William E. Langlois
Michel O. Deville

Slow Viscous Flow

Second Edition

Springer

Slow Viscous Flow

William E. Langlois • Michel O. Deville

Slow Viscous Flow

Second edition

Springer

William E. Langlois
San Jose City College
Division of Science and Mathematics
San Jose, CA, USA

Michel O. Deville
Ecole Polytechnique Fédérale de Lausanne
STI DO
MXD 130
Station 12
Lausanne, Switzerland

ISBN 978-3-319-03834-6 ISBN 978-3-319-03835-3 (eBook)
DOI 10.1007/978-3-319-03835-3
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014933672

Mathematics Subject Classification (2010): 76A02, 76M10, 76D05, 76D08

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To our wives Cecile and Christina and our families

. . . above the altar was carved the statue of Madame St. Mary, and this minstrel did come before this image right humbly.

"Sweet Lady,"said he,"scorn not the thing I know, for with the help of God I will essay to serve you in good faith, even as I may. I cannot read your hours nor chant your praise, but at the least I can set before you what art I have."

Then commenced this minstrel his merry play, leaping low and small, tall and high, over and under. Then he knelt upon his knees before the statue, and meekly bowed his head.

"Most gracious Queen" said he, "of your pity and charity scorn not this my service" Again he leaped and played, and for holiday and festival, made the somersault of Metz. Afterwards he did the Spanish vault, springing and bounding, then the vaults they love in Britanny, and all of these feats he did as best as he was able. Then he walked on his two hands, with his feet in the air, and his head near the ground. Thus long did this minstrel leap and play, till at last, nigh fainting with weariness, he could stand no longer on his feet, but fell to his knees.

"Lady," said he,"I worship you with heart, with body, feet and hands, for this I can neither add to nor take away. Now I am your minstrel."

Then he smote upon his breast, he sighed and wept, since he knew no better prayer than tears, nor no better worship than his art . . .

From a medieval French legend.

Preface to the Second Edition

The scientific tripod is based on the three subjects (in alphabetical order to avoid a value judgment): computation, experiment, and theory. It is obvious at the very beginning of this millennium that the analytical and theoretical approaches have lost their impact and influence on research developments. The exponential growth of computer power has made simulations and visualization tools the effective and relevant devices to elaborate new models, new designs, and new technologies.

However, the domain covered by the fluid mechanics discipline is quite large and turbulence in flows is still a challenging problem in classical physics. Therefore the need for theoretical analyses of simple situations is still required in order to build up a line of reasoning helping the scientist to understand the basic phenomena. In the realm of reductionism that decomposes complicated matters into simpler problems, the simple solutions form the background of more complex cases with our intellectual abilities and educational biases to practice linear superposition. As soon as we tackle nonlinear physics we have to rely on the toolbox provided with those instruments in such a way that we can "think out of the box." Even though the Chinese Wise Man wrote "A picture is better than a thousand words" an image does not yield any explanation nor insight views. Leonardo wrote "Mechanics is the paradise of the mathematical sciences, because by means of it one comes to the fruits of mathematics"; replace "Mechanics" by "Fluid mechanics" and here we are.

The author of the first edition, although still teaching, has been away from research for 20 years. Therefore, to bring the book up to date, a second author who is more current has been added.

Half a century later, the preface to the first edition seems prophetic. Aspects of hydrodynamics once considered offbeat have indeed become important. The authors, for example, have worked on problems where variations in viscosity and surface tension cannot be ignored. Also, the advent of nanotechnology has expanded interest in the hydrodynamics of thin films, and hydromagnetic effects and radiative heat transfer are now routinely encountered in materials processing.

This monograph addresses the basic principles of fluid mechanics and solves fluid flow problems where viscous effects are the dominant physical phenomena.

Readers who are interested in aerodynamics and turbulence applications are invited to refer to the large body of literature that cover those fields.

M.O.D. thanks the Presses Polytechniques et Universitaires Romandes for giving permission of borrowing material from the book on finite elements: Azaïez et al. (2011).

Changes in the New Edition

Many misprints and some outright errors in the first edition have been corrected here. Topics new to the second edition include the second principle of thermodynamics, Boussinesq approximation, time dependent flows, Marangoni convection, Kovasznay flow, plane periodic solutions, Hele-Shaw cells, Stokeslets, rotlets, finite element methods, Wannier flow, corner eddies, and analysis of the Stokes operator. In keeping with the spirit of the first edition, we seek to supplement the existing literature, not to compete with it. Since 1964, for example, there has been an enormous literature about slow flow past obstacles. We discuss it but make no attempt to replicate it.

The bibliography is no longer presented at the end of each chapter. References are collected at the end of the monograph.

San Jose, USA W.E. Langlois
Lausanne, Switzerland Michel O. Deville
February 2014

Preface to the First Edition

A book entitled *Slow Viscous Flow* could be anything from a treatise on weakly elliptic systems of partial differential equations to a plumbers manual. To locate the present book on the spectrum, regard it as a monograph on theoretical hydrodynamics, written in the language of applied mathematics.

Certain key applications of hydrodynamics involve high-speed flow—a subject which raises many interesting questions. Hence the typical modern treatise on viscous flow deals primarily with boundary layer theory, heat transfer, turbulence. Its section on slow flow is brief, for neither the author nor the intended reader is especially interested. Unfortunately the same dispatch is given to the *fundamenlals* of hydrodynamics. The author knows from experience which subtle points must be stressed and which can be glossed—in the usual applications. As technology expands, however, aspects of hydrodynamics once considered off-beat become increasingly important.

This book is intended to supplement the existing literature, not to compete with it. When continuity requires that a topic be included even though it is adequately treated elsewhere, a different viewpoint is taken here. Consequently this volume does not offer a complete course in hydrodynamics, only the first trimester. For the theory of high-speed flow—a vital subject—and for experimental and historical aspects, the student must look elsewhere.

Background Requirements

The reader is assumed familiar with elementary differential equations and with the manipulation of multiple integrals; the required physics background includes elementary mechanics and the rudiments of thermodynamics. More extensive preparation makes for easier going, but is not essential.

One can go quite deeply into hydrodynamics without encountering anything more involved than sines and cosines. The subject offers mathematical content free of mathematical complexity—a feature which our esoteric brethren found so appealing in topology and algebra when those subjects were young. Complexity there is, especially in applied hydrodynamics, but this, a basic monograph, usually avoids it. In a few instances involved calculations are included, for they illustrate important points.

Two short sections (Chap. 5, Sect. 5.4; Chap. 7, Sects. 7.1–7.4) assume that the reader has been exposed to complex function theory. They can be skipped without loss of continuity.

The reader versed in Cartesian tensor analysis can skip Chap. 1. For the reader who has mastered general tensors, Chap. 3 need only be scanned to ensure that the notation is compatible with his own.

Bibliographical Scheme

The nine chapters could be expanded to nine volumes. However most readers want to get on with it, and so do I. If a chapter evokes special interest, the bibliography at its end indicates appropriate collateral reading; a review comment accompanies each entry. References pertaining to specific points of analysis are given in footnotes.

San Jose, USA
January 1964

W.E. Langlois

Contents

Chapter 1
Cartesian Tensors

Abstract The notation and methodology of Cartesian tensor analysis is introduced. The focus is on that part of the subject which is useful in developing the equations of hydrodynamics.

Tensor notation will be used quite freely throughout this monograph. Since many of the quantities encountered in viscous flow theory are tensorial in character, avoiding tensor notation would introduce needless complexity.

In this first chapter we shall introduce the subject of **Cartesian tensor analysis**, which, as we shall see in the next chapter, is a useful tool for the formulation of the equations governing viscous flow. Later on we shall be concerned with obtaining solutions to these equations—a process often simplified by the use of curvilinear coordinate systems. Therefore Chap. 3 will cover a bit of **general tensor analysis**, which allows us to pass easily from one coordinate system to another.

In a sense it is superfluous to treat Cartesian coordinates as a special case. Excellent articles and monographs have been written on various aspects of continuum mechanics with general tensor notation used throughout. However those readers who are unfamiliar with tensor analysis may find it somewhat easier to learn the subject in two steps rather than one. Many monographs and textbooks cover tensor analysis. With no intent of being exhaustive we refer the reader to Aris (1962), Jeffreys (1931), Rivlin (1957), and Spencer (2004).

1.1 The Classical Notation

Cartesian tensor analysis deals with the problems involved in passing from one rectangular Cartesian coordinate system to another. Such problems can be resolved by straightforward algebraic methods. However with the tensor approach the amount of manipulation is considerably reduced by using an apparently elaborate, but actually quite simple, suffix notation.

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_1,

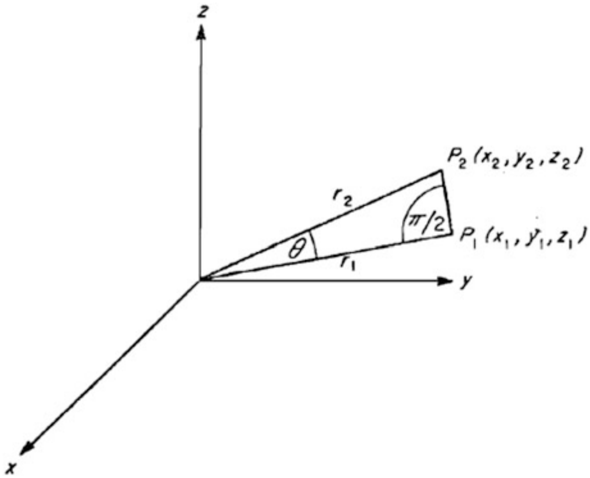


Fig. 1.1 Cartesian coordinate system with two arbitrary points P_1 and P_2

In order to set the stage for the suffix notation, let us discuss briefly the more traditional component notation. Consider in $x-y-z$ space a directed straight line which has direction cosines l, m, n. Any parallel line directed in the same sense also has direction cosines l, m, n. Even though we have the **normalization relation**

$$l^2 + m^2 + n^2 = 1\ , \tag{1.1}$$

two direction cosines of a line do not determine the third: there is an indeterminacy in sign associated with the two possible senses.

If two lines meet, the angle θ between them can be expressed in terms of l_1, m_1, n_1 and l_2, m_2, n_2, the direction cosines of the two lines respectively. With the construction in Fig. 1.1,

$$\begin{aligned} \overline{P_1P_2}^2 &= (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2; \\ x_1 &= r_1l_1, \quad y_1 = r_1m_1, \quad z_1 = r_1n_1; \\ x_2 &= r_2l_2, \quad y_2 = r_2m_2, \quad z_2 = r_2n_2\ . \end{aligned} \tag{1.2}$$

Hence

$$\begin{aligned} \overline{P_1P_2}^2 &= (r_2l_2 - r_1l_1)^2 + (r_2m_2 - r_1m_1)^2 + (r_2n_2 - r_1n_1)^2 \\ &= (l_1^2 + m_1^2 + n_1^2)r_1^2 + (l_2^2 + m_2^2 + n_2^2)r_2^2 \\ &\quad -2(l_1l_2 + m_1m_2 + n_1n_2)r_1r_2 \\ &= r_1^2 + r_2^2 - 2(l_1l_2 + m_1m_2 + n_1n_2)r_1r_2\ . \end{aligned} \tag{1.3}$$

However

$$r_1 = r_2\cos\theta, \quad \overline{P_1P_2} = r_2\sin\theta\ . \tag{1.4}$$

Substituting (1.4) into (1.3) and dividing by r_2^2 gives us the relation

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2 \,. \tag{1.5}$$

When the lines are at right angles, this reduces to the **orthogonality condition**

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \,. \tag{1.6}$$

A point $P(x, y, z)$ will have different coordinates in another rectangular Cartesian system $x'-y'-z'$ having the same origin and coordinate length scale as the $x-y-z$ system. Suppose that the direction cosines of the primed axes with respect to the unprimed axes are as follows:

$$x')l_1, m_1, n_1; \quad y')l_2, m_2, n_2; \quad z')l_3, m_3, n_3 \,. \tag{1.7}$$

The nine direction cosines so defined must satisfy the three normalization relations

$$\begin{aligned} l_1^2 + m_1^2 + n_1^2 &= 1 \,, \\ l_2^2 + m_2^2 + n_2^2 &= 1 \,, \\ l_3^2 + m_3^2 + n_3^2 &= 1 \,. \end{aligned} \tag{1.8}$$

Since the primed axes are mutually perpendicular, the direction cosines also satisfy the orthogonality conditions[1]

$$\begin{aligned} l_2 l_3 + m_2 m_3 + n_2 n_3 &= 0 \,, \\ l_3 l_1 + m_3 m_1 + n_3 n_1 &= 0 \,, \\ l_1 l_2 + m_1 m_2 + n_1 n_2 &= 0 \,. \end{aligned} \tag{1.9}$$

Despite the six relationships provided by (1.8) and (1.9), it is necessary, because of indeterminacies in sign, to specify six direction cosines in order to determine the orientation of a coordinate system. Three direction cosines are needed to specify the x' coordinate direction; when these have been selected, two degrees of freedom remain for the y' direction which must be normal to the x'-axis; when the five direction cosines defining the x' and y' directions have been designated, the locus

[1] The orthogonality conditions hold even if the $x-y-z$ system and the $x'-y'-z'$ system have different coordinate length scales, i.e., if the transformation of coordinates involves a stretching or shrinking as well as a rotation. For transformations of this sort, the normalization relations do not apply.

Coordinate transformations with a change of scale fall within the framework of Chap. 3. For the present we shall continue to assume that the change of coordinates involves only a rotation, and possibly a reflexion, so that the normalization relations (1.8) and the orthogonality conditions (1.9) both apply.

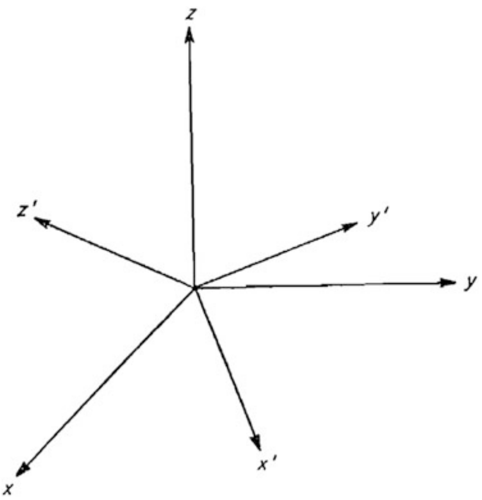


Fig. 1.2 Coordinate transformations

of the z'-axis is completely determined, but its sense is not. Thus a sixth direction cosine is needed to specify whether the $x' - y' - z'$ system is right-handed or left-handed (Fig. 1.2).

The direction cosines of the unprimed axes with respect to the primed axes are given by

$$x)l_1, l_2, l_3; \quad y)m_1, m_2, m_3; \quad z)n_1, n_2, n_3 \, . \tag{1.10}$$

Consequently the direction cosines satisfy relationships which can be considered the duals of (1.8) and (1.9):

$$\begin{aligned} l_1^2 + l_2^2 + l_3^2 &= 1 \, , \\ m_1^2 + m_2^2 + m_3^2 &= 1 \, , \\ n_1^2 + n_2^2 + n_3^2 &= 1 \, ; \end{aligned} \tag{1.11}$$

$$\begin{aligned} m_1 n_1 + m_2 n_2 + m_3 n_3 &= 0 \, , \\ n_1 l_1 + n_2 l_2 + n_3 l_3 &= 0 \, , \\ l_1 m_1 + l_2 m_2 + l_3 m_3 &= 0 \, . \end{aligned} \tag{1.12}$$

The relations (1.11) and (1.12) are equivalent to (1.8) and (1.9), but a direct algebraic establishment is somewhat awkward. Consequently we shall defer the proof until we have tensor notation at our disposal.

Consider a point $P = (x, y, z) = (x', y', z')$. Let $\overline{OP} = r$; let the line $\overrightarrow{OP}$ have direction cosines l, m, n in the $x - y - z$ system and l', m', n' in the $x' - y' - z'$ system. We have

$$x = rl, \quad y = rm, \quad z = rn\ ; \tag{1.13}$$

$$x' = rl', \quad y' = rm', \quad z' = rn'\ . \tag{1.14}$$

Also, with Eq. (1.5),

$$\begin{aligned} l &= l'l_1 + m'l_2 + n'l_3\ , \\ m &= l'm_1 + m'm_2 + n'm_3\ , \\ n &= l'n_1 + m'n_2 + n'n_3\ , \end{aligned} \tag{1.15}$$

Substituting (1.15) into (1.13) and then using (1.14), we obtain

$$\begin{aligned} x &= l_1x' + l_2y' + l_3z'\ , \\ y &= m_1x' + m_2y' + m_3z'\ , \\ z &= n_1x' + n_2y' + n_3z'\ . \end{aligned} \tag{1.16}$$

The dual of (1.16) is easily obtained. We can solve (1.16) for x', y', z' in terms of x, y, z, or we can repeat the procedure which led to (1.16), this time using the obvious dual of (1.15). Either way we obtain

$$\begin{aligned} x' &= l_1x + m_1y + n_1z\ , \\ y' &= l_2x + m_2y + n_2z\ , \\ z' &= l_3x + m_3y + n_3z\ . \end{aligned} \tag{1.17}$$

The coordinates of P in a system $x'' - y'' - z''$ oriented in the same manner as the $x' - y' - z'$ system, but with origin at (a, b, c) in the $x' - y' - z'$ system, are given by

$$\begin{aligned} x'' &= x' - a\ , \\ y'' &= y' - b\ , \\ z'' &= z' - c\ . \end{aligned} \tag{1.18}$$

With (1.17),

$$\begin{aligned} x'' &= l_1x + m_1y + n_1z - a\ , \\ y'' &= l_2x + m_2y + n_2z - b\ , \\ z'' &= l_3x + m_3y + n_3z - c\ . \end{aligned} \tag{1.19}$$

Equivalently,

$$\begin{aligned} x &= l_1(x'' + a) + l_2(y'' + b) + l_3(z'' + c) \,, \\ y &= m_1(x'' + a) + m_2(y'' + b) + m_3(z'' + c) \,; \\ z &= n_1(x'' + a) + n_2(y'' + b) + n_3(z'' + c) \,. \end{aligned} \tag{1.20}$$

1.2 Suffix Notation

The results obtained so far can be written much more succinctly if we set

$$\begin{aligned} x = x_1, \quad & y = x_2, \quad z = x_3 \,, \\ x' = x'_1, \quad & y' = x'_2, \quad z' = x'_3 \,, \\ x'' = x''_1, \quad & y'' = x''_2, \quad z'' = x''_3 \,, \end{aligned} \tag{1.21}$$

and denote the direction cosines of the primed axes with respect to the unprimed axes by

$$x'_1) a_{11}, a_{12}, a_{13}; \quad x'_2) a_{21}, a_{22}, a_{23}; \quad x'_3) a_{31}, a_{32}, a_{33} \,. \tag{1.22}$$

Equations (1.16) then become

$$\begin{aligned} x_1 &= a_{11}x'_1 + a_{21}x'_2 + a_{31}x'_3 \,, \\ x_2 &= a_{12}x'_1 + a_{22}x'_2 + a_{32}x'_3 \,, \\ x_3 &= a_{13}x'_1 + a_{23}x'_2 + a_{33}x'_3 \,. \end{aligned} \tag{1.23}$$

This system of three equations can be written on a single line:

$$x_i = a_{1i}x'_1 + a_{2i}x'_2 + a_{3i}x'_3 \quad (i = 1, 2, 3) \,. \tag{1.24}$$

More succinctly yet

$$x_i = \sum_{j=1}^{3} a_{ji}x'_j \quad (i = 1, 2, 3) \,. \tag{1.25}$$

In a similar manner (1.17) can be written

$$x'_i = \sum_{j=1}^{3} a_{ij}x_j \quad (i = 1, 2, 3) \,. \tag{1.26}$$

Thus there begins to emerge one of the advantages of suffix notation: economy of space.

Quite generally we shall adopt the convention that any symbol to which a subscript is attached represents the set of three quantities obtained by giving the subscript the values $1, 2, 3$ unless some other meaning is explicitly specified. Thus u_k represents u_1, u_2, u_3. Further, any symbol or product of symbols to which *n different* subscripts are attached denotes the set of 3^n quantities obtained by giving these subscripts the values $1, 2, 3$. Thus, t_{rs} represents the set of nine quantities $t_{11}, t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}$; $u_i v_j$ represents the nine quantities $u_1v_1, u_1v_2, u_1v_3, u_2v_1, u_2v_2, u_2v_3, u_3v_1, u_3v_2, u_3v_3$; $u_i t_{pq}$ represents the 27 quantities u_1t_{11}, u_1t_{12}, etc.; $\partial u_i/\partial y_j$ represents the set of nine partial derivatives of u_1, u_2, u_3 with respect to y_1, y_2, y_3, and so on.

Meaning is assigned to expressions formed by the addition or subtraction of symbols or products of symbols to which subscripts are attached, provided that the terms which are added or subtracted carry the same subscripts. Thus $u_i + v_i$ denotes the three quantities $u_1 + v_1, u_2 + v_2, u_3 + v_3$; $u_{ij} + v_{ij}$ denotes the nine quantities $u_{11} + v_{11}, u_{12} + v_{12}$, etc.; $u_i s_{jk} - t_{ijk}$ denotes the 27 quantities $u_1s_{11} - t_{111}, u_1s_{12} - t_{112}$, etc. Expressions such as $u_i + v_k, u_{ij} + v_{mn}, u_{ij} - v_i w_k$ are not given any meaning in the suffix notation and will not occur in calculations employing it.

Meaning can be attached to an equation involving symbols with subscripts if the subscripts occurring on both sides of the equation are the same. If n different subscripts occur on each side, then the equation represents the 3^n equations obtained by giving these subscripts the values $1, 2, 3$ independently. For example

$$u_i = av_i \tag{1.27}$$

represents the three equations

$$u_1 = av_1,\ u_2 = av_2,\ u_3 = av_3 \tag{1.28}$$

and

$$u_i v_j = w_{ij} \tag{1.29}$$

represents the nine equations obtained by giving i and j the values $1, 2, 3$. Equations such as $u_i = v_j, u_i v_j = w_{ik}$ have no meaning in the suffix notation. Equations with fewer different subscripts on one side than on the other can be given a meaning, but this is almost never done. The only case which occurs with any frequency is that in which one side of the equation is explicitly zero. Thus

$$u_i = 0 \tag{1.30}$$

represents the three equations

$$u_1 = 0, u_2 = 0, u_3 = 0 \tag{1.31}$$

and

$$t_{ij} = 0 \tag{1.32}$$

represents the nine equations

$$\begin{aligned} t_{11} &= 0, \quad t_{12} = 0, \quad t_{13} = 0\,, \\ t_{21} &= 0, \quad t_{22} = 0, \quad t_{23} = 0\,, \\ t_{31} &= 0, \quad t_{32} = 0, \quad t_{33} = 0\,. \end{aligned} \tag{1.33}$$

The meaning of an equation is unchanged if subscripts are altered consistently throughout the equation. Thus

$$u_k = av_k, u_p v_q = w_{pq}, t_{rs} = 0 \tag{1.34}$$

have the same meanings as Eqs. (1.27), (1.29), and (1.32), respectively. We assume that the symbols with subscripts represent quantities which obey the commutative, associative, and distributive laws of algebra when the subscripts are assigned values chosen from among $1, 2, 3$. It follows immediately that the symbols themselves obey these laws. Thus

$$\begin{aligned} u_i + v_i &= v_i + u_i\;, \\ s_{ij} + t_{ij} &= t_{ij} + s_{ij}\;, \\ u_i v_k &= v_k u_i\;, \\ u_i t_{jk} &= t_{jk} u_i\;, \\ u_i + (v_i + w_i) &= (u_i + v_i) + w_i\;, \\ u_i (v_j w_k) &= (u_i v_j) w_k\;, \\ u_i (v_j + w_j) &= u_i v_j + u_i w_j\;, \end{aligned} \tag{1.35}$$

and so on.

1.3 The Summation Convention

An important extension of the suffix notation already introduced gives meaning to symbols and products of symbols in which the same subscript occurs precisely twice. In such an expression the repeated subscript is given the values $1, 2, 3$ and the three terms so formed are summed. Thus

$$u_{ii} = \sum_{i=1}^{3} u_{ii}\;,$$

$$u_j v_j = \sum_{j=1}^{3} u_j v_j \ , \tag{1.36}$$

$$t_{iik} = \sum_{i=1}^{3} t_{iik} \quad (k = 1, 2, 3) \ ,$$

$$a_{ij} b_{ji} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} b_{ji} \ ,$$

and so on.

With this **summation convention**, the equations for rotation of coordinates become

$$x_i = a_{ji} x'_j \ , \tag{1.37}$$

$$x'_i = a_{ij} x_j \ , \tag{1.38}$$

which are a bit more concise than (1.25) and (1.26) and represent a considerable shortening of the original systems (1.16) and (1.17). Changing a repeated subscript to a different subscript which does not occur elsewhere in the symbol or in the product of symbols does not alter the meaning. Thus

$$u_{ii} = u_{jj}, \quad u_j v_j = u_p v_p, \quad u_i v_{ji} = u_k v_{jk} \ . \tag{1.39}$$

Repeated subscripts are therefore known as **dummy suffixes**. Subscripts which are not repeated in a symbol or product of symbols are known as **free suffixes**. An equation has meaning only if the free suffixes on its left side match those on its right side. On the other hand different dummy suffixes may occur on the two sides of an equation. Alternately, dummy suffixes may occur on one side of the equation and not on the other.

Terms in which the same subscript appears more than twice have no meaning in the suffix notation.

The process of setting two free suffixes equal, thereby converting them to a pair of dummy suffixes, is known as **contraction**. Each contraction reduces by two the number of free indices.

1.4 The Kronecker Delta and the Alternating Tensor

A two-suffix symbol frequently encountered is the **Kronecker delta** δ_{ij} (after Leopold Kronecker 1823–1891) which takes the value unity when $i = j$ and zero when $i \neq j$. Thus, δ_{ij} is defined by

$$\delta_{11} = \delta_{22} = \delta_{33} = 1, \quad \delta_{23} = \delta_{32} = \delta_{31} = \delta_{13} = \delta_{12} = \delta_{21} = 0 \ . \tag{1.40}$$

It is immediately evident that

$$\delta_{ij} = \delta_{ji} \ , \tag{1.41}$$

$$\delta_{ii} = 3 \ . \tag{1.42}$$

If u_i is any one-suffix symbol, it is easily verified that

$$\delta_{ij} u_i = u_j \ . \tag{1.43}$$

Similarly if t_{rs} is any two-suffix symbol,

$$\begin{aligned} \delta_{pr} t_{rs} &= t_{ps} \ , \\ \delta_{ps} t_{rs} &= t_{rp} \ , \\ \delta_{rs} t_{rs} &= t_{ss} \ , \end{aligned} \tag{1.44}$$

and so on. Because of these properties, δ_{ij} is sometimes called the **substitution tensor**.

The Kronecker delta, or substitution tensor, can be regarded as representing the elements of the 3×3 unit matrix:

$$(\delta_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{1.45}$$

Somewhat analogous to the Kronecker delta is a three-suffix symbol ε_{ijk}, called the **alternating tensor**. This quantity is defined as having the value 1 when i, j, k is an even permutation of $1, 2, 3$, the value -1 when i, j, k is an odd permutation of $1, 2, 3$, and the value zero when two or more of the subscripts are equal. Thus

$$\begin{aligned} &\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \ , \\ &\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1 \ , \\ &\varepsilon_{111} = \varepsilon_{112} = \varepsilon_{113} = \varepsilon_{121} = \ldots = \varepsilon_{323} = \varepsilon_{333} = 0 \ . \end{aligned} \tag{1.46}$$

If v_{rs} is any two-suffix symbol and $\det(v_{rs})$ is its determinant,[2] it follows directly from the definition of a determinant that

$$\varepsilon_{ijk} \det(v_{rs}) \equiv \varepsilon_{ijk} \begin{vmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{vmatrix} = \varepsilon_{pqr} v_{ip} v_{jq} v_{kr} \ . \tag{1.47}$$

[2]A function of an n-suffix symbol is not necessarily an n-suffix symbol. In particular, $\det(v_{rs})$ is a zero-suffix symbol.

The following relationships between the alternating tensor and the Kronecker delta are easily established by direct evaluation of both sides:

$$\varepsilon_{ijm}\varepsilon_{ijn} = 2\delta_{mn} \ , \tag{1.48}$$

$$\varepsilon_{ijk}\varepsilon_{ipq} = \begin{vmatrix} \delta_{jp} & \delta_{jq} \\ \delta_{kp} & \delta_{kq} \end{vmatrix} \equiv \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp} \ . \tag{1.49}$$

By using the Kronecker delta, we can write the normalization relations (1.8) and the orthogonality conditions (1.9) quite succinctly. Denoting the direction cosines by a_{ij}, as designated by (1.22), we obtain the **orthonormality conditions**

$$a_{ik}a_{jk} = \delta_{ij} \ . \tag{1.50}$$

This appears to represent nine equations, in place of the six we had before, but there is a duplicity: the orthogonality conditions are given twice. Similarly Eqs. (1.11) and (1.12) can be written

$$a_{ki}a_{kj} = \delta_{ij} \ , \tag{1.51}$$

in which the orthogonality conditions are again given twice.

1.5 Orthogonal Transformations

We indicated earlier that (1.51) is equivalent to (1.50). Let us now turn our attention to the proof. We shall establish that (1.50) implies (1.51); the converse follows *mutatis mutandis*.

First we note that $a_{ik}a_{jk}$ represents the elements of the matrix product of (a_{ij}) and its transpose. Since the determinant of a matrix product is equal to the product of the determinants,

$$\det(a_{ik}a_{jk}) = \left[\det(a_{ij})\right]\left[\det(a_{ji})\right] = \left[\det(a_{ij})\right]^2 \ . \tag{1.52}$$

With (1.50), however,

$$\det(a_{ik}a_{jk}) = \det(\delta_{ij}) = 1 \ . \tag{1.53}$$

Consequently

$$\det(a_{ij}) = \pm 1 \ . \tag{1.54}$$

We now seek nine quantities c_{ki} which solve the nine linear equations

$$c_{ki}a_{kj} = \delta_{ij} \ . \tag{1.55}$$

The determinant of this system is $[\det(a_{ij})]^3$. Since (1.54) guarantees that this does not vanish, (1.55) has a unique solution.

Satisfied that there is exactly one set of c_{ki} which solves the system (1.55), we proceed formally. Multiplying both sides of (1.50) by c_{im} yields

$$c_{im}a_{ik}a_{jk} = c_{im}\delta_{ij} = c_{jm} \; . \tag{1.56}$$

However the c_{im} satisfy (1.55). Consequently

$$c_{im}a_{ik}a_{jk} = \delta_{mk}a_{jk} = a_{jm} \; . \tag{1.57}$$

Thus $c_{jm} = a_{jm}$, and substitution into (1.55) yields (1.51).

We have noted that the proof for the equivalence of formulas (1.37) and (1.38) for rotation of coordinates is fairly easy to carry out with the classical notation. With suffix notation it is trivial. Multiplying both sides of (1.38) by a_{ik}, we obtain

$$a_{ik}x_i' = a_{ik}a_{ij}x_j = \delta_{kj}x_j = x_k \; , \tag{1.58}$$

which is (1.37). Similarly (1.37) implies (1.38). A transformation

$$x_i' = a_{ij}x_j \tag{1.59}$$

in which the relation

$$a_{ki}a_{kj} = \delta_{ij} \tag{1.60}$$

and its equivalent

$$a_{ik}a_{jk} = \delta_{ij} \tag{1.61}$$

are satisfied, so that the inverse transformation is

$$x_i = a_{ji}x_j' \; , \tag{1.62}$$

is called an **orthogonal transformation**.

In proving the equivalence of (1.50) and (1.51), we called attention to the fact that

$$\det(a_{ij}) = \pm 1 \; . \tag{1.63}$$

If the x_i-system is right-handed, the plus sign corresponds to the x_i'-system being right-handed and the minus sign corresponds to the x_i'-system being left-handed. If the x_i-system is left-handed, the opposite holds. Two rectangular Cartesian coordinate systems with common origin can be brought into coincidence by a solid-body rotation if and only if they are either both right-handed or both left-handed.

Otherwise a mirror reflexion must be used. Transformations with $\det(a_{ij}) = +1$ are called **proper orthogonal transformations**. Let us suppose that an x_i'' system is defined in terms of the x_i' system by the orthogonal transformation

$$x_i'' = b_{ij} x_j' \ . \tag{1.64}$$

With (1.59),

$$x_i'' = h_{ij} x_j \,, \tag{1.65}$$

where

$$h_{ij} = a_{kj} b_{ik} \ . \tag{1.66}$$

Since (1.59) and (1.64) both represent orthogonal transformations,

$$\begin{aligned} h_{ik} h_{jk} &= a_{mk} b_{im} a_{nk} b_{jn} \\ &= a_{mk} a_{nk} b_{im} b_{jn} \\ &= \delta_{mn} b_{im} b_{jn} \\ &= b_{in} b_{jn} \\ &= \delta_{ij} \ . \end{aligned} \tag{1.67}$$

Hence (1.65) also represents an orthogonal transformation.

We have called (1.62) *the* inverse of (1.59), but we have not yet formally established the uniqueness. Let us suppose that the x_i'-system is given in terms of the x_i-system according to (1.59) and that there exists a set of nine quantities d_{ij} such that for each x_i and x_j'

$$x_i = d_{ji} x_j' \ . \tag{1.68}$$

We then have

$$x_i' = a_{ik} d_{jk} x_j' \ , \tag{1.69}$$

so that

$$a_{im} x_i' = a_{im} a_{ik} d_{jk} x_j' = \delta_{mk} d_{jk} x_j' = d_{jm} x_j' = d_{im} x_i' \ . \tag{1.70}$$

This result must hold for all x_i' and this can be only if $d_{im} = a_{im}$. Hence (1.62) is the only inverse to (1.59). The geometrical significance of this needs clarification; e.g., if (1.59) represents a plane rotation, then its inverse can be achieved by rotating backward, or by rotating forward to“complete the circle”—or many circles. All of these processes, however, generate the same set of direction cosines a_{ji}.

Given one rectangular Cartesian coordinate system x_i, we observe that a rectangular Cartesian coordinate system with the same origin can be formed from the x_i-system by rotation, by reflexion, or by a combination of rotation and reflexion. However, we must not overlook the simplest thing we can do to the x_i-system to generate a rectangular Cartesian system: we can leave it alone. The **identity transformation**

$$x'_i = x_i \tag{1.71}$$

can also be expressed

$$x'_i = \delta_{ij} x_j \ ; \tag{1.72}$$

as is geometrically obvious, the direction cosines of the identity transformation are the components of the Kronecker delta. It is equally evident that the orthogonal transformation generated by superimposing the identity transformation upon some other orthogonal transformation is simply the other transformation.

In summary, the set of all possible orthogonal transformations satisfies three basic theorems:

1. It is closed under the binary operation of superposition, i.e., superimposing one orthogonal transformation upon another yields still another orthogonal transformation (not necessarily a new one);
2. There is an identity transformation which, when superimposed upon any orthogonal transformation, produces no additional change;
3. Each orthogonal transformation has a unique inverse which, when superimposed upon it, yields the identity transformation.

A fourth property is too evident to be called a theorem:

4. Superposition of orthogonal transformations obeys the associative law.

Readers acquainted with the rudiments of algebra will recognize these four properties as the defining axioms of a **group**. Thus the set of all possible orthogonal transformations form a group, called the **full orthogonal group**, under the binary operation of superposition. As a consequence many important properties of orthogonal transformations can be deduced from group-theoretic considerations.

It can be verified, geometrically or algebraically, that the *proper* orthogonal transformations also form a group, called the **proper orthogonal group**, or the **full rotation group**, under the binary operation of superposition. It is then said that these form a **subgroup** of the full orthogonal group. Other subgroups exist—some, like the full rotation group, have infinitely many members; others consist of a finite number of orthogonal transformations. These other subgroups play an important role in the physics of materials of lower symmetry than isotropic. Since we shall deal only with viscous fluids, which are inherently isotropic, we need not pursue these considerations further.

1.6 Basic Properties of Cartesian Tensors

Let x_i and x'_i be two rectangular Cartesian coordinate systems related by the orthogonal transformation (1.59). Let v_i be three quantities defined in the x_i-system; let v'_i be three quantities defined in the x'_i-system which are related to the v_i in the same way that the coordinates in the primed system of a given point are related to the coordinates in the unprimed system of the same point, viz.,

$$v'_i = a_{ij} v_j \; . \tag{1.73}$$

Then, the v_i are said to be the components of a **vector**, or **Cartesian tensor of rank** 1, in the x_i-system and the v'_i are the components of the *same* vector in the x'_i-system. The inverse of (1.73) follows directly from the orthonormality conditions. Thus

$$v_i = \delta_{ik} v_k = a_{ji} a_{jk} v_k = a_{ji} v'_j \; . \tag{1.74}$$

Going one step further, let t_{ij} be nine quantities defined in the x_i-system and let t'_{ij} be nine quantities defined in the x'_i-system such that

$$t'_{ij} = a_{im} a_{jn} t_{mn} \; . \tag{1.75}$$

The t_{ij} are then said to be the components in the x_i-system of a **Cartesian tensor of rank** 2, and the t'_{ij} are the components of this tensor in the x'_i- system. We find that (1.75) has the inverse

$$t_{ij} = a_{mi} a_{nj} t'_{mn}. \tag{1.76}$$

Quite generally let $A_{i_1 i_2 \ldots i_n}$ denote 3^n quantities defined in the x_i-system and let $A'_{i_1 i_2 \ldots i_n}$ denote 3^n quantities defined in the x'_i-system such that

$$A'_{i_1 i_2 \ldots i_n} = a_{i_1 j_1} a_{i_2 j_2} \ldots a_{i_n j_n} A_{j_1 j_2 \ldots j_n} \; , \tag{1.77}$$

or, equivalently,

$$A_{i_1 i_2 \ldots i_n} = a_{j_1 i_1} a_{j_2 i_2} \ldots a_{j_n i_n} A'_{j_1 j_2 \ldots j_n} \; , \tag{1.78}$$

Then $A_{i_1 i_2 \ldots i_n}$ and $A'_{i_1 i_2 \ldots i_n}$ are said to be the components, in the x_i-system and x'_i-system, respectively, of a **Cartesian tensor of rank** n.

Finally let us suppose there is a single quantity s which has the same value in all rectangular Cartesian coordinate systems. Such a quantity is called a **scalar**, or a **Cartesian tensor of rank zero**.

It is sometimes convenient to refer to a Cartesian tensor by its components in a generic rectangular Cartesian coordinate system. Thus the tensor with components $v_{ij...r}$ in the x_i-system might be referred to as the tensor $v_{ij...r}$.

Let us briefly consider second rank tensors, with regard to their symmetry properties. We see from (1.75) that if $t_{ij} = t_{ji}$, then $t'_{ij} = t'_{ji}$. The tensor is then said to be **symmetric**. Similarly, if $t_{ij} = -t_{ji}$, then $t'_{ij} = -t'_{ji}$, and the tensor is said to be **skew-symmetric**.

Any Cartesian tensor of rank 2 can be uniquely represented as the sum of a symmetric tensor and a skew-symmetric tensor. That such a representation always exists is trivial:

$$t_{ij} = \frac{1}{2}(t_{ij} + t_{ji}) + \frac{1}{2}(t_{ij} - t_{ji}) \ . \tag{1.79}$$

To show the uniqueness, assume

$$t_{ij} = A_{ij} + B_{ij} \ , \tag{1.80}$$

where A_{ij} is symmetric and B_{ij} is skew-symmetric. Then

$$\begin{aligned} t_{ij} + t_{ji} &= A_{ij} + A_{ji} + B_{ij} + B_{ji} \\ &= 2A_{ij} \ . \end{aligned} \tag{1.81}$$

Hence $A_{ij} = \frac{1}{2}(t_{ij} + t_{ji})$; similarly $B_{ij} = \frac{1}{2}(t_{ij} - t_{ji})$.

It is apparent that the sum of two Cartesian tensors of rank n is also a Cartesian tensor of rank n. No meaning is assigned to the sum of tensors of differing rank.

