le \lambda_n \le \beta_n \le A + \varepsilon,$$

which combined with (2.2.8) implies $\lim_{n\to+\infty} \lambda_n = A = 0$.

(2) Since $y'(t) \in L^1([0, +\infty))$, we infer from the Cauchy convergence criterion that for any $\varepsilon > 0$, there exists a constant $M_2 > 0$ such that for $t_1 \ge t_2 \ge M_2$, we have

$$|y(t_1) - y(t_2)| = |\int_{t_1}^{t_2} y'(t)dt| < \varepsilon,$$

which implies that $\lim_{t\to+\infty} y(t)$ exists. Thus (2.2.2) follows from (2.2.1).

(3) Since $y(t)$ is uniformly continuous on $[0, +\infty)$, we infer that for any $\varepsilon > 0$, there is a constant $\delta > 0$ such that for any $t_1, t_2 \in [0, +\infty)$ with $|t_1 - t_2| < \delta$,

$$|y(t_1) - y(t_2)| < \varepsilon. \tag{2.2.11}$$

Since $y(t) \in L^1([0, +\infty))$, we obtain that for $\varepsilon_1 = \delta\varepsilon$, there exists a constant $M_3 > 0$ such that for all $t > M_3$,

$$\left| \int_t^{t+\delta} y(s)ds \right| < \delta\varepsilon. \tag{2.2.12}$$

Now consider the integral $\int_t^{t+\delta} y(s)ds$. When $t < s < t + \delta$, we conclude from (2.2.11) that

$$y(s) - \varepsilon < y(t) < y(s) + \varepsilon,$$

which combined with (2.2.12) gives

$$\int_t^{t+\delta} y(s)ds - \delta\varepsilon \leq \int_t^{t+\delta} y(t)ds \leq \int_t^{t+\delta} y(s)ds + \delta\varepsilon,$$

i.e.,

$$\left| \int_t^{t+\delta} y(t)ds - \int_t^{t+\delta} y(s)ds \right| \leq \delta\varepsilon. \tag{2.2.13}$$

Thus it follows from (2.2.12) and (2.2.13) that for all $t > M_3$,

$$\begin{aligned} |y(t)| &= \frac{1}{\delta}\left| \int_t^{t+\delta} y(t)ds \right| \\ &\leq \frac{1}{\delta}\left[\left| \int_t^{t+\delta} y(t)ds - \int_t^{t+\delta} y(s)ds \right| + \left| \int_t^{t+\delta} y(s)ds \right| \right] \leq 2\varepsilon, \end{aligned}$$

which implies (2.2.3).

(4) Without loss of generality, we assume that $y(t)$ is monotonically decreasing on $[0, +\infty)$. Then

$$y(t) \geq 0.$$

Otherwise, if there exists a $t = b > 0$ such that $y(t) = y(b) < 0$, then for all $t > b$, we have

$$y(t) \leq y(b) < 0,$$

which implies that

$$\int_0^{+\infty} y(t)dt = \int_0^b y(t)dt + \int_b^{+\infty} y(t)dt$$

is divergent, a contradiction.

It follows from $y(t) \in L^1([0, +\infty))$ that for any $\varepsilon > 0$, there is a constant $M_4 > 0$ such that for all $t > M_4$, we have

$$\varepsilon/2 > \int_{t/2}^t y(s)ds \geq y(t) \int_{t/2}^t ds = ty(t)/2$$

that is, for all $t > M_4$,

$$0 < ty(t) \leq \varepsilon$$

which gives (2.2.4) and (2.2.5). □

Note that the above lemma provides the asymptotic behavior of $y(t)$ for large time.

The next theorem is related to the uniform Bellman–Gronwall inequality and was first established by Shen and Zheng [863] in 1993 (see, e.g., Zheng [998]). It is very useful and powerful in dealing with the global well-posedness and asymptotic behavior of solutions to some evolutionary differential equations.

Theorem 2.2.2 (The Shen–Zheng Inequality [863, 998])**.** *Let $0 < T \le +\infty$, and assume that $y(t), h(t)$ are non-negative continuous functions on $[0, T]$ satisfying the following conditions:*

$$\begin{cases} y'(t) \le A_1 y^2(t) + A_2 + h(t), & (2.2.14) \\ \int_0^T y(t)dt \le A_3, \quad \int_0^T h(t)dt \le A_4 & (2.2.15) \end{cases}$$

where A_i $(i = 1, 2, 3, 4)$ are given non-negative constants. Then for any $r > 0$ with $0 < r < T$, the following estimate holds for all $t \in (0, T - r)$,

$$y(t + r) \le \left(\frac{A_3}{r} + A_2 r + A_4\right) e^{A_1 A_3}. \tag{2.2.16}$$

Furthermore, if $T = +\infty$, then

$$\lim_{t\to+\infty} y(t) = 0. \tag{2.2.17}$$

Proof. Assume that $0 \le t \le s \le t + r < T$ for any given $0 < r < T$. We multiply (2.2.14) by $\exp\left(-\int_t^s A_1 y(\tau)d\tau\right)$ and obtain the relation

$$\begin{aligned} &\frac{d}{ds}\left(y(s)\exp\left\{-\int_t^s A_1 y(\tau)d\tau\right\}\right) \\ &\quad \le (A_2 + h(s))\exp\left\{-\int_t^s A_1 y(\tau)d\tau\right\} \le A_2 + h(s). \end{aligned} \tag{2.2.18}$$

Integration between s and $t + r$ yields

$$\begin{aligned} y(t + r) &\le y(s)\exp\left\{\int_s^{t+r} A_1 y(\tau)d\tau\right\} + (A_2 r + A_4)\exp\left\{\int_t^{t+r} A_1 y(\tau)d\tau\right\} \\ &\le (y(s) + A_2 r + A_4)\exp(A_1 A_3). \end{aligned} \tag{2.2.19}$$

Next, integrating (2.2.19) with respect to s between t and $t + r$ we obtain (2.2.16). From (2.2.14) and (2.2.16) we conclude that

$$\begin{aligned} y'(t) &\le A_1\left[\left(\frac{A_3}{r} + A_2 r + A_4\right) e^{A_1 A_3}\right]^2 + A_2 + h(t) \\ &= A_r + h(t), \quad \text{for all } t \ge r, \end{aligned} \tag{2.2.20}$$

where we denote

$$A_r \equiv A_1\left[\left(\frac{A_3}{r} + A_2 r + A_4\right) e^{A_1 A_3}\right]^2 + A_2.$$

To prove (2.2.17), we use a contradiction argument. Suppose it is not true. Then there exist a monotonically increasing sequence $\{t_n\}$ and a constant $a > 0$ such that for all $n \in \mathbb{N}$,

$$\begin{cases} t_n \geq r + \dfrac{a}{4A_r}, \quad t_{n+1} \geq t_n + \dfrac{a}{4A_r}, & (2.2.21) \\ \lim\limits_{n\to+\infty} t_n = +\infty, & (2.2.22) \\ y(t_n) \geq \dfrac{a}{2} > 0. & (2.2.23) \end{cases}$$

On the other hand, from (2.2.20) we know that for all $t_n - \frac{a}{4A_r} \leq t < t_n$,

$$y(t_n) - y(t) \leq A_r(t_n - t) + \int_t^{t_n} h(\tau)d\tau. \tag{2.2.24}$$

Combining (2.2.21)–(2.2.22) and (2.2.24) yields that for all $t_n - \frac{a}{4A_r} \leq t < t_n$,

$$a/2 - y(t) \leq y(t_n) - y(t) \leq a/4 + \int_{t_n - \frac{a}{4A_r}}^{t_n} h(\tau)d\tau. \tag{2.2.25}$$

Thus we deduce from (2.2.23) and (2.2.25) that for all $t_n - \frac{a}{4A_r} \leq t < t_n$,

$$y(t) + \int_{t_n - \frac{a}{4A_r}}^{t_n} h(\tau)d\tau \geq a/4. \tag{2.2.26}$$

Let

$$n_T = \max\left\{n : n \in \mathbb{N}, r + \frac{a}{4A_r} \leq t_n \leq T\right\}. \tag{2.2.27}$$

Then

$$\lim_{T\to+\infty} n_T = +\infty. \tag{2.2.28}$$

Now it follows from (2.2.15) that for all $T > 0$,

$$\begin{aligned} A_3 + \frac{aA_4}{4A_r} &\geq \int_0^T y(\tau)d\tau + \frac{a}{4A_r}\int_0^T h(\tau)d\tau \\ &\geq \sum_{1\leq n\leq n_T} \left[\int_{t_n - \frac{a}{4A_r}}^{t_n} y(\tau)d\tau + \frac{a}{4A_r}\int_{t_n - \frac{a}{4A_r}}^{t_n} h(\tau)d\tau\right] \geq \frac{a^2 n_T}{16A_r}, \end{aligned}$$

which contradicts (2.2.28). □

Corollary 2.2.1 (The Shen–Zheng Inequality [863]). *If we replace A_2 and A_3 by $\max(A_2, A_3)$ in (2.2.16), then for any $t \geq 1$,*

$$y(t) \leq 2\Big(\max(A_2, A_3) + A_4\Big)e^{A_1 \max(A_2, A_3)}. \tag{2.2.29}$$

The following result may be considered as a variant with a variable coefficient $a_0 + a(t)$ of the above inequality (see, e.g., Ducomet and Zlotnik [226]).

Theorem 2.2.3 (The Ducomet–Zlotnik Inequality [226]). *Let* $a_0 =$ constant ≥ 0 *and* $a(t), h(t) \in L^1(\overline{\mathbb{R}^+})$. *If the function* $y(t)$ *satisfies* $y(t) \geq 0$ *on* $\overline{\mathbb{R}^+}$, $y \in W^{1,1}(0,T)$ *for any* $T > 0$, *and*

$$\frac{dy(t)}{dt} + (a_0 + a(t))y(t) \leq h(t), \tag{2.2.30}$$

then the following upper bound and the stabilization property hold:

$$\begin{cases} \sup\limits_{t\geq 0} y(t) \leq \left(y(0) + \|h\|_{L^1(\mathbb{R}^+)}\right) \exp\left\{\|a\|_{L^1(\mathbb{R}^+)}\right\}, & (2.2.31) \\ \lim\limits_{t\to+\infty} y(t) = 0. & (2.2.32) \end{cases}$$

Proof. By (2.2.30), we have

$$\frac{d}{dt}\left(y(t)\exp\left\{\int_0^t (a_0 + a(s))ds\right\}\right) \leq h(t)\exp\left\{\int_0^t (a_0 + a(s))ds\right\}. \tag{2.2.33}$$

Integrating (2.2.33) and using the assumptions on a_0 and a, h, we derive

$$\begin{aligned} y(t) &\leq y(0)\exp\left\{-\int_0^t (a_0 + a(s))ds\right\} + \int_0^t h(\tau)\exp\left\{-\int_\tau^t (a_0 + a(s))ds\right\}d\tau \\ &\leq y(0)e^{-a_0 t}\exp\left\{\int_0^t a(s)ds\right\} + \int_0^t h(\tau)\exp\left\{-\int_\tau^t (a_0 + a(s))ds\right\}d\tau \\ &\leq \left(y(0) + \|h\|_{L^1(\mathbb{R}^+)}\right)\exp\left\{\|a\|_{L^1(\mathbb{R}^+)}\right\}, \end{aligned}$$

which yields us (2.2.31). On the other hand, under our assumptions

$$y(t), y'(t) \in L^1(\mathbb{R}^+). \tag{2.2.34}$$

Thus (2.2.32) follows from (2.2.34) and (2) in Lemma 2.2.1, and the proof is complete. □

Later on, the above result was extended by Străskraba and Zlotnik [900] to the following form.

Theorem 2.2.4 (The Străskraba–Zlotnik Inequality [900]). *Let*

$$Y : \overline{\mathbb{R}^+} = [0, +\infty) \to \overline{\mathbb{R}^+}, \quad Y \in W^{1,1}(0,T)$$

for any $T > 0$ *and satisfy for all* $t \in \overline{\mathbb{R}^+}$,

$$Y'(t) + (a_0 + a_1(t) + a_2(t))Y(t) \leq G(t), \tag{2.2.35}$$

where $a_0 > 0, a_1 \in L^1(\overline{\mathbb{R}^+}), a_2 \in L^q(\overline{\mathbb{R}^+}),$ *and* $G \in L^s(\overline{\mathbb{R}^+}), G \geq 0$ *for some* $q \in (1, +\infty)$ *and* $s \in [1, +\infty]$. *Then for any* $\varepsilon \in (0,1)$, *for all* $t \in \overline{\mathbb{R}^+}$, *the following estimate holds,*

$$Y(t) \leq C_{\varepsilon,q}\left(e^{-a_0(1-\varepsilon)t}Y(0) + \int_0^t e^{-a_0(1-\varepsilon)(t-\tau)}G(\tau)d\tau\right), \tag{2.2.36}$$

with

$$C_{\varepsilon,q} = \exp\left(\|a_1\|_{L^1(\mathbb{R}^+)} + q^{-1}\left(a_0 q'\varepsilon\right)^{-(q-1)}\|a_2\|^q_{L^q(\mathbb{R}^+)}\right), \quad q' = q/(q-1).$$

As a consequence of estimate (2.2.36), *the following uniform estimate holds,*

$$\|Y\|_{C(\overline{\mathbb{R}^+})} \leq C_{\varepsilon,q}\left(Y(0) + [a_0(1-\varepsilon)s']^{-1/s'}\|G\|_{L^s(\mathbb{R}^+)}\right) \tag{2.2.37}$$

where $s' = s/(s-1)$ *and* $[a_0(1-\varepsilon)s']^{-1/s'} = 1$ *in the case* $s = 1$ (*i.e.*, $s' = +\infty$). *Moreover, if* $s < +\infty$, *then as* $t \to +\infty$,

$$Y(t) \to 0. \tag{2.2.38}$$

Proof. Set $(I_0 a)(t) := \int_0^t a(\tau)d\tau, a := a_1 + a_2$. Multiplying inequality (2.2.35) by $e^{I_0(a_0+a)}$, and applying I_0 to the result, we easily get

$$Y(t) \leq e^{-(a_0 t + I_0 a)}Y(0) + \int_0^t e^{-[a_0(t-\tau)+\int_\tau^t a(\theta)d\theta]}G(\tau)d\tau. \tag{2.2.39}$$

Using the Hölder and Young inequalities, we get for any $0 \leq \tau \leq t$ and $\varepsilon > 0$,

$$\begin{aligned}\left|\int_\tau^t a(\theta)d\theta\right| &\leq \|a_1\|_{L^1(\mathbb{R}^+)} + (t-\tau)^{1/q'}\|a_2\|_{L^q(\mathbb{R}^+)} \leq \|a_1\|_{L^1(\mathbb{R}^+)} + a_0\varepsilon(t-\tau)\\ &\quad + q^{-1}(a_0 q'\varepsilon)^{-(q-1)}\|a_2\|^q_{L^q(\mathbb{R}^+)} \leq a_0\varepsilon(t-\tau) + \log C_{\varepsilon,q}.\end{aligned} \tag{2.2.40}$$

Inserting (2.2.40) into (2.2.39), we obtain (2.2.36). The rest of the proof follows from the Hölder inequality and the well-known relation for $\alpha > 0$ and $1 \leq s < +\infty$,

$$\lim_{t\to+\infty}\int_0^t e^{-\alpha(t-\tau)}|G(\tau)|d\tau = 0, \ \text{ for all } G \in L^s(\mathbb{R}^+). \qquad \square$$

The following result can be viewed as a variant of Theorem 2.2.3 that was used by Zlotnik [1024].

Theorem 2.2.5 (The Zlotnik Inequality [1024]). *Let* $y(t) \in C[0,T]$ *be a function such that* $y \geq 0$, $\frac{d(y^q)}{dt} \in L^1(0,T)$ *for some constant* $q \geq 1$, *and*

$$\frac{d(y^q)}{dt} + q\alpha y^q \leq a f_1 y^{q-1} + f_2^q \ \text{ on } \ (0,T) \tag{2.2.41}$$

where $\alpha = \text{constant} \geq 0$, $f_1(t) \in L^1(0,T)$, $f_2(t) \in L^q(0,T)$, $f_1(t) \geq 0$ *and* $f_2(t) \geq 0$. *Then, for all* $t \in [0,T]$,

$$y(t) \leq \exp(-\alpha t)\left\{y(0) + \int_0^t \exp(\alpha\tau) f_1(\tau)d\tau + \left[\int_0^t (\exp(\alpha\tau)f_2(\tau))^q d\tau\right]^{1/q}\right\}, \tag{2.2.42}$$

and if $T = +\infty$, *then we further have*

$$\begin{cases} y(t) \leq C_{\varepsilon,q}^{1/q}\left[e^{-q\alpha(1-\varepsilon)t}y^q(0) + \displaystyle\int_0^t e^{-q\alpha(1-\varepsilon)(t-\tau)}(f_1(\tau) + f_2^q(\tau))d\tau\right]^{1/q}, & (2.2.43)\\ \|y(t)\|_{C(\overline{\mathbb{R}^+})} \leq C_{\varepsilon,q}\left(y(0) + \|f_1\|_{L^1(\mathbb{R}^+)} + \|f_2\|^q_{L^q(\mathbb{R}^+)}\right) & (2.2.44)\end{cases}$$

and as $t \to +\infty$

$$y(t) \to 0 \tag{2.2.45}$$

with

$$C_{\varepsilon,q} = \exp\left((q-1)\|f_1\|_{L^1(\mathbb{R}^+)}\right). \tag{2.2.46}$$

Proof. Obviously, multiplying inequality (2.2.41) by $\exp(q\alpha t)$ gives us the same inequality for the functions $\exp(\alpha t)y(t)$ and $\exp(\alpha t)f_k(t)$ instead of $y(t)$ and $f_k(t)$, respectively ($k = 1,2$), but for $\alpha = 0$. Thus we can restrict our considerations to the case $\alpha = 0$. Moreover, it suffices to derive the estimate (2.2.42) for $t = T$.

Set $I_0(y)(t) = \int_0^t y(\tau)d\tau$. When $\alpha = 0$ in (2.2.41), integrating (2.2.41), we obtain $y^q(t) \leq C + qI_0(f_1y^{q-1})$ on $[0,T]$, where $C = y^q(0) + \|f_2\|^q_{L^q(0,T)}$. We introduce a non-negative function $z_\varepsilon \in C[0,T]$ such that $z_\varepsilon^q(t) = C+\varepsilon+qI_0(f_1z_\varepsilon^{q-1})$ on $[0,T], \varepsilon > 0$. First, it is obvious that $z_\varepsilon^q(t) \geq C+\varepsilon > 0$ and $\frac{d(z_\varepsilon^q)}{dt} = qf_1z_\varepsilon^{q-1}$ on $[0,T]$, whence it follows that $z_\varepsilon(t) = (C+\varepsilon)^{1/q}+I_0(f_1)$ on $[0,T]$. Second, note that $y^q(0)+\varepsilon \leq z_\varepsilon^q(0)$, hence it follows that $y^q(t) < z_\varepsilon^q(t)$ on $[0,t_0)$ for some $t_0 \in (0,T]$. Therefore,

$$z_\varepsilon^q(t_0) - y^q(t_0) \geq \varepsilon + qI_0(f_1(z_\varepsilon^{q-1} - y^{q-1}))\mid_{t=t_0} \geq \varepsilon.$$

Hence we easily see that $y^q(t)+\varepsilon \leq z_\varepsilon^q(t)$ on $[0,T]$. Therefore, $y(T) \leq (z_\varepsilon^q(T) - \varepsilon)^{1/q}$, and the estimate

$$y(T) \leq C^{1/q} + \|f_1\|_{L^1(0,T)} \leq y(0) + \|f_1\|_{L^1(0,T)} + \|f_2\|_{L^q(0,T)}$$

as $\varepsilon \to 0^+$ follows. The proof of (2.2.42) is hence complete. To prove (2.2.43)–(2.2.45), let $z(t) = y^q(t)$. Then using the Young inequality we have

$$qf_1y^q + f_2^q = qf_1z^{(q-1)/q} + f_2^q \leq qf_1(\frac{q-1}{q}z + 1/q) + f_2^q = (q-1)f_1z + f_1 + f_2^q$$

which, when inserted into (2.2.41), gives

$$z_t + [q\alpha - (q-1)f_1]z \leq f_1 + f_2^q \tag{2.2.47}$$

which satisfies (2.2.35) in Theorem 2.2.3 for $a_0 = q\alpha > 0, a_1 = -(q-1)f_1 \in L^1(0,T), a_2 = 0$ and $G(t) = f_1 + f_2^q \in L^1(0,T)$ with $s = 1$. Applying Theorem 2.2.3 to (2.2.47) yields

$$z(t) \le C_{\varepsilon,q}\left[e^{-q\alpha(1-\varepsilon)t}z(0) + \int_0^t e^{-q\alpha(1-\varepsilon)(t-\tau)}(f_1(\tau) + f_2^q(\tau))d\tau\right],$$

whence

$$y(t) \le C_{\varepsilon,q}^{1/q}\left[e^{-q\alpha(1-\varepsilon)t}y^q(0) + \int_0^t e^{-q\alpha(1-\varepsilon)(t-\tau)}(f_1(\tau) + f_2^q(\tau))d\tau\right]^{1/q},$$

with $C_{\varepsilon,q}$ given by (2.2.46). Applying Theorem 2.2.3 again to (2.2.47) yields (2.2.44) and (2.2.45) immediately. □

We should point out here that Străskraba and Zlotnik [901] also established another related result on the asymptotic behavior of a non-negative function $y(t)$ (i.e., $\lim_{t\to+\infty} y(t) = 0$), which is not based on a differential inequality or an integral inequality, but on an equation or a decomposition of $y(t)$. We now reproduce this as follows.

Theorem 2.2.6 (The Străskraba–Zlotnik Inequality [901]). *Assume that* $y(t) \in W^{1,1}_{\text{loc}}(\overline{\mathbb{R}^+})$ *satisfies for all* $t \in \overline{\mathbb{R}^+}$,

$$\begin{cases} y(t) = y_1'(t) + y_2(t), & (2.2.48)\\ |y_2(t)| \le \sum_{i=1}^n \alpha_i, \quad |y'(t)| \le \sum_{i=1}^n \beta_i, & (2.2.49)\end{cases}$$

where $y_1(t) \in W^{1,1}_{\text{loc}}(\overline{\mathbb{R}^+})$ *and* $\lim_{s\to+\infty} y_1(s) = 0$, *and* $\alpha_i, \beta_i \in L^{p_i}(\overline{\mathbb{R}^+})$ *for some* $p_i \in [1,+\infty)$, $i = 1, 2, \ldots, n$. *Then*

$$\lim_{t\to+\infty} y(t) = 0. \tag{2.2.50}$$

Proof. Using Sobolev's embedding theorem $W^{1,1} \hookrightarrow L^\infty$, we have

$$\begin{aligned}|y(t)| &\le C\left|\int_t^{t+1} y(s)ds\right| + C\int_t^{t+1}|y'(s)|ds\\ &\le C|y_1(t+1) - y_1(t)| + C\sum_{i=1}^n \int_t^{t+1}(\alpha_i + \beta_i)ds\\ &\le C|y_1(t+1) - y_1(t)| + C\sum_{i=1}^n\left[\|\alpha_i\|_{L^{p_i}(t,t+1)} + \|\beta_i\|_{L^{p_i}(t,t+1)}\right]\\ &\to 0, \quad \text{as} \quad t\to+\infty,\end{aligned}$$

which completes the proof. □

In 2006, Basov [73] proved a similar result.

Theorem 2.2.7 (The Basov Inequality [73]). *Let $y(t), z(t) \in L^1(\overline{\mathbb{R}^+})$, for all $t \in \overline{\mathbb{R}^+}$, $y(t) \geq 0$, and*

$$y'(t) \leq z(t). \tag{2.2.51}$$

Then as $t \to +\infty$,

$$y(t) \to 0. \tag{2.2.52}$$

Proof. Let us start with the following representation:

$$y(t) = \int_{t-1}^{t} y(s)ds + \int_{t-1}^{t} \int_{s}^{t} y'(\tau)d\tau ds. \tag{2.2.53}$$

Since $y \in L^1(\overline{\mathbb{R}^+})$, the first integral on the right-hand side goes to 0 as $t \to +\infty$. To estimate the second summand, we note that it is non-negative, $y' \leq z$ and $z \in L^1(\overline{\mathbb{R}^+})$. That is, we have

$$0 \leq \int_{t-1}^{t} \int_{s}^{t} y'(\tau)d\tau ds \leq \int_{t-1}^{t} \int_{s}^{t} |z(\tau)|d\tau ds \to 0$$

as $t \to +\infty$, which completes the proof. □

The next result can be found in Wang [946].

Theorem 2.2.8 (The Wang Inequality [946]). *Assume that $h(t) \in C([0, +\infty))$ satisfies*

$$\lim_{t_1 > t_2 \to +\infty} \int_{t_2}^{t_1} |h(t)|dt = 0, \tag{2.2.54}$$

or

$$\lim_{t \to +\infty} h(t) = 0, \tag{2.2.55}$$

and that $g(t) \in C^1([0, +\infty))$, $g(t) \geq 0$, satisfies for all $t \geq 0$,

$$g'(t) \leq -\alpha g(t) + h(t), \tag{2.2.56}$$

with a constant $\alpha > 0$.

Then

$$\lim_{t \to +\infty} g(t) = 0. \tag{2.2.57}$$

Proof. By (2.2.56), we have

$$g(t) \leq g(0)e^{-\alpha t} + \int_0^t h(\tau)e^{-\alpha(t-\tau)d\tau}. \tag{2.2.58}$$

Let $v(t) := \int_0^t h(\tau)e^{-\alpha(t-\tau)}d\tau$. Then it suffices to prove

$$\lim_{t \to +\infty} v(t) = 0. \tag{2.2.59}$$

Indeed, if (2.2.54) holds, then for any $\varepsilon > 0$, there exists a constant $t_0 > 0$, such that for $t_1 > t_2 > t_0$,

$$\left| \int_{t_2}^{t_1} h(t)dt \right| \le \int_{t_2}^{t_1} |h(t)|dt < \varepsilon$$

which implies that

$$|h(t)| \in L^1(\mathbb{R}^+). \tag{2.2.60}$$

Obviously, we have

$$\begin{aligned} v(t) &= \int_0^{t/2} h(\tau)e^{-\alpha(t-\tau)}d\tau + \int_{t/2}^t e^{-\alpha(t-\tau)}h(\tau)d\tau \\ &\equiv I_1(t) + I_2(t). \end{aligned} \tag{2.2.61}$$

It follows from (2.2.60)–(2.2.61) that as $t \to +\infty$, we have

$$\begin{cases} |I_1(t)| \le e^{-\alpha t/2} \int_0^{t/2} |h(\tau)|d\tau \le e^{-\alpha t/2} \int_0^{+\infty} |h(\tau)|d\tau \to 0, & (2.2.62) \\ |I_2(t)| \le \int_{t/2}^t |h(\tau)|d\tau \to 0. & (2.2.63) \end{cases}$$

Therefore, (2.2.57) follows from (2.2.58) and (2.2.62)–(2.2.63). If (2.2.55) holds, then by the l'Hospital Rule,

$$\begin{aligned} \lim_{t\to+\infty} v(t) &= \lim_{t\to+\infty} \frac{\int_0^t h(\tau)e^{\alpha\tau}d\tau}{e^{\alpha t}} = \lim_{t\to+\infty} \frac{h(t)e^{\alpha t}}{\alpha e^{\alpha t}} \\ &= \frac{1}{\alpha} \lim_{t\to+\infty} h(t) = 0 \end{aligned}$$

which, together with (2.2.58), gives (2.2.57), and completes the proof. □

Krejčí and Sprekels [462] in 1998 extended the Shen–Zheng inequality (i.e., Theorem 2.2.1) when $T = +\infty$ to the following result (see also, Zheng [998]), which can be also considered as a nonlinear generalization of the Bellman–Gronwall inequality in Theorem 2.1.3.

Theorem 2.2.9 (The Krejčí–Sprekels Inequality [462]). *Assume that $y(t)$ is absolutely continuous in $[0, +\infty)$, $y(t) \ge 0, y'(t) \in L^1_{\text{loc}}(\mathbb{R}^+)$ and satisfies the following conditions for all a.e. $t \in (0, +\infty)$,*

$$\begin{cases} \int_0^{+\infty} y(t)dt \le C_1 < +\infty, & (2.2.64) \\ y'(t) \le f(y(t)) + h(t), & (2.2.65) \end{cases}$$

where $h(t) \geq 0$ *is such that*

$$\int_0^{+\infty} h(t)dt \leq C_2 < +\infty, \tag{2.2.66}$$

and let f *be a non-decreasing function from* $\overline{\mathbb{R}^+}$ *into* $\overline{\mathbb{R}^+}$. *Then*

$$\lim_{t \to +\infty} y(t) = 0. \tag{2.2.67}$$

If, moreover, there exist constants $A_1, A_2 \geq 0$ *such that for any* $y \geq 0$,

$$f(y) \leq A_1 y^2 + A_2, \tag{2.2.68}$$

then

$$y(t) \leq \begin{cases} e^{A_1 C_1}[y(0) + C_2 + A_2], & \textit{for all } t < 1, \\ e^{A_1 C_1}[C_1 + C_2 + A_2/2], & \textit{for all } t \geq 1. \end{cases} \tag{2.2.69}$$

Proof. Assume that there exist a constant $\alpha > 0$ and a sequence $t_n \uparrow +\infty$, such that for all $n \in \mathbb{N}$,

$$y(t_n) \geq 2\alpha. \tag{2.2.70}$$

We may assume (passing to a subsequence if necessary) that the following inequality holds for every $n \in \mathbb{N}$,

$$t_{n+1} - t_n > 2C_1/\alpha + \beta \tag{2.2.71}$$

where

$$\beta := \frac{\alpha}{2f(2\alpha)}, \quad t_1 > C_1/\alpha. \tag{2.2.72}$$

By (2.2.64) and (2.2.66), the sets

$$A_n := \{t \in [t_n - C_1/\alpha, t_n] : y(t) < \alpha\} \tag{2.2.73}$$

are non-empty and we may set for all $n \in \mathbb{N}$,

$$a_n := \sup A_n \tag{2.2.74}$$

and similarly,

$$\begin{cases} b_n := \inf\{t \in [t_n, t_n + C_1/\alpha] : y(t) < \alpha\}, & (2.2.75) \\ s_n := \min\{t \in [a_n, b_n] : y(t) \geq 2\alpha\}. & (2.2.76) \end{cases}$$

By construction, we have for all $n \in \mathbb{N}$,

$$\begin{cases} a_n < s_n \leq t_n < b_n < a_{n+1}, & (2.2.77) \\ a_{n+1} - b_n > \beta, & (2.2.78) \\ y(a_n) = y(b_n) = \alpha, y(s_n) = 2\alpha, y(t) \geq \alpha, \quad for \ \ all \ \ t \in [a_n, b_n]. & (2.2.79) \end{cases}$$

We now define an auxiliary function $z(t)$ by the formula

$$z(t) := \begin{cases} y(t) - \alpha, & \text{for all} \quad t \in \bigcup_{n=1}^{\infty} [a_n, b_n], \\ 0, & \text{otherwise}. \end{cases} \tag{2.2.80}$$

Then $z(t)$ is non-negative, absolutely continuous, and for a.e. $t > 0$, we have

$$z'(t) \leq f(z(t) + \alpha) + h(t), \quad z(t) \leq y(t). \tag{2.2.81}$$

Moreover, for a.e. $t \in [s_n - \beta, s_n]$, we have

$$z(t) \leq \alpha, \tag{2.2.82}$$

and integrating (2.2.81) from t to s_n, we obtain

$$\begin{aligned} \alpha - z(t) &\leq \int_t^{s_n} \left(f(z(\tau) + \alpha) + h(\tau)\right) d\tau \\ &\leq \beta f(2\alpha) + \int_{s_n - \beta}^{s_n} h(\tau) d\tau. \end{aligned} \tag{2.2.83}$$

For all $t \in [s_n - \beta, s_n]$, we thus have

$$\alpha/2 \leq z(t) + \int_{s_n - \beta}^{s_n} h(\tau) d\tau, \tag{2.2.84}$$

and integrating (2.2.84) over $[s_n - \beta, s_n]$ once more, we conclude that for all $n \in \mathbb{N}$,

$$\frac{1}{2}\alpha\beta \leq \int_{s_n - \beta}^{s_n} \Big(z(\tau) + \beta h(\tau)\Big) d\tau, \tag{2.2.85}$$

which is a contradiction, since both z and h are integrable and the intervals $(s_n - \beta, s_n)$ are pairwise disjoint.

In order to prove (2.2.69), it suffices to rewrite (2.2.65) with (2.2.68) in the form

$$\frac{d}{dt}\left(y(t) e^{-A_1 \int_0^t y(\tau) d\tau}\right) \leq \left(A_2 + h(t)\right) e^{-A_1 \int_0^t y(\tau) d\tau}. \tag{2.2.86}$$

Hence, for every $0 \leq s < t$, we have

$$\begin{aligned} y(t) &\leq y(s) e^{A_1 \int_s^t y(\tau) d\tau} + \int_s^t [A_2 + h(\tau)] e^{A_1 \int_\tau^t y(\sigma) d\sigma} d\tau \\ &\leq e^{A_1 C_1} \left(y(s) + C_2 + A_2(t - s)\right). \end{aligned} \tag{2.2.87}$$

For any $t \leq 1$, we simply put $s = 0$; for all $t \geq 1$, we integrate (2.2.87) with respect to s from $t - 1$ to t, which completes the proof. $\square$

The following example, presented in [462], shows that we cannot expect any a priori pointwise bound for $y(t)$ if $f(y)$ grows faster than y^2.

Example 2.2.1 ([462])**.** Let $\varepsilon \in (0,1)$ be given. For $n > 1$, we set

$$y_n(t) \le \begin{cases} |t-1|^{\varepsilon-1}, & \text{for all } t \in [0,2] \setminus [1-1/n, 1+1/n], \\ n^{1-\varepsilon}, & \text{for all } t \in [1-1/n, 1+1/n], \\ e^{2-\varepsilon}, & \text{for all } t > 2. \end{cases} \tag{2.2.88}$$

Then y_n are absolutely continuous, $\int_0^{+\infty} y_n(t)dt \le 1 + 2/\varepsilon$, $y_n(0) = 1$, $y_n'(t) \le (1-\varepsilon)y_n^{2+\varepsilon/(1-\varepsilon)}(t)$ a.e., and the sequence $\{y_n(1)\}$ is unbounded.

Later on, Zheng [999] showed the following strong version of the inequality (2.2.65).

Theorem 2.2.10 (The Zheng Inequality [999])**.** *Assume that $y(t)$ is a continuous non-negative function on $[0,+\infty)$ satisfying the following conditions:*

$$\begin{cases} \displaystyle\int_0^{+\infty} y(t)dt \le C_1 < +\infty, & (2.2.89) \\ \displaystyle y(t) - y(s) \le \int_s^t (f(y(\tau)) + h(\tau))d\tau, \ \textit{for all } 0 \le s < t < +\infty, & (2.2.90) \end{cases}$$

where f and h satisfy the same assumptions as in Theorem 2.2.8. Then

$$\lim_{t \to +\infty} y(t) = 0. \tag{2.2.91}$$

Proof. We shall again use a contradiction argument, exactly as in Theorem 2.2.8. By the continuity and non-negativity of $y(t)$, and condition (2.2.89), it is easy to see that if (2.2.91) is not true, then there exist a constant $\beta > 0$ and a sequence $t_n, t_n \to +\infty$, such that $y(t_n) = \beta$ and $t_{n+1} - t_n > \beta/(2f(\beta))$ for all n. Let m_n be the minimum of y on $[t_n, t_n - \beta/2f(\beta), t_n]$. We now estimate the lower bound of m_n.

If $m_n < \beta$, let the minimum be achieved at t_n^* $(t_n^* < t_n)$ and let $t_n^{**} = \sup\{t \mid t_n^* < t, y(t) < \beta\}$. Then $y(t_n^{**}) = \beta$. It then follows from (2.2.91) that

$$\begin{aligned} \beta - m_n = y(t_n^{**}) - y(t_n^*) &\le \int_{t_n^*}^{t_n^{**}} \left(f(y(\tau)) + h(\tau)\right) d\tau \\ &\le f(\beta) \cdot \frac{\beta}{2f(\beta)} + \int_{t_n - \beta/(2f(\beta))}^{t_n} h(t)dt. \end{aligned} \tag{2.2.92}$$

Thus, for all $t \in [t_n - \beta/(2f(\beta)), t_n]$,

$$\begin{aligned} \beta/2 &\le m_n + \int_{t_n - \beta/(2f(\beta))}^{t_n} h(t)dt \\ &\le y(t) + \int_{t_n - \beta/(2f(\beta))}^{t_n} h(t)dt, \end{aligned} \tag{2.2.93}$$

which clearly holds when $m_n = \beta$. Thus, integrating (2.2.93) with respect to t and summing with respect to n, we reach a contradiction to condition (2.2.89). □

Define the space

$$C_{\mathrm{loc}} = \left\{u : [0, +\infty) \to \mathbb{R}; u|_{[0,T]} \in C[0, T] \text{ for all } T > 0\right\},$$

which is equipped with the seminorm

$$\|u\|_{[0,t]} := \max_{0\le\tau\le t} |u(\tau)|,$$

where the function $u|_{[0,T]}$ denotes the restriction of u to $[0, T]$.

The next result (see, e.g., Theorem 2.2.10, Krejčí, Sprekels and Zheng [464]) generalizes Theorem 2.2.1 and Theorems 2.2.8–2.2.9 which have been stated in the case of continuous functions, while Theorem 2.2.10 ([464]) below extends them to the discontinuous case. The proof is based on the following lemma.

Lemma 2.2.11 ([464]). *Let $T > 0, g \in C_{\mathrm{loc}}, p \in L^1(0, T)$, and $y \in BV(0, T)$ be given such that*

(i) $g(u) \ge 0, p(t) \ge 0, y(t) \ge 0$ *for every* $u \ge 0$ *and* (*almost*) *every* $t \in [0, T]$,

(ii) *the function* $q(t) := \int_0^t p(\tau)d\tau - y(t)$ *is non-decreasing in* $[0, T]$.

For $u \ge 0$, put $G(u) := \int_0^u g(v)dv$. Then the function

$$Q(t) := \int_0^t g(y(\tau))p(\tau)d\tau - G(y(t)) \tag{2.2.94}$$

is non-decreasing on $[0, T]$.

Proof. Indeed, for any $n \in \mathbb{N}$, we construct the equidistant partition $0 = s_0 < s_1 < \cdots < s_n = T$ of the interval $[0, T]$, $s_k := Tk/n$ for $k = 0, 1, \ldots, n$. We approximate the functions p, y by piecewise constant and piecewise linear interpolants, respectively. To this end, we define

$$\begin{cases} p_n(t) := \dfrac{n}{T}\displaystyle\int_{s_{k-1}}^{s_k} p(\tau)d\tau, \\ y_n(t) := y(s_{k-1}) + \dfrac{n}{T}(t - s_{k-1})\left(y(s_k) - y(s_{k-1})\right), \end{cases}$$

for all $t \in [s_{k-1}, s_k),\ k = 1, \ldots, n$, continuously extended to $t = T$, and a function $Q_n : [0, T] \to \mathbb{R}$ by

$$Q_n(t) := \int_0^t g(y_n(\tau))p_n(\tau)d\tau - G(y_n(t)).$$

By hypothesis (ii), we get for all $t \in (s_{k-1}, s_k)$,

$$y_n'(t) = \frac{n}{T}(y(s_k) - y(s_{k-1})) \le \frac{n}{T}\int_{s_{k-1}}^{s_k} p(\tau)d\tau = p_n(\tau),$$

hence $Q_n'(t) = g(y_n(t))(p_n(t) - y_n'(t)) \geq 0$. We now know that $p_n \to p$ strongly in $L^1(0,T)$ as $n \to +\infty, y_n(t) \to y(t)$ a.e., hence $Q_n(t) \to Q(t)$ a.e. Since Q_n are non-decreasing for every n, the function Q is also non-decreasing and hence the proof is complete. □

Theorem 2.2.12 (The Krejčí–Sprekels–Zheng Inequality [464]). *Let $f \in C_{\text{loc}}, h \in L^1(0,+\infty)$, and $y \in BV_{\text{loc}}(0,+\infty) \cap L^1(0,+\infty)$ satisfy*

(i) $f(u) \geq 0,\ h(t) \geq 0,\ y(t) \geq 0$ *for every* $u \geq 0$ *and (almost) every* $t \geq 0$,

(ii) $\int_0^{+\infty} h(t)dt =: H,\quad \int_0^{+\infty} y(t)dt =: Y$,

(iii) *the function* $q_1(t) := \int_0^t (f(y(\tau)) + h(\tau))d\tau - y(t)$ *is non-decreasing on* $(0,+\infty)$.

Then

$$\lim_{t\to+\infty} y(t) = 0. \tag{2.2.95}$$

If, further, the function $F(u) := \int_0^u (\max(1, f(v)/v))^{-1}\, dv$ *for all* $u \geq 0$ *satisfies the condition*

$$\lim_{u\to+\infty} F(u) = +\infty, \tag{2.2.96}$$

then

$$\begin{cases} y(t) \leq \overline{Y} := F^{-1}(F(y(0)) + Y + H), \ \textit{for all } t \geq 0, & (2.2.97)\\ y(t) \leq F^{-1}(2Y + H), \ \textit{for all } t \geq 1, & (2.2.98)\\ \operatorname{Var}_{[0,+\infty)}(y^2) \leq y^2(0) + 4(Y\|f\|_{[0,\overline{Y}]} + H\overline{Y}). & (2.2.99)\end{cases}$$

Proof. First, we assume that (2.2.96) is valid and for all $t > 0$, let $p(t) := f(y(t)) + h(t)$. Then we know that the hypotheses of Lemma 2.2.2 are satisfied for any $T > 0$, and so the function

$$Q_1(t) := \int_0^t \frac{y(\tau)(f(y(\tau)) + h(\tau))}{\max\{y(\tau), f(y(\tau))\}} d\tau - F(y(t)) \tag{2.2.100}$$

is non-decreasing in $[0,+\infty)$. Obviously, we know that for every $t \geq s \geq 0$,

$$\begin{aligned} F(y(t)) - F(y(s)) &\leq \int_s^t \frac{y(\tau)f(y(\tau)) + y(\tau)h(\tau)}{\max\{y(\tau), f(y(\tau))\}} d\tau \\ &\leq \int_s^t (y(\tau) + h(\tau))d\tau \leq Y + H \end{aligned}$$

which, by taking $s = 0$, gives (2.2.97). Note that the following inequality,

$$F(y(t)) \leq F(y(s)) + Y + H \leq y(s) + Y + H,$$

which, integrated over $[t-1, t]$ for $t \geq 1$, yields (2.2.98) immediately.

If now we define

$$q_2(t) := \int_0^t (\|f\|_{[0,\bar{Y}]} + h(\tau))d\tau - y(t), \tag{2.2.101}$$

then by assumption (iii) and inequality (2.2.97), we conclude that the function q_2 is non-decreasing, and Lemma 2.2.2 for $g(u) = u$ and arbitrary $T > 0$ implies that the function

$$Q_2(t) := \int_0^t y(\tau)(\|f\|_{[0,\bar{Y}]} + h(\tau))d\tau - \frac{1}{2}y^2(t) \tag{2.2.102}$$

is non-decreasing in $[0, +\infty)$.

Let $S := \{t_j\}_{j=0}^n, 0 = t_0 < t_1 < \cdots < t_n$ be an arbitrary sequence, and define

$$V(S) := \sum_{j=1}^n |y^2(t_j) - y^2(t_{j-1})|. \tag{2.2.103}$$

Note that in the above definition of $V(S)$, we first eliminate monotone parts of the sequence $\{y^2(t_j)\}$. To this end, we may choose $j_0 := 0$, and for $k \geq 1$, we define by induction the sets M_k of all indices $i \geq j_{k-1}$ such that the sequence $\{y^2(t_j)\}_{j=j_{k-1}}^i$ is monotone. Then we set $j_k : \max M_k$ until $j_k = n$ for some $k = n'$. Therefore the sequence $\{y^2(t_{j_k})\}_{k=0}^{n'}$ is alternating, that is,

$$\left(y^2(t_{j_{k+1}}) - y^2(t_{j_k})\right)\left(y^2(t_{j_k}) - y^2(t_{j_{k-1}})\right) < 0 \quad for \quad k = 1, \ldots, n' - 1 \tag{2.2.104}$$

and there holds that

$$V(S) = \sum_{k=1}^{n'} |y^2(t_{j_k}) - y^2(t_{j_{k-1}})|. \tag{2.2.105}$$

We now either have $(-1)^k \left(y^2(t_{j_k}) - y^2(t_{j_{k-1}})\right) > 0$ for every $k = 1, \ldots, n'$ and

$$V(S) = y^2(0) - y^2(t_n) + 2\sum_{i=1}^{k'} \left(y^2(t_{j_{2i}}) - y^2(t_{j_{2i-1}})\right), \quad k' = \left[\frac{n'}{2}\right],$$

or for every $k = 1, 2, \ldots, n'$,

$$(-1)^k \left(y^2(t_{j_k}) - y^2(t_{j_{k-1}})\right) < 0,$$

and $k'' = [\frac{n'-1}{2}]$,

$$V(S) = y^2(0) - y^2(t_n) + 2\sum_{i=0}^{k''} \left(y^2(t_{j_{2i+1}}) - y^2(t_{j_{2i}})\right).$$

Noting that the function (2.2.102) is non-decreasing, we conclude in both cases

$$\begin{aligned} V(S) &\leq y^2(0) + 4\int_0^{t_n} y(\tau)\left(\|f\|_{[0,\bar{Y}]} + h(\tau)\right)d\tau \\ &\leq y^2(0) + 4\left(Y\|f\|_{[0,\bar{Y}]} + H\overline{Y}\right). \end{aligned} \tag{2.2.106}$$

Since the sequence S is arbitrary, (2.2.99) follows. In particular, the function $y^2(t)$ tends to a finite limit as $t \to +\infty$. Note that $y(t)$ is integrable, this limit must be zero.

Assume now f is an arbitrary non-negative continuous function. Then if for all $u \geq 0$, we take

$$g(u) := \max\left(0, \min(1, 2-u)\right), \quad G(u) := \int_0^u g(v)dv,$$

then from Lemma 2.2.2, it follows that

$$Q(t) := \int_0^t g(y(\tau))(f(y(\tau)) + h(\tau))d\tau - G(\tau)$$

is non-decreasing in $[0, +\infty)$. If now for all $t \geq 0$, we set $y^*(t) := G(y(t)) \leq y(t), q_1^*(t) := \int_0^t (F^* + h(\tau))d\tau - y^*(t)$, where $F^* := \|f\|_{[0,2]}$, then for every $t > s > 0$, we obtain $q_1^*(t) - q_1^*(s) \geq Q(t) - Q(s) \geq 0$, hence q_1^* is non-decreasing.

We are hence now in the previous situation, with $y^*(t), q_1^*(t), f^*(t) \equiv F^*$ instead of $y(t), q_1(t), f(u)$, respectively, which yields that $\lim_{t\to+\infty} y^*(t) = 0$, so that there exists a time $T > 0$ such that $y^*(t) \leq 1$ for all $t \geq T$. Thus $y(t) = y^*(t)$ for all $t \geq T^*$ and the proof is now complete. □

Remark 2.2.1 ([464]). In Theorem 2.2.10, if we remove the condition (2.2.96), then we no longer have an a priori bound for $y(t)$. Indeed, it suffices to consider any continuous function $f(u) \geq \max\{1, u\}$ such that $F_\infty := \int_0^{+\infty} v/f(v)dv < +\infty$. If we let $\Phi(u) := \int_0^u 1/f(v)dv$, we then have

$$\Phi_\infty := \int_0^{+\infty} 1/f(v)dv \leq \int_0^1 1/f(v)dv + \int_1^{+\infty} 1/f(v)dv \leq 1 + F_\infty - F(1).$$

For an arbitrary $\varepsilon \in (0, \Phi_\infty)$, there is a constant $R_\varepsilon > 0$ such that $\Phi_\infty - \Phi(R_\varepsilon) = \int_{R_\varepsilon}^{+\infty} 1/f(v)dv = \varepsilon$ and define the function

$$y_\varepsilon(t) := \begin{cases} \Phi^{-1}(t), \text{ for all } t \in [0, \Phi_\infty - \varepsilon], \\ 0, \text{ for all } t > \Phi_\infty - \varepsilon. \end{cases}$$

Then $y_\varepsilon(\Phi_\infty - \varepsilon) = R_\varepsilon$ and $y_\varepsilon'(t) = f(y_\varepsilon(t))$ for all $t \in (0, \Phi_\infty - \varepsilon)$, hence

$$\int_0^{+\infty} y_\varepsilon(t)dt = \int_0^{\Phi_\varepsilon - \varepsilon} y_\varepsilon'(t)y_\varepsilon(t)/f(y_\varepsilon(t))dt = F(R_\varepsilon) \leq F_\infty$$

and

$$\int_0^t f(y_\varepsilon(\tau))d\tau - y_\varepsilon(t) = \begin{cases} 0, \text{ for all } t \in [0, \Phi_\infty - \varepsilon], \\ R_\varepsilon + tf(0), \text{ for all } t > \Phi_\infty - \varepsilon. \end{cases}$$

Hence, the hypotheses (i)–(iii) of Theorem 2.2.10 hold with $h \equiv 0$ and $Y = F_\infty$ independently of ε, while $y_\varepsilon(\Phi_\infty - \varepsilon) \to +\infty$ as $\varepsilon \to 0$.

The next result, which can be found in Wang [946], is concerned with the method of Lyapunov functionals. Since it is a useful tool in dealing with the asymptotic behavior of global solutions to evolutionary differential equations, we still state it here.

Theorem 2.2.13 ([946]). *Let $a > 0$ be a constant, and $V(t) \in C^1([a, +\infty))$ satisfy*

(1) *$V(t)$ is bounded from below;*

(2)
$$\begin{cases} \dfrac{dV(t)}{dt} \le -Kg(t) + h(t), & (2.2.107) \\ g(t) \in C^1[a, +\infty),\ g(t) \ge 0,\ \displaystyle\int_a^{+\infty} h(t)dt < +\infty, & (2.2.108) \end{cases}$$

where $K > 0$ is a constant. If there exists a constant β such that

$$\frac{dg(t)}{dt} \le \beta, \tag{2.2.109}$$

then

$$\lim_{t \to +\infty} g(t) = 0. \tag{2.2.110}$$

Proof. If (2.2.110) is not valid, then without loss of generality, we may assume that $\beta > 0$, and there is a constant $\varepsilon_0 > 0$ and a sequence $\{t_i\}, t_i \to +\infty$ satisfying

$$a = t_0 < t_1 < t_2 < \cdots < t_i < \cdots, \quad t_i - t_{i-1} > 1,\ i = 1, 2, \ldots \tag{2.2.111}$$

such that

$$g(t_i) > \varepsilon_0. \tag{2.2.112}$$

Now choose an $\alpha > 0$ so small that

$$t_{i-1} < t_i - \alpha, \quad \alpha\beta < \varepsilon_0/2. \tag{2.2.113}$$

With the help of (2.2.109), we derive for all $t \in [t_i - \alpha, t_i]$,

$$g(t_i) - g(t) \le \beta(t_i - t). \tag{2.2.114}$$

Then it follows from (2.2.112) and (2.2.114) that

$$g(t) \ge g(t_i) + \beta(t - t_i) > \varepsilon_0 - \beta\alpha > \varepsilon_0/2, \quad t_i - \alpha \le t \le t_i. \tag{2.2.115}$$

Exploiting (2.2.107) and (2.2.115), we deduce that for all $N \in \mathbb{N}$ and for all $t \ge t_N$,

$$\begin{aligned} V(t) &\le V(a) - K\int_a^{+\infty} g(t)dt + \int_a^{+\infty} h(t)dt \\ &\le V(a) - K\sum_{i=1}^{N}\int_{t_i-\alpha}^{t_i} g(t)dt + \int_a^{+\infty} h(t)dt \\ &\le V(a) - \varepsilon_0 NK\alpha/2 + \int_a^{+\infty} h(t)dt. \end{aligned} \tag{2.2.116}$$

Thus it follows from (2.2.108) by letting $N \to +\infty$ that in (2.2.116),

$$\lim_{t\to+\infty} V(t) = -\infty,$$

which contradicts the assumption (1). The proof is thus complete. □

To end this section, we are going to introduce a conjecture due to Cruz-Sampedro [178] which describes the exact asymptotic behavior of solution at infinity to abstract second-order differential inequalities in Hilbert spaces and extends previous work of Jäger [392] of Agmon.

Assume $\mathbb{H}$ is a Hilbert space over the complex numbers $\mathbb{C}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We consider functions ω on $\mathbb{R}^+$ with values in $\mathbb{H}$ satisfying the following form of second-order differential inequalities

$$\| -\omega''(r) + B_r\omega(r) - k^2\omega(r)\| \le Cr^{-1-\delta}\|\omega(r)\| + g(r), \quad r > 0, \tag{2.2.117}$$

where $C \ge 0$ and $\delta > 0$ are constants, g is a function in $L^2(\mathbb{R}^+)$ with compact support, and k is a complex number satisfying $\Im k > 0$, and as in [392], we assume that B_r are non-negative linear operators on $\mathbb{H}$ with a common domain D, independent of r and dense in $\mathbb{H}$, and satisfy the following condition:

Condition I. For every $h \in D$, the map $r \mapsto B_r h$ from $\mathbb{R}^+$ to $\mathbb{H}$ is continuous and the function $r \mapsto (B_r h, h)$ is of class C^1 on $\mathbb{R}^+$. Moreover, for some constant $\beta > 1$, for all $h \in D$ and some $r_0 \ge 0$,

$$-\frac{d}{dr}(B_r h, h) \ge \frac{\beta}{r}(B_r h, h), \quad r > r_0. \tag{2.2.118}$$

Thus we are in a position to state main results from [178] as follows.

Theorem 2.2.14 ([178]). *If $\omega \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{H})$ is a solution to* (2.2.117)*, with $\Im k > 0$, satisfying*

$$\int_0^{+\infty} \|\omega(r)\|^2 dr < +\infty, \tag{2.2.119}$$

then there exists a $\psi \in \mathbb{H}$ such that

$$\lim_{r\to+\infty} \exp(-ikr)\omega(r) = \psi \ \textit{ strongly in } \ \mathbb{H}. \tag{2.2.120}$$

It follows from Theorem 2.2.12 that any square integrable solution ω of (2.2.117) with $\Im k > 0$ satisfies as $r \to +\infty$

$$\|\omega(r)\| = O(\exp(-\Im kr)). \tag{2.2.121}$$

It follows from the following unique continuation "at infinity", due to Cruz-Sampedro [178], that this upper bound is optimal.

Theorem 2.2.15 ([178]). *Assume that* ω *is defined as in Theorem* 2.2.12 *and*

$$\lim_{r\to+\infty} \|\exp(-ikr)\omega(r)\| = 0. \tag{2.2.122}$$

Then ω *has compact support.*

Recall that the exact asymptotic behavior at infinity of solutions to (2.2.117) with $k \in \mathbb{R}\backslash\{0\}$ has been studied by Jäger [390, 391, 392] who proved not only in [392] an analogue of Theorem 2.2.12 for $\delta > 1/2$, but also in [390, 391] that if ω is a solution to (2.2.117) satisfying $\liminf_{r\to+\infty}\|\omega(r)\| = 0$, then ω has compact support. We also note that, when $k \in \mathbb{C}$ with $\Im k > 0$, Agmon [11] proved Theorem 2.2.12 for the class of operators $B_r = p(r)S$, where S is a fixed non-negative operator on $\mathbb{H}$ and $p(r)$ a positive continuously differentiable function satisfying for some constant $\beta > 1$ and for all $r > r_0$

$$(r^\beta p(r))' \leq 0. \tag{2.2.123}$$

Furthermore, Agmon [11] also proved Theorem 2.2.13 for this latter class of operators for $\beta = 2$ and conjectured that his result holds for any $\beta > 1$. In fact, Theorem 2.2.13 ([178]) has confirmed that Agmon's conjecture is correct.

It is known that the usual concrete form of operators B_r satisfying **Condition I** is given by $B_r = -\Lambda/r^2$ with Λ being the Laplace–Beltrami operator on the unit sphere S^{n-1} (see, e.g., [11, 12, 390, 391]). We also find the concrete operators B_r satisfying **Condition I** without the form $B_r = p(r)S$ in [15, 178].

First, Theorem 2.2.12 is a direct corollary of the abstract results in Theorems 2.2.14 and 2.2.15 below. Since the proofs of these results are simple variants of Agmon's proofs [11] of the similar results for $B_r = p(r)S$, we shall omit the detailed proof here. In order to prove Theorem 2.2.13, it merely suffices to follow Agmon [11] (see, e.g., Cruz-Sampedro [178]) closely and introduce some new techniques of the Bessel-function arguments of Agmon to prove his conjecture. Next, to prove Theorem 2.2.13 we need some auxiliary estimates and definitions.

Consider the following abstract second-order differential operator L

$$L = -\frac{d^2}{dr^2} + 2A_r\frac{d}{dr} + B_r, \tag{2.2.124}$$

where the operator L acts on $\mathbb{H}$-valued functions on $\mathbb{R}^+$; for all $r > 0$, A_r and B_r are linear operators on $\mathbb{H}$ with a common domain D dense in $\mathbb{H}$ and independent of r.

We first make the following assumptions (see Cruz-Sampedro [178]).

(A) There exists a constant $\kappa > 0$ such that for all $h \in D$ and $r > 0$,

$$\Re(A_r h, h) \geq \kappa\|h\|^2. \tag{2.2.125}$$

(B) The operators B_r are non-negative, and for every $h \in D$, the map $r \mapsto B_r h$ from $\mathbb{R}^+$ to $\mathbb{H}$ is continuous and the function $r \mapsto (B_r h, h)$ is of class C^1 on

$\mathbb{R}^+$. Moreover, for some $\mu > 0$, $r_0 \geq 0$ and $r > r_0$, we have for all $h \in D$,

$$-\frac{d}{dr}(B_r h, h) \geq \frac{\mu}{r}(B_r h, h). \tag{2.2.126}$$

Next, we define the domain of the operator L.

Definition 2.2.1 ([178]). By W_L we denote the class of $\mathbb{H}$-valued functions v on $\mathbb{R}^+$ satisfying:

i) v belongs to the Sobolev space $W^{2,2}_{\text{loc}}(\mathbb{R}^+;\mathbb{H})$, i.e., v is strongly differentiable and v' is absolutely continuous on compact subintervals of $\mathbb{R}^+$, with $v'' \in L^2_{\text{loc}}(\mathbb{R}^+;\mathbb{H})$.
ii) For almost all $r > 0$, $v(r)$ and $v'(r)$ belong to D.
iii) The map $r \mapsto B_r v(r)$ belongs to $L^2_{\text{loc}}(\mathbb{R}^+;\mathbb{H})$ and the map $r \mapsto (B_r v(r), v(r))$ is absolutely continuous on compact sub-intervals of $\mathbb{R}^+$.

Now taking $v \in W_L$ and noting that the operators B_r are non-negative, for any r, s in $\mathbb{R}^+$,

$$\begin{aligned}&(B_r v(r), v(r)) - (B_s v(s), v(s))\\ &\quad \geq (B_r v(s), v(s)) - (B_s v(s), v(s)) + 2\Re(B_r v(s), v(r) - v(s)).\end{aligned} \tag{2.2.127}$$

Using (2.2.127) and the assumptions on v, we may conclude that for almost all $r > 0$,

$$\frac{d}{dr}(B_r v(r), v(r)) = (\dot{B}_r v(r), v(r)) + 2\Re(B_r v(r), v'(r)), \tag{2.2.128}$$

where the prime at v denotes differentiation with respect to r and

$$(\dot{B}_r v(r), v(r)) = \frac{d}{dt}(B_t v(r), v(r))|_{t=r}. \tag{2.2.129}$$

Noting that the following three theorems are straightforward generalizations of similar results proved by Agmon [11] for $B_r = p(r)S$, we omit their proofs here. Indeed, Theorem 2.2.14 below can be derived as in [11] from Theorem 2.2.15 which follows directly from Theorem 2.2.14.

Theorem 2.2.16 ([178]). *Assume that v be an $\mathbb{H}$-valued function on $\mathbb{R}^+$ such that $v \in W_L$ and $v(r) = 0$ for all $0 \leq r \leq r_0$, where r_0 is as in (2.2.126), and further v satisfies, for all $r \geq r_0$,*

$$\|Lv(r)\| \leq Cr^{-1-\delta}\|v(r)\| + \varphi(r), \tag{2.2.130}$$

where $C \geq 0$ and $\delta > 0$ are constants and φ is a function in $L^2_{\text{loc}}(\mathbb{R}^+)$ satisfying

$$\int_0^{+\infty} r^\alpha \varphi^2(r)dr < +\infty \tag{2.2.131}$$

for some constant α *such that* $0 \le \alpha \le \mu$ *and* $\alpha < 1 + 2\delta$*. The constant* μ *is as defined in* (2.2.126)*. Assume, in addition, that there holds*

$$\int_0^{+\infty} \|v'(r)\|^2 \exp(-4\eta r)dr < +\infty \tag{2.2.132}$$

for some constant η *satisfying* $0 \le \eta < \kappa$*, where* κ *is as in* (2.2.125)*. Then the following inequality holds:*

$$\begin{aligned}\kappa \int_0^{+\infty} r^\alpha \|v'(r)\|^2 dr + (\mu - \alpha) \int_0^{+\infty} r^{\alpha-1}(B_r v(r), v(r))dr \\ \le \kappa^{-1} \left(\int_0^{+\infty} r^\alpha \varphi^2(r)dr + C_1^2 \int_0^T r^{\alpha-2\delta} \|v'(r)\|^2 dr \right),\end{aligned} \tag{2.2.133}$$

where

$$C_1 = 2C/(1 + 2\delta - \alpha) \quad \text{and} \quad T = (C_1/\kappa)^{1/\delta}.$$

Theorem 2.2.17 ([178])**.** *Assume now that* v *satisfies the assumptions of Theorem* 2.2.14 *and that the parameter* μ *in* (2.2.126) *is larger than* 1*. Then it follows that there exists a* $\psi \in \mathbb{H}$ *such that*

$$\lim_{r \to +\infty} v(r) = \psi \tag{2.2.134}$$

strongly in $\mathbb{H}$*.*

We point out here that the following result plays a crucial role in the proof of Theorem 2.2.13.

Theorem 2.2.18 ([178])**.** *Assume that* L *is defined as in* (2.2.124) *and* v *is an* $\mathbb{H}$*-valued function on* $\mathbb{R}^+$ *such that* $v \in W_L$ *and* $v(r) = 0$ *for all* $0 \le r \le r_0$*, where* r_0 *is as in* (2.2.118)*, that* v *satisfies, for all* $r \ge r_0$*,*

$$\|Lv(r) + \gamma r^{-\alpha} v(r)\| \le C r^{-1-\delta} \|v(r)\| + \varphi(r), \tag{2.2.135}$$

where $C \ge 0$ *and* $\delta > 0$ *are constants,* γ *is a real number,* α *satisfies* $1 < \alpha \le \beta$*, and* φ *is a function in* $L^2_{\text{loc}}(\mathbb{R}^+)$ *satisfying*

$$\int_0^{+\infty} r^\alpha |\varphi(r)|^2 dr < +\infty. \tag{2.2.136}$$

Moreover, assume that

$$\int_0^{+\infty} \|v'(r)\|^2 \exp(-4\eta r)dr < +\infty \tag{2.2.137}$$

for some constant η *satisfying* $0 \le \eta < \kappa$, *where* κ *is as in* (2.2.125). *Then*

$$\begin{aligned}\int_0^{+\infty} & r^{\alpha-2}\|v(r)\|^2 dr \\ & \le \frac{4}{(\kappa(\alpha-1))^2}\left(\int_0^{+\infty} r^{\alpha}|\varphi(r)|^2 dr + C^2\int_0^{R_0} r^{\alpha-2-2\delta}\|v(r)\|^2 dr\right),\end{aligned} \tag{2.2.138}$$

where

$$R_0 = (2C/\kappa(\alpha-1))^{1/\delta}.$$

Remark 2.2.2. ([178]) We observe that the condition that R_0 is independent of γ plays a key role in proving Theorem 2.2.13.

Now define on $\mathbb{R}^+$ for all $\nu > 0$, $\mu > 0$ and $r > 0$,

$$G_{\nu,\mu}(r) = (p(r/\nu))^{-1/4}\exp\left(-\nu\int_1^{r/\nu} p(t)^{1/2}dt\right), \tag{2.2.139}$$

where $p(r) \equiv -k^2 + r^{-\mu}$. Observe that $\Im k > 0$, $p(r)$ never vanishes, so that the square root of $p(r)$ is chosen to satisfy $\Re(p(r)^{1/2}) > 0$. Since the functions $G_{\nu,\mu}$ are the WKB approximations (see [706]) to the solutions of the Bessel-like equation

$$-u'' + \left(-k^2 + \frac{\nu^\mu}{r^\mu}\right)u = 0, \quad r > 0,$$

we can employ (2.2.139) to get

$$G'_{\nu,\mu}(r) = -\left((p(r/\nu))^{1/2} + \frac{1}{4\nu}\frac{p'(r/\nu)}{p(r/\nu)}\right)G_{\nu,\mu}(r) \tag{2.2.140}$$

and

$$G''_{\nu,\mu}(r) = \left(-k^2 + \frac{\nu^\mu}{r^\mu} + \frac{1}{r^2}g_\nu(r)\right)G_{\nu,\mu}(r), \tag{2.2.141}$$

where the constant C_1 depends only on k and μ such that for all $r > 0$ and all $\nu > 0$,

$$|g_\nu(r)| \le C_1. \tag{2.2.142}$$

Thus $g_\nu(r)/r^2$ is short-range uniformly in ν, which is crucial to apply Theorem 2.2.14 in the proof. Next, if we set $A_\nu(r) = -G'_{\nu,\mu}(r)/G_{\nu,\mu}(r)$, then we can show that there exists a $\nu_0 > 0$, depending on μ, such that for all $r > 1$ and for all $\nu > \nu_0$,

$$\Re A_\nu(r) \ge \frac{1}{2}(Im\ k)^{3/2}|k|^{-1/2}. \tag{2.2.143}$$

In fact, we may conclude

$$A_\nu(r) = (-k^2 + (\nu/r)^\mu)^{1/2} - q_\nu(r)/r, \tag{2.2.144}$$

where

$$q_\nu(r) = \frac{\mu}{4}\frac{(\nu/r)^\mu}{(\nu/r)^\mu - k^2}.$$

Since $\Im k > 0$, there is a constant $C_2 > 0$ independent of ν such that $|q_\nu(r)| \le C_2$ for all $r > 0$. Let $\theta = \mathrm{Arg}(-ik), |\theta| < \pi/2$. Thus using some basic estimates as in [11], we can derive that

$$\begin{aligned} \Re\left((\nu/r)^\mu - k^2\right)^{1/2} &\ge \cos\theta|(\nu/r)^\mu - k^2|^{1/2} \\ &\ge (\cos\theta)^{3/2}((\nu/r)^\mu + |k|^2)^{1/2} \\ &\ge \frac{1}{\sqrt{2}}(\cos\theta)^{3/2}\left((\nu/r)^{\mu/2} + |k|\right), \end{aligned} \tag{2.2.145}$$

from which it follows that there exists a $\nu_0 > 0$ such that for all $\nu > \nu_0$ and for all $r > 1$,

$$\Re\, A_\nu(r) \ge \frac{1}{2}(\cos\theta)^{3/2}|k| = \frac{1}{2}(\Im k)^{3/2}|k|^{-1/2}.$$

This hence proves (2.2.143). Finally, set $G_{\nu,\mu}(r) = \exp(-\rho\nu + i\theta\nu)$, where ρ_ν and θ_ν are real-valued functions on $\mathbb{R}^+$, and note that $y_\nu(r) \equiv \rho'_\nu(r)$ satisfies $y_\nu(r) = \Re\, A_\nu(r)$ and for all $r > 0$,

$$y'_\nu(r) = (y_\nu)^2 - (\theta'_\nu)^2 - (\Im\, k)^2 + (\Re\, k)^2 - (\nu/r)^\mu - \Re\, g_\nu(r)/r^2. \tag{2.2.146}$$

After having established Theorems 2.4.14–2.4.16, we can prove Theorem 2.2.13, see [178] for details.

2.3 Differential and difference inequalities leading to decay rates

From Section 2.2 we know only that the non-negative function ($y(t)$, say) goes to zero as time tends to infinity. We have no information on the decay rate of $y(t)$. In fact, the decay rate of $y(t)$ depends on some factors which include some terms in the inequality. This can be clearly seen from the following two theorems, which indicate that when the differential inequality involves a decay term $h(t)$, the corresponding non-negative function $y(t)$ has a similar decay rate.

The next result was obtained in Qin, Ren and Wei [803] to prove the decay rate of global solutions.

Theorem 2.3.1 (The Qin–Ren–Wei Inequality [803]). *Assume that $y(t) \in C^1(\mathbb{R}^+)$, $y(t) \ge 0$ on $\mathbb{R}^+$, and satisfies for all $t > 0$,*

$$y'(t) \le -C_0 y(t) + \lambda(t), \tag{2.3.1}$$

where $0 \le \lambda(t) \in L^1(\mathbb{R}^+)$ *and* $C_0 > 0$ *is a constant. Then*

$$\lim_{t\to+\infty} y(t) = 0. \tag{2.3.2}$$

Furthermore,

(1) *if* $\lambda(t) \le C_1 e^{-\alpha_0 t}$ *for all* $t > 0$, *with* $C_1 > 0, \alpha_0 > 0$ *being constants, then for all* $t > 0$,

$$y(t) \le C_2 e^{-\alpha t}, \tag{2.3.3}$$

where $C_2 > 0$ *and*$\alpha > 0$ *are constants.*

(2) *if* $\lambda(t) \le C_3(1+t)^{-p}$ *for all* $t > 0$, *where* $p > 1$ *and* $C_3 > 0$ *are constants, then for all* $t > 0$,

$$y(t) \le C_4(1+t)^{-p+1}, \tag{2.3.4}$$

with a constant $C_4 > 0$.

Proof. Multiplying (2.3.1) by $e^{C_0 t}$ and integrating the resulting inequality, we have

$$y(t) \le y(0)e^{-C_0 t} + e^{-C_0 t}\int_0^t \lambda(s)e^{C_0 s}ds. \tag{2.3.5}$$

Noting that $\lambda(t) \in L^1(\mathbb{R}^+)$, we get

$$\begin{aligned} e^{-C_0 t}\int_0^t \lambda(s)e^{C_0 s}ds &= \int_0^{t/2} \lambda(s)e^{-C_0(t-s)}ds + \int_{\frac{t}{2}}^t \lambda(s)e^{-C_0(t-s)}ds \\ &\le e^{-(C_0/2)t}\int_0^{+\infty} \lambda(s)ds + \int_{\frac{t}{2}}^t \lambda(s)ds \to 0, \text{ as } t \to +\infty. \end{aligned} \tag{2.3.6}$$

Now (2.3.2) follows from (2.3.5) and (2.3.6).

(1) If $\lambda(t) \le C_1 e^{-\alpha_0 t}$, for all $t > 0$, with $C_1 > 0, \alpha_0 > 0$ being constants, then it follows from (2.3.5)–(2.3.6) that

$$\begin{aligned} y(t) &\le y(0)e^{-C_0 t} + C_1' e^{-(C_0/2)t} + C_1\int_{\frac{t}{2}}^t e^{-\alpha_0 s}ds \\ &\le y(0)e^{-C_0 t} + C_1' e^{-(C_0/2)t} + \frac{C_1}{\alpha_0}e^{-(\alpha_0/2)t} \\ &\le C_2 e^{-\alpha t}, \end{aligned}$$

with $C_1' = \int_0^{+\infty} \lambda(s)ds$, $C_2 = \max\left\{y(0),\ C_1',\ \frac{C_1}{\alpha_0}\right\} > 0$ and $\alpha = \frac{1}{2}\min\{C_0,\ \alpha_0\} > 0$. This proves (2.3.3).

(2) If $\lambda(t) \le C_3(1+t)^{-p}$, for all $t > 0$, with $p > 1, C_3 > 0$ being constants, similarly to case (1), it follows from (2.3.5)–(2.3.6) that for all $t > 0$,

$$\begin{aligned} y(t) &\le y(0)e^{-C_0 t} + C_1' e^{-(C_0/2)t} + C_3 \int_{\frac{t}{2}}^{t} \frac{1}{(1+s)^p} ds \\ &\le y(0)e^{-C_0 t} + C_1' e^{-(C_0/2)t} + \frac{C_3}{p-1}\left(1 + \frac{t}{2}\right)^{-p+1} \\ &\le C_4(1+t)^{-p+1}, \end{aligned}$$

for some constant $C_4 = C_4(y(0),\ C_1',\ C_3,\ p) > 0$ depending only on $y(0),\ C_1',\ C_3$ and p. This gives us (2.3.4) and hence the proof is complete. □

For case (1) in Theorem 2.3.1, there is another proof, due to Muñoz Rivera [645], which will be stated in the next corollary.

Corollary 2.3.1 (The Muñoz Rivera Inequality [645])**.** *Assume that*

$$y(t) \in C^1(\mathbb{R}^+), \quad y(t) \ge 0 \ \ on \ \ \mathbb{R}^+,$$

and satisfies for all $t > 0$,

$$y'(t) \le -C_0 y(t) + C_1 e^{-\alpha_0 t} \tag{2.3.7}$$

where C_0, C_1, *and* α_0 *are positive constants. Then there exist positive constants* C_2 *and* α *such that for all* $t > 0$,

$$y(t) \le C_2 e^{-\alpha t}. \tag{2.3.8}$$

Proof. Let

$$F(t) = y(t) + \frac{2C_1}{\alpha_0} e^{-\alpha_0 t}.$$

Then from (2.3.7) we have

$$\begin{aligned} F'(t) &= y'(t) - 2C_1 e^{-\alpha_0 t} \le -C_0 y(t) - C_1 e^{-\alpha_0 t} \\ &\le -\alpha F(t), \end{aligned} \tag{2.3.9}$$

where $\alpha = \min(C_0, \alpha_0/2) > 0$. Consequently, for all $t > 0$,

$$F(t) \le C_2 e^{-\alpha t},$$

with $C_2 = F(0) = y(0) + \frac{2C_1}{\alpha_0}$, which implies (2.3.8). □

Recently, Qin and Ren [802] extended Theorem 2.3.1 to the next form.

Theorem 2.3.2 (The Qin–Ren Inequality [802]). *Suppose that $y(t) \in C^1(\mathbb{R}^+)$, $y(t) \geq 0$ on $\mathbb{R}^+$, and $h(t)$ is a positive function satisfying $\int_0^{+\infty} h(t)dt = +\infty$.*

(1) *If $y(t)$ satisfies for all $t > 0$,*

$$y'(t) \leq -C_0 h(t)y(t) + \lambda(t), \tag{2.3.10}$$

where $0 \leq \lambda(t) \in L^1(\mathbb{R}^+)$, $\lim\limits_{t\to+\infty} \frac{\lambda(t)}{h(t)} = 0$, and $C_0 > 0$ is a constant, then

$$\lim_{t\to+\infty} y(t) = 0. \tag{2.3.11}$$

Furthermore,

(2) *if $\lambda(t) \leq C_1 h(t)e^{-\alpha_0 \int_0^t h(s)ds}$ for all $t > 0$, with $C_1 > 0$, $\alpha_0 > 0$ being constants, then there exists a constant $R_1 > 0$, such that as $t \geq R_1$, for all $t > 0$,*

$$y(t) \leq C_2 e^{-\alpha \int_0^t h(s)ds}, \tag{2.3.12}$$

with $C_2 > 0$, $\alpha > 0$ being constants.

Proof. (1) Multiplying (2.3.10) by $e^{C_0 \int_0^t h(\omega)d\omega}$ and integrating the resulting inequality, we have

$$y(t) \leq y(0)e^{-C_0 \int_0^t h(\omega)d\omega} + e^{-C_0 \int_0^t h(\omega)d\omega} \int_0^t \lambda(s)e^{C_0 \int_0^s h(\omega)d\omega} ds. \tag{2.3.13}$$

Noting that $\lambda(t) \in L^1(\mathbb{R}^+)$, we have

$$\begin{aligned} e^{-C_0 \int_0^t h(\omega)d\omega} \int_0^t \lambda(s)e^{C_0 \int_0^s h(\omega)d\omega} ds &= \int_0^t \lambda(s)e^{-C_0 \int_s^t h(\omega)d\omega} ds \\ &= \frac{\int_0^t \lambda(s)e^{C_0 \int_0^s h(\omega)d\omega} ds}{e^{C_0 \int_0^t h(\omega)d\omega}}. \end{aligned} \tag{2.3.14}$$

By the l'Hospital rule, we can derive in case (1) that

$$\lim_{t\to+\infty} \frac{\int_0^t \lambda(s)e^{C_0 \int_0^s h(\omega)d\omega} ds}{e^{C_0 \int_0^t h(\omega)d\omega}} = \frac{1}{C_0} \lim_{t\to+\infty} \frac{\lambda(t)}{h(t)} = 0 \tag{2.3.15}$$

which, together with (2.3.13) and (2.3.14), gives us (2.3.11).

(2) If $\lambda(t) \leq C_1 h(t)e^{-\alpha_0 \int_0^t h(\omega)d\omega}$ for all $t > 0$, then from (2.3.14) it follows that

$$0 \leq \frac{1}{C_0} \lim_{t\to+\infty} \frac{\lambda(t)}{h(t)} \leq \frac{C_1}{C_0} \lim_{t\to+\infty} e^{-\alpha_0 \int_0^t h(\omega)d\omega} = 0,$$

whence

$$\lim_{t\to+\infty} \frac{\lambda(t)}{h(t)} = 0. \tag{2.3.16}$$

We claim that there exists a large constant $R_1 > 0$ such that if $t \geq R_1$, then

$$\frac{\int_0^t \lambda(s)e^{C_0 \int_0^s h(\omega)d\omega} ds}{e^{C_0 \int_0^t h(\omega)d\omega}} \leq e^{-\alpha \int_0^t h(\omega)d\omega}, \tag{2.3.17}$$

where $0 < \alpha < \min\{C_0,\ \alpha_0\}$.

In fact,

$$\begin{aligned}\lim_{t\to+\infty} \frac{\int_0^t \lambda(s)e^{C_0 \int_0^s h(\omega)d\omega} ds}{e^{(C_0-\alpha) \int_0^t h(\omega)d\omega}} &= \frac{1}{C_0 - \alpha} \lim_{t\to+\infty} \frac{\lambda(t)e^{C_0 \int_0^t h(\omega)d\omega} ds}{e^{(C_0-\alpha) \int_0^t h(\omega)d\omega} h(t)} \\ &\leq \frac{C_1}{C_0 - \alpha} \lim_{t\to+\infty} e^{(\alpha-\alpha_0) \int_0^t h(\omega)d\omega} = 0.\end{aligned}$$

Then there exists a large constant $R_1 > 0$ such that for $t \geq R_1$,

$$\frac{\int_0^t \lambda(s)e^{C_0 \int_0^s h(\omega)d\omega} ds}{e^{(C_0-\alpha) \int_0^t h(\omega)d\omega}} \leq 1, \tag{2.3.18}$$

which is (2.3.17). By (2.3.13), (2.3.17), (2.3.18), for all $t \geq R_1$,

$$y(t) \leq y(0)e^{-C_0 \int_0^t h(\omega)d\omega} + e^{-\alpha \int_0^t h(\omega)d\omega} \leq C_2 e^{-\alpha \int_0^t h(\omega)d\omega}, \tag{2.3.19}$$

with $C_2 = \max\{y(0),\ 1\} > 0$. This proves (2.3.12). □

The next result is simple and can be proved directly.

Theorem 2.3.3 (The Qin–Ren Inequality [802]). *Let $0 < y(t) \in C^1([0, +\infty))$ and satisfy for all $t > 0$,*

$$y'(t) + ay^p(t) \leq 0, \quad y(0) > 0, \tag{2.3.20}$$

where $p \geq 1$ and $a > 0$ is a constant. Then for all $t > 0$,

$$\begin{cases} y(t) \leq y(0)e^{-at}, & \text{if } p = 1, \quad (2.3.21) \\ y(t) \leq 1/[y^{1-p}(0) + a(p-1)t]^{1/(p-1)}, & \text{if } p > 1. \quad (2.3.22) \end{cases}$$

Note that when $p = 1$, the estimate (2.3.21) reduces to (2.3.8) with $C_1 = 0$ in Corollary 2.3.1, and (2.3.22) establishes the polynomial decay of $y(t)$.

We may find the following result in Muñoz Rivera [645].

Theorem 2.3.4 (The Muñoz Rivera Inequality [645]). *Assume that $y(t) \in C^1(\mathbb{R}^+)$, $y(t) \geq 0$ on $\mathbb{R}^+$, and satisfies for all $t > 0$,*

$$y'(t) \leq -K_0[y(t)]^{1+1/p} + \frac{K_1}{(1+t)^{1+p}} \tag{2.3.23}$$

where $K_0 > 0, K_1 > 0$ and $p > 1$ are constants. Then there exists a constant $K_2 > 0$ such that for all $t > 0$,

$$y(t) \leq \frac{K_2[py(0) + 2K_1]}{(1+t)^p}. \tag{2.3.24}$$

Proof. Let $h(t) = \frac{2K_1}{p(1+t)^p}$ and $g(t) = y(t) + h(t)$. Then (2.3.23) yields

$$\begin{aligned} g'(t) &= y'(t) - \frac{2K_1}{(1+t)^{p+1}} \\ &\leq -K_0 \Big\{ [y(t)]^{1+1/p} + K_0^{-1} K_1 (1+t)^{-(1+p)} \Big\} \\ &\leq -K_0 \left\{ [y(t)]^{1+1/p} + \left(\frac{p}{2}\right)^{1+1/p} \frac{1}{K_0 K_1^{1/p}} [h(t)]^{1+1/p} \right\}. \end{aligned} \tag{2.3.25}$$

Thus taking $a_0 = \min\left(1, (\frac{p}{2})^{1+1/p} \frac{1}{K_0 K_1^{1/p}}\right)$, we can obtain

$$g'(t) \leq -K_0 a_0 \Big\{ [y(t)]^{1+1/p} + [h(t)]^{1+1/p} \Big\}. \tag{2.3.26}$$

Noting that

$$[g(t)]^{1+1/p} = [f(t) + g(t)]^{1+1/p} \leq a_1 \left([f(t)]^{1+1/p} + [g(t)]^{1+1/p} \right)$$

with $a_1 = 2^{p-1}$, we can conclude from (2.3.26) that

$$g'(t) \leq \frac{-K_0 a_0}{a_1} [g(t)]^{1+1/p},$$

which implies

$$\begin{aligned} g(t) &\leq p^p g(0) \left\{ p + \frac{K_0 a_0}{a_1} [g(0)]^{1/p} t \right\}^{-p} \\ &\leq \frac{p^{p-1}[py(0) + 2K_1]}{a_2^p (1+t)^p} \end{aligned} \tag{2.3.27}$$

with $a_2 = \min\left(p, \frac{K_0 a_0}{a_1} [g(0)]^{1/p}\right)$. Therefore, (2.3.24) follows from (2.3.27) with $K_2 = p^{p-1}/a_2^p$. □

The following result is a generalization of Theorem 2.3.4 (see, e.g., [652]).

Theorem 2.3.5 (The Muñoz Rivera–Racke Inequality [652])**.** *Assume that the function $f(t) \geq 0$ is differentiable on $\overline{\mathbb{R}^+}$ and satisfies for all $t > 0$,*

$$f'(t) \leq \frac{-c_1}{f(0)^{1/\alpha}} f(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^\beta} f(0) \tag{2.3.28}$$

with positive constants $\alpha > 0$, $c_1, c_2, f(0) > 0$ and

$$\beta \geq \alpha + 1. \tag{2.3.29}$$

Then there exists a constant $c_3 > 0$ such that for all $t \geq 0$

$$f(t) \leq \frac{c_3}{(1+t)^\alpha} f(0). \tag{2.3.30}$$

Proof. Let for all $t > 0$,

$$F(t) := f(t) + \frac{2c_2}{\alpha}(1+t)^{-\alpha} f(0).$$

Then from (2.3.29), it follows

$$\begin{aligned} F'(t) &= f'(t) - 2c_2(1+t)^{-(\alpha+1)} f(0) \\ &\leq -\frac{c_1}{f(0)^{1/\alpha}} f^{1+\frac{1}{\alpha}}(t) - c_2(1+t)^{-(\alpha+1)} f(0), \end{aligned}$$

which yields

$$\begin{aligned} F'(t) &\leq -\frac{c}{f(0)^{1/\alpha}} \left(f^{1+\frac{1}{\alpha}}(t) + (1+t)^{-(\alpha+1)} f(0)^{1+\frac{1}{\alpha}} \right) \\ &\leq -\frac{c}{F(0)^{1/\alpha}} F^{1+\frac{1}{\alpha}}(t). \end{aligned} \tag{2.3.31}$$

Integration of (2.3.31) with respect to t yields

$$F(t) \leq \frac{F(0)}{(1+ct)^{\alpha}} \leq \frac{c}{(1+t)^{\alpha}} f(0)$$

which gives (2.3.30) for some constant $c_3 > 0$. □

The next result is another criterion for the polynomial decay rate which was obtained by Kim [439] in terms of a difference inequality.

Theorem 2.3.6 (The Kim Inequality [439])**.** *Let H be a continuous, positive, decreasing real function on $[0, +\infty)$ and bounded by 1. Assume that there are four constants $C_1 > 1$ and $C_2, \beta, \gamma > 0$, such that for all $s > 0$,*

$$H(s) \leq C_1 \left(\frac{1}{H(s)} \right)^{\beta} \left[H(s) - H\left(\left(\frac{C_2}{H(s)} \right)^{\gamma} + s \right) \right]. \tag{2.3.32}$$

Then there exist constants $C > 0$ and $\delta > 0$, such that for any $t > 0$,

$$H(t) \leq Ct^{-\delta}. \tag{2.3.33}$$

Proof. Let $t > 0$. We distinguish two cases:

(1) if $\left(\frac{H(s)}{C_2}\right)^{\gamma} < \frac{1}{t}$, then $H(s) \leq \frac{C_2}{t^{1/\gamma}}$;

(2) if $\left(\frac{H(s)}{C_2}\right)^{\gamma} \geq \frac{1}{t}$, then $\left(\frac{C_2}{H(s)}\right)^{\gamma} + s \leq t + s$, thus

$$H(t+s) \leq H\left(\left(\frac{C_2}{H(s)} \right)^{\gamma} + s \right) \tag{2.3.34}$$

and

$$H(s) \leq C_1^{1/(\beta+1)} \left\{ H(s) - H(t+s) \right\}^{1/(\beta+1)}. \tag{2.3.35}$$

Consequently, for all $s, t > 0$,

$$H(s) \leq C_1^{1/(\beta+1)} \Big\{ H(s) - H(t+s) \Big\}^{1/(\beta+1)} + \frac{C_2}{t^{1/\gamma}}. \tag{2.3.36}$$

Let

$$\Psi_t(s) = \left[\left(\frac{C_1 t}{s} \right)^{1/(\beta+1)} + \frac{C_2}{t^{1/\gamma}} \right]^{-1}.$$

Now we again distinguish two cases:

(i) if $H(s) - H(t+s) \leq \frac{t}{t+s}$, then $H(s) \leq (\frac{C_1 t}{t+s})^{1/(\beta+1)} + \frac{C_2}{t^{1/\gamma}}$, and thus $\Psi_t(t+s)H(t+s) \leq 1$;

(ii) if $\frac{t}{t+s} < H(s) - H(t+s)$, then $\frac{tH(s)}{t+s} \leq \frac{t}{t+s} < H(s) - H(t+s)$, and therefore

$$\begin{aligned} \Psi_t(t+s)H(t+s) &< \frac{s}{t+s} H(s)\Psi_t(t+s) = \Psi_t(s)H(s) \left(\frac{\frac{\Psi_t(t+s)}{t+s}}{\frac{\Psi_t(s)}{s}} \right) \\ &< \Psi_t(s)H(s), \end{aligned} \tag{2.3.37}$$

where we have used the fact that $\xi \longmapsto \Psi_t(\xi)\xi$ is decreasing. Consequently, we have proved that for any $s, t > 0$, we have either $\Psi_t(t+s)H(t+s) \leq 1$, or $\Psi_t(t+s)H(t+s) < \Psi_t(s)H(s)$. In particular, we deduce that for any $t > 0$ and $0 \neq n \in \mathbb{N}$, either $\Psi_t((n+1)t)H((n+1)t) \leq 1$, or $\Psi_t((n+1)t)H((n+1)t) < \Psi_t(nt)H(nt)$. Then by induction, we have

$$\Psi_t((n+1)t)H((n+1)t) \leq \max\Big(1, \Psi_t(t)H(t)\Big) = 1 \tag{2.3.38}$$

where we have used that $C_1 > 1$ and $H \leq 1$. Hence for all $t > 0$ and $0 \neq n \in \mathbb{N}$,

$$H\left((n+1)t\right) \leq \left(\frac{C_1}{n+1} \right)^{1/(\beta+1)} + \frac{C_2}{t^{1/\gamma}}. \tag{2.3.39}$$

We choose n such that $n+1 \leq t < n+2$ to obtain that for all $t \geq 2$,

$$H(t^2) \leq \left(\frac{2C_1}{t} \right)^{1/(\beta+1)} + \frac{C_2}{t^{1/\gamma}}. \tag{2.3.40}$$

The desired result now follows immediately. □

We give below an integro-differential inequality, due to Galdi [295], which also provides an exponential decay rate.

Theorem 2.3.7 (The Galdi Inequality [295]). *Assume that* $\beta \leq +\infty$ *and* $y(t)$ *is a real, non-negative continuous function in* $[0, \beta)$ *such that*

$$y(t) \in C^1(0, \beta), \quad \lim_{t \to \beta} y(t) = 0. \tag{2.3.41}$$

Then if $y(t)$ *satisfies the integro-differential inequality for all* $t \in (0, \beta)$,

$$y'(t) + a \int_t^\beta y(s)ds \le by(t), \tag{2.3.42}$$

with constants $a > 0$ *and* $b \in \mathbb{R}$, *then for all* $t \in (0, \beta)$,

$$y(t) \le ky(0)\exp(-\sigma t), \tag{2.3.43}$$

where $k = \sigma - 1/[\sigma_1 - \delta + \delta e^{-\sigma_1 \beta}] = \sqrt{b^2 + 4a}/\sigma$, $\sigma = (\sqrt{b^2 + 4a} - b)/2$, $\sigma_1 = \sqrt{b^2 + 4a}$, $\delta = (b + \sqrt{b^2 + 4a})/2$.

Proof. Let $\Psi(t) = y(t)e^{-bt}$. Then by (2.3.42)

$$\Psi'(t) + a \int_t^\beta e^{b(t-s)}\Psi(s)ds \le 0. \tag{2.3.44}$$

Set

$$F(t) = \Psi(t) + \delta \int_t^\beta e^{-b(t-s)}\Psi(s)ds. \tag{2.3.45}$$

Then from (2.3.44) it follows that

$$\begin{aligned} F'(t) + \delta F(t) = \Psi'(t) + a \int_t^\beta e^{-b(t-s)}\Psi(s)ds \\ + (\delta^2 - \delta b - a) \int_t^\beta e^{-b(t-s)}\Psi(s)ds \le 0, \end{aligned} \tag{2.3.46}$$

provided that we choose δ as the positive root to the equation $\delta^2 - \delta b - a = 0$, that is, $2\delta = b + \sqrt{b^2 + 4a}$. Integrating (2.3.42) yields

$$F(t) \le F(0)e^{-\delta t},$$

which can be equivalently rewritten as

$$y(t) + \delta \int_t^\beta y(s)ds \le F(0)e^{(b-\delta)t}. \tag{2.3.47}$$

We now estimate $F(0)$ in terms of $y(0)$. From (2.3.43), setting $\sigma_1 = 2\delta - b$, it follows that

$$-\frac{d}{dt}\left(e^{-\delta t} \int_t^\beta y(s)ds\right) \le F(0)e^{-\sigma_1 t}$$

which, upon integrating from 0 to β, gives

$$\int_0^\beta y(s)ds \le F(0)\frac{1 - e^{-\sigma_1 \beta}}{\sigma_1}. \tag{2.3.48}$$

Substituting the value of $F(0)$ into (2.3.44), we obtain

$$\int_0^\beta y(s)ds \le y(0)\frac{1-e^{-\sigma_1\beta}}{\sigma_1-\delta(1-e^{-\sigma_1\beta})}$$

which implies

$$F(0)=y(0)+\delta\int_0^\beta y(s)ds \le \frac{y(0)\sigma_1}{\sigma_1-\delta+\delta e^{-\sigma_1\beta}}. \tag{2.3.49}$$

Therefore, taking into account (2.3.47), we finally obtain (2.3.43). □

Remark 2.3.1 ([295]). Note that if $b<0$, (2.3.42) immediately implies (2.3.43) for $k=1$ and $\sigma=-b$.

Remark 2.3.2 ([295]). Since $k\le k_1=\sigma_1/(\sigma_1-\delta)$, it follows from (2.3.47) and (2.3.49) that (2.3.43) implies for all $t\in(0,\beta)$,

$$y(t)\le k_1 y(0)\exp(-\sigma t). \tag{2.3.50}$$

The following result is due to Agmon and Nirenberg [13, 14].

Theorem 2.3.8 (The Agmon–Nirenberg inequality). *If $\alpha(t),\beta(t)$ are continuous non-negative decreasing functions for all $t\ge 0$ satisfying for all $t\ge 1$,*

$$\alpha(t)\le\beta(t)+c'\left(\alpha(t-1)-\alpha(t)\right), \tag{2.3.51}$$

with some positive constant c', then for any $t\ge 1$,

$$e^{\sigma t}\alpha(t)\le c_1\left(\alpha(0)-\alpha(1)\right)+c_1\int_1^{+\infty} e^{\sigma t}\beta(t)dt+c_1\beta(1) \tag{2.3.52}$$

where c_1,σ are positive constants depending only on c', provided that the right-hand side of (2.3.52) *is finite.*

Proof. The proof is easy and is left to the reader as an exercise. □

Now we shall introduce a series of Nakao inequalities (see, e.g., Nakao [681, 663, 666, 680]). These inequalities are connected with difference inequalities, which are not only very important for the study of asymptotic behavior of global solutions, but also seem to be interesting in their own right. One of advantages of the Nakao inequalities is that any of them can furnish a decay rate.

The following two theorems can be found in [663, 684].

Theorem 2.3.9 (The Nakao Inequality [663]). *Let $\phi(t)$ be a bounded positive function on $\mathbb{R}^+$ satisfying, for some constants k and $\alpha>0$, and for all $t\ge 0$,*

$$k\phi^{\alpha+1}(t)\le\phi(t)-\phi(t+1). \tag{2.3.53}$$

Then for all $t\ge 1$,

$$\phi(t)\le\left\{\alpha k(t-1)+M^{-\alpha}\right\}^{-1/\alpha}, \tag{2.3.54}$$

where $M=\sup_{t\in[0,1]}\phi(t)$.

Proof. Setting $\phi^{-\alpha}(t) = y(t)$, then from (2.3.53) it follows

$$\begin{aligned} y(t+1) - y(t) &= \int_0^1 \frac{d}{d\theta}[\theta\phi(t+1) + (1-\theta)\phi(t)]^{-\alpha} d\theta \\ &= -\alpha \int_0^1 [\theta\phi(t+1) + (1-\theta)\phi(t)]^{-\alpha-1} d\theta \cdot [\phi(t+1) - \phi(t)] \\ &\geq \alpha k \phi^{\alpha+1}(t) \int_0^1 [\phi(t)]^{-\alpha-1} d\theta \\ &= \alpha k. \end{aligned} \tag{2.3.55}$$

For any $t \geq 1$, choose an integer n as $n \leq t < n+1$. Then from (2.3.55) it follows that

$$y(t) \geq y(t-n) + n\alpha k \geq y(t-n) + (t-1)\alpha k, \tag{2.3.56}$$

whence

$$\phi^{-\alpha}(t) \geq (t-1)\alpha k + \phi^{-\alpha}(t-n), \tag{2.3.57}$$

or

$$\begin{aligned} \phi(t) &\leq [\alpha k(t-1) + \phi(t-n)^{-\alpha}]^{-1/\alpha} \\ &\leq [\alpha k(t-1) + M^{-\alpha}]^{-1/\alpha}. \end{aligned} \tag{2.3.58}$$

□

Corollary 2.3.2 (The Nakao Inequality [663]). *Let $\phi(t)$ be a non-negative non-increasing function on $[0, T]$ for any $T >> 1$ such that for all $0 \leq t \leq T-1$,*

$$\phi^{\alpha+1}(t) \leq C_0(\phi(t) - \phi(t+1)),$$

with constants $C_0 > 0$ and $\alpha > 0$. Then for all $1 \leq t \leq T$,

$$\phi(t) \leq \{\phi^{-\alpha}(0) + \alpha C_0^{-1}(t-1)\}^{-1/\alpha}.$$

Theorem 2.3.10 (The Nakao Inequality [663]). *Let $\phi(t)$ be as in Theorem 2.3.9, which satisfies (2.3.53) for $\alpha = 0$. Then we have for all $t \geq 1$,*

$$\phi(t) \leq M e^{-k't}, \tag{2.3.59}$$

where $k' = -\log(1-k) > 0$.

Proof. By (2.3.53) for $\alpha = 0$, we have

$$\phi(t+1) \leq (1-k)\phi(t) \quad \text{(which implies } k < 1). \tag{2.3.60}$$

Therefore, for all $t \geq 1$, we have for an integer n with $n \leq t < n+1$,

$$\begin{aligned} \phi(t) &\leq \frac{1}{1-k}\phi(t-1) \leq \left(\frac{1}{1-k}\right)^n \phi(t-n) \\ &\leq M(1-k)^{-t} = Me^{t\log(1-k)} \end{aligned}$$

which proves (2.3.59). □

The next result was given in [663, 666, 676, 662].

Theorem 2.3.11 (The Nakao Inequality [663, 666, 676, 662]). *Assume that $\phi(t)$ is a bounded non-negative function on $\mathbb{R}^+$ satisfying for all $t \geq 0$,*

$$\max_{s\in[t,t+1]} \phi^{1+\alpha}(s) \leq K_0(\phi(t) - \phi(t+1)) + g(t), \tag{2.3.61}$$

where $K_0 > 0$ is a constant, $g(t)$ a non-negative function, α a non-negative constant. Then we have

(i) *if $\lim_{t\to+\infty} g(t) = 0$, then*

$$\lim_{t\to+\infty} \phi(t) = 0. \tag{2.3.62}$$

Moreover,

(ii) *if we assume that $\alpha > 0$ and $g(t) \leq K_1|t|^{-\theta-1}$ for constants $\theta > 1/\alpha, K_1 \geq 0$, then for all $t > 0$,*

$$\phi(t) \leq C_3 t^{-1/\alpha}, \tag{2.3.63}$$

and

(iii) *if $\alpha = 0$ and $g(t) \leq K_2 e^{-\theta t}$ for constants $\theta > 0, K_2 \geq 0$, then for all $t > 0$,*

$$\phi(t) \leq C_4 e^{-\theta_1 t} \tag{2.3.64}$$

where $\theta_1 = \min\left(\theta, \log \frac{K_0}{K_0-1}\right)$, and C_3, C_4 are positive constants depending on other constants known from the involved data.

Proof. First, we prove (2.3.62) in (i). Suppose (2.3.62) does not hold. Then there exist a real sequence $\{t_n\}_{n=1}^{\infty}$ and a constant $\varepsilon_0 > 0$ such that

$$t_n > 2n, \quad \phi(t_n) \geq \varepsilon_0 > 0. \tag{2.3.65}$$

Also by our assumption on $g(t)$ in (i), we can choose an integer $N \geq 1$, which may be as large as wanted, such that for all $t \geq N$,

$$g(t) \leq \frac{1}{2}\varepsilon_0^{1+\alpha}. \tag{2.3.66}$$

By (2.3.61), (2.3.65), and (2.3.66), we have

$$\varepsilon_0^{1+\alpha} \leq K_0(\phi(t_N - 1) - \phi(t_N)) + g(t_N - 1) \tag{2.3.67}$$

and

$$0 < \frac{1}{2}\varepsilon_0^{1+\alpha} \leq K_0(\phi(t_N - 1) - \phi(t_N)). \tag{2.3.68}$$

Therefore, we can use again (2.3.61), (2.3.65), and (2.3.66) to obtain

$$\frac{1}{2}\varepsilon_0^{1+\alpha} \leq K_0(\phi(t_N - 2) - \phi(t_N - 1)). \tag{2.3.69}$$

Repeating this procedure, it follows that for $j = 1, \dots, N$,

$$\frac{1}{2}\varepsilon_0^{1+\alpha} \le K_0(\phi(t_N - j) - \phi(t_N - j + 1)). \tag{2.3.70}$$

Summing up the above inequalities over $j = 1, \dots, N$ yields

$$\frac{1}{2}N\varepsilon_0^{1+\alpha} \le K_0\{\phi(t_N - N) - \phi(t_N)\}, \tag{2.3.71}$$

which is impossible, because the left-hand side tends to $+\infty$ as N goes to $+\infty$, while the right-hand side remains bounded thanks to the boundedness of $\phi(t)$.

Next, we prove (2.3.63) in (ii). Set $\phi_0(t) = vt^{-\theta}, v > 0$, and $w(t) = \phi(t) + \phi_0(t)$. Then we have, for any $t > 0$,

$$\begin{aligned}\max_{s\in[t,t+1]} |w(s)|^{1+\alpha} &= \max_{s\in[t,t+1]} |\phi(s) + \phi_0(s)|^{1+\alpha} \\ &\le 2^{1+\alpha} \max_{s\in[t,t+1]} \left[\phi^{1+\alpha}(s) + \phi_0^{1+\alpha}(s)\right] \\ &\le 2^{1+\alpha}K_0\left[w(t) - w(t+1)\right] + I(t),\end{aligned} \tag{2.3.72}$$

where

$$I(t) = 2^{1+\alpha}\left[-K_0vt^{-\theta} + K_0v(t+1)^{-\theta} + v^{1+\alpha}t^{-\theta(1+\alpha)} + g(t)\right]. \tag{2.3.73}$$

We shall show $I(t) < 0$ for sufficiently large t. Indeed, we write

$$I(t) = vK_0 2^{1+\alpha}(t+1)^{-\theta}\left(1 - \left(\frac{t+1}{t}\right)^{\theta} + \frac{v^{\alpha}}{K_0}(t+1)^{\theta}t^{-\theta(1+\alpha)} + \frac{1}{vK_0}(t+1)^{\theta}g(t)\right). \tag{2.3.74}$$

Here it is easily seen that there exists a positive integer $T_1 > 0$ such that for all $t > T_1$,

$$\left(\frac{t+1}{t}\right)^{\theta} - 1 \ge \frac{1}{2}\theta t^{-1}. \tag{2.3.75}$$

Therefore, by (2.3.75) and the assumption on $g(t)$ in (ii), we have, for any $t \ge T_1$,

$$I(t) \le C(t+1)^{-\theta}t^{-1}\left[-\frac{\theta}{2} + \frac{1}{vK_0K_1} + \frac{v^{\alpha}}{K_0}(t+1)^{\theta}t^{-\theta(1+\alpha)+1}\right]. \tag{2.3.76}$$

Furthermore, since $\theta > \frac{1}{\alpha}$, we have

$$\lim_{t\to+\infty}(t+1)^{\theta}t^{-\theta(1+\alpha)+1} = 0. \tag{2.3.77}$$

Therefore, if we choose v so large that $(vK_0K_1)^{-1} < \frac{\theta}{2}$, and choose T $(\ge T_1)$ sufficiently large, then we have for all $t > T$,

$$I(t) < 0. \tag{2.3.78}$$

Consequently, for any $t > T$, we derive from (2.3.72)

$$\max_{s\in[t,t+1]} w(s)^{1+\alpha} \le 2^{1+\alpha} K_0 \left[w(t) - w(t+1)\right]. \tag{2.3.79}$$

Now, putting $w^{-\alpha}(t) = y(t)$, and noting that

$$\max_{s\in[t,t+1]} w^{1+\alpha}(s) \int_0^1 \left\{\theta w(t+1) + (1-\theta) w(t)\right\}^{-1-\alpha} d\theta \ge 1,$$

we can deduce from (2.3.79) that for all $t > T$,

$$\begin{aligned} y(t+1) - y(t) &= \int_0^1 \frac{d}{d\theta}\left\{\theta w(t+1) + (1-\theta)w(t)\right\}^{-\alpha} d\theta \\ &= \alpha \int_0^1 \left\{\theta w(t+1) + (1-\theta)w(t)\right\}^{-1-\alpha} d\theta \quad (w(t) - w(t+1)) \\ &\ge \alpha 2^{-1-\alpha} K_0^{-1}. \end{aligned} \tag{2.3.80}$$

Hence, for all $t > T$ and any integer n such that $n + T \le t < n + T + 1$, we obtain

$$\begin{aligned} y(t) &\ge y(t-n) + n\alpha 2^{-1-\alpha} K_0^{-1} \\ &\ge \min_{s\in[T,T+1]} y(s) + n\alpha 2^{-1-\alpha} K_0^{-1} \end{aligned} \tag{2.3.81}$$

or

$$w^{-\alpha}(t) \ge \left(\max_{s\in[T,T+1]} w(s)\right)^{-\alpha} + n\alpha 2^{-1-\alpha} K_0^{-1} \tag{2.3.82}$$

whence

$$w(t) \le \left[\frac{2^{1+\alpha} K_0}{2^{1+\alpha} K_0 [\max_{s\in[T,T+1]} w(s)]^{-\alpha} + \alpha(t - T - 1)}\right]^{1/\alpha}. \tag{2.3.83}$$

From the definition of $w(t)$ and estimate (2.3.83), we thus obtain (2.3.63).

Finally we consider the case $\alpha = 0$ in (iii). If $K_0 \le 1$, then (2.3.61) yields

$$\phi(t+1) \le g(t) \le K_2 e^{-\theta t} \tag{2.3.84}$$

which needs nothing to prove. Hence, we may suppose $K_0 > 1$. Then from (2.3.61), it follows

$$\phi(t+1) \le \frac{K_0 - 1}{K_0} \phi(t) + \frac{K_2}{K_0} e^{-\theta t} \tag{2.3.85}$$

and by induction,

$$\begin{aligned} \phi(t) &\le \left(\frac{K_0-1}{K_0}\right)^n \phi(t-n) + \sum_{i=1}^n \left(\frac{K_0-1}{K_0}\right)^{i-1} \frac{K_2}{K_0} e^{-\theta(t-i)} \\ &\le \left(\frac{K_0-1}{K_0}\right)^n \phi(t-n) + \frac{K_2}{K_0} e^{-\theta(t-1)} \left(1 - (\frac{K_0-1}{K_0} e^{\theta})^{n-1}\right) \left(1 - \frac{K_0-1}{K_0} e^{\theta}\right)^{-1} \end{aligned} \tag{2.3.86}$$

where n is the integer such that $t \le n < t+1$. Furthermore, by (2.3.86), there exists a constant $C' > 0$, such that

$$\phi(t) \le C\left(e^{-(\log \frac{K_0}{K_0-1})t} + e^{-\theta t}\right) \le C' e^{-\theta_1 t}, \tag{2.3.87}$$

which hence proves (2.3.64). □

The following result is a corollary of Theorem 2.3.11 for $r = 0, g(t) = \text{const.} > 0$.

Corollary 2.3.3 (The Nakao Inequality [663, 666, 676, 662])**.** *Assume that $\phi(t)$ is a non-negative continuous function on $[0, T)$, $T > 1$, possibly $T = +\infty$, satisfying for all $t > 0$,*

$$\sup_{s\in[t,t+1]} \phi^{1+\gamma}(s) \le C_0(\phi(t) - \phi(t+1)) + K, \tag{2.3.88}$$

with some constants $C_0 > 0, K > 0$ and $\gamma > 0$. Then for all $0 \le t < T$,

$$\phi(t) \le \left\{C_0^{-1}\gamma(t-1)^+ + \left(\sup_{0\le s\le 1}\phi(s)\right)^{-\gamma}\right\}^{-1/\gamma} + K^{1/(\gamma+1)}, \tag{2.3.89}$$

with $(t-1)^+ = \max(t-1, 0)$. If (2.3.88) holds with $\gamma = 0$, then instead of (2.3.89) we have for all $0 \le t < T$,

$$\phi(t) \le \sup_{0\le s\le 1}\phi(s)\left(\frac{C_0}{1+C_0}\right)^{[t]} + K, \tag{2.3.90}$$

where $[t]$ denotes the integral part of t.

Proof. Obviously, it is trivial that (2.3.89) is valid for all $0 \le t \le 1$. Set

$$\beta(t) = \left\{C_0^{-1}\gamma(t-1)^+ + \left(\sup_{0\le s\le 1}\phi(s)\right)^{-\gamma}\right\}^{-1/\gamma}.$$

Thus to prove our assertion, it suffices to show that if for some $t \ge 0$,

$$\phi(t) \le \beta(t) + K^{1/(\gamma+1)}, \tag{2.3.91}$$

then the following inequality holds,

$$\phi(t+1) \le \beta(t+1) + K^{1/(\gamma+1)}. \tag{2.3.92}$$

Assume that (2.3.92) is not true. Then

$$\phi(t+1) > \beta(t+1) + K^{1/(\gamma+1)} > K^{1/(\gamma+1)}. \tag{2.3.93}$$

We first claim that $\phi(t) > K^{1/(\gamma+1)}$. Indeed, if $\phi(t) \le K^{1/(\gamma+1)}$, we have $\phi(t) \le \phi(t+1)$ and hence inequality (2.3.88) implies $\phi^{\gamma+1}(t+1) \le K$, i.e., $\phi(t+1) \le K^{1/(\gamma+1)}$, which contradicts (2.3.93). Thus

$$\tilde{\phi}(t) \equiv \phi(t) - K^{1/(\gamma+1)} > 0, \ \ \tilde{\phi}(t+1) \equiv \phi(t+1) - K^{1/(\gamma+1)} > 0.$$

Consequently,

$$\tilde{\phi}^{1+\gamma}(t) + K \le \left(\tilde{\phi}(t) + K^{1/(\gamma+1)}\right)^{1/(\gamma+1)} = \phi^{1+\gamma}(t) \tag{2.3.94}$$

whence, by (2.3.88),

$$\tilde{\phi}^{1+\gamma}(t) \le C_0 \left(\tilde{\phi}(t) - \tilde{\phi}(t+1)\right). \tag{2.3.95}$$

Now set

$$\tilde{\phi}^{-\gamma}(t) = \psi(t).$$

Then it follows that, by using (2.3.95),

$$\begin{aligned}\psi(t+1) - \psi(t) &= -\int_0^1 \frac{d}{d\eta}\left(\eta\tilde{\phi}(t) + (1-\eta)\tilde{\phi}(t+1)\right)^{-\gamma} d\eta \\ &= \gamma \int_0^1 \left(\eta\tilde{\phi}(t) + (1-\eta)\tilde{\phi}(t+1)\right)^{-\gamma-1} d\eta \cdot \left(\tilde{\phi}(t) - \tilde{\phi}(t+1)\right) \\ &\ge \gamma\tilde{\phi}^{-\gamma-1}(t)\left(\tilde{\phi}(t) - \tilde{\phi}(t+1)\right) \ge \gamma C_0^{-1},\end{aligned}$$

which, thus, gives us

$$\tilde{\phi}^{-\gamma}(t+1) = \psi(t+1) \ge \psi(t) + \gamma C_0^{-1} = \tilde{\phi}^{-\gamma}(t) + \gamma C_0^{-1}. \tag{2.3.96}$$

Also from (2.3.96) and (2.3.91), we can derive

$$\begin{aligned}\phi(t+1) = \tilde{\phi}(t+1) + K^{1/(\gamma+1)} &\le \left(\tilde{\phi}^{-\gamma}(t) + \gamma C_0^{-1}\right)^{-1/\gamma} + K^{1/(\gamma+1)} \\ &\le \left(\gamma C_0^{-1}(t-1)^+ + \left(\sup_{0\le s\le 1} \phi(s)\right)^{-\gamma} + \gamma C_0^{-1}\right)^{-1/\gamma} + K^{1/(\gamma+1)} \\ &\le \beta(t+1) + K^{1/(\gamma+1)}\end{aligned}$$

which contradicts (2.3.93). Hence the proof of (2.3.89) is complete. The proof of (2.3.90) is easy. □

The next result is a natural generalization of Theorem 2.3.11 (see, e.g., Nakao [669, 670]).

Theorem 2.3.12 (The Nakao Inequality [663, 666, 676, 662]). *Assume that $\phi(t)$ is a bounded non-negative function on $\mathbb{R}^+$ satisfying for all $t > 0$,*

$$\sup_{s\in[t,t+1]} \phi^{1+\alpha}(s) \le C_0(1+t)^r(\phi(t)-\phi(t+1)) + g(t) \tag{2.3.97}$$

where $C_0 > 0$ is a constant, $g(t)$ is a non-negative function, and α a non-negative constant. Then the following assertion hold:

(i) *if $\alpha > 0, r = 1$, and $\lim_{t\to+\infty} (\log t)^{1+1/\alpha} g(t) = 0$, then for all $t > 0$,*

$$\phi(t) \le C_1\Big(\log(1+t)\Big)^{-1/\alpha}; \tag{2.3.98}$$

(ii) *if $\alpha > 0, 0 \le r < 1$, and $\lim_{t\to+\infty} t^{(1-r)(1+1/\alpha)} g(t) = 0$, then for all $t > 0$,*

$$\phi(t) \le C_2 t^{-(1-r)/\alpha}; \tag{2.3.99}$$

(iii) *if $\alpha = 0, r = 1$, and $g(t) \le K_1 t^{-\theta-1}$ for constants $\theta > 0, K_1 \ge 0$, then for all $t > 0$,*

$$\phi(t) \le C_3(1+t)^{-\theta'} \tag{2.3.100}$$

where $\theta' = \min(\theta, C_0^{-1})$;

(iv) *if $\alpha = 0, 0 \le r < 1$, and $g(t) \le K_2 t^{-\theta} \exp\left(-\frac{1}{(C_0+1)(1-r)}(t+1)^{1-r}\right)$ for $\theta > 1$, then for all $t > 0$,*

$$\phi(t) \le C_4 \exp\left(-\frac{1}{(C_0+1)(1-r)}t^{1-r}\right). \tag{2.3.101}$$

The above C_i $(i = 1, 2, 3, 4)$ are constants depending on $\phi(0)$ and other constants known from the involved data.

Proof. The basic idea of the proof is the same as that of Theorem 2.3.11, where the case $r = 0$ is treated. We now give the proofs of (i)–(iv) separately.

(i) Set

$$\psi(t) = \phi(t) + \nu(\log(1+t))^{-1/\alpha}$$

where ν is a positive constant to be determined later on. Then by (2.3.97), we have

$$\begin{aligned}\sup_{s\in[t,t+1]} \psi^{1+\alpha}(s) &\le 2^{1+\alpha}\left\{\max_{s\in[t,t+1]} \phi^{1+\alpha}(s) + \nu^{1+\alpha}(\log(1+t))^{-(1+\alpha)/\alpha}\right\}\\ &\le C\left\{(1+t)\left(\psi(t)-\psi(t+1)\right)I_1(t)\right\}\end{aligned} \tag{2.3.102}$$

where

$$\begin{aligned}I_1(t) = {}&(1+t)\left[\nu\left(\log(2+t)\right)^{-1/\alpha} - \nu\left(\log(1+t)\right)^{-1/\alpha}\right]\\ &+ g(t) + \nu^{1+\alpha}\left(\log(t+1)\right)^{-(1+\alpha)/\alpha}.\end{aligned}$$

We shall show $I_1(t) \le 0$ if t is sufficiently large. Noting that for large $t > 0$,

$$I_1(t) = \nu\,(\log(1+t))^{-1-1/\alpha}\left[(1+t)\log(t+1)\left\{\left[\frac{\log(t+2)}{\log(t+1)}\right]^{-1/\alpha} - 1\right\}\right.$$
$$\left. + \frac{1}{\nu}[\log(t+1)]^{1+1/\alpha}g(t) + \nu^\alpha\right],$$

$$\left[\frac{\log(t+2)}{\log(t+1)}\right]^{-1/\alpha} - 1 = \left(\frac{\log(t+2)-\log(t+1)}{\log(t+1)} + 1\right)^{-1/\alpha} - 1$$
$$\le -(2\alpha)^{-1}\Big(\log(t+2) - \log(t+1)\Big)\,(\log(t+1))^{-1},$$

we have for large $t > 0$,

$$I_1(t) \le \nu(\log(t+1))^{-1-1/\alpha}$$
$$\times\left\{-(2\alpha)^{-1}(1+t)\log[(t+2)/(t+1)] + \nu^{-1}\,(\log(t+1))^{1+1/\alpha}\,g(t) + \nu^\alpha\right\}.$$

Now, by the assumption on $g(t)$, the second term in the brackets in the right-hand side tends to 0 as $t \to +\infty$. Moreover, we can easily check that

$$\lim_{t\to+\infty}(1+t)\log\left(\frac{t+2}{t+1}\right) = 1.$$

Therefore, there exists a $T_1 > 0$ such that for all $t \ge T_1$, we have

$$I_1(t) \le \nu\Big(\log(t+1)\Big)^{-1-1/\alpha}\left(-(4\alpha)^{-1} + \frac{1}{2}\nu^\alpha\right) < 0$$

where we have chosen $\nu > 0$ so that $\nu < (2\alpha)^{-1/\alpha}$. Thus for all $t > T_1$ and small ν, we have

$$\sup_{s\in[t,t+1]}\psi^{1+\alpha}(s) \le C_0'(1+t)\,[\psi(t) - \psi(t+1)]\,. \tag{2.3.103}$$

Setting $\psi^{-\alpha}(s) = w(s)$, we obtain

$$w(t) - w(t+1) = \int_0^1 \frac{d}{d\theta}[\theta\psi(t) + (1-\theta)\psi(t+1)]^{-\alpha}d\theta$$
$$= -\alpha\int_0^1\{\theta\psi(t) + (1-\theta)\psi(t+1)\}^{-1-\alpha}d\theta\;\;[\psi(t) - \psi(t+1)]$$
$$\le -\alpha C_0'^{-1}(t+1)^{-1}. \tag{2.3.104}$$

Therefore, for the integer n with $n + T_1 \le t < n + 1 + T_1$,

$$w(t-n) - w(t) \le -\alpha C_0'^{-1}\sum_{i=0}^{n-1}\frac{1}{t-i} \le -\alpha C_0'^{-1}\int_0^{n-1}\frac{1}{t-x}dx$$
$$\le -\alpha C_0'^{-1}(\log t - \log(t+1-n)),$$

whence

$$w(t) \geq \inf_{s\in[0,1]} w(s+T_1) + \alpha C_0'^{-1} \log t - \alpha C_0'^{-1} \log(T_1+2)$$

which immediately yields

$$\phi(t) \leq \psi(t) \leq \left\{ \inf_{s\in[T_1,T_1+1]} w(s) + \alpha C_0'^{-1} \log t - \alpha C_0'^{-1} \log(T_1+2) \right\}^{-1/\alpha}. \tag{2.3.105}$$

Therefore, for all $0 < t \leq T_1$, we easily derive from (2.3.97) that

$$\phi(t) \leq \max\left(g(t), [C_0\phi(0) + g(0)]^{1/(1+\alpha)} \right). \tag{2.3.106}$$

The estimates (2.3.105)–(2.3.106) immediately imply (i).

(ii) In this case, we may set

$$\psi(t) = \phi(t) + \nu t^{-(1-r)/\alpha}.$$

Then, as in (2.3.100), we can obtain

$$\sup_{s\in[t,t+1]} \psi^{1+\alpha}(s) \leq 2^{1+\alpha}(C_0(1+t)^r[\psi(t) - \psi(t+1)] + I_2(t)), \tag{2.3.107}$$

where

$$I_2(t) = \nu t^{-(1-r)/\alpha} \left\{ C_0(1+t)^r \left[\left(\frac{1+t}{t} \right)^{-(1-r)/\alpha} - 1 \right] \right.$$
$$\left. + \nu^{-1} t^{(1-r)/\alpha} g(t) + \nu^\alpha t^{-(1-r)} \right\}.$$

Using the assumption on $g(t)$ and the following inequality for large $t > 0$,

$$\left(\frac{t+1}{t} \right)^{-(1-r)/\alpha} - 1 \leq -\frac{1-r}{2\alpha} t^{-1}$$

we may obtain for large $t > 0$,

$$I_2(t) = \nu t^{-(1-r)/\alpha + r - 1} \left(-C_0(1+t^{-1})^r(1-r)/2\alpha + \nu^\alpha/2 \right) < 0,$$

where we have chosen ν sufficiently small.

Thus there exists a $T_2 > 0$ such that for all $t \geq T_2$,

$$\sup_{s\in[t,t+1]} \psi^{1+\alpha}(s) \leq 2^{1+\alpha} C_0(1+t)^r(\psi(t) - \psi(t+1))$$

which implies, as in the proof of (i), for any positive integer n with $C_0' = 2^{1+\alpha}C_0$,

$$\begin{aligned} w(t-n) - w(t) &\leq -\alpha C_0'^{-1} \int_0^{n-1} \frac{1}{(t-x)^r} dx \\ &\leq \alpha C_0'^{-1}(1-r)^{-1} \left((t-n+1)^{1-r} - t^{1-r} \right). \end{aligned}$$

This immediately yields (ii).

(iii) The proofs of (iii) and (iv) are almost the same, so we only show (iv). By (2.3.97), we derive

$$\phi(t+1) \le \frac{C_0(1+t)^r}{C_0(1+t)^r+1}\phi(t) + g(t)$$

whence, by induction,

$$\begin{aligned}\phi(t+1) &\le \prod_{i=0}^{n} \frac{C_0(t+1-i)^r}{C_0(1+t-i)^r+1}\phi(t-n) + \sum_{j=0}^{n}\prod_{i=0}^{j} \frac{C_0(t+1-i)^r}{C_0(t+1-i)^r+1} g(t-j) \\ &= I_1 + I_2.\end{aligned}$$

Fix the integer n such that $n \le t < n+1$. Then it is readily seen that

$$\begin{aligned}\log(I_1) \le & -\sum_{i=0}^{n} \frac{1}{C_0(t+1-i)^r+1} + \sup_{s\in[0,1]} \log\phi(s) \\ \le & -\int_0^n \frac{1}{C_0(t+1-x)^r+1} + \sup_{s\in[0,1]} \log\phi(s) \\ \le & -\frac{1}{(C_0+1)(1-r)}(1+t)^{1-r} + \frac{1}{(C_0+1)(1-r)}(t+1-n)^{1-r} \\ & + \log\{C_0\phi(0)+g(0)\}^{1/(1+\alpha)}\end{aligned}$$

where in the above second inequality, we assume that $\sup_{s\in[t,t+1]}\phi(s) > 0$. Thus this gives us immediately

$$I_1 \le C_1 \exp\left(-\frac{1}{(C_0+1)(1-r)}(1+t)^{1-r}\right).$$

In the same manner, we conclude that

$$\begin{aligned}I_2 &\le \sum_{j=0}^{n-1} \exp\left(-\int_0^j \frac{1}{C_0(1+t-x)^r+1}dx\right) g(t-j) \\ &\le C_2 \exp\left(-\frac{1}{(C_0+1)(1-r)}(1+t)^{1-r}\right).\end{aligned}$$

The proof is hence complete. □

From the proof of Theorem 2.3.12, we easily get the next result, due to Nakao [685].

Corollary 2.3.4 (The Nakao Inequality [685]). *Assume $\phi(t)$ that is a non-negative non-increasing function satisfying* (2.3.97) *for $g(t) \equiv 0$ and $C_0 \ge 1$. Then the following assertions hold:*

(i) *If* $\alpha > 0$ *and* $r < 1$, *then for all* $t > 0$,

$$\phi(t) \leq \left\{ \phi^{-\alpha}(0) + \frac{\alpha}{C_0} \int_0^{[t-1]^+} (1+t-s)^{-r} ds \right\}^{-1/\alpha}$$

where $[a]^+ := \max\{a, 0\}$.

(ii) *If* $\alpha = 0$ *and* $r < 1$, *then for all* $t > 0$,

$$\phi(t) \leq \phi(0) \exp\left\{ \frac{1}{(C_0+1)(1-r)} \right\} \exp\left\{ -\frac{1}{(C_0+1)(1-r)} (1+t)^{1-r} \right\}.$$

The following result, due to Nakao [670], extends from the case $\theta > 1$ in Theorem 2.3.12 (iv) to the case of $\theta = 1$.

Theorem 2.3.13 (The Nakao Inequality [670]). *Assume that* (2.3.97) *holds for* $\phi(t)$ *in Theorem* 2.3.12. *If* $\alpha = 0, 0 \leq r < 1$ *and* $\lim_{t\to+\infty}(1+t)\exp(kt^{1-r})g(t) = 0$ *for some constant* $k > 0$, *then for all* $t > 0$,

$$\phi(t) \leq C' \exp\left(-Ct^{1-r}\right), \tag{2.3.108}$$

with constants $C', C > 0$.

The following result is a generalization of Corollary 2.3.4 (see also [663], [667], [678], [675]).

Theorem 2.3.14 (The Nakao Inequality [670]). *Let* $\phi(t)$ *be a non-negative function on* $[0, +\infty)$ *satisfying for all* $t > 0$,

$$\sup_{t\leq s\leq t+T} \phi^{1+\alpha}(s) \leq g(t)[\phi(t) - \phi(t+T)] \tag{2.3.109}$$

for constants $T > 0, \alpha > 0$, *and a non-decreasing function* $g(t)$. *Then* $\phi(t)$ *has the decay property for all* $t \geq T$,

$$\phi(t) \leq \left(\phi^{-\alpha}(0) + \alpha \int_T^t g^{-1}(s) ds \right)^{-1/\alpha}. \tag{2.3.110}$$

In particular, if $\alpha = 0$ *and* $g(t) =$ *constant in the above assumption, then for all* $t \geq T$,

$$\phi(t) \leq C\phi(0) \exp\{-\lambda t\} \tag{2.3.111}$$

with some constant $\lambda > 0$.

Later on, Kawashima, Nakao, and Ono [423] improved (ii) in Theorem 2.3.12, which can be stated as a theorem.

Theorem 2.3.15 (The Kawashima–Nakao–Ono Inequality [423]). *Let $\phi(t)$ be a non-negative function on $[0, +\infty)$, satisfying (2.3.97) for some constants C_0, $\alpha > 0, 0 \le r < 1$, and let $g(t)$ satisfy for all $t > 0$,*

$$0 \le g(t) \le k_1(1+t)^{-\beta} \tag{2.3.112}$$

with some constants k_1 and $\beta > 0$. Then $\phi(t)$ decays, for all $t > 0$, as

$$\phi(t) \le k_2(1+t)^{-\theta}, \quad \theta = \min\left(\frac{1-r}{\alpha}, \frac{\beta}{1+\alpha}\right) \tag{2.3.113}$$

where $k_2 > 0$ is a constant depending on $\phi(0)$ and other data involved in the assumptions.

Proof. In fact, we may assume $\beta/(1+\alpha) \le (1-r)/\alpha$, i.e., $0 < \beta \le (1+\alpha)(1-r)/\alpha$. Suppose (2.3.113) is not true. Then for any large $K > 0$, there exists some constant $T > 1$ such that for all $0 \le t \le T - 1/2$,

$$\phi(t) \le K(1+t)^{-\beta/(1+\alpha)},$$
$$\phi(T) \ge K(1+T)^{-\beta/(1+\alpha)}.$$

Here we can easily prove $\phi^{1+\alpha}(t) \le \max\{ck_1, c'C_0\phi(0) + c''k_1\} < +\infty$. Thus taking $t = T - 1$ in (2.3.97), we get

$$K^{1+\alpha}(1+T)^{-\beta} \le C_0T^r\left(KT^{-\beta/(1+\alpha)} - K(1+T)^{-\beta/(1+\alpha)}\right) + k_1(1+T)^{-\beta}$$

and, taking K so large that $K > \min\{2k_1, 1\}$, and for some constant $C_* > 0$,

$$\begin{aligned} K^{1+\alpha}(1+T)^{-\beta} &\le 2k_0T^r\left(T^{-\beta/(1+\alpha)} - (1+T)^{-\beta/(1+\alpha)}\right) \\ &\le 2k_0K(1+T)^{r-\beta/(1+\alpha)}\left((1+T^{-1})^{\beta/(1+\alpha)} - 1\right) \\ &\le C_*K(1+T)^{r-\beta/(1+\alpha)-1}. \end{aligned} \tag{2.3.114}$$

Since $\beta \le (1+\alpha)(1-r)/\alpha$, (2.3.114) yields

$$K^\alpha \le C_*(1+T)^{r-\alpha\beta/(1+\alpha)-1} \le C_*$$

which, in fact, is a contradiction if we choose $K > C_*^{1/\alpha}$. □

Let us now state some generalizations of Theorem 2.3.15 (see, e.g., [680]).

Theorem 2.3.16 (The Nakao Inequality [680]). *Assume that $\phi(t)$ is a non-negative continuous non-increasing function on $\mathbb{R}^+$ satisfying the inequality for all $t \ge 0$,*

$$\phi(t+T) \le C\sum_{i=1}^{2}(1+t)^{\theta_i}[\phi(t) - \phi(t+T)]^{\varepsilon_i}, \tag{2.3.115}$$

with some constants $T > 0, C > 0, 0 < \epsilon_i \leq 1$ *and* $\theta_i \leq \epsilon_i$ $(i = 1, 2)$. *Then* $\phi(t)$ *has the following decay properties:*

(i) *If* $0 < \varepsilon_i < 1$ *with* $\epsilon_1 + \epsilon_2 < 1$ *and* $\theta_i < \varepsilon_i$, $i = 1, 2$, *then for all* $t \geq 0$,

$$\phi(t) \leq C_0(1+t)^{-\alpha} \tag{2.3.116}$$

with $\alpha = \min_{i=1,2}\{(\varepsilon_i - \theta_i)/(1-\varepsilon_i)\}$, *where we put* $(\varepsilon_i - \theta_i)/(1-\varepsilon_i) = +\infty$ *if* $\varepsilon_i = 1$.

(ii) *If* $\theta_1 = \varepsilon_1 < 1$ *and* $\theta_2 < \varepsilon_2 \leq 1$, *then for all* $t \geq 0$,

$$\phi(t) \leq C_0\left[\log(2+t)\right]^{-\varepsilon_1/(1-\varepsilon_1)}. \tag{2.3.117}$$

(iii) *If* $\theta_1 = \varepsilon_1 < 1$ *and* $\varepsilon_2 = \theta_2 \leq 1$, *then for all* $t \geq 0$,

$$\phi(t) \leq C_0\left[\log(2+t)\right]^{-\tilde{\alpha}} \tag{2.3.118}$$

with $\tilde{\alpha} = \min_{i=1,2}\{\varepsilon_i/(1-\varepsilon_i)\}$.

(iv) *If* $\varepsilon_1 = \varepsilon_2 = 1$, *then we have for all* $t \geq 0$,

$$\begin{cases} \phi(t) \leq C_0 \exp\{-\lambda t^{1-\theta}\}, \quad if \ \ \theta < 1, & (2.3.119) \\ \phi(t) \leq C_0(1+t)^{-\alpha}, \quad if \ \ \theta = 1 & (2.3.120) \end{cases}$$

with some constants $\lambda > 0, \alpha > 0$, *where we set* $\theta = \min\{\theta_1, \theta_2\}$. *In the above,* C_0 *denotes constants depending on* $\phi(0)$ *and other known constants.*

Proof. Note that the case: $\varepsilon_1 = \varepsilon_2$ and $\eta_1 = \eta_2$ was proved in [667] in a more detailed form. The proof, however, is not applicable to our situation and we employ a different technique, here, due to Nakao [680].

(i) To prove (2.3.116), we may take $M \geq \max_{0\leq s\leq 1}\phi(s) \equiv \phi(0)$ and assume that for some $T \geq 1$,

$$\sup_{0\leq t\leq T} \phi(t)(1+t)^{\alpha} = \phi(T)(1+T)^{\alpha} = M. \tag{2.3.121}$$

Then, by inequalities (2.3.115), we have

$$\begin{aligned} M(1+T)^{-\alpha} = \phi(T) &\leq C\sum_{i=1}^{2} T^{\theta_i} M^{\varepsilon_i}(T^{-\alpha} - (1+T)^{-\alpha})^{\epsilon_i} \\ &\leq C\sum_{i=1}^{2} T^{\theta_i} M^{\varepsilon_i} \alpha T^{-(\alpha+1)\varepsilon_i} \end{aligned} \tag{2.3.122}$$

and

$$M \leq C\sum_{i=1}^{2} T^{\theta_i-(\alpha+1)\varepsilon_i-\alpha} M^{\varepsilon_i}\alpha^{\varepsilon_i} = C\sum_{i=1}^{2} M^{\varepsilon_i}\alpha^{\varepsilon_i} \tag{2.3.123}$$

which is a contradiction if we take $M > 0$ large enough. This means that there exists a constant $C_0 = C(\phi(0)) > 0$ such that for all $t \geq 0$,

$$\phi(t)(1+t)^{\alpha} \leq C_0, \tag{2.3.124}$$

which proves (2.3.116).

Next, we consider the case (ii). Here we assume, for $M \geq \phi(0)$ and $T \geq 1$, that, for $\alpha = \frac{\varepsilon_1}{1-\varepsilon_1}$,

$$\sup_{0\leq t\leq T} \phi(t)\left[\log(2+t)\right]^{\alpha} = \phi(T)\left[\log(2+T)\right]^{\alpha} = M. \tag{2.3.125}$$

Then we have again by (2.3.116),

$$\begin{aligned} M\left[\log(2+T)\right]^{-\alpha} = \phi(T) &\leq C\sum_{i=1}^{2} T^{\theta_i} M^{\varepsilon_i}\left[(\log(1+T))^{-\alpha} - (\log(2+T))^{-\alpha}\right]^{\varepsilon_i} \\ &\leq C\sum_{i=1}^{2} T^{\theta_i} M^{\varepsilon_i} \alpha^{\varepsilon_i} (1+T)^{-\varepsilon_i}\left[\log(1+T)\right]^{-(\alpha+1)\varepsilon_i} \end{aligned} \tag{2.3.126}$$

whence,

$$M \leq C\left\{\left[\log(2+T)\right]^{\alpha-(\alpha+1)\varepsilon_1} M^{\varepsilon_1} + M^{\varepsilon_2}\right\} = C\sum_{i=1}^{2} M^{\varepsilon_i}$$

which is again a contradiction if we choose $M > 0$ large enough. This implies (2.3.117). The proof of (2.3.118) in the case (iii) is essentially included in the above case. Finally, in the case (iv), we have

$$\phi(t) \leq C(1+t)^{\theta}[\phi(t) - \phi(t+1)]$$

which finally implies (2.3.119)–(2.3.120). □

Remark 2.3.3 ([680])**.** It is clear from the proof that Theorem 2.3.16 admits a generalization as a difference inequality of the form for all $t \geq 0$,

$$\phi(t+1) \leq C\sum_{i=1}^{m}(1+t)^{\eta_i}\left(\phi(t) - \phi(t+1)\right)^{\varepsilon_i}. \tag{2.3.127}$$

For example, if $0 < \varepsilon_i < 1$ and $\eta_i < \varepsilon_i$, we conclude from the above inequality that for $\alpha = \min\left\{\frac{\varepsilon_i - \eta_i}{1-\varepsilon_i}\right\}$,

$$\phi(t) \leq C_0(1+t)^{-\alpha}. \tag{2.3.128}$$

Remark 2.3.4 ([680])**.** When $\varepsilon_1 = \varepsilon_2$ and $\theta_1 = \theta_2$, more detailed results are proved in Nakao [664, 667].

Remark 2.3.5 ([680]). The above theorem can be easily generalized to the difference inequality of the form for all $t > 0$,

$$\phi(t+1) \leq C \sum_{i=1}^{m} (1+t)^{\theta_i} [\phi(t) - \phi(t+1)]^{\varepsilon_i}. \tag{2.3.129}$$

For example, if $0 < \varepsilon_i < 1$ and $\theta_i < \varepsilon_i$, we obtain from (2.3.129) that for all $t > 0$,

$$\phi(t) \leq C_0(1+t)^{-\eta}, \tag{2.3.130}$$

where $\eta = \min_{1 \leq i \leq m}\{(\varepsilon_i - \theta_i)/(1-\varepsilon_i)\}$.

Corollary 2.3.5 ([680]). *Let $\phi(t)$ be a non-negative function on $\overline{\mathbb{R}^+} \equiv [0, +\infty)$ satisfying for all $t > 0$,*

$$\sup_{t \leq s \leq t+1} \phi^{1+\gamma}(s) \leq K_0(1+t)^{\beta}(\phi(t) - \phi(t+1)) \tag{2.3.131}$$

with some constants $K_0 > 0, \gamma > 0, 0 \leq \beta < 1$. Then $\phi(t)$ has the decay properties for all $t > 0$,

$$\phi(t) \leq C_0(1+t)^{-(1-\beta)/\gamma}, \tag{2.3.132}$$

and if $\gamma = 0$, then for all $t > 0$,

$$\phi(t) \leq C_0 \exp\{-\lambda t^{1-\beta}\} \tag{2.3.133}$$

where $C_0 > 0, \lambda > 0$ are constants.