Let $s_{i_1 i_2 \dots i_m}$ and $t_{j_1 j_2 \dots j_n}$ be the components in the x_i-system of two Cartesian tensors and let $s'_{i_1 i_2 \dots i_m}$ and $t'_{j_1 j_2 \dots j_n}$ be their components in the x'_i-system. The expressions $s_{i_1 i_2 \dots i_m} t_{j_1 j_2 \dots j_n}$ and $s'_{i_1 i_2 \dots i_m} t'_{j_1 j_2 \dots j_n}$ each denote 3^{m+n} quantities. Moreover,

$$\begin{aligned} & s'_{i_1 i_2 \dots i_m} t'_{j_1 j_2 \dots j_n} \\ &= \left(a_{i_1 p_1} a_{i_2 p_2} \dots a_{i_m p_m} s_{p_1 p_2 \dots p_m}\right) \left(a_{j_1 q_1} a_{j_2 q_2} \dots a_{j_n q_n} t_{q_1 q_2 \dots q_n}\right) \\ &= a_{i_1 p_1} a_{i_2 p_2} \dots a_{i_m p_m} a_{j_1 q_1} a_{j_2 q_2} \dots a_{j_n q_n} s_{p_1 p_2 \dots p_m} t_{q_1 q_2 \dots q_n} \ . \end{aligned} \tag{1.82}$$

Thus $s_{i_1 i_2 \dots i_m} t_{j_1 j_2 \dots j_n}$ is a Cartesian tensor of rank $(m+n)$, called the **outer product** of $s_{i_1 i_2 \dots i_m}$ and $t_{j_1 j_2 \dots j_n}$.

It is easily established that contraction of a tensor of rank R yields a tensor of rank $R-2$. Further contractions each diminish the rank by 2, and yield a Cartesian tensor at each step. On page 71 of G. K. Batchelor's monograph (Batchelor 1953), a tensor of rank 2 is expressed as a tensor of rank 32, 15 times contracted. The particular contraction $s_{i_1 i_2 \dots i_{m-1} k} t_{k j_2 j_2 j_3 \dots j_n}$ is called the **inner product** of $s_{i_1 i_2 \dots i_m}$ and $t_{j_1 j_2 \dots j_n}$.

1.7 Isotropic Tensors

Toward the end of the next chapter we shall introduce a postulate which entails that the fluids we consider be isotropic, by which we mean that they exhibit no preferred directions. This coincides rather well with our intuitive feelings about how a fluid should behave. By contrast, it is not always reasonable to base the mechanics of deformable solids upon a postulate implying isotropy. Materials such as wood, in which the direction of grain is important; crystals, which are characterized by the crystallographic axes; or fibers, which have a direction of draw, must be approached from less restrictive assumptions.

In isotropic materials, the physical properties are independent of the orientation of the coordinate system in which the physical laws are set out. In formulating such laws, we make use of the fact that it is possible to write down the most general Cartesian tensor, of any given rank, which has the same components in all rectangular Cartesian coordinate systems related by proper orthogonal transformations. Cartesian tensors which transform in this way are called **isotropic tensors**.

If the components of a tensor are given in any rectangular Cartesian system, then their components in any other rectangular Cartesian coordinate system can be found from the appropriate transformation law. Let us examine specifically that tensor which has components δ_{ij} in the x_i-system. By setting $t_{mn} = \delta_{mn}$ in (1.75) and using the orthonormality conditions (1.50), we obtain

$$t'_{ij} = a_{im}a_{jn}\delta_{mn} = a_{im}a_{jm} = \delta_{ij} \; . \tag{1.83}$$

Thus the Kronecker delta is an isotropic tensor.

For third-rank tensors the transformation law (1.77) becomes

$$A'_{ijk} = a_{ip}a_{jq}a_{kr}A_{pqr} \; . \tag{1.84}$$

If we set $A_{pqr} = \varepsilon_{pqr}$ and use (1.47),

$$A'_{ijk} = a_{ip}a_{jq}a_{kr}\varepsilon_{pqr} = \varepsilon_{ijk}\,\det(a_{ij}) \; . \tag{1.85}$$

Since $\det(a_{ij}) = 1$ for a proper orthogonal transformation, the alternating tensor is an isotropic tensor. Note, however, that the components of this tensor change sign when the orthogonal transformation includes a reflexion.

By considerations of this sort, we find that $\delta_{ik}\delta_{mp}, \delta_{im}\delta_{kp}$ and $\delta_{ip}\delta_{km}$ are all isotropic tensors of rank 4. Each of them is unchanged even if the orthogonal transformation includes a reflexion.

We have thus found that isotropic tensors do exist. Let us now inquire whether there are others, substantially different from those already found.

First we note that by definition, every Cartesian tensor of rank zero, i.e., every scalar, is isotropic. Next, we check to see if there are any isotropic vectors: let us assume that e_i is such a vector. Then, according to (1.73),

$$e_i = a_{ij}e_j \; , \tag{1.86}$$

for any orthogonal transformation. However consider a rotation with x_3-axis fixed. If θ is the angle of rotation,

$$(a_{ij}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1.87}$$

The $i = 1, 2$ components of Eq. (1.86) require that

$$\begin{aligned} (1-\cos\theta)e_1 - \sin\theta\, e_2 &= 0 , \\ \sin\theta\, e_1 + (1-\cos\theta)e_2 &= 0 . \end{aligned} \tag{1.88}$$

Unless $\theta = 2n\pi$, so that the determinant of coefficients vanishes, the unique solution to (1.88) is

$$e_1 = e_2 = 0 . \tag{1.89}$$

By considering a rotation about one of the other coordinate axes, we find that $e_3 = 0$. Thus the only isotropic tensor of rank 1 is the **null vector**, all components of which are zero.

Considerations of this sort lead to the following conclusions:

1. The only isotropic tensors of rank 2 are scalar multiples of the Kronecker delta.
2. The only isotropic tensors of rank 3 are scalar multiples of the alternating tensor.
3. The most general isotropic tensor of rank 4 is $A\delta_{ik}\delta_{mp} + B\delta_{im}\delta_{kp} + C\delta_{ip}\delta_{km}$ where A, B, C are scalars.

The proof of these statements is straight-forward, but the manipulation gets fiercer as the rank increases. The details involved in calculating the most general isotropic tensor of rank 4 are set out on page 68 of Jeffreys (1931).

We shall have no need to consider isotropic tensors of rank higher than 4. There is, however, a quite general result, called **the fundamental theorem of isotropic tensors**, which states that any isotropic tensor of any rank can be expressed as the sum of outer products of Kronecker deltas and alternating tensors with scalar coefficients. If the rank is even, a typical term in the sum is $A\delta_{hi}\delta_{jk}\ldots\delta_{rs}$, where A is a scalar and the subscripts can be permuted in any way; if the rank is odd, a typical term is $A\delta_{hi}\delta_{jk}\ldots\delta_{pq}\varepsilon_{rst}$. Rivlin and Smith (1957) have described an elegant method which leads to this result and, more generally, provides a technique for finding those tensors whose components are unchanged under transformation groups of lower symmetry.

Chapter 2
The Equations of Viscous Flow

Abstract The equations of viscous hydrodynamics are developed in detail. Limiting assumptions are introduced only at the points where they become necessary. Boundary conditions for various applications are set out. Incompressible fluids, the Boussinesq approximation, and vorticity transfer are discussed. Similarity rules for steady and unsteady flow are presented.

In this chapter we turn our attention to the development of the equations and boundary conditions which govern the motion of a viscous fluid. Some of the preliminary results apply to a more general class of continuous media, but we shall proceed from the general to the specific, introducing the various assumptions and postulates only when they are needed.

Many books and monographs are devoted to fluid mechanics and hydrodynamics. Without any pretense of exhaustiveness, we may refer the reader to Batchelor (1992), Eringen (1962), Guyon et al. (2012), Happel and Brenner (2013), Landau and Lifshitz (1997), Leal (2007), Panton (1984), Pozrikidis (2009), Serrin (1959), Tritton (1980), Truesdell (1952), and Truesdell and Toupin (1960).

2.1 Kinematics of Flow

We begin by considering the motion of a continuous medium, without regard to the forces which may be acting within it or upon its boundaries. We shall, however, focus our attention on those kinematical quantities which will play a role when we turn to dynamical considerations, especially those which are known to be important in hydrodynamic theory.

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_2,

2.1.1 Description of Deformation in a Fixed Coordinate System

Consider a mass of material to be undergoing deformation. We specify the position of each particle at time t by its coordinates x_i in a fixed rectangular Cartesian system. We further stipulate that at some reference time T the particle is at X_i. Thus we can identify each material particle by its coordinate X_i at time T and consider x_i to be a function of X_i and t. Hence the functional relationship

$$x_i = x_i(X_j, t) \tag{2.1}$$

describes the deformation completely.

The velocity of a particle is given by

$$v_i = \frac{Dx_i}{Dt}, \tag{2.2}$$

where D/Dt denotes differentiation with respect to time, keeping X_i constant. Most writers in hydrodynamics refer to the operation denoted by D/Dt as **material differentiation**, and we shall follow this practice. Other terms in use are **substantial differentiation, particle differentiation, differentiation following the fluid**, and **differentiation following the particle**.

As we shall see later, the reference coordinates X_i have no significance when the material under consideration is a viscous fluid. Hence it is usually convenient to treat the various kinematic and dynamic quantities we shall encounter as being functions of the spatial coordinates x_i and time t, rather than as functions of the material coordinates X_i and time. No conceptual difficulty is involved here, for the impenetrability of matter requires that at any instant of time the mapping (2.1) be one-to-one. Hence the inverse mapping exists, and any function of X_i and t can equally well be regarded as a function of x_i and t.

However we don't want to abandon completely the idea of following the motion of a specified particle. In particular we shall often find it convenient to use material differentiation. Let us suppose that f is some flow variable. As mentioned previously, it is convenient to regard f as a function of x_i and t. The material derivative of f, however, is its partial time derivative, regarding X_i and t as the independent variables. Therefore

$$\frac{Df(x_i,t)}{Dt} = \frac{\partial f(x_i,t)}{\partial t} + \frac{\partial f(x_i,t)}{\partial x_j}\frac{Dx_j(X_k,t)}{Dt}, \tag{2.3}$$

or, using (2.2),

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j}. \tag{2.4}$$

Although the mathematical derivation of (2.4) is rather straightforward, the physical meaning of Df/Dt may be somewhat elusive. If so, it will be helpful to consider again the physical origin of Df/Dt: It is the time rate of change of the quantity f associated with a given particle, as the particle moves through space. If the particle happens to be stationary, or moving normal to the spatial gradient of f, the only contribution to Df/Dt arises from the time rate of change of f at the point in space occupied by the particle, i.e., from $\partial f/\partial t$. On the other hand, even if the values of f at the various points of space do not vary with time, the spatial distribution of f may be non-homogeneous and the motion of the particle such that it is being convected in a direction of changing f. This possibility leads to the $v_j\, \partial f/\partial x_j$ term in (2.4). In general f will vary with both time and position, so that both terms in (2.4) must be retained.

Equation (2.4) was derived formally as though f were a scalar. It is evident, however, that the derivation is equally valid if f represents a component of a tensor of any order. In fact one quantity whose material derivative we shall encounter quite often is a vector: the velocity v_i. The acceleration of a particle is given by

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \, . \tag{2.5}$$

We see from (2.5) that the acceleration is non-linear in the velocity components. It is precisely this nonlinearity which is the principal mathematical difficulty pervading classical hydrodynamics.

In addition to describing the deformation of the medium, the three relations represented by (2.1) are parametric equations for a family of space curves, the **particle paths**. For fixed X_i and varying t, Eq. (2.1) describe the locus of the particle which, at time T, occupies the point X_i. If the velocity components v_i are known functions of X_i and t, these same particle paths can be obtained by an immediate integration of (2.2). If v_i are known functions of x_i and t, however, the particle paths must be obtained by integration of the differential equations

$$\frac{Dx_i}{Dt} = v_i(x_i, t) \, , \tag{2.6}$$

subject to the initial conditions

$$x_i(X_i, T) = X_i \, . \tag{2.7}$$

Since in this consideration the X_i are fixed parameters, (2.6) represents a system of ordinary differential equations.

Now let us consider, not a specific particle, but a specific instant of time. A curve drawn so that its tangent at each point is in the direction of the fluid velocity at that point is called a **streamline**. The aggregate of all streamlines at a given instant constitutes the **flow pattern** at that instant. The differential equations for the streamlines are

$$\frac{dx_2}{dx_3} = \frac{v_2}{v_3}\,,$$
$$\frac{dx_3}{dx_1} = \frac{v_3}{v_1}\,, \tag{2.8}$$
$$\frac{dx_1}{dx_2} = \frac{v_1}{v_2}\,,$$

any one of which is derivable from the other two.

By its definition the streamline passing through a given point at a specific instant of time is tangent to the path of the particle which occupies the point at this instant. If the motion is **steady**, so that the velocity at any point is independent of time, the streamlines and particle paths coincide. Thus far we have been concerned with the kinematics of individual particles, or of the kinematical conditions obtaining at individual points in space. The variations from particle to particle, or from point to point, have concerned us only incidentally. The very nature of viscous fluids forces us to consider these variations in some detail. In particular the spatial derivatives of the velocity vector play a central role in viscous flow theory: As we shall see later, the resistance to flow presented by the viscous medium depends upon the quantities $\partial v_i/\partial x_j$. These quantities are the nine components of a Cartesian tensor,[1] which may be decomposed into its symmetric and skew-symmetric parts according to

$$\frac{\partial v_i}{\partial x_j} = e_{ij} + \omega_{ij}\,, \tag{2.9}$$

where e_{ij} is the **rate of deformation tensor**, defined by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\,, \tag{2.10}$$

and ω_{ij} is the **vorticity tensor**, defined by

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)\,. \tag{2.11}$$

The names given these tensors would seem to connote that a physical significance can be assigned to each of them. We can see that this is in fact the case by considering two neighboring particles, which, at time t, are at the locations x_i and $x_i + dx_i$, respectively. The distance ds between these particles is given by

$$(ds)^2 = dx_i dx_i\,. \tag{2.12}$$

[1]It is easily verified that the quantities obtained by taking first derivatives of the components of a tensor of rank n comprise a tensor of rank $n + 1$.

Let us now calculate the time rate of change of $(ds)^2$ as the particles move through the fluid. We have

$$\frac{D}{Dt}(ds)^2 = 2dx_i \frac{D}{Dt}(dx_i) \ . \tag{2.13}$$

However,

$$dx_i = \frac{\partial x_i}{\partial X_k} dX_k \ , \tag{2.14}$$

so that

$$\frac{D}{Dt}(ds)^2 = 2dx_i \left[\frac{\partial x_i}{\partial X_k} \frac{D}{Dt}(dX_k) + dX_k \frac{D}{Dt}\left(\frac{\partial x_i}{\partial X_k}\right) \right] . \tag{2.15}$$

In calculating a material derivative, X_k and t are independent variables, so that

$$\frac{D}{Dt}(dX_k) = 0 \ . \tag{2.16}$$

Also

$$\frac{D}{Dt}\left(\frac{\partial x_i}{\partial X_k}\right) = \frac{\partial}{\partial X_k}\left(\frac{Dx_i}{Dt}\right) = \frac{\partial v_i}{\partial X_k} = \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial X_k} \ . \tag{2.17}$$

Therefore

$$\frac{D}{Dt}(ds)^2 = 2dx_i dX_k \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial X_k} = 2dx_i dx_j \frac{\partial v_i}{\partial x_j} \ . \tag{2.18}$$

Using (2.9), this can be written

$$\frac{D}{Dt}(ds)^2 = 2dx_i dx_j (e_{ij} + \omega_{ij}) \ . \tag{2.19}$$

Since both i and j are dummy variables in Eq. (2.19), they may be interchanged wherever convenient. Thus

$$\frac{D}{Dt}(ds)^2 = 2dx_i dx_j e_{ij} + dx_i dx_j (\omega_{ij} + \omega_{ji}) \ . \tag{2.20}$$

However the vorticity tensor is skew-symmetric. The second term on the right side of (2.20) therefore vanishes, and we have

$$\frac{D}{Dt}(ds)^2 = 2dx_i dx_j e_{ij} \ . \tag{2.21}$$

Thus the rate of deformation tensor provides a measure of the local and instantaneous rate at which the shape of the medium is changing. A necessary and sufficient condition that the fluid motion be one of which a rigid body is susceptible is that e_{ij} vanish throughout the medium. We may anticipate, therefore, that the dynamical behavior of a fluid medium will be intimately associated with the rate of deformation tensor.

What then of ω_{ij}? If the change of shape of the medium affects only e_{ij}, the vorticity tensor must arise from the rigid-body part of the fluid motion. Since all the components of $\partial v_i/\partial x_j$ vanish in a pure translation, ω_{ij} must arise from the instantaneous rotation of the particle at $x_i + dx_i$ about the particle at x_i. We shall abandon discussion of the vorticity tensor for the present, but shall return to it later when we investigate the description of fluid motion in a moving coordinate system.

Before leaving the description of motion with reference to a stationary coordinate system, however, we shall formulate the principle of conservation of mass, as applied to continuous media. We focus our attention on the set of particles which, at the reference time T, lie within a volume $\mathbf{V}$. Let us suppose that at time t this same set of particles occupies a volume $\mathbf{V}^*$. If the fluid density is denoted by ρ, conservation of mass requires that

$$\begin{aligned}\iiint\limits_{\mathbf{V}} \rho(X_i, T) dX_1 dX_2 dX_3 &= \iiint\limits_{\mathbf{V}^*} \rho(x_i, t) dx_1 dx_2 dx_3 \\ &= \iiint\limits_{\mathbf{V}} \rho(x_i, t) \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} dX_1 dX_2 dX_3 \, ,\end{aligned} \tag{2.22}$$

where $\partial(x_1, x_2, x_3)/\partial(X_1, X_2, X_3)$ denotes the Jacobian of the transformation (2.1), i.e.,

$$\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} \, . \tag{2.23}$$

Taking the material derivative of both sides of (2.22), we have

$$\iiint\limits_{\mathbf{V}} \left[\frac{D\rho}{Dt} \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} + \rho \frac{D}{Dt} \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \right] dX_1 dX_2 dX_3 = 0 \, . \tag{2.24}$$

Equation (2.24) must hold for an arbitrary choice of the volume $\mathbf{V}$. This can be so if and only if the integrand vanishes identically. Hence

$$\frac{D\rho}{Dt} \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} + \rho \frac{D}{Dt} \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = 0 \, . \tag{2.25}$$

In order to simplify this awkward form of the conservation of mass postulate, let us evaluate the material derivative of the Jacobian determinant. It can be shown by a straightforward but rather long induction proof that the derivative of a determinant of any order is the sum of a number of determinants, each of which is obtained from the original determinant by replacing the elements of one row by their derivatives. Thus with (2.2) we have

$$\frac{D}{Dt}\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}=\frac{\partial(v_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}+\frac{\partial(x_1,v_2,x_3)}{\partial(X_1,X_2,X_3)}+\frac{\partial(x_1,x_2,v_3)}{\partial(X_1,X_2,X_3)}\ . \tag{2.26}$$

However expanding in the minors of the first row,

$$\frac{\partial(v_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}=\frac{\partial v_1}{\partial X_1}\frac{\partial(x_2,x_3)}{\partial(X_2,X_3)}-\frac{\partial v_1}{\partial X_2}\frac{\partial(x_2,x_3)}{\partial(X_1,X_3)}+\frac{\partial v_1}{\partial X_3}\frac{\partial(x_2,x_3)}{\partial(X_1,X_2)}\ , \tag{2.27}$$

and

$$\frac{\partial v_1}{\partial X_i}=\frac{\partial v_1}{\partial x_j}\frac{\partial x_j}{\partial X_i}\ . \tag{2.28}$$

Hence

$$\begin{aligned}\frac{\partial(v_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}&=\frac{\partial v_1}{\partial x_j}\left[\frac{\partial x_j}{\partial X_1}\frac{\partial(x_2,x_3)}{\partial(X_2,X_3)}-\frac{\partial x_j}{\partial X_2}\frac{\partial(x_2,x_3)}{\partial(X_1,X_3)}+\frac{\partial x_j}{\partial X_3}\frac{\partial(x_2,x_3)}{\partial(X_1,X_2)}\right]\\&=\frac{\partial v_1}{\partial x_j}\frac{\partial(x_j,x_2,x_3)}{\partial(X_1,X_2,X_3)}\ .\end{aligned} \tag{2.29}$$

Since a determinant with two rows equal vanishes,

$$\frac{\partial(x_j,x_2,x_3)}{\partial(X_1,X_2,X_3)}=\delta_{1j}\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}\ . \tag{2.30}$$

Therefore

$$\frac{\partial(v_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}=\frac{\partial v_1}{\partial x_1}\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}\ . \tag{2.31}$$

By similar arguments

$$\frac{\partial(x_1,v_2,x_3)}{\partial(X_1,X_2,X_3)}=\frac{\partial v_2}{\partial x_2}\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}\ , \tag{2.32}$$

$$\frac{\partial(x_1,x_2,v_3)}{\partial(X_1,X_2,X_3)}=\frac{\partial v_3}{\partial x_3}\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}\ . \tag{2.33}$$

Thus

$$\frac{D}{Dt}\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} = \frac{\partial v_i}{\partial x_i}\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} . \tag{2.34}$$

Since the impenetrability of matter prohibits the vanishing of the Jacobian $\partial(x_1, x_2, x_3)/\partial(X_1, X_2, X_3)$, substitution of (2.34) into (2.25) yields the **equation of continuity**

$$\frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i} = 0 . \tag{2.35}$$

An alternative derivation of (2.35) can be obtained by considering a volume fixed in space. The rate of change of mass within this volume is equated to the rate of flow of mass across its boundaries. Gauss' divergence theorem is then used to convert the surface integrals to volume integrals and the result

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 , \tag{2.36}$$

which is identical with (2.35), is obtained immediately.

If Gauss' theorem is assumed known, this method of proof is much shorter than ours. However, the result (2.34) is quite useful in its own right—in fact, we'd have to derive it sooner or later anyway—and the step from (2.34) to the continuity equation (2.35) is trivial.

For a fluid with constant density,[2] we have

$$\frac{\partial v_i}{\partial x_i} = 0 , \tag{2.37}$$

so that the velocity field is solenoidal, i.e. divergence free.

Let us consider the case of constant-density flow under conditions such that v_3 is independent of x_3. Equation (2.37) then becomes

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 . \tag{2.38}$$

This equation is identically satisfied by introducing a **stream function** $\psi(x_1, x_2, t)$ such that

$$v_1 = \frac{\partial\psi}{\partial x_2} ,$$

$$v_2 = -\frac{\partial\psi}{\partial x_1} . \tag{2.39}$$

[2]The concept of incompressibility, of which constant density is a special case, will be discussed in Sect. 2.4.

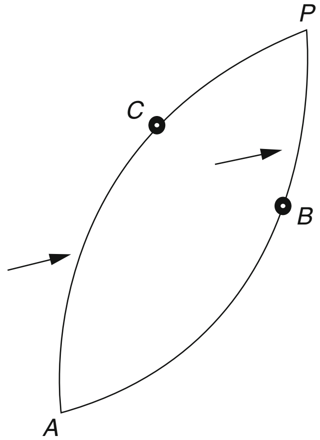


Fig. 2.1 Flux across two curves

We note that, according to (2.39), adding an arbitrary function of time to ψ does not change its significance.

In order to obtain a physical interpretation of the stream function, let us consider the case of **two-dimensional flow**, defined by

$$\begin{aligned} v_1 &= v_1(x_1, x_2, t)\ , \\ v_2 &= v_2(x_1, x_2, t)\ , \\ v_3 &= 0. \end{aligned} \tag{2.40}$$

In Fig. 2.1, let A be a fixed point in the plane of motion and ABP, ACP two curves, also in the plane of motion joining A to an arbitrary point P. We denote the coordinates in this plane of A and P by (a_1, a_2) and (x_1, x_2), respectively. Since the fluid is incompressible, and since no fluid is created or destroyed in the region bounded by these arcs, the flux of fluid clockwise across ACP is equal to the flux clockwise across ABP. Thus once the reference point A has been specified, the flux F across any path joining A and P depends only upon the coordinates of P and time. In particular, we could, as in Fig. 2.2, choose the path to consist of two straight lines: one parallel to the x_1-axis, one parallel to the x_2-axis. Then

$$F = -\int_{a_1}^{x_1} v_2(x_1, a_2, t)dx_1 + \int_{a_2}^{x_2} v_1(x_1, x_2, t)dx_2\ . \tag{2.41}$$

With Eqs. (2.39),

$$\begin{aligned} F &= \int_{a_1}^{x_1} \left[\frac{\partial}{\partial x_1}\psi(x_1, a_2, t)\right] dx_1 + \int_{a_2}^{x_2} \left[\frac{\partial}{\partial x_2}\psi(x_1, x_2, t)\right] dx_2 \\ &= [\psi(x_1, a_2, t) - \psi(a_1, a_2, t)] + [\psi(x_1, x_2, t) - \psi(x_1, a_2, t)] \\ &= \psi(x_1, x_2, t) - \psi(a_1, a_2, t)\ . \end{aligned} \tag{2.42}$$

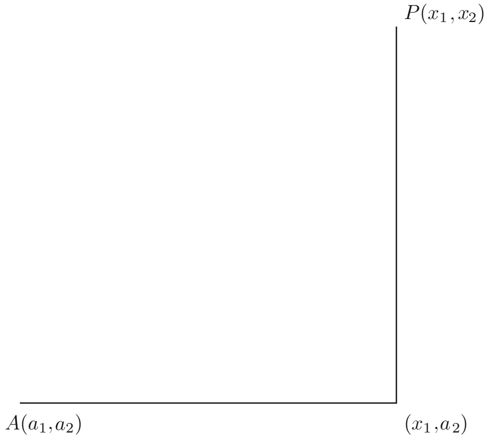


Fig. 2.2 Flux across the path *AP* made of two straight lines

Thus, if the stream function vanishes at a reference point A, its value at any other point P is the clockwise flux through any curve drawn from A to P. The arbitrary function of time which can be added to the stream function is related to the selection of the reference point.

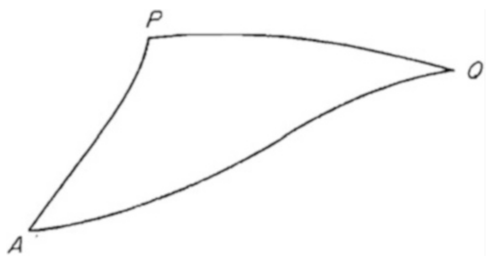


Fig. 2.3 Special case with P and Q on the same streamline

In Fig. 2.3, consider now another point Q, which, at time t, is on the same streamline as P. Since the fluid motion at each point on the streamline *PQ* is parallel to the streamline, there is no flux across *PQ*. Hence the stream function has the same value at Q as it does at P, and we see that *the stream function is constant along a streamline*.

Equations (2.40) are sufficient, but not necessary, conditions for the existence of a stream function in the flow of a fluid with constant density. As mentioned earlier, we need only

$$\begin{aligned} v_1 &= v_1(x_1, x_2, x_3, t) \ , \\ v_2 &= v_2(x_1, x_2, x_3, t) \ , \\ v_3 &= v_3(x_1, x_2, t) \ , \end{aligned} \tag{2.43}$$

but the physical interpretation is now somewhat obscured.

A stream function can also be introduced for steady compressible flow whenever (ρv_3) is independent of x_3. We then let

$$v_1 = \frac{1}{\rho}\frac{\partial\psi}{\partial x_2} ,$$
$$v_2 = -\frac{1}{\rho}\frac{\partial\psi}{\partial x_1} , \tag{2.44}$$

whence, with the vanishing of $\partial\rho/\partial t$ for steady flow, the equation of continuity (2.36) is identically satisfied.

There can be no stream function for unsteady, compressible flow.

The mathematical advantage of introducing a stream function is that the number of dependent variables is immediately reduced by one. As we shall see later, this often opens the way to still further simplification of problems in viscous flow.

2.1.2 Description of Deformation in a Moving Coordinate System

It is not always convenient to describe the deformation of a mass of material with respect to a fixed coordinate system. For instance, in geophysical applications of hydrodynamics, the fluid motion is usually referred to a coordinate system rotating with the earth. Again, an analysis of the motion of a solid object through a fluid is often facilitated by choosing a coordinate system moving with the object.

Of more immediate concern to us, however, is the formulation of physical laws governing the motion of a viscous fluid. One of the basic requirements is that the form of each of these laws be independent of the motion of the coordinate system to which the fluid motion is referred. This is our principal reason for studying moving coordinate systems.

Consider, therefore, a rectangular Cartesian coordinate system y_i which moves in a specified manner with respect to the fixed system x_i: At time t, the point which is at x_i in the fixed system is at y_i in the moving system, where

$$y_i = a_{ij}(x_j - b_j) . \tag{2.45}$$

In (2.45), a_{ij} and b_i are functions of time, such that, at each instant, a_{ij} is the cosine of the angle between the axes y_i and x_j, while b_1, b_2, b_3 , are the coordinates in the x_i system of the origin of the y_i system. Since both systems are rectangular Cartesian, the direction cosines must satisfy the orthonormality conditions

$$a_{ik}a_{jk} = a_{ki}a_{kj} = \delta_{ij} . \tag{2.46}$$

We shall assume that both systems are right-handed, so that

$$\det a_{ij} = 1 . \tag{2.47}$$

The transformation (2.45) has the inverse

$$x_i = b_i + a_{ji} y_j \ . \tag{2.48}$$

Let us now return our attention to the kinematics of deformation. We shall ignore relativistic considerations completely, even though this is not strictly justified. Properly speaking, we should account for the intimate link which exists between kinematics and dynamics in general relativity theory. In a very few astronomical investigations relativistic effects are important, so that now and then there appears in the literature a paper discussing some aspect of relativistic hydrodynamics. In the overwhelming majority of applications, however, the relative velocities encountered are tiny fractions of the speed of light, so that a non-relativistic treatment is operationally quite adequate. We may plead also the counsel of despair: To treat relativistic hydrodynamics properly, we would first have to develop a rather large portion of general relativity theory, postponing considerably our study of hydrodynamics. According to (2.1), the location x_i in the fixed system occupied by a particle at time t is a function of X_i and t; according to (2.45), the location y_i in the moving system occupied by the same particle at the same time is a function of x_i and t. Thus

$$y_i = y_i(X_j, t) \ . \tag{2.49}$$

It is therefore meaningful to define velocity components u_i, relative to the moving coordinate system, of the material particle, which at time t is at x_i in the fixed system and y_i in the moving system, according to

$$u_i = \frac{Dy_i}{Dt} \ . \tag{2.50}$$

We can use (2.2) and (2.45) to express this in terms of the velocity components v_i, relative to the fixed system:

$$u_i = a_{ij}\left(v_j - \frac{Db_j}{Dt}\right) + (x_j - b_j)\frac{Da_{ij}}{Dt} \ . \tag{2.51}$$

On the other hand we can use the inverse transformation (2.48) to express v_i in terms of u_i:

$$v_i = \frac{Dx_i}{Dt} = \frac{Db_i}{Dt} + a_{ji}u_j + y_j\frac{Da_{ji}}{Dt} \ . \tag{2.52}$$

It is evident that (2.51) and (2.52) are equivalent. In fact it is not difficult to derive either one from the other.

In a similar manner the acceleration relative to the moving system can be expressed in terms of the acceleration relative to the fixed system. We have

$$\frac{Du_i}{Dt} = a_{ij}\left(\frac{Dv_j}{Dt} - \frac{D^2 b_j}{Dt^2}\right) + 2\left(v_j - \frac{Db_j}{Dt}\right)\frac{Da_{ij}}{Dt} + (x_j - b_j)\frac{D^2 a_{ij}}{Dt^2} \ . \tag{2.53}$$

A more important form of this relationship expresses the acceleration relative to the fixed system in terms of Du_i/Dt. Differentiating (2.52),

$$\frac{Dv_i}{Dt} = a_{ji}\frac{Du_j}{Dt} + \frac{D^2 b_i}{Dt^2} + 2u_j\frac{Da_{ji}}{Dt} + y_j\frac{D^2 a_{ji}}{Dt^2} \ . \tag{2.54}$$

If the y_i system were stationary, so that the transformation (2.45) would merely describe the difference in orientation and location-of-origin between the x_i reference frame and the y_i-frame, the last three terms would vanish from Eq. (2.54). They would also vanish if the motion of the y_i frame were a steady translation. In the dynamical equations of hydrodynamics, these terms give rise to D'Alembert forces when the motion is referred to an accelerated coordinate system.

Let us now return our attention to the spatial variation of velocity. Since a_{ij} and b_j depend only upon time, differentiation of (2.51) yields

$$\frac{\partial u_i}{\partial y_m} = a_{ik}\frac{\partial v_k}{\partial x_j}\frac{\partial x_j}{\partial y_m} + \frac{\partial x_j}{\partial y_m}\frac{Da_{ij}}{Dt} \ . \tag{2.55}$$

However (2.48) implies

$$\frac{\partial x_j}{\partial y_m} = a_{kj}\frac{\partial y_k}{\partial y_m} = a_{kj}\delta_{km} = a_{mj} \ . \tag{2.56}$$

Therefore

$$\frac{\partial u_i}{\partial y_m} = a_{ik}a_{mj}\frac{\partial v_k}{\partial x_j} + a_{mj}\frac{Da_{ij}}{Dt} \ . \tag{2.57}$$

In terms of its symmetric and antisymmetric parts

$$\frac{\partial u_i}{\partial y_m} = e'_{im} + \omega'_{im} \ , \tag{2.58}$$

where

$$e'_{im} = \frac{1}{2}\left[a_{mj}\frac{Da_{ij}}{Dt} + a_{ij}\frac{Da_{mj}}{Dt} + (a_{ik}a_{mj} + a_{mk}a_{ij})\frac{\partial v_k}{\partial x_j}\right] \tag{2.59}$$

and

$$\omega'_{im} = \frac{1}{2}\left[a_{mj}\frac{Da_{ij}}{Dt} - a_{ij}\frac{Da_{mj}}{Dt} + (a_{ik}a_{mj} - a_{mk}a_{ij})\frac{\partial v_k}{\partial x_j}\right] \ . \tag{2.60}$$

The expression for e'_{ij} can be simplified. We first note that

$$a_{mj}\frac{Da_{ij}}{Dt} + a_{ij}\frac{Da_{mj}}{Dt} = \frac{D}{Dt}(a_{mj}a_{ij}) \ . \tag{2.61}$$

The orthonormality condition (2.46) requires that this vanish. Also, from (2.10),

$$\frac{1}{2}(a_{ik}a_{mj} + a_{mk}a_{ij})\frac{\partial v_k}{\partial x_j} = a_{ik}a_{mj}e_{kj} \ . \tag{2.62}$$

Thus

$$e'_{im} = a_{ik}a_{mj}e_{kj} \ , \tag{2.63}$$

which may be recognized as the transformation rule for a Cartesian tensor of rank 2. By its very definition we expected e_{kj} to transform as a Cartesian tensor from one fixed coordinate system to another fixed system. That it transforms according to the same rule from a fixed system to a moving system reflects again the fact that the rate of deformation tensor does not depend upon rigid body motions of the medium.

By contrast, it is possible to define, for any differentiable velocity field v_i in the neighborhood of a point, a particular motion of the coordinate system y_i so that ω'_{im} vanishes. Using (2.46),

$$\frac{1}{2}\left(a_{mj}\frac{Da_{ij}}{Dt} - a_{ij}\frac{Da_{mj}}{Dt}\right) = a_{mj}\frac{Da_{ij}}{Dt} \ . \tag{2.64}$$

Therefore ω'_{ij} vanishes if the a_{ij} satisfy the differential equations

$$a_{mj}\frac{Da_{ij}}{Dt} = \frac{1}{2}(a_{mk}a_{ij} - a_{ik}a_{mj})\frac{\partial v_k}{\partial x_j} \ , \tag{2.65}$$

which can be simplified by multiplying both sides by $a_{in}a_{mp}$ and using (2.46):

$$a_{in}\frac{Da_{ip}}{Dt} = \frac{1}{2}a_{in}a_{mp}(a_{mk}a_{ij} - a_{ik}a_{mj})\frac{\partial v_k}{\partial x_j} \tag{2.66}$$

$$= \frac{1}{2}(\delta_{nj}\delta_{pk} - \delta_{nk}\delta_{pj})\frac{\partial v_k}{\partial x_j}$$

$$= \frac{1}{2}\left(\frac{\partial v_p}{\partial x_n} - \frac{\partial v_n}{\partial x_p}\right) = \omega_{pn} \ . \tag{2.67}$$

We have already observed that the right side of (2.67), the vorticity tensor, is associated with local rotations in the medium. The left side, since it does not involve b_i, is associated in some way with the rotation of the moving coordinate system. We can be more specific: let us consider the motion of a particle attached to the moving frame. For such a particle $u_i = 0$ and, according to (2.52), $v_i = w_i$, where

$$w_i = \frac{Db_i}{Dt} + y_j \frac{Da_{ji}}{Dt} . \tag{2.68}$$

Using (2.45),

$$w_i - \frac{Db_i}{Dt} = a_{jk} \frac{Da_{ji}}{Dt} (x_k - b_k) . \tag{2.69}$$

The left side of (2.69) represents the velocity of a particle attached to a moving frame, with the translational motion of the frame subtracted. It can therefore be expressed in terms of the angular velocity Ω_i of the moving frame;

$$w_i - \frac{Db_i}{Dt} = \varepsilon_{ijk} \Omega_j (x_k - b_k) . \tag{2.70}$$

Note that the right side of (2.70) is Cartesian tensor notation for the i^{th} component of $\boldsymbol{\Omega} \times (\mathbf{x} - \mathbf{b})$.

Since we are considering the motion of a particle attached to an arbitrary point on the moving frame, Eqs. (2.69) and (2.70) must be identical for arbitrary values of $(x_k - b_k)$. If we write both equations in component form and compare coefficients of $(x_k - b_k)$ we find, with the help of (2.46),

$$\begin{aligned} \Omega_1 &= a_{j2} \frac{Da_{j3}}{Dt} , \\ \Omega_2 &= a_{j3} \frac{Da_{j1}}{Dt} , \\ \Omega_3 &= a_{j1} \frac{Da_{j2}}{Dt} . \end{aligned} \tag{2.71}$$

Comparing with (2.67), we find

$$\begin{aligned} \Omega_1 &= \omega_{32} = -\omega_{23} , \\ \Omega_2 &= \omega_{13} = -\omega_{31} , \\ \Omega_3 &= \omega_{21} = -\omega_{12} . \end{aligned} \tag{2.72}$$

Thus we have seen that, for every point in space and for every instant of time, it is possible to define the motion of the y_i coordinate system so that ω'_{ij} vanishes. The y_i-system need only rotate with angular velocity components Ω_i related to the off-diagonal components of the vorticity tensor according to (2.72).

We are now in a position to investigate more deeply the connexion between the vorticity tensor and the local and instantaneous rotations within the medium. Let us denote the components of the curl of the velocity by ξ_i. Translating the usual vector definition $\boldsymbol{\xi} = \nabla \times \mathbf{v}$ into Cartesian tensor notation,

$$\xi_i = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} \ . \tag{2.73}$$

Comparing with (2.11), we find

$$\xi_i = -\varepsilon_{ijk}\omega_{jk} = \varepsilon_{ijk}\omega_{kj} \ . \tag{2.74}$$

Thus ξ_i can be expressed in terms of the components of the vorticity tensor. It is therefore not surprising that $\boldsymbol{\xi}$ is called the **vorticity vector**, or more simply the **vorticity**. If the vorticity vanishes throughout the medium, we say that the velocity field is **irrotational**. If the vorticity vanishes at a point in space, but not necessarily throughout the medium, we shall say that the velocity field is **locally irrotational** in the neighborhood of the point.

As we have seen, the velocity field measured with respect to a coordinate system rotating with angular velocity Ω_i is locally irrotational in the neighborhood of a point at which the vorticity tensor is related to Ω_i according to (2.72). A comparison of (2.72) with (2.74) reveals that

$$\Omega_i = \frac{1}{2}\xi_i \ . \tag{2.75}$$

We may therefore summarize much of this section with the statement:

The velocity field is locally irrotational at a given point when it is measured with respect to a coordinate system rotating with angular velocity equal to half the vorticity at that point.

The rather lengthy derivations leading to this statement do not lend themselves readily to physical interpretation. The matter becomes quite clear, however, when we investigate directly the physical significance of the vorticity. Let us consider again the two neighboring particles which, at time t, are at x_p and $x_p + dx_p$, respectively. Their relative velocity is

$$\begin{aligned} v_i(x_p + dx_p) - v_i(x_p) &= \frac{\partial v_i}{\partial x_j} dx_j \\ &= e_{ij}dx_j + \omega_{ij}dx_j \ . \end{aligned} \tag{2.76}$$

As we have seen in Sect. 2.1, e_{ij} measures the rate at which the particles are moving apart. The term $e_{ij}dx_j$ in (2.76) is therefore the radial component of their relative motion. The tangential component is $\omega_{ij}dx_j$. However

$$\begin{aligned} \omega_{ij}dx_j &= \frac{1}{2}(\omega_{ij} - \omega_{ji})dx_j \\ &= \frac{1}{2}(\delta_{js}\delta_{ir} - \delta_{jr}\delta_{is})\omega_{rs}dx_j \ . \end{aligned} \tag{2.77}$$

With the identity (1.49),

$$\begin{aligned} \omega_{ij} dx_j &= -\frac{1}{2}\varepsilon_{kji}\varepsilon_{krs}\omega_{rs} dx_j \\ &= \frac{1}{2}\varepsilon_{ikj}\xi_k dx_j \ . \end{aligned} \tag{2.78}$$

But the right side of (2.78) is Cartesian tensor notation for $\frac{1}{2}\boldsymbol{\xi} \times \mathbf{dx}$. Hence the particle at $x_i + dx_i$ is rotating about the particle at x_i with an angular velocity equal to half the vorticity. An observer rotating with this same angular velocity will say that the particles are not rotating about each other. As we shall see later, such an observer is in an enviable situation for formulating the physical laws governing viscous flow.

This concludes our kinematical treatment of the deformation of a continuous medium. We may observe that we have nowhere *required* the medium to be a fluid. All references to viscous flow theory have been anticipatory in character. In fact we have not even yet defined what we mean by a fluid. Everything presented so far applies equally well to fluids, elastic solids, plastic solids, viscoelastic materials of various sorts—to all continuous media. Of course the quantities we have chosen to discuss are those which will pertain to the theory of viscous fluids. If we were about to undertake the study of some other type of material, we would be obliged to introduce other quantities, and perhaps could leave off some of those discussed here.

2.2 Dynamics of Flow

Hydrodynamics is often concerned with the motion arising in a fluid mass due to the application of external forces. Such forces may be applied at the bounding surface of the fluid or be distributed throughout the volume. Forces applied at the boundaries are called **surface tractions**, a term taken over from solid mechanics, where it is somewhat more appropriate. The surface tractions may vary from point to point of the boundary. We shall assume, however, that such variation is continuous. Thus we can characterize the surface tractions by associating with each point of the boundary a vector, the direction of which is that of the force acting at the point and the magnitude of which is the magnitude of the force acting per unit area of the boundary.

Forces distributed throughout the volume of the fluid are called **body forces**. They may vary from point to point of the volume and again we assume that the variation is continuous. We can then characterize the body forces by associating with each point of the volume a vector, the direction of which is the direction of the body force acting at the point and the magnitude of which is the magnitude of the body force acting per unit mass of the fluid.

So far we have discussed only the external forces applied to the fluid. As an illustration of what we mean, let us consider the forces acting upon the water in a lake. Atmospheric pressure and wind forces apply surface tractions to the free-surface; gravitational body forces act on each particle in the lake; the lake bed imposes surface tractions as a reaction to the weight of the water; fish swimming through the lake propel themselves by applying other surface tractions to the water. In addition to these forces, each layer of water feels the weight of the water above it. This, however, is not an external force. True, it *results* from an external force—the gravitational body force acting on the upper layers. However it is transmitted to the lower layers by *internal forces*, through which the various parts of a fluid mass may act and react on one another. It is toward these internal forces that we shall now turn our attention.

Let us consider a fixed surface **S** drawn through the volume occupied by the fluid. In general there will be a flow of fluid across **S**. Also, and more to the point, forces may act across **S**. At any instant of time these forces may be characterized by a vector associated with each point of **S**, having as its direction the direction of the force acting on an element of area at the point and as its magnitude the magnitude of the force per unit area of the surface. This vector is called the **stress vector**.

It is conceivable that force couples may act across the surface **S**. In fact the formulation of an approximate continuum theory for the mechanics of a solid composed of crystallites relies heavily on the concept of **couple stresses**. Since we are dealing with fluid materials only, we postulate the complete absence of couple stresses.

The characterization of the stress vector given above is not quite complete. The force acting across **S** acts on the material on both sides of **S**. In fact the action upon the material on one side is precisely the reaction to the action upon the material on the other side. Since these actions are equal and opposite, to speak of "the force acting on an element of area" leaves an ambiguity of sign. Also it is clear that the stress vector at a point depends on the orientation, at that point, of the imaginary surface on which it acts.

Both of these matters can be cleared up by considering a fixed element of volume which, at a specific instant of time, lies within the space occupied by the fluid. In Fig. 2.4, following a line of reasoning due to Augustin-Louis Cauchy (1789–1857), we choose the element of volume in the shape of an infinitesimal tetrahedron *OPQR*, the edges *OP*, *OQ*, *OR* of which are parallel to the axes of a rectangular Cartesian coordinate system x_i. Let the face *PQR* of this tetrahedron have an area dS and let the normal to it have direction-cosines n_i in the x_i coordinate system. Let $F_i\,dS$ be the components, in the x_i-system, of the resultant force acting on the fluid inside the tetrahedron across the face *PQR*. We now calculate the forces acting on the fluid inside the tetrahedron across each of its other three faces.

Consider first the face *OPR*, normal to the x_1-axis. The area of this face is $n_1 dS$. We denote by T_{1i} the components of the stress vector acting across this face, with the ambiguity of sign resolved in such a way that the resultant force acting on the material inside the tetrahedron is $-T_{1i}n_1 dS$. This choice of sign makes tensions positive and pressures negative. Using the analogous terminology, we have resultant forces acting on the material inside the tetrahedron across *OPQ* and *OQR* respectively given by $-T_{2i}n_2 dS$ and $-T_{3i}n_3 dS$.

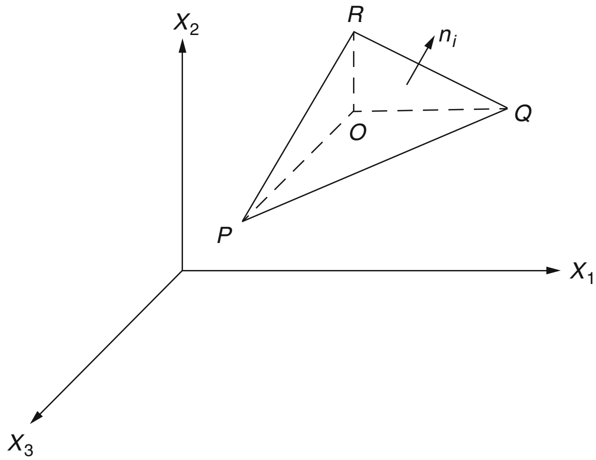


Fig. 2.4 Cauchy tetrahedron

Further, let f_i be the body force per unit mass acting at the point O. Then the resultant body force acting on the element of volume is $\rho f_i dV$, where dV is the volume of the element and ρ is the density of the fluid in the neighborhood of O. Applying Newton's second law to the motion of the fluid within the elementary tetrahedron, we have

$$F_i dS - T_{ji} n_j dS + \rho f_i dV = \rho \frac{Dv_i}{Dt} dV . \tag{2.79}$$

We now shrink the dimensions of the tetrahedron to zero without altering its shape, so that $dV/dS \to 0$. Equation (2.79) then becomes

$$F_i = T_{ji} n_j . \tag{2.80}$$

Thus if the stress vectors acting on elements of area at O normal to the three coordinate axes are known, the stress vector on an element of area of arbitrary orientation can be found. The nine quantities T_{ij} are called the **components of stress** in the x_i-system. The diagonal components T_{11}, T_{22}, T_{33} are called the **normal stresses**; the other components are called the **tangential**, or **shear**, stresses.

Let us consider another rectangular Cartesian coordinate system y_i, defined by

$$y_i = a_{ij}(x_j - b_j) , \tag{2.81}$$

where a_{ij} and b_i are either constants or functions of time only. According to the definition just given, the components of stress in the y_i-system are, at any specific instant of time, the components $T'_{1k}, T'_{2k}, T'_{3k}$ in the y_i-system of the three stress vectors acting on the elements of area normal to the axes y_1, y_2, y_3 respectively. The direction-cosines of the y_1-axis in the x_i-system are a_{1j}. Consequently the stress vector on an element of area normal to the y_1-axis has components F_i in the x_i-system, obtained by replacing n_i with a_{1j} in Eq. (2.80), i.e.,

$$F_i = a_{1j} T_{ji} \ . \tag{2.82}$$

However T'_{1k} are the components in the y_i-system of this same vector, so that at any specific time the transformation rule for a vector implies

$$T'_{1k} = a_{ki} F_i = a_{1j} a_{ki} T_{ji} \ . \tag{2.83}$$

In a similar manner analogous formulas for T'_{2k} and T'_{3k} can be obtained. Together with (2.83), these may be written as the single equation

$$T'_{mk} = a_{mj} a_{ki} T_{ji} \ , \tag{2.84}$$

which is the transformation rule for a second-rank Cartesian tensor. Thus it is established that the components of stress in the various rectangular Cartesian coordinate systems are the components of a Cartesian tensor of rank 2, the Cauchy **stress tensor**.

2.2.1 Conservation of Momentum

We have already formulated the principle of conservation of mass as it applies to continuous media. A somewhat similar technique can be used to formulate the principle of conservation of momentum.

We consider again the set of particles which, at the reference time T, lie within the volume $\mathbf{V}$. It is erroneous to stipulate that the total momentum of these particles remains the same at all times, for they may be acted upon by forces arising outside $\mathbf{V}$. Rather, we note that at any instant of time t the rate of change of the total momentum possessed by these particles is equal to the net force acting upon them.

We again suppose that at time t the particles occupy a volume $\mathbf{V}^*$. We require that $\mathbf{V}^*$ be completely immersed in the fluid, so that no surface tractions act upon the particles within $\mathbf{V}^*$. The net force transmitted across the boundary $\mathbf{S}^*$ of $\mathbf{V}^*$ is therefore

$$\iint\limits_{\mathbf{S}^*} T_{ji} n_j \, \mathbf{dS}^* \ , \tag{2.85}$$

where n_i are the components of the outward drawn normal at a generic point of $\mathbf{S}^*$. The particles within $\mathbf{V}^*$ may also be acted upon by body forces f_i per unit mass, which have net resultant

$$\iiint\limits_{\mathbf{V}^*} \rho f_i \, \mathbf{dV}^* \ . \tag{2.86}$$

The total momentum of the particles within V^* is

$$\iiint\limits_{\mathbf{V}^*} \rho v_i \, \mathbf{dV}^* \, . \tag{2.87}$$

Since $\mathbf{V}^*$ always includes the same particles, there is no momentum transfer across $\mathbf{S}^*$. Conservation of momentum then requires that

$$\iint\limits_{\mathbf{S}^*} T_{ji} n_j \, \mathbf{dS}^* + \iiint\limits_{\mathbf{V}^*} \rho f_i \, \mathbf{dV}^* = \frac{D}{Dt} \iiint\limits_{\mathbf{V}^*} \rho v_i \, \mathbf{dV}^* \, . \tag{2.88}$$

The surface integral appearing in (2.88) can be converted to a volume integral by using Gauss' divergence theorem, we have

$$\iint\limits_{\mathbf{S}^*} T_{ji} n_j \, \mathbf{dS}^* = \iiint\limits_{\mathbf{V}^*} \frac{\partial T_{ji}}{\partial x_j} \mathbf{dV}^* \, . \tag{2.89}$$

Equation (2.88) then becomes

$$\iiint\limits_{\mathbf{V}^*} \left[\frac{\partial T_{ji}(x_k, t)}{\partial x_j} + \rho(x_k, t) f_i(x_k, t) \right] \mathbf{dV}^* = \frac{D}{Dt} \iiint\limits_{\mathbf{V}^*} \rho(x_k, t) v_i(x_k, t) \mathbf{dV}^* \tag{2.90}$$

or equivalently,

$$\iiint\limits_{\mathbf{V}} \left[\frac{\partial T_{ji}(x_k, t)}{\partial x_j} + \rho(x_k, t) f_i(x_k, t) \right] \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \mathbf{dV}$$
$$= \frac{D}{Dt} \iiint\limits_{\mathbf{V}} \rho(x_k, t) v_i(x_k, t) \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \mathbf{dV} \, . \tag{2.91}$$

Carrying out the indicated material differentiation with the help of (2.34) and omitting the obvious arguments of the various functions, we have

$$\iiint\limits_{\mathbf{V}} \left(\frac{\partial T_{ji}}{\partial x_j} + \rho f_i - \rho \frac{D v_i}{Dt} - v_i \frac{D\rho}{Dt} - \rho v_i \frac{\partial v_j}{\partial x_j} \right) \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \mathbf{dV} = 0 \, . \tag{2.92}$$

Using the equation of continuity (2.35),

$$\iiint\limits_{\mathbf{V}} \left(\frac{\partial T_{ji}}{\partial x_j} + \rho f_i - \rho \frac{D v_i}{Dt} \right) \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \mathbf{dV} = 0 \,. \tag{2.93}$$

The three equations represented by (2.93) must hold for any volume $\mathbf{V}$ satisfying the requirement that, at time t, $\mathbf{V}^*$ is completely immersed in the fluid. Consequently the integrands must vanish identically. We then arrive at the **equations of motion**, which must be satisfied at each point within a continuous medium:

$$\rho \frac{D v_i}{Dt} = \rho f_i + \frac{\partial T_{ji}}{\partial x_j} \,. \tag{2.94}$$

If the medium is in equilibrium, the left side of (2.94) vanishes. The resulting **equations of equilibrium**

$$\frac{\partial T_{ji}}{\partial x_j} + \rho f_i = 0 \tag{2.95}$$

are more important in elasticity theory than in fluid mechanics, but they do form the basis for the theory of hydrostatics.

2.2.2 Conservation of Angular Momentum

There is one more purely mechanical principle to formulate: the rate of change of the angular momentum, about any point, possessed by a set of particles must equal the net torque about the point.

Consider once more the set of particles which, at the reference time T, occupies the volume $\mathbf{V}$ and which, at time t, occupies the volume $\mathbf{V}^*$. We again require that $\mathbf{V}^*$ be completely immersed in the fluid, so that no surface tractions act upon the particles within $\mathbf{V}^*$.

We first calculate the net torque, about a generic point a_i, which acts on the particles within $\mathbf{V}^*$. The body forces acting throughout $\mathbf{V}^*$ and the stresses acting across its boundary $\mathbf{S}^*$ will, in general, both contribute to this torque. It is conceptually possible that there is another contribution: external force couples may act on the material.

No force couple acting on the surface of the fluid can contribute to the torque on $\mathbf{V}^*$, which was chosen to be completely immersed. If force couples act throughout the volume of the fluid, we shall assume they are continuously distributed. We can thus characterize them by a vector M_i, the force couple per unit mass. It is assumed that M_i has no resultant force, any such resultant being incorporated into the body force f_i.

In the study of elastic solids subjected to magnetic fields, it is quite important to account for a distribution of force couples throughout the medium. Textbooks on hydrodynamics, however, usually ignore the possibility of force couples acting throughout the fluid. Indeed, if one associates such couples with magnetic effects, the lack of oriented elements within a fluid would seem to preclude their existence. The choice of language in this last statement is deliberately cautious, because the point is not universally conceded. In any case such considerations are properly the domain of physics, not hydrodynamics. We must either account for the possible existence of force couples, or postulate that they do not act on the materials we shall consider. Ultimately we shall adopt the latter course of action. However there's no need to do it just yet, and at this stage the former course of action is more appealing.

Since we assume that no couple stresses act, the net torque about a_i which acts on the particles within $\mathbf{V}^*$ is

$$\iint\limits_{\mathbf{S}^*} \varepsilon_{ijk}(x_j - a_j)T_{mk}n_m \mathbf{dS}^* + \iiint\limits_{\mathbf{V}^*} \rho\varepsilon_{ijk}(x_j - a_j)f_k \mathbf{dV}^* + \iiint\limits_{\mathbf{V}^*} \rho M_i \mathbf{dV}^* . \tag{2.96}$$

This must be equated to the instantaneous rate of change of angular momentum

$$\frac{D}{Dt}\iiint\limits_{\mathbf{V}^*} \rho\varepsilon_{ijk}(x_j - a_j)v_k \mathbf{dV}^* . \tag{2.97}$$

Gauss' divergence theorem yields

$$\iint\limits_{\mathbf{S}^*} \varepsilon_{ijk}(x_j - a_j)T_{mk}n_m \mathbf{dS}^* = \iiint\limits_{\mathbf{V}^*} \varepsilon_{ijk}\left[(x_j - a_j)\frac{\partial T_{mk}}{\partial x_m} + T_{jk}\right]\mathbf{dV}^* . \tag{2.98}$$

Equating (2.96)–(2.97) and using (2.98), we then have

$$\iiint\limits_{\mathbf{V}^*} \varepsilon_{ijk}(x_j - a_j)\left(\frac{\partial T_{mk}}{\partial x_m} + \rho f_k\right)\mathbf{dV}^* + \iiint\limits_{\mathbf{V}^*} (\varepsilon_{ijk}T_{jk} + \rho M_i)\mathbf{dV}^*$$
$$= \frac{D}{Dt}\iiint\limits_{\mathbf{V}^*} \rho\varepsilon_{ijk}(x_j - a_j)v_k \mathbf{dV}^* . \tag{2.99}$$

However

$$\frac{D}{Dt}\iiint\limits_{\mathbf{V}^*} \rho\varepsilon_{ijk}(x_j - a_j)v_k \mathbf{dV}^* = \frac{D}{Dt}\iiint\limits_{\mathbf{V}} \rho\varepsilon_{ijk}(x_j - a_j)v_k \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}\mathbf{dV} . \tag{2.100}$$

Carrying out the indicated differentiation in (2.100), substituting into (2.99), converting the integrals on the left side of (2.99) into integrals over **V**, and noting that the resulting equation must hold for an arbitrary choice of the volume **V**, we find

$$
\begin{aligned}
&\left[\varepsilon_{ijk}(x_j - a_j)\left(\frac{\partial T_{mk}}{\partial x_m} + \rho f_k\right) + \varepsilon_{ijk}T_{jk} + \rho M_i\right]\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)} \\
&= \rho\varepsilon_{ijk}v_j v_k \frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)} + \rho\varepsilon_{ijk}(x_j - a_j)\frac{Dv_k}{Dt}\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)} \\
&+\varepsilon_{ijk}(x_j - a_j)v_k\left[\frac{D\rho}{Dt}\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)} + \rho\frac{D}{Dt}\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}\right]
\end{aligned}
\tag{2.101}
$$

Equation (2.25) reduces this to

$$
\varepsilon_{ijk}(x_j - a_j)\left(\frac{\partial T_{mk}}{\partial x_m} + \rho f_k - \rho\frac{Dv_k}{Dt}\right) + \varepsilon_{ijk}T_{jk} + \rho M_i = \rho\varepsilon_{ijk}v_j v_k \ . \tag{2.102}
$$

The right side of (2.102) vanishes, for it represents the ith component of $\mathbf{v}\times\mathbf{v}$. Using the momentum equation (2.94), we therefore have

$$
\varepsilon_{ijk}T_{jk} + \rho M_i = 0 \ , \tag{2.103}
$$

which, with the identity (1.49), can be rewritten

$$
T_{qp} - T_{pq} = \rho\varepsilon_{pqi}M_i \ . \tag{2.104}
$$

Thus conservation of angular momentum leads us to an important conclusion:

When no force couples act throughout the volume of the fluid, the stress tensor is symmetric.

This is about as far as it is convenient to go without specifying the mechanical properties of the fluids we want to consider. Everything presented so far refers to any continuous medium; we now begin to be specific.

2.2.3 The Constitutive Equation for a Newtonian Viscous Fluid

The concept of a quantitative relationship between the internal forces in a moving fluid and the kinematical quantities describing the motion is due to Newton. In the hypothesis just before Prop. LI, Lib. II of his famous work *Philosophiae Naturalis Principia Mathematica* he takes

Resistentiam quae oritur ex defectu lubricitatis partium fluidi, caeteris paribus, proportionatem esse velocitati, qua partes fluidi separatur ab invicem.[3]

This formulation embodies most of the principles which govern the modern theory of hydrodynamics. Many fluids have been found empirically to behave in accord with Newton's hypothesis over a wide range of flow conditions.

In Sect. 2.1.1 we demonstrated that the rate at which neighboring particles are moving apart is measured by the rate of deformation tensor. Thus Newton's hypothesis can be interpreted as stipulating that the viscous contribution to the stress tensor be proportional to the rate of deformation tensor. However it is not necessary to adopt so restrictive an assumption, which, as we shall see, can be derived as a result of a more general postulate. This more general postulate, which we shall take as the defining axiom for a **Newtonian viscous fluid**, is:

The components of stress at a given point in the fluid and at a specific instant of time are linear functions of the first spatial derivatives of the velocity components, evaluated at the same point and at the same instant. They have no other explicit dependence upon the kinematic variables describing the fluid motion.

Observe that the material coordinates X_i play no role in the defining axiom. This implies, among other things, that a Newtonian viscous fluid has no intrinsic preferred directions, i.e., that it is isotropic.

The assumption that the stress components T_{ij} at a place x_i in the fixed coordinate system and at a time t are functions only of the velocity gradients $\partial v_m/\partial x_n$, evaluated at the same place and time, can be stated in equation form:

$$T_{ij} = F_{ij}\left(\frac{\partial v_m}{\partial x_n}\right) . \tag{2.105}$$

However a physical postulate should contain no reference, explicit or implicit, to the motion of the coordinate system in which the various quantities appearing in the postulate are measured. Hence the stress components T'_{ij} at a point y_i in the moving coordinate system and at a time t are related to the velocity gradients $\partial u_m/\partial y_n$ by the functional relationship (2.105). Thus

$$T'_{ij} = F_{ij}\left(\frac{\partial u_m}{\partial y_n}\right) . \tag{2.106}$$

We now note that

$$\begin{aligned} T_{pq} &= a_{ip}a_{jq}T'_{ij} \\ &= a_{ip}a_{jq}F_{ij}\left(\frac{\partial u_m}{\partial y_n}\right) . \end{aligned} \tag{2.107}$$

---

[3]The resistance arising from imperfect slipping between fluid particles to be proportional to the velocity with which the particles are moving apart, other things being equal.

This relationship between the stress components in the fixed system and the velocity gradients in the moving system must hold for any choice of the functions of time a_{ij} and b_j which define the moving system. In particular it must hold if the y_i-system is moving so that, at time t:

(i) It has an instantaneous angular velocity equal to half the vorticity at the place x_i;
(ii) Its directions coincide instantaneously with those of the fixed coordinate system x_i.

It was shown in Sect. 2.1.2 that the fluid motion is locally irrotational when observed in a coordinate system moving in accordance with (i), so that

$$\frac{\partial u_m}{\partial y_n} = e'_{mn} = a_{mr}a_{ns}e_{rs} \; . \tag{2.108}$$

Substituting this result into (2.107), we have

$$T_{pq} = a_{ip}a_{jq}F_{ij}(a_{mr}a_{ns}e_{rs}) \; . \tag{2.109}$$

With the y_i-system oriented according to (ii),

$$a_{ij} = \delta_{ij} \; , \tag{2.110}$$

so that (2.109) becomes

$$T_{pq} = F_{pq}(e_{mn}) \; . \tag{2.111}$$

Thus we find as a consequence of our defining axiom for a Newtonian fluid: **The components of the stress tensor at a given point in space-time depend only on the components of the rate of deformation tensor at the same point in space-time.**

Moreover the axiom requires that this dependence be *linear*. Thus

$$T_{ij} = A_{ij} + B_{ijmn}e_{mn} \; , \tag{2.112}$$

where A_{ij} and B_{ijmn} do not depend explicitly on the kinematic variables describing the fluid motion. They may depend upon other physical quantities associated with the state of the medium.

lt is not obvious a priori that A_{ij} and B_{ijmn} are Cartesian tensors. However Eq. (2.112) determines T_{ij} for all values of e_{mn}, and we know that T_{ij} and e_{mn} are both Cartesian tensors. Setting e_{mn} equal to zero demonstrates immediately that A_{ij} is a Cartesian tensor, the stress at equilibrium. To show that B_{ijmn} is a Cartesian tensor, we consider the transformation of (2.112) into an arbitrary Cartesian coordinate system x'_i, according to

$$x'_i = a_{ij}x_j \tag{2.113}$$

with

$$a_{ik}a_{jk} = a_{ki}a_{kj} = \delta_{ij} \tag{2.114}$$

and denote by primed symbols the components of various quantities in the x_i'-system. Thus

$$\begin{aligned} B'_{ijmn}e'_{mn} &= T'_{ij} - A'_{ij} \\ &= a_{ir}a_{js}(T_{rs} - A_{rs}) \ . \end{aligned} \tag{2.115}$$

With (2.112), therefore,

$$B'_{ijmn}e'_{mn} = a_{ir}a_{js}B_{rspq}e_{pq} \ . \tag{2.116}$$

However

$$e'_{mn} = a_{mp}a_{nq}e_{pq} \tag{2.117}$$

so that

$$(a_{mn}a_{nq}B'_{ijmn} - a_{ir}a_{js}B_{rspq})e_{pq} = 0 \ . \tag{2.118}$$

Since (2.118) must hold for an arbitrary choice of e_{pq},

$$a_{mp}a_{nq}B'_{ijmn} = a_{ir}a_{js}B_{rspq} \ . \tag{2.119}$$

Multiplying both sides by $a_{gp}a_{hq}$ and using (2.114), we find

$$B'_{ijgh} = a_{ir}a_{js}a_{gp}a_{hq}B_{rspq} \ , \tag{2.120}$$

which is the transformation rule for a Cartesian tensor of rank 4.[4]

The constitutive equation of a fluid should be independent of the orientation of the coordinate system in which it is observed. Hence A_{ij} and B_{ijmn} are isotropic tensors of rank 2 and rank 4 respectively. Since the most general isotropic tensor of rank 2 is a scalar multiple of the Kronecker delta, and since the most general isotropic tensor of rank 4 is

$$\kappa\delta_{in}\delta_{jm} + \lambda\delta_{ij}\delta_{mn} + \gamma\delta_{im}\delta_{jn} \ , \tag{2.121}$$

where κ, λ, γ are scalars, Eq. (2.112) can be written

$$T_{ij} = -p\delta_{ij} + \kappa e_{ji} + \lambda\delta_{ij}e_{kk} + \gamma e_{ij} \ . \tag{2.122}$$

[4]This proves a special case of the **quotient rule for tensors**. See Jeffreys (1931).

where $p, \kappa, \lambda, \gamma$ do not depend explicitly on the kinematic variables. They may depend upon state (non-kinematic) variables. The minus sign is included in the first term as a convenience, since it is sometimes possible to assign this term the significance of a hydrostatic pressure; we adopted earlier a sign convention which makes pressures negative.

The symmetry of the rate of deformation tensor allows us to write (2.122) in a form which we shall call **the constitutive equation for a Newtonian viscous fluid**:

$$T_{ij} = -p\delta_{ij} + \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \; . \tag{2.123}$$

where

$$2\mu = \kappa + \gamma \; . \tag{2.124}$$

It is interesting to note from (2.123) that the symmetry of e_{ij} guarantees the symmetry of T_{ij}. Thus, if the stress tensor is not symmetric,—implying the presence of force couples—it must depend upon something other than the first power of the velocity gradients. Since we have excluded this possibility, we shall henceforth assume that the stress tensor is symmetric.

In terms of the velocity gradients, Eq. (2.123) becomes

$$T_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial v_k}{\partial x_k} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \; . \tag{2.125}$$

If we substitute this result into the momentum equation (2.94), we obtain the **Navier-Stokes equation**

$$\begin{aligned} \rho\frac{Dv_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \lambda\frac{\partial^2 v_j}{\partial x_i \partial x_j} + \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial^2 v_j}{\partial x_i \partial x_j}\right) \\ + \frac{\partial v_j}{\partial x_j}\frac{\partial \lambda}{\partial x_i} + \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\frac{\partial \mu}{\partial x_j} \; . \end{aligned} \tag{2.126}$$

If the fluid is in equilibrium, so that the various velocity derivatives vanish and p becomes the hydrostatic pressure, (2.126) reduces to the **hydrostatic equation**

$$\frac{\partial p}{\partial x_i} = \rho f_i \; . \tag{2.127}$$

Let us discuss briefly the new quantities introduced in this section. The parameters μ and λ, which may depend upon the state variables but not upon kinematic variables, are called the **shear or dynamic coefficient of viscosity** and the **coefficient of volume viscosity**, respectively; they have dimensions of impulse per unit area.

The quantity $\lambda + 2\mu/3$ is called the **bulk viscosity**. It has often been assumed that the **Stokes relation**

$$3\lambda + 2\mu = 0 \tag{2.128}$$

is at least approximately correct if the fluid under consideration is a gas. This relation follows from a tacit assumption used in the first order kinetic theory of gases, viz., that $\overline{T}$ the mean value of the normal stresses, does not depend explicitly on the kinematic variables. To show the connexion, contract the constitutive equation (2.123):

$$\overline{T} \equiv \frac{T_{ii}}{3} = -p + (3\lambda + 2\mu)\frac{e_{ii}}{3} \,. \tag{2.129}$$

Neither the theoretical foundation nor the experimental verification of the Stokes relation is especially convincing. Also, Truesdell (1952) remarked on page 229 that "The Stokes relation implies the anomalous result that a spherical mass of fluid may perform symmetrical oscillations in perpetuity, without frictional loss". Stokes himself never took the relation very seriously, and it is now generally conceded to be invalid, except for monatomic gases, with the hard-to-obtain experimental data leniently interpreted.

For many fluids an accurate and useful approximation is obtained by assuming that λ and μ do not depend upon the state variables at all, but are empirical constants. The Navier-Stokes equation (2.126) then reduces to

$$\rho\frac{Dv_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \lambda\frac{\partial^2 v_j}{\partial x_i \partial x_j} + \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial^2 v_j}{\partial x_i \partial x_j}\right) \,. \tag{2.130}$$

Up to this point the quantity p has been considered merely as "a function of the state variable". In most published work on the theory of viscous compressible fluids, the function used is the thermostatic equation of state. It is universally recognized that this is not a logically valid procedure because, in general, the fluid is not in thermodynamic equilibrium. Since nonequilibrium thermodynamics has not been sufficiently developed to be used successfully in the theory of fluid dynamics, the thermostatic equation of state is used to provide an approximate expression for p. It is then hoped that it is a good approximation, and, except for certain extreme conditions, this has been found to be the case.

It is relevant to note that the constitutive equation can be thought of as a nonequilibrium equation of state. Let us consider a static fluid system for which the pressure is a function only of density and temperature. If the same fluid is in motion, pressure loses its meaning, and must be replaced by the more general quantity, stress. The components of stress then depend on density and temperature, but they also depend on the rate of deformation and probably upon other state variables which vanish at equilibrium, e.g., the material derivative of temperature. As the fluid is allowed to approach a state of equilibrium, the constitutive equation tends to the thermostatic equation of state.

In the hydrodynamics literature one sometimes sees reference to “the dependence of viscosity upon pressure”. This terminology is not really erroneous, but it is somewhat loose. It refers to experiments in which viscosity measurements are carried out within a sealed vessel (or equivalent setup) so that the *ambient* pressure can be regulated. What is then measured is the dependence of viscosity upon density. If the coefficient of viscosity really did depend explicitly on the stress, the constitutive equation would be non-linear and the fluid would be non-Newtonian.

2.2.4 The Constitutive Equation for a Non-Newtonian Viscous Fluid

Unlike air and water which are standard examples of Newtonian fluids, many other fluids, present in nature and technological applications, are non-Newtonian. Typical examples are blood, molten polymers, muds, paints, toothpaste, etc. The branch of continuum mechanics which comprises the study of such materials is called rheology.

Blood is made of aqueous plasma, which behaves essentially as a viscous Newtonian fluid, and of white and red cells and platelets. The presence of these cells modifies the characteristics of the rheology, leading to a class of non-Newtonian fluids which will be the subject of this section.

Molten polymers are more complicated. The long molecular chains affect the flow in complicated ways. The stress-deformation relation is not instantaneous, but depends on the kinematical history of the material, leading to effects such as creep and stress relaxation. Molten polymers are not merely non-Newtonian but viscoelastic. They have memory. Such complex behavior is beyond the scope of this monograph. Interested readers are referred to the book by Deville and Gatski (2012).

An experimenter attempting to measure the viscosity of a non-Newtonian fluid finds that it depends on the rate of shear. However this is only part of the story. Non-Newtonian fluids exhibit some unexpected and counterintuitive physical effects. Consider, for example, the circular Couette flow between two concentric cylinders, with the apparatus not fully filled with the experimental fluid, so that a free surface is in contact with the ambient air. We fix the outer cylinder and rotate the inner one at constant speed. If the fluid is Newtonian, the free surface deforms into a parabolic shape as the fluid is propelled against the outer cylinder by the centrifugal force. In case of a non-Newtonian fluid, the fluid climbs along the inner cylinder, in some cases draining the container as is shown in Fig. 2.5.

This is an example of a **normal stress effect**, also called a **Weissenberg effect**. Because these effects exhibit themselves normal to the direction of shear, it is insufficient to characterize a non-Newtonian fluid solely by an apparent viscosity which depends on the rate of shear. Weissenberg effects may, however, be explained by incorporating non-linear terms into the constitutive equation.

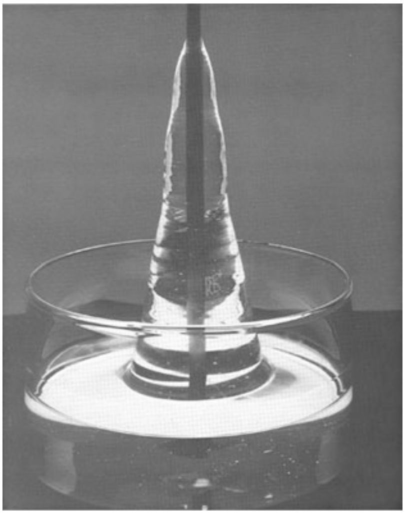


Fig. 2.5 Weissenberg effect with fluid rising up the inner cylinder of a Couette device (Reprinted with permission from: Boger and Walters (1993))

A **non-Newtonian**, but not viscoelastic, **viscous fluid** is therefore defined by the following axiom:

The components of the stress tensor at a given point in space-time depend nonlinearly on the components of the rate of deformation tensor at the same point in space-time.

Let us start from Eq. (2.111). To construct the functional expression of F_{pq}, the principle of material frame-indifference is needed. In continuum mechanics, it is usual that the description of a physical quantity depends on the observer who refers to a coordinate system and a clock. The frame is linked to the space coordinate system x_i and time t. The concept of objectivity or frame-indifference is central in the development of constitutive equations. It is easily understood that the distance between two material points is objective as this scalar is seen the same by two different observers. However the velocity is not frame-indifferent as it results directly from the observations.

Suppose there are two frames with position and time denoted by (x_i, t) and (x_i^*, t^*), respectively. The two frames are in relative motion such that the most general transformation between the observations (x_i, t) and (x_i^*, t^*) of the same phenomenon or event is given by

$$x_i^* = Q_{ij}(t)x_j + c_i(t), \quad t^* = t - \alpha \,, \tag{2.131}$$

where $Q_{ij}(t)$ is an orthogonal rotation tensor, $c_i(t)$ a translation vector, and α a scalar constant. Equation (2.131) is the most general description of a **rigid body motion**. We note that imposing **objectivity** of the stress tensor will be equivalent to considering that rigid body motion does not affect the stress.

Let us now consider two simultaneous events registered as (x_i^1, t) and (x_i^2, t) by the first observer, and (x_i^{1*}, t) and (x_i^{2*}, t) by the second observer. For these two events, the relative positions seen by the observers are $u_i = x_i^2 - x_i^1$ and $u_i^* = x_i^{2*} - x_i^{1*}$, respectively. From (2.131), one obtains

$$u_i^* = Q_{ij} u_j \quad \text{or} \quad \mathbf{u}^* = \mathbf{Q}\mathbf{u}\,. \tag{2.132}$$

Vector fields which transform according to (2.132) are called **objective** or frame-indifferent.

With the help of the definition of an objective vector, we define an objective second order tensor. The first observer spots the two vectors v_i and w_i that are linked by the second order tensor L_{ij}

$$w_i = L_{ij} v_j\ . \tag{2.133}$$

As v_i and w_i are objective vectors, the starred observer sees them as $w_i^* = Q_{ij} w_j$ and $v_i^* = Q_{ij} v_j$. This observer considers the second order tensor as L_{ij}^* such that $w_i^* = L_{ij}^* v_j^*$. The link between L_{ij} and L_{ij}^* is computed

$$w_i^* = Q_{ij} w_j = Q_{ij} L_{jk} v_k = Q_{ij} L_{jk} Q_{lk} v_l^*\ , \tag{2.134}$$

i.e.

$$\mathbf{w}^* = \mathbf{Q}\mathbf{L}\mathbf{Q}^T \mathbf{v}^*\ , \tag{2.135}$$

where the upper index T indicates the transpose of the tensor. This last relation yields

$$\mathbf{L}^* = \mathbf{Q}\mathbf{L}\mathbf{Q}^T\ . \tag{2.136}$$

The tensor fields that are transformed according to (2.136) by a change of frame are called objective tensors.

Let us examine how the velocity gradient tensor is transformed by the change of frame. Taking the material time derivative of (2.131) one gets

$$v_i^* = \dot{c}_i + \dot{Q}_{ij} x_j + Q_{ij} v_j\ , \tag{2.137}$$

where the overdots denote material differentiation. The components of the starred velocity gradient are

$$\frac{\partial v_i^*}{\partial x_j^*} = \frac{\partial v_i^*}{\partial x_k} \frac{\partial x_k}{\partial x_j^*}\,. \tag{2.138}$$

Equation (2.131) yields

$$\frac{\partial x_j^*}{\partial x_k} = Q_{jk}\,.$$

The inverse $\partial x_k/\partial x_j^*$ is $Q_{kj}^{-1} = Q_{kj}^T$. The evaluation of $\partial v_i^*/\partial x_k$ is carried out via Eq. (2.137). We have

$$\frac{\partial v_i^*}{\partial x_k} = \dot{Q}_{ik} + Q_{il}\frac{\partial v_l}{\partial x_k} .$$

Combining the various relations, one writes

$$(\nabla \mathbf{v})^* = (\mathbf{Q}\nabla \mathbf{v} + \dot{\mathbf{Q}})\mathbf{Q}^T , \tag{2.139}$$

where the tensor $\dot{\mathbf{Q}}\mathbf{Q}^T$ is antisymmetric. Indeed, as $\mathbf{Q}$ is orthogonal, one has $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$. Taking the time derivative of this relation, one concludes that

$$\dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = 0 .$$

From Eqs. (2.10) and (2.139) it is easy to show that

$$e_{ij}^* = Q_{i\ell}\frac{\partial v_\ell}{\partial x_k}Q_{jk} . \tag{2.140}$$

The rate of deformation tensor is therefore an objective tensor.

The constitutive equation has to satisfy the principle of material frame indifference, which states that the functional F_{pq} is invariant for each continuous change of frame. In mathematical terms, the principle demands

$$T_{pq}^* = F_{pq}(e_{mn}^*) . \tag{2.141}$$

The function F_{pq} must verify the identity

$$Q_{ij}F_{jk}(e_{mn})Q_{lk} = F_{jk}(Q_{im}e_{mn}Q_{jn}) , \tag{2.142}$$

or

$$\mathbf{QF(e)Q}^T = \mathbf{F}(\mathbf{QeQ}^T) . \tag{2.143}$$

This relation expresses that the function $F_{pq}(e_{mn})$ is an isotropic function of the symmetric tensor e_{mn}. The following theorem due to Rivlin and Ericksen (1955) gives the constitutive equation for the non-Newtonian viscous fluid

Theorem 2.1. $\mathbf{T}$ *is an isotropic tensor function of* $\mathbf{e}$ *if and only if*

$$T_{ij}(e_{mn}) = \varphi_0\delta_{ij} + \varphi_1 e_{ij} + \varphi_2 e_{ik}e_{kj} , \tag{2.144}$$

where $\varphi_0, \varphi_1, \varphi_2$ *are scalar functions of the invariants of* e_{mn}.

An easy proof of the theorem may be found in the appendix of Spencer (2004).

Recall that the invariants of a tensor **e** are defined as

$$\mathrm{I}_1(\mathbf{e}) = \mathrm{tr}\,\mathbf{e} = e_{ii} \,, \tag{2.145}$$

$$\mathrm{I}_2(\mathbf{e}) = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{e})^2 - \mathrm{tr}\,\mathbf{e}^2\right] \,, \tag{2.146}$$

$$\mathrm{I}_3(\mathbf{e}) = \det\mathbf{e} \,, \tag{2.147}$$

where the notation tr denotes the **trace** of a tensor. From the **Cayley-Hamilton theorem**, which states that a tensor satisfies its own characteristic equation, **e** then satisfies

$$\mathbf{e}^3 - \mathrm{I}_1(\mathbf{e})\,\mathbf{e}^2 + \mathrm{I}_2(\mathbf{e})\,\mathbf{e} - \mathrm{I}_3(\mathbf{e})\,\mathbf{I} = 0 \,. \tag{2.148}$$

Therefore **e** is now expressible as a matrix polynomial in terms of lower degree matrices. In general, all matrices $\mathbf{e}^n, n \geq 3$, can be expressed as linear combinations of $\mathbf{e}^2$, **e** and **I**. This is the reason why the constitutive equation (2.144) involves **e** up to the power 2.