To close this section, we shall introduce some results, due to Vărvărucă [933], on exact rates of convergence as time goes to infinity for solutions of nonlinear evolution equations. To this end, we need to introduce some basic concepts on sub-differential mappings.

Definition 2.3.1. Let X be a real Banach space with dual space X^*. We say that φ is a normal convex function on X if φ is a convex function from X to $(-\infty, +\infty]$, but $\varphi \not\equiv +\infty$.

Definition 2.3.2. A function $\varphi: X \to (-\infty, +\infty]$ is called lower semi-continuous on X if for all $x \in X$, $\liminf_{y \to x} \varphi(y) \geq \varphi(x)$.

Definition 2.3.3. Let φ be a normal convex function on a Banach space X with dual space X^* and let $x \in X$, we define

$$\partial\varphi(x) = \{x^* \in X^* : \varphi(x) \leq \varphi(y) + (x-y, x^*) \text{ for all } y \in X\}.$$

We call $x^* \in X^*$ a sub-gradient of a function φ at $x \in X$. We call $\partial\varphi(x)$ the sub-differential of φ at x. Here $(\cdot,\cdot)$ denotes the duality pairing between X and X^*. When X is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, then we can take $(\cdot,\cdot) = \langle\cdot,\cdot\rangle$.

We can verify from the definition of $\partial\varphi(x)$ that $\partial\varphi: \ X \to X^*$ is a monotonic operator, usually called a multi-valued operator, and $\partial\varphi(x)$ is a possibly empty closed convex set in X. Moreover, we recall that

$$\varphi(x) = \min\{\varphi(y) : y \in X\}$$

if and only if $0 \in \partial\varphi(x)$.

Now we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, and $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous convex and bounded from below function, and has a minimum on H. Without loss of generality, we may assume that

$$\min_{u \in H} \varphi(u) = 0.$$

We shall first introduce some results, due to Vărvărucă [933], on the asymptotic behavior of solutions of the following differential inequalities

$$\begin{cases} |u'(t) + v(t)|^2 \le a(t)\varphi(u(t)), \ \text{for a.e. } t \in (0, +\infty), \\ v(t) \in A(u(t)), \ \text{for a.e. } t \in (0, +\infty), \\ a \in L^1(0, +\infty), \end{cases} \tag{2.3.134}$$

where $A := \partial\varphi$ is the sub-differential of φ, and the real-valued function a is non-negative. Second, we shall introduce some results for the Cauchy problem (2.3.134). Obviously, when a is identically zero on $(0, +\infty)$, (2.3.134) readily reduces to the following problem

$$\begin{cases} u'(t) + A(u(t)) \ni 0, \ \text{for a.e. } t \in (0, +\infty), \\ u(0) = u_0, \ u_0 \in \overline{\text{Dom } (A)}. \end{cases} \tag{2.3.135}$$

To proceed, we need the following definition.

Definition 2.3.4. By a solution of (2.3.134), we understand a function

$$u \in W^{1,2}_{\text{loc}}([0, +\infty); H)$$

such that $u(t) \in \text{Dom}(A)$ for a.e. $t \in (0, +\infty)$ and there exists a

$$v \in L^{1,2}_{\text{loc}}([0, +\infty); H)$$

such that (2.3.134) holds, where A denotes the sub-differential of φ.

The following result is due to Brézis [116]; a proof can also be found in [70].

Lemma 2.3.17 ([116]). *Let $[a, b]$ be an interval, and $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous convex function. Let $u \in W^{1,2}([a, b]; H)$ be such that $u(t) \in$* $\text{Dom}(\partial\varphi)$ *for a.e. $t \in [a, b]$ and there exists a $v \in L^2(a, b; H)$ with $v(t) \in \partial\varphi(u(t))$ for a.e. $t \in [a, b]$. Then the mapping $t \mapsto \varphi(u(t))$ is absolutely continuous on $[a, b]$ and its derivative is given by*

$$\frac{d}{dt}\varphi(u) = \langle v, u' \rangle, \quad \text{a.e. in } (a, b).$$

The following results (see [933]) show that the long-time behavior of solutions of (2.3.134) is in many respects similar to that for the solutions of (2.3.135).

Lemma 2.3.18 ([933]). *For any initial data in* $\overline{\mathrm{Dom}(A)}$, *if the solution of problem* (2.3.135) *has a strong limit in* H *as* $t \to +\infty$, *then any solution of* (2.3.134) *converges strongly as* $t \to +\infty$ *to an element of* F.

Proof. Let $f(t) := u'(t) + v(t)$ for all $t \in (0, +\infty)$. Assume first that u satisfies (2.3.134) and $f \in L^2_{\mathrm{loc}}([0, +\infty); H)$. If we can prove that $f \in L^1(0, +\infty; H)$, then this readily implies the strong convergence of u as $t \to +\infty$ by a standard argument (see, e.g., [638]). By (2.3.134) and noting that

$$|f(t)|^2 \le a(t)\varphi(u(t)), \quad \text{for a.e. } t \in (0, +\infty),$$

and $a \in L^1(0, +\infty)$, it suffices to prove that $t \mapsto \varphi(u(t))$ is also in $L^1(0, +\infty)$. Assume that z_0 is an arbitrary element of F. Then for a.e. $t \in (0, +\infty)$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u - z_0|^2 + \langle v, u - z_0\rangle \\ &\quad = \langle u - z_0, u' + v\rangle \le |u - z_0|\sqrt{a\varphi(u)} \le \frac{1}{2}a|u - z_0|^2 + \frac{1}{2}\varphi(u). \end{aligned} \tag{2.3.136}$$

By the definition of a sub-differential, we have $\langle v, u - z_0\rangle \ge \varphi(u)$. Hence, we conclude from (2.3.136) that

$$\frac{d}{dt}|u - z_0|^2 + \varphi(u) \le a|u - z_0|^2, \quad \text{a.e. in } (0, +\infty),$$

and, since $a \in L^1(0, +\infty)$, it follows from the Bellman–Gronwall inequality (e.g., Theorem 1.2.1) that $t \mapsto \varphi((u(t))$ is in $L^1(0, +\infty)$. This completes the proof of the strong convergence of u. □

Lemma 2.3.19 ([933]). *If* u *is any solution to problem* (2.3.134), *then*

$$\lim_{t \to +\infty} \varphi(u(t)) = 0. \tag{2.3.137}$$

Proof. It follows obviously from Lemma 2.3.1 that

$$\frac{d}{dt}\varphi(u) = \langle v, u'\rangle \le \frac{1}{4}|u' + v|^2 \le \frac{1}{4}a\varphi(u), \quad \text{a.e. in } (0, +\infty), \tag{2.3.138}$$

which, by the Bellman–Gronwall inequality (e.g., Theorem 1.2.1), implies that $\varphi(u(t))$ has a finite limit as $t \to +\infty$. Noting that $t \mapsto \varphi(u(t))$ is in $L^1(0, +\infty)$ and using Lemma 2.3.2, we obtain $\lim_{t\to+\infty} \varphi(u(t)) = 0$. □

Let $z \in F$ and $p \ge 2$. The function φ is said to be locally sub-homogeneous of degree p with respect to z if there exists an open set D_z containing z such that, for any $u \in D_z \cap \mathrm{Dom}(\varphi)$, the mapping

$$t \mapsto \frac{\varphi(z + t(u - z))}{t^p}$$

is non-decreasing on $(0, 1]$.

A function φ is said to satisfy the condition (SH_p) with respect to z if there exists an open set D_z containing z such that the following holds: for all $u \in D_z \cap \mathrm{Dom}(A)$, for all $v \in A(u)$,

$$\langle v, u - z\rangle \geq p\varphi(u).$$

The following result assume that the relation between these two conditions.

Lemma 2.3.20 ([933]). *Assume that $p \geq 2$ and $z \in F$. If φ is locally sub-homogeneous of degree p with respect to z, then it satisfies (SH_p) with respect to z.*

Proof. It follows from the definition of a sub-differential and the local sub-homogeneity of φ, for all $u \in D_z \cap \mathrm{Dom}(A)$, for all $v \in A(u)$, for all $t \in (0, 1)$, that

$$\langle v, z - u\rangle \leq \frac{1}{1-t}\Big[\varphi(z + t(u - z)) - \varphi(u)\Big] \leq \frac{t^p - 1}{1-t}\varphi(u), \tag{2.3.139}$$

and the result follows by letting here $t \nearrow 1$. □

The next theorem is due to Vărvărucă [933].

Theorem 2.3.21 ([933]). *Let $z \in F$ and let φ satisfy (SH_p) with respect to z, for some $p \geq 2$. Then, for any solution u of* (2.3.134) *the following holds: if $u(0) \neq z$, then $u(t) \neq z$ for all $t \geq 0$. In this case, if, in addition, $u(t)$ belongs to the set D_z in (SH_p) for all sufficiently large t, then there exists a non-negative constant Λ^∞ such that*

$$\lim_{t\to+\infty} \frac{p\varphi(u(t))}{|u(t) - z|^p} = \Lambda^\infty. \tag{2.3.140}$$

Moreover,

$$\lim_{t\to+\infty} \frac{-\log|u(t) - z|}{t} = \Lambda^\infty, \quad \text{if } p = 2, \tag{2.3.141}$$

and

$$\lim_{t\to+\infty} \frac{|u(t) - z|^{2-p}}{(p-2)t} = \Lambda^\infty, \quad \text{if } p > 2. \tag{2.3.142}$$

Proof. Since the general case can be done by a translation argument, without loss of generality, we may assume that $z = 0$. Now let u be any solution of (2.3.134) with $u(0) \neq 0$. By noting that φ satisfies (SH_p) with respect to the origin for some $p \geq 2$, it necessarily satisfies (SH_2) with respect to 0. Thus, it follows from Theorem 2.1 and Remark 2.1 in [932] that $u(t) \neq 0$ for all $t \geq 0$.

Without loss of generality, we now assume that $u(t)$ belongs to D_0 in (SH_p) for all $t \geq 0$. Consider the function

$$\Lambda(t) := \frac{p\varphi(u(t))}{|u(t)|^p},$$

which, by Lemma 2.3.1., is well defined and absolutely continuous on compact intervals. On the other hand, by a similar argumentation to that in [932] (e.g., Theorem 2.1 and Theorem 2.2), we conclude

$$\frac{1}{p}\frac{d\Lambda}{dt}+\frac{1}{2}\frac{1}{|u|^p}\left|v-\frac{p\varphi(u)}{|u|^2}u\right|^2\leq\frac{a}{2p}\Lambda,\ \text{a.e. in } (0,+\infty), \tag{2.3.143}$$

or

$$\frac{1}{p}\frac{d\Lambda}{dt}+\frac{1}{2}|u|^{p-2}\left|\frac{v}{|u|^{p-1}}-\Lambda\frac{u}{|u|}\right|^2\leq\frac{a}{2p}\Lambda,\ \text{a.e. in } (0,+\infty). \tag{2.3.144}$$

Note that in [932], the case $p = 2$ was only considered with a slightly different definition of Λ. Since $a \in L^1(0,+\infty)$, it follows from the Bellman–Gronwall inequality (e.g., Theorem 1.2.1) that $\Lambda(t)$ has a finite limit, denoted by Λ^∞, as $t \to +\infty$.

For the last part of the theorem, noting that the estimate claimed for $p = 2$ is a restatement of [932] (see Theorem 2.2), thus it suffices only to consider the case $p > 2$. As $u(t) \neq 0$ for all $t \geq 0$, the mapping $x(t) := \frac{1}{p-2}|u(t)|^{2-p}$ is well defined and absolutely continuous on compact intervals, with derivative given by

$$\frac{dx}{dt}=-\frac{\langle u,u'\rangle}{|u|^p}=\frac{1}{|u|^p}\left[-\langle u,u'+v\rangle+\langle v-\frac{p\varphi(u)}{|u|^2}u,u\rangle\right]+\Lambda,\ \text{a.e. in } (0,+\infty). \tag{2.3.145}$$

Using the Schwarz inequality, it follows from (2.3.145) that

$$\left|\frac{dx}{dt}-\Lambda\right|\leq\frac{1}{|u|^{p-1}}\left[|u'+v|+\left|v-\frac{P\varphi(u)}{|u|^2}u\right|\right]=c\sqrt{x},\ \text{a.e. in } (0,+\infty), \tag{2.3.146}$$

where the real-valued function c is given by

$$c:=\sqrt{p-2}\frac{1}{|u|^{p/2}}\left[|u'+v|+\left|v-\frac{p\varphi(u)}{|u|^2}u\right|\right],\ \text{a.e. in } (0,+\infty).$$

Using (2.3.134), the boundedness of Λ on $[0,+\infty)$, and (2.3.144), we conclude that $c \in L^2(0,+\infty)$. The required estimate now follows from Lemma 2.3.5 below. This completes the proof of Theorem 2.3.17. □

We have noted that Theorem 2.3.17 yields optimal estimates only for $\Lambda^\infty \neq 0$, which is not necessarily true. However, the condition on φ only in a neighborhood of $z \in F$ guarantee that any non-constant solution of (2.3.134) does not reach z, and also yields lower bounds for $|u(t) - z|$, only if that the solution $u(t)$ lies in that neighborhood for all sufficiently large t.

Lemma 2.3.22 ([933]). *Assume that $x : [0,+\infty) \to [0,+\infty)$ is a function which is absolutely continuous on compact intervals and satisfies the differential inequality*

$$-c\sqrt{x}+\Lambda\leq\frac{dx}{dt}\leq c\sqrt{x}+\Lambda,\ \textit{a.e. in } (0,+\infty), \tag{2.3.147}$$

where Λ *is non-negative a.e., and has a finite limit* Λ^∞ *as* $t \to +\infty$*, and* $c \in L^2(0,+\infty), c \geq 0$ *a.e. on* $(0,+\infty)$*. Then*

$$\lim_{t\to+\infty} \frac{x(t)}{t} = \Lambda^\infty. \tag{2.3.148}$$

Proof. Indeed, for each $\varepsilon > 0$, using Young's inequality, we can derive from (2.3.147)

$$-\frac{c^2}{4\varepsilon}x + (\Lambda - \varepsilon) \leq \frac{dx}{dt} \leq \frac{c^2}{4\varepsilon}x + (\Lambda + \varepsilon), \text{ a.e. in } (0,+\infty). \tag{2.3.149}$$

Noting that $\frac{1}{4\varepsilon}c^2 \in L^1(0,+\infty)$ and $\lim_{t\to+\infty} \Lambda(t) = \Lambda^\infty$, we can derive (2.3.148) from (2.3.149) and Theorem 1.2.1 immediately. □

In order to establish the desired exact rates of convergence, now we need to obtain, for solutions u which converge to z, values for Λ^∞ which are strictly positive. This can be easily done when the set F is a single point. The following result then follows readily from Theorem 2.3.17, and further generalizes the abstract results in [302], [303] and [306] from the homogeneous case to the case when φ is locally sub-homogeneous.

Theorem 2.3.23 ([933])**.** *Assume that* $p \geq 2$ *and* $F = \{\bar{z}\}$*. If* φ *satisfies* (SH_p) *with respect to* $\bar{z}$*, and* (C_p)*:*

(C_p) *there exists* $m > 0$ *such that* $\varphi(u) \geq m|u - \bar{z}|^p$*, for all* $u \in \mathrm{Dom}(\varphi)$*,*

then any solution u *of* (2.3.134) *converges strongly to* $\bar{z}$ *in* H *as* $t \to +\infty$*, and the estimates in Theorem* 2.3.17 *are valid with* $\Lambda^\infty > 0$*.*

Proof. It follows obviously that (C_p) implies the strong convergence to $\bar{z}$ as $t \to +\infty$ of all the solutions of (2.3.135) and from Lemma 2.3.2 that the same is true for any solution of (2.3.134). Thus, the estimates in Theorem 2.3.17 hold in this case, and (C_p) guarantees that $\Lambda^\infty > 0$. □

When F is not a single point, we may use Theorem 2.3.17 to derive the required results. To this end, we may assume that φ satisfies conditions which guarantee that all the solutions of (2.3.134) converge strongly in H as $t \to +\infty$. Note that for any given solution of (2.3.134) the value of its limit in F is not known a priori, so we need to impose a condition to describe the behavior of φ in a neighborhood of F.

The following is a global version of (SH_p): for $p \geq 2$, we say that φ satisfies the condition (SH_pF) if there exists an open set D_F containing F, such that for all $z \in F$, for all $u \in D_F \cap \mathrm{Dom}(A)$, for all $v \in A(u)$,

$$\langle v, u - z\rangle \geq p\varphi(u).$$

We observe that such a condition is easily satisfied for some applications, we refer to some examples in Section 12 of Chapter 7 ([933]).

Since the set F of minimizers of φ is non-empty, closed and convex, we can consider the standard projection operator onto F, denoted by P. For $p \geq 2$, let (C_pF) denote the following condition:

there exists an $m > 0$ such that: $\varphi(u) \geq m|u - Pu|^p$, for all $u \in \mathrm{Dom}(\varphi)$.

The following lemma generalizes slightly Lemma 2.4 in [739] with the same proof.

Lemma 2.3.24 ([739]). *Assume that $u : [a, b] \to H$ is absolutely continuous, and let F be a non-empty closed and convex subset of H. Let P denote the projection operator onto F. Then the mapping $t \to Pu(t)$ is absolutely continuous, and therefore differentiable a.e. on $[a, b]$, with*

$$\left\langle \frac{d}{dt}Pu, u - Pu \right\rangle = 0, \quad \text{a.e. in } (a, b). \tag{2.3.150}$$

The following result also generalizes the inequality that for all $t \geq 0$,

$$|u(t) - \bar{z}| \leq 2|u(t) - Pu(t)|, \tag{2.3.151}$$

and improves some estimates in [638] (Theorem 1.2, p. 73).

Lemma 2.3.25 ([638]). *Assume that $A : H \to H$ is a (possibly multi-valued) maximal monotone operator with $F := A^{-1}(0) \neq \emptyset$, and denote by P the projection operator onto the closed convex set F. Assume further that u satisfies for a.e. $t \in (0, +\infty)$,*

$$u'(t) + A(u(t)) \ni f(t), \tag{2.3.152}$$

where $f \in L^1(0, +\infty; H)$, and u has a strong limit $\bar{z}$ as $t \to +\infty$, where $\bar{z} \in F$. Then for all $t \geq 0$,

$$|u(t) - \bar{z}| \leq 2|u(t) - Pu(t)| + \int_t^{+\infty} |f(\tau)|d\tau. \tag{2.3.153}$$

Proof. Note that the following result is well known: If u_1, u_2 are solutions of

$$u_i'(t) + A(u_i(t)) \ni f_i(t), \text{ for a.e. } t \in (0, +\infty), \quad i = 1, 2,$$

where $f_1, f_2 \in L^1_{\mathrm{loc}}(0, +\infty; H)$, then for all $t, h \geq 0$,

$$|u_1(t + h) - u_2(t + h)| \leq |u_1(t) - u_2(t)| + \int_t^{t+h} |f_1(\tau) - f_2(\tau)|d\tau,$$

from which the required estimate (2.3.153) and the conclusion:

if u satisfies (2.3.152), then for all $t, h \geq 0$,

$$|u(t + h) - Pu(t)| \leq |u(t) - Pu(t)| + \int_t^{t+h} |f(\tau)|d\tau, \tag{2.3.154}$$

follow.

Moreover, we can derive from (2.3.154) and the triangle inequality that for all $t, h \geq 0$,

$$|u(t+h) - u(t)| \leq 2|u(t) - Pu(t)| + \int_t^{t+h} |f(\tau)|d\tau. \tag{2.3.155}$$

Therefore letting $h \to +\infty$ in (2.3.155) for any $t \geq 0$, we can obtain (2.3.153). □

Remark 2.3.6 ([933]). If A is the sub-differential of a lower semi-continuous convex function φ, then the set F in Lemma 2.3.7 coincides with the set of minimizers of φ.

Lemma 2.3.26 ([933]). *Let $x : [0, +\infty) \to (0, +\infty)$ be a continuous function.*

(i) *If $p > 2$ and there exists a constant $l > 0$ such that*

$$\lim_{t\to+\infty} \frac{x(t)}{t^{-1/(p-2)}} = l, \tag{2.3.156}$$

then there exists a constant $M > 0$ such that for all $t \geq 0$,

$$\int_t^{+\infty} x^p(\tau)d\tau \leq Mx^2(t). \tag{2.3.157}$$

(ii) *If there exists a constant $l > 0$ such that*

$$\lim_{t\to+\infty} \frac{-\log x(t)}{t} = l, \tag{2.3.158}$$

then there exist a constant $M > 0$ and a sequence $\{t_n\}$ with $t_n \nearrow +\infty$ such that for all $n \geq 1$,

$$\int_{t_n}^{+\infty} x^2(\tau)d\tau \leq Mx^2(t_n). \tag{2.3.159}$$

Proof. An easy calculation gives the following identities for all $t \geq 0$,

$$\int_t^{+\infty} \tau^{-p/(p-2)}d\tau = (p/2 - 1)t^{-2/(p-2)},$$
$$\int_t^{+\infty} e^{-c\tau}d\tau = \frac{e^{-ct}}{c}, \text{ for all } c > 0$$

which readily implies the desired results. Here we leave the details to the reader. □

Theorem 2.3.27 ([933]). *Let $p \geq 2$ and let φ satisfy (C_pF) and (SH_p) with respect to all the points of F. Then for any solution u of (2.3.134), the following holds: if $u(0) \notin F$, then for all $t \geq 0$,*

$$u(t) \notin F. \tag{2.3.160}$$

In this case, u converges strongly in H as $t \to +\infty$, and if by $\bar{z}$ we denote its limit, then there exists a non-negative constant Λ^∞ such that

$$\lim_{t\to+\infty} \frac{p\varphi(u(t))}{|u(t)-\bar{z}|^p} = \Lambda^\infty. \tag{2.3.161}$$

Moreover,

$$\begin{cases} \displaystyle\lim_{t\to+\infty} \frac{-\log|u(t)-\bar{z}|}{t} = \Lambda^\infty, \quad \text{if } p=2, & (2.3.162)\\ \displaystyle\lim_{t\to+\infty} \frac{|u(t)-\bar{z}|^{2-p}}{(p-2)t} = \Lambda^\infty, \quad \text{if } p>2. & (2.3.163)\end{cases}$$

If, either $a \equiv 0$ on $(0,+\infty)$, or (SH_pF) holds, then

$$\Lambda^\infty > 0. \tag{2.3.164}$$

Proof. Let u be any solution of (2.3.134) with $u(0) \notin F$. For all the points of F, we get that (SH_p) is valid. Then the first part of Theorem 2.3.17 yields $u(t) \notin F$, for all $t \geq 0$. Due to that (C_pF) implies the "uniform convergence condition" of Pazy, it then follows that every solution of problem (2.3.135) converges strongly in H as $t \to +\infty$, (see [739], Theorem 2.2). Therefore, by Lemma 2.3.2, the strong convergence also holds for u; let $\bar{z}$ be its limit, where $\bar{z} \in F$. Then the result of Theorem 2.3.17 obviously holds with respect to $\bar{z}$.

To prove that $\Lambda^\infty > 0$, consider first the case when u satisfies (2.3.135). By a result of Pazy [739] (Lemma 4.6), a more general version of Lemma 2.3.7, (2.3.151) holds. Thus, it follows immediately from (2.3.151) and (C_pF) that $\Lambda^\infty > 0$.

When u is a solution of (2.3.134) and φ satisfies (SH_pF), it suffices to use Theorem 2.3.20 to consider now the case when (SH_pF) holds and u satisfies (2.3.134). In fact, Theorem 2.3.19 holds under the present assumptions. Hence, by Hölder's inequality, it follows from Lemma 2.3.7 that for all $t \geq 0$,

$$\begin{aligned} |u(t)-\bar{z}| &\leq 2|u(t)-Pu(t)| + \int_t^{+\infty} \sqrt{a(\tau)\varphi(u(\tau))}\,d\tau \\ &\leq 2|u(t)-Pu(t)| + C_1\left(\int_t^{+\infty} \varphi(u(\tau))\,d\tau\right)^{1/2}, \end{aligned} \tag{2.3.165}$$

where $C_1 := \left(\int_0^{+\infty} a(\tau)d\tau\right)^{1/2} \geq 0$. Thus we can derive from Theorem 2.3.17 and (2.3.165) that for all $t \geq 0$,

$$|u(t)-\bar{z}| \leq 2|u(t)-Pu(t)| + C_2\left(\int_t^{+\infty} |u(\tau)-Pu(\tau)|^p d\tau\right)^{1/2}, \tag{2.3.166}$$

where $C_2 > 0$ is a constant independent of $t \geq 0$.

To prove that $\Lambda^\infty > 0$, by (C_pF) it suffices to prove that there exist a constant $C > 0$ and a sequence $\{t_n\}$ with $t_n \nearrow +\infty$ such that for all $n \geq 1$,

$$|u(t_n) - \bar{z}| \leq C|u(t_n) - Pu(t_n)|. \tag{2.3.167}$$

Thus, (2.3.167) follows immediately from (2.3.166) if we can prove that there exists a constant $C_3 > 0$ and a sequence $\{t_n\}$ with $t_n \nearrow +\infty$ such that for all $n \geq 1$,

$$\left(\int_{t_n}^{+\infty} |u(\tau) - Pu(\tau)|^p d\tau\right)^{1/2} \leq C_3|u(t_n) - Pu(t_n)| \tag{2.3.168}$$

which, by Theorem 2.3.20 (see below), is a straightforward corollary of Lemma 2.3.8. The proof is thus complete. □

Now we shall show the next result when F is not a single point and provides optimal rates of decay for the distance between the solution u and the set F of minimizers of φ. Denote by (B_p) the following condition: there exists an open set D_F containing F such that for all $u \in D_F \cap \mathrm{Dom}(A)$, for all $v \in A(u)$,

$$\langle v, u - Pu\rangle \geq p\varphi(u).$$

Obviously, (SH_pF) implies (B_p).

Theorem 2.3.28 ([933]). *Assume $p \geq 2$ and φ satisfies (B_p) and (C_pF). Then, for any solution u of problem* (2.3.134), *there holds that if $u(0) \notin F$, then for all $t \geq 0$,*

$$u(t) \notin F. \tag{2.3.169}$$

If this case holds, then u strongly converges in H as $t \to +\infty$ and there exists a constant $\Gamma^\infty > 0$ such that

$$\lim_{t\to+\infty} \frac{p\varphi(u(t))}{|u(t) - Pu(t)|^p} = \Gamma^\infty. \tag{2.3.170}$$

Moreover,

$$\begin{cases} \lim\limits_{t\to+\infty} \dfrac{-\log|u(t) - Pu(t)|}{t} = \Gamma^\infty, \quad \text{if } p = 2, & (2.3.171)\\[2ex] \lim\limits_{t\to+\infty} \dfrac{|u(t) - Pu(t)|^{2-p}}{(p-2)t} = \Gamma^\infty, \quad \text{if } p > 2. & (2.3.172)\end{cases}$$

Proof. Let u be any solution of (2.3.134) with $u(0) \notin F$. Noting that the proof of Theorem 2.3.19, (C_pF) guarantees the strong convergence of u and applying Lemma 2.3.6 when F is the set of minimizers of φ, we can obtain that the mapping $t \mapsto |u(t) - Pu(t)|^2$ is absolutely continuous on compact intervals, with derivative given by

$$\frac{d}{dt}|u - Pu|^2 = 2\langle u', u - Pu\rangle, \text{ a.e. in } (0, +\infty), \tag{2.3.173}$$

which enables us to argue as in the proof of Theorem 2.3.17 by only using "$u(t) - Pu(t)$" to replace the difference "$u(t) - z$" everywhere and (B_p) instead of (SH_p). Thus by assumption (C_pF), we conclude readily that $\Gamma^\infty > 0$. The proof is thus complete. □

In order to obtain a characterization for the set of possible values of Λ^∞ in Theorem 2.3.19, we introduce the following assumptions for $p \geq 2$:

(H_1) : φ is locally sub-homogeneous of degree p with respect to all the points of F;

(H_2) : there exists a reflexive Banach space $(V, \|\cdot\|)$ densely, continuously and compactly embedded in H such that $\mathrm{Dom}(\varphi) \subseteq V$ and for all $u \in \mathrm{Dom}(\varphi)$, and $z \in F$, there exists an $\eta > 0$, such that

$$\varphi(u) + |u - z|^p \geq \eta \|u - z\|^p;$$

(H_3) : for each $z \in F$, the function ψ_z defined on the set

$$K_z = \{t(y - z) : t \geq 0, y \in \mathrm{Dom}(\varphi)\}$$

by the relations $\psi_z(0) = 0$, and $\psi_z(u) = \lim_{t \searrow 0} \frac{\varphi(z+tu)}{t^p}$ for all $u \in K_z$ admits an extension $\tilde{\psi}_z : H \to [0, +\infty]$ such that:

(i) $\mathrm{Dom}(\tilde{\psi}_z)$ is the closure in V of K_z;
(ii) $\tilde{\psi}_z$ is lower semi-continuous on H;
(iii) for every $u \in \mathrm{Dom}(\tilde{\psi}_z)$, there exists a sequence $\{u_n\}$ of elements of K_z such that $u_n \to u$ strongly in H as $n \to +\infty$ and

$$\tilde{\psi}_z(u) = \lim_{n \to +\infty} \psi_z(u_n).$$

For the applications, we note that these hypotheses can imply the convexity of $\tilde{\psi}_z$, for all $z \in F$ and are easy to determine explicitly ψ_z for all $z \in F$, as well as the extensions $\tilde{\psi}_z$.

Theorem 2.3.29 ([933]). *For $p \geq 2$, let φ satisfy $(H_1), (H_2), (H_3)$ and (C_pF). Then the result of Theorem 2.3.19 holds for $\Lambda^\infty \geq 0$. Moreover, if u is any non-stationary solution of problem (2.3.134) with limit $\bar{z}$, then Λ^∞ in Theorem 2.3.19 satisfies: there exist an $\omega \in \mathrm{Dom}(\partial\tilde{\psi}_{\bar{z}})$ with $|\omega| = 1$ and a sequence $\{t_n\}$ with $t_n \nearrow +\infty$ such that*

$$\omega = \lim_{n \to +\infty} \frac{u(t_n) - \bar{z}}{|u(t_n) - \bar{z}|} \quad \text{and} \quad \partial\tilde{\psi}_{\bar{z}}(\omega) \ni \Lambda^\infty \omega. \tag{2.3.174}$$

Proof. Under the present hypotheses, Theorem 2.3.19 is valid. Hence it clearly follows from Lemma 2.3.4. Now let u be any non-stationary solution of (2.3.134), with limit $\bar{z}$, where $\bar{z} \in F$. Without loss of generality, we assume that $\bar{z} = 0$; the general case can be done by a translation argument.

We first assert that there exists a sequence $\{t_n\}$ with $t_n \nearrow +\infty$ such that

$$\begin{cases} \frac{v(t_n)}{|u(t_n)|^{p-1}} - \Lambda(t_n)\frac{u(t_n)}{|u(t_n)|} \to 0, \text{ strongly in } H, \\ v(t_n) \in Au(t_n), \text{ for all } n \geq 1. \end{cases} \tag{2.3.175}$$

Without loss of generality, we assume that for all $t \geq 0, u(t) \in D_0$ in the definition of local sub-homogeneity of φ with respect to 0, which ensures that the estimates in the proof of Theorem 2.3.17 are valid. In particular, (2.3.144) implies that

$$\int_0^{+\infty} |u(\tau)|^{p-2}\left|\frac{v(\tau)}{|u(\tau)|^{p-1}} - \Lambda(\tau)\frac{u(\tau)}{|u(\tau)|}\right|^2 d\tau < +\infty. \tag{2.3.176}$$

If $p = 2$, (2.3.175) follows immediately from (2.3.176), while for $p > 2$, we may use Theorem 2.3.19 to deduce from (2.3.176) that for all $\alpha > 0$,

$$\int_\alpha^{+\infty} \frac{1}{\tau}\left|\frac{v(\tau)}{|u(\tau)|^{p-1}} - \Lambda(\tau)\frac{u(\tau)}{|u(\tau)|}\right|^2 d\tau < +\infty. \tag{2.3.177}$$

Since $\int_\alpha^{+\infty} \frac{1}{\tau} d\tau = +\infty$ for all $\alpha > 0$, (2.3.175) follows immediately in the case $p > 2$.

Since $\lim_{t\to+\infty} \Lambda(t) = \Lambda^\infty$, we can derive from (2.3.175) that

$$\begin{cases} \frac{v(t_n)}{|u(t_n)|^{p-1}} - \Lambda^\infty\frac{u(t_n)}{|u(t_n)|} \to 0, \text{ strongly in } H, \\ v(t_n) \in Au(t_n), \text{ for all } n \geq 1. \end{cases} \tag{2.3.178}$$

Now for all $n \geq 1$, let

$$u_n := u(t_n), v_n := v(t_n), \quad \omega_n := \frac{u_n}{|u_n|}, \quad \alpha_n := |u_n|.$$

Then we obtain that $|\omega_n| = 1$ for all $n \geq 1$, and that $\alpha_n \to 0$ as $n \to +\infty$.

Next, since Λ is bounded on $[0, +\infty)$ and (H_2) holds with respect to the origin, hence the set $\{\frac{u(t)}{|u(t)|} : t \geq 0\}$ is bounded in V. As V is reflexive and compactly embedded in H, we know that $\{\omega_n\}$ contains a subsequence which converges weakly in V and strongly in H to an element $\omega \in V$ with $|\omega| = 1$. For simplicity, we use the original sequence instead of this convergent subsequence. Thus

$$\omega_n \to \omega, \text{ strongly in } H, \tag{2.3.179}$$

which, along with (2.3.178), yields

$$\begin{cases} \frac{v_n}{\alpha_n^{p-1}} \to \Lambda^\infty\omega, \quad \text{strongly in } H, \\ v_n \in A(\alpha_n\omega_n), \quad \text{for all } n \geq 1. \end{cases} \tag{2.3.180}$$

Since $\omega_n = \frac{1}{\alpha_n}$, where $u_n = u(t_n) \in \mathrm{Dom}(\varphi)$ and $\alpha_n > 0$, we can derive that ω_n belongs to the set K_0 in (H_3) for all $n \geq 1$. Since ω_n converges to ω weakly in V as $n \to +\infty$, and K_0 is convex, (H_3) shows that ω belongs to $\mathrm{Dom}(\tilde{\psi}_0)$. Since ω was constructed as the limit of $\omega_n = \frac{u(t_n)}{|u(t_n)|}$, it only suffices to prove that $\omega \in \mathrm{Dom}(\partial\tilde{\psi}_0)$ and $\partial\tilde{\psi}_0(\omega) \ni \Lambda^\infty\omega$.

Assume that y is an arbitrary element of K_0. Since $\{\alpha_n\}$ converges to 0 as $n \to +\infty$, and $\mathrm{Dom}(\varphi)$ is a convex set containing the origin, we derive that there exists some n_0 such that $\alpha_n y \in \mathrm{Dom}(\varphi)$ for all $n \geq n_0$. Noting that $v_n \in A(\alpha_n\omega_n)$, and using the definition of a sub-differential, we conclude that for all $n \geq n_0$,

$$\langle v_n, \alpha_n y - \alpha_n\omega_n\rangle \leq \varphi(\alpha_n y) - \varphi(\alpha_n\omega_n),$$

which further yields that for all $n \geq n_0$,

$$\left\langle \frac{v_n}{\alpha_n^{p-1}}, y - \omega_n \right\rangle \leq \frac{\varphi(\alpha_n y)}{\alpha_n^p} - \frac{\varphi(\alpha_n\omega_n)}{\alpha_n^p}. \tag{2.3.181}$$

Now passing to the limit as $n \to +\infty$ in (2.3.181), using (2.3.179), (2.3.180), the definition of ψ_0 in (H_3), and Theorem 2.3.19, we obtain

$$\langle \Lambda^\infty\omega, y - \omega\rangle \leq \psi_0(y) - \frac{\Lambda^\infty}{p}. \tag{2.3.182}$$

On the other hand, it follows from (H_3) that (2.3.182) is valid not only for $y \in K_0$, but also for all $y \in \mathrm{Dom}(\tilde{\psi}_0)$, i.e., for all $y \in \mathrm{Dom}(\tilde{\psi}_0)$,

$$\langle \Lambda^\infty\omega, y - \omega\rangle \leq \tilde{\psi}_0(y) - \frac{\Lambda^\infty}{p}. \tag{2.3.183}$$

Thus, choosing $y := \omega$, with $\omega \in \mathrm{Dom}(\tilde{\psi}_0)$, we obtain that $\tilde{\psi}_0(\omega) \geq \frac{1}{p}\Lambda^\infty$. On the other hand, since $\tilde{\psi}_0$ is lower semi-continuous on H, from (2.3.179), the local sub-homogeneity of φ with respect to 0, and Theorem 2.3.19, it follows that

$$\begin{aligned}\tilde{\psi}_0(\omega) &\leq \lim_{n\to+\infty}\inf \psi_0(\omega_n) \leq \lim_{n\to+\infty}\inf \frac{\varphi(\alpha_n\omega_n)}{\alpha_n^p}\\ &= \lim_{n\to+\infty}\frac{\Lambda(t_n)}{p} = \frac{\Lambda^\infty}{p}\end{aligned} \tag{2.3.184}$$

which further shows that $\tilde{\psi}_0(\omega) = \frac{1}{p}\Lambda^\infty$. Therefore, (2.3.183) implies that for all $y \in \mathrm{Dom}(\tilde{\psi}_0)$,

$$\langle \Lambda^\infty\omega, y - \omega\rangle \leq \tilde{\psi}_0(y) - \tilde{\psi}_0(\omega) \tag{2.3.185}$$

which, therefore, implies that $\omega \in \mathrm{Dom}(\partial\tilde{\psi}_0)$ and $\partial\tilde{\psi}_0(\omega) \ni \Lambda^\infty\omega$. The proof is complete. □

Remark 2.3.7 ([933])**.** In fact, a special case of Theorem 2.3.21 is that $F = \{0\}$ and there exists a reflexive Banach space $(V, \|\cdot\|)$ which is further densely, continuously and compactly embedded in H such that $\mathrm{Dom}(\varphi) = V$, and there exists an $\eta > 0$ satisfying for all $u \in V$,

$$\varphi(u) \geq \eta \|u\|^p, \tag{2.3.186}$$

with φ homogeneous of degree p with respect to the origin. For this case, $\tilde{\psi}_0$ coincides with φ and thus, in Theorem 2.3.21, we recover the classical results in [303] and [306] that Λ^∞ is an eigenvalue for the operator $A := \partial\varphi$.

Remark 2.3.8 ([933])**.** In fact, we do not know whether or not it is true that in Theorem 2.3.21, there exists an $\omega \in H$ with $|\omega| = 1$ and $\partial\tilde{\psi}_{\bar{z}}(\omega) \ni \Lambda^\infty \omega$, such that

$$\omega = \lim_{t\to+\infty} \frac{u(t) - \bar{z}}{|u(t) - \bar{z}|}.$$

However, such a result is known in the linear case, see, e.g., [303].

Here we shall consider the second-order case and modify the methods to obtain exact rates of convergence for solutions of the second-order problem

$$\begin{cases} u''(t) \in A(u(t)), \ \text{for a.e. } t \in (0, +\infty), \\ u(0) = u_0, \ u_0 \in \overline{\mathrm{Dom}(A)}, \\ \sup\limits_{t\geq 0} |u(t)| < +\infty, \end{cases} \tag{2.3.187}$$

which indeed generalizes the corresponding results of Biler [106]. Such a result is due to Vărvărucă [933].

Consider now a solution u of problem (2.3.187). By the above classical existence and regularity results, e.g., in [628] and [629], without loss of generality, we may assume that $u \in W^{2,2}_{\mathrm{loc}}((0, +\infty); H)$. By v we denote the second derivative of u. Thus, we have that $v \in L^2_{\mathrm{loc}}([0, +\infty); H)$, and $v(t) \in Au(t)$ for a.e. $t \in (0, +\infty)$.

The next result, an analogue of Theorem 2.3.19, is also due to [933] and generalizes the results of Biler [106] by using some ideas from Mitidieri [628].

Theorem 2.3.30 ([933])**.** *Let $p \geq 2$ and let φ satisfy (C_pF) and (SH_p) with respect to all the points of F. Then for any solution u of* (2.3.187)*, the following holds: if $u_0 \notin F$, then $u(t) \notin F$ for all $t \geq 0$.*

If this is the case, then u converges strongly in H as $t \to +\infty$, and if we denote by $\bar{z}$ its limit, then there exists a constant $\Lambda^\infty > 0$ such that

$$\lim_{t\to+\infty} \frac{-\log|u(t) - \bar{z}|}{t} = \Lambda^\infty, \quad \textit{if } \ p = 2, \tag{2.3.188}$$

and

$$\lim_{t\to+\infty} \frac{|u(t) - \bar{z}|^{1-p/2}}{(p/2-1)t} = \Lambda^\infty, \quad \textit{if } \ p > 2. \tag{2.3.189}$$

Proof. Assume u satisfies (2.3.187) with $u_0 \in \overline{F}$. Clearly we may derive that $u(t) \notin F$ for all $t \geq 0$ readily from the following estimates (2.3.190)–(2.3.192). So we may assume that $u(t) \notin F$ for all $t \geq 0$. Since (C_pF) holds. It follows from [628] (Theorem 3.1) that u converges strongly as $t \to +\infty$ to some point of F, denoted by $\bar{z}$. To prove these estimates, we assume without loss of generality, that $\bar{z} = 0$; the general case can be done by a translation argument.

Noting that the identities

$$\begin{cases} \dfrac{d}{dt}\left(-\log|u|\right) = \dfrac{-\langle u, u'\rangle}{|u|^2}, \text{ a.e. in } (0, +\infty), & (2.3.190) \\ \dfrac{d}{dt}\left(\dfrac{|u|^{1-p/2}}{p/2-1}\right) = \dfrac{-\langle u, u'\rangle}{|u|^{1+p/2}}, \text{ a.e. in } (0, +\infty), \quad p > 2, & (2.3.191) \end{cases}$$

we need the study of the function $\Lambda : [0, +\infty) \to \mathbb{R}$ defined by

$$\Lambda := \frac{-\langle u, u'\rangle}{|u|^{1+p/2}}.$$

Indeed, we can show

$$\frac{d\Lambda}{dt} + \frac{1+p/2}{|u|^{1+p/2}}\left|u' - \frac{\langle u, u'\rangle}{|u|^2}u\right|^2 = \frac{p\varphi(u) - \langle v, u\rangle}{|u|^{1+p/2}}, \text{ a.e. in } (0, +\infty), \tag{2.3.192}$$

by using the identity $2\varphi(u) = |u'|^2$, which is valid for the solutions of (2.3.187) (see, e.g., [106] and [629]).

Since (SH_p) now holds with respect to 0, $\Lambda(t)$ is non-increasing on $[0, +\infty)$, and therefore has a limit, denoted by Λ^∞. We can easily check that the mapping $t \mapsto |u(t)|^2$ is convex and bounded, it is necessarily non-increasing. Hence, its first derivative is non-positive, and so $\Lambda(t) \geq 0$ for all $t \geq 0$, and this proves that Λ^∞ is finite and $\Lambda^\infty \geq 0$. The required estimates now follow from (2.3.190) and from (2.3.191), respectively.

Therefore, it suffices to show that $\Lambda^\infty > 0$. We shall use the contradiction argument. To this end, assume that $\Lambda^\infty = 0$. It hence follows from (2.3.192) that

$$\int_0^{+\infty} |u(\tau)|^{p/2-1} \frac{1}{|u(\tau)|^p}\left|u'(\tau) - \frac{\langle u(\tau), u'(\tau)\rangle}{|u(\tau)|^2}u(\tau)\right|^2 d\tau < +\infty.$$

Using an argument similar to the proof of Theorem 2.3.21, we conclude that there exists a sequence $\{t_n\}$ with $t_n \nearrow +\infty$ such that

$$\frac{u'(t_n)}{|u(t_n)|^{p/2}} + \Lambda(t_n)\frac{u(t_n)}{|u(t_n)|} \to 0, \text{ strongly in } H.$$

Since $\Lambda^\infty = 0$ by assumption, it follows that

$$\frac{u'(t_n)}{|u(t_n)|^{p/2}} \to 0, \text{ strongly in } H$$

which together with the equality $2\varphi(u) = |u'|^2$ implies that as $n \to +\infty$,

$$\frac{\varphi(u(t_n))}{|u(t_n)|^p} \to 0.$$

Thus this contradicts (C_pF) and the fact that (2.3.151) also holds in the present setting (see, e.g., [628], Theorem 3.1). Therefore, $\Lambda^\infty > 0$. The proof is complete. □

2.4 Differential inequalities for non-existence of global solutions

In this section, we present some differential inequalities which yield the non-existence of global solutions to certain evolutionary partial differential equations.

We begin with the following two simple differential inequalities (see, e.g., Levine [508]).

Theorem 2.4.1 (The Levine Inequality [508]). *Let $F : [0, T) \to [0, +\infty)$ be a twice differentiable function satisfying for all $t \in [0, T)$,*

$$F''(t) \geq -\lambda F(t) + g_1(F(t)), \quad F'(0) > 0, \quad F(0) > s_1, \tag{2.4.1}$$

where $\lambda > 0$ is a constant, and $g_1 : \mathbb{R} \to \mathbb{R}$ is a convex function, H_1 is any indefinite integral of $g_1(s)$: for all $s \in \mathbb{R}$,

$$H_1'(s) = g_1(s), \tag{2.4.2}$$

and there exists an $s_1 \in \mathbb{R}$ such that $H_1(s) - \frac{1}{2}\lambda s^2$ is non-decreasing on $(s_1, +\infty)$ and, for every $\varepsilon > 0$,

$$\int_{s_1}^{+\infty} \left\{H_1(s) - \frac{1}{2}\lambda s^2 - [H_1(s_1) - \frac{1}{2}\lambda s_1^2] + \varepsilon\right\}^{-1/2} ds < +\infty, \tag{2.4.3}$$

that is, $[H_1(s) - \frac{1}{2}\lambda s^2]^{-1/2}$ is integrable at infinity. Then

$$T \leq \frac{\sqrt{2}}{2} \int_{F(0)}^{+\infty} \left\{H_1(s) - \frac{1}{2}\lambda s^2 - [H_1(s_1) - \frac{1}{2}\lambda s_1^2] + \frac{1}{2}(F'(0))^2\right\}^{-1/2} ds \tag{2.4.4}$$

and

$$\lim_{t \uparrow T} F(t) = +\infty. \tag{2.4.5}$$

Proof. Since $F'(0) > 0$, it follows that $F'(t) > 0$ on an interval $[0, \eta) \subset [0, T)$. Let η be the largest number such that $F'(t) > 0$ on $[0, \eta)$. Then $F'(\eta) = 0$ for $\eta < T$

or $\eta = T$. If $\eta < T$, then multiplying (2.4.1) by $F'(t)$ for $t \in [0, \eta)$ and integrating the resulting inequality over $(0, \eta)$, we obtain

$$-\frac{1}{2}(F'(0))^2 > H_1(F(\eta)) - \frac{1}{2}\lambda F^2(\eta) - [H_1(F(0)) - \frac{1}{2}\lambda F^2(0)]. \tag{2.4.6}$$

In view of the fact $F(\eta) \geq F(0) > s_1$, the right-hand side of inequality (2.4.6) is non-negative. Since the left-hand side of (2.4.6) is negative, we reached a contradiction. This shows $\eta = T$. Thus $F'(t) > 0$ on the existence interval $[0, T)$ and $F(t) \geq F(0) > s_1$ for any $t \in [0, T)$.

It follows, by standard arguments, that for any $t \in [0, T)$,

$$T \leq \frac{\sqrt{2}}{2} \int_{F(0)}^{+\infty} \left\{ H_1(s) - \frac{1}{2}\lambda s^2 - \left[H_1(s_1) - \frac{1}{2}\lambda s_1^2 \right] + \frac{1}{2}(F'(0))^2 \right\}^{-1/2} ds.$$

Therefore, T is bounded from above by the (finite) integral on the right-hand side, i.e.,

$$\lim_{t \uparrow T} F(t) = +\infty.$$

The proof is thus complete. □

Theorem 2.4.2 (The Levine Inequality [508]). *Assume that $F(t)$ satisfies for all $t > 0$,*

$$F''(t) + \alpha F'(t) \geq C F^r(t), \; F(0) = F_0, \;\; F'(0) = F_1 \tag{2.4.7}$$

where $\alpha > 0, C > 0$ and $r > 0$ are all real constants. If $F_0 \geq 0, F_1 > 0$, and

$$k = \int_{F_0}^{+\infty} \left[\frac{2C}{r+1}(y^{r+1} - F_0^{r+1}) + F_1^2 \right]^{-1/2} dy < 1, \tag{2.4.8}$$

then for all $t > 0$,

$$F(t) > 0, \;\; F'(0) > 0, \tag{2.4.9}$$

and there is a constant $\tilde{T} \leq T^* = -\ln(1-k)/\alpha$, such that as $t \to \tilde{T}^-$

$$F(t) \to +\infty. \tag{2.4.10}$$

Proof. Multiplying (2.4.7) by $e^{\alpha t}$, then integrating the resulting expression over $(0, t)$, we have for all $t > 0$,

$$F'(t) \geq e^{-\alpha t} \left[F_1 + C \int_0^t e^{\alpha \tau} F^r(\tau) d\tau \right]. \tag{2.4.11}$$

If there is a constant $t_0 > 0$ such that $F(t) > 0$, for all $t \in (0, t_0)$, while $F(t_0) = 0$, then for all $t \in (0, t_0)$, (2.4.11) implies that $F'(t) > 0$. Hence, $F(t)$ is strictly increasing on $[0, t_0]$ and thus $F(t_0) > F_0 \geq 0$, which contradicts the assumption $F(t_0) = 0$. Therefore $F(t) > 0$ for $t > 0$, which together with (2.4.11) implies that $F'(t) > 0$ for $t > 0$. Now multiplying (2.4.7) by $e^{2\alpha t}F'(t)$, we get for all $t > 0$,

$$\frac{d}{dt}\left[e^{2\alpha t}F'(t)\right] \geq \frac{2C}{r+1}\frac{d}{dt}F^{r+1}(t). \tag{2.4.12}$$

Integrating (2.4.12) over $(0,t)$, we deduce that for all $t > 0$,

$$F'(t) \geq e^{-\alpha t}\left[\frac{2C}{r+1}(F^{r+1}(t) - F_0^{r+1}) + F_1^2\right]^{1/2}, \tag{2.4.13}$$

whence for all $t > 0$,

$$e^{-\alpha t} \leq \left[\frac{2C}{r+1}(F^{r+1}(t) - F_0^{r+1}) + F_1^2\right]^{-1/2}. \tag{2.4.14}$$

Integrating (2.4.14) over $(0,T)$, we can obtain

$$1 - e^{-\alpha T} \leq \alpha \int_{F_0}^{+\infty}\left[\frac{2C}{r+1}(y^{r+1} - F_0^{r+1}) + F_1^2\right]^{-1/2} dy \equiv k$$

which completes the proof. □

The following result, due to Levine[508], is a counterpart of differential inequalities of the first order.

Theorem 2.4.3 (The Levine Inequality [508])**.** *Let $F : [0,T) \to [0,+\infty)$ be a differentiable function satisfying for all $t \in [0,T]$,*

$$F'(t) \geq -\lambda F(t) + g_2(F(t)), \qquad F'(0) > 0, F(0) > s_2 \tag{2.4.15}$$

where $\lambda > 0$ is a constant, and $g_2 : \mathbb{R} \to \mathbb{R}$ is a convex function with the property that there exists an $s_2 > 0$ such that $g_2(s) - \lambda s$ is positive on $(s_2, +\infty)$, and for every $\varepsilon > 0$,

$$\int_{s_2+\varepsilon}^{+\infty} [g_2(s) - \lambda s]^{-1}\, ds < +\infty. \tag{2.4.16}$$

Then

$$T \leq \int_{F(0)}^{+\infty} (g_2(s) - \lambda s)^{-1} ds \tag{2.4.17}$$

and

$$\lim_{t\uparrow T} F(t) = +\infty. \tag{2.4.18}$$

Proof. Indeed, from (2.4.15) we conclude that $F'(t) > 0$ as soon as $F(t) > s_2$. By an argument analogous to that used in Theorem 2.4.2, $F'(t) > 0$ on $[0,T)$ and consequently $F(t) > s_2$ on $[0,T)$.

From (2.4.15), we easily find that for any $t \in [0,T)$, (2.4.17) and (2.4.18) hold. The proof is now complete. □

The next result may be viewed as a corollary of the above theorem (see, e.g., Kaplan [413], also Quittner and Souplet [818]).

Theorem 2.4.4 (The Kaplan–Quittner–Souplet Inequality [413, 818]). *Let $y(t)$ be a non-negative C^1 function on $[0,T)$ satisfying for some $\lambda \in \mathbb{R}$ and for $p > 1$,*

$$y'(t) \geq y^p(t) - \lambda y(t). \tag{2.4.19}$$

(i) *If $y(0) > 0$ and $\lambda \leq 0$, then*

$$T \leq y^{1-p}(0)/(p-1), \tag{2.4.20}$$

while

(ii) *if $\lambda > 0$ and $y(0) > \lambda^{1/(p-1)}$, then*

$$T \leq \int_{y(0)}^{+\infty} (\sigma^p - \lambda\sigma)^{-1}\, d\sigma. \tag{2.4.21}$$

Proof. (i) Let $y(0) > 0, \lambda \leq 0$, and set $T_1 = \sup\{t : y(s) > 0 \text{ on } [0,t)\}$. Then for any $t \in [0,T_1) \subseteq [0,T)$, we have from (2.4.19)

$$y'(t) \geq y^p(t),$$

whence

$$y(t) \geq \left[y^{1-p}(0) - (p-1)t\right]^{1/(p-1)} \to +\infty \tag{2.4.22}$$

as $t \nearrow T_{\max} \equiv y^{1-p}(0)/(p-1)$. Thus (2.4.20) is valid, since otherwise $y(t)$ will go to $+\infty$ at $T_{\max} < T$ because of (2.4.22).

(ii) Let $\lambda > 0$ and $y(0) > \lambda^{1/(p-1)}$, and set $y(t) = F(t), g_2(s) = s^p,\ p > 1$ and $s_2 = \lambda^{1/(p-1)}$. Then (2.4.21) is a consequence result of Theorem 2.4.3. The proof is thus complete. □

However, assertion (ii) in Theorem 2.4.4 can be extended to the following result (see, e.g., Bandle and Levine [66]), which can be also viewed as a generalization of Theorem 2.4.4.

Theorem 2.4.5 (The Bandle–Levine Inequality [66]). *Let $y(t)$ be a non-negative C^1function on $[0,T)$ which satisfies for some constant $\lambda \in \mathbb{R}$,*

$$y'(t) \geq f(y(t)) - \lambda y(t), \tag{2.4.23}$$

where $p > 1$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz function with the following properties:

(i) *$f(0) = 0, f(s) > 0$ for all $s > 0$;*
(ii) *$\lim_{s\to 0} f(s)/s^p = a > 0$ for some $p > 1$;*
(iii) *$\int_s^{+\infty} 1/f(\sigma)d\sigma < +\infty$ for all $s > 0$;*
(iv) *f is convex.*

For $\lambda > 0$, denote by s_λ the positive solution of $f(s) - \lambda s = 0$, if $y(0) > s_\lambda$. Then $y(t)$ cannot exist beyond

$$\tau_\lambda = \int_{y(0)}^{+\infty} \frac{ds}{f(s) - \lambda s}. \tag{2.4.24}$$

The next result is an immediate consequence of the above theorem.

Corollary 2.4.1. *Let $y(t) \in C^1([0, +\infty))$ be a non-negative differential function satisfying*

$$\begin{cases} y'(t) + C_1 y(t) \geq C_2 y^p(t), \text{ for all } t \in [0, +\infty), & (2.4.25) \\ y(0) = y_0 > 0 & (2.4.26) \end{cases}$$

where $C_1 \geq 0, C_2 > 0$ and $p > 1$ are constants. Then for all $t \geq 0$,

$$y(t) \leq C_3 :\equiv (C_1/C_2)^{1/(p-1)}. \tag{2.4.27}$$

Proof. The proof is left to the reader, which is easier. □

The following result is a generalization of Theorem 2.4.4, and improves the result of Glassey [318] that holds for $\alpha > 1, \ \ \beta \leq 1, \ \ \gamma \leq 1$.

Theorem 2.4.6 (The Glassey–Qin Inequality [318])**.** *Let $F : [0, +\infty) \to (0, +\infty)$ be a non-negative C^1 function satisfying for all $t \geq 0$,*

$$F'(t) \geq C_0 F^\alpha(t)(k + t)^{-\beta} \log^{-\gamma}(1 + k + t), \tag{2.4.28}$$

such that

$$F(0) > 0 \tag{2.4.29}$$

where $\alpha > 1, k > 0, \beta < 1, \gamma \geq 0$ or $\beta = 1, \gamma \leq 1$ are constants. Then $F(t)$ blows up in a finite time.

Proof. From (2.4.28)–(2.4.29) it follows that for all $t \geq 0$,

$$F(t) \geq F(0) > 0. \tag{2.4.30}$$

Integrating (2.4.28) we obtain

$$F^{1-\alpha}(t) \leq F^{1-\alpha}(0) - C_0(\alpha - 1) \int_0^t (k + s)^{-\beta} \log^{-\gamma}(1 + k + s) ds. \tag{2.4.31}$$

Now we compute

$$\begin{aligned} I &\equiv \int_0^t (k + s)^{-\beta} \log^{-\gamma}(1 + k + s) ds \\ &\geq \begin{cases} \log^{-\gamma}(1 + k + t) \int_0^t (k + s)^{-\beta} ds, & \text{if } \ \beta < 1, \ \gamma \geq 0 \ \text{ or } \ \beta = 1, \gamma < 1, \\ \int_0^t (1 + k + s)^{-1} \log^{-1}(1 + k + s) ds, & \text{if } \ \beta = 1, \ \ \gamma = 1 \end{cases} \\ &\geq \begin{cases} \frac{\log^{-\gamma}(1+k+t)}{1-\beta} [(1 + k + t)^{1-\beta} - (1 + k)^{1-\beta}], & \text{if } \ \beta < 1 \ \ \gamma \geq 0; \\ \log^{-\gamma}(1 + k + t)[\log(1 + k + t) - \log(k + t)], & \text{if } \ \beta = 1, \ \ \gamma < 1; \\ \log\log(1 + k + t) - \log\log(1 + k), \ \text{ if } \ \beta = 1, \ \ \gamma = 1. \end{cases} \end{aligned} \tag{2.4.32}$$

Obviously, there is a large $t_0 = t_0(k) > 0$ such that for $t \geq t_0$,

$$\begin{cases} (1+k+t)^{1-\beta} - (1+k)^{1-\beta} \\ \qquad \geq \frac{1}{2}(1+k+t)^{1-\beta}, \text{ if } \beta < 1, \ \gamma \geq 0, & (2.4.33) \\ \log(1+k+t) - \log(1+k) \\ \qquad \geq \frac{1}{2}\log(1+k+t), \text{ if } \beta = 1, & (2.4.34) \\ \log\log(1+k+t) - \log\log(1+k) \\ \qquad \geq \frac{1}{2}\log\log(1+k+t), \text{ if } \beta = 1, \ \gamma = 1. & (2.4.35) \end{cases}$$

Thus it follows from (2.4.31)–(2.4.35) and the desired inequality $\ln(1+x) < 1+x$ for all $x > 0$ that for $t \geq t_0$,

$$F(t) \geq \left\{1/\left[F^{1-\alpha}(0) - C_0(\alpha-1)\int_0^t (k+s)^{-\beta}\log^{-\gamma}(1+k+s)ds\right]\right\}^{1/(\alpha-1)}$$

$$= \begin{cases} \left\{1/\left[F^{1-\alpha}(0) - C_0(\alpha-1)\log^{-\gamma}(1+k+t)(1+k+t)^{1-\beta}/[2(1-\beta)]\right]\right\}^{1/(\alpha-1)}, \\ \quad \text{if } \beta<1, \gamma \geq 0, \\ \left\{1/\left[F^{1-\alpha}(0) - C_0(\alpha-1)\log^{-\gamma}(1+k+t)\log(1+k+t)/2\right]\right\}^{1/(\alpha-1)}, \\ \quad \text{if } \beta=1, \ \gamma=1, \\ \left\{1/\left[F^{1-\alpha}(0) - C_0(\alpha-1)\log\log(1+k+t)/2\right]\right\}^{1/(\alpha-1)}, \\ \quad \text{if } \beta=1, \ \gamma=1, \end{cases}$$

$$\geq \begin{cases} \left\{1/\left[F^{1-\alpha}(0) - C_0(\alpha-1)(1+k+t)^{1-\beta-\gamma}/[2(1-\beta)]\right]\right\}^{1/(\alpha-1)}, \\ \quad \text{if } \beta<1, \ \gamma \geq 0, \\ \left\{1/\left[F^{1-\alpha}(0) - C_0(\alpha-1)\log^{1-\gamma}(1+k+t)/2\right]\right\}^{1/(\alpha-1)}, \\ \quad \text{if } \beta=1, \ \gamma<1, \\ \left\{1/\left[F^{1-\alpha}(0) - C_0(\alpha-1)\log\log(1+k+t)/2\right]\right\}^{1/(\alpha-1)}, \\ \quad \text{if } \beta=1, \ \gamma=1. \end{cases} \tag{2.4.36}$$

This shows that there is some $T_0 > 0$ such that

$$\begin{cases} F^{1-\alpha}(0) - \frac{C_0(\alpha-1)}{2(1-\beta)}(1+k+T_0)^{1-\beta-\gamma} = 0, \text{ if } \beta < 1, \ \gamma \geq 0, \\ F^{1-\alpha}(0) - \frac{C_0(\alpha-1)}{2}\log^{1-\gamma}(1+k+T_0) = 0, \text{ if } \beta = 1, \ \gamma < 1, \\ F^{1-\alpha}(0) - \frac{C_0(\alpha-1)}{2}\log\log(1+k+T_0) = 0, \text{ if } \beta = \gamma = 1 \end{cases} \tag{2.4.37}$$

and the desired result follows from (2.4.36)–(2.4.37). □

Remark 2.4.1. Theorem 2.4.6 states that if $\gamma = 0$, then Theorem 2.4.6 holds for $\beta \leq 1$. Now we rewrite (2.4.28) as

$$F'(t) \geq C_0 F^{\alpha}(t)(k+t)^{-(\beta+\gamma)} \tag{2.4.38}$$

which, taking into account the above case for $\gamma \geq 0$, implies that the conclusion of Theorem 2.4.6 still holds for $\alpha > 1, \beta + \gamma \leq 1$.

Using the above result, we can easily show the following result (see, e.g., Glassey [318]).

Theorem 2.4.7 (The Glassey Inequality [318]). *Let $F : [0, +\infty) \to (0, +\infty)$ be a non-negative C^2 function satisfying for all $t > 0$,*

$$\begin{cases} F''(t) \geq C_0 F^\alpha(t)(k+t)^{-\beta} \log^{-\gamma}(1+k+t), & (2.4.39) \\ F(0) > 0, \quad F'(0) > 0 & (2.4.40) \end{cases}$$

where $k > 0$, $\alpha > 1$, $0 \leq \gamma$, $\beta < 2$ or $\gamma \leq 2$, $\beta = 2$ are constants. Then $F(t)$ blows up in a finite time.

Proof. By (2.4.39), $F''(t) > 0$ for any $t \in [0, T)$, hence $F'(t) > F'(0) > 0$ and $F(t) > F(0)$ for any $t \in [0, T)$. Multiplying (2.4.39) by $F'(t)$, we may obtain

$$\begin{aligned} \frac{1}{2}\left[(F'(t))^2\right]' &\geq \frac{C_0}{\alpha+1}[F^{\alpha+1}(t)]'(k+t)^{-\beta}\log^{-\gamma}(1+k+t) \\ &\geq \frac{C_0}{\alpha+1}\frac{d}{dt}\left[F^{\alpha+1}(t)(k+t)^{-\beta}\log^{-\gamma}(1+k+t)\right] \end{aligned}$$

i.e.,

$$\left[(F'(t))^2\right]' \geq \frac{2C_0}{\alpha+1}\frac{d}{dt}\left[F^{\alpha+1}(t)(k+t)^{-\beta}\log^{-\gamma}(1+k+t)\right]. \tag{2.4.41}$$

Integrating (2.4.41) with respect to t gives us

$$\begin{aligned} [F'(t)]^2 \geq{} & [F'(0)]^2 + \frac{2C_0}{\alpha+1}F^{\alpha+1}(t)(k+t)^{-\beta}\log^{-\gamma}(1+k+t) \\ & - \frac{2C_0}{\alpha+1}F^{\alpha+1}(0)k^{-\beta}\log^{-\gamma}(1+k) \\ \equiv{} & C_3(C_0) + \frac{2C_0}{\alpha+1}F^{\alpha+1}(t)(k+t)^{-\beta}\log^{-\gamma}(1+k+t). \end{aligned} \tag{2.4.42}$$

Let $C_* = C_0$ be such that $C_3(C_0) = 0$. When $C_0 \leq C_*$, then

$$C_3(C_0) \geq 0. \tag{2.4.43}$$

From (2.4.42) and (2.4.43) it follows that

$$F'(t) \geq \sqrt{\frac{2C_0}{\alpha+1}}F^{(\alpha+1)/2}(t)(k+t)^{-\beta/2}\log^{-\gamma/2}(1+k+t). \tag{2.4.44}$$

Applying Theorem 2.4.6 to (2.4.44) yields the conclusion of the theorem. When $C_0 > C_*$, we may rewrite (2.4.39) with C_0 replaced by C_* and argue again as above to be able to complete the proof. □

If we assume $\alpha = 0$ in (2.4.39), then we can obtain the following theorem.

Theorem 2.4.8 (The Qin Inequality)**.** *Let $F : [0, +\infty) \to (0, +\infty)$ be a non-negative C^2 function satisfying*

$$\begin{cases} F''(t) \geq C_0(k+t)^{-\beta} \log^{-\gamma}(1+k+t), & (2.4.45) \\ F(0) > 0, \quad F'(0) > 0 & (2.4.46) \end{cases}$$

where $C_0 > 0$ and $\beta \leq 1, \gamma \geq 0$ are constants. Then $F(t)$ blows up in a finite time.

Proof. Obviously, on the one hand, the conditions (2.4.45)–(2.4.46) show that $F'(t) > F'(0) > 0$ and $F(t) > F(0) > 0$ for any $t \geq 0$. On the other hand, we deduce from (2.4.45) that

$$\frac{d}{dt}\left(\log^{\gamma}(1+k+t)F'(t)\right) \geq C_0(k+t)^{-\beta}. \tag{2.4.47}$$

Integrating (2.4.47) with respect to t, we get

$$\log^{\gamma}(1+k+t)F'(t) \geq \begin{cases} \log^{\gamma}(1+k)F'(0) + \frac{C_0}{1-\beta}[(k+t)^{1-\beta} - k^{1-\beta}], & \text{if } \beta < 1, \\ \log^{\gamma}(1+k)F'(0) + C_0 \ln \frac{k+t}{k}, & \text{if } \beta = 1. \end{cases} \tag{2.4.48}$$

Noting that

$$\frac{d}{dt}\left(\log^{\gamma}(1+k+t)F(t)\right) \geq \log^{\gamma}(1+k+t)F'(t),$$

it thus follows from (2.4.47)–(2.4.48) that,

$$\begin{aligned}
&\log^{\gamma}(1+k+t)F(t) \\
&\geq \log^{\gamma}(1+k)F(0) + \int_0^t \log^{\gamma}(1+k+s)F'(s)ds \\
&\geq \begin{cases} \log^{\gamma}(1+k)F(0) + \int_0^t \left[\log^{\gamma}(1+k)F'(0) + \frac{C_0}{1-\beta}[(k+s)^{1-\beta} - k^{1-\beta}]\right] ds, \\ \quad \text{if } \beta < 1, \\ \log^{\gamma}(1+k)F(0) + \int_0^t [\ln^{\gamma}(1+k)F'(0) + C_0(\ln(k+s) - \ln k)]\, ds, \\ \quad \text{if } \beta = 1, \end{cases} \\
&= \begin{cases} I_1(C_0) + I_2(C_0)t + \frac{C_0}{(1-\beta)(2-\beta)}(k+t)^{2-\beta}, & \text{if } \beta < 1, \\ I_3(C_0) + I_4(C_0)t + C_0(k+t)\ln(t+k), & \text{if } \beta = 1, \end{cases}
\end{aligned} \tag{2.4.49}$$

where

$$\begin{cases} I_1(C_0) = \log^{\gamma}(1+k)F(0) - \frac{C_0}{(1-\beta)(2-\beta)}k^{2-\beta}, \\ I_2(C_0) = \log^{\gamma}(1+k)F'(0) - \frac{C_0}{1-\beta}k^{1-\beta}, \\ I_3(C_0) = \log^{\gamma}(1+k)F(0) - C_0 k \log k, \\ I_4(C_0) = \log^{\gamma}(1+k)F'(0) - C_0 \log k - C_0. \end{cases}$$

Case 1: $\beta < 1$: Let

$$C_1 = \frac{(1-\beta)(2-\beta)\log^\gamma(1+k)F(0)}{k^{2-\beta}} > 0, \quad C_2 = \frac{(1-\beta)F'(0)\log^\gamma(1+k)}{k^{2-\beta}} > 0.$$

Then

$$I_1(C_1) = 0, \quad I_2(C_2) = 0.$$

(1) When $C_0 \leq C_3 \equiv \min[C_1, C_2] > 0$, we know that

$$I_1(C_0) \geq 0, \qquad I_2(C_0) \geq 0. \tag{2.4.50}$$

Thus (2.4.49) reduces (due to $\log(1+x) \leq x$ for all $x > 0$), as $t \to +\infty$, to

$$F(t) \geq \frac{C_0}{(1-\beta)(2-\beta)}(k+t)^{2-\beta}\log^{-\gamma}(1+k+t) \to +\infty \tag{2.4.51}$$

as long as $\beta < 1$. This implies that $F(t)$ blows up in a finite time.

(2) When $C_0 > C_3$, $I_1(C_0)$ and $I_2(C_0)$ cannot vanish simultaneously. But we may rewrite (2.4.45) with C_0 replaced by C_3 and repeat the same argument as above to be able to complete the proof.

Case 2: $\beta = 1$: We can also obtain the following estimate for large $t \to +\infty$ *as* $t \to +\infty$

$$\begin{aligned} F(t) &\geq \frac{C_0(t+k)\log(t+k)}{\log^\gamma(1+k+t)} + \frac{I_3(C_0) + I_4(C_0)t}{\log^\gamma(1+k+t)} \\ &\geq \frac{C_0(k+t)\log(k+t) - C'(k+t)}{\log^\gamma(1+k+t)} \end{aligned} \tag{2.4.52}$$

for a constant $C' > 0$, which implies that $F(t)$ blows up in a finite time. The proof is complete. □

The following is an analogue of Theorem 2.4.7 for $\gamma = 0$, which was established in Kato [419] in order to prove the non-existence of a global solution of nonlinear wave equations.

Corollary 2.4.2 (The Kato Inequality [419]). *If $p > 1, a, b, R > 0$ are constants, then the differential inequality for all $t \geq R > 0$,*

$$w''(t) \geq bt^{-1-p}w^p(t), \tag{2.4.53}$$

has no global solution such that for sufficiently large $t \geq R > 0$,

$$w'(t) \geq a > 0, \quad w(t) \geq at. \tag{2.4.54}$$

Proof. Arguing by contradiction, let $w(t)$ be a global solution. Inequality (2.4.53) for $w(t) \geq at$ implies that $w''(t) \geq ba^p/t$. Hence $w'(t) \geq K\log t$ eventually (i.e., for sufficiently large t), where K will denote different positive constants in the sequel. Therefore, eventually,

$$w(t) \geq Kt\log t. \tag{2.4.55}$$

Next, we note that

$$\begin{aligned}&[(w'(t))^2 - 2b(1+p)^{-1}t^{-1-p}w^{1+p}(t)]' \\ &\quad = 2w'(t)\left(w''(t) - bt^{-1-p}w^p(t)\right) + 2bt^{-2-p}w^{1+p}(t) > 0,\end{aligned}$$

whence

$$\left(w'(t)\right)^2 \geq Kt^{-1-p}w^{1+p}(t) - K. \tag{2.4.56}$$

Since, by (2.4.55), $w(t)/t \geq K\log t$ eventually, we may omit the last term $-K$ in (2.4.56) for large t, with the first K modified. Since

$$t^{-1-p}w^{1+p}(t) = (w(t)/t)^{p-1}\left(w(t)/t\right)^2 \geq K(\log t)^{p-1}\left(w(t)/t\right)^2,$$

(2.4.56) yields

$$w'(t) \geq K(\log t)^{(p-1)/2}\left(w(t)/t\right), \text{ eventually.} \tag{2.4.57}$$

Since $K(\log t)^{(p-1)/2}$ is arbitrarily large for large t, the linear differential inequality (2.4.57) for w implies

$$w(t) \geq Kt^k, \text{ eventually,} \tag{2.4.58}$$

for any positive number k. Returning to (2.4.56), we then obtain

$$w'(t) \geq Kt^{-(1+p)/2}w^{(p-1)/4}(t)w^{(p+3)/4}(t) \geq Kw^{(p+3)/4}(t), \tag{2.4.59}$$

because we can choose $k = 2(p+1)/(p-1)$ in (2.4.58). Since $(p+3)/4 > 1$, (2.4.59) is a nonlinear differential inequality in $w(t)$ for which no global solution with $w(t) > 0$ exists. This contradiction proves the theorem. □

While studying the blow-up phenomenon of a nonlinear wave equation with a critical exponent, Todorova and Yordanov [919] proved the following theorem.

Theorem 2.4.9 (The Todorova–Yordanov Inequality [919])**.** *Let $0 \geq A > -1$ and $r > 0$. Assume that $F(t)$ is a twice continuously differentiable solution of the inequality for all $t > 0$,*

$$F''(t) + F'(t) \geq C_0(t+k)^A|F(t)|^{1+r}, \tag{2.4.60}$$

with a constant $C_0 > 0$, such that

$$F(0) > 0, \quad F'(0) > 0. \tag{2.4.61}$$

Then $F(t)$ blows up in a finite time. The blow-up time can be estimated by

$$T_0 = \left\{\frac{2(A+1)F^{-r/2}(0)}{\delta\alpha} + k^{A+1}\right\}^{1/(A+1)} - k \tag{2.4.62}$$

where $\delta > 0$ is a small constant satisfying $\delta < F'(0)/[k^A F^{1+r/2}(0)]$.

Proof. We now consider the auxiliary initial value problem

$$Y'(t) = \nu(t+k)^A[Y(t)]^{1+r/2}, \;\; Y(0) \equiv F(0) > 0 \tag{2.4.63}$$

where $\nu > 0$ is a small number to be chosen later on. Since

$$Y(t) = \left\{[Y(0)]^{-r/2} - \frac{\nu r}{2(A+1)}[(t+k)^{A+1} - k^{A+1}]\right\}^{-2/r} \tag{2.4.64}$$

and $A > -1$, the solution $Y(t)$ of the above problem blows up at a finite time T_0 in (2.4.62) and satisfies for all $0 \le t < T_0$,

$$Y(t) > Y(0) > 0. \tag{2.4.65}$$

We now may compute

$$\begin{aligned} Y''(t) &= \nu(1+r/2)(t+k)^A[Y(t)]^{r/2} + \nu A(t+k)^{A-1}[Y(t)]^{1+r/2} \\ &\le \nu^2(1+r/2)(t+k)^{2A}[Y(t)]^{1+r}, \end{aligned} \tag{2.4.66}$$

where we have used that $A \le 0$ and that $Y(t)$ satisfies (2.4.63). Adding (2.4.63) and (2.4.66), and observing that $2A \le A$ and $[Y(t)]^{1+r/2} < [Y(0)]^{-r/2}[Y(t)]^{1+r/2}$, we thus obtain

$$\begin{aligned} Y''(t) + Y'(t) &\le \nu^2(1+r/2)(t+k)^{2A}[Y(t)]^{1+r} + \nu(t+k)^A[Y(t)]^{1+r/2} \\ &\le B(t+k)^A[Y(t)]^{1+r}, \end{aligned} \tag{2.4.67}$$

where $B = \nu^2(1+r/2) + \nu A[Y(0)]^{-r/2}$. Furthermore, we may choose $\nu > 0$ so small that

$$B = \nu^2(1+r/2) + \nu A[Y(0)]^{-r/2} < C_0, \;\; Y'(0) = \nu k^A[Y(0)]^{1+r/2} < F'(0).$$

Therefore from (2.4.67) we can obtain the inequality

$$Y''(t) + Y'(t) \le C_0(t+k)^A[Y(t)]^{1+r} \tag{2.4.68}$$

and the initial conditions

$$Y(0) \le F(0), \quad Y'(0) < F'(0). \tag{2.4.69}$$

We can now show that $F(t) \ge Y(t)$ for all $0 \le t < T_0$, so that $F(t)$ also blows up in a finite time. From $F'(0) > Y'(0)$, we have $F'(t) > Y'(0)$ for $t > 0$ small enough; consequently,

$$t_0 = \sup\{t \in [0, T_0) \mid F'(\tau) > Y'(\tau) \;\text{ for }\; 0 \le \tau < t\},$$

and assume that $t_0 < T_0$, where T_0 is the blowup time for $Y(t)$, thus for all $t \in [0, t_0)$,

$$F'(t) > Y'(t), \qquad F'(t_0) = Y'(t_0). \tag{2.4.70}$$

Since $F'(t) - Y'(t) > 0$, the function $F(t) - Y(t)$ is strictly increasing in the interval $0 \le t < t_0$. In particular, $F(t) - Y(t) > F(0) - Y(0) = 0$ for all $t \in [0, t_0)$. Moreover, $F(t_0) > Y(t_0)$, because if $F(t_0) = Y(t_0)$, then the function $F(t) - Y(t)$ will have zeros at 0 and at t_0, so by Rolle's Theorem, its derivative will vanish between 0 and t_0, i.e., $F'(t_1) = Y'(t_1)$ for some $0 < t_1 < t_0$, which is impossible by the definition of t_0. Therefore,

$$F(t_0) > Y(t_0), \quad F'(t_0) = Y'(t_0). \tag{2.4.71}$$

On the other hand, subtracting (2.4.68) from (2.4.60), we obtain

$$\begin{aligned} +[F'(t) - Y'(t)] &\ge C(t+k)^{2A}\left\{[F(t)]^{1+r} - [Y(t)]^{1+r}\right\} \\ &\ge 0 \end{aligned} \tag{2.4.72}$$

for all $0 \le t \le t_0$. We now can rewrite (2.4.72) in the form

$$\frac{d}{dt}\left\{e^t[F'(t) - Y'(t)]\right\} \ge 0$$

which, integrated over $[0, t_0)$, yields

$$e^{t_0}[F'(t_0) - Y'(t_0)] \ge F'(0) - Y'(0) > 0,$$

i.e.,

$$F'(t_0) - Y'(t_0) > 0. \tag{2.4.73}$$

This is a contradiction to (2.4.71). Thus, $t_0 \ge T_0$, and the proof is hence complete. □

The next result is due to the Quittner–Souplet Inequality [818].

Theorem 2.4.10 (The Quittner–Souplet Inequality [818])**.** *Assume that* $0 < r < 1 < p$ *and* $k, \lambda \ge 0$. *Let the functions* $y(t), z(t) \in C^1((0,T))$ *satisfy* $y(t) \ge 0, z(t) > 0$ *and the following system of differential inequalities on* $(0,T)$*:*

$$z'(t) \ge y^p(t), \quad y'(t) + \lambda y(t) + k(z'(t))^r \ge z(t). \tag{2.4.74}$$

Then $T < +\infty$.

Proof. By translating the origin of time, we may assume that actually $y(t), z(t) \in C^1([0,T))$ and $z(0) > 0$. Fix $\gamma > 0$ such that $\max(r, 1/p) < \gamma < 1$. It follows from the first inequality in (2.4.74) that, for all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$C_\varepsilon\left(z'(t)\right)^\gamma \ge y^{p\gamma}(t) + (3\lambda+1)y(t) - \varepsilon, \quad C_\varepsilon\left(z'(t)\right)^\gamma \ge 3k\left(z'(t)\right)^\gamma - \varepsilon. \tag{2.4.75}$$

Hence

$$2C_\varepsilon\left(z'(t)\right)^\gamma + 3y'(t) \ge 3\left(y'(t) + \lambda y(t) + k\left(z'(t)\right)^r\right) + y^{p\gamma}(t) + y(t) - 2\varepsilon.$$

By the second inequality in (2.4.74),

$$2C_\varepsilon\,(z'(t))^\gamma + 3y'(t) \geq 3z(t) + y^{p\gamma}(t) + y(t) - 2\varepsilon. \tag{2.4.76}$$

Next, choosing $m \in (0,\gamma)$ and using Young's inequality, we have

$$2C_\varepsilon\,(z'(t))^\gamma = 2C_\varepsilon \frac{(z'(t))^\gamma}{z^m(t)} z^m(t) \leq \varepsilon z^{m/(1-\gamma)}(t) + C_\varepsilon \frac{z'(t)}{z^{m/\gamma}(t)}.$$

Hence

$$C''_\varepsilon \left(z^\theta(t)\right)' + \varepsilon z^{m/(1-\gamma)}(t) \geq 2C_\varepsilon\,(z'(t))^\gamma, \tag{2.4.77}$$

where $\theta = 1 - m/\gamma \in (0,1)$, and for a large constant $C''_\varepsilon > 0$.

Now assume further that $m < 1-\gamma$ and define

$$\phi = C''_\varepsilon z^\theta(t) + 3y(t).$$

By combining (2.4.75) and (2.4.76), for $0 < \varepsilon < 1$, we get

$$\begin{aligned}\phi'(t) &\geq 3z(t) + y^{p\gamma}(t) + y(t) - 2\varepsilon - \varepsilon z^{m/(1-\gamma)}(t)\\ &\geq 2z(t) + y^{p\gamma}(t) + y(t) - 3\varepsilon, \quad 0 \leq t < T.\end{aligned}$$

Choosing $\varepsilon < z(0)/3$, setting $\nu = \min(p\gamma, 1/\theta) > 1$, and using the fact that $z(t)$ is non-decreasing, we then obtain on $(0,T)$,

$$\phi'(t) \geq z(t) + y^{p\gamma}(t) + y(t) \geq [z(0)]^{1-\theta\nu} z^{\theta\nu}(t) + y^\nu(t) \geq C\phi^\nu(t)$$

with some constant $C > 0$, which, by virtue of Theorem 2.4.4, implies that $T < +\infty$. □

The following three results are generalizations of Theorem 2.4.10, which are due to Quittner and Souplet [818].

Theorem 2.4.11 (The Quittner–Souplet Inequality [818]). *Let $p, q, \varepsilon > 0, pq > 1$, and $0 < T \leq +\infty$. Assume that $0 \leq y(t), z(t) \in C^1(0,T), (y(t), z(t)) \not\equiv (0,0)$, and that $(y(t), z(t))$ solves on $(0,T)$*

$$y'(t) \geq \varepsilon z^p(t), \quad z'(t) \geq \varepsilon y^q(t). \tag{2.4.78}$$

Then $T < +\infty$ and

$$y(t) \leq C_1(T-t)^{-\alpha/2}, \quad z(t) \leq C_1(T-t)^{-\beta/2}, \;\; 0 < t < T, \tag{2.4.79}$$

with constants $C_1 = C_1(p,q,\varepsilon) > 0, \alpha > 0, \beta > 0$.

Proof. From (2.4.78) we deduce that

$$\varepsilon^{-p-1}y(t) \geq \varepsilon^{-p}\int_0^t z^p(s)ds \geq \int_0^t \left(\int_0^s y^q(\sigma)d\sigma\right)^p ds =: h(t). \tag{2.4.80}$$

Therefore,

$$\begin{aligned}[(h'(t))^{(p+1)/p}]' &= (p+1)\left(\int_0^t y^q(s)ds\right)^p y^q(t)\\ &\geq (p+1)\varepsilon^{q(p+1)}h'(t)h^q(t) = C\left(h^{q+1}(t)\right)'.\end{aligned} \tag{2.4.81}$$

Noting that $h(0) = h'(0) = 0$, (2.4.81) yields

$$(h'(t))^{(p+1)/p} \geq Ch^{q+1}(t). \tag{2.4.82}$$

Moreover, since $(y(t), z(t)) \not\equiv (0,0)$, we can assume that $h > 0$ on (t_0, T) for some $t_0 \in (0,T)$. Putting $\gamma = p\frac{q+1}{p+1} > 1$, we get

$$\left[h^{1-\gamma}(t)\right]' = -(\gamma-1)h'(t)h^{-\gamma}(t) \leq -C < 0. \tag{2.4.83}$$

Integrating (2.4.83) over (t,s) for $t_0 < t < s < T$, we may obtain

$$h^{1-\gamma}(t) \geq h^{1-\gamma}(s) + C(s-t) \geq C(s-t). \tag{2.4.84}$$

From (2.4.84) it follows that $T < +\infty$. By letting $s \to T$, we obtain

$$h(t) \leq C(T-t)^{-1/(\gamma-1)} = C(T-t)^{-\alpha/2}, \;\; t_0 < t < T. \tag{2.4.85}$$

Next, fix $t_0 < t < T$ and let $\tau = (T-t)/4$. Since $y'(t) \geq 0$, we have

$$\begin{aligned}h(t+2\tau) &= \int_0^{t+2\tau}\left[\int_0^s y^q(\sigma)d\sigma\right]^p ds\\ &\geq \tau\left[\int_0^{t+\tau} y^q(\sigma)d\sigma\right]^p \geq \tau\left[\tau y^q(t)\right]^p = \tau^{p+1}y^{pq}(t).\end{aligned} \tag{2.4.86}$$

In view of (2.4.85), we deduce that

$$\begin{aligned}y^{pq}(t) &\leq \tau^{-(p+1)}h(t+2\tau) \leq C\tau^{-(p+1)}(T-t-2\tau)^{-(p+1)/(pq-1)}\\ &= C(T-t)^{-pq(p+1)/(pq-1)},\end{aligned} \tag{2.4.87}$$

hence the estimate of $y(t)$ on (t_0, T) holds. A similar estimate of $z(t)$ follows in the same way. Since the constant C is independent of t_0 and $y = z = 0$ in $(0,t)$ if $h(t) = 0$, the above estimates obtained in (t_0, T) remain true on $(0,T)$. □

Theorem 2.4.12 (The Quittner–Souplet Inequality [818]). *Let $p, q > 0$ satisfy $pq > 1$, and $0 < T < +\infty$. Assume that $0 \le y(t), z(t)$ are locally absolutely continuous and non-decreasing on $(0, T)$, and that $(y(t), z(t))$ solves a.e. on $(0, T)$*

$$y'(t) \le z^p(t), \quad z'(t) \le y^q(t). \tag{2.4.88}$$

Assume also that $\sup_{t\in(0,T)} (y(t) + z(t)) = +\infty$ and (2.4.79) holds for constants $C_1 > 0, \alpha > 0, \beta > 0$. Then

$$y(t) \ge \eta(T-t)^{-\alpha/2}, \quad z(t) \ge \eta(T-t)^{-\beta/2}, \ T-\eta < t < T, \tag{2.4.89}$$

with a constant $\eta = \eta(p, q, C_1) > 0$.

Proof. We first observe that for suitable a, b (depending on p, q), the functions

$$\bar{y}(t) = a(T-t)^{-\alpha/2}, \ \ \bar{z}(t) = b(T-t)^{-\beta/2} \tag{2.4.90}$$

satisfy on $(0, T)$,

$$\bar{y}'(t) = \bar{z}^p(t), \quad \bar{z}'(t) = \bar{y}^q(t). \tag{2.4.91}$$

We deduce that, for each $t \in (0, T)$,

$$\textit{either} \ \ y(t) \ge \bar{y}(t) \ \text{ or } \ z(t) \ge \bar{z}(t). \tag{2.4.92}$$

Indeed, if this failed for some $t \in (0, T)$, then we would have $y(t) < \bar{y}(t-\eta)$ and $z(t) < \bar{z}(t-\eta)$ for some $\eta > 0$ so that, by a simple comparison argument, we would conclude that $y(s) \le \bar{y}(s-\eta)$ and $z(s) \le \bar{z}(s-\eta), \ t \le s < T$, contradicting the fact that (y, z) is unbounded on $(0, T)$.

In order to reach a contradiction, we assume that there exist sequences $\eta_i \to 0^+$ and $t_i \to T$ such that

$$z(t_i) \le \eta_i (T - t_i)^{-\beta/2}. \tag{2.4.93}$$

Fix $k > 1$ and put $t_i' := t_i - k(T - t_i)$. Then (2.4.88), (2.4.93) and $z'(t) \ge 0$ guarantee that, for large i,

$$\begin{aligned} a(T-t_i)^{-\alpha/2} &\le y(t_i) \le y(t_i') + \int_{t_i'}^{t_i} z^p(s)ds \\ &\le C_1(T-t_i')^{-\alpha/2} + k\eta_i^p (T-t_i)^{1-p(\beta/2)}. \end{aligned} \tag{2.4.94}$$

Using $1 - p(\beta/2) = -\alpha/2$ and noting that $T - t_i' = (1+k)(T-t_i)$, we can get

$$a \le C_1(1+k)^{-\alpha/2} + k\eta_i^p. \tag{2.4.95}$$

Letting $i \to +\infty$, we get a contradiction for $k = k(p, q, a)$ large enough. Consequently, there exists a $\eta = \eta(p, q) > 0$ such that $z(t) \ge \eta(T-t)^{-\beta/2}$ on $[T-\eta, T)$. The estimate for $y(t)$ follows in an analogous manner. □

Theorem 2.4.13 (The Quittner–Souplet Inequality [818]). *Let* $p, q > 0$ *satisfy* $pq > 1$ *and* $\lambda > 0$. *Then there exists a constant* $k = k(p, q) > 0$ *such that the system of differential inequalities for all* $t \geq 0$

$$y'(t) \geq z^p(t) - \lambda y(t), \quad z'(t) \geq y^q(t) - \lambda z(t), \tag{2.4.96}$$

has no global non-negative solution $(y(t), z(t)) \in C([0, +\infty)) \cap C^1((0, +\infty))$ *with* $y(0) \geq k\lambda^{\alpha/2}$.

Proof. Set $\tau = \lambda^{-1}$ and assume that $(y(t), z(t))$ exists on $[0, \tau]$. Then there exists a $C_1 = C_1(q) > 0$ such that

$$y(\tau) \geq C_1 y(0), \ z(\tau) \geq C_1 \lambda^{-1} y^q(0). \tag{2.4.97}$$

Indeed, from (2.4.96) we derive that $\big(y(t)e^{\lambda t}\big)' \geq 0$, and hence $y(t) \geq y(0)e^{-\lambda t} \geq y(0)e^{-1}$ on $[0, \tau]$. This implies $\big(z(t)e^{\lambda t}\big)' \geq e^{\lambda t} y^q(t) \geq e^{-q} y^q(0)$ on $[0, \tau]$, hence

$$z(\tau) \geq e^{-(q+1)} \lambda^{-1} y^q(0), \tag{2.4.98}$$

which in turn yields (2.4.97). Next, since $pq > 1$, we may choose $A, B > 1$ depending only on p, q, such that $p(B - 1) > A$ and $q(A - 1) > B$. We claim now that if for some t_0 there exist two constants $a, b > 0$ such that

$$y(t_0) > a, \ z(t_0) > b, \quad b^p > A\lambda a, \quad a^q > B\lambda b, \tag{2.4.99}$$

then $(y(t), z(t))$ cannot exist globally.

To this end, we argue a contradiction and assume that (y, z) exists for all $t > 0$. By a time shift, we may assume $t_0 = 0$. Let $(\tilde{y}, \tilde{z})$ be the unique, positive local solution of

$$\begin{cases} \tilde{y}'(t) = \tilde{z}^p(t) - \lambda \tilde{y}(t), & t \geq 0, \qquad (2.4.100) \\ \tilde{z}'(t) = \tilde{y}^q(t) - \lambda \tilde{z}(t), & t \geq 0, \qquad (2.4.101) \\ \tilde{y}(0) = a, & \tilde{z}(0) = b. \qquad (2.4.102) \end{cases}$$

By an easy comparison argument (using the fact that $z \mapsto z^p$ and $y \mapsto y^q$ are increasing functions), it follows that $(\tilde{y}, \tilde{z})$ exists for all $t > 0$ and we have $y(t) \geq \tilde{y}(t) > 0$ and $z(t) \geq \tilde{z}(t) > 0$. Set

$$\phi(t) = \tilde{z}^p(t) - A\lambda \tilde{y}(t), \quad \psi(t) = \tilde{y}^q(t) - B\lambda \tilde{z}(t).$$

By (2.4.99), $\phi(0) > 0$ and $\psi(0) > 0$. Assume now that $\phi(t), \psi(t) > 0$ on $[0, T]$ for some $T > 0$. Then it holds on $(0, T]$,

$$\tilde{y}'(t) \geq (A - 1)\lambda \tilde{y}(t), \quad \tilde{z}'(t) \geq (B - 1)\lambda \tilde{z}(t). \tag{2.4.103}$$

On the other hand, for all $t \in (0, T]$, we derive

$$\phi'(t) = p\tilde{z}^{p-1}(t)\tilde{z}'(t) - A\lambda \tilde{y}'(t) \geq [p(B - 1) - A]\, \lambda \tilde{z}^p(t) > 0 \tag{2.4.104}$$

and

$$\psi'(t) = q\tilde{y}^{q-1}(t)\tilde{y}'(t) - B\lambda\tilde{z}'(t) \geq [q(A-1) - B]\,\lambda\tilde{y}^q(t) > 0. \tag{2.4.105}$$

We deduce that $\phi, \psi > 0$ on $[0, +\infty)$. Consequently,

$$\tilde{y}'(t) \geq C\tilde{z}^p(t), \quad \tilde{z}'(t) \geq C\tilde{y}^q(t) \tag{2.4.106}$$

with $C = 1 - \max(A^{-1}, B^{-1}) > 0$. However, as a consequence of Theorem 2.4.11, this implies that $(\tilde{y}(t), \tilde{z}(t))$ cannot exist for all $t > 0$. This contradiction proves the claim.

Let us now show that, for suitable $k, \varepsilon, \eta > 0$ (independent of λ), the relation $y(0) \geq k\lambda^{\alpha/2}$ guarantees that $a := \varepsilon\lambda^{\alpha/2}$ and $b := \eta\lambda^{\beta/2}$ satisfy (2.4.99) for $t_0 = \tau$. In view of the latter claim, this will prove the lemma. The last two conditions in (2.4.99) are equivalent to

$$\begin{cases} \eta^p\lambda^{\frac{p(q+1)}{pq-1}} > A\lambda\varepsilon^{\frac{p+1}{pq-1}} = A\varepsilon\lambda^{\frac{p(q+1)}{pq-1}}, \\ \varepsilon^q\lambda^{\frac{q(p+1)}{pq-1}} > B\lambda\eta^{\frac{q+1}{pq-1}} = B\eta\lambda^{\frac{q(p+1)}{pq-1}}, \end{cases}$$

that is, $\eta^p > A\varepsilon$ and $\varepsilon^q > B\eta$; such $\eta, \varepsilon > 0$ certainly exist since $pq > 1$. By virtue of (2.4.97), the first two conditions in (2.4.99) are satisfied if

$$\varepsilon\lambda^{\frac{p+1}{pq-1}} < C_1k\lambda^{\frac{p+1}{pq-1}}, \quad \eta\lambda^{\frac{q+1}{pq-1}} < C_1\lambda^{-1}k^q\lambda^{\frac{q(p+1)}{pq-1}} = C_1k^q\lambda^{\frac{q+1}{pq-1}}.$$

It thus suffices to choose $k > \max\left[C_1^{-1}\varepsilon, C_1^{-1/q}\eta^{1/q}\right]$. □

Now, we need the next comparison theorem which is due to Čaplygin [130] (see also, Beckenbach and Bellman [83]).

Theorem 2.4.14 (The Čaplygin Inequality [130]). *If $u(t), v(t) \in C^2([0, +\infty))$ satisfy the inequalities*

$$\begin{cases} u''(t) + p(t)u'(t) - q(t)u(t) > 0, \quad t \geq 0, & (2.4.107) \\ v''(t) + p(t)v'(t) - q(t)v(t) = 0, \quad t \geq 0, & (2.4.108) \\ q(t) \geq 0, \quad t \geq 0, & (2.4.109) \\ u(0) = v(0), \ u'(0) = v'(0), & (2.4.110) \end{cases}$$

then for all $t > 0$,

$$u(t) > v(t). \tag{2.4.111}$$

Proof. In fact, subtracting (2.4.108) from (2.4.107), we obtain

$$w''(t) + p(t)w'(t) - q(t)w(t) > 0 \tag{2.4.112}$$

where $w = u - v$ with $w(0) = w'(0) = 0$. It thus follows that $w(t) > 0$ in some initial interval $(0, t_0]$. Assume that $w(t)$ eventually becomes negative, so that w must have a local maximum at some point t_1. At this point, we would have $w' = 0, w > 0$, and therefore, by (2.4.112), $w'' > 0$. This, however, contradicts the assumption that t_1 is a local maximum. □

The next result, due to Agmon and Nirenberg [14], is a generalization of Theorem 2.4.14.

Theorem 2.4.15 (The Agmon–Nirenberg Inequality [14]). *Let $f(t)$ and $g(t)$ be twice differentiable functions in an interval $[t_0, t_1)$ such that*

$$\begin{cases} f''(t) + a(t)|f'(t)| + b(t) \geq 0, & (2.4.113) \\ g''(t) + a(t)|g'(t)| + b(t) = 0, & (2.4.114) \end{cases}$$

where $a(t), b(t)$ are continuous functions in $[t_0, t_1)$, $a \geq 0$. If $f(t_0) \geq g(t_0)$ and $f'(t_0) \geq g'(t_0)$, then for all $t_0 \leq t < t_1$,

$$f(t) \geq g(t).$$

Proof. Obviously, the function $h(t) = f(t) - g(t)$ satisfies

$$\begin{aligned} h''(t) + a(t)|h'(t)| &= f''(t) - g''(t) + a|f'(t) - g'(t)| \\ &\geq f''(t) - g''(t) + a|f'(t)| - a|g'(t)| \\ &= f''(t) + a|f'(t)| + b - (g''(t) + a|g'(t)| + b) \geq 0. \end{aligned}$$

Moreover, $h(t_0) \geq 0$ and $h'(t_0) \geq 0$. We have to show that $h(t) \geq 0$. Clearly, it suffices to show that $h'(t) \geq 0$. Now if this does not hold for all t, then (since $h'(t_0) \geq 0$) there exists a subinterval (α, β) such that $h'(t) < 0$ for all $\alpha < t < \beta$, while $h'(\alpha) = 0$. But then by the above observations, in this subinterval

$$\frac{d}{dt}\left(\exp\left\{-\int_\alpha^t a(s)ds\right\} h'(t)\right) \geq 0$$

and since $h'(\alpha) = 0$, we reached a contradiction for all $\alpha < t < \beta$:

$$\exp\left\{-\int_{t_0}^t a(s)ds\right\} h'(t) \geq 0.$$

This completes the proof. □

Theorem 2.4.16 (The Agmon–Nirenberg Inequality [14]). *Let $f(t)$ be a (scalar) twice differentiable function in the interval $[t_0, t_1)$. Assume that f satisfies a differential inequality of the form, for all $t_0 \leq t < t_1$,*

$$f''(t) + a(t)|f'(t)| + b(t) \geq 0, \tag{2.4.115}$$

where $a(t)$ and $b(t)$ are non-negative measurable functions which are bounded in any compact subinterval contained in $[t_0, t_1)$. Then the following estimate holds for all $t_0 \leq t < t_1$:

$$\begin{aligned} f(t) \geq f(t_0) &+ \min\{0, f'(t_0)\} \int_{t_0}^t \exp\left(\int_{t_0}^s a(r)dr\right) ds \\ &- \int_{t_0}^t \exp\left(\int_{t_0}^s a(r)dr\right) \left[\int_{t_0}^s b(\sigma) \exp\left(-\int_{t_0}^\sigma a(\rho)d\rho\right) d\sigma\right] ds. \end{aligned} \tag{2.4.116}$$

Proof. Assume now first that $a(t)$ and $b(t)$ are continuous. Let $g(t)$ be the solution of the initial value problem

$$g''(t) - a(t)g'(t) + b(t) = 0, \quad g(t_0) = f(t_0), \; g'(t_0) = \min\{0, f'(t_0)\}. \tag{2.4.117}$$

Observe that, since $a, b \geq 0$ and since $g'(t_0) \leq 0$, we have $g'(t) \leq 0$ for all t. Thus g is also a solution of the equation

$$g''(t) + a(t)|g'(t)| + b(t) \geq 0 \tag{2.4.118}$$

and $g(t_0) = f(t_0), g'(t_0) \leq f'(t_0)$. Now apply Theorem 2.4.15 to the function f in Theorem 2.4.16 and to the solution of g of (2.4.117). This yields for all $t_0 \leq t < t_1$,

$$f(t) \geq g(t).$$

This is simply the inequality (2.4.116) with explicitly given g. In the general case when a, b are not assumed to be continuous, but only measurable and locally bounded, we may consider an arbitrary subinterval $[t_0, t']$ with $t_0, t' < t_1$. Since a and b are measurable and bounded in the subinterval, there exist two sequences $\{a_n(t)\}$ and $\{b_n(t)\}$ of continuous functions in $[t_0, t']$ such that $a_n(t) \geq a(t), b_n(t) \geq b(t)$, and

$$a_n \to a \text{ in } L^1[t_0, t'], \quad b_n \to b \text{ in } L^1[t_0, t_1']. \tag{2.4.119}$$

Clearly, for all $t_0 \leq t \leq t'$ and $n = 1, 2, \ldots$, we have

$$f''(t) + a_n(t)|f'(t)| + b_n(t) \geq 0.$$

Thus by the preceding consideration (since a_n and b_n are continuous), the inequality (2.4.116) holds with a and b replaced by a_n and b_n, respectively. Letting $n \to +\infty$ in the latter inequality, from (2.4.119) it follows that (2.4.116) holds for all $t \in [t_0, t']$ and consequently also for all $t \in [t_0, t_1)$, since t' is an arbitrary number such that $t_0 < t' < t_1$. The proof is complete. □

The following result is related to Theorems 2.4.15–2.4.16.

Theorem 2.4.17 (The Agmon–Nirenberg Inequality [14]). *Let $l(t)$ be a twice differentiable function in $[t_1, t_2]$ satisfying the differential inequality* (2.4.115), *where $a(t)$ and $b(t)$ are non-negative bounded measurable functions in the interval. Then the following inequality holds for all $t_0 \leq t \leq t_1$:*

$$\begin{aligned} l(t) \leq l(t_0) &\frac{\int_t^{t_1} \exp\{\mp \int_{t_0}^s a(r)dr\}ds}{\int_{t_0}^{t_1} \exp\{\mp \int_{t_0}^s a(r)dr\}ds} + l(t_1)\frac{\int_{t_0}^{t} \exp\{\mp \int_{t_0}^s a(r)dr\}ds}{\int_{t_0}^{t_1} \exp\{\mp \int_{t_0}^s a(r)dr\}ds} \\ &+ \int_{t_0}^{t_1} (s - t_0)b(s)ds \; \exp\left\{2\int_{t_0}^{t_1} a(\tau)d\tau\right\} \end{aligned} \tag{2.4.120}$$

where we take the negative sign if $l(t_0) \leq l(t_1)$ and the positive sign if $l(t_0) \geq l(t_1)$.

The next result, due to Agmon–Nirenberg [14], is an equivalent form to Theorem 2.4.17.

Theorem 2.4.18 (The Agmon–Nirenberg Inequality [14]). *Let $f(t)$ be a twice differentiable function in a closed interval $[t_0, t_1]$ satisfying the differential inequality*

$$f''(t) + a(t)f'(t) + b(t) \geq 0 \tag{2.4.121}$$

where $a(t), b(t)$ are bounded measurable functions in the interval. Then the following inequality holds for all $t_0 < t < t_1$,

$$\begin{aligned} f(t) \leq f(t_0)\frac{\int_t^{t_1} \exp\{\mp \int_{t_0}^s |a(r)|dr\}ds}{\int_{t_0}^{t_1} \exp\{\mp \int_{t_0}^s |a(r)|dr\}ds} + f(t_1)\frac{\int_{t_0}^{t} \exp\{\mp \int_{t_0}^s |a(r)|dr\}ds}{\int_{t_0}^{t_1} \exp\{\mp \int_{t_0}^s |a(r)|dr\}ds} \\ + \int_{t_0}^{t_1} (s-t_0)|b(s)|ds \ \exp\left\{2\int_{t_0}^{t_1} |a(s)|ds\right\} \end{aligned} \tag{2.4.122}$$

where we have to take the inequality with negative sign if $f(t_0) \leq f(t_1)$, and with the positive sign if $f(t_0) \geq f(t_1)$.

Proof. We first observe that it suffices to establish the theorem when both $a(t)$ and $b(t)$ are continuous in $[t_0, t_1]$. Indeed if the theorem holds in this case, we easily deduce its validity for measurable and bounded coefficients a, b as follows. Let $\{a_n(t)\}$ be a sequence of continuous functions in $[t_0, t_1]$ such that $a_n \to a$ in $L^1[t_0, t_1]$. Set $c_n(t) = (a(t) - a_n(t))f'(t) + b(t)$. Since $c_n(t)$ is measurable and bounded, there exists a sequence $\{b_n(t)\}$ of continuous functions in $[t_0, t_1]$ such that $b_n(t) \geq c_n(t)$ and $b_n(t) - c_n(t) \to 0$ in $L^1[t_0, t_1]$. Clearly,

$$\begin{aligned} f''(t) + a_n(t)f'(t) + b_n(t) &\geq f''(t) + a_n(t)f'(t) + c_n(t) \\ &= f''(t) + a(t)f'(t) + b(t) \geq 0. \end{aligned} \tag{2.4.123}$$

Since a_n and b_n are continuous, it follows from (2.4.123), and from the assertion of the theorem in this case, that the inequality (2.4.122) holds with a and b replaced by a_n and b_n, respectively. Letting $n \to +\infty$ in (2.4.123), and using the fact that $a_n \to a$ and $b_n \to b$ in $L^1[t_0, t_1]$, we can obtain (2.4.122) in the general case.

Assuming, from now on, that a and b are continuous, we denote by $g(t)$ the unique solution in $C^2[t_0, t_1]$ of the boundary value problem

$$g''(t) + a(t)g'(t) + b(t) = 0, \quad g(t_0) = f(t_0), \quad g(t_1) = f(t_1). \tag{2.4.124}$$

Obviously, we have for all $t_0 \leq t \leq t_1$,

$$f(t) \leq g(t). \tag{2.4.125}$$

Indeed, $h(t) = f(t) - g(t)$ satisfies the relations

$$h''(t) + a(t)h'(t) \geq 0, \quad h(t_0) = h(t_1) = 0 \tag{2.4.126}$$

and (2.4.125) follows from the maximum principle which holds for functions h satisfying (2.4.126), i.e., the maximum of h in the interval is attained at one of the end points.

Next, let

$$g(t) = g_0(t) + g_1(t), \tag{2.4.127}$$

where g_0 is a solution of the homogeneous differential equation $g_0'' + ag_0' = 0$ with $g_0(t_0) = f(t_0), g_0(t_1) = f(t_1)$, while $g_1(t)$ is a solution of the non-homogeneous differential equation (2.4.124) with zero boundary conditions. For g_1, we have the integral representation

$$g_1(t) = \int_{t_0}^{t_1} G(t,s)b(s)ds \tag{2.4.128}$$

with Green's kernel $G(t,s)$ given by for $t \leq s$,

$$G(t,s) = \frac{\left[\int_{t_0}^{t} \exp\left(-\int_{t_0}^{r} a(\rho)d\rho\right) dr\right]\left[\int_{s}^{t_1} \exp\left(-\int_{s}^{r} a(\rho)d\rho\right) dr\right]}{\int_{t_0}^{t_1} \exp\left(-\int_{t_0}^{r} a(\rho)d\rho\right) dr}$$

and for $t \geq s$,

$$G(t,s) = \frac{\left[\int_{t_0}^{s} \exp\left\{-\int_{t_0}^{r} a(\rho)d\rho\right\} dr\right]\left[\int_{t}^{t_1} \exp\left(-\int_{s}^{r} a(\rho)d\rho\right) dr\right]}{\int_{t_0}^{t_1} \exp\left(-\int_{t_0}^{r} a(\rho)d\rho\right) dr}.$$

From the above formulas, we can derive

$$\begin{cases} |G(t,s)| \leq \exp\left(2\displaystyle\int_{t_0}^{t_1} |a(r)|dr\right) \dfrac{(t-t_0)(t_1-s)}{t_1-t_0}, & \text{for } t \leq s, \\ |G(t,s)| \leq \exp\left(2\displaystyle\int_{t_0}^{t_1} |a(r)|dr\right) \dfrac{(s-t_0)(t_1-t)}{t_1-t_0}, & \text{for } t \geq s. \end{cases}$$

Using (2.4.128), we can derive g_1

$$\begin{aligned} |g_1(t)| &\leq \exp\left(2\int_{t_0}^{t_1} |a(r)|dr\right)\left[\int_{t_0}^{t} |b(s)|(s-t_0)ds + \int_{t}^{t_1} |b(s)|(t-t_0)ds\right] \\ &\leq \exp\left(2\int_{t_0}^{t_1} |a(r)|dr\right)\int_{t_0}^{t_1} |b(s)|(s-t_0)ds. \end{aligned}$$

Since

$$f(t) \leq g(t) \leq g_0(t) + |g_1(t)|,$$

we can see that the proof of (2.4.122) will be complete if we show that

$$g_0(t) \leq f(t_0)\frac{\int_{t}^{t_1} \exp\left(\mp\int_{t_0}^{s} |a(r)|dr\right) ds}{\int_{t_0}^{t_1} \exp\left(\mp\int_{t_0}^{s} |a(r)|dr\right) ds} + f(t_1)\frac{\int_{t_0}^{t} \exp\left(\mp\int_{t_0}^{s} |a(r)|dr\right) ds}{\int_{t_0}^{t_1} \exp\left(\mp\int_{t_0}^{s} |a(r)|dr\right) ds} \tag{2.4.129}$$

where in the inequality we take the negative sign if $f(t_0) \le f(t_1)$ and the positive sign if $f(t_0) \ge f(t_1)$.

We shall only establish (2.4.129) for the case $f(t_0) \le f(t_1)$ since the proof in the other case is similar. Denote the right-hand side of (2.4.129) (with the negative sign) by $h(t)$. We have readily

$$h''(t) + |a(t)|h'(t) = 0, \quad h(t_0) = 0, \quad h(t_1) = f(t_1). \tag{2.4.130}$$

Note also that h' is of constant sign. Since $h(t_0) \le h(t_1)$, we have $h'(t) \ge 0$. Set $k(t) = g_0(t) - h(t)$. Since g_0 satisfies the equation $g_0'' + ag_0' = 0$, and h satisfies equation (2.4.130), it follows that

$$k''(t) + a(t)k'(t) = (|a(t)| - a(t))h'(t) \ge 0,$$

and consequently the maximum principle (since $k(t_0) = k(t_1) = 0$) gives us $k(t) \le 0$ or $g_0(t) \le h(t)$. This yields (2.4.129), which thus completes the proof of the theorem. □

The following theorem (see, e.g., Ladyzhenskaya, Solonnikov and Ural'ceva [472], Levine [508]) is very useful in establishing the non-existence of global solutions to some differential equations.

Theorem 2.4.19 (The Ladyzhenskaya – Solonnikov – Ural'ceva – Levine Inequality [472, 508])**.** *Assume that the twice differentiable positive function $\Phi(t)$ satisfies for all $t > 0$ the inequality*

$$\Phi(t)\Phi''(t) - (1+\gamma)(\Phi'(t))^2 \ge -2C_1\Phi(t)\Phi'(t) - C_2\Phi^2(t), \tag{2.4.131}$$

where $\gamma > 0$ and $C_1, C_2 \ge 0$. Then

(i) *if $\Phi(0) > 0, \Phi'(0) + \gamma_2\gamma^{-1}\Phi(0) > 0$, and $C_1 + C_2 > 0$, we have*

$$\Phi(t) \to +\infty \tag{2.4.132}$$

as

$$t \to t_1 \le t_2 = \frac{1}{2\sqrt{C_1^2 + \gamma C_2}} \ln\left(\frac{\gamma_1\Phi(0) + \gamma\Phi'(0)}{\gamma_2\Phi(0) + \gamma\Phi'(0)}\right), \tag{2.4.133}$$

where $\gamma_1 = -C_1 + \sqrt{C_1^2 + \gamma C_2}$, $\gamma_2 = -C_1 - \sqrt{C_1^2 + \gamma C_2}$;

(ii) *if $\Phi(0) > 0, \Phi'(0) > 0$, and $C_1 = C_2 = 0$, then*

$$\Phi(t) \to +\infty \tag{2.4.134}$$

as

$$t \to t_1 \le t_2 = \frac{\Phi(0)}{\gamma\Phi'(0)}. \tag{2.4.135}$$

Proof. (i) Let $y(t) = \Phi^{-\gamma}(t)$. From (2.4.131) it follows readily that $y(t)$ solves the inequality

$$y''(t) + 2C_1 y'(t) - \gamma C_2 y(t) \le 0. \tag{2.4.136}$$

Let $z(t)$ solve the second-order differential equation

$$\begin{cases} z''(t) + 2C_1 z'(t) - \gamma C_2 z(t) = 0, & (2.4.137) \\ z(0) = y(0), \quad z'(0) = y'(0), & (2.4.138) \end{cases}$$

that is,

$$z(t) = ae^{\gamma_1 t} + be^{\gamma_2 t}, \tag{2.4.139}$$

with

$$a = -\frac{\gamma\Phi'(0) + \gamma_2\Phi(0)}{2\sqrt{C_1^2 + \gamma C_2}\Phi^{\gamma+1}(0)} < 0, \quad b = \frac{\gamma\Phi'(0) + \gamma_1\Phi(0)}{2\sqrt{C_1^2 + \gamma C_2}\Phi^{\gamma+1}(0)}.$$

It is easy to verify that

$$z(t_2) = 0. \tag{2.4.140}$$

On the other hand, by a simple comparison principle,

$$0 \le y(t) \le z(t)$$

which, along with (2.4.140), yields $\lim_{t\uparrow t_2} y(t) = 0$. Hence (2.4.133) follows immediately.

(ii) When $C_1 = C_2 = 0$, we deduce from (2.4.131) that

$$\frac{d^2(\Phi^{-\gamma}(t))}{dt^2} = -\gamma\Phi^{-(\alpha+2)}(t)\left[\Phi(t)\Phi''(t) - (1+\gamma)(\Phi'(t))^2\right] \le 0,$$

whence

$$\Phi(t) \ge \Phi(0) \Big/ \left[1 - \gamma t\Phi'(0)\Phi^{-1}(0)\right]^{1/\gamma}. \tag{2.4.141}$$

Since $1 - \gamma t_2\Phi'(0)\Phi^{-1}(0) = 0$, then there exists some time $t_1 \le t_2$ such that (2.4.134) holds. □

Theorem 2.4.21 below shows that a conclusion similar to (i) of Theorem 2.4.19 holds when $C_1 = 0, C_2 < 0$. To this end, we show the following result which is due to Knops, Levine and Payne [440].

Theorem 2.4.20 (The Knops–Levine–Payne Inequality [440])**.** *If* $y(t) : [0, +\infty) \to [0, +\infty)$ *is a* C^1 *function satisfying*

$$y'(t) + y^2(t) + a^2 \le 0, \quad y(0) > 0 \tag{2.4.142}$$

with a constant $a > 0$*, then there exists some time*

$$t_1 \le t_2 = a^{-1}\left[\pi/2 + \tan^{-1}[y(0)/a]\right]$$

such that as $t \to t_1^-$*,*

$$y(t) \to +\infty. \tag{2.4.143}$$

Proof. In fact, solving (2.4.142) we get

$$\tan^{-1}[y(t)/a] \leq \tan^{-1}[y(0)/a] - at. \tag{2.4.144}$$

It is clear that this inequality cannot hold for all time since the right-hand side of (2.4.144) tends to $-\infty$, while the left-hand side remains bounded as t tends to $+\infty$. Obviously, the solution necessarily blows up at some time t_1 satisfying

$$t_1 \leq t_2 = a^{-1}\{\pi/2 + \tan^{-1}[y(0)/a]\}.$$

This completes the proof. □

Using Theorem 2.4.20, we can easily prove the following result, which corresponds to the case of $C_2 < 0$ in Theorem 2.4.19, and is due to Knops, Levine and Payne [440].

Theorem 2.4.21 (The Knops–Levine–Payne Inequality [440]). *Assume that the twice differentiable positive function $\Phi(t)$ satisfies for all $t > 0$,*

$$\Phi(t)\Phi''(t) - (1+\gamma)(\Phi'(t))^2 \geq a^2\Phi^2(t), \tag{2.4.145}$$

where $\gamma > 0, a > 0$ are constants, and $\Phi(0)\Phi'(0) < 0$. Then there exists some time $t_1 \leq t_2$ such that

$$\Phi(t) \to +\infty \tag{2.4.146}$$

as

$$t \to t_1 \leq t_2 = (\sqrt{\gamma}a)^{-1}\left[\frac{pi}{2} + \tan^{-1}\left(\frac{-\sqrt{\gamma}\Phi'(0)}{\Phi(0)a}\right)\right]. \tag{2.4.147}$$

Proof. Let $y(t) = \frac{d(\Phi^{-\gamma}(t))}{dt}/\Phi^{-\gamma}(t) = -\gamma\Phi'(t)/\Phi(t)$. Then (2.4.145) becomes

$$y'(t) + y^2(t) + a_1^2 \leq 0 \tag{2.4.148}$$

with $a_1 = \sqrt{\gamma}a > 0$. Applying Theorem 2.4.20 to (2.4.148) completes the proof immediately. □

Glassey [317, 318] used the following theorem to establish the blowup of solutions to nonlinear wave equations.

Theorem 2.4.22 (The Glassey Inequality [317, 318]). *Let $\phi(t) \in C^2([0,+\infty))$ satisfy for all $t \geq 0$,*

$$\phi''(t) \geq h(\phi(t)), \tag{2.4.149}$$

with

$$\phi(0) = \alpha > 0, \quad \phi'(0) = \beta > 0. \tag{2.4.150}$$

Assume that $h(s) \geq 0$ for all $s \geq \alpha$. Then

(i) $\phi'(t) > 0$ *wherever* $\phi(t)$ *exists;*

(ii) *we have*

$$t \le \int_{\alpha}^{\phi(t)} \left[\beta^2 + 2\int_{\alpha}^{s} h(\xi)d\xi\right]^{-1/2} ds. \tag{2.4.151}$$

Proof. Assuming that (i) is false, let $t = t_1$ be the first point where $\phi'(t_1) = 0$. Then integrating the differential inequality (2.4.149) over $[0, t_1]$ we obtain

$$\begin{aligned} 0 = \phi'(t_1) &\ge \phi'(0) + \int_0^{t_1} h(\phi(s))ds \\ &\ge \beta + \int_0^{t_1} h(\phi(s))ds. \end{aligned}$$

By the definition of t_1, we know that $\phi'(s) > 0$ for all $0 \le s \le t_1$, hence $\phi(s) \ge \phi(0) = \alpha$ for all $0 \le s \le t_1$. Thus the integral term above is non-negative, and the resulting contradiction proves (i). To prove (2.4.151) in (ii), we use the assertion in (i) and multiply the differential inequality (2.4.149) by $\phi'(t)$ to get

$$\phi'(t)\phi''(t) \ge \phi'(t)h(\phi(t)),$$

or

$$\frac{d}{dt}\left(\frac{1}{2}(\phi')^2 - \int_{\alpha}^{\phi} h(\xi)d\xi\right) \ge 0. \tag{2.4.152}$$

Thus integrating (2.4.152) over $[0, t]$ gives us

$$(\phi'(t))^2 \ge \beta^2 + 2\int_{\alpha}^{\phi(t)} h(\xi)d\xi$$

and, since $\phi'(t) > 0$, we may separate variables and integrate to obtain (2.4.151) in (ii) immediately. □

The following two results are used to prove the non-existence of global solutions. The simple proofs are left to the reader (see Qin [770]).

Theorem 2.4.23. *Let* $c(t)$ *and* $y(t)$ *be two non-negative functions on* $[0, +\infty)$ *and* $\alpha > 0$. *Assume that* $c(t) \in L^1(0, T)$ *for any* $T > 0$ *and that* $y(t)$ *is absolutely continuous and satisfies for all* $t > 0$,

$$y'(t) + c(t)y^{1+\alpha}(t) \le 0. \tag{2.4.153}$$

Then

$$y(t) \le \left\{C\int_0^t c(s)ds\right\}^{-1/\alpha}. \tag{2.4.154}$$

Theorem 2.4.24. *Let $\alpha, C > 0$ be two constants. Let $y(t)$ be a non-negative absolutely continuous function on $[0, +\infty)$ satisfying for any $t > 0$,*

$$y'(t) + Cy^{1-\alpha}(t) \leq 0. \tag{2.4.155}$$

Then

$$y(t) \leq \Big(y^{\alpha}(0) - \alpha Ct\Big)^{1/\alpha}. \tag{2.4.156}$$

The following result is due to Knops and Payne [441].

Theorem 2.4.25 (The Knops–Payne Inequality [441])**.** *Let $F : [0, T) \to [0, +\infty)$, for $0 < T \leq +\infty$, be a C^2 function satisfying*

$$F(t)F''(t) - (F'(t))^2 \geq -C_1 F(t) \tag{2.4.157}$$

with a constant $C_1 \in \mathbb{R}$.

(i) *If $C_1 \leq 0, F(0) = 0$, then for all $t \in [0, T)$,*

$$F(t) \equiv 0. \tag{2.4.158}$$

(ii) *If $C_1 \leq 0, F(0) > 0$, then for all $t \in [0, T)$,*

$$F(t) \geq F(0) \exp\left[F'(0)t/F(0)\right], \tag{2.4.159}$$

or, equivalently, if there exists a $t_0 \in [0, T)$ such that $F(t_0) > 0$, then

$$F(t) \geq F(t_0) \exp\left[F'(t_0)(t - t_0)/F(t_0)\right],$$

and for any $0 < T' \leq T$, we have for all $0 \leq t \leq T'$,

$$F(t) \leq [F(0)]^{1-t/T'} [F(T')]^{t/T'}, \tag{2.4.160}$$

or, equivalently, for all $0 \leq t \leq T'$,

$$F(t) \leq F(0) \exp\left[\frac{t}{T'} \ln \frac{F(T')}{F(0)}\right]. \tag{2.4.161}$$

If, in addition, $F'(0) > 0$, then

$$T < +\infty. \tag{2.4.162}$$

If further $T = +\infty, \lim_{t\to+\infty} \ln F(t)/t = 0$, then for all $t \in [0, +\infty)$,

$$F(t) \leq F(0). \tag{2.4.163}$$

(iii) *If $C_1 > 0, F(0) > 0, F'(0) > 0$, then*

(1) *When* $F'(0) > \sqrt{2C_1F(0)}$, *we have*

$$\begin{cases} F(t) \geq F(0)e^{\sqrt{A}\,t}, & (2.4.164) \\ T < +\infty, & (2.4.165) \end{cases}$$

for $A^2 = [F'(0)/F(0)]^2 - 2C_1/F(0) > 0$*;*

(2) *When* $F'(0) = \sqrt{2C_1F(0)}$ *(i.e.,* $A^2 = 0$*), we have*

$$\begin{cases} F(t) \geq F(0) + 2(F(0)C_1/2)^{1/2}t + C_1t^2/2, & (2.4.166) \\ T < +\infty. & (2.4.167) \end{cases}$$

(3) *When* $F'(0) < \sqrt{2C_1F(0)}$, *we have*

$$F(t) \geq \left[F^{1/2}(0)e^{-\sqrt{-A}t/2} + \sqrt{2C_1/(-A)}(1 - e^{-\sqrt{-A}t/2})\right]^2, \quad (2.4.168)$$

where $A < 0$.

Proof. (i) If $C_1 \leq 0, F(0) = 0$, then by continuity it follows that either $F(t) \equiv 0$ for any $t \in [0, T)$ or there exists an open interval $(t_1, t_2) \subseteq [0, T)$ such that for all $0 \leq t_1 < t < t_2 \leq T$,

$$F(t) > 0. \quad (2.4.169)$$

The first case is what we wish to prove, so without loss of generality, we may assume that (2.4.169) holds. Then for all $0 \leq t_1 < t < t_2 \leq T$,

$$\frac{d^2}{dt^2}(\ln F(t)) \geq 0. \quad (2.4.170)$$

Jensen's inequality or expansion for $\ln F(t)$ in a finite Taylor series may now be used according to (2.4.170) to yield, for all $0 \leq t_1 < t < t_2 \leq T$,

$$F(t) \leq [F(t_1)]^{(t_2-t)/(t_2-t_1)}[F(t_2)]^{(t-t_1)/(t_2-t_1)}. \quad (2.4.171)$$

Let either $t_1 = 0$, or, by continuity, $F(t_1) = 0$. We consider the second alternative. Then (2.4.171) gives us at once $F(t) \equiv 0$ for all $0 \leq t < t_2 \leq T$, and so by continuity we have $F(t_2) = 0$, and hence $F(t) = 0$ for all $0 \leq t \leq T$. When $t_1 = 0$, a repetition of the above argument leads to the same conclusion.

(ii) If $C_1 \leq 0, F(0) > 0$, then it follows from (2.4.157) that for all $0 \leq t < T$,

$$\frac{d^2}{dt^2}(\ln F(t)) \geq 0,$$

which implies that $\ln F(t)$ is a convex function. Then expanding $\ln F(t)$ in a finite Taylor series around $t = 0$ gives us (2.4.159), and (2.4.160)–(2.4.161) are easily derived from Jensen's inequality for $\ln F(t)$. When further $F'(0) > 0$, and if $T =$

$+\infty$, then the right-hand side of (2.4.159) will tend to $+\infty$, and hence $F(t) \to +\infty$ as $t \to +\infty$; this is a contradiction, since $F(t)$ is a C^2 function on $[0, +\infty)$. This proves (2.4.162). If, further, $T = +\infty, \lim_{t\to+\infty} \ln F(t)/t = 0$, then letting $T' \to +\infty$ in (2.4.160), we obtain (2.4.163).

(iii) If $C_1 > 0$, $F(0) > 0$, $F'(0) > 0$, then by continuity we conclude that there exists a time $t_1 \le T$ such that

$$F'(t) > 0, \quad 0 \le t < t_1; \quad F(t_1) = 0. \tag{2.4.172}$$

Hence $F(t) > F(0) > 0$ for all $0 \le t < t_1$, and further, we can deduce from (2.4.157) that for all $0 \le t < t_1$,

$$\frac{d}{dt}\left[\frac{F'(t)}{F(t)}\right] \ge -\frac{C_1}{F(t)}. \tag{2.4.173}$$

Multiplying (2.4.173) by $F'(t)/F(t)$ and integrating the resulting inequality, we may obtain

$$[F'(t)/F(t)]^2 \ge A + 2C_1/F(t), \quad 0 \le t < t_1, \tag{2.4.174}$$

where $A = [F'(0)/F(0)]^2 - 2C_1/F(0)$. If $A \ge 0$, then it follows from (2.4.174) that $F'(t) > 0$ for all $t \in [0, t_1]$, thus contradicting (2.4.172). Hence $F'(t) \ge 0$ for all $t \in [0, T)$.

(1) If, further, $F'(0) > \sqrt{2C_1F(0)}$, i.e., $\sqrt{A} > 0$, then we obtain from (2.4.174) that for all $t \in [0, T)$,

$$F'(t) \ge \sqrt{A}F(t),$$

which hence gives us (2.4.164) and (2.4.165).

(2) If $A = 0$, then we obtain from (2.4.174),

$$F^{-1/2}(t)F'(t) \ge \sqrt{2C_1}$$

which readily implies (2.4.166) and (2.4.167).

(3) If $A < 0$, then from (2.4.174) we derive

$$[F'(t)/F(t) + \sqrt{-A}]^2 \ge (F'(t)/F(t))^2 - A \ge 2C_1/F(t)$$

whence

$$F'(t)/F(t) + \sqrt{-A} \ge \sqrt{2C_1/F(t)}$$

i.e.,

$$F'(t) \ge -\sqrt{-A}F(t) + \sqrt{2C_1}F^{1/2}(t), \tag{2.4.175}$$

with $F(0) > 0, F'(0) > 0$. Denoting $G(t) = F^{1/2}(t)$, we obtain from (2.4.175) that

$$2G'(t)+\sqrt{-A}G(t) \ge \sqrt{2C_1}, \quad G(0) = F^{1/2}(0), \quad G'(0) = F'(0)/(2F(0)). \tag{2.4.176}$$

Thus solving the inequality (2.4.176) yields (2.4.168). □

We next present a series of results on the non-existence of global solutions to nonlinear differential inequalities. These results were obtained by Alaa and Guedda [16].

Consider the differential inequality

$$w''(t) + g(w'(t)) \geq h(w(t)) \tag{2.4.177}$$

subjected to initial conditions $w(0)$ and $w'(0)$. Assume that the functions g and h are continuous and satisfy

$$\begin{cases} h(r) > 0, \text{ for all } r > r_0, & (2.4.178) \\ \liminf\limits_{r \to +\infty} \dfrac{h(r)}{r^p} > 0, \quad p > 1, & (2.4.179) \\ g(r) \leq Kr, \text{ for all } r \in \mathbb{R}, & (2.4.180) \end{cases}$$

where $r_0 > 0, p > 1$ and K is a real constant. Then the following theorem holds.

Theorem 2.4.26 ([16]). *There is no global solution $w \in C^2$ to* (2.4.177) *such that*

$$w(0) > r_0, \quad w'(0) \geq 0. \tag{2.4.181}$$

For the proof of the above theorem, see [16]). □

Remark 2.4.2 ([16]). If there exists a time $t_0 > 0$ such that $w(t_0) > r_0$ and $w'(t_0) \geq 0$, Theorem 2.4.26 remains valid.

Corollary 2.4.3 ([16]). *Let $p > 1$, $p > q > 0$, $\delta, \beta \in \mathbb{R}$ and let α be a positive real number. Then there is no global solution $w(t)$ to*

$$w''(t) + \delta w'(t) \geq \alpha w^p(t) - \beta w^q(t) \tag{2.4.182}$$

such that

$$w(0) > (\beta^+/\alpha)^{1/(p-q)}, \;\; w'(0) \geq 0 \tag{2.4.183}$$

where $\beta^+ = \max\{\beta, 0\}$.

Remark 2.4.3 ([16]). We find that there exists a global solution to (2.4.182) such that $\beta > 0$ and

$$0 < w(0) < (\beta/\alpha)^{1/(p-q)}. \tag{2.4.184}$$

Indeed, the function

$$w(t) = \frac{r}{(1 + e^{st})^{2/(p-1)}}$$

satisfies on $\mathbb{R}^+$

$$w''(t) + \frac{2}{p-1} sw'(t) = \alpha w^p(t) - \beta w^{(p+1)/2}(t), \tag{2.4.185}$$

where

$$r = \left(\frac{\beta}{\alpha}\right)^{1/(p-q)}, \quad s^2 = \frac{\alpha(p-1)^2}{2(p+1)}(\beta/\alpha)^{(p-1)/(p-q)}, \quad q = (p+1)/2. \tag{2.4.186}$$

The following result shows that solutions to (2.4.177) may blow up at a finite time when $w'(0) < 0$.

If we set

$$H(r) = \int_0^r h(s)ds, \tag{2.4.187}$$

then we have the following result.

Theorem 2.4.27 ([16]). *Assume, in addition, that $rg(r) \geq 0$ for any $r \leq 0$. Then inequality* (2.4.177) *has no global solution w such that*

$$\frac{w_1^2}{2} - H(w_0) \leq -H(r_0), \tag{2.4.188}$$

where $w_0 = w(0) > r_0, \quad w_1 = w'(0) < 0$.

For the proof of Theorem 2.4.27, we refer the reader to [16]. □

Corollary 2.4.4 ([16]). *Let $p > 1, p > q > 0, \delta \geq 0, 1 \geq s > 0$ and let α, β be positive real numbers. There is no global solution $w(t)$ to the following problem*

$$\begin{cases} w''(t) + \delta|w'(t)|^{s-1}w'(t) \geq \alpha w^p(t) - \beta w^q(t), & (2.4.189)\\ w(0) = w_0, \quad w'(0) = w_1, & (2.4.190)\end{cases}$$

such that

$$\begin{aligned}&\frac{w_1^2}{2} - \frac{\alpha}{p+1}w_0^{p+1} + \frac{\beta}{q+1}w_0^{q+1} \\ &\quad \leq \frac{p-q}{(p+1)(q+1)}\alpha^{-(q+1)/(p-q)}\beta^{(p+1)/(p-q)},\end{aligned} \tag{2.4.191}$$

where

$$w_0 > r_0 = (\beta/\alpha)^{1/(p-q)}, \quad w_1 < 0. \tag{2.4.192}$$

Let us give an example. Consider the problem

$$w''(t) + \delta w'(t) = \alpha e^{\tau t}|w(t)|^{p-1}w(t) + \beta w(t), \tag{2.4.193}$$

where $\alpha > 0, \beta \geq 0$ and $\tau \geq 0$ are constants.

When $\delta = 1$, the change of variable $y = e^t$ transforms (2.4.193) into

$$\frac{d}{dy}\left(y^2\frac{dz}{dy}\right) = \alpha y^\tau |z|^{p-1}z + \beta z, \quad y \geq 1,$$

which is an equation of Emden–Fowler type. For the case $\tau = 0$, we have the following theorem.

Theorem 2.4.28 ([16]). *Let $\tau = 0$, $p > 1$, $\delta \in \mathbb{R}$, and let α, β be positive real numbers. For any $\Gamma \neq 0$, there exists a unique global non-trivial solution, w_Γ, to* (2.4.193)*, such that $w_\Gamma(0) = \Gamma$. The function w_Γ does not change sign and satisfies the relation*

$$\lim_{t\to+\infty} \frac{w_\Gamma'(t)}{w_\Gamma(t)} = -\frac{\delta + \sqrt{\delta^2 + 4\beta}}{2}. \tag{2.4.194}$$

If, in addition, $\delta = 0$, we have

$$w_\Gamma(t) = \Gamma \left(\frac{2\sqrt{\beta(p+1)}}{Ae^{(p-1)\sqrt{\beta}t} + B} \right)^{2/(p-1)} e^{\sqrt{\beta}t}, \tag{2.4.195}$$

where

$$\begin{cases} A := \sqrt{\beta(p+1) + 2\alpha|\Gamma|^{p-1}} + \sqrt{\beta(p+1)}, \\ B := -\sqrt{\beta(p+1) + 2\alpha|\Gamma|^{p-1}} + \sqrt{\beta(p+1)}. \end{cases}$$

For the proof of Theorem 2.4.28, we refer the reader to [16]. □

Remark 2.4.4 ([16]). For the case of $\beta = 0$, we can derive that if $\delta \geq 0$, any global positive solution satisfies

$$w(t) \leq \frac{\Gamma}{\left[1 + \Gamma^{(p-1)/2}\frac{p-1}{2}\sqrt{\frac{2\alpha}{p+1}}t\right]^{2/(p-1)}} = w_\Gamma(t),$$

where $\Gamma = w(0)$. Note that if $\delta = 0$, w_Γ is the unique global solution to the following problem

$$\begin{cases} w''(t) + \delta w'(t) = \alpha|w(t)|^{p-1}w(t) + \beta w(t), & (2.4.196) \\ w(0) = \Gamma, \quad w'(0) = \Gamma_0. & (2.4.197) \end{cases}$$

such that $w_\Gamma(0) = \Gamma$.

Now we discuss the case of $\tau > 0$ and $\beta = 0$ and we have the following results.

Theorem 2.4.29 ([16]). *Let $\tau, \alpha > 0$ and $\delta > -\tau/2$. Assume that $p > 1$, then any global solution w to* (2.4.193) *is monotone and satisfies, for all $t \geq t_0$,*

$$0 < |w(t)| \leq \frac{|w(t_0)|}{\left\{1 + \frac{p-1}{2}\sqrt{\frac{2\alpha}{p+1}}|w(t_0)|^{(p-1)/2}[e^{\tau t/2} - e^{\tau t_0/2}]\right\}^{2/(p-1)}} \tag{2.4.198}$$

where t_0 is large enough.

For the case of $\delta = -\tau/2$, we know $E = \text{const.}$, and hence we have the following corollary.

Corollary 2.4.5 ([16]). *Assume that* $\delta = -\tau/2$ *and* $\beta = 0$. *Then problem* (2.4.193) *has a unique family of global solutions*

$$w(t) = \frac{w(0)}{[1 + \frac{p-1}{2}\sqrt{\frac{2\alpha}{p+1}}|w(0)|^{(p-1)/2}[e^{\tau t/2} - 1]]^{2/(p-1)}}. \tag{2.4.199}$$

Remark 2.4.5. Estimate (2.4.198) is still valid if in (2.4.193) the equality "=" is replaced by "≥". Note that $|w(t)| \leq Ce^{-[\tau/(p-1)]t}$ for large t and the function $Ke^{-[\tau/(p-1)]t}$ satisfies (2.4.193) for any $\delta \in \mathbb{R}$ and any $\beta \geq 0$, where τ and K satisfy the conditions

$$\tau > \frac{p-1}{2}\left[\delta + \sqrt{\delta^2 + 4\beta}\right] \tag{2.4.200}$$

and

$$\tau^2 - (p-1)\delta\tau - (p-1)^2\beta = (p-1)^2\alpha|K|^{p-1}. \tag{2.4.201}$$

Using Theorem 2.4.27 and the function

$$E_0(t) = \frac{1}{2}e^{-\tau t}\Big\{[w'(t)]^2 - \beta w^2(t)\Big\} - \frac{\alpha}{p+1}w^{p+1}(t) \tag{2.4.202}$$

as well as the following theorem, we can complete the proof of Theorem 2.4.29, for its details, see [16]. □

Theorem 2.4.30 ([16]). *Let* $\tau, \alpha > 0$, $\beta \leq 0$ *and* $\delta > -\tau/2$. *Assume that* $p > 1$. *Then there is no global solution to* (2.4.196)–(2.4.197) *such that*

$$[w'(0)]^2 < \beta[w(0)]^2 + \frac{2\alpha}{p+1}|w(0)|^{p+1}. \tag{2.4.203}$$

Now to close this chapter, we introduce some results due to Li and Zhou [537], which were used to prove the breakdown of solutions to the Cauchy problem for semilinear dissipative wave equations.

The first one is the following comparison theorem, which can be viewed as a generalization of Theorems 2.4.16–2.4.18.

Theorem 2.4.31 (The Li–Zhou Inequality [537]). *Assume that* $k(t)$ *and* $h(t)$ *are twice differentiable functions satisfying the inequalities for all* $t \geq 0$,

$$\begin{cases} a(t)k''(t) + k'(t) \geq b(t)k^{1+\alpha}(t), & (2.4.204)\\ a(t)h''(t) + h'(t) \leq b(t)h^{1+\alpha}(t), & (2.4.205)\end{cases}$$

where $\alpha \geq 0$ *is a constant and for all* $t \geq 0$,

$$a(t) > 0, \;\; b(t) > 0. \tag{2.4.206}$$

Assume further that

$$k(0) > h(0), \;\; k'(0) \geq h'(0). \tag{2.4.207}$$

Then for all $t > 0$,

$$k'(t) > h'(t). \tag{2.4.208}$$

Proof. Without loss of generality, we may suppose that

$$k'(0) > h'(0). \tag{2.4.209}$$

In fact, if $k'(0) = h'(0)$, from (2.4.204)–(2.4.205) it easily follows that

$$k''(0) > h''(0). \tag{2.4.210}$$

Hence, there exists a constant $\delta_0 > 0$ so small that $k'(t) > h'(t)$ holds for all $0 < t \leq \delta_0$; then $k'(\delta_0) > h'(\delta_0)$ and $k(\delta_0) > h(\delta_0)$. Thus, we can take $t = \delta_0$ as the initial time for establishing the desired estimate.

By (2.4.209) and continuity, if (2.4.208) fails, then there exists a time $t^* > 0$ such that for all $0 \leq t < t^*$,

$$k'(t) > h'(t), \quad k'(t^*) = h'(t^*), \tag{2.4.211}$$

and then

$$k''(t^*) \leq h''(t^*). \tag{2.4.212}$$

On the other hand, by (2.4.207) and (2.4.211), we have

$$k(t^*) > h(t^*),$$

so taking $t = t^*$ in (2.4.204)–(2.4.205) gives

$$a(t^*)(k''(t^*) - h''(t^*)) \geq b(t^*)(k^{1+\alpha}(t^*) - h^{1+\alpha}(t^*)) > 0,$$

which contradicts (2.4.212). This finishes the proof. □

In a similar way, we can prove the following theorem and corollaries which are also due to Li and Zhou [537].

Theorem 2.4.32 (The Li–Zhou Inequality [537])**.** *If hypothesis* (2.4.207) *in Theorem* 2.4.31 *is replaced by*

$$k(0) \geq h(0), \;\; k'(0) > h'(0), \tag{2.4.213}$$

then for all $t > 0$,

$$k'(t) > h'(t). \tag{2.4.214}$$

By taking $h(t) \equiv 0$ in Theorems 2.4.31–2.4.32 respectively, we get the following two corollaries.

Corollary 2.4.6 (The Li–Zhou Inequality [537])**.** *Assume that* (2.4.206) *holds. If* $k(t)$ *satisfies* (2.4.204) *and*

$$k(0) > 0, \;\; k'(0) \geq 0, \tag{2.4.215}$$

then for all $t > 0$,

$$k'(t) > 0. \tag{2.4.216}$$

Corollary 2.4.7 (The Li–Zhou Inequality [537]). *If* (2.4.215) *in Corollary* 2.4.6 *is replaced by*

$$k(0) \geq 0, \;\; k'(0) > 0, \tag{2.4.217}$$

then for all $t \geq 0$,

$$k'(t) > 0. \tag{2.4.218}$$

Theorem 2.4.33 (The Li–Zhou Inequality [537]). *Assume that* $a(t)$ *satisfies* (2.4.206) *and* $\alpha \geq 0$. *If the function* $h(t)$ *satisfies*

$$a(t)h''(t) + h'(t) = C_0 h^{1+\alpha}(t) \tag{2.4.219}$$

for a constant $C_0 > 0$, *and*

$$h(0) > 0, \;\; h'(0) = 0, \tag{2.4.220}$$

then for all $t \geq 0$,

$$h''(t) > 0. \tag{2.4.221}$$

Proof. The relations (2.4.219)–(2.4.220) imply $h''(0) > 0$; then if (2.4.221) fails, by continuity, there exists a constant $t^* > 0$ such that for all $0 \leq t < t^*$,

$$h''(t) > 0, \;\; h''(t^*) = 0, \tag{2.4.222}$$

whence

$$h'''(t^*) \leq 0. \tag{2.4.223}$$

Using (2.4.222) and (2.4.220), we get for all $0 < t \leq t^*$,

$$h'(t) > 0, \;\; h(t) > 0.$$

In particular, we have

$$h(t^*), \; h'(t^*) > 0. \tag{2.4.224}$$

Differentiating (2.4.219) with respect to t, taking $t = t^*$, and using (2.4.222) and (2.4.224), we get

$$a(t^*)h'''(t^*) = C_0(1+\alpha)h^\alpha(t^*)h'(t^*) > 0$$

which contradicts (2.4.223). The proof is complete. □

Theorem 2.4.34 (The Li–Zhou Inequality [537]). *Assume that the function* $I(t)$ *satisfies the inequality for all* $t > 0$,

$$I''(t) + I'(t) \geq C_0 \frac{I^{1+\alpha}(t)}{(1+t)^\beta} \tag{2.4.225}$$

with

$$I(0) > 0, \;\; I'(0) \geq 0, \tag{2.4.226}$$

where $C_0 > 0, \alpha > 0$ *are constants. When* $0 \le \beta \le 1$*,* $I = I(t)$ *necessarily blows up in a finite time. Moreover, if*

$$I(0) = \varepsilon \tag{2.4.227}$$

where $\varepsilon > 0$ *is a small parameter, then the lifespan* $\tilde{T}(\varepsilon)$ *has the upper bound:*

$$\tilde{T}(\varepsilon) \le \begin{cases} \exp\{a\varepsilon^{-\alpha}\}, & \text{if } \beta = 1, \\ b\varepsilon^{-\alpha/(1-\beta)}, & \text{if } 0 \le \beta < 1, \end{cases} \tag{2.4.228}$$

where a, b *are positive constants independent of* $\varepsilon > 0$*.*

Proof. Without loss of generality, we may suppose that

$$I'(0) > 0. \tag{2.4.229}$$

Indeed, if $I'(0) = 0$, then by (2.4.225)–(2.4.226) it is easy to see that $I''(0) > 0$, and so there exists a constant $\delta_0 > 0$ so small that $I(\delta_0) > 0$ and $I'(\delta_0) > 0$. Thus we may take $t = \delta_0$ as initial time for establishing the desired estimate.

We first prove that when $\beta = 0$, the solution $I = I(t)$ to (2.4.225)–(2.4.226) necessarily blows up in a finite time. In this case, (2.4.225) can be written as

$$I''(t) + I'(t) \ge C_0 I^{\alpha+1}(t). \tag{2.4.230}$$

For this purpose, it suffices to show that for $\eta > 0$ small enough, the solution $I = I_1(t)$ to the Cauchy problem

$$I_1'(t) = \eta I_1^{1+\alpha/2}(t), \quad I = I_1(0), \tag{2.4.231}$$

where

$$0 < I_1(0) < I(0), \tag{2.4.232}$$

satisfies

$$I(t) > I_1(t) \tag{2.4.233}$$

in the existence domain. Indeed, for $\eta > 0$ small enough we have

$$I_1'(0) = \eta I_1^{1+\alpha/2}(0) < I'(0). \tag{2.4.234}$$

Moreover,

$$I_1''(t) = \eta(1+\alpha/2) I_1^{\alpha/2}(t) I_1'(t) = \eta^2 (1+\alpha/2) I_1^{1+\alpha}(t).$$

Then, noting that $I_1(t) \ge I_1(0)$, for $\eta > 0$ small enough we get

$$\begin{aligned} I_1''(t) + I_1'(t) &= \eta^2(1+\alpha/2) I_1^{1+\alpha}(t) + \eta I_1^{1+\alpha/2}(t) \\ &= \left[\eta^2(1+\alpha/2) + \eta/I_1^{\alpha/2}(t)\right] I_1^{1+\alpha}(t) \\ &\le \left[\eta^2(1+\alpha/2) + \eta/I_1^{\alpha/2}(0)\right] I_1^{1+\alpha}(t) \le C_0 I_1^{1+\alpha}(t). \end{aligned} \tag{2.4.235}$$

Hence, taking into account (2.4.230), (2.4.235), (2.4.232) and (2.4.234), Theorem 2.4.31 yields for all $t \geq 0$,

$$I'(t) > I_1'(t),$$

then, because of (2.4.232), we obtain (2.4.233).

We now use a scaling argument to estimate the lifespan $\tilde{T}(\varepsilon)$ of $I = I(t)$ for $\beta = 0$, provided that (2.4.232) holds.

Let

$$I_2(t) = \varepsilon^{-1} I(\varepsilon^{-\alpha} t). \tag{2.4.236}$$

From (2.4.229) and (2.4.230) it follows that $I_2(t)$ satisfies

$$\begin{cases} \varepsilon^\alpha I_2''(t) + I_2'(t) \geq C_0 I_2^{1+\alpha}(t), & (2.4.237) \\ I_2(0) = 1, \ I_2'(0) > 0. & (2.4.238) \end{cases}$$

Let $I_3 = I_3(t)$ solve the following problem:

$$\begin{cases} \varepsilon^\alpha I_3''(t) + I_3(t) = C_0 I_3^{1+\alpha}(t), & (2.4.239) \\ I_3(0) = 1, \ I_3'(0) = 0. & (2.4.240) \end{cases}$$

By Theorem 2.4.32, for all $t \geq 0$,

$$I_2'(t) > I_3'(t), \tag{2.4.241}$$

and so for all $t > 0$,

$$I_2(t) > I_3(t). \tag{2.4.242}$$

By Theorem 2.4.33, we obtain for all $t \geq 0$,

$$I_3''(t) > 0. \tag{2.4.243}$$

Then, for $\varepsilon > 0$ small enough, it follows from (2.4.239) that

$$I_3''(t) + I_3'(t) \geq C_0 I_3^{1+\alpha}(t). \tag{2.4.244}$$

Noting (2.4.240), according to the conclusion obtained in the previous discussion, $I = I_3(t)$ must blow up in a finite time and the lifespan of $I = I_3(t)$ has an upper bound independent of ε. Thus, by (2.4.236) and (2.4.242), the lifespan $\tilde{T}(\varepsilon)$ of $I = I(t)$ satisfies

$$\tilde{T}(\varepsilon) \leq b\varepsilon^{-\alpha}, \tag{2.4.245}$$

where b is a positive constant independent of ε. The estimate (2.4.245) just concerns the case $\beta = 0$.

We next consider case $\beta = 1$, for which (2.4.225) takes the form

$$I''(t) + I'(t) \geq C_0 \frac{I^{1+\alpha}(t)}{1+t}. \tag{2.4.246}$$

Let

$$J(t) = I(e^t - 1). \tag{2.4.247}$$

By (2.4.246) and (2.4.226), it is easy to see that $J(t)$ solves

$$\begin{cases} e^{-t}J''(t) + (1 - e^{-t})J'(t) \geq C_0 J^{1+\alpha}(t), & (2.4.248) \\ J(0) > 0, \quad J'(0) \geq 0. & (2.4.249) \end{cases}$$

Furthermore, by Corollary 2.4.6, we have for all $t > 0$,

$$I'(t) > 0, \tag{2.4.250}$$

whence for all $t > 0$,

$$J'(t) > 0. \tag{2.4.251}$$

Thus from (2.4.248) it follows that

$$e^{-t}J''(t) + J'(t) \geq C_0 J^{1+\alpha}(t), \tag{2.4.252}$$

which, by Theorem 2.4 31, implies for all $t > 0$,

$$J'(t) > J_1'(t). \tag{2.4.253}$$

Consequently, for all $t \geq 0$,

$$J(t) > J_1(t), \tag{2.4.254}$$

where $J_1 = J_1(t)$ solves the problem

$$\begin{cases} e^{-t}J_1''(t) + J_1'(t) = C_0 J_1^{1+\alpha}(t), & (2.4.255) \\ J_1(0) = J(0)/2 > 0, \quad J_1'(0) = 0. & (2.4.256) \end{cases}$$

By Theorem 2.4.32, for all $t \geq 0$,

$$J_1''(t) > 0, \tag{2.4.257}$$

and then from (2.4.255) it follows that

$$J_1''(t) + J_1'(t) \geq C_0 J_1^{1+\alpha}(t). \tag{2.4.258}$$

Taking into account (2.4.256), for $J_1 = J_1(t)$, we can use the result obtained in the case $\beta = 0$. Then when $\beta = 1$, by (2.4.247) and (2.4.254), the function $I = I(t)$ must blow up in a finite time. Moreover, if (2.4.232) holds, noting (2.4.245), the lifespan $\tilde{T}(\varepsilon)$ of $I = I(t)$ satisfies

$$\tilde{T}(\varepsilon) \leq \exp\{a\varepsilon^{-\alpha}\}. \tag{2.4.259}$$

where a is a positive constant independent of ε. The estimate (2.4.225) is nothing else than (2.4.226) in the case $\beta = 1$.

Finally, we consider the case $0 < \beta < 1$. Let

$$J(t) = I((t+1)^{1/(1-\beta)} - 1). \tag{2.4.260}$$

It follows from (2.4.225)–(2.4.226) that $J(t)$ solves the problem

$$\begin{cases} (1-\beta)^2(t+1)^{-\beta/(1-\beta)}J''(t) \\ \qquad +(1-\beta)[1-\beta(t+1)^{-1/(1-\beta)}]J'(t) \geq C_0 J^{1+\alpha}(t), & (2.4.261) \\ J(0) > 0, \quad J'(0) \geq 0. & (2.4.262) \end{cases}$$

By Corollary 2.4.6, we obtain (2.4.250) and (2.4.251). Hence, it follows from (2.4.261) that

$$(1-\beta)^2(t+1)^{-\beta/(1-\beta)}J''(t) + J'(t) \geq C_0 J^{1+\alpha}(t), \tag{2.4.263}$$

and then, by Theorem 2.4.31, the relations (2.4.253)–(2.4.254) hold, where $J_1 = J_1(t)$ satisfies the relations (2.4.256) and

$$(1-\beta)^2(t+1)^{-\beta/(1-\beta)}J_1''(t) + J_1'(t) = C_0 J_1^{1+\alpha}(t). \tag{2.4.264}$$

By Theorem 2.4.32, we have (2.4.257) and then (2.4.264) implies the relation (2.4.258). Thus, the result obtained in the case $\beta = 0$ is still valid for $J_1 = J_1(t)$. Therefore, when $0 < \beta < 1$, by (2.4.260) and (2.4.254), the function $I = I(t)$ must blow up in a finite time. Moreover, if (2.4.232) holds, then because of (2.4.245), the lifespan $\tilde{T}(\varepsilon)$ of $I = I(t)$ satisfies

$$\tilde{T}(\varepsilon) \leq b\varepsilon^{-\alpha/(1-\beta)}, \tag{2.4.265}$$

where b is a positive constant independent of ε. This gives us (2.4.233) in the case $0 < \beta < 1$. □

Corollary 2.4.8 (The Li–Zhou Inequality [537]). *Assume that $v = v(t) \geq 0$ satisfies the estimate*

$$v(t) \geq C_1 + C_2 \left\{ \int_{t-2}^{t} (t-\tau)\frac{v^{1+\alpha}(\tau)}{\tau^\beta}d\tau + 2\int_{t_0}^{t-2} \frac{v^{1+\alpha}(\tau)}{\tau^\beta}d\tau \right\} \tag{2.4.266}$$

where $\alpha > 0, 0 \leq \beta \leq 1, t_0 \geq 0$ and C_1, C_2 are positive constants. Then $v = v(t)$ necessarily blows up in a finite time. Moreover, for

$$C_1 = \varepsilon > 0 \tag{2.4.267}$$

small enough, the lifespan $\tilde{T}(\varepsilon)$ of $v = v(t)$ satisfies the estimate (2.4.233).

Proof. Let $J = J(t)$ satisfy

$$J(t) = C_1 + C_2 \left\{ \int_{t-2}^{t} (t-\tau)\frac{J^{1+\alpha}(\tau)}{\tau^\beta}d\tau + 2\int_{t_0}^{t-2} \frac{J^{1+\alpha}(\tau)}{\tau^\beta}d\tau \right\}. \tag{2.4.268}$$

We have for all $t \geq t_0$,

$$v(t) \geq J(t). \tag{2.4.269}$$

Moreover, it follows from (2.4.268) that $J(t)$ solves the problem

$$\begin{cases} J(t_0) = C_1 > 0, \;\; J'(t) = C_2 \displaystyle\int_{t-2}^{t} \frac{J^{1+\alpha}(\tau)}{\tau^\beta} d\tau, & (2.4.270) \\ J''(t) = C_2 \left(\dfrac{J^{1+\alpha}(t)}{t^\beta} - \dfrac{J^{1+\alpha}(t-2)}{(t-2)^\beta} \right), & (2.4.271) \end{cases}$$

which implies that for all $t \geq t_0$,

$$J(t) > 0, \;\; J'(t) > 0. \tag{2.4.272}$$

Thus we can choose $\mu > 0$ so large that for all $t \geq t_0 + 2$,

$$\begin{aligned} J''(t) + \mu J'(t) &= C_2 \left(\frac{J^{1+\alpha}(t)}{t^\beta} - \frac{J^{1+\alpha}(t-2)}{(t-2)^\beta} + \mu \int_{t-2}^{t} \frac{J^{1+\alpha}(\tau)}{\tau^\beta} d\tau \right) \\ &\geq C_2 \left\{ \frac{J^{1+\alpha}(t)}{t^\beta} + [\frac{\mu}{t^\beta} - \frac{1}{(t-2)^\beta}] J^{1+\alpha}(t-2) \right\} \\ &\geq C_2 \frac{J^{1+\alpha}(t)}{t^\beta}. \end{aligned} \tag{2.4.273}$$

Let

$$I(t) = J(\mu^{-1} t). \tag{2.4.274}$$

We obtain for all $t \geq t_0 + 2$,

$$I''(t) + I'(t) \geq C_2 \mu^{\beta-2} \frac{I^{1+\alpha}(t)}{t^\beta} \geq C_0 \frac{I^{1+\alpha}(t)}{(1+t)^\beta}, \tag{2.4.275}$$

where $C_0 > 0$ is a constant. Because of (2.4.272), Theorem 2.4.31 can be applied to $I = I(t)$, and then using the relations (2.4.269) and (2.4.274) we can reach the desired conclusion. □

Chapter 3

Attractors for Evolutionary Differential Equations

In this chapter, we prove the existence of global (uniform) attractors for some evolutionary differential equations. The chapter includes four sections. In Section 3.1, we shall use Theorems 1.1.2 and 2.1.3 to establish the existence of global attractors for a nonlinear reaction-diffusion equation. We refer to Babin and Vishik [54], Chepyzhov and Vishik [154], and Temam [915] for related concepts of infinite-dimensional dynamical systems such as absorbing sets, maximal (global) attractors, etc. In Section 3.2, we employ Theorem 1.5.20 to show the existence of attractors for differential equations with delay. In Section 3.3, we exploit Theorem 2.1.15 to study the global attractors for the Cahn–Hilliard equation in H^2 and H^3. In Section 3.4, we use Theorems 2.1.7 and 2.3.2 to establish the global existence, asymptotic behavior of the solution, and the existence of uniform attractors for a non-autonomous linear viscoelastic equation with linear damping and a delay term. All inequalities applied in this chapter are very important in deriving the existence of global (uniform) attractors. In particular, they may be used to derive the existence of (uniform) absorbing sets.

3.1 Maximal attractors for nonlinear reaction-diffusion equations

In this section, we shall use Theorem 1.1.2 and Theorem 2.1.3 to establish the existence of global attractors for a nonlinear reaction-diffusion equation. We borrow these results from Temam [915].

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3.1.1 An initial boundary value problem

We shall consider the following initial boundary value problem

$$\begin{cases} u_t - d\Delta u + g(u) = 0 \quad \text{in } \Omega \times \mathbb{R}^+, & (3.1.1) \\ u = 0 \quad \text{on } \Gamma, & (3.1.2) \\ u(x,0) = u_0(x), \quad x \in \Omega, & (3.1.3) \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^n$ with smooth boundary Γ, $d > 0$ is a constant, $u = u(x,t)$ is a scalar function with initial data $u_0(x)$, and g is a polynomial of odd degree with positive leading coefficient:

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_{2p-1} > 0. \tag{3.1.4}$$

Let $H = L^2(\Omega), V = H_0^1(\Omega)$. Then we can use the general results on existence and uniqueness of solutions to parabolic equations (see, e.g., Friedman [272], Lions [546]) to conclude the following theorem.

Theorem 3.1.1 ([272, 546, 915]). *For any given $u_0 \in H$, there exists a unique global solution u of problem* (3.1.1)–(3.1.3) *such that*

$$\begin{cases} u \in L^2(0,T; H_0^1(\Omega)) \cap L^{2p}(0,T; L^{2p}(\Omega)), \quad \text{for all } T > 0, & (3.1.5) \\ u \in C(\mathbb{R}^+; H). & (3.1.6) \end{cases}$$

The mapping $u_0 \mapsto u(t)$ is continuous in H. If, in addition, $u_0 \in H_0^1(\Omega)$, then for all $T > 0$,

$$u \in C([0,T]; V) \cap L^2(0,T; H^2(\Omega)). \tag{3.1.7}$$

Proof. We refer to [915] for a detailed proof. □

It follows obviously from Theorem 3.1.1 that we can define a semigroup by

$$S(t) : u_0 \in H \mapsto u(t) \in H. \tag{3.1.8}$$

Now the next result, due to Temam [915], reads as follows.

Theorem 3.1.2 ([915]). *Assume Ω is an open bounded set of $\mathbb{R}^n$ and g satisfies* (3.1.4). *The semigroup $S(t)$ associated with problem* (3.1.1)–(3.1.3) *possesses a maximal attractor $\mathcal{A}$ which is bounded in $H_0^1(\Omega)$, and compact and connected in $L^2(\Omega)$. Its basin of attraction is the whole space $L^2(\Omega)$: $\mathcal{A}$ attracts all bounded set of $L^2(\Omega)$.*

Proof. We first prove that there exist absorbing sets in $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Using the Young inequality, we obtain from (3.1.4) the existence a constant $C_1' > 0$ such that for all $s \in \mathbb{R}$,

$$\left| \sum_{j=0}^{2p-2} b_j s^{j+1} \right| \le \frac{1}{2} b_{2p-1} s^{2p} + C_1',$$

which gives that for all $s \in \mathbb{R}$,

$$\frac{1}{2} b_{2p-1} s^{2p} - C_1' \leq g(s)s \leq \frac{3}{2} b_{2p-1} s^{2p} + C_1'. \tag{3.1.9}$$

Multiplying (3.1.1) by $u = u(x,t)$, integrating over Ω, and using (3.1.2) and the Green formula, we get

$$\frac{1}{2}\frac{d}{dt}|u|^2 + d\|u\|^2 + \int_\Omega g(u)u dx = 0, \tag{3.1.10}$$

which, together with (3.1.9), yields

$$\frac{d}{dt}|u|^2 + 2d\|u\|^2 + \int_\Omega b_{2p-1} u^{2p} dx \leq 2C_1'|\Omega|, \tag{3.1.11}$$

where $|\Omega|$ is the measure (volume) of Ω. By the Poincaré inequality, there exists a constant $C_0 = C_0(\Omega) > 0$, such that for all $u \in H_0^1(\Omega)$,

$$|u| \leq C_0\|u\|. \tag{3.1.12}$$

Setting $C_2' = 2C_1'|\Omega|$, we hence conclude from (3.1.11) that

$$\frac{d}{dt}|u|^2 + \frac{2d}{C_0^2}|u|^2 \leq C_2'. \tag{3.1.13}$$

Multiplying (3.1.13) by $\exp\left(\frac{2d}{C_0^2}t\right)$ and using the classical Bellman–Gronwall inequality (e.g., Theorem 1.1.2), we arrive at

$$|u(t)|^2 \leq |u_0|^2 \exp\left(-\frac{2d}{C_0^2}t\right) + \frac{C_2'C_0^2}{2d}\left(1 - \exp(-\frac{2d}{C_0^2}t)\right), \tag{3.1.14}$$

whence

$$\limsup_{t\to+\infty} |u(t)| \leq \rho_0, \quad \rho_0^2 = C_2'C_0^2/(2d). \tag{3.1.15}$$

Therefore, we know that there exists an absorbing set B_0 in H, namely, any ball of H centered at 0 of radius $\rho_0' > \rho_0$. If B is a bounded set of H included in a ball $B(0,R)$ of H of radius R centered at 0, then $S(t)B \subseteq B(0,\rho_0')$ for all $t \geq t_0 = t_0(B;\rho_0')$ with

$$t_0 = \frac{C_0^2}{2d}\log\left(\frac{R^2}{(\rho_0')^2 - \rho_0^2}\right). \tag{3.1.16}$$

On the other hand, we also derive from (3.1.11), after integration in t, that for all $r > 0$,

$$2d\int_t^{t+r} \|u(s)\|^2 ds + \int_t^{t+r}\int_\Omega b_{2p-1}u^{2p} dx ds \leq rC_2' + |u(t)|^2. \tag{3.1.17}$$

By (3.1.15), we conclude that for all $r > 0$,

$$\lim\sup_{t\to+\infty}\left\{2d\int_t^{t+r}\|u(s)\|^2ds+\int_t^{t+r}\int_\Omega b_{2p-1}u^{2p}dxds\right\}\le rC_2'+\rho_0^2, \quad (3.1.18)$$

and if $u_0 \in B \subseteq B(0,R)$ and for all $t \geq t_0(B,\rho_0')$, then we have

$$2d\int_t^{t+r}\|u(s)\|^2ds+\int_t^{t+r}\int_\Omega b_{2p-1}u^{2p}dxds\le rC_2'+(\rho_0')^2. \quad (3.1.19)$$

Next, we shall prove that there exists an absorbing set in $V = H_0^1(\Omega)$ and the semigroup $S(t)$ is uniformly compact. In fact, multiplying (3.1.1) by $-\Delta u$, integrating over Ω, and using (3.1.2) and the Green formula, we get

$$-\int_\Omega \Delta u\frac{\partial u}{\partial t}dx=\sum_{i=1}^n\int_\Omega\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial t\partial x_i}dx=\frac{1}{2}\frac{d}{dt}\|u\|^2, \quad (3.1.20)$$

$$-\int_\Omega \Delta u g(u)dx=\int_\Omega\nabla g(u)\nabla udx=\sum_{i=1}^n\int_\Omega g'(u)\left(\frac{\partial u}{\partial x_i}\right)^2dx. \quad (3.1.21)$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+d|\Delta u|^2+\int_\Omega g'(u)|\nabla u|^2dx=0. \quad (3.1.22)$$

Similarly to (3.1.9), we may prove by the repeated use of Young's inequality that there exists a constant $C_3' > 0$ such that for all $s \in \mathbb{R}$,

$$\frac{2p-1}{2}b_{2p-1}s^{2p-2}-C_3'\le g'(s)=\sum_{i=1}^{2p-1}jb_js^{j-1}\le\frac{3}{2}(2p-1)b_{2p-1}s^{2p-2}+C_3'. \quad (3.1.23)$$

We also obtain from general results on the Dirichlet problem in Ω that on $H_0^1(\Omega)\cap H^2(\Omega)$ $|\Delta u|$ is a norm equivalent to that induced by $H^2(\Omega)$. Therefore, there exists a constant $C_1 = C_1(\Omega) > 0$, depending on Ω, such that for all $u \in H_0^1(\Omega)$,

$$\|u\|\le C_1|\Delta u|. \quad (3.1.24)$$

Setting $C_4' = \frac{1}{2}(2p-1)b_{2p-1} > 0$, we then deduce from (3.1.23) that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+(d-C_3')\|u\|^2+C_4'\int_\Omega u^{2p-2}|\nabla u|^2dx\le 0. \quad (3.1.25)$$

In particular,

$$\frac{d}{dt}\|u\|^2\le 2C_3'\|u\|^2. \quad (3.1.26)$$

If u_0 is in $V = H_0^1(\Omega)$, then the classical Bellman–Gronwall inequality (see Theorem 2.1.3) shows that for all $t > 0$,

$$\|u(t)\|^2\le\|u_0\|^2\exp(2C_3't). \quad (3.1.27)$$

A bound valid for all $t \in \mathbb{R}^+$ is obtained by using the uniform Bellman–Gronwall inequality (e.g., Theorem 2.1.3); for an arbitrary fixed $r > 0$, we obtain for all $t \geq t_*$,

$$\|u(t+r)\|^2 \leq \frac{\kappa}{r} \exp(2C_3' r), \tag{3.1.28}$$

provided that for all $t \geq t_*$,

$$\int_t^{t+r} \|u(s)\|^2 ds \leq \kappa. \tag{3.1.29}$$

An explicit value of κ can be derived from (3.1.11) and the above computation, when $t_* = 0$. Hence (3.1.28) provides a uniform bound of $\|u(t)\|$ for all $t \geq r$, while (3.1.27) provides a uniform bound for $\|u(t)\|$ for all $0 \leq t \leq r$. For our purpose, it is simpler and sufficient to set $t_* = t_0$ (as in (3.1.16)). In this case, the value of κ is given in (3.1.19), by

$$\kappa = \frac{1}{2d}(rC_2' + \rho_0'^2). \tag{3.1.30}$$

Thus it follows that the ball of V of radius ρ_1 centered at 0 is absorbing in V, where

$$\rho_1^2 = \frac{\kappa}{r} \exp(2C_3' r), \tag{3.1.31}$$

with κ as in (3.1.30), and if u_0 belongs to the ball $B(0,R)$ of H of radius r centered at 0, then $u(t)$ enters this absorbing set, denoted by B_1, at a time $t \leq t_0 + r$, and remains in it for all $t \geq t_0 + r$. At the same time, this result provides the uniform compactness of $S(t)$: any bounded set B of H is contained in such a ball $B(0,R)$, and for any $u_0 \in B$ and for all $t \geq t_0 + r$, t_0, r as above, $u(t)$ belongs to B_1, which is bounded in V and relatively compact in $H = L^2(\Omega)$. Therefore, by the standard theory on the existence of the maximal attractor, we have proved the existence of the maximal attractor in H. □

3.2 Attractors for autonomous differential equations with delay

In this section, we shall borrow results from Caraballo, Rubin and Valero [131] as an application of Theorem 1.5.20 to establish the existence of attractors for differential equations with delay.

We shall consider the following system of equations:

$$\begin{cases} x'(t) = F(x(t), x(t-h)) = f(x_t), \quad t > 0, \\ \quad x_0 = \psi \in X, \end{cases} \tag{3.2.1}$$

where $F(F_1, \ldots, F_n), F_i : \mathbb{R}^{2n} \to \mathbb{R}, h > 0, X = C([-h,0], L) \subset \mathcal{E}$ (L is a closed subset of $\mathbb{R}^n$), and F is continuous, and $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the scalar product and norm in $\mathbb{R}^n$, respectively.

We assume the following conditions hold:

(1) For each $\psi \in X$, there exists at least one solution $x(t)$ of problem (3.2.1) such that $x(t) \in X$ for all $t \geq 0$.

(2) There exists a constant $K > 0$ such that for any $\varepsilon > 0$, there exists a constant $\delta(\varepsilon) > 0$ such that for all $x, y \in L$, if $|x|, |y| \geq K + \varepsilon$,

$$\langle F(x,y), x\rangle \leq -\delta.$$

(3) There exist constants $C > 0, 0 < \alpha \leq 2$ such that for all $x, y \in L$,

$$\langle F(x,y), x\rangle \leq C(1 + |x|^{\alpha}).$$

First, let us introduce some notation. Let $h > 0$ be a given positive number (the delay time) and denote by $\mathcal{E}$ the Banach space $C([-h, 0]; \mathbb{R}^n)$ endowed with the norm $\|\psi\| = \sup_{\sigma\in[-h,0]} |\psi(\sigma)|$, which is the usual phase space when we deal with delay differential equations. However, it is sometimes useful to consider the solutions as mappings from $\mathbb{R}$ into $\mathbb{R}^n$ (we shall consider $\mathbb{R}^n$ in its usual Euclidean topology and denote by $\langle\cdot,\cdot\rangle$, $|\cdot|$ its scalar product and norm, respectively). For the case of infinite delay, we shall need a more careful choice of the phase space (e.g., [53, 341]), but omit those details here. By x_t we shall denote the element in $\mathcal{E}$ given by $x_t(s) = x(t + s)$ for all $s \in [-h, 0]$, and set $\mathbb{R}_{\mathrm{d}} = \{(t, s) \in \mathbb{R}^2, t \geq s\}$.

Remark 3.2.1 ([131]).

(i) It follows obvious from condition (2), that for all $|x|, |y| > K$, $x, y \in L$,

$$\langle F(x,y), x\rangle < 0.$$

(ii) If $X = \mathcal{E}$, then condition (1) is not necessary.

(iii) The most usual case in the applications seems to be $L = \mathbb{R}^n_+$.

We may refer to [131] for the definition of m-semiflow.

Theorem 3.2.1 ([131]). *Under above conditions* (1) *and* (3), *m-semiflow G is well defined and bounded for any $t \geq 0$.*

Proof. Obviously, condition (1) and Lemma 13 in [131] imply that G is well defined. Let $x(t)$ be an arbitrary solution. We shall derive an estimate on any interval $[0, T]$. Multiplying the equality in (3.2.1) by $x(t)$ and using condition (3), we can get

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 \leq C(1 + |x(t)|^{\alpha}),$$

which readily implies for all $t \in [0, T]$,

$$|x(t)|^2 \leq |x(0)|^2 + 2CT + 2\int_0^t C|x(s)|^{\alpha} ds. \tag{3.2.2}$$

Therefore, applying Theorem 1.5.20 to the above inequality (3.2.2) for $y(t) = |x(t)|$ gives us for all $t \in [0, T]$,

$$\begin{cases} |x(t)| \leq \Big((|x(0)|^2 + 2CT)^{(2-\alpha)/2} + (2-\alpha)CT\Big)^{1/(2-\alpha)} & \text{if } \alpha < 2, \\ |x(t)| \leq \Big(|x(0)|^2 + 2CT\Big)^{1/2} \exp(CT) & \text{if } \alpha = 2. \end{cases} \tag{3.2.3}$$

We have obtained that any solution exists globally in time in view of Corollary 6 in [131] (note that the continuity of F implies that f is bounded). Hence, in the case $X = \mathcal{E}$, the semiflow G is well defined without using condition (1). Finally, it follows from (3.2.3) that $G(t, \cdot)$ is bounded for any $t \geq 0$. □

3.3 H^2 estimates for the Cahn–Hilliard equation

In this section, we shall employ Theorem 2.1.15 to study the H^2 estimates for the Cahn–Hilliard equation. We adopt these results from Dlotko [213].

We shall consider the following Cahn–Hilliard equation in H^2:

$$u_t = -\varepsilon^2 \Delta^2 u + \Delta(f(u)), \quad x \in \Omega \subset \mathbb{R}^n, \ t \geq 0 \tag{3.3.1}$$

subject to the initial and boundary conditions

$$\begin{cases} u(0, x) = u_0(x), & \text{for all } x \in \Omega, \quad (3.3.2) \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial(\Delta u)}{\partial n} = 0, & \text{for all } x \in \partial\Omega. \quad (3.3.3) \end{cases}$$

The function f is assumed to be a polynomial of degree $2p-1$:

$$f(u) = \sum_{j=1}^{2p-1} a_j u^j, \quad p \in \mathbb{N}, \ p \geq 2, \tag{3.3.4}$$

with leading coefficient $a_{2p-1} > 0$. Moreover, $p = 2$ if $n = 3$. It follows obviously from (3.3.4) that $f(0) = 0$ and

$$|\Omega|^{-1} \int_\Omega u(t)dx = |\Omega|^{-1} \int_\Omega u_0 dx = \bar{u}_0. \tag{3.3.5}$$

First, integrating of (3.3.1) over Ω and using (3.3.3), we have for all $t \geq 0$. Second, multiplying (3.3.1) by $-\varepsilon^2 \Delta^2 u + f(u)$, we obtain the existence of the Lyapunov functional for problem (3.3.1)–(3.3.3),

$$L(u(t)) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u(t)|^2 dx + \int_\Omega \int_0^{u(t)} f(z)dzdx, \tag{3.3.6}$$

with $(d/dt)L(u(t)) = -\|\nabla[-\varepsilon^2\Delta u + f(u)]\|_{L^2}^2 \le 0$. It is easy to see that under the assumption that $a_{2p-1} > 0$, the primitive of f,

$$F(s) = \int_0^s f(z)dz,$$

is bounded from below for all $s \in \mathbb{R}$: $F(s) \ge M$. Therefore, from that (3.3.6) we derive for sufficiently smooth solutions:

$$\frac{\varepsilon^2}{2}\int_\Omega \sum_{i=1}^n u_{x_i}^2(t)dx \le \frac{\varepsilon^2}{2}\int_\Omega \sum_{i=1}^n u_{0x_i}^2 dx + \int_\Omega F(u_0)dx - M|\Omega|,$$

which, together with (3.3.5), guarantees that H^1 norm of $u(t)$:

$$\|u(t)\|_{H^1} = \left(\|\nabla u(t)\|_{L^2}^2 + \bar{u}_0^2\right)^{1/2} \le c \tag{3.3.7}$$

where $c = c(\|u_0\|_{H^1}, \bar{u}_0) > 0$ is a constant depending on $\|u_0\|_{H^1}$ and $\bar{u}_0$. Moreover, under our assumption on n and p, we have $H^1 \subset L^{2p}$ and F is a polynomial of degree $2p$. The estimate (3.3.7) was used by Temam (see, e.g., [915], (4.105), p. 156) to obtain the estimate

$$\|\Delta f(u(t))\|_{L^2}^2 \le k(1 + \|\Delta^2 u(t)\|_{L^2}^{2\sigma}), \tag{3.3.8}$$

where $\sigma \in [0,1)$ is a constant and k is a function $k = k(\|\nabla u(t)\|_{L^2}, \bar{u}_0)$ increasing with respect to both the first argument and the absolute value of the second argument. Due to (3.3.7), we shall define its finite majorant as $K = k(c(\|\nabla u_0\|_{L^2}, \bar{u}_0), \bar{u}_0)$.

In the sequel, we shall use a general semigroup approach to study the problem (3.3.1)–(3.3.3). Assume that $\partial\Omega \in C^5$. We shall show that the operator $A = \varepsilon^2\Delta^2$, considered on the Hilbert space

$$H = H^1(\Omega), \tag{3.3.9}$$

is sectorial (see, e.g., [355], p. 18).We take the domain of A as follows

$$D(A) = \left\{\phi \in H^5(\Omega) : \frac{\partial\phi}{\partial n} = \frac{\partial(\Delta\phi)}{\partial n} = 0 \quad on \ \ \partial\Omega\right\}. \tag{3.3.10}$$

It has been proved in Dlotko [213] that $D(A)$ is dense in H.

Defining in H the scalar product

$$(\phi,\psi)_H = (\nabla\phi, \nabla\psi)_{L^2} + (\phi,\psi)_{L^2}, \tag{3.3.11}$$

and integrating by parts, and using the boundary conditions, we easily check that for all $\phi, \psi \in D(A)$,

$$(A\phi,\psi)_H = (\phi, A\psi)_H, \tag{3.3.12}$$

which implies A is symmetric in H. We also note that A is a positive operator: indeed, for all $\phi, \ \ \psi \in D(A)$,

$$(A\phi, \phi)_H = \varepsilon^2(\nabla\Delta\phi, \nabla\Delta\phi)_{L^2} + \varepsilon^2(\Delta\phi, \Delta\phi)_{L^2} \geq 0. \tag{3.3.13}$$

Therefore, for arbitrary $\delta > 0$, $A + \delta I$ is bounded from below, i.e., for all $\phi, \ \ \psi \in D(A)$,

$$((A + \delta I)\phi, \phi)_H \geq \delta\|\phi\|_H^2. \tag{3.3.14}$$

Moreover, applying Theorem 4.9.1 in [922] (see also Theorem 5.5.1 in [922] and Theorem 19.3 in [271]), we know that for a sufficiently large number $\rho_0 \geq 0$, the operators $A + \rho I$, $\rho \geq \rho_0$, realize an isomorphism of the space $D(A)$ onto $H^1(\Omega)$, which together with (3.3.13) shows that $A + \rho I$, $\rho \geq \rho_0$, are self-adjoint and bounded from below, and hence are sectorial. Therefore, by the definition of the sectorial operator [355], A is sectorial itself.

Now using Temam's estimate (3.3.8), we may obtain an estimate, which is global in time of the solution of problem (3.3.1)–(3.3.3) in H^2.

It is obvious that on the set $\{\phi \in H^2(\Omega) : \ \ \partial\phi/\partial n = 0 \text{ on } \partial\Omega\}$ containing $D(A^{1/2})$, with $A^{1/2} = -\varepsilon\Delta$([922], Theorem 4.3.3; 8, Problem 18.5) and

$$D(A^{1/2}) = \left\{\phi \in H^3(\Omega) : \frac{\partial\phi}{\partial n} = 0 \ \text{ on } \ \partial\Omega\right\}. \tag{3.3.15}$$

We may define an equivalent norm to the H^2 norm as

$$\left(\|\Delta\phi\|_{L^2}^2 + |\bar{\phi}|^2\right)^{1/2}. \tag{3.3.16}$$

We should note that if we take $u_0 \in D(A^{1/2})$ in the following Theorem 3.3.1 and Lemma 3 in [213], then $u(t)$ belongs to $D(A)$ (see (3.3.10)) for all $t > 0$.

We have the following theorem (see [213]).

Theorem 3.3.1. *The following estimate of the solution of problem* (3.3.1)–(3.3.3) *in H^2 holds for all $t \geq 0$,*

$$\|\Delta u(t)\|_{L^2}^2 \leq \max\{c_1 z_1, \ \|\Delta u_0\|_{L^2}^2\}, \tag{3.3.17}$$

where z_1 is the positive root of the equation

$$\phi(z) := -\varepsilon^2 z + \frac{K}{\varepsilon^2}(1 + z^\sigma) = 0 \tag{3.3.18}$$

with K, σ defined as in (3.3.8).

Proof. Since $u(t) \in D(A)$ for all $t > 0$, we can multiply (3.3.1) by $\Delta^2 u$ and integrate over Ω to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\Delta u)^2 dx = -\varepsilon^2\int_\Omega (\Delta^2 u)^2 dx + \int_\Omega \Delta(f(u))\Delta^2 u dx$$

which, together with (3.3.8) and the Cauchy inequality, gives

$$\begin{aligned}\frac{d}{dt}\int_\Omega (\Delta u)^2 dx &\le -\varepsilon^2 \int_\Omega (\Delta^2 u)^2 dx + \frac{1}{\varepsilon^2}\int_\Omega \Delta(f(u))^2 dx \\ &\le -\varepsilon^2 \int_\Omega (\Delta^2 u)^2 dx + \frac{k}{\varepsilon^2}\left\{1 + \left(\int_\Omega (\Delta^2 u)^2 dx\right)^\sigma\right\},\end{aligned} \tag{3.3.19}$$

with $\sigma \in [0,1)$ and $k = k(\|\nabla u(t)\|_{L^2}, \bar{u}_0)$. By the Poincaré and Smoller inequalities (see [874]), it readily follows that

$$\int_\Omega g^2 dx \le c_0 \int_\Omega |\nabla g|^2 dx \le c_1 \int_\Omega (\Delta g)^2 dx, \tag{3.3.20}$$

where $g \in H^2$ fulfills the relations $\int_\Omega g dx = 0, \partial g/\partial n = 0$ on $\partial\Omega$. Moreover, noting the boundary conditions (3.3.3) and using $u(t) \in D(A)$ for all $t > 0, \int_\Omega \Delta u dx = 0$, (3.3.19) for $g = \Delta u$, yields

$$\int_\Omega (\Delta u)^2 dx \le c_1 \int_\Omega (\Delta^2 u)^2 dx, \tag{3.3.21}$$

which, by Theorem 2.1.15 for $y(t) = \|\Delta u\|^2_{L^2},\ z(t) = \|\Delta^2 u\|^2_{L^2}$, and using (3.3.16), gives us the desired (3.3.17). □

3.4 Global existence, exponential stability and uniform attractors for a non-autonomous wave equation

In this section, we shall use Theorems 2.1.7 and 2.3.2 to establish the global existence, exponential stability of the solutions, and the existence of uniform attractors for a non-autonomous linear viscoelastic equation with linear damping and a delay term. We present these results, found in Qin, Ren and Wei [803].

3.4.1 Global existence and exponential stability

We shall consider the following problem for a non-autonomous linear viscoelastic equation with linear damping and a delay term,

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds & \\ \qquad +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = f(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \ge 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \end{cases} \tag{3.4.1}$$

where $u = u(x,t)$, $t \ge 0$, $x \in \Omega$, Δ denotes the Laplace operator with respect to the x variable, Ω is a regular and bounded domain of $\mathbb{R}^N$, $N \ge 1$, μ_1, μ_2

are positive constants, $\tau > 0$ represents the time delay and u_0, u_1, f_0 are given functions belonging to suitable spaces. We shall study the global existence and the asymptotic behavior for the solution of problem (3.4.1) with a delay term appearing in the control term of the equation.

A wave equation with acoustic and memory boundary conditions on a part of the boundary of the domain Ω was also investigated recently in [935], where the existence and uniqueness of global solutions have been proved. An exponential delay result was proved by Mustafa in [653], who, in fact, used the energy method to show that the damping effect through heat conduction is strong enough to uniformly stabilize the system. Andrade, Ma, and Qin [45] studied the existence of uniform attractors for a nonlinear non-autonomous viscoelastic equation in a bounded domain $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$ by establishing the uniform asymptotic compactness of the semi-process generated by the global solutions. Ma and Qin [577] proved the existence of uniform attractors for a nonlinear non-autonomous damped extensible plate or beam equation in a bounded or unbounded domain $\Omega \subseteq \mathbb{R}^n$, by establishing the uniformly asymptotic compactness of the semi-process generated by the global solutions.

Recently, Qin, Ren and Wei [803] considered a non-autonomous thermoelastic system with boundary delay and using the contractive mapping method and multiplier techniques established the asymptotic behavior and the existence of a uniform attractor of the system. Kirane and Said-Houari [437] considered problem (3.4.1) with $f = 0$ and obtained exponential decay results, who used the energy method and Lyapunov functionals to show that the energy of the solution decreases exponentially as time tends to infinity.

For the relaxation function g, we assume

(H1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying

$$g(0) > 0, \quad 1 - \int_0^{+\infty} g(s)ds = l > 0. \tag{3.4.2}$$

(H2) There exists a positive non-increasing differentiable function $\eta(t)$ such that for all $t \geq 0$,

$$-g'(t) \geq \eta(t)g(t), \quad \int_0^{+\infty} \eta(t)dt = +\infty. \tag{3.4.3}$$

Now we introduce the following notations

$$\begin{cases} (\phi * \psi)(t) := \int_0^t \phi(t-\tau)\psi(\tau)d\tau, \\ (\phi \diamond \psi)(t) := \int_0^t \phi(t-\tau)|\psi(t) - \psi(\tau)|d\tau, \\ (\phi \circ \psi)(t) := \int_0^t \phi(t-\tau) \int_\Omega |\psi(t) - \psi(\tau)|^2 dxd\tau. \end{cases}$$

Following [437, 619, 649], we need the following lemmas which will be used later in order to define the new modified functional energy of problem (3.4.1).

Lemma 3.4.1. *For any function* $\phi \in C^1(\mathbb{R})$ *and any* $\psi \in H^1(0,T)$,

$$\begin{aligned}(\phi * \psi)(t)\psi_t(t) &= -\frac{1}{2}\phi(t)|\psi(t)|^2 + \frac{1}{2}(\phi' \diamond \psi)(t) \\ &= -\frac{1}{2}\frac{d}{dt}\left\{(\phi \diamond \psi)(t) - \left(\int_0^t \phi(\tau)d\tau\right)|\psi(t)|^2\right\}.\end{aligned} \tag{3.4.4}$$

Proof. See, e.g., [649]. □

Lemma 3.4.2. *For any* $u \in H_0^1(\Omega)$,

$$\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 \leq (1-l)C_*^2(g \circ \nabla u)(t), \tag{3.4.5}$$

where $C_* > 0$ *is the Poincaré constant and* l *is given in* (H1).

Proof. See, e.g., [619]. □

Now following [437], we shall prove the global existence and uniqueness of the solution of problem (3.4.1). In order to state our main result, we introduce the new variable

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad x \in \Omega, \quad \rho \in (0,1), \quad t > 0. \tag{3.4.6}$$

Then

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in} \quad \Omega \times (0,1) \times (0,+\infty). \tag{3.4.7}$$

Then problem (3.4.1) is equivalent to the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds & \\ \quad +\mu_1 u_t(x,t) + \mu_2 z(x,1,t) = f(x,t), & x \in \Omega,\ t > 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in \Omega,\ \rho \in (0,1),\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ z(x,0,t) = u_t(x,t), & x \in \Omega,\ t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ z(x,\rho,0) = f_0(x, t-\tau), & x \in \Omega,\ t \in (0,\tau). \end{cases} \tag{3.4.8}$$

Next, we shall give a sufficient condition that guarantees that this problem is globally well posed. Let ξ and ϵ be positive constants such that

$$\tau\mu_2 - 2\tau\epsilon < \xi < \tau(2\mu_1 - \mu_2). \tag{3.4.9}$$

Now we can state the following global existence result.

Theorem 3.4.3 ([803])**.** *For any* $T > 0$, *assume that* $f(x,t) \in L^2(0,T;L^2(\Omega))$, $\mu_1 \leq \mu_2$, *and* (H1), (H2) *are satisfied. Then for any* $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Omega \times (0,1))$, *problem* (3.4.9) *admits a unique weak solution* (u,z) *on* $(0,T)$ *such that*

$$\begin{aligned} &u \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)), \\ &u_t \in L^2(0,T; H_0^1(\Omega)) \cap L^2((0,T) \times \Omega). \end{aligned} \tag{3.4.10}$$

Proof. Similarly to [437] and [546], noting that $f(x,t) \in L^2(0,T;L^2(\Omega))$, we can easily prove the theorem. $\square$

We now use the energy method and suitable Lyapunov functionals to show that the energy of the solution of problem (3.4.1) decreases exponentially as time tends to infinity. Following [437], we shall discuss two cases: $\mu_2 < \mu_1$ and $\mu_2 = \mu_1$.

Case 1: $\boldsymbol{\mu_2 < \mu_1}$.

In this case, we shall show that under the assumption $\mu_2 < \mu_1$, the solution of problem (3.4.1) is exponentially stable. For this purpose, we need to establish first several lemmas.

Lemma 3.4.4 ([803])**.** *Let* (u, z) *be a solution of problem* (3.4.8)*. Suppose* (H1) *and* (H2) *are satisfied and* ξ *satisfies the inequality* (3.4.9)*. Then the energy functional defined by*

$$\begin{aligned} E(t) = E(t,z,u) = &\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 \\ &+ \frac{1}{2}(g \circ \nabla u)(t) + \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \end{aligned} \tag{3.4.11}$$

satisfies, for some positive constants M_1 *and* C_ε*,*

$$\begin{aligned} \frac{dE(t)}{dt} \le &- M_1\left(\int_\Omega u_t^2(x,t)dx + \int_\Omega z^2(x,1,t)dx\right) \\ &+ \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 + C\int_\Omega f^2dx. \end{aligned} \tag{3.4.12}$$

Proof. Similarly to [437], we can easily obtain

$$\begin{aligned} \frac{dE(t)}{dt} = &-\left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} - \widetilde{\varepsilon}\right)\|u_t\|_2^2 - \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_\Omega z^2(x,1,t)dx \\ &+ \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 + C\int_\Omega f^2dx \end{aligned} \tag{3.4.13}$$

where $\widetilde{\varepsilon}$ and $C = C(\widetilde{\varepsilon})$ are positive constants with $\widetilde{\varepsilon}$ small enough. Then using (3.4.9), we obtain (3.4.12). $\square$

Lemma 3.4.5 ([803])**.** *Let* (u, z) *be the solution of problem* (3.4.8)*, the function* F_1 *defined by*

$$F_1(t) = \int_\Omega u_t u dx \tag{3.4.14}$$

satisfies, for any $\delta > 0$*,*

$$\begin{aligned} \frac{dF_1(t)}{dt} \le &\left(1 + \frac{\mu_1}{4\delta}\right)\|u_t\|_2^2 - \left(\frac{l}{2} - \delta C_*^2(\mu_1 + \mu_2 + 1)\right)\|\nabla u\|_2^2 \\ &+ \frac{1}{4\delta}\int_\Omega\left[z^2(x,1,t) + f^2(x,t)\right]dx + \frac{(1-l)}{2}(g \circ \nabla u)(t). \end{aligned} \tag{3.4.15}$$

Proof. By the first equation in (3.4.8) and a direct computation, we have

$$\begin{aligned} F_1'(t) = \|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)\nabla u(s)dsdx \\ - \mu_1 \int_\Omega u_t u dx - \mu_2 \int_\Omega z(x,1,t)udx + \int_\Omega fudx. \end{aligned} \tag{3.4.16}$$

By the Young and Poincaré inequalities, we have for any $\delta > 0$,

$$\int_\Omega f(x,t)udx \le \delta C_*^2 \int_\Omega |\nabla u(t)|^2 dx + \frac{1}{4\delta}\int_\Omega f^2(x,t)dx. \tag{3.4.17}$$

Combining the estimates in [437], we can easily derive the desired result (3.4.15). □

Lemma 3.4.6. *Let* (u, z) *be the solution of problem* (3.4.8). *Then the function* F_2 *defined by*

$$F_2(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} z^2 d(x,\rho,t)d\rho dx \tag{3.4.18}$$

satisfies

$$\frac{dF_2(t)}{dt} \le -\rho F_2(t) + \frac{1}{2\tau}\int_\Omega u_t^2(x,t)dx - \frac{c}{2\tau}\int_\Omega z^2(x,1,t)dx. \tag{3.4.19}$$

Proof. See, e.g., [437]. □

We shall employ Theorem 2.3.2 to establish the following result. To this end, we need to construct a suitable Lyapunov functional $\mathcal{L}$.

Theorem 3.4.7 ([803])**.** *Let* u *be the solution of problem* (3.4.1). *Assume that* $\mu_2 < \mu_1$, g *and* η *satisfies* (H1) *and* (H2).

(1) *If* $f(x,t) \in L^2(\mathbb{R}^+, L^2(\Omega))$ *and* $\lim_{t\to+\infty} \frac{\|f\|_2^2}{\eta(t)} = 0$, *then*

$$\lim_{t\to+\infty} E(t) = 0. \tag{3.4.20}$$

Furthermore,

(2) *if* f *satisfies* $\|f\|_2^2 \le C_1 e^{-\alpha_0 \int_0^t \eta(s)ds}$ *for all* $t \ge 0$, *where* $C_1 > 0, \alpha_0 > 0$ *are constants, then there exists a constant* $R_1 > 0$, *such that for* $t \ge R_1$,

$$E(t) \le C_2 e^{-\alpha \int_0^t \eta(s)ds} \tag{3.4.21}$$

where $C_2 > 0$, $\alpha > 0$ *are constants.*

Proof. Note that we need to construct a functional $\mathcal{L}(t)$, equivalent to the energy $E(t)$, satisfying for all $t \geq 0$,

$$\frac{d\mathcal{L}(t)}{dt} \leq -\sigma\mathcal{L}(t) + \sigma_0 \parallel f \parallel_2^2, \tag{3.4.22}$$

where σ, σ_0 are positive constants. Specifically, we may define the Lyapunov functional

$$\mathcal{L}(t) := E(t) + NF_1(t) + NF_2(t) \tag{3.4.23}$$

where N is a positive real number which will be chosen later. Letting

$$\mathcal{F}(t) = \eta(t)\mathcal{L}(t) + 2\gamma_2 E(t), \tag{3.4.24}$$

which is equivalent to $E(t)$, similarly to [437], we easily obtain

$$\mathcal{F}'(t) \leq -\gamma_1\eta(t)E(t) + \eta(0)\tilde{C}\|f\|_2^2 \leq -\gamma_3\eta(t)\mathcal{F}(t) + \eta(0)\tilde{C}\|f\|_2^2. \tag{3.4.25}$$

Applying Theorem 2.3.2 to (3.4.25), with $y(t) = \mathcal{F}(t)$, $\lambda(t) = \eta(0)\tilde{C}\|f\|_2^2$ and $h(t) = \eta(t)$, we can conclude (3.4.20) and (3.4.21). □

Case 2: $\boldsymbol{\mu_2 = \mu_1}$.

Below we shall show that under assumption $\mu_2 = \mu_1$, the solution of problem (3.4.1) is exponentially stable.

Lemma 3.4.8 ([803])**.** *Let* (u, z) *be a solution of problem* (3.4.8)*. Suppose* (H1) *and* (H2) *are satisfied. Then the energy functional defined by* (3.4.11) *satisfies for any* $\widetilde{\varepsilon} > 0$ *and for all* $t \geq 0$*,*

$$\frac{dE(t)}{dt} \leq \widetilde{\varepsilon}\|u_t\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 + C\int_\Omega f^2 dx. \tag{3.4.26}$$

Proof. The proof is an immediate consequence of Lemma 3.4.2. Choosing $\xi = \tau\mu$, we can easily complete the proof. □

Lemma 3.4.9 ([803])**.** *Let* (u, z) *be the solution of problem* (3.4.8)*. Then the function* F_3 *defined by*

$$F_3(t) = -\int_\Omega u_t \int_0^t g(t-s)(u(t) - u(s))dsdx \tag{3.4.27}$$

satisfies

$$\begin{aligned} \frac{dF_3(t)}{dt} \leq\ & \left(r + 2r(1-l)^2\right)\|\nabla u\|_2^2 + \left(r_1(1+r) - \int_0^t g(s)ds\right)\|u_t\|_2^2 \\ & + \left(\frac{(1-l)}{2r} + 2r(1-l) + \frac{rC_*^2}{4r_1} + \frac{rC_*^2}{4r_2} + \frac{rC_*^2}{4r_3}\right)(g \circ \nabla u)(t) \\ & + rr_2\int_\Omega z^2(x,1,t)dx - \frac{g(0)}{4r_1}C_*^2(g' \circ \nabla u)(t) + r_3\int_\Omega f^2(x,t)dx, \end{aligned} \tag{3.4.28}$$

where r*,* r_1*,* r_2*, and* r_3 *are positive constants.*

Proof. Differentiating (3.4.27) with respect to t and using the first equation in (3.4.8), we have

$$\begin{aligned}F_3'(t) = &\int_\Omega \nabla u(t)\cdot\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\\
&-\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\\
&-\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g(s)ds\right)\|u_t\|_2^2\\
&-\mu_1\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))dsdx \qquad (3.4.29)\\
&-\mu_2\int_\Omega z(x,1,t)\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&+\int_\Omega f(x,t)\int_0^t g(t-s)(u(t)-u(s))dsdx.\end{aligned}$$

Using Young's inequality in the last term of (3.4.29), we have for any $r_3>0$,

$$\int_\Omega f(x,t)\int_0^t g(t-s)(u(t)-u(s))dsdx \le r_3\int_\Omega f^2(x,t)dx+\frac{C_*^2}{4r_3}(g\circ\nabla u)(t). \quad (3.4.30)$$

Combining this with the estimates in [437] with (3.4.30), we can complete the proof. □

Theorem 3.4.10 ([803])**.** *Let u be the solution of problem* (3.4.1)*. Assume that $\mu_2=\mu_1$, and that g satisfies* (H1) *and* (H2)*.*

(1) *If $f(x,t)\in L^2(\mathbb{R}^+,L^2(\Omega))$, $\lim_{t\to+\infty}\frac{\|f\|_2^2}{\eta(t)}=0$, then*

$$\lim_{t\to+\infty}E(t)=0. \quad (3.4.31)$$

Furthermore,

(2) *if f satisfies $\|f\|_2^2\le K_1e^{-k_0\int_0^t\eta(s)ds}$ for all $t\ge 0$, where $K_1>0$, $k_0>0$ are constants, then there exists a constant $R_3>0$, such that for $t\ge R_3$,*

$$E(t)\le Ke^{-k\int_0^t\eta(s)ds}, \quad (3.4.32)$$

with $K>0$, $k>0$ being constants.

Proof. We first define the Lyapunov function as

$$\widetilde{\mathcal{L}}(t):=aE(t)+a_1F_1(t)+a_2F_2(t)+F_3(t), \quad (3.4.33)$$

where a, a_1. and a_2 are positive real numbers which will be chosen later. Since the function g is positive, continuous, and satisfies $g(0) > 0$, for any fixed $t_0 > 0$, as $t \geq t_0 \geq 0$,

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds \equiv g_0. \tag{3.4.34}$$

Similarly to [437], we get for all $t \geq t_0$,

$$\widetilde{\mathcal{L}}'(t) \leq -\alpha' \widetilde{\mathcal{L}}(t) + \alpha_2 (g \circ \nabla u)(t) + \alpha_3 \|f\|_2^2 \tag{3.4.35}$$

where α', α_2 and α_3 are positive constants; α_3 is independent of initial data.

Arguing as in the proof of Theorem 3.4.2, we obtain the results. □

3.4.2 Uniform attractors

In this subsection, we shall establish the existence of uniform attractors for the system (3.4.8) with right-hand side $f(x,t)$.

Setting $v = u_t, \mathbb{R}_\epsilon = [\epsilon, +\infty), \epsilon \geq 0$, we consider the system

$$\begin{cases} u_t(x,t) - v(x,t) = 0, & (x,t) \in \Omega \times (\epsilon, +\infty), \\ u_{tt}(x,t) - \Delta u(x,t) + \int_0^1 g(t-s)\Delta u(x,s)ds & \\ \qquad +\mu_1 u_t(x,t) + \mu_2 z(x,1,t) = f(x,t), & (x,t) \in \Omega \times (\epsilon, +\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & (x,\rho,t) \in \Omega \times (0,1) \times (\epsilon, +\infty), \end{cases} \tag{3.4.36}$$

subject to the conditions

$$\begin{cases} u(x,\epsilon) = u_\epsilon(x), \quad u_t(x,\epsilon) = u_{1\epsilon}(x), & \\ \qquad u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega, \quad \epsilon \geq 0, & (3.4.37) \\ z(\rho,t) = z_\epsilon(\rho), \qquad \rho \in (0,1), & (3.4.38) \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq \epsilon, & (3.4.39) \\ z(x,0,t) = u_t(x,t), \quad x \in \Omega, \quad t \geq \epsilon. & (3.4.40) \end{cases}$$

Let

$$F = (0,f,0)^T \in X \equiv L^2\left(\mathbb{R}_\epsilon, (L^2(\Omega))^3\right), \;\; \mathcal{H} = H_0^1(\Omega) \times L^2(\Omega). \tag{3.4.41}$$

Then the energy of problem (3.4.36) is given by

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{\xi}{2}\int_\Omega \int_0^1 z^2(x,\rho,t)d\rho dx.$$

For any $(u_\epsilon, u_{1\epsilon}) \in \mathcal{H}$ and any $F \in X$, we define for all $t \geq \epsilon, \epsilon \geq 0$,

$$U_F(t,\epsilon) : (u_\epsilon, u_{1\epsilon}) \in \mathcal{H} \mapsto (u(t), u_t(t)) = U_F(t,\epsilon)(u_\epsilon, u_{1\epsilon})$$

where $(u(t), u_t(t))$ is the solution of problem (3.4.36).

For the proof of our result on the uniform attractors in $\mathcal{H}$, we define the hull of F_0 as

$$\Sigma = \mathcal{H}(F_0) = [F_0(t+h)|h \in \mathbb{R}^+]_Y \tag{3.4.42}$$

where $[\cdot]_Y$ denotes the closure in some Banach space Y.

We note that

$$F_0 \in X \subseteq \widehat{X} = L^2_{\text{loc}}(\mathbb{R}^+, (L^2(\Omega))^3),$$

where F_0 is a translation compact function in $\widehat{X}$ in the weak topology, which means that $\mathcal{H}(F_0)$ is compact in $\widehat{X}$. We now consider the Banach space $L^p_{\text{loc}}(\mathbb{R}^+, X_1)$ of functions $\sigma(s)$, $s \in \mathbb{R}^+$, with values in a Banach space X_1 that are locally p-power integrable in the Bochner sense. In particular, for any time interval $[t_1, t_2] \subseteq \mathbb{R}^+$,

$$\int_{t_1}^{t_2} \| \sigma(s) \|_{X_1}^p \, ds < +\infty.$$

Let $\sigma(s) \in L^p_{\text{loc}}(\mathbb{R}^+, X_1)$, and consider the quantity

$$\kappa_\sigma(h) = \sup_{t \in \mathbb{R}^+} \int_t^{t+h} \| \sigma(s) \|_{X_1}^p \, ds.$$

Lemma 3.4.11 ([803])**.** *Let Σ be defined as before and $F_0 \in X$. Then*

(1) *F_0 is a translation compact function in $\widehat{X}$ and any $F \in \Sigma = \mathcal{H}(F_0)$ is also a translation compact function in $\widehat{X}$. Moreover, $\mathcal{H}(F) \subseteq \mathcal{H}(F_0)$;*
(2) *the set $\mathcal{H}(F_0)$ is bounded in $L^2(\mathbb{R}^+, (L^2(\Omega))^3)$ such that for all $F \in \Sigma$,*

$$\kappa_F(h) \leq \kappa_{F_0}(h) < +\infty.$$

Proof. See, e.g., Chepyzhov and Vishik [154]. □

Similarly to Theorem 3.4.1, we have the following existence and uniqueness result.

Theorem 3.4.12 ([803])**.** *Let $\Sigma = [F_0(t+h)|h \in \mathbb{R}^+]_X$, where $F_0 \in X$ is an arbitrary but fixed symbol function. Then for any $F \in \Sigma$ and any $(u_\epsilon, u_{1\epsilon}) \in \mathcal{H}, \epsilon \geq 0$, problem* (3.4.36) *admits a unique global solution $(u(t), u_t(t)) \in \mathcal{H}$, which generates a unique family of semiprocesses $\{U_F(t,\epsilon)\}$ $(t \geq \epsilon, \epsilon \geq 0)$ on $\mathcal{H}$, namely a two-parameter family of operators, such that for any $t \geq \epsilon, \epsilon \geq 0$,*

$$U_F(t,\epsilon)(u_\epsilon, u_{1\epsilon}) = (u(t), u_t(t)) \in \mathcal{H},$$
$$u(t) \in C\left(\mathbb{R}_\epsilon, H_0^1(\Omega)\right), \;\; u_t \in C\left(\mathbb{R}_\epsilon, L^2(\Omega)\right).$$

In order to prove our results, we shall introduce some basic concepts and basic lemmas.

Definition 3.4.13. Let Y be a Banach space and $\widehat{\Sigma}$ be a parameter set. The operators $\{U_\sigma(t,\epsilon)\}$ $(t \geq \epsilon, \epsilon \geq 0, \sigma \in \widehat{\Sigma})$ are said to form a family of semi-processes in Y with symbol space $\widehat{\Sigma}$, if for any $\sigma \in \widehat{\Sigma}$

$$U_\sigma(t,s)U_\sigma(s,\epsilon) = U_\sigma(t,\epsilon), \quad \text{for all } t \geq s \geq \epsilon, \ \ \epsilon \geq 0, \tag{3.4.43}$$

$$U_\sigma(\epsilon,\epsilon) = \mathrm{Id} \ \text{(identity operator), for all } \epsilon \geq 0. \tag{3.4.44}$$

Definition 3.4.14. A set B_0 is said to be a uniformly (with respect to (w.r.t.) $F \in \widehat{\Sigma}$) absorbing set for the family of semi-processes $\{U_F(t,\epsilon)\}$ $(F \in \widehat{\Sigma}, t \geq \epsilon, \epsilon \geq 0)$, if for every bounded set B of X and any $\epsilon \geq 0$, there exists some time $t_0 = t_0(B,\epsilon) \geq \epsilon$, such that for all $t > t_0$,

$$\bigcup_{F \in \widehat{\Sigma}} U_F(t,\epsilon)B \subseteq B_0. \tag{3.4.45}$$

Definition 3.4.15. The family of semi-processes $\{U_F(t,\epsilon)\}$ $(F \in \widehat{\Sigma}, t \geq \epsilon, \epsilon \geq 0)$ is said to be asymptotically compact in Y if $\left\{U_{F_n}(t_n,\epsilon)(u_\epsilon^{(n)}, u_{1\epsilon}^{(n)})\right\}$ is pre-compact in Y whenever $\left\{u_\epsilon^{(n)}, u_{1\epsilon}^{(n)}\right\}$ is bounded in Y, $F^{(n)} \subset \widehat{\Sigma}$, and $\{t_n\} \subset \mathbb{R}_\epsilon, t_n \to +\infty$ as $n \to +\infty$.

Definition 3.4.16. A closed set $\mathcal{A} \subseteq Y$ is called uniformly (w.r.t. $F \in \widehat{\Sigma}$) if for any bounded set P of B and any fixed $\epsilon \in \mathbb{R}^+$,

$$\lim_{t \to +\infty} \left(\sup_{F \in \widehat{\Sigma}} \mathrm{dist} \left(U_F(t,\epsilon)P, \mathcal{A} \right) \right) = 0; \tag{3.4.46}$$

here $\mathrm{dist}(\cdot,\cdot)$ stands for the usual Hausdorff semi-distance between two sets P_1 and P_2 in Y, defined as $dist(P_1,P_2) = \sup\limits_{x_1 \in P_1} \inf\limits_{x_2 \in P_2} \|x_1 - x_2\|_Y$.

Definition 3.4.17. A closed set $\mathcal{A} \subseteq Y$ is called the uniform (w.r.t. $F \in \widehat{\Sigma}$) attractor of the semi-processes $\{U_F(t,\epsilon)\}$ $(F \in \widehat{\Sigma}, t \geq \epsilon, \epsilon \geq 0)$ acting on Y if $\mathcal{A}$ is a uniformly attracting set and $\mathcal{A}$ satisfies the following minimality property: $\mathcal{A}$ belongs to any closed uniformly attracting set of the semi-processes $\{U_F(t,\epsilon)\}$ $(F \in \widehat{\Sigma}, t \geq \epsilon, \epsilon \geq 0)$.

Definition 3.4.18. Let Y be a Banach space and P be a bounded subset of Y, $\widehat{\Sigma}$ be a symbol space. We call a function $\phi(\cdot,\cdot;\cdot,\cdot)$, defined on $(Y \times Y) \times (\widehat{\Sigma} \times \widehat{\Sigma})$, a contractive function on $P \times P$ if, for any sequence $\{x_n\}_{n=1}^\infty \subset P$ and any $\{\sigma_n\} \subseteq \widehat{\Sigma}$, there are subsequences $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ and $\{\sigma_{n_k}\}_{k=1}^\infty \subset \{\sigma_n\}_{n=1}^\infty$ such that

$$\lim_{k \to +\infty} \lim_{l \to +\infty} \phi(x_{n_k}, x_{n_l}; \sigma_{n_k}, \sigma_{n_l}) = 0.$$

We denote the set of all contractive functions on $Y \times Y$ by $\mathrm{Contr}(P, \widehat{\Sigma})$.

More details on the subject of uniform attractors can be found in the books [45, 154, 577, 770, 773, 774].

Lemma 3.4.19. *Let* $\{U_F(t,\epsilon)\}\,(F\in\widehat{\Sigma}, t\geq\epsilon,\epsilon\geq 0)$ *be a family of semi-processes that satisfies the translation identities* (3.4.43) *and* (3.4.44) *on a Banach space* Y*, and has a bounded uniformly (w.r.t.* $F\in\widehat{\Sigma}$*) absorbing set* $B_0\subseteq Y$*. Moreover, assuming that for any* $\varepsilon>0$*, there exist a time* $T=T(B_0,\varepsilon)>0$ *and a contractive function* ϕ_T *on* $B_0\times B_0$ *such that for all* $x,y\in B_0$*, for all* $F_1,F_2\in\widehat{\Sigma}$*,*

$$\| U_{F_1}(T,0)x-U_{F_2}(T,0)y \|\leq\varepsilon+\phi_T(x,y;F_1,F_2).$$

Then $\{U_F(t,\epsilon)\}\,(F\in\widehat{\Sigma}, t\geq\epsilon,\epsilon\geq 0)$ *is uniformly* $\big($*w.r.t.* $F\in\widehat{\Sigma}\big)$ *asymptotically compact in* Y*.*

Proof. See, e.g., [154]. □

First we shall establish that the family of semi-processes $\{U_F(t,\epsilon)\}$ has a bounded uniformly absorbing set.

Theorem 3.4.20 ([803])**.** *Under the assumption* (3.4.41)*, the family of semi-processes* $\{U_F(t,\epsilon)\}$ $(F\in\Sigma, t\geq\epsilon,\epsilon\geq 0)$*, corresponding to problem* (3.4.36)–(3.4.40) *has a bounded uniformly (w.r.t.* $F\in\Sigma$*) absorbing set* B_0 *in* $\mathcal{H}$*.*

Proof. Similarly to the proof of Theorem 3.4.2, we have

$$\frac{dE(t)}{dt}\leq-\gamma\eta(t)E(t)+C_1\parallel f(t)\parallel_2^2,$$

where γ, C_1 are two positive constants independent of initial data.

In the following, C denotes a general positive constant independent of initial data, which may be different in different estimates.

Obviously,

$$E(t)\leq E(\epsilon)e^{-\gamma\int_\epsilon^t\eta(w)dw}+C\int_\epsilon^t\parallel f(s)\parallel_2^2 e^{-\gamma\int_s^t\eta(w)dw}ds. \tag{3.4.47}$$

Applying Theorem 2.1.7 to (3.4.49), we can conclude that

$$\begin{aligned}E(t)&\leq E(\epsilon)e^{-\gamma\int_\epsilon^t\eta(w)dw}+C\int_\epsilon^t\parallel F(s)\parallel_2^2 e^{-\gamma\int_s^t\eta(w)dw}ds\\&\leq E(\epsilon)e^{-\gamma\int_\epsilon^t\eta(w)dw}+C\frac{1}{1-e^{-\gamma\eta(0)}}\sup_{t\geq\epsilon}\int_t^{t+1}\parallel F(s)\parallel_2^2 ds\\&\leq E(\epsilon)e^{-\gamma\int_\epsilon^t\eta(w)dw}+C\frac{1}{1-e^{-\gamma\eta(0)}}\kappa_{F_0}(1).\end{aligned}$$

Now for any bounded set $B_0\subseteq\mathcal{H}$, for any $(u_\epsilon,u_{1\epsilon})\in B_0$, and $\epsilon\geq 0$, there exists a constant $C_{B_0}>0$ such that $E(\epsilon)\leq C_{B_0}\leq C$. Take

$$R_0^2=2\left(2C\frac{\kappa_{F_0}(1)}{1-e^{-\gamma\eta(0)}}+1\right),$$

$$t_0=t_0(\epsilon,F_0),\quad\int_\epsilon^{t_0}\eta(w)dw=-\gamma^{-1}\log\left(\frac{C\kappa_{F_0}(1)+1}{CC_{B_0}(1-e^{-\gamma\eta(0)})}\right).$$

Then for any $t \geq t_0 \geq \epsilon$, we have

$$E(t) \leq C\frac{\kappa_{F_0}(1)}{1-e^{-\gamma\eta(0)}} + CC_{B_0}e^{-\gamma\int_\epsilon^{t_0}\eta(w)dw} \leq \frac{R_0^2}{2},$$

which thus gives us

$$\| (u(t), u_t(t)) \|_{\mathcal{H}}^2 \leq 2E(t) \leq R_0^2.$$

Then, $B_0(0, R_0) = \{u(t), u_t(t) \in \mathcal{H} : \| (u(t), u_t(t)) \|_{\mathcal{H}} \leq R_0\} \subseteq \mathcal{H}$ is a uniformly absorbing set for any $F \in \Sigma$, i.e., for any bounded subset B in $\mathcal{H}$, there exists a time $t_0 = t_0(\epsilon, F_0) \geq \epsilon$, such that for all $t \geq t_0$,

$$\bigcup_{F\in\Sigma} U_F(t, \epsilon)B \subseteq B_0.$$

The proof is thus complete. □

Now following [803], we can prove that the family of semi-processes $\{U_F(t, \epsilon)\}$ ($F \in \Sigma$, $t \geq \epsilon$, $\epsilon \geq 0$), corresponding to (3.4.37) is uniformly (w.r.t. $F \in \Sigma$) asymptotically compact in $\mathcal{H}$. Hence, by the standard existence theory of uniform attractors, we can establish the existence of uniformly (w.r.t. $F \in \Sigma$) compact attractor $\mathscr{A}_F$.

Chapter 4

Global Existence and Uniqueness for Evolutionary PDEs

In this chapter, we present some results on global existence and uniqueness of solutions to evolutionary PEDs obtained by application of analytic inequalities in Chapters 1 and 2. This chapter consists of four sections. In Section 4.1, we use the simultaneous singular Bellman–Gronwall inequality, i.e., Theorem 1.3.2, to discuss the local existence, regularity, and continuous dependence on initial data of solutions to a weakly coupled parabolic system for non-regular initial data. In Section 4.2, we use Theorem 1.4.9 to study some properties of solutions to the Cauchy problem for multi-dimensional conservation laws with anomalous diffusion. In Section 4.3, we use Theorem 2.1.19 to investigate the blow-up of solutions of semilinear heat equations. In Section 4.4, we exploit Theorems 2.1.17 and 2.1.18 to establish the global existence, L^∞ estimates, and decay estimates of solutions for the quasilinear parabolic system. Inequalities used in this chapter are crucial in obtaining the global solutions.

4.1 A weakly coupled Parabolic system

In this section, we use Theorem 1.3.2 to establish the local existence, regularity, and continuous dependence on initial data of solutions to a weakly coupled parabolic system for non-regular initial data. These results are taken from Dickstein and Loayza [210].

We shall consider the following weakly coupled parabolic system for non-regular initial data

$$\begin{cases} u_t - \Delta u = v^p \quad \text{in } \ (0,T) \times \mathbb{R}^N, & (4.1.1) \\ v_t - \Delta v = u^q \quad \text{in } \ (0,T) \times \mathbb{R}^N, & (4.1.2) \\ u(0) = u_0, \ \ v(0) = v_0 \ \text{ in } \ \mathbb{R}^N, & (4.1.3) \end{cases}$$

where $p, q \geq 1$, $pq > 1$, and $u_0, v_0 \in C_0(\mathbb{R}^N)$.

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It is well known (see [210]) that the problem (4.1.1)–(4.1.3) has a unique classical solution $w(t) = (u(t), v(t))$ defined on a maximal interval $[0, T)$, $T \le +\infty$. Escobedo and Herrero [237] proved that when $T < +\infty$, then $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \to +\infty$ and $\|v(t)\|_{L^\infty(\mathbb{R}^N)} \to +\infty$ as $t \to T^-$. In this case, we say that w blows up at the blow-up time T. We also know that blow-up phenomena for semilinear parabolic systems in $\mathbb{R}^N$ have been studied by several authors, see, e.g., Andreucci, Herrero ad Velázquez [47], Dickstein and Loayza [210], Deng and Levine [201], Escobedo and Herrero [237], Levine [514], Quittner and Souplet [817], Zaag [987], etc.

We need to introduce first some notations. L^r stands for the Lebesgue space $L^r(\mathbb{R}^N)$ and $\|\cdot\|_{L^r}$ denotes its usual norm. Thus by the results from Brézis and Cazenave [118] and Weissler [955], the following problem is locally well posed in $L^r(\mathbb{R}^N)$ when $r \ge 1$ and $2r > N(p-1)$ or $r > 1$ and $2r = N(p-1)$, and the solution to the following nonlinear problem is classical for all $t > 0$:

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u \ \text{ in } \ (0,T) \times \mathbb{R}^N, & (4.1.4) \\ \qquad u(0) = u_0 \qquad \text{ in } \ \mathbb{R}^N. & (4.1.5) \end{cases}$$

Now following [210], consider $E^{r,s} \equiv L^r + L^s$, which is the Banach space endowed with the standard norm

$$\|u\|_{r,s} = \inf \left\{ \|u_r\|_{L^r} + \|u_s\|_{L^s},\ u = u_r + u_s,\ u_r \in L^r, u_s \in L^s \right\}.$$

Recall that Dickstein [208] showed problem (4.1.4)–(4.1.5) is also locally well posed in $E^{r,s}$ provided that $r \ge 1$ and $2r > N(p-1)$. Existence of solutions (regular for all $t > 0$) was also obtained for initial data $u_0 \in \mathcal{M}$, the space of finite measures in $\mathbb{R}^N$.

The first fact we need is that the heat semigroup $S(t)$ is defined in $E^{r,s}$ and satisfies if $1 \le r \le s \le \gamma$,

$$\sup_{0<t\le T} \|S(t)u\|_{L^\gamma} \le Ct^{-(N/2)(1/r-1/\gamma)}\|u\|_{r,s}, \tag{4.1.6}$$

where $C = \max\left(1, T^{(N/2)(1/r-1/s)}\right)$.

We also need the following lemma.

Lemma 4.1.1 ([210]). *Assume that* $0 < p, 0 < q$ *such that* $pq > 1$, *and* $\gamma_{1,0} > 0, \gamma_{2,0} > 0$, $0 < c_1 < d_1, 0 < c_2 < d_2$ *such that*

$$\frac{p}{\gamma_{2,0}} - \frac{1}{\gamma_{1,0}} = c_1, \quad \frac{q}{\gamma_{1,0}} - \frac{1}{\gamma_{2,0}} = c_2, \tag{4.1.7}$$

for any $n \in \mathbb{N}$, *define* $\gamma_{1,0}, \gamma_{2,0}$ *recursively by*

$$\frac{p}{\gamma_{2,n}} - \frac{1}{\gamma_{1,n+1}} = d_1, \quad \frac{q}{\gamma_{1,n}} - \frac{1}{\gamma_{2,n+1}} = d_2. \tag{4.1.8}$$

Then $\{\gamma_{1,n}\}_{n\in\mathbb{N}}, \{\gamma_{2,n}\}_{n\in\mathbb{N}}$ *are increasing sequences such that* $\frac{p}{\gamma_{2,N}} < d_1$, $\frac{q}{\gamma_{1,N}} < d_2$ *for some* $N \in \mathbb{N}$.

For the proof of Lemma 4.1.1, see [210]. □

We now discuss the admissibility of $E^{r_1,s_1} \times E^{r_2,s_2}$ as a space of solutions of problem (4.1.1)–(4.1.3). For $w = (u,v) \in E^{r_1,s_1} \times E^{r_2,s_2}$, we set $\|w\|_{r_1,s_1,r_2,s_2} = \|u\|_{r_1,s_1} + \|v\|_{r_2,s_2}$.

Theorem 4.1.2 ([210]). *Assume that $p \geq 1$, $q \geq 1$, $1 \leq r_1 \leq s_1 \leq qr_2$, $1 \leq r_2 \leq s_2 \leq pr_1$ such that*

$$\max\Big(p/r_2 - 1/r_1, q/r_1 - 1/r_2\Big) < 2/N. \tag{4.1.9}$$

Then for any $w_0 \in E^{r_1,s_1} \times E^{r_2,s_2}$, there exist a time $T > 0$ and a unique classical solution $w \in C([0,T); E^{r_1,s_1} \times E^{r_2,s_2})$ to problem (4.1.1)–(4.1.3) in $(0,T)$.

Let B_M be the closed ball of radius $M > 0$ of $E^{r_1,s_1} \times E^{r_2,s_2}$. For any $w_0, z_0 \in E^{r_1,s_1} \times E^{r_2,s_2}$, denote by $w = (u_w, v_w), z = (u_z, v_z)$ the corresponding solutions. Given $M > 0$, there exists a time $T > 0$ such that for all $w_0, z_0 \in B_M$ and for any $\gamma_1 \in [s_1, +\infty], \gamma_2 \in [s_2, +\infty]$, we can find a constant $C > 0$ such that for all $t \in (0,T)$,

$$\begin{aligned} &t^{(N/2)(1/r_1 - 1/\gamma_1)}\|u_w(t) - u_z(t)\|_{L^{\gamma_1}} + t^{(N/2)(1/r_2 - 1/\gamma_2)}\|v_w(t) - v_z(t)\|_{L^{\gamma_2}} \\ &\quad \leq C\|w_0 - z_0\|_{r_1,s_1,r_2,s_2}. \end{aligned} \tag{4.1.10}$$

Proof. Since the proof is only based on the fixed point argument due to Weissler [955], we shall just present the details of the novelties appearing here, which can be divided into the following several steps.

Step 1. Existence of a Weak Solution: For any $w_0 \in E^{r_1,s_1} \times E^{r_2,s_2}$, $T < 1$, and $M > 0$ such that $\|u_0\|_{r_1,s_1} + \|v_0\|_{r_2,s_2} \leq M$, let

$$\alpha_1 = \frac{N}{2}\Big(1/r_1 - 1/(qr_2)\Big), \quad \alpha_2 = \frac{N}{2}\Big(1/r_2 - 1/(pr_1)\Big), \tag{4.1.11}$$

and

$$W = \left\{ \begin{array}{c} w = (u,v) \in L^\infty((0,T); L^{qr_2} \times L^{pr_1}), \\ \|w\|_W \equiv \sup\limits_{t \in (0,T)} \Big(t^{\alpha_1}\|u(t)\|_{L^{qr_2}} + t^{\alpha_2}\|v(t)\|_{L^{pr_1}}\Big) \end{array} \right\}. \tag{4.1.12}$$

W obviously is a Banach space with the norm $\|\cdot\|_W$. For the rest of this part, we refer to [210].

Step 2. Regularity: See the details in [210].

Step 3. Uniqueness: See the details in [210].

Step 4. Continuous Dependence: Consider $w_0 = (u_{w,0}, v_{w,0})$, $z_0 = (u_{z,0}, v_{z,0})$ and assume $w(t) = (u_w(t), v_w(t))$, $z(t) = (u_z(t), v_z(t))$ is the corresponding solution defined in $[0,T)$ with initial data w_0 and z_0, respectively.

Letting

$$\phi(t) = t^{\alpha_1}\|u_w(t) - u_z(t)\|_{L^{qr_2}}, \quad \psi(t) = t^{\alpha_2}\|v_w(t) - v_z(t)\|_{L^{pr_1}}.$$

Then it follows (for the details, see [210])

$$\phi(t) \le C\left(\|u_{w,0} - u_{z,0}\|_{r_1,s_1} + t^{\alpha_1}\int_0^t (t-s)^{-\alpha_1} s^{-p\alpha_2}\psi(s)ds\right), \tag{4.1.13}$$

$$\psi(t) \le C\left(\|v_{w,0} - v_{z,0}\|_{r_2,s_2} + t^{\alpha_2}\int_0^t (t-s)^{-\alpha_2} s^{-q\alpha_1}\phi(s)ds\right). \tag{4.1.14}$$

Therefore, applying Theorem 1.3.2 to (4.1.13)–(4.1.14), we can conclude

$$t^{\alpha_1}\|u_w(t) - u_z(t)\|_{L^{qr_2}} + t^{\alpha_2}\|v_w(t) - v_z(t)\|_{L^{pr_1}} \le C\|w_0 - z_0\|_{r_1,s_1,r_2,s_2}$$

which is just (4.1.10) for $\gamma_1 = qr_2$ and $\gamma_2 = pr_1$.

For the rest of (4.1.10), see [210]. □

Remark 4.1.1 ([210]). In fact, for different values of $r_1,\ s_1,\ r_2,\ s_2,\ p,\ q$, we can also obtain other existence results of solutions to problem (4.1.1)–(4.1.3) for $w_0 \in E^{r_1,s_1} \times E^{r_2,s_2}$. For example, Theorem 4.1.1 still applies the case when $r_1 = s_1,\ r_2 = s_2$ and $qr_2 = r_1$. Indeed, if $qr_2 < r_1$, there exists a unique solution $w(t) = (u(t), v(t)) \in C((0,T); L^{r_1} \times L^{r_1})$ of problem (4.1.1)–(4.1.3), $0 < t < T$, such that $u(t) - S(t)u_0, v(t) - S(t)v_0 \in C((0,T); L^{r_1})$.

4.2 A convection equation with anomalous diffusion

In this section, we shall use Theorem 1.4.9 to study the Cauchy problem for a multi-dimensional conservation law with anomalous diffusion. These results are taken from Brandolese and Karch [115].

We shall consider the following problem

$$\begin{cases} u_t + (-\Delta)^{\alpha/2}u + \nabla \cdot f(u) = 0, \quad x \in \mathbb{R}^d,\ t > 0, & (4.2.1)\\ u(x,0) = u_0, & (4.2.2)\end{cases}$$

where we always assume $1 < \alpha < 2$, and that the C^1 vector field $f(u) = (f_1(u), \ldots, f_d(u))$ is of a polynomial growth, namely, it satisfies the usual estimates

$$|f(u)| \le C|u|^q, \quad |f(u) - f(v)| \le C|u-v|\left(|u|^{q-1} + |v|^{q-1}\right) \tag{4.2.3}$$

for some constants $C > 0$ and $q > 1$ and for all $u, v \in \mathbb{R}$.

The notation used here is as follows: The L^p-norm of a Lebesgue measurable, real-valued function $v \in L^p(\mathbb{R}^d)$ defined on $\mathbb{R}^d$ is denoted by $\|v\|_{L^p}$. In the sequel, we shall use the weighted L^∞ space

$$L^\infty_\theta = \left\{v \in L^\infty(\mathbb{R}^d) : \|v\|_{L^\infty_\theta} \equiv \operatorname{ess}\sup_{x\in\mathbb{R}^d} |v(x)|(1+|x|)^\theta < +\infty\right\} \tag{4.2.4}$$

for any $\theta \geq 0$, and its homogeneous counterpart is

$$\dot{L}^\infty_\theta = \Big\{ v \in L^\infty(\mathbb{R}^d \backslash \{0\}) : \|v\|_{L^\infty_\theta} \equiv \operatorname{ess}\sup_{x\in\mathbb{R}^d} |v(x)||x|^\theta < +\infty \Big\}. \tag{4.2.5}$$

By the same letter C independent of x we denote the universal constants which may vary from line to line. Sometimes, we write, e.g., $C = C(T)$ when we want to emphasize the dependence of C on a parameter T.

We need the following lemmas.

Lemma 4.2.1 ([115]). *For any $u_0 \in L^1(\mathbb{R}^d)$ and $1 < \alpha \leq 2$, the initial value problem* (4.2.1)–(4.2.2) *has a unique solution $u \in C([0, +\infty); L^1(\mathbb{R}^d))$ such that for every $p \in [1, +\infty]$,*

$$u \in C((0, +\infty); W^{1,p}(\mathbb{R}^d)) \tag{4.2.6}$$

and the following estimate holds true for all $t > 0$,

$$\|u(t)\|_{L^p} \leq Ct^{-(d/\alpha)(1-1/p)}\|u_0\|_{L^1} \tag{4.2.7}$$

with a constant $C > 0$ independent of t and u_0. Moreover, if $u_0 \in L^p(\mathbb{R}^d)$, then the corresponding solution u satisfies

$$u \in C([0, +\infty); L^p(\mathbb{R}^d)), \quad \|u(t)\|_{L^p} \leq \|u_0\|_{L^p}. \tag{4.2.8}$$

Proof. We refer to Biler, Karch, and Woyczyński [107], Droniou, Gallouet, and Vovelle [217] and Droniou and Imbert [218] for the details of the proof. □

In following lemma, we shall give some space-time estimates for the solution of the linear problem

$$v_t + (-\Delta)^{\alpha/2}v = 0 \tag{4.2.9}$$

with

$$v(x, 0) = v_0 \tag{4.2.10}$$

where

$$v(x, t) = S_\alpha(t)v_0(x) = p_\alpha(\cdot, t) * v_0(x). \tag{4.2.11}$$

This lemma contains a direct generalization to $\mathbb{R}^d$ of estimates from Hayashi, Kaikina, Naumkin, and Shishmarev ([354], Lemma 1.40). Thus, we only sketch its proof.

Lemma 4.2.2 ([115]). *Assume that $v_0 \in L^\infty_{\alpha+d}$. There exists a constant $C > 0$, independent of v_0 and t, such that*

$$\|S_\alpha(t)v_0\|_{L^\infty} \leq C\min\{t^{-d/\alpha}\|v_0\|_{L^1}, \|v_0\|_{L^\infty}\}, \tag{4.2.12}$$

$$\|S_\alpha(t)v_0\|_{L^\infty_{\alpha+d}} \leq C(1+t)\|v_0\|_{L^\infty_{\alpha+d}}, \tag{4.2.13}$$

$$\|\nabla S_\alpha(t)v_0\|_{L^\infty_{\alpha+d}} \leq Ct^{-1/\alpha}\|v_0\|_{L^\infty_{\alpha+d}} + Ct^{1-1/\alpha}\|v_0\|_{L^1}. \tag{4.2.14}$$

Proof. We refer to [115] for details of the proof. □

Recall that the authors in [354] proved that if $u_0 \in L^\infty_{\alpha+d}$, then the corresponding solution to problem (4.2.1)–(4.2.2) satisfies $u \in C([0,T], L^\infty_{\alpha+d})$ for every $T > 0$.

We also note the critical exponent $\tilde{q} \equiv 1+\frac{\alpha-1}{d}$ plays a crucial role for studying the large-time behavior of solutions to problem (4.2.1)–(4.2.2). Indeed, using the terminology of Biler, Karch and Woyczyńshi [107], the behavior of solutions as $t \to +\infty$ is genuinely nonlinear when $q = \tilde{q}$, is weakly nonlinear when $q > \tilde{q}$, and is (expected to be) hyperbolic when $1 < q < \tilde{q}$.

Brandolese and Karch [115] improved the above-mentioned space-time estimates in [354] by considering the supercritical case $q > \tilde{q}$ and the critical case $q = \tilde{q}$. Such a result can be stated in the next theorem.

Theorem 4.2.3 ([115]).

(i) *Let $\alpha \in (1,2)$. Assume that $u = u(x,t)$ is a solution of the Cauchy problem* (4.2.1)–(4.2.2), *where the nonlinearity f satisfies* (4.2.3) *with $q > \tilde{q} = 1 + (\alpha-1)/d$ and $u_0 \in L^\infty_{\alpha+d}$. There exists a constant $C > 0$, depending on u_0 but independent of x and t, such that for all $x \in \mathbb{R}^d$, $t > 0$,*

$$|u(x,t)| \le Cp_\alpha(x, 1+t). \tag{4.2.15}$$

The same conclusion holds true for $q = \tilde{q}$ provided that $\|u_0\|_{L^1}$ is sufficiently small.

(ii) *Under the more stringent assumption*

$$u_0 \in E_{\alpha+d} \equiv \left\{ v \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^d) : \|v\|_{E_{\alpha+d}} \equiv \|v\|_{L^\infty_{\alpha+d+1}} + \|\nabla v\|_{L^\infty_{\alpha+d+1}} < +\infty \right\},$$

we also have for all $t > 0$,

$$\|\nabla u(t)\|_{L^\infty_{\alpha+d+1}} \le C(1+t). \tag{4.2.16}$$

Proof. First we recall estimates (4.2.7) and (4.2.8) with $p = +\infty$. The solution satisfies

$$\|u(t)\|_{L^\infty} \le C(1+t)^{-d/\alpha}. \tag{4.2.17}$$

Hence, in order to establish (4.2.15), it suffices to show that

$$\|u(t)\|_{L^\infty_{\alpha+d}} \le C(1+t). \tag{4.2.18}$$

Indeed, using the basic estimate

$$g(x,t) \le (1+t)^{-d/\alpha} \min\{1, |x(1+t)^{-1/\alpha}|^{-\alpha-d}\}$$

and the asymptotic formula (4.2.13) (implying that $\min\{1, |x|^{-\alpha-d}\} \le CP_\alpha(x)$ for all $x \in \mathbb{R}^d$ and a constant $C > 0$), we easily obtain

$$g(x,t) \equiv \min\left\{(1+t)^{-d/\alpha}, (1+t)/(1+|x|)^{\alpha+d}\right\} \le Cp_\alpha(x, t+1).$$

To prove (4.2.16), we only use the following integral equation which follows from the Duhamel principle

$$u(t) = S_\alpha u_0 - \int_0^t \nabla S_\alpha(t-s) \cdot f(u)(s)ds, \tag{4.2.19}$$

where $S_\alpha u_0(x) = p_\alpha(x,t) * u_0(x)$ is the solution of the linear equation (4.2.4) subject to the initial datum u_0. Hence it follows from (4.2.19) and (4.2.3) that

$$\begin{aligned}&\|\nabla S_\alpha(t-\tau)f(u(\tau))\|_{L^\infty_{\alpha+d}} \\ &\quad \le C(t-\tau)^{-1/\alpha}\|u(\tau)\|^{q-1}_{L^\infty}\|u(\tau)\|_{L^\infty_{\alpha+d}} + C(t-\tau)^{1-1/\alpha}\|u(\tau)\|^q_{L^q}.\end{aligned}$$

Moreover, since by (4.2.7) and (4.2.8) with $p = q$, the solution satisfies

$$\|u(\tau)\|^q_{L^q} \le C(1+\tau)^{-d(q-1)/\alpha} \tag{4.2.20}$$

where we have used the inequalities

$$\int_0^t (t-\tau)^{1-1/\alpha}\|u(\tau)\|^q_{L^q}d\tau \le C\int_0^t (t-\tau)^{1-1/\alpha}(1+\tau)^{-d(q-1)/\alpha}d\tau \le C(1+t),$$

which holds for $1/\alpha + d(q-1)/\alpha \ge 1$. Thus, by computing the $L^\infty_{\alpha+d}$-norm of (4.2.19) and using (4.2.13), we arrive at

$$\|u(t)\|_{L^\infty_{\alpha+d}} \le C(1+t) + C\int_0^t (t-\tau)^{-1/\alpha}(1+\tau)^{-d(q-1)/\alpha}\|u(\tau)\|_{L^\infty_{\alpha+d}}d\tau. \tag{4.2.21}$$

Similarly, for the time-critical case $1/\alpha + d(q-1)/\alpha = 1$ (i.e., for $q = \tilde{q}$), we can use the following estimate to obtain the desired result:

$$\|u(\tau)\|_{L^\infty} \le C\tau^{-d/\alpha}\|u_0\|_{L^1} \tag{4.2.22}$$

with a constant $C > 0$ independent of u_0 and $t > 0$. Hence, we obtain the counterpart of the inequality of (4.2.21):

$$\|u(t)\|_{L^\infty_{\alpha+d}} \le C(1+t)+C\|u_0\|^{q-1}_{L^1}\int_0^t (t-\tau)^{-1/\alpha}\tau^{-d(q-1)/\alpha}\|u(\tau)\|_{L^\infty_{\alpha+d}}d\tau. \tag{4.2.23}$$

Finally, applying the singular Bellman–Gronwall inequality of Theorem 1.4.9 to inequalities (4.2.21) and (4.2.23), we can complete the proof of (4.2.16). □

4.3 Estimates on solutions for semilinear heat equations

In this section, we use Theorem 2.1.19 to investigate the blow-up of solutions of the semilinear heat equations. We took these results from Giga and Kohn [312].

We shall consider the following semilinear problem of heat equations

$$\begin{cases} u_t - \Delta u - |u|^{p-1}u = 0 \quad \text{in} \quad \Omega \times [0,T), & (4.3.1) \\ u = 0 \quad \text{on} \quad \partial\Omega \times [0,T) & (4.3.2) \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is a domain, possibly unbounded, with $C^{2,\alpha}$ boundary, u is a scalar-valued function and $p > 1$.

We assume that the function u is a classical solution to problem (4.3.1)–(4.3.2), with

$$u, \nabla u, \nabla^2 u, \text{ and } u_t \text{ bounded and continuous on } \overline{\Omega} \times [0, \tau) \text{ for every } \tau < T \tag{4.3.3}$$

and T is the blow-up time: as $t \to T^-$,

$$\sup_{x \in \Omega} |u(x,t)| \to +\infty.$$

If $\Omega \neq \mathbb{R}^n$, then (4.3.1) imposes a Dirichlet condition (4.3.2) at $\partial\Omega$, i.e., if D is unbounded, then (4.3.3) includes the condition that u stays bounded as $|x| \to +\infty$. For the simplicity, we may assume that the initial data satisfy

$$u(x,0) = u_0(x) \in C^2(\overline{\Omega}). \tag{4.3.4}$$

We shall prove some estimates on solutions to problem (4.3.1)–(4.3.3) in this section with the help of Theorem 2.1.19.

For any $a \in \overline{\Omega}$, writing the solution in "similarity variables about (a, T)" means considering the function $w_a(y,s)$, defined by

$$w_a(y,s) = (T-t)^{1/(p-1)} u(x,t),\ y = (x-a)(T-t)^{-1/2},\ s = -\log(T-t) \tag{4.3.5}$$

which solves

$$w_s - \frac{1}{\rho}\nabla \cdot (\rho \nabla w) + \frac{1}{p-1} w - |w|^{p-1} w = 0 \tag{4.3.6}$$

with $\rho(y) = \exp(-\frac{1}{4}|y|^2)$, on the space-time domain

$$W_a = \Big\{(y,s) : s > s_0, e^{-s/2} y + a \in \Omega\Big\}, \quad s_0 = -\log T. \tag{4.3.7}$$

The slice of W_a at time s will be denoted by $\Omega_a(s)$:

$$\Omega_a(s) = e^{s/2}(\Omega - a). \tag{4.3.8}$$

We define the energy $E(w_a)$ by

$$E(w_a) = \int_{W_a} \rho(y) \left[\frac{1}{2}|\nabla w_a|^2 + \frac{1}{2(p-1)}|w_a|^2 - \frac{1}{p+1}|w_a|^{p+1}\right] dy \tag{4.3.9}$$

and shall often suppress the subscript a, writing w for w_a, $\Omega(s)$ for $\Omega_a(s)$, etc.

Theorem 4.3.1 ([310]). *Let $a \in \overline{\Omega}$, and suppose that Ω is star-shaped with respect to a. Let $E_0 = E(w_a)(s_0)$ denote the initial energy for w_a, and assume that $E_0 \leq 1$. Then $w = w_a$ satisfies*

$$\int\int_{W_a} |w_s|^2 \rho dy ds \leq E_0, \tag{4.3.10}$$

and for every $s \geq s_0$, we have

$$\begin{cases} \displaystyle\int_{\Omega(s)} |w|^2 \rho dy \leq C(n,p) E_0^{1/p}, & (4.3.11) \\ \displaystyle\int_s^{s+1} \left(\int_{\Omega(\tau)} |w|^{p+1} \rho dy \right)^2 d\tau \leq C(n,p) E_0^{(p+1)/p}, & (4.3.12) \\ \displaystyle\int_s^{s+1} \left(\int_{\Omega(\tau)} (|\nabla w|^2 + |w|^2) \rho dy \right)^2 d\tau \leq C(n,p) E_0^{(p+1)/p}. & (4.3.13) \end{cases}$$

Proof. The a priori estimates for w in (4.3.10)–(4.3.13) are more or less the same as those in Propositions 2.2 of [310], except that their dependence on initial energy is more explicit.

The first assertion (4.3.10) is identical to (2.20) of [310]. For (4.3.11), we recall (2.24) of [310]: if $g(s) = \left(\int_{\Omega(s)} |w|^2 \rho dy \right)^{1/2}$, then

$$gg' + 2E[w](s) \geq C g^{p+1} \tag{4.3.14}$$

with a constant $C = C(n,p) > 0$. Since $E(w)$ is decreasing (this is (2.23) of [310]), it follows from (4.3.14) that

$$C g^{p+1} \leq gg' + 2E_0. \tag{4.3.15}$$

Applying Theorem 2.1.19 to (4.3.15) shows that (4.3.10) and (4.3.15) imply (4.3.12). To prove (4.3.12), we recall (2.28) of [310]:

$$\left(\int_{\Omega(s)} |w|^{p+1} \rho dy \right)^2 \leq C(p) \Big\{ g^2(s) \int_{\Omega(s)} |w_s|^2 \rho dy + E_0^2 \Big\}. \tag{4.3.16}$$

Integrating with respect to s, making use of (4.3.10) and (4.3.11), and noting that $E_0^2 \leq E_0^{(p+1)/p}$ since $p > 1$ and $E_0 \leq 1$, we get (4.3.12). To get the last result (4.3.13), we derive from the definition of $E[w]$ in (4.3.9)

$$\frac{1}{2} \int_{\Omega(s)} \left(|\nabla w|^2 + \frac{1}{p-1} |w|^2 \right) \rho dy = E[w](s) + \frac{1}{p+1} \int_{\Omega(s)} |w|^{p+1} \rho dy. \tag{4.3.17}$$

Since $E[w](s) \leq E_0$, it follows from (4.3.17) that

$$\int_{\Omega(s)} \left(|\nabla w|^2 + |w|^2 \right) \rho dy \leq C(p) \Big[E_0 + \int_{\Omega(s)} |w|^{p+1} \rho dy \Big]. \tag{4.3.18}$$

Squaring both sides of (4.3.18), integrating with respect to s, and making use of (4.3.12), we easily obtain (4.3.13).

The proof is thus complete. □

4.4 Global existence decay estimates for a quasilinear parabolic system

In this section, we shall exploit Theorem 2.1.18 to establish L^∞ decay estimates on solutions for a quasilinear parabolic system. We choose these results from Chen [140].

We shall consider the quasilinear parabolic system

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^m \nabla u) + f(u,v), & x \in \Omega,\ t > 0, \\ v_t = \operatorname{div}(|\nabla v|^m \nabla v) + g(u,v), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \ \ v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \end{cases} \tag{4.4.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N > 1)$ with smooth boundary $\partial\Omega$ and $m > 0$.

For $m = 0, f(u,v) = u^\alpha v^p, g(u,v) = u^q v^\beta$ and $u_0(x), v_0(x) \geq 0$, the existence and non-existence of solutions (see, e.g., [238, 209, 947, 144] and the references therein) for problem (4.4.1) were studied.

We now recall some known results in the literature.

Assume that the initial data $u_0(x), v_0(x) \geq 0$ and $u_0, v_0 \in L^\infty(\Omega)$. Then we have

(A_1): if $\alpha > 1$ or $\beta > 1$ or $s_0 = (1-\alpha)(1-\beta) - pq < 0$, then problem (4.4.1) admits a global solution for small initial data and this solution necessarily blows up in a finite time for large initial data;

(A_2): all solutions of problem (4.4.1) exist globally if $\alpha, \beta \leq 1$ and $s_0 \geq 0$.

Next, we study problem (4.4.1) for general initial data and give the L^∞ decay estimates for solutions of problem (4.4.1), that is, the behavior of solutions as $t \to 0^+$ and $t \to +\infty$. To deal with such a problem, one usually bases on the comparison principle (see, e.g., [238, 209, 947, 144]), while the method used here is very different by using an improved Moser's technique as in [683, 141]. Indeed, we have neither the restriction of the non-negativity nor the bounded on $u_0(x)$ and $v_0(x)$.

Definition 4.4.1. A pair of measurable functions $(u(x,t), v(x,t))$ on $\Omega \times \mathbb{R}^+$ is said to be a global weak solution of problem (4.4.1) if

$$u(x,t), v(x,t) \in L^\infty_{\text{loc}}(\overline{\mathbb{R}^+}, W_0^{1,m+1}) \cap L^{m+1}_{\text{loc}}(\overline{\mathbb{R}^+}, W_0^{m+1})$$

and the equalities

$$\int_0^t \int_\Omega \big\{ -u\phi_t + |\nabla u|^m \nabla u \nabla \phi - f(u,v)\phi \big\} dxdt$$
$$= \int_\Omega \big\{ u_0(x)\phi(x,0) - u(x,t)\phi(x,t) \big\} dx,$$

$$\int_0^t\int_\Omega \{ - v\phi_t + |\nabla v|^m \nabla u \nabla \phi - g(u,v)\phi\} dxdt$$
$$= \int_\Omega \{v_0(x)\phi(x,0) - v(x,t)\phi(x,t)\}dx$$

are valid for any $t > 0$ and any $\phi \in C^1(\overline{\mathbb{R}^+}, C_0^1(\Omega))$, where $\mathbb{R}^+ = (0,+\infty), \overline{\mathbb{R}^+} = [0,+\infty)$.

Lemma 4.4.1 ([140])**.** *Assume that*

(H1) *The functions* $f(u,v), g(u,v) \in C(\mathbb{R}^2) \cap C^1(\mathbb{R}^2 \setminus \{(0,0)\})$ *and for all* $(u,v) \in \mathbb{R}^2$,

$$|f(u,v)| \le k_1|u|^\alpha|v|^p, \quad |g(u,v)| \le k_2|u|^q|v|^\beta, \tag{4.4.2}$$

where the parameters α, β, p, q *satisfy*

$$0 \le \alpha, \beta < 1+m;\ m,p,q > 0;\ s_m = (m+1-\alpha)(m+1-\beta)-pq > 0. \tag{4.4.3}$$

(H2) $u_0(x) \in L^{p_0}(\Omega), v_0(x) \in L^{q_0}(\Omega)$ *with*

$$p_0 > \max\{1, q+1-\alpha\}, \quad q_0 > \max\{1, p+1-\beta\}.$$

Then problem (4.4.1) *admits a global weak solution* $(u(x,t), v(x,t))$ *satisfying*

$$u(x,t) \equiv u(t) \in L^\infty(\overline{\mathbb{R}^+}, L^{p_0}), \quad v(x,t) \equiv v(t) \in L^\infty(\overline{\mathbb{R}^+}, L^{q_0})$$

and for any $T > 0$ *and for all* $t \in (0,T]$,

$$||u(t)||_{L^\infty} \le Ct^{-\sigma}, \quad ||v(t)||_{L^\infty} \le Ct^{-\sigma}, \tag{4.4.4}$$

$$||u(t)||_{L^{m+2}}^{m+2} + ||v(t)||_{L^{m+2}}^{m+2} \le C\left(t^{-1-\sigma} + t^{1-2(p+\alpha)\sigma} + t^{1-2(q+\beta)\sigma}\right), \tag{4.4.5}$$

where the constants

$$C = C(T, ||u_0||_{L^{p_0}}, ||v_0||_{L^{q_0}}) > 0,$$
$$\sigma = \min\left\{\frac{N}{p_0(m+2)+mN}, \frac{N}{q_0(m+2)+mN}\right\} > 0.$$

In order to show Theorem 4.4.1, we also need the following lemmas.

Lemma 4.4.2 ([672])**.** *Let* $\beta \ge 0$, $N > p \ge 1$, $\beta + 1 \ge q$, *and* $1 \le r \le q \le (\beta+1)Np/(N-p)$. *Then for* $|u|^\beta u \in W^{1,p}(\Omega)$, *we have*

$$||u||_{L^q} \le C^{1/(\beta+1)}||u||_{L^r}^{1-\theta}|||u|^\beta u||_{W^{1,p}}^{\theta/(\beta+1)}, \tag{4.4.6}$$

with $\theta = (\beta+1)(r^{-1}-q^{-1})/(N^{-1}-p^{-1}+(\beta+1)r^{-1})^{-1}$, *where* $C > 0$ *is a constant depending only on* N, p *and* r.

For $j = 1, 2, \ldots,$ we choose $f_j(u,v), g_j(u,v) \in C^1$ as follows: $f_j(u,v) = f(u,v), g_j(u,v) = g(u,v)$ when $u^2+v^2 \geq j^{-2}, |f_j(u,v)| \leq \eta, |g_j(u,v)| \leq \eta$ when $u^2+v^2 \leq j^{-2}$ with some $\eta > 0$ and $(f_j(u,v), g_j(u,v)) \to (f(u,v), g(u,v))$ uniformly in $\mathbb{R}^2$ as $j \to +\infty$.

Let $(u_{0,j}, v_{0,j}) \in C_0^2(\Omega)$ and $u_{0,j} \to u_0$ in $L^{p_0}(\Omega)$, $v_{0,j} \to v_0$ in $L^{q_0}(\Omega)$ as $j \to +\infty$. We consider the following approximation of problem (4.4.1):

$$\begin{cases} u_t = \operatorname{div}((|\nabla u|^2 + j^{-1})^{m/2}\nabla u) + f_j(u,v), & x \in \Omega,\ t > 0, \\ v_t = \operatorname{div}((|\nabla v|^2 + j^{-1})^{m/2}\nabla v) + g_j(u,v), & x \in \Omega,\ t > 0, \\ u(x,0) = u_{0,j}(x),\ \ v(x,0) = v_{0,j}(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t \geq 0. \end{cases} \tag{4.4.7}$$

The approximate problem (4.4.8) is a standard quasilinear parabolic system and admits a unique smooth solution $(u_j(x,t), v_j(x,t))$ on $[0,T)$ for each $j = 1, 2, \ldots,$ (see, e.g., [546, 471]). Furthermore, if $T < +\infty$, then

$$\limsup_{t\to T^-}(||u(t)||_{L^\infty} + ||v(t)||_{L^\infty}) = +\infty.$$

In what follows, we always write u (v) instead of u_j (v_j) and u^p (v^p) for $|u|^{p-1}u$ $(|v|^{p-1}v)$ where $p > 0$. Also, let C and C_i be the generic constants independent of j and p changeable from line to line.

Lemma 4.4.3 ([140])**.** *Let* (H1) *and* (H2) *hold. If* $(u(x,t), v(x,t))$ *is the solution of problem* (4.4.7)*, then*

$$u(t) \in L^\infty(\overline{\mathbb{R}^+}, L^{p_0}(\Omega)),\ v(t) \in L^\infty(\overline{\mathbb{R}^+}, L^{q_0}(\Omega)). \tag{4.4.8}$$

We now derive the L^∞ decay estimates for $u(t)$ and $v(t)$ based on the above Lemmas 4.4.1–4.4.3.

Theorem 4.4.4 ([140])**.** *Under the assumption of Lemma* 4.4.2 *and for any* $T > 0$*, the solution* $(u(t), v(t))$ *also satisfies for all* $0 < t \leq T$*,*

$$||u(t)||_{L^\infty} \leq Ct^{-a}, \quad ||v(t)||_{L^\infty} \leq Ct^{-b}, \tag{4.4.9}$$

where the constant C *depends on* T, $||u_0||_{L^{p_0}}$, $||v_0||_{L^{q_0}}$*, and* $a = N/(p_0(m+2) + mN)$*,* $b = N/(q_0(m+2) + mN)$*.*

Proof. For the details of the following arguments, see [140]. Following [140], we have

$$\begin{aligned} &y_n'(t) + C_3||u||_{L^{\lambda_n}}^{\lambda_n+t_n}\big(\xi_{n-1}t^{-m_{n-1}}\big)^{m-t_n} + C_3||v||_{L^{\mu_n}}^{\mu_n+s_n}\big(\eta_{n-1}t^{-r_{n-1}}\big)^{m-s_n} \\ &\quad \leq C(\lambda_n + \mu_n)\big(\lambda_n^{\sigma_1}||u(t)||_{L^{\lambda_n}}^{\lambda_n} + \mu_n^{\sigma_2}||v(t)||_{L^{\mu_n}}^{\mu_n}\big) \end{aligned} \tag{4.4.10}$$

where for all $t \geq 0$,

$$y_n(t) = ||u(t)||_{L^{\lambda_n}}^{\lambda_n} + ||v(t)||_{L^{\mu_n}}^{\mu_n}$$

and for $A_{n-1}, \alpha_n, \beta_n, \tau_n$ and σ_0, see [140]. This gives us that for all $0 < t \leq T$,

$$y_n'(t) + C_3 A_{n-1}^{\alpha_n} t^{\beta_n} y_n^{1+\tau_n}(t) \leq C\lambda_n + C\lambda_n^{\sigma_0+1} y_n(t) + CA_{n-1}^{\alpha_n} T^{\beta_n}. \tag{4.4.11}$$

Applying Theorem 2.1.18 to (4.4.11), we have for all $0 < t \leq T$,

$$y_n(t) \leq B_n t^{-(1+\beta_n)/\tau_n}, \tag{4.4.12}$$

where

$$B_n = 2(C_3 A_{n-1}^{\alpha_n})^{-1/\tau_n} \left(C_3 \lambda_n^{\sigma_0+1} + \frac{1+\beta_n}{\tau_n} \right)^{1/\tau_n} + 2C\lambda_n \left(C\lambda_n^{\sigma_0+1} + \frac{1+\beta_n}{\tau_n} \right)^{-1}.$$

For the rest of the proof of (4.4.9), see [140]. □

Chapter 5

Global Existence and Uniqueness for Abstract Evolutionary Differential Equations

In this chapter, we discuss the global existence and uniqueness for some abstract models and ODEs. The chapter consists of three sections. In Section 5.1, we apply Corollary 1.4.4 to prove the global-in-time existence of solutions to an abstract evolutionary equation written below. In Section 5.2, we employ Theorem 2.1.23 and Theorem 2.4.16 to show the uniqueness of the solution to the Cauchy problem for vector-valued functions satisfying a differential inequality. In Section 5.3, we apply a new Gronwall-type inequality in Theorem 2.1.21 to a viscoelastic system with small dissipation. The inequalities applied here are specially selected so that they play a very important role in deriving the global existence and uniqueness of solutions.

5.1 Global existence of solutions to abstract evolutionary equations

In this section, we shall apply Corollary 1.4.4 to prove the global-in-time existence of solutions to an abstract evolutionary equation. These results are taken from Ye and Li [980].

We shall consider the abstract evolutionary equation

$$\begin{cases} \dfrac{du}{dt} + Au = f(t,u), \quad t > t_0, & (5.1.1) \\ u(t_0) = x. & (5.1.2) \end{cases}$$

We assume that

(H1) X is a Banach space with the norm $\|\cdot\|$. X is a sectorial operator on X such that the fractal power of $A_1 = A + \alpha I$ is well defined, and for any

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$\alpha \geq 0$, the graph space $X^\alpha = D(A_1^\alpha)$ with graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$ is also well defined. X_s equipped with the norm $\|\cdot\|_s$ is a linear subspace of X and constitutes a Banach space, U_s is an open set in $\mathbb{R} \times X_s$, $f : U_s \to X$ is well defined. We only discuss the case $X_s = X$.

(H2) There exists $\alpha \in [0, 1)$ such that f maps some open set U in $\mathbb{R} \times X^\alpha$ into X, and f is locally continuous with respect to t in U, and locally Lipschitz continuous with respect to u, i.e., if for all $(t^*, u^*) \in U$, there exists a neighborhood $V \subseteq U$ of (t^*, u^*) such that for any $(t, u) \in V, (s, v) \in V$,

$$\|f(t,u) - f(s,v)\| \leq L\left(|t-s|^\mu + \|u-v\|_\alpha\right), \tag{5.1.3}$$

where L and $\mu \in (0, 1)$ are positive constants.

Definition 5.1.1. A function $u(t)$ is said to be a solution on $[t_0, t_1)$ to problem (5.1.1)–(5.1.2) if $u(t)$ is a continuous function from $[t_0, t_1) \to X_s$ satisfying

(1) $u(t_0) = x$,

(2) for all $t \in (t_0, t_1), (t, u(t)) \in X_s, u(t) \in D(A)$, $\dfrac{du}{dt}$ exists and satisfies (5.1.1),

(3) for all $t \in (t_0, t_1)$, $t \mapsto f(t, u(t))$ is locally Hölder continuous and there exists some constant $\rho > 0$ such that $\int_{t_0}^{t_0+\rho} \|f(t, u(t))\| dt < +\infty$.

Here X_s equipped with the norm $\|\cdot\|_s$ is a linear subspace of X and a Banach space, U_s is an open set in $\mathbb{R} \times X$, and $f : U_s \to X$ is well defined.

5.1.1 Equivalent solutions

In this subsection, we prove the equivalence of solutions to problem (5.1.1)–(5.1.2) to the integral equation

$$u(t) = e^{-(t-t_0)A}x + \int_{t_0}^{t} e^{-(t-s)A} f(s, u(s))ds. \tag{5.1.4}$$

The following Lemmas are due to Ye and Li [980].

Lemma 5.1.1 ([980])**.** *Under the above assumptions* (H1) *and* (H2)*, the following assertions hold true:*

(1) *If u solves problem* (5.1.1)–(5.1.2) *on* $[t_0, t_1)$*, then u also solves* (5.1.4).

(2) *If $u(t) : [t_0, t_1) \to X^\alpha$ is a continuous function, and for some $\rho > 0$, $\int_{t_0}^{t_0+\rho} \|f(s, u(s))\| ds < +\infty$, and for all $t \in [t_0, t_1)$, u solves* (5.1.4)*, then $u(t)$ solves problem* (5.1.1)–(5.1.2) *on* $[t_0, t_1)$.

Applying the contraction mapping principle to the integral equation (5.1.4), we can show the following local existence theorem.

Lemma 5.1.2 ([980])**.** *Assume that above assumptions* (H1) *and* (H2) *hold. Then for any $(t_0, x) \in U$, there exists some time $T = T(t_0, x) > 0$ such that problem* (5.1.1)–(5.1.2) *admits a unique solution on* $[t_0, t_0 + T)$.

We shall use the continuation method to show the global existence of solutions based on a local solution established in Lemma 5.1.2. We also need the following lemma.

Lemma 5.1.3 ([980]). *Assume that the above assumptions* (H1) *and* (H2) *hold, and for any bounded closed set $B \subset U$, $f(B)$ is bounded in X. If u is a solution to problem* (5.1.1)–(5.1.2) *on* $[t_0, t_1)$ *with the maximal time t_1, then either $t_1 = +\infty$, or there exists a sequence $t_n \to t_1^-$ (as $n \to +\infty$) such that $(t_n, u(t_n)) \to \partial U$ (if U is unbound, then $+\infty \in \partial U$).*

Theorem 5.1.4. ([980]) *Assume that* (H1) *and* (H2) *hold, with $U = (\tau, +\infty) \times X^\alpha$, and for any $(t, u) \in U$, f satisfies*

$$\|f(t,u)\| \le K(t)(1 + \|u\|_\alpha) \tag{5.1.5}$$

where $K(\cdot)$ is continuous on $(\tau, +\infty)$. If $t_0 > \tau, x \in X^\alpha$, then for any $t \ge t_0$, problem (5.1.1)–(5.1.2) *admits a unique global solution $u(t)$.*

Proof. If the assertion does not hold, then by Lemma 5.1.3, there exist some time $t_1 < +\infty$ and a sequence $t_n \to t_1-$ such that

$$\|u(t_n)\|_\alpha \to +\infty. \tag{5.1.6}$$

However, from (5.1.4)–(5.1.5) it follows that

$$\begin{aligned} \|u(t)\|_\alpha &\le \|e^{-(t-t_0)A}x\|_\alpha + \int_{t_0}^t \|A_1^\alpha e^{-(t-s)A}\| K(s)(1 + \|u(s)\|_\alpha) ds \\ &\le C_1\|x\|_\alpha + \int_{t_0}^t C_2(t-s)^{-\alpha}(1 + \|u(s)\|_\alpha) ds \\ &\le a + b\int_{t_0}^t (t-s)^{-\alpha}\|u(s)\|_\alpha ds. \end{aligned} \tag{5.1.7}$$

Now applying Corollary 1.4.4 to (5.1.7), we can conclude that $\|u(t)\|_\alpha \le M$, which contradicts (5.1.6). The proof is now complete. □

5.2 Uniqueness of solutions for differential equations in a Hilbert space

In this section, we shall employ Theorem 2.1.23 and Theorem 2.4.16 to show the uniqueness of the solution to the Cauchy problem for vector-valued functions satisfying a differential inequality. These results are taken from Agmon and Nirenberg [14].

We shall consider the Cauchy problem for vector-valued functions $u(t)$ satisfying the differential inequality

$$\left\| \frac{du}{dt} - B(t)u(t) \right\| \le \Phi(t) \left\{ \|u(t)\|^2 + \int_t^T \omega(\tau)\|u(\tau)\|^2 d\tau \right\}^{1/2} \tag{5.2.1}$$

where $B(t)$ (for each t) is a linear operator in H with domain $D_{B(t)}$. We assume here that $u(t) \in D_{B(t)}$, that $u \in C^1([0,T);H)$, and that $B(t)u(t) \in C([0,T);H)$. $\Phi(t)$ denotes a non-negative measurable function which is bounded in every finite interval $[0,T']$ with $T' < T$, $\omega(t)$ is a non-negative continuous function on $[0,T)$ satisfying

$$\int_0^T \omega(\tau)\|u(\tau)\|^2 d\tau < +\infty. \tag{5.2.2}$$

Moreover, we shall derive lower bounds for such functions which show that if $u(t)$ satisfies (5.2.1) in an infinite interval $[0,+\infty)$, then $\|u(t)\|$ cannot tend too rapidly to zero when $t \to +\infty$, unless u is identically zero.

Consider $u(t)$ satisfying the differential inequality (5.2.1) in an interval $[0,T)$ (assuming, as before, that $u \in C^1([0,T);H), u(t) \in D_{B(t)}$, $B(t)u(t) \in C([0,T);H)$). Assume that hypotheses (1)–(3) (see (2.1.127)–(2.1.129)) in Section 2.1 hold, and set

$$\gamma(t) = \max_{i=1,2,3} \gamma_i(t), \quad \beta(t) = \max_{i=1,2,3} \beta_i(t).$$

We have the following uniqueness result for problem (5.2.1).

Theorem 5.2.1 ([14]).

(i) *If $u(t) = 0$ on some interval $(t_0, T), 0 < t_0 < T$, then for all $t \in [0,T)$,*

$$u(t) \equiv 0. \tag{5.2.3}$$

(ii) *In the special case when $\omega(t) \equiv 0$, if $u(t_0) = 0$ for some t_0, then for all $t \in [0,T)$,*

$$u(t) \equiv 0. \tag{5.2.4}$$

Proof. Assume that the vector-valued function $u(t)$ satisfying (5.2.1) is not identically zero. As in (2.1.132), let

$$q(t) = \|u(t)\|^2 + \int_t^T \omega(\tau)\|u(\tau)\|^2 d\tau. \tag{5.2.5}$$

Consider some subinterval $[t_0, t_1)$, $0 \le t_0 < t_1 \le T$, where $q(t) > 0$. Following Theorem 2.1.22, we define on this subinterval a function $l(t)$ by

$$l(t) = \log q(t) - \int_{t_0}^t \psi(\tau) d\tau \tag{5.2.6}$$

where

$$\psi(t) = \frac{2Re(u' - Bu, u)}{q(t)} - \omega(t)\frac{\|u(t)\|^2}{q(t)}.$$

Noting that $\psi(t)$ is continuous in $[t_0, t_1)$ and using (5.2.1), we conclude

$$|\psi(t)| \le \frac{2\|u' - Bu\|\|u\|}{q(t)} + \omega(t)\frac{\|u(t)\|^2}{q(t)} \le 2\Phi(t) + \omega(t). \tag{5.2.7}$$

Now, by Theorem 2.1.23, $l(t)$ is twice differentiable and satisfies in $[t_0, t_1)$ the differential inequality

$$l''(t) + a(t)|l'(t)| + b(t) \ge 0, \tag{5.2.8}$$

where $a(t)$ and $b(t)$ are non-negative locally bounded measurable functions given by (2.1.137). Also, we note that

$$l(t_0) = \log q(t_0),\ l'(t_0) = q'(t_0)/q(t_0) - \psi(t_0) \ge q'(t_0)/q(t_0) - 2\Phi(t_0) - \omega(t_0). \tag{5.2.9}$$

Applying Theorem 2.4.16 to the function $l(t)$, it follows from (2.4.116), (5.2.9), (5.2.6) and (5.2.7) that

$$\begin{aligned} \log q(t) &\ge l(t) - \int_{t_0}^{t} |\psi(\tau)| d\tau \ge l(t) - 2\int_{t_0}^{t} \Phi(\tau)d\tau - \int_{t_0}^{t} \omega(\tau)d\tau \\ &\ge \log q(t_0) + \min\{0, q'(t_0)/q(t_0) - 2\Phi(t_0) - \omega(t_0)\} \int_{t_0}^{t} \exp\left(\int_{t_0}^{s} a(r)dr\right) ds \\ &\quad - \int_{t_0}^{t} \exp\left(\int_{t_0}^{s} a(r)dr\right) \left[\int_{t_0}^{s} b(\sigma)\exp\left(-\int_{t_0}^{\sigma} a(\rho)d\rho\right) d\sigma\right] ds \\ &\quad - 2\int_{t_0}^{t} \Phi(\tau)d\tau - \int_{t_0}^{t} \omega(\tau)d\tau. \end{aligned} \tag{5.2.10}$$

We also note the following lower bound for $\log q(t)$, which holds for all $t_0 \le t \le t' < t_1$,

$$\begin{aligned} \log q(t) \ge\ & \log q(t') + \min\left(0, q'(t')/q(t') - 2\Phi(t') - \omega(t')\right) \int_{t}^{t'} \exp\left(\int_{s}^{t'} a(r)dr\right) ds \\ & - \int_{t}^{t'} \exp\left(\int_{s}^{t'} a(r)dr\right) \left[\int_{s}^{t'} b(\sigma)\exp\left(-\int_{\sigma}^{t'} a(\rho)d\rho\right) d\sigma\right] ds \\ & - 2\int_{t}^{t'} \Phi(\tau)d\tau - \int_{t}^{t'} \omega(\tau)d\tau \end{aligned} \tag{5.2.11}$$

which can be obtained in a straightforward way from the preceding considerations by replacing t by $-t$. Following [14], we can show that (5.2.10) and (5.2.11) imply that if u is not identically zero (which we have assumed), then $q(t) > 0$ for all $t \in [0,T)$. To this end, we show that if $q(\bar{t}) = 0$ for some $\bar{t}$, then we get a contradiction. Indeed, in this case q is not identically zero in at least one of the two intervals $[0,\bar{t})$ and $[\bar{t},T)$. Assume, for instance, that q is not identically zero in the first interval. Clearly, in this case there exists a subinterval $[t_0,t_1)$ with $0 \le t_0 < t_1 \le \bar{t}$ such that $q(t) > 0$ for all $t_0 \le t < t_1$ while $q(t_1) = 0$. We apply estimate (5.2.20) to $\log q(t)$ on this subinterval. Since $[t_0,t_1]$ is a bounded interval contained in $[0,T)$, the functions a, b, Φ, ω are all bounded by some constant M in $[t_0,t_1)$. Thus it follows from (5.2.20) that for all $t_0 \le t < t_1$,

$$\log q(t) \ge \log q(t_0) - C,$$

for some constant $C > 0$. Hence, $q(t) \ge q(t_0)e^{-C}$ for all $t_0 \le t < t_1$, so that by continuity we also have $q(t_1) \ge q(t_0)e^{-C} > 0$, a contradiction. Similarly, using (5.2.21) we may derive a contradiction when q is not identically zero in $[\bar{t},T)$. Thus, we have proved that $q > 0$ for all t if $u \not\equiv 0$. This implies, in particular, the assertions (i) and (ii) of the theorem, since in the case of (i), we have $q(t) = 0$ for all $t \ge t_0$, while in the case of (ii), we have $q(t_0) = \|u(t_0)\|^2 = 0$, so that in both cases we must have $u \equiv 0$. □

5.3 Dissipative estimates for PDEs

In this section, we shall apply Theorem 2.1.21 to a viscoelastic system with small dissipation. These results are due to Gatti, Pata, and Zelik [297].

It is well known that dissipative partial differential equations play a crucial role in modern mathematical physics which can be usually reformulated as the Cauchy problem in a suitable Banach space $(X, \|\cdot\|)$ (the phase space) of the form

$$\begin{cases} \dfrac{d}{dt}\xi(t) = A(\xi(t),t), \qquad t > 0, \\ \xi(0) = x \in X, \end{cases}$$

where, for every $t \ge 0, A(\cdot,t)$ is some operator densely defined on X. The global well-posedness for all initial data $x \in X$ defines the solution operator $S(t)$, namely, a one-parameter family of operators $S(t) : X \to X$ such that $S(t)x = \xi(t)$ is the unique solution at time $t \ge 0$ to the Cauchy problem with initial datum x. Moreover, further continuity properties of the solutions will reflect in the analogous continuity properties of $S(t)$. In the autonomous case (i.e., when A does not depend explicitly on t), the maps $S(t)$ form a semigroup of operators.

We shall consider the following evolution system arising in the theory of isothermal viscoelasticity (see, e.g., [174, 298])

$$\begin{cases} u_{tt} - \Delta u - \int_0^{+\infty} \mu(s)\Delta\eta(s)ds + g(u) = f, \\ \partial_t\eta = -\partial_s\eta + \partial_t u, \end{cases} \tag{5.3.1}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, and $u = u(x,t) : \Omega \times [0,+\infty) \to \mathbb{R}$, $\eta = \eta^t(x,s) : \Omega \times \mathbb{R}^+ \times [0,+\infty) \to \mathbb{R}$, subject to the boundary and initial conditions

$$\begin{cases} u(t)|_{\partial\Omega} = \eta^t|_{\partial\Omega} = \eta^t(0) = 0, \\ u(0) = u_0, \quad \partial_t u(0) = \nu_0, \quad \eta^0(s) = \eta_0(s), \end{cases} \tag{5.3.2}$$

with $u_0, \nu_0, \eta_0(s)$ being the given data. Here, $\mu : \overline{\mathbb{R}}^+ \to [0,+\infty)$ is a summable absolutely continuous function, with $\mu'(s) < 0$ almost everywhere. Moreover, the inequality

$$\mu(s+\sigma) \le \Theta e^{-\delta\sigma}\mu(s) \tag{5.3.3}$$

holds for some constants $\Theta \ge 1$ and $\delta > 0$, every $\sigma \ge 0$, and almost every $s \in \mathbb{R}^+$. Without loss of generality, we also assume that

$$\int_0^{+\infty} \mu(s)ds = 1.$$

In fact, the Cauchy problem (5.3.1)–(5.3.2) is called in the so-called memory setting (see, e.g., [180, 182]), and is equivalent to (see, e.g., [326]) the following example of an integro-differential equation arising in the theory of isothermal viscoelasticity

$$u_{tt} - 2\Delta u + \int_0^{+\infty} \mu(s)\Delta u(t-s)ds + g(u) = f, \tag{5.3.4}$$

subject to boundary and initial conditions

$$u(t)|_{\partial\Omega} = 0, \tag{5.3.5}$$

$$u(0) = u_0, \quad u(t)|_{t<0} = u_0 - \eta_0(-t), \quad \partial_t u(0) = \nu_0. \tag{5.3.6}$$

We set $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ with the usual inner products, and interpret $-\Delta$ as a positive self-adjoint operator on H with domain $H^2(\Omega) \cap H_0^1(\Omega)$. We also consider the Hilbert space $M = L^2_\mu(\mathbb{R}^+; V)$ of square-summable functions on $\mathbb{R}^+$ with values in V, with respect to the measure $\mu(s)ds$. To explain the boundary conditions on η, we may view $-\partial_s$ as the linear operator on M with domain

$$\mathrm{Dom}(T) = \{\eta = \eta(s) \in M : \partial_s \eta \in M, \quad \eta(0) = 0\}.$$

Then, $-\partial_s$ is the infinitesimal generator of the right-translation semigroup on M. Now, we introduce the product Hilbert space $X = V \times H \times M$.

Choosing $f \in H$ independent of time and $g \in C^2(\mathbb{R})$, with $g(0) = 0$, such that the growth condition

$$|g''(u)| \le C(1+|u|) \tag{5.3.7}$$

and the dissipation condition

$$\lim_{|u|\to+\infty} \inf \frac{g(u)}{u} > -\lambda \tag{5.3.8}$$

are satisfied, here $\lambda > 0$ is the first eigenvalue of $-\Delta$, problem (5.3.1)–(5.3.2) generates a strongly continuous semigroup $S(t)$ on X which yields the global attractor $\mathcal{A}$ (see [298]). As a byproduct, this establishes the existence of an absorbing set $\mathcal{B}$, but no information is available about the actual entrance time in $\mathcal{B}$, starting from a bounded subset of X. On the other hand, due to the very low dissipation, it seems out of reach to prove the existence of $\mathcal{B}$ by means of standard estimates, without using the gradient-system structure (except in the simpler case when g is sub-linear).

However, if we give a further assumption on the nonlinearity, then applying Theorem 2.1.21, we may find an absorbing set in a direct way.