Note that for an incompressible fluid, $\mathrm{I}_1(\mathbf{e})$ vanishes and the functions $\varphi_i, i = 0, 1, 2$ are only functions of the second and third invariants. Furthermore, if the fluid is at rest in hydrostatic equilibrium, the only stress remaining in the fluid is the hydrostatic pressure. Therefore the function φ_0 is identified with the pressure field and the constitutive equation for the non-Newtonian viscous incompressible fluid reads

$$T_{ij}(e_{mn}) = -p\,\delta_{ij} + \varphi_1(\mathrm{I}_2(\mathbf{e}), \mathrm{I}_3(\mathbf{e}))e_{ij} + \varphi_2(\mathrm{I}_2(\mathbf{e}), \mathrm{I}_3(\mathbf{e}))e_{ik}e_{kj} \,, \tag{2.149}$$

where p is a scalar pressure field which results from the incompressibility constraint. This model is the Reiner-Rivlin fluid, a particular case of the Rivlin-Ericksen fluid of second-order. Note also that if $\varphi_2 \equiv 0$ and $\varphi_1 = 2\mu$, the Newtonian fluid is recovered.

2.3 Energy Considerations

Looking back, we find that we have formulated viscous hydrodynamics in terms of ten unknown quantities:

Six independent components of stress;
Three components of velocity;
One state variable (the density).

Connecting these quantities, we have:

The constitutive equation (2.123)-6 component equations;
The Navier-Stokes equation (2.126)-3 component equations;
The equation of continuity (2.35).

Everything seems in order, but look further: the parameters p, λ, μ will, in general, depend upon other state variables besides the density. These state variables may, in turn, depend upon the ten quantities listed above—perhaps explicitly, perhaps implicitly. Thus in order to complete the formulation of hydrodynamics, we must look into the relationships between the ten quantities and state variables, i.e., we must consider energy transfer.

We shall make no attempt to consider all possible forms of energy which may be present in a flowing viscous fluid. Rather, we shall restrict ourselves to the relationships between mechanical energy and thermal energy, so that the only relevant state variables are the fluid density and temperature. We thereby exclude from our scope certain interesting and important areas of hydrodynamics—flow of chemically reacting fluids, magnetohydrodynamics, flow of dissociating gases. To treat these subjects, we would have to require that the readers (and the authors!) have a much broader background in physics and chemistry than is needed for the more traditional hydrodynamics which forms the subject of this book.

2.3.1 *Conservation of Energy in Continuous Media*

In classical mechanics, the kinetic energy of a particle having mass m and speed v is defined as $\frac{1}{2}mv^2$. Using an obvious generalization, we define the **kinetic energy** K of a fluid mass by

$$K = \frac{1}{2} \iiint_{\mathbf{V}^*} \rho v_j v_j \, \mathbf{dV}^* \, , \tag{2.150}$$

where $\mathbf{V}^*$ is the volume occupied by the fluid mass at the instant of time under consideration. If at the reference time T the fluid mass occupied the volume $\mathbf{V}$, (2.150) can be written

$$K = \frac{1}{2} \iiint_{\mathbf{V}} \rho(x_i, t) v_j(x_i, t) v_j(x_i, t) \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \mathbf{dV} \, . \tag{2.151}$$

If we take the material derivative of both sides and use Eq. (2.25), we find

$$\frac{DK}{Dt} = \iiint_{\mathbf{V}} \rho(x_i, t) v_j(x_i, t) \frac{Dv_j}{Dt} \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \mathbf{dV} \, , \tag{2.152}$$

or, equivalently,

$$\frac{DK}{Dt} = \iiint\limits_{\mathbf{V}^*} \rho v_i \frac{Dv_i}{Dt} \mathbf{dV}^* \, . \tag{2.153}$$

The kinetic energy of a fluid mass is only part of its total energy. The remainder is called the **internal energy**, and is denoted here by E. It can be expressed in terms of an **internal energy density** e, according to

$$E = \iiint\limits_{\mathbf{V}^*} \rho e \, \mathbf{dV}^* \, . \tag{2.154}$$

By a calculation similar to the one just carried out for the kinetic energy,

$$\frac{DE}{Dt} = \iiint\limits_{\mathbf{V}^*} \rho \frac{De}{Dt} \mathbf{dV}^* \, . \tag{2.155}$$

Conservation of energy requires that the material derivative of the total energy $K + E$ be equal to the sum of the rate at which mechanical work is done on the fluid mass and the rate at which thermal energy enters the fluid mass.

If we require that $\mathbf{V}^*$ be entirely immersed in the fluid, so that no surface tractions act on its boundary, mechanical work is done on the fluid within $\mathbf{V}^*$ only by the body forces and by the stress vectors acting on $\mathbf{S}^*$, the boundary of $\mathbf{V}^*$. The rate at which work is being done on a particle is equal to the dot product of the particles velocity and the force acting on the particle. Therefore the rate at which work is done on the fluid within $\mathbf{V}^*$ is

$$\iiint\limits_{\mathbf{V}^*} \rho f_i v_i \mathbf{dV}^* + \iint\limits_{S^*} T_{ji} n_j v_i \mathbf{dS}^* = \iiint\limits_{\mathbf{V}^*} \left[\rho f_i v_i + \frac{\partial (T_{ji} v_i)}{\partial x_j} \right] \mathbf{dV}^* \, , \tag{2.156}$$

where Gauss's divergence theorem is used to obtain the right side from the left.

Let us now turn our attention to the rate at which thermal energy enters the fluid within $\mathbf{V}^*$. First, heat may be conducted across $\mathbf{S}^*$. To account for this, we introduce a new quantity, the **heat flux vector** q_i, which denotes the amount of heat per unit area per unit time which crosses a surface normal to the x_i-axis in the increasing x_i-direction. Thus the heat flux per unit area across a surface having direction cosines n_i is $q_i n_i$ in the increasing n_i-direction. The rate at which heat is conducted into the fluid within $\mathbf{V}^*$ is, therefore,

$$-\iint\limits_{\mathbf{S}^*} q_i n_i \mathbf{dS}^* = -\iiint\limits_{\mathbf{V}^*} \frac{\partial q_i}{\partial x_i} \mathbf{dV}^* \, . \tag{2.157}$$

Next, there may be heat sources within $\mathbf{V}^*$. We assume that any heat sources which exist are continuously distributed, so that we can characterize them by associating with each point of the volume a scalar Q, the **heat supplied** per unit volume per unit time. Therefore the rate at which heat sources supply energy to the fluid within $\mathbf{V}^*$ is

$$\iiint\limits_{\mathbf{V}^*} Q\,\mathbf{dV}^* \,. \tag{2.158}$$

Finally, radiant energy may be entering or leaving the fluid within $\mathbf{V}^*$. This would be of importance if, for instance, different parts of the fluid were at significantly different temperatures. In this case the equations of hydrodynamics become coupled to the equations of radiative transfer. The resultant theory is quite useful in the study of stellar atmospheres, and in certain other problems where large temperature differences obtain, but we shall consider it to be beyond our scope. We shall postulate that radiative transfer of energy between different parts of the fluid does not occur.

We need not, however, exclude the possibility that the fluid receives radiant energy from an external agency, for such energy can be considered part of the heat source Q. This approach can be used to study an important problem: the effect of insolation (absorption of radiant solar energy) on oceanographic and meteorological fluid motions.

We have now accounted for all the energy being added to the fluid within $\mathbf{V}^*$. Note that this is not at all the same as the energy being added to the volume $\mathbf{V}^*$ itself. Fluid is moving into and out of $\mathbf{V}^*$, and will, in general, carry energy with it. If we wanted to obtain an energy balance based upon the rate of change of energy within the fixed volume $\mathbf{V}$, we would have to account for this convected energy. Although this can be done, it is slightly easier to keep track of the energy possessed by the fluid mass which, at time t, happens to occupy the volume $\mathbf{V}^*$. We are now in a position to state the law for energy conservation constituting the first principle of thermodynamics.

First principle of thermodynamics *The material time derivative of total energy in* $\mathbf{V}^*$ *is equal to the sum of the power of volume and contact forces and the rate of heat received by the volume under consideration.*

Adding up the contributions from (2.156)–(2.158), we have

$$\frac{D}{Dt}(K+E) = \iiint\limits_{\mathbf{V}^*} \left[\rho f_i v_i + \frac{\partial(T_{ji}v_i)}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + Q\right]\mathbf{dV}^* \,. \tag{2.159}$$

If we now insert (2.153) and (2.155) into (2.159) and note that the resulting equation must hold for an arbitrary choice of the volume $\mathbf{V}^*$, subject only to the restriction that $\mathbf{V}^*$ be immersed in the fluid, we obtain

$$\rho v_i \frac{Dv_i}{Dt} + \rho \frac{De}{Dt} = \rho f_i v_i + \frac{\partial (T_{ji} v_i)}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + Q \,. \tag{2.160}$$

This can be simplified by using Eqs. (2.9) and (2.94). We have

$$\begin{aligned} \frac{\partial (T_{ji} v_i)}{\partial x_j} &= T_{ji} \frac{\partial v_i}{\partial x_j} + v_i \frac{\partial T_{ji}}{\partial x_j} \\ &= T_{ji}(e_{ij} + \omega_{ij}) + \rho v_i \left(\frac{Dv_i}{Dt} - f_i \right) \\ &= T_{ji} e_{ij} + \rho v_i \left(\frac{Dv_i}{Dt} - f_i \right) . \end{aligned} \tag{2.161}$$

so that (2.160) becomes

$$\rho \frac{De}{Dt} = T_{ji} e_{ij} - \frac{\partial q_i}{\partial x_i} + Q \,, \tag{2.162}$$

the **Fourier-Kirchhoff-Neumann energy equation**.

The scalar $T_{ji} e_{ij}$ is the **stress power**, the rate at which internal mechanical work is being done per unit volume.

2.3.2 The Energy Equation for a Newtonian Viscous Fluid

It should be noted that nowhere in the derivation of (2.162) was it assumed that we were dealing with a Newtonian fluid. Equation (2.162) is valid for all continuous media. By inserting the constitutive equation (2.125) and the relationship (2.10), we obtain

$$\rho \frac{De}{Dt} = -p \frac{\partial v_i}{\partial x_i} + \Phi - \frac{\partial q_i}{\partial x_i} + Q \,, \tag{2.163}$$

where Φ is the **dissipation function**, denoted by

$$\Phi = \lambda \left(\frac{\partial v_i}{\partial x_i} \right)^2 + \mu \frac{\partial v_i}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) . \tag{2.164}$$

If we employ **Fourier's law of heat conduction**,

$$q_i = -k \frac{\partial \Theta}{\partial x_i} \,, \tag{2.165}$$

where k is the **coefficient of heat conductivity** and Θ is the absolute temperature, (2.163) becomes

$$\rho \frac{De}{Dt} = -p \frac{\partial v_i}{\partial x_i} + \Phi + \frac{\partial}{\partial x_i}\left(k \frac{\partial \Theta}{\partial x_i}\right) + Q \ . \tag{2.166}$$

2.3.3 Second Principle of Thermodynamics

From statistical physics considerations, we know that physical phenomena evolve in an irreversible way. This irreversibility is taken into account by the physical quantity called entropy, which is a measure of the disorder in a system. In the continuum mechanics approach, the **Clausius-Duhem inequality** expresses the second principle of thermodynamics in terms of **entropy**.

Second principle of thermodynamics *The rate of change of the entropy in* $\mathbf{V}^*$ *is greater than or equal to the sum of the volume distribution of entropy sources and the entropy flux across the boundary.*

We write

$$\frac{d}{dt} \iiint_{\mathbf{V}^*} \rho s \, \mathbf{dV}^* \geq \iiint_{\mathbf{V}^*} \frac{Q}{\Theta} \mathbf{dV}^* - \iint_{\mathbf{S}^*} \frac{q_i n_i}{\Theta} \mathbf{dS}^* \ , \tag{2.167}$$

where s is the entropy density per unit mass. We observe that the heat received by the fluid is decomposed into two terms: the first one for the volume contribution and the second one for the surface heat exchange.

Carrying through the algebra, this principle gives the Clausius-Duhem inequality

$$\rho \frac{Ds}{Dt} \geq \frac{1}{\Theta}\left(\rho \frac{De}{Dt} - T_{ij} e_{ij}\right) + \frac{1}{\Theta^2} q_i \frac{\partial \Theta}{\partial x_i} \ , \tag{2.168}$$

which must be satisfied for every thermodynamical process.

At this stage, we have two more unknown quantities, namely the internal energy density e and the entropy density s. It is customary to consider them depending only on the temperature field and the density.

Let us examine the consequences of the Clausius-Duhem inequality. To this end, we define the deviatoric rate of deformation tensor as

$$e_{ij}^d = e_{ij} - \frac{1}{3} e_{mm} \delta_{ij} \ , \tag{2.169}$$

so that $e_{ii}^d = 0$. We then assume

Postulate *The Clausius-Duhem inequality (2.168) is always satisfied for arbitrary and independent histories of the thermodynamic variables, i.e. the temperature* Θ*, the density* ρ*, the deviatoric rate of deformation* e^d_{ij}*, and the temperature gradient* $(\partial\Theta/\partial x_i)$*.*

Here history means the time evolution of the variables, for a given material position, and is called the **thermodynamic process**. We notice that the constitutive equations express T_{ij}, q_i, e, s according to the thermodynamic process. The history of ρ will provide the history of e_{ii} by the law of mass conservation. Let us define η by

$$\eta = 3\lambda + 2\mu \,. \tag{2.170}$$

The constitutive relationship for the stress tensor (2.123) is written as

$$T_{ij} = -p\delta_{ij} + \eta e_{kk}\delta_{ij} + 2\mu e^d_{ij} \,. \tag{2.171}$$

The Clausius-Duhem inequality (2.168) yields the following expression valid for all viscous Newtonian fluids

$$\rho\left(\frac{De}{Dt} - \Theta\frac{Ds}{Dt}\right) - \frac{p}{\rho}\,\frac{D\rho}{Dt} \le \eta\, e^2_{mm} + 2\mu\, e^d_{ij}\, e^d_{ij} + \frac{k}{\Theta}\left(\frac{\partial\Theta}{\partial x_i}\right)\left(\frac{\partial\Theta}{\partial x_i}\right). \tag{2.172}$$

With the postulate about the thermodynamic process, the left hand side of this vanishes:

$$\rho\left(\frac{De}{Dt} - \Theta\frac{Ds}{Dt}\right) - \frac{p}{\rho}\,\frac{D\rho}{Dt} = 0 \,. \tag{2.173}$$

The reader is referred to Botsis and Deville (2006) for a detailed proof of (2.173). Replacing the material time derivative in (2.173) by differentials, one finds the classical state relation

$$\rho(de - \Theta\, ds) - \frac{p}{\rho}\, d\rho = 0 \,. \tag{2.174}$$

In order that the right hand side of the inequality (2.172) be always positive, it is necessary and sufficient that

$$\eta \ge 0, \quad \mu \ge 0, \quad k \ge 0 \,. \tag{2.175}$$

Therefore the inequality right hand side is composed of linear combination of squares of independent terms, which must be a quadratic positive definite form with positive coefficients.

We note that the Stokes relation (2.128) is a special case of (2.175).

If we assume that the viscous fluid is such that the internal energy does not depend on the density ρ, then $e = e(\Theta)$ and this is the model of an ideal gas. The equation of state for the pressure is

$$p = \rho R \Theta \,, \tag{2.176}$$

where the constant R is that of the ideal gas with SI units $Jkg^{-1}K^{-1}$. For air, $R = 287Jkg^{-1}K^{-1}$. Note that the experimental evidence shows that μ, λ and k do not depend on the temperature for an ideal gas.

2.4 Incompressible Fluids

In order to simplify the solution of the hydrodynamic equations, it is often assumed that the fluid under investigation is incompressible. However our intuitive concept of incompressibility involves more than is evident at first. Let us examine this concept in reference to the hydrostatic case, where it can be discussed in terms of pressure, rather than stress.

It is sometimes said that an incompressible material is characterized by the equation of state

$$p = f(\Theta); \tag{2.177}$$

the possibility that $f(\Theta)$ is constant is not excluded.

In the sense of Sect. 2.2.3, this is not an equation of state at all; it's a constraint. Given ρ and Θ, the equation of state should determine a unique value of the hydrostatic pressure. Thus an equation of state for a fluid satisfying (2.177) would be a functional relationship determining a unique value for p for each value of Θ, over some range of Θ.

However, if there were such an equation of state, it would not be possible for an incompressible fluid to be simultaneously in static equilibrium and thermal equilibrium when body forces are present.[5] Although this does not contradict any of the assumptions introduced previously, it certainly fails to reflect our intuitive concept of how an incompressible fluid should behave. Surely a tower of water can stand at equilibrium in a gravity field, yet we feel that the compressibility of water should usually be negligible.

[5] For thermal equilibrium, the temperature is uniform. Hence, by (2.177), the density is also uniform. For static equilibrium the hydrostatic equation (2.127) then predicts that p varies along the body force field. Hence, if each value of Θ were to determine a unique value of p, the temperature could not be uniform after all.

Therefore *an incompressible fluid has no thermal equation of state*. The assumption of incompressibility is a constraint, embodied by (2.177). The pressure is the reaction to that constraint, cf. Langlois (1971).[6] As such it is a *primitive unknown* to be determined, for conditions of static equilibrium, from the hydrostatic equation and, more generally, to be determined along with the velocity components and temperature from the equations of viscous flow. The fluid density, and hence the two coefficients of viscosity, are uniquely determined from the temperature via (2.177).

Another way to look at incompressibility consists in linking incompressibility to isochoric motion which imposes $\partial v_i/\partial x_i = 0$. Consequently by mass conservation $D\rho/Dt = 0$ and $\rho =$ *constant*.

Actually the simplification of the hydrodynamic equations usually associated with the incompressibility assumption is obtained only when we introduce a second assumption: that ρ and μ are constant. For this case, the equation of continuity (2.36) and the Navier-Stokes equation (2.126) are not coupled to the energy equation. The four dependent variables, $p, v_i (i = 1, 2, 3)$, are determined from four equations obtained from (2.36) and (2.126):

$$\frac{\partial v_i}{\partial x_i} = 0 \,, \tag{2.178}$$

$$\rho \frac{Dv_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 v_i \,, \tag{2.179}$$

subject to boundary conditions which we shall develop later. Observe that the volume viscosity λ does not appear in (2.179). As we showed in Sect. 2.1.1, for v_3 independent of x_3 the continuity equation (2.178) is automatically satisfied by introducing a stream function ψ, such that

$$v_1 = \frac{\partial \psi}{\partial x_2} \,, \qquad v_2 = -\frac{\partial \psi}{\partial x_1} \,. \tag{2.180}$$

We now introduce (2.180) into the three equations represented by (2.179). For economy of subscripts, we let

$$x_1 = x, \quad x_2 = y \,, \qquad x_3 = z, \quad v_3 = w \,. \tag{2.181}$$

[6] Truesdell (1966), pages 42–44, treats the matter for materials much more general than Newtonian fluids.

With (2.180) and (2.181), Eq. (2.179) become

$$\rho\left(\frac{\partial^2\psi}{\partial t\partial y}+\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}+w\frac{\partial^2\psi}{\partial y\partial z}\right)=\rho f_1-\frac{\partial p}{\partial x}+\mu\nabla^2\left(\frac{\partial\psi}{\partial y}\right), \tag{2.182}$$

$$\rho\left(\frac{\partial^2\psi}{\partial t\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}+w\frac{\partial^2\psi}{\partial x\partial z}\right)=-\rho f_2+\frac{\partial p}{\partial y}+\mu\nabla^2\left(\frac{\partial\psi}{\partial x}\right), \tag{2.183}$$

$$\rho\left(\frac{\partial w}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial w}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial w}{\partial y}\right)=\rho f_3-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right). \tag{2.184}$$

We now differentiate (2.182) with respect to y, differentiate (2.183) with respect to x, and add the resulting equations. We thus eliminate the pressure and obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\\&+\left(\frac{\partial w}{\partial x}\frac{\partial}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial}{\partial y}+w\frac{\partial^2}{\partial x^2}+w\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi}{\partial z}\\&=\left(\frac{\partial f_1}{\partial y}-\frac{\partial f_2}{\partial x}\right)+\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\psi+\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2\psi}{\partial z^2},\end{aligned} \tag{2.185}$$

where $\nu=\mu/\rho$ is the **kinematic coefficient of viscosity**.

This result is not especially useful as its stands. However, if the flow is two-dimensional, so that w is zero and ψ is independent of z, a further simplification results: Eq. (2.184) becomes trivial and Eq. (2.185) reduces to

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\\&=\left(\frac{\partial f_1}{\partial y}-\frac{\partial f_2}{\partial x}\right)+\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\psi.\end{aligned} \tag{2.186}$$

In this special case, then, the flow field is determined from one partial differential equation in one dependent variable.

2.4.1 The Boussinesq Approximation

An important matter forces us to reconsider the simplification of incompressibility. We know that gradients of temperature or solute concentration in a liquid can lead to significant buoyant convection, even if the resulting density differences are only a few hundreths of a percent. The way out is to use the **Boussinesq approximation** (after Joseph Valentin Boussinesq 1842–1929): fluid density changes are ignored except in the body force term. This is widely useful in practice. The logical inconsistency of accounting for density changes "here" but not "there" can be circumvented by taking a slightly different view: regard the liquid as an incompressible fluid acted upon by a body force which depends on the temperature and/or the concentration of a solute.

A note about terminology: the words "advection" and "convection" are sometimes used as synonyms. However in some branches of applied hydrodynamics, notably those pertaining to the earth sciences, they are assigned different meanings. "Advection" is generic, simply referring to transport of mass, heat, momentum, etc. carried about by a moving fluid. "Convection" is reserved for transport when the fluid motion originates from, or is strongly influenced by, density variations as discussed in this subsection. Thus, a meteorologist will refer to "advection fog" but "cumulus convection".

For the incompressible fluid, the internal energy density e depends in general on the entropy s and the temperature Θ, i.e. $e = e(s, \Theta)$, cf. Eq. (2.174). The heat capacity c is defined by the relation

$$c = \frac{\partial e}{\partial \Theta} \ . \tag{2.187}$$

Here the heat capacity c is unique because the difference between heat capacity at constant pressure and constant volume, c_p and c_v, respectively, vanishes, cf. Panton (1984). In the Boussinesq approximation, the density is assumed to be a constant everywhere in the momentum and energy equations, except in the body force term. There, the density varies according to the following equation of state

$$\rho = \rho_0[1 - \alpha(\Theta - \Theta_0)] \ , \tag{2.188}$$

where α is the volume expansion coefficient and $\rho_0 = \rho(\Theta_0)$ with Θ_0 a reference temperature. For natural convection, the body force term f_i is equal to the gravity acceleration g_i. The energy equation within the Boussinesq approximation becomes

$$\rho_0 c \frac{D\Theta}{Dt} = \Phi + \frac{\partial}{\partial x_i}\left(k \frac{\partial \Theta}{\partial x_i}\right) + Q \ , \tag{2.189}$$

the solution of which will yield the temperature field.

2.5 The Hydrodynamic Equations in Summary

We have observed that under very general conditions the flow of a Newtonian fluid is described by the partial differential equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 , \tag{2.190}$$

$$\rho \frac{D v_i}{D t} = \rho f_i - \frac{\partial p}{\partial x_i} + \lambda \frac{\partial^2 v_j}{\partial x_i \partial x_j} + \mu \left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial^2 v_j}{\partial x_i \partial x_j} \right) + \frac{\partial v_j}{\partial x_j} \frac{\partial \lambda}{\partial x_i} + \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \frac{\partial \mu}{\partial x_j} , \tag{2.191}$$

$$\rho \frac{De}{Dt} = -p \frac{\partial v_i}{\partial x_i} + \frac{\partial}{\partial x_i} \left(k \frac{\partial \Theta}{\partial x_i} \right) + Q + \lambda \left(\frac{\partial v_i}{\partial x_i} \right)^2 + \mu \frac{\partial v_i}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) . \tag{2.192}$$

Thus we have five equations of motion, three of which are represented by the vector equation (2.191). The five dependent variables are: the density ρ; the absolute temperature Θ; the three velocity components v_i. The parameters λ, μ, k, p, e are usually assumed to be known functions of ρ and Θ, although nothing in our development prohibits k and e from depending upon kinematic variables. The quantities f_i and Q are specified functions of time and space.

If the fluid is incompressible, the governing equations not only become simpler but also change their fundamental character: if the density does not depend on the temperature, so that the continuity equation (2.190) becomes

$$\frac{\partial v_i}{\partial x_i} = 0 , \tag{2.193}$$

all the terms involving the volume viscosity λ drop out of the Navier-Stokes equation (2.191); the pressure p is no longer a specified function of the state variables, but is a primitive unknown; if the coefficient of viscosity μ does not depend upon temperature,[7] the dependent variables v_i and p can be determined without recourse to the energy equation (2.192), which now is used only to determine the temperature distribution once the flow field is obtained. The equations determining the flow are then

[7] This monograph is concerned almost exclusively with incompressible, constant-density, constant-viscosity flow. Somewhat imprecisely, but more succinctly, and more in keeping with common practice, we shall refer to a fluid flowing under these conditions as an **incompressible fluid**.

$$\frac{\partial v_i}{\partial x_i} = 0 \,, \tag{2.194}$$

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 v_i \,, \tag{2.195}$$

where $\nu = \mu/\rho$.

2.5.1 Boussinesq Equations

With the assumptions given in Sect. 2.4.1 and with Eqs. (2.188) and (2.189), we write the Boussinesq equations

$$\frac{\partial v_i}{\partial x_i} = 0 \,, \tag{2.196}$$

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = g_i \left[1 - \alpha(\Theta - \Theta_0)\right] - \frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu_0 \nabla^2 v_i \,, \tag{2.197}$$

$$\rho_0 c \left(\frac{\partial \Theta}{\partial t} + v_j \frac{\partial \Theta}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(k \frac{\partial \Theta}{\partial x_i}\right) + Q + \mu \frac{\partial v_i}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \,, \tag{2.198}$$

where $\nu_0 = \mu/\rho_0$ and g_i is the i-th component of the gravitational acceleration.

2.6 Boundary Conditions

A system of partial differential equations is not sufficient to determine the dependent variables in a mathematical problem: we need to impose initial conditions and boundary conditions. The specification of initial conditions is usually quite obvious and will not be treated here. Suffice it to say that, for an incompressible fluid, the initial velocity field must be divergence free. However in years past, the boundary conditions of hydrodynamics were the subject of considerable debate. The issue is now fairly well settled, except for certain aspects which we'll mention presently.

2.6.1 The No-Slip Condition

Under most conditions of importance the layer of fluid in contact with a solid body has the same velocity as the body. This statement cannot be proved from hydrodynamic considerations, but has been quite convincingly demonstrated by many experiments. Also statistical mechanical investigations of fluids tend to support its correctness. It is now generally recognized as the boundary condition to be imposed on the velocity components v_i, except under certain extreme conditions. For the first century or so after Newton, however, the kinematic conditions at solid-fluid interfaces were not understood at all. Most authors of this period included slip terms in their solutions to the hydrodynamic equations. As the nineteenth century progressed, however, more and more experimental evidence piled up in favor of the no-slip condition. For a viscous fluid, we write

$$v_{i,fluid} = v_{i,wall} \ . \tag{2.199}$$

If we were examining the case of an inviscid fluid, this condition would be relaxed as the fluid would no longer stick to the wall. The boundary condition imposes the continuity of normal velocity components

$$v_{i,fluid} n_i = v_{i,wall} n_i \ , \tag{2.200}$$

where n_i are the components of the unit normal vector to the wall.

In order to anticipate the circumstances under which the no-slip condition breaks down, let us discuss briefly and qualitatively its physical origin. First there is the effect of surface asperities: the fluid tends to become trapped in the tiny pockets and crevasses which are present on any solid surface. Also there is adhesion: there are very real attractive forces between the molecules of the solid and those of the fluid.

We may therefore expect the no-slip condition to break down whenever the important geometrical lengths traversed by the fluid molecules are nearly as short as their mean free path. Although this does not usually happen, there are at least three conditions of importance where it does. The fluid under consideration may be a rarified gas: treatment of problems involving flight through the upper atmosphere cannot be based on the no-slip condition. The region occupied by the fluid may be very short in one or more directions: this can be of importance in certain lubrication problems. The fluid may be flowing at extremely high speeds, so that the molecules move a long way between collisions: the branch of hydrodynamics called **hypersonics** deals with flow under this condition. A relevant reference is Hayes and Probstein (2004).

The no-slip condition also breaks down when new fluid-solid interfaces are being formed, so that molecular forces have not yet had time to become established. This can happen, for instance, in spreading problems. Even a casual reading of the literature on spreading is sufficient to convince the reader that he has discovered a poorly understood aspect of hydrodynamics.

The same arguments used to establish the no-slip condition for fluid-solid interfaces lead to the conclusion that the velocity is continuous across fluid-fluid interfaces, and the arguments are subject to the same reservations.

The no-slip condition is not always satisfactory as some physical phenomena occurring at the micro scale fail to be modeled with this boundary condition. The same experimental evidence holds also for viscoelastic fluids. A **slip condition** is set up which allows the fluid to flow without full adherence along the wall. If τ_i are the components of the unit tangent vector to the boundary, the slip condition reads

$$v_i \tau_i = \chi T_{ij} n_j \tau_i \ , \tag{2.201}$$

where χ is the slip coefficient chosen, most of the time, on the basis of experimental results. Condition (2.201) states that the tangential velocity is proportional to the tangential component of the stress vector.

A very interesting problem, both from the theoretical and experimental point of view, is related to the **moving contact line**. This occurs when the interface of a viscous fluid with air, considered as an inviscid fluid, is in contact with a moving solid wall. Coating is the industrial process that applies a very thin liquid layer on a solid support, involving very often the presence of contact lines, see e.g. Weinstein and Ruschak (2004). Dynamic wetting is a key issue of the problem. The reader is referred to a recent review for the latest experimental and theoretical developments on the subject, cf. Snoeijer and Andreotti (2013).

2.6.2 Force Boundary Conditions

It may happen that the velocities of the surfaces bounding a fluid are not all specified, but that the surface tractions acting on some of these boundaries are known. It would appear superficially that Newton's law of action and reaction requires the stress vector at a point on the fluid surface to equal the surface traction applied at the same point. This is not quite correct, however, because of the fluid's surface tension. The free surface of a liquid possesses a finite energy τ per unit area; τ is called the **coefficient of surface tension**. Consequently work must be done just to increase the surface area of a body of liquid. In this section, and in many applications, τ is assumed constant. That assumption will be relaxed in the next section.

2.6.2.1 Free Surface Conditions

When the viscous fluid (medium I) is in contact with a gas (medium II), one assumes that the contact forces are in equilibrium and one writes the relation:

$$F_{I,i} + F_{II,i} = 0 \ , \tag{2.202}$$

with the definition (2.80) for the stress vector. With (2.202) and the equality $n_{I,i} = -n_{II,i}$, one obtains

$$T_{I,ji} \, n_{I,j} = T_{II,ji} \, n_{I,j} \ . \tag{2.203}$$

The free surface conditions are projected on the unit normal and tangent vectors to yield

$$T_{I,ji}\, n_{I,j}\, n_{I,i} = T_{II,ji}\, n_{I,j}\, n_{I,i} \ , \tag{2.204}$$

$$T_{I,ji}\, n_{I,j}\, \tau_{I,i} = T_{II,ji}\, n_{I,j}\, \tau_{I,i} \ . \tag{2.205}$$

Taking $\mathbf{n}_I$ the normal vector pointing outward of the viscous fluid as geometrical reference, we then have $T_{II,ij} = -p_{gas}\delta_{ij}$, where p_{gas} is the pressure in the air considered as an inviscid fluid. Omitting the fluid index I for the sake of simplicity, the conditions (2.204) and (2.205) become

$$T_{ji}\, n_j\, n_i = -p_{gas}(n_i n_i) = -p_{gas} \ , \tag{2.206}$$

$$T_{ji}\, n_j\, \tau_i = 0 \ . \tag{2.207}$$

Equation (2.207) indicates that at the free surface, the tangential component of the contact force must vanish as the inviscid air is unable to sustain a shear stress. The free surface conditions (2.206) and (2.207) imply knowledge of the surface shape. But the shape of the surface is itself part of the problem solution via the unit normal vector **n**. Therefore free surface flows constitute one of the major difficulties in fluid mechanics as an intrinsically nonlinear problem. To tackle it, one very often resorts to the Lagrangian representation where a given initial configuration of the surface deforms with elapsing time.

2.6.2.2 Flow with Surface Tension

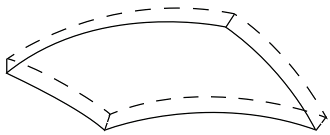
Consider first the hydrostatic case, so that we can talk about pressure, rather than stress. There is a pressure jump across the liquid surface at any point where the surface is curved. The magnitude of this pressure jump can be calculated by using virtual work. As an element of surface is given a virtual displacement outward (cf. Fig. 2.6), the surface area increases. The energy required for this comes from the unbalanced pressure forces acting through the virtual displacement. Letting the element of surface shrink to a point then leads to the result Adam (1968)

$$p_{fluid} - p_{ambient} = \tau \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \ , \tag{2.208}$$

where R_1 and R_2 are the principal radii of curvature of the surface. The fluid pressure exceeds ambient when the surface is concave inward. The pressure discontinuity across a fluid-fluid interface is given by an expression analogous to (2.208), where τ now represents the **coefficient of interfacial tension**.

The hydrodynamic case is not substantially different. Account must be taken of energy changes going on within an elemental volume of fluid bounded on one side by the element of area considered in the hydrostatic case. However as this

Fig. 2.6 Movement of the liquid surface



elemental volume shrinks to a point, volume effects approach zero as the cube of a linear dimension; consequently they are negligible compared with surface effects, which approach zero as the square of the linear dimension. Thus the surface traction and stress vector at a surface differ by the vector $\tau(1/R_1 + 1/R_2)n_i$, where n_i is the outward normal. The tangential components of the stress vector equal the tangential components of the surface traction; it is only the normal components which differ. If the surface of the fluid is concave outward, the stress vector has the larger component; if it is concave inward, the opposite occurs; if the mean curvature is zero, the stress vector equals the surface traction.

We may note that most fluid surfaces and interfaces have a low tension coefficient. Therefore unless curvatures are large, a good approximation is obtained by setting the stress vector at fluid surfaces equal to the surface traction, and by assuming the stress vector to be continuous across fluid-fluid interfaces. This would be a poor approximation, for example, in a problem involving the motion of small ripples or small drops.

2.6.3 Thermocapillary Flow

In the processing of molten metals, and in other flows of hot liquids, the surface tension, which depends on the temperature, can vary significantly from point to point. The surface traction and stress vector now also differ by a tangential component as well as a normal component. The difference is now

$$\tau(1/R_1 + 1/R_2)n_i + (\nabla_S \tau)_i \ , \tag{2.209}$$

where $\nabla_S \tau$ is the surface gradient of the surface tension, i.e.,

$$(\nabla_S \tau)_i = \frac{\partial \tau}{\partial x_i} - n_i n_j \frac{\partial \tau}{\partial x_j} \ . \tag{2.210}$$

As a result the variation of surface tension can drive a flow even in the absence of any other driving mechanisms.

When the surface tension variation is caused by temperature variation, the effect is called **thermocapillary convection**. In materials processing it can be one of several important driving mechanisms, cf. Langlois (1985). The resulting flow is often quite complicated and its treatment wasn't computational feasible until about 1980.

If the variation of surface tension is caused by a variation of solute concentration, the resulting flow is called **solutocapillary convection**. It was correctly cited by James Thomson, the elder brother of Lord Kelvin, in 1855 as the explanation for the phenomenon of "wine tears".

The generic term for flow caused by variation of surface tension, whatever the underlying cause, is the eponym **Marangoni convection** (after Carlo Giuseppe Matteo Marangoni 1840–1925).

2.6.4 Other Boundary Conditions

It may happen that a flowing fluid is subject to boundary conditions which do not fall into either of the categories discussed so far. Without attempting to exhaust the possibilities, we shall cite a few examples.

If the fluid is compressible, so that the energy equation is coupled to flow equations, it may be necessary to impose boundary conditions on a state variable, usually the temperature. It may be specified, for instance, that part of the fluid boundary is thermally insulated, or perhaps maintained at a specific temperature.

In the study of flow through pipes, we may want to impose a **volume flow condition**. The net flow of fluid across a fixed area, e.g., the cross section of the pipe, may be specified.

If the fluid has a free surface, the **kinematical free surface condition** applies. This stipulates that the free surface is a **material surface**: it always consists of the same particles. At first glance this may seem not quite right: Is not a large part of physical chemistry concerned with the diffusion of molecules toward and away from a free surface? True enough, but hydrodynamics completely ignores microscopic motions within the fluid. A basic assumption, which will be taken throughout this book, is that the fluid motion can be represented as a smooth function of time and space. With this assumption all surfaces in the fluid must remain intact, and this extends, in particular, to free surfaces. If the motion is insufficiently smooth, the free surface might not be a material surface. For example, the fluid mass might fracture, as in a breaking wave. In the paper Dussan (1976), E. B. Dussan V develops the mathematical criteria and supplies a critique of the subject's history.

Let us suppose that a free surface is described by

$$S(x_i, t) = 0 , \tag{2.211}$$

and focus our attention on a particle lying on this surface. Since it must remain on the surface, it must move in such a way that $S(x_i, t)$, evaluated at the point x_i which the particle occupies at time t, must always be zero. Thus on the free surface

$$\frac{DS}{Dt} = 0 . \tag{2.212}$$

This can be written

$$\frac{\partial S}{\partial t} + v_i \frac{\partial S}{\partial x_i} = 0 \, . \tag{2.213}$$

It is necessary to use this boundary condition, and also the appropriate force boundary condition, in the study of waves on a liquid surface.

2.7 Similarity Considerations

We shall soon turn our attention to the solution of boundary value problems involving the flow of viscous fluids. In order to carry out these solutions with some degree of generality, it is important to know the **scaling rules** (more often called the **similarity rules**) which may allow us to relate the solutions of two different flow problems pertaining to configurations which are geometrically similar. The reader is referred to the monograph by Barenblatt (2003) for a complete description of a modern treatment of dimensional analysis and physical similarity.

Intuition tells us that such scaling rules may not always be available. Consider, for example, the problems an oceanographer would have in building a scale model of San Francisco bay. Let us suppose that the bay, 65 km long, were modeled in a laboratory 12 m long. For geometrical similarity, the 10 m depth would be represented by a film of fluid 2 mm thick, and the sloughs would correspond to tiny threads of fluid 3 mm wide and half a millimeter thick. In studying fluid motions within the bay, it would be reasonable to neglect surface tension, and the fluid viscosity would play a minor role. The large-scale motions would be analyzed by a Bernoulli's law sort of approach. In the model, however, surface tension and viscosity would be the dominant effects. The scaling would be better if, in the model, a fluid were used which was much less viscous than sea water and had a much lower coefficient of surface tension. Liquid helium perhaps?

Other illustrations of this type are easily devised. In general, geometrical scaling is not enough for the construction of a realistic model. It may be necessary to change the time scale, and this may induce a change in the velocity scale. As in the example above, even this may not be enough. Whether scaling can be carried out should somehow be reflected in the mathematics of the situation. So far the governing equations and boundary conditions look the same for all geometrically similar situations, so that scaling rules must be ascertained from the solutions. Since these solutions are not always obtainable by analytic methods, it is quite important to rewrite the differential equations so that similarity considerations can be investigated directly from them. This is done by writing the differential equations in terms of dimensionless variables, obtained by normalizing the various quantities with respect to mass, length, time, and temperature scales inherent in the problem.

2.7.1 Similarity Rules for Steady, Incompressible Flow Without Body Forces When No Free Surface Is Present

Let us consider in detail the case of steady, incompressible flow without body forces in a region bounded only by rigid obstacles. The region is not necessarily limited in extent, but may extend to infinity in one or more directions. In addition to the no-slip condition on the boundaries the velocity may be required to assume a specified value "at infinity", or to produce a specified flux through some given surface. This monograph will deal almost (but not quite) exclusively with problems which fall into this category. Such problems are governed by the differential equations

$$\frac{\partial v_i}{\partial x_i} = 0 \,, \tag{2.214}$$

$$\rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 v_i \ , \tag{2.215}$$

subject to appropriate boundary conditions.