We take $g \in C^1(\mathbb{R})$, with $g(0) = 0$. Instead of (5.3.8), we assume the slightly less general condition

$$\liminf_{|u|\to+\infty} g'(u) > -\lambda. \tag{5.3.9}$$

Moreover, we replace (5.3.7) by

$$|g(u)|^{6/5} \leq C\mathcal{G}(u) + C, \tag{5.3.10}$$

where

$$\mathcal{G} = \int_0^u g(y)dy.$$

Remark 5.3.1 ([297]). In fact, for instance, any function of the form $g(u) = u|u|^p + g_0(u), p \in (0, 4]$, with $g_0(0) = 0$ and for all $q < p$, $|g_0'(u)| \leq C(1 + |u|^q)$. This includes the physically significant case of the derivative of the double-well potential $g(u) = u^3 - u$ can satisfy conditions (5.3.9)–(5.3.10).

Remark 5.3.2 ([297]). Obviously, it is easy to check that (5.3.10) yields the bound

$$|g(u)| \leq C(1 + |u|^5). \tag{5.3.11}$$

In fact, (5.3.9) and (5.3.11) suffice to ensure the existence of (possibly non-unique) solutions for all initial data $x \in X$, using a standard Galerkin approximation scheme. We agree to call Galerkin solutions those solutions obtained as limits in the approximation scheme, for which formal estimates apply. We also note that it seems impossible to obtain dissipative estimates for Galerkin solutions in the presence of a nonlinearity of supercritical growth rate (that is, with reference to the remark above, when $p > 2$) using the Lyapunov function approach, since the asymptotic compactness of such solutions is not known.

For any $u = u(x) \in V$, we may define

$$G(u) = \int_\Omega \mathcal{G}(u(x))dx.$$

Then following lemma is a straightforward consequence of (5.3.9) (see [297]).

Lemma 5.3.1 ([297]). *There exist constants $\kappa \in (0,1)$ and $C \geq 0$ such that for every $u \in V$,*

$$\langle g(u), u\rangle_H \geq G(u) - \frac{\kappa}{2}\|u\|_V^2 - C, \quad G(u) \geq -\frac{\kappa}{2}\|u\|_V^2 - C. \tag{5.3.12}$$

Give a Galerkin solution $\xi(t) = (u(t), \partial_t u(t), \eta^t)$, with $\xi(0) = (u_0, \nu_0, \eta_0)$, we may also define the corresponding energy by

$$E(t) = \frac{1}{2}\|\xi(t)\|^2.$$

Then we have the following main result due to Gatti, Pata and Zelik [297].

Theorem 5.3.2 ([297]). *Assume that (5.3.9) and (5.3.10) hold, then there exists a constant $R_0 > 0$ such that, for every $R > 0$ and every $x = (u_0, \nu_0, \eta_0) \in X$ with $\|x\| \leq R$, the energy $E(t)$ of a corresponding Galerkin solution fulfills the relation, for all $t \geq t_R$,*

$$E(t) \leq R_0 \tag{5.3.13}$$

with some time $t_R \geq 0$ depending only on R. Both R_0 and t_R can be explicitly computed.

Proof. To simplify the calculations, we may assume that (5.3.3) holds with $\Theta = 1$. In this case, (5.3.3) is equivalent to the following inequality

$$\mu'(s) + \delta\mu(s) \leq 0. \tag{5.3.14}$$

Obviously, the (Lyapunov) functional

$$L(t) = E(t) + G(u(t)) - \langle f, u(t)\rangle_H$$

satisfies the equality

$$\frac{dL(t)}{dt} = -2I(t), \tag{5.3.15}$$

with

$$I(t) = -\int_0^{+\infty} \mu'(s)\|\eta^t(s)\|_V^2 ds \geq \delta\|\eta^t\|_M^2,$$

where the latter inequality follows from (5.3.14). Following [731], choosing now $\nu > 0$ small and $s_\nu > 0$ such that $\int_0^{s_\nu} \mu(s)ds \leq \nu/2$, and putting

$$\mu_\nu(s) = \mu(s_\nu)\chi_{(0,s_\nu]}(s) + \mu(s)\chi_{(s_\nu,+\infty)}(s),$$

and introducing the further functionals

$$\begin{cases} \Phi_1(t) = -\int_0^{+\infty} \mu_\nu(s)\langle\partial_t u(t), \eta^t(s)\rangle_H ds, \\ \Phi_2(t) = \langle\partial_t u(t), u(t)\rangle_H, \end{cases}$$

we can conclude, by exploiting (5.3.11) and (5.3.14), the inequalities (cf., [298], [731])

$$\begin{cases} \dfrac{d}{dt}\Phi_1(t) \le \varepsilon_\nu \|u\|_V^2 - (1-\varepsilon_\nu)\|\partial_t u\|_H^2 + c_\nu I + c_\nu \\ \qquad\qquad + \displaystyle\int_0^{+\infty} \mu(s) \mid \langle g(u), \eta(s)\rangle_H \mid ds, \\ \dfrac{d}{dt}\Phi_2(t) \le \; -(1-\kappa-\varepsilon_\nu)\|u\|_V^2 + \|\partial_t u\|_H^2 - \dfrac{\kappa}{2}\|u\|_V^2 - G(u) + c_\nu I + c_\nu, \end{cases}$$

for some constants $c_\nu \ge 0$ and $\varepsilon_\nu > 0$ such that $\varepsilon_\nu \to 0$ as $\nu \to 0$ (both c_ν and ε_ν can be explicitly computed). Therefore, it follows that by fixing ν small enough, the functional $\Phi(t) = 2\Phi_1(t) + \Phi_2(t)$ satisfies

$$\begin{aligned} &\frac{d}{dt}\Phi(t) + 2\omega E(t) + \frac{\kappa}{2}\|u\|_V^2 + G(u) \\ &\quad \le cI + 2\int_0^{+\infty} \mu(s)|\langle g(u), \eta(s)\rangle_H| ds + c, \end{aligned} \tag{5.3.16}$$

with some constant $\omega > 0$. Hereinafter in this section, $c \ge 0$ stands for a generic constant, independent of the initial data. Hence, for any $\varepsilon \in (0, \varepsilon_0]$ and $\kappa \ge 0$, we may set

$$\Lambda(t) = L(t) + \varepsilon\Phi(t) + \kappa.$$

On the other hand, it follows from (5.3.12) that, if $\varepsilon_0 > 0$ is small enough and κ is large enough, then we have

$$\frac{1-\kappa}{2}E \le \Lambda \le 2E + G(u) + c. \tag{5.3.17}$$

In particular, using again (5.3.12), we may arrive at

$$2\omega E + \frac{\kappa}{2}\|u\|_V^2 + G(u) \ge \omega\Lambda - c.$$

Thus combining (5.3.15) and (5.3.16) and using (5.3.14), up to further reducing ε_0 is needed, we can obtain

$$\frac{d}{dt}\Lambda(t) + \omega\varepsilon\Lambda(t) \le -\delta\|\eta\|_M^2 + 2\varepsilon\int_0^{+\infty} \mu(s) \mid \langle g(u), \eta(s)\rangle_H \mid ds + c.$$

Moreover, it follows from (5.3.12) and (5.3.17) that

$$|G(u)| \le G(u) + \kappa E + c \le \Lambda + c.$$

Therefore, using (5.3.10), we obtain

$$\begin{aligned}2\varepsilon \int_0^{+\infty} \mu(s)|\langle g(u), \eta(s)\rangle_H| ds &\le c\varepsilon \int_0^{+\infty} \mu(s)\|g(u)\|_{L^{6/5}}\|\eta(s)\|_V ds \\ &\le c\varepsilon\|\eta\|_M\|g(u)\|_{L^{6/5}} \\ &\le c\varepsilon\|\eta\|_M + c\varepsilon\|\eta\|_M|G(u)|^{5/6} \\ &\le \delta\|\eta\|_M^2 + c\varepsilon^2\Lambda^{5/3} + c,\end{aligned}$$

which implies that for every $\varepsilon \in (0, \omega\varepsilon_0]$,

$$\frac{d}{dt}\Lambda(t) + \varepsilon\Lambda(t) \le c\varepsilon^2\Lambda^{5/3} + c.$$

By (5.3.11) and (5.3.17),

$$E(t) \le \frac{2}{1-\kappa}\Lambda(t), \qquad \Lambda(0) \le c(1+R^6). \tag{5.3.18}$$

Therefore, applying Theorem 2.1.21 to (5.3.18), we can obtain the desired result. □

Chapter 6

Global Existence and Asymptotic Behavior for Equations of Fluid Dynamics

In this chapter, we prove the global existence and asymptotic behavior of solutions to fluid models. The chapter includes five sections. In Section 6.1, we exploit Theorems 1.4.11 and 1.5.1 to show the asymptotic behavior of the solutions to the Navier–Stokes equations in 2D exterior domains. In Section 6.2, we use Theorems 2.2.1 and Theorem 2.2.9 to investigate the asymptotic behavior of the L^α norm ($\alpha > 2$) of strong solutions to the initial value problem for the nonstationary Navier–Stokes equations in the whole space. In Section 6.3, we use Theorem 2.2.3 to study an equation for a 1D viscous compressible barotropic fluid. In Section 6.4, we exploit Theorems 1.1.2, 2.1.13 and 2.2.4 to establish uniform estimates for symmetric solutions to s system of quasilinear equations. In Section 6.5, we use Theorems 1.2.1, 2.1.13, and 2.2.2 to establish the pointwise and the stabilization for 1D compressible Navier–Stokes equations. In the nuclear fluid case, we also justify the sharpness of the main condition on the "self-gravitation" force. Inequalities used in this chapter are crucial and typical in establishing the global existence and large-time behavior of global solutions to fluid models.

6.1 Asymptotic behavior for the 2D homogeneous incompressible Navier–Stokes equations

In this section, we shall exploit Theorems 1.4.11 and 1.5.1 to show the asymptotic behavior of the incompressible Navier–Stokes equations in 2D exterior domains. These results are chosen from Bae and Jin [57] as applications of Theorems 1.4.11 and 1.5.1.

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6.1.1 Introduction

We shall consider the following incompressible Navier–Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u + \nabla p = 0, \quad \text{in } (x,t) \in \Omega \times (0,+\infty), & (6.1.1) \\ \nabla \cdot u = 0, \quad \text{in } (x,t) \in \Omega \times (0,+\infty), & (6.1.2) \end{cases}$$

subject to no slip boundary and initial conditions

$$\begin{cases} u(x,t) = 0, \quad \text{on } \partial\Omega, & (6.1.3) \\ u(x,0) = u_0(x), & (6.1.4) \end{cases}$$

where u and p are the velocity and pressure, respectively, of the incompressible fluid in an exterior domain Ω of a simply connected set B in $\mathbb{R}^2$ which contains the origin and is contained in a bounded ball, for example, the unit ball. Furthermore, we assume that the boundary ∂B is smooth enough to get the unique existence of the solution.

We now introduce the following definition.

Definition 6.1.1. Let $C^\infty_{0,\sigma}(\Omega)$ denote the set of all C^∞-real vector functions $\phi = (\phi^1, \phi^2)$ with compact support in Ω, such that $\operatorname{div}\phi = 0$. $L^2_\sigma(\Omega)$ is the closure of $C^\infty_{0,\sigma}$, with respect to the L^2-norm $||\cdot||_{L^2}$. We use the usual notation that $L^p_\sigma(\Omega)$ means the set of measurable functions in $L^p(\Omega)$ with divergence free.

Let P_Ω be the Leray projection from $L^2(\Omega) \to L^2_\sigma(\Omega)$, and let $-A_\Omega = P_\Omega\Delta$ be the infinitesimal generator of semigroup $e^{-A_\Omega t}$. Let us recall the following well-known decay estimates of the Stokes semigroup in a 2D exterior domain (see [194, 867]).

Lemma 6.1.1 ([57])**.** *Let $f \in L^r_\sigma(\Omega)$. Then we have for all $t > 0$,*

$$||e^{-A_\Omega(t)} f||_{L^q} \le Ct^{-(\frac{1}{r}-\frac{1}{q})}\|f\|_{L^r}, \tag{6.1.5}$$

with $1 < r \le q \le +\infty,\ 1 < r < q = +\infty$; *and for all* $t > 0$,

$$\|\nabla e^{-A_\Omega(t)} f\|_{L^q} \le Ct^{-(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}}\|f\|_{L^r}, \tag{6.1.6}$$

with $1 < r \le q \le 2$.

Assume that u is a solution of the Navier–Stokes equations in a 2D domain with the initial velocity $u_0 \in L^2_\sigma(\Omega)$. Dan and Shibata [193] showed the following lemma (see also, e.g., Kozonon and Ogawa [458]).

Lemma 6.1.2 ([57])**.** *There holds that, for $2 \le q < +\infty$, as $t \to +\infty$,*

$$||u(t)||_{L^q} = o(t^{-(1/2-1/q)}), \quad \|\nabla u(t)\|_{L^2} = o(t^{-1/2}). \tag{6.1.7}$$

Moreover, we have

$$\|u(t)\|_{L^\infty} = o(t^{-1/2}). \tag{6.1.8}$$

Lemma 6.1.3 ([57]). *Assume that* u *is a solution of problem* (6.1.1)–(6.1.4) *satisfying Lemma* 6.1.2. *Then we have, for all* $t > s > 0$, *and for* $2 \leq q \leq +\infty$,

$$||e^{-A_\Omega(t-s)}[P_\Omega(u\cdot\nabla)u](s)||_{L^q} \leq C(t-s)^{-1/2}||u\otimes u||_{L^q}. \tag{6.1.9}$$

Proof. Let $\phi \in C^\infty_{0,\sigma}(\Omega)$. We note that for $\frac{1}{q'} = 1 - \frac{1}{q}$, by (6.1.6),

$$\begin{aligned}&\langle e^{-A_\Omega(t-s)}[P_\Omega(u\cdot\nabla)u](s),\phi\rangle = \langle u\otimes u, \nabla e^{-A_\Omega(t-s)}\phi\rangle\\ &\quad\leq C||u\otimes u||_{L^q}||\nabla e^{-A_\Omega(t-s)}\phi||_{L^{q'}} \leq C||u\otimes u||_{L^q}(t-s)^{-1/2}||\phi||_{L^{q'}}.\end{aligned}$$

Consequently,

$$||e^{-A_\Omega(t-s)}[P_\Omega(u\cdot\nabla)u](s)||_{L^q} \leq C(t-s)^{-1/2}||u\otimes u||_{L^q}. \qquad \square$$

For the proof of our main theorem, we need the following lemma.

Lemma 6.1.4 ([57]). *For all* $t > 0$, *we have, if* $u_0 \in L^2_\sigma(\Omega)$ *for* $q \geq 2$, *then*

$$||u(t)||_{L^q} \leq Ct^{-1/2+1/q},$$

and if $u_0 \in L^q(\Omega)\cap L^2_\sigma(\Omega)$ *for* $1 < q < 2$, *then*

$$||u(t)||_{L^q} \leq Ct^{-1/2+1/q}.$$

Proof. It is well known that $u(t) \in L^q(\Omega)$ for $2 \leq q \leq +\infty$ if $u_0 \in L^2_\sigma(\Omega)$. Moreover, $||u(t)||_{L^q} \leq Ct^{-1/2+1/q}$ for $2 \leq q \leq +\infty$. Hence, we have only to show that $u(t) \in L^q(\Omega)$ for $1 < q < 2$, if $u_0 \in L^q(\Omega)\cap L^2_\sigma(\Omega)$. By the Duhamel principle, u can be represented as

$$u(t) = e^{-A_\Omega t}u_0 + \int_0^t e^{-A_\Omega(t-s)}[P_\Omega(u\cdot\nabla)u](s)ds = I + II. \tag{6.1.10}$$

By (6.1.5), I and II are estimated by

$$||I||_{L^q} \leq C||u_0||_{L^q}, \quad ||II||_{L^q} \leq C\int_0^t ||(u\cdot\nabla)u(s)||_{L^q}ds.$$

By the Hölder inequality,

$$||(u\cdot\nabla)u(s)||_{L^q} \leq ||u(s)||_{L^{2q/(2-q)}}||\nabla u(s)||_{L^2},$$

and from (6.1.7), it follows

$$||u(s)||_{L^{2q/(2-q)}} \leq Cs^{-1/2+(2-q)/2q} = Cs^{-1+1/q}, \quad ||\nabla u(s)||_{L^2} \leq Cs^{-1/2}.$$

Hence,

$$||II||_{L^q} \leq C\int_0^t s^{-3/2+1/q}ds \leq Ct^{-1/2+1/q}.$$

By the estimates of I and II, we conclude that, for $1 < q < 2$, if $u_0 \in L^q(\Omega)\cap L^2_\sigma(\Omega)$, for all $t > 0$,

$$||u(t)||_{L^q} \leq Ct^{-1/2+1/q}. \qquad \square$$

Remark 6.1.1 ([57]). We note that $u_0 \in L^2_\sigma(\Omega) \cap L^r(\Omega),\ 1 < r \le q < 2$, implies $u_0 \in L^q(\Omega)$, by interpolation techniques. We note that if $0 < \varepsilon < \min\{1/q, 1/2\}$, then

$$\lim_{t\to 0^+} t^{-\varepsilon} \int_0^t s^{-1/2}||u(s)||_{L^q} ds = 0,$$

since by Lemma 6.1.2, for all $q \ge 2$,

$$\int_0^t s^{-1/2}||u(s)||_{L^q} ds \le C \int_0^t s^{-1/2} s^{-(1/2)+(1/q)} ds \le Ct^{1/q},$$

and for all $1 < q < 2$,

$$\int_0^t s^{-1/2}||u(s)||_{L^q} ds \le C \int_0^t s^{-1/2}(1+s)^{-1/2+1/q} \le Ct^{1/2}(1+t)^{-1/2+1/q}.$$

Theorem 6.1.5 ([57]). *Let $1 < r \le q < +\infty$ or $1 < r < q = +\infty$. Assume $u_0 \in L^2_\sigma(\Omega) \cap L^r(\Omega)$. Let u be the solution of the Navier–Stokes equations in a two-dimensional exterior domain Ω with the initial velocity u_0, so that estimates* (6.1.7)–(6.1.8) *hold. Moreover, assume that for a given small $\varepsilon > 0$, for all $t > 0$,*

$$||u(t)||_{L^\infty} \le \varepsilon t^{-1/2}. \tag{6.1.11}$$

Then, for $1 < r \le q < +\infty$ or $1 < r < q = +\infty$,

$$||u(t)||_{L^q} = O\left(t^{-1/r+1/q}\right). \tag{6.1.12}$$

Moreover, for $1 < r \le q \le 2$,

$$||\nabla u(t)||_{L^q} = O\left(t^{-1/r+1/q-1/2}\right). \tag{6.1.13}$$

Proof. By the Duhamel principle, u can be written as $u = I + II$, where I and II are defined in (6.1.10). By (6.1.5), we have, for $1 < r \le q \le +\infty$ or $1 < r < q = +\infty$, if $u_0 \in L^r(\Omega)$,

$$||I||_{L^q} \le Ct^{-(1/r-1/q)}. \tag{6.1.14}$$

By (6.1.6), we also get, for $1 < r \le q \le 2$, whenever $u_0 \in L^r(\Omega)$,

$$\|\nabla I\|_{L^q} \le Ct^{-1/r+1/q-1/2}.$$

Step 1. We shall prove the inequality (6.1.12) for $1 < r \le q \le +\infty$, $\frac{2q}{q+2} < r$ (since $\frac{2q}{q+2} \le 2$).

By the hypotheses,

$$||u \otimes u||_{L^q} \le C||u(s)||_{L^\infty}||u(s)||_{L^q} \le C\varepsilon s^{-1/2}||u(s)||_{L^q}.$$

If we apply (6.1.5) to II, then

$$||II||_{L^q} \leq C\varepsilon \int_0^t (t-s)^{-1/2} s^{-1/2} ||u(s)||_{L^q} ds. \tag{6.1.15}$$

Combining (6.1.14) and (6.1.15), we derive for $1 < r \leq q \leq +\infty$,

$$||u(t)||_{L^q} \leq Ct^{-(1/r-1/q)} + C\varepsilon \int_0^t (t-s)^{-1/2} s^{-1/2} ||u(s)||_{L^q} ds. \tag{6.1.16}$$

Let $X(t) = ||u(t)||_{L^q}$; then (6.1.16) can be rewritten as

$$X(t) \leq Ct^{-(1/r-1/q)} + C\varepsilon \int_0^t (t-s)^{-1/2} s^{-1/2} X(s) ds. \tag{6.1.17}$$

By Theorem 1.4.11,

$$X(t) \leq Ct^{-(\frac{1}{r}-\frac{1}{q})} + C\varepsilon t^{-1/2} \int_0^t s^{-1/2} X(s) ds. \tag{6.1.18}$$

Recall Remark 6.1.1 that if $\varepsilon < \min\left\{\frac{1}{q}, \frac{1}{2}\right\}$, $1 < q \leq +\infty$, we get

$$\lim_{t \to 0^+} t^{-\varepsilon} \int_0^t s^{-1/2} X(s) ds = 0.$$

Hence, applying Theorem 1.5.1, we conclude that there is a constant $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, for all $t > 0$, for $\frac{1}{r} - \frac{1}{q} < \frac{1}{2}$, i.e., for $\frac{2q}{q+2} < r \leq q < +\infty$,

$$X(t) \leq Ct^{-(\frac{1}{r}-\frac{1}{q})}. \tag{6.1.19}$$

Step 2. We shall prove the inequality (6.1.12) for $1 < r < \frac{2q}{2+q}$, $r \leq q < +\infty$.

Taking $p \in (1, r)$, and using (6.1.6),

$$\begin{aligned} ||e^{-A_\Omega(t-s)}[P_\Omega(u \cdot \nabla)u](s)||_{L^q} &\leq C(t-s)^{-1/p+1/q} ||P_\Omega(u \cdot \nabla)u](s)||_{L^p} \\ &\leq C(t-s)^{-1/p+1/q} ||(u \cdot \nabla)u||_{L^p} \leq C(t-s)^{-1/p+1/q} ||u(s)||_{L^{2p/(2-p)}} ||\nabla u(s)||_{L^2}. \end{aligned} \tag{6.1.20}$$

Let $m = \frac{2p}{2-p}$. Since $m > r > p = \frac{2m}{2+m}$, the result of **Step 1**, implies that, if $u_0 \in L^r(\Omega) \cap L^2_\sigma(\Omega)$,

$$||u||_{L^m} \leq Ct^{-1/r+1/m}.$$

Hence, II can be estimated as, for all $t > 0$,

$$||II||_{L^q} \leq C \int_0^t (t-s)^{-1/p+1/q} s^{-1/r+1/m} s^{-1/2} ds \leq Ct^{-1/r+1/q}.$$

Step 3. We shall prove the inequality (6.1.12) for $1 < r < q = +\infty$.

Taking $p \in (1,2)$ with $\frac{2r}{2+r} < p < r$, and using (6.1.5), we get

$$||e^{-A_\Omega(t-s)}[P_\Omega(u\cdot\nabla)u](s)||_{L^\infty} \le C(t-s)^{-1/p}||u(s)||_{L^{2p/(2-p)}}||\nabla u(s)||_{L^2}.$$

Let $m = \frac{2p}{2-p}$. Since $m > r > p = \frac{2m}{2+m}$, the result of **Step 1** implies that, if $u_0 \in L^r(\Omega) \cap L^2_\sigma(\Omega)$,

$$||u||_{L^m} \le Ct^{-1/r+1/m}.$$

Hence, II can be estimated as, for all $t > 0$,

$$||II||_{L^\infty} \le C\int_0^t (t-s)^{-1/p}s^{-1/r+1/m}s^{-1/2}ds \le Ct^{-1/r}.$$

Step 4. We shall prove the inequality (6.1.13) for $1 < r \le q \le 2$ (since that $\frac{2q}{q+2} \le 1$).

Choosing $1 < p < r$, and using (6.1.6), we have

$$||\nabla e^{-A_\Omega(t-s)}[P_\Omega(u\cdot\nabla)u](s)||_{L^q} \le C(t-s)^{-1/p+1/q-1/2}||P_\Omega(u\cdot\nabla)u(s)||_{L^p}.$$

We note that

$$||P_\Omega(u\cdot\nabla)u(s)||_{L^p} \le C||(u\cdot\nabla)u(s)||_{L^p} \le C||u(s)||_{L^{2p/(2-p)}}||\nabla u(s)||_{L^2}.$$

Let $m = \frac{2p}{2-p}$. Note that $2 < m < +\infty$. By (6.1.19) of **Step 1**, if $u_0 \in L^r(\Omega)\cap L^2_\sigma(\Omega)$,

$$||u(s)||_{L^m} \le Cs^{-1/r+1/m}.$$

Hence, ∇II can be estimated as

$$||\nabla II||_{L^q} \le C\int_0^t (t-s)^{-1/p+1/q-1/2}s^{-1/r+(2-p)/2p}s^{-1/2}ds \le Ct^{-1/r+1/q-1/2}. \quad \square$$

6.2 Large-time behavior for nonhomogeneous incompressible Navier–Stokes equations

In this section, we shall use Theorems 2.2.1 and 2.2.9 to investigate the asymptotic behavior of the L^α norm ($\alpha > 2$) of strong solutions to the initial value problem for the nonstationary Navier–Stokes equations in the whole space. We present the results from Zheng [999] as applications of Theorems 2.2.1 and 2.2.9. We shall consider the following initial value problem for the nonstationary Navier–Stokes equations in the whole space $\mathbb{R}^n$:

$$\begin{cases} v_t - \mu\Delta v + (v\cdot\nabla)v + \nabla p = f \text{ in } [0,+\infty)\times\mathbb{R}^n, \\ \nabla\cdot v = 0 \text{ in } [0,+\infty)\times\mathbb{R}^n, \\ v|_{t=0} = a(x) \text{ in } \mathbb{R}^n, \end{cases} \tag{6.2.1}$$

where $n \geq 2$ and μ is a given positive constant, $a(x)$ is the initial velocity, $f(x,t)$ is the external force, and $v(x,t)$, $p(x,t)$ are the vector field, the scalar field respectively. The pressure p is determined up to a constant by the vector field v.

Throughout this section, we always assume that a and f are weakly divergence free, i.e.,

$$\begin{cases} \nabla \cdot f = 0, \quad \text{a.e. in } (0,+\infty), & (6.2.2) \\ \nabla \cdot a = 0 & (6.2.3) \end{cases}$$

are satisfied in the sense of distributions.

By a solution of problem (6.2.1) on $[0,T]$ we mean a divergence-free vector field $v(t,x) \in L^1(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^n))$ such that

$$\int_0^T \int_{\mathbb{R}^n} [v \cdot \phi' + \mu v \cdot \Delta\phi + (v \cdot \nabla)\phi \cdot v + f \cdot \phi] dxdt = -\int_{\mathbb{R}^n} a\phi|_{t=0} dx \tag{6.2.4}$$

for every regular divergence-free vector field $\phi(t,x)$, which has compact support in the space variables and satisfies $\phi(T,x) \equiv 0$.

First, we shall use Theorem 2.2.1 to prove the following theorem, due to Zheng [999].

Theorem 6.2.1 ([999]). *Let $n \geq 3$ and $\alpha \geq n$. Assume that $a \in L^\alpha(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $f \in L^1(0,+\infty; L^\alpha(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)) \cap L^\infty(0,+\infty; L^\alpha(\mathbb{R}^n))$ and satisfy* (6.2.2) *and* (6.2.3). *If $||a||_{L^\alpha(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)}$, $||f||_{L^1(0,+\infty;L^2(\mathbb{R}^n))}$, and $||f||_{L^\infty(0,+\infty;L^\alpha(\mathbb{R}^n))}$ are sufficiently small, then when $\alpha > n$, problem* (6.2.1) *admits a unique solution $v \in C([0,+\infty); L^2(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)) \cap L^2(0,+\infty; H^1(\mathbb{R}^n))$. Moreover, as $t \to +\infty$,*

$$||v(t)||_{L^\alpha(\mathbb{R}^n)} \to 0. \tag{6.2.5}$$

Proof. We follow the proof due to [999]. The case $\alpha > n$: The global existence and uniqueness of strong solutions were proved in [87]. The proof of (6.2.5) consists of the following four steps. In the first two steps, we essentially use the same argumentation as in [87].

Step 1. Let $T \in (0,+\infty)$ and v be a solution of problem (6.2.1) in the class

$$v(t) \in L^1(0,T; W^{2,\alpha}(\mathbb{R}^n)), v_t(t) \in L^1(0,T; L^\alpha(\mathbb{R}^n)). \tag{6.2.6}$$

Then it follows from Theorem 1.4 and Lemma 3.1 in [87] that $y(t) = ||v(t)||_{L^\alpha}$ satisfies the following differential inequality (see formula (3.6) in [87]):

$$\begin{cases} y'(t) \leq -C_8 \left(C_9 \mu K^{-\beta} - \mu^{-\frac{n+\alpha}{\alpha-n}} y^\gamma \right) y^{1+\beta} + ||f||_{L^\alpha}, \quad t \in (0,T), \\ y(0) = ||a||_{L^\alpha}, \end{cases} \tag{6.2.7}$$

where

$$\begin{cases} K = ||a||_{L^2} + \int_0^{+\infty} ||f(\tau)||_{L^2} d\tau, & (6.2.8) \\ \beta = \frac{4\alpha}{(\alpha-2)n}, \quad \gamma = \frac{2\alpha^2(n-2)}{n(\alpha-2)(\alpha-n)} & (6.2.9) \end{cases}$$

and C_8, C_9 are positive constants independent of t, v, a, f, as described in [87].

Step 2. Let $z(t)$ be the solution to the initial value problem

$$\begin{cases} z'(t) = -C_8[C_9\mu K^{-\beta} - \mu^{-\frac{n+\alpha}{\alpha-n}} z^\gamma(t)] z^{1+\beta}(t) + ||f(t)||_{L^\alpha}, \\ z(0) = ||a||_{L^\alpha}. \end{cases} \tag{6.2.10}$$

Then, as proved in [87], when the smallness conditions on a and f are satisfied, problem (6.2.10) admits a global solution $z(t) \in C([0,+\infty))$, $z(t) \geq 0$, $z'(t) \in L^1_{\mathrm{loc}}((0,+\infty))$. Moreover, for all $t \geq 0$,

$$0 \leq z^\gamma(t) \leq \frac{C_9}{2}\mu^{2\alpha/(\alpha-n)}K^{-\beta}. \tag{6.2.11}$$

Step 3. It turns out that, by (6.2.10)–(6.2.11), the following holds: for all $t \geq 0$,

$$\begin{cases} z'(t) + C_{10} z^{1+\beta}(t) \leq ||f(t)||_{L^\alpha}, \\ z(0) = ||a||_{L^\alpha}, \end{cases} \tag{6.2.12}$$

with

$$C_{10} = \frac{C_8 C_9}{2}\mu K^{-\beta}. \tag{6.2.13}$$

Integrating the equation in (6.2.12) with respect to t yields for all $t \geq 0$,

$$\int_0^t z^{1+\beta}(\tau)d\tau \leq \frac{1}{C_{10}}\left(\int_0^{+\infty} ||f||_{L^\alpha}d\tau + ||a||_{L^\alpha}\right) \equiv: \tilde{C}. \tag{6.2.14}$$

Now let $w(t) = z^{1+\beta}(t)$. It thus follows from (6.2.12) and Young's inequality that

$$\begin{aligned} w'(t) &= (1+\beta)z^\beta(t)z'(t) \leq (1+\beta)z^\beta(t)||f(t)||_{L^\alpha} \leq \beta z^{1+\beta}(t) + \frac{1}{1+\beta}||f(t)||_{L^\alpha}^{1+\beta} \\ &\leq \frac{\beta}{2}w^2(t) + \frac{\beta}{2} + \frac{1}{1+\beta}||f(t)||_{L^\alpha}^{1+\beta}. \end{aligned} \tag{6.2.15}$$

By the assumption on f, $||f||_{L^\alpha}^{1+\beta} \in L^1(\mathbb{R}^+)$. By Theorem 2.2.1, we conclude from (6.2.14)–(6.2.15) that as $t \to +\infty$,

$$w(t) \to 0, \tag{6.2.16}$$

i.e., as $t \to +\infty$,

$$z(t) \to 0. \tag{6.2.17}$$

On the other hand, it follows from (6.2.10), (6.2.7), and the comparison principle for ODEs that for all $t \geq 0$,

$$0 \leq y(t) \leq z(t). \tag{6.2.18}$$

Step 4. By approximating a and f by regular vector fields, then passing to the limit, as shown in [87], we can deduce that for the unique global solution v whose existence is stated in Theorem 6.2.1, as well as in Theorem 0.2 in [87], $||v||_{L^\alpha}$ also satisfies (6.2.18). Thus the proof is complete by combining (6.2.18) with (6.2.17). □

Second, we shall use Theorem 2.2.9 to show the next result also due to Zheng [999].

Theorem 6.2.2 ([999]). *Let $n = 2$ and $\alpha > n = 2$. Assume that $a \in L^\alpha(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $f \in L^1(0, +\infty; L^\alpha(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$. Then problem* (6.2.1) *admits a unique solution*

$$v(t) \in C([0, +\infty); L^2(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)) \cap L^2(0, +\infty; H^1(\mathbb{R}^n)).$$

Moreover, as $t \to +\infty$,

$$||v(t)||_{L^\alpha} \to 0. \tag{6.2.19}$$

Proof. The proof consists of the following steps.

Step 1. For the case $n = 2$, we know that for $a \in L^2(\mathbb{R}^n)$, $f \in L^1([0, +\infty); L^2(\mathbb{R}^n))$, problem (6.2.1) admits a unique solution

$$v(t) \in C([0, +\infty); L^2(\mathbb{R}^n)) \cap L^2([0, +\infty); H^1(\mathbb{R}^n)),\ v_t(t) \in L^1([0, +\infty); H^1(\mathbb{R}^n))$$

(see, e.g., [914]). On the other hand, under the additional assumptions on a and f, the local existence and uniqueness result of Theorem 2.2 in [87] states that the above solution $v(t)$ belongs to $C([0, T_0); L^\alpha(\mathbb{R}^n))$ for some $T_0 > 0$. Therefore, to prove that $v(t) \in C([0, +\infty); L^\alpha(\mathbb{R}^n))$, it suffices to show that $||v(t)||_{L^\alpha}$ is uniformly bounded. It is easy to see from the energy inequality that for all $t \geq 0$,

$$||v(t)||_{L^2} \leq ||a||_{L^2} + \int_0^{+\infty} ||f(s)||_{L^2} ds, \tag{6.2.20}$$

and for all $t \geq 0$,

$$\mu \int_0^t ||\nabla v(s)||^2_{L^2} ds \leq \Big(||a||_{L^2} + \int_0^{+\infty} ||f(s)||_{L^2} ds \Big) \int_0^{+\infty} ||f(s)||_{L^2} ds. \tag{6.2.21}$$

By the Gagliardo–Nirenberg inequality for $n = 2$, we have

$$||v(t)||_{L^\alpha} \leq C ||\nabla v(t)||^{(\alpha-2)/\alpha}_{L^2} ||v(t)||^{2/\alpha}_{L^2}. \tag{6.2.22}$$

Then it follows from (6.2.20)–(6.2.22) that $v(t)$ belongs to $L^q(0, +\infty; L^\alpha(\mathbb{R}^2))$ with $q = \frac{2\alpha}{\alpha-2}$. Now $2/q + n/\alpha = 1$ $(n = 2)$. Then it is well known (see, e.g., [87]) that $||v(t)||_{L^\alpha}$ is uniformly bounded. Therefore, combining this with the local existence and uniqueness result, we can conclude that $v(t) \in C([0, +\infty); L^\alpha(\mathbb{R}^2))$. Furthermore, it follows that

$$\int_0^{+\infty} ||v(t)||^q_{L^\alpha} dt \leq C_q \tag{6.2.23}$$

with any q, such that $2\alpha/(\alpha - 2) \leq q < +\infty$.

Step 2. We now want to show that $y(t) = ||v(t)||^p_{L^\alpha},\ p = \alpha^2/(\alpha-2) \geq 2\alpha/(\alpha-2)$ satisfies condition (2.2.90) in Theorem 2.2.9. To this end, we approximate a and f by sequences of smooth divergence-free vector fields a_k and f_k with compact support in x such that

$$\begin{cases} a_k \to a \ \text{ strongly in } \ L^\alpha(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), & (6.2.24) \\ f_k \to f \ \text{ strongly in } \ L^1([0,+\infty); L^\alpha(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)) & (6.2.25) \end{cases}$$

and

$$f_k \to f \ \text{ weakly in } \ L^\infty([0,+\infty); L^\alpha(\mathbb{R}^2)). \tag{6.2.26}$$

Then for each a_k and f_k, since $n = 2$, problem (6.2.1) has a unique global smooth solution v_k. Multiplying the first equation in (6.2.1) by $|v_k|^{\alpha-2}v_k$, integrating the result with respect to x, and using integration by parts we obtain

$$\frac{1}{\alpha}\frac{d}{dt}||v_k||^\alpha_{L^\alpha} + \frac{\mu}{2}N_\alpha(v_k) + 4\mu\frac{\alpha-2}{\alpha}M_\alpha(v_k) = I_1 + I_2, \tag{6.2.27}$$

where

$$\begin{cases} N_\alpha(v_k) = \displaystyle\int_{\mathbb{R}^2} |\nabla v_k|^2 |v_k|^{\alpha-2} dx, & (6.2.28) \\ M_\alpha(v_k) = \displaystyle\int_{\mathbb{R}^2} |\nabla |v_k|^{\alpha/2}|^2 dx, & (6.2.29) \end{cases}$$

and I_1 and I_2 are integrals involving p_k and f_k, respectively. Following the same proof as for Lemmas 1.1 and 1.2 in [87], we can derive the following estimates

$$|I_1| \leq C\frac{(\alpha-2)^2}{\mu}||v_k||^{\alpha+2}_{L^{\alpha+2}}, \tag{6.2.30}$$

and

$$|I_2| \leq ||f_k||_{L^\alpha}||v_k||^{\alpha-1}_{L^\alpha}. \tag{6.2.31}$$

Let $g = |v_k|^{\alpha/2}$. By the Gagliardo–Nirenberg inequality for $n = 2$,

$$||g||_{L^{2(\alpha+2)/\alpha}} \leq C||\nabla g||^{2/(\alpha+2)}_{L^2}||g||^{\alpha/(\alpha+2)}_{L^2}, \tag{6.2.32}$$

i.e.,

$$||v_k||^{\alpha+2}_{L^{\alpha+2}} \leq C||v_k||^\alpha_{L^\alpha}\left(M_\alpha(v_k)\right)^{2/\alpha}. \tag{6.2.33}$$

Note that

$$\left|\nabla|v_k|^{\alpha/2}\right| \leq \frac{\alpha}{2}|v_k|^{(\alpha/2)-1}|\nabla v_k| \quad \text{a.e. in } \mathbb{R}^2. \tag{6.2.34}$$

Combining (6.2.32) with (6.2.33) yields, for any $\varepsilon > 0$,

$$||v_k||^{\alpha+2}_{L^{\alpha+2}} \leq C||v_k||^\alpha_{L^\alpha}\left(N_\alpha(v_k)\right)^{2/\alpha} \leq \varepsilon N_\alpha(v_k) + C_\varepsilon||v_k||^{\alpha^2/(\alpha-2)}_{L^\alpha}. \tag{6.2.35}$$

Then it follows from (6.2.27), (6.2.30), (6.2.31) and (6.2.35) that

$$\begin{aligned}\frac{1}{\alpha}\frac{d}{dt}||v_k||_{L^\alpha}^\alpha + \frac{\mu}{4}N_\alpha(v_k) + 2\mu\frac{\alpha-2}{\alpha}M_\alpha(v_k)\\ \leq C||v_k||_{L^\alpha}^{\alpha^2/(\alpha-2)} + ||f_k||_{L^\alpha}||v_k||_{L^\alpha}^{\alpha-1}.\end{aligned} \tag{6.2.36}$$

Therefore, for $p = \alpha^2/(\alpha-2)$,

$$\frac{d}{dt}||v_k||_{L^\alpha}^p \leq C\left(||v_k||_{L^\alpha}^{(\alpha^2+2\alpha)/(\alpha-2)} + ||f_k||_{L^\alpha}||v_k||_{L^\alpha}^{p-1}\right). \tag{6.2.37}$$

Integrating (6.2.37) with respect to t yields that for any $0 \leq s < t < +\infty$,

$$\begin{aligned}&||v_k(t)||_{L^\alpha}^p - ||v_k(s)||_{L^\alpha}^p\\ &\leq C\int_s^t \left(||v_k(\tau)||_{L^\alpha}^{(\alpha^2+2\alpha)/(\alpha-2)} + ||f_k(\tau)||_{L^\alpha}||v_k(\tau)||_{L^\alpha}^{p-1}\right)d\tau\\ &\leq C\int_s^t \left(||v_k(\tau)||_{L^\alpha}^{p(\alpha+2)/\alpha} + ||v_k(\tau)||_{L^\alpha}^p + ||f_k(\tau)||_{L^\alpha}^p\right)d\tau.\end{aligned} \tag{6.2.38}$$

On the other hand, by the energy estimates it is easy to conclude from the first equation of problem (6.2.1) that

$$v_k \to v \ \text{ in } \ C([0,+\infty); L^2(\mathbb{R}^2)) \tag{6.2.39}$$

and

$$v_k \to v \ \text{ in } \ L^2([0,+\infty); H^1(\mathbb{R}^2)). \tag{6.2.40}$$

Indeed, we deduce from the first equation of problem (6.2.1) and the Gagliardo–Nirenberg inequality that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||v-v_k||_{L^2}^2 + \mu||\nabla(v-v_k)||_{L^2}^2 \qquad (6.2.41)\\ &\leq C||v-v_k||_{L^4}^2||\nabla v_k||_{L^2} + C||f-f_k||_{L^2}||v-v_k||_{L^2}\\ &\leq C||v-v_k||_{L^2}||\nabla(v-v_k)||_{L^2}||\nabla v_k||_{L^2} + C||f-f_k||_{L^2}||v-v_k||_{L^2}\\ &\leq \frac{\mu}{2}||\nabla(v-v_k)||_{L^2}^2 + C||v-v_k||_{L^2}^2||\nabla v_k||_{L^2}^2 + C||f-f_k||_{L^2}||v-v_k||_{L^2},\end{aligned}$$

whence

$$\frac{d}{dt}||v-v_k||_{L^2} \leq C||\nabla v_k||_{L^2}^2||v-v_k||_{L^2} + C||f-f_k||_{L^2}. \tag{6.2.42}$$

It follows from (6.2.42), (6.2.21) that as $k \to +\infty$, for all $t > 0$,

$$\begin{aligned}&||v-v_k(t)||_{L^2} \qquad (6.2.43)\\ &\leq \left(||a-a_k||_{L^2} + C\int_0^t ||f-f_k||_{L^2}d\tau\right)\exp\left(\int_0^t ||\nabla v_k(\tau)||_{L^2}^2 d\tau\right) \to 0.\end{aligned}$$

Integrating (6.2.41) with respect to t and combining the result with (6.2.43) yields that for any $t > 0$, as $k \to +\infty$,

$$\int_0^t ||\nabla(v - v_k)||_{L^2}^2 d\tau \to 0. \tag{6.2.44}$$

By the Gagliardo–Nirenberg inequality (6.2.22), we have

$$v_k \to v \text{ in } L^q([0,+\infty); L^\alpha(\mathbb{R}^2)) \tag{6.2.45}$$

with $q = 2\alpha/(\alpha - 2)$. Since $||v_k||_{L^\alpha}$ and $||v||_{L^\alpha}$ are uniformly bounded, it follows from (6.2.45) that for any $T > 0$, for any q, $1 \le q < +\infty$,

$$||v_k||_{L^\alpha} \to ||v||_{L^\alpha} \text{ in } L^q(0,T), \tag{6.2.46}$$

which also implies that for almost all $t \in (0,T)$, $||v_k(t)||_{L^\alpha} \to ||v(t)||_{L^\alpha}$. Taking the limit in (6.2.38) yields

$$||v(t)||_{L^\alpha}^p - ||v(s)||_{L^\alpha}^p \le C \int_s^t \Big(||v(\tau)||_{L^\alpha}^{p(\alpha+2)/\alpha} + ||v(\tau)||_{L^\alpha}^p + ||f(\tau)||_{L^\alpha}^p\Big) d\tau. \tag{6.2.47}$$

By continuity of $||v(t)||_{L^\alpha}^p$, (6.2.47) holds for any $0 \le s < t < +\infty$. Thus combining (6.2.47) with (6.2.23) and applying Theorem 2.2.9 yields, as $t \to +\infty$,

$$||v(t)||_{L^\alpha} \to 0,$$

which completes the proof. □

6.3 A uniform lower Bound for density of a 1D viscous compressible barotropic fluid equation

In this section, we shall use Theorem 2.2.3 to give a uniform lower bound for density of a 1D viscous compressible barotropic fluid equation. These results are adopted from Straŝkraba and Zlotnik [900].

We consider the following system of equations describing 1D flow of a viscous compressible barotropic fluid:

$$\begin{cases} \rho_t + (\rho u)_x = 0, & (6.3.1) \\ (\rho u)_t + (\rho u^2)_x - \big(\mu u_x - p(\rho)\big)_x = \rho f, & (6.3.2) \end{cases}$$

in the domain $Q = \Omega \times \mathbb{R}^+ = (0,l) \times (0,+\infty)$ with the boundary and initial conditions

$$u\,|_{x=0,l} = 0; \quad \rho\,|_{t=0} = \rho^0(x), \quad u\,|_{t=0} = u^0(x) \text{ in } \Omega. \tag{6.3.3}$$

As in [900], we make the following assumptions:

1) The mass force
$$f(x,t) = f_\infty(x) + \Delta f(x,t) \tag{6.3.4}$$
with $f_\infty \in L^\infty(\Omega)$ and $\Delta f = f_1 + f_2$ with $f_1 \in L^{\infty,1}(Q) \cap L^{\infty,2}(Q_T)$ for all $T > 0$ and $f_2 \in L^{\infty,2}(Q)$. Here $Q_T = \Omega \times (0,T)$, we use the anisotropic Lebesgue space $L^{q,s}(Q)$ equipped with the norm $\|w\|_{L^{q,s}(Q)} := \big\| \|w\|_{L^q(\Omega)} \big\|_{L^s(\mathbb{R}^+)}$.

2) The initial data satisfy $\rho^0, u^0 \in H^1(\Omega)$ and $0 < \underline{\rho}^0 \le \rho^0, u^0 \mid_{x=0,l} = 0$.

3) The state function p is continuous and increasing function on $\overline{\mathbb{R}^+} = [0,+\infty)$ such that
$$\begin{cases} p(0) = 0, \quad p(+\infty) = +\infty, \quad p'(r) \in L^\infty_{\text{loc}}(\mathbb{R}^+), & (6.3.5) \\ rp'(r) = O(1) \text{ as } r \to 0^+, & (6.3.6) \\ p(r) = O(r^{\gamma_0}) \text{ as } r \to 0^+, & (6.3.7) \end{cases}$$
for some constant $0 < \gamma_0 \le 1$. (Obviously, these conditions are satisfied for the most popular state functions $p(r) = p_1 r^\gamma$ with $p_1 > 0$ and $\gamma > 0$.)

4) Assume also that the viscosity coefficient $\mu = \text{const.} > 0$.

It readily follows from the results in [430, 1022, 1023] and conditions (6.3.5)–(6.3.6) and the above conditions 1), 2) on f and ρ^0, u^0 that the strong generalized solution exists and is unique.

Introduce the integration operators $Iw(x) := \int_0^x w(\xi)d\xi$ and $I^*w(x) := \int_x^l w(\xi)d\xi$ and the mean value $\langle w\rangle := \frac{1}{l}\int_0^l w(x)dx$ for $w \in L^1(\Omega)$. Set also $I^{\langle 1\rangle}w := Iw - \langle Iw\rangle = -I^*w + \langle I^*w\rangle$. Note that for all $w \in L^1(\Omega)$,
$$\| I^{\langle 1\rangle} w \|_{C(\overline{\Omega})} \le \| w \|_{L^1(\Omega)} . \tag{6.3.8}$$

We recall some well-known results on uniform estimates (with respect to time) of the solution to the problem (6.3.1)–(6.3.3) and its asymptotic behavior as $t \to +\infty$. Obviously, we have the mass and energy conservation laws
$$\begin{cases} \displaystyle\int_\Omega \rho(x,t)dx = \int_\Omega \rho^0(x)dx =: m, & (6.3.9) \\ \displaystyle\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho u^2 + P(\rho) - \rho F\right)dx + \mu\int_\Omega u_x^2 dx = \int_\Omega \rho\Delta f u dx, & (6.3.10) \end{cases}$$
with the functions $P(r) := r\int_1^r \frac{p(s)-p(1)}{s^2}ds \ge 0$ for $r > 0$ and $F := If_\infty$. Let $N > 1$ be an arbitrarily large parameter and $K^{(i)}(N), K_i = K_i(N), (i = 1,2,\dots)$ be positive non-decreasing functions of N which may also depend on μ, p, and m, l.

Lemma 6.3.1 ([900]). *Assume that conditions* (6.3.4)–(6.3.7) *and the following conditions hold*
$$\begin{cases} 0 < \rho^0 \le N, \ \|u^0\|_{L^2(\Omega)} \le N, \ \|f_\infty\|_{L^\infty(\Omega)} \le N, & (6.3.11) \\ \|f_1\|_{L^{\infty,1}(Q)} + \|f_2\|_{L^{\infty,2}(Q)} \le N, \|P(\rho^0)\|_{L^1(\Omega)} \le N. & (6.3.12) \end{cases}$$

Then we have

(i) $$\|\sqrt{\rho}u\|_{L^{2,\infty}(Q)} + \|P(\rho)\|_{L^{1,\infty}(Q)} + \|u_x\|_{L^2(Q)} \le K^{(1)}(N); \tag{6.3.13}$$

(ii) $$\rho(x,t) \le \bar{\rho} =: K^{(2)}(N) \tag{6.3.14}$$

(iii) *the total kinetic energy tends to zero, i.e., as* $t \to +\infty$,

$$\frac{1}{2}\int_\Omega (\rho u^2)(x,t)dx \to 0. \tag{6.3.15}$$

Proof. We refer to Straŝkraba and Zlotnik [900] for the detailed proof. □

Now consider the stationary problem

$$\begin{cases} p(\rho_\infty)_x = \rho_\infty f_\infty \quad \text{on } \Omega, & (6.3.16)\\ \displaystyle\int_\Omega \rho_\infty(x)dx = m. & (6.3.17)\end{cases}$$

Here we mainly discuss the case of positive solutions, more exactly, functions $\rho_\infty \in C(\overline{\Omega})$ such that $\rho_\infty > 0$ on $\overline{\Omega}$, and $p(\rho_\infty)_x \in L^\infty(\Omega)$. In order to get the necessary and sufficient conditions for its existence, we introduce the quantities:

$$F_{\min} := \min_{\overline{\Omega}} F(x), \quad F_{\max} := \max_{\overline{\Omega}} F(x), \quad C_p := \int_0^1 \frac{p(r)}{r^2}dr \le +\infty.$$

Lemma 6.3.2 ([900])**.** *Assume that condition* (6.3.4) *holds and* $f_\infty \in L^\infty(\Omega)$*. Then the positive solution* ρ_∞ *to problem* (6.3.16)–(6.3.17) *exists if and only if*

$$\begin{cases} C_p < +\infty, \\ C_p = +\infty \quad or \quad F_{\max} - F_{\min} < \Psi(+\infty), \\ \frac{1}{m}\int_\Omega \Psi^{-1}(F(x) - F_{\min})dx < 1, \end{cases} \tag{6.3.18}$$

where $\Psi(r) := \frac{p(r)}{r} + \int_0^r \frac{p(s)}{s^2}ds$ *for* $r > 0$ *and* $\Psi(0) = 0$*, with* Ψ^{-1} *being the inverse of* Ψ*. Moreover, for* $C_p < +\infty$*, the function* Ψ *is required to be continuous and increasing on* $\overline{\mathbb{R}^+}$*. In addition, the positive solution is unique.*

Obviously, if $\Psi(+\infty) = +\infty$, then the condition $F_{\max} - F_{\min} < \Psi(+\infty)$ is automatically satisfied. Recall that the positive solutions were studied in [88] in the case $p \in C^1(\mathbb{R}^+), p' > 0$, and $f_\infty \in L^1(\Omega)$. The generalization of conditions from [88] for $p \in C(\mathbb{R}^+), p$ increasing, as well as a simplification for $f_\infty \in L^\infty(\Omega)$ have been done in [1020, 1021, 1024].

Recall that if $p(r) = p_1 r^\gamma$ with $p_1 > 0$, then $C_p = +\infty$ for $0 < \gamma \le 1$. While, for $\gamma > 1$, we have $C_p < +\infty$ and $\Psi(r) = p_1\gamma' r^{\gamma-1}, \Psi(+\infty) = +\infty$, and $\Psi^{-1}(s) = \left(\frac{s}{p_1\gamma'}\right)^{1/(\gamma-1)}$ with $\gamma' = \frac{\gamma}{\gamma-1}$.

It also follows from (6.3.16) that under condition (6.3.5) and for $f_\infty \in L^\infty(\Omega)$, the positive solution ρ^∞ belongs to $W^{1,\infty}(\Omega)$.

Let $BV(\overline{\Omega})$ be the space of functions of the bounded variation on $\overline{\Omega}$, with the norm $\|w\|_{BV(\overline{\Omega})} := \sup_{\overline{\Omega}}|w(x)| + \mathrm{Var}_{\overline{\Omega}} w$.

Lemma 6.3.3 ([900]). *Assume conditions* (6.3.4)–(6.3.6) *and* (6.3.18) *hold and* $f_\infty \in BV(\overline{\Omega})$. *Then the density stabilizes to the stationary one in* $L^q(\Omega)$*-norm, i.e., for all* $q \in [1, +\infty)$, *as* $t \to +\infty$,

$$\|\rho(\cdot, t) - \rho_\infty(\cdot)\|_{L^q(\Omega)} \to 0. \tag{6.3.19}$$

Moreover, as $t \to +\infty$,

$$\|p^\infty[\rho] - p(\rho_\infty)\|_{C(\overline{\Omega})} \to 0, \tag{6.3.20}$$

with

$$p^{(\infty)}[\rho] := -I^*(\rho f_\infty) + \langle p(\rho) + I^*(\rho f_\infty)\rangle = \langle p(\rho)\rangle + I^{<1>}(\rho f_\infty). \tag{6.3.21}$$

Proof. We refer to Straŝkraba and Zlotnik [900] for the detailed proof. □

After preparation in Lemmas 6.3.1–6.3.3, we are in a position to prove a uniform lower bound for the density in the next theorem.

Theorem 6.3.4 ([900]). *Assume conditions* (6.3.4)–(6.3.7), (6.3.11)–(6.3.12), (6.3.18) *and* $N^{-1} \le \rho^0, f_\infty \in BV(\overline{\Omega})$ *hold. Then the density is globally uniformly bounded from below, i.e.,*

$$0 < \underline{\rho} < \rho(x, t) \quad \text{in } \overline{Q}, \tag{6.3.22}$$

where $\underline{\rho} > 0$ *is a constant.*

Proof. We exploit and develop the approach from Proposition 6.1 in [255]. Dividing the equation (6.3.1) by ρ, we get

$$(\log \rho)_t + u(\log \rho)_x + u_x = 0. \tag{6.3.23}$$

Applying the operator $I^{\langle 1\rangle}$ to equation (6.3.2) and using the boundary conditions (6.3.3), we get

$$\alpha_t + \rho u^2 - \mu u_x + p(\rho) - \langle \rho u^2 + p(\rho)\rangle = I^{\langle 1\rangle}(\rho f) \tag{6.3.24}$$

with the function $\alpha := I^{\langle 1\rangle}(\rho u)$.

Equations (6.3.23) and (6.3.24) together with the formula $\rho u^2 = u(I^{\langle 1\rangle}(\rho u))_x$ yield the equality

$$(\mu \log \eta - \alpha)_t + u(\mu \log \eta - \alpha)_x + \langle \rho u^2 + p(\rho)\rangle + I^{\langle 1\rangle}(\rho f) = p(\rho), \tag{6.3.25}$$

with $\eta := \frac{1}{\rho}$. Using the function $p^{(\infty)}[\rho]$, in (6.3.21), and introducing the functions $y := \beta\eta^{\gamma_0}$ and $\beta := e^{-(\gamma_0/\mu)\alpha}$ where γ_0 is taken from condition (6.3.7), we can rewrite (6.3.25) as the following linear first-order partial differential equation for y:

$$y_t + uy_x + \frac{\gamma_0}{\mu}\left[\langle\rho u^2\rangle + p^{(\infty)}[\rho] + I^{\langle 1\rangle}(\rho\Delta f)\right] y = \frac{\gamma_0}{\mu}\beta\rho^{-\gamma_0}p(\rho). \qquad (6.3.26)$$

First, (6.3.8) implies the inequality

$$(\mu\log\eta - \alpha)_t + u(\mu\log\eta - \alpha)_x \leq p(\bar{\rho}) + m\|f\|_{L^\infty(\Omega)}.$$

Passing to Lagrangian coordinates and integrating the result with respect to t, we have

$$\max_{\overline{\Omega}}(\mu\log\eta - \alpha) \leq \max_{\overline{\Omega}}((\mu\log\eta - \alpha)|_{t=0}) + I_0(p(\bar{\rho}) + m\|f\|_{L^\infty(\Omega)}).$$

Using (6.3.8), the mass conservation law and the energy estimates (6.3.9)–(6.3.10), we conclude

$$\|\alpha\|_{C(\overline{Q})} \leq \|\rho u\|_{L^{1,\infty}(Q)} \leq m^{1/2}\|\sqrt{\rho}u\|_{L^{2,\infty}(Q)} \leq m^{1/2}K^{(1)} =: K_3, \qquad (6.3.27)$$

By estimate (6.3.27) and assumptions 1) and 2) on ρ_0 and f, we obtain, for all $T > 0$,

$$\eta(x,t) \leq K_{1,T} := K_2 \exp\frac{1}{\mu}(p(\bar{\rho}) + 2mN)(T+1) \quad \text{in } \overline{Q_T}. \qquad (6.3.28)$$

On the other hand, it follows from (6.3.27) that

$$K_3^{-1} \leq \beta \leq K_3 = e^{(\gamma_0/\mu)m^{1/2}K^{(1)}}, \quad K_3^{-1}y \leq \eta^{\gamma_0} \leq K_3 y \ \ in \ \ \overline{Q}. \qquad (6.3.29)$$

Exploiting (6.3.4)–(6.3.7) and the uniform upper bound (6.3.14), we have

$$\|\rho^{-\gamma_0}p(\rho)\|_{C(\overline{Q})} \leq K_4. \qquad (6.3.30)$$

Using (6.3.20) in Lemma 6.3.3, we get

$$0 < \frac{1}{2}p(\underline{\rho}_\infty) \leq p^{(\infty)}[\rho] \quad \text{in } Q \setminus Q_{T_0} \qquad (6.3.31)$$

for sufficiently large $T_0 > 0$. Thus it follows from equation (6.3.26) and (6.3.29)–(6.3.31) that y satisfies the differential inequality

$$\begin{aligned} &y_t + uy_x + \frac{\gamma_0}{\mu}\Big(\frac{1}{2}p(\underline{\rho}_\infty) - m\|f_1\|_{L^\infty(\Omega)} - m\|f_2\|_{L^\infty(\Omega)}\Big)y \\ &\leq \frac{\gamma_0}{\mu}K_3K_4 =: K_5 \quad \text{in } Q \setminus Q_{T_0}. \end{aligned} \qquad (6.3.32)$$

By (6.3.28), in particular, $\|y(\cdot, T_0)\|_{C(\overline{\Omega})} \leq K_{6,T_0} = K_3 K_{1,T_0}^{\gamma_0}$. Multiplying (6.3.32) by ρy^{k-1} with $k > 1$, integrating the result over Ω, and using the elementary formula

$$\rho(y_t + uy_x)y^{k-1} = \frac{1}{k}[(\rho y^k)_t + (\rho u y^k)_x], \tag{6.3.33}$$

we obtain that the function $Y_k = (\int_\Omega \rho y^k dx)^{1/k} > 0$, satisfies the differential inequality

$$\frac{1}{k}\frac{d}{dt}(Y_k^k) + (a_0 + a_1 + a_2)Y_k^k \leq K_5 \int_\Omega \rho y^{k-1} dx \leq K_5 m^{1/k} Y_k^{k-1} \quad \text{on } (T_0, +\infty). \tag{6.3.34}$$

Here $a_0 = \frac{\gamma_0}{2\mu} p(\underline{\rho}_\infty), a_s = -\frac{\gamma_0}{\mu} m \|f_s\|_{L^\infty(\Omega)}$ for $s = 1, 2$. It follows from (6.3.34) that $Y = Y_k$ satisfies also inequality (2.2.35) in Theorem 2.2.3 over $(T_0, +\infty)$, with $G = K_5 m^{1/k}$. Therefore, using estimate (2.2.37) in Theorem 2.2.3 for $s = +\infty$,

$$\|Y_k\|_{C([T_0, +\infty))} \leq K_7 \left(Y_k(T_0) + \frac{2}{a_0} K_5 m^{1/k} \right) \leq K_{8,T_0} m^{1/k} \tag{6.3.35}$$

with $K_7 = \exp \frac{\gamma_0}{\mu} \left(mN + \frac{1}{p(\underline{\rho}_\infty)} (mN)^2 \right)$ and $K_{8,T_0} = K_7 \Big(K_{6,T_0} + \frac{2}{a_0} K_5 \Big)$, but $Y_k(t) \to \max_{\overline{\Omega}} y(x,t)$ as $k \to +\infty$, so estimate (6.3.35) yields

$$y \leq K_{8,T_0} \quad \text{in} \quad \overline{Q} \setminus \overline{Q}_{T_0},$$

which, together with (6.3.28)–(6.3.29) yields

$$\eta \leq \bar{\eta} = \max \left\{ K_{1,T_0}, (K_3 K_{8,T_0})^{1/\gamma_0} \right\}.$$

Setting $\underline{\rho} = \bar{\eta}^{-1}$, we thus complete the proof. □

6.4 Uniform bounds on specific volume for symmetrically quasilinear viscous barotropic fluid equations

In this section, we exploit Theorems 1.1.2, 2.1.13, and 2.2.4 to establish uniform estimates on specific volume for symmetrically quasilinear equations. We took these results from Zlotnik [1024].

We shall consider the following symmetrically quasilinear equations

$$\begin{cases} \eta_t = (r^m u)_x, \quad \eta = 1/\rho, & (6.4.1) \\ u_t = r^m [v(\eta)\rho(r^m u)_x - p(\eta)]_x + g[r], & (6.4.2) \\ r_t = u, & (6.4.3) \end{cases}$$

in the domain $Q = \Omega \times \mathbb{R}^+$, and with the boundary and initial conditions

$$u|_{x=0,M} = 0, \;\; \eta|_{t=0} = \eta^0(x), \;\; u|_{t=0} = u^0(x), \;\; r|_{t=0} = r^0(x) \quad \text{on} \quad \Omega \tag{6.4.4}$$

with $\eta^0 > 0$ and $r^0 > 0$ on Ω satisfying

$$(r^0(x))^{m+1} = (m+1)\int_0^x (\eta^0)(x')dx' + a^{m+1} \text{ on } \quad \overline{\Omega}, \;\; a > 0, \tag{6.4.5}$$

where the functions η, u, and r are the specific volume, the velocity, and the Euler coordinate, the functions ρ, p, and v are the density, pressure, and viscosity coefficient, the function g is the mass force, $\Omega = (0, M)$, the constant M is the total mass of the gas and $g[r](x,t) = g(r(x,t),t)$. The values $m = 0, 1, 2$ correspond to the plane, cylinder, and spherical symmetry, respectively. In fact, problem (6.4.1)–(6.4.5) describes symmetric solutions of the equations of motion of a viscous barotropic gas (a compressible fluid) in a closed volume. This system can be written in Lagrange material coordinates x and t.

We introduce the mean value on Ω and integration operators:

$$\langle v \rangle = M^{-1}\int_\Omega v(x)dx, \;\; (Iv)(x) = \int_0^x v(x')dx', \;\; (I_t y)(t) = \int_0^t y(\tau)d\tau,$$

and the antiderivatives as follows:

$$E(\zeta) = \int_1^\zeta [-p(\xi)]d\xi, \;\; A(\zeta) = \int_1^\zeta [v(\xi)/\xi]d\xi, \;\; G(\chi) = \int_1^\chi g_s(\chi')d\chi'.$$

We set $(v, w) = \int_\Omega v(x)w(x)dx$ and $E_+(\zeta) = \max\{E(\zeta), 0\}$. As usual, we use the classical Lebesgue $L^q(\Omega)$ together with their anisotropic version $L^{q,r}(Q)$, for $q, r \in [1, +\infty)$, equipped with the associated norm by

$$\| \cdot \|_{L^{q,r}(Q)} = \big\| \, \| \cdot \|_{L^q(\Omega)} \big\|_{L^r(\mathbb{R}^+)}.$$

We also use the abbreviation $\| \cdot \|_\Omega$ for $\| \cdot \|_{L^2(\Omega)}$. Let also $V_2(Q)$ be the standard space of functions w having finite (parabolic) energy $\|w\|_{V_2(Q)} = \|w\|_{L^{2,\infty}(Q)} + \|w_x\|_{L^2(Q)}$. By $H^1(\Omega)$ (resp. $H^{2,1}(Q_T)$), we denote the standard Sobolev space equipped with the norm $\|\varphi\|_{H^1(\Omega)} = \|\varphi\|_{L^2(\Omega)} + \|\varphi_x\|_{L^2(\Omega)}$ (resp. $\|w\|_{H^{2,1}(Q_T)} = \|w\|_{L^{2,\infty}(Q_T)} + \|w_x\|_{V_2(Q_T)} + \|w_t\|_{L^2(Q_T)}$). Hereinafter $Q_T = \Omega \times (0,T)$, $Q = \Omega \times \mathbb{R}^+$.

For brevity, we shall consider regular generalized solutions of problem (6.4.1)–(6.4.5) such that $\eta \in W_2^1(Q_T)$, $u \in W_2^{2,1}(Q_T)$, and $r \in W_2^1(Q_T)$. Moreover, $\rho = 1/\eta \in L^\infty(Q_T)$ for all $T > 0$. Note that by (6.4.1) and (6.4.3), $\eta, \; r \in C(\overline{Q_T})$ for all $T > 0$.

First it readily follows from equation (6.4.1) and (6.4.4) that

$$||\eta(\cdot,t)||_{L^1(\Omega)} = V \equiv ||\eta^0||_{L^1(\Omega)} \quad on \quad \overline{\mathbb{R}^+} = [0, +\infty). \tag{6.4.6}$$

By (6.4.1) and (6.4.3), we have

$$\big(\eta - (m+1)^{-1}(r^{m+1})_x\big)_t = 0,$$

which, together with (6.4.5), gives us

$$(m+1)^{-1}(r^{m+1})_x = \eta \quad \text{in} \quad \overline{Q}. \tag{6.4.7}$$

Thus noting that $r_t|_{x=0} = 0$ and $r^0(0) = a$, we arrive at

$$r^{m+1} = (m+1)I\eta + a^{m+1} \qquad \text{in} \qquad \overline{Q} \tag{6.4.8}$$

which, together with (6.4.6), readily implies

$$a \le r \le R = \big((m+1)V + a^{m+1}\big)^{1/(m+1)} \qquad \text{in} \qquad \overline{Q}.$$

Assume that

1) the functions p and v are defined and continuous on $\mathbb{R}^+$,

$$v(\zeta) > 0 \ \text{ on } \ \mathbb{R}^+;$$

2) $$g(\chi, t) = g_s(\chi) + \Delta g(\chi, t),$$

where $g_s \in L^\infty(a,R), \Delta g$ is a measurable function on $(a,R) \times \mathbb{R}^+$, and $|\Delta g(\chi,t)| \le \Delta\bar{g}(t) = \bar{g}_1(t) + \bar{g}_2(t)$ for $\chi \in (a,R)$ and $t > 0$; 3) $\bar{g}_k \in L^k(\mathbb{R}^+)$ and $\bar{g}_k \ge 0, k = 1, 2$. We set

$$\tilde{g}_s(\chi) = \chi^{-m} g_s(\chi), \qquad \Delta g[r](x,t) = \Delta g(r(x,t),t).$$

We also use the notation as follows: The symbol K with indices stands for positive non-decreasing functions of the parameter $N > 1$ which may also depend on $a, M, m,$ and some other parameters, but are independent of the functions $\eta^0, u^0,$ and g. In the proofs, we shall omit the argument N. The symbol c with indices is the generic positive constant.

Lemma 6.4.1 ([1024]). *Assume the following conditions hold:*

$$\begin{cases} ||\eta^0||_{L^1(\Omega)} + ||E_+(\eta^0)||_{L^1(\Omega)} + ||u^0||_\Omega + ||g_s||_{L^\infty(a,R)} \\ \quad \le N, \ ||\bar{g}_1||_{L^1(\mathbb{R}^+)} + ||\bar{g}_2||_{L^2(\mathbb{R}^+)} \le N, & (6.4.9) \\ 0 < v_0 \le v(\zeta) \ \ \textit{on } \mathbb{R}^+, & (6.4.10) \\ 0 < E^{(1)}(\zeta) \equiv E(\zeta) + c^{(0)}(\zeta+1) \\ \quad \textit{on } \mathbb{R}^+ \textit{ for some constant } c^{(0)} > 0. & (6.4.11) \end{cases}$$

Then there holds that

$$||E^{(1)}(\eta)||_{L^{1,\infty}(Q)} + ||u||_{L^{2,\infty}(Q)} ||(v(\eta)\rho)^{1/2}(r^m u)_x||_Q \le K^{(1)}(N). \tag{6.4.12}$$

Proof. By (6.4.2) and the boundary condition in (6.4.4), we have

$$(r^{-m}u_t, \varphi) + (v(\eta)\rho(r^m u)_x - p(\eta), \varphi_x) - (\tilde{g}_s(r), \varphi) = (r^{-m}\Delta g[r], \varphi) \quad (6.4.13)$$

with an arbitrary function φ such that $\varphi, \varphi_x \in L^2(Q_T)$ for all $T > 0$ and $\varphi|_{x=0,M} = 0$. Setting $\varphi = r^m u$, using (6.4.1) and (6.4.3), the relations $(E(\eta))_t = -p(\eta)\eta_t$ and $(G(r))_t = g_s(r)r_t$, and (6.4.6), we can obtain that, for any $\varepsilon > 0$,

$$\begin{aligned} &M\left(E^{(1)}(\eta) + (1/2)u^2 - G(r)\right)_t + ||[v(\eta)\rho]^{1/2}(r^m u)_x||_\Omega^2 \\ &\quad = (\Delta g[r], u) \le M^{1/2}g_1||u||_\Omega + (2\varepsilon)^{-1}M^2 g_2^2 + (\varepsilon/2)||u||_{L^\infty(\Omega)}^2. \end{aligned} \quad (6.4.14)$$

Obviously, we have

$$||u||_{L^\infty(\Omega)} \le a^{-m}||(r^m u)_x||_{L^1(\Omega)} \le a^{-m}V^{1/2}||\rho^{1/2}(r^m u)_x||_\Omega. \quad (6.4.15)$$

Thus choosing $\varepsilon = a^{2m}\nu_0/V$ in (6.4.14), using assumptions (6.4.9)–(6.4.11), and applying the Bellman–Gronwall inequality in Theorem 1.1.2, we can derive (6.4.12).

Note that condition (6.4.10) implies

$$A(0^+) \equiv \lim_{\zeta\to 0^+} A(\zeta) = -\infty, \quad A(+\infty) \equiv \lim_{\zeta\to+\infty} A(\zeta) = +\infty. \quad (6.4.16)$$

We set $\Lambda\omega = I\omega - (\overline{V})^{-1}\langle\eta I\omega\rangle$ for $\omega \in L^1(\Omega)$, where $\overline{V} = V/M$. We can readily verify that $\Lambda f_x = f - (\overline{V})^{-1}\langle\eta f\rangle$ for $f \in W_1^1(\Omega)$ and $||\Lambda\omega||_{C(\overline{\Omega})} \le ||\omega||_{L^1(\Omega)}$. □

Lemma 6.4.2 ([1024]). *There holds that*

$$(A(\eta))_t = p(\eta) - d + (\Lambda(r^{-m}u))_t + d_1, \quad (6.4.17)$$

where

$$d = (\overline{V})^{-1}\langle\eta p(\eta)\rangle + \Lambda(\tilde{g}_s(r))$$

and

$$d_1 = m\Lambda(r^{-m-1}u^2) - V^{-1}||u||_\Omega^2 - \overline{V}^{-1}\langle\nu(\eta)(r^m u)_x\rangle - \Lambda(r^{-m}\Delta g[r]).$$

Proof. Setting $\sigma = \nu(\eta)\rho(r^m u)_x - p(\eta)$ and using (6.4.3), we may transform (6.4.2) to the form

$$(r^{-m}u)_t + mr^{-m-1}u^2 = \sigma_x + r^{-m}g[r]. \quad (6.4.18)$$

Applying the operator Λ to this equation (6.4.18), we obtain

$$(\Lambda(r^{-m}u))_t - \bar{V}^{-1}\langle u^2\rangle + m\Lambda(r^{-m-1}u^2) = \sigma - \bar{V}^{-1}\langle\eta\sigma\rangle + \Lambda(r^{-m}g[r])$$

where we have used the relation (see (6.4.1))

$$\langle(\eta_t)I(r^{-m}u)\rangle = -\langle u^2\rangle.$$

Thus noting that $\sigma = (A(\eta))_t - p(\eta)$, we can prove (6.4.17). □

Obviously, if $\nu(\zeta) \equiv$ const., then $\langle \nu(\eta)(r^m u)_x \rangle = 0$, and the expression for d_1 in (6.4.17) can be simplified.

The next theorem concerns a uniform lower bound for the function η.

Theorem 6.4.3 ([1024]). *If assumptions* (6.4.9)–(6.4.10) *and the following conditions hold,*

$$\begin{cases} N^{-1} \leq \eta^0(x) \quad on \quad \overline{\Omega}, & (6.4.19) \\ \zeta\nu(\zeta) \leq c_0(E_+(\zeta) + \zeta + 1) \quad on \quad \mathbb{R}^+, & (6.4.20) \\ p(0^+) = +\infty; \quad \zeta p(\zeta) = O(E(\zeta)) \text{ as } \zeta \to 0^+; & \\ \quad \overline{\lim}_{\zeta \to +\infty} p(\zeta) < +\infty, & (6.4.21) \end{cases}$$

then we have

$$K^{(2)}(N)^{-1} \leq \eta(x,t) \quad in \quad \overline{Q}. \tag{6.4.22}$$

Proof. First, by (6.4.4) and the continuity of the functions p and ν, the function $A^{(-1)}$, its inverse A, and the function $f(y) = p(A^{(-1)}(y))$ are defined and continuous on $\mathbb{R}^+$. Second, by assumption (6.4.20), we also know $f(-\infty) = +\infty$. Since p is continuous on $\mathbb{R}^+$, it follows from (6.4.21) that for sufficiently large $c^{(0)}$, c_1, and c_2,

$$\begin{cases} 0 < E_+(\zeta) + \zeta + 1 \leq c_1 E^{(1)}(\zeta) & \\ \quad = c_1\Big(E(\zeta) + c^{(0)}(\zeta+1)\Big) \quad \text{on} \quad \mathbb{R}^+, & (6.4.23) \\ \zeta p(\zeta) \leq c_2 E^{(1)}(\zeta) \quad \text{on} \quad \mathbb{R}^+. & (6.4.24) \end{cases}$$

Thus it remains to estimate terms on the right-hand side of (6.4.17). Indeed, using (6.4.24) and (6.4.11), we derive

$$\begin{cases} d \leq V^{-1} c_2 ||E^{(1)}(\eta)||_{L^1(\Omega)} + Ma^{-m}||g_s(r)||_{L^\infty(\Omega)} & \\ \quad \leq V^{-1} c_2 K^{(1)} + Ma^{-m} N, & (6.4.25) \\ ||\Lambda(r^{-m}u)||_{C(\overline{\Omega})} \leq a^{-m}||u||_{L^1(\Omega)} \leq M^{1/2} a^{-m} K^{(1)}. & (6.4.26) \end{cases}$$

On the one hand, we can derive from (6.4.20), (6.4.23), (6.4.13) and (6.4.11) that

$$\begin{aligned} ||d_1||_{C(\overline{\Omega})} &\leq (ma^{-m-1} + V^{-1})||u||^2_\Omega \\ &+ V^{-1}\Big(c_0 c_1 ||E^{(1)}(\eta)||_{L^1(\Omega)}\Big)^{1/2} ||[\nu(\eta)\rho]^{1/2}(r^m u)_x||_\Omega \\ &+ Ma^{-m}||\Delta g[r]||_{L^\infty(\Omega)} \leq (ma^{-m-1} + V^{-1})MVa^{-2m}||\rho^{1/2}(r^m u)_x||^2_\Omega \\ &+ V^{-1}\Big(c_0 c_1 K^{(1)}\Big)^{1/2} ||[\nu(\eta)\rho]^{1/2}(r^m u)_x||_\Omega + Ma^{-m}(\overline{g}_1 + \overline{g}_2). \end{aligned}$$

On the other hand, using (6.4.11), the second condition in (6.4.9), and (6.4.10), we conclude that for all $\varepsilon > 0$ and for all $0 \le t_1 < t_2$,

$$\begin{aligned}\int_{t_1}^{t_2} ||d_1||_{C(\overline{\Omega})} dt \le c_3\big[(\nu_0^{-1} + \varepsilon^{-1}K^{(1)})||[\nu(\eta)\rho]^{1/2}u_x||_Q \\ + ||\overline{g}_1||_{L^1(\mathbb{R}^+)} + \varepsilon^{-1}||\overline{g}_2||^2_{L^2(\mathbb{R}^+)}\big] \\ + \varepsilon(t_2 - t_1) \le (1+\varepsilon^{-1})K_1 + \varepsilon(t_2 - t_1),\end{aligned} \tag{6.4.27}$$

with a constant $c_3 = c_3(a, M, V, m)$.

Therefore, applying assertion (1) in Theorem 2.1.13 with $y = A(\eta)$ and $b = -I_t d + \Lambda(r^{-m}u) + I_t d_1$ to (6.4.17) and using (6.4.25)–(6.4.27) with $\varepsilon = 1$, we can obtain

$$\min\{A(\eta^0(x)), -K_2\} - K_3 \le A(\eta(x,t)) \text{ in } \overline{Q},$$

which, together with (6.4.19) and $A(0^+) = -\infty$, readily implies (6.4.22). □

Remark 6.4.1 ([1024])**.** Let $\overline{g}_2 = 0$, i.e., $||\Delta\overline{g}||_{L^1(\mathbb{R}^+)} \le N$. Then condition (6.4.10) can be omitted in Lemma 6.4.1 and weakened in Theorem 6.4.1 by replacing it by the conditions $\zeta/\nu(\zeta) = O(E(\zeta))$ as $\zeta \to 0^+$, $A(0^+) = -\infty$, and $\underline{\lim}_{\zeta\to+\infty}\nu(\zeta) > 0$. Under these conditions, we have $\zeta/\nu(\zeta) \le cE(1)(\zeta)$ (see (6.4.23)), and hence,

$$||u||_{L^\infty(\Omega)} \le a^{-m}||(r^m u)_x||_{L^1(\Omega)} \le a^{-m}\left(cK^{(1)}\right)^{1/2}||[\nu(\eta)\rho]^{1/2}(r^m u)_x||_\Omega.$$

Moreover, $A(+\infty) = +\infty$.

The next result is concerned with a uniform upper bound for the function η under a particular additional condition on the absolute value of the function $\tilde{g}_s$.

Theorem 6.4.4 ([1024])**.** *Assume conditions* (6.4.9), (6.4.10), (6.4.20) *and the following conditions hold:*

$$\begin{cases} \eta^0(x) \le N \quad \textit{on} \quad \overline{\Omega}, & (6.4.28)\\ p_\gamma(\zeta) \equiv c^{(1)}\zeta^{-\gamma} \le p(\zeta) & \\ \quad \textit{on } \mathbb{R}^+ \textit{ for some constants } c^{(1)} > 0 \textit{ and } \gamma \ge 1, & (6.4.29)\\ p(+\infty) = 0. & (6.4.30)\end{cases}$$

If $\tilde{g}_s$ is a function such that

$$M||\tilde{g}_s||_{L^\infty(a,R)}/p_\gamma(\overline{V}) \le 1 - N^{-1}, \tag{6.4.31}$$

then

$$\eta(x,t) \le K^{(3)}(N) \quad \textit{in} \quad \overline{Q}. \tag{6.4.32}$$

Proof. Obviously, by the assumptions on p and ν, condition (6.4.11) and inequality (6.4.22) are valid. Therefore, we can use the energy estimate (6.4.12) and estimates (6.4.26) and (6.4.27).

By assumption (6.4.29) and the Jensen inequality, we have

$$\langle \eta p(\eta) \rangle \geq \langle c^{(1)} \eta^{1-\gamma} \rangle \geq c^{(1)} (\overline{V})^{1-\gamma},$$

which, along with condition (6.4.30) yields

$$-d \leq -p_\gamma(\overline{V}) + M||\tilde{g}_s||_{L^\infty(\Omega)} \leq -N^{-l} p_\gamma(\overline{V}) \leq -K_1^{-1}.$$

Therefore, applying assertion (2) in Theorem 2.1.13 to (6.4.17) (by (6.4.16) and (6.4.30), in this case, $f(+\infty) = p(A^{(-1)}(+\infty)) = 0$) and using the estimate $-d \leq -K_1^{-1}$and (6.4.26) and (6.4.27) with $\varepsilon = (2K_1)^{-1} > 0$, we can obtain the estimate $A(\eta(x,t)) \leq \max\{A(\eta^0(x), K_2\} + K_3$ in $\overline{Q}$, which, together with condition (6.4.28) and the property $A(+\infty) = +\infty$, implies (6.4.32). □

In the sequel, we shall prove the stabilization of the function u to zero in the norm of $L^q(\Omega)$ as $t \to +\infty$.

To this end, we first consider the auxiliary linear non-uniformly parabolic problem

$$\begin{cases} bv_t = (\kappa D v - \psi)_x + a_0 v + f, & (6.4.33) \\ v|_{x=0,M} = 0, \qquad v|_{t=0} = v^0(x) \quad \text{on} \quad \Omega. & (6.4.34) \end{cases}$$

Suppose that the following properties hold for all $T > 0$: (1) $b, \kappa \in L^\infty(Q_T), b > 0, \kappa > 0, a_0, b_t \in L^1(Q_T), \psi \in L^2(Q_T)$; (2) $f = f_0 + \Delta f$, and $|\Delta f| \leq |f_1| + |f_2|$; (3) $f, f_k \in L^1(Q_T), k = 0, 1, 2$, and $v^0 \in L^1(\Omega)$.

We now consider generalized solutions with $v \in L^\infty(Q_T), v_x \in L^2(Q_T)$, and $v_t \in L^1(Q_T)$ for all $T > 0$.

Theorem 6.4.5 ([1024]). *Let $b \in L^\infty(Q)$, $||1/\kappa||_{L^{1,\infty}(Q)} \leq N_\kappa$ and $q \in [2, +\infty)$ with $q' = q/(q-1)$. If*

$$d_1 = ||(b^0)^{1/q} v^0||_{L^q(\Omega)} + ||b^{-1/q'} f_1||_{L^{q,1}(Q)} + q||f_2||_{L^{1,q}(Q)} < +\infty, \tag{6.4.35}$$

$$\psi = f_0 = 0, \tag{6.4.36}$$

and

$$||b^{-1/q'}(a_0 + q^{-1} b_t) v||_{L^{q,1}(Q)} \leq N_1,$$

then we have

$$|||v|||_q \equiv ||b^{1/q} v||_{L^{q,\infty}(Q)} + ||v||_{L^{\infty,q}(Q)} \leq c_1(N_\kappa)(d_1 + N_1) \tag{6.4.37}$$

and as $t \to +\infty$,

$$||(b^{1/q} v)(\cdot, t)||_{L^q(\Omega)} \to 0. \tag{6.4.38}$$

Proof. We know that the solution v of problem (6.4.33)–(6.4.34) satisfies the integral identity

$$(bv_t, \varphi) + (\kappa v_x - \psi, \varphi_x) = (a_0 v + f, \varphi) \tag{6.4.39}$$

for all $\varphi \in H_0^1$. Choosing $\varphi = q|v|^{q-2}v$ in (6.4.39), we obtain

$$\begin{aligned}&\big(||b|v|^q||_{L_1(\Omega)}\big)_t + 4(q')^{-1}||\kappa^{1/2}(|v|^{q/2-1}v)_x||_\Omega^2\\&\quad = q(\psi, (|v|^{q-2}v)_x) + q(a_q v + f, |v|^{q-2}v),\end{aligned} \tag{6.4.40}$$

where $a_q = a_0 + q^{-1}b_t$. It is easy to see that

$$\begin{aligned}&\big(M||b||_{L^\infty(\Omega)}\big)^{-1} ||b|v|^q||_{L^1(\Omega)}\\&\quad \leq ||v||_{L^\infty(\Omega)}^q \leq ||1/\kappa||_{L^1(\Omega)}||\kappa^{1/2}(|v|^{q/2-1}v)_x||_\Omega^2.\end{aligned} \tag{6.4.41}$$

Therefore, using (6.4.36) and the inequality on the right-hand side of (6.4.41), the Hölder inequality, and the estimate

$$\begin{aligned}q(f_2, |v|^{q-2}v) &\leq q||f_2||_{L^1(\Omega)}||v||_{L^\infty(\Omega)}^{q-1}\\&\leq \varepsilon(q')^{-1}||v||_{L^\infty(\Omega)}^q + \varepsilon^{-(q-1)}q^{-1}(q||f_2||_{L^1(\Omega)})^q\end{aligned}$$

with an arbitrary $\varepsilon > 0$, we can derive from (6.4.40),

$$\begin{aligned}&\Big(||b^{1/q}v||_{L^\infty(\Omega)}^q\Big)_t + 3(N_\kappa q')^{-1}||v||_{L^\infty(\Omega)}^q\\&\quad \leq q||b^{-1/q'}(a_q v + f_1)||_{L^q(\Omega)}||b^{1/q}v||_{L^q(\Omega)}^{q-1} + (N_\kappa q)^{q-1}||f_2||_{L^1(\Omega)}^q.\end{aligned} \tag{6.4.42}$$

Using the inequality on the left-hand side of (6.4.41), we can derive a differential inequality for $||b^{1/q}v||_{L^q(\Omega)}$. Note the following fact: Let $\alpha \geq 0$ and let $1 \leq s < +\infty$. From the Hölder inequality, we obtain, for all $t > 0$ and $y \in L^s(\mathbb{R}^+)$,

$$\int_0^t \exp(\alpha(\tau - t))|y(\tau)|d\tau \leq \alpha^{-(s-1)/s}\int_0^t \exp(\alpha(\tau - t))|y(\tau)|^s d\tau \tag{6.4.43}$$

and it was proved in [49] that for all $y \in L^1(\mathbb{R}^+)$, as $t \to +\infty$,

$$\int_0^t \exp(\alpha(\tau - t))|y(\tau)|d\tau \to 0. \tag{6.4.44}$$

Applying Theorem 2.2.4, (6.4.43)–(6.4.44) imply (6.4.38) and the estimate

$$\begin{aligned}||b^{1/q}v||_{L^{q,\infty}(\Omega)} \leq \tilde{d}_1 = ||(b^0)^{1/q}v^0||_{L^q(\Omega)} + ||b^{-1/q'}a_q v||_{L^{q,1}(\Omega)} + ||b^{-1/q'}f_1||_{L^{q,1}(\Omega)}\\ + \ (N_\kappa q)^{1-1/q}||f_2||_{L^{q,1}}.\end{aligned}$$

Integrating inequality (6.4.42), we obtain

$$||v||_{L^{\infty,q}(Q)} \leq (N_\kappa q'/3)^{1/q}q^{1/q}\tilde{d}_1,$$

which gives us (6.4.37). □

6.5 Stabilization for the 1D compressible Navier–Stokes equations

In this section, we shall use Theorems 1.2.1, 2.1.13, and 2.2.2 to establish the stabilization for the 1D compressible Navier–Stokes equations. These results are due to Ducomet and Zlotnik [226].

We shall consider the following system of quasilinear differential equations governing the 1D motions of viscous compressible heat-conducting media

$$\begin{cases} u_t = v_x, & (6.5.1)\\ v_t = \sigma_x + g, & (6.5.2)\\ e[u,\theta]_t = \sigma v_x + \pi_x, & (6.5.3)\end{cases}$$

subject to the following boundary and initial conditions:

$$\begin{cases} v|_{x=0} = 0, \quad \sigma|_{x=M} = -p_\Gamma, \quad \theta|_{x=0} = \theta_\Gamma, \quad \pi|_{x=M} = 0, & (6.5.4)\\ u|_{t=0} = u^0(x), \quad v|_{t=0} = v^0(x), \quad \theta|_{t=0} = \theta^0(x), & (6.5.5)\end{cases}$$

where an outer pressure p_Γ = const. and a given temperature θ_Γ = const. > 0, $(x,t) \in Q \equiv \Omega \times \mathbb{R}^+ = (0,M) \times (0,+\infty)$ are the Lagrangian mass coordinates, and M is the total mass of the medium. The unknown quantities $u > 0$ and $\theta > 0$ are the specific volume, the velocity, and the absolute temperature. We also denote by $\rho = \frac{1}{u}$ the density, $\sigma = \nu\rho v_x - p[u,\theta]$ the stress, $e(u,\theta)$ the internal energy, and $-\pi = -\kappa[u,\theta]\rho\theta_x$ the heat flux. Hereinafter, the notation $\lambda[u,\theta](x,t) = \lambda(u(x,t),\theta(x,t))$, for $\lambda = e, p, \kappa$, etc., is used.

We first define the Helmholtz free energy

$$\Psi(u,\theta) = -c_V\theta\log\theta - P_0(u) - P_1(u)\theta,$$

where c_V = const. > 0. Then thermodynamics indicates that

$$p(u,\theta) = -\Psi_u(u,\theta) = p_0(u) + p_1(u)\theta, \tag{6.5.6}$$

with $p_0 = P_0'$ and $p_1 = P_1'$, and

$$e(u,\theta) = \Psi(u,\theta) - \theta\Psi_\theta(u,\theta) = -P_0(u) + c_V\theta, \tag{6.5.7}$$

where $\Psi_u = \frac{\partial\Psi}{\partial u}$ and $\Psi_\theta = \frac{\partial\Psi}{\partial\theta}$.

Now, we consider the more difficult case of the nuclear fluid and assume that the functions $p_0, p_1 \in C^1(\overline{\mathbb{R}^+})$ satisfy

$$\begin{cases} \lim_{u\to0^+} p_0(u) = +\infty, \quad \lim_{u\to+\infty} p_0(u) = 0, & (6.5.8)\\ p_1(u) \geq 0, \quad up_1(u) = O(1) \ \text{as}\ u \to +\infty, & (6.5.9)\end{cases}$$

and that the viscosity and heat conductivity coefficients satisfy that

$$\nu = \text{const.} > 0 \text{ and } \kappa \in C^1(\overline{\mathbb{R}^+} \times \overline{\mathbb{R}^+}),$$

with $0 < \underline{\kappa} \le \kappa(u, \theta) \le \overline{\kappa}$, where $\underline{\kappa}$ and $\overline{\kappa}$ are given constants. We do not impose any growth conditions on the derivatives of κ.

Assume that the initial data are such that $u^0 \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega u^0 > 0, v^0 \in L^4(\Omega), \theta^0 \in L^2(\Omega), \log\theta^0 \in L^1(\Omega)$ with $\theta^0 > 0$. Moreover, we assume $g \in L^1(\Omega)$ is the so-called "self-gravitation force".

Indeed, these dynamical boundary conditions (6.5.6) mean that we impose a fixed stress on the right boundary (fixed external pressure in the fluid context, or stress-free condition for $p_\Gamma = 0$ in the solid context) and consider the fixed boundary left. For the thermal boundary conditions, we suppose that the temperature is known on the fixed boundary and the flux is zero on the free one.

Obviously, this monotonicity is not valid in a number of physical situations, for example, the case of the two-term pressure

$$p(u, \theta) = p_0(u) + p_1(u)\theta, \tag{6.5.10}$$

which is linear in θ, but is with complicated non-monotone $p_0(u)$, this is crucial important for nuclear fluid models (see [221, 223, 222] and references therein).

We shall borrow here the notations in Section 6.4 and use the integration operators $I^*\varphi(x) = \int_x^M \varphi(\xi)d\xi$, for $\varphi \in L^1(\Omega)$, and $I_0 a(t) = \int_0^t a(\tau)d\tau$, for $a \in L^1(0, T)$, and define the function for all $x \in \overline{\Omega}$,

$$p_S(x) := p_\Gamma - \int_x^M g(\xi)d\xi,$$

which will play the role of a stationary pressure, and set $\underline{p}_S := \min_{\overline{\Omega}} p_S$ and $\bar{p}_S := \max_{\overline{\Omega}} p_S$. Obviously, $\underline{p}_S \le p_\Gamma \le \bar{p}_S$. Let $N > 1$ be an arbitrarily large parameter and $K_i = K_i(N)$ and $K^{(i)} = K^{(i)}(N), i = 0, 1, 2, \ldots$, be positive non-decreasing functions of N, which may also depend on $M, v, \underline{\kappa}, \bar{\kappa}$, etc; but neither on the initial data nor on g.

To simplify the presentation, we shall only discuss the case of so-called regular weak (or strong) solutions (see, e.g., [49]) such that $u \in L^\infty(Q_T), u_x, u_t \in L^{2,\infty}(Q_T), \min_{\bar{Q}_T} u > 0$, and $v, \theta \in H^{2,1}(Q_T), \min_{\bar{Q}_T} \theta > 0$ for any $T > 0$.

Now we are in a position to state our main result due to [226].

Theorem 6.5.1 ([226]). *Assume that the initial data, p_Γ, and g satisfy*

$$\begin{cases} N^{-1} \le u^0 \le N, \quad \|v^0\|_{L^4(\Omega)} + \|\log\theta^0\|_{L^1(\Omega)} + \|\theta^0\|_{L^2(\Omega)} \le N, & (6.5.11)\\ \|g\|_{L^1(\Omega)} \le N, \quad N^{-1} \le \underline{p}_S. & (6.5.12)\end{cases}$$

Then the following estimates in Q together with the $L^2(\Omega)$-stabilization property hold:

$$0 < K_1^{-1} = \underline{u} \le u(x,t) \le \bar{u} = K_2 \qquad in \quad \overline{Q}, \tag{6.5.13}$$

$$\|v\|_{V_2(Q)} + \|v^2\|_{V_2(Q)} + \|\log\theta\|_{L^{1,\infty}(Q)} + \|(\log\theta)_x\|_{L^2(Q)} \\ + \|\theta - \theta_\Gamma\|_{V_2(Q)} \le K_3, \tag{6.5.14}$$

$$\|p[u,\theta] - p_S\|_{L^2(Q)} \le K_4, \tag{6.5.15}$$

$$\|v^2(\cdot,t)\|_{L^2(\Omega)} + \|\theta(\cdot,t) - \theta_\Gamma\|_{L^2(\Omega)} + \|p[u,\theta](\cdot,t) \\ - p_S(\cdot)\|_{L^2(\Omega)} \to 0 \text{ as } t \to +\infty. \tag{6.5.16}$$

Remark 6.5.1 ([226])**.**

(1) The second set of conditions (6.5.12) implies that $N^{-1} \le p_\Gamma$.

(2) For non-monotone $p(u,\theta_\Gamma)$, if there exist two points $0 < u^{(1)} < u^{(2)}$ such that

$$\underline{p}_S < p^{(1)} := p(u^{(1)},\theta_\Gamma) < p^{(2)} := p(u^{(2)},\theta_\Gamma) < \bar{p}_S$$

and

$$\begin{cases} p^{(1)} \le p(u,\theta_\Gamma), & \text{for } 0 < u \le u^{(1)}, \\ p^{(1)} \le p(u,\theta_\Gamma) \le p^{(2)}, & \text{for } u^{(1)} < u < u^{(2)}, \\ p(u,\theta_\Gamma) \le p^{(2)}, & \text{for } u^{(2)} \le u, \end{cases}$$

then necessarily $u_S \notin C(\overline{\Omega})$.

Note that the second condition (6.5.12) is essential in Theorem 6.5.1.

The proof of Theorem 6.5.1 can be divided by a series of Lemmas 6.5.1–6.5.6 providing necessary a priori estimates and stabilization properties. Sometimes, we use the abbreviation $\|\cdot\|_G$ for $\|\cdot\|_{L^2(G)}$.

Lemma 6.5.2 ([226])**.** *The following energy estimates hold*

$$\begin{cases} \|u\|_{L^{1,\infty}(Q)} + ||v||_{L^{2,\infty}(Q)} + \|\theta\|_{L^{1,\infty}(Q)} \\ \quad + \|\log\theta\|_{L^{1,\infty}(Q)} \le K^{(1)}, & (6.5.17) \\ \left\|\sqrt{\frac{\rho}{\theta}} v_x\right\|_Q + \left\|\frac{\sqrt{\rho}}{\theta}\theta_x\right\|_Q \le K^{(2)}. & (6.5.18) \end{cases}$$

Proof. An easy calculation shows equations (6.5.1)–(6.5.7) imply

$$\begin{cases} \left(\frac{1}{2}v^2 + e[u,\theta]\right)_t = (\sigma v + \pi)_x + gv, & (6.5.19) \\ c_V\theta_t = \pi_x + (\nu\rho v_x - p_1[u]\theta)v_x. & (6.5.20) \end{cases}$$

Hereinafter we use the notation $\lambda[u](x,t) = \lambda(u(x,t))$, for $\lambda = p_i, P_i, i = 0, 1$, etc.

Multiplying (6.5.20) by θ_Γ/θ and subtracting the result from (6.5.19), we obtain

$$\left(\frac{1}{2}v^2 + e[u,\theta] - c_V\theta_\Gamma \log\frac{\theta}{\theta_\Gamma} - \theta_\Gamma P_1[u] + p_\Gamma u\right)_t + \theta_\Gamma \nu \frac{\rho}{\theta} v_x^2$$
$$= \Big((\sigma + p_\Gamma)v\Big)_x + \left(1 - \frac{\theta_\Gamma}{\theta}\right)\pi_x + gv.$$

Next, setting $P(u,\theta) := P_0(u) + P_1(u)\theta$, integrating the above equality over Ω and using the formula

$$\int_\Omega gv dx = \int_\Omega (I^* g) v_x dx = \frac{d}{dt}\int_\Omega (I^* g) u dx,$$

we get, for any constant C,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left[\frac{1}{2}v^2 + c_V\theta_\Gamma\left(\frac{\theta}{\theta_\Gamma} - \log\frac{\theta}{\theta_\Gamma}\right) + p_S u - P[u,\theta_\Gamma] + C\right] dx \\ &\quad + \theta_\Gamma \int_\Omega \left(\nu\frac{\rho}{\theta}v_x^2 + \kappa[u,\theta]\frac{\rho}{\theta^2}\theta_x^2\right) dx = 0.\end{aligned} \tag{6.5.21}$$

Conditions (6.5.8) and (6.5.9) imply the property for all $\varepsilon > 0$,

$$P(u,\theta_\Gamma) \le \varepsilon u + C_\varepsilon \qquad \text{on } \mathbb{R}^+.$$

Integrating (6.5.21) over $(0,T)$ for any $T > 0$, applying conditions (6.5.11) and (6.5.12), and choosing $\varepsilon := \frac{1}{2}\underline{p}_S$, we can obtain (6.5.17) and (6.5.18). Here we used the inequality $\frac{1}{2}\alpha \le \alpha - \log\alpha + \log 2 - 1$. □

Lemma 6.5.3 ([226]). *The following uniform lower bound holds:*

$$0 < \underline{u} = \left(K^{(3)}\right)^{-1} \le u(x,t) \quad \text{in } \overline{Q}. \tag{6.5.22}$$

Proof. Acting of the operator I^* on (6.5.2), we have

$$I^* v_t = -\nu\rho v_x + p[u,\theta] - p_S, \tag{6.5.23}$$

which, together with the relation $\rho v_x = (\log u)_t$, gives us

$$(\nu \log u)_t = p[u,\theta] - p_S - I^* v_t. \tag{6.5.24}$$

Now putting $y := \nu\log u$, using the fact that $p_1[u]\theta \ge 0$ and fixing any $x \in \overline{\Omega}$, we get

$$\frac{dy}{dt} \ge p_0\left(\exp\frac{y}{\nu}\right) - \bar{p}_S - \frac{d}{dt}I^* v.$$

The function $f(z) := p_0(\exp \frac{z}{\nu}) - \bar{p}_S$ has the property $f(-\infty) = +\infty$ (see, e.g., (6.5.8)). Moreover, due to the energy estimate (6.5.17), we get

$$\left| I^* v\Big|_\tau^t \right| \leq 2 \sup_{\overline{Q}} |I^* v| \leq 2M^{1/2} \|v\|_{L^{2,\infty}(Q)} \leq K_0. \tag{6.5.25}$$

Thus Claim (i) in Theorem 2.1.13 (with $N_1 = 0$) implies

$$\min\{\nu \log u^0(x), \nu \log \breve{u}\} - K_0 \leq y(x,t),$$

with a number $\breve{u}$ such that $p_0(u) - \bar{p}_S \geq 0$, for any $0 < u \leq \breve{u}$. Then

$$\underline{u} := \min\{N^{-1}, \breve{u}\} \exp\left(-\frac{K_0}{\nu}\right) \leq u(x,t) \qquad in\ \overline{Q}. \qquad \square$$

Lemma 6.5.4 ([226])**.** *The following uniform upper bound holds:* $u(x,t) \leq \bar{u} = K^{(4)}$ *in* $\bar{Q}$.

Proof. First we rewrite (6.5.1) as

$$u_t = \frac{1}{\nu}(\sigma + \delta)u + \frac{1}{\nu}u(p[u,\theta] - \delta),$$

where $\delta > 0$ is a parameter. Now we may consider this as an ordinary differential equation with respect to u and obtain the formula

$$u = \exp\left(\frac{1}{\nu} I_0(\sigma+\delta)\right)\left\{u^0 + \frac{1}{\nu} I_0\left[\exp\left(-\frac{1}{\nu} I_0(\sigma+\delta)\right) u(p[u,\theta]-\delta)\right]\right\}. \tag{6.5.26}$$

Applying the operator I_0 to (6.5.23), we get

$$I_0 \sigma = -p_S t - I^*(v - v^0).$$

Thus, choosing $\delta := \frac{1}{2}\underline{p}_S$ and using estimate (6.5.25) we obtain on $\overline{\Omega}$, for all $0 \leq \tau \leq t$,

$$\frac{1}{\nu} I_0(\sigma+\delta)\Big|_\tau^t = -\frac{1}{\nu}(p_S - \delta)(t-\tau) - \frac{1}{\nu} I^* v\Big|_*^t \leq -\alpha(t-\tau) + K_1,$$

with $\alpha := \frac{1}{2\nu}\underline{p}_S > 0$. Conditions (6.5.8) and (6.5.9) on p_0 and p_1, together with the lower bound $\underline{u} \leq u$, give us

$$u(p[u,\theta]-\delta) \leq u \max\{p_0[u]-\delta, 0\} + u p_1[u]\theta \leq K_2 + K_3\theta.$$

Therefore, it follows from (6.5.26) that

$$\hat{u}(t) := \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha t}\left[1 + \int_0^t e^{\alpha\tau}(1 + \|\theta(\cdot,\tau)\|_{L^\infty(\Omega)}) d\tau\right]. \tag{6.5.27}$$

Setting $a := \|\frac{\sqrt{\rho}}{\theta}\theta_x\|_\Omega^2$, using the inequalities (see, e.g., [44, 49]) for all $\varepsilon > 0$,

$$\|\theta\|_{L^\infty(\Omega)} \le \theta_\Gamma + \|\theta_x\|_{L^1(\Omega)} \le \theta_\Gamma + \left(a\|\theta\|_{L^1(\Omega)}\|\theta\|_{L^\infty(\Omega)}\hat{u}\right)^{1/2}$$
$$\le \varepsilon\|\theta\|_{L^\infty(\Omega)} + \theta_\Gamma + \frac{1}{4\varepsilon}a\|\theta\|_{L^1(\Omega)}\hat{u},$$

and the estimate $\|\theta\|_{L^{1,\infty}(\Omega)} \le K^{(1)}$, we conclude

$$\|\theta\|_{L^\infty(\Omega)} \le K_5(1 + a\hat{u}).$$

Thus, by estimate (6.5.27), the function $z(t) := e^{\alpha t}\hat{u}(t)$ satisfies

$$z(t) \le K_6\left(e^{\alpha t} + \int_0^t a(\tau)z(\tau)d\tau\right) \qquad \text{on } \mathbb{R}^+.$$

Since $\|a\|_{L^1(\mathbb{R}^+)} \le (K^{(2)})^2$ according to Lemma 6.5.1, Theorem 1.2.1 yields

$$z(t) \le K_6 \exp\left(\alpha t + K_6(K^{(2)})^2\right) = K^{(4)}e^{\alpha t} \qquad \text{on } \mathbb{R}^+,$$

whence $u \le \hat{u} \le \bar{u} := K^{(4)}$ in $\overline{Q}$. □

Corollary 6.5.1 ([226]). *For v, the following estimates hold*

$$\frac{1}{\sqrt{M}}\|v\|_Q \le \|v\|_{L^{\infty,2}(Q)} \le (K^{(1)})^{1/2}\left\|\frac{v_x}{\sqrt{\theta}}\right\|_Q \le K^{(5)}, \tag{6.5.28}$$
$$\|(\log\theta)_x\|_Q \le \bar{u}^{1/2}K^{(2)}. \tag{6.5.29}$$

Proof. In fact, we derive from Lemma 6.5.1

$$\|v\|_{C(\overline{\Omega})} \le \|v_x\|_{L^1(\Omega)} \le \|\theta\|_{L^1(\Omega)}^{1/2}\left\|\frac{v_x}{\sqrt{\theta}}\right\|_\Omega \le (K^{(1)})^{1/2}\left\|\frac{v_x}{\sqrt{\theta}}\right\|_\Omega \tag{6.5.30}$$

and

$$\left\|\frac{v_x}{\sqrt{\theta}}\right\|_Q \le \bar{u}^{1/2}\left\|\sqrt{\frac{\rho}{\theta}}v_x\right\|_Q \le \bar{u}^{1/2}K^{(2)}.$$

Similarly, we can prove (6.5.29). □

Lemma 6.5.5 ([226]). *The following estimates hold for v^2 and $\theta - \theta_\Gamma$:*

$$\begin{cases} \|v^2\|_{V_2(Q)} + \|\theta - \theta_\Gamma\|_{V_2(Q)} \le K^{(6)}, \\ \|v^2(\cdot,t)\|_\Omega + \|\theta(\cdot,t) - \theta_\Gamma\|_\Omega \to 0 \qquad \textit{as } t \to +\infty. \end{cases} \tag{6.5.31}$$

Proof. In fact, we may rewrite (6.5.19) as

$$\left(\frac{1}{2}v^2 + c_V(\theta - \theta_\Gamma)\right)_t = (\sigma v + \pi)_x + p_0[u]v_x + gv.$$

Thus, taking the $L^2(\Omega)$-inner product with $\frac{1}{2}v^2 + c_V(\theta - \theta_\Gamma)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{1}{2}v^2 + c_V(\theta - \theta_\Gamma)\right)^2 dx \\
&\qquad + \int_\Omega \Big[(\nu\rho v_x - p[u,\theta])v + \kappa[u,\theta]\rho\theta_x\Big](vv_x + c_V\theta_x)dx \\
&\quad = \int_\Omega (p_0[u]v_x + gv)\left(\frac{1}{2}v^2 + c_V(\theta - \theta_\Gamma)\right) dx \\
&\qquad - p_\Gamma\left(v\left(\frac{1}{2}v^2 + c_V(\theta - \theta_\Gamma)\right)\right)\bigg|_{x=M}.
\end{aligned} \tag{6.5.32}$$

Now taking the $L^2(\Omega)$-inner product of (6.5.2) with v^3, we get

$$\frac{1}{4}\frac{d}{dt}\int_\Omega v^4 dx + 3\int_\Omega (\nu\rho v_x - p[u,\theta])v^2 v_x dx = \int_\Omega gv^3 dx - p_\Gamma v^3\Big|_{x=M}.$$

Adding equality (6.5.32) to the last one multiplied by a parameter $\delta \geq 1$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left[\left(\frac{1}{2}v^2 + c_V(\theta - \theta_\Gamma)\right)^2 + \frac{\delta}{2}v^4\right] dx \\
&\qquad + \int_\Omega [(1+3\delta)\nu\rho v^2 v_x^2 dx + c_V\kappa[u,\theta]\rho\theta_x^2]dx \\
&\quad = -\int_\Omega (\nu c_V + \kappa[u,\theta])\rho v v_x\theta_x dx \\
&\qquad + \int_\Omega \left[p_0[u]v_x\left(\frac{1}{2}v^2 + c_V(\theta - \theta_\Gamma)\right) + p[u,\theta]((1+3\delta)v^2 v_x + c_V v\theta_x)\right] dx \\
&\qquad + \int_\Omega gv\left(\left(\frac{1}{2} + \delta\right)v^2 + c_V(\theta - \theta_\Gamma)\right) dx \\
&\qquad - p_\Gamma\left(v\left(\left(\frac{1}{2} + \delta\right)v^2 + c_V(\theta - \theta_\Gamma)\right)\right)\bigg|_{x=M} \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{6.5.33}$$

Now we need to estimate the summands in the last equality. First, using the two-sided bounds $\underline{u} \leq u \leq \bar{u}$ and $\underline{\kappa} \leq \kappa \leq \bar{\kappa}$, we obtain

$$K_1^{-1}\left(\delta\|vv_x\|_\Omega^2 + \|\theta_x\|_\Omega^2\right) \leq \int_\Omega [(1+3\delta)\nu\rho v^2 v_x^2 + c_V\kappa[u,\theta]\rho\theta_x^2]dx, \tag{6.5.34}$$

and for all $\varepsilon > 0$,

$$|I_1| \leq K_2 \|vv_x\|_\Omega \|\theta_x\|_\Omega \leq \frac{K_2^2}{4\varepsilon} \|vv_x\|_\Omega^2 + \varepsilon \|\theta_x\|_\Omega^2. \tag{6.5.35}$$

Second, using the estimates $|p_0[u]| \leq K_3$ and the inequality

$$|p[u,\theta]| = |p[u,\theta_\Gamma] + p_1[u](\theta - \theta_\Gamma)| \leq K_4(1 + |\theta - \theta_\Gamma|),$$

we have

$$\begin{aligned} |I_2| &\leq K_5 \left[\int_\Omega (\delta v^2 |v_x| + |v\theta_x|) dx + \int_\Omega |\theta - \theta_\Gamma| (|v_x| + \delta v^2 |v_x| + |v\theta_x|) dx \right] \\ &=: K_5(I_{21} + I_{22}). \end{aligned} \tag{6.5.36}$$

Furthermore, the following estimates hold, for any $\varepsilon > 0$,

$$I_{21} \leq \delta \|vv_x\|_\Omega \|v\|_\Omega + \|v\|_\Omega \|\theta_x\|_\Omega \leq \varepsilon \Big(\delta \|vv_x\|_\Omega^2 + \|\theta_x\|_\Omega^2 \Big) + \frac{\delta + 1}{4\varepsilon} \|v\|_\Omega^2 \tag{6.5.37}$$

and

$$\begin{aligned} I_{22} &\leq \|\theta - \theta_\Omega\|_{L^\infty(\Omega)} \left\| \frac{v_x}{\sqrt{\theta}} \right\|_\Omega \|\theta\|_{L^1(\Omega)}^{1/2} + \|\theta - \theta_\Gamma\|_\Omega \|v\|_{L^\infty(\Omega)} \Big(\delta \|vv_x\|_\Omega + \|\theta_x\|_\Omega \Big) \\ &\leq \varepsilon \left(\frac{\delta}{2} \|vv_x\|_\Omega^2 + \|\theta_x\|_\Omega^2 \right) + \frac{MK^{(1)}}{2\varepsilon} \left\| \frac{v_x}{\sqrt{\theta}} \right\|_\Omega^2 + \frac{\delta}{\varepsilon} \|v\|_{L^\infty(\Omega)}^2 \|\theta - \theta_\Gamma\|_\Omega^2. \end{aligned} \tag{6.5.38}$$

Third,

$$\begin{aligned} |I_3| + |I_4| &\leq \Big(\|g\|_{L^1(\Omega)} + p_\Gamma \Big) \|v\|_{C(\overline{\Omega})} M^{1/2} \|(1 + 2\delta) vv_x + c_V \theta_x\|_\Omega \\ &\leq \varepsilon \Big(\delta \|vv_x\|_\Omega^2 + \|\theta_x\|_\Omega^2 \Big) + \frac{K_6 \delta}{\varepsilon} \|v\|_{C(\overline{\Omega})}^2. \end{aligned} \tag{6.5.39}$$

All the above quantities K_i, $1 \leq i \leq 6$, are independent of δ and ε.

Now, choosing $\varepsilon := K_7^{-1}$ small enough, $\delta := K_8$ large enough, and setting

$$y := \int_\Omega \left[\left(\frac{1}{2} v^2 + c_V(\theta - \theta_\Gamma) \right)^2 + \frac{\delta}{2} v^4 \right] dx,$$

we derive from (6.5.33)–(6.5.39)

$$\frac{dy}{dt} + K_9^{-1} (\|vv_x\|_\Omega^2 + \|\theta_x\|_\Omega^2) \leq K_{10}(ay + h), \tag{6.5.40}$$

with $a := \|v\|_{L^\infty(\Omega)}^2$ and $h := \|\frac{v_x}{\sqrt{\theta}}\|_\Omega^2$ (see (6.5.30)), which, along with the inequalities,

$$K_{11}^{-1} \left(\frac{1}{2} \|v^2\|_\Omega^2 + \|\theta - \theta_\Gamma\|_\Omega^2 \right) \leq y \leq K_{11} \left(\frac{1}{2} \|v^2\|_\Omega^2 + \|\theta - \theta_\Gamma\|_\Omega^2 \right),$$

gives us

$$\frac{dy}{dt} + K_{12}^{-1} y \le K_{10}(ay + h),$$

with $K_{12} := K_9 K_{11} M^2$. By Corollary 6.5.1, we have

$$\|a\|_{L^1(\mathbb{R}^+)} \le K^{(1)} \|h\|_{L^1(\mathbb{R}^+)} \le (K^{(5)})^2. \tag{6.5.41}$$

Therefore, by Theorem 2.2.2, we conclude as $t \to +\infty$,

$$\sup_{t\ge 0} y(t) \le K_{13}, \qquad y(t) \to 0.$$

On the other hand, integrating inequality (6.5.40) over $\mathbb{R}^+$, we also obtain

$$K_9^{-1}(\|vv_x\|_Q^2 + \|\theta_x\|_Q^2) \le y(0) + K_{10}\left(\|a\|_{L^1(\mathbb{R}^+)} \sup_{t\ge 0} y + \|h\|_{L^1(\mathbb{R}^+)}\right),$$

whence $\|vv_x\|_Q + \|\theta_x\|_Q \le K_{14}$. □

Lemma 6.5.6 ([226]). *The following estimate holds:*

$$\|v_x\|_Q \le K^{(7)}. \tag{6.5.42}$$

Proof. Taking L^2-inner product of the second equation (6.5.2) with v, we get (cf. (6.5.21))

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}v^2 + p_S u - P[u,\theta_\Gamma]\right) dx + \int_\Omega \nu\rho v_x^2 dx = \int_\Omega p_1[u](\theta-\theta_\Gamma)v_x dx.$$

Integrating this equality over $(0,T)$ and exploiting the bounds $\underline{u} \le u \le \bar{u}$, we can get

$$\|v_x\|_{Q_T}^2 \le K_1(1 + \|\theta-\theta_\Gamma\|_{Q_T}\|v_x\|_{Q_T}).$$

Thus $\|v_x\|_{Q_T} \le K_1^{1/2} + K_1\|\theta-\theta_\Gamma\|_{Q_T} \le K_1^{1/2} + K_1 M\|\theta_x\|_{Q_T}$, for any $T>0$, and the result follows from Lemma 6.5.4. □

Now we establish additional properties of $p[u,\theta] - p_S$.

Lemma 6.5.7 ([226]). *The following estimates together with the stabilization property hold*

$$\begin{cases} \|p[u,\theta] - p_S\|_Q \le K^{(8)}, & (6.5.43) \\ \|p[u,\theta](\cdot,t) - p_S(\cdot)\|_\Omega \to 0 \qquad \textit{as } t \to +\infty. & (6.5.44) \end{cases}$$

Proof. (1) Obviously, from equation (6.5.22) it follows that for any $T>0$,

$$\|p[u,\theta]-p_S\|_{Q_T}^2 + \|I^* v_t\|_{Q_T}^2 = \|\nu\rho v_x\|_{Q_T}^2 + 2\int_{Q_T}(p[u,\theta]-p_S)I^* v_t dxdt. \tag{6.5.45}$$

Using elementary transformations and the bounds $\underline{u} \le u \le \bar{u}$, we have

$$\begin{aligned}
&\int_\Omega (p[u,\theta] - p_S) I^* v_t dxdt \\
&= \int_{Q_T} \Big(p[u,\theta_\Gamma] - p_S\Big) I^* v_t dxdt + \int_{Q_T} p_1[u](\theta - \theta_\Gamma) I^* v_t dxdt \\
&= \int_\Omega (p[u,\theta_\Gamma] - p_S) I^* v dx|_0^T - \int_{Q_T} p_u[u,\theta_\Gamma] u_t I^* v dxdt \\
&\quad + \int_{Q_T} p_1[u](\theta - \theta_\Gamma) I^* v_t dxdt \\
&\le K_1(\|v(\cdot,T)\|_\Gamma + \|v^0\|_\Omega + \|v_x\|_{Q_T}\|v\|_{Q_T} + \|\theta - \theta_\Gamma\|_{Q_T}\|I^* v_t\|_{Q_T}),
\end{aligned}$$

whence

$$\begin{aligned}
&\|p[u,\theta] - p_S\|_{Q_T}^2 + \frac{1}{2}\|I^* v_t\|_{Q_T}^2 \qquad (6.5.46)\\
&\quad \le \nu \underline{u}^{-2}\|v_x\|_{Q_T}^2 + K_1(\|v(\cdot,T)\|_\Omega + \|v^0\|_\Omega + M\|v_x\|_{Q_T}^2) + (K_1 M^2)\|\theta_x\|_{Q_T}^2.
\end{aligned}$$

Therefore, estimate (6.5.44) follows from Lemmas 6.5.1, 6.5.4 and 6.5.5.

(2) First, instead of property (6.5.44), let us prove that, as $t \to +\infty$,

$$\|p[u,\theta_\Gamma](\cdot,t) - p_S(\cdot)\|_\Omega \to 0. \qquad (6.5.47)$$

Using the estimates $\underline{u} \le u$ (see (6.5.43)) and $\|\theta_x\|_Q \le K^{(6)}$, we have

$$\|p[u,\theta_\Gamma] - p_S\|_Q \le \|p[u,\theta] - p_S\|_Q + \|p_1[u]\|_{L^\infty(Q)}\|\theta - \theta_\Gamma\|_Q \le K_2. \qquad (6.5.48)$$

Therefore

$$\begin{aligned}
&\int_0^{+\infty} \left|\frac{d}{dt}\|p[u,\theta_\Gamma] - p_S\|_\Omega^2\right| dt = 2\int_0^{+\infty}\left|\int_\Omega p_u[u,\theta_\Gamma]u_t(p[u,\theta_\Gamma] - p_S)dx\right| dt\\
&\quad \le 2\|p_u[u,\theta_\Gamma]\|_{L^\infty(Q)}\|v_x\|_Q\|p[u,\theta_\Gamma] - p_S\|_Q \le K_3 \qquad (6.5.49)
\end{aligned}$$

which, along with estimates (6.5.48), implies property (6.5.47).

Finally, by the bounds $\underline{u} \le u \le \bar{u}$ and the stabilization property (6.5.31), as $t \to +\infty$,

$$\begin{aligned}
&\Big|\|p[u,\theta] - p_S\|_\Omega^2 - \|p[u,\theta_\Gamma] - p_S\|_\Omega^2\Big| \\
&\quad \le [2M^{1/2}(\|p[u,\theta_\Gamma]\|_{L^\infty(\Omega)} + \bar{p}_S) \\
&\qquad + \|p_1[u]_{L^\infty(\Omega)}\|\theta - \theta_\Gamma\|_\Omega]\|p_1[u]\|_{L^\infty(\Omega)}\|\theta - \theta_\Gamma\|_\Omega \\
&\quad \le K_4(1 + \|\theta - \theta_\Gamma\|_\Omega)\|\theta - \theta_\Gamma\|_\Omega \to 0,
\end{aligned}$$

which, together with (6.5.47), implies (6.5.44). □

Chapter 7

Asymptotic Behavior of Solutions for Parabolic and Elliptic Equations

In this chapter, we shall study the asymptotic behavior for parabolic and elliptic equations. This chapter embraces three sections. In Section 7.1, we shall use Theorems 2.1.14 and 2.3.7 to establish the uniform and decay estimates for flows in a semi-infinite straight channel. In Section 7.2, we shall exploit Theorems 2.3.17–2.3.21 to establish exact rates of convergence for nonlinear PDEs. In Section 7.3, we shall apply Theorem 2.2.11 and the Lyapunov functional method (i.e., Lemma 2.5.3 of [121]) to prove the large-time behavior of solutions to the initial boundary value problem of (semilinear) parabolic equations. All inequalities applied in this chapter are specially selected and crucial in proving the asymptotic behavior of solutions to some parabolic and elliptic equations.

7.1 Decay estimates for flows in a semi-infinite straight channel

In this section, we shall use Theorems 2.1.14 and 2.3.7 to establish the uniform and decay estimates for the flows in a semi-infinite straight channel. We choose these results from Galdi [295].

7.1.1 Uniform estimates

We shall consider in this subsection flows occurring in a straight cylinder $\Omega = \{x_n > 0\} \times \Sigma$, where the cross section Σ is a C^∞ smooth, bounded and simply connected to a more general class of domains, $\mathbf{n}$ is its unit outward normal vector. The cross section at distance a from the origin is denoted by $\Sigma(a)$, despite all cross sections having the same shape and size. Denote by (u, τ) a solution to the

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problem

$$\begin{cases} \Delta u = \nabla \tau, & \text{in } \Omega, \qquad (7.1.1) \\ \nabla \cdot u = 0, & \text{in } \Omega, \qquad (7.1.2) \\ u = 0, & \text{on } \partial\Omega - \Sigma(0), \qquad (7.1.3) \\ \int_\Sigma u \cdot \mathbf{n} dx_n = 0. & \qquad (7.1.4) \end{cases}$$

For simplicity, we assume (u, τ) is regular, that is, indefinitely differentiable in the closure of any bounded subset of Ω. We also note, however, that the same conclusions may be reached merely assuming (u, τ) to possess the same regularity of generalized solutions to Leray's problem (see, e.g., Galdi [295], Chapter V.I.). We shall first show that every regular solution to problem (7.1.1)–(7.1.4) with u satisfying a general "growth" condition as $|x| \to +\infty$ has, in fact, square summable gradients over the whole of Ω. Next, we further prove that these solutions decay exponentially fast in the Dirichlet integral, i.e.,

$$\|u\|_{H^1(\Omega^R)} \le C\|u\|_{H^1(\Omega)} \exp(-\sigma R) \tag{7.1.5}$$

where $\Omega^a = \{x \in \Omega : x_n > a\}$ and C, σ are positive constants depending on Σ. Now we use Theorem 2.1.14 to prove the following uniform estimate.

Theorem 7.1.1 ([295]). *Let (u, τ) be a regular solution to problem* (7.1.1)–(7.1.4) *with*

$$\liminf_{x_n \to +\infty} \left(\int_0^{x_n} \left[\int_{\Sigma(\xi)} \nabla u : \nabla u \, d\Sigma \right] d\xi \right) e^{-a x_n} = 0 \tag{7.1.6}$$

where $a^{-1} \equiv (1/2 + C_0)\sqrt{\mu}$, $C_0 > 0$ is a constant specified and $\mu > 0$ is the Poincaré constant for Σ. Then

$$\|u\|_{H^1(\Omega)} < +\infty. \tag{7.1.7}$$

Proof. In fact, multiplying both sides of (7.1.1) by u and integrating by parts over $(0, x_n) \times \Sigma$, we have

$$\begin{aligned} G(x_n) &\equiv \int_0^{x_n} \left[\int_{\Sigma(\xi)} \nabla u : \nabla u \, d\Sigma \right] dx_n \\ &= \int_{\Sigma(x_n)} \left(\tau u_n - \frac{1}{2} \frac{\partial u^2}{\partial x_n} \right) d\Sigma - \int_{\Sigma(0)} \left(\tau u_n - \frac{1}{2} \frac{\partial u^2}{\partial x_n} \right) d\Sigma. \end{aligned} \tag{7.1.8}$$

Integrating this relation from t to $t+1, t \ge 0$, we obtain

$$\int_t^{t+1} G(x_n) dx_n = \int_{\Omega_{t,t+1}} \left(\tau u_n - \frac{1}{2} \frac{\partial u^2}{\partial x_n} \right) dx + b \tag{7.1.9}$$

where $\Omega_{t,t+1} = \Omega \cap \{x \in \mathbb{R}^n : t < x_n < t+1\}$ and $b \equiv -\int_{\Sigma(0)}(\tau u_n - \frac{1}{2}\frac{\partial u^2}{\partial x_n})d\Sigma$. Consider the problem

$$\begin{cases} \nabla \cdot w = u_n, \quad \text{in } \Omega_{t,t+1}, & (7.1.10)\\ w \in H_0^1(\Omega_{t,t+1}), & (7.1.11)\\ \|w\|_{H^1(\Omega_{t,t+1})} \le C_0 \|u_n\|_{L^2(\Omega_{t,t+1})}. & (7.1.12) \end{cases}$$

Since $\int_{\Omega_{t,t+1}} u_n dx = 0$, problem (7.1.10)–(7.1.12) admits a solution with a constant $C_0 > 0$ independent of t (see Theorem III 3.1 and Lemma III 3.3 in Galdi [295]). Thus it follows from (7.1.9)–(7.1.12) that

$$\begin{aligned} \int_t^{t+1} G(x_n)dx_n &= \int_{\Omega_{t,t+1}} \left(-\nabla\tau \cdot w - \frac{1}{2}\frac{\partial u^2}{\partial x_n}\right) dx + b \\ &= \int_{\Omega_{t,t+1}} \left(-\nabla u : \nabla w - \frac{1}{2}\frac{\partial u^2}{\partial x_n}\right) dx + b \qquad (7.1.13)\\ &\le \left(C_0 + \frac{1}{2}\right) \|u\|_{L^2(\Omega_{t,t+1})} \|u\|_{H^1(\Omega_{t,t+1})} + b. \end{aligned}$$

We next observe that, since u vanishes at $\partial\Omega$, there exists $\mu = \mu(\Sigma) > 0$ (the Poincaré constant for Σ) such that

$$\|u\|^2_{L^2(\Sigma)} \le \mu \|\nabla u\|^2_{L^2(\Sigma)}. \qquad (7.1.14)$$

In fact, we may give estimates for μ (see (II.4.4) and Exercise II.4.2 in Galdi [295]): $\mu \le \frac{1}{2}|\Sigma|$ if $n = 3$; $\mu \le \frac{(2d)^2}{\pi^2}$ if $n = 2$.

By (7.1.14), we get

$$y(t) \equiv \int_t^{t+1} G(x_n)dx_n \le \sqrt{\mu}\left(C_0 + \frac{1}{2}\right) \|u\|^2_{H^1(\Omega_{t,t+1})} + b. \qquad (7.1.15)$$

Since $\|u\|^2_{H^1(\Omega_{t,t+1})} = \frac{dy}{dt}$, (7.1.15) can be rewritten as

$$ay(t) \le b + \frac{dy(t)}{dt} \qquad (7.1.16)$$

where a is defined in the theorem. Thus (7.1.16) implies that $y(t)$ satisfies (2.1.58) in Theorem 2.1.14 with $b = |b|$, and furthermore, it is easy to show that, in view of (7.1.15), $y(t)$ verifies (2.1.59) in Theorem 2.1.14 and hence Theorem 2.1.14 implies for all $t > 1$,

$$\int_t^{t+1} G(x_n)dx_n \le \frac{|b|}{a}, \qquad (7.1.17)$$

which yields

$$l \equiv \lim_{x_n \to +\infty} G(x_n) = \|u\|_{H^1(\Omega)} < +\infty. \qquad (7.1.18)$$

In fact, since $G(x_n)$ is monotonically increasing in x_n, l exists (either finite or infinite).

If we assume $l = +\infty$, then by the monotone convergence theorem, we deduce

$$\lim_{x_n \to +\infty} \int_t^{t+1} G(x_n) dx_n = \int_0^1 G(\xi + t) d\xi = +\infty$$

which contradicts (7.1.17). Therefore, (7.1.18) holds and hence the proof is complete. □

7.1.2 Exponential decay

In this subsection, we shall derive the exponential estimate (7.1.5) by using Theorem 2.3.7. The next theorem is the second result in this section, which is also due to Galdi [295].

Theorem 7.1.2 ([295]). *Let* (u, τ) *be a regular solution to problem* (7.1.1)–(7.1.4) *satisfying* (7.1.6). *Then* (7.1.7) *holds and for all* $R > 0$, *the inequality* (7.1.5) *holds with*

$$C = [2(C_0^2 + 2)^{1/2}]/[(C_0^2 + 2)^{1/2} - C_0], \quad \sigma = [(C_0^2 + 2)^{1/2} - C_0]/\mu$$

where $C_0 > 0$ *is the constant in* (7.1.12) *and* μ *is the Poincaré constant for* Σ. *Moreover, for all* $|\alpha| \geq 0$, (u, τ) *satisfies the pointwise estimate*

$$|D^\alpha u(x)| + |D^\alpha \nabla \tau(x)| \leq C_2 \|u(x)\|_{H^1(\Omega)} \exp(-\sigma x_n), \tag{7.1.19}$$

for every $x \in \Omega$ *with* $x_n \geq 1$.

Proof. From Theorem 7.1.1, it follows that (7.1.7) holds, and so it suffices to show that (7.1.5) is valid. For the sake of simplicity, we shall only treat the case $n = 3$ and Cartesian coordinates will be denoted by x_1, x_2, x_3 and x, y, z, indifferently. Proceeding as in the proof of Theorem 7.1.1, we may write the identity

$$\begin{aligned} &- \int_z^{z_1} \left(\int_{\Sigma(\xi)} \nabla u : \nabla u d\Sigma \right) d\xi \\ &= \int_{\Sigma(z_1)} \left(\tau u_3 - \frac{1}{2} \frac{\partial u^2}{\partial z} \right) d\Sigma - \int_{\Sigma(z)} \left(\tau u_3 - \frac{1}{2} \frac{\partial u^2}{\partial z} \right) d\Sigma. \end{aligned} \tag{7.1.20}$$

By virtue of Theorems 7.1.1 and 2.3.7, we know that $u, \nabla \tau \in H^m(\Omega)$ for all $m \geq 0$ and so, in particular, it easily follows that as $z_1 \to +\infty$,

$$i(z_1) \equiv \int_{\Sigma(z_1)} \tau u_3(x', z_1) dx' = o(1). \tag{7.1.21}$$

In fact, setting

$$\bar{\tau}(z_1) = \frac{1}{|\Sigma|}\int_\Sigma \tau(x', z_1)dx',$$

it follows from (7.1.4) and the Poincaré inequality that

$$|i(z_1)| = \left|\int_{\Sigma(z_1)} (\tau - \bar{\tau})u_3(x', z_1)dx'\right| \leq C\|\tau\|_{H^1(\Sigma)}\|u\|_{L^2(\Sigma)} \tag{7.1.22}$$

and (7.1.21) becomes a consequence of

$$\begin{cases} |D^\alpha u(x)| \to 0 \text{ as } |x| \to +\infty \text{ in } \Omega, & (7.1.23)\\ |D^\alpha \nabla\tau(x)| \to 0 \text{ as } |x| \to +\infty \text{ in } \Omega. & (7.1.24)\end{cases}$$

Thus, letting $z_1 \to +\infty$ in (7.1.20), we derive from (7.1.21) and (7.1.23),

$$H(z) \equiv \int_z^{+\infty}\left(\int_{\Sigma(\xi)} \nabla u : \nabla u d\Sigma\right)d\xi = \int_{\Sigma(z)}\left(\tau u_3 - \frac{1}{2}\frac{\partial u^2}{\partial z}\right)d\Sigma. \tag{7.1.25}$$

Integrating both sides of (7.1.25) between $t+l$ and $t+l+1$ with l a non-negative integer, we obtain

$$\int_{t+l}^{t+l+1} H(z)dz = \int_{t+l}^{t+l+1}\int_{\Sigma(z)} \tau u_3 d\Sigma dz - \frac{1}{2}\int_{\Sigma(t+l+1)} u^2 d\Sigma + \frac{1}{2}\int_{\Sigma(t+l)} u^2 d\Sigma. \tag{7.1.26}$$

By writing $u_3 = \nabla\cdot w$ with w being a solution to (7.1.12) and by arguing as in the proof of Theorem 7.1.1, from (7.1.26) it follows that

$$\int_{t+l}^{t+l+1} H(z)dz \leq C_0\sqrt{\mu}\|u\|^2_{H^1(\Omega_{t+l,t+l+1})} - \frac{1}{2}\int_{\Sigma(t+l+1)} u^2 d\Sigma + \frac{1}{2}\int_{\Sigma(t+l+1)} u^2 d\Sigma. \tag{7.1.27}$$

Summing both sides of (7.1.27) from $l = 0$ to $l = +\infty$ and observing that $\lim_{z\to+\infty}\int_{\Sigma(z)} u^2(x', z)d\Sigma = 0$, we get

$$\int_t^{+\infty} H(z)dz \leq C_0\sqrt{\mu}H(t) + \frac{1}{2}\int_{\Sigma(t)} u^2 d\Sigma. \tag{7.1.28}$$

Since, by (7.1.14), we have

$$\int_{\Sigma(t)} u^2 d\Sigma \leq \mu\int_{\Sigma(t)} \nabla u : \nabla u d\Sigma = -\mu H'(t)$$

which, along with (7.1.28), gives us

$$H'(t) + \frac{2}{\mu}\int_t^{+\infty} H(s)ds \leq \frac{2C_0}{\sqrt{\mu}}H(t). \tag{7.1.29}$$

Thus applying now Theorem 2.3.7 to the inequality (7.1.29), we can complete the proof of (7.1.5) and hence of the theorem. □

7.2 Exact rates of convergence for nonlinear PDEs

In this section, we shall exploit Theorems 2.3.17–2.3.21 to establish exact rates of convergence for nonlinear PDEs. We pick these results from Vărvărucă [933].

7.2.1 Nonlinear diffusion equations and porous medium equations

Let Ω be an open and bounded subset of $\mathbb{R}^n, n \geq 1$, whose boundary $\partial\Omega$ is a C^2 manifold. Let $\beta \subseteq \mathbb{R} \times \mathbb{R}$ be a maximal monotone graph with $\beta^{-1}(0) \neq \emptyset$, and $j : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous convex function, such that $\beta = \partial j$. We assume that $\min_{r\in\mathbb{R}} j(r) = 0$.

First, we introduce a definition.

Definition 7.2.1 ([933]). We say that a function j is of type (T_p) if $j^{-1}(0)$ is a finite interval, denoted by $[\beta_1, \beta_2]$, there exists a constant $\varepsilon > 0$ such that $(\beta_1 - \varepsilon, \beta_2 + \varepsilon) \subseteq \mathrm{Dom}(j)$, and the following conditions hold

$$\begin{cases} r \mapsto \frac{j(r)}{(\beta_1 - r)^p} \quad \text{non-increasing on } (-\infty, \beta_1) \cap \mathrm{Dom}(j), \\ \lim_{r \nearrow \beta_1} \frac{j(r)}{(\beta_1 - r)^p} := b_1 > 0, \\ r \mapsto \frac{j(r)}{(r - \beta_2)^p} \quad \text{non-increasing on } (\beta_2, +\infty) \cap \mathrm{Dom}(j), \\ \lim_{r \searrow \beta_2} \frac{j(r)}{(r - \beta_2)^p} := b_2 > 0. \end{cases} \tag{7.2.1}$$

Note that these functions are (globally) sub-homogeneous of degree p with respect to all the points of $[\beta_1, \beta_2]$. An important example for the following form in this class is called of type (E_p),

$$j(r) = \begin{cases} b_1(\beta_1 - r)^p, & \text{for} \quad r < \beta_1, \\ 0, & \text{for} \quad \beta_1 \leq r \leq \beta_2, \\ b_2(r - \beta_2)^p, & \text{for} \quad r > \beta_2. \end{cases} \tag{7.2.2}$$

We now consider some problems of nonlinear PDEs to which Theorems 2.3.17–2.3.20 in Chapter 2 can be applied.

Problem 7.2.1 ([933]). The problem for the nonlinear diffusion equations

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta\beta(u) = 0, & \text{a.e. in } \Omega \times (0, +\infty), \\ \beta(u) = 0, & \text{a.e. in } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x), & \text{a.e. in } \Omega. \end{cases} \tag{7.2.3}$$

For the problem (7.2.3), we assume:

1) $\lim_{|r|\to+\infty} \frac{j(r)}{|r|} = +\infty$,
2) the operator β is single-valued and continuous on $\mathbb{R}$.