We assume that the flow region can be characterized by one geometrical length L. This could represent, for example, the breadth of an obstacle, or the diameter of a tube through which the fluid flows. In some problems, e.g., flow past a semi-infinite plate, there is no geometrical length scale; in others, notably fluid-film lubrication, there is more than one. Such cases require special treatment, and their study will be deferred. We normalize the length variables with respect to L. Thus, we define new spatial variables x_i' according to

$$x_i' = \frac{x_i}{L} \ . \tag{2.216}$$

Since the flow is steady, there is no time scale per se. However the velocity scale imposes one. If the kinematic boundary conditions can be characterized by a single speed U, the inherent time scale of the problem is L/U, the inertial time t_{inert}, i.e. the time it takes a particle moving with speed U to traverse the distance L. We introduce normalized velocity components

$$v_i' = \frac{v_i}{U} \ . \tag{2.217}$$

Since temperature considerations are irrelevant, there remains only to choose the mass scale. The natural choice is ρL^3, the mass of fluid within a cube of side equal to the characteristic length. This choice imposes the force scale $\rho L^2 U^2$, and hence the pressure scale ρU^2, which those readers familiar with hydraulics will recognize as twice the dynamic head.[8] Consequently we set

$$p' = \frac{p}{\rho U^2} \, . \tag{2.218}$$

In terms of the normalized variables, Eqs. (2.214) and (2.215) become, respectively,

$$\frac{\partial v_i'}{\partial x_i'} = 0 \, , \tag{2.219}$$

$$v_j' \frac{\partial v_i'}{\partial x_j'} = -\frac{\partial p'}{\partial x_i'} + \left(\frac{1}{\mathcal{R}e}\right) \frac{\partial^2 v_i'}{\partial x_j' \partial x_j'} \, , \tag{2.220}$$

where $\mathcal{R}e$ is the **Reynolds number**, a dimensionless grouping defined by

$$\mathcal{R}e = \frac{\rho L U}{\mu} \, . \tag{2.221}$$

With this dimensionless formulation, we see that similarity rules can be established between two geometrically similar problems only if their Reynolds numbers are the same, for otherwise they are governed by different systems of differential equations. This condition is also sufficient, provided the boundary conditions are the same in the dimensionless formulation of both problems, i.e., provided the problems are kinematically similar.

In addition to the relation it bears to similarity considerations, the Reynolds number plays another role in viscous hydrodynamics. It measures the relative importance of fluid inertia and fluid viscosity: ρU measures fluid momentum per unit volume and μ/L measures the impulse per unit volume arising from viscous forces.

Either the Reynolds number or its reciprocal can serve as a perturbation parameter. If $\mathcal{R}e$ is extremely large, fluid inertia dominates the problem. By dropping the term involving $(1/\mathcal{R}e)$ from Eq. (2.220), we obtain the equation of motion for an inviscid fluid. In doing so, we eliminate the highest derivatives and consequently lose some ability to satisfy boundary conditions. Inviscid flow theory proceeds on the assumption that only relative *normal* velocity need vanish on solid obstacles;

[8] Actually we could present a good case for doing things the other way around. This monograph deals with *slow* viscous flow, so that a pressure scale related to the dynamic head is somewhat fictitious. We would be better off choosing a pressure scale related to typical viscous stresses, *viz.*, $\mu U/L$, which leads to the mass scale $\mu L^2/U$. However the end result is the same, and every other hydrodynamics book which treats the subject uses, in effect, ρL^3 as mass scale; perversity for the sake of consistency seems unfair to the reader. In the next section, we do, in essence, adopt the alternative choice.

nothing is required about the tangential velocity. The solution of problems in high Reynolds number flow can often be carried out by assuming inviscid flow theory to be applicable, except in thin films of fluid surrounding solid obstacles. Within these films the viscous flow equations must be used. However the adjustment to no-slip at the boundary takes place so steeply that derivatives across the film dominate the differential equations. Hence a simplification can be obtained, and the resulting **boundary layer theory** has been extensively studied.

Much of this book will be concerned with the opposite situation. If $\mathcal{R}e$ is sufficiently small, viscosity dominates. As an approximation (or, better, as an asymptotic result for $\mathcal{R}e$ approaching zero), the inertia term can be dropped from (2.220), unless, as sometimes happens, this term is needed to provide a mechanism for nonlinear exchange of momentum (more about this in Chap. 6). For small $\mathcal{R}e$ the normalization (2.218) is quite misleading, as indicated in the footnote on page 72. Although the pressure term in (2.220) appears to be of the same order as the inertia term, the viscous forces will generate pressures of order $\mu L^2/U$. Thus the pressure term must be retained. In terms of the physical (dimensional) variables we obtain, for vanishingly small Reynolds number, the **equations of creeping viscous flow**

$$\frac{\partial v_i}{\partial x_i} = 0 \,,$$
$$\frac{\partial p}{\partial x_i} = \mu \nabla^2 v_i \,. \tag{2.222}$$

These equations could also be obtained from (2.214) and (2.215) by postulating that the fluid has zero density.

2.7.2 Similarity Rules for Unsteady, Incompressible Flow Without Body Forces When No Free Surface Is Present

In this case we start from the unsteady Navier-Stokes equation for a viscous incompressible fluid, namely

$$\frac{\partial v_i}{\partial x_i} = 0 \,, \tag{2.223}$$

$$\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 v_i \,. \tag{2.224}$$

As before, we normalize space by Eq. (2.216) and velocity by (2.217). This choice automatically induces the time normalization

$$t' = \frac{Ut}{L} \,. \tag{2.225}$$

The pressure is still given by (2.218) and in terms of normalized variables, Eqs. (2.223) and (2.224) become, respectively,

$$\frac{\partial v_i'}{\partial x_i'} = 0\,, \tag{2.226}$$

$$\frac{\partial v_i'}{\partial t'} + v_j' \frac{\partial v_i'}{\partial x_j'} = -\frac{\partial p'}{\partial x_i'} + \left(\frac{1}{\mathcal{R}e}\right) \frac{\partial^2 v_i'}{\partial x_j' \partial x_j'}\,, \tag{2.227}$$

where $\mathcal{R}e$ is the Reynolds number as defined by Eq. (2.221). This normalization procedure is favored by aerodynamicists, because when $\mathcal{R}e \to \infty$, the limit equations are the **Euler equations** only valid for inviscid fluids. This limit corresponds to flows where the characteristic velocity U is very high or the viscosity very low.

If we tackle highly viscous fluids or very slow flows, one may resort to the normalization procedure rheologists use. The dimensionless time is scaled by the viscous diffusion time $t_{visc} = L^2/\nu$

$$t' = \frac{\nu t}{L^2}\,. \tag{2.228}$$

The pressure is no longer scaled by the dynamic pressure but instead by the viscous velocity gradient

$$p' = \frac{p}{\frac{\mu U}{L}}\,. \tag{2.229}$$

The dimensionless equations are

$$\frac{\partial v_i'}{\partial x_i'} = 0\,, \tag{2.230}$$

$$\frac{\partial v_i'}{\partial t'} + \mathcal{R}e \left(v_j' \frac{\partial v_i'}{\partial x_j'} \right) = -\frac{\partial p'}{\partial x_i'} + \frac{\partial^2 v_i'}{\partial x_j' \partial x_j'}\,. \tag{2.231}$$

The Reynolds number zero limit yields the dimensional **Stokes momentum equations**

$$\frac{\partial v_i}{\partial x_i} = 0\,, \tag{2.232}$$

$$\rho \frac{\partial v_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}\,. \tag{2.233}$$

The Stokes momentum equation (2.233) is, unlike the Navier-Stokes equation, a linear equation so that every combination of solutions is itself a solution.

This property is very useful for obtaining closed form solutions of simple flows. Note that the Reynolds number may be seen as the ratio of two characteristic times

$$\mathcal{R}e = \frac{t_{visc}}{t_{inert}} \,. \tag{2.234}$$

However the range of the Reynolds number which occurs in practice is very large. For values of $\mathcal{R}e$ too large for (2.232) and (2.233) to apply but still small, laminar flow results. For moderate values of $\mathcal{R}e$, around a few dozen the flow becomes unstable and time dependent physics takes place. This kind of instability very often goes through a Hopf bifurcation where the time dependency is linked to a single frequency. Then the instability becomes more and more complex, generating harmonics and giving rise to a more populated spectrum. At higher values of the Reynolds number, around 10^6—10^7, the flow reaches fully developed turbulence, a branch of physics that is not yet completely understood.

The flow may be unsteady because it is submitted to a time dependent forcing. There might be an oscillating pressure gradient. Or there might be motion of a bounding surface, like an oscillating boundary wall or an oscillating body inside the fluid. The angular frequency ω of the periodicity provides a natural time scale: the dimensionless time becomes

$$t' = \omega t \,. \tag{2.235}$$

Keeping the previous dimensionless variables for space, velocity and pressure, namely Eqs. (2.216)–(2.218), respectively, the reduced Navier-Stokes equations are

$$\frac{\partial v_i'}{\partial x_i'} = 0 \,, \tag{2.236}$$

$$St \frac{\partial v_i'}{\partial t'} + v_j' \frac{\partial v_i'}{\partial x_j'} = -\frac{\partial p'}{\partial x_i'} + \left(\frac{1}{\mathcal{R}e}\right) \frac{\partial^2 v_i'}{\partial x_j' \partial x_j'} \,. \tag{2.237}$$

The Strouhal number

$$St = \frac{\omega L}{U} \tag{2.238}$$

compares two characteristic time scales of the physical phenomenon, the time associated with the oscillating forcing and the inertial or advection time L/U. This number is used for example to quantify the von Kármán street behind a circular cylinder where the vortex shedding corresponds to a Hopf bifurcation with a single temporal period.

Another set of reduced Navier-Stokes equations can be obtained using the following normalizations: Eq. (2.216) for space, Eq. (2.235) for time, Eq. (2.217) for the velocity and Eq. (2.229) for the pressure. We find

$$\frac{\partial v_i'}{\partial x_i'} = 0 \, , \tag{2.239}$$

$$\mathcal{R}e_{osc} \frac{\partial v_i'}{\partial t'} + \mathcal{R}e_{tr} v_j' \frac{\partial v_i'}{\partial x_j'} = -\frac{\partial p'}{\partial x_i'} + \frac{\partial^2 v_i'}{\partial x_j' \partial x_j'} \, , \tag{2.240}$$

with a **translational Reynolds number** $\mathcal{R}e_{tr}$ and an **oscillatory Reynolds number** $\mathcal{R}e_{osc}$ defined by the relationships

$$\mathcal{R}e_{tr} = \frac{UL}{\nu}, \quad \mathcal{R}e_{osc} = \frac{\omega L^2}{\nu} \, . \tag{2.241}$$

Here we have adopted the notation proposed by Shankar (2007). Note that $\mathcal{R}e_{osc}$ is the ratio of the frequency time ω^{-1} and the viscous diffusion time L^2/ν. This dimensionless group is also named **Stokes number** by some authors, see e.g. Fung (1984).

2.8 Vorticity Transfer

One of the most noticeable features which distinguishes viscous flow from inviscid flow is the presence of vorticity. It is a well-known result of inviscid fluid theory that a flow field once irrotational remains irrotational unless acted upon by non-conservative body forces. Thus motion generated from rest by impulsive motion of solid walls, or by the application of conservative body forces, is always irrotational.

By contrast, in a viscous fluid—even one of low viscosity—the presence of vorticity is usually quite pronounced. This becomes visually evident, for example, in the stirring of a martini. The experimental evidence indicates quite strongly that for low Reynolds number flow the vorticity tends to diffuse through the flow field; at high Reynolds number it tends to concentrate in wakes and boundary layers.

The question of vorticity transfer is quite important in problems of high-speed flow, especially where turbulence is a possibility. Although of less import for the slow-flow case, which is the principal subject matter of this book, it still enters the picture often enough to warrant at least a brief discussion. To keep the treatment concise, we consider only the incompressible case.

The vorticity is defined as the curl of the velocity, the tensor expression being supplied by Eq. (2.73). To obtain an equation for the vorticity, we therefore take the curl of the Navier-Stokes equation (2.195):

$$\begin{aligned} \frac{\partial \xi_i}{\partial t} + v_s \frac{\partial \xi_i}{\partial x_s} + \varepsilon_{ijk} \frac{\partial v_s}{\partial x_j} \frac{\partial v_k}{\partial x_s} &= \varepsilon_{ijk} \left(\frac{\partial f_k}{\partial x_j} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x_j \partial x_k} \right) + \nu \nabla^2 \xi_i \\ &= \varepsilon_{ijk} \frac{\partial f_k}{\partial x_j} + \nu \nabla^2 \xi_i \, . \end{aligned} \tag{2.242}$$

The first two terms on the left side of this equation comprise $D\xi_i/Dt$ and the third term can be simplified. Making use of the identity (1.49) and the observation that $\varepsilon_{ijk}(\partial v_s/\partial x_j)(\partial v_s/\partial x_k) = 0$, we obtain

$$\begin{aligned}
\varepsilon_{ijk}\frac{\partial v_s}{\partial x_j}\frac{\partial v_k}{\partial x_s} &= \varepsilon_{ijk}\left(\frac{\partial v_k}{\partial x_s} - \frac{\partial v_s}{\partial x_k}\right)\frac{\partial v_s}{\partial x_j} \\
&= \varepsilon_{kij}(\delta_{qs}\delta_{pk} - \delta_{qk}\delta_{sp})\frac{\partial v_p}{\partial x_q}\frac{\partial v_s}{\partial x_j} \\
&= \varepsilon_{kij}\varepsilon_{kts}\varepsilon_{tpq}\frac{\partial v_p}{\partial x_q}\frac{\partial v_s}{\partial x_j} \\
&= (\delta_{it}\delta_{js} - \delta_{is}\delta_{jt})\xi_t\frac{\partial v_s}{\partial x_j} \\
&= \xi_i\frac{\partial v_j}{\partial x_j} - \xi_j\frac{\partial v_i}{\partial x_j} \\
&= -\xi_j\frac{\partial v_i}{\partial x_j}\ .
\end{aligned} \tag{2.243}$$

Consequently Eq. (2.242) reduces to the **vorticity equation**

$$\frac{D\xi_i}{Dt} = \xi_j\frac{\partial v_i}{\partial x_j} + \varepsilon_{ijk}\frac{\partial f_k}{\partial x_j} + \nu\nabla^2\xi_i\ . \tag{2.244}$$

If the body forces are conservative, they can be derived from a potential:

$$f_i = -\frac{\partial \Phi}{\partial x_i}\ . \tag{2.245}$$

For this important case the body force term drops out of the vorticity equation and we obtain

$$\frac{D\xi_i}{Dt} = \xi_j\frac{\partial v_i}{\partial x_j} + \nu\nabla^2\xi_i\ . \tag{2.246}$$

If the flow is two-dimensional, the vorticity vector is perpendicular to the plane of the motion so that $\xi_j\,\partial v_i/\partial x_j$ vanishes. If we choose the $x_1 - x_2$ plane as the plane of motion, ξ_1 and ξ_2 both vanish and setting $i = 3$ in (2.246) gives us

$$\frac{D\xi_3}{Dt} = \nu\nabla^2\xi_3\ , \tag{2.247}$$

in which ∇^2 denotes $\partial^2/\partial^2 x_1^2 + \partial^2/\partial^2 x_2^2$. Thus for two-dimensional flow the only non-zero component of vorticity obeys the equation for the conduction of heat in a moving medium. If, as the experimental evidence indicates, vorticity is generated mainly where the fluid passes over solid surfaces, the analogy with heat conduction illustrates vividly why in high-speed flow the vorticity is confined mostly to the wake and boundary layer.

One should bear in mind that the analogy between vorticity transfer and heat conduction has its limitations, even in the two-dimensional case where the equations are the same. Heat is a quantity which may exist independently of the flow field. If the medium is stationary, the material derivative reduces to the partial time derivative and we recover the classical heat conduction equation. Motion of the medium merely superimposes a convective transfer of heat upon the conductive transfer. On the other hand the vorticity is a *kinematical* quantity, defined in terms of the velocity components. Thus it exists in intimate relation with the flow field, not independently of it. No flow, no vorticity. While it is sometimes useful to think of the vorticity as something carried about by the flow field, we must bear in mind that this is not precisely the true picture.

For two-dimensional flow, computational fluid dynamics is sometimes carried out using the vorticity-streamfunction formulation, see for example Peyret and Taylor (1983). Streamfunction is written as one word to avoid hyphenation difficulties. The three dependent variables v_1, v_2, p are replaced by the two variables ω and Ψ. The fact that vorticity is pseudo-conserved, i.e., behaves much like heat, is useful in this approach. Stable computation is often facilitated by using conservative differencing schemes, cf. Sengupta (2013) and Versteeg and Malalasekera (2007), which are constructed in such a way that truncation error doesn't lead to false production of a conserved quantity. The partial analogy between heat and vorticity makes it practical to use this approach in the vorticity-streamfunction formulation.

When rewritten in terms of the vorticity, the Navier-Stokes equation (2.195) admits of an interesting interpretation. We begin by rewriting the nonlinear term:

$$\begin{aligned} v_j \frac{\partial v_i}{\partial x_j} &= \frac{\partial}{\partial x_i}(\frac{1}{2} v_j v_j) - v_j (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \frac{\partial v_q}{\partial x_p} \\ &= \frac{\partial}{\partial x_i}(\frac{1}{2} v_j v_j) - \varepsilon_{kij}\varepsilon_{kpq} v_j \frac{\partial v_q}{\partial x_p} \\ &= \frac{\partial}{\partial x_i}(\frac{1}{2} v_j v_j) - \varepsilon_{ijk} v_j \xi_k \ . \end{aligned} \tag{2.248}$$

If we assume conservative body forces, so that (2.245) applies, Eq. (2.195) then becomes

$$\frac{\partial v_i}{\partial t} - \varepsilon_{ijk} v_j \xi_k = -\frac{\partial}{\partial x_i}\left(\Phi + \frac{p}{\rho} + \frac{1}{2} v_j v_j\right) + \nu \nabla^2 v_i \ , \tag{2.249}$$

or in vector notation

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \boldsymbol{\xi} = -\nabla \left(\Phi + \frac{p}{\rho} + \frac{1}{2} v^2 \right) + \nu \nabla^2 \mathbf{v} \tag{2.250}$$

For steady-state irrotational flow of an inviscid fluid, **Bernoulli's law**

$$\Phi + \frac{p}{\rho} + \frac{1}{2} v^2 = const. \tag{2.251}$$

is derived by integrating (2.249). Consider further. If the flow is irrotational, it is derivable from a potential:

$$v_i = \frac{\partial \varphi}{\partial x_i} \,. \tag{2.252}$$

In order that the continuity equation (2.194) be satisfied, the potential φ must be harmonic, i.e.,

$$\nabla^2 \varphi = 0 \,. \tag{2.253}$$

However, if φ is harmonic, (2.252) reveals that each component of the velocity is harmonic, so that the viscous term $\nu \nabla^2 v_i$ in (2.249) vanishes. By choosing the pressure to obey Bernoulli's law, we satisfy (2.249) by the potential flow (2.252). In general, however, it is not possible to satisfy the no-slip condition at solid boundaries by a potential flow. As indicated earlier, viscous flow near solid surfaces tends to be rotational.

Note that the "Bernoulli term" in Eq. (2.249) includes part of the fluid inertia, viz.,$-\frac{1}{2}\partial(v_j v_j)/\partial x_i$. This becomes significant when we obtain a solution (v_i, p) to Eqs. (2.222), which govern steady creeping flow, and seek to determine its value as an approximation to the flow of a fluid with finite density. The approximation can always be improved by modifying the pressure according to

$$p_{modified} = p - \frac{1}{2} \rho v_j v_j \;, \tag{2.254}$$

i.e., by exchanging pressure head for dynamic head. Therefore, if creeping flow becomes in some sense a bad approximation, it must be because the rotational term $\varepsilon_{ijk} v_j \xi_k$ becomes too large. For a posteriori calculations of this sort, it is convenient to note that

$$\varepsilon_{ijk} v_j \xi_k = v_j \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \right) \,. \tag{2.255}$$

Chapter 3
Curvilinear Coordinates

Abstract General tensor analysis relevant to viscous hydrodynamics is presented. The covariant and contravariant forms of the Navier-Stokes equation are set out. The complete set of equations, in terms of physical components, is given for cylindrical and spherical coordinate systems.

We now set about writing the equations of viscous flow, developed in the last chapter, in a form appropriate for use with curvilinear coordinate systems. For any given curvilinear system this can be done directly, by writing the various partial derivatives with respect to the Cartesian coordinates in terms of the partials with respect to the curvilinear coordinates. Although this is a messy procedure, one might argue that it need only be carried out once for any given coordinate system. However, in order to obtain a formulation in terms of an *arbitrary* curvilinear system, it is best to employ general tensor analysis. The real power of this method is appreciated when materials more general than the Newtonian fluid are studied: without a concise, and completely general, method of passing from one coordinate system to another, only the most trivial boundary value problems for these materials would be solved.

This chapter may be considered an introduction to the use of general tensor analysis in continuum mechanics. What is set out here will suffice for our purposes. Geometrical interpretations are scarcely touched, and the reader interested in them is encouraged to consult McConnell (1957), Sokolnikoff (1951), Ericksen (1960).

3.1 General Tensor Analysis

In Chap. 1 we treated quantities called Cartesian tensors, whose components in one Cartesian coordinate system can be calculated from those in another according to the transformation rule (1.77). We shall soon find that when we consider transformations from one curvilinear system to another, *two* transformation rules

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_3,

need be considered. Consequently we shall find it convenient to modify our suffix notation somewhat: both subscripts and superscripts will be used, to denote which of the two transformation rules applies. Moreover we shall find that *whenever the summation convention applies, the repeated index appears once as a subscript, once as a superscript.*

In the ensuing presentation we shall consider relationships between two coordinate systems, one or both of which may be curvilinear. Those readers who are unfamiliar with general tensor notation may find it helpful, at various stages of the proceedings, to specialize the analysis to a familiar example, e.g., the relationship between a Cartesian system and a cylindrical polar system.

3.1.1 Coordinate Transformations

Let x^i denote the coordinates of a point in one of the systems and let $\bar{x}^i$ denote the coordinates of the same point in the other system. We will use superscripts for the coordinates since, when we discuss the transformation rules, we shall find that differentials of the coordinates transform according to the rule traditionally associated with superscripts. We assume that there is a functional relationship

$$\bar{x}^i = \bar{x}^i(x^m) \tag{3.1}$$

which possesses derivatives of any order required, except perhaps at isolated points or along isolated arcs. The transformation (3.1) will have an inverse

$$x^i = x^i(\bar{x}^m) \,, \tag{3.2}$$

provided the Jacobian determinant

$$J = \left| \frac{\partial \bar{x}^i}{\partial x^m} \right| \tag{3.3}$$

is different from zero. In certain coordinate transformations of practical interest J vanishes at points or along arcs; for an example of each consider transformation from spherical coordinates and from cylindrical coordinates into a Cartesian system. We shall assume that J never vanishes over a region.

We adopt the convention that *in expressions involving the operator $\partial/\partial x^i$, the suffix i is to be considered a subscript, not a superscript* (this may, at first, appear confusing, but there is a mnemonic: the suffix appears "below the line", where a subscript should). With this convention the differentials dx^i and $d\bar{x}^i$ are related according to

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j \,, \tag{3.4}$$

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^j} d\bar{x}^j \ , \tag{3.5}$$

Equation (3.4) is the prototype of one of the two transformation rules important in tensor theory. In general let A^i be three quantities defined in the x^i-system, and let $\bar{A}^i$ be three quantities, defined in the $\bar{x}^i$-system, related to the A^i according to

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \ . \tag{3.6}$$

Then A^i and $\bar{A}^i$ are said to be the components, in the x^i-system and the $\bar{x}^i$-system, respectively, of a **contravariant vector**, or **contravariant tensor of rank** 1. The inverse of (3.6) is easily calculated. Solving for A^i, we find

$$A^i = J^{-1} \gamma^i_j \bar{A}^j \ , \tag{3.7}$$

where γ^i_j is the cofactor of $\partial \bar{x}^i / \partial x^j$ in J. By well-known rules for calculating partial derivatives in an inverse transformation (see, for example, Widder (1989)), we obtain

$$A^i = \frac{\partial x^i}{\partial \bar{x}^j} \bar{A}^j \ . \tag{3.8}$$

To generalize the definition, let $A^{i_1 i_2 \dots i_n}$ denote 3^n quantities defined in the x^i-system and let $\bar{A}^{i_1 i_2 \dots i_n}$ denote 3^n quantities defined in the $\bar{x}^i$-system such that

$$\bar{A}^{i_1 i_2 \dots i_n} = \frac{\partial \bar{x}^{i_1}}{\partial x^{j_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{j_2}} \cdots \frac{\partial \bar{x}^{i_n}}{\partial x^{j_n}} A^{j_1 j_2 \dots j_n} \tag{3.9}$$

or, equivalently,

$$A^{i_1 i_2 \dots i_n} = \frac{\partial x^{i_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{i_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{i_n}}{\partial \bar{x}^{j_n}} \bar{A}^{j_1 j_2 \dots j_n} \ . \tag{3.10}$$

Then $A^{i_1 i_2 \dots i_n}$ and$\bar{A}^{i_1 i_2 \dots i_n}$ are said to be the components, in the x^i-system and $\bar{x}^i$-system, respectively, of a **contravariant tensor of rank** n.

The prototype of the other transformation rule is suggested by considering the first partial derivatives of a scalar. Precisely as in Cartesian tensor analysis, a **scalar**, or **tensor of rank zero**, is a quantity which has the same value in all coordinate systems. If K is any differentiable scalar, then

$$\frac{\partial K}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial K}{\partial x^j} \ . \tag{3.11}$$

Now let A_i be three quantities, defined in the x^i-system, which transform to three quantities $\bar{A}_i$ defined in the $\bar{x}^i$-system, in the same way that $\partial K / \partial x^i$ transforms to

$\partial K/\partial \bar{x}^i$, i.e.,

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \ . \tag{3.12}$$

Then A_i and $\bar{A}_i$ are said to be the components, in the x^i-system and the $\bar{x}^i$- system, respectively, of a **covariant vector**, or **covariant tensor of rank** 1. The generalization to covariant tensors of rank n is obvious:

$$\bar{A}_{i_1 i_2 \ldots i_n} = \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} \cdots \frac{\partial x^{j_n}}{\partial \bar{x}^{i_n}} A_{j_1 j_2 \ldots j_n} \ , \tag{3.13}$$

or, equivalently,

$$A_{i_1 i_2 \ldots i_n} = \frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{j_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{j_n}}{\partial x^{i_n}} \bar{A}_{j_1 j_2 \ldots j_n} \ . \tag{3.14}$$

We shall sometimes encounter sets of quantities which transform partially according to the contravariant rule, partially according to the covariant rule. We say that the 3^{m+n} quantities $A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n}$ are the components in the x^i-system of a **mixed tensor of rank (m + n), m-times contravariant, n-times covariant**, if they transform to 3^{m+n} quantities $\bar{A}^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n}$ according to

$$\bar{A}^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n} = \frac{\partial \bar{x}^{i_1}}{\partial x^{r_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{r_2}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{r_m}} \frac{\partial x^{s_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{s_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{s_n}}{\partial \bar{x}^{j_n}} A^{r_1 r_2 \ldots r_m}_{s_1 s_2 \ldots s_n} \ . \tag{3.15}$$

Since no ambiguity can arise, we refer to a mixed tensor of rank 2, once contravariant, once covariant simply as a **mixed tensor of rank** 2.

In Chap. 1 we found that the Kronecker delta transforms as a Cartesian tensor which has the same components in any Cartesian coordinate system. This result can be extended to general tensor analysis by introducing a mixed tensor of rank 2 which, in the x^i-system, say, has the same components as the Kronecker delta. This tensor is denoted by δ^i_j and, for the purposes of general tensor analysis, *it* is called the Kronecker delta. Let us suppose that δ^i_j transforms to c^i_j in the $\bar{x}^i$-system. By (3.15)

$$c^i_j = \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^j} \delta^r_s = \frac{\partial \bar{x}^i}{\partial x^r} \frac{\partial x^r}{\partial \bar{x}^j} = \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \delta^i_j \ . \tag{3.16}$$

The substitution properties, discussed in Sect. 1.4 for the Cartesian Kronecker delta, are also possessed by δ^i_j.

The distinction between covariance and contravariance does not arise in Cartesian tensor analysis, for the transformation rules coincide. Consider two coordinate systems related by the orthogonal transformation (1.59) and its inverse (1.62). We have

$$\frac{\partial x_i'}{\partial x_j} = a_{ij} = \frac{\partial x_j}{\partial x_i'} \ . \tag{3.17}$$

3.1.2 The Metric Tensors

We now turn our attention to the following question: Given a contravariant vector A^i, can we find an **associated covariant vector** A_i which has the same components as A^i in an arbitrary Cartesian system of given length scale?

Let the Cartesian system be denoted by ξ^i and let x^i denote a coordinate system related to ξ^i through a transformation with non-vanishing Jacobian. Associated with the x^i-system, we define a symmetric set of nine quantities g_{ij} according to

$$g_{ij} = \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^k}{\partial x^j} \ . \tag{3.18}$$

If we have another coordinate system $\bar{x}^i$, with quantities $\overline{g}_{ij}$ defined by

$$\overline{g}_{ij} = \frac{\partial \xi^k}{\partial \bar{x}^i} \frac{\partial \xi^k}{\partial \bar{x}^j} \ , \tag{3.19}$$

then the g_{ij} and $\overline{g}_{ij}$ are related according to

$$\begin{aligned} \overline{g}_{ij} &= \frac{\partial \xi^k}{\partial x^r} \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial \xi^k}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^j} \\ &= \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} g_{rs} \ , \end{aligned} \tag{3.20}$$

which is the transformation rule for a covariant tensor of rank 2.

Somewhat imprecisely, g_{ij} is called the **covariant metric tensor** for the x^i-system. Actually, it is a set of nine quantities *defined only for the x^i-system* which are related to nine quantities, analogously defined for another coordinate system, according to the transformation rule for a covariant tensor of rank 2.

It is evident that the covariant metric tensor for any Cartesian system with the same length scale as the reference ξ_i-system has the same components as the Kronecker delta.

Given the contravariant vector A^i, we can show that

$$A_i = g_{ij} A^j \ , \tag{3.21}$$

which has the same components as A^i in any Cartesian system with the same length scale as the ξ_i-system, transforms as a covariant vector:

$$
\begin{aligned}
\frac{\partial x^j}{\partial \bar{x}^i} A_j &= \frac{\partial x^j}{\partial \bar{x}^i} g_{jk} A^k \\
&= \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial \xi^m}{\partial x^j} \frac{\partial \xi^m}{\partial x^k} A^k \\
&= \frac{\partial \xi^m}{\partial \bar{x}^i} \frac{\partial \xi^m}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^j} \bar{A}^j \\
&= \frac{\partial \xi^m}{\partial \bar{x}^i} \frac{\partial \xi^m}{\partial \bar{x}^j} \bar{A}^j \\
&= \bar{g}_{ij} \bar{A}^j = \bar{A}_i \ .
\end{aligned} \tag{3.22}
$$

Quite generally, if $A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \cdots j_n}$ is a tensor of rank $(m+n)$, m-times contravariant, n-times covariant, then $g_{ki_r} A^{i_1 i_2 \ldots i_r \ldots i_m}_{j_1 j_2 \cdots j_n}$ is a tensor of rank $(m+n)$, $(m-1)$-times contravariant, $(n+1)$-times covariant.

In a similar manner we associate with the x^i-system a **contravariant metric tensor** g^{ij} which raises indices in much the same way that the covariant metric tensor lowers them:

$$
g^{ij} = \frac{\partial x^i}{\partial \xi^m} \frac{\partial x^j}{\partial \xi^m} \ . \tag{3.23}
$$

By reasoning similar to that set out in display (3.16), we find that, if A_i is a covariant vector, then

$$
A^i = g^{ij} A_j \tag{3.24}
$$

is a contravariant vector, and the result generalizes to tensors of any rank.

By comparing (3.4) and (3.8), or by appealing directly to the methods used for calculating partial derivatives in an inverse transformation, Widder (1989) we find

$$
g^{ij} = g^{-1} c^{ij} \ , \tag{3.25}
$$

where

$$
g = \det(g_{ij}) = \left[\det \left(\frac{\partial \xi^i}{\partial x^j} \right) \right]^2 \tag{3.26}
$$

and c^{ij} is the cofactor of g_{ij} in g.

As a direct consequence of (3.25), we find

$$
g^{ik} g_{kj} = \delta^i_j \ . \tag{3.27}
$$

Thus the "mixed metric tensor" of any coordinate system is the Kronecker delta.

It is readily verified that if $A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_m}$ is any tensor of rank $2m$, m-times contravariant, m-times covariant, then contracting it completely to $A^{i_1 i_2 \ldots i_m}_{i_1 i_2 \ldots i_m}$ yields a scalar. In particular

$$(ds)^2 = g_{ij} dx^i dx^j \tag{3.28}$$

is a scalar; by referring to a Cartesian system, we see that it is the square of the distance between two points whose coordinates, in the x^i-system, are x^i and $x^i + dx^i$.

3.1.3 The Christoffel Symbols: Covariant Differentiation

It would be convenient if the first partial derivatives of a tensor represented the components of another tensor, but this is not the case. Consider, for example, a covariant vector with components A_i in the x^i-system and components $\bar{A}_i$ in the $\bar{x}^i$ system:

$$\begin{aligned} \frac{\partial \bar{A}_i}{\partial \bar{x}^j} &= \frac{\partial}{\partial \bar{x}^j} \left(\frac{\partial x^r}{\partial \bar{x}^i} A_r \right) \\ &= \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial A_r}{\partial \bar{x}^j} + A_r \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j} \\ &= \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial A_r}{\partial x^s} + A_r \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j} \; . \end{aligned} \tag{3.29}$$

If the transformation from the x^i-system to the $\bar{x}^i$-system is linear, so that $\partial^2 x^r / \partial \bar{x}^i \partial \bar{x}^j$ vanishes, (3.29) reduces to the transformation rule for a covariant tensor of rank 2. In general, however, (3.29) is not a tensor transformation.

The form of (3.29) suggests a possible method of procedure. Perhaps a set of quantities, themselves not comprising a tensor, can be added to $\partial A_i / \partial x_j$ such that: (1) the sum transforms as a covariant tensor of rank 2; (2) the added terms vanish in Cartesian coordinate systems.

For the x^i-system we define a set of 27 quantities $\Gamma_{k;ij}$, called the **Christoffel symbols of the first kind** (after Elwin Bruno Christoffel, 1829–1900), according to

$$\Gamma_{k;ij} = \left(\frac{1}{2} \right) \left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) \; . \tag{3.30}$$

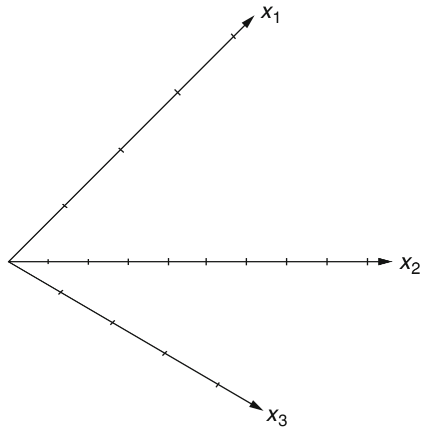


Fig. 3.1 Oblique rectilinear coordinate system

We then define another set of 27 quantities Γ_{ij}^k, called the **Christoffel symbols of the second kind**,[1] according to

$$\Gamma_{ij}^k = g^{km}\Gamma_{m;ij} . \tag{3.31}$$

Note that $\Gamma_{k;ij}$ and Γ_{ij}^k are symmetric with respect to the indices i and j. Also, *the Christoffel symbols vanish in any Cartesian system*, for the g_{ij} are then all constant. In fact they vanish in any coordinate system obtainable from a Cartesian system by an affine (linear) transformation; the most general such system is an oblique rectilinear system with different, but constant, scales in the three coordinate directions, as illustrated in Fig. 3.1.

Let us see how the Christoffel symbols transform. Let $\overline{\Gamma}_{k;ij}$ denote the symbols of the first kind calculated for the $\bar{x}^i$-system. Then

$$\begin{aligned}
\overline{\Gamma}_{k;ij} &= \frac{1}{2}\left(\frac{\partial \overline{g}_{kj}}{\partial \bar{x}^i} + \frac{\partial \overline{g}_{ik}}{\partial \bar{x}^j} - \frac{\partial \overline{g}_{ij}}{\partial \bar{x}^k}\right) \\
&= \frac{1}{2}\left[\frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^j} g_{rs}\right) + \frac{\partial}{\partial \bar{x}^j}\left(\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^k} g_{rs}\right) - \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j} g_{rs}\right)\right] \\
&= \frac{1}{2}\left[\frac{\partial x^t}{\partial \bar{x}^i}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^j} + \frac{\partial x^t}{\partial \bar{x}^j}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^k} - \frac{\partial x^t}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\right]\frac{\partial g_{rs}}{\partial x^t} \\
&\quad + \frac{1}{2}\left[\frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^j}\right) + \frac{\partial}{\partial \bar{x}^j}\left(\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^k}\right) - \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\right)\right] g_{rs}
\end{aligned} \tag{3.32}$$

[1] Some authors prefer the notation $[ij, k]$ for $\Gamma_{k;ij}$ and $\left\{ \begin{matrix} k \\ ij \end{matrix} \right\}$ for Γ_{ij}^k, principally to stress that the Christoffel symbols are *not* tensors.

$$= \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^t}{\partial \bar{x}^k}\Gamma_{t;rs} + \frac{1}{2}\left(\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} + \frac{\partial x^s}{\partial \bar{x}^k}\frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}\right) g_{rs}$$

$$= \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^t}{\partial \bar{x}^k}\Gamma_{t;rs} + \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial x^s}{\partial \bar{x}^k} g_{rs} \; .$$

The last step in this derivation follows from the symmetry of g_{rs}. After this, determining the transformation rule for the symbols of the second kind seems easy:

$$\begin{aligned}
\overline{\Gamma}^k_{ij} &= \overline{g}^{km}\overline{\Gamma}_{m;ij} \\
&= \frac{\partial \bar{x}^k}{\partial x^p}\frac{\partial \bar{x}^m}{\partial x^q} g^{pq}\left(\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^t}{\partial \bar{x}^m}\Gamma_{t;rs} + \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial x^s}{\partial \bar{x}^m} g_{rs}\right) \\
&= \delta^t_q \frac{\partial \bar{x}^k}{\partial x^p}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j} g^{pq}\Gamma_{t;rs} + \delta^s_q \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial \bar{x}^k}{\partial x^p} g^{pq} g_{rs} \\
&= \frac{\partial \bar{x}^k}{\partial x^p}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\Gamma^p_{rs} + \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j}\frac{\partial \bar{x}^k}{\partial x^p} \; .
\end{aligned} \tag{3.33}$$

An alternate form of (3.33) is sometimes useful. Multiplying both sides by $\partial x^m / \partial \bar{x}^k$ and contracting with respect to the index k, we obtain the **Christoffel formula**

$$\frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j} = \frac{\partial x^m}{\partial \bar{x}^k}\overline{\Gamma}^k_{ij} - \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\Gamma^m_{rs} \; . \tag{3.34}$$

We now return our attention to the covariant vector A_i but this time we consider a set of nine quantities $A_{i,j}$ defined by

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \Gamma^k_{ij} A_k \; . \tag{3.35}$$

If corresponding quantities $\bar{A}_{i,j}$ are defined in the $\bar{x}^i$-system, we have, with (3.29),

$$\begin{aligned}
\bar{A}_{i,j} &= \frac{\partial \bar{A}_i}{\partial \bar{x}^j} - \overline{\Gamma}^k_{ij}\bar{A}_k \\
&= \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial A_r}{\partial x^s} + \left(\frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j} - \frac{\partial x^m}{\partial \bar{x}^k}\Gamma^k_{ij}\right) A_m \; .
\end{aligned} \tag{3.36}$$

With the Christoffel formula (3.34),

$$\bar{A}_{i,j} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\left(\frac{\partial A_r}{\partial x^s} - \Gamma^m_{rs} A_m\right) = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j} A_{r,s} \; . \tag{3.37}$$

Thus $A_{i,j}$, which is called the **covariant derivative** of A_i, transforms as a covariant tensor of rank 2. Moreover in a Cartesian system it reduces to the partial derivative.