Then it follows from [70, 116] that the problem can be written in the form (2.3.135) in the space $H^{-1}(\Omega)$, endowed with the inner product: for all u, $v \in H^{-1}(\Omega)$,

$$\langle u, v \rangle = (J^{-1}u, v),$$

where $J = -\Delta$ is the canonical isomorphism (duality mapping) from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ and $(\cdot,\cdot)$ is the usual pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, with the function φ given by

$$\varphi(u) = \begin{cases} \int_\Omega j(u(x))dx, & \text{if } u \in L^1(\Omega) \text{ and } j(u) \in L^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

The set F of minimizers of φ is given by

$$F = \left\{ u \in L^1(\Omega) : u(x) \in [\beta_1, \beta_2] \text{ a.e. } x \in \Omega \right\}.$$

It is easy to show that if j is of type (T_p), then φ is (globally) sub-homogeneous of degree p with respect to all the points of F. Let us prove that (C_pF) (see Section 2.3 in Chapter 2 for its definition) also holds. To this end, we may consider the operator $\tilde{P} : L^p(\Omega) \to F$, defined a.e. in Ω, by

$$[\tilde{P}u](x) = \begin{cases} \beta_1, & \text{for } u(x) < \beta_1, \\ u(x), & \text{for } \beta_1 \le u(x) \le \beta_2, \\ \beta_2, & \text{for } u(x) > \beta_2. \end{cases}$$

Thus the following inequalities prove (C_pF): for all $u \in \mathrm{Dom}(\varphi) \subseteq L^p(\Omega)$,

$$|u - Pu|^p \le |u - \tilde{P}u|^p \le K_1 \|u - \tilde{P}u\|^p \le K_2 \varphi(u),$$

where $\|\cdot\|$ denotes the $L^p(\Omega)$ norm, and K_1, K_2 are positive constants. Therefore, if j is of type (T_p), then we can apply Theorems 2.3.19–2.3.20 to reach the conclusions.

Our analysis can be divided into different cases as follows.

Case I: The porous medium equation for the function j of type (E_p) with $\beta_1 = \beta_2 = 0$ and $b_1 = b_2 = 1/p$, where $p > 2$.

Note that Theorem 2.3.17 implies that solutions converge to 0 at an exact algebraic rate in both the $H^{-1}(\Omega)$ and $L^p(\Omega)$ norms.

Obviously, as in Remark 2.3.7, Theorem 2.3.18 can also be applied with $V := L^p(\Omega)$. Actually, since φ is homogeneous of degree p, the problem (7.2.3) is covered by the result in [306], although not mentioned there. Related results have been obtained for the problem (7.2.3) by completely different methods in [51].

Case II: The two-phase Stefan problem in the form (7.2.3) in **Problem 7.2.1** for a function j of type (E_2), with $\beta_1 \neq \beta_2$.

This problem was mentioned in [116], and analyzed in detail in [70], whose physical description was also given.

We note that the functions u and $\beta(u)$ represent the enthalpy, and respectively the temperature of a system composed of water and ice, where phase transition can take place at the temperature 0; the boundary of the body is being kept at this critical value of the temperature. In fact, there is a “huge” number of equilibrium states for the enthalpy.

We would like to mention the significant process in [384] has been made on the problem of identifying the limit of a solution for this equation in terms of its initial data. Theorem 2.3.18 asserts that the enthalpy converges to an equilibrium state at a rate which is exactly of exponential type in the $H^{-1}(\Omega)$ norm, and also shows that the equilibrium is never reached if the initial state is not one of equilibrium.

It follows easily from Theorem 2.3.19 that, if $p = 2$, then

$$\lim_{t\to+\infty} \frac{-\log\sqrt{\varphi(u(t))}}{t} = \Lambda^{\infty}, \tag{7.2.4}$$

and, because of the particular form of φ in the Stefan problem, we deduce from (7.2.4) that

$$\lim_{t\to+\infty} \frac{-\log\|\beta(u(t))\|}{t} = \Lambda^{\infty} \tag{7.2.5}$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm, and thus the convergence of the temperature to 0 is also exactly at an exponential rate.

However, Theorem 2.3.21 seems not to apply in general in **Problem 7.2.1** if j is of type (T_p), but only when $\beta_1 = \beta_2 =: \beta$. In this case, we can show that Theorem 2.3.21 holds with $V := L^p(\Omega)$ and

$$\tilde{\psi}_\beta(u) = \begin{cases} \displaystyle\int_\Omega j_0(u(x))dx, & \text{if } u \in L^p(\Omega),\\ +\infty, & \text{otherwise,}\end{cases} \tag{7.2.6}$$

where $j_0(r) = b_1(r^-)^p + b_2(r^+)^p$, and the real number β is identified with the function taking the value β almost everywhere in Ω.

Problem 7.2.2 ([933])**.** The parabolic equation governed by the pseudo-p-Laplace operator, $p \geq 2$, with nonlinear Neumann boundary conditions

$$\begin{cases} \frac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}\right) = 0, & \text{a.e. in } \Omega\times(0,+\infty),\\ -\displaystyle\sum_{i=1}^{n} |\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i}\cos(\overrightarrow{n},\overrightarrow{e_i}) \in \beta(u), & \text{a.e. in } \partial\Omega\times(0,+\infty),\\ u(x,0) = u_0(x), & \text{a.e. in } \Omega,\end{cases} \tag{7.2.7}$$

where $\overrightarrow{n}$ is the outward normal on $\partial\Omega$, and $\{\overrightarrow{e_i} : i \in \{1,\ldots,n\}\}$ is the canonical basis in $\mathbb{R}^n$.

From [70, 116] for $p = 2$, and [932], (Example 1.5.4, p. 18) for $p > 2$, it follows that this equation in (7.2.7) can be written in the form (2.3.135) in the space $L^2(\Omega)$ with the usual inner product, and with φ given by

$$\varphi(u) = \begin{cases} \displaystyle\frac{1}{p}\sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx + \int_{\partial\Omega} j(u(x))d\sigma, & \\ \qquad\qquad \text{if } u \in W^{1,p}(\Omega) \text{ and } j(u) \in L^1(\partial\Omega), & \\ +\infty, \qquad \text{otherwise.} & \end{cases} \tag{7.2.8}$$

Then

$$F = \{u \in W^{1,p}(\Omega) : \exists \;\; c \in [\beta_1, \beta_2] \;\text{ such that } u(x) = c, \;\; \text{a.e. } x \in \Omega\}. \qquad (7.2.9)$$

We may show that if j is of type (T_p) for the same value of p as in the pseudo-p-Laplace operator, then it follows from Wirtinger's and Friedrichs' Inequalities that φ is (globally) sub-homogeneous of degree p with respect to all the points of F, and (C_pF) also holds. Thus, if j is of type (T_p), then Theorems 2.3.19–2.3.20 hold for solution u to the problem (7.2.7). Moreover, we may prove that the hypotheses of Theorem 2.3.21 are also satisfied for $V := W^{1,p}(\Omega)$ and, for $\beta_1 \neq \beta_2$, hence we can derive from Theorem 2.3.21 that

(1) for $z \in (\beta_1, \beta_2)$:

$$\tilde{\psi}_z(u) = \begin{cases} \dfrac{1}{p}\displaystyle\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx, & \text{if } u \in W^{1,p}(\Omega), \\ +\infty, & \text{otherwise,} \end{cases} \qquad (7.2.10)$$

(2) for $z = \beta_1$:

$$\tilde{\psi}_z(u) = \begin{cases} \dfrac{1}{p}\displaystyle\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx + \int_{\partial\Omega} j_1(u(x))d\sigma, & \text{if } u \in W^{1,p}(\Omega), \\ +\infty, & \text{otherwise,} \end{cases} \qquad (7.2.11)$$

(3) for $z = \beta_2$:

$$\tilde{\psi}_z(u) = \begin{cases} \dfrac{1}{p}\displaystyle\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx + \int_{\partial\Omega} j_2(u(x))d\sigma, & \text{if } u \in W^{1,p}(\Omega), \\ +\infty, & \text{otherwise,} \end{cases} \qquad (7.2.12)$$

where $j_1(r) = b_1(r^-)^p$, and $j_2(r) = b_2(r^+)^p$, for $r \in \mathbb{R}$. For the case $\beta_1 = \beta_2 =: \beta$, we have

$$\tilde{\psi}_\beta(u) = \begin{cases} \dfrac{1}{p}\displaystyle\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx + \int_{\partial\Omega} j_0(u(x))d\sigma, & \text{if } u \in W^{1,p}(\Omega), \\ +\infty, & \text{otherwise,} \end{cases} \qquad (7.2.13)$$

where $j_0(r) = b_1(r^-)^p + b_2(r^+)^p$.

A typical example arising in physics with the form considered here is the thermostat control process, where the function j involved is of type (E_2) with $\beta_1 \neq \beta_2$, we may refer to [70] for details.

We should point out here that Theorem 2.3.21 can be also applied in some cases where j is not necessarily of type (T_p), namely:

- Dirichlet boundary conditions: $\mathrm{Dom}(j) = \{0\}, j(0) = 0$,
- Neumann boundary conditions: $j(r) = 0$, for all $r \in \mathbb{R}$,
- the Signorini problem: $\mathrm{Dom}(j) = [0, +\infty), j(r) = 0$, for all $r \geq 0$.

The first two above are easy to handle, while in the third case, in Theorem 2.3.21, we only take $V := W^{1,p}(\Omega)$, and with $\tilde{\psi}_0 = \varphi$ and respectively, for all $z \in (0, +\infty)$,

$$\tilde{\psi}_z(u) = \begin{cases} \frac{1}{p}\sum_{i=1}^{n}\int_\Omega |\frac{\partial u}{\partial x_i}|^p dx, & \text{if } u \in W^{1,p}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{7.2.14}$$

Remark 7.2.1 ([933])**.** It was considered in [306] that the parabolic problem governed by the standard p-Laplace operator, $p > 2$, with Dirichlet boundary conditions (i.e., equation (2.3.135)) with $H := L^2(\Omega)$ and φ given by

$$\varphi(u) = \begin{cases} \frac{1}{p}\int_\Omega |\nabla u|^p dx, & \text{if } u \in W_0^{1,p}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{7.2.15}$$

This example fits into the framework described in Remark 2.3.7 with $V := W_0^{1,p}(\Omega)$.

7.3 Large-time behavior of solutions for parabolic equations

In this section, in order to apply Theorem 2.2.11, we shall use the Lyapunov functional method (i.e., Lemma 2.5.3 of [121]) to prove the large-time behavior of solutions to the initial boundary value problem of (semilinear) parabolic equations. We choose these results from Wang [946].

7.3.1 Large-time behavior for semilinear parabolic equations

We shall consider in this subsection the following semilinear wave equations

$$\begin{cases} u_{it} - d_i \Delta u_i = f_i(x, t, u_1, \ldots, u_m), & x \in \Omega, \quad t > 0, & (7.3.1) \\ u_i(x, t) = 0, & x \in \partial\Omega, \ t > 0, & (7.3.2) \\ u_i(x, 0) = \phi_i(x), & x \in \overline{\Omega}, \quad i = 1, 2, \ldots, m & (7.3.3) \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, ν denotes the unit outward normal to $\partial\Omega$.

We first introduce the following lemma.

Lemma 7.3.1 ([121]). *Let $f_i \in C^1(\overline{\Omega}\times\mathbb{R}^+\times\mathbb{R}^m)$, $i = 1, 2, \ldots, m$, and let $u(x,t) = (u_1(x,t), \ldots, u_m(x,t))$ be a global solution to problem* (7.3.1)–(7.3.3) *such that there exists a constant $K > 0$ verifying*

$$|u_i(x,t)| \le K, \quad x \in \overline{\Omega}, \quad t > 0,\ i = 1, 2, \ldots, m. \tag{7.3.4}$$

Then there exists a constant $M > 0$ such that for all $t \ge 1$, $i = 1, 2, \ldots, m$,

$$\|u_i(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} \le M, \tag{7.3.5}$$

with $0 < \alpha < 1$.

Proof. See, e.g., Brown, Dunne and Darduer [121]. □

Lemma 7.3.2 ([946]). *Let $u(x,t) = (u_1(x,t), \ldots, u_m(x,t))$ be a bounded solution to problem* (7.3.1)–(7.3.3), *$u_i^*(x) \in C(\overline{\Omega})$, $i = 1, 2, \ldots, m$. Set*

$$g(t) = \int_\Omega \sum_{i=1}^m [(u_i(x,t) - u_i^*(x))^2 + |\nabla u_i(x,t)|^2] dx. \tag{7.3.6}$$

If $f_i(x,t,u) \in C^1(\overline{\Omega}\times\mathbb{R}^+\times\mathbb{R}^m)$, $(i = 1, 2, \ldots, m)$, then there exists a constant β such that for all $0 \le t < +\infty$,

$$\frac{dg(t)}{dt} \le \beta. \tag{7.3.7}$$

Proof. By the divergence theorem and Lemma 7.3.1, we derive that there exist constants β_i $(i = 1, 2, \ldots, m)$ such that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla u_i(x,t)|^2 dx &= 2\int_\Omega \nabla u_i \cdot \nabla(u_{it}) dx \\ &= -2\int_\Omega u_{it}\Delta u_i dx + 2\int_\Omega u_{it}\frac{\partial u_i}{\partial \nu} ds \\ &= -2\int_\Omega \Delta u_i(d_i\Delta u_i + f_i(x,t,u))dx \le \beta_i, \end{aligned} \tag{7.3.8}$$

$$\frac{d}{dt}\int_\Omega |u_i(x,t) - u_i^*(x)|^2 dx = 2\int_\Omega (u_i - u_i^*)(d_i\Delta u_i + f_i(x,t,u))dx \le \beta_i. \tag{7.3.9}$$

The proof is hence complete. □

We now state the first main result, due to Wang [946], in this section.

Theorem 7.3.3 ([946]). *Under the assumptions of Lemma* 7.3.2, *let*

$$g(t) = \int_\Omega \sum_{i=1}^m |\nabla u_i(x,t)|^2 dx.$$

If $u_i f_i(x,u) \le 0,\ i = 1,2,\ldots,m,$ *then*

$$\lim_{t\to+\infty}\int_\Omega \sum_{i=1}^m \left(|\nabla u_i(x,t)|^2 + u_i^2(x,t)\right) dx = 0 \tag{7.3.10}$$

or equivalently,

$$\lim_{t\to+\infty} \|u\|_{H^1(\Omega)} = 0. \tag{7.3.11}$$

Proof. Similarly to the proof of Lemma 7.3.2, we can show that there exists a constant β such that

$$\frac{dg(t)}{dt} \le \beta. \tag{7.3.12}$$

Let

$$V(t) = \int_\Omega \sum_{i=1}^m u_i^2(x,t)dx.$$

Then

$$\begin{aligned}\frac{dV(t)}{dt} &= 2\int_\Omega \sum_{i=1}^m u_i u_{it} dx = 2\int_\Omega \sum_{i=1}^m u_i(d_i \Delta u_i + f_i(x,t,u))dx \\ &\le 2\int_\Omega \sum_{i=1}^m d_i u_i \nabla u_i dx = -2\int_\Omega \sum_{i=1}^m d_i |\nabla u_i|^2 dx \\ &\le -2d\int_\Omega \sum_{i=1}^m |\nabla u_i(x,t)|^2 dx = -2dg(t)\end{aligned} \tag{7.3.13}$$

with $d = \min(d_1, \ldots, d_m) > 0$. Therefore, we conclude from Theorem 2.2.11

$$\lim_{t\to+\infty}\int_\Omega \sum_{i=1}^m |\nabla u_i|^2 dx = 0. \tag{7.3.14}$$

Thus by the Poincaré inequality, it follows from Theorem 2.2.11 that

$$\lim_{t\to+\infty}\int_\Omega \sum_{i=1}^m u_i^2 dx = 0$$

which, together with (7.3.14), gives us (7.3.11). The proof is now complete. □

7.3.2 Large-time behavior of solutions for parabolic equations

In this subsection, in order to apply Theorem 2.2.7, we shall consider the following Cauchy problem for a linear non-homogeneous heat equation

$$\begin{cases} u_t - d\Delta u = f(x,t), & x \in \Omega, \quad t > 0, & (7.3.15) \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, & (7.3.16) \\ u(x,0) = \phi(x), & x \in \overline{\Omega} & (7.3.17) \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and unit outward normal ν.

The next is the second result of this section due to Wang [946].

Theorem 7.3.4 ([946]). *Assume that* $f(x,t) \in C(\overline{\Omega} \times [0,+\infty)), \partial\Omega \in C^\alpha, \phi(x) \in C^\alpha(\overline{\Omega})$ *such that*

$$\int_\Omega f(x,t)dx = 0, \quad \int_\Omega \phi(x)dx = 0, \tag{7.3.18}$$

and that $u(x,t)$ *is a solution to the problem* (7.3.15)–(7.3.17). *If there exists a number* $p > n/2$, $p \geq 1$ *such that*

$$\lim_{t\to+\infty} \int_\Omega |f(x,t)|^p dx = 0, \tag{7.3.19}$$

then there is a number $p' > n/2$ *such that*

$$\lim_{t\to+\infty} \int_\Omega |u(x,t)|^{p'} dx = 0. \tag{7.3.20}$$

Proof. Set $\bar{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$ with $|\Omega|$ being the volume of Ω. Noting the following relations from (7.3.15)–(7.3.16) and (7.3.18),

$$\frac{d\bar{u}(t)}{dt} = \frac{1}{|\Omega|}\int_\Omega f(x,t)dx = 0, \ \ \bar{u}(0) = \frac{1}{|\Omega|}\int_\Omega \phi(x)dx = 0,$$

we can derive

$$\bar{u}(t) \equiv 0. \tag{7.3.21}$$

Now multiplying (7.3.15) by u in $L^2(\Omega)$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx = -d\int_\Omega |\nabla u|^2 dx + \int_\Omega uf dx. \tag{7.3.22}$$

Thus by the Hölder inequality, we have, for any $\varepsilon > 0$,

$$\int_\Omega uf dx \leq C(\varepsilon)\left(\int_\Omega |f|^p dx\right)^{2/p} + \varepsilon\left(\int_\Omega |u|^q dx\right)^{2/q} \tag{7.3.23}$$

with $q = p/(p-1)$. Since $p > n/2, p \geq 1$, when $n > 2$, by the embedding theorem, we know that $p > 2n/(n+2)$ and $H^1(\Omega) \hookrightarrow L^q(\Omega)$; when $n = 2, p > 1$, we also see, by the embedding theorem, that $H^1(\Omega) \hookrightarrow L^q(\Omega)$; when $n = 1$, we still have $H^1(\Omega) \hookrightarrow C(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$. In one word, we always have

$$H^1(\Omega) \hookrightarrow L^q(\Omega)$$

which gives us

$$\left(\int_\Omega |u|^q dx\right)^{2/q} \leq C\left(\int_\Omega |\nabla u|^2 dx\right). \tag{7.3.24}$$

Inserting (7.3.23)–(7.3.24) into (7.3.22) implies

$$\frac{d}{dt}\int_\Omega u^2dx \le -2d\int_\Omega |\nabla u|^2dx + C\varepsilon\int_\Omega |\nabla u|^2dx + C(\varepsilon)\left(\int_\Omega |f|^pdx\right)^{2/p}. \quad (7.3.25)$$

Picking $\varepsilon > 0$ small enough in (7.3.25) and using the Poincaré inequality, we conclude

$$\frac{d}{dt}\int_\Omega u^2dx \le -\alpha\int_\Omega |\nabla u|^2dx + C(\varepsilon)\|f\|^2_{L^p(\Omega)}. \quad (7.3.26)$$

By assumption (7.3.19) and Theorem 2.2.7, we can derive

$$\lim_{t\to+\infty}\int_\Omega u^2dx = 0. \quad (7.3.27)$$

Thus when $n \le 3$, we know that $p' = 2 > n/2$ and hence (7.3.20) follows from (7.3.27); when $n > 3$, multiplying (7.3.15) by $|u|^{m-1}u$ $(m > 1)$ and integrating the result over Ω, we deduce

$$\begin{aligned}\frac{1}{m+1}\frac{d}{dt}\int_\Omega |u|^{m+1}dx &= -\frac{4md}{(1+m)^2}\int_\Omega \left|\nabla |u|^{(m+1)/2}\right|^2dx + \int_\Omega fu|u|^{m-1}dx \\ &\le -\frac{4md}{m+1}\int_\Omega \left|\nabla |u|^{(m+1)/2}\right|^2dx + \int_\Omega |f||u|^mdx. \quad (7.3.28)\end{aligned}$$

Let $w = |u|^{(m+1)/2}$, then we derive from (7.3.28) that for all $\varepsilon > 0$,

$$\begin{cases}\dfrac{1}{m+1}\dfrac{d}{dt}\displaystyle\int_\Omega w^2dx \le -\dfrac{4md}{(m+1)^2}\int_\Omega |\nabla w|^2dx + \int_\Omega |f|w^{2m/(m+1)}dx, & (7.3.29)\\ \displaystyle\lim_{t\to+\infty}\int_\Omega w^{4/(m+1)}dx = \lim_{t\to+\infty}\int_\Omega u^2dx = 0, & (7.3.30)\\ \displaystyle\int_\Omega |f|w^{2m/(m+1)}dx \le C(\varepsilon)\left(\int_\Omega |f|^pdx\right)^{2/p} + \varepsilon\left(\int_\Omega w^{2mq/(m+1)}dx\right)^{2/q}. & (7.3.31)\end{cases}$$

Set $q_0 = 2mq/(m+1)$. Then we have

$$\left(\int_\Omega w^{2mq/(m+1)}dx\right)^{2/q} = \|w\|^{4m/(m+1)}_{L^{q_0}(\Omega)}. \quad (7.3.32)$$

By the Hölder inequality, we have

$$\|w\|_{L^{q_0}(\Omega)} \le \|w\|^\theta_{L^l(\Omega)}\|w\|^{1-\theta}_{L^r(\Omega)}, \ \ 1/q_0 = \theta/l + (1-\theta)/r, \ \ \theta \in (0,1). \quad (7.3.33)$$

Hence from (7.3.33) it follows that for all $\varepsilon > 0$,

$$\begin{aligned}\|w\|^{4m/(m+1)}_{L^{q_0}(\Omega)} &\le C\|w\|^{4m\theta/(m+1)}_{L^l(\Omega)}\|w\|^{4m(1-\theta)/(m+1)}_{L^r(\Omega)} \\ &\le C(\varepsilon)\|w\|^{8m\theta/(m+1)}_{L^l(\Omega)} + \varepsilon\|w\|^{8m(1-\theta)/(m+1)}_{L^r(\Omega)}.\end{aligned} \quad (7.3.34)$$

In (7.3.34), choosing $l = 4/(m+1), r = 2n/(n-2)$ and θ such that $4m(1-\theta) = m+1$, we derive

$$\frac{1+m}{2mq} = \frac{(1+m)(p-1)}{2mp} = \frac{(1+m)(3m-1)}{16m)} + \frac{(1+m)(n-2)}{8mn}$$

which gives us

$$m = \frac{1}{3}\left[1 + 8\left((p-1)/p - (n-2)/(4n)\right)\right]. \tag{7.3.35}$$

Since $p > n/2$, we easily verify from (7.3.35) that $m > 1$. Let $a = (p-1)/p - (n-2)/(4n)$. Then

$$m = \frac{1}{3}(1+8a), \quad 1+m = \frac{1}{3}(4+8a). \tag{7.3.36}$$

Since $H^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega) = L^r(\Omega)$, we get

$$\begin{aligned}\|w\|^2_{L^r(\Omega)} &\leq C\int_\Omega |\nabla w|^2 dx + C\int_\Omega w^2 dx \\ &\leq C\int_\Omega |\nabla w|^2 dx + 2C\int_\Omega |w-\overline{w}|^2 dx + 2C\int_\Omega \overline{w}^2 dx \\ &\leq C\int_\Omega |\nabla w|^2 dx + C\int_\Omega \overline{w}^2 dx.\end{aligned} \tag{7.3.37}$$

Inserting (7.3.31)–(7.3.34) and (7.3.37) into (7.3.29), we can conclude

$$\begin{aligned}\frac{1}{m+1}\frac{d}{dt}\int_\Omega w^2 dx \leq &-\frac{4md}{(m+1)^2}\int_\Omega |\nabla w|^2 dx + C(\varepsilon)\|f\|^2_{L^p(\Omega)} \\ &+ C(\varepsilon)\|w\|^{8m\theta/(1+m)}_{L^l(\Omega)} + \varepsilon\left(\int_\Omega |\nabla w|^2 dx + \int_\Omega \overline{w}^2 dx\right).\end{aligned} \tag{7.3.38}$$

Choosing $\varepsilon > 0$ small enough in (7.3.38) and applying the Poincaré inequality,

$$\int_\Omega |\nabla w|^2 dx \geq C\int_\Omega w^2 dx - C\int_\Omega \overline{w}^2 dx,$$

we obtain from (7.3.38) that for some constant $\alpha > 0$,

$$\frac{d}{dt}\int_\Omega w^2 dx \leq -\alpha\int_\Omega w^2 dx + h(t), \tag{7.3.39}$$

where

$$h(t) = C\left(\int_\Omega \overline{w}^2 dx + \|f\|^2_{L^p(\Omega)} + \|w\|^{8m\theta/(m+1)}_{L^l(\Omega)}\right).$$

Since

$$\begin{aligned}\overline{w} &= \frac{1}{|\Omega|}\int_\Omega w dx = \frac{1}{|\Omega|}\int_\Omega |u|^{(m+1)/2} dx \\ &= \frac{1}{|\Omega|}\int_\Omega |u|^{(2+4a)/3} dx \leq C\left(\int_\Omega u^2 dx\right)^{(1+2a)/3}\end{aligned}$$

and using (7.3.30) and assumption (7.3.19), we can get

$$\lim_{t\to+\infty} h(t) = 0. \tag{7.3.40}$$

By Theorem 2.2.7 and (7.3.39)–(7.3.40), we have $\int_\Omega w^2 dx = 0$, i.e.,

$$\lim_{t\to+\infty} \int_\Omega |u|^{m+1} dx = 0. \tag{7.3.41}$$

Multiplying (7.3.15) by $|u|^{m_1-1}u$ and integrating the result over Ω, we can get the corresponding (7.3.28) where m should be replaced by m_1. Let $w = |u|^{(m_1+1)/2}$, the corresponding (7.3.29)–(7.3.34) hold where m should be replaced by m_1. Choosing $l = 2(1+m_1)/(1+m_1), q_0 = 2qm_1/(1+m_1), \theta \in (0,1)$ such that $4m_1(1-\theta) = 1+m_1, r = 2n/(n-2)$ in (7.3.33) to get

$$m_1 = \frac{1}{3}\left(1 + 16a/3 + 32a^2/3\right).$$

Hence

$$1 + m_1 = 4/3 + (4/3)^2 a + 2(4/3)^2 a^2.$$

Similarly to (7.3.41), we can get

$$\lim_{t\to+\infty} \int_\Omega |u|^{m_1+1} dx = 0. \tag{7.3.42}$$

Repeating the same process as the proof of (7.3.41), we can derive

$$\lim_{t\to+\infty} \int_\Omega |u|^{m_k+1} dx = 0 \tag{7.3.43}$$

where m_k verifies

$$\begin{aligned} 1 + m_k &= 4/3 + (4/3)^2 a + (4/3)^3 a^2 + \cdots + (4/3)^{k+1} a^k + 2(4/3)^{k+1} a^{k+1} \\ &> 4/3 + (4/3)^2 a + \cdots + (4/3)^{k+1} a^k \equiv g(k,a). \end{aligned}$$

Obviously, as $4a/3 \geq 1$, $g(k,a)$ is divergent to $+\infty$; while as $4a/3 < 1$, $g(k,a)$ converges to

$$\frac{4}{3}\left(\frac{1}{1-4a/3}\right) = \frac{4}{3-4a} = \frac{4}{3-4(1-1/p-1/4+1/(2n))}$$

and $\frac{4}{3-4(1-1/p-1/4+1/(2n))} > n/2$ is equivalent to

$$8 > 3n - 4n\left(1 - 1/p - 1/4 + 1/(2n)\right) = 4n\left(1/p - 1/(2n)\right)$$

which is further equivalent to $n/p < 2 + 1/2$. Since $p > n/2$, we have $n/p < 2+1/2$. Hence when k is large sufficiently, $1 + m_k > n/2$. The proof is thus complete. □

Chapter 8

Asymptotic Behavior of Solutions to Hyperbolic Equations

This chapter mainly studies the asymptotic behavior of solutions to some hyperbolic equations. This chapter includes seven sections. In Section 8.1, we shall use Theorem 2.3.11 to study the decay of solutions to 1D nonlinear wave equations. In Section 8.2, we shall exploit Theorem 2.3.14 to investigate the decay property of the solutions to the initial boundary value problem for a wave equation with a dissipative term. In Section 8.3, we shall apply Theorem 2.3.6 to establish the polynomial decay rate for nonlinear wave equations. In Section 8.4, we shall employ Theorems 1.5.13–1.5.14 and Corollary 1.5.2 to establish the decay rate estimates for the wave equation damped with a boundary nonlinear velocity feedback $\rho(u_t)$. In Section 8.5, we shall apply Theorem 2.3.14 to study the large-time behavior of energy for a N-dimensional dissipative anisotropic elastic system. In Section 8.6, we shall use Theorem 1.5.12 to study the stabilization of weakly coupled evolution equations. In Section 8.7, we shall use Theorem 1.5.9 to study the energy decay rates of nonlinear hyperbolic systems by a nonlinear feedback which can be localized on a part of the boundary or locally distributed. Inequalities used in this chapter are specially chosen and very important in proving the decay rates of global solutions to some hyperbolic equations.

8.1 Estimates on approximated solutions for 1D nonlinear wave equations

In this section, we shall use Theorem 2.3.11 to study the decay of solutions to 1D nonlinear wave equations. We choose these results from Nakao [665].

We shall consider the following nonlinear wave equations:

$$u_{tt} - u_{xx} + \alpha(x)u_t + \rho(x, u_t) + \beta(x, u) = f(x, t) \quad \text{on} \ [0, \pi] \times \overline{\mathbb{R}^+} \tag{8.1.1}$$

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with initial boundary conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad u(0,t) = u(\pi,t) = 0, \tag{8.1.2}$$

where $\overline{\mathbb{R}^+} = [0, +\infty)$.

Throughout this section, we shall make the following assumptions:

(A1) There exist some constants $\alpha_0,\ \alpha_1 > 0$ such that $\alpha(x) \in C^2[0,\pi]$ and $0 < \alpha_0 < \alpha(x) \leq \alpha_1$, for all $x \in [0,\pi]$.

(A2) $\rho(x,s) \in C^3([0,\pi] \times \mathbb{R})$ and satisfies

$$\begin{cases} k_0 \sum_{i=1}^{2} |s|^{r_i+2} \leq \rho(x,s)s \leq k_1 \sum_{i=1}^{2} (1+|s|^{r_i})s^2, & (8.1.3) \\ k_0 \sum_{i=1}^{2} |s|^{r_i} \leq \dfrac{\partial \rho(x,s)}{\partial s} \leq k_1 \sum_{i=1}^{2} (1+|s|^{r_i}) & (8.1.4) \end{cases}$$

for some non-negative constants r_i $(i = 1, 2)$.

(A3) $\beta(x,u) \in C^3([0,\pi] \times \mathbb{R})$ and satisfies $\beta(x,u)u \geq 0$ and $\int_0^u \beta(x,s)ds \leq k_2 \beta(x,u)u$ for some constant $k_2 > 0$.

Assume initial data $(u_0, u_1) \in V_0 \times V_0$. Finally, we assume

(A4) $f \in C^3(\overline{\mathbb{R}^+}; L^2[0,\pi]) \cap \bigcap_{k=1}^{2} C^{3-k}(\overline{\mathbb{R}^+}; H^k[0,\pi] \cap \dot{H}^{k-1}[0,\pi]) (\subset C^2([0,\pi] \times \overline{\mathbb{R}^+}))$ and

$$\delta_i(t) := \left(\int_t^{t+1} \| D_t^i f(s) \|_{L^2}^2 \, ds \right)^{1/2} \leq \nu_i e^{-\lambda_i t} \tag{8.1.5}$$

where $\nu_i,\ \lambda_i$ are positive constants, $i = 0, 1, 2, 3$.

In what follows, we denote by D_t and D_x the partial derivatives $\partial/\partial t$ and $\partial/\partial x$, respectively. We refer to Lions [546] for standard function spaces and norms used here.

Lemma 8.1.1 ([546]). *The Sobolev space $H^1 \equiv H^1([0,\pi])$ is embedded continuously into $C^{1/2}([0,\pi])$, in particular, into $L^p \equiv L^p([0,\pi])$ $0 < p \leq +\infty$, we have for all $u \in H^1$,*

$$\| u \|_{L^p} + \| u \|_{C^{1/2}} \leq C \| u \|_{H^1},$$

or for all $u \in H^1$,

$$\| u \|_{L^p} + \| u \|_{C^{1/2}} \leq C' \| u \|_{\dot{H}^1},$$

where C, C' are positive constants, we define $\| u \|_{\dot{H}^1} = \| \nabla u \|_{L^2}$, which is equivalent to $\| u \|_{H^1} = (\| u \|_{L^2}^2 + \| \nabla u \|_{L^2})^{1/2}$ for $u \in \dot{H}^1$, $\dot{H}^1$ is the homogeneous Sobolev space.

We now employ Galerkin's method to construct the solutions. Let $\{\phi(x)\}_{k=1}^{\infty}$ be the eigenfunctions of the operator $-(d/dx)^2$ in L^2 with the zero boundary condition, that is, $\phi_k(x) = \sin kx$. We introduce the subspace V_0 of L^2 as follows:

$$V_0 = \text{closed linear extension of the eigenfunctions } \{\phi_k\} \textit{ in } H^4\}.$$

We assume the following expansion as an mth approximate solution

$$u_m = u_m(x,t) = \sum_{k=1}^{m} \lambda_k^m(t)\phi_k(x) \tag{8.1.6}$$

where the functions $\lambda_k^m(t)$ $(k = 1, 2, \ldots, m)$ are differentiable real-valued function for $t \in \overline{\mathbb{R}^+}$ which are determined by the system of ordinary differential equations

$$\begin{aligned}(D_t^2 u_m, \phi_k) &+ (D_x u_m, D_x \phi_k) + (\alpha(x) D_t u_m, \phi_k)\\ &+ (\rho(\cdot, D_t u_m) + \beta(\cdot, u_m), \phi_k) = (f, \phi_k), \quad k = 1, 2, \ldots, m.\end{aligned} \tag{8.1.7}$$

We assume the initial data $(u_0, u_1) \in V_0 \times V_0$ and the initial values $\lambda_k^m(0)$ and $D_t\lambda_k^m(0)$ for the system (8.1.7) are chosen in such a way that as $m \to +\infty$, we have

$$\begin{cases} u_{m_0} = u_m(x,0) = \sum_{k=1}^m \lambda_k^m(0)\phi_k(x) \\ \quad \to u_0(x) = u_0 \quad \text{strongly in } H^4, \\ (D_t u_m)_0 = D_t u_m(x,0) = \sum_{k=1}^m D_t\lambda_k^m(0)\phi_k(x) \\ \quad \to (D_t u)_0(x) \equiv u_1(x) \quad \text{strongly in } H^4. \end{cases} \tag{8.1.8}$$

The global existence of functions λ_k^m $(k = 0, 1, 2, \ldots, m)$ follows from the standard energy estimate of $u_m(t)$ and the theory of ordinary differential equations. We denote by C_i $(i = 0, 1, 2, \ldots)$ various constants which may depend on other known constants.

Now we can use Theorem 2.3.11 to prove the following theorem.

Theorem 8.1.2 ([665]). *Under assumptions* (A1)–(A4), *we have, for all* $t \geq 0$,

$$E(u_m(t)) \leq K_0(t), \tag{8.1.9}$$

and

$$\sum_{i=1}^{2} \int_t^{t+1} \| D_t u_m(s) \|_{L^{r_i+2}}^{r_i+2} ds \leq C_0 \left(K_0^2(t) + \delta_0^2(t)\right) \tag{8.1.10}$$

where

$$E(u(t)) = \left(\| D_t u(t) \|_{L^2}^2 + \| u(t) \|_{\dot{H}_1}^2 + 2\int_0^{\pi} \int_0^{u(t)} \beta(x,s) ds dx \right)^{1/2}$$

and $K_0(t) > 0$ *is a constant such that* $K_0(t) \leq \text{const.} e^{-\lambda_0' t}$ *for some constant* $\lambda_0' > 0$.

Proof. In fact, the proof is essentially included in Nakao [679] and we only brief it. From (8.1.6), it follows

$$E^2(u_m(t+1)) - E^2(u_m(t)) + 2\int_t^{t+1} \Big\{(\rho(x, D_t u_m), D_t u_m) + (\alpha D_t u_m, D_t u_m)\Big\} ds$$
$$= 2\int_t^{t+1} (f(s), D_t u_m(s)) ds \tag{8.1.11}$$

which, together with the assumptions (A1)–(A2), gives us

$$\alpha_0 \int_t^{t+1} \| D_t u_m(s) \|_{L^2}^2 \, ds + 2k_0 \sum_{i=0}^{2} \int_t^{t+1} \| D_t u_m(s) \|_{L^{r_i+2}}^{r_i+2} \, ds$$
$$\leq \frac{1}{\alpha_0}\delta_0^2(t) + E^2(u_m(t)) - E^2(u_m(t+1)) \quad (\equiv A^2(t)). \tag{8.1.12}$$

Hence there exist two points $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\alpha_0 \| D_t u_m(t_i) \|_{L^2}^2 \leq 4A^2(t), \quad i = 1, 2. \tag{8.1.13}$$

Multiplying (8.1.7) by $\lambda_k^m(t)$, summing over k from 1 to m and integrating over $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} \Big\{ \| u_m(s) \|_{\dot{H}^1}^2 + (\beta(\cdot, u_m(s)), u_m(s))\Big\} ds$$
$$\leq \sum_{i=1}^{2} | (D_t u_m(t_i), u_m(t_i)) | + \int_{t_1}^{t_2} \| D_t u_m(s) \|_{L^2}^2 \, ds$$
$$+ \int_{t_1}^{t_2} \Big\{\alpha_1 | (D_t u_m(s), u_m(s)) | + | (\rho(\cdot, D_t u_m(s)), u_m(s)) |$$
$$+ | (f(s), u_m(s)) | \Big\} ds \tag{8.1.14}$$
$$\leq C_1 \left(A(t) + \delta_0(t) + \sum_{i=1}^{2} A(t)^{2(r_i+1)/(r_i+2)} \right) \max_{s\in[t,t+1]} \| u_m(s) \|_{\dot{H}^1} + \alpha_0^{-1} A^2(t)$$

where we have used the assumption (A2) and the inequality (8.1.4). From (8.1.12) and (8.1.14), it follows that there exists a point $t^* \in [t_1, t_2]$ such that

$$\| u_m(t^*) \|_{\dot{H}^1}^2 + \Big(\beta(\cdot, u_m(t^*)), u_m(t^*)\Big) + \alpha_0 \| D_t u_m(t^*) \|_{L^2}^2$$
$$\leq 2C_1 \left(A(t) + \delta_0(t) + \sum_{i=1}^{2} A(t)^{2(r_i+1)/(r_i+2)} \right) \max_{s\in[t,t+1]} \| u_m(s) \|_{\dot{H}^1}$$
$$+ 2(1 + \alpha_0^{-1}) A^2(t)$$

which, together with (A3), implies

$$E(u_m(t^*))^2 \tag{8.1.15}$$
$$\leq C_2 \left\{ \left(A(t) + \delta_0(t) + \sum_{i=1}^{2} A(t)^{2(r_i+1)/(r_i+2)} \right) \max_{s\in[t,t+1]} \| u_m(s) \|_{\dot{H}^1} + A^2(t) \right\}.$$

Therefore, using a similar equality as (8.1.11), we get

$$\begin{aligned}
&\max_{s\in[t,t+1]} E^2(u_m(s)) \\
&\quad \leq E^2(u_m(t^*)) + \int_t^{t+1} 2\Big\{ (\rho(x, D_t u_m(s)), D_t u_m(s)) \\
&\qquad + |\ (\alpha D_t u_m, D_t u_m)\ | + |\ (f, D_s u_m)\ | \Big\} ds \\
&\quad \leq C_3 \Bigg\{ \left(A(t) + \delta_0(t) + \sum_{i=1}^{2} A(t)^{2(r_i+1)/(r_i+2)} \right) \max_{s\in[t,t+1]} E(u_m(s)) \\
&\qquad + A^2(t) + \delta_0^2(t) \Bigg\}
\end{aligned}$$

which yields, by the Young inequality,

$$\max_{s\in[t,t+1]} E^2(u_m(s)) \leq C_4 \left(A^2(t) + \delta_0^2(t) + \sum_{i=1}^{2} A(t)^{4(r_i+1)/(r_i+2)} \right). \tag{8.1.16}$$

If we assume $E(u_m(t)) \leq E(u_m(t+1))$ for some $t > 0$, then $A^2(t) \leq (1/\alpha_0)\delta_0^2(t)$ and we derive from (8.1.16)

$$\max_{s\in[t,t+1]} E^2(u_m(s)) \leq C_5 \left(\delta_0^2(t) + \sum_{i=1}^{2} \delta_0(t)^{4(r_i+1)/(r_i+2)} \right)$$

which implies for all $t \geq 0$,

$$E^2(u_m(t)) \leq \max \left(\max_{s\in[0,1]} E^2(u_m(s)), \max_t C_5 \left(\delta_0^2(t) + \sum_{i=1}^{2} \delta_0(t)^{4(r_i+1)/(r_i+2)} \right) \right).$$

Thus we know

$$\max_{s\in[0,1]} E^2(u_m(s)) \leq E^2(u_m(0)) + \frac{1}{\alpha_0} \delta_0^2(0) \leq C_6(\| u_0 \|_{\dot{H}^1}, \| u_1 \|_{L^2}, \delta_0(0))$$

where C_6 denotes a constant depending on $\| u_0 \|_{\dot{H}^1}$, $\| u_1 \|_{L^2}$ and $\delta_0(0)$. Thus we obtain for all $t \geq 0$,

$$E^2(u_m(t)) \leq C_6 + C_5 \max_t \left(\delta_0^2(t) + \sum_{i=1}^{2} \delta_0(t)^{4(r_i+1)/(r_i+2)} \right) \tag{8.1.17}$$

which implies the boundedness of $E^2(u_m(t))$ on $\overline{\mathbb{R}^+}$. Since $A(t)$ is bounded by (8.1.17), we deduce from (8.1.6)

$$\begin{aligned}\max_{s\in[t,t+1]} E^2(u_m(s)) &\le C_7\Big(A(t)^2+\delta_0^2(t)\Big)\\ &\le C_8\Big(E^2(u_m(t))-E^2(u_m(t+1))+\delta_0^2(t)\Big)\end{aligned}$$

which, together with Theorem 2.3.11, implies (8.1.10). The inequality (8.1.11) follows immediately from (8.1.10). □

8.2 Estimates on approximated solutions of wave equations

In this section, we shall exploit Theorem 2.3.14 to establish the decay estimates of the solutions to the initial boundary value problem for a dissipative wave equation. These results are adopted from Nakao [678].

Consider the following initial boundary value problem for the dissipative wave equation

$$\begin{cases} u_{tt}-\Delta u+\rho(x,u_t)=0, \quad \text{in} \quad \Omega\times[0,+\infty), & (8.2.1)\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \quad \text{in} \ \Omega & (8.2.2)\\ u(x,t)=0, \quad \text{on} \ \partial\Omega & (8.2.3)\end{cases}$$

where $\rho(x,v)$ is a function like $\rho(x,v)=a(x)|v|^r v,\ -1<r<+\infty$.

Here we shall treat the so-called strong solutions rather than energy finite solutions. To this end, we assume that $\partial\Omega$ is smooth (say, C^2 class), and restrict the growth order of $\rho(x,v)$ as $|v|\to+\infty$. Note that when $\rho(x,v)=a(x)|v|^r v$, Nakao [678] proved the following result: if $0<r\le 2/(N-2)^+$, then

$$E(t)\le C_1(1+t)^{-2\eta},$$

with $\eta=\min\{1/r,\ 2(r+1)/r(N-2)^+\}$ and if $-1<r<0$, then

$$E(t)\le C_1(1+t)^{-2\tilde\eta},$$

with $\tilde\eta=\min\{-(r+1)/r,\ -2/r(N-2)^+\}$, where $C_1>0$ is a constant depending on $||u_0||_{H^2}+||\nabla u_1||$. We recall that $\eta=1/r$ if $1\le N\le 4$, and the decay rate $\tilde\eta$ in the latter case coincides with the result in [673], where the case $\rho(x,v)=|v|^r v,\ -1<r<0$ was treated.

We now make the following hypotheses on $\rho(x,v)$:

(H) $\rho(x,v)\in C(\overline{\Omega}\times\mathbb{R})$ is monotonically increasing in v and satisfies the following assumptions:

(i) $a(x)|v|^{r+2} \le \rho(x,v)v \le k_0 a(x)\Big\{|v|^{r+2} + |v|^2\Big\}$ if $|v| \le 1$ with some $-1 < r < +\infty$,

(ii) $a(x)|v|^{p+2} \le \rho(x,v)v \le k_1 a(x)\{|v|^{p+2} + |v|^2\}$ if $|v| \ge 1$ with some $-1 \le p \le 2/(N-2)$ ($-1 \le p < +\infty$ if $N = 1,2$), where k_0, k_1 are positive constants and $a(x)$ is a bounded function satisfying the condition (8.2.6), i.e.,

$$a(x) \ge \varepsilon_0 > 0 \quad on \ \ \omega,$$

where ω is a neighborhood in $\overline{\Omega}$ of $\Gamma(x_0)$ for some $x_0 \in \mathbb{R}^N$.

In fact, there are indeed some functions of $\rho(x,v)$ to satisfy the assumption (H). For example, when $\rho(x,v) = a(x)|v|^r v$, we can choose $\rho = r$ and the case $-1 < r \le 2/(N-2)^+$ is included; while when $\rho(x,v) = a(x)v/\sqrt{1+v^2}$, we can choose $r = 0$ and $\rho = -1$.

In the following, we shall use Theorem 2.3.14 to show the next result due to Nakao [678].

Theorem 8.2.1 ([678]). *Assume that $\partial\Omega$ is C^2 class, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$. Then under the assumption* (H), *the problem* (8.2.1)–(8.2.3) *admits a unique solution*

$$u(t) \in W^{2,\infty}([0,+\infty), L^2(\Omega)) \cap W^{1,\infty}([0,+\infty),\, H_0^1) \cap L^\infty([0,+\infty),\, H^2(\Omega))$$

satisfying the decay property,

$$E(t) \le C_1(1+t)^{-2\eta_i},\ i = 1,2,3,4, \tag{8.2.4}$$

where $C_1 > 0$ is a constant depending on $\|u_0\|_{H^2} + \|u_1\|_{H^1}$ and the decay rates η_i $(i = 1,2,3,4)$ are given as follows corresponding to the different cases.

(1) *If $r \ge 0$ and $0 \le p \le 2/(N-2)$ ($0 \le p < +\infty$ if $N = 1,2$), then we have*

$$\eta_1 = \min\left\{\frac{1}{r}, \frac{2(p+1)}{p(N-2)^+}\right\}.$$

(2) *If $r \ge 0$ and $-1 \le p < 0$, then we have*

$$\eta_2 = \min\left\{\frac{1}{r}, \frac{-2}{p(N-2)^+}\right\}.$$

(3) *If $-1 < r < 0$, $0 \le p \le 2/(N-2)$ ($0 \le p < +\infty$ if $N = 1,2$), then we have*

$$\eta_3 = \min\left\{\frac{-(r+1)}{r}, \frac{2(p+1)}{p(N-2)^+}\right\}.$$

(4) *If $-1 < r < 0$ and $-1 \le p < 0$, then we have*

$$\eta_4 = \min\left\{\frac{-(r+1)}{r}, \frac{-2}{p(N-2)^+}\right\}.$$

Remark 8.2.1 ([678])**.**

(i) If $N = 2$ and $r = 0$, then the rates η_1, η_2 can be chosen arbitrarily large.

(ii) If $N = 1$ and $r = 0$, then $E(t) \le C_1 e^{-\lambda t}$.

(iii) If $p = 0$, then the constant C_1 in (8.2.7) can be replaced by $C_0 = C(E(0))$ and the result is valid for the energy finite solutions. In particular, if $p = r = 0$, we have the usual exponential decay as in Zuazua [1028].

(iv) More precisely, the estimate (8.2.7) may be written as, for a fixed $T > 0$,

$$E(t) \le \left\{ E(0)^{-1/(2\eta_i)} + C_1^{-1}(t-T)^+ \right\}^{-2\eta_i}.$$

First, by the standard method (see Lions and Strauss [544], and [674]) we can prove the existence and uniqueness of global solutions. Next, we show the decay estimate (8.2.7). For the proof of Theorem 8.2.1, we need the following lemmas.

Lemma 8.2.2 (The Gagliardo–Nirenberg Interpolation Inequality)**.** *Let* $1 \le r < p \le +\infty$,$1 \le q \le p$ *and* $0 \le m$. *Then we have, for all* $v \in W^{m,p} \cap L^r$,

$$||v||_{W^{k,p}} \le C||v||^{\theta}_{W^{m,q}}||v||^{1-\theta}_r, \tag{8.2.5}$$

with some constant $C > 0$ *and*

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1}$$

provided that $0 < \theta \le 1$ $(0 < \theta < 1$ *if* $p = +\infty$ *and* $mq = N)$.

Proof of Theorem 8.2.1. Assume that $(u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ and $u(t)$ is the strong solution obtained in Theorem 8.2.1.

Multiplying the equation (8.2.1) by u_t and integrating it over $[t, t+T] \times \Omega$, $T > 0$, we have

$$\int_t^{t+T}\int_\Omega \rho(x, u_t)u_t dxds = E(t) - E(t+T) \equiv D^{r+2}(t). \tag{8.2.6}$$

Multiplying the equation (8.2.1) by u and integrating the result, we get

$$\begin{aligned}&\int_t^{t+T}\int_\Omega \{|u_t|^2 - |\nabla u|^2\}dxds \\ &\quad= -\int_t^{t+T}\int_\Omega \rho(x,u_t)udxds + (u_t(t+T), u(t+T)) - (u_t(t), u(t)).\end{aligned} \tag{8.2.7}$$

Multiplying the equation (8.2.1) by $(x-x_0)\cdot\nabla u$, we can obtain

$$\begin{aligned}&\frac{N}{2}\int_t^{t+T}\int_\Omega\{|u_t|^2-|\nabla u|^2\}dxds+\int_t^{t+T}\int_\Omega|\nabla u|^2dxds\\&\quad+\int_t^{t+T}\int_\Omega\rho(x,u_t)(x-x_0)\cdot\nabla udxds\\&=(u_t(t+T),(x-x_0)\cdot\nabla u(t+T))-(u_t(t),(x-x_0)\cdot\nabla u)\\&\quad+\frac{1}{2}\int_t^{t+T}\int_{\partial\Omega}(x-x_0)\cdot\nu\left|\frac{\partial u}{\partial\nu}\right|^2d\sigma ds.\end{aligned}\tag{8.2.8}$$

Now we take a function $\eta\in W^{1,\infty}(\Omega)$ such that

$$\eta=1\ \text{ on }\ \tilde{\omega}\ ,\ \eta\geq 0\ \text{ and }\eta=0\ \text{ on }\overline{\Omega}\backslash\omega$$

where $\tilde{\omega}$ is an open set in $\overline{\Omega}$ with $\Gamma(x_0)\subset\tilde{\omega}\subset\omega$.

Multiplying the equation (8.2.1) by ηu and integrating the resulting equation, we arrive at

$$\begin{aligned}&\int_t^{t+T}\int_\Omega\eta|\nabla u|^2dxds=(u_t(t)-\eta u(t))-(u_t(t+T),\eta u(t+T))\\&+\int_t^{t+T}\int_\Omega\nabla\eta\cdot u\nabla udxds+\int_t^{t+T}\int_\Omega\eta|u_t|^2dxds-\int_t^{t+T}\int_\Omega\rho(x,u_t)u\,dxds\end{aligned}\tag{8.2.9}$$

which is valid for any $\eta\in W^{1,\infty}(\Omega)$.

Finally, we take a vector field $h(x)$ such that

$$h=\nu\ \text{ on }\ \Gamma(\mathrm{x}_0),\ \mathrm{h}\cdot\nu\geq 0\ \text{ on }\ \partial\Omega\ \text{ and }\ \mathrm{h}=0\ \text{ on }\ \Omega\backslash\hat{\omega}$$

where $\hat{\omega}$ is an open set in $\mathbb{R}^N$ with the property

$$\Gamma(x_0)\subset\hat{\omega}\cap\overline{\Omega}\subset\omega\ \ .$$

Then multiplying the equation (8.2.1) by $h\cdot\nabla u$ and integrating the resulting equation, we can conclude

$$\begin{aligned}&\frac{1}{2}\int_t^{t+T}\int_{\partial\Omega}h\cdot\nu\left|\frac{\partial u}{\partial\nu}\right|^2dxd\sigma\\&=(u_t(t),h\cdot\nabla u(t))-(u_t(t+T),h\cdot\nabla u(t+T))\\&\quad-\frac{1}{2}\int_t^{t+T}\int_\Omega\nabla\cdot h|u_t|^2dxds+\sum_{i,j}\int_t^{t+T}\int_\Omega\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial h_i}{\partial x_j}dxds\\&\quad+\int_t^{t+T}\int_\Omega\rho(x,u_t)h\cdot\nabla udxds\end{aligned}\tag{8.2.10}$$

which is valid for any vector field $h(x)$.

Then from (8.2.7) and (8.2.8), it follows that

$$\begin{aligned}
&\int_t^{t+T} E(s)ds \\
&\leq C\Big\{ ||u_t(t+T)||||\nabla u(t+T)|| + ||u_t(t)||||\nabla u(t)|| \\
&\quad + \int_t^{t+T}\int_\Omega |\rho(x,u_t)|(|u|+|\nabla u|)dxds + \int_t^{t+T}\int_{\Gamma(x_0)} (x-x_0)\cdot\nu|\frac{\partial u}{\partial \nu}|^2 dxds\Big\} \\
&\leq C\Big\{ D^{r+2}(t) + E(t+T) \\
&\quad + \int_t^{t+T}\int_\Omega |\rho(x,u_t)|(|u|+|\nabla u|)dxds + \int_t^{t+T}\int_{\Gamma(x_0)} |\frac{\partial u}{\partial \nu}|^2 dxds\Big\}. \qquad (8.2.11)
\end{aligned}$$

To estimate the last term in (8.2.11), we shall use (8.2.7) and (8.2.8). First, we derive from (8.2.8)

$$\begin{aligned}
\int_t^{t+T}\int_\Omega \eta|\nabla u|^2 dxds &\leq C\Big\{ E(t+T) + E(t) + \int_t^{t+T}\int_\omega |u|^2 dxds \\
&\quad + \int_t^{t+T}\int_\omega |u_t|^2 dxds + C\int_t^{t+T}\int_\Omega |\rho(x,u_t)u|dxds\Big\} \qquad (8.2.12)
\end{aligned}$$

where we have used the fact that $|\nabla\eta|^2/\eta$ is bounded. Second, we infer from (8.2.10)

$$\begin{aligned}
&\int_t^{t+T}\int_{\Gamma(x_0)} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma ds \leq \int_t^{t+T}\int_{\partial\Omega} h\cdot\nu \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma ds \\
&\quad \leq C\Big\{ E(t+T) + E(t) + \int_t^{t+T}\int_\omega |\nabla u|^2 dxds \qquad (8.2.13) \\
&\quad\quad + \int_t^{t+T}\int_\omega |u_t|^2 dxds + \int_t^{t+T}\int_\omega |\rho(x,u_t)||\nabla u|dxds\Big\}.
\end{aligned}$$

It follows from (8.2.12) and (8.2.13) (note that $\partial\Omega$ is C^2 class) that

$$\begin{aligned}
&\int_t^{t+T}\int_{\Gamma(x_0)} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma ds \\
&\quad \leq C\Big\{ E(t+T) + E(t) + \int_t^{t+T}\int_\omega |u(s)|^2 dxds \qquad (8.2.14) \\
&\quad\quad + \int_t^{t+T}\int_\omega |u_t|^2 dxds\Big\} + \int_t^{t+T}\int_\omega |\rho(x,u_t)|(|u|+|\nabla u|)dxds.
\end{aligned}$$

From (8.2.11) and (8.2.14), we may conclude

$$
\begin{aligned}
TE(t+T) &\le \int_t^{t+T} E(s)ds \qquad (8.2.15)\\
&\le C\bigg\{ D^{r+2}(t) + E(t+T) + \int_t^{t+T}\int_\omega |u(s)|^2 dxds \\
&\quad + \int_t^{t+T}\int_\omega |u_t|^2 dxds + \int_t^{t+T}\int_\omega |\rho(x,u_t)|(|u|+|\nabla u|)dxds\bigg\}.
\end{aligned}
$$

Thus, taking a large $T>0$ and using (8.2.6), we can obtain

$$
\begin{aligned}
E(t) \le C\bigg\{ D^{r+2}(t) &+ \int_t^{t+T}\int_\omega |u(s)|^2 dxds + \int_t^{t+T}\int_\omega |u_t|^2 dxds \\
&+ \int_t^{t+T}\int_\Omega |\rho|(|u|+|\nabla u|)dxds\bigg\}. \qquad (8.2.16)
\end{aligned}
$$

Now we fix a large $T>0$ in the sequel. In order to estimate the last term in (8.2.14), we first note that by assumption on ρ, we have

$$
\begin{aligned}
&\int_t^{t+T}\int_\Omega |\rho(x,u_t)|(|u|+|\nabla u|)dxds \\
&\le C\bigg\{ \int_t^{t+T}\int_{\Omega_1} a(x)(|u_t|+|u_t|^{r+1})(|u|+|\nabla u|)dxds \qquad (8.2.17)\\
&\quad + \int_t^{t+T}\int_{\Omega_2} a(x)(|u_t|+|u_t|^{p+1})(|u|+|\nabla u|)dxds\bigg\} \equiv I_1 + I_2,
\end{aligned}
$$

where we set for each $t\ge 0$,

$$
\Omega_1 = \Omega_1(t) = \Big\{x\in\Omega \Big| |u_t(x,t)|\le 1\Big\} \text{ and } \Omega_2 = \Omega\backslash\Omega_1.
$$

Second, we note that $u(t)$ satisfies, at least formally,

$$
u_{ttt} - \Delta u_t + \frac{\partial}{\partial v}\rho(x,u_t)u_{tt} = 0. \qquad (8.2.18)
$$

Since $\frac{\partial}{\partial v}\rho(x,v)\ge 0$ by the monotonicity, multiplying (8.2.16) by u_{tt} and integrating the resulting equation, we get for all $t\ge 0$,

$$
\begin{aligned}
||u_{tt}(t)||^2 + ||\nabla u_t(t)||^2 &\le ||u_{tt}(0)||^2 + ||\nabla u(0)||^2 \\
&\le ||\Delta u_0 + \rho(x,u_1)||^2 + ||\nabla u_1||^2.
\end{aligned}
$$

Here, because $-1 \le p \le 2/(N-2)$ ($-1 \le p < +\infty$ if $N = 1,2$),

$$||\rho(x,u_1)||^2 \le C \int_\Omega (|u_1| + |u_1|^2 + |\nabla u_1|^{2(p+1)})dx$$
$$\le C\Big(\|u_1\| + \|u_1\|^2 + \|\nabla u_1\|^{2(p+1)}\Big).$$

Hence, we have

$$||u_{tt}(t)||^2 + ||\nabla u_t(t)||^2 \le C_1 < +\infty. \tag{8.2.19}$$

If $\rho(x,v)$ is not differentiable in the above, we only approximate it appropriately by smooth non-decreasing functions $\rho_\varepsilon(x,v)$, because the solution $u(t)$ is given as a limit function of the approximate solutions $u_\varepsilon(t)$, which satisfies (8.2.17) corresponding to $\rho_\varepsilon(x,u_t)$. By (8.2.17), we also know

$$\begin{aligned} ||\Delta u(t)||^2 &= ||u_{tt}(t) + \rho(x,u_t)||^2 \\ &\le C\Big\{||u_{tt}(t)||^2 + ||u_t(t)|| + ||u_t(t)||^2 + ||\nabla u_t(t)||^{2(p+1)}\Big\} \\ &\le C_1 < +\infty, \end{aligned} \tag{8.2.20}$$

where we have used again the assumption $p \le 2/(N-2)^+$. When ρ is independent of x or $a(x) \ge \varepsilon_0 > 0$ on Ω, we can derive (8.2.20) without the restriction on p by multiplying the equation by $-\Delta u_t$ and integrating, though in the latter case, we must make an additional assumption on $\rho(x,v)$ with respect to x dependence. Once the estimate (8.2.20) has been established, the argument below is valid for p such that $-1 \le p \le 4/(N-2)^+$. To estimate I_i ($i = 1,2$), in (8.2.17) and derive difference inequalities on $E(t)$, we need to consider the cases separately.

Case (1): $r \ge 0$ and $0 \le p \le 2/(N-2)$ ($0 \le p < +\infty$ if $N = 1,2$). By Poincaré's inequality, we may get,

$$\begin{aligned} I_1 &\le C\left(\int_t^{t+T}\int_{\Omega_1} a(x)|u_t|^2 dxds\right)^{1/2}\left(\int_t^{t+T} ||\nabla u(s)||^2 ds\right)^{1/2} \\ &\le C\left(\int_t^{t+T}\int_{\Omega_1} a(x)|u_t|^{r+2} dxds\right)^{1/(r+2)} \sqrt{E(t)} \\ &\le C\left(\int_t^{t+T}\int_{\Omega_1} \rho(x,u_t)u_t dxds\right)^{1/(r+2)} \sqrt{E(t)} \\ &\le CD(t)\sqrt{E(t)} \end{aligned} \tag{8.2.21}$$

and

$$I_2 \le C\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{p+1}(|u| + |\nabla u|)dxds \tag{8.2.22}$$

$$\le C\left(\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{p+2} dxds\right)^{(p+1)/(p+2)} \Big(\int_t^{t+T}\int_\Omega |\nabla u|^{p+2} dxds\Big)^{1/(p+2)}$$

where we have noted that by (8.2.20) and Lemma 8.2.1,

$$||\nabla u||_{p+2} \le C||\nabla u||^{1-\theta}||\Delta u||^{\theta} \le C_1 E^{(1-\theta)/2}(t)$$

with $\theta = pN/(2(p+2))$ (≤ 1).

Hence

$$I_2 \le C_1(D(t))^{(r+2)(p+1)/(p+2)}(E(t))^{(4+2p-Np)/(4(p+2))}. \tag{8.2.23}$$

It follows from (8.2.16), (8.2.21) and (8.2.23) that

$$E(t) \le C\left(A_1^2(t) + \int_t^{t+T}\int_\omega (|u|^2 + |u_t|^2)dxds\right) \tag{8.2.24}$$

where

$$A_1^2(t) = C_1\Big(D^{r+2}(t) + D^2(t) + (D(t))^{4(r+2)(p+1)/(4+2p+Np)}\Big). \tag{8.2.25}$$

Case (2): $0 \le r$ and $-1 \le p < 0$. Instead of (8.2.22), we have,

$$\begin{aligned} I_2 &\le C\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|(|u| + |\nabla u)|)dxds \\ &\le C\left(\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^2dxds\right)^{1/2}\left(\int_t^{t+T} ||\nabla u(s)||^2ds\right)^{1/2} \\ &\le C\left(\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{p+2}dxds\right)^{\alpha/(2(p+2))} \\ &\quad \times \left(\int_t^{t+T}\int_{\Omega} |u_t|^{2N/(N-2)^+}dxds\right)^{(2-\alpha)(N-2)^+/4N} \sqrt{E(t)} \\ &\le C_1\left(\int_t^{t+T}\int_{\Omega} \rho(x,u_t)u_tdxds\right)^{\alpha/(2(p+2))} \sqrt{E(t)} \\ &\le C_1(D(t))^{\alpha(r+2)/(2(p+2))}\sqrt{E(t)} \end{aligned} \tag{8.2.26}$$

with

$$\alpha = 4(p+2)/(4 - pN + 2p), \;\; \alpha = p + 2 - \varepsilon, \; 0 < \varepsilon \ll 1, \;\; \text{if } N = 2.$$

Hence, it follows that

$$E(t) \le C\left(A_2^2(t) + \int_t^{t+T}\int_\omega (|u|^2 + |u_t|^2)dxds\right), \tag{8.2.27}$$

where

$$A_2^2(t) = C_1 \left(D^{r+2}(t) + D^2(t) + (D(t))^{4(r+2)/(4-pN+2p)} \right). \tag{8.2.28}$$

Here we have noted that the last term in (8.2.28) should be replaced by

$$(D(t))^{r+2+\varepsilon},\ 0 < \varepsilon \ll 1,$$

when $N = 2$.

Case (3): $-1 < r < 0$ and $0 \le p \le 2/(N-2)$ ($0 \le p < +\infty$ if $N = 1, 2$). Then, instead of (8.2.21), we have,

$$\begin{aligned} I_1 &\le C \int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^{r+1}(|u| + |\nabla u|)dxds \\ &\le C \left(\int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^{r+2}dxds \right)^{(r+1)/(r+2)} \left(\int_t^{t+T} \int_{\Omega} |\nabla u|^{r+2}dxds \right)^{1/(r+2)} \\ &\le C(D(t))^{r+1}\sqrt{E(t)}. \end{aligned} \tag{8.2.29}$$

Hence, we obtain (8.2.24) with $A_1^2(t)$ replaced by

$$A_3^2(t) = C_1 \Big\{ D^{r+2}(t) + (D(t))^{2(r+1)} + (D(t))^{4(r+2)(p+1)/(4+pN+2p)} \Big\}. \tag{8.2.30}$$

Case (4): $-1 < r < 0$ and $-1 \le p < 0$. We obtain, by the above arguments, the inequality (8.2.24) with $A_1^2(t)$ replaced by

$$A_4^2(t) = C_1 \Big\{ D^{r+2}(t) + (D(t))^{2(r+1)} + (D(t))^{4(r+2)/(4-pN+2p)} \Big\}. \tag{8.2.31}$$

When $N = 2$, a modification is needed in the last term. Thus we obtain

$$E^2(t) \le C \left\{ A_i^2(t) + \int_t^{t+T} \int_{\omega} (|u|^2 + |u_t|^2)dxds \right\} \tag{8.2.32}$$

for $i = 1, 2, 3, 4$, which corresponds to the cases (1), (2), (3) and (4), respectively.

To achieve the desired difference inequality on $E(t)$, we need further to estimate the last term in (8.2.30). For the second term on the right-hand side in (8.2.32), we can prove the following claim:

There exists a constant $C > 0$ such that

$$\int_t^{t+T} ||u(s)||^2 ds \le C \left\{ A_i^2(t) + \int_t^{t+T} \int_{\omega} |u_t|^2 dxds \right\}. \tag{8.2.33}$$

In fact, we prove (8.2.33) by contradiction. If (8.2.33) were false, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ and a sequence of solutions $\{u_n\}_{n=1}^{\infty}$ such that

$$\int_{t_n}^{t_n+T} ||u_n(s)||^2 ds \le n \left\{ A_i^2(t_n) + \int_{t_n}^{t_n+T} \int_{\omega} |u_{nt}|^2 dx ds \right\} \tag{8.2.34}$$

for $n = 1, 2, 3, \ldots$, where $A_i(t)$ should be defined with $u(t)$ replaced by $u_n(t)$. In the sequel, for simplicity, we write $u(t)$ for $u_n(t)$, never changing the feature of the proof.

Setting

$$\lambda_n^2 = \int_{t_n}^{t_n+T} ||u(s)||^2 ds$$

and

$$v_n(t) = u(t + t_n)/\lambda_n, \ 0 < t \le T,$$

we have from (8.2.34)

$$Q_n^2 \equiv \frac{1}{\lambda_n^2}\Big\{A_i^2(t_n) + \int_{t_n}^{t_n+T} \int_{\omega} |u_t(s)|^2 dx ds\Big\} \to 0 \ \text{ as } \ \text{n} \to +\infty \tag{8.2.35}$$

and

$$\int_0^T ||v_n(s)||^2 ds = 1. \tag{8.2.36}$$

Thus, we deduce from the inequality (8.2.32) with $t = t_n$, for sufficiently large n, and for all $0 \le t \le T$,

$$\begin{aligned} \frac{1}{2}(||v_n(t)||^2 + ||\nabla v_n(t)||^2) &\le \frac{1}{2}(||v_n(0)||^2 + ||\nabla v_n(0)||^2) \\ &\le C\{{Q_n}^2 + 1\} \le 2C < +\infty. \end{aligned} \tag{8.2.37}$$

To take a limit of $\{v_n(t)\}$, we shall first check that

$$\lim_{n\to+\infty} \frac{1}{\lambda_n} \rho(x, u_t(t + t_n)) = 0 \ \ in \ \ L^1([0, T] \times \Omega).$$

Indeed, for the case (1), we know (cf. (8.2.21), (8.2.22) and (8.2.24))

$$\begin{aligned} \int_t^{t+T} \int_{\Omega} |\rho(x, u_t(s))| dx ds &\le C_1\Big\{D(t) + (D(t))^{(r+2)(p+1)/(p+2)}\Big\} \\ &\le C\Big(A_1(t) + A_1^{\beta}(t)\Big) \end{aligned} \tag{8.2.38}$$

with

$$\beta = 1 + \frac{Np}{2(p+2)} > 1.$$

Hence, as $n \to +\infty$,

$$\frac{1}{\lambda_n}\int_{t_n}^{t_n+T}\int_{\Omega} |\rho(x,u_t)|dxds \le C\Big\{Q_n + Q_n^{\beta}\lambda_n^{\beta-1}\Big\} \to 0. \tag{8.2.39}$$

For the case (2), we see (cf. (8.2.26))

$$\int_t^{t+T}\int_{\Omega} |\rho(x,u_t)|dxds \le C_1\Big\{D(t) + (D(t))^{2(r+2)/(4-pN+2p)}\Big\} \le C_1 A_2(t)$$

and as $n \to +\infty$,

$$\frac{1}{\lambda_n}\int_{t_n}^{t_n+T}\int_{\Omega} |\rho(x,u_t)|dxds \le CQ_n \to 0. \tag{8.2.40}$$

For the above cases (3) and (4), we can also prove easily the same conclusions as those in (8.2.39) and (8.2.40), respectively. Thus, we have proved that, as $n \to +\infty$

$$\frac{1}{\lambda_n}\rho(x, u_t(t+t_n)) \to 0 \text{ in } \mathrm{L}^1([0,\mathrm{T}]\times\Omega). \tag{8.2.41}$$

Therefore, along a subsequence, we have

$$v_n(t) \to v(t) \text{ weakly star in } W^{1,\infty}([0,T],\, L^2(\Omega)) \cap L^\infty([0,T],\, H_0^1(\Omega))$$
$$\text{and strongly in } L^2([0,T]\times\Omega)$$

and the limit function $v(t) \in W^{1,\infty}([0,T];L^2) \cap L^\infty([0,T],\, H_0^1)$ satisfies, by (8.2.35), (8.2.36) and (8.2.41),

$$v_{tt} - \Delta v = 0 \ \text{ on } \ [0,\mathrm{T}]\times\Omega, \tag{8.2.42}$$

$$\int_0^T\int_{\omega} |v_t|^2 dxds = 0 \tag{8.2.43}$$

and

$$\int_0^T ||v(t)||^2 dt = 1. \tag{8.2.44}$$

By a standard theory, we know that $v(t)$ belongs, in fact, to $C^1([0,T],L^2)\cap C([0,T];H_0^1)$ and the equation (8.2.42), together with the condition (8.2.43), implies

$$v(x,t) \equiv 0 \ \text{ on } \ [0,\mathrm{T}]\times\Omega.$$

Indeed, we can apply the inequality (8.2.32) to $u(t) \equiv v_t(t)$, i.e., $u_\varepsilon \equiv v_t * \eta_\varepsilon(t)$, $\eta_\varepsilon(t)$ being the mollifier in t, (note that $\rho \equiv 0$ in this situation). Then, (8.2.43) implies $E(0) = 0$ for $u(t) = v_t(t)$, that is, $v_t(0) = v_{tt}(0) = 0$ in Ω and hence, $v_t(x,t) = 0$ or $v(x,t) = v(x)$, independent of t, on Ω which, together with the boundary condition, implies the above result. For more general result on the unique continuation property of wave equations, we refer to [1028]. This contradicts to another condition (8.2.44). Thus (8.2.33) is valid.

Now by (8.2.33) and the inequality (8.2.32), we can conclude

$$E(t) \leq C\left(A_i^2(t) + \int_t^{t+T}\int_\omega |u_t|^2 dxds\right) \tag{8.2.45}$$

for each case $i = 1, 2, 3$ and 4.

Indeed, we shall estimate the last term in (8.2.45) in terms of four cases (1)–(4). For the case (1), we get

$$\begin{aligned}
&\int_t^{t+T}\int_\omega |u_t|^2 dxds \leq C\int_t^{t+T}\int_\Omega a(x)|u_t|^2 dxds \\
&\leq C\left(\int_t^{t+T}\int_{\Omega_1} a(x)|u_t|^{r+2}dxds\right)^{2/(r+2)} + C\int_t^{t+T}\int_{\Omega_2} a(x)|u_t|^{p+2}dxds \\
&\leq C\left\{\left(\int_t^{t+T}\int_\Omega \rho(x,u_t)u_t dxds\right)^{2/(r+2)} + \int_t^{t+T}\int_\Omega \rho(x,u_t)u_t dxds\right\} \\
&\leq C\left(D^2(t) + D^{r+2}(t)\right).
\end{aligned} \tag{8.2.46}$$

Thus it follows from (8.2.25), (8.2.32) and (8.2.46) that

$$E(t) \leq C_1\Big(D^2(t) + (D(t))^{4(r+2)(p+1)/(4+2p+Np)}\Big) \leq C_1(D(t))^{(r+2)\kappa_1}$$

with

$$\kappa_1 = \min\left\{\frac{2}{r+2}, \frac{4(p+1)}{4+2p+Np}\right\},$$

where we have used the fact that $E(t)$ is bounded. Recalling the definition of $D(t)$, we get

$$E^{1/\kappa_1}(t) \leq C_1\Big(E(t) - E(t+T)\Big). \tag{8.2.47}$$

Therefore, applying Theorem 2.3.14 to (8.2.47), we obtain

$$E(t) \leq C_1(1+t)^{-2\eta_1} \tag{8.2.48}$$

with

$$\eta_1 = \min\left\{\frac{1}{r}, \frac{2(p+1)}{(N-2)^+p}\right\}.$$

Obviously, if $p = r = 0$, we have the exponential decay of $E(t)$, which is, in fact, valid for energy finite solutions. When $r > 0$, the above decay rate is valid even for $N = 2$ since we can take $\varepsilon > 0$ small enough. But, if $r = 0$ and $N = 2$, we should replace η_1 by an arbitrarily large number.

For the case (2): $0 \le r$ and $-1 \le p < 0$, we have (cf. (8.2.26))

$$\int_t^{t+T} \int_{\Omega_2} a(x)|u_t|^2 dxds \le C_1 (D(t))^{4(r+2)/(4-Np+2p)}$$

with a modification for $N = 2$, and hence, instead of (8.2.46), we get

$$\int_t^{t+T} \int_{\omega} |u_t|^2 dxds \le C_1 \Big\{ D^2(t) + (D(t))^{4(r+2)/(4-Np+2p)} \Big\}. \tag{8.2.49}$$

Thus from (8.2.28), (8.2.45) and (8.2.49), we can derive

$$E(t) \le C_1 \Big\{ D^2(t) + (D(t))^{4(r+2)/(4-Np+2p)} \Big\} \le C_1 (D(t))^{(r+2)\kappa_2}$$

and

$$E^{1/\kappa_2}(t) \le C_1 \Big(E(t) - E(T+t) \Big) \tag{8.2.50}$$

with

$$\kappa_2 = \min \left\{ \frac{2}{r+2}, \frac{4}{4-Np+2p} \right\}.$$

Applying Theorem 2.3.14 to (8.2.50), we obtain

$$E(t) \le C_1 (1+t)^{-2\eta_2}$$

with

$$\eta_2 = \min \left\{ \frac{1}{r+1}, \frac{-2}{p(N-2)^+} \right\}, \quad -1 \le p < 0.$$

When $r = 0$ and $N = 2$, we should understand that η_2 denotes an arbitrarily large number.

For the case (3), we may get

$$\begin{aligned} \int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^2 dxds &\le C \int_t^{t+T} \int_{\Omega_1} a(x)|u_t|^{r+2} dxds \\ &\le \int_t^{t+T} \int_{\Omega_1} \rho(x, u_t) u_t dxds \le C D^{r+2}(t) \end{aligned} \tag{8.2.51}$$

whence,

$$\int_t^{t+T} \int_{\omega} |u_t|^2 dxds \le C_1 D^{r+2}(t).$$

Thus noting (8.2.30) and (8.2.45), we obtain

$$E(t) \le C_1 \Big\{ D(t)^{2(r+1)} + D(t)^{4(r+2)(p+1)/(4+Np+2p)} \Big\} \tag{8.2.52}$$

which, by Theorem 2.3.14, implies,

$$E(t) \le C_1(1+t)^{-2\eta_3}$$

with

$$\eta_3 = \min\left\{\frac{-(r+1)}{r}, \frac{2(p+1)}{p(N-2)^+}\right\}.$$

For $N=2$, we also obtain the same result by taking sufficiently small $\varepsilon > 0$.

Finally, for the case (4), we have

$$\int_t^{t+T}\int_\omega |u_t|^2 dxds \le C_1\Big\{D^{r+2}(t) + (D(t))^{4(r+2)/(4-Np+2p)}\Big\}.$$

and hence, by (8.2.31) and (8.2.45),

$$E(t) \le C_1\Big\{(D(t))^{2(r+1)} + (D(t))^{4(r+2)/(4-Np+2p)}\Big\}$$

which, together with Theorem 2.3.14, yields

$$E(t) \le C_1(1+t)^{-2\eta_4}$$

with

$$\eta_4 = \min\left\{\frac{-(r+1)}{r}, \frac{-2}{p(N-2)^+}\right\}.$$

Therefore the proof of Theorem 8.2.1 is now complete. □

8.3 Polynomial decay rate for nonlinear wave equations

In this section, we shall apply Theorem 2.3.6 to establish the polynomial decay rate for nonlinear wave equations. Such a result is chosen from Kim [439].

We shall consider the following wave equation with a localized linear dissipation in a three-dimensional bounded domain Ω on which there exists a trapped ray

$$\begin{cases} w_{tt} - \Delta w + \alpha(x) w_t = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ w = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ (w(\cdot,0), w_t(\cdot,0)) = (w_0, w_1) & \text{in } \Omega, \end{cases} \tag{8.3.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with a boundary $\partial\Omega$ at least Lipschitz, and α is a non-negative function in $L^\infty(\Omega)$ and depends on a non-empty proper subset ω of Ω on which $1/\alpha \in L^\infty(\omega)$ (in particular, $\{x \in \Omega : \alpha(x) > 0\}$ is a non-empty open set).

For non-identically zero initial data $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$, we define the energy of the solution w of problem (8.3.1) at time t by

$$E(w,t) = \int_\Omega \left(|w_t(x,t)|^2 + |\nabla w(x,t)|^2\right) dx.$$

We note that $E(w,t)$ is a continuous decreasing function of time and we have, for any $0 \le t_0 < t_1$,

$$E(w,t_1) = E(w,t_0) - 2\int_{t_0}^{t_1}\int_\Omega \alpha(x)|w_t(x,t)|^2\,dxdt. \tag{8.3.2}$$

We shall use Theorem 2.3.6 to study the energy decay rates for the damped wave equation (8.3.1).

To achieve this result, we shall construct a geometry (Ω,ω) with a trapped ray (the geometric control condition is then not fulfilled) and establish a polynomial decay rate, therefore better than the logarithmic one when $(w_0,w_1) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

Now we assume $(w_0,w_1) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

Let us now introduce our working geometry and explain why there is a trapped ray.

First, we set $D(r_1,r_2) = \{(x_1,x_2) \in \mathbb{R}^2 : |x_1| < r_1, |x_2| < r_2\}$ where $r_1, r_2 > 0$. Next, let $m_1, m_2, \rho > 0$. We choose Ω a connected open set in $\mathbb{R}^3$ bounded by $\Gamma_1, \Gamma_2, \Upsilon$ where

$$\begin{cases} \Gamma_1 = \overline{D(m_1,m_2)} \times \{\rho\}, \text{ with boundary } \partial\Gamma_1, \\ \Gamma_2 = \overline{D(m_1,m_2)} \times \{-\rho\}, \text{ with boundary } \partial\Gamma_2, \\ \Upsilon \text{ is a surface with boundary} \partial\Upsilon = \partial\Gamma_1 \cup \partial\Gamma_2. \end{cases}$$

Therefore, the boundary of Ω is $\partial\Omega = \partial\Gamma_1 \cup \partial\Gamma_2 \cup \Upsilon$.

Second, we assume that either $\partial\Omega$ is C^2 with $\Upsilon \subset (\mathbb{R}^2 \backslash D(m_1,m_2)) \times \mathbb{R}$ (in particular, $\Upsilon \in C^2$) or Ω is convex (in particular Υ is Lipschitz).

Third, we choose $\omega = \Omega \cap \Theta$ where Θ is a small neighborhood of Υ in $\mathbb{R}^3$ such that $\Theta \cap D(M_1,M_2) \times [-\rho,\rho] = \emptyset$ for some $M_1 \in (0,m_1)$ and $M_2 \in (0,m_2)$.

Now we recall that the bicharacteristics associated to $\partial_t^2 - \Delta$ in the whole space are curves in the space-time variables and their Fourier variables described by

$$\begin{cases} x(s) = x_0 + 2\xi(s)s, \\ t(s) = t_0 - 2\tau(s)s, \end{cases} \quad \text{and} \quad \begin{cases} \xi(s) = \xi_0, \\ \tau(s) = \tau_0, \end{cases}$$

with $|\xi(s)|^2 - \tau^2(s) = 0$ for all $s \in [0,+\infty)$, when $(x_0,t_0,\xi_0,\tau_0) \in \mathbb{R}^4 \times (\mathbb{R}^4\backslash\{0\})$. The rays associated to $\partial_t^2 - \Delta$ in the whole space are the projection of the bicharacteristics on the space-time domain. In particular, for all $s \in [0,+\infty)$,

$$\begin{cases} x(s) - x_0 - 2\xi_0 s = 0, \\ t(s) + 2\tau_0 s = 0, \\ |\xi_0|^2 - \tau_0^2 = 0, \end{cases} \tag{8.3.3}$$

with, $t_0 = 0$ and $\tau_0 \neq 0$. For the definition of generalized bicharacteristics, we refer to [614].

Recall that the boundary $\partial\Omega$ (and more precisely Υ) is Lipschitz or of class C^2 will not create difficulties. Following the idea in [439], we may define the ray starting at $x_0 \in \Omega$ with direction $\xi_0 \in S^2$ (S is the unit sphere in $\mathbb{R}^3$, i.e., $|\xi_0| = 1$) by a continuous curve $x(s)$ parametrized by s satisfying the following rules: it is the solution of problem (8.3.3) with initial data $x(0) = x_0$ for $s \in [0, s_0]$ until it hits the boundary $\partial\Omega$ at $x(s_0)$; if for some $s_1 > 0, x(s_1) \in \overline{\Upsilon}$, the parametrization of the curve $x(s)$ stops; if for some $s_1 > 0, x(s_1) \in (\Gamma_1 \cup \Gamma_2) \backslash (\partial\Gamma_1 \cup \partial\Gamma_2)$, the curve $x(s)$ is reflected like a billiard ball following the rule of geometric optics "angle of incidence = angle of reflection" until it hits the boundary $\partial\Omega$ at $x(s_2)$ for some $s_2 > s_1$. We shall only consider the above geometry (Ω, ω). Recall that the real function $\alpha \in L^\infty(\omega)$ satisfies $\alpha \geq 0$ and $1/\alpha \in L^\infty(\Omega)$.

We use Theorem 2.3.6 to prove the next theorem due to [439].

Theorem 8.3.1 ([439]). *There exist constants $C > 0$ and $\delta > 0$ such that for any $t > 0$ and any initial data $(w_0, w_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, the solution of problem* (8.3.1) *satisfies*

$$\int_\Omega \left(|w_t(x,t)|^2 + |\nabla w(x,t)|^2\right) dx \leq \frac{C}{t^\delta} \|(w_0, w_1)\|^2_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)}. \tag{8.3.4}$$

Remark 8.3.1 ([439]). The polynomial decay rate for the damped wave equations holds in particular for the following two choices of a non-negative real function $\alpha : \alpha > 0$ a.e. on X where X is a neighborhood of Υ in $\mathbb{R}^3$; $\alpha \in C(\overline{\Omega})$ such that $\alpha > 0$ on $\overline{\Upsilon}$. Indeed, with such a choice of α, we may choose ω as above.

Remark 8.3.2 ([439]). In a two-dimensional square domain, the polynomial decay rate for the damped wave equations was established by Liu and Rao [490], which was generalized recently by Burq and Hitrik [123] for partially rectangular planar domain by using resolvent estimates. In the one-dimensional case, a sharp polynomial decay rate was established by Zhang and Zuazua [995] for a wave-heat coupled system where the dissipation acts through the heat equation on a proper sub-domain.

To achieve the above polynomial decay rate, we need to establish a kind of observability estimates for the wave equations, i.e., we have the following result.

Theorem 8.3.2 ([439]). *The following two statements are equivalent.*

(i) *There exist constants $C > 0$ and $\delta > 0$ such that for any non-identically zero initial data $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, the solution u of the wave equation*

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ (u(\cdot,0), u_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega \end{cases} \tag{8.3.5}$$

satisfies

$$\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \le C\int_0^{C\Lambda^{1/\delta}}\int_\Omega \alpha(x)\left(|u_t(x,t)|^2+|u(x,t)|^2\right)dxdt, \tag{8.3.6}$$

where $\Lambda = \frac{\|(u_0,u_1)\|^2_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)}}{\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}$.

(ii) *There exist constants* $C>0$ *and* $\delta>0$ *such that for any non-identically zero initial data* $(w_0,w_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$*, the solution* w *of problem* (8.3.1) *satisfies*

$$E(w,t)\le \frac{C}{t^\delta}\Big\|(w_0,w_1)\Big\|^2_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)}.$$

Clearly, the proof of Theorem 8.3.1 now reduces to (8.3.6) of Theorem 8.3.2. We also note that (8.3.6) looks like an observability estimate where the time of observability depends on the quantity Λ which can be seen as a measure of the frequency of the initial data (u_0,u_1).

In fact, by an easy minimization technique, (8.3.6) is equivalent to that there exists a constant $C>0$ such that for any $h>0$ sufficiently small, and for any u, solution of (8.3.5) with initial data $(u_0,u_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$, there holds that

$$\begin{aligned}\|(u_0,u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \le C\int_0^{C(1/h)^{1/\delta}}\int_\Omega \alpha(x)\left(\Big|u_t(x,t)\Big|^2+\Big|u(x,t)\Big|^2\right)dxdt\\ +h\Big\|(u_0,u_1)\Big\|^2_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)}.\end{aligned}$$

Following [439], we shall divide the proof of (8.3.6) into two steps:

1) We shall choose ω_0 an adequate subset of $D(m_1,m_2)\times(-\frac{\varrho}{4},\frac{\varrho}{4})$ such that any ray starting at any $x_0\in\Omega$ with any direction $\xi_0\in S^2$ will meet a suitable compact set in $\omega_0\cup\omega$, this will imply an observability estimate with $\omega_0\cup\omega\times(0,T)$ being the domain of observation for some $T>0$.
2) Since $1/\alpha\in L^\infty(\omega)$, we only need to establish a kind of Hölder interpolation estimate against the fact that if $u=0$ on $\omega\times(0,C(1/h)^{1/\delta})$ for any $h>0$ sufficiently small, then $u=0$ on $\omega_0\cup\omega\times(0,T)$.

To show this fact, by a classical trace inequality valid for any solution of the wave equation with homogeneous Dirichlet boundary condition, we only replace the term

$$u\big|_{\omega\times(0,C(1/h)^{1/\delta})}\quad\text{by}\quad \partial_\nu u\Big|_{\Upsilon\times(-C(1/h)^{1/\delta},C(1/h)^{1/\delta})}$$

in the above Hölder interpolation estimate (see Theorem 8.3.3) where ν is the unit outward normal to $\partial\Omega$.

The notation in this section is as follows: c denotes a positive constant which only may depend on (m_1, m_2, ρ), and γ will denote an absolute constant larger than one. The value of $c > 0$ and $\gamma > 1$ may change from line to line.

Proof of Theorem 8.3.2. The proof uses many classical techniques for hyperbolic systems (see, e.g., [548]) as a decomposition argument in order to deal with the wave equation with a second member and as a useful transformation for deriving estimates with weaker norms from a stronger game of norms.

We now prove (ii) ⇒ (i). First, let $(u_0, u_1) = (w_0, w_1)$. Next, we combine the polynomial decay rate for $E(w, t)$ and the formula (8.3.2) applied with $t_0 = 0$ and $t_1 = t$, in order to get by choosing

$$t = \left(\frac{2C\|(u_0, u_1)\|^2_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)}}{\|(u_0, u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}} \right)^{1/\delta},$$

the following inequality

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq 4 \int_0^{(2C\Lambda)^{1/\delta}} \int_\Omega \alpha(x)|w_t(x,t)|^2\, dxdt.$$

Since $u - w$ solves a damped wave equation with a second member αu_t and with identically zero initial data, we conclude that

$$\|(u_0, u_1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq 16 \int_0^{(2C\Lambda)^{1/\delta}} \int_\Omega \alpha(x)|u_t(x,t)|^2\, dxdt \tag{8.3.7}$$

holds with the same δ as the one of the statement (ii).

Next, we prove (i) ⇒ (ii). We divide the proof into three steps.

Step 1. Let $(w_0, w_1) = (u_0, u_1)$. Then by using the above similar decomposition argument, it follows from (i) that for the solution w of problem (8.3.1), there holds that

$$E(w, 0) \leq 2C\left(1 + c\|\alpha\|_{L^\infty(\Omega)}C^2\Lambda^{2/\delta}\right)\int_0^{C\Lambda^{1/\delta}}\int_\Omega \alpha(x)\left(|w_t(x,t)|^2 + |w(x,t)|^2\right)dxdt,$$

which also holds for any non-identically zero initial data $(w_0, w_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$.

Step 2. We only need to apply the previous inequality to $\partial_t\tilde{w}$ where $\tilde{w}$ is a solution of problem (8.3.1) with non-identically zero initial data $(\tilde{w}(\cdot, 0), \tilde{w}_t(\cdot, 0)) = (\tilde{w}_0, \tilde{w}_1) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$ satisfying the compatibility condition $\Delta\tilde{w}_0 - \alpha\tilde{w}_1 \in H_0^1(\Omega)$. Noticing that $E(\tilde{w}, 0) \leq c(1 + \|\alpha\|_{L^\infty(\Omega)})E(\tilde{w}_t, 0)$, there exists some constant $c_1 > 0$ depending on $(\Omega, \omega, \alpha, \delta)$ such that

$$E(\tilde{w}, 0) + E(\tilde{w}_t, 0) \leq c_1\tilde{\Lambda}^{2/\delta}\int_0^{c_1\tilde{\Lambda}^{1/\delta}}\int_\Omega \alpha(x)\left(\left|\tilde{w}_{tt}(x,t)\right|^2 + \left|\tilde{w}_t(x,t)\right|^2\right)dxdt,$$

with

$$\tilde{\Lambda} = \frac{E(\tilde{w}_t, 0) + \|(\tilde{w}_1, \Delta\tilde{w}_0 - \alpha\tilde{w}_1)\|^2_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)}}{E(\tilde{w}, 0) + E(\tilde{w}_t, 0)}.$$

This, by a translation on the time variable and (8.3.2), implies that there is some constant $c_1 > 0$ depending on $(\Omega, \omega, \alpha, \delta)$ such that it holds that for all $s \geq 0$,

$$\mathcal{H}(s) \leq c_1 \left(\frac{1}{\mathcal{H}(s)}\right)^{2/\delta} \left[\mathcal{H}(s) - \mathcal{H}\left(c_1 \left(\frac{1}{\mathcal{H}(s)}\right)^{1/\delta} + s\right)\right], \tag{8.3.8}$$

where

$$\mathcal{H}(s) = \sigma \frac{E(\tilde{w}, s) + E(\tilde{w}_t, s)}{E(\tilde{w}_t, 0) + \|(\tilde{w}_1, \Delta\tilde{w}_0 - \alpha\tilde{w}_1)\|^2_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)}}$$

and $\sigma > 0$ is a constant depending on (Ω, ω, α) such that $\mathcal{H}$ is bounded by one. Applying Theorem 2.3.6 to (8.3.8), we conclude that there are constants $C > 0$ and $\delta > 0$ such that for all $t > 0$,

$$E(\tilde{w}_t, t) \leq \frac{C}{t^\delta} \left(E(\tilde{w}_t, 0) + \|(\tilde{w}_1, \Delta\tilde{w}_0 - \alpha\tilde{w}_1)\|^2_{(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)}\right).$$

Step 3. We may use a well-known transformation in order to deduce the desired statement (ii) from Step 2. Indeed, we apply the previous inequality to

$$\tilde{w}(\cdot, t) = \int_0^t w(\cdot, \ell)\, d\ell - (-\Delta)^{-1}(w_1 + \alpha w_0) \quad \text{in } \Omega.$$

This thus completes the proof of Theorem 8.3.2. □

8.4 Decay rate estimates for dissipative wave equations

In this section, we shall employ Theorems 1.5.13–1.5.14 and Corollary 1.5.2 to establish the decay rate estimates for the wave equation damped with a boundary nonlinear velocity feedback $\rho(u_t)$. We adopt these results from Martinez [587].

We shall study the decay property of the solutions of the wave equation damped by a nonlinear boundary feedback

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \text{in} \quad \Gamma \times \mathbb{R}^+, & (8.4.1) \\ u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, & (8.4.2) \\ \partial_\nu u + m \cdot \nu \rho(u_t) = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, & (8.4.3) \\ u(0) = u^0, \ u_t(0) = u^1 & (8.4.4) \end{cases}$$

where Ω is a bounded open domain of class C^2 in $\mathbb{R}^N$ and let $\{\Gamma_0, \Gamma_1\}$ be a partition of its boundary Γ. By ν we denote the outward unit normal vector to Γ and fix x_0 in $\mathbb{R}^N$, we define

$$m(x) = x - x_0.$$

Usually, we can define the energy of the problem (8.4.1)–(8.4.4) by

$$E(t) = \frac{1}{2}\int_{\Omega}\left(u_t^2 + |\nabla u|^2\right)dx.$$

In the sequel, we shall use Theorems 1.5.13–1.5.14 and Corollary 1.5.2 to show an explicit decay estimate of the energy even if ρ has not a polynomial behavior in zero.

Assume $\rho : \mathbb{R} \to \mathbb{R}$ is a non-decreasing continuous function such that $\rho(0) = 0$. Moreover, we assume that

$$\begin{cases} \Gamma_0 \neq \emptyset, \quad \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset, & (8.4.5)\\ m\cdot\nu \le 0 \quad \text{on} \quad \Gamma_0, \quad \text{and} \quad \mathrm{m}\cdot\nu \ge 0 \quad \text{on} \quad \Gamma_1. & (8.4.6)\end{cases}$$

Indeed, there exist some examples to verify (8.4.5)–(8.4.6), e.g., if $\Omega = \Omega_1 \backslash \Omega_0$, where Ω_0 and Ω_1 are convex sets such that $\overline{\Omega}_0 \subset \Omega_1$, then (8.4.5)–(8.4.6) are satisfied with $\Gamma_0 = \partial\Omega_0, \Gamma_1 = \partial\Omega_1$, and $x_0 \in \Omega_0$.

We also assume that there exist a strictly increasing and a odd function g of class C^1 on $[-1, 1]$ and two positive constants c_1 and c_2 such that

$$\begin{cases} |g(y)| \le |\rho(y)| \le |g^{-1}(y)|, \text{ for all } y \in [-1, 1],\\ c_1|y| \le |\rho(y)| \le c_2|y|, \text{ for all } |y| \ge 1, \end{cases} \tag{8.4.7}$$

where g^{-1} denotes the inverse function of g. Set

$$G(y) = yg(y), \qquad H(y) = g(y)/y, \tag{8.4.8}$$

where $H(0) = g'(0)$. As usual, denote

$$H^1_{\Gamma_0}(\Omega) := \left\{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_0\right\}.$$

The following standard theorem due to Komornik [449] concerns the existence and the regularity of the solutions.

Theorem 8.4.1 ([449]). *Assume* (8.4.5)–(8.4.6) *hold.*

(i) *If* $(u^0, u^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, *then problem* (8.4.1)–(8.4.4) *has a unique solution such that*

$$u \in C(\mathbb{R}^+, H^1_{\Gamma_0}(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega)). \tag{8.4.9}$$

The energy of the solution u defined by

$$E(t) = \frac{1}{2}\int_{\Omega}\left(u_t^2 + |\nabla u|^2\right)dx \tag{8.4.10}$$

is non-increasing.

(ii) *Moreover, if* ρ *is globally Lipschitz,* $(u_0, u_1) \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega)$ *satisfying*

$$\partial_\nu u^0 + m \cdot \nu \rho(u^1) = 0 \quad on \quad \Gamma_1, \tag{8.4.11}$$

then the solution of problem (8.4.1)–(8.4.4) *has the stronger regularity property*

$$\begin{cases} u \in L^\infty(\mathbb{R}^+, H^2(\Omega)), \quad u_t \in L^\infty(\mathbb{R}^+, H^1_{\Gamma_0}(\Omega)), & (8.4.12) \\ u_{tt} \in L^\infty(\mathbb{R}^+, L^2(\Omega)). & (8.4.13) \end{cases}$$

We shall use Theorems 1.5.13–1.5.14 and Corollary 1.5.2 to show the next result due to Martinez [587].

Theorem 8.4.2 ([587]). *Assume that* (8.4.5)–(8.4.7) *hold. Then for any* $(u^0, u^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$*, the solution* u *to problem* (8.4.1)–(8.4.4) *satisfies the estimate for all* $t \geq 1$,

$$E(t) \leq C\left(G^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{8.4.14}$$

with a constant $C > 0$ *only depending on the initial energy* $E(0)$ *and in a continuous way. Moreover if* $H(0) = 0$ *and* H *is non-decreasing on* $[0, \eta]$ *for some constant* $\eta > 0$*, then we have the following better estimate for all* $t \geq 1$,

$$E(t) \leq C\left(g^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{8.4.15}$$

with a constant $C > 0$ *only depending on the initial energy* $E(0)$ *in a continuous way.*

Note that the case where ρ has a polynomial behavior in zero corresponds to the case $g(y) = cy^p$ for $y \in [0, 1]$. We refer to [587] for some examples.

Proof. To prove Theorem 8.4.2, following [587], we can divide three parts in the following.

(I) First, we need to use the multiplier method to give the following three lemmas whose proofs can be found in [587].

Lemma 8.4.3 ([587]). *The function* $E : \mathbb{R}^+ \to \mathbb{R}^+$ *is non-increasing, locally absolutely continuous and*

$$E'(t) = -\int_{\Gamma_1} m \cdot \nu u_t \rho(u_t) ds. \tag{8.4.16}$$

Lemma 8.4.4 ([587]). *Let* $\phi : \mathbb{R}^+ \to \mathbb{R}$ *be a function of class* C^2*. For all* $0 \leq S < T < +\infty$*, letting*

$$M(u) = 2m \cdot \nabla u + (N - 1)u, \tag{8.4.17}$$

then we have

$$
\begin{aligned}
2 & \int_S^T E^2(t)\phi'(t)dt \\
& = \int_S^T E\phi' \int_\Gamma \big(M(u)\partial_\nu u + m\cdot\nu(u_t^2 - |\nabla u|^2)\big)\, dsdt \\
& \quad + \int_S^T (E'\phi' + E\phi'') \int_\Omega u_t M(u)dxdt - \left[E\phi' \int_\Omega u_t M(u)dx\right]_S^T .
\end{aligned}
\tag{8.4.18}
$$

Now assume that ϕ is a strictly increasing concave function. Therefore, ϕ' is a bounded function on $\mathbb{R}^+$. Denote λ the maximum of ϕ'.

Lemma 8.4.5 ([587])**.** *There exists a positive constant c such that for all* $0 \le S < T$*, we have*

$$
\int_S^T E^2(t)\phi'(t)dt \le cE^2(S) + c\int_S^T E(t)\phi'(t)\left(\int_{\Gamma_1} m\cdot\nu\Big(u_t^2 + \rho(u_t^2)\Big)ds\right) dt.
\tag{8.4.19}
$$

(II) Next, we show first estimate on the decay rate of the energy.

Assume now that ϕ is a strictly increasing concave function of class C^2 on $[0, +\infty)$ such that as $t \to +\infty$,

$$
\phi(t) \to +\infty, \ \ \phi'(t) \to 0,
\tag{8.4.20}
$$

(e.g., $t \mapsto \ln(1+t)$). Now introduce for all $t \ge 1$,

$$
h(t) = g^{-1}(\phi'(t)),
\tag{8.4.21}
$$

such that h is a decreasing positive function satisfying as $t \to +\infty$,

$$
h(t) \to 0.
$$

Following [587], we can prove the next two lemmas.

Lemma 8.4.6 ([587])**.** *There exists a positive constant c such that for all* $1 \le S < T < +\infty$*,*

$$
\int_S^T E(t)\phi'(t)\int_{\Gamma_1} m\cdot\nu\, u_t^2 dsdt \le cE^2(S) + cE(S)\int_S^T \phi'(t)\Big(g^{-1}(\phi'(t))\Big)^2 dt.
\tag{8.4.22}
$$

Lemma 8.4.7 ([587])**.** *There exists a positive constant c independent of the constant of Lipschitz of* ρ *such that for all* $1 \le S < T < +\infty$*,*

$$
\int_S^T E(t)\phi'(t)\int_{\Gamma_1} m\cdot\nu\rho^2(u_t)dsdt \le cE^2(S) + cE(S)\int_S^T \phi'(t)\Big(g^{-1}(\phi'(t))\Big)^2 dt.
\tag{8.4.23}
$$

Now we assume that ϕ satisfies the following additional property:

$$\int_1^{+\infty} \phi'(t)\Big(g^{-1}(\phi'(t))\Big)^2 dt \ \text{ converges,} \tag{8.4.24}$$

which, indeed, is closely related to the behavior of g near 0 and the decay rate of ϕ' at infinity. Then it follows from [587] that there exists a constant $c > 0$ such that for all $1 \le S < T$,

$$\int_S^T E^2(t)\phi'(t)dt \le cE^2(S) + cE(S)\int_S^{+\infty} \phi'(t)\Big(g^{-1}(\phi'(t))\Big)^2 dt, \tag{8.4.25}$$

which, in particular, implies that for all $S \ge 1$,

$$\int_S^{+\infty} E^2(t)\phi'(t)dt \le cE(S). \tag{8.4.26}$$

Define $F(t) := E(t+1)$ and $\tilde{\phi}(t) := \phi(t+1)$ on $[0, +\infty)$. Thus we can apply the Bellman–Gronwall inequality in Theorem 1.5.13 with $\sigma = 1$ to obtain a decay rate estimate on F, so on E : there exists a constant $C > 0$ depending on $E(1)$ in a continuous way such that for all $t \ge 1$,

$$E(t) \le \frac{C}{\phi(t)}, \tag{8.4.27}$$

which readily gives us a first estimate of the decay rate of the energy.

Now we define ψ for all $t \in [1, +\infty)$ by

$$\psi(t) = 1 + \int_1^t \frac{1}{g(\frac{1}{\tau})} d\tau. \tag{8.4.28}$$

Then ψ is a strictly increasing function of class C^2 on $[1, +\infty)$ such that as $t \to +\infty$,

$$\psi'(t) = \frac{1}{g(\frac{1}{t})} \to +\infty,$$

which gives us as $t \to +\infty$,

$$\begin{aligned} &\psi(t) \to +\infty, \\ &\int_1^{+\infty} \left(g^{-1}\left(\frac{1}{\psi'(\tau)}\right)\right)^2 d\tau = \int_1^{+\infty} \frac{1}{\tau^2} d\tau < +\infty. \end{aligned}$$

Define ϕ for all $t \in [1 + \infty)$ by

$$\phi(t) = \psi^{-1}(t). \tag{8.4.29}$$

Obviously, ϕ is a strictly increasing concave function of class C^2 on $[1, +\infty)$ that satisfies all the special assumptions (see [587] for details). With this special modified ϕ, it follows from (8.4.25) that

$$\begin{aligned} \int_S^T E^2(t)\phi'(t)dt &\le cE^2(S) + cE(S)\int_S^{+\infty} \phi'(t)\Big(g^{-1}(\phi'(t))\Big)^2 dt \\ &\le cE^2(S) + cE(S)\int_{\phi(S)}^{+\infty} \Big(g^{-1}(\phi'(\phi^{-1}(\tau)))\Big)^2 d\tau \\ &\le cE^2(S) + cE(S)\int_{\phi(S)}^{+\infty} \left(g^{-1}\left(\frac{1}{(\phi^{-1})(\tau)}\right)\right)^2 d\tau \\ &\le cE^2(S) + cE(S)\int_{\phi(S)}^{+\infty} \frac{1}{\tau^2} d\tau = cE^2(S) + c\frac{E(S)}{\phi(S)}. \end{aligned} \tag{8.4.30}$$

Then we can apply Theorem 1.5.14 with $\sigma = \sigma' = 1$ to obtain that for all $t \ge 1$,

$$E(t) \le \frac{C}{\phi^2(t)}, \tag{8.4.31}$$

which is clearly a better estimate than (8.4.27). For the rest of the proof of (8.4.14), we may refer the reader to [587]. □

(**III**) Third, we shall derive the second estimate on the decay rate of the energy.

The proof of this part can be found in [587]. Thus the proof of Theorem 8.4.2 is complete. □

8.5 Energy decay for a dissipative anisotropic elastic system

In order to apply Theorem 2.3.14 in this section, we shall study the large-time behavior of energy for a N-dimensional dissipative anisotropic elastic system. These results are chosen from Qin, Liu and Deng [787].

We denote by Ω an open bounded domain of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. If $\phi = \phi(x, t)$ is the displacement vector field, then the system in question reads:

$$\phi_i'' - [A_{ijkl}\phi_{k,l}]_{,j} + \hat{f}^i\phi_i' = 0 \ \ in \ \ \Omega \times [0, +\infty) \tag{8.5.1}$$

where $i, j, k, l = 1, 2, \ldots, N$ and $\phi(x, t) = (\phi_1, \ldots, \phi_N)$, $\omega' = \partial\omega/\partial t$, $\omega'' = \partial^2\omega/\partial^2 t$, $\omega_{,j} = \partial\omega/\partial x_j$, and we use Einstein's convention on summing over repeated lower indices, and $\hat{f}^i = \hat{f}^i(x)$ $(i = 1, 2, \ldots, N)$ are smooth non-negative functions on $\overline{\Omega}$ which may vanish somewhere on $\overline{\Omega}$.

The initial conditions and boundary conditions are given by

$$\begin{cases} \phi(x,0) = \phi^0(x), \quad \phi'(x,0) = \phi^1(x), & (8.5.2)\\ \quad \phi|_{\partial\Omega} = 0. & (8.5.3) \end{cases}$$

The functions $A_{ijkl}(x)$ $(i, j, k, l = 1, 2, \dots, N)$ are sufficiently smooth functions satisfying

$$A_{ijkl} = A_{jikl} = A_{klij} \tag{8.5.4}$$

and there are two positive constants α and β such that for any $N \times N$ matrix $(\xi_{ij})_{N\times N}$, there holds on $\overline{\Omega}$

$$\begin{cases} A_{ijkl}\xi_{ij}\xi_{kl} \geq \alpha\xi_{ij}\xi_{ij}, & (8.5.5)\\ (A_{ijkl} - q_\mu A_{ijkl,\mu})\xi_{ij}\xi_{ij} \geq \beta A_{ijkl}\xi_{ij}\xi_{kl} & (8.5.6) \end{cases}$$

where $\mu = 1, 2, \dots, N$.

Recall that when $N = 3$ and $\hat{f}^i = \hat{g}^i \equiv 0$, Beale [74, 80, 112] proved the global existence and regularity of solutions for linear homogeneous wave equations with acoustic boundary in a Hilbert space of data with finite energy by means of semigroup methods, and the asymptotic behavior was obtained in [80], but no decay rate was given there. Muñoz Rivera and Qin [651] also studied the same model as in [74, 80, 112] and obtained the polynomial decay estimate of the energy. For the following general case

$$\begin{cases} \phi_i'' - [A_{ijkl}\phi_{k,l}]_{,j} + \hat{f}^i\phi_i' = 0, \ \text{ in } \ \Omega \times [0, +\infty) & (8.5.7)\\ \hat{f}^i \geq 0, \ \ i = 1, 2, \dots, N, \ \ \hat{f}(x) = (\hat{f}^1, \dots, \hat{f}^N) \not\equiv 0, \ \text{ for all } x \in \Omega, & (8.5.8)\\ m_i(x)\varepsilon_i''(x,t) + d_i(x)\varepsilon_i'(x,t) + k_i(x)\varepsilon_i(x,t) = -\rho\phi_i'(x,t), \ \text{ on } \ \Gamma_0, & (8.5.9)\\ \varepsilon_i'(x,t) + g_i = A_{ijkl}\phi_{k,l}\nu_j, \ \text{ on } \ \Gamma_0, & (8.5.10)\\ g_i(x,t) = -\hat{g}^i\phi_i'(x,t), \ \ i = 1, 2, \dots, N & (8.5.11) \end{cases}$$

when $\hat{f}^i = \hat{g}^i \equiv 0$, Qin and Muñoz Rivera [797] established the polynomial decay of energy for problem (8.5.17)–(8.5.21). The object of this section is to use Theorem 2.3.13, by combining the methods in [677, 1028], to prove the decay property of energy for problem (8.5.1)–(8.5.3).

We only use standard function spaces and omit their definitions, but we note that $||\cdot||$ denotes the L^2 norm on Ω.

Define the space

$$\mathcal{H} = (H_0^1(\Omega))^N \times (L^2(\Omega))^N, \quad \text{with} \quad H_0^1(\Omega) = \{u : u \in H^1(\Omega), \ u|_{\partial\Omega} = 0\}.$$

It follows that $\mathcal{H}$ together with the inner product

$$(u, w) = \int_\Omega (u_{N+j}w_{N+j} + A_{ijkl}u_{i,j}w_{k,l})dx \tag{8.5.12}$$

is a Hilbert space, where $u = (u_1, u_2, \ldots, u_{2N})^T,\ w = (w_1, w_2, \ldots, w_{2N})^T \ \in \mathcal{H}$. Here the superscript "$T$" denotes the transposition of a vector. Thus from (8.5.5) it follows that the induced norm on $\mathcal{H}$ by the above inner product

$$|u|^2_{\mathcal{H}} = \int_\Omega (u_{N+j}u_{N+j} + A_{ijkl}u_{i,j}u_{k,l})dx$$

is equivalent to the usual norm on $\mathcal{H}$

$$||u||_{\mathcal{H}} = \left(\int_\Omega (u_{N+j}u_{N+j} + u_{i,j}u_{i,j})dx\right)^{1/2}$$

for any $u = (u_1, u_2, \ldots, u_{2N})^T \ \in \mathcal{H}$.

Define an operator $\mathcal{A}$ on $\mathcal{H}$ so that for smooth

$$U = (\phi, \phi_t) = (\phi_1, \ldots, \phi_N, \phi_1', \ldots, \phi_N') \in \mathbb{R}^{2N},$$

equations (8.5.1)–(8.5.3) are equivalent to $U(t) \in D(\mathcal{A})$ and

$$U_t = \mathcal{A}U$$

where $U = (u_1, u_2, \ldots, u_{2N})^T$ satisfies

$$u_i = \phi_i,\ u_{N+i} = \phi_i',\ i = 1, 2, \ldots, N \tag{8.5.13}$$

and

$$\begin{aligned} D(\mathcal{A}) = \Big\{U = (u_1, u_2, \ldots, u_{2N})^T \in \mathcal{H} : u_{N+i} \in H_0^1(\Omega),& \\ (A_{ijkl}u_{k,l})_{,j} - \hat{f}^i u_{N+i} \in L^2(\Omega)\Big\}.& \end{aligned} \tag{8.5.14}$$

Obviously, we can derive from (8.5.1)–(8.5.3) and (8.5.14) that

$$\begin{cases} u_i' = u_{N+i} & (8.5.15) \\ u_{N+i}' = (A_{ijkl}u_{k,l})_{,j} - \hat{f}^i u_{N+i} & (8.5.16) \end{cases}$$

where $i = 1, 2, \ldots, N$.

For any $U \in D(\mathcal{A})$, it follows from (8.5.12)–(8.5.16) and the definition of $D(\mathcal{A})$ that

$$\begin{aligned} (\mathcal{A}U, U) &= \int_\Omega A_{ijkl}u_{N+k,l}u_{i,j} + [(A_{ijkl}u_{k,l})_{,j} - \hat{f}^i u_{N+i}]u_{N+i}dx \\ &= \int_\Omega A_{ijkl}u_{N+k,l}u_{i,j}dx + \int_{\partial\Omega} A_{ijkl}u_{k,l}\nu_j u_{N+i}dS \\ &\quad - \int_\Omega A_{ijkl}u_{N+i,j}u_{k,l}dx - \int_\Omega \hat{f}^i u_{N+i}u_{N+i}dx \\ &= -\int_\Omega \hat{f}^i u_{N+i}u_{N+i}dx \le 0 \end{aligned} \tag{8.5.17}$$

which implies that $\mathcal{A}$ is a dissipative operator on $\mathcal{H}$. Thus, similar to the proofs in [74, 80, 112, 798], we can obtain the following results on the global existence and regularity of global solutions.

Theorem 8.5.1 ([787]). *The operator $\mathcal{A}$ defined on $\mathcal{H}$ is closed, densely defined and dissipative. It generates a C_0-semigroup on $\mathcal{H}$.*

Theorem 8.5.2 ([787]). *Assume that $U^0 \in \mathcal{H}$ is C^∞ and vanishes near $\partial\Omega$, let $U(t)$ be the solution of $U'(t) = \mathcal{A}U(t),\ t \geq 0$, with $U(0) = U^0$. Then $u_1(t), \ldots, u_{2N}(t) \in C^\infty(\overline{\Omega})$ for any $t \geq 0$.*

We now introduce the following energy functions

$$\begin{cases} E_0(t;\phi) = \dfrac{1}{2}\displaystyle\int_\Omega (\phi_i'\phi_i' + A_{ijkl}\phi_{i,j}\phi_{k,l})dx, & (8.5.18)\\ E_h(t;\phi) = E_0(t;\partial_t^h\phi), \quad h = 1, 2, \ldots, m. & (8.5.19)\end{cases}$$

If $\hat{f}^i(x)$ $(i = 1, 2, \ldots, N)$ are smooth functions and $\phi(x,t) = (\phi_1(x,t), \ldots, \phi_N(x,t))$ is a smooth solution of problem (8.5.1)–(8.5.3), then $\frac{\partial^k}{\partial t^k}\phi_i$ $(k = 0, 1, \ldots, m)$ necessarily vanish on the boundary of Ω.

The following is our main result in this section.

Theorem 8.5.3 ([787]). *Assume that $\hat{f}^i(x) \geq 0$ $(i = 1, 2, \ldots, N)$ on Ω and there exist a point $x_0 \in \mathbb{R}^N$ and a neighborhood ω of $\Gamma(x_0)$ such that for $i = 1, 2, \ldots, N$,*

$$\hat{f}^i(x) \geq \varepsilon_0 > 0 \quad \text{on } \omega \quad \text{and} \quad \int_\omega \frac{1}{(\hat{f}^i(x))^{p_i}}dx < +\infty \tag{8.5.20}$$

for some $0 < p_i < 1$. Furthermore, assume that $\hat{f}^i(x)$ $(i = 1, 2, \ldots, N)$ belong to $C^{m-1}(\overline{\Omega})$ and (ϕ^0, ϕ^1) satisfies the compatibility condition of the m order with m satisfying

$$m > \frac{N}{2}.$$

Then the solution $\phi(x,t)$ of problem (8.5.1)–(8.5.3) satisfies the decay property: for $h = 0, 1, 2, \ldots, m$,

$$E_h(t) \leq \Big(E_h(0)^{-N/2mp} + C(t-T)^+\Big)^{-2mp/N}, \tag{8.5.21}$$

where $0 \leq t < +\infty$ with some time $T > 0$ independent of (ϕ^0, ϕ^1), and $\alpha^+ = \max\{\alpha, 0\}$.

Proof. In order to prove this theorem, we need Theorem 2.3.14 and the techniques in [677, 1028].

First, multiplying the equation (8.5.1) by ϕ_i' and integrating the result on $[t, t+T] \times \Omega,\ t > 0, T > 0$, we have

$$\int_t^{t+T}\int_\Omega \hat{f}^i\phi_i'\phi_i'dxds = E_0(t) - E_0(t+T) \equiv D^2(t). \tag{8.5.22}$$

Next, multiplying the equation (8.5.1) by ϕ_i and integrating the result, we arrive at

$$\begin{aligned}&\int_t^{t+T}\int_\Omega (A_{ijkl}\phi_{i,j}\phi_{k,l} - \phi_i'\phi_i')dxds \\ &\quad = -\int_t^{t+T}\int_\Omega \hat{f}^i\phi_i'\phi_i dxds - (\phi_i'(t+T), \phi_i(t+T)) + (\phi_i'(t), \phi_i(t))\end{aligned} \tag{8.5.23}$$

where $(\cdot,\cdot)$ denotes the inner product in L^2.

We shall derive the inequality

$$\int_t^{t+T} E(s)ds \le C\left\{E(t+T) + D^2(t) + \int_t^{t+T}\int_\omega (\phi_i'\phi_i' + \phi_i\phi_i)dxds\right\}. \tag{8.5.24}$$

Multiplying the equation (8.5.1) by $q_\mu\phi_{i,\mu}$, and integrating the result, we obtain

$$\begin{aligned}\int_t^{t+T}\int_\Omega \phi_i'' q_\mu\phi_{i,\mu}dxds - \int_t^{t+T}\int_\Omega [A_{ijkl}\phi_{k,l}]_{,j} q_\mu\phi_{i,\mu}dxds \\ = -\int_t^{t+T}\int_\Omega \hat{f}^i\phi_i' q_\mu\phi_{i,\mu}dxds.\end{aligned} \tag{8.5.25}$$

Thus from (8.5.4), we can derive

$$(A_{ijkl}\phi_{k,l}\phi_{i,j})_{,\mu} = A_{ijkl,\mu}\phi_{k,l}\phi_{i,j} + 2A_{ijkl}\phi_{k,l}\phi_{i,j\mu} \tag{8.5.26}$$

or

$$A_{ijkl}\phi_{k,l}\phi_{i,j\mu} = \frac{1}{2}\Big[(A_{ijkl}\phi_{kl}\phi_{i,j})_{,\mu} - A_{ijkl,\mu}\phi_{k,l}\phi_{i,j}\Big]. \tag{8.5.27}$$

By (8.5.27), we arrive at

$$\begin{aligned}&\int_t^{t+T}\int_\Omega [A_{ijkl}\phi_{k,l}]_{,j} q_\mu\phi_{i,\mu}dxds \\ &\quad = \int_t^{t+T}\int_{\partial\Omega} A_{ijkl}\nu_j q_\mu\phi_{k,l}\phi_{i,j}d\sigma ds \\ &\qquad - \int_t^{t+T}\int_\Omega (A_{ijkl}q_{\mu,j}\phi_{k,l}\phi_{i,\mu} + A_{ijkl}\phi_{k,l}q_\mu\phi_{i,j\mu})dxds \\ &\quad = \int_t^{t+T}\int_{\partial\Omega} A_{ijkl}\nu_j q_\mu\phi_{k,l}\phi_{i,j}d\sigma ds - \int_t^{t+T}\int_\Omega A_{ijkl}q_{\mu,j}\phi_{k,l}\phi_{i,\mu}dxds \\ &\qquad - \frac{1}{2}\int_t^{t+T}\int_\Omega \{(A_{ijkl}\phi_{k,l}\phi_{k,l})_{,\mu} - A_{ijkl,\mu}\phi_{i,j}\phi_{k,l}\}dxds\end{aligned}$$

$$
\begin{aligned}
&= \int_t^{t+T}\int_{\partial\Omega} A_{ijkl}\nu_j q_\mu \phi_{k,l}\phi_{i,j} d\sigma ds - \int_t^{t+T}\int_\Omega A_{ijkl} q_{\mu,j}\phi_{k,l}\phi_{i,\mu} dxds \\
&\quad - \frac{1}{2}\int_t^{t+T}\int_{\partial\Omega} A_{ijkl}\phi_{k,l}\phi_{i,j}\nu_\mu q_\mu d\sigma ds + \frac{1}{2}\int_t^{t+T}\int_\Omega A_{ijkl}\phi_{k,l}\phi_{i,j} q_{\mu,\mu} dxds \\
&\quad - \frac{1}{2}\int_t^{t+T}\int_\Omega A_{ijkl,\mu}\phi_{i,j}\phi_{k,l} dxds. \qquad (8.5.28)
\end{aligned}
$$

Noting that $\phi|_{\partial\Omega} = 0$, it holds that on $\partial\Omega$

$$\nu_j \phi_{i,\mu} = \nu_\mu \phi_{i,j}. \qquad (8.5.29)$$

By (8.5.25), (8.5.28) and (8.5.29), we deduce

$$
\begin{aligned}
&\frac{N}{2}\int_t^{t+T}\int_\Omega (\phi_i'\phi_i' - A_{ijkl}\phi_{i,j}\phi_{k,l}) dxds + \int_t^{t+T}\int_\Omega A_{ijkl}\phi_{i,j}\phi_{k,l} dxds \\
&= -\int_t^{t+T}\int_\Omega \hat{f}^i \phi_i' q_\mu \phi_{i,\mu} dxds - (\phi_i'(t+T), q_\mu\phi_{i,\mu}(t+T)) + (\phi_i'(t), q_\mu \phi_{i,\mu}(t)) \\
&\quad + \frac{1}{2}\int_t^{t+T}\int_\Omega A_{ijkl,\mu}\phi_{k,l}\phi_{i,j} q_\mu dxds + \frac{1}{2}\int_t^{t+T}\int_{\partial\Omega} A_{ijkl}\phi_{k,l}\phi_{i,\mu} q_\mu \nu_j d\sigma ds \\
&\quad + \frac{1}{2}\int_t^{t+T}\int_\Omega A_{ijkl}\phi_{k,l}\phi_{i,j} q_{\mu,\mu} dxds. \qquad (8.5.30)
\end{aligned}
$$

Therefore, it follows from (8.5.6), (8.5.23) and (8.5.30) that

$$
\begin{aligned}
&\left(\frac{N}{2} - \gamma\right)\int_t^{t+T}\int_\Omega \phi_i'\phi_i' dxds + \left[\gamma - \frac{2N-(1+\beta)}{2}\right]\int_t^{t+T}\int_\Omega A_{ijkl}\phi_{k,l}\phi_{i,j} dxds \\
&\le C\left(\int_t^{t+T}\int_\Omega \hat{f}^i \phi_i'\phi_i' dxds\right)^{1/2}\left(\int_t^{t+T}\int_\Omega \phi_{i,\mu}\phi_{i,\mu} dxds\right)^{1/2} \\
&\quad + C\{E_0(t) + E_0(t+T)\} + C\int_t^{t+T}\int_{\Gamma(x_0)} A_{ijkl}\phi_{i,\nu_j}\phi_{i,\nu_j} d\sigma ds \qquad (8.5.31)
\end{aligned}
$$

for any constant $\gamma > 0$.

Now taking $\frac{2N-(1+\beta)}{2} < \gamma < \frac{N}{2}$ and using (8.5.22), we can get

$$\int_t^{t+T} E_0(s) ds \le C\{E_0(T+t) + D^2(t)\} + C\int_t^{t+T}\int_{\Gamma(x_0)} A_{ijkl}\phi_{i,\nu_j}\phi_{i,\nu_j} d\sigma ds. \qquad (8.5.32)$$

To estimate the last term on the right-hand side of (8.5.32), we take a function $\eta_i \in C^1(\overline{\Omega})$ such that

$$0 \le \eta_i \le 1, \quad \eta_i = 1 \text{ on } \hat{\omega}, \quad \eta_\mathrm{i} = 0 \text{ on } \Omega\setminus\omega \quad \text{and} \quad \frac{\eta_{\mathrm{i,j}}\eta_{\mathrm{i,j}}}{\sqrt{\eta_\mathrm{i}}\sqrt{\eta_\mathrm{i}}} \in \mathrm{C}(\overline{\Omega}) \qquad (8.5.33)$$

where $\hat{\omega}$ is an open set in $\overline{\Omega}$ with $\Gamma(x_0) \subset \hat{\omega} \subset \omega$.

Now multiplying equation (8.5.1) by $\eta_i\phi_i$ and integrating, we can derive

$$\int_t^{t+T}\int_\Omega A_{ijkl}\phi_{i,j}\phi_{k,l}dxds \\ \leq C\left\{E_0(t)+E_0(t+T)+\int_t^{t+T}\int_\omega(\phi_i'\phi_i'+\phi_i\phi_i)dxds\right\} \tag{8.5.34}$$

where, by the Hölder inequality and (8.5.33), we have used the inequality

$$|(\phi_i,\eta_{i,j}\phi_{i,j})| = |\int_\Omega \phi_i\frac{\eta_{i,j}}{\sqrt{\eta_i}}\sqrt{\eta_i}\phi_{i,j}dx| \leq C\left(\int_\Omega \phi_i\phi_i dx\right)^{1/2}\left(\int_\Omega \sqrt{\eta}_i\phi_{i,j}\phi_{i,j}dx\right)^{1/2}. \tag{8.5.35}$$

Furthermore, we take an open set $\tilde{\omega}$ in $\mathbb{R}^N$ with $\tilde{\omega}\cap\partial\Omega\subset\hat{\omega}$ and C^1 vector field s_i such that $s_i=\nu_i$ *on* $\Gamma(x_0)$, $s_i\cdot\nu_i\geq 0$ *on* $\partial\Omega$ and $s_i=0$ *on* $\Omega\backslash\tilde{\omega}$.

Now multiplying the equation (8.5.1) by $s_i\cdot\phi_{i,j}$ and integrating the result, we can conclude

$$\int_t^{t+T}\int_{\Gamma(x_0)} A_{ijkl}\phi_{i,\nu_j}\phi_{i,\nu_j}dxds \leq \int_t^{t+T}\int_{\partial\Omega} s_i\cdot\nu_i A_{ijkl}\phi_{i,\nu_j}\phi_{i,\nu_j}d\sigma ds \\ \leq C\int_t^{t+T}\int_{\tilde{\omega}}(\phi_i'\phi_i'+A_{ijkl}\phi_{i,j}\phi_{k,l})dxds + C\{E_0(t)+E_0(t+T)\}. \tag{8.5.36}$$

Thus from (8.5.32), (8.5.34) and (8.5.36), it follows readily

$$\int_t^{t+T}E_0(s)ds \leq C\left\{E_0(t+T)+D^2(t)+\int_t^{t+T}\int_\omega(\phi_i'\phi_i'+\phi_i\phi_i)dxds\right\}. \tag{8.5.37}$$

Noting that

$$TE_0(t+T)\leq\int_t^{t+T}E_0(s)ds,$$

we can derive from (8.5.37) that if we take $T>2C$, then we get

$$E_0(t+T)\leq C\left\{D^2(t)+\int_t^{t+T}\int_\omega(\phi_i'\phi_i'+\phi_i\phi_i)dxds\right\}. \tag{8.5.38}$$

We now estimate the last two terms on the right-hand side of (8.5.38). To treat the last term, we need the following inequality.

Lemma 8.5.4 ([787])**.** *For a large* $T>0$*, there exists a constant* $C>0$*, independent of* (ϕ^0,ϕ^1)*, such that the estimate*

$$\int_t^{t+T}\int_\Omega\phi_i(s)\phi_i(s)dxds\leq C\int_t^{t+T}\int_\Omega\hat{f}^i\phi_i'\phi_i'dxds+\int_t^{t+T}\int_\omega\phi_i'\phi_i'dxds \tag{8.5.39}$$

holds for any energy finite solution of problem (8.5.1)–(8.5.3).

Proof. We prove (8.5.39) by the contradiction argument. If (8.5.39) were false, there exist a sequence $\{t_n\}_{n=1}^{\infty}$ and a sequence of solutions $\{\phi_{in}\}_{n=1}^{\infty}$ such that

$$\begin{cases} \int_{t_n}^{t_n+T}\int_\Omega \phi_{in}(s)\phi_{in}(s)dxds = 1, \\ \int_{t_n}^{t_n+T}\int_\Omega \hat{f}^i\phi'_{in}\phi'_{in}dxds + \int_{t_n}^{t_n+T}\int_\omega \phi'_{in}\phi'_{in}dxds \to 0 \text{ as } n\to+\infty. \end{cases} \tag{8.5.40}$$

We note that inequality (8.5.38) remains valid by homogeneity even if we replace $\phi_i(t)$ by $\phi_{in}(t)$.

Thus, setting $\psi_{in}(t) = \phi_{in}(t+t_n),\ 0 < t \le T$, from (8.5.40)–(8.5.41) it follows that

$$\begin{cases} \int_0^T\int_\Omega \psi_{in}(s)\psi_{in}(s)dxds = 1, & (8.5.41) \\ M_n^2 \equiv \int_0^T\int_\Omega \hat{f}^i\psi'_{in}(s)\psi'_{in}(s)dxds \\ \qquad + \int_0^T\int_\omega \psi'_{in}(s)\psi'_{in}(s)dxds \to 0 \text{ as } n\to+\infty & (8.5.42) \end{cases}$$

and by (8.5.22) and (8.5.38),

$$\begin{aligned} &\sup_{0\le t\le T}\Big\{\int_\Omega (\psi'_{in}(s)\psi'_{in}(s) + \psi_{in,j}(s)\psi_{in,j}(s))dx\Big\} = 2E_0(\psi_{in}(0)) \\ &= 2\Big\{E_0(\psi_{in}(T)) + \int_0^T\int_\Omega \hat{f}^i\psi'_{in}(s)\psi'_{in}(s)dxds\Big\} \\ &\le C\{M_n^2+1\} \le 2C < +\infty \end{aligned} \tag{8.5.43}$$

for large n, where $C > 0$ is a constant independent of (ϕ^0, ϕ^1).

Therefore, $\{\psi_{in}(t)\}$ converges along a subsequence to a function $\psi_i(t) \in C([0,T]; H_0^1) \cap C^1([0,T]; L^2)$ in appropriate topologies, which is a solution of the problem

$$\begin{cases} \psi''_i - [A_{ijkl}\psi_{i,l}]_{,j} = 0 \text{ in } \Omega\times[0,\mathrm{T}], \\ \psi|_{\partial\Omega} = 0. \end{cases}$$

Thus by (8.5.42)–(8.5.43), we conclude,

$$\begin{cases} \int_0^T\int_\Omega \hat{f}^i\psi'_i\psi'_i dxds = 0, & (8.5.44) \\ \int_0^T\int_\Omega \psi_i\psi_i dxds = 1. & (8.5.45) \end{cases}$$

This is a contradiction if we take a large $T > 0$ ($T > d(\Omega)$, diameter of Ω, is sufficiently large), because the condition (8.5.44) implies $\psi_i \equiv 0$ for a solution of the elastic system above.

Now, by (8.5.38) and Lemma 8.5.1, we have

$$E_0(t+T) \le C\left\{D^2(t) + \int_t^{t+T}\int_\omega \phi_i'\phi_i' dxds\right\}. \tag{8.5.46}$$

Finally, by the assumption on $\hat{f}^i(x)$ and Lemma 8.5.1, we can obtain

$$\begin{aligned}
&\int_t^{t+T}\int_\omega \phi_i'\phi_i' dxds \\
&\quad \le \left\{\int_t^{t+T}\int_\omega \hat{f}^i\phi_i'\phi_i' dxds\right\}^{p_i/(p_i+1)} \left\{\int_t^{t+T}\int_\omega \hat{f}^i(x)^{-p_i} dxds\right\}^{1/(p_i+1)} \\
&\qquad \times \sum_{i=1}^N \sup_{t\le s\le t+T} ||\phi_i'(s)||_{L^\infty}^{2/(p_i+1)} \\
&\quad \le \sum_{i=1}^N CD(t)^{2p_i/(p_i+1)} \sup_{t\le s\le t+T} ||\phi_i'(s)||^{2(1-(N/2m))/(p_i+1)} ||\phi_i'(s)||_{H^m}^{N/(m(p_i+1))} \\
&\quad \le \sum_{i=1}^N CD(t)^{2p_i/(p_i+1)} E_0(t)^{(2m-N)/2m(p_i+1)} \equiv A^2(t).
\end{aligned} \tag{8.5.47}$$

Thus we may derive from (8.5.46) that

$$E_0(t+T) \le C[D^2(t) + A^2(t)]$$

which, together with identity (8.5.22), implies

$$E_0(t) \le C[D^2(t) + A^2(t)].$$

Thus, recalling the definition of $A^2(t)$ and using Young's inequality, we arrive at

$$E_0(t) \le CD^2(t) + CD(t)^{4mp/(2mp+N)}, \tag{8.5.48}$$

or

$$E_0(t)^{1+\frac{N}{2mp}} \le C\Big\{E_0(t) - E_0(t+T)\Big\}, \tag{8.5.49}$$

where $p = \min\{p_1, \dots, p_N\} > 0$.

Now, applying Theorem 2.3.14 to inequality (8.5.49), we obtain the decay estimate

$$E_0(t) \le \Big\{E_0(0)^{-N/2mp} + C(t-T)^+\Big\}^{-2mp/N} \tag{8.5.50}$$

for $0 \le t < +\infty$ with some $T > 0$ independent of (ϕ^0, ϕ^1), where we have used the notation $\alpha^+ = \max\{\alpha, 0\}$.

Similarly, keeping in mind that equations (8.5.1)–(8.5.3) are all linear in t, we have that for $h = 0, 1, \ldots, N$,

$$E_h(t) \leq \left\{ E_h(0)^{-N/2mp} + C(t-T)^+ \right\}^{-2mp/N}. \tag{8.5.51}$$

The proof is thus complete. □

□

8.6 Stabilization of weakly coupled wave equations

In this section, we shall use Theorem 1.5.12 to study the stabilization of weakly coupled wave equations. These results are chosen from Alabau, Cannarsa and Komornik [19].

In this section, we shall study the stability of the system

$$\begin{cases} u_{tt} + A_1 u + B u_t + \alpha v = 0, \\ v_{tt} + A_2 v + \alpha u = 0, \end{cases} \tag{8.6.1}$$

in a separable real Hilbert space H with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$, where A_1, A_2 and B are self-adjoint positive linear operators in H. In addition, we also assume that B is a bounded operator. By $\mathcal{L}(H)$ we shall denote the Banach algebra of all bounded linear operators $B : H \to H$ equipped with the usual norm

$$\|B\| = \sup\{|Bx| : x \in H, |x| \leq 1\}.$$

We shall use the notation $A : D(A) \subset H \to H$ for any linear operator on H with domain $D(A)$.

We now recall basic notions of semigroup theory, a classical topic in functional analysis (see, e.g., [235]). Let $A : D(A) \subset H \to H$ be the infinitesimal generator of a C_0-semigroup of bounded linear operators on H denoted by $\{e^{tA}\}_{t\geq 0}$, or, e^{tA}. It is well known that the H-valued function $U(t) := e^{tA}x$ is continuous for any $x \in H$, of class $C^1([0, +\infty), H)$ for any $x \in D(A)$, and that, in the latter case, U solves the Cauchy problem

$$\begin{cases} U'(t) = AU(t), & \text{for all } t \geq 0, \\ U(0) = x. \end{cases} \tag{8.6.2}$$

Moreover, for any $k \in \mathbb{N}$,

$$x \in D(A^k) \Rightarrow U^{(k)}(t) = e^{tA}A^k x = A^k e^{tA} x.$$

We also recall that a semigroup e^{tA} is said to be strongly stable if for all $x \in H$,

$$\lim_{t \to +\infty} e^{tA} x = 0.$$

If there exist two constants $M, \omega > 0$ such that for all $t \geq 0$,

$$\|e^{tA}\| \leq Me^{-\omega t},$$

then e^{tA} is called exponentially stable.

Now we consider the weakly coupled system of second-order evolution equations in a Hilbert space H,

$$\begin{cases} u_{tt} + A_1 u + Bu_t + \alpha v = 0, \\ v_{tt} + A_2 v + \alpha u = 0, \end{cases} \tag{8.6.3}$$

and assume the following condition hold:

(H1) For $i = 1, 2, A_i : D(A_i) \subset H \to H$ is a densely defined closed linear operator such that

$$A_i = A_i^*, \qquad \langle A_i x, x\rangle \geq \omega_i |x|^2, \text{ for all } x \in D(A_i), \tag{8.6.4}$$

for some constant $\omega_i > 0,\ i = 1, 2$.