In a similar manner, we find that if A^i is a contravariant vector, then

$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{jk} A^k \tag{3.38}$$

transforms as a mixed tensor of rank 2.

The general result is as follows. Let $A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n}$ be a tensor of rank $(m+n)$, m-times contravariant, n-times covariant. Then

$$\begin{aligned} A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n, k} = & \frac{\partial A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n}}{\partial x^k} \\ & - \Gamma^p_{j_1 k} A^{i_1 i_2 \ldots i_m}_{p j_2 \ldots j_n} - \Gamma^p_{j_2 k} A^{i_1 i_2 \ldots i_m}_{j_1 p j_3 \ldots j_n} - \ldots - \Gamma^p_{j_n k} A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_{n-1} p} \\ & + \Gamma^{i_1}_{kp} A^{p i_2 \ldots i_m}_{j_1 j_2 \ldots j_n} + \Gamma^{i_2}_{kp} A^{i_1 p i_3 \ldots i_m}_{j_1 j_2 \ldots j_n} + \ldots + \Gamma^{i_m}_{kp} A^{i_1 i_2 \ldots i_{m-1} p}_{j_1 j_2 \ldots j_n} \end{aligned} \tag{3.39}$$

transforms as a tensor of rank $(m + n + 1)$, m-times contravariant, $(n + 1)$- times covariant, and is called the covariant derivative of $A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n}$. The rules for covariant differentiation of sums and products of tensor are identical with the usual rules of differentiation, and the operations of contraction and covariant differentiation commute.

Higher covariant derivatives, e.g., $A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n, rs}$ are obtained by repeated application of (3.39). By referring to a Cartesian system, in which the second covariant derivative reduces to the second partial, we obtain

$$A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n, rs} = A^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots j_n, sr} , \tag{3.40}$$

subject to the usual continuity conditions. This result is a consequence of our space being Euclidean, so that it admits of Cartesian coordinate systems. In more general spaces (3.40) does not necessarily apply. Even in classical physics and geometry this point is by no means academic. It is sometimes convenient to consider a curved surface imbedded in Euclidean 3-space as a two-dimensional, non-Euclidean space and to define a two-dimensional tensor analysis upon it. Throughout this monograph we shall restrict our studies to Euclidean spaces, so that (3.40) always applies.

3.1.4 Ricci's Lemma

An important result, known as **Ricci's lemma** (after Gregorio Ricci-Curbastro, 1853–1925), simplifies considerably the manipulations associated with covariant differentiation:

The covariant derivative of either metric tensor is zero.

The proof is direct. Taking the covariant derivative of g_{ij}, we obtain

$$\begin{aligned} g_{ij,k} &= \frac{\partial g_{ij}}{\partial x^k} - \Gamma^p_{ik} g_{pj} - \Gamma^p_{jk} g_{ip} \\ &= \frac{\partial g_{ij}}{\partial x^k} - g_{pj} g^{pq} \Gamma_{q;ik} - g_{ip} g^{pq} \Gamma_{q;jk} \\ &= \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\delta^q_j \left(\frac{\partial g_{qk}}{\partial x^i} + \frac{\partial g_{iq}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^q} \right) - \frac{1}{2}\delta^q_i \left(\frac{\partial g_{qk}}{\partial x^j} + \frac{\partial g_{jq}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^q} \right) \\ &= 0\,. \end{aligned} \tag{3.41}$$

Similarly

$$\begin{aligned} g^{ij}_{,k} &= \frac{\partial g^{ij}}{\partial x^k} + \Gamma^i_{pk} g^{pj} + \Gamma^j_{pk} g^{ip} \\ &= \frac{\partial g^{ij}}{\partial x^k} + (g^{pj} g^{iq} + g^{ip} g^{jq}) \Gamma_{q;pk} \\ &= \frac{\partial g^{ij}}{\partial x^k} + g^{ip} g^{jq} (\Gamma_{q;pk} + \Gamma_{p;qk}) \\ &= \frac{\partial g^{ij}}{\partial x^k} + g^{ip} g^{jq} \frac{\partial g_{pq}}{\partial x^k} \\ &= \frac{\partial g^{ij}}{\partial x^k} + \frac{\partial}{\partial x^k} (g^{ip} g^{jq} g_{pq}) - g_{pq} \frac{\partial}{\partial x^k} (g^{ip} g^{jq}) \\ &= 2\frac{\partial g^{ij}}{\partial x^k} - \delta^i_q \frac{\partial g^{jq}}{\partial x^k} - \delta^j_p \frac{\partial g^{ip}}{\partial x^k} \\ &= 0\,. \end{aligned} \tag{3.42}$$

As a corollary, we have

$$\delta^i_{j,k} = (g^{ip} g_{pj})_{,k} = g^{ip} g_{pj,k} + g_{pj} g^{ip}_{,k} = 0\,. \tag{3.43}$$

An important consequence of Ricci's lemma is that metric tensors can be taken outside the sign of covariant differentiation. Thus, the operations of raising and lowering indices commute with covariant differentiation.

3.2 The Hydrodynamic Equations in General Tensor Form

The equations governing the motion of a Newtonian viscous fluid were summarized, in Cartesian tensor notation, in Sect. 2.5. We now set out to rewrite each of these equations in a form which (1) transforms according to the transformation rule for a tensor of some rank and variance and (2) reduces to the equations of Chap. 2 when the coordinate system is Cartesian.

We begin with the scalar equation of continuity (2.190). Our two requirements are satisfied by the equation

$$\frac{\partial \rho}{\partial t} + (\rho v^i)_{,i} = 0\,, \tag{3.44}$$

where v^i is a contravariant vector, called the **contravariant velocity**, which coincides with the physical velocity of the fluid particle when the coordinate system is Cartesian. The associated covariant vector v_i is called the **covariant velocity**.

The contravariant velocity has a kinematical significance. Let a fixed coordinate system y^i and a Cartesian system ξ^i, also fixed, be related by a transformation with non-vanishing Jacobian. Let $\hat{v}^i$ denote the components of velocity in the ξ^i-system. Then

$$v^i = \frac{\partial x^i}{\partial \xi_j}\hat{v}^j = \frac{\partial x^i}{\partial \xi_j}\frac{D\xi^j}{Dt} = \frac{Dx^i}{Dt}\,. \tag{3.45}$$

Thus the component v^i signifies the rate of change of the coordinate x^i seen from a particle as it moves through space. A typical example is the angular velocity of a particle whose motion is referred to a polar coordinate system. Note that the contravariant velocity need not have physical dimensions of speed.

The Navier-Stokes equation (2.191) is a vector equation, and we rewrite it in a form which transforms according to the contravariant rule:

$$\begin{aligned}\rho\left(\frac{\partial v^i}{\partial t} + v^j v^i_{,j}\right) &= \rho f^i - g^{ij}\frac{\partial p}{\partial x^j} + (\lambda+\mu)g^{ij}\frac{\partial}{\partial x^j}(v^k_{,k}) + \mu g^{jk} v^i_{,jk}\\ &\quad + g^{ij}\frac{\partial \lambda}{\partial x^j}v^k_{,k} + \frac{\partial \mu}{\partial x^j}(g^{jk}v^i_{,k} + g^{ik}v^j_{,k})\,,\end{aligned} \tag{3.46}$$

where the contravariant vector f^i reduces to the body force when the coordinate system is Cartesian. Corresponding to (3.46), we have the covariant equation

$$\begin{aligned}\rho\left(\frac{\partial v_i}{\partial t} + v^j v_{i,j}\right) &= \rho f_i - \frac{\partial p}{\partial x^i} + (\lambda+\mu)\frac{\partial}{\partial x^i}(v^k_{,k}) + \mu g^{jk} v_{i,jk}\\ &\quad + \frac{\partial \lambda}{\partial x^i}v^k_{,k} + \frac{\partial \mu}{\partial x^j}(g^{jk}v_{i,k} + v^j_{,i})\,.\end{aligned} \tag{3.47}$$

The energy equation (2.192) is a scalar equation. In general tensor notation,

$$\rho\left(\frac{\partial e}{\partial t} + v^i\frac{\partial e}{\partial x^i}\right) = -pv^i_{,i} + g^{ij}\left(k\frac{\partial \Theta}{\partial x^i}\right)_{,j} + Q + \lambda(v^i_{,i})^2 + 2\mu e^i_j e^j_i\,, \tag{3.48}$$

where $e^i_{\;j}$ denotes the mixed components of the rate of deformation tensor, whose covariant components are

$$e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \; . \tag{3.49}$$

Equations (2.194) and (2.195), which govern the motion of an incompressible fluid, generalize to

$$v^i_{,i} = 0 \; , \tag{3.50}$$

$$\frac{\partial v^i}{\partial t} + v^j v^i, j = f^i - \frac{g^{ij}}{\rho}\frac{\partial p}{\partial x^j} + \nu g^{jk} v^i_{,jk} \; . \tag{3.51}$$

The covariant form equivalent to (3.51) is

$$\frac{\partial v_i}{\partial t} + v^j v_{i,j} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x^i} + \nu g^{jk} v_{i,jk} \; . \tag{3.52}$$

Generalizing the constitutive equation (2.123), we obtain an expression for the mixed stress tensor:

$$T^i_j = -p\delta^i_j + \lambda \delta^i_j e^k_k + 2\mu e^i_{\;j} \; , \tag{3.53}$$

from which expressions for T_{ij} and T^{ij} are readily obtained.

3.3 Orthogonal Curvilinear Coordinates: Physical Components of Tensors

Consider a point P which has coordinates x^i_0 in the curvilinear x^i-system. The three surfaces

$$x^i = x^i_0 \tag{3.54}$$

meet and form a trihedral at P in Fig. 3.2. If for every point P these three surfaces meet at right angles, we say that x^i is an **orthogonal coordinate system**.[2] The distance ds from P to a point with coordinates $x^i_0 + dx^i$ is then given by the Pythagorean theorem.

We have

[2]This terminology is not intended to connote that such coordinate systems are obtained from Cartesian systems by orthogonal transformation—which is the case only if the x^i-system is itself Cartesian.

$$(ds)^2 = \left(\frac{dx^i}{h_i}\right)^2 , \tag{3.55}$$

where h_i are scale factors for the three coordinate directions. Their values will, in general, be functions of the point P. Comparing with Eq. (3.28), we see that for orthogonal coordinate systems

$$\begin{aligned} g_{ii} &= \frac{1}{h_i^2} \quad \text{(no summation convention)} , \\ g_{ij} &= 0 \quad (i \neq j) . \end{aligned} \tag{3.56}$$

Consequently

$$g = \det(g_{ij}) = \frac{1}{h_1^2 h_2^2 h_3^2} , \tag{3.57}$$

so that, in view of Eq. (3.25),

$$\begin{aligned} g^{ii} &= h_i^2 \quad \text{(not summed)} , \\ g^{ij} &= 0 \quad (i \neq j) . \end{aligned} \tag{3.58}$$

It is then easily verified that the Christoffel symbols of the second kind are given by

$$\begin{aligned} \Gamma^k_{ij} &= 0 , \\ \Gamma^j_{ii} &= \left(\frac{h_j^2}{h_i^3}\right) \frac{\partial h_i}{\partial x^j} , \\ \Gamma^i_{ij} &= \Gamma^i_{ji} = -\left(\frac{1}{h_i}\right) \frac{\partial h_i}{\partial x^j} , \\ \Gamma^i_{ii} &= -\left(\frac{1}{h_i}\right) \frac{\partial h_i}{\partial x^i} , \end{aligned} \tag{3.59}$$

where i, j, k are unequal and the summation convention is not used.

Let us now consider again the generic point P in space. We define a Cartesian coordinate system ξ^i with origin at P and with the increasing ξ^i-axis directed along dx^i. If the length scale of the ξ^i-system is chosen so that

$$(ds)^2 = (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 , \tag{3.60}$$

comparison with (3.55) reveals

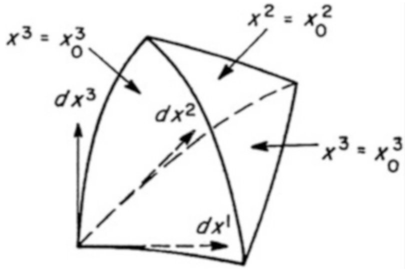


Fig. 3.2 Orthogonal coordinate system

$$\frac{\partial \xi^i}{\partial x^i} = \frac{1}{h_i} \quad \text{(not summed)}\ ,$$

$$\frac{\partial \xi^i}{\partial x^j} = 0 \quad (i \neq j)\ . \tag{3.61}$$

Therefore, according to the transformation rule (3.6), a contravariant vector with components A^i in the x^i-system will have components

$$\hat{A}^i = \frac{A^i}{h_i} \quad \text{(not summed)} \tag{3.62}$$

in the ξ^i-system. Because of the intimate relation between the ξ^i-system and the x^i-system, we may think of the $\hat{A}^i$ as a new set of components, defined in the x^i-system, of the vector A^i. These components, *which do not form a tensor*, are called the **physical components** of the vector. Since they are defined for a local Cartesian system, they have the physical significance, and dimensions, which characterize the vector in the usual sense of vector analysis. By way of example, if v^i is the contravariant velocity of the particle at P, then $\sqrt{(\hat{v}^1)^2 + (\hat{v}^2)^2 + (\hat{v}^3)^2}$ is the speed of the particle and $\hat{v}^i$ is its component of velocity along the direction dx^i.

Since the physical components of velocity do not form a tensor, it is customary to adopt for them some notation which cannot be confused with tensor notation. Our own preference is to attach subscripts signifying the descriptive labels of the curvilinear coordinates. Thus for a spherical polar system we shall denote the physical components of velocity by v_r, v_θ, v_ϕ. For the generic curvilinear system x^i, there are no descriptive labels. Hence we use the letters α, β, γ, associated in an obvious manner with the x^1, x^2, x^3-directions. Thus Eq. (3.62) is rewritten

$$A_\alpha = \frac{A^1}{h_1},\ A_\beta = \frac{A^2}{h_2},\ A_\gamma = \frac{A^3}{h_3}\ . \tag{3.63}$$

By using Eqs. (3.24) and (3.58), we can express the physical components in terms of the covariant components:

$$A_\alpha = h_1 A_1,\ A_\beta = h_2 A_2,\ A_\gamma = h_3 A_3\ . \tag{3.64}$$

Physical components can be defined for tensors of any rank. For the important case of rank 2, we have

$$\begin{aligned} A_{\alpha\alpha} &= \frac{A^{11}}{h_1^2} = A_1^1 = h_1^2 A_{11} \ , \\ A_{\alpha\beta} &= \frac{A^{12}}{h_1 h_2} = \left(\frac{h_1}{h_2}\right) A_1^2 = \left(\frac{h_2}{h_1}\right) A_2^1 = h_1 h_2 A_{12} \ , \text{etc.} \end{aligned} \tag{3.65}$$

The equations of hydrodynamics, referred to a generic orthogonal coordinate system, can be written in terms of the physical velocity components. Since this only involves substituting the results of this section into the general tensor equations of flow, it is not difficult. However it is messy and serves little purpose. Instead we shall carry out the details for the two curvilinear systems most used in practice.

3.3.1 Cylindrical Polar Coordinates

A cylindrical polar system x^i is usually designated by the descriptive labels r, θ, z. In terms of a reference Cartesian system ξ^i,

$$\begin{aligned} r &= x^1 = \sqrt{(\xi^1)^2 + (\xi^2)^2} \ , \\ \theta &= x^2 = \arctan(\xi^2/\xi^1), \\ z &= x^3 = \xi^3 \ . \end{aligned} \tag{3.66}$$

For this system the metric tensors are given by

$$\left(g^{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \left(g_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.67}$$

Thus the cylindrical polar system is orthogonal.

The Christoffel symbols of the second kind all vanish except

$$\Gamma_{22}^1 = -r, \ \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r} \ . \tag{3.68}$$

The physical components of velocity are related to the tensor components according to

$$\begin{aligned} v_r &= v^1 = v_1 \ , \\ v_\theta &= r v^2 = \frac{v_2}{r} \ , \end{aligned} \tag{3.69}$$

$$v_z = v^3 = v_3 \; .$$

Substituting these results into the equation of continuity (3.44), we obtain

$$\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \frac{v_\theta}{r}\frac{\partial \rho}{\partial \theta} + v_z \frac{\partial \rho}{\partial z} + \rho \Delta = 0 \; , \tag{3.70}$$

where the **dilation** Δ is given by

$$\Delta = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \; . \tag{3.71}$$

The Navier-Stokes equation (3.47) yields the three component equations

$$\begin{aligned}
&\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = \rho f_r - \frac{\partial p}{\partial r} \\
&+ (\lambda + \mu) \frac{\partial \Delta}{\partial r} + \mu \left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \right) \\
&+ \Delta \frac{\partial \lambda}{\partial r} + 2 \left(e_{rr} \frac{\partial \mu}{\partial r} + \frac{1}{r} e_{r\theta} \frac{\partial \mu}{\partial \theta} + e_{rz} \frac{\partial \mu}{\partial z} \right) \; , \\
&\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = \rho f_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \\
&+ \frac{(\lambda + \mu)}{r} \frac{\partial \Delta}{\partial \theta} + \mu \left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} \right) \\
&+ \frac{\Delta}{r} \frac{\partial \lambda}{\partial \theta} + 2 \left(e_{\theta r} \frac{\partial \mu}{\partial r} + \frac{1}{r} e_{\theta\theta} \frac{\partial \mu}{\partial \theta} + e_{\theta z} \frac{\partial \mu}{\partial z} \right) \; , \\
&\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \rho f_z - \frac{\partial p}{\partial z} + (\lambda + \mu) \frac{\partial \Delta}{\partial z} + \mu \nabla^2 v_z \\
&+ \Delta \frac{\partial \lambda}{\partial z} + 2 \left(e_{zr} \frac{\partial \mu}{\partial r} + \frac{1}{r} e_{z\theta} \frac{\partial \mu}{\partial \theta} + e_{zz} \frac{\partial \mu}{\partial z} \right) \; ,
\end{aligned} \tag{3.72}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \; , \tag{3.73}$$

and the physical components of the rate of deformation tensor are given by

$$e_{rr} = \frac{\partial v_r}{\partial r} \; ,$$

$$
\begin{aligned}
e_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) , \\
e_{zz} &= \frac{\partial v_z}{\partial z} , \\
2e_{\theta z} = 2e_{z\theta} &= \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} , \\
2e_{zr} = 2e_{rz} &= \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} , \\
2e_{r\theta} = 2e_{\theta r} &= \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta} .
\end{aligned}
\tag{3.74}
$$

From the energy equation (3.48), we obtain

$$
\begin{aligned}
\rho\left(\frac{\partial e}{\partial t} + v_r\frac{\partial e}{\partial r} + \frac{v_\theta}{r}\frac{\partial e}{\partial \theta} + v_z\frac{\partial e}{\partial z}\right) &= -p\Delta + \frac{\partial}{\partial r}\left(k\frac{\partial \Theta}{\partial r}\right) + \frac{k}{r}\frac{\partial \Theta}{\partial r} \\
+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial \Theta}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial \Theta}{\partial z}\right) &+ Q + \lambda\Delta^2 + 2\mu\sum_{\alpha,\beta=r,\theta,z} e_{\alpha\beta}^2 .
\end{aligned}
\tag{3.75}
$$

The physical components of stress are given by

$$
\begin{aligned}
T_{\alpha\alpha} &= -p + \lambda\Delta + 2\mu e_{\alpha\alpha} \quad (\alpha = r, \theta, z) \\
T_{\alpha\beta} &= 2\mu e_{\alpha\beta} \quad (\alpha \neq \beta) .
\end{aligned}
\tag{3.76}
$$

For an incompressible fluid with constant ρ and μ, Eqs. (2.223) and (2.224) apply. When the motion is referred to the cylindrical polar system, we have

$$
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 , \tag{3.77}
$$

$$
\begin{aligned}
\rho(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}) \\
= \rho f_r - \frac{\partial p}{\partial r} + \mu(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}) ,
\end{aligned}
\tag{3.78}
$$

$$
\begin{aligned}
\rho(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}) \\
= \rho f_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}) ,
\end{aligned}
\tag{3.79}
$$

$$
\rho(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}) = \rho f_z - \frac{\partial p}{\partial z} + \mu\nabla^2 v_z . \tag{3.80}
$$

For two dimensional flow, in which v_z is zero and all $\partial/\partial z$ terms vanish,[3] this system can be somewhat simplified by introducing a stream function ψ such that

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta},\ v_\theta = -\frac{\partial \psi}{\partial r}\ . \tag{3.81}$$

Equations (3.77) and (3.80) are automatically satisfied and p can be eliminated from (3.78), (3.79), yielding

$$\rho\left[\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{1}{r}\frac{\partial(\psi,\nabla^2\psi)}{\partial(r,\theta)}\right] = \frac{\rho}{r}\left[\frac{\partial f_r}{\partial \theta} - \frac{\partial}{\partial r}(r f_\theta)\right] + \mu\nabla^4\psi\ , \tag{3.82}$$

in which ∇^2 denotes the two-dimensional Laplacian, i.e.,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\ . \tag{3.83}$$

For fluid motion which is symmetrical about the z-axis, so that $\partial/\partial\theta$ terms vanish, we introduce the **Stokes stream function** Ψ, i.e. a stream function for flow in the meridional planes, such that

$$v_z = \frac{1}{r}\frac{\partial \Psi}{\partial r},\ v_r = -\frac{1}{r}\frac{\partial \Psi}{\partial z} \tag{3.84}$$

and proceed as for two-dimensional flow. If v_θ is not zero, the resulting equations are usually expressed in terms of the **swirl** Ω, defined by

$$\Omega = r v_\theta\ . \tag{3.85}$$

Since $\partial v_\theta/\partial\theta = 0$, use of (3.84) automatically satisfies (3.77). Eliminating p from (3.78) and (3.80), we obtain

$$\frac{\partial}{\partial t}(E^2\Psi) + \frac{1}{r^2}\frac{\partial}{\partial z}(\Omega^2) + \frac{1}{r}\frac{\partial(\Psi, E^2\Psi)}{\partial(r,z)} + \frac{2}{r^2}\frac{\partial \Psi}{\partial z}E^2\Psi = r\left(\frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z}\right) + \nu E^4\Psi\ , \tag{3.86}$$

where E^2 is a second order differential operator, defined by

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\ . \tag{3.87}$$

[3] Except that, if f_z is a function of z alone, we permit the pressure p to include an additive term depending on z, so that (3.80) can be satisfied hydrostatically by setting $\partial p/\partial z = \rho f_z$.

In terms of the swirl, Eq. (3.79) becomes

$$\frac{\partial \Omega}{\partial t} + \frac{1}{r}\frac{\partial(\Psi, \Omega)}{\partial(r, z)} = r f_\theta + \nu E^2 \Omega \,. \tag{3.88}$$

In Sect. 2.8 it was pointed out that, in two-dimensional incompressible flow, the vorticity is "pseudo-conserved", in the sense that its equation closely resembles that of a truly conserved physical quantity. In axisymmetric flow, that isn't true for $\omega = \partial v_z/\partial r - \partial v_r/\partial z$. However a related quantity, the **Svanberg vorticity** $S = \omega/r$, is pseudo-conserved. It was so named Langlois (1981) because a very old result, Svanberg's vorticity theorem Truesdell and Toupin (1960), sheds light on why this is so. It is a better choice than ω for the vorticity-streamfunction formulation of axisymmetric incompressible flow. The governing equation for S is

$$\begin{aligned} &\frac{\partial S}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r\, S) + \frac{\partial}{\partial z}(v_z\, S) + \frac{\partial}{\partial z}(\frac{\Omega^2}{r^4}) \\ &\quad = \frac{1}{r}\left(\frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z}\right) + \frac{\nu}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r^2\, S)\right] + \nu\frac{\partial^2 S}{\partial z^2} \,. \end{aligned} \tag{3.89}$$

3.3.2 Spherical Polar Coordinates

It is sometimes advantageous to refer fluid motion to a spherical polar coordinate system (r, θ, ϕ), defined in terms of a Cartesian system ξ^i according to

$$\begin{aligned} r &= x^1 = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2} \,, \\ \theta &= x^2 = \arctan\frac{\sqrt{(\xi^1)^2 + (\xi^2)^2}}{\xi^3}, \\ \phi &= x^3 = \arctan\left(\frac{\xi^2}{\xi^1}\right) \,. \end{aligned} \tag{3.90}$$

The metric tensors of the spherical polar system are given by

$$\left(g^{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2 \sin^2\theta \end{pmatrix}, \quad \left(g_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}, \tag{3.91}$$

so that the system is orthogonal. The non-vanishing Christoffel symbols of the second kind are

$$\Gamma^1_{22} = -r,\; \Gamma^1_{33} = -r\, \sin^2\theta \,,$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r},\ \Gamma_{33}^2 = -\sin\theta\cos\theta\ , \tag{3.92}$$
$$\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r},\ \Gamma_{23}^3 = \Gamma_{32}^3 = \cot\theta\ .$$

The physical components of velocity are related to the tensor components according to

$$\begin{aligned} v_r &= v^1 = v_1\ , \\ v_\theta &= r v^2 = \frac{v_2}{r}\ , \\ v_\phi &= (r\ \sin\theta) v^3 = \frac{v_3}{r\sin\theta}\ . \end{aligned} \tag{3.93}$$

Substituting these results into the equation of continuity (3.44), we obtain

$$\frac{\partial\rho}{\partial t} + v_r\frac{\partial\rho}{\partial r} + \frac{v_\theta}{r}\frac{\partial\rho}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial\rho}{\partial\phi} + \rho\Delta = 0\ , \tag{3.94}$$

where

$$\Delta = \frac{\partial v_r}{\partial r} + \frac{2v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\theta\cot\theta}{r} + \frac{1}{r\ \sin\theta}\frac{\partial v_\phi}{\partial\phi}\ . \tag{3.95}$$

The Navier-Stokes equation (3.47) yields

$$\begin{aligned} &\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = \rho f_r - \frac{\partial p}{\partial r} \\ &+ (\lambda+\mu)\frac{\partial\Delta}{\partial r} + \mu\left(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2v_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right) \\ &+ \Delta\frac{\partial\lambda}{\partial r} + 2\left(e_{rr}\frac{\partial\mu}{\partial r} + \frac{1}{r}e_{r\theta}\frac{\partial\mu}{\partial\theta} + \frac{1}{r\sin\theta}e_{r\phi}\frac{\partial\mu}{\partial\phi}\right)\ , \\ &\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right) = \rho f_\theta - \frac{1}{r}\frac{\partial p}{\partial\theta} \\ &+ \frac{(\lambda+\mu)}{r}\frac{\partial\Delta}{\partial\theta} + \mu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right) \\ &+ \frac{\Delta}{r}\frac{\partial\lambda}{\partial\theta} + 2\left(e_{\theta r}\frac{\partial\mu}{\partial r} + \frac{1}{r}e_{\theta\theta}\frac{\partial\mu}{\partial\theta} + \frac{1}{r\sin\theta}e_{\theta\phi}\frac{\partial\mu}{\partial\phi}\right)\ , \\ &\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi\cot\theta}{r}\right) \end{aligned}$$

$$= \rho f_\phi - \frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} + \frac{(\lambda+\mu)}{r\sin\theta}\frac{\partial\Delta}{\partial\phi}$$
$$+ \mu\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}\right)$$
$$+ \frac{\Delta}{r\sin\theta}\frac{\partial\lambda}{\partial\phi} + 2\left(e_{\phi r}\frac{\partial\mu}{\partial r} + \frac{1}{r}e_{\phi\theta}\frac{\partial\mu}{\partial\theta} + \frac{1}{r\sin\theta}e_{\phi\phi}\frac{\partial\mu}{\partial\phi}\right), \tag{3.96}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{3.97}$$

and the physical components of the rate of deformation tensor are given by

$$\begin{aligned}
e_{rr} &= \frac{\partial v_r}{\partial r}, \\
re_{\theta\theta} &= \frac{\partial v_\theta}{\partial\theta} + v_r, \\
r\ \sin\theta\, e_{\phi\phi} &= \frac{\partial v_\phi}{\partial\phi} + v_r\sin\theta + v_\theta\cos\theta, \\
2r\sin\theta\, e_{\theta\phi} &= 2r\sin\theta\, e_{\phi\theta} = \sin\theta\frac{\partial v_\phi}{\partial\theta} - v_\phi\cos\theta + \frac{\partial v_\theta}{\partial\phi}, \\
2r\sin\theta\, e_{\phi r} &= 2r\sin\theta\, e_{r\phi} = \frac{\partial v_r}{\partial\phi} + r\sin\theta\frac{\partial v_\phi}{\partial r} - v_\phi\sin\theta, \\
2re_{r\theta} &= 2re_{\theta r} = r\frac{\partial v_\theta}{\partial r} - v_\theta + \frac{\partial v_r}{\partial\theta}.
\end{aligned} \tag{3.98}$$

The energy equation (3.48) becomes

$$\begin{aligned}
\rho\left(\frac{\partial e}{\partial t} + v_r\frac{\partial e}{\partial r} + \frac{v_\theta}{r}\frac{\partial e}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial e}{\partial\phi}\right) &= -p\Delta + \frac{\partial}{\partial r}\left(k\frac{\partial\Theta}{\partial r}\right) + \frac{2k}{r}\frac{\partial\Theta}{\partial r} \\
&+ \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(k\frac{\partial\Theta}{\partial\theta}\right) + \frac{k\cot\theta}{r^2}\frac{\partial\Theta}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial\Theta}{\partial\phi}\right) \\
&+ Q + \lambda\Delta^2 + 2\mu\sum_{\alpha,\beta=r,\theta,\phi} e^2_{\alpha\beta}.
\end{aligned} \tag{3.99}$$

The physical components of stress are given by

$$\begin{aligned}
T_{\alpha\alpha} &= -p + \lambda\Delta + 2\mu e_{\alpha\alpha} \quad (\alpha = r,\theta,\phi), \\
T_{\alpha\beta} &= 2\mu e_{\alpha\beta} \quad (\alpha \neq \beta).
\end{aligned} \tag{3.100}$$

Equations (2.223) and (2.224), which govern an incompressible fluid with constant ρ and μ, become

$$\frac{\partial v_r}{\partial r} + \frac{2v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta \cot\theta}{r} + \frac{1}{r\,\sin\theta}\frac{\partial v_\phi}{\partial \phi} = 0, \quad (3.101)$$

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = \rho f_r - \frac{\partial p}{\partial r}$$
$$+\mu\left(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right), \quad (3.102)$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right) = \rho f_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\mu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right), \quad (3.103)$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi\cot\theta}{r}\right) = \rho f_\phi$$
$$-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right). \quad (3.104)$$

If the flow is symmetrical about the axis $\theta = 0$, so that all ϕ-derivatives vanish, the continuity equation (3.101) is automatically satisfied by expressing v_r and v_θ in terms of a Stokes stream function:

$$v_r = \frac{1}{r^2\,\sin\theta}\frac{\partial \Psi}{\partial \theta},\; v_\theta = -\frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r}. \quad (3.105)$$

The velocity component v_ϕ is related to the swirl according to

$$v_\phi = \frac{\Omega}{r\sin\theta}. \quad (3.106)$$

Eliminating p from Eqs. (3.92) and (3.103), we obtain

$$\frac{\partial}{\partial t}(E^2\Psi) + \frac{2\Omega}{r^2\sin^2\theta}\left(\frac{\partial \Omega}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \Omega}{\partial \theta}\sin\theta\right) - \frac{1}{r^2\sin\theta}\frac{\partial(\Psi, E^2\Psi)}{\partial(r,\theta)}$$
$$+\frac{2E^2\Psi}{r^2\sin^2\theta}\left(\frac{\partial \Psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \Psi}{\partial \theta}\sin\theta\right) = \sin\theta\left[\frac{\partial f_r}{\partial \theta} - \frac{\partial}{\partial r}(rf_\theta)\right] + \nu E^4\Psi, \quad (3.107)$$

where E^2 is a second order differential operator, defined by

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} \ . \tag{3.108}$$

In terms of the swirl, Eq. (3.104) becomes

$$\frac{\partial \Omega}{\partial t} - \frac{1}{r^2 \sin\theta}\frac{\partial(\Psi, \Omega)}{\partial(r, \theta)} = r f_\phi \sin\theta + \nu E^2 \Omega \ . \tag{3.109}$$

Chapter 4
Exact Solutions to the Equations of Viscous Flow

Abstract A collection of exact solutions to the equations of viscous hydrodynamics is presented, along with one for non-Newtonian flow and one which uses the Boussinesq approximation to treat a problem in natural convection.

In this chapter we present some of the very few known cases for which the equations of viscous flow can be solved without approximation. We consider only incompressible flow, since there are virtually no exact solutions known for the flow of a viscous, compressible fluid. However for a few exceptions, see Goldstein (1960), von Mises (2004) and Chap. 9 in Panton (1984).

An impressive review of exact and approximate solutions of the Navier-Stokes equation was compiled by Berker (1963).

We shall assume throughout that the body forces are zero. However the solutions are easily modified to cover the important case of a conservative body force field. For this case the covariant components of body force are derivable from a potential:

$$f_i = \frac{\partial \phi}{\partial x^i} \ .$$

Consequently, if we set

$$p^* = p - \rho\phi \ ,$$

the grouping $(\rho f_i - \partial p/\partial x^i)$ in the covariant Navier-Stokes equation (3.47) is replaced by $-\partial p^*/\partial x^i$. If the body force is gravity, we might describe this procedure as incorporating the gravitational head into the pressure head.

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_4,

4.1 Rectilinear Flow Between Parallel Plates

In Fig. 4.1, consider that an incompressible fluid is contained in the region between two parallel infinite plates, which move steadily in their own planes. We choose a Cartesian coordinate system with x_3-axis normal to the plates and with origin midway between them, so that the plates correspond, say, to $x_3 = \pm h$. Further we let the coordinate system translate with the average velocity of the plates. Hence if the plate at $x_3 = +h$ moves with velocity components U, V in the x_1- and x_2-directions, respectively, then the plate at $x_3 = -h$ moves with velocity components $-U, -V$. Thus, the no-slip condition requires that

$$\begin{aligned} v_1 &= \pm U \quad \text{at } x_3 = \pm h \ , \\ v_2 &= \pm V \quad \text{at } x_3 = \pm h \ , \\ v_3 &= 0 \qquad \text{at } x_3 = \pm h \ . \end{aligned} \tag{4.1}$$

Since the coordinate system is not accelerated, Eqs. (2.194) and (2.195) apply. We seek a solution of the form

$$\begin{aligned} v_1 &= u(x_3) \ , \\ v_2 &= v(x_3) \ , \\ v_3 &= 0 \ , \\ p &= p(x_1) \ . \end{aligned} \tag{4.2}$$

Such a solution does, in fact, exist: the continuity equation (2.194) and the $i = 3$ component of the momentum equation (2.195) are automatically satisfied; the other two component equations reduce to

$$\mu \frac{\partial^2 u}{\partial x_3^2} = \frac{\partial p}{\partial x_1} \ , \tag{4.3}$$

$$\mu \frac{\partial^2 v}{\partial x_3^2} = 0 \ , \tag{4.4}$$

which are easily solved subject to the boundary conditions (4.1). According to (4.2), u is a function of x_3, only and p is a function of x_1 only. Consequently both sides of (4.3) are equal to the same constant, say $-G$. Thus

$$\begin{aligned} u &= U\frac{x_3}{h} + \left(\frac{Gh^2}{2\mu}\right)\left(1 - \frac{x_3^2}{h^2}\right) \ , \\ v &= \frac{Vx_3}{h} \ , \\ p &= C - Gx_1 \ , \end{aligned} \tag{4.5}$$

where C is a constant of integration.

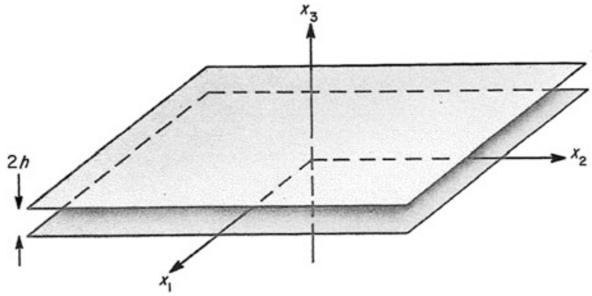


Fig. 4.1 Geometry for flow between parallel plates

A particular case corresponds to the flow between stationary plates, i.e. $U = V = 0$ in the presence of a pressure gradient. The velocity field reduces to a parabola

$$u_P = \left(\frac{Gh^2}{2\mu}\right)\left(1 - \frac{x_3^2}{h^2}\right) , \tag{4.6}$$

and this flow is know as **channel flow** or **plane Poiseuille flow** (after Jean Louis Marie Poiseuille, 1797–1869).

The **flux** Q of the flow in the x_1-direction per unit length in the x_2 direction is defined by

$$Q = \int_{-h}^{h} u \, dx_3 . \tag{4.7}$$

Integrating the first of Eqs. (4.5), we obtain

$$Q = 2G\frac{h^3}{3\mu} . \tag{4.8}$$

Thus the pressure gradient is proportional to the flux, to which the motion of the plates does not contribute.

If there is no pressure gradient, we can select an orientation of the coordinate system so that (4.5) reduces to the **Couette flow** (after Maurice Marie Alfred Couette, 1858–1943)

$$\begin{aligned} u &= U\frac{x_3}{h} , \\ v &= 0 , \\ p &= C . \end{aligned} \tag{4.9}$$

This solution is sometimes called **homogeneous shear flow**.

The general case (4.5) corresponds to Couette flow with a pressure flow superimposed; the two basic flows may be oblique to each other. Since, for the assumed form of solution (4.2), the nonlinear terms drop out of the Navier-Stokes equation, the two flows do not couple.

4.2 Plane Shear Flow of a Non-Newtonian Fluid

Let us consider the simple shear flow of the Reiner-Rivlin fluid (Eq. (2.149)) in a Cartesian orthonormal coordinate system such that

$$v_1 = \dot{\gamma}\, x_2,\; v_2 = v_3 = 0\,, \tag{4.10}$$

where $\dot{\gamma}$ is called the shear rate. The components e_{ij} of the tensor $\mathbf{e}$ vanish except for $e_{12} = e_{21} = \frac{\dot{\gamma}}{2}$, which then leads to $\mathrm{I}_2\,(\mathbf{e}) = -\dot{\gamma}^2/4$ and $\mathrm{I}_3\,(\mathbf{e}) = 0$. We will denote the matrices associated with tensors by their symbol within square brackets. Therefore the matrices $[e]$, $[e^2]$ are given by

$$[e] = \begin{pmatrix} 0 & \frac{\dot{\gamma}}{2} & 0 \\ \frac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \left[e^2\right] = \begin{pmatrix} \frac{\dot{\gamma}^2}{4} & 0 & 0 \\ 0 & \frac{\dot{\gamma}^2}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.11}$$

The corresponding stress components are then

$$T_{11} = T_{22} = -p + \varphi_2\,\frac{\dot{\gamma}^2}{4}, \quad T_{33} = -p\,, \tag{4.12}$$

$$T_{12} = T_{21} = \varphi_1\,\frac{\dot{\gamma}}{2} = \tau(\dot{\gamma})\,, \tag{4.13}$$

$$T_{13} = T_{23} = 0\,. \tag{4.14}$$

Let us introduce the **first normal stress differences** defined by the relations

$$N_1 = T_{11} - T_{22}, \quad N_2 = T_{22} - T_{33}\,. \tag{4.15}$$

Here we obtain

$$N_1 = 0 \quad \text{and} \quad N_2 = \varphi_2\,\frac{\dot{\gamma}^2}{4}\,. \tag{4.16}$$

However, this does not correspond to the physical reality as experimental data show neither N_1 nor N_2 vanishes. We therefore need a different kind of model, namely the second-order fluid, to resolve the right normal stress differences. This model uses Rivlin-Ericksen tensors, a concept described in Deville and Gatski (2012). It can be shown that all three functions N_1, N_2, τ depend on the nature of the fluid, and are called the **viscometric functions** of the material. Note also that in the Newtonian case $N_2 = 0$, indicating that for this kind of fluid, there is no imbalance in the first normal stress differences.

4.3 The Flow Generated by an Oscillating Plate

Let us now suppose that an infinite flat plate at the bottom of an infinitely deep sea of fluid executes linear harmonic motion parallel to itself. We let the plate lie in the $x_3 = 0$ plane of a Cartesian coordinate system, so oriented that the oscillation is along the x_1-axis. The location of the origin is unimportant, but the x_i-system is fixed in space, not in the oscillating plate. The motion of the plate generates in the fluid a rectilinear flow, partially in-phase, partially out-of-phase, with the plate. The pressure, however, remains constant.

If the velocity-amplitude and frequency of the plate motion are denoted, respectively, by A and ω, the no-slip condition requires that

$$v_1 = = A\cos\omega t \quad \text{at } x_3 = 0\,, \tag{4.17}$$

$$v_2 = v_3 = 0 \quad \text{at } x_3 = 0\,. \tag{4.18}$$

If we set

$$\begin{aligned} v_1 &= f(x_3)\cos\omega t + g(x_3)\sin\omega t\,, \\ v_2 &= v_3 = 0\,, \\ p &= constant\,, \end{aligned} \tag{4.19}$$

the equations of motion (2.194), (2.195) are satisfied, provided only that

$$(\omega g - \nu f'')\cos\omega t = (\omega f + \nu g'')\sin\omega t\,. \tag{4.20}$$

Equation (4.20) can hold for all values of t only if both sides vanish independently. Thus

$$\begin{aligned} \omega g - \nu f'' &= 0\,, \\ \omega f + \nu g'' &= 0\,. \end{aligned} \tag{4.21}$$

It is readily verified that the general solution of this system is

$$\begin{aligned} f(x_3) &= e^{-kx_3}(c_1\cos kx_3 + c_2\sin kx_3) + e^{+kx_3}(c_3\cos kx_3 + c_4\sin kx_3)\,, \\ g(x_3) &= e^{-kx_3}(c_1\sin kx_3 - c_2\cos kx_3) - e^{+kx_3}(c_3\sin kx_3 - c_4\cos kx_3)\,, \end{aligned} \tag{4.22}$$

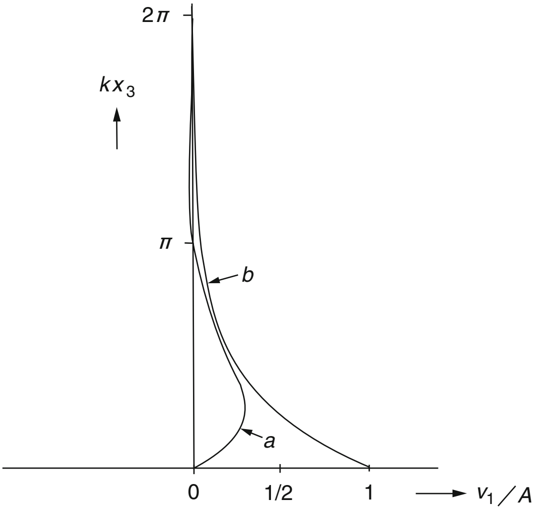


Fig. 4.2 Velocity distribution above an oscillating plate. (*a*) Plate at maximum displacement. (*b*) Plate at midcycle

where

$$k = \sqrt{\frac{\omega}{2\nu}} \, . \tag{4.23}$$

Since the growing exponentials are not physically admissible, we set

$$c_3 = c_4 = 0 \, . \tag{4.24}$$

The remaining constants are evaluated from the boundary condition (4.17):

$$c_1 = A, \; c_2 = 0 \, . \tag{4.25}$$

We then have

$$\begin{aligned} v_1 &= Ae^{-kx_3}(\cos kx_3 \cos \omega t + \sin kx_3 \sin \omega t) \\ &= Ae^{-kx_3} \cos(\omega t - kx_3) \, . \end{aligned} \tag{4.26}$$

Thus the oscillating plate sets up a corresponding oscillation in the fluid. As we move away from the plate, the amplitude decays exponentially and the phase lag with respect to the plate motion varies linearly. Two fluid layers a distance $2\pi/k$ apart oscillate in phase. This distance, which, by (4.23), is equal to $2\pi\sqrt{2\nu/\omega}$, is called the **depth of penetration** of the harmonic motion. That it increases with viscosity and decreases with frequency is not surprising: if we slowly oscillate a flat plate in a sticky fluid, we expect to drag large masses of fluid along with the plate;

on the other hand, if we move the plate rapidly in a fluid of low viscosity, we expect the fluid essentially to ignore the plate, except in a thin boundary-layer.

The velocity profile above an oscillating plate is illustrated in Fig. 4.2.

4.4 Transient Flow in a Semi-infinite Space

Let us consider the case in which the plate executes a motion more general than steady-state oscillation. To make it definite, let us suppose that the plate, and the fluid above it, are at rest until time zero, when the plate begins to move in the x_1-direction[1] with velocity $V(t)$. If we set

$$\begin{aligned} v_1 &= v(x_3, t) \ , \\ v_2 &= v_3 = 0 \ , \\ p &= \text{constant} \ , \end{aligned} \tag{4.27}$$

the equations of motion (2.194), (2.195) are satisfied if

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x_3^2} \ . \tag{4.28}$$

Thus the velocity generated by the moving plate diffuses through the fluid according to the heat equation. This equation can be solved subject to the initial condition

$$v(x_3, 0) = 0 \ , \tag{4.29}$$

and to the boundary condition

$$v(0, t) = V(t) \ , \tag{4.30}$$

by straightforward application of the Laplace transform. Some instructive cases are worked out in Schlichting (1960); unsteady motion between parallel plates is also considered. The reader is also referred to Dowty (1963).

If we assume $V(t) = V$, with V a constant, then we can work out easily a closed form solution. The initial and boundary conditions are recapitulated as

$$t < 0, \quad v = 0, \quad \forall \ x_3 \ , \tag{4.31}$$

$$t \geqslant 0, \quad v = V, \quad \text{at } x_3 = 0 \ , \tag{4.32}$$

$$v = 0, \quad \text{at } x_3 = \infty \ . \tag{4.33}$$

[1] Motion in the x_2-direction could be superimposed; the resulting fluid motions do not couple.

Equation (4.28) is a diffusion type equation similar to the heat equation. We will transform this partial differential equation into an ordinary differential equation by a change of variables that is based on similarity considerations. As the problem has no other space variable than x_3 and no other time scale than t, one combines them to form the dimensionless group

$$\eta = \frac{x_3}{2\sqrt{\nu t}} \ . \tag{4.34}$$

This change of variable will produce an ordinary differential equation whose solution is a function of η. This solution is called a **self-similar solution** as the velocity profile with respect to x_3 is similar at any time t. Setting

$$v = V\, f(\eta) \ , \tag{4.35}$$

Eq. (4.28) becomes

$$f'' + 2\eta f' = 0 \ , \tag{4.36}$$

with the conditions

$$\eta = 0,\ f = 1; \quad \eta = \infty, f = 0 \ . \tag{4.37}$$

Integrating (4.36), one obtains

$$f = A \int_0^{\eta} e^{-\eta'^2} d\eta' + B \ . \tag{4.38}$$

With the conditions (4.37), one gets for $\eta = 0$, $B = 1$ and for $\eta = \infty$, $A = -2/\sqrt{\pi}$. In terms of the error function $\text{erf}(x)$ defined by

$$\text{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-\tau^2} d\tau \ , \tag{4.39}$$

which makes $\text{erf}(\infty) = 1$, Eq. (4.38) becomes

$$f = 1 - \text{erf}\,\eta \ . \tag{4.40}$$

The velocity profile for $t > 0$ is

$$v = V \left[1 - \text{erf}(\frac{x_3}{2\sqrt{\nu t}}) \right] \ . \tag{4.41}$$

and is shown in Fig. 4.3.

The penetration depth of the plate movement in the semi-infinite space is related to the question: for t fixed, what is the distance to the plate where the velocity

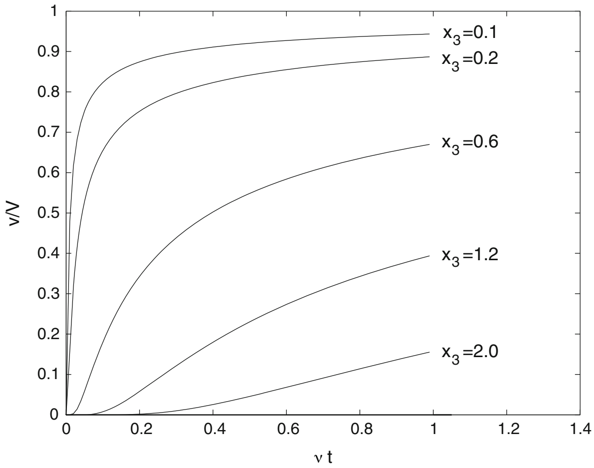


Fig. 4.3 Transient flow in a semi-infinite space

reaches, for example, 1 % of the V value? From numerical erf values, the function $1 - \text{erf}(\eta)$ is 0.01 for $\eta \sim 2$. The penetration depth δ is consequently given by

$$\eta_\delta = \frac{\delta}{2\sqrt{\nu t}} \simeq 2, \quad \delta \simeq 4\sqrt{\nu t}\ . \tag{4.42}$$

It is proportional to the square root of the kinematic viscosity and time. If the viscosity is very small, the fluid "sticks" less to the wall and the effect of the wall presence is reduced. If t goes to infinity, the velocity at each position in the semi-infinite space goes to V.

4.5 Channel Flow with a Pulsatile Pressure Gradient

Blood flow in the vascular tree is driven by the pulsating pressure gradient produced by the heart that is acting as a pump. In order to avoid (temporarily) the geometrical complexity of cylindrical coordinates appropriate for blood flow in the arteries, we will tackle a simplified version of the problem, namely the plane channel flow under an oscillating pressure gradient.

Recall that the standard Poiseuille flow with a steady constant pressure gradient denoted by G gives rise to the parabolic velocity profile (4.6). Let us add now an oscillating component characterized by the pulsation ω such that

$$-\frac{1}{\rho}\frac{\partial p}{\partial x_1} = -G - C\cos\omega t\ , \tag{4.43}$$

with C a constant obtained from experimental data, for example. For the sake of simplicity in the analytical treatment, it is customary to resort to Fourier representation and use the following relation

$$-\frac{1}{\rho}\frac{\partial p}{\partial x_1} = -G - \Re(Ce^{i\omega t}) \,, \tag{4.44}$$

where $\Re$ means the real part. As a steady state oscillating solution is sought for the velocity field, the solution is written as a complex function

$$v_1 = u_P + \Re\left(u(\omega, x_3)e^{i\omega t}\right) \,, \tag{4.45}$$

where the Poiseuille solution u_P given by Eq. (4.6) corresponds to constant pressure gradient.

The Navier-Stokes equations lead to the relation

$$\frac{\partial v_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\frac{\partial^2 v_1}{\partial x_3^2} \,. \tag{4.46}$$

With Eqs. (4.44) and (4.45), Eq. (4.46) gives

$$i\omega\, u = -C + \nu\frac{\partial^2 u}{\partial x_3^2} \,. \tag{4.47}$$

The boundary conditions are

$$u(h) = 0, \quad \frac{\partial u}{\partial x_3}(0) = 0 \,. \tag{4.48}$$

The solution of (4.47) is

$$u = \Re\left[\frac{iC}{\omega}\left(1 - \frac{\cosh\sqrt{\frac{i\omega}{\nu}}x_3}{\cosh\sqrt{\frac{i\omega}{\nu}}h}\right)\right] \,. \tag{4.49}$$

Taking the relation $i^{1/2} = (1+i)/\sqrt{2}$ into account, the real part of (4.49) yields the velocity field

$$v_1 = u_P - \frac{C}{\omega}\left[\left(1 - \frac{f_1(\omega, x_3)}{f_3(kh)}\right)\sin\omega t - \frac{f_2(\omega, x_3)}{f_3(kh)}\cos\omega t\right] \,, \tag{4.50}$$

where the various notations are defined as follows

$$k = \sqrt{\frac{\omega}{2\nu}} \,,$$

$$cc(x) = \cos(x)\cosh(x) \,,$$

$$ss(x) = \sin(x)\sinh(x) \,,$$

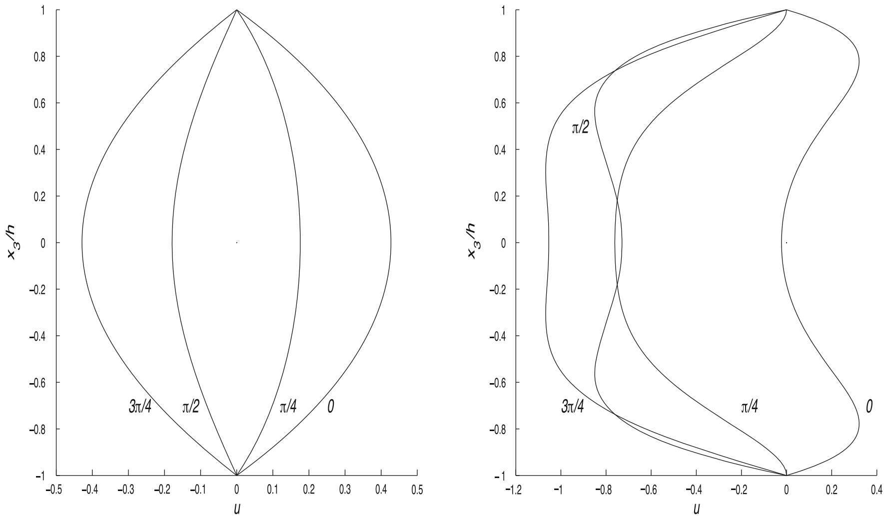


Fig. 4.4 Pulsating velocity field with $\omega = 1$; *left*: $k = 1/\sqrt{2}$; *right*: $k = 5/\sqrt{2}$

$$
\begin{aligned}
f_1(\omega, x_3) &= cc(kx_3)cc(kh) + ss(kx_3)ss(kh) , \\
f_2(\omega, x_3) &= cc(kx_3)ss(kh) - ss(kx_3)cc(kh) , \\
f_3(\omega) &= cc^2(\omega) + ss^2(\omega) .
\end{aligned}
\tag{4.51}
$$

Figure 4.4 shows the time evolution of the velocity profile for two different values of k. The left part represents the flow for a low frequency case or when the viscous forces are important, i.e. $hk \ll 1$, whereas the right part corresponds to high frequency forcing or to a fluid with low viscosity. The low frequency solution may be obtained by taking the limit of Eq. (4.50) when $k \to 0$. Since $cc(x) \to 1$ and $ss(x)$ is asymptotic to x^2 as $x \to 0$, one has

$$
v_1 = u_P + \frac{Ch^2}{2\nu} \cos \omega t \left(1 - (\frac{x_3}{h})^2\right) , \tag{4.52}
$$

so that the pulsating term is still a parabola with a modified amplitude which is given by the oscillating part of the pressure gradient in (4.43). The high frequency case or the equivalent inviscid fluid may be treated with the approximation $hk \gg 1$. Then, since $cc(x)$ and $ss(x)$ are asymptotic, respectively, to $1/2\, e^x \cos x$ and $1/2\, e^x \sin(x)$ as $x \to \infty$, the limit solution reads

$$
v_1 = u_P - \frac{C}{\omega} (\sin \omega t - \sin(\omega t - \eta) e^{-\eta}) , \tag{4.53}
$$

where the new variable η measuring the distance from the upper wall is defined as

$$\eta = k(h - x_3) = \frac{h - x_3}{\sqrt{2\nu/\omega}} \, . \tag{4.54}$$

Note that the first term of the oscillating part of (4.53) is the response of the inviscid fluid ($\nu = 0$) to the pressure gradient. As soon as we are inside the fluid the second oscillating term goes to zero leaving the flow oscillating under the influence of the pressure gradient.

4.6 Poiseuille Flow

Probably the most important exact solution in applied viscous hydrodynamics is the Poiseuille solution for pressure flow through a straight circular pipe of uniform diameter. Let a cylindrical polar coordinate system be defined with z-axis along the axis of the pipe; in view of the axial symmetry of the situation, the specific orientation of the $\theta = 0$ direction is unimportant. If the radius of the pipe is denoted by R, the no-slip condition requires

$$v_r = v_\theta = v_z = 0 \quad \text{at } r = R \, . \tag{4.55}$$

We seek a solution to Eqs. (3.77) through (3.80) in the form

$$\begin{aligned} v_r &= v_\theta = 0 \, , \\ v_z &= u(r) \, , \\ p &= p(z) \, . \end{aligned} \tag{4.56}$$

The continuity equation (3.77) is automatically satisfied, as are the momentum equations (3.78) and (3.79). There remains only (3.80), which reduces to

$$\mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) = \frac{\partial p}{\partial z} \, . \tag{4.57}$$

As for the flow between plates, both sides must be equal to the same constant, say $-G$. With the boundary conditions (4.55) and the restriction that the velocity be finite at the tube axis,

$$u = \left(\frac{GR^2}{4\mu} \right) \left(1 - \frac{r^2}{R^2} \right) \, , \tag{4.58}$$

$$p = C - Gz \, , \tag{4.59}$$

where C is a constant of integration.

If we denote by Q the volume rate of flow through the pipe, so that

$$Q = 2\pi \int_0^R ru\,dr\,, \tag{4.60}$$

integration of (4.58) yields

$$Q = \pi \frac{GR^4}{8\mu}\,. \tag{4.61}$$

We can superimpose a swirl without disturbing the parabolic velocity profile (4.58). Let us suppose that we rotate the pipe about its own axis, not necessarily with constant angular velocity. The boundary conditions (4.55) must then be modified:

$$v_r = v_z = 0 \quad \text{at } r = R\,, \tag{4.62}$$

$$v_\theta = R\omega(t) \quad \text{at } r = R. \tag{4.63}$$

If we set

$$\begin{aligned} v_r &= 0, \\ v_\theta &= v(r,t)\,, \\ v_z &= u(r)\,, \end{aligned} \tag{4.64}$$

where $u(r)$ is the Poiseuille solution (4.58), the continuity equation (3.77) is automatically satisfied.

The pressure field (4.59) will not quite suffice. As evidenced by Eq. (3.78), an r-dependence must be added to balance the centrifugal force generated by v_θ. We set

$$p(z,r,t) = C - Gz + \rho \int_0^r r^{-1}[v(r,t)]^2 dr\,. \tag{4.65}$$

With (4.64) and (4.65), Eqs. (3.78) and (3.80) are satisfied. There remains only (3.79), which reduces to

$$\frac{\partial v}{\partial t} = \nu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} \right)\,. \tag{4.66}$$

By way of example, suppose that the pipe wall undergoes steady-state torsional oscillation, so that

$$\omega(t) = A \cos nt\,. \tag{4.67}$$

We expect that the resulting swirl will have an in-phase and an out-of-phase component, so that

$$v(r,t) = f(r)\cos nt + g(r)\sin nt\,. \tag{4.68}$$

Substituting into (4.66),

$$(\cos nt)\left(\frac{ng}{\nu} - f'' - \frac{f'}{r} + \frac{f}{r^2}\right) = (\sin nt)\left(\frac{nf}{\nu} + g'' + \frac{g'}{r} - \frac{g}{r^2}\right) . \qquad (4.69)$$

If (4.69) is to hold for all values of t, both sides must vanish independently. Thus

$$\frac{ng}{\nu} - f'' - \frac{f'}{r} + \frac{f}{r^2} = 0 , \qquad (4.70)$$

$$\frac{nf}{\nu} + g'' + \frac{g'}{r} - \frac{g}{r^2} = 0 . \qquad (4.71)$$

If we multiply (4.70) by i and subtract from (4.71), we obtain

$$\frac{n}{\nu}(f - ig) + \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)(g + if) = 0 . \qquad (4.72)$$

By setting

$$F(r) = f(r) - ig(r) , \qquad (4.73)$$

(4.72) can be written as Bessel's equation of order unity:

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + \left(\frac{i^3 n}{\nu} - \frac{1}{r^2}\right)F = 0 . \qquad (4.74)$$

Since $F(0)$ must be finite, we reject the Neumann function as an admissible solution. Therefore

$$F(r) = cJ_1(i^{3/2} r\sqrt{n/\nu}) , \qquad (4.75)$$

where c is a (complex) constant of integration. Comparing (4.63), (4.64), (4.67), (4.68), (4.73), we see that

$$F(R) = RA , \qquad (4.76)$$

so that

$$c = \frac{\mathrm{ber}_1 R\sqrt{n/\nu} - i\,\mathrm{bei}_1 R\sqrt{n/\nu}}{(\mathrm{ber}_1 R\sqrt{n/\nu})^2 + (\mathrm{bei}_1 R\sqrt{n/\nu})^2} RA , \qquad (4.77)$$

where $\mathrm{ber}_1 z$ and $\mathrm{bei}_1 z$ denote, respectively, the real and imaginary parts of $J_1(i^{3/2}z)$. With (4.73) and (4.75), we then obtain

$$f(r) = \frac{RA}{(\mathrm{ber}_1 R\sqrt{n/\nu})^2 + (\mathrm{bei}_1 R\sqrt{n/\nu})^2}(\mathrm{ber}_1 R\sqrt{n/\nu}\mathrm{ber}_1 r\sqrt{n/\nu} + \mathrm{bei}_1 R\sqrt{n/\nu}\mathrm{bei}_1 r\sqrt{n/\nu}) \, ,$$

$$g(r) = \frac{RA}{(\mathrm{ber}_1 R\sqrt{n/\nu})^2 + (\mathrm{bei}_1 R\sqrt{n/\nu})^2}(\mathrm{bei}_1 R\sqrt{n/\nu}\mathrm{ber}_1 r\sqrt{n/\nu} - \mathrm{ber}_1 R\sqrt{n/\nu}\mathrm{bei}_1 r\sqrt{n/\nu}) \, . \quad (4.78)$$

The remarks in Sect. 4.3 concerning the penetration of the boundary motion into the fluid carry over *mutatis mutandis* to the torsional oscillation of a circular pipe. Also in analogy with Sect. 4.3, more general rotary motion of the pipe can be treated by applying the Laplace transform to Eq. (4.66). Unsteady longitudinal motion of the pipe can also be treated.

In the next chapter we shall consider the exact solution of the problem of steady-state flow through a pipe of non-circular cross section.

4.7 Starting Transient Poiseuille Flow

We will examine the transient flow in a circular pipe where the fluid starts from rest to reach the Poiseuille steady parabolic profile (4.58). The only non vanishing velocity component is v_z and the pressure gradient goes instantaneously at $t = 0$ from zero to the value $-G$ everywhere. The dynamic equation is from (3.80)

$$G + \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right) = \rho\frac{\partial v_z}{\partial t} \, , \quad (4.79)$$

with the initial condition

$$v_z(r,0) = 0, \quad 0 \leq r \leq R \, , \quad (4.80)$$

and the boundary condition

$$v_z(R,t) = 0, \forall t \, . \quad (4.81)$$

In order to render (4.79) homogeneous, let us change variables

$$w(r,t) = \frac{G}{4\mu}\left(R^2 - r^2\right) - v_z(r,t) \, . \quad (4.82)$$

The new variable will be solution of the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{\nu}\frac{\partial w}{\partial t} \, , \quad (4.83)$$

with the initial condition

$$w(r,0) = \frac{G}{4\mu}\left(R^2 - r^2\right), \tag{4.84}$$

and the boundary condition

$$w(R,t) = 0, \quad \forall t . \tag{4.85}$$

Through the transient phase, the velocity v_z will increase till the steady state (4.58) is reached, whereas the transient perturbation $w(r,t)$ will decay to zero. To solve (4.83), we proceed by separation of variables

$$w(r,t) = f(r)g(t) . \tag{4.86}$$

Substituting in (4.83), one gets

$$\frac{dg(t)}{dt} + C\nu\, g(t) = 0 , \tag{4.87}$$

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + C\, f = 0 , \tag{4.88}$$

where C is an arbitrary constant. The solution of (4.87) reads

$$g(t) = B\exp(-C\nu t) . \tag{4.89}$$

As $w(r,t)$ decreases with respect to time, we assume that C will involve only positive values so that C can be written λ^2/R^2. This will ease the next computations, as we will observe. Equation (4.88) then becomes

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \frac{\lambda^2}{R^2}\, f = 0 . \tag{4.90}$$

The change of variable $\lambda r/R = z$ leads (4.90) to the canonical form of the Bessel equation

$$\frac{d^2 f}{dz^2} + \frac{1}{z}\frac{df}{dz} + \left(1 - \frac{k^2}{z^2}\right) f = 0 , \tag{4.91}$$

whose general solution is given by

$$f = C_1 J_k(z) + C_2 Y_k(z) , \tag{4.92}$$

where the functions J_k and Y_k are Bessel functions of first and second kind, respectively, of order k and C_1, C_2 are arbitrary constants. Consequently, the solution of (4.90) is

$$f = C_1 J_0(\frac{\lambda r}{R}) + C_2 Y_0(\frac{\lambda r}{R}) . \tag{4.93}$$

As Y_0 goes to $-\infty$ when $r \to 0$, one concludes that $C_2 = 0$ for w to remain finite on the axis. The general solution of (4.83) becomes

$$w(r,t) = C_3 J_0(\frac{\lambda r}{R}) \exp(-\frac{\lambda^2}{R^2}\nu t) . \tag{4.94}$$

The solution (4.94) verifies condition (4.85) for λ values, denoted λ_n, given by the zeroes of the Bessel function J_0

$$J_0(\lambda_n) = 0 . \tag{4.95}$$

The solution is obtained as

$$w(r,t) = \frac{G}{4\mu} \sum_{n=1}^{\infty} c_n \, J_0(\frac{\lambda_n r}{R}) \exp(-\frac{\lambda_n^2}{R^2}\nu t), \tag{4.96}$$

and the coefficients c_n are determined by (4.84):

$$R^2 - r^2 = \sum_{n=1}^{\infty} c_n \, J_0(\frac{\lambda_n r}{R}) . \tag{4.97}$$

To solve Eq. (4.97), let us recall the orthogonality properties of Bessel functions as expressed by Lommel integrals

$$\int_0^1 z\, J_n(\lambda_i z) J_n(\lambda_j z) dz = 0, \quad \lambda_i \neq \lambda_j , \tag{4.98}$$

$$\int_0^1 z\, J_n^2(\lambda_i z) dz = \frac{1}{2}[J_n'(\lambda_i)]^2 . \tag{4.99}$$

Solution of (4.97) is obtained with $z = r/R$ as

$$c_n = \frac{2R^2}{[J_0'(\lambda_n)]^2} \int_0^1 (1 - z^2) z\, J_0(\lambda_n z) dz . \tag{4.100}$$

The evaluation of the two integrals in (4.100) is carried out using successively the recurrence relationships (4.102) and then (4.101) Abramowitz and Stegun (1972) with $\ell = 2, m = 0$

$$\int z^{\ell+1} J_m(z)dz = z^{\ell+1} J_{m+1}(z) + (\ell - m)z^{\ell} J_m(z) - (\ell^2 - m^2) \int z^{\ell-1} J_m(z)dz \;, \tag{4.101}$$

$$\int_{z_0}^{z} z^m J_{m-1}(z)dz = [z^m J_m(z)]_{z_0}^{z} \;, \tag{4.102}$$

$$zJ_m'(z) = mJ_m(z) - zJ_{m+1}(z) \;. \tag{4.103}$$

This yields

$$\int_0^1 z\, J_0(\lambda_n z)dz = \frac{1}{\lambda_n} J_1(\lambda_n) \;, \tag{4.104}$$

$$\begin{aligned}\int_0^1 z^3\, J_0(\lambda_n z)dz &= \frac{1}{\lambda_n^4}\left[\lambda_n^3 J_1(\lambda_n) + 2\lambda_n^2 J_0(\lambda_n) - 4\lambda_n J_1(\lambda_n)\right] \;, \\ &= \frac{1}{\lambda_n} J_1(\lambda_n) - \frac{4}{\lambda_n^3} J_1(\lambda_n) \;.\end{aligned} \tag{4.105}$$

One finds with the help of the relation $[J_0'(\lambda_n)]^2 = [J_1(\lambda_n)]^2$:

$$c_n = \frac{8R^2}{\lambda_n^3 J_1(\lambda_n)} \;. \tag{4.106}$$

Taking Eqs. (4.82), (4.96) and (4.106) into account, the velocity profile is

$$v_z(r,t) = \frac{G}{4\mu}\left(R^2 - r^2\right) - \frac{2GR^2}{\mu} \sum_{n=1}^{\infty} \frac{J_0(\frac{\lambda_n r}{R})}{\lambda_n^3 J_1(\lambda_n)} \exp(-\frac{\lambda_n^2}{R^2}\nu t) \;. \tag{4.107}$$

Figure 4.5 shows the velocity variation with respect to time.

4.8 Pulsating Flow in a Circular Pipe

We come back to the blood flow in arteries. Assuming that arteries are rigid circular pipes, —an assumption far from the physiological phenomena as arterial walls deform and move under pressure waves Zamir (2000)—, we are faced with a time-periodic pressure gradient driving the Poiseuille flow. The cardiac cycle is indeed time-periodic and therefore the pressure gradient should be represented by a Fourier series. For the sake of simplicity we will consider only a Fourier in such a way that

$$\frac{\partial p}{\partial z} = -Ce^{i\omega t} \;, \tag{4.108}$$

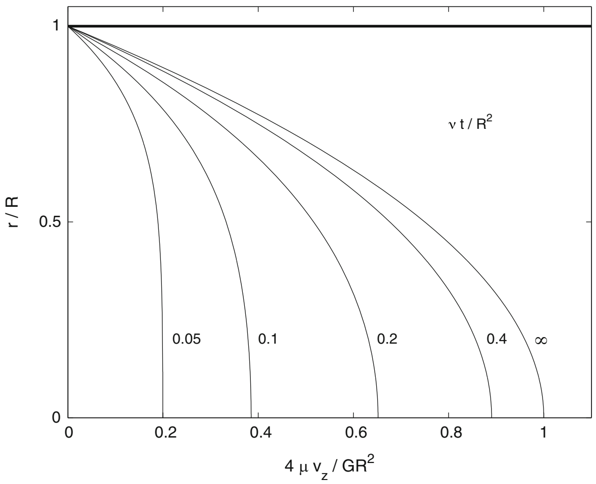


Fig. 4.5 Transient Poiseuille flow in a circular pipe

with ω the angular frequency. The flow governing equation is obtained from (3.80):

$$Ce^{i\omega t} + \mu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} \right) = \rho \frac{\partial v_z}{\partial t} \ , \tag{4.109}$$

and a solution is sought in terms of the Fourier representation

$$v_z = u(r)e^{i\omega t} \ . \tag{4.110}$$

The combination of Eqs. (4.109) and (4.110) generates the solution

$$u = \frac{C}{i\rho\omega} \left(1 - \frac{J_0(i^{3/2}\alpha r/R)}{J_0(i^{3/2}\alpha)} \right) \ , \tag{4.111}$$

in which there appears a dimensionless number α called the **Womersley number** defined as

$$\alpha = R\sqrt{\frac{\omega}{\nu}} \ . \tag{4.112}$$

Note that the Womersley number is the square root of the oscillatory Reynolds number (2.241).

The solution (4.111) was obtained with the boundary conditions

$$u(R) = 0, \quad \frac{du}{dr}(r = 0) = 0 \ . \tag{4.113}$$

The function $J_0(i^{3/2}\sqrt{\frac{\omega}{\nu}}r) = J_0(i^{3/2}\alpha r/R)$ is the Kelvin function of order 0.

As the Womersley number is the ratio of the radius to the penetration depth, it is a characteristic feature of pulsatile blood flow. Typical values of α in the aorta range from 20 for a human in good health to 8 for a cat. Another way of interpreting the Womersley number consists in estimating the distance from the rigid wall, say δ, where the viscous forces and the inertia are of equal magnitude. Near the wall, viscosity is dominant and a rough estimate of the viscous forces is $\mu U/\delta^2$. Near the symmetry axis, inertia dominates and yields the estimate $\rho\omega U$. Equating the two forces leads to the definition

$$\delta = \frac{\nu}{\omega} . \tag{4.114}$$

If α is large, the viscous effects are confined to a region very close to the wall. In the centre of the flow, the dynamics will be driven by the equilibrium of inertia and pressure forces, resulting in a velocity profile that will be more blunt than the parabolic profile that comes from the balance of viscous and pressure forces.

4.9 Helical Flow in an Annular Region

4.9.1 The Newtonian Case

In this section we treat the motion of fluid contained between two concentric circular pipes of constant radii R_1 and R_2 with, say, $R_1 < R_2$ as indicated in Fig. 4.6. The pipes rotate about their common axis with constant angular velocities ω_1 and ω_2, respectively. In addition the pipes may translate steadily, parallel to their common axis; let us say that the outer pipe translates with velocity U relative to the inner.

We define a cylindrical coordinate system r, θ, z which translates with the inner pipe but does not rotate with it. The z-axis lies along the common axis of the pipes; because of the axial symmetry, the orientation of the $\theta = 0$ axis is unimportant.

Since the cylindrical coordinate system is not accelerated, the fluid motion is governed by Eqs. (3.77) through (3.80). The no-slip condition requires that

$$\begin{aligned} &v_r = v_z = 0, \quad v_\theta = R_1\omega_1 \quad \text{at } r = R_1 ; \\ &v_r = 0, \quad v_\theta = R_2\omega_2, \quad v_z = U \quad \text{at } r = R_2 . \end{aligned} \tag{4.115}$$

From our experiences with the exact solutions found in previous sections, we expect that

$$\begin{aligned} v_r &= 0 , \\ v_\theta &= v(r) , \\ v_z &= u(r) , \\ p &= C - Gz + \rho \int_0^r r^{-1}[v(r)]^2 dr . \end{aligned} \tag{4.116}$$

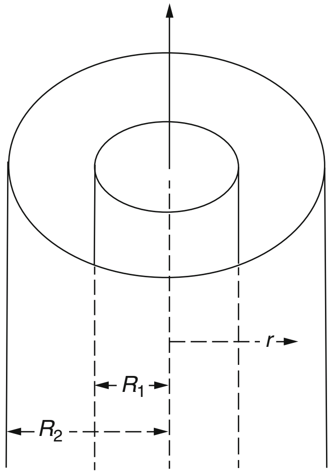


Fig. 4.6 Helical flow geometry

The continuity equation is indeed satisfied, as is Eq. (3.78). Equation (3.79) reduces to

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0\,, \tag{4.117}$$

and Eq. (3.80) becomes

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} = -\frac{G}{\mu}\,. \tag{4.118}$$

Thus the rotary flow and the axial flow do not couple. Integrating (4.117) subject to the boundary conditions on v_θ yields the **axisymmetric Couette flow**

$$v = \left(\frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2}\right) r + \left(\frac{\omega_1 - \omega_2}{R_2^2 - R_1^2}\right)\frac{R_1^2 R_2^2}{r}\,. \tag{4.119}$$

Integrating (4.118) subject to the boundary conditions on v_r yields

$$u = \frac{G}{4\mu}\left[R_1^2 - r^2 + \frac{(R_2^2 - R_1^2)\ln(r/R_1)}{\ln(R_2/R_1)}\right] + U\frac{\ln(r/R_1)}{\ln(R_2/R_1)}\,. \tag{4.120}$$

If the pipes do not translate relative to one another, so that $U = 0$, Eq. (4.120) describes **pressure flow in a coaxial pipe**. The opposite case, for which $G = 0$ but $U \neq 0$, is referred to as **drag flow**, but the term is not in common use. The general case described by (4.119) and (4.120) might be termed pressure flow with superimposed Couette flow and drag flow.

As in Poiseuille's problem, the exact solution presented here can be generalized by permitting unsteady motion of the pipes parallel to themselves.

4.9.2 The Non-Newtonian Circular Couette Flow

Circular Couette flow occurs in the gap between two rotating concentric cylinders. The inner cylinder of radius R_1 has the angular velocity ω_1 while the outer cylinder of radius R_2 spins at ω_2. The apparatus has a height H which is much larger than the radius of either cylinder so that the apparatus height is supposed infinite. Referring to the previous cylindrical coordinates system r, θ, z, the steady state velocity field is such that

$$v_r = 0, \quad v_\theta = v_\theta(r), \quad v_z = 0\,. \tag{4.121}$$

This v_θ velocity field is then determined from the integration of the θ-momentum equation

$$\begin{aligned}\rho(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}) \\ = \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2T_{r\theta}}{r}\,,\end{aligned} \tag{4.122}$$

which reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\,T_{r\theta}\right) = 0\,. \tag{4.123}$$

The stress component $T_{r\theta}$ of (2.149) is given by the relation

$$T_{r\theta} = \frac{\varphi_1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)\,. \tag{4.124}$$

Taking (4.124) into account, the integration of (4.123) with the boundary conditions $v_\theta(R_1) = R_1\omega_1$ and $v_\theta(R_2) = R_2\omega_2$ yields the velocity field (4.119).

In the case of a fixed outer cylinder $\omega_2 = 0$ and the velocity is given by

$$v_\theta = Ar + \frac{B}{r} = \frac{\omega_1 R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r} - r\right)\,. \tag{4.125}$$

The r-momentum equation

$$\begin{aligned}\rho(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}) \\ = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r}\end{aligned}$$

simplifies and gives

$$\frac{d\,T_{rr}}{dr} + \frac{1}{r}\left(T_{rr} - T_{\theta\theta}\right) = -\rho\frac{v_\theta^2}{r}\,. \tag{4.126}$$

With the stress component

$$T_{rr} = -p + \frac{\varphi_2}{4}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2,$$

Eq. (4.126) yields

$$-\frac{\partial p}{\partial r} + \varphi_2\frac{\partial}{\partial r}\left(\frac{B^2}{r^4}\right) = -\rho\frac{v_\theta^2}{r}\,. \tag{4.127}$$

As $T_{zz} = -p$, one obtains

$$-\,T_{zz} = p = \varphi_2\,\frac{B^2}{r^4}\,\Big|_{R_1}^{r} + \int_{R_1}^{r}\rho\frac{v_\theta^2}{r}\,dr + C \tag{4.128}$$

$$= p(R_1) + \varphi_2\,\frac{B^2}{r^4} + \int_{R_1}^{r}\rho\frac{v_\theta^2}{r}\,dr\,. \tag{4.129}$$

If the fluid is Newtonian, $\varphi_2 = 0$ and the pressure increases from the inner to the outer cylinder. The fluid rises along the outer cylinder under centrifugal forces. For the non-Newtonian fluid, if $\varphi_2 > 0$ and if B is sufficiently large under a high shear due to a small gap between the cylinders, the pressure increases when one approaches the inner cylinder and this produces the rod-climbing effect as shown in Fig. 2.5.

4.10 Hamel's Problem: Flow in a Wedge-Shaped Region

The exact solutions presented so far have all been somewhat degenerate. In every case the form we assumed for the velocity profile caused the nonlinear inertia terms in the Navier-Stokes equation either to vanish completely or to produce only a centrifugal force, easily balanced by a pressure gradient. Since the mechanism of non-linear momentum transfer is best studied through exact solutions in which the non-linear terms play an important role, it is worthwhile to seek out such solutions.

In Fig. 4.7 consider that an incompressible fluid is contained in the trough between two non-parallel walls. Consider further that a line-source (or sink) of uniform output Q per unit length lies along the line of intersection of the walls. Let a

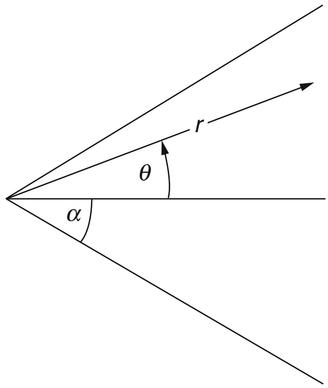


Fig. 4.7 Wedge geometry

cylindrical polar coordinate system r, θ, z be defined so that the walls correspond to $\theta = \pm\alpha$. The velocity components must, then, satisfy the no-slip condition

$$v_r = v_\theta = v_z = 0 \quad \text{at } \theta = \pm\alpha, \tag{4.130}$$

along with the volume flow condition

$$\int_{-\alpha}^{\alpha} r v_r \, d\theta = Q \ . \tag{4.131}$$

We expect a priori that the flow will be two-dimensional. Moreover we suspect that a purely radial pattern of flow may satisfy the hydrodynamic equations. Therefore we seek a solution with

$$v_\theta = v_z = 0 \ . \tag{4.132}$$

The continuity equation (3.77) then requires that

$$v_r = \frac{1}{r} f(\theta) \ , \tag{4.133}$$

so that Eq. (3.79) becomes

$$\frac{\partial p}{\partial \theta} = \frac{2\mu}{r^2} f'(\theta) \ . \tag{4.134}$$

Thus

$$p = \frac{2\mu}{r^2} f(\theta) + g(r) \ . \tag{4.135}$$

Substituting (4.133) and (4.135) into (3.78) yields

$$r^3 g'(r) = \mu[f''(\theta) + 4f(\theta)] + \rho[f(\theta)]^2 \,. \tag{4.136}$$

Since the left side of (4.136) is a function of r alone and the right side is a function of θ alone, both sides must equal some constant, call it $-\mu K$. Then (4.135) yields

$$p = \frac{\mu}{2r^2}[4f(\theta) + K] + p_a \,, \tag{4.137}$$

where p_a is the pressure at $r = \infty$. Also (4.136) becomes a differential equation for $f(\theta)$:

$$f'' + 4f + \frac{f^2}{\nu} + K = 0 \,. \tag{4.138}$$

Integration of this equation introduces two new constants, which can be eliminated by use of the boundary conditions (4.130). The volume flow condition (4.131) then determines K.