(H2) B is a bounded linear operator on H such that

$$B = B^*, \qquad \langle Bx, x\rangle \geq \beta |x|^2, \qquad \text{for all } x \in H, \tag{8.6.5}$$

for some constant $\beta > 0$.

(H3) The parameter α is a real number such that

$$0 < |\alpha| < \sqrt{\omega_1 \omega_2}. \tag{8.6.6}$$

Thus we can rewrite system (8.6.3) with the initial conditions,

$$\begin{cases} u(0) = u^0, \quad u_t(0) = u^1, \\ v(0) = v^0, \quad v_t(0) = v^1, \end{cases} \tag{8.6.7}$$

as an abstract Cauchy problem of type (8.6.2) in a standard way with the product space

$$\mathcal{H} := D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H.$$

We can also write an element $U \in \mathcal{H}$ as

$$U = (u, p, v, q), \quad \text{where} \quad u \in D(A_1^{1/2}), \quad v \in D(A_2^{1/2}), \qquad p,\, q \in H.$$

Thus it follows from assumption (8.6.4) that $\mathcal{H}$ is a Hilbert space with the scalar product

$$(U|\hat{U}) := \langle A_1 u, \hat{u}\rangle + \langle p, \hat{p}\rangle + \langle A_2 v, \hat{v}\rangle + \langle q, \hat{q}\rangle, \text{ for all } U, \hat{U} \in \mathcal{H}.$$

Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be the operator defined as follows

$$D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}),$$
$$\mathcal{A}U = (p, -A_1 u - Bp - \alpha v, q, -A_2 v - \alpha u), \qquad \text{for all } U \in D(\mathcal{A}).$$

Then problem (8.6.3) takes the equivalent form

$$U'(t) = \mathcal{A}U(t), \qquad U(0) = U_0, \tag{8.6.8}$$

where $U_0 = (u^0, u^1, v^0, v^1)$.

On the other hand, from the classical results (see, e.g., [740]) it follows that $\mathcal{A}$ generates a C_0-semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$. In fact, $\mathcal{A}$ is a bounded perturbation of a maximal dissipative operator. It is also easy to check that $e^{t\mathcal{A}}U_0 = (u(t), p(t), v(t), q(t))$, where the pair (u, v) is the solution of system (8.6.3) with initial conditions (8.6.7).

In general, we do not know how to characterize the domains of $\mathcal{A}^n$ for large n. However, under suitable assumptions, we can show that the subspace of $\mathcal{H}$ given by for all $n \geq 0$,

$$\mathcal{H}_n = D(A_1^{(n+1)/2}) \times D(A_1^{n/2}) \times D(A_2^{(n+1)/2}) \times D(A_2^{n/2}),$$

is contained in, and sometimes equal to, $D(\mathcal{A}^n)$.

In order to apply Theorem 1.5.12, we need the following lemmas, which have been proved in [19].

Lemma 8.6.1 ([19]). *Assume that* (8.6.4) *and* (8.6.5) *hold. Let* $n \geq 1$ *be such that*

$$\begin{cases} BD(A_1^{(k+1)/2}) \subseteq D(A_1^{k/2}), & (8.6.9) \\ D(A_1^{(k/2)+1}) \subseteq D(A_2^{k/2}), & (8.6.10) \\ D(A_2^{(k/2)+1}) \subseteq D(A_1^{k/2}), & (8.6.11) \end{cases}$$

for every integer k *satisfying* $0 < k \leq n-1$ (*no assumption if* $n = 1$). *Then* $\mathcal{H}_k \subset D(\mathcal{A}^k)$ *for every* $0 \leq k \leq n$.

Remark 8.6.1 ([19]). In the same manner, we may derive the equality

$$\mathcal{H}_k = D(\mathcal{A}^k), \;\; 0 \leq k \leq n,$$

provided that conditions (8.6.10) and (8.6.11) are replaced by the stronger assumptions

$$\begin{cases} D(A_1^{(k+1)/2}) \subseteq D(A_2^{k/2}), \\ D(A_2^{(k+1)/2}) \subseteq D(A_1^{k/2}), \quad 0 < k \leq n-1. \end{cases}$$

Notice that the above assumptions hold trivially whenever $A_1 = A_2$.

We recall that the energies associated with operators A_1, A_2 are given by

$$E_i(u,p) = \frac{1}{2}(|A_i^{1/2}u|^2 + |p|^2), \qquad \text{for all } (u,p) \in D(A_i^{1/2}) \times H \quad (i = 1,2).$$

Now we also define the total energy of the system as

$$\mathcal{E}(U) := E_1(u,p) + E_2(v,q) + \alpha\langle u, v\rangle. \tag{8.6.12}$$

thus, from assumption (H1) it follows that, for $i = 1,2$, for all $u \in D(A_i^{1/2})$, for all $p \in H$,

$$|u|^2 \leq \frac{2}{\omega_i} E_i(u,p). \tag{8.6.13}$$

Applying the Cauchy–Schwarz inequality, it follows that under hypotheses (H1) and (H3), the total energy $\mathcal{E}$ controls the energies of the components: if

$$U = (u,p,v,q) \in D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H,$$

then

$$\mathcal{E}(U) \geq v(\alpha)\Big[E_1(u,p) + E_2(v,q)\Big] \tag{8.6.14}$$

with $v(\alpha) = 1 - |\alpha|(\omega_1\omega_2)^{-1/2} > 0$.

Obviously, the following lemma shows that (8.6.8) is a dissipative system.

Lemma 8.6.2 ([19])**.** *Assume that* (8.6.4)–(8.6.5) *hold and let* $U = (u,p,v,q) = (u,u_t,v,v_t)$ *be the solution of problem* (8.6.8) *with* $U_0 \in D(\mathcal{A})$*. Then for all* $t \geq 0$*,*

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u_t(t)|^2. \tag{8.6.15}$$

In particular, $t \mapsto \mathcal{E}(U(t))$ *is non-increasing on* $[0,+\infty)$*.*

In order to apply the polynomial decay criterion of Theorem 1.5.12 to problem (8.6.8), we have to bound the integral of the total energy of U, on any time interval $[0,T]$, by a linear combination of the energies of the derivatives of U at 0. The next lemma is the first step towards such a final goal.

Lemma 8.6.3 ([19])**.** *Assume that* (8.6.4)–(8.6.6) *hold and let* $U = (u,u_t,v,v_t)$ *be the solution of problem* (8.6.8) *with* $U_0 \in D(\mathcal{A})$*. Then for some constant* $c \geq 0$ *and every* $T \geq 0$*,*

$$\int_0^T \mathcal{E}(U(t))dt \leq \int_0^T |v_t(t)|^2 dt + c\mathcal{E}(U(0)). \tag{8.6.16}$$

We note that the above result shows that the main technical difficulty encountered here is to control the integral term $\int_0^T |v_t|^2 dt$ by the total energy of U (and of a finite number of its derivatives) at 0. As simple as it may appear, such an estimate cannot be taken for granted as there is no direct dissipation term in the second equation of (8.6.3). In fact, for its validity, we shall need an extra assumption on the problem and a delicate iteration argument.

As we have already noted, in addition to assumptions (H1)–(H3), we need to impose further restrictions on the data, that is, we shall assume that, for some integer $j \geq 2$, and for all $u \in D(A_2^{j/2})$,

$$|A_1 u| \leq c|A_2^{j/2} u|. \tag{8.6.17}$$

Remark 8.6.2 ([19])**.** In fact, condition (8.6.17) can be rewritten equivalently as a domain inclusion such that for all $u \in H$,

$$D(A_2^{j/2}) \subseteq D(A_1), \qquad |A_1 A_2^{-j/2} u| \leq c|u|. \tag{8.6.18}$$

Theorem 8.6.4 ([19])**.** *Assume that* (8.6.4)–(8.6.6) *and* (8.6.17) *hold.*

(i) *If* $U_0 \in D(\mathcal{A}^{nj})$ *for some integer* $n \geq 1$*, then the solution* U *of problem* (8.6.8) *satisfies for all* $t > 0$*,*

$$\mathcal{E}(U(t)) \leq \frac{c_n}{t^n} \sum_{k=0}^{nj} \mathcal{E}(U^{(k)}(0)), \tag{8.6.19}$$

for some constant $c_n > 0$*.*

(ii) *For every* $U_0 \in \mathcal{H}$*, we have as* $t \to +\infty$*,*

$$\mathcal{E}(U(t)) \to 0.$$

Proof. Obviously, part (ii) of this theorem follows readily from (i) by using the density of $D(\mathcal{A}^j)$ in $\mathcal{H}$. Furthermore, by Theorem 1.5.15, part (i) will follow if we have established the estimate

$$\int_0^T \mathcal{E}(U(t))dt \leq c \sum_{k=0}^{j} \mathcal{E}(U^{(k)}(0)) \tag{8.6.20}$$

for every fixed $T > 0$, with a constant $c > 0$ depending only on α. Thus the rest of this subsection shall prove (8.6.20) for which we need some intermediate steps presented below as separate lemmas without recalling the standing assumptions (8.6.4)–(8.6.6) and (8.6.17). The reader should note that all constants, labeled c, may depend on α but not on T and blow up as $|\alpha| \downarrow 0$ or $|\alpha| \uparrow \sqrt{\omega_1 \omega_2}$. □

8.7 Energy decay rates of nonlinear dissipative hyperbolic systems

In this section, we shall use Theorem 1.5.9 to study the energy decay rates of nonlinear hyperbolic systems by a nonlinear feedback which can be localized on a part of the boundary or locally distributed. We adopt these results from Alabau [18].

We shall consider the following second-order equation

$$\begin{cases} u_{tt}(t)(\cdot) + Au(t)(\cdot) + B(\cdot, u_t(t)(\cdot)) = 0, \quad t > 0, & (8.7.1) \\ u(0)(\cdot) = u^0(\cdot), \ \ u_t(\cdot) = u^1(\cdot) & (8.7.2) \end{cases}$$

where A is a coercive self-adjoint densely defined linear unbounded operator in H, with domain $D(A)$, and Ω is a bounded open subset of $\mathbb{R}^N$ with a smooth boundary denoted by Γ and ω is an open subset of Ω of positive measure.

Let $H = L^2(\Omega)$ and by $|\cdot|_H$ we denote the L^2-norm on Ω. Note that the above abstract equation can include the Petrovsky equation or the system of linear elasticity. Let $V = D(A^{1/2})$. The operator B is the monotone continuous operator defined from $\overline{\Omega} \times H$ on H by

$$B(\cdot, v) = \rho(\cdot, v). \tag{8.7.3}$$

We assume the feedback function ρ satisfies the following hypotheses:

(H) $\rho \in C(\overline{\Omega} \times \mathbb{R})$ and is monotone increasing with respect to the second variable, there exist a function $a \in C(\overline{\Omega})$, $a \geq 0$ on Ω, and a strictly increasing function $g \in C^1(\mathbb{R})$ such that

$$\begin{cases} a(x)|v| \leq |\rho(x,v)| \leq Ca(x)|v|, \ \text{ for all } x \in \Omega, \ \ if \ \ |v| \geq 1, & (8.7.4) \\ a(x)g(|v|) \leq |\rho(x,v)| \leq Ca(x)g^{-1}(|v|), \ \text{ for all } x \in \Omega, \ \ if \ \ |v| \leq 1, & (8.7.5) \\ a(x) \geq a_- > 0, \quad \text{for all } x \in \omega, & (8.7.6) \end{cases}$$

where g^{-1} denotes the inverse function of g and where C is a positive constant.

We recall the following classical existence and regularity result (see, e.g., [479] and [350] for the proof) using the theory of the maximal nonlinear monotone operator.

Theorem 8.7.1 ([18])**.** *Assume hypothesis* **(H)** *holds. Then for all* $(u^0, u^1) \in V \times H$, *the problem* (8.7.1)–(8.7.2) *has a unique solution*

$$u \in C([0,+\infty), V) \cap C^1([0,+\infty), H).$$

Moreover, for all $(u^0, u^1) \in D(A) \times V$, *the solution* u *of problem* (8.7.1)–(8.7.2) *belongs to the class* $L^\infty([0,+\infty), D(A)) \cap W^{1,\infty}([0,+\infty), V) \cap W^{2,\infty}([0,+\infty), H)$ *and its energy, defined by*

$$E(t) = \frac{1}{2}\Big(|u_t(t)|_H^2 + |A^{1/2}u(t)|_H^2\Big), \tag{8.7.7}$$

satisfies the following dissipation relation

$$E'(t) = -\int_\Omega u_t(t,x)\rho(x, u_t(t,x))dx \leq 0. \tag{8.7.8}$$

In this section, we shall prove the following main result, due to Alabau [18], by making use of Theorem 1.5.9.

Theorem 8.7.2 ([18]). *Assume hypothesis* **(H)** *holds. Assume that there exists a constant* $r_0 \in (0,1)$ *with* $g(r_0) < 1$, *such that* $g \in C^2([0, r_0])$ *and the function* H *defined by* (1.5.170) *is strictly convex on* $[0, r_0^2]$. *Let* δ_i *for* $i = 1, 2, 3$ *and* $(u^0, u^1) \in D(A) \times V$, *satisfying* $0 < |u^1|_H^2 + |A^{1/2}u^0|_H^2$, *be given and let* f *be the non-negative* C^1 *and strictly increasing function defined from* $[0, r_0^2)$ *onto* $[0, +\infty)$ *by, for all* $s \in [0, 2\beta r_0^2)$,

$$f(s) = F^{-1}\left(\frac{s}{2\beta}\right), \tag{8.7.9}$$

where F *is given by* (1.5.173) *and where* $\beta = \beta_{E(0)}$ *depends on* $E(0)$ *in the following manner:*

$$\beta_{E(0)} = \max\{\eta_1, \eta_2 E(0)\} \tag{8.7.10}$$

where η_1 *and* η_2 *are independent of* $E(0)$. *We assume that the energy* $E(t)$ *defined by* (8.7.7) *associated to the solution of problem* (8.7.1)–(8.7.2) *satisfies*

$$\int_S^T f(E(t))E(t)dt \le \delta_1 E(S) f(E(S)) + \delta_2 \int_S^T f(E(t)) \left(\int_\Omega |\rho(x, u_t(t)(x))|^2 dx\right) dt$$
$$+ \delta_3 \int_S^T f(E(t)) \left(\int_\omega |u_t(t)(x)|^2 dx\right) dt. \tag{8.7.11}$$

Then $E(t)$ *satisfies the estimate for all* $t \ge T_0/H'(r_0^2)$,

$$E(t) \le 2\beta_{E(0)} z^2(t) \frac{z(t)g'(z(t)) - g(z(t))}{z(t)g'(z(t)) + g(z(t))}, \tag{8.7.12}$$

where z *is given by*

$$z(t) = \phi^{-1}(t/T_0) \tag{8.7.13}$$

with ϕ *being the strictly decreasing and onto function defined from* $(0, r_0]$ *onto* $[1/H'(r_0^2), +\infty)$ *by*

$$\phi(v) = \frac{2v}{vg'(v) + g(v)} + 4\alpha(v) \tag{8.7.14}$$

where α *is defined on* $(0, r_0]$ *by the following integral expression*

$$\alpha(\tau) = \int_\tau^{r_0} \frac{g(u)(u^2 g''(u) + ug'(u) - g(u))}{(ug'(u) + g(u))^2 (ug'(u) - g(u))} du. \tag{8.7.15}$$

Proof. Let $\varepsilon_0 = g(r_0)$. From the assumption on r_0, we have $0 < \varepsilon_0 < 1$. Thus from the hypothesis **(H)**, it follows that for all $x \in \Omega$ if $|v| \le \varepsilon_0$,

$$a(x)g(|v|) \le |\rho(x, v)| \le Ca(x)g^{-1}(|v|).$$

For $\varepsilon_0 \le |v| \le 1$, we obtain, noting that g^{-1} is increasing on $\mathbb{R}$,

$$r_0 \le g^{-1}(|v|) \le g^{-1}(1).$$

Hence, using (H), we get for all $x \in \Omega$ and for all $\varepsilon_0 \le |v| \le 1$,

$$|\rho(x,v)| \le Ca(x)\frac{g^{-1}(|v|)}{|v|}|v| \le Ca(x)\frac{g^{-1}(1)}{\varepsilon_0}|v|,$$

and

$$|\rho(x,v)| \ge a(x)\frac{g(|v|)}{|v|}|v| \ge a(x)g(\varepsilon_0)|v|.$$

Hence ρ satisfies the following inequalities for all $x \in \Omega$ and for all $|v| \le \varepsilon_0$,

$$c_1 a(x)|v| \le |\rho(x,v)| \le c_2 a(x)|v|, \tag{8.7.16}$$

and for all $x \in \Omega$, for all $|v| \le \varepsilon_0$,

$$c_1 a(x) g(|v|) \le |\rho(x,v)| \le c_2 a(x) g^{-1}(|v|). \tag{8.7.17}$$

Let, for all fixed $t \ge 0, \Omega_1^t = \{x \in \Omega : |u_t(t)(x)| \le \varepsilon_0\}$, and

$$c_g = \frac{1}{c_2\|a\|_{L^\infty}}. \tag{8.7.18}$$

Thus, using the definition of c_g and (8.7.17), we can derive for all $x \in \Omega_1^t$,

$$c_g^2|\rho(x,u_t(t)(x))|^2 \le r_0^2.$$

Hence, noting that

$$\frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g^2|\rho(x,u_t(t)(x))|^2dx \in [0,r_0^2],$$

which is the domain of convexity of H, and by Jensen's inequality, we deduce

$$\begin{aligned} &H\left(\frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g^2|\rho(x,u_t(t)(x))|^2dx\right)\\ &\quad\le \frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} H(c_g^2|\rho(x,u_t(t)(x))|^2)dx \\ &\quad\le \frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g|\rho(x,u_t(t)(x))|g(c_g|\rho(x,u_t(t)(x))|)dx. \end{aligned} \tag{8.7.19}$$

However, from (8.7.17), we can infer that on Ω_1^t,

$$c_g|\rho(x,u_t(t)(x))| \le g^{-1}(|u_t(t)(x)|).$$

Hence, noting that g is increasing, we may obtain

$$g(c_g|\rho(x,u_t(t)(x))|) \le |u_t(t)(x)| \quad \text{on } \Omega_1^{\mathrm{t}}$$

which, together with (8.7.19), implies

$$H\left(\frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g^2|\rho(x,u_t(t)(x))|^2dx\right) \le \frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g|u_t(t)(x)\rho(x,u_t(t)(x))|dx. \tag{8.7.20}$$

On the other hand, using (8.7.17), we may obtain

$$\frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g|u_t(t)(x)\rho(x,u_t(t)(x))|dx \le \frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} \varepsilon_0 g^{-1}(\varepsilon_0)dx = H(r_0^2) \tag{8.7.21}$$

which yields

$$H^{-1}\left(\frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g u_t(t)(x)\rho(x,u_t(t)(x))dx\right) \in [0,r_0^2]. \tag{8.7.22}$$

Now let f be an arbitrary non-negative strictly increasing function defined from $[0,\eta)$ onto $[0,+\infty)$. Then, it follows from (8.7.20) that

$$\begin{aligned}&\int_S^T f(E(t))\int_{\Omega_1^t}|\rho(x,u_t(u)(x))|^2dxdt\\ &\le \int_S^T \frac{|\Omega_1^t|}{c_g^2}f(E(t))H^{-1}\left(\frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g u_t(t)(x)\rho(x,u_t(t)(x))dx\right)dt.\end{aligned} \tag{8.7.23}$$

We now define $\hat{H}$ as in (1.5.171). Then $\hat{H}$ is a convex and proper function. Hence, we may apply Young's inequality to any numbers A and B in $\mathbb{R}$, that is,

$$AB \le \hat{H}^*(A) + \hat{H}(B). \tag{8.7.24}$$

We can thus apply the above inequality (8.7.24) to $A = A(t) = f(E(t))$ and

$$B = B(t) = H^{-1}\left(\frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g u_t(t)(x)\rho(x,u_t(t)(x))dx\right).$$

Since $B(t) \in [0,r_0^2]$, and using (8.7.8), we readily deduce

$$\begin{aligned}&\frac{|\Omega_1^t|}{c_g^2}f(E(t))H^{-1}\left(\frac{1}{|\Omega_1^t|}\int_{\Omega_1^t} c_g u_t(t)(x)\rho(x,u_t(t)(x))dx\right)\\ &\quad\le \frac{|\Omega_1^t|}{c_g^2}\hat{H}^*(f(E(t))) + \frac{1}{c_g}\int_{\Omega_1^t} u_t(t)(x)\rho(x,u_t(t)(x))dx\\ &\quad\le \frac{|\Omega|}{c_g^2}\hat{H}^*(f(E(t))) + \frac{1}{c_g}(-E'(t)).\end{aligned} \tag{8.7.25}$$

Inserting (8.7.25) into (8.7.23) gives us, for all $0 \le S \le T$,

$$\begin{aligned}&\int_S^T f(E(t)) \int_{\Omega_1^t} |\rho(x, u_t(t)(x))|^2 dxdt \\ &\quad \le \frac{|\Omega|}{c_g^2} \int_S^T \hat{H}^*(f(E(t)))dt + \frac{1}{c_g} E(S).\end{aligned} \tag{8.7.26}$$

We also note $|u_t(t)| \ge \varepsilon_0$ on $\Omega \backslash \Omega_1^t$. Hence, from (8.7.16) it follows that for all $x \in \Omega \backslash \Omega_1^t$,

$$|\rho(x, u_t(t)(x))|^2 \le \frac{1}{c_g} u_t(t)(x)\rho(x, u_t(t)(x)), \tag{8.7.27}$$

which further implies

$$\int_S^T f(E(t)) \int_{\Omega \backslash \Omega_1^t} |\rho(x, u_t(t)(x))|^2 dxdt \le \frac{1}{c_g} \int_S^T (-E'(t)) f(E(t))dt. \tag{8.7.28}$$

To estimate the term $\int_S^T f(E(t)) \int_\omega |u_t(t)|^2 dxdt$, we set

$$r_1^2 = H^{-1}(c_1 a_- c_g H(r_0^2)) \tag{8.7.29}$$

and

$$\varepsilon_1 = \min\Big(r_0, g(r_1)\Big). \tag{8.7.30}$$

Then we have $\varepsilon_1 \le \varepsilon_0$.

We now define, for fixed $t \ge 0$, the set $\omega_1^t = \{x \in \omega : |u_t(t)(x)| \le \varepsilon_1\}$. Using **(H)**, we have $a(x) \ge a_-$ for all $x \in \omega$. Thus, using (8.7.17), we may obtain for all $x \in \omega_1^t$,

$$u_t(t)(x)g(u_t(t)(x)) \le \frac{1}{c_1 a_-} u_t(t)(x)\rho(x, u_t(t)(x)). \tag{8.7.31}$$

On the other hand, noting that

$$\frac{1}{|\omega_1^t|} \int_{\omega_1^t} |u_t(t)(x)|^2 dx \le |\varepsilon_1|^2 \le r_0^2,$$

and, using Jensen's inequality together with (8.7.31), we conclude

$$\begin{aligned}H\left(\frac{1}{|\omega_1^t|} \int_{\omega_1^t} |u_t(t)(x)|^2 dx\right) &\le \frac{1}{|\omega_1^t|} \int_{\omega_1^t} H(|u_t(t)(x)|^2) dx \\ &\le \frac{1}{|\omega_1^t|} \int_{\omega_1^t} u_t(t)(x) g(u_t(t)(x)) dx \\ &\le \frac{1}{|\omega_1^t| c_1 a_-} \int_{\omega_1^t} u_t(t)(x)\rho(x, u_t(t)(x)) dx.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_S^T f(E(t))\int_{\omega_1^t}|u_t(t)(x)|^2dxdt\\ \le \int_S^T |\omega_1^t|f(E(t))H^{-1}\left(\frac{1}{|\omega_1^t|c_1a_-}\int_{\omega_1^t}u_t(t)(x)\rho(x,u_t(t)(x))dx\right)dt.\end{aligned} \tag{8.7.32}$$

Setting $A = A(t) = f(E(t))$ and

$$B = B(t) = H^{-1}\left(\frac{1}{|\omega_1^t|c_1a_-}\int_{\omega_1^t}u_t(t)(x)\rho(x,u_t(t)(x))dx\right),$$

and by the choice of $\varepsilon_1 > 0$, we can derive

$$\begin{aligned}\frac{1}{|\omega_1^t|c_1a_-}\int_{\omega_1^t}u_t(t)(x)\rho(x,u_t(t)(x))dx &\le \frac{c_2\|a\|_{L^\infty}}{c_1a_-}\varepsilon_1 g^{-1}(\varepsilon_1)\\ &\le H(r_1^2)\frac{c_2\|a\|_{L^\infty}}{c_1a_-} = H(r_0^2).\end{aligned}$$

Hence, $B(t) \in [0, r_0^2]$. Now applying Young's inequality (8.7.24) with this choice of A and B, taking into account (8.7.8) in (8.7.32), we conclude, for all $0 \le S \le T$,

$$\int_S^T f(E(t))\int_{\omega_1^t}|u_t(t)(x)|^2dxdt \le |\omega|\int_S^T \hat{H}^*(f(E(t)))dt + \frac{1}{c_1a_-}E(S). \tag{8.7.33}$$

We have $|u_t(t)| \ge \varepsilon_1$ on $\omega\backslash\omega_1^t$. For $|u_t(t)| \ge \varepsilon_0$, (8.7.16) holds. Hence we just need to prove that similar inequalities hold for $\varepsilon_1 \le |u_t(t)| \le \varepsilon_0$. For this purpose, due to (8.7.17), ρ satisfies for all $x \in \Omega$ and $\varepsilon_1 \le |v| \le \varepsilon_0$,

$$\tilde{c}_1a(x)|v| \le |\rho(x,v)| \le \tilde{c}_2a(x)|v|, \tag{8.7.34}$$

where, since $g(\varepsilon_1) \le \varepsilon_0$, we have $\tilde{c}_1 = c_1g(\varepsilon_1/\varepsilon_0) \le c_1$ and $\tilde{c}_2 = c_2g^{-1}(\varepsilon_0/\varepsilon_1) \ge c_2$. Thus, (8.7.16) also holds on $\omega\backslash\omega_1^t$ with the constants c_1 and c_2 replaced by $\tilde{c}_1$ and $\tilde{c}_2$ respectively. Obviously, we have for all $x \in \omega\backslash\omega_1^t$,

$$|u_t(t)(x)|^2 \le \frac{1}{\tilde{c}_1a_-}u_t(t)(x)\rho(x,u_t(t))(x)), \tag{8.7.35}$$

which implies

$$\int_S^T f(E(t))\int_{\omega\backslash\omega_1^t}|u_t(t)(x)|^2dxdt \le \frac{1}{\tilde{c}_1a_-}\int_S^T(-E'(t))f(E(t))dt. \tag{8.7.36}$$

Now inserting (8.7.26), (8.7.28), (8.7.33) and (8.7.36) in (8.7.11), we can get

$$\int_S^T f(E(t))E(t)dt \le \delta_1 E(S)f(E(S)) + \left(\frac{\delta_2}{c_g} + \frac{\delta_3}{c_1 a_-}\right) E(S) \tag{8.7.37}$$
$$+ \left(\frac{\delta_2}{c_g} + \frac{\delta_3}{c_1 a_-}\right) \int_S^T (-E'(t))f(E(t))dt + \left(\delta_2 \frac{|\Omega|}{c_g^2} + \delta_3|\omega|\right) \int_S^T \hat{H} * (f(E(t)))dt.$$

We define F by (1.5.172), and recall that F is a strictly increasing function from $[0,+\infty)$ onto $[0, r_0^2)$ (see, e.g., Lemma 1.5.4). We also choose a real number $\beta = \beta_{E(0)}$ as follows:

$$\beta = \max\left(\delta_3|\omega| + \frac{\delta_2|\Omega|}{c_g^2}, \frac{E(0)}{2F(H'(r_0^2))}\right). \tag{8.7.38}$$

We now choose the weighted function f as follows: for all $s \in [0, 2\beta r_0^2)$,

$$f(s) = F^{-1}(\frac{s}{2\beta}). \tag{8.7.39}$$

Then f is a strictly increasing function from $[0, 2\beta r_0^2)$ onto $[0,+\infty)$ and f satisfies the relation for all $s \in [0, 2\beta r_0^2)$,

$$\beta \hat{H}^*(f(s)) = \frac{1}{2} s f(s).$$

Since E is non-increasing, we may get for all $t \ge 0$,

$$E(t) \le E(0) < E(0)\frac{r_0^2}{F(H'(r_0^2))} \le 2\beta r_0^2.$$

Hence, we have, in particular, for all $t \ge 0$,

$$\beta \hat{H}^*(f(E(t))) = \frac{1}{2}E(t)f(E(t)). \tag{8.7.40}$$

Note that with this choice of β and f, the last term on the right-hand side of (8.7.37) is bounded above by

$$\frac{1}{2}\int_S^T E(t)F(E(t))dt. \tag{8.7.41}$$

On the other hand, we recall that $-E'$ is non-negative on $[0,+\infty)$, E is non-negative and non-increasing on $[0,+\infty)$ whereas f is non-negative and increasing on $[0, 2\beta r_0^2)$. Thus, the third term on the right-hand side of (8.7.37) is bounded above by

$$\left(\frac{\delta_2}{c_g} + \frac{\delta_3}{\tilde{c}_1 a_-}\right) \int_S^T (-E'(t))f(E(t))dt \le \left(\frac{\delta_2}{c_g} + \frac{\delta_3}{\tilde{c}_1 a_-}\right) E(S)F^{-1}\left(\frac{E(S)}{\beta}\right). \tag{8.7.42}$$

Thus inserting (8.7.41) and (8.7.42) into (8.7.37), yields

$$\int_S^T E(t)F^{-1}\left(\frac{E(t)}{2\beta}\right)dt$$
$$\leq 2\beta\left(\delta_1\frac{\delta_2}{c_g}+\frac{\delta_3}{\tilde{c}_1 a_-}\right)E(S)F^{-1}\left(\frac{E(S)}{2\beta}\right)+2\left(\frac{\delta_2}{c_g}+\frac{\delta_3}{c_1 a_-}\right)E(S).$$

Hence, the energy E satisfies the estimate for all $0 \leq S \leq T$,

$$\int_S^T E(t)F^{-1}\left(\frac{E(t)}{2\beta}\right)dt \leq T_0 E(S), \tag{8.7.43}$$

where T_0 is independent of $E(0)$ and, with our choice of β, is given by

$$T_0 = 2\left(\frac{\delta_2}{c_g}+\frac{\delta_3}{c_1 a_-}+\left(\delta_1+\frac{\delta_2}{c_g}+\frac{\delta_3}{\tilde{c}_1}a_-\right)H'(r_0^2)\right). \tag{8.7.44}$$

Therefore, the functions g, H, E and β satisfies the hypotheses of Theorem 1.5.19. Therefore, applying Theorem 1.5.19 to (8.7.43), we conclude that E satisfies the desired estimate (8.7.12), which completes the proof. □

Chapter 9

Asymptotic Behavior of Solutions to Thermoviscoelastic, Thermoviscoelastoplastic and Thermomagnetoelastic Equations

In this chapter, we shall establish the asymptotic behavior for thermoviscoelastic, thermoviscoelastoplastic and thermomagnetoelastic equations. This chapter consists of three sections. In Section 9.1, we shall first employ Lemma 1.5.4 to extend the decay results in [620] for a viscoelastic system to those for the thermoviscoelastic system (9.1.1) and then establish the existence of the global attractor for the homogeneous thermoviscoelastic system (9.1.54). In Section 9.2, we shall employ Theorem 2.2.8 to investigate weak stabilization for a thermoviscoelastoplastic system with hysteresis. In Section 9.3, we shall apply Theorems 2.3.1, 2.3.4–2.3.5 and Corollary 2.3.1, we consider initial boundary value problems for some linear thermomagnetoelastic models describing elastic materials where reciprocal effects of the temperature, the magnetic field and the elastic displacement are taken into account. Inequalities used in this chapter are carefully selected and crucial in deriving the large-time behavior (including decay rates) of solutions to some thermoviscoelastic, thermoviscoelastoplastic and thermomagnetoelastic equations.

9.1 Large-time behavior for thermoviscoelastic systems

In this section, we shall first employ Lemma 1.5.4 to extend the decay results in [620] for a viscoelastic system to those for the thermoviscoelastic system (9.1.1) and then establish the existence of the global attractor for the homogeneous thermoviscoelastic system (9.1.54). These results here are picked from Qin and Ma [794].

Y. Qin, *Analytic Inequalities and Their Applications in PDEs*, Operator Theory: Advances and Applications 241, DOI 10.1007/978-3-319-00831-8_9

We shall consider the following thermoviscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + \nabla\,\theta = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ \theta_t - \Delta\theta + \operatorname{div} u_t = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ \theta = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u = 0, \quad (x,t) \in \Gamma_0 \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-\tau)\frac{\partial u}{\partial \nu}d\tau + h(u_t) = 0, \quad (x,t) \in \Gamma_1 \times \mathbb{R}^+, \end{cases} \tag{9.1.1}$$

with the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \Omega, \tag{9.1.2}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint, with meas $(\Gamma_0) > 0$, ν is the unit outward normal to $\partial\Omega$, $u(x,t)$ and $\theta(x,t)$ represent displacement vector and temperature derivations, respectively, g, h are specific functions.

We assume that the basic conditions on the relaxation function $g(t)$ hold

(H1) $g \in C^1([0,+\infty)) \cap L^1((0,+\infty))$;
(H2) $g(t) \geq 0$, $g'(t) \leq 0$, for all $t > 0$;
(H3) $l = 1 - \int_0^{+\infty} g(t)\,dt > 0$.

Note that condition (H3) simply states that the static modulus of elasticity is positive. This restriction is quite natural. In addition, conditions (H1) and (H2) imply

$$g(+\infty) = \lim_{t\to+\infty} g(t) = 0. \tag{9.1.3}$$

In the sequel, we denote by $\|\cdot\|$ the norm of $L^2(\Omega)$.

The energy $E(u,\theta,t)$ of problem (9.1.1) can be defined by

$$E(t) \equiv E(u,\theta,t) = \frac{1}{2}\int_\Omega \left(u_t^2 + \theta^2 + \left(1 - \int_0^t g(s)ds\right)|\nabla u|^2\right)dx + \frac{1}{2}(g \circ \nabla u)(t), \tag{9.1.4}$$

where, for all $v \in L^2(\Omega)$,

$$(g \circ v)(t) = \int_\Omega \int_0^t g(t-s) \mid v(t) - v(s) \mid^2 dsdx. \tag{9.1.5}$$

By a straightforward calculation, under assumption (A_2) (see below), we have

$$\frac{dE(t)}{dt} = \frac{1}{2}(g_t \circ \nabla u) - \frac{1}{2}g(t)\int_\Omega \mid \nabla u \mid^2 dx - \int_\Omega \mid \nabla\theta \mid^2 dx - \int_{\Gamma_1} u_t h(u_t) d\Gamma \leq 0, \tag{9.1.6}$$

which indicates that the energy $E(u,\theta,t)$ decreases on $(0,+\infty)$.

Indeed, if the relaxation function $g(t)$ satisfies conditions (H1), (H2) and (H3), Navarro [686] proved the asymptotic stability for system (9.1.1), Liu and

Zheng [567] proved that the energy decays to zero exponentially. However, the most interesting question is whether the energy decays exponentially or polynomially as $t \to +\infty$. In the case of higher space dimension, the problem is more complicated. Recently, Liu [558] proved the exponential stability with a boundary feedback. In order to see such complexity, we look at some special cases: thermoelastic systems and viscoelastic systems.

We make the following assumptions on the general decay of the kernel g and the function h (see Messaoudi [617]).

(A_1) $g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) > 0$ is a differentiable function satisfying (H1)–(H3) and that there exists a non-increasing differentiable function η such that

$$g'(t) \leq -\eta(t)g(t),\ t \geq 0 \quad \text{and} \quad \int_0^{+\infty} \eta(t)dt = +\infty.$$

(A_2) $h : \mathbb{R} \to \mathbb{R}$ is a non-decreasing C^0 function such that there exists a strictly increasing function $h_0 \in C^1([0, +\infty))$ with $h_0(0) = 0$, and positive constants c_1, c_2, and ε such that

$$\begin{aligned} h_0(|s|) \leq \mid h(s) \mid &\leq h_0^{-1}(|s|) \quad \text{for all} \quad \mid s \mid \leq \varepsilon, \\ c_1|s| \leq \mid h(s) \mid &\leq c_2|s| \qquad\quad \text{for all} \quad \mid s \mid \geq \varepsilon. \end{aligned}$$

Hypothesis (A_2) implies that $sh(s) > 0$, for all $s \neq 0$.
Set

$$V = \Big\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0 \Big\},$$

with an equivalent norm

$$\| v \|_V^2 = \frac{1}{2} \| \nabla v \|^2 + \frac{1}{2} \int_{\Gamma_1} uh(u)d\Gamma. \tag{9.1.7}$$

Using a standard semigroup approach, we may easily prove the following global existence result.

Lemma 9.1.1 ([794]). *Assume that* (A_1)–(A_2) *hold. Then*

(i) *for every initial condition* $(u_0, u_1, \theta_0) \in V \times L^2(\Omega) \times L^2(\Omega)$, *problem* (9.1.1) *has a unique global mild solution* $(u(t), v(t))$ *satisfying*

$$u(t) \in C(\overline{\mathbb{R}^+}, V) \cap C^1(\overline{\mathbb{R}^+}, L^2(\Omega)), \quad \theta(t) \in C(\overline{\mathbb{R}^+}, L^2(\Omega));$$

(ii) *for every initial condition* $(u_0, u_1, \theta_0) \in (H^2(\Omega) \cap V) \times V \times (H^2(\Omega) \cap H_0^1(\Omega))$, *problem* (9.1.1) *has a unique global classical solution* $(u(t), v(t))$ *satisfying*

$$\begin{gathered} u(t) \in C(\overline{\mathbb{R}^+}, H^2(\Omega) \cap V) \cap C^1(\overline{\mathbb{R}^+}, V) \cap C^2(\overline{\mathbb{R}^+}, L^2(\Omega)), \\ \theta(t) \in C(\overline{\mathbb{R}^+}, H^2(\Omega) \cap H_0^1(\Omega)) \end{gathered}$$

with $\overline{\mathbb{R}^+} = [0, +\infty)$.

Proof. In order to use the theory of semigroups, we introduce the new variable

$$w^t(x,s) = u(x,t) - u(x,t-s) \tag{9.1.8}$$

and extend the solution u to negative times, setting $u(t) = 0$ for $t < 0$ and $v = u_t$. Then problem (9.1.1) can be transformed into the system

$$\begin{cases} u_t - v = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ v_t - l\Delta u - \int_0^{+\infty} g(s)\Delta w^t(t-s)ds + \operatorname{div}\theta = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \theta_t - \Delta\theta + \operatorname{div} u_t = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ w_t^t = -w_s^t + u_t, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \theta = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u = 0, & (x,t) \in \Gamma_0 \times \mathbb{R}^+, \\ l\frac{\partial u}{\partial \nu} + \displaystyle\int_0^t g(t-\tau)\frac{\partial w^t}{\partial \nu}d\tau + h(u_t) = 0, & (x,t) \in \Gamma_1 \times \mathbb{R}^+, \end{cases} \tag{9.1.9}$$

where g satisfies (A_1). In view of (A_1), let $L_g^2(\mathbb{R}^+, V)$ be the Hilbert space of V-valued functions on $\mathbb{R}^+$, endowed with the norm

$$\| w \|^2_{L_g^2(\mathbb{R}^+,V)} = \int_0^{+\infty} g(s) \| w(s) \|_V^2 \, ds.$$

We consider problem (9.1.9) in the following Hilbert space

$$\mathcal{H} = V \times L^2 \times L^2 \times L_g^2(\mathbb{R}^+, V)$$

with the energy norm

$$\|(u,v,\theta,\theta_t,w)\|_{\mathcal{H}} = \left\{ l\|u\|_V^2 + \frac{1}{2}\left(\|v\|^2 + \| \theta \|^2 \right) + \int_0^{+\infty} g(s)\|w(s)\|_V^2 \, ds \right\}^{1/2} \tag{9.1.10}$$

and define a linear unbounded operator A on $\mathcal{H}$ by

$$A(u,v,\theta,w) = (v, B(u,w) - \nabla\theta, \Delta\theta - \nabla v, v - w_s) \tag{9.1.11}$$

where $w_s^t = \frac{\partial w^t}{\partial s}$ and

$$B(u,w) = l\Delta u + \int_0^{+\infty} g(s)\Delta w^t(s)ds.$$

Then problem (9.1.9) can be formulated as an abstract Cauchy problem

$$\Phi' = A\Phi, \quad \Phi = (u,v,\theta,w)$$

on the Hilbert space $\mathcal{H}$ for an initial condition $\Phi(0) = (u_0, u_1, \theta_0, w_0)$. The domain of A is given by:

$$\begin{aligned} D(A) = \Big\{ &(u, v, \theta, w) \in \mathcal{H} \ : \theta \in H^2(\Omega) \cap H_0^1(\Omega), v \in V, \\ &lu + \int_0^{+\infty} g(s) w^t(s)\, ds \in H^2(\Omega) \cap V, \\ &w^t(s) \in H_g^1(\mathbb{R}^+,\, V),\ w^t(0) = 0, \\ &l\frac{\partial u}{\partial \nu} + \int_0^{+\infty} g(t-\tau) \frac{\partial w^t}{\partial \nu} d\tau + h(u_t) = 0 \quad \text{on } \Gamma_1 \Big\}. \end{aligned} \tag{9.1.12}$$

It is clear that $D(A)$ is dense in $\mathcal{H}$.

First, we prove that A is dissipative and closed. By a straightforward calculation, it follows from Lemma 3.2 in [558] that

$$\begin{aligned} &\langle A(u, v, \theta, w), (u, v, \theta, \theta_t, w) \rangle_{\mathcal{H}} \\ &= (v, u)_V + \frac{1}{2}(B(u, w) - \nabla\theta, v) + \frac{1}{2}(\Delta\theta - \operatorname{div} v, \theta) + (v - w_s, w)_{L_g^2(\mathbb{R}^+, V)} \\ &= -\frac{1}{2}\int_{\Gamma_1} u h(u) d\Gamma - \frac{1}{2} \parallel \nabla\theta \parallel^2 - g(s) \parallel w(s) \parallel_V^2 \Big|_0^{+\infty} \\ &\quad + \int_0^{+\infty} g'(s) \parallel w(s) \parallel^2_{(H^1_{\Gamma_1}(\Omega))^n} ds \\ &\leq 0. \end{aligned}$$

Thus A is dissipative. In order to prove that A is closed, let $(u_n, v_n, \theta_n, w_n) \in D(A)$ be such that

$$\begin{cases} (u_n, v_n, \theta_n, w_n) \to (u, v, \theta, w) & \text{in } \mathcal{H}, \\ A(u_n, v_n, \theta_n, w_n) \to (a, b, c, d) & \text{in } \mathcal{H}. \end{cases}$$

Then we have

$$\begin{cases} u_n \to u \quad \text{in } V, & (9.1.13) \\ v_n \to v \quad \text{in } L^2(\Omega), & (9.1.14) \\ \theta_n \to \theta \quad \text{in } H_0^1(\Omega), & (9.1.15) \\ w_n \to w \quad \text{in } L_g^2(\mathbb{R}^+, V), & (9.1.16) \\ v_n \to a \quad \text{in } V, & (9.1.17) \\ B(u_n, w_n) - \nabla\theta_n \to b \quad \text{in } L^2(\Omega), \Delta\theta_n - \operatorname{div} v_n \to c \quad \text{in } L^2(\Omega), & (9.1.18) \\ v_n - w_{ns} \to d \quad \text{in } L_g^2(\mathbb{R}^+, V). & (9.1.19) \end{cases}$$

By (9.1.14) and (9.1.17), we can obtain

$$\begin{cases} v_n \to v \quad \text{in } V, & (9.1.20) \\ v = a \in V. & (9.1.21) \end{cases}$$

By (9.1.19) and (9.1.21), we deduce

$$\Delta\theta_n \to c + \operatorname{div} v \quad \text{in } L^2(\Omega), \tag{9.1.22}$$

and noting that Δ is an isomorphism from $H^2(\Omega) \cap H_0^1(\Omega)$ onto $L^2(\Omega)$, it follows from (9.1.15) that

$$\theta_n \to \theta \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega). \tag{9.1.23}$$

It therefore follows from (9.1.19) and (9.1.24) that

$$c = \Delta\theta - \operatorname{div} v, \quad \theta \in H^2(\Omega) \cap H_0^1(\Omega). \tag{9.1.24}$$

From (9.1.16), (9.1.20) and (9.1.21), we derive

$$\begin{cases} w_n \to w \quad \text{in } H_g^1(\mathbb{R}^+, V), & (9.1.25) \\ d = v - w_s, \quad w \in H_g^1(\mathbb{R}^+, V),\ w(0) = 0. & (9.1.26) \end{cases}$$

In addition, it follows from (9.1.13), (9.1.16) and (9.1.24) that

$$B(u_n, w_n) - \nabla\theta \to B(u, w) - \nabla\theta \tag{9.1.27}$$

in the sense of distribution. It therefore follows from (9.1.18) and (9.1.28) that

$$b = B(u, w) - \nabla\theta, \quad B(u, w) \in L^2(\Omega) \tag{9.1.28}$$

and

$$lu + \int_0^{+\infty} g(s)w(s)ds \in H^2(\Omega) \cap V, \tag{9.1.29}$$

since $\mu\Delta + (\lambda + \mu)\nabla\text{div}$ is an isomorphism from $H^2(\Omega) \cap V$ onto $L^2(\Omega)$. Moreover, by (9.1.21), (9.1.30) and the trace theorem, we conclude

$$l\frac{\partial u}{\partial \nu} + \int_0^{+\infty} g(t - \tau)\frac{\partial w}{\partial \nu}d\tau + h(u_t) = 0 \quad \text{on } \Gamma_1.$$

Thus, by (9.1.22), (9.1.25), (9.1.27), (9.1.29) and (9.1.30), we conclude

$$A(u, v, \theta, w) = (a, b, c, d), \quad (u, v, \theta, w) \in D(A).$$

Hence, A is closed.

Next, we prove that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent of the operator A. For any $G = (g_1, g_2, g_3, g_4) \in \mathcal{H}$, consider

$$A\Phi = G, \tag{9.1.30}$$

i.e.,

$$\begin{cases} v = g_1, & \text{in} \quad V, & (9.1.31) \\ B(u, w) - \nabla\theta = g_2, & \text{in} \quad L^2(\Omega), & (9.1.32) \\ \Delta\theta - \operatorname{div} v = g_3, & \text{in} \quad L^2(\Omega), & (9.1.33) \\ v - w_s = g_4, & \text{in} \quad L_g^2(\mathbb{R}^+, V). & (9.1.34) \end{cases}$$

Inserting $v = g_1$ obtained from (9.1.31) into (9.1.33), we can obtain

$$\Delta\theta = g_3 + \operatorname{div} g_1 \in L^2(\Omega). \tag{9.1.35}$$

By the standard theory for linear elliptic equations, we have a unique $\theta \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying (9.1.36).

We plug $v = g_1$ obtained from (9.1.31) into (9.1.34) to get

$$w_s = g_1 - g_4 \in L_g^2(\mathbb{R}^+, V). \tag{9.1.36}$$

Applying the standard theory for the linear elliptic equations again, we know that there exists a unique $w \in H_g^1(\mathbb{R}^+, V)$ satisfying (9.1.36). Then plugging θ and w just obtained from solving (9.1.35), (9.1.36) respectively into (9.1.32) and applying the standard theory for the linear elliptic equations again yields the unique solvability of $u \in D(A)$ for (9.1.32), and such that $lu + \int_0^{+\infty} g(s)w(s)\,ds \in H^2(\Omega) \cap V$. Thus the unique solvability of (9.1.30) follows. It is clear from the regularity theory for the linear elliptic equations that $\|\Phi\|_{\mathcal{H}} \le K\|G\|_{\mathcal{H}}$ with K being a positive constant independent of Φ. From the semigroup theory (see, e.g., [558, 717, 1000]), we can complete the proof. □

Now we state and prove our main result. First we establish several lemmas in the following.

Using the Cauchy–Schwartz and Poincaré's inequalities, we can obtain the following lemma (see, e.g., [617]) immediately.

Lemma 9.1.2 ([617]). *There exists a constant $c > 0$ such that for all $u \in V$,*

$$\int_\Omega \Big(\int_0^t g(t-s)(u(t)-u(s))ds \Big)^2 dx \le c(g \circ \nabla u)(t).$$

Now we are going to construct a Lyapunov functional $\mathcal{F}(t)$ equivalent to $E(t)$. To this end, we define several functionals which allow us to obtain the required estimates.

Lemma 9.1.3 ([794]). *Under the assumptions* (A_1)–(A_2)*, the function F_1 defined by*

$$F_1(t) = \int_\Omega u u_t dx$$

satisfies the estimate

$$\begin{aligned} F_1'(t) \le & -\frac{l}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega u_t^2 dx + c(g\circ \nabla u)(t) \\ & + \frac{\lambda_0^2}{4\varepsilon}\int_\Omega |\nabla\theta|^2\, dx + c\int_{\Gamma_1} h^2(u_t) d\Gamma \end{aligned} \tag{9.1.37}$$

where $\lambda_0 > 0$ is the best constant in Poincaré's inequality.

Proof. By a straightforward calculation, using (9.1.1), we obtain

$$
\begin{aligned}
F_1'(t) &= \int_\Omega u_t^2 dx + \int_\Omega u\Delta u dx - \int_\Omega u\nabla\theta dx - \int_\Omega u\left(\int_0^t g(t-\tau)\Delta u(\tau)d\tau\right)dx \\
&\le \int_\Omega u_t^2 dx - l\int_\Omega |\nabla u|^2 dx - \int_\Omega u\nabla\theta dx \qquad (9.1.38)\\
&\quad + \int_\Omega \nabla u\left(\int_0^t g(t-\tau)(\nabla u(\tau)-\nabla u(t))d\tau\right)dx - \int_{\Gamma_1} uh(u_t)d\Gamma.
\end{aligned}
$$

Using Young's inequality and Lemma 9.1.2, we get for any $\varepsilon > 0$,

$$
\begin{aligned}
&\int_\Omega \nabla u\left(\int_0^t g(t-\tau)(\nabla u(\tau)-\nabla u(t))d\tau\right)dx \\
&\quad \le \varepsilon\int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon}\int_\Omega\left(\int_0^t g(t-\tau)\,|\nabla u(\tau)-\nabla u(t)|\,d\tau\right)^2 dx \\
&\quad \le \varepsilon\int_\Omega |\nabla u|^2 dx + \frac{c}{\varepsilon}(g\circ\nabla u)(t). \qquad (9.1.39)
\end{aligned}
$$

Using Poincaré's and Young's inequalities, we find that

$$
-\int_\Omega u\nabla\theta dx \le \varepsilon\int_\Omega |\nabla u|^2 dx + \frac{\lambda_0^2}{4\varepsilon}\int_\Omega |\nabla\theta|^2 dx. \qquad (9.1.40)
$$

Similarly, using the Trace Theorem, we conclude for any $\varepsilon > 0$,

$$
-\int_{\Gamma_1} uh(u_t)d\Gamma \le \varepsilon\int_\Omega |\nabla u|^2 dx + \frac{\lambda_0^2}{4\varepsilon}\int_{\Gamma_1} h^2(u_t)d\Gamma. \qquad (9.1.41)
$$

Combining (9.1.39)–(9.1.41) and choosing $\varepsilon > 0$ small enough, we can show (9.1.37). □

Lemma 9.1.4 ([794])**.** *Under the assumptions* (A_1)–(A_2)*, the function* F_2 *defined by*

$$
F_2(t) = -\int_\Omega u_t\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx
$$

satisfies the estimate

$$
\begin{aligned}
F_2'(t) \le &-\left(\int_0^t g(s)ds - \varepsilon\right)\int_\Omega u_t^2 dx + \varepsilon\int_\Omega (|\nabla u|^2 + |\nabla\theta|^2)dx \\
&+ \frac{c}{\varepsilon}(g\circ\nabla u)(t) - \frac{c}{\varepsilon}(g'\circ\nabla u)(t) + c\int_{\Gamma_1} h^2(u_t)d\Gamma
\end{aligned} \qquad (9.1.42)
$$

for any $0 < \varepsilon < 1$.

Proof. By exploiting (9.1.1) and integrating by parts, we can get

$$\begin{aligned}F_2'(t) = &\left(1-\int_s^t g(s)ds\right)\int_\Omega \nabla u\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)dx\\&+\int_{\Gamma_1}\left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)h(u_t)d\Gamma\\&-\int_\Omega u_t\left(\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau\right)dx\\&+\int_\Omega \nabla\theta\left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)dx\\&-\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2dx+\int_\Omega\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)^2dx.\end{aligned}$$

Using Young's and Poincaré's inequalities and Lemma 9.1.2, we obtain for any $\varepsilon > 0$,

$$\begin{aligned}&\left(1-\int_s^t g(s)ds\right)\int_\Omega \nabla u\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)dx\\&\qquad\qquad\le \varepsilon\int_\Omega |\nabla u|^2\,dx+\frac{c}{\varepsilon}(g\circ\nabla u)(t),\\&\int_{\Gamma_1}\left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)h(u_t)d\Gamma\le c(g\circ\nabla u)(t)+c\int_{\Gamma_1}h^2(u_t)d\Gamma,\\&-\int_\Omega u_t\left(\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau\right)dx\le\varepsilon\int_\Omega u_t^2dx-\frac{\lambda_0^2}{4\varepsilon}(g'\circ\nabla u)(t),\\&\int_\Omega\nabla\theta\left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)dx\le\varepsilon\int_\Omega|\nabla\theta|^2\,dx+\frac{\lambda_0^2}{4\varepsilon}(g\circ\nabla u)(t).\end{aligned}$$

Combining all above estimates, we can obtain (9.1.42). □

Now for $N_1, N_2 > 1$, let

$$\mathcal{F}(t) = N_1E(t) + N_2F_2(t) + F_1(t)$$

and set $g_0 = \int_0^{t_0} g(s)ds > 0$ for some fixed $t_0 > 0$. By combining (9.1.6), (9.1.37) and (9.1.42), taking $\varepsilon > 0$ small enough, and N_1, N_2 large enough, we arrive at

$$\begin{aligned}\mathcal{F}'(t)\le &-\frac{l}{4}\int_\Omega |\nabla u|^2\,dx-k\int_\Omega u_t^2dx\\&-k_1\int_\Omega|\nabla\theta|^2\,dx+c(g\circ\nabla u)(t)+c\int_{\Gamma_1}h^2(h_t)d\Gamma,\end{aligned}$$

where

$$k=\left(N_2g_0-\frac{l}{4}-1\right)>0,\quad k_1=(N_1-\varepsilon N_2)>0,$$

which yields, for all $t \geq t_0$,

$$\mathcal{F}'(t) \leq -\beta_1 E(t) + c(g \circ \nabla u)(t) + c \int_{\Gamma_1} h^2(h_t) d\Gamma, \tag{9.1.43}$$

or

$$E(t) \leq -\beta_2 \mathcal{F}'(t) + c(g \circ \nabla u)(t) + c \int_{\Gamma_1} h^2(h_t) d\Gamma. \tag{9.1.44}$$

On the other hand, we can choose N_1 so large that $\mathcal{F}(t)$ is equivalent to $E(t)$, i.e.,

$$\mathcal{F}(t) \sim E(t). \tag{9.1.45}$$

Now our main result, due to Qin and Ma [794], reads as follows.

Theorem 9.1.5 ([794]). *Assume that* (A_1)–(A_2) *hold. Then there exists a constant* $C > 0$ *such that, for* t *large enough, the solution of problem* (9.1.1) *satisfies*

$$E(t) \leq C \left(H_0^{-1} \left(\frac{1}{\int_0^t \eta(s) ds} \right) \right)^2, \tag{9.1.46}$$

where $H_0(s) = s h_0(s)$. *Moreover, if* K *defined by* $K(s) = \frac{h_0(s)}{s}$ *is strictly increasing with* $K(0) = 0$, *then we have the improved estimate*

$$E(t) \leq C \left(h_0^{-1} \left(\frac{1}{\int_0^t \eta(s) ds} \right) \right)^2. \tag{9.1.47}$$

Proof. We borrow some ideas from [620] for a viscoelastic system. Define $\phi(t) = 1 + \int_1^t \frac{1}{h_0(1/s)} ds$. Then

$$\phi'(t) = \frac{1}{h_0(1/t)} > 0, \quad \text{for all } t \geq 1, \;\; \phi'(t) \to +\infty \quad \text{as} \quad t \to +\infty,$$

and $\phi'(t)$ is strictly increasing. Thus ϕ is convex and strictly increasing C^2 function, with $\phi(t) \to +\infty$ as $t \to +\infty$. If we put $\sigma_0 = \phi^{-1}$, then σ_0 is strictly increasing, $\sigma_0'(t) = h_0(1/t)$ is decreasing, and $\sigma_0(t) \to +\infty$ as $t \to +\infty$.

Now we define

$$\sigma(t) := \sigma_0 \left(\int_0^t \eta(s) ds \right),$$

for some $t_1 \geq t_0$ with $\int_0^{t_1} \eta(s) ds \geq 1$. Using the properties of σ_0 and η, we easily check that σ is a strictly increasing and concave twice differentiable function, such that $\sigma(t) \to +\infty$ as $t \to +\infty$. It follows from (9.1.44) and (9.1.45) that for all

$T \geq s \geq t_1$,

$$\begin{aligned}
\int_s^T \sigma'(t)E^2(t)dt &\leq c\int_s^T \sigma'(t)E(t)\mathcal{F}(t)dt \\
&\leq -c_1\int_s^T \sigma'\mathcal{F}\mathcal{F}'dt + c_2\int_s^T \sigma'\mathcal{F}(g\circ\nabla u)dt + c_2\int_s^T \sigma'\mathcal{F}\left(\int_{\Gamma_1} h^2(u_t)d\Gamma\right)dt \\
&\leq cE^2(s) + c\int_s^T \sigma' E(g\circ\nabla u)dt + c\int_s^T \sigma' E\left(\int_{\Gamma_1} h^2(u_t)d\Gamma\right)dt. \qquad (9.1.48)
\end{aligned}$$

Now we need to estimate the last two terms of (9.1.48) as follows

$$I_1 = \int_s^T \sigma' E(g\circ\nabla u)dt, \quad I_2 = \int_s^T \sigma' E\left(\int_{\Gamma_1} h^2(u_t)d\Gamma\right)dt.$$

Using (A_1), the definition of σ and noting the fact that σ_0' and η are non-increasing, we derive

$$\begin{aligned}
\sigma'(t)g(t-\tau) = \sigma_0'\left(\int_0^t \eta(s)ds\right)\eta(t)g(t-\tau) &\leq c\eta(t)g(t-\tau) \\
&\leq c\eta(t-\tau)g(t-\tau) \leq -cg'(t-\tau)
\end{aligned}$$

which implies

$$\begin{aligned}
I_1 &\leq -c\int_s^T E\left(\int_\Omega\int_0^t g'(t-\tau)\mid\nabla u(t)-\nabla u(\tau)\mid^2 d\tau dx\right)dt \\
&\leq -c\int_s^T EE'dt \leq cE^2(s).
\end{aligned} \qquad (9.1.49)$$

For I_2, we consider the following partition of Γ_1,

$$\begin{cases}
\Gamma_{11} = \{x\in\Gamma_1 :\mid u_t\mid > \varepsilon\}, \\
\Gamma_{12} = \left\{x\in\Gamma_1 :\mid u_t\mid \leq \varepsilon \text{ and } \mid u_t\mid \leq \sigma_0'\left(\int_0^t\eta(s)ds\right)\right\}, \\
\Gamma_{13} = \left\{x\in\Gamma_1 :\mid u_t\mid \leq \varepsilon \text{ and } \mid u_t\mid > \sigma_0'\left(\int_0^t\eta(s)ds\right)\right\}.
\end{cases} \qquad (9.1.50)$$

Using (A_2), (9.1.6), and the properties of σ, σ_0 and η, we obtain

$$\begin{aligned}
&\sigma'(t)\int_{\Gamma_{11}} h^2(u_t)d\Gamma \leq c\sigma'(t)\int_{\Gamma_{11}} u_t h(u_t)d\Gamma \leq -cE'(t), \\
&\sigma'(t)\int_{\Gamma_{12}} h^2(u_t)d\Gamma \leq \eta(t)\sigma_0'\left(\int_0^t\eta(s)ds\right)\int_{\Gamma_{12}}\left(h_0^{-1}(|u_t|)\right)^2 d\Gamma \\
&\qquad\qquad \leq \eta(t)\sigma_0'\left(\int_0^t\eta(s)ds\right)\left(h_0^{-1}\left(\sigma_0'\left(\int_0^t\eta(s)ds\right)\right)\right)^2,
\end{aligned}$$

$$\sigma'(t)\int_{\Gamma_{13}} h^2(u_t)d\Gamma = \eta(t)\sigma_0'\left(\int_0^t \eta(s)ds\right)\int_{\Gamma_{13}} h^2(u_t)d\Gamma$$
$$\leq \eta(t)h_0^{-1}(\varepsilon)\int_{\Gamma_{13}} u_t h(u_t)d\Gamma \leq -cE'(t)$$

which imply

$$\sigma'(t)\int_{\Gamma_1} h^2(u_t)d\Gamma \leq -cE'(t) + c\eta(t)\sigma_0'\left(\int_0^t \eta(s)ds\right)\left(h_0^{-1}\left(\sigma_0'\left(\int_0^t \eta(s)ds\right)\right)\right)^2.$$

Hence

$$I_2 \leq cE^2(s) + cE(s)\int_s^T \eta(t)\sigma_0'\left(\int_0^t \eta(s)ds\right)\left(h_0^{-1}\left(\sigma_0'\left(\int_0^t \eta(s)ds\right)\right)\right)^2 dt. \tag{9.1.51}$$

Therefore a combination of (9.1.48), (9.1.49) and (9.1.51) yields

$$\begin{aligned}
&\int_s^T \sigma'(t)E^2(t)dt \\
&\leq cE^2(s) + cE(s)\int_s^{+\infty} \eta(t)\sigma_0'\left(\int_0^t \eta(s)ds\right)\left(h_0^{-1}\left(\sigma_0'\left(\int_0^t \eta(s)ds\right)\right)\right)^2 dt \\
&= cE^2(s) + cE(s)\int_{\int_0^s \eta(s)ds}^{+\infty} \sigma_0'(\tau)(h_0^{-1}(\sigma_0'(\tau)))^2 d\tau \\
&= cE^2(s) + cE(s)\int_{\sigma_0(\int_0^s \eta(s)ds)}^{+\infty} \left(h_0^{-1}\left(h_0(1/s)\right)\right)^2 ds \\
&= cE^2(s) + \frac{cE(s)}{\sigma_0(\int_0^s \eta(s)ds)} = cE^2(s) + \frac{cE(s)}{\sigma(s)}.
\end{aligned}$$

Thus by Lemma 1.5.4 with $p = q = 1$, we conclude for all $t \geq t_1$,

$$E(t) \leq \frac{c}{\sigma(t)^2} = \frac{c}{\left(\sigma_0\left(\int_0^s \eta(s)ds\right)\right)^2}. \tag{9.1.52}$$

To obtain (9.1.46), we may take s_0 such that $h_0(1/s_0) \leq 1$. Since h_0 is increasing and $H_0(s) = sh_0(s)$, we may derive for all $s \geq s_0$,

$$\sigma_0^{-1}(s) \leq 1 + (s-1)\frac{1}{h_0(1/s)} \leq \frac{s}{h_0(1/s)} = \frac{1}{H_0(1/s)}.$$

Hence, with $t = \frac{1}{H_0(1/s)}$, we easily obtain for all $t \geq 1$,

$$\frac{1}{\sigma_0(t)} \leq H_0^{-1}(1/t),$$

which yields (9.1.46) by virtue of (9.1.52).

To prove (9.1.47), we take

$$\sigma_0 = \phi^{-1} \quad \text{where} \quad \phi(t) = 1 + \int_1^t \frac{1}{K(1/s)} ds, \quad t \geq 1,$$

and replace (9.1.50) by

$$\begin{cases} \Gamma_{11} = \{x \in \Gamma_1 :| u_t |> \varepsilon\}, \\ \Gamma_{12} = \Big\{x \in \Gamma_1 :| u_t | \leq \varepsilon \text{ and } h_0^{-1}(| u_t |) \leq K^{-1}\Big(\sigma_0'\Big(\int_0^t \eta(s)ds\Big)\Big)\Big\}, \\ \Gamma_{13} = \Big\{x \in \Gamma_1 :| u_t | \leq \varepsilon \text{ and } h_0^{-1}(| u_t |) > K^{-1}\Big(\sigma_0'\Big(\int_0^t \eta(s)ds\Big)\Big)\Big\}. \end{cases}$$

Then repeating similar computations, we obtain (9.1.47) immediately. □

9.2 A thermoviscoelastoplastic system with hysteresis

In this section, we shall employ Theorem 2.2.8 to investigate weak stabilization for a thermoviscoelastoplastic system with hysteresis. We choose these results from Krejčí and Sprekels [462].

9.2.1 Thermoelastoplastic constitutive laws

We recall that the observation of the uniaxial load-deformation experiments for many materials reveals that the stress-strain ($\sigma - \varepsilon$) relations strongly depend on the absolute (Kelvin) temperature θ and exhibits a strong plastic behavior confirmed by the occurrence of rate-independent hysteresis loops ([462]).

If such a relation involves a hysteresis, it is impossible to be expressed in terms of simple-valued functions since the latter are certainly not able to give a correct account of the inherent memory structure that are responsibly for the complicated looping in the interior of experimentally observed hysteresis loops.

To avoid these difficulties, the Russian group in the seventies (see, e.g., [459]) introduced the notion of hysteresis operators with which Krejčí and Sprekels [461] proposed a different approach to thermoelastoplastic hysteresis. Assume that the temperature-dependent plastic stress σ^p satisfies the following form of an operator $\mathcal{P}$ of Prandtl–Ishlinskii type,

$$\sigma^p = \mathcal{P}[\varepsilon, \theta] := \int_0^{+\infty} \varphi(r, \theta) s_r[\varepsilon] dr \tag{9.2.1}$$

where s_r denotes the so-called stop operator or elastic-plastic element with threshold $r > 0$ (to be defined below), and $\varphi(\cdot, \theta) \geq 0$ is a density function with respect to $r > 0$ and the absolute temperature θ. The integral formula (9.2.1) corresponds to an infinite rheological combination in parallel of elements s_r.

We recall that the stop operator $s_r : W^{1,1}(0,T) \to W^{1,1}(0,T)$ in equation (9.2.1) is defined as the solution operator $\sigma_r = s_r[\varepsilon]$ of the variational inequality

$$|\sigma_r(t)| \leq r,\ (\dot{\varepsilon}(t) - \dot{\sigma}_r(t))(\sigma_r(t) - \tilde{\sigma}) \geq 0 \quad \text{for a.e. } t \in [0,T], \ \text{for all } \tilde{\sigma} \in [-r, r], \tag{9.2.2}$$

(where $\dot{\varepsilon}(t) = \frac{d}{dt}\varepsilon(t)$) with initial condition

$$\sigma_r(0) = \operatorname{sign}(\varepsilon(0)) \min\{r, |\varepsilon(0)|\} \tag{9.2.3}$$

which describes the strain-stress law of Prandtl's model for elastic-perfectly plastic materials with a unit elasticity modulus and yield point r.

For the given density function φ in (9.2.1), then it can be identified by letting ε monotonically increase for fixed temperature θ starting from the origin. Therefore, the corresponding formula reads (see [460])

$$\Phi(\varepsilon, \theta) = \int_0^{\varepsilon} \int_s^{+\infty} \varphi(r, \theta) dr ds. \tag{9.2.4}$$

We now only consider the case when φ is non-negative, i.e., the initial loading curves at each constant temperature are concave and non-decreasing.

The operator s_r has following properties for whose proof, we refer to [120, 460].

Lemma 9.2.1. *Let $r > 0$ be given. Then*

(i) *For every $\varepsilon \in W^{1,1}(0,T)$, we have, for a.e. $t \in [0,T]$,*

$$\left(\frac{d}{dt} s_r[\varepsilon]\right)^2 = \dot{\varepsilon} \frac{d}{dt} s_r[\varepsilon]. \tag{9.2.5}$$

(ii) *For every $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$, we have, for a.e. $t \in [0,T]$,*

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\left(s_r[\varepsilon_1] - s_r[\varepsilon_2]\right)^2 \leq (\dot{\varepsilon}_1 - \dot{\varepsilon}_2)\left(s_r[\varepsilon_1] - s_r[\varepsilon_2]\right), & (9.2.6) \\ \displaystyle\int_0^T \left|\frac{d}{dt}(s_r[\varepsilon_1] - s_r[\varepsilon_2])(t)\right| dt \leq |\varepsilon_1(0) - \varepsilon_2(0)| + 2\int_0^T |\dot{\varepsilon}_1 - \dot{\varepsilon}_2|(t) dt, & (9.2.7) \\ |(s_r[\varepsilon_1] - s_r[\varepsilon_2])(t)| \leq 2 \max\limits_{0 \leq \tau \leq t} |\varepsilon_1(\tau) - \varepsilon_2(\tau)|. & (9.2.8) \end{cases}$$

(iii) *For every $r,\ q > 0$ and $\varepsilon \in W^{1,1}(0,T)$, we have, for all $t \in [0,T]$,*

$$|(s_r[\varepsilon] - s_q[\varepsilon])(t)| \leq |r - q|. \tag{9.2.9}$$

In fact, it follows from the inequalities (9.2.8)–(9.2.9) that the stop operator s_r is Lipschitz continuous in $W^{1,1}(0,T)$ and admits a Lipschitz continuous extension onto $C[0,T]$. Moreover, by definition we immediately know that s_r is a causal operator, that is, the following implication holds for every $t \in [0,T]$,

$$\varepsilon_1(\tau) = \varepsilon_2(\tau), \quad \text{for all } \tau \in [0,t] \ \Rightarrow \ s_r[\varepsilon_1] = s_r[\varepsilon_2](t) \tag{9.2.10}$$

which indicates that the output values at time t depend only on past values of the input. Due to this reason, we need to consider s_r as a family of operators acting in the spaces $C[0,t]$ for all $t \in [0,T]$.

The following corollary immediately follows from inequality (9.2.8).

Corollary 9.2.1 ([462]). *For all* $\varepsilon,\ \varepsilon_1,\ \varepsilon_2 \in W^{1,1}(0,T)$, *we have*

$$\begin{cases} s_r[\varepsilon]\left(\dot{\varepsilon} - \dfrac{d}{dt}s_r[\varepsilon]\right) \geq 0 \ \ \textit{a.e. in} \quad [0,T]\ (\textit{energy inequality}), & (9.2.11) \\ |(s_r[\varepsilon_1] - s_r[\varepsilon_2])(t)| \leq |\varepsilon_1(0) - \varepsilon_2(0)| + \displaystyle\int_0^t |\dot{\varepsilon}_1 - \dot{\varepsilon}_2|(\tau)d\tau, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for all } t \in [0,T]. & (9.2.12) \end{cases}$$

In this section, we shall consider the one-dimensional equation of motion

$$\rho u_{tt} = \sigma_x + f, \tag{9.2.13}$$

where $\rho > 0$ is a constant referential density, u is the displacement, σ is the total un-axial stress and f is the volume force density.

We assume that σ can be decomposed into the sum

$$\sigma = \sigma^p + \sigma^e + \sigma^v + \sigma^d, \tag{9.2.14}$$

where

$$\sigma^e = \gamma(\varepsilon), \tag{9.2.15}$$

is the (nonlinear) kinematic hardening component with a given non-decreasing Lipschitz continuous function $\gamma : \mathbb{R} \to \mathbb{R},\ \gamma(0) = 0$,

$$\sigma^v = \mu\dot{\varepsilon} \tag{9.2.16}$$

with a constant $\mu > 0$ is the viscous component,

$$\sigma^d = -\beta\theta \tag{9.2.17}$$

is the thermic dilation component with a constant $\beta \in \mathbb{R}$ and σ^p is the thermoplastic component given by (9.2.1). Equation (9.2.16) can be interpreted rheologically as a combination in parallel of the above components (see, e.g., [529]). Assume the stop operator s_r acts on functions of x and t satisfying the formula

$$s_r[\varepsilon](x,t) := s_r[\varepsilon(x,\cdot)](t), \tag{9.2.18}$$

i.e., x plays the role of a parameter. The equation of motion (9.2.15) is now coupled with the energy balance equation

$$U_t = \sigma\varepsilon_t - q_x + g, \tag{9.2.19}$$

where U is the total internal energy, q is the heat flux and g is the heat source density. The model is thermodynamically consistent provided the temperature θ and the entropy S satisfy the inequalities

$$\begin{cases} \theta > 0, & (9.2.20) \\ S_t \geq \frac{g}{\theta} - \left(\frac{q}{\theta}\right)_x \quad \text{(the Clausius–Duhem inequality)}, & (9.2.21) \end{cases}$$

in an appropriate sense.

The authors in [461] derived the following expression for thermoplastic parts of internal energy U^p and entropy S^p in operator form corresponding to the constitutive law (9.2.1),

$$\begin{cases} U^p = \mathcal{V}[\varepsilon, \theta] := \frac{1}{2}\int_0^{+\infty} (\varphi(r,\theta) - \theta\varphi_\theta(r,\theta))s_r^2[\varepsilon]dr, & (9.2.22) \\ S^p = \mathcal{S}[\varepsilon, \theta] := -\frac{1}{2}\int_0^{+\infty} \varphi_\theta(r,\theta))s_r^2[\varepsilon]dr. & (9.2.23) \end{cases}$$

Combining (9.2.16), (9.2.22), (9.2.23), we put

$$\begin{cases} U := C_V\theta + \mathcal{V}[\varepsilon,\theta] + \Gamma(\varepsilon) + V_0, & (9.2.24) \\ S := C_V \log\theta + \mathcal{S}[\varepsilon,\theta] + \beta\varepsilon, & (9.2.25) \end{cases}$$

where $C_V > 0$, the purely caloric part of the specific heat is a constant, $V_0 > 0$ is a constant which is chosen in order to ensure that $U \geq 0$ according to Hypothesis (H2) below, and $\Gamma(\varepsilon) := \int_0^\varepsilon \gamma(s)ds$. For the heat flux, we assume Fourier's law

$$q = -\kappa\theta_x \tag{9.2.26}$$

with a constant heat conduction coefficient $\kappa > 0$. The system (9.2.13), (9.2.19) is coupled with the small deformation hypothesis

$$\varepsilon = u_x \tag{9.2.27}$$

and the system (9.2.13)–(9.2.19) can be rewritten as the form

$$\begin{cases} \rho u_{tt} - (\gamma(u_x) + \mathcal{P}[u_x,\theta] + \mu u_{xt} - \beta\theta)_x = f(\theta,x,t), & (9.2.28) \\ (C_V\theta + \mathcal{V}[u_x,\theta])_t - \kappa\theta_{xx} = (\mathcal{P}[u_x,\theta] + \mu u_{xt} - \beta\theta)\, u_{xt} + g(\theta,x,t) & (9.2.29) \end{cases}$$

where $x \in [0,T]$, $t \in [0,T]$ with $T > 0$, $\mu > 0$, $C_V > 0$, $\beta \in \mathbb{R}$ being fixed constants, $\gamma : \mathbb{R} \to \mathbb{R}$, $f, g : [0,+\infty) \times [0,1] \times [0,T] \to \mathbb{R}$ are given functions, and $\mathcal{P}, \mathcal{V}$ are the operators defined by (9.2.1), (9.2.22) with a given distribution function $\varphi : ([0,+\infty))^2 \to [0,+\infty)$ satisfying Hypothesis (H2) below.

Consider problem (9.2.28)–(9.2.29). Assume that the volume force and heat source densities are given functions of x and t which may also depend on the instantaneous value of θ. Then by rescaling the units so that $\rho \equiv \kappa \equiv 1$, the system (9.2.28)–(9.2.29) reduces to the following boundary and initial conditions

$$\begin{cases} u(0,t) = u(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \text{ for all } t \geq 0, & (9.2.30) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ \theta(x,0) = \theta^0(x). & (9.2.31) \end{cases}$$

Here we assume the data to satisfy the following conditions

Hypothesis (H1).

(i) $u^0, u^1 \in H^2(0,1) \cap H_0^1(0,1)$, $\theta^0 \in H^1(0,1)$, and there exists a constant $\delta > 0$ such that for all $x \in [0,1]$,

$$\theta^0(x) \geq \delta. \tag{9.2.32}$$

(ii) $\gamma : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function, $\gamma(0) = 0$, and there exists a constant $\gamma_0 > 0$ such that

$$0 \leq \frac{d\gamma(\varepsilon)}{d\varepsilon} \leq \gamma_0 \text{ a.e. in } \mathbb{R}. \tag{9.2.33}$$

(iii) The functions f, g are measurable, $f(\cdot, x, t)$, $g(\cdot, x, t)$ are absolutely continuous in $[0, +\infty)$ for a.e. $(x,t) \in [0,1] \times [0,T]$. Moreover, there exists a constant $K > 0$ and functions f_0, $g_0 \in L^2([0,1] \times [0,T])$ such that

$$\begin{cases} g(0,x,t) = g_0(x,t) \geq 0, \\ \qquad \text{a.e. } (x,t) \in [0,1] \times [0,T], & (9.2.34) \\ |f(\theta,x,t)| + |f_t(\theta,x,t)| \leq f_0(x,t), \\ \qquad \text{a.e. } (\theta,x,t) \in [0,+\infty) \times [0,1] \times [0,T], & (9.2.35) \\ |f_\theta(\theta,x,t)| + |g_\theta(\theta,x,t)| \leq K, \\ \qquad \text{a.e. } (\theta,x,t) \in [0,+\infty) \times [0,1] \times [0,T]. & (9.2.36) \end{cases}$$

Hypothesis (H2).

The function $\varphi : ([0,+\infty))^2 \to [0,+\infty)$ is measurable, $\varphi(r,\cdot)$, $\varphi_\theta(r,\cdot)$ are absolutely continuous for a.e. $r > 0$, and there exist constants $L > 0$, $V_0 > 0$ such that for a.e. $\theta > 0$, the following inequalities hold.

$$\begin{cases} \displaystyle\int_0^{+\infty} \varphi(r,\theta)dr \leq L, & (9.2.37) \\ \displaystyle\int_0^{+\infty} |\varphi_\theta(r,\theta)|dr \leq L, & (9.2.38) \\ \displaystyle\int_0^{+\infty} \theta|\varphi_{\theta\theta}(r,\theta)|r^2 dr \leq C_V, & (9.2.39) \end{cases}$$

where $C_V > 0$ is the constant introduced in (9.2.24)–(9.2.25),

$$\frac{1}{2}\int_0^{+\infty} |\varphi(r,\theta) - \theta\varphi_\theta(r,\theta)|(1 + r^2)dr \leq V_0. \tag{9.2.40}$$

The next result is the existence result in [463] stated as follows.

Theorem 9.2.2 ([462]). *Let Hypotheses* (H1), (H2) *hold. Then there exists a unique solution* $(u(t), \theta(t))$ *to the problem* (9.2.28)–(9.2.31) *such that*

$$\begin{cases} u_{tt}, u_{xx}, u_{xxt}, \theta_x \in L^\infty(0,T; L^2(0,1)), & (9.2.41) \\ u_{xtt}, \theta_t, \theta_{xx} \in L^2([0,T] \times [0,T]), & (9.2.42) \\ \theta, u, u_x, u_{xt} \in C([0,1] \times [0,T]). & (9.2.43) \end{cases}$$

In addition, there exists a constant $c_0 > 0$ *depending only on the given data such that for all* $t \in [0,T]$ *and* $x \in [0,1]$, *we have*

$$\theta(x,t) \geq \delta e^{-c_0 t} > 0, \tag{9.2.44}$$

and problem (9.2.28)–(9.2.31) *are satisfied almost everywhere.*

Corollary 9.2.2 ([462]). *The solution from Theorem* 9.2.1 *satisfies the Clausius–Duhem inequality* (9.2.21) *with* S *defined by* (9.2.25), (9.2.23) *almost everywhere in* $[0,1] \times [0,T]$.

Now we begin to prove that the velocity tends to 0 in L^2 as $t \to +\infty$, but due to the technical reason, we do not know the asymptotic behaviour in time for the velocity gradient and the temperature.

Theorem 9.2.3 ([462]). *Assume the hypotheses of Theorem* 9.2.1 *hold. Assume, moreover, that* $\gamma(\varepsilon) = \gamma_0 \varepsilon$ *for some* $\gamma_0 > 0$ *and that* $f(\theta, x, t) = g(\theta, x, t) = 0$ *for all* $\theta > 0$ *and a.e.* $x \in [0,1]$, $t > 0$. *Then the solution* $(u(t), \theta(t))$ *of problem* (9.2.28)–(9.2.31) *satisfies*

$$\lim_{t \to +\infty} \int_0^1 u_t^2(x,t) dx = 0. \tag{9.2.45}$$

Proof. In the sequel, by $C_1, C_2, \dots$ we denote universal constants depending only on the initial conditions. The proof can be split into six steps.

Step 1. Multiplying (9.2.28) by u_t, and adding the result to (9.2.29) and integrating with respect to x over $[0,1]$, we have the global balance identity

$$\frac{d}{dt} \int_0^1 \left(\frac{1}{2} u_t^2 + \frac{\gamma_0}{2} u_x^2 + C_V \theta + \mathcal{V}[u_x, \theta] \right) (x,t) dx = 0. \tag{9.2.46}$$

Step 2. Multiplying (9.2.29) by $-1/\theta$, we may rewrite the result in the form

$$\begin{aligned} &\left(-C_V \log \theta + \frac{1}{2} \int_0^{+\infty} \varphi_\theta(r,\theta) s_r^2[u_x] dr \right)_t + \frac{1}{\theta}(\theta_{xx} + \mu u_{xt}^2) \\ &\quad + \frac{1}{\theta} \int_0^{+\infty} \varphi(r,\theta) s_r[u_x](u_x - s_r[u_x])_t dr + \beta u_{xt} = 0, \end{aligned} \tag{9.2.47}$$

and integrating with respect to x and t, we can obtain from (9.2.11) that

$$\begin{aligned}\int_0^1 \Big(- C_V \log\theta + \frac{1}{2}\int_0^{+\infty} \varphi_\theta(r,\theta) s_r^2[u_x] dr\Big)(x,t)dx \\ + \int_0^t \int_0^1 \left(\frac{\theta_x^2}{\theta^2} + \mu \frac{u_{xt}^2}{\theta}\right) dx d\tau \le C_1.\end{aligned} \tag{9.2.48}$$

On the other hand, using the relations

$$\begin{aligned}&\int_0^{+\infty} \varphi_\theta(r,\theta) s_r^2[u_x] dr \\ &= \int_0^{+\infty} \left[(\varphi_\theta(r,\theta) - \varphi_\theta(r,1)) + (\varphi_\theta(r,1) - \varphi(r,1)) + \varphi(r,1)\right] s_r^2[u_x] dr \\ &\ge - \int_0^{+\infty} \left[|\varphi_\theta(r,\theta) - \varphi_\theta(r,1)| + |\varphi_\theta(r,1) - \varphi(r,1)|\right] r^2 dr\end{aligned} \tag{9.2.49}$$

the left-hand side of (9.2.48) can be estimated. In fact, from Hypothesis (H2) it follows that

$$\int_0^{+\infty} |\varphi_\theta(r,1) - \varphi(r,1)| r^2 dr \le 2V_0, \tag{9.2.50}$$

$$\begin{aligned}&\int_0^{+\infty} |\varphi_\theta(r,\theta) - \varphi_\theta(r,1)| r^2 dr \\ &\quad \le \left| \int_1^\theta \int_0^{+\infty} |\varphi_{\theta\theta}(r,\theta')| dr d\theta' \right| \le C_V |\log\theta|.\end{aligned} \tag{9.2.51}$$

Using now the trivial inequality

$$|\log\theta| \le \max\{\theta, -\log\theta\}, \tag{9.2.52}$$

and

$$\int_0^1 \theta(x,t) dx \le C_2, \tag{9.2.53}$$

(which follows from (9.2.46)), we conclude that

$$\int_0^1 |\log\theta(x,t)| dx + \int_0^t \int_0^1 \left(\frac{\theta_x^2}{\theta^2} + \frac{u_{xt}^2}{\theta}\right) dx d\tau \le C_3. \tag{9.2.54}$$

Step 3. For every x and t, we have

$$|u_t(x,t)| \le \int_0^1 |u_{xt}(\xi,t)| d\xi \le \int_0^1 \frac{|u_{xt}|}{\sqrt{\theta}} \sqrt{\theta} d\xi \le \sqrt{C_2} \left(\int_0^1 \frac{u_{xt}^2}{\theta} d\xi \right)^{1/2}. \tag{9.2.55}$$

Hence

$$\int_0^t \max_{x\in[0,1]} |u_t(x,\tau)|^2 d\tau \le C_4. \tag{9.2.56}$$

Step 4. Analogously, for every x, y and t, we get

$$\begin{aligned}\sqrt{\theta(x,t)} &\le \sqrt{\theta(y,t)} + \frac{1}{2}\int_0^1 \frac{|\theta_x|}{\sqrt{\theta}}(\xi,t)d\xi \\ &\le \sqrt{\theta(y,t)} + \frac{1}{2}\left(C_2\int_0^1 \frac{\theta_x^2}{\theta^2}(\xi,t)d\xi\right)^{1/2}.\end{aligned} \tag{9.2.57}$$

Hence

$$\max_{x\in[0,1]}\theta(x,t) \le C_5\left(1+\int_0^1 \frac{\theta_x^2}{\theta^2}(\xi,t)d\xi\right). \tag{9.2.58}$$

Step 5. Multiplying (9.2.28) by u_t and integrating the result over x, we can obtain from (9.2.41)

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u_t^2(x,t)dx + \mu\int_0^1 u_{xt}^2(x,t)dx \le \int_0^1 (V_0+|\beta|\theta+\gamma_0|u_x|)|u_{xt}|(x,t)dx, \tag{9.2.59}$$

which, by Hölder's inequality, together with (9.2.46) and (9.2.58), leads to

$$\begin{aligned}\frac{d}{dt}\int_0^1 u_t^2(x,t)dx + \int_0^1 u_{xt}^2(x,t)dx &\le C_6\left(1+\int_0^1 \theta^2(x,t)dx\right) \\ &\le C_7\left(1+\max_{x\in[0,1]}\theta(x,t)\right) \\ &\le C_8\left(1+\int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)dx\right).\end{aligned} \tag{9.2.60}$$

Step 6. For any $t>0$, if we set

$$y(t) := \int_0^1 u_t^2(x,t)dx,\ h(t) := C_8\int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)dx, \tag{9.2.61}$$

then from (9.2.54)–(9.2.56), it follows

$$\int_0^t y(\tau)d\tau \le C_4,\ \int_0^t h(\tau)d\tau \le C_8C_3, \tag{9.2.62}$$

and further (9.2.60) can be rewritten in the form

$$y'(t)+y(t) \le C_8+h(t), \quad \text{a.e. } t\in[0,+\infty). \tag{9.2.63}$$

Therefore, applying Theorem 2.2.8 to (9.2.63), we can complete the proof. □

9.3 Asymptotic behavior for a linear thermomagnetoelastic system

In order to apply Theorems 2.3.1, 2.3.4–2.3.5 and Corollary 2.3.1, we shall consider in this section initial boundary value problems for some linear thermo-magneto-elastic models describing elastic materials where reciprocal effects of the temperature, the magnetic field and the elastic displacement are taken into account. These results introduced here are chosen from Munõz Rivera and Racke [652].

We shall consider the initial boundary value problems for linear differential equations of the homogeneous, isotropic case in three space dimensions (see Munõz Rivera and Racke [652])

$$\begin{cases} u_{tt} - Eu - \alpha[\nabla \times h] \times \overrightarrow{H} + \gamma\nabla\theta = 0, & (9.3.1) \\ h_t - \Delta h - \beta\nabla \times [u_t \times \overrightarrow{H}] = 0, & (9.3.2) \\ \theta_t - \kappa\Delta\theta + \gamma \operatorname{div} u_t = 0, & (9.3.3) \end{cases}$$

subject to the initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad h(0,x) = h_0(x), \quad \theta(0,x) = \theta_0(x), \quad (9.3.4)$$

and the boundary conditions for h and θ

$$\nabla \times h \times \nu = 0, \quad h \cdot v = 0, \quad \theta = 0 \text{ on } \Gamma \quad (9.3.5)$$

and the memory type boundary condition for u,

$$u = 0 \text{ on } \Gamma_0, \quad u + r * \partial_\nu u = 0 \text{ on } \Gamma_1, \quad (9.3.6)$$

where $u = (u^1, u^2, u^3)' = u(t,x)$ is the displacement vector depending on the time variable $t \geq 0$ and on $x \in \mathbb{R}^3$, $h = (h^1, h^2, h^3)' = h(t,x)$ is the magnetic field, $\theta = \theta(t,x)$ is the temperature difference with respect to a fixed reference temperature, and $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$ and $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 \neq \emptyset$, and $\partial_\nu u$ is an abbreviation,

$$\partial_\nu u := (C_{ijkl} u^k_{,l} v^j)_{i=1,2,3} - \alpha H h^3 \nu + \alpha H \nu^3 h \quad (9.3.7)$$

which is the natural Neumann type boundary operator for equation (9.3.1), the sign $*$ denotes the convolution in time, i.e.,

$$(r * f)(t) := \int_0^t r(t-s) f(s) ds.$$

We assume that there is a point $x_0 \in \Omega$ such that

$$\begin{cases} \Gamma_0 = \{x \in \Gamma | (x - x_0) \cdot \nu(x) \leq 0\}, & (9.3.8) \\ \Gamma_1 = \{x \in \Gamma | (x - x_0) \cdot \nu(x) \geq a > 0\} & (9.3.9) \end{cases}$$

for some constant $a > 0$ and $\nu = \nu(x)$ denoting the exterior normal vector in $x \in \Gamma$. E is the elasticity operator

$$Eu = [(C_{ijkl} u^k_{,l})_{,j}]_{i=1,2,3}\,, \tag{9.3.10}$$

where C_{ijkl} $(i,j,k,l = 1,2,3)$ are the elastic moduli being constant here and leading in the homogeneous isotropic case under consideration to

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) \quad (\delta_{ij}\text{: Kronecker delta}) \tag{9.3.11}$$

whence

$$Eu = \mu\Delta u + (\lambda + \mu)\nabla \mathrm{div}\, u, \tag{9.3.12}$$

with positive constants λ and μ. Parameters α, β with $\alpha\beta > 0$, $\gamma \neq 0$ and $\kappa > 0$ are constants, $\overrightarrow{H} = (0,0,H)'$ is a constant vector with $H \neq 0$ distinguishing the x_3-direction. The notation "$,j$" means differentiation with respect to x_j, and superscript $'$ stands for the transposition of a vector or a matrix.

Differentiating the boundary condition on Γ_1, we can get

$$\partial_\nu u + \frac{r'}{r(0)} * \partial_\nu u = -\frac{u_t}{r(0)} \tag{9.3.13}$$

or, in terms of the associated resolvent kernel g, we have

$$\partial_\nu u = -\tau u_t - \tau g * u_t \tag{9.3.14}$$

with

$$\tau := \frac{1}{r(0)} > 0. \tag{9.3.15}$$

First, we shall assume that g essentially decays exponentially, i.e., for all $t \geq 0$,

$$\begin{cases} 0 < g(t) \leq c_0 e^{-g_0 t}, \\ -c_1 g(t) \leq g'(t) \leq -c_2 g(t), \\ -c_3 g'(t) \leq g''(t) \leq -c_4 g'(t), \end{cases} \tag{9.3.16}$$

with positive constants $g_0, c_0, c_1, c_2, c_3, c_4$. The classical example $\tilde{g}$ is

$$\tilde{g}(t) = c_0 e^{-g_0 t}.$$

The exponential type kernel together with the "damping" boundary condition on Γ_1 will lead to an exponential decay result for $(u, h, \theta)(t)$.

Second, we shall consider polynomially decaying kernels satisfying for any $t \geq 0$,

$$\begin{cases} 0 < g(t) \leq b_0(1+t)^{-p}, \\ -b_1 g(t)^{(p+1)/p} \leq g'(t) \leq -b_2 g(t)^{(p+1)/p}, \\ -b_3 |g'(t)|^{(p+2)/(p+1)} \leq g''(t) \leq -b_4 |g'(t)|^{(p+2)/(p+1)}, \end{cases} \tag{9.3.17}$$

with positive constants b_0, b_1, b_2, b_3, b_4 and $p \geq 1$. The typical example $\overline{g}$ is

$$\overline{g}(t) = b_0(1+t)^{-p}.$$

The required result will be a polynomial decay for the solution $(u, h, \theta)(t)$.

We assume that the initial magnetic field h_0 satisfy

$$\operatorname{div} h_0 = 0 \tag{9.3.18}$$

which implies by (9.3.2), for all $t \geq 0$,

$$\operatorname{div} h(t, \cdot) = 0.$$

Concerning the resolvent kernel g, we assume either condition (9.3.16) or (9.3.17). Since we have formulated condition in terms of the resolvent kernel, we point out the following relationship between the decay of a kernel and the decay of the associated resolvent kernel.

Let

$$b(t) := -\frac{r'(t)}{r(0)}.$$

Then b and g satisfy

$$b + g = -b * g.$$

Lemma 9.3.1 ([652]).

(i) *If g satisfies that there exist a constant $\gamma > 0$ and a constant $c_g > 0$ such that for all $t \geq 0$:*

$$|g(t)| \leq c_g e^{-\gamma t},$$

and if for some constant $0 < \varepsilon < \gamma$,

$$c_g < \gamma - \varepsilon$$

holds, then we have for all $t \geq 0$,

$$|b(t)| \leq \frac{c_g(\gamma - \varepsilon)}{\gamma - \varepsilon - c_g} e^{-\varepsilon t}.$$

(ii) *If g satisfies that there exist a number $p > 1$ and a constant $c_g > 0$, such that for all $t \geq 0$:*

$$|g(t)| \leq c_g(1+t)^{-p},$$

and if

$$\frac{1}{c_g} > c_p := \sup_{0 \leq t < +\infty} \int_0^t (1+t)^p (1+t-\tau)^{-p}(1+\tau)^{-p} d\tau$$

holds, then we have for all $t \geq 0$,

$$|b(t)| \leq \frac{c_g}{1 - c_g c_p}(1+t)^{-p}.$$

Proof. (i) Let

$$\tilde{g}(t) := e^{\varepsilon t} g(t), \ \ \tilde{b}(t) := e^{\varepsilon t} b(t).$$

Then

$$\tilde{g} + \tilde{b} = -\tilde{g} * \tilde{b}.$$

Note that the operator G given by

$$G(h) := \tilde{g} * h,$$

acting on $h \in C[0,T]$, $T > 0$ arbitrary, but fixed, has the norm less than or equal to $\frac{c_g}{\gamma - \varepsilon}$. Hence

$$\sup_{0 \le t \le T} |\tilde{b}(t)| \le \frac{1}{1 - c_g/(\gamma - \varepsilon)} \sup_{0 \le t \le T} |\tilde{g}(t)| \le \frac{c_g(\gamma - \varepsilon)}{\gamma - \varepsilon - c_g},$$

which implies the assertion in (i).

(ii) Let

$$\tilde{g}(t) := (1+t)^p g(t), \ \ \tilde{b}(t) := (1+t)^p b(t).$$

Then

$$\tilde{g} + \tilde{b} = -k[g] * b$$

with kernel

$$k[g](t, \tau) := \tilde{g}(t - \tau)(1+t)^p (1 + t - \tau)^{-p} (1 + \tau)^{-p}.$$

The operator $K[g]$ acting on $C[0,T]$ as

$$K[g](h) := k[g] * h$$

has the norm less than or equal to

$$c_g \sup_{0 \le t < +\infty} \int_0^t (1+t)^p (1+t-\tau)^{-p} (1+\tau)^{-p} d\tau \le c_g c_p.$$

Hence

$$\sup_{0 \le t \le T} |\tilde{b}(t)| \le \frac{1}{1 - c_g c_p} \sup_{0 \le t \le T} |\tilde{g}(t)| \le \frac{c_g}{1 - c_g c_p},$$

which gives us the assertion (ii). □

Remark 9.3.1 ([652])**.** For the finiteness of c_p, compare Lemma 7.4 in [822] in a more general setting.

Remark 9.3.2 ([652])**.** Since the resolvent kernel of the resolvent kernel is the original kernel, it is clear that in (ii) of Lemma 9.3.1, no stronger uniform polynomial decay can be obtained. In this sense, the characterization is sharp and shows that exponentially decaying kernels correspond to exponentially decaying resolvents, and polynomial decaying kernels correspond to polynomially decaying resolvents.

Taking the boundary condition (9.3.6) for u on Γ_1 in the form (9.3.14), and performing an integration by parts, we may get

$$\begin{aligned}\partial_\nu u &= -\tau u_t - \tau g * u_t \\ &= -\tau u_t - \tau g(0)u - \tau g' * u + \tau g u_0.\end{aligned} \tag{9.3.19}$$

As a non-negative energy function, we can define

$$\begin{aligned}F(t) := \frac{1}{2}\int_\Omega \left(|u_t|^2 + C_{ijkl}u^k_{,l}u^i_{,j} + \frac{\alpha}{\beta}|h|^2 + |\theta|^2\right)(t,x)dx \\ - \frac{\tau}{2}\int_{\Gamma_1}(g'\Box u)(t,z)dz + \frac{\tau}{2}g(t)\int_{\Gamma_1}|u|^2(t,z)dz,\end{aligned} \tag{9.3.20}$$

where

$$(f\Box\phi)(t) := \int_0^t f(t-s)|\phi(t)-\phi(s)|^2 ds.$$

Lemma 9.3.2 ([652]). *For $f, \phi \in C^1([0,+\infty), \mathbb{R})$, we have*

$$2(f*\phi)(t)\phi_t(t) = (f'\Box\phi)(t) + \frac{d}{dt}\left\{\int_0^t f(s)ds|\phi(t)|^2 - (f\Box\phi)(t)\right\} - f(t)|\phi(t)|^2.$$

Proof. Obviously, we have

$$\begin{aligned}\frac{d}{dt}(f\Box\phi)(t) &= f'\Box\phi + 2\int_0^t f(t-s)(\phi(t)-\phi(s))ds\ \phi_t \\ &= f'\Box\phi - 2f*\phi\ \phi_t + 2\int_0^t f(s)ds\ \phi\phi_t \\ &= f'\Box\phi - 2f*\phi\ \phi_t + \frac{d}{dt}\left\{\int_0^t f(s)ds|\phi|^2\right\} - f|\phi|^2.\end{aligned}$$

□

Lemma 9.3.3 ([652]). *The following estimate holds*

$$\begin{aligned}\frac{dF(t)}{dt} \le &- \frac{\alpha}{\beta}\int_\Omega |\nabla\times h|^2 dx - \kappa\int_\Omega (\nabla\theta)^2 dx - \frac{\tau}{2}\int_{\Gamma_1}|u_t|^2 dz \\ &- \frac{\tau}{2}\int_{\Gamma_1} g''\Box u dz + \frac{\tau g'}{2}\int_{\Gamma_1}|u|^2 dz + \tau g^2\int_{\Gamma_1}|u_0|^2 dz.\end{aligned}$$

Proof. Multiplying equation (9.3.1) by u_t, equation (9.3.2) by $\frac{\alpha}{\beta}h$ and (9.3.3) by θ, and integrating and summing the resulting equalities, we obtain

$$\frac{1}{2}\frac{d}{dt}M(t) = \int_{\Gamma_1}\partial_\nu u u_t dz - \frac{\alpha}{\beta}\int_\Omega |\nabla\times h|^2 dx - \kappa\int_\Omega |\nabla\theta|^2 dz \tag{9.3.21}$$

with

$$M = M(t) =: \int_\Omega \left(|u_t|^2 + C_{ijkl}u^k_{,l}u^i_{,j} + \frac{\alpha}{\beta}|h|^2 + |\theta|^2\right)dx.$$

Using (9.3.19) and Lemma 9.3.2, we can get

$$\begin{aligned}\frac{dM(t)}{dt} = & -\frac{\alpha}{\beta}\int_{\Omega}|\nabla\times h|^2dx - \kappa\int_{\Omega}|\nabla\theta|^2dx - \tau\int_{\Gamma_1}|u_t|^2dz \\ & -\tau g(0)\int_{\Gamma_1}uu_tdz - \tau\int_{\Gamma_1}(g'*u)u_tdz + \tau g\int_{\Gamma_1}u_0u_tdz \\ = & -\frac{\alpha}{\beta}\int_{\Omega}|\nabla\times h|^2dx - \kappa\int_{\Omega}|\nabla\theta|^2dx - \tau\int_{\Gamma_1}|u_t|^2dz \\ & -\frac{1}{2}\frac{d}{dt}\left\{\tau g\int_{\Gamma_1}|u|^2dz - \tau\int_{\Gamma_1}g'\Box udz\right\} \\ & -\frac{\tau}{2}\int_{\Gamma_1}g''\Box udz + \frac{\tau g'}{2}\int_{\Gamma_1}|u|^2dz + \tau g\int_{\Gamma_1}u_0u_tdz\end{aligned}$$

which yields the assertion. $\Box$

Define

$$q(x) := x - x_0.$$

Lemma 9.3.4 ([652]). *Let*

$$f := \alpha(\nabla\times h)\times\overrightarrow{H} - \gamma\nabla\theta.$$

Then we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}u_tq^ku_{,k}dx = & \int_{\Gamma}\frac{\partial u}{\partial\nu_A}q^ku_{,k}dz - \int_{\Omega}C_{ijml}u^m_{,l}q^k_{,j}u^i_{,k}dx \\ & -\frac{1}{2}\int_{\Gamma}q^k\nu^kC_{ijml}u^m_{,l}u^i_{,j}dz + \frac{1}{2}\int_{\Gamma}q^k\nu^k|u_t|^2dz \\ & +\frac{1}{2}\int_{\Omega}q^k_{,k}C_{ijml}u^m_{,l}u^i_{,j}dx - \frac{1}{2}\int_{\Omega}q^k_{,k}|u_t|^2dx + \int_{\Omega}fq^ku_{,k}dx.\end{aligned}$$

Proof. Multiplying (9.3.1) by $q^ku_{,k}$ and integrating the result, we arrive at

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}u_tq^ku_{,k}dx = & \int_{\Omega}u_{tt}q^ku_{,k}dx + \int_{\Omega}u_tq^ku_{t,k}dx \\ = & \int_{\Omega}(C_{ijml}u^m_{,l})_{,j}q^ku^i_{,k}dx + \frac{1}{2}\int_{\Omega}|u_t|^2_{,k}q^kdx + \int_{\Omega}fq^ku_{,k}dx \\ = & \int_{\Gamma}C_{ijml}u^m_{,l}v^jq^ku^i_{,k}dz - \int_{\Omega}C_{ijml}u^m_{,l}\partial_j(q^ku^i_{,k})dx \\ & +\frac{1}{2}\int_{\Gamma}q^k\nu^k|u_t|^2dz - \frac{1}{2}\int_{\Omega}q^k_{,k}|u_t|^2dx + \int_{\Omega}fq^ku_{,k}dx.\end{aligned}$$

Using the symmetry of the moduli C_{ijml}, we conclude

$$\frac{d}{dt}\int_\Omega u_t q^k u_{,k} dx = \int_\Gamma \frac{\partial u}{\partial v_A} q^k u_{,k} dz - \int_\Omega C_{ijml} u^m_{,l} q^k_{,j} u^i_{,k} dx$$
$$- \frac{1}{2}\int_\Omega q^k (C_{ijml} u^m_{,l} u^i_{,j})_{,k} dx + \frac{1}{2}\int_\Gamma q^k \nu^k |u_t|^2 dz$$
$$- \frac{1}{2}\int_\Omega q^k_{,k} |u_t|^2 dx + \int_\Omega f q^k u_{,k} dx. \qquad \square$$

From Lemma 9.3.4, we get by using $q^k = x^k - x^k_0$,

$$\frac{d}{dt}\int_\Omega u_t q^k u_{,k} dx = \int_\Gamma \frac{\partial u}{\partial v_A} q^k u_{,k} dz - \int_\Omega C_{ijml} u^m_{,l} u^i_{,k} dx$$
$$- \frac{1}{2}\int_\Omega q^k \nu^k C_{ijml} u^m_{,l} u^i_{,j} dx + \frac{1}{2}\int_\Gamma q^k \nu^k |u_t|^2 dz$$
$$+ \frac{3}{2}\int_\Omega C_{ijml} u^m_{,l} u^i_{,j} dx - \frac{3}{2}\int_\Omega |u_t|^2 dx + \int_\Omega f q^k u_{,k} dx$$

which thus implies

$$\frac{d}{dt}\int_\Omega u_t q^k u_{,k} dx = \int_\Gamma \frac{\partial u}{\partial \nu_A} q^k u_{,k} dz - \frac{1}{2}\int_\Omega (|u_t|^2 + C_{ijml} u^m_{,l} u^i_{,k})_k dx$$
$$+ \int_\Omega (C_{ijml} u^m_{,l} u^i_{,k} - |u_t|^2) dx - \frac{1}{2}\int_\Gamma q^k v^k C_{ijml} u^m_{,l} u^i_{,k} dz$$
$$+ \frac{1}{2}\int_\Gamma q^k v^k |u_t|^2 dz + \int_\Omega f q^k u_{,k} dx. \tag{9.3.22}$$

Using the differential equation (9.3.1), we obtain

$$\frac{d}{dt}\int_\Omega u_t u dx = \int_\Omega u u_{tt} dx + \int_\Omega |u_t|^2 dx$$
$$= \int_\Omega u^i \{C_{ijml} u^m_{,l})_{,j} + f^i\} dx + \int_\Omega |u_t|^2 dx$$
$$= \int_\Gamma u \frac{\partial u}{\partial \nu_A} dz - \int_\Omega (C_{ijml} u^m_{,l} u^i_{,j} - |u_t|^2) dx + \int_\Omega u f dx$$

which yields

$$\int_\Omega (C_{ijml} u^m_{,l} u^i_{,j} - |u_t|^2) dx = -\frac{d}{dt}\int_\Omega u_t u dx + \int_\Gamma u \frac{\partial u}{\partial \nu_A} dz + \int_\Omega u f dx.$$

Thus substituting this identity into (9.3.52), we get

$$\begin{aligned}\frac{d}{dt}\chi(t) = &\int_\Gamma \frac{\partial u}{\partial \nu_A} q^k u_{,k} dx - \frac{1}{2}\int_\Omega (|u_t|^2 + C_{ijml} u^m_{,l} u^i_{,j}) dz \\ &+ \int_\Gamma u \frac{\partial u}{\partial \nu_A} dz + \int_\Omega uf dx - \frac{1}{2}\int_\Gamma q^k \nu^k C_{ijml} u^m_{,l} u^i_{,j} dz \\ &+ \frac{1}{2}\int_\Gamma q^k \nu^k |u_t|^2 dz + \int_\Omega f q^k u_{,k} dx\end{aligned}$$

with

$$\chi(t) =: \int_\Omega u_t q^k u_{,k} dx + \int_\Omega u_t u dx.$$

Since $u = 0$ on Γ_0, we have

$$\begin{cases} \displaystyle\int_{\Gamma_0} \frac{\partial u}{\partial \nu} q^k u_{,k} dz = \int_{\Gamma_0} \left|\frac{\partial u}{\partial \nu}\right|^2 q^k \nu^k dz, & (9.3.23) \\ \displaystyle\int_{\Gamma_0} \operatorname{div} u \nu q^k u_{,k} dz = \int_{\Gamma_0} q^k \nu^k |\operatorname{div} u|^2 dz, & (9.3.24) \\ \displaystyle\int_{\Gamma_0} (\nabla u)\nu q^k u_{,k} dz = \int_{\Gamma_0} q^k \nu^k |\operatorname{div} u|^2 dz. & (9.3.25) \end{cases}$$

Observing that (9.3.11) yields

$$\frac{\partial u}{\partial \nu_A} = \lambda \operatorname{div} u\nu + \mu \frac{\partial u}{\partial \nu} + \mu(\nabla u)\nu,$$

we conclude from (9.3.23)–(9.3.25)

$$\int_\Gamma \frac{\partial u}{\partial \nu_A} q^k u_{,k} dz = \int_{\Gamma_0} q^k \nu^k \left\{\mu \left|\frac{\partial u}{\partial \nu}\right|^2 + (\mu+\lambda)|\operatorname{div} u|^2\right\} dz + \int_{\Gamma_1} \frac{\partial u}{\partial v_A} q^k u_{,k} dz$$

which implies

$$\begin{aligned}\frac{d}{dt}\chi(t) = &\int_{\Gamma_0} q^k \nu^k \left\{\mu \left|\frac{\partial u}{\partial \nu}\right|^2 + (\mu+\lambda)|\operatorname{div} u|^2\right\} dx \\ &+ \int_{\Gamma_1} \frac{\partial u}{\partial \nu_A} q^k u_{,k} dx - \frac{1}{2}\int_\Omega \{|u_t|^2 + C_{ijml} u^m_{,l} u^i_{,j}\} dx \\ &- \frac{1}{2}\int_\Gamma q^k \nu^k C_{ijml} u^m_{,l} u^i_{,j} dx + \int_\Gamma u \frac{\partial u}{\partial \nu_A} dz \\ &+ \int_\Omega uf dz + \frac{1}{2}\int_{\Gamma_1} q^k \nu^k |u_t|^2 dz + \int_\Omega f q^k u_{,k} dx.\end{aligned} \tag{9.3.26}$$

Since on Γ_1,

$$q^k \nu^k \geq a > 0,$$

we can get

$$\begin{cases} \int_{\Gamma_1} \frac{\partial u}{\partial \nu_A} q^k u_{,k} dz \le c \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu_A} \right|^2 dz + \frac{1}{8} \int_{\Gamma_1} q^k \nu^k C_{ijml} u^m_{,l} u^i_{,j} dz, & (9.3.27) \\ \int_{\Gamma_1} \frac{\partial u}{\partial \nu_A} u dz \le c \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu_A} \right|^2 dz + \frac{1}{8} \int_{\Omega} C_{ijml} u^m_{,l} u^i_{,j} dx, & (9.3.28) \end{cases}$$

where $c > 0$ denotes various positive constants.

Noting that on Γ_0,

$$C_{ijml} u^m_{,l} u^i_{,j} = \mu |\frac{\partial u}{\partial \nu}|^2 + (\mu + \lambda) |\operatorname{div} u|^2$$

and using (9.3.26)–(9.3.28), we obtain

$$\begin{aligned} \frac{d}{dt}\chi(t) \le & -\frac{1}{4} \int_{\Omega} (|u_t|^2 + C_{ijml} u^m_{,l} u^i_{,j}) dx + c \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu_A} \right|^2 dz \\ & + \frac{1}{2} \int_{\Gamma_1} q^k \nu^k |u_t|^2 dz - \frac{1}{4} \int_{\Gamma_1} q^k \nu^k C_{ijml} u^m_{,l} u^i_{,j} dz \\ & + \int_{\Omega} u f dx + \int_{\Omega} f q^k u_{,k} dx. \end{aligned}$$

On Γ_1, using (9.3.19), we have

$$\begin{aligned} \frac{\partial u}{\partial \nu_A} &= \partial_v u + \alpha H h^3 \nu + \alpha H h \nu^3 \\ &= -\tau u_t - \tau g u - \tau \int_0^t g'(t-s) \left(u(s, \cdot) - u(t, \cdot) \right) ds \\ &\quad + \tau g u_0 + \alpha H h^3 \nu - \alpha H h \nu^3. \end{aligned}$$

Noting that

$$\int_0^t g'(t-s) \left(u(s, \cdot) - u(0, \cdot) \right) ds \le \left(\int_0^t |g'(s)| ds \right)^{1/2} \left(|g'| \Box u \right)^{1/2},$$

we arrive at

$$\int_{\Omega} |f|^2 dx \le c \int_{\Omega} |\nabla \times h|^2 dx + c \int_{\Omega} |\nabla \theta|^2 dx,$$

$$\int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu_A} \right|^2 dz \le c \int_{\Gamma_1} (|u_t|^2 + g^2 |u|^2 + |g'| \Box u) dx + c g^2 F(0) + c \int_{\Omega} |\nabla \times h|^2 dx.$$

Using

$$\int_{\Omega} |h|^2 dx + \int_{\Omega} |\nabla h|^2 dx \le c \int_{\Omega} |\nabla \times h|^2 dx$$

in our situation because of the properties of h and the fact that Ω is simply connected (see page 358 in [211] or page 157 in [498]), we thus have proved the following lemma.

Lemma 9.3.5 ([652]). *The following estimate holds,*

$$\begin{aligned}\frac{d}{dt}\chi(t) \leq & -\frac{1}{2}\int_\Omega (|u_t|^2 + C_{ijml}u^m_{,l}u^i_{,j})dx - \frac{1}{4}\int_{\Gamma_1} C_{ijml}u^m_{,l}u^i_{,j}dz \\ & + c\int_\Omega (|\nabla\times h|^2 + |\nabla\theta|^2)dx + c\int_{\Gamma_1}|u_t|^2 dz \\ & + c\int_{\Gamma_1}|g'|\Box u dz + cg^2F(0).\end{aligned}$$

Now we can prove the following main results which is due to [652].

Theorem 9.3.6 ([652]). *Let g be an exponentially decaying resolvent kernel as in* (9.3.16)*, and assume* (9.3.18) *holds. Then the energy F defined in* (9.3.20)*, which is associated to the solution of the initial boundary value problem* (9.3.1)–(9.3.6)*, decays exponentially, i.e., there exist constants $d_0 > 0, d_1 > 0$ such that for all $t \geq 0$,*

$$F(t) \leq d_0 e^{-d_1 t}F(0). \tag{9.3.29}$$

Proof. Let

$$L(t) := NF(t) + \chi(t), \tag{9.3.30}$$

with $N > 0$ sufficiently large.

Then there are positive constants k_0, k_1 such that for all $t \geq 0$,

$$k_0F(t) \leq L(t) \leq k_1F(t). \tag{9.3.31}$$

Moreover, for N large enough, using Lemmas 9.3.3 and 9.3.5,

$$\frac{d}{dt}L(t) \leq -k_2F(t) + cg^2F(0)$$

with a constant $k_2 > 0$. Here we have used the assumption (9.3.16) in order to conclude the following estimates

$$-\frac{\tau}{2}\int_{\Gamma_1} g''\Box u dz \leq \tilde{c}\int_{\Gamma_1} g'\Box u dz$$

and

$$\frac{\tau}{2}\int_{\Gamma_1} g'|u|^2 dz \leq -\tilde{c}\int_{\Gamma_1} g|u|^2 dz$$

for the corresponding two terms appearing in Lemma 9.3.3, where $\tilde{c} > 0$ is a constant.

Thus we obtain

$$\frac{d}{dt}L(t) \le -\frac{k_2}{k_1}L(t) + cg^2F(0).$$

Using the exponential decay of g, we conclude from Theorem 2.3.1 or Corollary 2.3.1 that there exist constants $\tilde{d}_0, \tilde{d}_1 > 0$ such that for all $t \ge 0$:

$$L(t) \le \tilde{d}_0 e^{-\tilde{d}_1 t} L(0)$$

which implies (9.3.29) by using (9.3.31) again. □

Finally, we consider the case where g decays polynomially as in (9.3.17).

Theorem 9.3.7 ([652]). *Let g be a polynomial decaying resolvent kernel as in* (9.3.17), *and assume* (9.3.18) *holds. Then the energy F defined in* (9.3.20), *which is associated to the initial boundary value problem* (9.3.1)–(9.3.6), *decays polynomially, i.e., there exists a constant $d_2 > 0$ such that for all $t \ge 0$,*

$$F(t) \le \frac{d_2}{(1+t)^{p+1}}F(0). \tag{9.3.32}$$

Proof. We define the functional $L(t)$ as in (9.3.30) and we have the equivalence to the energy $F(t)$ as given in (9.3.31). A negative term

$$-cg(t)\int_{\Gamma_1} |u|^2(t,z)dz$$

can be obtained from Lemma 9.3.5 and the estimate

$$g(t)\int_{\Gamma_1} |u|^2(t,z)dz \le c\int_{\Omega} C_{ijml}u^m_{,l}u^i_{,j}(t,x)dx.$$

From Lemmas 9.3.3 and 9.3.5, using the properties of g'' from the assumption (9.3.17) for the term

$$-\frac{\tau}{2}\int_{\Gamma_1} g''\Box u dz,$$

we can obtain

$$\begin{aligned}\frac{d}{dt}L(t) \le & -k_3\left(M(t) + g(t)\int_{\Gamma_1}|u|^2dz + N\int_{\Gamma_1}|g'|^{1+\frac{1}{1+p}}\Box u dz\right)\\ & + k_4\int_{\Gamma_1}|g'|\Box u dz + cg^2F(0),\end{aligned} \tag{9.3.33}$$

where k_3, k_4 denote positive constants and $M = M(t)$ was defined in (9.3.21). □

To continue the proof, we need several inequalities collected in the next two lemmas which are based on those from [648], partially extending those.

Lemma 9.3.8 ([652]). *Let m and h be integrable functions, and let $0 \le r < 1$ and $q > 0$. Then for any $t \ge 0$, we have*

$$\int_0^t |m(t-\tau)h(\tau)|d\tau \tag{9.3.34}$$
$$\le \left(\int_0^t |m(t-\tau)|^{1+\frac{1-r}{q}}|h(\tau)|d\tau\right)^{q/(q+1)} \left(\int_0^t |m(t-\tau)|^r|h(\tau)|d\tau\right)^{1/(q+1)}.$$

Proof. Let

$$v(\tau) := |m(t-\tau)|^{1-(r/(q+1)}|h(\tau)|^{q/(q+1)},$$
$$w(\tau) := |m(t-\tau)|^{r/(1+q)}|h(\tau)|^{1/(q+1)}.$$

Then for any fixed $t \ge 0$, we obtain

$$|m(t-\tau)h(\tau)| = |v(\tau)h(\tau)|.$$

Applying the Hölder's inequality with exponents

$$\delta = \frac{q}{q+1} \text{ for } v \text{ and } \delta^* = q+1 \text{ for } w,$$

we get the desired estimate (9.3.34). □

Lemma 9.3.9 ([652]). *Let $p > 1$, $0 \le r < 1$ and $t \ge 0$. Then we have, for $0 < r < 1$,*

$$\int_{\Gamma_1} |g'|\Box u dz \le 2\left(\int_0^t |g'(\tau)|^r ||u||^2_{L^\infty((0,t),L^2(\Gamma_1))}\right)^{1/(1+(1-r)(p+1))}$$
$$\times \left(\int_{\Gamma_1} |g'|^{1+\frac{1}{p+1}}\Box u dz\right)^{(1-r)(p+1)/(1+(1-r)(p+1))}, \tag{9.3.35}$$

and for $r = 0$,

$$\int_{\Gamma_1} |g'|\Box u dz \le 2\left(\int_0^t ||u(\tau,\cdot)||^2_{L^2(\Gamma_1)}d\tau + t||u(\tau,\cdot)||^2_{L^2(\Gamma_1)}\right)^{1/(p+2)}$$
$$\times \left(\int_{\Gamma_1} |g'|^{1+\frac{1}{p+1}}\Box u dz\right)^{(p+1)/(p+2)}. \tag{9.3.36}$$

Proof. Applying Lemma 9.3.6 with $m(\tau) := |g'(\tau)|, h(\tau) := \int_{\Gamma_1} |u(t) - u(\tau)|^2 dz$ and $q := (1-r)(p+1)$ ($t > 0$ fixed) proves the lemma immediately. □

Applying Lemma 9.3.7 with $1 > r > 0$, we get

$$\int_{\Gamma_1} |g'|^{1+(1/(p+1))}\Box u dz \ge \frac{c}{\left(\int_0^t |g'|^r(\tau)dz\right)^{1+(1/(1-r)(p+1))} F(0)^{1/(1-r)(p+1)}}$$
$$\times \left(\int_{\Gamma_1} |g'|\Box u dz\right)^{1+(1/(1-r)(p+1))}$$

(with $c = c(r)$ as long as r is not yet fixed).

On the other hand, we may have

$$\left(cg\int_{\Gamma_1}|u|^2dz+M\right)^{1+\frac{1}{(1-r)(p+1)}}\leq cF(0)^{\frac{1}{(1-r)(p+1)}}\left(M+cg\int_{\Gamma_1}|u|^2dz\right).$$

We conclude from (9.3.33) using the last two inequalities:

$$\frac{d}{dt}L(t)\leq\frac{-c}{F(0)^{\frac{1}{(1-r)(p+1)}}}\left[\left(cg\int_{\Gamma_1}|u|^2dz+M\right)^{1+\frac{1}{(1-r)(p+1)}}\right.$$
$$\left.+\left(\int_{\Gamma_1}|g'|\Box u dz\right)^{1+\frac{1}{(1-r)(p+1)}}\right]+cg^2F(0)$$

if $r>1/(p+1)$ such that $\int_0^{+\infty}|g'|^r(\tau)d\tau<+\infty$.

This implies, using (9.3.32),

$$\frac{d}{dt}L(t)\leq\frac{-\tilde{c}}{L(0)^{\frac{1}{(1-r)(p+1)}}}L(t)^{1+\frac{1}{(1-r)(p+1)}}+cg^2(t)L(0) \tag{9.3.37}$$

with some constant $\tilde{c}>0$. Thus Theorem 2.3.4 with $f=L,\alpha=(1-r)(p+1)$ and $\beta=p^2$ gives us

$$L(t)\leq\frac{c}{(1+t)^{(1-r)(p+1)}}L(0). \tag{9.3.38}$$

Choosing $1\geq r>1/(p+1)$ such that

$$\alpha=(1-r)(p+1)>1$$

or, equivalently,

$$\frac{1}{1+p}<r<\frac{p}{1+p},$$

we obtain from the inequality (9.3.38)

$$\int_0^{+\infty}F(\tau)d\tau\leq c\int_0^{+\infty}L(\tau)d\tau\leq cL(0) \tag{9.3.39}$$

and

$$t||u(t,\cdot)||^2_{L^2(\Gamma_1)}\leq ctL(t)\leq cL(0) \tag{9.3.40}$$

as well as

$$\int_0^t||u(\tau,\cdot)||^2_{L^2(\Gamma_1)}d\tau\leq c\int_0^{+\infty}L(\tau)d\tau\leq cL(0). \tag{9.3.41}$$

From the estimates (9.3.39)–(9.3.41), we conclude, using Lemma 9.3.7 again, now with $r=0$:

$$\int_{\Gamma_1}|g'|^{1+1/(p+1)}\Box u dz\geq\frac{c}{F(0)^{1/(p+1)}}\left(\int_{\Gamma_1}|g'|\Box u dz\right)^{1+1/(p+1)},$$

and hence, with the same arguments as in the derivation of (9.3.37),

$$\frac{d}{dt}L(t) \le \frac{-\overline{c}}{L(0)^{1/(p+1)}}L(t)^{1+(1/(p+1))} + cg^2(t)L(0)$$

This implies by applying Theorems 2.3.4–2.3.5 again with $\alpha = p+1$, $\beta = 2p$ $(\ge p+1)$

$$L(t) \le \frac{c}{(1+t)^{p+1}}L(0)$$

and hence, by (9.3.31) and for some constant $d_2 > 0$ such that

$$F(t) \le \frac{d_2}{(1+t)^{p+1}}F(0)$$

which completes the proof of Theorem 9.3.2. □

Remark 9.3.3 ([652]). We note that for a thermoviscoelastic system discussed in [648], it was shown that a polynomial relaxation function cannot lead to an exponential decay. This gives us a hint for the conjecture that the polynomial decay rate obtained here can not be replaced by an exponential decay result. Indeed, for the system of pure elasticity, with the memory type boundary as discussed here, the exponential decay for exponential kernels was shown in [20]. A merely polynomial kernel there can not lead to a general exponential decay result, which can be seen as follows.

To end this section, we give an example.

In the one-dimensional case, the system of equations of elasticity in $\Omega := (0,1)$ with memory type boundary condition reduces to the problem

$$\begin{cases} u_{tt} - \alpha u_{xx} = 0, & (9.3.42) \\ u\mid_{t=0} = 0, \quad u_t\mid_{t=0} = u_1, & (9.3.43) \\ u\mid_{x=0} = 0, \quad (u + (u_x * r))\mid_{x=1} = 0, & (9.3.44) \end{cases}$$

where $\alpha > 0$ is a constant. The energy is given by

$$\mathcal{E}(t) := \int_0^1 (|u_t|^2 + \alpha|u_x|^2)(t,x)dx.$$

We assume

$$u_0 = 0, \quad u_1 \in C_0^\infty(\Omega)\backslash\{0\} \tag{9.3.45}$$

and for the kernel b with $-r'(t)/r(0)$:

$$b(t) = \frac{1}{(1+t)^p}$$

for some constant $p > 1$. Now the assumption that there exist constants $c > 0$ and $\delta > 0$ such that for all $t \geq 0$, we have

$$\mathcal{E}(t) \leq ce^{-\delta t}\mathcal{E}(0) \tag{9.3.46}$$

which will lead to a contradiction in the sequel.

Observe that $v := u_t$ satisfies the same system (9.3.42)–(9.3.44) as u due to the choice of the initial data in (9.3.45). Hence also the energy associated to v decays exponentially, which implies, using the differential equation and Sobolev's imbedding theorem, that there is a constant $c_0 > 0$ depending on the initial data such that for all $t \geq 0$:

$$|u(t,1)| + |u_x(t,1)| \leq c_0 e^{-\delta t}. \tag{9.3.47}$$

The boundary condition (9.3.43) can also be stated as (cf. (9.3.13))

$$(-b * u_x + u_x + \tau u)\ |_{x=1} = 0,$$

which implies by (9.3.47)

$$\left| \int_0^t \frac{1}{(1+t-s)^p} u_x(s,1)ds \right| \leq c_0 e^{-\delta t}. \tag{9.3.48}$$

On the other hand, by dividing the integral from 0 to t into two parts from 0 to $\frac{t}{2}$ and from $\frac{t}{2}$ to t, it can be easily seen that for any $m > 1$,

$$\left| \int_0^t \frac{1}{(1+t-s)^m} u_x(s,1)ds \right| \leq \frac{c_0}{(1+t)^m}. \tag{9.3.49}$$

For all $t \geq 0$ and $\beta \geq 0$, let

$$G_\beta(t) = \int_{t+\beta}^{+\infty} u_x(s,1)ds.$$

Then

$$\begin{aligned} \int_0^t \frac{1}{(1+t-s)^p} u_x(s,1)ds &= \left[\frac{1}{(1+s)^p} G_\beta(t-s) \right]_{s=0}^{s=t} + p \int_0^t \frac{1}{(1+s)^p} G_\beta(t-s)ds \\ &= \frac{G_\beta(0)}{(1+t)^p} - G_\beta(t) + O\left(\frac{1}{(1+t)^{p+1}} \right), \end{aligned} \tag{9.3.50}$$

where we have used (9.3.49) for $m = p+1$.

Case 1: There exists a $\tilde{\beta} \in [0, +\infty)$ such that $G_{\tilde{\beta}}(0) \neq 0$. Thus from (9.3.50), it follows

$$\lim_{t \to +\infty} \left| \int_0^t \frac{1}{(1+t-s)^p} u_x(s,1)ds \right| (1+t)^p = G_{\tilde{\beta}}(0) \neq 0$$

which is a contraction to (9.3.48).

Case 2: For all $\beta \in [0, +\infty) : \quad G_\beta(0) = 0$. This implies that for all $t \geq 0$,

$$u_x(t, 1) = 0,$$

and hence, using the boundary condition and the initial condition $u_0 = 0$, we see that u satisfies

$$\begin{cases} u_{tt} - \alpha u_{xx} = 0, \\ \qquad u\,|_{t=0} = 0, \quad u_t\,|_{t=0} = u_1, \\ \qquad u\,|_{x=0} = 0, \quad u\,|_{x=1} = 0, \end{cases}$$

which yields that the energy

$$\mathcal{E}(t) = \int_0^1 |u_1(x)|^2 dx$$

is a constant, this is a contraction to assumption (9.3.46).

Chapter 10

Blow-up of Solutions to Nonlinear Hyperbolic Equations and Hyperbolic-Elliptic Inequalities

In this chapter, we shall consider the blow-up of solutions to some nonlinear hyperbolic equations and hyperbolic-elliptic inequalities. This chapter consists of seven sections. In Section 10.1, we apply Theorem 2.4.6 to investigate the blow-up of solutions to semilinear wave equations. In Section 10.2, we shall employ Theorem 2.4.22 to study the blow-up of solutions to semilinear wave equations. In Section 10.3, we shall employ Theorem 2.4.7 or Corollary 2.4.2 to establish the blow-up of solutions of some nonlinear hyperbolic equations. In Section 10.4, we shall exploit Corollary 2.4.8 to study the blow-up and estimates of the lifespan of solutions to semilinear wave equations. In Section 10.5, we shall employ Theorem 2.4.9 to investigate the Cauchy problem for the dissipative nonlinear wave equations. In Section 10.6, we shall apply Theorem 2.4.4 to investigate the blow-up of solutions to wave equations with a nonlinear dissipation. In Section 10.7, we shall apply Theorem 2.4.26 to prove the non-existence of global non-negative solutions to the quasilinear hyperbolic-elliptic inequalities. Inequalities chosen in this chapter are specially selected and very important in deriving the blow-up of solutions to nonlinear hyperbolic equations and hyperbolic-elliptic inequalities.

10.1 Blow-up of solutions for nonlinear wave equations

In this section, we apply Theorem 2.4.6 to study the blow-up of solutions to semilinear wave equations. Here we shall choose the results due to Glassey [318].

Consider the following Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = |u|^p, \quad x \in \mathbb{R}^n, \ \ 0 < t \le T, & (10.1.1) \\ u(x,0) = f(x), \ \ u_t(x,0) = g(x), & (10.1.2) \end{cases}$$

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where Δ denotes the Laplacian. We expect "small" solutions to exist globally if a given sufficiently regular datum is small at infinity and p is large enough (depending on n), while most solutions should blow up in a finite time if p is "too small" (but greater than one). Thus we can guess that there exists a critical value of p say $p_0(n)$ such that global existence of all small solutions holds if $p > p_0(n)$, while most solutions blow up in a finite time if $1 < p < p_0(n)$.

The following theorem is due to Glassey [318].

Theorem 10.1.1 ([318]). *Let $0 < T, n \le 3$, and $u(x,t) \in C^2(\mathbb{R}^n \times [0,T))$ be a solution to problem* (10.1.1)–(10.1.2).

(1) *Assume that*
   (i) $f, g \in C_0^\infty(\mathbb{R}^n)$, *supp*$\{f,g\} \subseteq \{|x| \le k\}$;
   (ii) $c_f \equiv \int_{\mathbb{R}^n} f(x)dx > 0, \quad c_g \equiv \int_{\mathbb{R}^n} g(x)dx > 0$;
   (iii) $1 < p < \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$ $(1 < p < +\infty, \ \ if \ \ n = 1)$.
   Then $T < +\infty$, i.e., u blows up in a finite time.

(2) *Let $p = 1 + \sqrt{2}$ when $n = 3$, let the Cauchy data satisfy above* (i) *and* (ii), *then there exists a positive constant c_0, depending only on k, such that if $u \in C^2(\mathbb{R}^3 \times [0,T))$ is a solution to problem* (10.1.1)–(10.1.2), *then $T < +\infty$, provided that $c_f \ge c_0$ and $c_g \ge c_0$.*

Remark 10.1.1 ([318]).

(1) We do not discuss the case $n = 1$ explicitly since the result of it follows from Kato's theorem [418]. However, the proof given here extends easily to $n = 1$. We also note that there is no critical value of p for $n = 1$, because the free solution (the solution of $u_{tt} - u_{xx} = 0$ with the same data) does not decay uniformly to zero as $t \to +\infty$.

(2) We observe the critical value of p,

$$p_0(n) = \frac{1}{2(n-1)}[n + 1 + (n^2 + 10n - 7)^{1/2}] \tag{10.1.3}$$

is the large root of the quadratic

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

(3) For $n = 3$, we know from John's theorem [405] that the present blow-up result up to and including the critical value $p_0(3) = 1 + \sqrt{2}$ is sharp, so we also expect that $p_0(2) = \frac{1}{2}(3 + \sqrt{17})$ is sharp.

(4) We note that hypothesis (i) implies the existence of a unique classical local in time solution to problem (10.1.1) (for $n \le 3$). Such a solution u can be obtained by iteration; thus u has compact support in x for each fixed t. Moreover, it suffices to assume that $f \in C_0^3, g \in C_0^2$ over $\mathbb{R}^n; n = 2, 3$.

(5) Strauss in [898] discussed a similar quadratic equation for p arising in analyzing the asymptotic behavior of solutions to nonlinear Schrödinger equations of the form

$$iu_t = \Delta u + u^p,$$

and had also answered an early conjecture of the present result.

First we derive the lower bounds on L^p-norms of solutions of problem (10.1.1). Note that the free solution will be the solution u_0 of

$$\frac{\partial^2 u_0}{\partial t^2} - \Delta u_0 = 0 \tag{10.1.4}$$

with the same Cauchy data as u. By L^q-norms we denote

$$\|h\|_q = \left(\int_{\mathbb{R}^n} |h(x)|^q dx\right)^{1/q}, \quad (q \geq 1)$$

an integral without explicit limits is to be taken over all of $\mathbb{R}^n$, and the letters c are universal constants which may change from line to line.

The proof of the following lemma can be found in [318].

Lemma 10.1.2 ([318]). *Let $u \in C^2(\mathbb{R}^n \times [0,T))$ be a solution of problem* (10.1.1) *with the Cauchy data satisfying hypotheses* (i) *and* (ii) *of Theorem* 10.1.1. *Let $p > 1$ if $n = 3$; $p > 2$ if $n = 2$, then there exists a positive constant c, depending only on p, n, k, c and C_g such that*

$$\int |u|^p dx \geq c(k+t)^{\frac{-(n-1)(p-2)}{2}} ln^{-v}(1+k+t) \tag{10.1.5}$$

on $0 \leq t < T$, where

$$v = \begin{cases} \frac{1}{2}(p-2), & if \;\; n = 2, \\ v = 0, & if \;\; n = 3. \end{cases}$$

Corollary 10.1.1 ([318]). *Let $k, n, p > 2$ be fixed, then the constant c in* (10.1.5) *tends to infinity as c_f and c_g tend to infinity.*

Now we are in a position to prove Theorem 10.1.1.

Proof. Let u satisfy (10.1.1)–(10.1.2), we shall derive a contradiction by assuming that $T = +\infty$.

First assume that $1 < p \leq 1 + 2/n$ for any n. Define

$$F(t) = \int u dx. \tag{10.1.6}$$

Since u has the compact support, the integral extends only over the set $\{|x| < k+t\}$.

Integrating (10.1.1), we obtain

$$F''(t) = \int |u|^p dx = \int_{|x|<k+t} |u|^p dx. \tag{10.1.7}$$

By Hölder's inequality, we have

$$F(t) \leq c\|u(t)\|_p (k+t)^{n(\frac{p-1}{p})} \tag{10.1.8}$$

where a constant $c > 0$ depends only on n, p. Thus (10.1.7) implies

$$F''(t) \geq cF^p(t)(k+t)^{-n(p-1)} \tag{10.1.9}$$

where we have used the fact $F(t) > 0$ and $u \geq u_0 \quad on \quad \mathbb{R}^n \times [0, T)$,

$$F(t) \geq \int_{|x|<k+t} u_0 dx = c_f + tc_g > 0.$$

Using estimate (10.1.7) and the initial condition $F'(0) = c_g > 0$, we conclude that $F'(t) > 0$. Hence multiplying (10.1.9) by $F'(t)$, we may get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(F'(t)\right)^2 &\geq \frac{c}{p+1}(k+t)^{-n(p-1)}\frac{d}{dt}F^{p+1}(t)\\ &\geq \frac{d}{dt}\left[\frac{c}{p+1}F^{p+1}(t)(k+t)^{-n(p-1)}\right].\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2}\left(F'(t)\right)^2 \geq{}& \frac{c}{p+1}F^{p+1}(t)(k+t)^{-n(p-1)}\\ &+\frac{1}{2}\left(F'(0)\right)^2 - \frac{c}{p+1}F^{p+1}(0)k^{-n(p-1)}.\end{aligned} \tag{10.1.10}$$

If c in (10.1.9) satisfies

$$c \leq c_* \equiv \frac{p+1}{2}\left(F'(0)\right)^2 F^{-p-1}(0)k^{n(p-1)},$$

then (10.1.10) gives us

$$\left(F'(t)\right)^2 \geq \frac{2c}{p+1}(k+t)^{-n(p-1)}F^{p+1}(t). \tag{10.1.11}$$

If c in (10.1.9) satisfies $c > c_*$, we replace (10.1.9) by

$$F''(t) \geq c_* F^p(t)(k+t)^{-n(p-1)} \tag{10.1.12}$$

and obtain the same inequality as above (10.1.11) with c replaced by c_*, in either case, (10.1.11) yields the differential inequality

$$F''(t) \geq cF^\alpha(t) \cdot (k+t)^{-\beta} \tag{10.1.13}$$

where $\alpha = (p+1)/2, \beta = (n(p-1))/2, c > 0$. Since $\alpha > 1$, (10.1.13) implies that $F \to +\infty$ in a finite time, provided that $\beta \leq 1$, which holds iff $p \leq 1 + 2/n$.

Therefore, to proceed, we may assume that

$$1 + 2/n < p < \frac{1}{2(n-1)}[n + 1 + (n^2 + 10n - 7)^{1/2}] \equiv p_0(n), n = 2, 3. \quad (10.1.14)$$

Notice that the additional hypothesis in Lemma 10.1.1 ($p > 2$ if $n = 2$) is now superfluous. We treat the critical case $p = p_0(n)$ for $n = 3$ at the conclusion of this discussion.

Defining again $F(t)$ by

$$F(t) = \int_{|x|<k+t} u dx = \int u dx, \quad (10.1.15)$$

we have

$$F''(t) = \int |u|^p dx \quad (10.1.16)$$

so that applying Lemma 10.1.1 gives us

$$F''(t) = \int |u|^p dx \geq c(k+t)^{-\frac{(n-1)(p-2)}{2}} \ln^{-\nu}(1+k+t). \quad (10.1.17)$$

We recall from Lemma 10.1.1 and (10.1.14) that $\nu \geq 0$ for $n = 2$ and $n = 3$. Thus (10.1.17) implies

$$\frac{d}{dt}\left[\ln^{\nu}(1+k+t)F'(t)\right] - \nu(1+k+t)^{-1}\ln^{\nu-1}(1+k+t)F'(t) \geq c(k+t)^{-\frac{(n-1)(p-2)}{2}}$$

whence

$$\frac{d}{dt}\left[\ln^{\nu}(1+k+t)F'(t)\right] \geq c(k+t)^{-\frac{(n-1)(p-2)}{2}}$$

since $\nu \geq 0$ and $F'(t) \geq 0$ for all t.

Integrating once in time, we then have

$$\begin{aligned} &\ln^{\nu}(1+k+t)F'(t) \\ &\quad \geq c_g \ln^{\nu}(1+k) + c\left[1 - \frac{(n-1)(p-2)}{2}\right]^{-1}\left[(k+t)^{1-\frac{(n-1)(p-2)}{2}} - k^{1-\frac{(n-1)(p-2)}{2}}\right] \end{aligned}$$

since $F'(0) = c_g$. We note that

$$1 - \frac{(n-1)(p-2)}{2} > 0, \quad iff \;\; p < 2 + 2(n+1)^{-1}$$

which is true by (10.1.14). Since $\nu > 0$ and $F(t) \geq 0$ for all $t > 0$,

$$\frac{d}{dt}\left[\ln^{\nu}(1+k+t)F(t)\right] \geq \ln^{\nu}(1+k+t)F'(t).$$

Therefore, we can integrate once again to get

$$\begin{aligned}\ln^{\nu}(1+k+t)F(t) \geq c_f' + tc_g' - c\left(1 - \frac{(n-1)(p-2)}{2}\right)^{-1} k^{1-\frac{(n-1)(p-2)}{2}} t \\ + c\left[1 - \frac{(n-1)(p-2)}{2}\right]^{-1}\left[2 - \frac{(n-1)(p-2)}{2}\right]^{-1} \\ \times [(k+t)^{2-\frac{(n-1)(p-2)}{2}} - k^{2-\frac{(n-1)(p-2)}{2}}] \qquad (10.1.18)\end{aligned}$$

where $c_f' = c_f \ln^{\nu}(1+k), c_g' = c_g \ln^{\nu}(1+k)$.

Set

$$\begin{cases} c_3 = \left[1 - \dfrac{(n-1)(p-2)}{2}\right] c_g' k^{\frac{(n-1)(p-2)}{2}-1}, & (10.1.19) \\ c_4 = c_f'\left[1 - \dfrac{(n-1)(p-2)}{2}\right]\left[2 - \dfrac{(n-1)(p-2)}{2}\right] k^{\frac{(n-1)(p-2)}{2}-2}, & (10.1.20) \\ c_{**} = \min(c_3, c_4) > 0. & (10.1.21) \end{cases}$$

If c in (10.1.17) (which depends only p, n, k, c_f and c_g) satisfies $c \leq c_{**}$, then (10.1.18) shows that there exists a positive constant c such that

$$F(t) \geq c(k+t)^{2-\frac{(n-1)(p-2)}{2}} \cdot \ln^{-v}(1+k+t). \qquad (10.1.22)$$

If c in (10.1.17) satisfies $c > c_{**}$, we replace (10.1.17) by

$$F''(t) \geq c_{**}(k+t)^{\frac{-(n-1)(p-2)}{2}} \cdot \ln^{-v}(1+k+t) \qquad (10.1.23)$$

and again proceed to derive an inequality of the form (10.1.22).

In the inequality

$$c(k+t)^{n(p-1)}\|u(t)\|_p^p \geq F^P(t), \qquad (10.1.24)$$

we may write, for some $\theta \in (0,1)$, the right-hand side in the form

$$F^p(t) = F^{\theta p}(t)F^{(1-\theta)p}(t)$$

and estimate the second factor below using (10.1.22). Thus (10.1.24) implies

$$\|u(t)\|_p^p \geq cF^{\theta p}(t)(k+t)^{\frac{p(1-\theta)}{2}[4-(n-1)(p-2)]-n(p-1)} \ln^{-vp(1-\theta)}(1+k+t). \quad (10.1.25)$$

Inserting inequality (10.1.25) into (10.1.16), we can get

$$F''(t) \geq cF^{\theta p}(t) \cdot (k+t)^{-u} \ln^{-\varepsilon}(1+k+t) \qquad (10.1.26)$$

where

$$\begin{cases} \mu = n(p-1) - \dfrac{p(1-\theta)}{2}[4-(n-1)(p-2)], & (10.1.27) \\ \varepsilon = \nu p(1-\theta) \geq 0. & (10.1.28) \end{cases}$$

Multiplying (10.1.26) by $F'(t)$, we may get

$$\frac{1}{2}\frac{d}{dt}\left(F'(t)\right)^2 \geq \frac{d}{dt}\left[\frac{c}{\theta p+1}F^{\theta p+1}(t)\cdot(k+t)^{-\mu}\ln^{-\varepsilon}(1+k+t)\right] \tag{10.1.29}$$

provided that

$$\mu \geq 0. \tag{10.1.30}$$

Assume (10.1.30) for the moment; then (10.1.29) implies

$$\left(F'(t)\right)^2 \geq \frac{2c}{\theta p+1}F^{\theta p+1}(t)(k+t)^{-\mu}\ln^{-\varepsilon}(1+k+t) \tag{10.1.31}$$

provided that c in (10.1.26) is chosen sufficiently small exactly as in the argument yielding (10.1.11) from (10.1.9). Estimate (10.1.31) thus gives us the differential inequality

$$F'(t) \geq cF^{\alpha}(t)\cdot(k+t)^{-\beta}\ln^{-\frac{\varepsilon}{2}}(1+k+t) \tag{10.1.32}$$

where $\alpha = \frac{1}{2}(\theta p+1), \beta = \frac{1}{2}\mu, c > 0$. Applying Theorem 2.4.6 to (10.1.32) shows that $F \to +\infty$ in a finite time whenever

$$\alpha > 1,\ \ \beta < 1, 0 \leq \varepsilon,\ \ or\ \ \alpha > 1,\ \ \beta = 1,\ \ \varepsilon/2 \leq 1. \tag{10.1.33}$$

This analysis is based on the assumption that some $\theta \in (0,1)$ exists for which (10.1.30) and (10.1.33) are valid. Hence the final list of constraints is

$$(1)\ \ \alpha > 1,\ \ 0 \leq \mu < 2,\ \ 0 \leq \varepsilon, \qquad (2)\ \ \alpha > 1\ \ \mu = 2,\ \ \varepsilon \leq 2. \tag{10.1.34}$$

We now need to verify these inequalities are satisfied simultaneously. By (10.1.27) and (10.1.34), case (1) is equivalent to the list of constraints in the following:

$$\begin{cases} \theta > \dfrac{1}{p}, & (10.1.35) \\ 1 - \dfrac{2n(p-1)}{p[4-(n-1)(p-2)]} \leq \theta < 1 - \dfrac{2[n(p-1)-2]}{p[4-(n-1)(p-2)]} \\ \qquad = \dfrac{p[4-(n-1)(p-2)] - 2[n(p-1)-2]}{p[4-(n-1)(p-2)]}. & (10.1.36) \end{cases}$$

Hence case (1) holds iff

$$\begin{aligned} &\max\left\{p^{-1}, \frac{p[4-(n-1)(p-2)] - 2n(p-1)}{p[4-(n-1)(p-2)]}\right\} < \theta \\ &\quad < \frac{p[4-(n-1)(p-2)] - 2[n(p-1)-2]}{p[4-(n-1)(p-2)]}. \end{aligned} \tag{10.1.37}$$

A simple calculation shows that

$$\frac{1}{p} > \frac{p[4-(n-1)(p-2)] - 2n(p-1)}{p[4-(n-1)(p-2)]}$$

iff $p > \frac{2}{n-1}$, and this holds by assumption (10.1.14) on p for $n = 2, 3$, therefore, from (10.1.37) case (1) holds if

$$\frac{1}{p} < \theta < \frac{p[4-(n-1)(p-2)] - 2[n(p-1)-2]}{p[4-(n-1)(p-2)]} \tag{10.1.38}$$

which also shows that $\theta < 1$ if $n(p-1)-2 > 0$, i.e., if $p > 1+\frac{2}{n}$. This is guaranteed by (10.1.14). Thus by (10.1.38), we can require

$$4-(n-1)(p-2) < p[4-(n-1)(p-2)] - 2[n(p-1)-2],$$

i.e.,

$$(p-1)[4-(n-1)(p-2)] > 2[n(p-1)-2].$$

This is equivalent to

$$(n-1)p^2 - (n+1)p - 2 < 0 \tag{10.1.39}$$

which holds if

$$p < \frac{1}{2(n-1)}\left[n+1+(n^2+10n-7)^{1/2}\right]. \tag{10.1.40}$$

This is precisely hypothesis (iii) of the theorem. Similarly, case (2) holds if

$$\begin{cases} 1/p < \theta = 1 - \dfrac{2[n(p-1)-2]}{p[4-(n-1)(p-2)]} \\ \qquad\qquad = \dfrac{p[4-(n-1)(p-2)] - 2[n(p-1)-2]}{p[4-(n-1)(p-2)]}, & (10.1.41) \\ \theta \geq 1 - \dfrac{4}{p(p-2)}. & (10.1.42) \end{cases}$$

We may find a θ satisfying (10.1.41), which also implies (10.1.40). When $n = 3, \nu = 0$ (by Lemma 10.1.1), so $\varepsilon = 0$ in (10.1.28). Hence, (10.1.42)) holds automatically in three dimensions $n = 3$; when $n = 2$, by (10.1.28),

$$\varepsilon = \nu p(1-\theta) = \frac{1}{2}p(p-2)(1-\theta)$$

where we have used $\nu = \frac{1}{2}(p-2)$ from Lemma 10.1.1, thus (10.1.42) is the additional constraint imposed by case (2) when $n = 2$ by (10.1.55), and $p > 2$ by (10.1.14). Hence (10.1.41)–(10.1.42) are satisfied whenever

$$\frac{1}{p} > 1 - \frac{4}{p(p-2)}.$$

But this holds iff $p^2 - 3p - 2 < 0$ which is precisely (10.1.39) when $n = 2$. This completes the proof of the theorem. □

Now let $n = 3, p = p_0(3) = 1 + \sqrt{2}$. Thus

$$p^2 - 2p - 1 = 0. \tag{10.1.43}$$

In this case, the interval (10.1.38) degenerates to a point, and we obtain $\theta = 1/\rho,\ \alpha = 1,\ \beta = 1,\ \mu = 2$, i.e., we get the linear differential inequality

$$F'(t) \geq cF(t)(k+t)^{-1}$$

which implies power growth in t for F, with a rate depending on c. Since we have restricted c several times, we are not a priori clear that c can be made large by choosing large data, this, however, is the case that we establish with the following argument.

In fact, we shall show that the functional $F(t)$, defined by (10.1.6), satisfies an estimate of the form, for all $t > 0$,

$$F(t) \geq c(k+t)^d, \tag{10.1.44}$$

where $c > 0$ depends only on c_g, c_f and the fixed values of k and p, and where $d > 0$ can be made arbitrarily large, provided that c_f and c_g (defined in part (1) of the theorem) are sufficiently large.

To do this, consider again the inequality

$$F''(t) = \int |u|^p dx \geq c(k+t)^{-(p-2)}. \tag{10.1.45}$$

From Corollary 10.1.1, we derive that this constant $c > 0$ is proportional to c_1^p, where $c_1 = \min\left\{c_g, k^{-1}c_f\right\} > 0$. Consider now the Cauchy data for which c_1, now $p = 1 + \sqrt{2} > 2$, and the constants c_3, c_4, defined by (10.1.19), (10.1.20), depend linearly on c_f, c_g, hence, for large data, we may assume that the constant c in (10.1.17) above satisfies $c > c_{**}$, where c_{**} is defined by (10.1.21). Therefore we obtain an inequality of the form (10.1.23), which is then integrated to get (10.1.22), which will be written as, for all t,

$$F(t) \geq c_2(k+t)^{2-(p-2)}, \tag{10.1.46}$$

where $c_2 > 0$ is proportional to c_1 (since c_{**} is also proportional to c_1).

Next, the constant $c > 0$ in (10.1.8):

$$F(t) \leq c\|u(t)\|_p(k+t)^{3(\frac{p-1}{p})} \tag{10.1.47}$$

depends only on p; we write $c = c_p$.

Therefore, we conclude

$$\|u(t)\|_p^p \geq c_p \cdot F^p(t) \cdot (k+t)^{-3(p-1)}, \quad c_p > 0. \tag{10.1.48}$$

We now write $F^p(t) = F(t) \cdot F^{p-1}(t)$ and estimate the second factor below using (10.1.46), it follows that

$$\begin{aligned} \|u(t)\|_p^p &\geq c_p \cdot F(t) \cdot (k+t)^{-3(p-1)} c_2^{p-1} (k+t)^{\frac{p-1}{2}[4-2(p-2)]} \\ &= c_{p,k} c_1^{p-1} F(t)(k+t)^{-2} \end{aligned} \tag{10.1.49}$$

since c_2 is proportional to c_1, and p satisfies (10.1.43). Thus, this and (10.1.45) imply for all $t > 0$,

$$F''(t) \geq c_5 F(t)(k+t)^{-2}, \tag{10.1.50}$$

where $c_5 > 0$ is proportional to c_1^{p-1}.

From (10.1.50), we derive for all $t > 0$,

$$\frac{d}{dt}\left[F'^2(t) - \frac{c_5 F^2(t)}{(k+t)^2}\right] \geq 0, \tag{10.1.51}$$

whence

$$\begin{aligned} (F'(t))^2 &\geq \frac{c_5 F^2(t)}{(k+t)^2} + c_g^2 - \frac{c_5 c_f^2}{k^2} \\ &\geq \frac{c_5 F^2(t)}{2(k+t)^2} + c_g^2 - \frac{c_5 c_f^2}{k^2} + \frac{c_5 c_2^2}{2}(k+t)^{2-2(p-2)} \end{aligned} \tag{10.1.52}$$

where we have used (10.1.46). The exponent in the last term here, $2 - 2(p-2)$ is positive if $p < 3$, which is satisfied by $p = 1 + \sqrt{2}$. Hence, (10.1.52) shows that

$$(F'(t))^2 \geq c_6 F^2(t)(k+t)^{-2} \tag{10.1.53}$$

where $c_6 > 0$ is proportional to c_1^{p-1}. By increasing the initial time if necessary, we can therefore assume (10.1.53) holds for all $t > 0$. It then follows from (10.1.53) that for all $t > 0$,

$$F'(t) \geq dF(t)(k+t)^{-1}, \tag{10.1.54}$$

where $d > 0$ is proportional to $c_1^{(p-1)/2}$.

Estimate (10.1.54) implies, for all $t > 0$,

$$F(t) \geq F(0)k^{-d}(k+t)^d, \tag{10.1.55}$$

which is estimate (10.1.44), and $d > 0$ can be made arbitrarily large provided that both c_f and c_g are sufficiently large.

Returning to (10.1.9) and using (10.1.55), we have, for any $\theta' \in (0,1)$ and for all $t > 0$,

$$\begin{aligned} F''(t) &\geq c_p F^p(t)(k+t)^{-3(p-1)} \\ &= c_p F^{p\theta'}(t) F^{p(1-\theta')}(t)(k+t)^{-3(p-1)} \\ &\geq c_{p,k,d} F^{p\theta'}(t) c_f^{p(1-\theta')}(k+t)^{dp(1-\theta')-3(p-1)} \end{aligned} \tag{10.1.56}$$

which obviously will imply finite time blow-up of $F(t)$ as before, provided that we can choose a $\theta' \in (0,1)$, satisfying

$$p\theta' > 1 \tag{10.1.57}$$

and

$$0 \le 3(p-1) - dp(1-\theta') \le 2. \tag{10.1.58}$$

Thus we require that θ' satisfy

$$\max\left\{\frac{1}{p}, 1 - \frac{3(p-1)}{dp}\right\} \le \theta' < 1 - \frac{3p-5}{dp}. \tag{10.1.59}$$

Now choosing $\frac{1}{p} < 1 - \frac{3(p-1)}{dp}$ for $d > 3$, (10.1.59) always holds, indeed, this can be achieved by a sufficiently large choice of c_1. Finally (10.1.59) also implies that any such θ' satisfies $0 < \theta' < 1$, since $p = 1 + \sqrt{2} > 5/3$, and this completes the proof. □

Remark 10.1.2 ([318]). For $n > 3$, we can easily recover Kato's result [418] as follows, let u be a solution of (10.1.1) with finite speed of propagation. Then, noting that $\int u_0 dx \ge c_1(k+t)$, where $c_1 = \min\left\{c_g, k^{-1}c_f\right\}$, we get

$$F(t) = \int u dx \ge c_1(k+t).$$

This estimate holds in any dimension, regardless of the sign of the fundamental solution, thus it follows from (10.1.7)–(10.1.8) that, for $\theta \in (0,1)$,

$$\begin{aligned} F''(t) &= \int |u|^p dx \ge cF^p(t)(k+t)^{-n(p-1)} \\ &\ge cF^{\theta p}(t)(k+t)^{p(1-\theta)-n(p-1)}. \end{aligned}$$

Integrating this as before, we find that $F(t) \to +\infty$ in a finite time provided that

$$1 < p \le \frac{n+1}{n-1}$$

which is Kato's theorem [418] when specialized to (10.1.1).

10.2 Blow-up of solutions to semilinear wave equations

In this section, we shall employ Theorem 2.4.22 to study the blow-up of solutions to semilinear wave equations. These results are due to Glassey [316].

It is well known that the blow-up problem on a bounded domain is more relatively easier, since the method of proof, closely following that of Kaplan [413], is independent of both the spatial dimension and the Riemann function of the wave operator.

We shall consider the initial boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u = f(u) & x \in \Omega, \quad t > 0, \\ \quad u(x,0) = 0 \ \text{ for} & x \in \partial\Omega, \ t \geq 0, \\ \quad u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \quad t > 0, \end{cases} \tag{10.2.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with smooth boundary $\partial\Omega$.

Let $\psi(x)$ denote the first eigenfunction for the problem

$$\begin{cases} \Delta\psi + \mu\psi = 0, & x \in \Omega, \\ \qquad \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{10.2.2}$$

and let $\mu = \mu_1$ be the corresponding first eigenvalue.

By a classical theorem (see [176]), we may assume that $\psi(x) > 0$ in Ω.

We now give two assumptions. Let $S \subset \mathbb{R}^n$ and let $\lambda, \ \alpha, \beta$ denote non-negative constants. The first hypothesis $(H1)$ concerns the Cauchy data:

(H1) $u(x,0) \geq \alpha, \ \frac{\partial u}{\partial t}(x,0) \geq \beta$ for all $x \in S$.

Second, we specify the nature of the nonlinearity:

(H2) $f(s)$ is bounded from below by a locally Lipschitzian, convex function $g(s)$ satisfying
 (i) $g(s) - \lambda s$ is a non-negative, non-decreasing function for all $s \geq \alpha$;
 (ii) $g(s)$ grows fast enough as $s \to +\infty$ so that the integral

$$T_0 = \int_\alpha^{+\infty} \left[\lambda\alpha^2 + \beta^2 - \lambda s^2 + 2\int_\alpha^s g(\xi)\, d\xi\right]^{-1/2} ds \tag{10.2.3}$$

 converges.

We assume that

(i) $u_0(x) = u(x,0) \geq 0, \ u_1(x)u_t(x,0) \geq 0$ for all $x \in \Omega$, there exist two points $x_0, x_1 \in \Omega$ such that $u_0(x_0) = u(x_0,0) > 0, \ u_1(x_1) = u_t(x_1,0) > 0$.

(ii) (H2) holds with $\lambda = \mu$,

$$\alpha = \int_\Omega \psi(x)u(x_0,0)\, dx; \quad \beta = \int_\Omega \psi(x)u_t(x_1,0)\, dx.$$

Note that both α and β are positive by hypotheses.

We may now prove the following main result due to Glassey [316].

Theorem 10.2.1 ([316]). *Assume that $u(x,t)$ is a C^2 solution of problem* (10.2.1) *such that* (i) *and* (ii) *hold. Then*

$$\lim_{t\to t_0^-} \sup_{x\in\overline{\Omega}} |u(x,t)| = +\infty$$

for some finite time $t_0 \le T_0$, where T_0 is given by (10.2.3).

Proof. Let $\psi(x)$ be as defined by (10.2.2). Without loss of generality, we may assume that ψ is normalized:

$$\int_\Omega \psi(x)\,dx = 1.$$

Let

$$\phi(t) = \int_\Omega \psi(x)u(x,t)\,dx.$$

Then multiplying (10.2.1) by ψ and integrating over Ω, and noting that $u \in C^2$, we obtain

$$\int_\Omega \psi u_{tt}\,dx = \ddot{\phi} = \int_\Omega \psi\Delta u\,dx + \int_\Omega \psi f(u)\,dx.$$

By Jensen's inequality and (ii), we have, since ψ is normalized,

$$\int_\Omega \psi f(u)\,dx \ge \int_\Omega \psi g(u)\,dx \ge g\Big(\int_\Omega \psi u\,dx\Big) = g(\phi).$$

Now using

$$\psi\Delta u = \nabla\cdot(\psi\nabla u) - \nabla\cdot(u\nabla\psi) + u\Delta\psi;$$

and the boundary conditions satisfied by u and ψ, we have

$$\int_\Omega \psi\Delta u\,dx = \int_\Omega u\Delta\psi\,dx = -\mu\int_\Omega u\psi\,dx = -\mu\phi(t).$$

Thus we obtain

$$\ddot{\phi} + \mu\phi \ge g(\phi)$$

with

$$\phi(0) = \int_\Omega \psi(x)u(x,0)\,dx = \alpha > 0; \quad \dot{\phi}(0) = \int_\Omega \psi(x)u_t(x,0)\,dx = \beta > 0.$$

Hypothesis (ii) implies that Theorem 2.4.22 is applicable with $h(s) = g(s) - \mu s$, therefore

$$t \le \int_\alpha^{\phi(t)} \left[\mu\alpha^2 + \beta^2 - \mu s^2 + 2\int_\alpha^s g(\xi)\,d\xi\right]^{-1/2} ds,$$

and $\phi(t)$ develops a singularity in a finite time $t_0 \le T_0$, where

$$T_0 = \int_{\alpha}^{+\infty} \left[\mu\alpha^2 + \beta^2 - \mu s^2 + 2\int_{\alpha}^{s} g(\xi) d\xi \right]^{-1/2} ds.$$

Finally, since $\phi(t) > 0$, we have

$$\begin{aligned}\phi(t) = |\phi(t)| &= \left| \int_{\Omega} \psi(x) u(x,t)\, dx \right| \le \sup_{x\in\overline{\Omega}} |u(x,t)| \int_{\Omega} \psi(x)\, dx \\ &= \sup_{x\in\overline{\Omega}} |u(x,t)|,\end{aligned}$$

which thus proves the theorem. □

Corollary 10.2.1 ([316])**.** *For each p, $1 \le p \le +\infty$,*

$$\|u(t)\|_{L^p(\Omega)} = \left(\int_{\Omega} |u(x,t)|^p\, dx \right)^{1/p}$$

blows up in a finite time.

Proof. The proof of the corollary is simply to apply Hölder's inequality to the term $|\int_{\Omega} \psi(x) u(x,t)\, dx|$. □

Remark 10.2.1 ([316])**.** As Kaplan has noted (see, e.g., [413]), Δ may be replaced by any uniformly elliptic self-adjoint second-order operator

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right)$$

with smooth coefficients $a_{ij}(x)$.

Remark 10.2.2 ([316])**.** The same result of Theorem 10.2.1 holds if the boundary condition is of the form

$$u(x,t) = \Phi(x,t) \quad \text{for all} \quad x \in \partial\Omega,\ t \ge 0$$

provided that $\Phi(x,t) \ge 0$ for all $x \in \partial\Omega,\ t \ge 0$. To show this, we need only to prove that the term $-\int_{\Omega} \nabla\cdot(u\nabla\psi)\, dx$ is non-negative. Now ψ obeys the maximum (minimum) principle, and assumes its minimum value (zero) on $\partial\Omega$. It thus follows that $\Delta\psi$ is directed toward the interior of Ω, so that $\frac{\partial\psi}{\partial\nu} \le 0$ on $\partial\Omega$, where $\nu =$ outer normal to $\partial\Omega$. Then clearly

$$-\int_{\Omega} \nabla\cdot(u\nabla\psi)\, dx = -\int_{\partial\Omega} \Phi(x,t) \frac{\partial\psi}{\partial\nu}(x)\, dS_x \ge 0.$$

Remark 10.2.3 ([316]). The corresponding problem with general linear homogeneous boundary conditions on u can be treated similarly. We define $\psi(x)$ as the first eigenfunction of $\Delta\psi + \mu\psi = 0$ in Ω, satisfying the same boundary conditions as u on $\partial\Omega$. For example, if the boundary condition is

$$\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega,$$

we may choose

$$\psi(x) = \text{const.} = (\text{measure}(\Omega))^{-1} \equiv (\mathrm{m}(\Omega))^{-1}$$

so that

$$\phi(t) = \int_\Omega \psi(x)u(x,t)\,dx = \frac{1}{m(\Omega)}\int_\Omega u(x,t)\,dx.$$

We then easily obtain $\frac{d^2\phi}{dt^2} \equiv \ddot{\phi} \geq g(\phi)$, and proceed as above.

We now consider the Cauchy problem for the equation $(10.2.1)_1$ over $\mathbb{R}^n \times \mathbb{R}^+$ for $n \leq 3$, and only discuss the case $n = 3$, while the method is similar when $n = 1$ or 2.

To this end, for any $R > 0$, we define

$$\psi(x) = \frac{c}{r}\sin\frac{\pi r}{R}, \quad \text{for all} \quad |x| = r \leq R, \tag{10.2.4}$$

where $c > 0$ is chosen so that $\int_{|x|\leq R}\psi(x)\,dx = 1$. Let $\mu = \pi^2/R^2$ and we also assume that

(i) (H1) holds for arbitrary $\alpha > 0$, $\beta > 0$, with $S = \{x \in \mathbb{R}^3 : |x| \leq R + 2T_0\}$, where $T_0 > 0$ is given by (10.2.3) with $\lambda = \mu$;

(ii) $\Delta u(x,0) \geq 0$ for all $x \in S$;

(iii) (H2) holds with $\lambda = \mu$ in the following weakened form: the function $g(u)$ is assumed convex only for all $u \geq \alpha$.

Under the above assumptions, we can prove the following blow-up result.

Theorem 10.2.2 ([316]). *Let $u(x,t)$ be a C^2 solution to the Cauchy problem* $(10.2.1)_1$ *such that* (i)–(iii) *hold. Then*

$$\lim_{t\to t_0^-}\sup_{|x|\leq R}|u(x,t)| = +\infty \tag{10.2.5}$$

for some finite time $t_0 \leq T_0$.

Proof. The solution $u(x,t)$ to the Cauchy problem of the equation $(10.2.1)_1$ satisfies the following nonlinear integral equation

$$u(x,t) = u_0(x,t) + \frac{1}{4\pi}\int_0^t \frac{1}{t-\tau}\int_{|y-x|=t-\tau} f(u(y,\tau))\,dS_y d\tau$$

where $u_0(x,t)$ is the solution of the linear equation with the same data as that of u when $t=0$. Thus,

$$u_0(x,t) = \frac{t}{4\pi}\int_{|\omega|=1} \frac{\partial u}{\partial t}(x+\omega t, 0)\, d\omega + \frac{1}{4\pi}\int_{|\omega|=1} u(x+\omega t, 0)\, d\omega \\ + \frac{1}{4\pi t}\int_{|y-x|\le t} \Delta u(y,0)\, dy.$$

From (i) it clearly follows $u_0(x,t) \ge \alpha + \beta t$ for all $|x| \le R + T_0,\ 0 \le t \le T_0$.

We now claim that $u(x,t) \ge \alpha$ for all $|x| \le R,\ 0 \le t \le T_0$. Let $C(x_0, T_0)$ be any backward characteristic cone, the x-coordinate of whose vertex x_0 satisfies $|x_0| \le R$. We shall show that $u(x,t) \ge \alpha$ in $C(x_0,T_0)$ using Keller's method. Then, since x_0 was an arbitrary point in $|x| \le R$, we shall prove the claim. In fact, we assume the assertion $u(x,t) \ge \alpha$ in $C(x_0,T_0)$ is false. Let

$$t_1 = \inf\{t : u(x,t) < \alpha \ \text{ in } \quad C(x_0,T_0)\},$$

and let $(x_1, t_1+\varepsilon)$, with sufficiently small ε, be a point in $C(x_0,T_0)$ where $u(x_1, t_1+\varepsilon) < \alpha$. Then from the integral equation, it follows

$$u(x_1, t_1+\varepsilon) - \alpha = u_0(x_1, t_1+\varepsilon) - \alpha \\ + \frac{1}{4\pi}\int_0^{t_1+\varepsilon} \frac{1}{t_1+\varepsilon-\tau}\int_{|y-x_1|=t_1+\varepsilon-\tau} f(u(y,\tau)) dS_y d\tau.$$

Now $|x_1| \le R + T_0$, so that $u_0(x_1, t_1+\varepsilon) - \alpha \ge \beta t_1$ from the above equality. Using (H2) and the definition of the point t_1, we conclude

$$\begin{aligned} &u(x_1, t_1+\varepsilon) - \alpha \\ &\quad\ge \beta t_1 + \frac{1}{4\pi}\int_{t_1}^{t_1+\varepsilon} \frac{1}{t_1+\varepsilon-\tau}\int_{|y-x_1|=t_1+\varepsilon-\tau} g(u(y,\tau)) dS_y d\tau \\ &\quad\ge C + \frac{1}{4\pi}\int_{t_1}^{t_1+\varepsilon} \frac{1}{t_1+\varepsilon-\tau}\int_{|y-x_1|=t_1+\varepsilon-\tau} [g(u(y,\tau)) - g(\alpha)]\, dS_y d\tau \end{aligned}$$

where constant $C > 0$ depends on $\beta,\ t_1,\ \varepsilon$ and the positive value of $g(\alpha)$. We now split this integral into two components, one over the set $u \ge \alpha$; the other over the component of this set. Whenever $u \ge \alpha,\ g(u) - g(\alpha)$ is non-negative, so we may restrict our attention to the region $u < \alpha$.

Therefore, since g is Lipschitizian, we get

$$u(x_1, t_1+\varepsilon) - \alpha \ge \varepsilon K (u-\alpha)_{\min} + C,$$

where $(u-\alpha)_{\min}$ is the least value of $u-\alpha$ in the backward characteristic cone $C(x_1, t_1+\varepsilon)$ for all $t \ge t_1$, and where K is proportional to the Lipschitz constant for g. Taking $0 < \varepsilon < 1/K$ and applying the above to the point in $C(x_1, t_1 + \varepsilon)$ where $u - \alpha$ assumes its minimum, we may get

$$(u-\alpha)_{\min} \geq C + \varepsilon K (u-\alpha)_{\min}.$$

Thus

$$(u-\alpha)_{\min} \geq \frac{1}{1-\varepsilon K} > 0$$

which is impossible. Hence no such point t_1 exists, and the claim is thus proved. Now let $\psi(x)$ and μ be as defined above. Then

$$\Delta\psi + \mu\psi = 0 \quad \text{in} \quad |x| < R;$$

ψ vanishes on $|x| = R$, and

$$\left.\frac{\partial\psi}{\partial r}\right|_{|x|=R} < 0.$$

Multiplying $(10.2.1)_1$ by $\psi(x)$ and integrating over $|x| \leq R$; with

$$\phi(t) = \int_{|x|\leq R} \psi(x) u(x,t)\, dx,$$

we may obtain

$$\ddot{\phi} = \int_{|x|\leq R} \psi \Delta u \, dx + \int_{|x|\leq R} \psi f(u)\, dx.$$

Now $U(x,t) \geq \alpha$ for all $|x| \leq R,\ 0 \leq t \leq T_0$; thus, by the convexity of g and Jensen's inequality,

$$\int_{|x|\leq R} \psi f(u)\, dx \geq \int_{|x|\leq R} \psi g(u)\, dx \geq g(\phi).$$

Using the properties of ψ and the fact that u is a positive solution, we can get

$$\int_{|x|\leq R} \psi \Delta u\, dx = \int_{|x|\leq R} [\nabla\cdot(\psi\nabla u) - \nabla\cdot(u\nabla\psi) + u\Delta\psi] dx \geq -\mu\phi(t).$$

Therefore, $\ddot{\phi} + \mu\phi \geq g(\phi)$ with

$$\phi(0) = \int_{|x|\leq R} \psi(x) u(x,0)\, dx \equiv \alpha_1 \geq \alpha; \quad \phi(0) = \int_{|x|\leq R} \psi(x) u_t(x,0)\, dx \equiv \beta_1 \geq \beta.$$

Now applying Theorem 2.4.22 with $h(s) = g(s) - \mu s$; we can find that $\phi(t)$ blows up in a finite time $t_0 \leq T_1$, where

$$T_1 = \int_{\alpha_1}^{+\infty} \left[\mu\alpha_1^2 + \beta_1^2 - \mu s^2 + 2\int_{\alpha_1}^{s} g(\xi)\, d\xi\right]^{-1/2} ds.$$

It thus remains only to show that $T_1 \leq T_0$. For this purpose, set

$$T^* = T^*(\alpha, \beta) = \int_\alpha^{+\infty} \left[\mu\alpha^2 + \beta^2 - \mu s^2 + 2\int_\alpha^s g(\xi)\, d\xi \right]^{-1/2} ds$$
$$= \int_\alpha^{+\infty} \left[\beta^2 + 2\int_\alpha^s (g(\xi) - \mu\xi)\, d\xi \right]^{-1/2} ds.$$

Then from (iii) it follows that $T^*(\alpha, \beta)$ decreases as α, β increase. Hence since $\alpha_1 \geq \alpha,\ \beta_1 \geq \beta$, we obtain

$$T_1 = T^*(\alpha_1, \beta_1) \leq T^*(\alpha, \beta) = T_0.$$

Thus $\phi(t)$ blows up in a finite time $t_0 \leq T_0$. Then

$$\phi(t) = |\phi(t)| \leq \sup_{|x| \leq R} |u(x,t)| \int_{|x| \leq R} \psi(x)\, dx = \sup_{|x| \leq R} |u(x,t)|$$

which thus completes the proof. □

Remark 10.2.4 ([316]). When $n = 3$, Keller ([432]) assumes that on some set $|x - x_0| \leq T$, the data satisfy

$$u(x, 0) = \alpha = \text{const.}; \quad u_t(x, 0) \geq \beta = \text{const.},$$

which, when $\beta > 0$, is a special case of (H1).

Corollary 10.2.2 ([316]). *For each p, $1 \leq p \leq +\infty$, the expression*

$$\left(\int_{|x| \leq R} |u(x,t)|^p\, dx \right)^{1/p}$$

blows up in a finite time.

10.3 Blow-up of solutions to nonlinear hyperbolic equations

In this section, we shall employ Theorem 2.4.7 or Corollary 2.4.2 to establish the blow-up of solutions of some nonlinear hyperbolic equations. We choose these results from Kato [418].

We shall consider in this section the equation

$$u_{tt} + Au = Fu = f(t, x, u), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{10.3.1}$$

where A is an elliptic linear operator of the form

$$A = -\sum_{i,k=1}^{n} \partial_j a_{jk}(t,x) \partial_k - \sum_{j=1}^{n} \partial_j a_j(t,x), \quad \partial_j = \frac{\partial}{\partial x_j}.$$

An essential property of A is that $A^*1 = 0$, where A^* denotes the formal adjoint of A. We assume that f satisfies

$$\begin{cases} f(t,x,s) \geq \begin{cases} b|s|^{p_0}, & \text{for all} \quad |s| \leq 1, \\ b|s|^p, & \text{for all} \quad |s| \geq 1, \end{cases} & (10.3.2) \\ 1 < p \leq p_0 = (n+1)/(n-1). & (10.3.3) \end{cases}$$

If $n = 1, p_0$ may be any number greater than or equal to p.

However, condition (10.3.3) is not general enough to recover John's value $p = p_0 < 1 + 2^{1/2}$ for $n = 3$, but it is rather general in other respect since p may be any number larger than 1 and may be different from p_0.

The main result in Kato [418] reads as follows.

Theorem 10.3.1 ([418]). *Assume that* (10.3.2)–(10.3.3) *hold for* f. *Let* u *be a generalized solution of problem* (10.3.1) *on a time interval* $0 \leq t \leq T \leq +\infty$, *which is supported on a forward cone*

$$K_R = \{(t,x); t \geq 0, |x| \leq t + R\}, \qquad R > 0.$$

Moreover, assume that, for $w(t) = \int_{\mathbb{R}^n} u(t,x)dx$,

$$\text{either} \quad (a) \quad w_t(0) > 0, \quad \text{or} \quad (b) \quad w_t(0) = 0 \quad \text{and} \quad w(0) \neq 0. \qquad (10.3.4)$$

Then we necessarily have $T < +\infty$.

Remark 10.3.1 ([418]). We usually suppose that a generalized solution u of (10.3.1) on $0 \leq t < T$ at least satisfies the following conditions

$$\begin{cases} u \quad \text{and} \quad Fu \quad \text{are in} \quad C([0,T]; L^1_{\text{loc}}(\mathbb{R}^n)), & (10.3.5) \\ \dfrac{d^2}{dt^2}(u,\phi) + (u, A^*\phi) = (Fu,\phi) \quad \text{for every} \quad \phi \in C_0^\infty(\mathbb{R}^n). & (10.3.6) \end{cases}$$

It seems necessary to admit such generalized solutions u in the theorem, at least for $n \geq 4$, since condition (10.3.2) for small $|s|$ might conflict with the existence of classical solutions to problem (10.3.1).

Remark 10.3.2 ([418]). The condition that u is supported on K_R would require that A is uniformly elliptic and that $f(t,x,0) = 0$.

Remark 10.3.3. The assumptions on u are not absurd, in the sense that with certain additional assumptions such as Lipschitz continuity of $f(t,x,s)$ in s, the existence of solutions u of problem (10.3.1) satisfying these conditions can be proved at least on a finite time interval (see, e.g., Reed [832]).

Remark 10.3.4 ([418]). It is interesting to compare (10.3.3) with the condition used by Fujita [282] for positive solutions of the nonlinear heat equation $u_t - \Delta u = Fu$. Fujita's blow-up condition roughly corresponds to (10.3.2) with $1 < p \leq p_0 < (n+2)/n$, while global solutions are shown to exist for small initial data if $f(s) = s^q$ with $q > (n+2)/n$. The critical case $q = (n+2)/n$ was proved by Weissler [956] to belong to the blow-up case.

In order to prove Theorem 10.3.1, we need an elementary lemma as follows (see, e.g., [418]).

Lemma 10.3.2 ([418]). *Let*

$$g_0(s) = \inf \{|s|^{p_0}, |s|^p\}, \quad s \in \mathbb{R}. \tag{10.3.7}$$

Then there is a convex function g on $\mathbb{R}$ and a constant $b_1 > 0$ such that

$$f(t,x,s) \geq g(s) \geq b_1 g_0(s), \quad g(0) = 0. \tag{10.3.8}$$

Proof. Indeed, estimates (10.3.2)–(10.3.3) imply that $f \geq bg_0$. Hence it suffices to find a convex function g such that $bg_0 \geq g \geq b_1 g_0$ with some $b_1 > 0$. This is trivial since g_0 is piecewise convex with a finite derivative. □

Proof of Theorem 10.3.1. We make for simplicity some changes in notation. We shall normalize the Lebesgue measure dx so that the unit ball has unit volume. Also we shift the origin of time t so that the cone K_R has the form $|x| \leq t$ and the initial time is at $t = R$, hence this will change T into $T + R$, but we shall denote it again by T.

With the above modifications, we have

$$w_{tt} \geq (g(u), 1), \quad R \leq t < T. \tag{10.3.9}$$

To see this, we have only to apply (10.3.6) with a $\phi \geq 0$ such that $\phi = 1$ on $B_{R'}$, where $R' > R$. We denote by B_r the closed ball in $\mathbb{R}^n$ with center 0 and radius r. Then $A^*\phi = 0$ and $\phi = 1$ on the support of $u(t, \cdot)$ for all $t < R'$, which yields (10.3.9) for all $t < R'$ by (10.3.8). Since R' is arbitrary, we obtain (10.3.9) for all $t > T$.

Since g is convex and the integral in $(g(u), 1)$ may be taken on the ball B_t, which has volume t^n, it follows from (10.3.8),

$$\begin{aligned}(g(u), 1) &= t^n (g(u), t^{-n})_{B_t} \geq g((u, t^{-n})_{B_t}) \\ &= t^n (g(t^{-n} w) \geq b_1 t^n g_0(t^{-n} w),\end{aligned}$$

where $(\cdot,\cdot)_{B_t}$ denotes the scalar product in $L^2(B_t)$. Thus (10.3.9) gives us a differential inequality for w:

$$w_{tt} \geq b_1 t^n \inf \left\{t^{-np_0}|w|^{p_0}, t^{-np}|w|^p\right\}, \quad R \leq t < T. \tag{10.3.10}$$

We shall complete the proof by assuming that $T = +\infty$ in (10.3.10) and deducing a contradiction.

To this end, we first note that w_t is monotone non-decreasing because $w_{tt} > 0$, by (10.3.10). If $w_t(R) = a > 0$, it follows that $w_t \geq a$ for all $t \geq R$ and so $w \geq w(R) + at > 0$ for sufficiently large t. If $w_t(R) = 0$, then $w(R) \neq 0$ by hypothesis, so that $w_{tt}(R) > 0$ by (10.3.10) and $w_t(R + \varepsilon) > 0$ for some $\varepsilon > 0$. If we take $R+\varepsilon$ as the new initial time, we are in the same situation as above. In any case, there is a constant $R_1 \geq R, R_1 \geq 1$, such that for all $t \geq R_1 \geq 1$,

$$w_t \geq a > 0, \quad w \geq at > 0, \tag{10.3.11}$$

possibly with a modified $a > 0$.

Now (10.3.3) implies that $n - np_0 = -1 - p_0$ and $n - np \geq -1 - p$. For all $t \geq R_1 \geq 1$, therefore, (10.3.10) yields

$$\begin{aligned} w_{tt} &\geq b_1 \inf\left\{t^{-t-p_0} w^{p_0}, t^{-1-p} w^p\right\} \\ &= b_1 t^{-1-p} w^p \inf\left\{(w/t)^{p_0 - p}, 1\right\} \geq b_2 t^{-1-p} w^p, \end{aligned}$$

where $b_2 = b_1 \inf\{a^{p_0-p}, 1\} > 0$, because $w/t \geq a > 0$ by (10.3.11) and $p_0 - p \geq 0$. Thus applying Theorem 2.4.7 or Corollary 2.4.2 to the above inequality completes the proof. $\Box$

10.4 Breakdown of solutions to semilinear wave equations $\Box u + u_t = |u|^{1+\alpha}$

In this section, we shall exploit Corollary 2.4.8 to study the blow-up and estimates of the lifespan of solutions to semilinear wave equations. We here adopt the results from Li and Zhou [537].

We shall consider the following Cauchy problem for fully nonlinear wave equations with linear dissipation

$$\begin{cases} \Box u + u_t = F(u, Du, D_x Du), & (10.4.1) \\ t = 0: \ u = \varepsilon\varphi(x), \ u_t = \varepsilon\psi(x), & (10.4.2) \end{cases}$$

where

$$D_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \quad D = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \tag{10.4.3}$$

and $\varphi,\ \psi \in C_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$ is a small parameter.

Let

$$\hat{\lambda} = (\lambda; (\lambda_i),\ i = 0, 1, \ldots, n;\ (\lambda_{ij}),\ i, j = 0, 1, \ldots, n,\ i + j \geq 1). \tag{10.4.4}$$

Assume that in a neighborhood of $\hat{\lambda} = 0$, the nonlinear term $F = F(\hat{\lambda})$ in (10.4.1) is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}), \tag{10.4.5}$$

where α is an integer ≥ 1.

Note that Nishida [699] and Matsumura [593, 594] studied this kind of problem in some special cases. For all n and α with $n \geq 1$ and $\alpha \geq 1$, Li [540] obtained the following lower bound for the lifespan $\tilde{T}(\varepsilon)$ of classical solutions to the Cauchy problem (10.4.1)–(10.4.2),

$$\tilde{T}(\varepsilon) \geq \begin{cases} +\infty, & \text{if } n\alpha > 2, \\ \exp\{\bar{a}\varepsilon^{-\alpha}\}, & \text{if } n\alpha = 2, \\ \bar{b}\varepsilon^{-2\alpha/(2-n\alpha)}, & \text{if } n\alpha < 2, \end{cases} \tag{10.4.6}$$

where $\bar{a}$ and $\bar{b}$ are positive constants independent of ε.

In order to show the sharpness of estimate (10.4.6), we now consider the Cauchy problem (10.4.1)–(10.4.2) with $F = |u|^{1+\alpha}$ and for all $x \in \mathbb{R}^n$,

$$\varphi(x) \equiv 0, \ \psi(x) \geq 0, \tag{10.4.7}$$

and

$$\int_{\mathbb{R}^n} \psi(x)\, dx > 0. \tag{10.4.8}$$

Let

$$v = S(t)g \tag{10.4.9}$$

be the solution to the Cauchy problem

$$\begin{cases} \Box v + v_t = 0, & (10.4.10) \\ t = 0: \ v = 0, \ v_t = g(x). & (10.4.11) \end{cases}$$

Then the solution u to the Cauchy problem

$$\begin{cases} \Box u + u_t = F(t,x), & (10.4.12) \\ t = 0: \ u = f(x), \ u_t = g(x) & (10.4.13) \end{cases}$$

can be expressed as

$$u = \partial_t(S(t)f) + S(t)(f+g) + \int_0^t S(t-\tau)F(\tau,\cdot)\, d\tau. \tag{10.4.14}$$

When $n \leq 2$, we have (see, e.g., Chapter 6 in [176]) that

$$S(t)g = \frac{1}{2}e^{-t/2} \int_{|x-y|\leq t} I_0\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right) g(y)\, dy, \quad \text{if } n=1 \tag{10.4.15}$$

and

$$S(t)g = \frac{1}{2\pi}e^{-t/2} \int_{|x-y|\leq t} I_0 \frac{\operatorname{ch}\left(\frac{1}{2}\sqrt{t^2-|x-y|^2}\right)}{\sqrt{t^2-|x-y|^2}} g(y)\, dy, \quad \text{if } n=2 \tag{10.4.16}$$

where

$$I_0(y) = \sum_{m=0}^{+\infty} \frac{1}{(m!)^2} \left(\frac{y}{2}\right)^{2m} \tag{10.4.17}$$

is the Bessel function of order zero with imaginary argument, which satisfies

$$I_0''(y) + \frac{1}{y} I_0'(y) - I_0(y) = 0 \tag{10.4.18}$$

and the following asymptotic formula holds as $y \to +\infty$,

$$I_0(y) \approx \sqrt{\frac{1}{2\pi y}} e^y. \tag{10.4.19}$$

By the positivity of the fundamental solution to problem (10.4.1), (10.4.7)–(10.4.8) in the case $n \leq 2$ (see (10.4.15)–(10.4.16)), we have

$$u(t, x) \geq 0, \tag{10.4.20}$$

then $u = u(t, x)$ is a solution to the Cauchy problem

$$\begin{cases} \Box u + u_t = u^{1+\alpha}, \\ t = 0: \ u = 0, \ u_t = \varepsilon\psi(x). \end{cases} \tag{10.4.21}$$

When $\alpha \geq 1$ is an integer, (10.4.21) is a particular case of the Cauchy problem (10.4.1)–(10.4.2), then by Corollary 2.4.8, we can get the sharpness of (10.4.6).

10.5 Blow-up of solutions to nonlinear wave equations with damping

In this section, we shall employ Theorem 2.4.9 to investigate the Cauchy problem for the dissipative nonlinear wave equations. The results introduced here are from Todorova and Yordanov [919].

We shall study the global existence, blow-up and asymptotic behavior as $t \to +\infty$ for solutions of the following Cauchy problem for the dissipative nonlinear wave equation

$$\begin{cases} \Box u + u_t = |u(x,t)|^p, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, & (10.5.1) \\ t = 0 : u = \varepsilon u_0, \ u_t = \varepsilon u_1, & (10.5.2) \end{cases}$$

where $\Box = \partial_t^2 - \Delta_x$ is the wave operator, $\varepsilon > 0$, and (u_0, u_1) are compactly supported data from the energy space:

$$\begin{cases} u_0 \in H^1(\mathbb{R}^n), \quad u_1 \in L^2(\mathbb{R}^n), \\ \text{supp } u_i \subseteq B(K) \equiv \{x \in \mathbb{R}^n : |x| < K\}, \quad i = 0, 1, \end{cases}$$

and the so-called critical exponent $p_c(n)$ is the number defined by the following property (see also Section 10.4):

For $p_c(n) < p$, all small data solutions of problem (10.5.1)–(10.5.2) exist global in time, while for $1 < p < p_c(n)$, all solutions with data positive on average blow up in a finite time no matter how small the initial data are.

Recall that if the damping is missing, the critical exponent for the nonlinear wave equation $\Box u = |u|^p$ is the positive root $p_0(n)$ of the equation $(n-1)p^2 - (n+1)p - 2 = 0$, where $n \geq 2$ is the space dimension (for $p_0(1) = +\infty$, see Sideris [869]). This is known as Strauss' conjecture [870] which was solved for more than 20 years, beginning from Glassey [319], John [405], Sideris [870], Choquet-Bruhat [162], [163], Zhou [1013], Agemi, Kubota and Takmura [10], and ending with Lindblad and Sogge [544], Georgiev, Lindblad, and Sogge [299] and Tataru [910]. Moreover, Nakao and Ono in [684] also proved the global existence for the damped wave equation (10.5.1) for $p > 1 + 4/n$.

Todorova and Yordanov [919] solved the critical exponent case for problem (10.5.1)–(10.5.2) with a source and a linear damping term and proved that the damping is powerful enough to shift the critical exponent $p_0(n)$ of the wave equation to the left, i.e., the critical exponent $p_c(n)$ for equation (10.5.1) is strictly less than $p_0(n)$, which will be studied in the following result for the critical exponent $p_c(n) = 1 + 2/n$ of problem (10.5.1)–(10.5.2). Thus, $p_c(n) < p_0(n)$.

We shall use Theorem 2.4.9 to show the next blow-up result.

Theorem 10.5.1 ([919]). *Let $1 < p < 1 + 2/n$. If*

$$c_i = \int_{\mathbb{R}^n} u_i(x)\,dx > 0, \quad i = 0, 1,$$

then for any $\varepsilon > 0$, the solution of problem (10.5.1)–(10.5.2) *will blow up in a finite time.*

Proof. The proof will be split into two parts. First, we show the result for exponents in the smaller range $1 < p < 1 + 1/n$. Let

$$F(t) = \int_{\mathbb{R}^n} u(t,x)dx, \tag{10.5.3}$$

where u is the local solution of problem (10.5.1)–(10.5.2).

(1) For $1 < p < 1 + 1/n$. Following [919], we can derive

$$\ddot{F}(t) + \dot{F}(t) \geq C(t+K)^{-n(p-1)} \mid F(t) \mid^p . \tag{10.5.4}$$

We observe that from Theorem 2.4.9 the blowup of F follows immediately. Noting that $F(0) = \varepsilon c_0$ and $\dot{F}(0) = \varepsilon c_1$ are positive by assumptions and applying Theorem 2.4.9 to (10.5.4), we can conclude that F blows up in a finite time if $1 < p < 1 + 1/n$.

(2) For $1 + 1/n \le p < 1 + 2/n$. The details of arguments for this case can be found in [919].

The lower estimate for F is given by the next lemma.

Lemma 10.5.2 ([919]). *Under the assumptions of Theorem* 10.5.1 *hold, i.e.,* $1 < p < 1 + 2/n$ *and*

$$c_i \equiv \int_{\mathbb{R}^n} u_i(x)dx > 0, \quad i = 0, 1,$$

for each $B \ge 0$, *there exists a constant* $C_B > 0$ *such that for all* $t \ge 0$,

$$F(t) \ge C_B(t + K)^B. \tag{10.5.5}$$

In fact, we can now derive a stronger version of (10.5.4) by writing $\mid F \mid^p$ as $\mid F \mid^{(p-1)/2} \mid F \mid^{(p+1)/2}$ and using Lemma 10.5.1, the following modified inequality

$$\ddot{F}(t) + \dot{F}(t) \ge C(t + K)^{(p-1)(B/2-n)} |F(t)|^{(p+1)/2}. \tag{10.5.6}$$

Thus choosing $B = n$, applying Lemma 10.5.1 and Theorem 2.4.9, we can complete the proof of Theorem 10.5.1. □

To prove Lemma 10.5.1, we may refer the reader to [919] for details. □

10.6 Blow-up of solutions to wave equations with a nonlinear dissipation

In this section, we shall apply Theorem 2.4.4 to investigate the blow-up of solutions to wave equations with a nonlinear dissipation. We choose these results from Tatar [909].

We shall consider the following equation

$$u_{tt} + \lambda u + u_t(V_\gamma * u_t^2) = \Delta u + a \mid u \mid^{p-1} u \quad \text{in} \quad \mathbb{R}^N \times (0, +\infty), \tag{10.6.1}$$

subject to initial data

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^N, \tag{10.6.2}$$

where

$$(V_\gamma * u_t^2)(x, t) = \int_{\mathbb{R}^N} V_\gamma(x - y)u_t^2(y, t)dy \tag{10.6.3}$$

and

$$V_\gamma(x) = |x|^{-\gamma}, \ 0 < \gamma < N, \ \lambda \ge 0, \ a > 0, \ p > 1.$$

We shall use the usual $L^p, 1 \le p \le +\infty$ spaces and Sobolev spaces $H^k, k = 1, 2, \ldots.$

First we recall the following global existence result: If $a = 0$ and $(u_0, u_1) \in H^2(\mathbb{R}^N) \times (H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))$, $q = 6N/(3N-2\gamma)$, then problem (10.6.1)–(10.6.2) admits a global solution (see, e.g., [634], [639], [909]) satisfying

(i) $u(t) \in C([0, +\infty); E)$ and for any $t > 0$,

$$\| u(t) \|_E^2 + \int_0^t \int_{\mathbb{R}^N} u_t^2(x,s)dx \int_{\mathbb{R}^N} | x-y |^{-\gamma} u_t^2(y,s)dyds = \| u(0) \|_E^2 \tag{10.6.4}$$

where

$$E = \left\{ w = (w_1, w_2) : \| w \|_E = \frac{1}{2} \left(\int_{\mathbb{R}^N} [\lambda w_1^2 + |\nabla w_1|^2 + w_2^2] dx \right)^{1/2} < +\infty \right\}. \tag{10.6.5}$$

(ii) for any $T > 0$,

$$u(t) \in C([0,T]; L^2(\mathbb{R}^N)). \tag{10.6.6}$$

(iii) for any $T > 0$,

$$u_{tt}(t), \nabla u_t(t), \Delta u(t), u_t(t) \int_{\mathbb{R}^N} | x-y |^{-\gamma} u_t^2(y,t)dy \in L^\infty([0,T], L^2(\mathbb{R}^N)). \tag{10.6.7}$$

The following is the blow-up theorem due to Tatar [909].

Theorem 10.6.1 ([909]). *Let $p > 3$ and assume the above hypotheses hold. Then for any $T > 0$, we can find initial data $u_0(x)$ and $u_1(x)$ (of compact support) for which the corresponding solution $u(x,t)$ blows up at a finite time $T^* \le T$.*

Proof. Multiplying (10.6.1) by u_t, and integrating over $\mathbb{R}^N$, we have

$$\frac{dE(t)}{dt} = - \int_{\mathbb{R}^N} u_t^2 \int_{\mathbb{R}^N} V_\gamma(x-y) u_t^2(y,t)dydx \tag{10.6.8}$$

where

$$E(t) = \int_{\mathbb{R}^N} \left[\frac{1}{2}\lambda u^2 + \frac{1}{2} u_t^2 + \frac{1}{2} | \nabla u |^2 - \frac{a}{p+1} | u |^{p+1} \right] dx. \tag{10.6.9}$$

Noting that $dE(t)/dt \le 0$, we obtain for all $t \ge 0$,

$$E(t) \le E(0). \tag{10.6.10}$$

Let us introduce the functional

$$H(t) = \int_0^t \int_{\mathbb{R}^N} \left\{ \frac{a}{p+1} |u|^{p+1} - \frac{1}{2}\lambda u^2 - \frac{1}{2} u_t^2 - \frac{1}{2} | \nabla u |^2 \right\} dxds + (dt+l) \int_{\mathbb{R}^N} u_0^2 dx. \tag{10.6.11}$$

The positive constants d and l are to be chosen later on. A differentiation of this functional (with the above observation (10.6.10)) implies that

$$H'(t) = -E(t) + d\int_{\mathbb{R}^N} u_0^2 dx \geq d\int_{\mathbb{R}^N} u_0^2 dx - E(0). \tag{10.6.12}$$

We readily choose d so that

$$d\int_{\mathbb{R}^N} u_0^2 dx - E(0) = H'(0) > 0. \tag{10.6.13}$$

It follows from (10.6.12)–(10.6.13) that for all $t \geq 0$,

$$H'(t) \geq H'(0).$$

Moreover, the identity (10.6.8) yields

$$H'(0) - H'(t) = -\int_0^t \int_{\mathbb{R}^N} u_t^2 (V_\gamma * u_t^2) dxds \leq 0. \tag{10.6.14}$$

Now we choose a second auxiliary functional

$$L(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{2}\left(\int_{\mathbb{R}^N} u^2 dx - \int_{\mathbb{R}^N} u_0^2 dx\right)$$

with $\varepsilon > 0$ and $0 < \sigma = (p-3)/[6(p+1)] < 1$. We next show that $L(t)$ satisfies

$$L'(t) \geq CL^q(t), \quad q > 1$$

which will yield the blow-up of solutions in a finite time by Theorem 2.4.4.

A direct calculation gives us

$$\begin{aligned} L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_{\mathbb{R}^N} u_0 u_1 dx + \varepsilon\int_0^t \int_{\mathbb{R}^N} u_t^2 dxds \\ + \varepsilon\int_0^t \int_{\mathbb{R}^N} u u_{tt} dxds. \end{aligned} \tag{10.6.15}$$

The last term in (10.6.15) may be estimated by multiplying (10.6.1) by u and integrating the result over $\mathbb{R}^N \times (0,t)$. Indeed,

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^N} u u_{tt} dxds = & -\lambda\int_0^t \int_{\mathbb{R}^N} u^2 dxds - \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2 dxds \\ & + a\int_0^t \int_{\mathbb{R}^N} |u|^{p+1} dxds - \int_0^t \int_{\mathbb{R}^N} u u_t (V_\gamma * u_t^2) dxds. \end{aligned} \tag{10.6.16}$$

We have, by the Parseval equality and convolution property enjoyed by the kernel $V_\gamma(x)$ (see, e.g., [634], Chapters 7.1 and 3.4),

$$\int_0^t \int_{\mathbb{R}^N} uu_t \int_{\mathbb{R}^N} V_\gamma(x-y)u_t^2 dydxds \\ \leq \int_0^t \left[\int_{\mathbb{R}^N} (V_{\frac{N+\gamma}{2}} * u_t^2)^2 dx\right]^{1/2} \left[\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * (uu_t)\right)^2 dx\right]^{1/2} ds.$$

In fact,

$$\int_{\mathbb{R}^N} uu_t \int_{\mathbb{R}^N} V_\gamma(x-y)u_t^2 dydx = \int_{\mathbb{R}^N} \widehat{uu_t}\overline{\widehat{V_\gamma * u_t^2}}dx = \int_{\mathbb{R}^N} \widehat{uu_t} \mid \widehat{V_{\frac{N+\gamma}{2}}} \mid^2 \overline{\widehat{u_t^2}}dx \\ \leq \left[\int_{\mathbb{R}^N} (\widehat{V_{\frac{N+\gamma}{2}} * u_t^2})^2 dx\right]^{1/2} \left[\int_{\mathbb{R}^N} (\widehat{V_{\frac{N+\gamma}{2}} * (uu_t)})^2 dx\right]^{1/2}. \tag{10.6.17}$$

The notion $\widehat{\cdot}$ stands for the Fourier transform. Also, by the Cauchy–Schwarz inequality, we have

$$V_{\frac{N+\gamma}{2}} * (uu_t) = \int_{\mathbb{R}^N} V_{\frac{N+\gamma}{2}}(x-y)u_t u(y)dy \\ \leq \left(\int_{\mathbb{R}^N} V_{\frac{N+\gamma}{2}}(x-y)u_t^2(y)dy\right)^{1/2} \left(\int_{\mathbb{R}^N} V_{\frac{N+\gamma}{2}}(x-y)u^2(y)dy\right)^{1/2}.$$

That is,

$$V_{\frac{N+\gamma}{2}} * (uu_t) \leq \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^{1/2} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^{1/2}.$$

Therefore,

$$\int_0^t \int_{\mathbb{R}^N} uu_t \int_{\mathbb{R}^N} V_\gamma(x-y)u_t^2 dydxds \\ \leq \int_0^t \left[\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx\right]^{1/2} \left[\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)\left(V_{\frac{N+\gamma}{2}} * u^2\right) dx\right]^{1/2} ds \\ \leq \int_0^t \left[\int_{\mathbb{R}^N} (V_{\frac{N+\gamma}{2}} * u_t^2)^2 dx\right]^{3/4} \left[\int_{\mathbb{R}^N} (V_{\frac{N+\gamma}{2}} * u^2)^2 dx\right]^{1/4} ds.$$

By the Young inequality, we get for any $\delta > 0$,

$$\int_0^t \int_{\mathbb{R}^N} uu_t \int_{\mathbb{R}^N} V_\gamma(x-y)u_t^2 dydxds \\ \leq \delta \int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dxds + \frac{1}{4\delta^3} \int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dxds. \tag{10.6.18}$$

Inserting (10.6.16)–(10.6.18) to (10.6.15), we conclude

$$\begin{aligned}
L'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) &+ \varepsilon \int_{\mathbb{R}^N} u_0 u_1 dx + \varepsilon \int_0^t \int_{\mathbb{R}^N} u_t^2 dxds \\
&- \lambda\varepsilon \int_0^t \int_{\mathbb{R}^N} u^2 dxds - \varepsilon \int_0^t \int_{\mathbb{R}^N} \mid \nabla u \mid^2 dxds \\
&+ a\varepsilon \int_0^t \int_{\mathbb{R}^N} \mid u \mid^{p+1} dxds - \varepsilon\delta \int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dxds \\
&- \frac{\varepsilon}{4\delta^3} \int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dxds. \qquad (10.6.19)
\end{aligned}$$

By a similar argument to that in (10.6.17), we may obtain

$$\int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dx = \int_{\mathbb{R}^N} u_t^2 \left(V_\gamma * u_t^2\right) dx.$$

It follows then from (10.6.7) that

$$\int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u_t^2\right)^2 dxds = H'(t) - H'(0). \qquad (10.6.20)$$

The last term on the right-hand side of (10.6.19) may be handled as follows. By Hölder's inequality, we see that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dxds \\
&\quad = \int_0^t \int_{\mathbb{R}^N} u^2 \left(V_\gamma * u^2\right) dxds \qquad (10.6.21) \\
&\quad \leq \int_0^t \left(\int_{\mathbb{R}^N} \mid u \mid^{p+1} dx\right)^{2/(p+1)} \left(\int_{\mathbb{R}^N} (V_\gamma * u^2)^{(p+1)/(p-1)} dx\right)^{(p-1)/(p+1)} ds \\
&\quad \leq \left(\int_0^t \int_{\mathbb{R}^N} \mid u \mid^{p+1} dxds\right)^{2/(p+1)} \left(\int_0^t \int_{\mathbb{R}^N} (V_\gamma * u^2)^{(p+1)/(p-1)} dxds\right)^{(p-1)/(p+1)}.
\end{aligned}$$

Using the Hardy–Littlewood–Sobolev inequality, we get

$$\int_{\mathbb{R}^N} \left(V_\gamma * u^2\right)^{(p+1)/(p-1)} dx \leq A \left(\int_{\mathbb{R}^N} u^{2r} dx\right)^{(1/r)(p+1)/(p-1)}$$

with $A > 0$ and $r = N(p+1)/(2pN - \gamma(p+1))$.

Observe that when $p > 3$, we have $2r \leq p+1$ (in fact $2r < p+1$). Indeed, $p > 3$ implies $\gamma < N \leq 2N(p-1)/(p-1)$, then $2r < p+1$.

By Hölder's inequality, we have

$$\int_{\mathbb{R}^N} \left(V_\gamma * u^2\right)^{(p+1)/(p-1)} dx \le C(R+T)^{\mu_1} \left(\int_{\mathbb{R}^N} \mid u \mid^{p+1} dx\right)^{2/(p-1)}$$

for some positive constant C and $\mu_1 = (N/r(p-1))(p+1-2r)$. Therefore, when $p > 3$,

$$\begin{aligned}
&\left(\int_0^t \int_{\mathbb{R}^N} (V_\gamma * u^2)^{(p+1)/(p-1)} dxds\right)^{(p-1)/(p+1)} \\
&\le \widehat{C}(R+T)^{\mu_2} \left(\int_0^t \left(\int_{\mathbb{R}^N} \mid u \mid^{p+1} dx\right)^{2/(p-1)} ds\right)^{(p-1)/(p+1)} \\
&\le \widehat{C}(R+T)^{\mu_2} \left(\int_0^t 1^{(p-1)/(p-3)} ds\right)^{(p-3)/(p+1)} \left(\int_0^t \int_{\mathbb{R}^N} \mid u \mid^{p+1} dxds\right)^{2/(p+1)},
\end{aligned}$$

where $\widehat{C} = C^{(p-1)/(p+1)}$ and $\mu_2 = (N/r)(1-(2r/p+1))$. From now on, C will denote a generic positive constant which may change from line to line. Hence from (10.6.21) it follows that

$$\int_0^t \int_{\mathbb{R}^N} \left(V_{\frac{N+\gamma}{2}} * u^2\right)^2 dxds \le C(R+T)^{\mu_2} T^{(p-3)/(p+1)} \left(\int_0^t \int_{\mathbb{R}^N} |u|^{p+1} dxds\right)^{4/(p+1)}. \tag{10.6.22}$$

By (10.6.20) and (10.6.22), we obtain from (10.6.19) that

$$\begin{aligned}
L'(t) \ge\ & (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\delta H'(t) + \varepsilon\delta H'(0) + \varepsilon \int_{\mathbb{R}^N} u_0 u_1 dx \\
& + \varepsilon \int_0^t \int_{\mathbb{R}^N} u_t^2 dxds - \lambda\varepsilon \int_0^t \int_{\mathbb{R}^N} u^2 dxds - \varepsilon \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2 dxds \\
& + a\varepsilon \int_0^t \int_{\mathbb{R}^N} |u|^{p+1} dxds - \frac{\varepsilon}{4\delta^3} C(R+T)^{\mu_2} T^{(p-3)/(p+1)} \\
& \times \left(\int_0^t \int_{\mathbb{R}^N} |u|^{p+1} dxds\right)^{4/(p+1)}.
\end{aligned} \tag{10.6.23}$$

Selecting $\delta = MH^{-\sigma}(t)$, the inequality (10.6.23) becomes

$$\begin{aligned}
L'(t) \ge\ & \Big((1-\sigma) - \varepsilon M\Big) H^{-\sigma}(t)H'(t) + \varepsilon M H^{-\sigma}(t) H'(0) + \varepsilon \int_{\mathbb{R}^N} u_0 u_1 dx \\
& + \varepsilon \int_0^t \int_{\mathbb{R}^N} u_t^2 dxds - \lambda\varepsilon \int_0^t \int_{\mathbb{R}^N} u^2 dxds - \varepsilon \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2 dxds \\
& + a\varepsilon \int_0^t \int_{\mathbb{R}^N} |u|^{p+1} dxds - \frac{\varepsilon}{4M^3} C(R+T)^{\mu_2} T^{(p-3)/(p+1)} H^{3\sigma}(t) \\
& \times \left(\int_0^t \int_{\mathbb{R}^N} |u|^{p+1} dxds\right)^{4/(p+1)}.
\end{aligned} \tag{10.6.24}$$

Now we estimate the last term on the right-hand side of (10.6.24), from the definition of $H(t)$ it follows that

$$H^{3\sigma}(t) \le 2^{3\sigma-1}\left[\left(\frac{a}{p+1}\right)^{3\sigma}\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{3\sigma} + (dT+l)^{3\sigma}\left(\int_{\mathbb{R}^N}u_0^2dx\right)^{3\sigma}\right].$$

Therefore,

$$\begin{aligned} &H^{3\sigma}(t)\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{4/(p+1)} \\ &\quad\le 2^{3\sigma-1}\,(a/(p+1))^{3\sigma}\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{3\sigma+(4/(p+1))} \\ &\qquad + 2^{3\sigma-1}(dT+l)^{3\sigma}\left(\int_{\mathbb{R}^N}u_0^2dx\right)^{3\sigma}\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{4/(p+1)}. \end{aligned}$$

As $\sigma = (p-3)/[6(p+1)]$ and $p > 3$, we have $3\sigma + 4/(p+1) \le 1$. In this case, we have

$$\begin{aligned} &H^{3\sigma}(t)\left(\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right)^{4/(p+1)} \\ &\quad\le 2^{3\sigma-1}\left[(a/(p+1))^{3\sigma} + (dT+l)^{3\sigma}\left(\int_{\mathbb{R}^N}u_0^2dx\right)^{3\sigma}\right]\left(1+\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\right). \end{aligned}$$

Inserting this estimate in (10.6.24) and choosing $\varepsilon \le (1-\sigma)/M$, we may obtain

$$\begin{aligned} L'(t) \ge{}& \varepsilon\int_{\mathbb{R}^N}u_0u_1dx + \varepsilon\int_0^t\int_{\mathbb{R}^N}u_t^2dxds - \lambda\varepsilon\int_0^t\int_{\mathbb{R}^N}u^2dxds \\ &- \varepsilon\int_0^t\int_{\mathbb{R}^N}|\nabla u|^2dsdx + a\varepsilon\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds \\ &- \frac{\varepsilon}{M}B(T)\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds - \frac{\varepsilon}{M}B(T), \end{aligned}$$

where

$$\begin{aligned} B(T) ={}& 2^{3(\sigma-1)}C(R+T)^{\mu_2}T^{(p-3)/(p+1)} \\ &\times\left[\Big(a/(p+1)\Big)^{3\sigma} + (dT+l)^{3\sigma}\Big(\int_{\mathbb{R}^N}u_0^2dx\Big)^{3\sigma}\right]. \end{aligned}$$

For a positive constant K to be determined later on, we may also write

$$\begin{aligned} L'(t) \ge{}& KH(t) - \frac{aK}{p+1}\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds + \lambda\frac{K}{2}\int_0^t\int_{\mathbb{R}^N}u^2dxds \\ &+ \frac{K}{2}\int_0^t\int_{\mathbb{R}^N}u_t^2dxds + \frac{K}{2}\int_0^t\int_{\mathbb{R}^N}|\nabla u|^2dxds - K(dT+l)\int_{\mathbb{R}^N}u_0^2dx \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\int_{\mathbb{R}^N} u_0u_1dx - \frac{\varepsilon}{M}B(T) + \varepsilon\int_0^t\int_{\mathbb{R}^N} u_t^2dxds - \lambda\varepsilon\int_0^t\int_{\mathbb{R}^N} u^2dxds\\
&-\varepsilon\int_0^t\int_{\mathbb{R}^N}|\nabla u|^2dxds + a\varepsilon\int_0^t\int_{R^N}|u|^{p+1}dxds\\
&-\frac{\varepsilon}{M}B(T)\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds.
\end{aligned}
$$

That is,

$$
\begin{aligned}
L'(t) \geq KH(t) &+ \left[\varepsilon\left(a - \frac{B(T)}{M}\right) - \frac{aK}{p+1}\right]\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\\
&+\lambda\left(\frac{K}{2}-\varepsilon\right)\int_0^t\int_{\mathbb{R}^N}u^2dxds + \left(\frac{K}{2}-\varepsilon\right)\int_0^t\int_{\mathbb{R}^N}|\nabla u|^2dxds\\
&+\left(\frac{K}{2}+\varepsilon\right)\int_0^t\int_{\mathbb{R}^N}u_t^2dxds + \varepsilon\int_{\mathbb{R}^N}u_0u_1dx - K(dT+l)\\
&\times\int_{\mathbb{R}^N}u_0^2dx - \frac{\varepsilon}{M}B(T).
\end{aligned}
$$

Putting $K = 2\varepsilon$, we conclude that

$$
\begin{aligned}
L'(t) \geq 2\varepsilon H(t) &+ \varepsilon\left[a\frac{p-1}{p+1} - \frac{B(T)}{M}\right]\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds\\
&+2\varepsilon\int_0^t\int_{\mathbb{R}^N}u_t^2dxds + \varepsilon\left[\int_{\mathbb{R}^N}u_0u_1dx - 2(dT+l)\int_{\mathbb{R}^N}u_0^2dx - \frac{B(T)}{M}\right].
\end{aligned}
$$

Choose u_0 and u_1 such that

$$
\int_{\mathbb{R}^N}u_0u_1dx - 2(dT+l)\int_{\mathbb{R}^N}u_0^2dx > 0, \qquad (10.6.25)
$$

indeed, since the set of initial data satisfying (10.6.6) and (10.6.25) is not empty, we can pick M so large that

$$
\int_{\mathbb{R}^n}u_0u_1dx - 2(dT+l)\int_{\mathbb{R}^n}u_0^2dx \geq \frac{B(T)}{M} > 0.
$$

The constant M must also be sufficiently large so that $a\frac{p-1}{p+1} > \frac{B(T)}{M}$. Once this is satisfied, we can select b such that

$$
a\frac{p-1}{p+1} - \frac{B(T)}{M} \geq b > 0.
$$

It follows that

$$
L'(t) \geq 2\varepsilon H(t) + \varepsilon b\int_0^t\int_{\mathbb{R}^N}|u|^{p+1}dxds + 2\varepsilon\int_0^t\int_{\mathbb{R}^N}u_t^2dxds. \qquad (10.6.26)
$$

Next, it is clear that

$$L^{1/(1-\sigma)}(t) \le 2^{1/(1-\sigma)} \left\{ H(t) + \varepsilon^{1/(1-\sigma)} \left(\int_0^t \int_{\mathbb{R}^N} u_t u dx ds \right)^{1/(1-\sigma)} \right\}. \quad (10.6.27)$$

By the Cauchy–Schwarz inequality and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^N} u_t u dx ds \\
&\quad \le \int_0^t \left(\int_{\mathbb{R}^N} u^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^N} u_t^2 dx \right)^{1/2} ds \\
&\quad \le C(R+T)^{\mu_3} \int_0^t \left(\int_{\mathbb{R}^N} u^{p+1} dx \right)^{1/(p+1)} \left(\int_{\mathbb{R}^N} u_t^2 dx \right)^{1/2} ds \\
&\quad \le C(R+T)^{\mu_3} \left\{ \int_0^t \left(\int_{\mathbb{R}^N} u^{p+1} dx \right)^{2/(p+1)} ds \right\}^{1/2} \left\{ \int_0^t \int_{\mathbb{R}^N} u_t^2 dx ds \right\}^{1/2},
\end{aligned}$$

where $\mu_3 = (N/2)\Big(p - 1/p + 1\Big) > 0$.

Therefore,

$$\begin{aligned}
&\left(\int_0^t \int_{\mathbb{R}^N} u_t u dx ds \right)^{1/(1-\sigma)} && (10.6.28) \\
&\quad \le C(R+T)^{\mu_4} T^{\alpha} \left\{ \int_0^t \int_{\mathbb{R}^N} u^{p+1} dx ds) \right\}^{1/(p+1)(1-\sigma)} \left\{ \int_0^t \int_{\mathbb{R}^N} u_t^2 dx ds) \right\}^{1/2(1-\sigma)} \\
&\quad \le C(R+T)^{\mu_4} T^{\alpha} \left\{ \int_0^t \int_{\mathbb{R}^N} u_t^2 dx ds + \left(\int_0^t \int_{\mathbb{R}^N} u^{p+1} dx ds \right)^{2/(p+1)(1-2\sigma)} \right\},
\end{aligned}$$

where

$$\mu_4 = \frac{\mu_3}{1-\sigma} = \frac{N}{2(1-\sigma)} \left(\frac{p-1}{p+1} \right) = \frac{N(p-1)}{p+3}, \quad \alpha = \frac{p-1}{p+3}.$$

We have used Young's inequality with $2(1-\sigma)$ and $2(1-\sigma/(1-2\sigma))$ in the last inequality.

Finally, it is easy to see, from (10.6.26) and (10.6.28), that we can find a sufficiently large constant $\tilde{C} > 0$ such that

$$L^{1/(1-\sigma)}(t) \le \tilde{C} L'(t).$$

Thus, integrating over $(0,t)$, we conclude

$$L^{\sigma/(1-\sigma)}(t) \ge \frac{1}{L(0)^{-\sigma/(1-\sigma)} - \frac{\sigma_t}{\tilde{C}(1-\sigma)}}$$

which implies that $L(t)$ blows up at a finite time

$$T^* \le \frac{(1-\sigma)\tilde{C}L(0)^{-\sigma/(1-\sigma)}}{\sigma}.$$

As $L(0) = H^{1-\sigma}(0) = \left(l \int_{\mathbb{R}^N} u_0^2 dx\right)^{1-\sigma}$, choosing l such that

$$l \ge \left(\frac{(1-\sigma)\tilde{C}}{\sigma T}\right)\left(\int_{\mathbb{R}^N} u_0^2 dx\right)^{-1},$$

we conclude that $T^* \le T$. □

10.7 Blow-up of solutions to the quasilinear hyperbolic-elliptic inequalities

In this section, we shall apply Theorem 2.4.26 to prove the non-existence of global non-negative solutions to the quasilinear hyperbolic-elliptic inequalities. These results are chosen from Alaa and Guedda [16].

We shall introduce some blow-up results, due to Alaa and Guedda [16], on the non-existence of global non-negative solutions to the following

$$\begin{cases} u_{tt} - a\Delta u + \varphi\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + \delta u_t \ge f(x,t), \text{ on } \Omega \times (0,T) & (10.7.1) \\ u = 0, \text{ in } \partial\Omega \times (0,T) & (10.7.2) \end{cases}$$

where Ω is a regular open subset of $\mathbb{R}^N$, $N \ge 1$, $a > 0$ and $f \in L^\infty(0,+\infty; L^2(\Omega))$. The function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is locally Lipschitz satisfying

$$\varphi(s) \ge bs^\gamma \tag{10.7.3}$$

where $b > 0$ and $\gamma > 0$.

Let Φ_1 be a positive eigenfunction of

$$-\Delta\Phi_1 = \lambda_1\Phi_1, \quad \Phi_1\Big|_{\partial\Omega} = 0,$$

where λ_1 is the first eigenvalue. We assume that

$$\|\Phi_1\|_{L^2(\Omega)} = 1. \tag{10.7.4}$$

Theorem 10.7.1 ([16]). *Let $\delta \in \mathbb{R}$ and $\gamma > 0$. Assume for all $t \ge 0$,*

$$\int_\Omega f(x,t)\Phi_1(x)dx \ge 0. \tag{10.7.5}$$

Then there exists no global solution to problem (10.7.1)–(10.7.2) *such that*

$$\int_\Omega u(x,0)\Phi_1(x)dx > \lambda_1^{-1/2}\left(\frac{a}{b}\right)^{1/2\gamma}, \quad \int_\Omega u_t(x,0)\Phi_1(x)dx \ge 0.$$

Proof. Let u be a global solution. Multiplying equation (10.7.1) by Φ_1, we can get

$$\omega'' + \delta\omega' \geq \lambda_1 \varphi\left(\int_\Omega |\nabla u|^2 dx\right)\omega - a\lambda_1\omega, \tag{10.7.6}$$

where

$$\omega(t) = \int_\Omega u(x,t)\Phi_1(x)dx \quad \textit{and} \quad \omega' := \frac{d\omega}{dt}.$$

As $\omega(0) > 0$, we obtain that $\omega(t) > 0$ on $(0, t_0)$ for $t_0 > 0$ small. On the other hand, we have

$$\varphi\left(\int_\Omega |\nabla u|^2 dx\right) \geq b\left(\int_\Omega |\nabla u|^2 dx\right)^\gamma,$$

whence

$$\varphi\left(\int_\Omega |\nabla u|^2 dx\right) \geq b\lambda_1^\gamma \left(\int_\Omega |u|^2 dx\right)^\gamma.$$

Next, using Hölder's inequality and (10.7.4), we deduce that

$$\omega^2(t) \leq \int_\Omega |u|^2 dx, \quad \text{whence} \quad \varphi\left(\int_\Omega |\nabla u|^2 dx\right) \geq b\lambda_1^\gamma \omega^{2\gamma}$$

which, substituted into (10.7.6), yields

$$\omega'' + \delta\omega' \geq b\lambda_1^{\gamma+1}\omega^{2\gamma+1} - a\lambda_1\omega, \tag{10.7.7}$$

on $[0, t_0)$.

Finally, since $\gamma > 0, \omega'(0) \geq 0$ and $\omega(0)$ is larger than $\left(a/(b\lambda_1^\gamma)\right)^{1/2\gamma}$, we deduce from Theorem 2.4.26 that ω is not global. This is a contradiction. This hence completes the proof. □

The following result is an immediate consequence of Corollary 2.4.4.

Corollary 10.7.1 ([16])**.** *Let* $\gamma > 0$ *and* $\delta \geq 0$*. Let* u *be a solution to problem* (10.7.1)–(10.7.2)*. Assume that*

$$B^2 + a\lambda_1 A^2 - \frac{\lambda_1^\gamma}{\gamma+1} b A^{2(\gamma+1)} \leq \frac{\gamma}{\gamma+1}\left(\frac{a^{\gamma+1}}{b}\right)^{1/\gamma}, \tag{10.7.8}$$

where

$$\begin{cases} A = \displaystyle\int_\Omega u(x,0)\Phi_1(x)dx \ > \lambda_1^{-1/2}\left(\frac{a}{b}\right)^{1/2\gamma}, \\ B = \displaystyle\int_\Omega u_t(x,0)\Phi_1(x)dx < 0. \end{cases} \tag{10.7.9}$$

Then the function

$$\omega(t) = \int_\Omega u(x,t)\Phi_1(x)dx$$

is not global.

Chapter 11

Blow-up of Solutions to Abstract Equations and Thermoelastic Equations

In this chapter, we shall study the blow-up of solutions to abstract equations and thermoelastic equations. This chapter consists of six sections. In Section 11.1, we shall employ Theorem 2.4.19 to prove the blow-up results of solutions to a class of abstract initial and initial boundary value problems. In Section 11.2, we shall employ Theorem 2.4.20 to study the blow-up of solutions to a class abstract nonlinear equations. In Section 11.3, we shall employ Theorem 2.4.19 to prove some abstract blow-up results. In Section 11.4, we shall employ Theorems 2.4.1 and 2.4.3 to study the blow-up results for two classes of evolutionary partial differential equations. In Section 11.5, we shall use Theorems 2.4.19–2.4.20 to study the blow-up phenomena of solutions to mixed problems. In Section 11.6, we shall employ Theorem 2.4.19 to establish the blow-up results for a nonlinear one-dimensional thermoelastic system with a non-autonomous forcing term and a thermal memory when the heat flux obeys both Fourier's law and Gurtin and Pipkin's law which extended those in [336]. Inequalities used in this chapter are very crucial in deriving the blow-up of solutions to some abstract equations and thermoelastic equations.

11.1 Blow-up of solutions to abstract nonlinear equations

In this section, we shall employ Theorem 2.4.19 to prove the blow-up results of solutions to a wide class of abstract initial and initial boundary value problems. These results are picked from Levine [505].

We shall consider the following initial and initial boundary value problems

$$P\frac{d^2u}{dt^2} = -A(t)u + \mathcal{F}(u), \quad t \in [0, T), \quad u(0) = u_0, u_t(0) = v_0, \tag{11.1.1}$$

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where u is a Hilbert space-valued function of t, $A(t)$ is a symmetric linear operator defined and non-negative for each $t \geq 0, P$ is strictly positive symmetric operator and $\mathcal{F}$ is a given nonlinearity.

Let u be a twice strongly continuously differentiable function satisfying (11.1.1) on $[0, T)$. Suppose $\mathcal{F}$ has a symmetric Fréchet derivative $\mathcal{F}_x$ so that the scalar-valued function

$$\mathcal{G}(x) \equiv \int_0^1 (\mathcal{F}(\rho x), x) d\rho \tag{11.1.2}$$

is an appropriate corresponding "potential" for $\mathcal{F}$. Assume further that there is a constant $\alpha > 0$ such that

$$(x, \mathcal{F}(x)) \geq 2(2\alpha + 1)\mathcal{G}(x) \tag{11.1.3}$$

for all x in the appropriate domain. Then, if

$$\mathcal{G}(u_0) > \frac{1}{2}\Big[(u_0, A(0)u_0) + (v_0, Pv_0)\Big] \equiv E(0), \tag{11.1.4}$$

then the interval of existence of u is bounded and, for some $T < +\infty$,

$$\lim_{t \to T^-} (u(t), Pu(t)) = +\infty \tag{11.1.5}$$

which implies that if the initial potential energy of the nonlinearity is larger than the total initial energy of the linear problem, then (11.1.1) does not admit global solutions.

Let H be a real Hilbert space, and let $D \subseteq H$ be a dense linear subspace. Let $(\cdot, \cdot)$ denote the scalar product on H and let $\| \cdot \|$ denote the corresponding norm.

We assume that for each $t \geq 0$:

(A1) $A(t) : D \to H$ is a symmetric linear operator;

(A2) $(x, A(t)x) \geq 0$ if $x \in D$. (Thus $A(t)$ has a self-adjoint extension, but we do not use this fact).

(A3) If $v : [0, +\infty) \to H$ is strongly continuously differentiable and if, for all $t \geq 0, v(t)$ and $dv(t)/dt \in D$, then $(v(t), A(t)v(t))$ is continuously differentiable and, for all $t \geq 0$,

$$Q_A(v, v)(t) \equiv (d/dt)(v(t), A(t)v(t)) - 2(dv(t)/dt, A(t)v(t)) \leq 0.$$

We assume further that

(P1) P is a symmetric linear operator, $P : D_P \to H$ and that $D \subseteq D_P \subseteq H$.

(P2) $(x, Px) > 0$ for all $x \in D_P, x \neq 0$.

Moreover, assume that D is a Hilbert space under a scalar product $(\cdot, \cdot)_D$. Assume that the injection from D into H is continuous as a mapping of Hilbert spaces, that is, there is a constant $c > 0$ such that $\|x\| \leq c\|x\|_D$ for all $x \in D$.

Then we assume that

(F1) $\mathcal{F} : D \to H$ is continuously differential as a function from D equipped with $\|\cdot\|_D$ into H, that the Fréchet derivative $\mathcal{F}_x$ is a symmetric, bounded linear operator on H and that $x \to \mathcal{F}_x$ is a strongly continuous map from D into $\mathcal{L}(H)$, the collection of bounded linear operators from H into itself.

(F2) Let $\mathcal{G}(x) \equiv \int_0^1 (\mathcal{F}(\rho x), x) d\rho$ denote the potential associated with $\mathcal{F}$, that is, $\mathcal{G} : D \to \mathbb{R}$ is the unique up to a constant scalar-valued function whose Fréchet derivative $\mathcal{G}(x)$ is defined by

$$\mathcal{G}_x y = (\mathcal{F}(x), y)$$

for all $x, y \in D$. Assume that for some constant $\alpha > 0$ and for all $x \in D$,

$$(x, \mathcal{F}(x)) \geq 2(2\alpha + 1)\mathcal{G}(x). \tag{11.1.6}$$

The verification of the action of $\mathcal{G}_x$ can be carried out directly from the definition.

Note that the following is a useful formula in the sequel, which is valid for $v : [0, T) \to D$ with a strongly continuous derivative v_t likewise taking values in D:

$$\mathcal{G}(v(t)) - \mathcal{G}(v(0)) = \int_0^t (\mathcal{F}(v(\eta)), v_\eta(\eta)) d\eta \tag{11.1.7}$$

where the strong continuity of v and v_t are taken in the sense of the norm on D. This follows directly from the chain rule and the action of $\mathcal{G}_x$. The following formal proof is nevertheless instructive. Suppressing the t argument, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{G}(v(t)) &= \int_0^1 \left[\rho(\mathcal{F}_{\rho v} v_t, v) + (\mathcal{F}(\rho v), v_t)\right] d\rho \\
&= \int_0^1 \left[\rho \frac{d}{d\rho}(\mathcal{F}(\rho v), v_t) + (\mathcal{F}(\rho v), v_t)\right] d\rho \\
&= \int_0^1 \frac{d}{d\rho}\left[\rho(\mathcal{F}(\rho v), v_t)\right] d\rho \\
&= (\mathcal{F}(v(t)), v_t(t)),
\end{aligned}$$

where we have used the symmetry of $\mathcal{F}_x$ in the second line.

Definition 11.1.1. We say that $u : [0, T) \to H$ is a solution to $Pu_{tt} = -A(t)u + \mathcal{F}(u)$ if, for each t, $u(t)$ and u_{tt} belong to D (u_t being the strong derivative of u in the norm $\|\cdot\|_D$ on D), u_{tt} exists and is strongly continuous in the sense of the norm on H and takes values in D_P, and the differential equation is satisfied in the classical sense.

We have the following result due to Levine [505].

Theorem 11.1.1 ([505]). *Let $u : [0,T) \to H$ be a solution to problem* (11.1.1) *in the sense of Definition* 11.1.1. *Let $P, \mathcal{F}$ and $A(\cdot)$ satisfy the above hypotheses. Then each of the following statements holds:*

(i) *If*

$$\beta_0 \equiv 2\Big\{\mathcal{G}(u_0) - \frac{1}{2}[(u_0, A(0)u_0) + (v_0, Pv_0)]\Big\} > 0, \tag{11.1.8}$$

then the solution can only exist on a bounded interval $[0,T)$ and

$$\lim_{t\to T^-}(u(t), Pu(t)) = +\infty$$

where

$$T \le T_{\beta_0} \equiv \alpha^{-1}\Big\{[\beta_0(u_0, Pu_0) + (u_0, Pv_0)^2]^{1/2} + (u_0, Pv_0)\Big\}^{-1}(u_0, Pu_0).$$

(ii) *If*

$$\begin{cases} \mathcal{G}(u_0) = \dfrac{1}{2}[(u_0, A(0)u_0) + (v_0, Pv_0)], & (11.1.9)\\ (v_0, Pv_0)/(v_0, Pu_0) = \lambda > 0, & (11.1.10)\end{cases}$$

then the solution can only exist on a bounded interval $[0,T)$ and

$$\lim_{t\to T^-}(u(t), Pu(t)) = +\infty$$

where

$$T \le (2a\lambda)^{-1}.$$

It clearly follows from Theorem 11.1.1 that if u_0 satisfies

$$\mathcal{G}(u_0) > \frac{1}{2}(u_0, A(0)u_0), \tag{11.1.11}$$

then there exist v_0's such that the corresponding solutions blow up in a finite time.

Remark 11.1.1 ([505]). Obviously, the solutions to (11.1.1) are not necessarily unique. If, however, $(x, Px) \ge \lambda(x, x)$ for all $x \in D$ and some $\lambda > 0$ and if $\mathcal{F}$ is such that the difference w of two solutions satisfies $\|w_{tt} + A(t)w\| \le K(t)(w, Pw)^{1/2}$ where $K(t)$ is a locally bounded function on $[0,T)$ depending upon u and v, then $w(0) = w_t(0) = 0$ implies $w \equiv 0$. For details, we refer to [502].

Corollary 11.1.1 ([505]). *Let $\mathcal{F}(sx) = s^{1+\delta}\mathcal{F}(x)$ for some constant $\delta > 0$ and for all $x \in D$. Let $(x_0, \mathcal{F}(x_0)) > 0$ for some point $x_0 \in D$. Then there are infinitely many vectors u_0 such that* (11.1.11) *holds.*

Proof. Choose s so large that

$$s^\delta \mathcal{G}(x_0) = s^\delta \int_0^1 (\mathcal{F}(\rho x_0), x_0) d\rho > \frac{1}{2}(x_0, A(0)x_0)$$

($s \geq s_0$, say). Then for any $u_0 = sx_0$ with $s \geq s_0$,

$$\mathcal{G}(u_0) = s^{\delta+2}\mathcal{G}(x_0) > \frac{1}{2}(u_0, A(0)u_0).$$

In fact, in most applications δ and α can be chosen as $\delta = 4\alpha$.

Therefore, we are now in a position to prove Theorem11.1.1. □

Proof of Theorem 11.1.1. Let

$$F(t) = (u(t), Pu(t)) + Q^2 + \beta(t+\tau)^2 \tag{11.1.12}$$

where Q, β and τ are non-negative constants to be determined later on. Then, using the symmetric of P, we have

$$F'(t) = 2(u_t, Pu) + 2\beta(t+\tau),$$

and

$$F''(t) = 2(u_t, Pu_t) + 2(u, Pu_{tt}) + 2\beta.$$

Thus it follows that

$$\begin{aligned} &F(t)F''(t) - (\alpha+1)(F'(t))^2 \\ &\quad = 4(\alpha+1)S^2 + 4(\alpha+1)Q^2[(u_t, Pu_t) + \beta] \\ &\qquad + 2F(t)\big\{(u, Pu_{tt}) - (2\alpha+1)[(u_t, Pu_t) + \beta)]\big\}, \end{aligned} \tag{11.1.13}$$

where

$$S^2 = \big[(u, Pu) + \beta(t+\tau)^2\big]\big[(u_t, Pu_t) + \beta\big] - \big[(u, Pu) + \beta(t+\tau)^2\big] \geq 0.$$

Define from (11.1.13),

$$\begin{aligned} H(t) &= (u, Pu_{tt}) - (2\alpha+1)[(u_t, Pu_t) + \beta] \\ &= -(u, Au) - (2\alpha+1)[(u_t, Pu_t) + \beta] + (u, \mathcal{F}(u)). \end{aligned} \tag{11.1.14}$$

Thus

$$\begin{aligned} H'(t) &= -[Q_A(u,u) + 2(u_t, Au) + 2(2\alpha+1)(u_t, Pu_{tt})] + d(u, \mathcal{F}(u))/dt \\ &= -Q_A(u,u) + 4\alpha(u_t, Au) + d(u, \mathcal{F}(u))/dt - 2(2\alpha+1)/(u_t, \mathcal{F}(u)) \end{aligned}$$

which implies

$$\begin{aligned} H'(t) = &-(2\alpha+1)Q_A(u,u) + 2\alpha d(u, Au)/dt \\ &+ d(u, \mathcal{F}(u))/dt - 2(2\alpha+1)(u_t, \mathcal{F}(u)) \end{aligned}$$

and using (i), (ii), the positive semi-definiteness of $A(t)$, (11.1.6) and (11.1.7),

$$\begin{aligned} H(t) &= H(0) + 2\alpha(u, Au) - 2\alpha(u_0, A(0)u_0) - (2\alpha+1)\int_0^t Q_A(u,u)d\eta \\ &\quad + (u, \mathcal{F}(u)) - (u_0, \mathcal{F}(u_0)) - 2(2\alpha+1)[\mathcal{G}(u) - \mathcal{G}(u_0)] \\ &\geq 2(2\alpha+1)\Big\{\mathcal{G}(u_0) - \frac{1}{2}[(u_0, A(0)u_0) + (v_0, Pv_0) + \beta]\Big\}. \end{aligned} \tag{11.1.15}$$

Now suppose (11.1.8) holds. Then with $Q^2 = 0$ and $\beta = \beta_0$, we find that $H(t) \geq 0$ and hence $(F^{-\alpha})'(t) \leq 0$. Also $F'(0) = 2(u_0, Pv_0) + 2\beta_0\tau > 0$ if τ is sufficiently large. Thus, by Theorem 2.4.19, the interval of existence cannot, in this case, exceed $T_{\beta_0} = F(0)/\alpha F'(0)$ in length, i.e.,

$$T \leq \frac{(u_0, Pu_0) + \beta_0\tau^2}{2\alpha((u_0, Pv_0) + \beta_0\tau)^2} = f(\tau).$$

On the other hand, we know $f(\tau)$ has a minimum, which is

$$(\alpha\beta_0)^{-1}\left\{-(u_0, Pv_0) + [(u_0, Pv_0)^2 + \beta_0(u_0, Pu_0)^{1/2}]\right\},$$

on the interval, $(-(u_0, Pv_0)/\beta_0, +\infty)$ at

$$\tau = \beta_0^{-1}\left\{-(u_0, Pu_0) + [(u_0, Pv_0)^2 + \beta_0(u_0, Pu_0)]^{1/2}\right\}.$$

Thus

$$T \leq T_{\beta_0} = \alpha^{-1}\Big\{[\beta_0(u_0, Pu_0) + (u_0, Pv_0)^2]^{1/2} + (u_0, Pu_0)\Big\}^{-1}(u_0, Pu_0).$$

If (11.1.9) and (11.1.10) hold, the proof is easier. Let $Q^2 = 0$ and $\beta = 0$ so that $[F^{-\alpha}(t)]'' \leq 0$ where $F(t) = (u(t), Pu(t))$ and $\lim_{t\to T^-}(u(t), Pu(t)) = +\infty$ where $T \leq F(0)/\alpha F'(0) = (u_0, Pu_0)/2\alpha(u_0, Pv_0) = 1/2\alpha\lambda$, by Theorem 2.4.19. □

11.2 Blow-up of solutions to a class of abstract nonlinear equations

In this section, we shall employ Theorems 2.4.19–2.4.20 to study the blow-up of solutions to a class of abstract nonlinear equations. These results are chosen from Knops, Levine and Payne [440].

We shall consider the following abstract nonlinear problem

$$\begin{cases} P\dfrac{d^2u}{dt^2} = -Nu + A^*\mathcal{F}(Au(t)), & t \in [0, T), \quad (11.2.1) \\ u(0) = u_0, \ \ \dfrac{du}{dt}(0) = v_0, & u_0, \ v_0 \in D \quad (11.2.2) \end{cases}$$

where H is a real Hilbert space with scalar product $(\cdot,\cdot)$ and norm $||\cdot||$, and D_1, D are dense linear subspaces with $D_1 \subseteq D$, by $\mathcal{L}(H)$ we denote the bounded linear operators from H into H, $A : D \to \mathcal{L}(H)$ is a given linear, one-to-one, but not necessarily bounded, transformation, $\mathcal{L}_A^1$ denotes the image of A restricted to D_1, A^* is the adjoint operator of A, that is, $A^* : \mathcal{L}_A^1 \to H$, such that for all $x, y \in D_1$,

$$\mathcal{B}(Ax, Ay) = -(A^*Ax, y). \tag{11.2.3}$$

Let $\mathcal{L}_A$ denote the image of A, i.e., $\mathcal{L}_A = \{X \in \mathcal{L}(H) | X = Ax \ for\ some\ x \in D\}$. Assume that there is a given positively definite form $\mathcal{B} : \mathcal{L}_A \times \mathcal{L}_A \to \mathbb{R}$. We use this form to make $\mathcal{L}_A$ into a Hilbert space in the usual manner.

Let $W : \mathcal{L}_A \to \mathbb{R}$ be a nonlinear functional which possesses at each point $X \in \mathcal{L}_A$ a Fréchet derivative W_X, which is continuous in the topology of $\mathcal{F}_A$ induced by $\mathcal{B}$. By the Riesz representation theorem, the action of W_X can be written as for all $Y \in \mathcal{L}_A$,

$$W_X \cdot Y = \mathcal{B}(Y, Z), \tag{11.2.4}$$

for some $Z \in \mathcal{L}_A$ which is uniquely determined by X. We shall denote the functional by $Z = \mathcal{F}(X)$, so that (11.2.4) reduces to

$$W_X \cdot Y = \mathcal{B}(Y, \mathcal{F}(X)). \tag{11.2.5}$$

Here $\mathcal{F}$ is a mapping from $\mathcal{L}_A$ into $\mathcal{L}_A$. For technical reasons, we shall assume that

$$X \in \mathcal{L}_A^1 \Rightarrow \mathcal{F}(X) \in \mathcal{L}_A^1. \tag{11.2.6}$$

Let P and N be symmetric linear operators from D_1 into H, that is, for all $x, y \in D_1$,

$$\begin{cases} (x, Py) = (Px, y), & (11.2.7) \\ (x, Ny) = (Nx, y). & (11.2.8) \end{cases}$$

We shall assume that there is a constant $\lambda > 0$ such that

$$(x, Nx) \geq \lambda(x, Px), \quad \text{for all } x \in D_1 \tag{11.2.9}$$

and

$$(x, Px) > 0, \ \text{for all } x \in D_1, \ x \neq 0. \tag{11.2.10}$$

We shall further assume that the bilinear forms (x, Py) and (x, Ny) can be extended to all of D in such a way that (11.2.9) and (11.2.10) hold on D. Let $u : [0, T) \to D$ be a Hilbert space-valued function belonging to $C^1([0, T), D)$, the space of once strongly continuously differentiable functions whose strong derivatives likewise take values in D.

We need the following definitions which were introduced in [440].

Definition 11.2.1 (Weak Solutions)**.** We say that $u \in C^1([0,T),D)$ is a weak solution of the Cauchy problem (11.2.1)–(11.2.2) provided that for every $\varphi \in C^\infty([0,T),D)$, u satisfies

$$\begin{cases} (\varphi(t), P\dfrac{du(t)}{dt}) - (\varphi(0), Pv_0) = \displaystyle\int_0^t \left[\left(\dfrac{d\varphi(\eta)}{d\eta}, P\dfrac{du(\eta)}{d\eta} \right) - (N\varphi(\eta), u(\eta)), \right. \\ \qquad\qquad\qquad\qquad\qquad \left. -\mathcal{B}(A\varphi(\eta), A\mathcal{F}(u(\eta))) \right] d\eta, & (11.2.11) \\ E(t) \equiv \dfrac{1}{2} \left[\left(\dfrac{du}{dt}, P\dfrac{du}{dt} \right) + (u(t), Nu(t)) \right] + W(Au(t)) \leq E(0). & (11.2.12) \end{cases}$$

We assume, moreover, that the scalar-valued functions, $(u(t), Pu(t))$ and

$$\frac{1}{2}\left[\left(\frac{du}{dt}, P\frac{du}{dt} \right) - (u, Nu) - 2\mathcal{B}(Au, \mathcal{F}(Au)) \right]$$

are continuous functions of $t \in [0,T)$.

Definition 11.2.2 (Strong Solutions)**.** We say that $u : [0,T) \to D$ is a strong solution of problem (11.2.1)–(11.2.2) if u, $\frac{du}{dt}$ and $\frac{d^2u}{dt^2}$ all exist and are continuous in the strong sense, take values in D_1, and problem (11.2.1)–(11.2.2) hold in the classical sense. Moreover, we assume that $Pu, P\frac{du}{dt}$ and Nu are strongly continuous.

We assume in all cases that

$$\frac{d}{dt}(u(t), Pu(t)) = 2\left(u(t), P\frac{du(t)}{dt} \right) \qquad (11.2.13)$$

and in the case of strong solutions it holds that

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{du(t)}{dt}, P\dfrac{du(t)}{dt} \right) = 2\left(\dfrac{d^2u(t)}{dt^2}, P\dfrac{du(t)}{dt} \right), & (11.2.14) \\ \dfrac{d}{dt}(u(t), Nu(t)) = 2\left(\dfrac{du(t)}{dt}, Nu(t) \right). & (11.2.15) \end{cases}$$

In the latter case, using (11.2.3), (11.2.6), and (11.2.1)–(11.2.2) together with (11.2.13)–(11.2.15), we may deduce (11.2.12) with equality.

We also assume that W is almost homogeneous from above of degree $2(1+2\alpha)$ for some constant $\alpha > 0$, i.e., for all $x \in D$,

$$2(1+2\alpha)W(Ax) \geq \mathcal{B}(Ax, \mathcal{F}(Ax)). \qquad (11.2.16)$$

Equation (11.2.16) may be regarded as a statement about the constitutive assumptions in applications to elasticity. Alternatively, we may simply assume that a particular weak solution to problem (11.2.1)–(11.2.2) satisfies, for all $t \in [0,T)$,

$$2(1+2\alpha)W(Au(t)) \geq \mathcal{B}(Au(t), \mathcal{F}(Au(t))), \qquad (11.2.17)$$

for some constant $\alpha > 0$.

We may find an example in [440] which satisfies all above assumptions of problem (11.2.1)–(11.2.2).

Theorem 11.2.1 ([440]). *If $E(0) < 0$, then, a weak solution $u(t)$ to problem* (11.2.1)–(11.2.2)*, cannot exist on* $[0, +\infty)$*, and, in fact, for some time $t_1 > 0$,*

$$(u(t), Pu(t)) \geq [t_1/(t_1 - t)]^{1/\alpha} F(0) - \beta(t + t_0)^2 \tag{11.2.18}$$

where $t_1 < t_0/\alpha$ and t_0 is given by (11.2.36) *below.*

Proof. Let β and t_0 be positive constants to be determined later, and let $F(t)$ be the function defined by

$$F(t) = (u, Pu) + \beta(t + t_0)^2, \quad t \in [0, T] \tag{11.2.19}$$

where u is the weak solution to problem (11.2.1)–(11.2.2) in Definition 11.2.1. We know that $F(t)$ is a real-valued continuous function of t which is also positively definite in the sense that

$$\begin{cases} \text{i)} \ F(t) \geq 0, \ \text{ for all } t, t_0 \ \textit{and} \ \beta \geq 0; & (11.2.20) \\ \text{ii)} \ F(t) = 0 \Leftrightarrow u = 0, \ \beta = 0, \ \text{for } t_0 > 0, \ t \in [0, T). & (11.2.21) \end{cases}$$

We shall show that $F(t)$ satisfies a second-order differential inequality on a class of weak solutions of problem (11.2.1)–(11.2.2) satisfying the inequality (11.2.16).
Now from (11.2.19), it readily follows

$$F'(t) = 2\left(u, P\frac{du}{dt}\right) + 2\beta(t + t_0). \tag{11.2.22}$$

Since (11.2.11) holds for the limit of smooth functions φ, (11.2.22) may be rewritten as

$$F'(t) = 2(u_0, Pv_0) - 2\int_0^t \left[(u, Nu) + \mathcal{B}(Au, \mathcal{F}(Au)) - \left(\frac{du}{d\eta}, P\frac{du}{d\eta}\right)\right] d\eta + 2\beta(t + t_0), \tag{11.2.23}$$

which, differentiated in t, leads to

$$F''(t) = 2\left(\frac{du}{dt}, P\frac{du}{dt}\right) - 2(u, Nu) - 2\mathcal{B}(Au, \mathcal{F}(Au)) + 2\beta. \tag{11.2.24}$$

On the other hand, in view of (11.2.11) and (11.2.12), we may rearrange (11.2.24) to arrive at

$$\begin{aligned} F''(t) = {} & 4(1+\alpha)\left(\frac{du}{dt}, P\frac{du}{dt}\right) + 4\alpha(u, Nu) \\ & + 2\left[2(1+2\alpha)W(Au) - \mathcal{B}(Au, \mathcal{F}(Au))\right] - 4(1+2\alpha)E(0) + 2\beta. \end{aligned} \tag{11.2.25}$$

Hence from (11.2.22) and (11.2.25), it follows readily

$$
\begin{aligned}
&F(t)F''(t) - (1+\alpha)(F'(t))^2 \\
&\quad = S^2(t) + 4\alpha(u, Nu)F(t) - 2(1+2\alpha)(\beta + 2E(0))F(t) \\
&\qquad + 2\Big[2(1+2\alpha)W(Au) - \mathcal{B}(Au, \mathcal{F}(Au))\Big]F(t),
\end{aligned}
\tag{11.2.26}
$$

where $\alpha > 0$ is the constant defined in (11.2.16) and

$$
\begin{aligned}
S^2(t) = 4(1+\alpha)\Bigg\{ &\left[(u, Pu) + \beta(t+t_0)^2\right]\left[\left(\frac{du}{dt}, P\frac{du}{dt}\right) + \beta\right] \\
&- \left[\left(u, P\frac{du}{dt}\right) + \beta(t+t_0)\right]^2 \Bigg\}.
\end{aligned}
\tag{11.2.27}
$$

Since $1 + \alpha \geq 0$, it follows from Schwarz' inequality that

$$
S^2(t) \geq 0. \tag{11.2.28}
$$

Thus, using of the assumption that the class of weak solutions is restricted by inequality (11.2.16), we derive from (11.2.26),

$$
F(t)F''(t) - (1+\alpha)(F'(t))^2 \geq 4\alpha(u, Nu)F(t) - 2(1+2\alpha)(\beta + 2E(0))F(t), \tag{11.2.29}
$$

which is our fundamental second-order differential inequality.

We now prove the Hölder continuity of solutions (within a certain class) upon the initial data.

To this end, we can define the class $\mathcal{M}$ of solutions considered here by

$$
\mathcal{M} = \{u : G(T) \leq M^2\}, \tag{11.2.30}
$$

where M is a constant and take $G(t) = (u, Pu)$. The initial data (u_0, v_0) is also required to satisfy

$$
E(0) \leq 0, \tag{11.2.31}
$$

which implies $W(Au_0) \leq 0$. Thus it follows immediately from (11.2.29), with $\beta = 0$, that inequality

$$
\frac{d^2}{dt^2}\left(G^{-\alpha}(t)\right) \leq 0 \tag{11.2.32}
$$

holds on the interval $[0, T)$, hence applying Jensen's inequality, we have

$$
G^{\alpha}(t) \leq [G(0)G(T)]^{\alpha}/[(1 - t/T)G^{\alpha}(T) + (t/T)G^{\alpha}(0)]. \tag{11.2.33}
$$

Obviously, this last expression indicates that the null solution is stable on compact subsets of $[0, T)$ in the class $\mathcal{M}$ under small perturbations of the initial data (u_0, v_0). Using logarithmic convexity, we can also observe that the restriction $E(0) \leq 0$ can easily be relaxed (see, e.g., [503]).

On the other hand, uniqueness of the null solution in the case when $W(0) < 0$ follows immediately from (11.2.33) since under homogeneous initial data $E(0) = W(0) < 0$ and the assertion of this theorem are applicable. Indeed, condition $E(0) < 0$ implies $W(Au_0) < 0$ and

$$\mathcal{B}(Au_0, \mathcal{F}(Au_0)) < 0.$$

Setting $\beta + 2E(0) = 0$ in (11.2.28), then we have for all $F(t) > 0,\ t \in [0, T)$ and

$$\frac{d^2(F^{-\alpha}(t))}{dt^2} \leq 0. \tag{11.2.34}$$

Using Theorem 2.4.19, in this case, T cannot be infinite, i.e., no weak solution can exist for all time. In fact, it follows from (11.2.34) by integration that

$$F^\alpha(t) \geq F^\alpha(0) \left[1 - \alpha t \,(F(0))^{-1} \frac{dF(0)}{dt}\right]^{-1}. \tag{11.2.35}$$

We readily claim that no matter what the values of u_0 and v_0 are, we may always choose t_0 so large that $dF(0)/dt > 0$, and, therefore, we conclude from (11.2.35) that there is a time t_1 such that $F(t) \to +\infty$ as $t \to t_1^-$. Hence, for the data $E(0) < 0$, any solution possesses finite blow-up time. The value of t_1 can be estimated by selecting the value t_0 which minimizes $F(0)/\frac{dF(0)}{dt}$. Thus we may take

$$t_0 = \frac{1}{2E(0)}\left[(u_0, Pv_0) - \{(u_0, Pv_0)^2 - 2E(0)(u_0, Pu_0)\}^{1/2}\right] \tag{11.2.36}$$

and then $\alpha t_1 \leq t_0$. Using (11.2.36), we obtain from (11.2.35)

$$F^\alpha(t) \geq \frac{t_1}{t_1 - t} F^\alpha(0),\ t < t_1 \tag{11.2.37}$$

which completes the proof. □

Theorem 11.2.2 ([440]).

(i) *If $E(0) = 0$ and $(u_0, Pv_0) > 0$, then $\lim_{t\to T^-}(u(t), Pu(t)) = +\infty$ for some time T, $0 < T < +\infty$.*

(ii) *If $E(0) = 0$ and $(u_0, Pv_0) = 0$, then $u(t) \equiv 0$ if and only if $W(0) = 0$ and $u_0 = v_0 = 0$. Otherwise, $\lim_{t\to T^-}(u(t), Pu(t)) = +\infty$ for some time T, $0 < T < +\infty$.*

(iii) *If $E(0) = 0$ and $(u_0, Pv_0) < 0$, then either the solution blows up in a finite time in the sense of the preceding statements or else*

$$(u_0, Pu_0) > (u(t), Pu(t)) > [1 - \alpha(u_0, Pv_0)/(u_0, Pu_0)]^{-1/\alpha}(u_0, Pu_0). \tag{11.2.38}$$

Moreover, both cases can occur.

Proof. We refer to [440] for details. □

In the following result, we shall assume

$$E(0) > 0, \tag{11.2.39}$$

and define

$$k_1 \equiv \alpha\lambda/(2\alpha+1)E(0) \tag{11.2.40}$$

and

$$J_1(0) = (u_0, Pu_0)^2\Big\{E(0)/(u_0, Pu_0) + \lambda(2\alpha+1)^{-1}[k_1(u_0, Pu_0)]^{2\alpha} - 1\Big\}. \tag{11.2.41}$$

Theorem 11.2.3 ([440])**.**

(i) *If $E(0) > 0$, $k_1(u_0, Pu_0) > 1$ and $(u_0, Pv_0) > 0$, then (u, Pu) becomes unbounded like $(t_1 - t)^{-k}$ for some positive constants k and t_1.*

(ii) *The above statement* (i) *holds if $(u_0, Pv_0) = 0$, the other conditions on the data being unchanged.*

(iii) *If $E(0) > 0$, $k_1(u_0, Pu_0) > 1$ and $(u_0, Pv_0) < 0$, but*

$$(u_0, Pv_0)^2 \le J_1(0),$$

then the assertions of the above two statements (i)–(ii) *still hold.*

Proof. For the proofs of (i) and (iii), we refer to [440] for details. Here we only give the proof of (ii) as application of Theorem 2.4.20.

To this end, we now take $\beta = 0$ and employ the function $G(t) = (u, Pu)$. If we also use the positive-definiteness conditions (11.2.9)–(11.2.10), the fundamental inequality (11.2.29) may be rewritten as

$$G(t)G''(t) - (1+\alpha)(G'(t))^2 \ge 4\alpha\lambda G^2(t) - 4(1+2\alpha)E(0)G(t). \tag{11.2.42}$$

(ii) When $(u_0, Pv_0) = 0$, $E(0) > 0$, $k_1(u_0, Pu_0) > 1$, i.e.,

$$\frac{\alpha\lambda G(0)}{(1+2\alpha)E(0)} > 1, \quad G'(0) = 0, \tag{11.2.43}$$

then there exists a constant $\delta > 0$ such that

$$k_1 G(0) \ge 1 + 2\delta, \quad k_1 = \alpha\lambda/[(1+2\alpha)E(0)], \tag{11.2.44}$$

while by continuity there exists an interval $[0, t_2]$ such that for all $t \in [0, t_2]$,

$$k_1 G(t) \ge 1 + \delta, \tag{11.2.45}$$

which, along with (11.2.45), implies that for all $t \in [0, t_2]$ (see [440]),

$$\frac{d^2(G^{-\alpha}(t))}{dt^2} \le \frac{-4\alpha^2\delta\lambda G^{-\alpha}(t)}{(1+\delta)}. \tag{11.2.46}$$

On discarding the right-hand side of (11.2.46) and integrating, we see that $dG(t)/dt \geq 0$, $t \in [0, t_2]$, and hence $G(t)$ cannot decrease on $[0, t_2]$.

Thus we may choose $t_2 = +\infty$ and obtain over the interval of existence

$$\frac{d^2(G^{-\alpha}(t))}{dt^2} \leq \frac{-8\alpha^2\delta\lambda G^{-\alpha}(t)}{(1+2\delta)}. \tag{11.2.47}$$

Setting

$$y(t) = \frac{d(G^{-\alpha}(t))}{dt} \Big/ G^{-\alpha}(t) = -\alpha \frac{dG(t)}{dt} \Big/ G(t), \tag{11.2.48}$$

then (11.2.47) reduces to

$$\frac{dy}{dt} + y^2 + a^2 \leq 0 \tag{11.2.49}$$

with

$$a^2 = \frac{8\alpha^2\delta\lambda}{(1+\delta)}. \tag{11.2.50}$$

Applying Theorem 2.4.20 to (11.2.49), we conclude inequality (11.2.47) cannot hold for all time, and hence the solution necessarily ceases to exist after a finite time. In fact, integrating (11.2.49), we have, since $y(0) = 0$,

$$G(t) \geq G(0)/(\cos at)^{\alpha/a^2}, \tag{11.2.51}$$

which implies $G(t)$ becomes unbounded at some $\hat{t}_0$ where $\hat{t}_0 \leq \pi/2a$. □

11.3 Blow-up of solutions to formally parabolic equations

In this section, we shall employ Theorem 2.4.19 to prove some non-existence "abstract" theorems, which are due to Levine [504].

We shall consider the following abstract problem

$$P\frac{du}{dt} = -Au + \mathscr{F}(u(t)),\ t \in [0,T),\ \mathscr{F}(0) = 0,\ u(0) = u_0 \tag{11.3.1}$$

where P and A are "positive" linear operators defined on a dense subdomain D of a real Hilbert space H, where $\mathscr{F}$ satisfies the following assumption:

(F1) $\mathscr{F} : D \to H$ has a symmetric Fréchet derivative $\mathscr{F}_x$ at each $x \in D, x \mapsto \mathscr{F}_x$ is strongly continuous, and the scalar-valued function $\mathscr{G} : D \to \mathbb{R}$, the potential associated with $\mathscr{F}$, defined by

$$\mathscr{G}(x) = \int_0^1 (\mathscr{F}(\rho x), x) d\rho \tag{11.3.2}$$

satisfies that for all $x \in D$ and some constant $\alpha > 0$,

$$2(\alpha + 1)\mathscr{G}(x) \le (x, \mathscr{F}(x))$$

and $\mathscr{F}(0) = 0$ for simplicity, with u_0 satisfying

$$\mathscr{G}(u_0) > \frac{1}{2}(u_0, Au_0), \tag{11.3.3}$$

then the existence interval $[0, T)$ of u is bounded, and the solution becomes arbitrarily large in the sense that both (u, Pu) and $\int_0^t (u, Pu)d\eta$ are unbounded in $[0, T)$. In the sequel, we assume that, P and $A(t)$ are symmetric linear operators defined on a dense domain $D \subseteq H$, $\dot{A}(t) = \frac{d}{dt}A(t)$ exists in the strong sense.

First we observe that for any $v : [0, T) \to D$ which is strongly continuously differentiable in the D norm, we derive from (11.3.2)

$$\frac{d}{dt}\mathscr{G}(v(t)) = (\mathscr{F}(v(t)), v_t(t)) \tag{11.3.4}$$

where $v_t \equiv dv/dt$. For convenience, we give a simple formal proof of (11.3.4), using the symmetry of $\mathscr{F}_x$,

$$\begin{aligned}
\frac{d}{dt}\mathscr{G}(v(t)) &= \frac{d}{dt}\int_0^1 (\mathscr{F}(\rho v), v)d\rho \\
&= \int_0^1 [\rho(\mathscr{F}_{\rho v} \cdot v, v_t) + (\mathscr{F}(\rho v), v_t)]d\rho \\
&= \int_0^1 \rho\frac{d}{d\rho}\Big((\mathscr{F}(\rho v), v_t) + (\mathscr{F}(\rho v), v_t)\Big)d\rho \\
&= \int_0^1 \frac{d}{d\rho}[\rho(\mathscr{F}(\rho v), v_t)]d\rho = (\mathscr{F}(v(t)), v_t(t)).
\end{aligned}$$

The next result is due to [504].

Theorem 11.3.1 ([504]). *Let $u : [0, T) \to D$ be a strongly continuously differentiable solution in the D norm to the problem* (11.3.1) *with $u_0 \in D$. Assume that, for each $t, A(t)$ and P are symmetric and*

(i) $(x, Px) > 0$ *for all* $x \in D, x \neq 0$;
(ii) $(x, A(t)x) \ge 0$ *for all* $x \in D$;
(iii) $(x, \dot{A}(t)x) \le 0$ *for all* $x \in D$;
(iv) *Condition* (F1) *is satisfied.*

Finally, let u_0 satisfy

$$\mathscr{G}(u_0) > \frac{1}{2}(u_0, A(0)u_0). \tag{11.3.5}$$

Then the existence interval $[0, T)$ *of* u *is bounded, and, i.e.,*

$$T \leq \left[(2\alpha+1)(u_0, Pu_0)/\alpha^2(2\alpha+2)\right]\left[\mathscr{G}(u_0) - \frac{1}{2}(u_0, A(0)u_0)\right]^{-1}, \qquad (11.3.6)$$

and

$$\lim_{t\to T^-}\int_0^t (u(\eta), Pu(\eta))d\eta = +\infty, \qquad (11.3.7)$$

$$\lim_{t\to T^-}\sup(u(t), Pu(t)) = +\infty. \qquad (11.3.8)$$

Proof. The proof used here is the so-called "concavity" arguments. Assume that $T = +\infty$, and for any $T_0 > 0, \beta > 0$ and $\tau > 0$, let, for all $t \in [0, T_0]$,

$$F(t) = \int_0^t (u, Pu)d\eta + (T_0 - t)(u_0, Pu_0) + \beta(t+\tau)^2. \qquad (11.3.9)$$

Since

$$\begin{aligned} F'(t) &= (u, Pu) - (u_0, Pu_0) + 2\beta(t+\tau) \\ &= 2\int_0^t (u, Pu_\eta)d\eta + 2\beta(t+\tau), \end{aligned} \qquad (11.3.10)$$

we can get that $F'(0) = 2\beta\tau > 0$ and that $F(t) > 0$ for all $t \in [0, T_0]$. Thus $F^{-\alpha}(t)$ is defined for any $\alpha > 0$. If we can show that $(F^{-\alpha}(t))'' \leq 0$, then, since a concave function must always lie below any tangent line, we can derive

$$F^{-\alpha}(t) \leq F^{-\alpha}(0) + [F^{-\alpha}(0)]'t$$

or

$$F(t) \geq F^{(1+1/\alpha)}(0)\left[F(0) - \alpha t F'(0)\right]^{-1/\alpha}. \qquad (11.3.11)$$

We shall note that for large enough τ, we may choose T_0 such that $T_0 \geq F(0)/\alpha F'(0) \equiv T_{\beta\tau}$. Thus, it follows from (11.3.1) and (11.3.11) that the existence interval of u must be contained in $[0, F(0)/\alpha F'(0))$ and that (11.3.7) and (11.3.8) hold if we can prove that $[F^{-\alpha}(t)]'' \leq 0$, which is in fact equivalent to the condition $F(t)F''(t) - (\alpha+1)(F'(t))^2 \geq 0$. Indeed, we have,

$$\begin{aligned} F''(t) &= 2\int_0^t (u_\eta, Pu)_{,\eta}d\eta + 2(u_t, Pu)_0 + 2\beta \\ &= 4(\alpha+1)\left[\int_0^t (u_\eta, P\eta)d\eta + \beta\right] + 2\int_0^t [(u_\eta, Pu)_{,\eta} - 2(\alpha+1)(u_\eta, Pu_\eta)]d\eta \\ &\quad + 2[(u_t, Pu)_0 - (2\alpha+1)\beta]. \end{aligned} \qquad (11.3.12)$$

Therefore, using (11.3.1), we find

$$\begin{aligned}
&F(t)F''(t)-(\alpha+1)(F'(t))^2\\
&\quad\geq 4(\alpha+1)S^2+2F(t)\left\{-\int_0^t[(u,Au)_{,\eta}-2(\alpha+1)(u_\eta,Au)]d\eta\right\}\\
&\qquad+2F\int_0^t[(u,\mathscr{F}(u)))_{,\eta}-2(\alpha+1)(u_\eta,\mathscr{F}(u))]d\eta\\
&\qquad+4(\alpha+1)(T_0-t)(u_0,Pu_0)\left[\int_0^t(u_\eta,Pu_\eta)d\eta+\beta\right]\\
&\qquad+2F[(u_t,Pu)_0-(2\alpha+1)\beta]
\end{aligned}$$

where, by Schwarz' inequality, we have

$$\begin{aligned}
S^2=&\left(\int_0^t(u,Pu)d\eta+\beta(t+\tau)^2\right)\left(\int_0^t(u_\eta,Pu_\eta)d\eta+\beta\right)\\
&-\left(\int_0^t(u_\eta,Pu)d\eta+\beta(t+\tau)\right)^2\geq 0.
\end{aligned}$$

Thus from (11.3.4) and assumption (iii) it follows

$$\begin{aligned}
&F(t)F''(t)-(\alpha+1)(F'(t))^2\\
&\quad\geq 4\alpha F(t)\int_0^t(u_\eta,Au)d\eta+2F(t)\left[(u,\mathscr{F}(u))-2(\alpha+1)\mathscr{G}(u)\right]\\
&\qquad+2F(t)\left[2(\alpha+1)\mathscr{G}(u_0)-(u_0,A(0)u_0)-(2\alpha+1)\beta\right].
\end{aligned}\tag{11.3.13}$$

Hence, from assumptions (i), (ii), (iii), (iv), (11.3.13) and a further computation, we derive that for all $t\in[0,T_0)$

$$\begin{aligned}
&F(t)F''(t)-(\alpha+1)(F'(t))^2\\
&\quad\geq 4(\alpha+1)F(t)\left[\mathscr{G}(u_0)-\frac{1}{2}(u_0,A(0)u_0)-(2\alpha+1)\beta/2(\alpha+1)\right].
\end{aligned}\tag{11.3.14}$$

Therefore, for any $\beta>0$ such that

$$(2\alpha+1)\beta=2(\alpha+1)\left[\mathscr{G}(u_0)-\frac{1}{2}(u_0,A(0)u_0)\right],$$

we have

$$F(t)F''(t)-(\alpha+1)(F'(t))^2\geq 0,\quad (F^{-\alpha}(t))''\leq 0.\tag{11.3.15}$$

Thus applying Theorem 2.4.19 (ii) to (11.3.15), we conclude that the existence interval of u is bounded. Let

$$\beta_0=2(\alpha+1)\left[\mathscr{G}(u_0)-\frac{1}{2}(u_0,A(0)u_0)\right]/(2\alpha+1)$$

and

$$T_{\beta\tau} = F(0)/\alpha F'(0) = [T_0(u_0, Pu_0) + \beta\tau^2]/2\alpha\beta\tau.$$

Since $T_0 \geq T_{\beta\tau}$, we have $(F^{-\alpha}(t))'' \leq 0$ even if we take $T_0 = T_{\beta\tau}$. This latter choice implies that $T_{\beta\tau} = \beta\tau^2[2\alpha\beta\tau - (u_0, Pu_0)]^{-1}$; thus we must choose τ so large that $2\alpha\beta\tau > (u_0, Pu_0)$. As a function of $\tau, T_{\beta\tau}$ has a minimum at $\tau = \tau(\beta) = (u_0, Pu_0)/(\alpha\beta)$ and $T_{\beta\tau(\beta)} = (u_0, Pu_0)/(\alpha^2\beta)$. This latter value attains its minimum when $\beta = \beta_0$ since β is restricted to $(0, \beta_0]$. Thus, T cannot exceed $(u_0, Pu_0)\alpha^{-2}\beta_0^{-1}$. □

The following corollary, due to [504], states that there may be many initial vectors u_0 such that the corresponding solutions to (11.3.1) have a finite blow-up time.

Corollary 11.3.1 ([504]). *Let $\mathscr{F}$ be homogeneous of degree $1 + \delta$ for some $\delta > 0$, that is, $\mathscr{F}(sx) = s^{1+\delta}\mathscr{F}(x)$ for all $s > 0$ and all $x \in D$. Assume that there exists an $x_0 \in D$ such that $(x_0, \mathscr{F}(x_0)) > 0$. Then there are an infinite number of initial vectors u_0 which satisfy condition* (11.3.5) *of Theorem* 11.3.1.

Proof. Let $u_0 = sx_0$ where s is so large that

$$s^\delta \mathscr{G}(x_0) = s^\delta \int_0^1 \mathscr{F}(\rho x_0), x_0)d\rho > \frac{1}{2}(x_0, A(0)x_0).$$

Then $\mathscr{G}(u_0) > \frac{1}{2}(x_0, A(0)x_0)$ for all

$$s > \left[\frac{1}{2}(1 + \delta)(x_0, A(0)x_0)(x_0, \mathscr{F}(x_0))^{-1}\right]^{1/\delta}. \qquad \square$$

Remark 11.3.1 ([504]). In fact, in most applications of Corollary 11.3.1, we may take $\delta = 2\alpha$.

Remark 11.3.2 ([504]). If $\mathscr{F}$ satisfies the hypothesis of Corollary 11.3.1, and all the other assumptions on $\mathscr{F}, P$ and $A(\cdot)$ hold, then the blow-up time $T = T(u_0) = T(sx_0) \to 0$ as $s \to +\infty$. To see this, we only note that

$$0 < T(sx_0) \leq \left[(2\alpha + 1)(x_0, Px_0)/2\alpha^2(\alpha + 1)\right] \left[\mathscr{G}(x_0)s^\delta - \frac{1}{2}(x_0, A(0)x_0)\right]^{-1}$$

and that the right-hand side of this inequality approaches zero as $s \to +\infty$. Thus, roughly speaking, the larger the initial value, the smaller the interval of existence.

Remark 11.3.3 ([504]). We note that Theorem 11.3.1 is false if (11.3.5) is not satisfied. Indeed, if we let $f \in C^2(0, \pi), f \neq 0$, satisfy

$$f'' + f^2 = 0, \qquad f(0) = f(\pi) = 0,$$

then we obtain that $u(x,t) = f(x)$ solves

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2 & \text{in} \quad (0,\pi) \times [0,+\infty), \\ u(x,0) = f(x), & x \in (0,\pi), \\ u(0,t) = u(\pi,t) = 0, & t \in [0,+\infty). \end{cases}$$

The nonlinearity $\mathscr{F}(u) = u^2$ satisfies hypothesis (F1) with $\alpha = \frac{1}{2}$ and

$$\mathscr{G}(f) = \frac{1}{3}\int_0^\pi f^3 dx.$$

However,

$$\mathscr{G}(f) = \frac{1}{3}(f, f^2) = \frac{1}{3}(f', f') < \frac{1}{2}(f', f')$$

which does not satisfy (11.3.5).

However, a similar result to Theorem 11.3.1 is true for weak solutions provided that we define a weak solution in an appropriate manner. To do this, let $D_* \supseteq D$ denote a dense subdomain of H upon which $P^* \equiv P^{1/2}, A^* \equiv P^{1/2}$ and $\mathscr{F}(\cdot)$ are defined. We assume that A is independent of t here.

Definition 11.3.1. We say that $u : [0,T) \to D_*$ is a weak solution of problem (11.3.1) if u, P^*u, A^*u and $\mathscr{F}(u)$ are strongly continuous, if u possesses a weak derivative (assumed locally integrable on $[0,T)$) which is D_*, valued, if P^*u possesses a weak derivative $(P^*u)_t$ and $(P^*u)_t = P^*u_t$ and if, for every $\phi : [0,T) \to D_*$ with these properties,

$$\begin{aligned}(P^*\phi, P^*u) = (P^*\phi(0), P^*u(0)) &+ \int_0^t (P^*\phi_\eta, P^*u)d\eta \\ &+ \int_0^t [(\phi, \mathscr{F}(u)) - (A^*\phi, A^*u)]d\eta.\end{aligned} \tag{11.3.16}$$

The conditions in the definition ensure that $\dfrac{d}{d\eta}(P^*u, P^*u) = 2(P^*u_\eta, P^*u)$.

Besides (iv) of Theorem 11.3.1 on $\mathscr{F}$, we further assume that weak solutions satisfy

$$2\int_0^t \|P^*u_\eta\|^2 d\eta + \|A^*u(t)\|^2 + 2\mathscr{G}(u_0) \le \|A^*u_0\|^2 + 2\mathscr{G}(u(t)). \tag{11.3.17}$$

In fact, this can easily be obtained formally by taking scalar products of both sides of (11.3.1) with u, integrating from 0 to t and using (11.3.4). Equation (11.3.17) is a kind of "energy" inequality.

Theorem 11.3.2 ([504])**.** *Let* $u : [0,T) \to D_*$ *be a weak solution of problem* (11.3.1) *in the sense of Definition* 11.3.1 *and assume* (11.3.17) *holds. If*

$$\mathscr{G}(u_0) > \frac{1}{2}\|A^* u_0\|^2, \tag{11.3.18}$$

then $T < +\infty$, *and*

$$\lim_{t\to T^-} \int_0^t \|P^* u\|^2 d\eta = +\infty \tag{11.3.19}$$

and consequently

$$\lim_{t\to T^-} \sup \|P^* u(t)\| = +\infty. \tag{11.3.20}$$

Proof. For arbitrary $T_0, \beta, \tau > 0$ and $t \in [0, T_0)$, let

$$F(t) \equiv \int_0^t \|P^* u\|^2 d\eta + (T_0 - t)\|P^* u_0\|^2 + \beta(t+\tau)^2. \tag{11.3.21}$$

Then from (11.3.16) with $u = \phi$ it follows

$$\begin{aligned} F'(t) &= \|P^* u\|^2 - \|P^* u_0\|^2 + 2\beta(t+\tau) \\ &= 2\int_0^t [(u, \mathscr{F}(u)) - \|A^* u\|^2] d\eta + 2\beta(t+\tau), \\ &= 2\int_0^t (P^* u, P^* u_\eta) d\eta + 2\beta(t+\tau), \end{aligned} \tag{11.3.22}$$

which gives us

$$\begin{aligned} F''(t) = 4(\alpha+1)&\left[\int_0^t \|P^* u_\eta\|^2 d\eta + \beta\right] \\ &+ 2\left[(u, \mathscr{F}(u)) - \|A^* u\|^2 - 2(\alpha+1)\int_0^t \|P^* u_\eta\|^2 d\eta - (2\alpha+1)\beta\right]. \end{aligned}$$

Hence from (11.3.17) and assumption (iv) it follows

$$\begin{aligned} F''(t) \geq 4(\alpha+1)&\left[\int_0^t \|P^* u_\eta\|^2 d\eta + \beta\right] \\ &+ 2\Big[2(\alpha+1)\mathscr{G}(u_0) - (\alpha+1)\|A^* u_0\|^2 + \alpha\|A^* u(t)\|^2 - (2\alpha+1)\beta\Big]. \end{aligned} \tag{11.3.23}$$

Thus combining (11.3.21)–(11.3.23) and (11.3.18), we may obtain

$$F(t)F''(t) - (\alpha+1)(F'(t))^2 \geq 0$$

if $0 < \beta \leq 2(\alpha+1)[\mathscr{G}(u_0) - \frac{1}{2}\|A^* u_0\|^2]/(2\alpha+1)$. Since $F'(0) = 2\beta\tau > 0$, by Theorem 2.4.19, the existence interval of u is finite and (11.3.19)–(11.3.20) hold, as we shall see from (11.3.11). An analysis similar to that used in the proof of Theorem 11.3.1 shows that

$$T \leq \left[(2\alpha+1)\|P^* u_0\|^2/(2\alpha^2(\alpha+1))\right]\left[\mathscr{G}(u_0) - \frac{1}{2}\|A^* u_0\|^2\right]^{-1}. \qquad \square$$

11.4 Blow-up of solutions to evolutionary PDEs

In this section, we shall employ Theorems 2.4.1 and 2.4.3 to investigate the blow-up results for two classes of evolutionary partial differential equations. These results are picked from Levine [508].

We shall consider the following two initial boundary value problems

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \mathcal{L}u + \mathcal{F}_1(u(x,t)), & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega \\ u_t(x,0) = v_0(x), & x \in \Omega \\ u(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,T), \\ \sum_{i,j=1}^n a_{ij} u_{,i}(x,t) v_j(x) + \beta u(x,t) = 0, & (x,t) \in \Gamma_2 \times (0,T) \end{cases} \tag{11.4.1}$$

and

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}u + \mathcal{F}_2(u(x,t)), & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega \\ u(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,T), \\ \sum_{i,j=1}^n a_{ij} u_{,i}(x,t) v_j(x) + \beta u(x,t) = 0, & (x,t) \in \Gamma_2 \times (0,T), \end{cases} \tag{11.4.2}$$

where β is a given constant and $\mathcal{F}, \mathcal{F}_1$ are given functions of a real-variable to be suitably restricted below. The solution $u(x,t)$ is a real-valued function, and all the other functions will be taken to be real-valued. The number T may be finite or infinite.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. Let

$$(\mathcal{L}f)(x) = \sum_{i,j=1}^n ((a_{i,j}(x) f_{,i}(x))) \quad (f_{,i} = \partial f / \partial x_i)$$

be a given second-order linear differential operator. Here the coefficients a_{ij} are assumed to be continuously differentiable with $a_{ij}(x) = a_{ji}(x), i,j = 1,2,\ldots,n$. $\mathcal{L}$ need not be elliptic.

Let $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ where Γ_1 and Γ_2 are piecewise smooth disjoint submanifolds of $\partial\Omega$. Let $v = (v_1, \ldots, v_n)$ denote the outer normal to $\partial\Omega$.

We first make the following assumptions on $\mathcal{L}$. The eigenvalue problem

$$\begin{cases} \mathcal{L}\psi + \lambda\psi = 0, & x \in \Omega, \\ \psi = 0, & x \in \Gamma_1, \\ \sum_{i,j=1}^n a_{ij} \psi_{,i} v_j + \beta\psi = 0, & x \in \Gamma_2 \end{cases} \tag{11.4.3}$$

possesses a positive solution ψ (on Ω) for some real number λ. We shall assume $\mathcal{L}$ is normalized so that

$$\int_\Omega \psi(x)dx = 1. \tag{11.4.4}$$

Now we further give the following assumptions on $\mathcal{F}_1, \mathcal{F}_2$.

1) There are convex functions $\mathcal{G}_1, \mathcal{G}_2 : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{F}_i(s) \geq \mathcal{G}_i(s); \quad i = 1, 2, \quad s \in \mathbb{R}.$$

2) Let $\mathcal{H}_1(s)$ be any indefinite integral of $\mathcal{G}_1(s)$, ($\mathcal{H}_1' = \mathcal{G}_1(s)$), and there is a number $s_1 \in \mathbb{R}$ such that

$$\mathcal{H}_1(s) - \frac{\lambda s^2}{2}$$

is non-decreasing on $(s_1, +\infty)$ and, for every $\varepsilon > 0$, there holds that

$$\int_{s_0}^{+\infty} \left(\mathcal{H}_1(s) - \frac{1}{2}\lambda s^2 - \left[\mathcal{H}_1(s_1) - \frac{1}{2}\lambda s_1^2\right] + \varepsilon\right)^{-1/2} ds < +\infty,$$

that is, $[\mathcal{H}_1(s) - \frac{1}{2}\lambda s^2]^{-1/2}$, is integrable at infinity.

3) There is a number $s_2 \in \mathbb{R}$ such that $\mathcal{G}_2(s) - \lambda s$ is positive on $(s_2, +\infty)$, and for any $\varepsilon > 0$, there holds that

$$\int_{s_2+\varepsilon}^{+\infty} [\mathcal{G}_2(s) - \lambda s]^{-1} ds < +\infty.$$

We now give a definition of a weak solution to problems (11.4.1) and (11.4.2).

Definition 11.4.1 ([508])**.** We say $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is a weak solution to problem (11.4.1) if, for each $t \in (0, T)$,

(i) $u(\cdot, t) \in L^1(\Omega), \mathcal{F}_1(u(\cdot, t)) \in L^1(\Omega)$,

(ii) $u_t(\cdot, t)$ exists and is in $L^1(\Omega)$,

(iii) u is sufficiently regular to satisfy the initial and boundary conditions $(11.4.1)_2$–$(11.4.1)_5$, and if for every $\phi : \overline{\Omega} \times [0, +\infty) \to \mathbb{R}$, twice continuously differentiable in $\Omega \times (0, T)$ and satisfying $(11.4.1)_4$–$(11.4.1)_5$, we have, for all $t \in [0, T)$,

$$\begin{aligned} \int_\Omega \phi(x, t) u_t(x, t) dx &= \int_\Omega \phi(x, 0) v_0(x) dx \\ &+ \int_0^t \int_\Omega \left[\frac{\partial \phi}{\partial \eta}(x, \eta) \frac{\partial u}{\partial \eta}(x, \eta) + (\mathcal{L}\phi)(x, \eta) u(x, \eta)\right] dx d\eta \\ &+ \int_0^t \int_\Omega \phi(x, \eta) \mathcal{F}_1(u(x, \eta)) dx d\eta. \end{aligned} \tag{11.4.5}$$

Definition 11.4.2 ([508])**.** We say $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is a weak solution to problem (11.4.2) if

(i) $u(\cdot, t) \in L^1(\Omega)$ for each $t \in [0, T)$,

(ii) u is sufficiently regular to satisfy $(11.4.2)_2$–$(11.4.2)_4$ and if, for all $\phi : \overline{\Omega} \times [0, T) \to \mathbb{R}$, ϕ twice continuously differentiable in x and continuously differentiable in t satisfying $(11.4.2)_3$–$(11.4.2)_4$, we have, for all $t \in [0, T)$,

$$\begin{aligned}\int_\Omega \phi(x,t)u(x,t)dx &= \int_\Omega \phi(x,0)u_0(x)dx \\ &+ \int_0^t \int_\Omega \left[\frac{\partial \phi}{\partial \eta}(x,\eta)u(x,\eta) + (\mathcal{L}\phi)(x,\eta)u(x,\eta)\right] dxd\eta \\ &+ \int_0^t \int_\Omega \phi(x,\eta)\mathcal{F}_2(u(x,\eta))dxd\eta.\end{aligned} \tag{11.4.6}$$

Remark 11.4.1 ([508])**.** Obviously, (11.4.5)–(11.4.6) can be obtained from $(11.4.1)_1$ and $(11.4.2)_1$ formally by multiplying them by $\mathcal{G}$, integrating the resulting expression over $\Omega \times [0, t)$ and performing integrations by parts.

Remark 11.4.2 ([508])**.** Note that the definitions of weak solutions for problems (11.4.1) and (11.4.2) do not require u to possess spacial derivatives except near the set $\Gamma_2 \times [0, T)$ and that only one t derivative for problem (11.4.1) or no t derivative need be required at all for problem (11.4.2). Thus $u_0(x)$ may have discontinuities away from Γ_2. This is in contrast to problems treated by the concavity or indirect lower bound methods where the existence of

$$E_1(t) = \frac{1}{2}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\sum_{i,j=1}^n \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx - \int_\Omega \mathcal{H}_1(u(x,t))dx$$

is required for problem (11.4.1) and the existence of

$$\begin{aligned}E_2(t) = &\int_0^t \int_\Omega \left(\frac{\partial u}{\partial \eta}\right)^2 dxd\eta + \frac{1}{2}\sum_{i,j=1}^n \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx \\ &- \int_\Omega \left(\int_0^{u(x,t)} \mathcal{G}_2(\sigma)d\sigma\right) dx\end{aligned}$$

is required for problem (11.4.2). Here the nonlinearities $\mathcal{F}_1$ and $\mathcal{F}_2$ are taken to be identical with $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

We now establish the following theorems.

Theorem 11.4.1 ([508])**.** *Let $u : \overline{\Omega} \times [0, T) \to \mathbb{R}$ be a weak solution to problem* (11.4.1) *in the sense of Definition* 11.4.1 *and let $\mathcal{L}, \mathcal{F}_1$ satisfy the above hypotheses. If*

$$\begin{cases} F_0 \equiv \displaystyle\int_\Omega \psi(x)u_0(x)dx > s_1, & (11.4.7) \\ F_0' \equiv \displaystyle\int_\Omega \psi(x)v_0(x)dx > 0, & (11.4.8) \end{cases}$$

then the interval $[0, T)$ *is necessarily bounded and*

$$\begin{cases} \lim_{t\to T^-} \left(\int_\Omega | u(x,t) |^p \, dx \right)^{1/p} = +\infty, \quad 1 \le p < +\infty, & (11.4.9) \\ \lim_{t\to T^-} \sup(\max_{x\in\overline{\Omega}} | u(x,t) |) = +\infty & (11.4.10) \end{cases}$$

where

$$T \le \frac{1}{2}\sqrt{2} \int_{F_0}^{+\infty} \left(\mathcal{H}_1(s) - \frac{1}{2}\lambda s^2 - \left(\mathcal{H}_1(s_1) - \frac{1}{2}\lambda s_1^2 \right) + \frac{1}{2}(F_0')^2 \right)^{-1/2} ds.$$

Proof. Let

$$F(t) = \int_\Omega \psi(x) u(x,t) dx.$$

Then

$$F'(t) = \int_\Omega \psi(x) u_t(x,t) dx.$$

If we now put $\phi(x,t) = \psi(x)$ in (11.4.5), we find, using (11.4.3) that

$$F'(t) = F'(0) - \lambda \int_0^t \int_\Omega \psi(x) u(x,\eta) dx d\eta + \int_0^t \int_\Omega \psi(x) \mathcal{F}_1(u(x,\eta)) dx d\eta. \quad (11.4.11)$$

Thus (11.4.11) implies that $F''(t)$ exists and

$$\begin{aligned} F''(t) &= -\lambda F(t) + \int_\Omega \psi(x) \mathcal{F}_1(u(x,t)) dx \\ &\ge -\lambda F(t) + \int_\Omega \psi(x) \mathcal{G}_1(u(x,t)) dx. \end{aligned}$$

Finally, by Jensen's inequality, we conclude

$$F''(t) \ge -\lambda F(t) + \mathcal{G}_1(F(t)). \quad (11.4.12)$$

Therefore, applying Theorem 2.4.1 to (11.4.12), we may complete the proof. □

Theorem 11.4.2 ([508])**.** *Let* $u : \Omega \times [0,T) \to \mathbb{R}$ *be a weak solution to problem* (11.4.2) *in the sense of Definition* 11.4.2 *and let* $\mathcal{L}, \mathcal{F}_2$ *satisfy the above hypotheses. If*

$$F_0 \equiv \int_\Omega \psi(x) u_0(x) dx > s_2, \quad (11.4.13)$$

then (11.4.9)–(11.4.10) *hold for some time* $T < +\infty$ *where*

$$T \le \int_{F_0}^{+\infty} [\mathcal{G}_2(s) - \lambda s]^{-1} ds.$$

Proof. Defining $F(t)$ as in the previous proof and setting $\mathcal{G}(x,t) = \mathcal{L}(x,t)$ in (11.4.6), we find that

$$F(t) = F(0) - \lambda \int_0^t \int_\Omega \psi(x)u(x,\eta)dxd\eta + \int_0^t \int_\Omega \psi(x)\mathcal{F}_2(u(x,\eta))dxd\eta$$

which, indeed, implies that $F'(t)$ exists. Hence it follows as before that

$$F'(t) \geq -\lambda F(t) + \mathcal{G}_2(F(t)). \tag{11.4.14}$$

Hence applying Theorem 2.4.3 again to (11.4.14), we can complete the proof. □

11.5 Blow-up of solutions to initial boundary value problems

In this section, we shall use Theorems 2.4.19–2.4.20 to study the blow-up phenomena of solutions to mixed problems. We adopt these results from Levine and Payne [520].

We shall consider the following two initial boundary value problems defined on the domain $D \times [0,T)$:

$$\begin{cases} \rho(\partial u/\partial t) = (-1)^m Mu, & \text{in } D \times [0,T), \\ Q_j(u) = (-1)^m f_j(N_j u), & j = 0,1,\ldots,m-1 \text{ on } \partial D \times [0,T), \\ u(x,0) = u_0(x), & \end{cases} \tag{11.5.1}$$

and

$$\begin{cases} \rho(\partial^2 u/\partial t^2) = (-1)^m Mu, & \text{in } D \times [0,T), \\ Q_j(u) = (-1)^m f_j(N_j u), & j = 0,1,\ldots,m-1 \text{ on } \partial D \times [0,T), \\ u(x,0) = u_0(x), & (\partial u/\partial t)(x,0) = v_0(x). \end{cases} \tag{11.5.2}$$

Here ρ, M, Q_j and N_j, are assumed to be independent of the parameter t. The function ρ is assumed to be positive in D, and the f_j are given locally functions.

Let M denote a formally self-adjoint differential operator of order $2m$ defined on a bounded domain D of $\mathbb{R}^n$. The coefficients of the operator are assumed to possess derivatives of the order indicated in the formal expression for the operator.

For elements u and v in the domain $D(M)$ of the operator M, we may write the following expression:

$$\int_D [vMu - uMv]\, dx = \sum_{j=0}^{m-1} \oint_{\partial D} \{N_j(v)Q_j(u) - N_j(u)Q_j(v)\}\, ds \tag{11.5.3}$$

where N_j and Q_j are linear differential operators on the boundary; in detail, we assume that N_j is a boundary operator containing derivatives up to order j while

Q_j is of order $2m-1-j$. The forms of N_j and Q_j are not uniquely determined by (11.5.3). For example, in the theory of elastic plates where $M = \Delta^2$ (the biharmonic operator), the boundary operators may depend on a parameter usually referred to as a Poisson ratio. The symbol ∂D denotes the boundary of D.

We assume further that the differential operators, M, N, and Q are such that the quantity $A(v,u)$ defined by

$$(-1)^m A(v,u) = -\int_D vMu\,dx + \sum_{j=0}^{m-1} \oint_{\partial D} N_j(v)Q_j(u)ds \tag{11.5.4}$$

is a symmetric positive semidefinite bilinear form.

In addition, they satisfy for some positive α, for a constant p to be specified later, and for all $m-1$ times continuously differentiable functions ϕ the condition

$$\sum_{j=0}^{m-1} \left\{ N_j(\phi)f_j(N_j(\phi)) - 2(p\alpha+1)\int_0^{N_j(\phi)} f_j(\sigma)\,d\sigma \right\} \geq 0. \tag{11.5.5}$$

It is easy to check that if for arbitrary Z_j, the point functions f_j are of the form

$$f_j(Z_j) = |Z_j|^{2p\alpha+1} b_j(Z_j), \tag{11.5.6}$$

where each b_j is a non-decreasing function of Z_j, then (11.5.5) is satisfied automatically.

By virtue of Theorem 2.4.19, we shall show that no solution with initial data in a certain class can exist for all time provided that (11.5.5), with p appropriately chosen, is satisfied. The method used here is a concavity method which has been extensively employed in the literature (see, e.g., Levine [504], [505], Levine and Payne [522] and Knops, Levine and Payne [440]).

First, we establish the following theorem for problem (11.5.1).

Theorem 11.5.1 ([520]). *Solutions of problem* (11.5.1) *cannot exist for all time with initial data satisfying the inequality*

$$\sum_{j=0}^{m-1} \oint_{\partial D} \int_0^{N_j(u_0(s))} f_j(\eta)d\eta ds > 1/2A(u_0,u_0) \tag{11.5.7}$$

if (11.5.5) *is satisfied for* $p=1$.

Proof. To apply Theorem 2.4.19, we need to find a twice continuously differentiable non-negative functional $F(t)$ defined on solutions of problem (11.5.1) such that

$$\begin{cases} F(t) = 0, \ u = 0, & (11.5.8) \\ \dfrac{d^2(F^{-\alpha}(t))}{dt^2} \leq 0, \quad \alpha > 0. & (11.5.9) \end{cases}$$

Here α will turn out to be the constant α in (11.5.5). Clearly, in order to establish (11.5.9), we need only show that $F(t)F''(t) - (\alpha+1)(F'(t))^2 \geq 0$.

To this end, we first assume that u exists on $D \times [0, +\infty)$ and then show that this leads to a contradiction. In this case, we may select an $F(t)$ of the following form for $0 \leq t \leq T_0 < +\infty$:

$$F(t) = \int_0^t \int_D \rho u^2 dx d\eta + (T_0 - t) \int_D \rho u_0^2 dx + \beta(t+\tau)^2 \tag{11.5.10}$$

where positive constants $T_0,\ \beta$ and τ will be determined later on. Then

$$\begin{aligned} F'(t) &= \int_D \rho u^2 dx - \int_{D_0} \rho u_0^2 dx + 2\beta(t+\tau) \\ &= 2\int_0^t \int_D \rho u \frac{\partial u}{\partial \eta} dx d\eta + 2\beta(t+\tau) \\ &= 2\int_0^t \int_D (-1)^m u M u dx d\eta + 2\beta(t+\tau) \\ &= 2\sum_{j=0}^{m-1} \oint_{\partial D} \int_0^t N_j(u) f_j(N_j(u)) ds d\eta - 2\int_0^t A(u,u) d\eta + 2\beta(t+\tau) \end{aligned} \tag{11.5.11}$$

and

$$F''(t) = 2\sum_{j=0}^{m-1} \oint_{\partial D} N_j(u) f_j(N_j(u)) ds - 2A(u,u) + 2\beta. \tag{11.5.12}$$

Thus it follows from (11.5.10)–(11.5.12) that

$$\begin{aligned} &F(t)F''(t) - (\alpha+1)\left(F'(t)\right)^2 \\ &\geq F(t)\Big\{ -2A(u,u) + 2\sum_{j=0}^{m-1} \oint_{\partial D} N_j(u) f_j(N_j(u)) ds + 2\beta \\ &\quad - 4(\alpha+1)\int_0^t \int_D \rho \left(\frac{\partial u}{\partial \eta}\right)^2 dx d\eta \Big\}. \end{aligned} \tag{11.5.13}$$

Now in (11.5.13), dropping the term

$$\left\{ 4(\alpha+1)(T_0 - t) \int_D \rho u_0^2\, dx \right\} \left\{ \int_0^t \int_D \rho \left(\frac{\partial u}{\partial \eta}\right)^2 dx d\eta + \beta \right\},$$

and using the Schwarz inequality in the expression for $(dF/dt)^2$, and inserting the following relation into (11.5.13),

$$
\begin{aligned}
2\int_0^t\int_D \rho\left(\frac{\partial u}{\partial\eta}\right)^2 dxd\eta &= 2(-1)^m\int_0^t\int_D Mu\frac{\partial u}{\partial\eta}dxd\eta\\
&= 2\sum_{j=0}^{m-1}\oint_{\partial D}\int_0^t \frac{\partial}{\partial\eta}(N_j(u))f_j(N_j(u))dsd\eta - 2\int_0^t A\left(u,\frac{\partial u}{\partial\eta}\right)d\eta\\
&= 2\sum_{j=0}^{m-1}\oint_{\partial D}\left\{\int_0^{N_j(u)} f_j(\sigma)\,d\sigma\right\}ds - A(u,u) + A(u_0,u_0)\\
&\quad - 2\sum_{j=0}^{m-1}\oint_{\partial D}\left\{\int_0^{N_j(u_0)} f_j(\sigma)\,d\sigma\right\}ds,
\end{aligned}
$$

we conclude from (11.5.13)

$$
\begin{aligned}
&\left[F(t)F''(t) - (\alpha+1)\left(F'(t)\right)^2\right]F^{-1}(t) \qquad (11.5.14)\\
&\geq 2\alpha A(u,u) + 2\left\{\sum_{j=0}^{m-1}\oint\left[N_j(u)f_j(N_j(u)) - 2(\alpha+1)\int_0^{N_j(u)} f_j(\sigma)d\sigma\right]ds\right\}\\
&\quad - 4(\alpha+1)\left[\frac{A(u_0,u_0)}{2} - \sum_{j=0}^{m-1}\oint_{\partial D}\left\{\int_0^{N_j(u_0)} f_j(\sigma)\,d\sigma\right\}ds\right] - 2(2\alpha+1)\beta.
\end{aligned}
$$

By hypothesis the first two terms on the right-hand side of (11.5.14) are non-negative, and since the data term is assumed to satisfy (11.5.7), we may select

$$
\beta = \frac{2(\alpha+1)}{2\alpha+1}\left\{\sum_{j=0}^{m-1}\oint_{\partial D}\left\{\int_0^{N_j(u_0)} f_j(\sigma)d\sigma\right\}ds - \frac{A(u_0,u_0)}{2}\right\} > 0. \qquad (11.5.15)
$$

Then we readily obtain

$$
F(t)F''(t) - (\alpha+1)(F'(t))^2 \geq 0 \qquad (11.5.16)
$$

which will lead to the non-existence result by Theorem 2.4.19. Since the consequences of (11.5.16) have been investigated in [440] and [504] (see Section 11.3), we merely sketch here the arguments. In fact, from the definition (11.5.10), we derive that $F(t) > 0$ in $[0,T]$ so that

$$
\frac{d^2(F^{-\alpha}(t))}{dt^2} \leq 0, \qquad (11.5.17)
$$

which gives us

$$
F^{-\alpha}(t) \leq F^{-\alpha}(0)[1 - \alpha(F'(0)/F(0))t], \quad 0 \leq t \leq T_0, \qquad (11.5.18)
$$

which shows that $F^{-\alpha}(t)$ necessarily decays to zero in a finite time

$$T \le F(0)/(\alpha F'(0))$$

if $F'(0) > 0$ and $T_0 \ge F(0)/(\alpha F'(0))$. But with the choice of F, we have $F'(0) = 2\beta\tau > 0$. A simple computation shows that $T_0 \ge F(0)/(\alpha F'(0))$ provided that τ is chosen to satisfy

$$\tau > \frac{1}{2}(\alpha\beta)^{-1}\int_D \rho u_0^2 dx \tag{11.5.19}$$

and T_0 is then taken so large that

$$T_0 \ge \beta\tau^2\left[2\alpha\beta\tau - \int_D \rho u_0^2 dx\right]^{-1}. \tag{11.5.20}$$

The blow-up time T cannot exceed the minimum value of the right-hand side of (11.5.19), considered as a function of τ. In fact, $T \le (\alpha^2\beta)^{-1}\int_D \rho u_0^2 dx$. □

Second, we establish the following theorem, due to [520], for problem (11.5.2).

Theorem 11.5.2 ([520]). *Solutions of problem* (11.5.2) *cannot exist for all time, with initial data satisfying the inequality*

$$E(0) = \frac{1}{2}\left[\int_D \rho v_0^2 dx + A(u_0, u_0)\right] - \sum_{j=0}^{m-1}\int_{\partial D}\left[\int_0^{N_j(u_0)} f_j(\sigma)d\sigma\right]ds < 0, \tag{11.5.21}$$

if (11.5.5) *is satisfied for* $p = 2$.

Proof. In fact, we may choose

$$F(t) = \int_D \rho u^2 dx + \beta(t+\tau)^2 \tag{11.5.22}$$

with β and τ again positive to be determined later. Then we have

$$\begin{cases} F'(t) = 2\int_D \rho u \dfrac{\partial u}{\partial t}dx + 2\beta(t+\tau), & (11.5.23) \\ F''(t) = 2\int_D \rho\left(\dfrac{\partial u}{\partial t}\right)^2 dx + 2(-1)^m\int_D uMudx + 2\beta & \\ \qquad = 2\int_D \rho\dfrac{\partial u}{\partial t}dx + 2\sum_{j=0}^{m-1}\oint_{\partial D} N_j(u)f_j(N_j(u))ds - 2A(u,u) + 2\beta. & (11.5.24) \end{cases}$$

Now if we define

$$E(t) = \frac{1}{2}\left[\int_D \rho\frac{\partial u}{\partial t}dx + A(u,u)\right] - \sum_{j=0}^{m-1}\oint_{\partial D}\left[\int_0^{N_j(u)} f_j(\sigma)d\sigma\right]ds, \tag{11.5.25}$$

then

$$\begin{aligned}E'(t) &= \int_D \rho \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dx + A(u, \frac{\partial u}{\partial t}) - \sum_{j=0}^{m-1} \oint_{\partial D} N_j \left(\frac{\partial u}{\partial t}\right) f_j(N_j(u)) ds \\ &= \int_D \frac{\partial u}{\partial t}\left[\rho \frac{\partial^2 u}{\partial t^2} - (-1)^m M u\right] dx = 0. \end{aligned} \tag{11.5.26}$$

Thus

$$E(t) = E(0), \tag{11.5.27}$$

and (11.5.24) may be rewritten, using (11.5.5) with $p = 2$, as

$$\begin{aligned}F''(t) \geq 2\int_D \rho \left(\frac{\partial u}{\partial t}\right)^2 dx + 4(2\alpha+1)\sum_{j=0}^{m-1}\oint_{\partial D}\left\{\int_0^{N_j(u)} f_j(\sigma)d\sigma\right\} ds \\ - 2A(u,u) + 2\beta,\end{aligned} \tag{11.5.28}$$

which yields, by using (11.5.25) and (11.5.27),

$$F''(t) \geq 4(\alpha+1)\int_D \rho \left(\frac{\partial u}{\partial t}\right)^2 dx + 4\alpha A(u,u) - 4(2\alpha+1)E(0) + 2\beta. \tag{11.5.29}$$

It then follows from (11.5.22), (11.2.23) and (11.5.29) that

$$\begin{aligned}&F(t)F''(t) - (\alpha+1)\left(F'(t)\right)^2 \\ &\quad \geq 4(\alpha+1)\left[\left\{\int_D \rho u^2 dx + \beta(t+\tau)^2\right\}\left\{\int_D \rho\left(\frac{\partial u}{\partial t}\right)^2 + \beta\right\}\right. \\ &\qquad\qquad \left. \times \left(\int_D \rho u \frac{\partial u}{\partial t} dx + \beta(t+\tau)\right)^2\right] \\ &\quad + 4\alpha F A(u,u) - 2(2\alpha+1)\big[2E(0)+\beta\big]F(t).\end{aligned} \tag{11.5.30}$$

Note that the first term on the right-hand side of (11.5.30) is non-negative by Schwarz' inequality. If we choose

$$\beta = -2E(0), \tag{11.5.31}$$

then we arrive at $\left(F^{-\alpha}(t)\right)'' \leq 0$, which leads to the breakdown of solution in a finite time provided that $F'(0) > 0$. But

$$F(0) = 2\left\{\int_D \rho u_0 v_0 dx - 2E(0)\tau\right\}, \tag{11.5.32}$$

and it is clear that because of condition (11.5.21), the constant τ may be chosen so large that $F'(0) > 0$ (provided that $\int_D \rho u_0 v_0 dx$ is finite). □

It is worth pointing out here that similar inequalities hold for appropriately defined weak solutions of problems (11.5.1) and (11.5.2).

Indeed, if $A(u,u)$ is positive definite and satisfies

$$A(u,u) \geq k^2 \int_D \rho u^2 dx, \tag{11.5.33}$$

then we can obtain somewhat different results for problem (11.5.2).

For instance, assume that in problem (11.5.2), $E(0) \leq 0$, β is chosen to be zero, and (11.5.33) is satisfied. Then (11.5.33) leads to

$$F(t)F''(t) - (\alpha+1)\left(F'(t)\right)^2 \geq 4k^2\alpha F^2(t) \tag{11.5.34}$$

or

$$\left(F^{-\alpha}(t)\right)'' \leq -4k^2\alpha F^{-\alpha}(t). \tag{11.5.35}$$

We may show that (11.5.35) which yields the solution of problem (11.5.2) blows up in a finite time.

If the contrary holds and the following substitution is made

$$y = \frac{d(F^{-\alpha}(t))/dt}{F^{-\alpha}(t)} = -\alpha\frac{dF(t)/dt}{F(t)}, \tag{11.5.36}$$

then (11.5.35) reduces to

$$dy/dt + y^2 + 4k^2\alpha \leq 0, \tag{11.5.37}$$

which yields

$$\tan^{-1}[y(t)/2k\sqrt{\alpha}] \leq \tan^{-1}[y(0)/2k\sqrt{\alpha}] - 2k\sqrt{\alpha}t. \tag{11.5.38}$$

Therefore applying Theorem 2.4.20 to (11.5.37), we can conclude that the solution must blow up in a finite time T satisfying

$$T \leq \left(2k\sqrt{\alpha}\right)^{-1}\{\pi/2 + \tan^{-1}[y(0)/2k\sqrt{\alpha}]\}. \tag{11.5.39}$$

Finally, as in [440, 504, 505, 522] under various other combinations of initial data assumptions, we can derive the blow-up of solutions in a finite time. □

11.6 Blow-up of solutions to the Cauchy problem in nonlinear one-dimensional thermoelasticity

In this section, we employ Theorem 2.4.19 to establish the blow-up results for a nonlinear one-dimensional thermoelastic system with a non-autonomous forcing term and a thermal memory when the heat flux obeys both Fourier's law. These results are chosen from Qin and Muñoz Rivera [800] (see also Qin [770]).

We shall consider the following Cauchy problem with a non-autonomous forcing term and a thermal memory

$$\begin{cases} u_{tt} = au_{xx} + b\theta_x + du_x - mu_t + f(t,u), & (11.6.1)\\ c\theta_t = \kappa\theta_{xx} + g*\theta_{xx} + bu_{xt} + pu_x + q\theta_x & (11.6.2)\end{cases}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad \text{for all } x \in \mathbb{R} \quad (11.6.3)$$

where by $u = u(x,t)$ and $\theta = \theta(x,t)$ we stand for the displacement and the temperature difference respectively, the function $g = g(t)$ is the relaxation kernel, the sign $*$ is the convolution product, i.e., $g * y(\cdot,t) = \int_0^t g(t-\tau)y(\cdot,\tau)d\tau$, the coefficients a, b, c are positive constants, while d, κ, p, q, m are non-negative constants and the function $f = f(t,u)$ is a non-autonomous forcing term.

Assume that for any fixed $t > 0$, $f(t,u)$ is the Fréchet derivative of some functional $F(t,u)$ such that

$$\frac{d}{dt}F(t,u) = F_t(t,u) + f(t,u)u_t \quad (11.6.4)$$

and there exists a constant $\alpha > 0$ such that

$$\tilde{g}(t) = e^{\alpha t}g(t) \quad (11.6.5)$$

is a positive definite kernel. Indeed, we find there exists a function $\tilde{g}(t)$ to satisfy (11.6.5). To this end, we need Lemma 7.2.1 in Qin [770].

In fact, if taking $g(t) \in C^1[0,+\infty)$ such that

$$g'(t) = -\gamma g(t) + c_0 e^{-\delta t}, \quad g(0) > c_0\gamma^{-1} \quad (11.6.6)$$

with $\delta > 0, \gamma > 0$ and $c_0 \geq 0$ being constants and defining

$$G(t) = g(t) + \frac{2c_0}{\delta}e^{-\delta t}, \quad (11.6.7)$$

then we get for $\gamma_0 = \min(\gamma, \delta/2)$,

$$G'(t) = -\gamma g(t) - c_0 e^{-\beta t} \leq -\gamma_0 G(t)$$

which implies

$$g(t) \leq G(t) \leq G(0)e^{-\gamma_0 t} \equiv: c_1 e^{-\gamma_0 t} \quad (11.6.8)$$

with $c_1 = g(0) + \frac{2c_0}{\delta} > 0$.

Letting now

$$\begin{cases} J_1(\omega) = \Re\hat{\tilde{g}}(t) = \int_0^{+\infty} e^{\alpha t}g(t)\cos\omega t dt,\\ J_2(\omega) = \Im\hat{\tilde{g}}(t) = \int_0^{+\infty} e^{\alpha t}g(t)\sin\omega t dt,\\ I_1(\omega) = \int_0^{+\infty} e^{(\alpha-\delta)t}\cos\omega t dt,\\ I_2(\omega) = \int_0^{+\infty} e^{(\alpha-\delta)t}\sin\omega t dt,\end{cases}$$

we can derive that for $0 < \alpha < \delta$,

$$I_1(\omega) = \frac{1}{\delta - \alpha} - \frac{\omega^2}{(\delta-\alpha)^2} I_2(\omega), \;\; I_2(\omega) = \frac{\omega}{\delta - \alpha} I_1(\omega),$$

i.e.,

$$I_1(\omega) = \frac{\delta - \alpha}{(\delta-\alpha)^2 + \omega^2}, \qquad I_2(\omega) = \frac{\omega}{(\delta-\alpha)^2+\omega^2}. \tag{11.6.9}$$

Using (11.6.6), (11.6.8) and integrating by parts, we can obtain

$$\begin{cases} \omega J_1(\omega) = (\gamma - \alpha) J_2(\omega) - c_0 I_2(\omega), & (11.6.10) \\ \omega J_2(\omega) = g(0) - (\gamma-\alpha) J_1(\omega) + c_0 I_1(\omega). & (11.6.11) \end{cases}$$

Inserting (11.6.9) into (11.6.10)–(11.6.11) implies that for $0 < \alpha < \min(\gamma_0, \delta)$,

$$\begin{aligned} J_1(\omega) &= \frac{g(0)(\gamma-\alpha)}{(\gamma-\alpha)^2+\omega^2} - \frac{c_0\omega}{(\gamma-\alpha)^2+\omega^2} I_2(\omega) + \frac{c_0(\gamma-\alpha)}{(\gamma-\alpha)^2+\omega^2} I_1(\omega) \\ &= \frac{[g(0)(\gamma-\alpha)-c_0]\omega^2 + g(0)(\gamma-\alpha)(\delta-\alpha)^2 + c_0(\gamma-\alpha)(\delta-\alpha)}{[(\gamma-\alpha)^2+\omega^2][(\delta-\alpha)^2+\omega^2]}. \end{aligned} \tag{11.6.12}$$

Thus choosing α so small that

$$0 < \alpha \le \min\Big\{\delta, \gamma_0, [g(0)\gamma - c_0]/g(0)\Big\},$$

then we conclude from (11.6.12)

$$J_1(\omega) \ge \frac{c}{1+\omega^2} > 0, \;\; \forall \omega \in (-\infty, +\infty)$$

which, together with Lemma 7.2.1 of [770], yields that $\tilde{g}(t)$ is a strongly positive definite kernel satisfying (11.6.5).

Next, in order to prove our desired results, we need to use Theorem 2.4.19 (see, e.g., Theorem 1.3.1 in [770] due to Kalantarov and Ladyzhenskaya [409]) which was also proved in [438, 440, 504].

Note that the energy for the system (11.6.1)–(11.6.2) is defined as

$$E(t) = \int_{-\infty}^{+\infty} \Big[u_t^2/2 + a u_x^2/2 - F(t,u) + c\theta^2/2\Big] dx. \tag{11.6.13}$$

Then by setting

$$v = e^{\alpha t} u, \; w = e^{\alpha t}\theta, \tag{11.6.14}$$

the problem (11.6.1)–(11.6.3) reduces to the following problem

$$\begin{cases} v_{tt} = a v_{xx} + b w_x + d v_x - (m - 2\alpha) v_t + (m-\alpha)\alpha v + \tilde{f}(t, v), & (11.6.15) \\ c w_t = \kappa w_{xx} + \tilde{g} * w_{xx} + b v_{xt} + (p - b\alpha) v_x + q w_x + c\alpha w, & (11.6.16) \\ t = 0 : v = u_0(x) \equiv v_0(x), v_t = u_1(x) + \alpha u_0(x) \equiv v_1(x), & \\ \qquad w = \theta_0(x) \equiv w_0(x) & (11.6.17) \end{cases}$$

with $\tilde{f}(t,v) = e^{\alpha t} f(t, e^{-\alpha t} v)$. The energy for problem (11.6.15)–(11.6.17) can be defined by

$$\tilde{E}(t) = \int_{-\infty}^{+\infty} \Big[-(m-\alpha)\alpha v^2/2 + v_t^2/2 + a v_x^2/2 - \tilde{F}(t,v) + c w^2/2 \Big] dx \quad (11.6.18)$$

with $\tilde{F}(t,v) = e^{2\alpha t} F(t, e^{-\alpha t} v)$. The main idea here is only to prove that the solution to the problem (11.6.15)–(11.6.17) blows up in a finite time, which further implies the blow-up of solutions of problem (11.6.1)–(11.6.3).

The following lemmas, due to Qin and Muñoz Rivera [800], concern the results on $\tilde{E}(t) \le \tilde{E}(0) \le 0$ when we assume that $\tilde{E}(0) \le 0$.

Lemma 11.6.1 ([800, 770]). *We assume that $\tilde{E}(0) \le 0$, and (11.6.5) holds. Then if the following assumptions hold, when $\alpha = \frac{m}{2} = \frac{p}{b} > 0, d = 0$ and $\kappa \ge 0$, it holds that for any $u \in \mathbb{R}$ and for all $t > 0$,*

$$\alpha u f(t,u) \le (\alpha - m)\alpha^2 u^2 + F_t(t,u). \quad (11.6.19)$$

Then for all $t > 0$,

$$\tilde{E}(t) \le \tilde{E}(0) \le 0. \quad (11.6.20)$$

Proof. Obviously, by an easy computation it follows from (11.6.5) that

$$\begin{aligned} \tilde{F}_t(t,v) &= 2\alpha \tilde{F}(t,v) + e^{2\alpha t} F_t(t,u) - \alpha v \tilde{f}(t,v) \\ &= e^{2\alpha t}[2\alpha F(t,u) + F_t(t,u) - \alpha u f(t,u)]. \end{aligned} \quad (11.6.21)$$

On the other hand, we can derive from (11.6.4), (11.6.14) and (11.6.21) that

$$\begin{aligned} \frac{d}{dt}\tilde{F}(t,v) &= 2\alpha e^{2\alpha t} F(t, e^{-\alpha t} v) + e^{2\alpha t} \frac{d}{dt} F(t,u) \\ &= 2\alpha \tilde{F}(t,v) + e^{2\alpha t} F_t(t,u) + e^{2\alpha t} f(t,u) u_t \\ &= 2\alpha \tilde{F}(t,v) + e^{2\alpha t} F_t(t,u) + e^{2\alpha t} f(t,u)[-\alpha e^{\alpha t} v + e^{\alpha t} v_t] \\ &= e^{2\alpha t}[2\alpha F(t,u) + F_t(t,u) - \alpha u f(t,u)] + \tilde{f}(t,v) v_t \\ &= \tilde{F}_t(t,v) + \tilde{f}(t,v) v_t. \end{aligned} \quad (11.6.22)$$

Thus using (11.6.13), (11.6.15)–(11.6.17) and (11.6.22), we get

$$\begin{aligned} \tilde{E}'(t) = d\int_{-\infty}^{+\infty} v_x v_t dx - (m - 2\alpha)\int_{-\infty}^{+\infty} v_t^2 dx - \int_{-\infty}^{+\infty} \tilde{F}_t(t,v) dx - \kappa \int_{-\infty}^{+\infty} w_x^2 dx \\ - \int_{-\infty}^{+\infty} \tilde{g} * w_x \, w_x dx + (p - b\alpha)\int_{-\infty}^{+\infty} v_x w dx + c\alpha \int_{-\infty}^{+\infty} w^2 dx. \end{aligned} \quad (11.6.23)$$

It is easy now to verify from (11.6.19) that

$$2\alpha \tilde{F}(t,v) - (\alpha - m)\alpha^2 v^2 - \tilde{F}_t(t,v) \le 0$$

which, along with (11.6.18), further implies

$$\begin{aligned}\tilde{E}'(t) =& -\kappa\int_{-\infty}^{+\infty} w_x^2 dx - \int_{-\infty}^{+\infty} \tilde{g} * w_x\ w_x dx + c\alpha\int_{-\infty}^{+\infty} w^2 dx - \int_{-\infty}^{+\infty} \tilde{F}_t(t,v)dx\\ \le& -\int_{-\infty}^{+\infty} \tilde{g} * w_x w_x dx + 2\alpha\tilde{E}(t) - \alpha\int_{-\infty}^{+\infty} (v_t^2 + av_x^2)dx\\ &+ \int_{-\infty}^{+\infty} [2\alpha\tilde{F}(t,v) - (\alpha - m)\alpha^2 v^2 - \tilde{F}_t(t,v)]dx\\ \le& -\int_{-\infty}^{+\infty} \tilde{g} * w_x\ w_x dx + 2\alpha\tilde{E}(t).\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{E}(t) &\le \tilde{E}(0) - \int_{-\infty}^{+\infty}\int_0^t \tilde{g} * w_x\ w_x d\tau dx + 2\alpha\int_0^t \tilde{E}(\tau)d\tau\\ &\le \tilde{E}(0) + 2\alpha\int_0^t \tilde{E}(\tau)d\tau.\end{aligned} \tag{11.6.24}$$

That is, for all $t > 0$,

$$\tilde{E}(t) \le \tilde{E}(0)e^{2\alpha t} \le 0,$$

which, together with (11.6.24), yields (11.6.20). □

Let

$$\Psi(t) = \int_{-\infty}^{+\infty} v^2(x,t)dx + \beta(t + t_0)^2$$

where $\beta \ge 0$ and $t_0 > 0$ are to be determined later on.

The next lemma, also due to Qin and Muñoz Rivera [800], plays a key role in showing that $\Psi(t)$ verifies the assumptions of Theorem 2.4.19 (or Theorem 1.3.1 of [770]) by choosing suitable $\beta \ge 0, t_0 > 0$ and initial data (u_0, u_1, θ_0).

Lemma 11.6.2 ([800, 770]). *We assume that for any $t \ge 0, \tilde{E}(t) \le \tilde{E}(0) \le 0$ and initial data*

$$u_0 \in H^2(\mathbb{R}),\ \ u_1 \in H^1(\mathbb{R}),\ \ \theta_0 \in H^1(\mathbb{R}) \tag{11.6.25}$$

and the following assumptions holds: when $\alpha = \frac{m}{2} = \frac{p}{b} > 0$ and $d = 0$, there exists a positive constant

$$\gamma \ge (\sqrt{1 + b^2c/a} - 1)/(4c),\ \ \text{for all } u \in \mathbb{R},\ t > 0 \tag{11.6.26}$$

verifying

$$uf(t,u) - 2(1 + 2\gamma)F(t,u) \ge 0,\ \ \text{for all } u \in \mathbb{R},\ t > 0 \tag{11.6.27}$$

and initial data satisfy

$$\begin{cases} \int_{-\infty}^{+\infty} u_0 u_1 dx > 0, & \text{if } \tilde{E}(0) < 0, \quad (11.6.28) \\ \int_{-\infty}^{+\infty} u_0 u_1 dx > 0, \ \int_{-\infty}^{+\infty} u_0^2 dx > 0, & \text{if } \tilde{E}(0) = 0, \quad (11.6.29) \end{cases}$$

Then for $\beta > 0$ small enough and suitable $t_0 > 0$ or $\beta = 0$, there exist constants $C_1 \geq 0$ and $C_2 \geq 0$ such that (i) *or* (ii) *in Theorem* 2.4.19 *holds.*

Proof. By a direct calculation, we have

$$\begin{cases} \Psi'(t) = 2\left[\int_{-\infty}^{+\infty} vv_t dx + \beta(t+t_0)\right], & (11.6.30) \\ \Psi''(t) = 2\left[\int_{-\infty}^{+\infty} (v_t^2 + vv_{tt}) dx + \beta\right]. & (11.6.31) \end{cases}$$

Using the Cauchy inequality and the Hölder inequality, we derive

$$\begin{aligned} &\left[\int_{-\infty}^{+\infty} vv_t dx + \beta(t+t_0)\right]^2 \\ &\leq \left\{\left(\int_{-\infty}^{+\infty} v^2 dx\right)^{1/2} \left(\int_{-\infty}^{+\infty} v_t^2 dx\right)^{1/2} + \sqrt{\beta}(t+t_0)\sqrt{\beta}\right\}^2 \\ &\leq \Psi(t)\left(\int_{-\infty}^{+\infty} v_t^2 dx + \beta\right) \end{aligned}$$

which, together with (11.6.30) and (11.6.31), yields

$$\begin{aligned} &\Psi(t)\Psi''(t) - (1+\gamma)(\Psi'(t))^2 \\ &= 2\Psi(t)\left[\int_{-\infty}^{+\infty} (v_t^2 + vv_{tt}) dx + \beta\right] - 4(1+\gamma)\left[\int_{-\infty}^{+\infty} vv_t dx + \beta(t+t_0)\right]^2 \\ &\geq 2\Psi(t)\left[-(1+2\gamma)\left(\int_{-\infty}^{+\infty} v_t^2 dx + \beta\right) + \int_{-\infty}^{+\infty} vv_{tt} dx\right]. \end{aligned} \quad (11.6.32)$$

Inserting (11.6.15) into (11.6.32), integrating by parts and using

$$\int_{-\infty}^{+\infty} v_t^2 dx = 2\tilde{E}(t) + \int_{-\infty}^{+\infty} [(m-\alpha)\alpha v^2 - av_x^2 + 2\tilde{F}(t,v) - cw^2] dx,$$

we conclude

$$\begin{aligned}&\Psi(t)\Psi''(t)-(1+\gamma)(\Psi'(t))^2 \qquad (11.6.33)\\ &\geq 2\Psi(t)\left[-(1+2\gamma)\left(\int_{-\infty}^{+\infty}v_t^2dx+\beta\right)-\int_{-\infty}^{+\infty}(av_x^2+bv_xw)dx\right.\\ &\left.-(m-2\alpha)\int_{-\infty}^{+\infty}vv_tdx+(m-\alpha)\alpha\int_{-\infty}^{+\infty}v^2dx+\int_{-\infty}^{+\infty}v\tilde{f}(t,v)dx\right].\end{aligned}$$

Noting that $(1+2\gamma)c-\varepsilon_1\geq 0$ for $\varepsilon_1=b^2/8a\gamma>0$, we derive from (11.6.33)

$$\begin{aligned}&\Psi(t)\Psi''(t)-(1+\gamma)(\Psi'(t))^2\\ &\geq 2\Psi(t)\left\{-2(1+2\gamma)\tilde{E}(t)-(1+2\gamma)\beta-2\gamma\alpha(m-\alpha)\int_{-\infty}^{+\infty}v^2dx-b\int_{-\infty}^{+\infty}v_xwdx\right.\\ &\left.+2a\gamma\int_{-\infty}^{+\infty}v_x^2dx-(m-2\alpha)\int_{-\infty}^{+\infty}v^2dx+\int_{-\infty}^{+\infty}[v\tilde{f}(t,v)-2(1+2\gamma)\tilde{F}(t,v)]dx\right\}\\ &\geq 2\Psi(t)\left\{-2(1+2\gamma)\tilde{E}(0)-(1+2\gamma)\beta-2\gamma\alpha^2\Psi(t)+[(1+2\gamma)c-\varepsilon_1]\int_{-\infty}^{+\infty}w^2dx\right.\\ &\left.+\left(2a\gamma-\frac{b^2}{4\varepsilon_1}\right)\int_{-\infty}^{+\infty}v_x^2dx+\int_{-\infty}^{+\infty}[v\tilde{f}(t,v)-2(1+2\gamma)\tilde{F}(t,v)]dx\right\}\\ &\geq 2\Psi(t)[-2(1+2\gamma)\tilde{E}(0)-(1+2\gamma)\beta-2\gamma\alpha^2\Psi(t)]. \qquad (11.6.34)\end{aligned}$$

If $\tilde{E}(0)<0$, we may pick $\beta>0$ and $t_0>0$ in (11.6.34) so small that

$$0<\beta\leq -2\tilde{E}(0),\ 0<t_0<\frac{1}{2}\left[1+\sqrt{1+4\alpha\beta^{-1}\int_{-\infty}^{+\infty}u_0u_1dx}\right] \qquad (11.6.35)$$

which, with (11.6.28), implies (1) of Theorem 2.4.19 with $C_1=0, C_2=4\gamma\alpha^2$ and $\gamma_1=2\gamma\alpha, \gamma_2=-2\gamma\alpha$.

If $\tilde{E}(0)=0$, then we take $\beta=0$ in (11.6.34) and can use (11.6.29) to derive (i) or (ii) of Theorem 2.4.19 with $C_1=0,\ C_2=4\gamma\alpha^2$ and $\gamma_1=2\gamma\alpha,\ \gamma_2=-2\gamma\alpha$. The proof is hence complete. □

Now we read our main result due to Qin and Muñoz Rivera [800] in this section.

Theorem 11.6.3 ([800, 770]). *We assume that assumptions in Lemma* 11.6.1 *and assumptions in Lemma* 11.6.2 *hold, then the solution* $v(t)$ *in* $L^2(\mathbb{R})$ *to problem* (11.6.15)–(11.6.17) *blows up in a finite time, that is, there exists some time* $t_1>0$ *such that*

$$\lim_{t\to t_1^-}\int_{-\infty}^{+\infty}v^2(x,t)dx=+\infty \qquad (11.6.36)$$

and further the solution $u(t)$ *in* $L^2(\mathbb{R})$ *to problem* (11.6.1)–(11.6.3) *blows up in a finite time, that is,*

$$\lim_{t \to t_1^-} \int_{-\infty}^{+\infty} u^2(x,t)dx = +\infty. \tag{11.6.37}$$

Proof. By Lemma 11.6.1, we can get

$$\tilde{E}(t) \leq \tilde{E}(0) \leq 0$$

which, together with Lemma 11.6.2, implies that (i) or (ii) of Theorem 2.4.19 holds. Thus we can derive (11.6.36)–(11.6.37) from (11.6.5) and Theorem 2.4.19. The proof is now complete. □

Chapter 12

Appendix: Basic Inequalities

In this chapter, we shall collect some basic inequalities which play a very crucial role in classical calculus. These inequalities include the Young inequality, the Hölder inequality, the Minkowski inequality, the Jensen inequality, and the Hausdorff–Young inequality, etc.

12.1 The Young inequalities

In this section we introduce the Young inequalities.

Theorem 12.1.1. *Let f be a real-valued, continuous and strictly increasing function on $[0, c]$ with $c > 0$. If $f(0) = 0, a \in [0, c]$ and $b \in [0, f(c)]$, then we have*

$$ab \le \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \tag{12.1.1}$$

where f^{-1} is the inverse function of f. Equality in (12.1.1) holds if and only if $b = f(a)$.

This is a classical result called "the Young inequality" whose proof can be found in Young [985].

If we take $f(x) = x^{p-1}$ with $p > 1$ in Theorem 12.1.1, then we conclude the following corollary.

Corollary 12.1.1. *There holds that*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \tag{12.1.2}$$

where $a, b \ge 0, p > 1$ and $1/p + 1/q = 1$.

If $0 < p < 1$, then

$$ab \ge \frac{a^p}{p} + \frac{b^q}{q}. \tag{12.1.3}$$

The equalities in (12.1.2) and (12.1.3) hold if and only if $b = a^{p-1}$.

Y. Qin, *Analytic Inequalities and Their Applications in PDEs*, Operator Theory: Advances and Applications 241, DOI 10.1007/978-3-319-00831-8_12

Hirzallah and Omar improved the above Young inequality (see, e.g., Kuang [466]).

Theorem 12.1.2 (The Hirzallah–Omar Inequality). *For* $a, b \geq 0, p > 1$ *and* $1/p + 1/q = 1$*, there holds that*

$$a^2 b^2 + (a^p - b^q)^2/r^2 \leq \left(\frac{a^p}{p} + \frac{b^q}{q}\right)^2 \tag{12.1.4}$$

with $r = \max(p, q)$.

In fact, the Young inequality has the following refinement.

Theorem 12.1.3. *Let* $1 < p < +\infty, p \vee q = \max(p, q), p \wedge q = \min(p, q), a, b \geq 0$. *Then we have*

$$\frac{1}{p \vee q}(\sqrt{a} - \sqrt{b})^2 \leq a/p + b/q - a^{1/p} b^{1/q} \leq \frac{1}{p \wedge q}(\sqrt{a} - \sqrt{b})^2. \tag{12.1.5}$$

In Corollary 12.1.1, if we consider a and b as εa and $\varepsilon^{-1} b$ respectively, we can conclude the following corollary.

Corollary 12.1.2. *For any* $\varepsilon > 0$*, we have*

$$ab \leq \frac{\varepsilon^p a^p}{p} + \frac{b^q}{q \varepsilon^q} \tag{12.1.6}$$

where $a, b \geq 0, p > 1$ *and* $1/p + 1/q = 1$.

The Young inequality has several variants in the following.

Corollary 12.1.3 (The Young inequality).

(1) *Let* $a, b > 0, 1/p + 1/q = 1, 1 < p < +\infty$. *Then*

(i)
$$a^{1/p} b^{1/q} \leq a/p + b/q; \tag{12.1.7}$$

(ii)
$$a^{1/p} b^{1/q} \leq a/(p\varepsilon^{1/q}) + b\varepsilon^{1/p}/q, \ \text{for all } \varepsilon > 0; \tag{12.1.8}$$

(iii)
$$a^\alpha b^{1-\alpha} \leq \alpha a + (1 - \alpha) b, \ 0 < \alpha < 1. \tag{12.1.9}$$

(2) *Let* $a_k \geq 0, p_k > 0, \sum_{k=1}^m p_k = 1$. *Then*

$$\prod_{k=1}^m a_k^{p_k} \leq \sum_{k=1}^m p_k a_k. \tag{12.1.10}$$

The Young inequality also has the following improved versions (see, e.g., Kuang [466]).

Theorem 12.1.4 (The Gerber Inequality).

(1) *Let* $0 < x < cy, c \geq 1, M = \frac{c^{\lambda+1}}{2y} \lambda(1 - \lambda)(x - y)^2$.

(i) *If* $0 < \lambda < 1$ *or* $\lambda > 2$*, then we have*

$$\frac{M}{c^3} < \lambda x + (1-\lambda)y - x^\lambda y^{1-\lambda} < M. \qquad (12.1.11)$$

(ii) *If* $\lambda < 0$ *or* $1 < \lambda < 2$*, then we have*

$$\frac{M}{c^3} > \lambda x + (1-\lambda)y - x^\lambda y^{1-\lambda} > M. \qquad (12.1.12)$$

(2) *Let* $r, x > 0, y > -1/r$*. Then*

$$xy \le x\left(\frac{x^r - 1}{r}\right) + \left(\frac{1+ry}{1+r}\right)^{1+1/r}. \qquad (12.1.13)$$

(3) *Let* $1 < p_k < +\infty, \sum_{k=1}^n \frac{1}{p_k} = 1$*. Then*

$$\prod_{k=1}^n |a_k| \le \sum_{k=1}^n \frac{1}{p_k}|a_k|. \qquad (12.1.14)$$

In 1932, Takahashi [906] proved the following inverse Young inequality.

Theorem 12.1.5 (The Takahashi Inequality)**.** *If for* $x \ge 0, f$ *and* g *are continuous and increasing functions such that* $f(0) = g(0) = 0, g^{-1}(x) \ge f(x)$ *for all* $x \ge 0$*, and if for every* $a > 0$ *and* $b > 0$*, we have*

$$ab \le \int_0^a f(x)dx + \int_0^b g(x)dx, \qquad (12.1.15)$$

then f *and* g *are invertible with* $f^{-1} = g, f = g^{-1}$*.*

In 1989, the Chinese mathematicians Lizhi Xu and Chunling Zou proved the following inverse theorem (see also Kuang [466]).

Theorem 12.1.6. *Let* f, g *be strictly increasing and continuous,* $f(0) = g(0) = 0$*,* f *be defined on* $[0, c]$*,* $b \in [0, f(c)]$*.*

(1) *If* $g^{-1}(x) \ge f(x)$*, for all* $x \in [0, c]$*, and for all* $a \in [0, c]$*, and for all* $b \in [0, f(c)]$*, there holds that*

$$ab \le \int_0^a f(x)dx + \int_0^b g(x)dx, \qquad (12.1.16)$$

then f *and* g *are invertible with* $f = g^{-1}$ *or* $g = f^{-1}$*.*

(2) *If for all* $x \in [0, c], g^{-1}(x) \le f(x)$ *and for all* $a \in [0, c]$*, and for all* $b \in [0, f(c)]$*, there holds*

$$\int_0^a f(x)dx + \int_0^b g(x)dx \le af(a) + bg(b) - f(a)f(b), \qquad (12.1.17)$$

then f *and* g *are invertible with* $f = g^{-1}$ *or* $g = f^{-1}$*.*

(3) *For* $a \in [0, c], b \in [0, f(c)]$, *there holds that*

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \le af(a) + bf^{-1}(b) - f(a)f^{-1}(b). \qquad (12.1.18)$$

The Young inequality has several generalizations as follows (see, e.g., Kuang [466]).

Theorem 12.1.7 (The Oppenheim Inequality)**.** *Let* $f_k(x)$ *be non-negatively continuous increasing,* $a_k \ge 0, k = 1, 2, \ldots, n$. *If there is at least one* k *such that* $f_k(0) = 0$, *then we have*

$$\prod_{k=1}^{n} f_k(a_k) \le \sum_{k=1}^{n} \int_0^{a_k} \prod_{j \ne k} f_j(x) df_k(x) \qquad (12.1.19)$$

and the equality in (12.1.19) *holds if and only if* $a_1 = \cdots = a_n$.

Theorem 12.1.8 (The Cooper Inequality)**.** *Let* $g_k(x)$ *be strictly increasing and continuous,* $g_k(0) = 0, a_k \ge 0, k = 1, 2, \ldots, n$. *If* $\prod_{k=1}^{n} g_k^{-1}(x) = x$, *then we have*

$$\prod_{k=1}^{n} a_k \le \sum_{k=1}^{n} \int_0^{a_k} \frac{g_k(x)}{x} dx \qquad (12.1.20)$$

and the equality (12.1.20) *holds if and only if* $g(a_1) = \cdots = g(a_n)$.

Theorem 12.1.9. *Let* f *be strictly increasing and continuous,* $f(0) = 0, f^{-1}$ *the inverse function of* f, $[x]$ *the maximal integral part of* x. *Then for any* $m, n \in \mathbb{N}$, *we have*

$$mn \le \sum_{k=0}^{m} [f(k)] + \sum_{k=0}^{n} [f^{-1}(k)]. \qquad (12.1.21)$$

In 1988, the Chinese mathematician Lizhi Xu proved the following result (see, e.g., Kuang [466]).

Theorem 12.1.10. *Let* $a, b > 0$ *and* f *be strictly increasing and continuous,* $f(0) = 0, f^{-1}$ *the inverse function of* f. *Then the following assertions hold.*

$$ab \le \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy \le af(a) + bf^{-1}(b) - f(a)f^{-1}(b). \qquad (12.1.22)$$

(i) *If* $f(x)$ *is convex, then when* $f''(x)[b - f(a)] \ge 0$, *we have*

$$ab + \frac{1}{2}[b - f(a)][f^{-1}(b) - a] \le \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy. \qquad (12.1.23)$$

(ii) *If* $f''(x)[b-f(a)] \le 0$*, then we have*

$$ab + \frac{1}{2}[b - f(a)][f^{-1}(b) - a] \ge \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy. \qquad (12.1.24)$$

If f' *is monotone, let* $h = (1/n)[f^{-1}(b) - a], n \ge 2,$

$$S_n = bf^{-1}(b) - h\left\{(f(a)+b)/2 + \sum_{k=1}^{n-1} f(a+kh)\right\}, \qquad (12.1.25)$$

then we have

$$\left|\int_0^a f(x)dx + \int_0^b f^{-1}(y)dy - S_n\right| \le (h^2/8)|f'(a) - f'[f^{-1}(b)]|. \qquad (12.1.26)$$

The following result, due to Zsolt, is related to the Young inequality (see, e.g., Kuang [466]).

Theorem 12.1.11. *Assume that* $f(x,y)$ *exists continuous partial derivatives of second order such that* $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) \ge 0, x, y \ge 0$*. Then for any non-negative numbers* x, y *and any Young function* ϕ*,*

$$f(x,y) \le f(0,0) + \int_0^x \frac{\partial}{\partial t} f[t, \phi(t)]dt + \int_0^y \frac{\partial}{\partial s} f[\phi^{(-1)}(s), s]ds \qquad (12.1.27)$$

where $\phi^{(-1)}$ *denotes the right inverse of* ϕ*.*

12.2 The Hausdorff–Young inequalities and the Young inequalities

Since the following inequalities will involve the concept of the Marcinkiewicz space or the weak L^p space denoted by L^p_*, we first introduce some basic concepts related to the Marcinkiewicz space.

Definition 12.2.1. Let (X, μ) be a measurable space with positive μ, $f(x)$ a μ-measurable function defined on X. If for any $\alpha > 0$, the set

$$E_\alpha = E_\alpha(f) = \{x : |f(x)| > \alpha\}$$

is measurable, then the function

$$f_*(\alpha) = \mu(E_\alpha)$$

is said to be the distributional function of f.

Clearly, $f_*(\alpha)$ is a non-negative function and it is easy to verify the following properties.

Lemma 12.2.1.

(1) $f_*(\alpha)$ *is a non-increasing, right continuous function;*
(2) *if* $|f(x)| \leq |g(x)|$, *then* $f_*(\alpha) \leq g_*(\alpha)$;
(3) *if* $\{f_m(x)\}$ *is a* μ*-measurable sequence, and*

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f_m(x) \cdots \uparrow f(x),\ m \to +\infty,$$

then we have

$$(f_m)_*(\alpha) \uparrow f_*(\alpha), \quad m \to +\infty;$$

(4) *if* $|f(x)| \leq |g(x)| + |h(x)|$, *then*

$$f_*(\alpha) \leq g_*(\alpha/2) + h_*(\alpha/2).$$

Now we are in a position to introduce the concept of the Marcinkiewicz space or the weak L^p space.

Definition 12.2.2. Let $1 \leq p < +\infty$. If

$$\|f\|_{p,w} \equiv \|f\|_{L^p_*} = [f]_p = \sup_{\alpha>0} \alpha f_*(\alpha)^{1/p} < +\infty, \tag{12.2.1}$$

then we call f to satisfy the Marcinkiewicz condition. All the functions satisfying the Marcinkiewicz condition (12.2.1) constitute a space which is called Marcinkiewicz space or the weak L^p space, denoted by L^p_*. In particular, when $p = +\infty$, we make a convention: $L^\infty_* = L^\infty$.

Remark 12.2.1.

(1) It is easy to see that $L^p \hookrightarrow L^p_*$.
(2) $\|\cdot\|_{L^p_*} = [\cdot]_p$ is not a norm, but due to

$$[f+g]_p \leq 2([f]_p + [g]_p),$$

L^p_* is a quasi-norm vector space with this quasi-norm.

For the Fourier convolution, we collect the famous Hausdorff–Young inequalities and Young inequalities for the convolution as follows. We can find its proof in some books of functional analysis, e.g., Yosida [986] and Belleni-Morante and McBride [91].

Theorem 12.2.2 (The Hausdorff–Young Inequality)**.** *If* $f \in L^p(\mathbb{R}^n)$ *and* $1 \leq p \leq 2$, $p^{-1} + q^{-1} = 1$. *Then we have*

$$\|\mathcal{F}f\|_q = \|\hat{f}\|_q \leq \|f\|_p \tag{12.2.2}$$

where $\mathcal{F}f = \hat{f}$ *is the Fourier transform of* f.

Theorem 12.2.3 (The Hausdorff–Young Inequality). *Let* $1 < p < 2$, $p^{-1} + q^{-1} = 1$. *Then we have*

$$\|\mathcal{F}f\|_{q,w} = \|\hat{f}\|_{q,w} \leq C_{p,q}\|f\|_{p,w} \tag{12.2.3}$$

with a constant $C_{p,q} > 0$.

Theorem 12.2.4 (The Discrete Hausdorff–Young Inequality). *Let* $f \in L^p[0, 2\pi]$, *and*

$$f(x) \sim \sum_{n=-\infty}^{+\infty} C_n e^{inx}. \tag{12.2.4}$$

Then we have

$$\left(\sum_{n=-\infty}^{+\infty} |C_n|^{p'}\right)^{1/p'} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{1/p} \tag{12.2.5}$$

with $1/p + 1/p' = 1$.

Theorem 12.2.5 (The Young Inequality). *Let* $K \in L^\rho(\mathbb{R}^n), \phi \in L^p(\mathbb{R}^n)$ *with* $1 < p < \rho'$, $1/\rho + 1/\rho' = 1$. *Then we have*

$$\|K * \phi\|_{L^q(\mathbb{R}^n)} \leq \|K\|_{L^\rho(\mathbb{R}^n)}\|\phi\|_{L^p(\mathbb{R}^n)} \tag{12.2.6}$$

where $1 + 1/q = 1/\rho + 1/p$ *and* $K * \phi$ *is the Fourier convolution of* K *and* ϕ *on* $\mathbb{R}^n$ *defined by* $(K * \phi)(x) = \int_{\mathbb{R}^n} K(x-y)\phi(y)dy$.

For the one-dimensional case, we have the following form of the Hausdorff–Young inequality (see, e.g., Belleni-Morante and McBride [91]).

Theorem 12.2.6 (The Young Inequality). *Let* $1 \leq p, q \leq +\infty$ *and* $f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})$. *Then we have*

$$f * g \in L^r(\mathbb{R}) \tag{12.2.7}$$

and

$$\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})} \tag{12.2.8}$$

where $1 + 1/r = 1/p + 1/q$ *and* $f * g$ *is the Fourier convolution of* f *and* g *on* $\mathbb{R}$ *defined by*

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy. \tag{12.2.9}$$

Remark 12.2.2. The important special cases in Theorem 12.2.5 embrace:

(1) $p = q = r = 1$ and (2) $q = 1, r = p$.

Theorem 12.2.7 (The Generalized Young Inequality). *If* $1 < p, q, r < +\infty, p^{-1} + q^{-1} = 1 + r^{-1}$. *Then we have*

$$\|f * g\|_r \leq C_{p,q}\|f\|_p\|g\|_{q,w}. \tag{12.2.10}$$

Theorem 12.2.8 (The Weak Young Inequality)**.** *If* $1 < p, q, r < +\infty, p^{-1} + q^{-1} = 1 + r^{-1}$. *Then we have*

$$\|f * g\|_{r,w} \le C_{p,q} \|f\|_{p,w} \|g\|_{q,w} \tag{12.2.11}$$

where $C_{p,q} > 0$ *is a constant.*

In the above, $\|\cdot\|_{p,w} = \|\cdot\|_{L^p_*}$ denotes the quasi-norm of space $L^p_*(\mathbb{R}^n)$, the weak $L^p(\mathbb{R}^n)$ space.

12.3 The Hölder inequalities

The following is the discrete Hölder inequality which was proved by Hölder in 1889 (see, e.g., Hölder [369]). However, as pointed out by Lech [493] that in fact it should be called the Roger inequality or Roger–Hölder inequality since Roger established the inequality (12.3.1) in 1888 earlier than Hölder did in 1889. However, we still call it the Hölder inequality.

Theorem 12.3.1. *If* $a_k \ge 0, b_k \ge 0$ *for* $k = 1, 2, \ldots, n$, *and* $1/p + 1/q = 1$ *with* $p > 1$, *then*

$$\sum_{k=1}^n a_k b_k \le \left(\sum_{k=1}^n a_k^p\right)^{1/p} \left(\sum_{k=1}^n b_k^q\right)^{1/q}. \tag{12.3.1}$$

If $0 < p < 1$, *then*

$$\sum_{k=1}^n a_k b_k \ge \left(\sum_{k=1}^n a_k^p\right)^{1/p} \left(\sum_{k=1}^n b_k^q\right)^{1/q}. \tag{12.3.2}$$

Here the equalities in (12.3.1)–(12.3.2) *hold if and only if* $\alpha a_k^p = \beta b_k^q$ *for* $k = 1, 2, \ldots, n$ *where* α *and* β *are real non-negative constants with* $\alpha^2 + \beta^2 > 0$.

Remark 12.3.1. If $p = 1$ or $p = +\infty$, we have the trivial case

$$\begin{cases} \displaystyle\sum_{k=1}^n a_k b_k \le \left(\sum_{k=1}^n a_k\right) \sup_{1\le k\le n} b_k, \quad \text{if } p = 1; & (12.3.3) \\ \displaystyle\sum_{k=1}^n a_k b_k \le \left(\sum_{k=1}^n b_k\right) \sup_{1\le k\le n} a_k, \quad \text{if } p = +\infty. & (12.3.4) \end{cases}$$

Remark 12.3.2. When $p = q = 2$, we call (12.3.1)–(12.3.2) to be the Cauchy inequality, or the Schwarz inequality or the Cauchy–Schwarz inequality or the Bunyakovskii inequality.

In 1992, Dragomir [215] gave a refinement of the Cauchy inequality.

Theorem 12.3.2 (The Dragomir Inequality). *Let a_k, b_k real numbers and $|a_k| + |b_k| \neq 0$. Then we have*

$$\left(\sum_k a_k b_k\right)^2 \le \left[\sum_k (a_k^2 + b_k^2)\right]\left[\sum_k \frac{a_k^2 b_k^2}{a_k^2 + b_k^2}\right] \le \left(\sum_k a_k^2\right)\left(\sum_k b_k^2\right). \tag{12.3.5}$$

For $a = \{a_k\}$ ($k \in \mathbb{N}$ is finite or infinite), if we define

$$\begin{cases} \|a\|_p = \left(\sum_k a_k^p\right)^{1/p} & \text{for } 0 < p < +\infty; \\ \|a\|_\infty = \sup_k |a_k| & \text{for } p = +\infty, \end{cases}$$

then we have the following refinement of the Hölder inequality.

Theorem 12.3.3. *Let $a_k, b_k > 0$, $1/p + 1/q = 1$, $1 < p < +\infty$, $p \vee q = \max(p, q)$, $p \wedge q = \min(p, q)$, $S_n = \frac{\sum_k a_k^{p/2} b_k^{q/2}}{(\sum_k a_k^p)^{1/2} (\sum_k b_k^q)^{1/2}}$. Then we have*

$$\frac{2}{p \vee q}(1 - S_n) \le 1 - \frac{\|ab\|_1}{\|a\|_p \|b\|_q} \le \frac{2}{p \wedge q}(1 - S_n). \tag{12.3.6}$$

The following result is a generalization with negative exponents.

Theorem 12.3.4. *If $a_k > 0, b_k > 0$ for $k = 1, 2, \ldots, n$, and $1/p + 1/q = 1$ with $p < 0$ or $q < 0$, then we have*

$$\left(\sum_{k=1}^n a_k^p\right)^{1/p}\left(\sum_{k=1}^n b_k^q\right)^{1/q} \le \sum_{k=1}^n a_k b_k \tag{12.3.7}$$

with equality in (12.3.7) holding if and only if $\alpha a_k^p = \beta b_k^q$ for $k = 1, 2, \ldots, n$ where α and β are real non-negative constants with $\alpha^2 + \beta^2 > 0$.

Jensen [394] proved the following generalization of the Hölder inequality.

Theorem 12.3.5 (The Jensen Inequality). *Let $a_{jk} > 0$, $p_j > 0$, ($j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$), $\sum_{j=1}^m \frac{1}{p_j} \ge 1$. Then we have*

$$\sum_{k=1}^n \left(\prod_{j=1}^m a_{jk}\right) \le \prod_{j=1}^m \left(\sum_{k=1}^n a_{jk}^{p_j}\right)^{1/p_j}. \tag{12.3.8}$$

If $p_j < 0, \sum_{j=1}^m \frac{1}{p_j} \le -1$, then we have

$$\sum_{k=1}^n \left(\prod_{j=1}^m a_{jk}\right) \ge \prod_{j=1}^m \left(\sum_{k=1}^n a_{jk}^{p_j}\right)^{1/p_j}. \tag{12.3.9}$$

The equalities in (12.3.8)–(12.3.9) hold if and only if $\sum_{j=1}^m 1/p_j = 1$ and all the column vectors of the matrix (a_{jk}) are proportional to each other.

The next result is a generalization of the above Jensen inequality.

Theorem 12.3.6. *Let* $a_{jk} > 0$ $(j = 1, 2, \dots, m;\ k = 1, 2, \dots, n)$, $r > 0$, $p_j > 0$, $\sum_j 1/p_j \geq 1/r$. *Then the following inequality holds*

$$\left(\sum_{k=1}^{n}\prod_{j=1}^{m} a_{jk}^{1/p_j}\right)^{1/r} \leq \prod_{j=1}^{m}\left(\sum_{k=1}^{n} a_{jk}^{1/r}\right)^{1/p_j}. \tag{12.3.10}$$

Here the equality in (12.3.10) *holds if and only if* $\sum_{j=1}^{m} 1/p_j = 1/r$ *and all the column vectors of the matrix* (a_{jk}) *are proportional to each other.*

We have also the following weighted Hölder inequality.

Theorem 12.3.7. *Under assumptions of Theorem* 12.3.6, *if*

$$1 < p_j < +\infty, \ \ \textstyle\sum_{j=1}^{n} 1/p_j = 1, \ \ \omega_k > 0,$$

then we have

$$\sum_{k=1}^{m}\left(\omega_k \prod_{j=1}^{n} a_{jk}\right) \leq \prod_{j=1}^{n}\left(\sum_{k=1}^{m} \omega_k a_{jk}^{p_j}\right)^{1/p_j}. \tag{12.3.11}$$

The following theorem is the converse theorem for the Hölder inequality.

Theorem 12.3.8. *Let* $p > 1$, $1/p + 1/q = 1$, $B > 0$. *Then for all* $a = \{a_k\}$ *satisfying*

$$\left(\sum_k |a_k|^p\right)^{1/p} \leq A, \tag{12.3.12}$$

there holds

$$\sum_k |a_k b_k| \leq AB \tag{12.3.13}$$

if and only if

$$\left(\sum_k |b_k|^q\right)^{1/q} \leq B. \tag{12.3.14}$$

By virtue of the discrete Hölder inequality (see Theorem 12.3.1), we easily obtain the following integral form of the Hölder inequality.

Theorem 12.3.9. *If* $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ *and* $\Omega \subseteq \mathbb{R}^n$ *is a smooth open set, then*

$$fg \in L^1(\Omega) \tag{12.3.15}$$

and

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \tag{12.3.16}$$

with $1 \le p \le +\infty, 1/p + 1/q = 1$ *and*

$$\begin{cases} \|f\|_{L^p(\Omega)} = \left\{\int_\Omega |f(x)|^p dx\right\}^{1/p}; & (12.3.17) \\ \|f\|_{L^\infty} = \operatorname{esssup}_{x\in\Omega} |f(x)|. & (12.3.18) \end{cases}$$

If $0 < p < 1$, *then we have*

$$\|fg\|_{L^1(\Omega)} \ge \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}. \tag{12.3.19}$$

The equalities in (12.3.16) *and* (12.3.19) *hold if and only if there exist* $\beta \in \mathbb{R}$ *and real numbers* C_1, C_2 *which are not all zeros such that* $C_1|f(x)|^p = C_2|g(x)|^q$ *and* $\arg(f(x)g(x)) = \beta$ *almost everywhere on* Ω *hold.*

Remark 12.3.3. We have the corresponding weighted Hölder inequality of the integral form. Let $1 < p < +\infty$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, $1/p + 1/q = 1$, $\omega(x) > 0$ on Ω. Then we have

$$\int_\Omega |fg|\omega(x)dx \le \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{1/p} \left(\int_\Omega |g(x)|^q \omega(x)dx\right)^{1/q}. \tag{12.3.20}$$

The following result is a variant of the integral form of the Hölder inequality which is due to Everitt [242].

Theorem 12.3.10. *Let* $p > 1$, $1/p+1/q = 1$. *Let* E *and* E' *with* $E' \subseteq E$ *be Lebesgue-measurable linear sets. If* f_1 *and* f_2 *are complex measurable functions such that* $f_1 \in L^p(E)$, $f_2 \in L^q(E)$, *define the function* H *by*

$$H(E) = \left(\int_E |f_1(x)|^p dx\right)^{1/p} \left(\int_E |f_2(x)|^q dx\right)^{1/q} - \left|\int_E f_1(x)f_2(x)dx\right|.$$

Then

$$0 \le H(E - E') \le H(E) - H(E'). \tag{12.3.21}$$

Pecaric (see also [466]) proved the following monotonic property of the Hölder inequality.

Theorem 12.3.11. *Let* $a_k, b_k \ge 0$, $u_k \ge v_k \ge 0$, $1/p + 1/q = 1$, $1 < p < +\infty$. *Then we have*

$$\begin{aligned} 0 &\le \left(\sum_{k=1}^n v_k a_k^p\right)^{1/p} \left(\sum_{k=1}^n v_k b_k^q\right)^{1/q} - \sum_{k=1}^n v_k a_k b_k \\ &\le \left(\sum_{k=1}^n u_k a_k^p\right)^{1/p} \left(\sum_{k=1}^n u_k b_k^q\right)^{1/q} - \sum_{k=1}^n u_k a_k b_k. \end{aligned} \tag{12.3.22}$$

If $0 < p < 1$, *then we have*

$$\begin{aligned}\left(\sum_{k=1}^{n} v_k a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} v_k b_k^q\right)^{1/q} - \sum_{k=1}^{n} v_k a_k b_k \\ \geq \left(\sum_{k=1}^{n} u_k a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} u_k b_k^q\right)^{1/q} - \sum_{k=1}^{n} u_k a_k b_k.\end{aligned} \tag{12.3.23}$$

Chinese mathematicians have made a great contribution in improving many famous inequalities including the Hölder inequality and the Minkowski inequality. In what follows, we shall introduce some of their elegant results on the Hölder inequality. Ke Hu (see, e.g., Kuang [466]) proved the following theorem.

Theorem 12.3.12 (The Hu Inequality)**.** *Let* $p \geq q \geq 0$, $1/p+1/q = 1$, $1-e_n+e_m \geq 0$, $a_n, b_n \geq 0$. *Then*

$$\begin{aligned}\sum_n a_n b_n \leq \left(\sum_n b_n^q\right)^{(1/q-1/p)} &\left\{ \left(\sum_n a_n^p\right)^2 \left(\sum_n b_n^q\right)^2 \right. \\ &\left. - \left[\left(\sum_n b_n^q e_n\right) \sum_n a_n^p - \left(\sum_n b_n^q\right)\left(\sum_n a_n^p e_n\right)\right]^2 \right\}^{1/(2p)}.\end{aligned} \tag{12.3.24}$$

The corresponding integral form of the above inequality is as follows.

Theorem 12.3.13. *Let* $p \geq q \geq 0$, $1/p+1/q = 1$, $1-\omega(x)+\omega(y) \geq 0$, $f(x), g(x) \geq 0$. *Then*

$$\left\{\begin{aligned}\int fg dx \leq \left(\int g^q dx\right)^{(1/q-1/p)} &\left\{ \left(\int f^p dx\right)^2 \left(\int g^q dx\right)^2 \right. \\ &\left. - \left[\left(\int g^q \omega dx\right) \int f^p dx - \left(\int g^q dx\right)\left(\int f^p \omega dx\right)\right]^2 \right\}^{1/(2p)}.\end{aligned}\right. \tag{12.3.25}$$

Corollary 12.3.1. *Let* $a_n \geq 0$, $p > 1$, $1 - e_n + e_m \geq 0$, $n, m = 1, 2, \ldots, N$. *Then we have*

$$\begin{aligned}\left(\sum_{k=1}^{N} a_k\right)^{2(2p-1)} \leq \left(\sum_{k=1}^{N} a_k^p\right)^2 &\left\{ N^2 \left(\sum_{k=1}^{N} a_k\right)^2 \right. \\ &\left. - \left[N \sum_{k=1}^{N} a_k e_k - \left(\sum_{k=1}^{N} a_k\right)\left(\sum_{k=1}^{N} e_k\right)\right]^2 \right\}^{p-1}.\end{aligned} \tag{12.3.26}$$

The corresponding integral form of the above corollary is the following corollary.

Corollary 12.3.2. *Let $f \geq 0$, $p > 1$, $1 - \omega(x) + \omega(y) \geq 0$, $x, y \in [a, b]$. Then we have*

$$\begin{cases} \left(\int_a^b f dx\right)^{2(2p-1)} \leq \left(\int_a^b f^p dx\right)^2 \Bigg\{(b-a)^2 \left(\int_a^b f dx\right)^2 - \Bigg[(b-a)\int_a^b f\omega dx \\ \qquad - \left(\int_a^b f dx\right)\left(\int_a^b \omega dx\right)\Bigg]^2\Bigg\}^{p-1}. \end{cases} \tag{12.3.27}$$

In 1998, Ke Hu (see, e.g., [466]) further proved the following theorem.

Theorem 12.3.14. *Let $f, g \geq 0$, $f \in L^p[0, b]$, $g \in L^q[0, b]$, $p \geq q > 1$, $1/p + 1/q = 1$, $|\bar{\omega}(x)\omega(y) - \omega(x)\bar{\omega}(y)| \leq 1$, $x, y \in [0, b]$. If we define*

$$\begin{aligned} F_s(t) = &\left\{\left(\int_0^t g^q d\tau\right)^{2/q-2/p}\right\}^s \Bigg\{\left(\left(\int_0^t f^p d\tau\right)^2 \left(\int_0^t g^q d\tau\right)^2 \right. \\ &- \left.\left[\left(\int_0^t f^p \omega d\tau\right)\left(\int_0^t g^q \bar{\omega} d\tau\right) - \left(\int_0^t f^p \bar{\omega} d\tau\right)\left(\int_0^t g^q \omega \tau\right)\right]^2\right\}^{s/p} \\ &- \left(\int_0^t fg d\tau\right)^{2s}, \end{aligned}$$

then for $0 \leq t_1 \leq t_2 \leq b$, $s = 1, 2, \ldots$, there holds that

$$F_s(t_2) \geq F_s(t_1) \geq 0. \tag{12.3.28}$$

In 2000, Yang [975] gave the following result.

Theorem 12.3.15. *Let*

$$a_{jk} > 0, \quad j = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, n,$$
$$p_j > 0, \quad \textstyle\sum_{j=1}^m 1/p_j = 1.$$

Set

$$h(t) = \prod_{i=1}^n \left\{\sum_{j=1}^m \left(\prod_{k=1}^n a_{jk}\right)^{1-t} (a_{ji}^{p_i})^t\right\}^{1/p_i},$$

then h is a monotone increasing function in t, particularly,

$$h(0) \leq h(1/2) \leq h(1) \tag{12.3.29}$$

is a refinement of the Hölder inequality.

In 1998, Liu [560] improved the Cauchy inequality when some additional assumptions are imposed.

Theorem 12.3.16. *Let* $0 < x_1 \le x_2/2 \le \cdots \le x_n/n, 0 < y_n \le y_{n-1} \le \cdots \le y_1$, *the Cauchy inequality can be improved as*

$$\begin{cases} \left(\sum_{k=1}^n x_k y_k\right)^2 \le \left(\sum_{k=1}^n y_k\right) \sum_{k=1}^n \left(x_k^2 - \frac{1}{4} x_{k-1} x_k\right) y_k, & (12.3.30) \\ \left(\sum_{k=1}^n x_k y_k\right)^2 \le \left(\sum_{k=1}^n y_k\right) \left\{\sum_{k=1}^n \left(\frac{7k+1}{8k} x_k^2 - \frac{k}{8(k-1)} x_{k-1}^2\right) y_k\right\} & (12.3.31) \end{cases}$$

and if and only if $x_k = kx_1$, $y_k = y_1$, *the equalities in* (12.3.30) *and* (12.3.31) *hold where* $x_0 = 0$.

Remark 12.3.4. In 1999, Alzer [26] further improved the above result as

$$\left(\sum_{k=1}^n x_k y_k\right)^2 \le \left(\sum_{k=1}^n y_k\right) \sum_{k=1}^n \left(\alpha + \frac{\beta}{k}\right) x_k^2 y_k \tag{12.3.32}$$

if $\alpha \ge 3/4, \beta \ge 1 - \alpha$.

In Torchinsky [920], we find the following extensive form.

Theorem 12.3.17. *Let* $1 < p < +\infty$, $1/p + 1/q = 1$, $0 \le x_n \le x_{n-1} \le \cdots \le x_1$, $0 \le y_n \le y_{n-1} \le \cdots \le y_1$, $B_n = \frac{1}{m-k} \sum_{j=k+1}^n y_j$. *Then we have*

$$\sum_{k=1}^n x_k y_k \le \left(\sum_{i=1}^m x_i^p\right)^{1/p} \left\{\sum_{j=1}^k y_j^q + (m-k) B_n^q\right\}^{1/q} \tag{12.3.33}$$

where $0 \le k \le m \le n$.

The following two theorems are related results on the Hölder inequality.

Theorem 12.3.18. *Let* $a_k, b_k > 0$, $1/p + 1/q = 1/r < 1$, $p, q > 0$. *Then*

$$2\|ab\|_1^{1/r} \le \|a\|_p \|b\|_q + \left(\sum_k a_k^{2-p} b_k^2\right)^{1/p} \left(\sum_k a_k^2 b_k^{2-q}\right)^{1/q}. \tag{12.3.34}$$

(1) (**The Daykin–Eliezer Inequality**) *If* $p, q > 0$ *or* $p > 0$, $q < 0$, $r < 0$, *then we have*

$$\|ab\|_r \le \|a\|_p \|b\|_q. \tag{12.3.35}$$

(2) (**The Acezel–Beckenbach Inequality**) *If* $p < 0$, $q < 0$ *or* $p > 0$, $q < 0$, $r > 0$, *then we have*

$$\|ab\|_r \ge \|a\|_p \|b\|_q. \tag{12.3.36}$$

In 1968, Daykin–Eliezer [197] proved the next result.

Theorem 12.3.19. *Let* $1/p+1/q=1/r$, $Q=\prod_{j,k=1}^{n}(a_ja_kb_jb_k)^{a_ja_kb_jb_k}$.

(1) *If* $0<a_k<1$, $0<b_k<1$ *or* $Q<1$ *and* $1/r<1$, *then*

$$\|ab\|_1^{1/r}\le\left\{\sum_k a_k^{2-p}b_k^2\right\}^{1/p}\left\{\sum_k a_k^2b_k^{2-q}\right\}^{1/q}. \tag{12.3.37}$$

(2) *If* $a_k>1$, $b_k>1$ *or* $Q>1$ *and* $1/r<1$, *then*

$$\|ab\|_1^{1/r}\le\|a\|_p\|b\|_q. \tag{12.3.38}$$

In 1992, Zhonglie Wang (see also Kuang [466]) improved the above result and proved the following result.

Theorem 12.3.20. *Let* $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$, $a_k,b_k>0$, $1\le k\le n$, $1/p+1/q-1=1/r$, $p,q>0$. *Then*

$$\|ab\|_1^{1+1/r}\le\left\{\|a\|_p\|b\|_q\|a^{2/p-1}b^{2/p}\|_p\|a^{2/q}b^{2/q-1}\|_q\right\}^{1/2} \tag{12.3.39}$$

and

$$f(x)=\|(ab)^{(1-x)/p}a^x\|_p\|(ab)^{(1-x)/q}b^x\|_q \tag{12.3.40}$$

is a logarithmically convex function. Particularly, when $r=+\infty$, *i.e.,* $1/p+1/q=1$, $f(x)$ *attains its minimum at* $x=0$.

To conclude this section, we shall review some results on the backward Hölder inequality whose definition is defined as follows.

Definition 12.3.1. The backward Hölder inequality means to seek a constant $C_{p,q}>0$ such that

$$\left(\sum_{k=1}^n a_k^p\right)^{1/p}\left(\sum_{k=1}^n b_k^q\right)^{1/q}\le C_{pq}\sum_{k=1}^n a_kb_k \tag{12.3.41}$$

where $1<p<+\infty$, $1/p+1/q=1$, $a_k,b_k\ge 0$, $k=1,2,\dots,n$.

In 1989, You made some improvements (see, e.g., Kuang [466]).

Theorem 12.3.21 (The Yau Inequality)**.** *Let* $0<m_1\le a_k\le M_1$, $0<m_2\le b_k\le M_2$, $1\le k\le n$, $1/p+1/q=1/r$.

(1) *If* $p,q>0$, *then we have*

$$\left(\frac{m_1}{M_1}\right)^{r/q}\left(\frac{m_2}{M_2}\right)^{r/p}\le\frac{\|ab\|_r}{\|a\|_p\|b\|_q}\le 1. \tag{12.3.42}$$

(2) *If* $p>0$, $q<0$, $r>0$, *then we have*

$$1\le\frac{\|ab\|_r}{\|a\|_p\|b\|_q}\le\left(\frac{m_1}{M_1}\right)^{r/q}\left(\frac{M_2}{m_2}\right)^{r/p}. \tag{12.3.43}$$

(3) *If* $p > 0$, $q < 0$, $r < 0$, *then we have*

$$1 \geq \frac{\|ab\|_r}{\|a\|_p\|b\|_q} \geq \left(\frac{m_1}{M_1}\right)^{r/q}\left(\frac{M_2}{m_2}\right)^{r/p}. \tag{12.3.44}$$

(4) *If* $p < 0$, $q < 0$, *then we have*

$$1 \leq \frac{\|ab\|_r}{\|a\|_p\|b\|_q} \leq \left(\frac{M_1}{m_1}\right)^{r/q}\left(\frac{M_2}{m_2}\right)^{r/p}. \tag{12.3.45}$$

Diaz, Goldman and Metcalf (see, e.g., [466]) showed the following theorem.

Theorem 12.3.22. *Let* $0 < m_1 \leq a_k \leq M_1$, $0 < m_2 \leq b_k \leq M_2$, $k = 1, 2, \ldots, n$, $1/p + 1/q = 1$. *Then*

$$\|a\|_p\|b\|_q \leq C_{p,q}\|ab\|_1 \tag{12.3.46}$$

where

$$C_{p,q} = \frac{C_1(p,q)}{C_2(p,q)}, \quad C_1(p,q) = M_1^p M_2^q - m_1^p m_2^q,$$

$$C_2(p,q) = \left\{[p(M_1 m_2 M_2^q - m_1 M_2 m_2^q)]^{1/p}\,[q(M_2 m_1 M_1^p - m_2 M_1 m_1^p)]\right\}^{1/q}.$$

Particularly, if $p = q = 2$, *then we have*

$$\|ab\|_1 \leq \|a\|_2\|b\|_2 \leq C_3\|ab\|_1. \tag{12.3.47}$$

Let

$$\beta_1 = \frac{M_1/m_1}{(M_1/m_1) + (M_2/m_2)}, \ \beta_2 = \frac{M_2/m_2}{(M_1/m_1) + (M_2/m_2)},$$

then the equality in (12.3.46) *holds if and only if* $k = \beta_1 n, l = \beta_2 n$ *are all integers, and* k*'s* a_j *agree with* m_1, *the rest* l*'s* ($l = n - k$) a_j *agree with* M_1, *and the corresponding* b_k *agree with* M_2, m_2 *respectively.*

Remark 12.3.5. In 1925, Polya–Szego (see, e.g., Agarwal and Pan [9] and Kuang [466]) proved in (12.3.47)

$$C_3 = \frac{1}{2}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right).$$

In 1964, Diaz and Metcalf (see [466]) improved the constant C_3:

$$\frac{m_1 M_1\|b\|_2^2 + m_2 M_2\|a\|_2^2}{m_1 m_2 + M_1 M_2} \leq \|ab\|_1.$$

In 1969, Baraes (see, e.g., [466]) proved the following result.

Remark 12.3.6. If $0 \le a_1 \le \cdots \le a_n$, $0 \le b_n \le \cdots \le b_1$ and $a_{k-1} + a_{k+1} \le 2a_k$, $b_{k-1} + b_{k+1} \le 2b_k$, $k = 1, 2, \ldots, n-1$, then $C_3 = (2n-1)/(n-2)$ in (12.3.47), and the equality in (12.3.47) holds if and only if $a_k = n-k$, $b_k = k-1$.

The corresponding integral form of Theorem 12.3.22 is the following theorem.

Theorem 12.3.23. *Let $f \in L^p[0,a], g \in L^q[0,a]$. If f, g are non-negative concave functions such that $0 < \|f\|_p < +\infty$, $0 < \|g\|_q < +\infty$, then we have*

$$\|f\|_p \|g\|_q \le C_{p,q} \|fg\|_1 \tag{12.3.48}$$

where

$$C_{p,q} = \frac{6}{(1+p)^{1/p}(1+q)^{1/q} a^{1-1/p-1/q}} \quad \text{if} \quad p > 1;$$
$$C_{p,q} = \frac{3}{(1+p)^{1/p}(1+q)^{1/q} a^{1-1/p-1/q}} \quad \text{if } |p| < 1,\ |q| < 1.$$

Remark 12.3.7. In 1986, Guangqing Chen (see, e.g., [466]) proved that: if $p, q > 0$, $1/p + 1/q = 1$, then the constant $C_{p,q}$ in (12.3.46) may be expressed as

$$C_{p,q} = \frac{(1/p)^{1/p}(1/q)^{1/q}[x_2 f(x_1) - x_1 f(x_2)]}{(x_2 - x_1)^{1/q}[f(x_1) - f(x_2)]^{1/p}}$$

where

$$f(x) = x^{-q/p},\ x_1 = \min_{1 \le k \le n} \left\{ \frac{|a_k|^p}{|a_k b_k|} \right\} < x_2 = \max_{1 \le k \le n} \left\{ \frac{|a_k|^p}{|a_k b_k|} \right\}.$$

The weighted form of (12.3.47) can be stated as (see, e.g., Kuang [466]) the following theorem.

Theorem 12.3.24. *Let $a_k, b_k > 0$, $\omega_k \ge 0$ be not all zero, $1 \le k \le n$. Then*

$$1 \le \frac{\left(\sum_{k=1}^n a_k^2 \omega_k\right)\left(\sum_{k=1}^n b_k^2 \omega_k\right)}{\left(\sum_{k=1}^n a_k b_k \omega_k\right)} \le \max_{j,k} \frac{(a_k b_j + a_j b_k)^2}{4 a_k a_j b_k b_j}. \tag{12.3.49}$$

The corresponding integral form is the following result (see, e.g., Kuang [466]).

Theorem 12.3.25 (The Zagier inequality)**.** *Let f, g be decreasing on $[a,b]$, $f(b) = g(b) = 0$, and the weighted function ω be integrable on $[a,b]$. Then*

$$\|f\|_{2,\omega}^2 \|g\|_{2,\omega}^2 \le \max\left\{ f(a)\int_a^b g\omega dx, g(a)\int_a^b f\omega dx \right\} \|fg\|_{1,\omega} \tag{12.3.50}$$

where $\|f\|_{2,\omega} = \int_a^b f^2 \omega dx$.

In 1991, Yutong Lou (see, e.g., [466]) proved the following analogue.

Theorem 12.3.26. *Let* $0 < m_1 \leq f(x) \leq M_1$, $0 < m_2 \leq g(x) \leq M_2$, $x \in E$, $f, g, \omega \in L(E)$, $\omega \geq 0$. *Then we have*

$$\|f\|_{2,\omega}\|g\|_{2,\omega} \leq C\|fg\|_{1,\omega} \tag{12.3.51}$$

whose discrete case is

$$\|a\|_{2,\omega}\|b\|_{2,\omega} \leq C\|ab\|_{1,\omega} \tag{12.3.52}$$

where $0 < m_1 \leq a_k \leq M_1$, $0 < m_2 \leq b_k \leq M_2$ *and* $C = \frac{m_1 m_2 + M_1 M_2}{2\sqrt{m_1 m_2 M_1 M_2}}$.

Remark 12.3.8. For the above discrete case, Xiaohua Liu (see, e.g., [466]) showed that (12.3.46) holds with

$$C = C_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right) \tag{12.3.53}$$

where the function takes the form

$$C_{p,q}(t) = (p^{1/p}q^{1/q})^{-1}\frac{1-t}{(1-t^{1/p})^{1/p}(1-t^{1/q})^{1/q}}.$$

The corresponding weighted form is as follows:

Let $0 < m \leq a_k/b_k \leq M$, $\omega_k \geq 0$. Then

$$\|a\|_{p,\omega}\|b\|_{p,\omega} \leq C_{p,q}(m/M)\|ab\|_{1,\omega} \tag{12.3.54}$$

with $\|a\|_{p,\omega} = \left(\sum_k a_k^p \omega_k\right)^{1/p}$.

Remark 12.3.9. The integral form of (12.3.54) is as follows. Let (X, Σ, μ) be a measurable space, f, g non-negative μ-measurable functions on X. If $0 < m \leq f(x)/g(x) \leq M$, a.e., $x \in X$, $f, g \in L(X)$, $1/p + 1/q = 1$, $p, q > 0$, then we have

$$\|f\|_{p,w}\|g\|_{q,w} \leq C_{p,q}\left(\frac{m}{M}\right)\|fg\|_{1,w} \tag{12.3.55}$$

where $\|f\|_{p,w} = (⨍_X f^p \omega dx)^{1/p}$.

Yadong Zhuang (see, e.g., [466]) also proved the following similar results.

Theorem 12.3.27.

(1) *Let* $0 < m_1 \leq a_k \leq M_1$, $0 < m_2 \leq b_k \leq M_2$, $k = 1, 2, \ldots, n$, $1/p + 1/q = 1$, $1 < p < +\infty$. *Then for any* $\alpha, \beta > 0$, *there holds*

$$\|a\|_p\|b\|_p \leq C_{p,q}\|ab\|_1 \tag{12.3.56}$$

with $C_{p,q} = (\alpha p)^{-1/p}(\beta q)^{-1/q}\max\left\{\frac{\alpha M_1^p + \beta m_2^q}{M_1 m_2}, \frac{\alpha m_1^p + \beta M_2^q}{M_2 m_1}\right\}$;

(2)
$$a^p/p + b^q/q \leq C_{p,q}ab \tag{12.3.57}$$

where $1 < p < +\infty$, $1/p + 1/q = 1$, $0 < m_1 \le a \le M_1$, $0 < m_2 \le b \le M_2$ *and*

$$\|f\|_p \|g\|_q \le C_{p,q} \|fg\|_1 \tag{12.3.58}$$

with $0 < m_1 \le f(x) \le M_1, 0 < m_2 \le g(x) \le M_2, x \in E$ *and*

$$C_{p,q} = \max\left\{(m_1^p/p + M_2^q/q)/(m_1 M_2), (M_1^p/p + m_2^q/q)/(M_1 m_2)\right\}.$$

In 1986, Jianbo Shao (see, e.g., [466]) showed the following result.

Theorem 12.3.28. *Let* $0 < m_1 \le |a_k| \le M_1$, $0 < m_2 \le |b_k| \le M_2$, $1 \le k \le n$. *Then*

$$\begin{aligned}&\sqrt{\frac{m_2 M_2}{m_1 M_1}}\left(\sum_{k=1}^{n} |a_k|^2\right) + \sqrt{\frac{m_1 M_1}{m_2 M_2}}\left(\sum_{k=1}^{n} |b_k|^2\right)\\ &\le \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)\left(\sum_{k=1}^{n} |a_k b_k|\right).\end{aligned} \tag{12.3.59}$$

The equality in (12.3.59) *holds if and only if* p*'s* a_k *agree with* m_1*, the rest* l*'s* a_k ($l = n - p$) *agree with* M_1*, the corresponding* b_k *agree with* M_2, m_2*, respectively.*

Zagier (see, e.g., Alzer [26]) proved the next backward Cauchy inequality.

Theorem 12.3.29. *Let* $a_1 \ge a_2 \ge \cdots \ge a_n > 0$, $b_1 \ge b_2 \ge \cdots \ge b_n > 0$. *Then*

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \le C_2 \sum_{k=1}^{n} a_k b_k \tag{12.3.60}$$

where $C_2 = \max\{a_1 \sum_{k=1}^{n} b_k, b_1 \sum_{k=1}^{n} a_k\}$. *The equality in* (12.3.60) *holds if and only if* $a_1 = \cdots = a_n$ *and* $b_1 = \cdots = b_n$.

In 1995, Pecaric (see, e.g., [466]) gave the weighted form of (12.3.60).

Theorem 12.3.30. *Let* $\{a_k\}, \{b_k\}, \{u_k\}, \{v_k\}$ *be decreasing sequence,* $\omega_k > 0$. *Then we have*

$$\left(\sum_{k=1}^{n} \omega_k a_k u_k\right)\left(\sum_{k=1}^{n} \omega_k b_k v_k\right) \le C_2 \sum_{k=1}^{n} \omega_k a_k b_k \tag{12.3.61}$$

where $C_2 = \max\{u_1 \sum_{k=1}^{n} \omega_k v_k, v_1 \sum_{k=1}^{n} \omega_k u_k\}$.

The corresponding integral form is the following theorem.

Theorem 12.3.31. *Assume that* $f, g \ge 0$ *are monotonically decreasing. Then for any integrable functions* $F, G : [0, +\infty) \to [0, 1]$*, there holds*

$$(f, F)(g, G) \le C_2(f, g) \tag{12.3.62}$$

where $C_2 = \max\{\int_0^{+\infty} f(x)dx, \int_0^{+\infty} g(x)dx\}$ *and* $(f, g) = \int_0^{+\infty} f(x)g(x)dx$.

12.4 The Minkowski inequalities

In 1896, Minkowski (see, e.g., [466]) established the following famous inequality.

Theorem 12.4.1. *Let $a = \{a_1, \dots, a_n\}$ or $a = \{a_1, \dots, a_n, \dots\}$ be a real sequence or a complex sequence. Define*

$$\begin{cases} \|a\|_p = \left(\sum_k |a_k|^p\right)^{1/p} & \text{if } 1 \le p < +\infty; \\ \|a\|_\infty = \sup_k |a_k| & \text{if } p = +\infty. \end{cases}$$

Then for $1 \le p \le +\infty$, we have

$$\|a+b\|_p \le \|a\|_p + \|b\|_p. \tag{12.4.1}$$

If $0 \ne p < 1$, then

$$\|a+b\|_p \ge \|a\|_p + \|b\|_p \tag{12.4.2}$$

where when $p < 0$, we require that $a_k, b_k, a_k + b_k \ne 0$ $(k = 1, 2, \dots)$.

Moreover, when $p \ne 0, 1$, the equality in (12.4.1) holds if the sequences a and b are proportional. When $p = 1$, the equality in (12.4.2) holds if and only if $\arg a_k = \arg b_k$ for all $k = 1, 2, \dots$.

Remark 12.4.1. If we replace p by $1/p$ in (12.4.1), we can obtain the following assertions:

(1) if $1 \le p < +\infty$, then we have

$$\left\{\sum_k |a_k + b_k|^{1/p}\right\}^p \ge \left(\sum_k |a_k|^{1/p}\right)^p + \left(\sum_k |b_k|^{1/p}\right)^p; \tag{12.4.3}$$

(2) if $0 < p < 1$, then we have

$$\left\{\sum_k |a_k + b_k|^{1/p}\right\}^p \le \left(\sum_k |a_k|^{1/p}\right)^p + \left(\sum_k |b_k|^{1/p}\right)^p. \tag{12.4.4}$$

We also have the following generalization.

Theorem 12.4.2. *Let $a_j = (a_{j1}, \dots, a_{jk}, \dots)$, $1 \le j \le m$. Then we have*

$$\left\|\sum_{j=1}^m a_j\right\|_p \le \sum_{j=1}^m \|a_j\|_p, \ 1 \le p \le +\infty. \tag{12.4.5}$$

Particularly, if $1 \le p < +\infty$, then we have

$$\left\{\sum_k \sum_{j=1}^m |a_{jk}|^p\right\}^{1/p} \le \sum_{j=1}^m \left(\sum_k |a_{jk}|^p\right)^{1/p}. \tag{12.4.6}$$

Remark 12.4.2. The inequality (12.4.6) has the following weighted form: Let $p_j, q_k > 0, 1 < p < +\infty$. Then there holds

$$\left\{\sum_{k=1}^{+\infty} q_k \left(\sum_{j=1}^{+\infty} p_j |a_{jk}|\right)^p\right\}^{1/p} \leq \sum_{j=1}^{+\infty} p_j \left(\sum_{k=1}^{+\infty} q_k |a_{jk}|^p\right)^{1/p}. \tag{12.4.7}$$

In the applications, the following integral form of the Minkowski inequality is frequently used.

Theorem 12.4.3. *Let Ω be a smooth open set in $\mathbb{R}^n$ and $f, g \in L^p(\Omega)$ with $1 \leq p \leq +\infty$. Then we have*

$$f + g \in L^p(\Omega) \tag{12.4.8}$$

and

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}. \tag{12.4.9}$$

If $0 < p < 1$, then we have

$$\|f + g\|_{L^p(\Omega)} \geq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}. \tag{12.4.10}$$

If $p > 1$, the equality in (12.4.9) holds if and only if there exist constants C_1 and C_2 which are not all zero such that $C_1 f(x) = C_2 g(x)$ a.e. in Ω.

If $p = 1$, then the equality in (12.4.9) holds if and only if $\arg f(x) = \arg g(x)$ a.e. in Ω or there exists a non-negative measurable function h such that $fh = g$ a.e. in the set $A = \{x \in \Omega | f(x)g(x) \neq 0\}$.

In what follows, we shall introduce some generalizations and improvements of the Minkowski inequality (see, e.g., Kuang [466]).

Theorem 12.4.4 (The Minkowski Inequality of Product Type)**.** *Let $a_k, b_k \geq 0$ ($k = 1, 2, \ldots, n$). Then we have*

$$\left\{\prod_{k=1}^{n} (a_k + b_k)\right\}^{1/n} \geq \left(\prod_{k=1}^{n} a_k\right)^{1/n} + \left(\prod_{k=1}^{n} b_k\right)^{1/n}. \tag{12.4.11}$$

Theorem 12.4.5 (The Determinant Minkowski Inequality)**.**

(1) *Assume that A, B are $n \times n$ positively definite matrices. Then we have*

$$|A|^{1/n} + |B|^{1/n} \leq |A + B|^{1/n} \tag{12.4.12}$$

where $|A| = \det A$ stands for the determinant of A.

(2) *Assume that A_k ($k = 1, \ldots, m$) are $n \times n$ positively definite matrices. Then for any $\lambda_k > 0$,*

$$\sum_{k=1}^{m} \lambda_k |A_k|^{1/n} \leq \left|\sum_{k=1}^{m} \lambda_k A_k\right|^{1/n} \tag{12.4.13}$$

and the equality in (12.4.13) holds if and only if for any two matrices A_j, A_k, there is a constant $C_{jk} > 0$ such that $A_j = C_{jk} A_k$ or $A_k = C_{jk} A_j$.

Ke Hu (see, e.g., [466]) improved the Minkowski inequality in the following way.

Theorem 12.4.6 (The Hu Inequality). *Let $a_k, b_k \geq 0$, $p \geq 1$. Then we have*

$$\left\{\sum_k (a_k + b_k)^p\right\}^{1/p} \leq \left(\sum_k a_k^p\right)^{1/p} + \left(\sum_k b_k^p\right)^{1/p} - \frac{1}{2} g(p) R^2(a,b) \quad (12.4.14)$$

where $g(p) = 1/(2p)$ for $p \geq 2$; $g(p) = (p-1)/(2p)$ for $1 \leq p < 2$,

$$R(a,b) = \frac{\sum_k (a_k^p + b_k^p)[\sum_k (a_k + b_k)^p e_k] - [\sum_k (a_k^p + b_k^p) e_k] \sum_k (a_k + b_k)^p}{\{\sum_k (a_k + b_k)^p\}^{3/2}}$$

and $1 - e_k + e_j \geq 0$.

In 1997, Xiehua Sun (see, e.g., [466]) established the following generalized Minkowski inequality.

Theorem 12.4.7 (The Sun Inequality). *Assume that real numbers p_j satisfy*

$$\textstyle\sum_{j=1}^m 1/p_j = 1/r, \quad r > 0, \quad m \geq 2,$$
$$a_{jk} > 0, \quad j = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, n.$$

Then when one of $\{p_j\}$ is positive ($p_j > 0$, say), other $p_k < 0$ ($k \neq j$), there holds

$$\prod_{j=1}^m \|a_{jk}\|_{p_j} \leq \left\| \prod_{j=1}^m a_{jk} \right\|_r . \quad (12.4.15)$$

The equality in (12.4.15) *holds if and only if*

$$a_{i1}^{p_i}/a_{j1}^{p_j} = a_{i2}^{p_i}/a_{j2}^{p_j} = \cdots = a_{in}^{p_i}/a_{jn}^{p_j} \quad (i, j = 1, 2, \ldots, m). \quad (12.4.16)$$

In 1948, Toyama (see, e.g., [466]) gave a backward Minkowski inequality.

Theorem 12.4.8. *Assume that not all $a_{jk} \geq 0$ are zero, and $0 < r < s$. Then we have*

$$\frac{[\sum_{k=1}^n (\sum_{j=1}^m a_{jk}^s)^{r/s}]^{1/r}}{[\sum_{j=1}^m (\sum_{k=1}^n a_{jk}^r)^{s/r}]^{1/s}} \leq (\min\{m, n\})^{1/r - 1/s} . \quad (12.4.17)$$

The upper bound is optimal.

To end this section, we shall introduce some inequalities for the integral average.

Definition 12.4.1. Let $0 < \mu(E) < +\infty$, $1 \leq p < +\infty$. Set

$$M_p(f) = \left\{ \frac{1}{\mu(E)} \int_E |f(x)|^p d\mu \right\}^{1/p} .$$

We call $M_p(f)$ to be the mean value of f with respect to p.

Concerning $M_p(f)$, we have the following properties.

Theorem 12.4.9.

(1) (**The Hölder Inequality**) $M_1(fg) \le M_p(f)M_q(g)$ *with* $1/p+1/q=1$, $1<p<+\infty$;
(2) (**The Minkowski Inequality**) $M_p(f+g) \le M_p(f)+M_p(g)$ *with* $1 \le p < +\infty$;
(3) $\lim_{p\to+\infty} M_p(f) = \|f\|_\infty$;
(4) $M_p(f)$ *is increasing in* p: $1 \le p_1 \le p_2 \Longrightarrow M_{p_1}(f) \le M_{p_2}(f)$;
(5) $M_p(f)$ *is increasing with respect to* E, *i.e., if* $E_1 \subseteq E_2$, *then*

$$\left\{\frac{1}{\mu(E_1)}\int_{E_1}|f|^p d\mu\right\}^{1/p} \le \left\{\frac{1}{\mu(E_2)}\int_{E_2}|f|^p d\mu\right\}^{1/p}.$$

(6) $(M_p(f))^p$ *is logarithmically convex in* p; $M_p(f)$ *is logarithmically convex in* $1/p$ *since we have*

$$M_r(f) \le (M_p(f))^{1-\alpha}(M_q(f))^{\alpha}$$

where $0<\alpha<1$ *and* $1/r=(1-\alpha)/p+\alpha/q$.

In 1989, Guangrong You (see, e.g., [466]) established the following backward Minkowski inequality.

Theorem 12.4.10. *Let* $0 < m_1 \le a_k \le M_1$, $0 < m_2 \le b_k \le M_2$, $1 \le k \le n$, $1/p+1/q=1$. *Then*

(1) *if* $p>1$, *then we have*

$$\begin{aligned}\left(\frac{m_1+m_2}{M_1+M_2}\right)^{1/p}&\left\{\left(\frac{m_1}{M_1}\right)^{1/q}\|a\|_p+\left(\frac{m_2}{M_2}\right)^{1/q}\|b\|_p\right\}\\ &\le \|a+b\|_p \le \|a\|_p+\|b\|_p;\end{aligned} \tag{12.4.18}$$

(2) *if* $0<p<1$, *then*

$$\begin{aligned}\|a\|_p+\|b\|_p &\le \|a+b\|_p\\ &\le \left(\frac{M_1+M_2}{m_1+m_2}\right)^{1/p}\left\{\left(\frac{m_1}{M_1}\right)^{1/q}\|a\|_p+\left(\frac{m_2}{M_2}\right)^{1/q}\|b\|_p\right\}.\end{aligned} \tag{12.4.19}$$

The following result was obtained by Sanja in 1995 (see, e.g., Kuang [466]).

Theorem 12.4.11. *Let* $f, g_k : [a,b] \to \mathbb{R}$ *be non-negative increasing and* $g_k \in L^p[a,b]$, $k=1,\ldots,n$. *Then for* $1<p<+\infty$, *the Minkowski inequality holds*

$$\sum_{k=1}^{n}\left\{\int_a^b (g_k^p(x))'f(x)dx\right\}^{1/p} \le \left\{\int_a^b\left(\sum_{k=1}^{n} g_k^p(x)\right)' f(x)dx\right\}^{1/p}. \tag{12.4.20}$$

If f *is decreasing and* $g_k(a)=0$, $k=1,2,\ldots,n$, *then the converse inequality holds.*

12.5 The Jensen inequalities

In this subsection, we shall introduce the Jensen inequality and the generalized Jensen inequalities due to Steffensen [890] and Ciesielski [165]. Since these inequalities will involve the concept of a convex function on a line segment, we first give the definition of a convex function on a line segment.

Definition 12.5.1. A function f is called convex on a line segment $I \subseteq \mathbb{R}$ if and only if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{12.5.1}$$

holds for all $x, y \in I$ and all real numbers $\lambda \in [0,1]$. A convex function f on I is said to be strictly convex if the strict inequality holds in (12.5.1) for $x \neq y$. If $-f$ is convex on I, then f is said to be concave on I.

There are several equivalent definitions in the literature. We collect them here.

Definition 12.5.2. A function f is called convex on a line segment $I \subseteq \mathbb{R}$ if and only if (12.5.1) holds with $\lambda = 1/2$. A convex function f on I is said to be strictly convex if the strict inequality holds in (12.5.1) for $\lambda = 1/2$, $x \neq y$.

Definition 12.5.3. A function f is called convex on a line segment $I \subseteq \mathbb{R}$ if and only if for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$, we have

$$\big(f(x_2) - f(x_1)\big)/(x_2 - x_1) \le \big(f(x_3) - f(x_2)\big)/(x_3 - x_2). \tag{12.5.2}$$

A convex function f on I is said to be strictly convex if strict inequality holds in (12.5.2) for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$.

Definition 12.5.4. A function f is called convex on a line segment $I \subseteq \mathbb{R}$ if and only if for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_1)}{x_3 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}. \tag{12.5.3}$$

A convex function f on I is said to be strictly convex if the strict inequalities hold in (12.5.3) for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$.

Definition 12.5.5. A function f is said to be convex on I if and only if the set $A = \{(x,y) : f(x) \le y, x \in I\}$ is a convex set in $\mathbb{R}^2$.

Definition 12.5.6. A function f is said to be convex on I if and only if for all $x_0 \in I$, $\phi(x) = [f(x) - f(x_0)]/(x - x_0)$ is an increasing function on I.

If f has a higher regularity, we have the following results.

Theorem 12.5.1. *A differentiable function f is said to be convex on I if and only if there exists a countable set $A \subset I$ such that f' is increasing on $I - A$.*

Theorem 12.5.2. *If f is a differentiable function of second order, then f is said to be convex on I if and only if $f''(x) \geq 0$, for all $x \in I$.*

Remark 12.5.1. If $f''(x) > 0$, for all $x \in I$, then f is strictly convex on I. But the converse is not true.

In the sequel, we shall introduce some related concepts of convex functions.

Definition 12.5.7. If f is a positive function on I and $\ln f(x)$ is a convex function, then f is said to be logarithmical convex. In this case,

$$\ln f[\alpha x + (1-\alpha)y] \leq \alpha \ln f(x) + (1-\alpha)\ln f(y) = \ln[f(x)^\alpha f(y)^{1-\alpha}]$$

or

$$f(\alpha x + (1-\alpha)y) \leq f(x)^\alpha f(y)^{1-\alpha}$$

for any $x, y \in I$ and all $\alpha \in [0,1]$.

For logarithmical convex functions, we have such a result.

Theorem 12.5.3. *A positive function f is logarithmical convex on I if and only if for any real number a, $f(x)e^{ax}$ is convex.*

Definition 12.5.8. If f is a positive function on I and for any $x_1, x_2 \in I$, there holds

$$\left[f\left(\frac{x_1+x_2}{2}\right)\right]^2 \leq f(x_1)f(x_2), \tag{12.5.4}$$

then f is said to be weakly logarithmical convex.

For weakly logarithmical convex functions, we have the next theorem.

Theorem 12.5.4. *Let f, g be weakly logarithmical functions on I. Then*

$$\left[f\left(\frac{x_1+x_2}{2}\right)\right]^2 + \left[g\left(\frac{x_1+x_2}{2}\right)\right]^2 \leq [f(x_1)+g(x_1)][f(x_2)+g(x_2)]. \tag{12.5.5}$$

Note that when $x_1, x_2 > 0$, we have from (12.5.4)

$$f(A(x_1,x_2)) \leq G(f(x_1), f(x_2))$$

where

$$A(x_1,x_2) = (x_1+x_2)/2,\ G(x_1,x_2) = \sqrt{x_1x_2}. \tag{12.5.6}$$

Thus motivated by (12.5.6), we now introduce

$$M_p(x,y) = \left[\frac{1}{2}(x^p + y^p)\right]^{1/p}, p \neq 0;\ \ M_0(x,y) = \lim_{p\to 0^+} M_p(x,y) = \sqrt{xy}.$$

Definition 12.5.9. If $f(M_p(x,y)) \leq M_p(f(x), f(y))$, then f is said to be p-power convex on I.

Definition 12.5.10. Let f be a real function on I. If for any $x, y \in I, 0 \leq \lambda \leq 1$, there holds that

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$$

then f is said to be a quasilinear convex function on I.

For a quasilinear convex function, we have the following theorem.

Theorem 12.5.5. *A function f is a quasi-linear function on I if and only if for any $x, y \in I$, $\lambda \in [0,1]$, $z = \lambda x + (1-\lambda)y$ and if $f(x) \leq f(y)$, then $f(z) \leq f(y)$. Particularly, if for any $x, y \in I$, $0 < \lambda < 1$, $z = \lambda x + (1-\lambda)y$, $f(x) < f(y)$, we have $f(z) < f(y)$. Then f is said to be pseudo-convex on I.*

Among the above inequalities for convex functions, the Jensen inequality should be the famous one which has the discrete form and integral form. The following is the discrete form (see, e.g., Jensen [394]).

Theorem 12.5.6. *Let $\phi(u) : [\alpha, \beta] \longrightarrow \mathbb{R}$ be a convex function. Assume that $a_k \geq 0$ $(k = 1, 2, \ldots, n)$ are non-negative constants verifying $\sum_{i=1}^n a_k > 0$, then for any $x_1, x_2, \ldots, x_n \in [\alpha, \beta]$, we have*

$$\phi\left(\frac{\sum_{k=1}^n a_k x_k}{\sum_{k=1}^n a_k}\right) \leq \frac{\sum_{k=1}^n a_k \phi(x_k)}{\sum_{k=1}^n a_k}. \tag{12.5.7}$$

Steffensen [890] extended the above Jensen inequality for convex functions.

Theorem 12.5.7. *If f is a convex function and x_k never decreases, and if c_k $(k = 1, \ldots, n)$ satisfies the conditions*

$$\sum_{k=\nu}^n c_k \leq \sum_{k=1}^n c_k \quad (\nu = 1, \ldots, n) \quad \textit{with} \quad \sum_{k=1}^n c_k > 0, \tag{12.5.8}$$

then

$$f\left(\frac{\sum_{k=1}^n c_k x_k}{\sum_{k=1}^n c_k}\right) \leq \frac{\sum_{k=1}^n c_k f(x_k)}{\sum_{k=1}^n c_k}. \tag{12.5.9}$$

The above result evidently generalizes the Jensen inequality in Theorem 12.5.6 since here we do require that the c_k $(k = 1, 2, \ldots, n)$ are non-negative.

There are some refinements of the Jensen inequality which we shall collect in the following seven theorems (see, e.g., Kuang [466]).

Theorem 12.5.8. *Let $F : D \to \mathbb{R}, x_k \in D, 1 \leq k \leq n$. Set*

$$\begin{aligned} f_{k,n} &= \frac{1}{C_n^k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{1}{k}\sum_{j=1}^k x_{i_j}\right), \\ g_{k,n} &= \frac{1}{C_{n+k-1}^k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{1}{k}\sum_{j=1}^k x_{i_j}\right). \end{aligned} \tag{12.5.10}$$

Then

$$\begin{cases} f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) = f_{n,n} \leq \cdots \leq f_{(k+1),n} \leq f_{k,n} \\ \qquad\qquad \leq \cdots \leq f_{1,n} = \frac{1}{n}\sum_{k=1}^{n} f(x_k), & (12.5.11) \\ f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \leq \cdots \leq g_{(k+1),n} \leq g_{k,n} \\ \qquad\qquad \leq \cdots \leq \frac{1}{n}\sum_{k=1}^{n} f(x_k), \quad k = 1, \ldots, n. & (12.5.12) \end{cases}$$

Dragomir [215] proved the following theorem

Theorem 12.5.9. *Let X be a linear space, A a convex subset of $X, f : A \to \mathbb{R}$ a convex function, $x_k \in A$, $\omega_k \geq 0$, $q_k \geq 0$, $1 \leq k \leq n$. Set*

$$G_n = \sum_{k=1}^{n} \omega_k > 0, \quad Q_n = \sum_{k=1}^{n} q_k > 0. \tag{12.5.13}$$

Then we have

(1)
$$\begin{aligned} f\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k x_k\right) &\leq \frac{1}{Q_n^k}\sum_{i_1,\ldots,i_k=1}^{n} q_{i_1}\cdots q_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) \\ &\leq \frac{1}{Q_n^k}\sum_{i_1,\ldots,i_k=1}^{n} q_{i_1}\cdots q_{i_k} f\left(\frac{1}{G_k}\sum_{j=1}^{k} \omega_j x_{i_j}\right) \leq \frac{1}{Q_n}\sum_{k=1}^{n} q_k f(x_k); \end{aligned} \tag{12.5.14}$$

(2)
$$\begin{aligned} \frac{1}{Q_n^2}\sum_{k=1}^{n}\sum_{j=1}^{n} q_k q_j f\left(\frac{x_k + x_j}{2}\right) &\leq \frac{1}{Q_n^2}\sum_{k=1}^{n}\sum_{j=1}^{n} q_k q_j \int_0^1 f(tx_k + (1-t)x_j)dt \\ &\leq \frac{1}{Q_n}\sum_{k=1}^{n} q_k f(x_k); \end{aligned} \tag{12.5.15}$$

(3)
$$\begin{aligned} f\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k x_k\right) &\leq \frac{1}{Q_n}\sum_{k=1}^{n} q_k f\left(tx_k + (1-t)\frac{1}{Q_n}\sum_{k=1}^{n} q_k x_k\right) \\ &\leq \frac{1}{Q_n^2}\sum_{k=1}^{n}\sum_{j=1}^{n} q_k q_j f(tx_k + (1-t)x_j) \leq \frac{1}{Q_n}\sum_{k=1}^{n} q_k f(x_k), \quad t \in [0,1]. \end{aligned} \tag{12.5.16}$$

In 1980, Vasic (see, e.g., [466]) proved the following refinement of the Jensen inequality.

Theorem 12.5.10. *Under assumptions of the above theorem, let* $m \le x_k \le M$, $\bar{x} = \frac{1}{Q_n}\sum_{k=1}^n q_k f(x_k)$. *Then*

$$\frac{1}{Q_n}\sum_{k=1}^n q_k f(x_k) \le \frac{1}{M-m}\Big\{(M-\bar{x})f(m) + (\bar{x}-m)f(M)\Big\}. \tag{12.5.17}$$

If $\frac{f(x)}{x-m}$ *is increasing on* $(m, M]$, *then*

$$\frac{1}{Q_n}\sum_{k=1}^n q_k f(x_k) \le \frac{1}{M-m}(\bar{x}-m)f(M). \tag{12.5.18}$$

If $\frac{f(x)}{M-x}$ *is increasing on* $[m, M)$, *then*

$$\frac{1}{Q_n}\sum_{k=1}^n q_k f(x_k) \le \frac{1}{M-m}(M-\bar{x})f(m). \tag{12.5.19}$$

Theorem 12.5.11 (The Lah–Ribaric Inequality)**.** *Let* f *be convex on* $[a,b], q_k \ge 0, \sum_{k=1}^n q_k = 1, x_k \in [a,b]$, *then we have*

$$\sum_{k=1}^n q_k f(x_k) \le \frac{b - \sum q_k x_k}{b-a} f(a) + \frac{\sum q_k x_k - a}{b-a} f(b). \tag{12.5.20}$$

Theorem 12.5.12 (The Dragomir–Ionescu Inequality)**.** *Let* f *be a differentiable convex function on* $(a,b), p_k \ge 0, x_k \in (a,b)$. *Then we have*

$$0 \le \sum_{k=1}^n p_k f(x_k) - f\left(\sum_{k=1}^n p_k x_k\right) \le \sum_{k=1}^n p_k x_k f'(x_k) - \left[\sum_{k=1}^n p_k x_k\right]\left[\sum_{k=1}^n p_k f'(x_k)\right]. \tag{12.5.21}$$

In 1985, Bingan Wang (see, e.g., [466]) gave another refinement of Theorem 12.5.6.

Theorem 12.5.13. *Let* ϕ *be a convex function on* D. *Then for any* $x_k \in D, p_k > 0, 1 \le k \le n$, *there holds*

$$\begin{aligned}\phi\left(\sum_{k=1}^n p_k x_k \Big/ \sum_{k=1}^n p_k\right) &\le \left(\sum_{j=1}^k p_j \Big/ \sum_{j=1}^n p_j\right)\phi\left(\sum_{j=1}^k p_j x_j \Big/ \sum_{j=1}^k p_j\right) \\ &\quad + \left(\sum_{j=k+1}^n p_j \Big/ \sum_{j=1}^n p_j\right)\phi\left(\sum_{j=k+1}^n p_j x_j \Big/ \sum_{j=k+1}^n p_j\right) \\ &\le \sum_{j=1}^n p_j f(x_j) \Big/ \sum_{j=1}^n p_j \end{aligned} \tag{12.5.22}$$

and the equalities in (12.5.22) *hold if and only if* $x_1 = \cdots = x_n$.

The Hadamard inequality can be also considered as a refinement of the Jensen inequality.

Theorem 12.5.14. *Let ϕ be a convex function on $[a,b]$. Then for any $x_1, x_2 \in [a,b]$, we have*

$$\phi\left(\frac{x_1+x_2}{2}\right) \le \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\phi(x)dx \le \frac{1}{2}[\phi(x_1)+\phi(x_2)]. \tag{12.5.23}$$

The equality in (12.5.23) *holds if and only if ϕ is a linear function.*

In 2000, Brnetic et al. also established the following inequality (see, e.g., Kuang [466]).

Theorem 12.5.15. *Assume that $f : [0,1] \to \mathbb{R}$ be a convex. Let*

$$h(t) = \frac{1}{n}\sum_{k=1}^{n} f[(1-t)x_k + tx_{k+1}].$$

Then $h(t)$ is a convex function on $[0,1]$ satisfying

$$f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \le h(t) \le \frac{1}{n}\sum_{k=1}^{n} f(x_k). \tag{12.5.24}$$

Similarly to the Hölder inequality, there is the backward Jensen inequality which was obtained by Slater in 1980.

Theorem 12.5.16 (The Slater Inequality)**.** *Let $\phi(t)$ be a convex and increasing function on (a,b). Then for any $x_k \in (a,b), p_k \ge 0, 1 \le k \le n, \sum_{k=1}^{n} p_k > 0, \sum_{k=1}^{n} p_k\phi'_+(x_k) > 0$, there holds that*

$$\frac{\sum_{k=1}^{n} p_k\phi(x_k)}{\sum_{k=1}^{n} p_k} \le \phi\left(\frac{\sum_{k=1}^{n} p_k\phi'_+(x_k)x_k}{\sum_{k=1}^{n} p_k\phi'_+(x_k)}\right) \tag{12.5.25}$$

where $\phi'_+(x_k)$ is the right derivative of ϕ at x_k.

The following is the integral form of Jensen's inequality.

Theorem 12.5.17. *Let $\phi(u) : [\alpha,\beta] \to \mathbb{R}$ be a convex function. Assume that $f : t \in [a,b] \to [\alpha,\beta]$, and $p(t)$ are continuous functions with $p(t) \ge 0, p(t) \not\equiv 0$. Then we have*

$$\phi\left(\frac{\int_a^b f(t)p(t)dt}{\int_a^b p(t)dt}\right) \le \frac{\int_a^b \phi(f(t))p(t)dt}{\int_a^b p(t)dt}. \tag{12.5.26}$$

A corresponding integral form of the Jensen inequality was also obtained (see, e.g., Steffensen [891]).

Theorem 12.5.18. *If f is a convex function and g never increases and h satisfies the conditions*

$$0 \le \int_0^\theta h(x)dx \le \int_0^1 h(x)dx \quad \textit{with } 0 \le \theta \le 1 \quad \textit{and} \quad \int_0^1 h(x)dx > 0, \tag{12.5.27}$$

then

$$f\left(\frac{\int_0^1 h(x)g(x)dx}{\int_0^1 h(x)dx}\right) \le \frac{\int_0^1 h(x)f(g(x))dx}{\int_0^1 h(x)dx}. \tag{12.5.28}$$

In the sequel, we would like to mention the results related to the above results of Steffensen's inequalities, which are due to Ciesielski [165]. The following is its discrete form.

Theorem 12.5.19. *Let $\{p_i\}$ be a sequence of real numbers such that*

$$\sum_{i=1}^{k} p_i \ge 0 \quad \textit{for } k = 1, 2, \ldots, n \quad \textit{and} \quad \sum_{i=1}^{n} |p_i| > 0. \tag{12.5.29}$$

Let $x_i \in [0, a]$ (where $a > 0$ is a constant) for $i = 1, \ldots, n$ and let $x_1 \ge x_2 \ge \cdots \ge x_n$. Furthermore, let f and f' be convex functions in $[0, a]$ and $f(0) \le 0$. Then we have

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} |p_i|}\right) \le \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} |p_i|}. \tag{12.5.30}$$

The corresponding integral form was also obtained (see, e.g., Ciesielski [165]) as follows.

Theorem 12.5.20. *Let the function g be non-increasing in $[\alpha, \beta]$ and let $a \ge g(t) \ge 0$ in $[\alpha, \beta]$. Let f and f' be convex in $[0, a]$ and let $f(0) \le 0$. Furthermore, let $p(t)$ be a function integrable in the Lebesgue sense in $[\alpha, \beta]$ such that*

$$\int_\alpha^x p(t)dt \ge 0 \quad \textit{for } x \in [\alpha, \beta] \quad \textit{and} \quad \int_\alpha^\beta |p(t)|dt > 0. \tag{12.5.31}$$

Then we have

$$f\left(\frac{\int_\alpha^\beta p(t)g(t)dt}{\int_\alpha^\beta |p(t)|dt}\right) \le \frac{\int_\alpha^\beta p(t)f(g(t))dt}{\int_\alpha^\beta |p(t)|dt}. \tag{12.5.32}$$

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Index

asymptotic behavior
differential inequalities leading to $\sim$, 106
for a linear thermomagnetoelastic system, 377
for the 2D homogeneous incompressible Navier–Stokes equations, 257
of solutions to hyperbolic equations, 307
attractors
for differential equations with delay, 213
for evolutionary differential equations, 209
global $\sim$ for the Cahn–Hilliard equation in H^2 and H^3, 215
maximal $\sim$ for nonlinear reaction-diffusion equations, 209
uniform $\sim$, 225
uniform $\sim$ for a non-autonomous wave equation, 218

blow-up of solutions
for nonlinear wave equations, 393
to abstract equations and thermoelastic equations, 429
to abstract nonlinear equations, 429, 434
to formally parabolic equations, 441, 448
to initial boundary value problems, 452, 458
to nonlinear hyperbolic equations, 410
to nonlinear hyperbolic equations and hyperbolic-elliptic inequalities, 393
to nonlinear wave equations with damping, 415
to semilinear wave equations, 403
to the quasilinear hyperbolic-elliptic inequalities, 426
to wave equations with a nonlinear dissipation, 417

convex functions
logarithmical $\sim$, 491
quasi-linear $\sim$, 492
weakly logarithmical $\sim$, 491

decay
energy $\sim$ for a dissipative anisotropic elastic system, 335
energy $\sim$ rates of nonlinear dissipative hyperbolic systems, 348
estimates for flows in a semi-infinite straight channel, 291
exponential $\sim$, 294
polynomial $\sim$ rate for nonlinear wave equations, 325
rate estimates for dissipative wave equations, 330
decay rates
differential and difference inequalities leading to $\sim$, 130

equations
Cahn–Hilliard $\sim$, 215

Y. Qin, *Analytic Inequalities and Their Applications in PDEs*, Operator Theory: Advances and Applications 241, DOI 10.1007/978-3-319-00831-8

Navier–Stokes ∼, 257, 262, 281
nonlinear diffusion ∼, 296
porous medium ∼, 296
thermomagnetoelastic ∼, 357
thermoviscoelastic ∼, 357
thermoviscoelastoplastic ∼, 357

estimates
decay ∼ for flows in a semi-infinite straight channel, 291
decay rate ∼ for dissipative wave equations, 330
dissipative ∼ for PDEs, 250
on approximated solutions for 1D nonlinear wave equations, 307

exact rates of convergence for nonlinear PDEs, 296

fluid equation
1D viscous compressible barotropic ∼, 268
symmetrically quasilinear viscous barotropic ∼, 273

global existence
and asymptotic behavior for equations of fluid Dynamics, 257
and uniqueness for abstract evolutionary differential equations, 245
and uniqueness for evolutionary PDEs, 231
for a quasilinear parabolic system, 240

inequalities
basic ∼, 467
differential ∼
leading to asymptotic behavior, 106
leading to uniform bounds, 79
differential and difference ∼, 79
leading to decay rates, 130
integral ∼ leading to upper bounds and decay rates, 51
some differential ∼ for non-existence of global solutions, 169

inequality
Acezel–Beckenbach ∼, 480
Agmon–Nirenberg ∼, 102, 139, 186–188
Alabau ∼, 62, 71, 75
Alabau–Cannarsa–Komornik ∼, 63
Ammari–Tucsnak ∼, 57
Babin–Vishik ∼, 82
Bae–Jin ∼, 43, 51
Bandle–Levine ∼, 172
Basov ∼, 115
Belleri–Pata ∼, 86
Bellman ∼, 3
Bellman–Gronwall ∼
classical, 2
generalized, 8
uniform, 81
Brandolese ∼, 38
Brézis–Pata–Prouse–Vishik ∼, 89
Caraballo–Rubin–Valero ∼, 78
Chen ∼, 95
Chepyzhov–Pata–Vishik ∼, 83, 85
Chi-Cheng Poon ∼, 16
Chu–Metcalf ∼, 14
Cooper ∼, 470
Daykin–Eliezer ∼, 480
Dickstein–Loayza ∼, 18
Dlotko ∼, 92
Dragomir ∼, 475
Dragomir–Ionescu ∼, 494
Ducomet–Zlotnik ∼, 111
Gagliardo–Nirenberg interpolation ∼, 314
Galdi ∼, 92, 137
Gatti–Pata–Zelik ∼, 98, 100
Gerber ∼, 468
Ghidaglia ∼, 90
Giga–Kohn ∼, 95

Glassey ∼, 175, 192
Glassey–Qin ∼, 173
Gollwitzer ∼, 5, 7
Greene ∼, 17
Gronwall ∼, 1
Hadamard ∼, 495
Haraux–Lagnese ∼, 58
Hausdorff–Young ∼, 471, 472
Henry ∼, 20–22, 24
with multiple integrals, 48
Henry–Gronwall–Bihari type integral ∼, 32
Hirzallah–Omar ∼, 468
Hölder ∼, 474
Hu ∼, 478, 488
Jensen ∼, 475, 490
Jones ∼, 4
Jones ∼, generalized, 5
Kaplan–Quittner–Souplet ∼, 172
Kato ∼, 177
Kawashima–Nakao–Ono ∼, 52, 151
Kim ∼, 136
Knops–Levine–Payne ∼, 191, 192
Knops–Payne ∼, 194
Komornik ∼, 59
Krejčí–Sprekels ∼, 116, 117
Krejčí–Sprekels–Zheng ∼, 121
Ladyzhenskaya–Solonnikov–Ural'ceva ∼, 56, 57, 81, 82, 190
Lah–Ribaric ∼, 494
Levine ∼, 169–171
Li–Zhou ∼, 201, 202
Martinez ∼, 64, 66, 68–70
Medveď ∼, 39, 47, 48
Minkowski ∼, 478, 486
backward, 488
determinant, 487
of product type, 487
Muñoz Rivera–Racke ∼, 135
Nagasawa ∼, 84
Nagumo ∼, 23
Nakao ∼, 139–141, 144, 146, 149–151, 154
Norbury–Stuart ∼, 15
Ohara ∼, 93
Oppenheim ∼, 470
Ou-Yang type integral ∼, 44
Pachapatte type integral ∼, 44
Pachpatte ∼, 6, 9, 10
Pata–Zelik ∼, 87, 88
Qin–Ren ∼, 85, 133
Qin–Ren–Wei ∼, 130
Quittner–Souplet ∼, 180, 181, 183, 184
Rivera ∼, 132, 134
Shen–Zheng ∼, 109
Slater ∼, 495
Străskraba–Zlotnik ∼, 111, 114
Sun ∼, 488
Todorova–Yordanov ∼, 178
Vaigant ∼, 53
Yang–Jin ∼, 96
Yau ∼, 481
Young ∼, 467, 468, 473
generalized, 473
inverse, 469
weak, 474
Zagier ∼, 483
Zheng ∼, 119
Zlotnik ∼, 91, 112
initial value problem for abstract evolutionary equations, 245

large-time behavior for
nonhomogeneous incompressible Navier–Stokes equations, 262
parabolic equations, 300
semilinear parabolic equations, 300
thermoviscoelastic systems, 357
local well-posedness, 246

solutions
equivalent ∼, 246
for semilinear heat equations, 237
of differential equations in a Hilbert space, 247
to convection equations with anomalous diffusion, 234
stabilization for the 1D compressible Navier–Stokes equations, 281

thermoelastoplastic constitutive laws, 369
thermomagnetoelastic equations, 357
thermoviscoelastic equations, 357
thermoviscoelastoplastic equations, 357
thermoviscoelastoplastic system with hysteresis, 369

wave equations
blow-up of solutions
for nonlinear ∼, 393
to ∼ with a nonlinear dissipation, 417
to nonlinear ∼ with damping, 415
to semilinear ∼, 403
decay rate estimates for dissipative ∼, 330
estimates on approximated solutions for 1D nonlinear ∼, 307
estimates on approximated solutions of ∼, 312
polynomial decay rate for nonlinear ∼, 325
stabilization of weakly coupled ∼, 344
uniform attractors for non-autonomous ∼, 218
weakly coupled parabolic system, 231