Before proceeding, let us consider the consequences of the non-linear term in (4.138). Were it not for this term, the flow would be reversible: if f were a solution, then $-f$ would be a solution to the equation obtained by replacing K with $-K$, i.e., by replacing the source with a sink. However, with the non-linear term, which results from fluid inertia, no such conclusion can be drawn. Because of its inertia, the fluid attempts to obey Bernoulli's law, which relates the pressure to the fluid speed, not to the direction of flow; it is prevented from doing so by viscosity, which always acts to oppose the flow. If either inertia or viscosity were negligible, the flow would be reversible (in the first case $p - p_a$ would reverse sign; in the second case it would not). With both effects present, source flow differs qualitatively from sink flow. Discussions of the nature of the difference are presented in Goldstein (1938) and Schlichting (1960).

In order to obtain a first integral to (4.138), multiply through by f' and integrate. Since the geometry of Hamel's problem is symmetric with respect to the $\theta = 0$ axis, $f'(0) = 0$. Hence

$$\frac{1}{2}f'^2 + 2(f^2 - f_1^2) + \frac{1}{3\nu}(f^3 - f_1^3) + K(f - f_1) = 0 \,, \tag{4.139}$$

where f_1/r is the midstream velocity. Thus an implicit relation between f and θ can be obtained in terms of an elliptic integral:

$$\theta = \pm\sqrt{\frac{3\nu}{2}} \int_f^{f_1} \frac{df}{\sqrt{f_1 - f}\sqrt{f^2 + (f_1 + 6\nu)f + f_1^2 + 6\nu f_1 + 3\nu K}} \,. \tag{4.140}$$

Because of the symmetry about $\theta = 0$, either sign may be retained. If we choose the plus sign, the no-slip condition at $\theta = \alpha$ gives us

$$\int_0^{f_1} \frac{df}{\sqrt{f_1 - f}\sqrt{f^2 + (f_1 + 6\nu)f + f_1^2 + 6\nu f_1 + 3\nu K}} = \alpha\sqrt{\frac{2}{3\nu}} \, . \tag{4.141}$$

The second relation required to determine the constants f_1 and K is provided by the volume flow condition (4.131).

4.10.1 The Axisymmetric Analog of Hamel's Problem

Having succeeded in finding an exact solution to the problem of source flow in a wedge, we might try seeking another exact solution by considering flow from a source at the apex of a cone. As we shall see, the search leads quickly to a frustration.

Let a spherical coordinate system be chosen with origin at the apex of the cone and with $\theta = 0$ along its axis. Since the problem is axially symmetric, the orientation of the $\phi = 0$ axis is unimportant.

If we assume that, as in Hamel's problem, the flow pattern is purely radial, the continuity equation (3.101) requires that

$$v_r = \frac{1}{r^2} f(\theta) \, . \tag{4.142}$$

Equation (3.103) then becomes

$$\frac{\partial p}{\partial \theta} = \frac{2\mu}{r^3} f'(\theta) \, , \tag{4.143}$$

so that

$$p = \frac{2\mu}{r^3} f(\theta) + g(r) \, . \tag{4.144}$$

Substituting (4.142) and (4.144) into Eq. (3.102) yields

$$r^4 g'(r) - \frac{2\rho}{r}[f(\theta)]^2 = \mu[f''(\theta) + \cot\theta f'(\theta) + 6f(\theta)] \, . \tag{4.145}$$

Since the left side depends upon r and the right side does not, both sides must equal some constant, say C. But consider further: setting the left side of (4.145) equal to C yields

$$2\rho[f(\theta)]^2 = r^5 g'(r) - Cr \, . \tag{4.146}$$

Once more we find that both sides must equal some constant, so that $f(\theta)$ itself is constant. However, $f(\theta)$ vanishes at $\theta = \alpha$, where α is the semi-vertical angle of the cone. Consequently $f(\theta)$ is identically zero.

Thus we have shown that there can be no purely radial flow in a cone, at least for an incompressible fluid without body forces. There must be a component of flow in the θ-direction, so that an eddy pattern results. For such a pattern the Navier-Stokes equation is quite complicated. Hence there is not much hope for exact solution—especially as people have been trying ever since Georg Karl Wilhelm Hamel (1877–1954) published his paper in 1916 Hamel (1916).[2]

Some insight as to why radial flow obtains in a wedge but not in a cone comes from dimensional considerations. For wedge flow the relevant physical parameters are the fluid density, its viscosity, the wedge half-angle, and the source output per unit length. The dimensions of these quantities are

$$\begin{aligned}
[\rho] &= ML^{-3} \,, \\
[\mu] &= ML^{-1}T^{-1} \,, \\
\alpha &: \text{ dimensionless} \,, \\
[Q] &= L^2T^{-1} \,.
\end{aligned} \tag{4.147}$$

No combination of these parameters yields a length. If source flow in a wedge were to produce a steady-state eddy pattern, the eddies would presumably be characterized by a length, (for example, the distance from the origin beyond which no back-flow occurs,) expressible in terms of the parameters of the problem (for otherwise we would reach the ridiculous conclusion that it is a fundamental constant of the universe). As we have seen, however, the parameters do not give us such a length. For flow in a cone, however, the source strength, say Q^*, has dimensions of volume per unit time. Hence, $Q^*\rho/\mu$ is a length. Moreover, it depends on Q^* the way one might expect: as the source gets stronger, the eddies are blasted farther and farther out from the origin.

4.11 Bubble Dynamics

Let us now suppose that a spherical bubble of inviscid gas is contained in an otherwise unlimited volume of liquid. Suppose further that the pressure p_g of the gas forming the bubble varies with time. As a consequence the radius R of the bubble will also vary with time. The pulsating bubble will generate a velocity field within the liquid which in turn generates a stress field.

---

[2]A translation of the paper exists: United States NACA Technical Memorandum 1342. Since Hamel's result is extensively discussed in the hydrodynamics literature, his original paper is now primarily of historical interest.

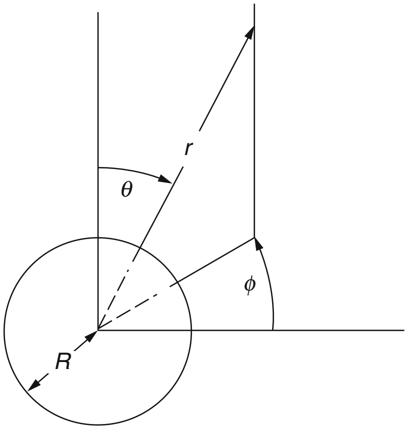


Fig. 4.8 Bubble geometry and spherical coordinates

The spherical symmetry of the situation makes it convenient to choose a spherical coordinate system with origin at the center of the bubble as in Fig. 4.8. The velocity field generated in the liquid will have only a radial component

$$v_r = v(r,t) \, , \tag{4.148}$$

so that the hydrodynamic equations (3.101) and (3.102) reduce to

$$\frac{\partial v}{\partial r} + \frac{2v}{r} = 0 \, , \tag{4.149}$$

$$\rho \left[\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} \right] = - \frac{\partial p}{\partial r} + \mu \left[\frac{\partial^2 v}{\partial r^2} + \frac{2}{r} \frac{\partial v}{\partial r} - \frac{2v}{r^2} \right] . \tag{4.150}$$

At the bubble wall, the liquid velocity must equal $\dot{R}(t)$, where an overdot denotes ordinary differentiation with respect to time. Thus integration of (4.149) yields

$$v = \frac{\dot{R} R^2}{r^2} \, . \tag{4.151}$$

Substituting this result into (4.150) and integrating, we obtain

$$\frac{(p - p_a)}{\rho} = \left(\frac{R}{r} \right) (R\ddot{R} + 2\dot{R}^2) - \left(\frac{R^4 \dot{R}^2}{2r^4} \right) , \tag{4.152}$$

where p_a is the pressure at infinity.

With Eqs. (3.98) and (3.100), we see that the physical components of stress are given by

$$
\begin{aligned}
T_{rr} &= -p - \left(\frac{4\mu R^2 \dot{R}}{r^3}\right) , \\
T_{\theta\theta} &= T_{\phi\phi} = -p + \left(\frac{2\mu R^2 \dot{R}}{r^3}\right) , \\
T_{\theta\phi} &= T_{\phi r} = T_{r\theta} = 0 .
\end{aligned} \tag{4.153}
$$

Within the bubble,

$$
\begin{aligned}
T_{rr} &= T_{\theta\theta} = T_{\phi\phi} = -p_g(t) , \\
T_{\theta\phi} &= T_{\phi r} = T_{r\theta} = 0 .
\end{aligned} \tag{4.154}
$$

The stress components $T_{\phi r}$ and $T_{r\theta}$ must be continuous across the bubble surface. A comparison of (4.153) and (4.154) reveals that this requirement is automatically satisfied. The stress component T_{rr} must experience a jump of magnitude $2\tau/R$, where τ is the coefficient of interfacial tension ; the value inside the bubble is lower. Comparing the first of Eqs. (4.153) with the first of Eqs. (4.154), we find that the pressure just outside the bubble wall is given by

$$
p(R+0,t) = p_g(t) - \frac{(2\tau + 4\mu \dot{R})}{R} . \tag{4.155}
$$

By setting $r = (R+0)$ in Eq. (4.152), we obtain an ordinary differential equation for the bubble radius as a function of time:

$$
R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{4\mu \dot{R}}{\rho R} + \frac{2\tau}{\rho R} = \frac{p_g(t) - p_a}{\rho} . \tag{4.156}
$$

Since (4.156) is an equation of second order, two initial conditions must be specified. Most simply, $R(0)$ and $\dot{R}(0)$ will be given.

The treatment given here has been restricted to spherical bubbles. In practice, the presence of a unidirectional gravitational field tends to destroy the spherical symmetry. It also causes the bubble to rise in the liquid, and our analysis does not account for streaming past the bubble. Thus Eq. (4.156) is virtually useless in the study of large-scale bubbles arising, say, from an underwater explosion.

However in certain physical problems the bubble is small enough so that interfacial tension causes it to remain essentially spherical. When streaming past the bubble is negligible, Eq. (4.156) can then be applied. This approach has been used to study the growth of vapor bubbles in superheated liquids, Plesset and Zwick (1954), where the variation of p_g with time is caused by thermal expansion of the gas due to the diffusion of heat into the bubble. Also the growth of small bubbles by diffusion of gas through the liquid has been studied by use of Eq. (4.156), cf. Barlow and Langlois (1962) and Langlois (1963). Cavitation bubbles can also be

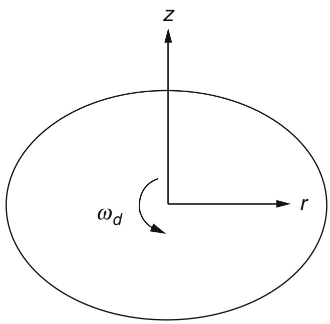


Fig. 4.9 Rotating disc

treated, but in the literature on cavitation in liquids, viscosity is usually neglected, so that the term $4\mu\dot{R}/\rho R$ is dropped from Eq. (4.156).

4.12 The Flow Generated by a Rotating Disc

As our next example of an exact solution to the equations of viscous hydrodynamics, we consider the flow generated by an infinite flat disc rotating in its own plane with constant angular velocity ω_d, as in Fig. 4.9. At first it would seem that purely rotary flow is generated, but, looking deeper, we see that this is not the case. First, solid body rotation is not an acceptable solution, for infinite pressures would be required to support the centrifugal forces generated by the rotating fluid. Therefore the fluid near the disc rotates faster than the fluid farther away. Consequently there is a variation of centrifugal force in the axial direction. The fluid near the disc is thrown outward more violently, so that other fluid must stream down the axis to replace it. Thus the motion is fully three-dimensional, albeit axisymmetric. By making a clever guess as to the form of the flow pattern, Theodore von Kármán (1891–1963) was able to reduce the hydrodynamic equations to a set of ordinary differential equations von Kármán (1921). He assumed that

$$v_r = ru(z), v_\theta = r\omega(z), v_z = v(z), p = p(z) \,. \tag{4.157}$$

Substituting these forms into Eqs. (3.77) through (3.80) yields

$$\begin{aligned} 2u + v' &= 0 \,, \\ u^2 - \omega^2 + u'v &= \nu u'' \,, \\ 2u\omega + \omega' v &= \nu\omega'' \,, \\ \rho v v' + p' &= \mu v'' \,. \end{aligned} \tag{4.158}$$

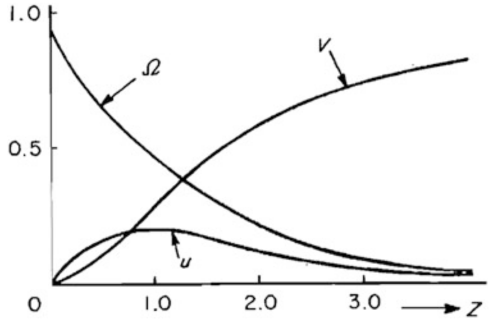


Fig. 4.10 Velocity components with respect to the normalized axial coordinate

These equations can be normalized by setting

$$z = \sqrt{\nu/\omega_d}\,Z, \quad u(z) = \omega_d U(Z), \quad \omega(z) = \omega_d \Omega(Z)\,.$$
$$v(z) = \sqrt{\nu\omega_d}\,V(Z), \quad p(z) = -\mu\omega_d P(Z)\,. \tag{4.159}$$

Thus

$$\begin{aligned} 2U + V' &= 0\,, \\ U^2 - \Omega^2 + U'V &= U''\,, \\ 2U\Omega + \Omega' V &= \Omega''\,, \\ VV' &= P' + V''\,. \end{aligned} \tag{4.160}$$

For boundary conditions, von Kármán assumed that the radial and azimuthal components of velocity approach zero as z approaches infinity. At $z = 0$, the no-slip condition applies. In terms of the normalized variables,

$$U = V = 0, \quad \Omega = 1 \quad \text{at } Z = 0\,,$$
$$U \to 0, \Omega \to 0 \quad \text{as } Z \to \infty\,. \tag{4.161}$$

von Kármán obtained an approximate solution to the system (4.160) subject to the boundary conditions (4.161). We shall not go into the details, nor into those of Cochran's numerical solution Cochran (1934) for they are set out in Goldstein (1938) and Schlichting (1960)

As shown in Fig. 4.10, the significant point is that the radial and azimuthal velocity components differ appreciably from zero only in a layer near the disc. The thickness of this layer is proportional to $\sqrt{\nu/\omega_d}$ which therefore plays the role of a depth of penetration for the flow generated by a rotating disc. As $z \to \infty$, the axial component of velocity approaches asymptotically the finite value $-0.886\sqrt{\nu\omega_d}$ so that the rotating disc acts as a centrifugal pump.

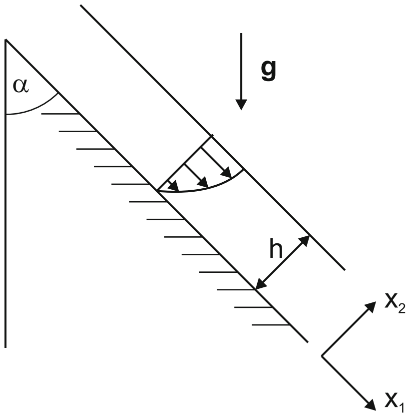


Fig. 4.11 Flow over an inclined plane

4.13 Free Surface Flow over an Inclined Plane

Taking the effect of gravity into account, consider the steady two-dimensional flow of a viscous fluid over a plane inclined with respect to the vertical direction by the angle α (cf. Fig. 4.11). The thickness of the fluid layer is uniform and equal to h. The fluid is in contact at the free surface with ambient air, which we will model as an inviscid fluid at pressure p_a. We assume that the air flow does not affect the viscous fluid flow. The flow is parallel as the trajectories of the fluid particles are parallel to the inclined plane. Therefore $\mathbf{v} = (v_1, 0, 0)$. By the incompressibility constraint, one obtains

$$\frac{\partial v_1}{\partial x_1} = 0, \tag{4.162}$$

and we deduce $v_1 = v_1(x_2)$. The only non zero component of the stress tensor is T_{12} or T_{21}. As pressure is uniform at the free surface, the pressure in the viscous fluid does not depend on the x_1 direction, but does depend on x_2. The first equation of (2.95) written in the x_1 direction yields

$$\frac{\partial T_{12}}{\partial x_2} + \rho g_1 = \frac{\partial T_{12}}{\partial x_2} + \rho g \cos\alpha = 0\,. \tag{4.163}$$

Integration of this relation yields

$$T_{12} = -\rho g\, x_2 \cos\alpha + C\ . \tag{4.164}$$

At the free surface $x_2 = h$, the shear stress must vanish as the inviscid fluid cannot sustain shear. One obtains

$$T_{12} = \rho g \cos\alpha (h - x_2)\ . \tag{4.165}$$

As $T_{12} = \mu\, dv_1/dx_2$, we may evaluate v_1 by integrating Eq. (4.165) with respect to x_2, with the boundary condition $v_1(x_2 = 0) = 0$. The velocity profile is given by

$$v_1 = \frac{\rho g \cos\alpha}{2\mu} x_2(2h - x_2)\ . \tag{4.166}$$

The Navier-Stokes equation in the x_2 direction gives the relation

$$-\frac{\partial p}{\partial x_2} + \rho g_2 = -\frac{\partial p}{\partial x_2} - \rho g \sin\alpha = 0\ . \tag{4.167}$$

Integrating with respect to x_2 and using the free surface condition $p(x_2 = h) = p_a$, we get

$$p = p_a - (\rho\, g \sin\alpha)(x_2 - h)\ . \tag{4.168}$$

The mass flux per unit length in the x_3 direction reads

$$Q = \int_0^h u\, dx_2 = \frac{\rho g \cos\alpha\, h^3}{2\mu}. \tag{4.169}$$

4.14 Natural Convection Between Two Differentially Heated Vertical Parallel Walls

We now consider the steady, two-dimensional, non-isothermal slow flow of a viscous incompressible fluid subjected to a variable temperature field. The fluid flows between two infinite vertical parallel walls at different temperatures, cf. Fig. 4.12, such that $\Theta_1 > \Theta_2$. We assume that the Boussinesq approximation is valid and the relevant equations are given by (2.196)–(2.198). The velocity field is a priori of the form $\mathbf{v} = (u(x_1, x_2), v(x_1, x_2), 0)$. However as the flow is invariant with respect to translation in the x_2 direction, one concludes that it depends only on the x_1 coordinate. With (2.196),

$$\frac{\partial u}{\partial x_1} = 0\ . \tag{4.170}$$

As $u = 0$ at the walls, $u = 0$ and $v = v(x_1)$. The temperature gradient is oriented in the horizontal direction, so that the temperature field is such that $\Theta = \Theta(x_1)$. Consequently Eq. (2.198) becomes

$$\frac{d^2\Theta}{dx_1^2} = 0\ . \tag{4.171}$$

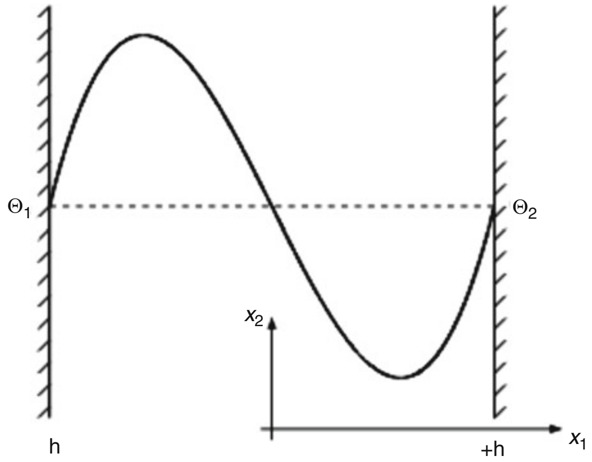


Fig. 4.12 Natural convection in an infinite plane channel

Integrating with the boundary conditions $\Theta = \Theta_1$ at $x_1 = -h$ and $\Theta = \Theta_2$ at $x = h$ yields

$$\Theta = \frac{\Theta_2 - \Theta_1}{2h} x_1 + \frac{\Theta_1 + \Theta_2}{2} = A x_1 + \frac{\Theta_1 + \Theta_2}{2} \, . \tag{4.172}$$

The momentum equation (2.197) gives

$$-\frac{\partial p}{\partial x_2} + \mu \frac{d^2 v}{dx_1^2} - \rho_0 g (1 - \alpha(\Theta - \Theta_0)) = 0 \tag{4.173}$$

The reference temperature is chosen such that $\Theta_0 = (\Theta_1 + \Theta_2)/2$, i.e. the mean temperature. As the flow is not driven by an exterior pressure gradient, the pressure is purely hydrostatic and results from the integration of

$$-\frac{\partial p}{\partial x_2} - \rho_0 g = 0 \, , \tag{4.174}$$

valid at equilibrium. Therefore the velocity field is driven by the buoyancy force and one solves

$$\mu \frac{d^2 v}{dx_1^2} + \rho_0 g \alpha A x_1 = 0 \, . \tag{4.175}$$

With the boundary conditions $v = 0$ at $x_1 = \pm h$,

$$v = \frac{g \alpha A}{6 \nu} x_1 (h^2 - x_1^2) \, . \tag{4.176}$$

It is easy to verify that this velocity profile corresponds to a vanishing flow rate across each horizontal section. A posteriori the velocity field is orthogonal to the temperature field; this leads to the vanishing of the transport term in the material derivative of Θ.

In the real world, it is impossible to build infinite walls. Therefore top and bottom walls confine the fluid and force it to form a convection cell. The flow we have analyzed is thus unstable Koschmieder (1993) and constitutes an idealization of the physical phenomena.

4.15 Flow Behind a Grid

Kovasznay (1948) examines the steady state two-dimensional exact solution of the Navier-Stokes equation for the laminar flow behind a periodic array of cylinders or rods. The velocity field is assumed to be such that $v_1 = U + u_1, v_2 = u_2$, where U is the mean velocity in the x_1 direction. The vorticity equation (2.247) yields

$$\frac{\partial \xi_3}{\partial t} + (U + u_1)\frac{\partial \xi_3}{\partial x_1} + u_2 \frac{\partial \xi_3}{\partial x_2} = \nu \nabla^2 \xi_3 \,. \tag{4.177}$$

Denoting the spacing of the grid by δ, we define the Reynolds number as $\mathcal{R}e = \delta U/\nu$. The dimensionless vorticity becomes $\omega = \xi_3\, \delta/U$. The other dimensionless variables are $x = x_1/\delta, y = x_2/\delta, \tau = tU/\delta, 1 + u = v_1/U, v = v_2/U$. The governing equation (4.177) is

$$\frac{\partial \omega}{\partial \tau} + (1 + u)\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{\mathcal{R}e}\nabla^2 \omega \,. \tag{4.178}$$

As steady state solutions are sought, the term $\partial\omega/\partial\tau$ vanishes. We are left with

$$\nabla^2 \omega - \mathcal{R}e\frac{\partial \omega}{\partial x} - \mathcal{R}e\left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right) = 0 \,. \tag{4.179}$$

To build up the analytical solution, the trick consists in finding an expression that cancels the nonlinear term. The streamfunction is introduced to satisfy the continuity equation

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \,, \tag{4.180}$$

and therefore the vorticity is

$$\omega = -\nabla^2 \psi \,. \tag{4.181}$$

Taking the periodicity into account, the streamfunction is set up such that

$$\psi = f(x)\sin 2\pi y \; . \tag{4.182}$$

With (4.182), the nonlinear term of (4.179) gives

$$f' f'' - f f''' = 0 \; . \tag{4.183}$$

Integrating (4.183) we obtain

$$f'' = k^2 f \; , \tag{4.184}$$

where k is a real or complex arbitrary constant. A further integration yields

$$f = Ce^{kx} \; . \tag{4.185}$$

With the stream function

$$\psi = Ce^{kx} \sin 2\pi y \tag{4.186}$$

canceling the nonlinear term in (4.179), we have to seek a solution of the equation

$$\nabla^2\omega - \mathcal{R}e\frac{\partial\omega}{\partial x} = 0 \; . \tag{4.187}$$

Setting

$$\omega = g(x)\sin 2\pi y \; , \tag{4.188}$$

we have

$$g'' - \mathcal{R}e\, g' - 4\pi^2 g = 0 \; , \tag{4.189}$$

the solution of which is

$$g(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} \; , \tag{4.190}$$

where

$$\lambda_{1,2} = \frac{\mathcal{R}e}{2} \pm \sqrt{\frac{\mathcal{R}e}{2} + 4\pi^2} \; . \tag{4.191}$$

Combining (4.188) and (4.190), the vorticity is

$$\omega = \left(Ae^{\lambda_1 x} + Be^{\lambda_2 x}\right)\sin 2\pi y \; , \tag{4.192}$$

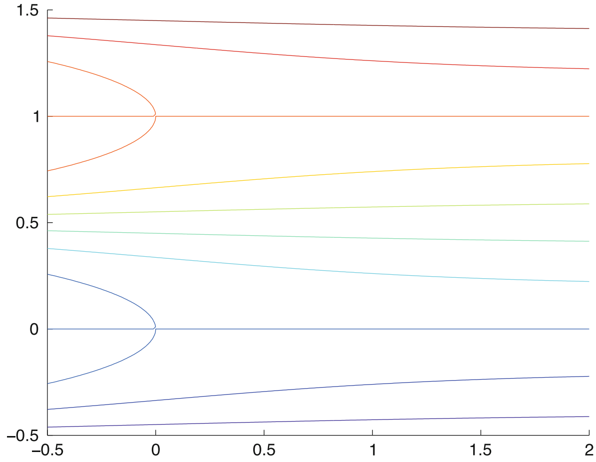


Fig. 4.13 Streamlines of the Kovasznay flow for $\mathcal{R}e = 40$

while Eqs. (4.181) and (4.186) give

$$\omega = C(4\pi^2 - k^2)e^{kx} \sin 2\pi y \; . \tag{4.193}$$

Comparison of (4.192) and (4.193) shows that two solutions are possible

$$k = \lambda_1, \quad A = -\mathcal{R}e\lambda_1 C, \quad B = 0, \tag{4.194}$$

$$k = \lambda_2, \quad A = 0, \quad B = -\mathcal{R}e\lambda_2 C, \tag{4.195}$$

With λ_2 and $\mathcal{R}e = 40$ the streamlines are shown in Fig. 4.13, with pairs of eddies generated behind the cylinders. The flow recovers uniformity downstream through the exponential term of the solution.

As the Kovasznay flow incorporates the nonlinear term, it is a good benchmark to test the numerical accuracy and space convergence of computational methods integrating the Navier-Stokes equation.

4.16 Plane Periodic Solutions

Many exact solutions of the Navier-Stokes equations are obtained for spatial periodic conditions. In this section we consider a two-dimensional (2D) solution due to Walsh (1992).

Let us first proof the following theorem

Theorem 4.1. *Let us consider a vector field* **u** *in the domain* Ω *that satisfies*

$$\nabla^2 \mathbf{u} = \lambda\, \mathbf{u}, \tag{4.196}$$

$$div\, \mathbf{u} = 0\, . \tag{4.197}$$

Then the velocity $\mathbf{v} = e^{\nu\lambda t}\mathbf{u}$ *satisfies the Navier-Stokes equation (2.178) and (2.179) with a pressure such that*

$$\nabla p = -\mathbf{v} \cdot \nabla\mathbf{v} \,. \tag{4.198}$$

The vector $\mathbf{v}$ is divergence free as is also $\mathbf{u}$. Furthermore,

$$\frac{\partial \mathbf{v}}{\partial t} = \nu\lambda\mathbf{v} = \nu\Delta\mathbf{v} \,. \tag{4.199}$$

It remains to prove that the nonlinear term is a gradient. This amounts to showing that

$$\frac{\partial}{\partial x_2}\left(v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2}\right) = \frac{\partial}{\partial x_1}\left(v_1\frac{\partial v_2}{\partial x_1} + v_2\frac{\partial v_2}{\partial x_2}\right) \,, \tag{4.200}$$

as $\mathbf{curl}\,\nabla = \mathbf{0}$. This is evident by incompressibility and relation (4.199).

In the 2D case, we resort to the streamfunction ψ, assuming that it is an eigenfunction of the Laplacian with eigenvalue λ. Consequently, $\mathbf{u} = (\partial\psi/\partial x_2, -\partial\psi/\partial x_1)$ satisfies (4.196) and (4.197)with the same λ. Therefore, $e^{\nu\lambda t}\psi$ is the streamfunction of the associated Navier-Stokes flow. If we have a periodic domain of size 2π, then the eigenfunctions λ are of the form $\lambda = -(k_{x_1}^2 + k_{x_2}^2)$, with k_{x_1} and k_{x_2} positive integers. For given k_{x_1}, k_{x_2}, the linearly independent eigenfunctions are

$$\cos(k_{x_1}x_1)\cos(k_{x_2}x_2), \sin(k_{x_1}x_1)\sin(k_{x_2}x_2) \,,$$
$$\cos(k_{x_1}x_1)\sin(k_{x_2}x_2), \sin(k_{x_1}x_1)\cos(k_{x_2}x_2) \,.$$

It is possible to build up complicated geometrical patterns by combination of the eigenfunctions named n, m eigenfunction by Walsh, with $\lambda = -(n^2 + m^2)$. A theorem in number theory shows that integers of the form p^{2i} and p^{2i+1}, where p is an integer number such that $p \equiv 1 \pmod 4$, may be written as sums of squares in exactly $i + 1$ manners. For example, $625 = 25^2 = 24^2 + 7^2 = 20^2 + 15^2$. Figure 4.14 displays the streamlines corresponding to $\psi = \sin(25x_1) + \cos(25x_2) - \sin(24x_1)\cos(7x_2) + \cos(15x_1)\cos(20x_2) - \cos(7x_1)\sin(24x_2)$.

4.17 Summary

The exact solutions presented in this chapter do not exhaust the list of those available, but they are fairly representative. A more comprehensive collection can be obtained by consulting the references.

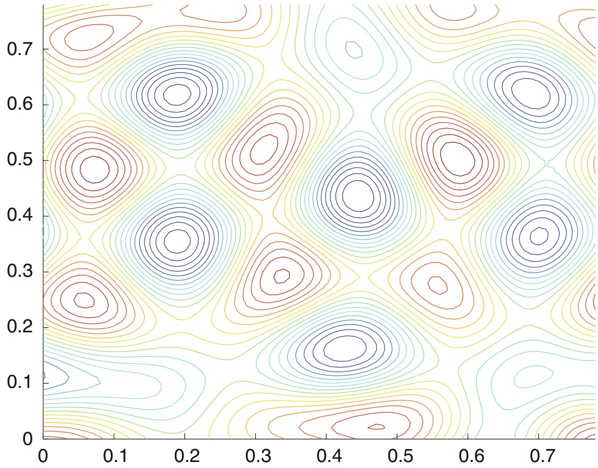


Fig. 4.14 ψ isocontours in the square $(0, \pi/4)^2$

Some flow problems such as the flow between parallel plates or in an annular region, are amenable to exact solution because the nonlinear inertia terms drop out of the hydrodynamic equations. Others, such as Hamel's problem, retain a nonlinear character, but enough nonlinear terms disappear so that the problem reduces to a differential equation whose solution can be recognized. Finally, there are flow problems, such as the flow generated by a rotating disc, which can be reduced to a system of normalized ordinary differential equations to be integrated numerically.

A semantical question arises: What is meant by "exact solution"? The answer probably varies from one era to another. In the mid-nineteenth century Hamel's solution probably would not have qualified, for it cannot be expressed in terms of functions well understood at that time. In the earlier twentieth century von Kármán's formulation of the rotating disc problem might not have been accepted as an exact solution because of the numerical labor that remained to be done.

Perhaps now we have come full cycle on von Kármán's problem: the student today might well ask if the numerical integration of four ordinary differential equations is any more an exact solution than would be numerical integration of the full hydrodynamic equations. However let us recall the state of computational art in the 1920s and 1930s. Numerical integration methods for both ordinary and partial differential equations were known, and the construction of analog computers was in sight. However high speed digital equipment, which makes practical the numerical treatment of partial differential equations, was still a generation away. Thus the reduction of a problem to ordinary differential equations really was a significant step.

Today much open source and commercial software is available. Visualization packages are also available to show the myriads of numerical results produced by simulation tools relying on high-performance computing. However the display of a result does not explain everything and simple (or simplified) models are still a source of understanding for what some people have named the incomprehensible Navier-Stokes equation.

Chapter 5
Pipe Flow

Abstract Steady flow of viscous fluids through straight pipes of noncircular cross section is treated by semi-inverse methods, separation of variables, and conformal mapping.

In this chapter we consider the general problem of steady-state flow through a straight pipe of uniform, but non-circular, cross section. As we shall see, the steady problem reduces to an exercise in potential theory; it is quite analogous to the problem of torsion of an elastic bar having the same cross section as the pipe.

5.1 Poisson's Equation for the Velocity

Choose in Fig. 5.1 a Cartesian coordinate system with x_3-axis parallel to the generators of the pipe. We assume that there exists within the fluid a constant pressure gradient, or equivalent conservative body force field, in the decreasing x_3-direction. We assume further that this pressure gradient generates a rectilinear flow field along the pipe. Thus we seek a solution to the hydrodynamic equations (2.194) and (2.195) in the form

$$\begin{aligned} v_1 &= v_2 = 0\,, \\ v_3 &= v(x_1, x_2)\,, \\ p &= C - \hat{G}x_3\,, \end{aligned} \tag{5.1}$$

where C and $\hat{G}$ are constants and $v(x_1, x_2)$ vanishes on the periphery of the pipe cross-section. It is evident that (2.194) is automatically satisfied, as are the $i = 1, 2$ components of (2.195), provided f_1 and f_2 are both zero. As in Chap. 4, a conservative body force field can be handled by modifying the pressure. If f_3 is

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_5,

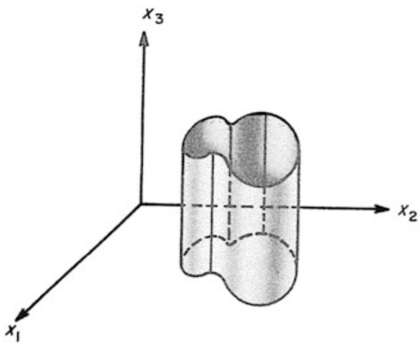


Fig. 5.1 Pipe geometry

constant, the $i = 3$ component reduces to

$$\nabla^2 v + \frac{G}{\mu} = 0\,, \tag{5.2}$$

where $G = \hat{G} + \rho f_3$, and ∇^2 is the two-dimensional Laplacian , i.e..

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\,. \tag{5.3}$$

Equation (5.2) is a special case of **Poisson's equation**, a linear, second-order partial differential equation of the elliptic type, which has been extensively studied in the literature of mathematics and mathematical physics. In this chapter we shall look for solutions which satisfy the boundary condition

$$v(x_1, x_2) = 0 \quad \text{on } \boldsymbol{\Gamma}\,, \tag{5.4}$$

where $\boldsymbol{\Gamma}$ is the periphery of the pipe cross section.

The boundary value problem represented by (5.2) and (5.4) is equivalent to the **Dirichlet problem**

$$\nabla^2 u = 0\,, \tag{5.5}$$

$$u = f(x_1, x_2) \quad \text{on } \boldsymbol{\Gamma}\,. \tag{5.6}$$

To show the equivalence, we need only set

$$v(x_1, x_2) = \frac{G}{2\mu}\left[u(x_1, x_2) - f(x_1, x_2)\right]\,, \tag{5.7}$$

where f is *any* function satisfying

$$\nabla^2 f = 2\,. \tag{5.8}$$

The analogy with the torsion problem of elasticity is reinforced by choosing

$$f(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) \; . \tag{5.9}$$

5.2 Polynomial Solutions

We begin with what might be termed a semi-inverse method. Evidently many polynomials have constant Laplacian; some of these, when equated to zero, yield the equation of a closed contour. Thus setting $v(x_1, x_2)$ equal to such a polynomial, multiplied by an appropriate constant, will satisfy (5.2) and (5.4), with Γ the contour on which the polynomial vanishes. We reject immediately all linear polynomials for two reasons: they vanish only on straight lines, not on closed contours; their Laplacians are not only constant, they vanish. Consideration of quadratic polynomials, however, bears fruit.

5.2.1 The Elliptical Pipe

We note first that the Laplacian of any quadratic polynomial is constant. We next recall that equating any quadratic polynomial to zero yields the equation of a conic section. Only the ellipses are of interest to us, for circles were treated in the last chapter and none of the other conic sections are closed contours.

The algebra is simplified by placing the centroid of the ellipse on the x_3-axis, and orienting the x_1- and x_2-axes along the axes of the ellipse (Fig. 5.2). Thus the most general ellipse can be represented by

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1 \; . \tag{5.10}$$

We find by inspection that a solution to (5.2) which vanishes on the ellipse (5.10) is provided by

$$v = \frac{G}{2\mu}\frac{a^2b^2}{a^2+b^2}\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}\right) \; . \tag{5.11}$$

The volume flow rate is easily calculated. We have

$$Q = \int\int v \, dx_1 dx_2 \; , \tag{5.12}$$

where the integration is carried out over the ellipse. With (5.11),

$$Q = \frac{\pi G}{4\mu}\frac{a^3b^3}{a^2+b^2} \; . \tag{5.13}$$

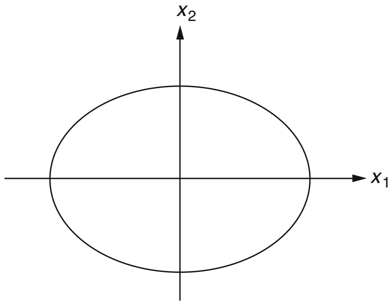


Fig. 5.2 Elliptical pipe

This result may be rewritten

$$Q = \frac{GA^2}{4\pi\mu}\frac{R}{1+R^2}\,, \tag{5.14}$$

where $A = \pi ab$ is the area of the ellipse and $R = a/b$ is the ratio of the semi-axes. We then obtain

$$\frac{\partial Q}{\partial R} = \frac{GA^2}{4\pi\mu}\frac{1-R^2}{(1+R^2)^2}\,. \tag{5.15}$$

Hence, for μ, G, A all fixed, Q has an extremum at $R = 1$, which is easily shown to be a maximum. We thus find that the circular pipe is more efficient than any elliptical pipe, in the sense that the circular pipe produces a greater volume flow for a given pressure gradient than does an elliptical pipe of the same cross-sectional area.This result can be generalized. In Pólya (1948) it is proved that the circular cross section has maximum torsional rigidity of all simply-connected cross sections with a given area. The hydrodynamic analogy is that the circular pipe is the most efficient.

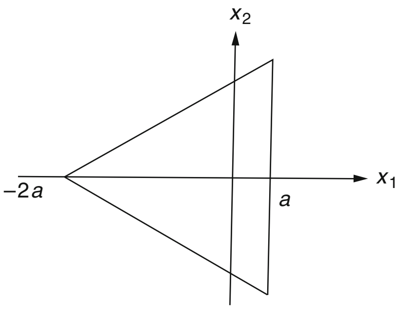
5.2.2 The Triangular Pipe

It seems surprising that we can express in closed form the flow through so unlikely a cross section as an equilateral triangle, but such is the way of semi-inverse methods.

If we place the origin at the intersection of the medians and let the negative x_1-axis pass through one vertex (Fig. 5.3), the equation of an equilateral triangle with altitude $3a$ becomes

$$(x_1 - a)(x_1 - \sqrt{3}\,x_2 + 2a)(x_1 + \sqrt{3}\,x_2 + 2a) = 0\,. \tag{5.16}$$

Fig. 5.3 Triangular pipe



Since

$$\nabla^2 \left[(x_1 - a)(x_1 - \sqrt{3}\,x_2 + 2a)(x_1 + \sqrt{3}\,x_2 + 2a) \right] = 12a \;, \tag{5.17}$$

we have

$$v = \left(\frac{G}{12a\mu} \right) (a - x_1)(x_1 - \sqrt{3}\,x_2 + 2a)(x_1 + \sqrt{3}\,x_2 + 2a) \;. \tag{5.18}$$

It is readily verified that the maximum velocity occurs at the origin, and that its value there is $Ga^2/3\mu$.

5.3 Separation of Variables: The Rectangular Pipe

The flow through a pipe of rectangular cross section is not so easily solved, for the equation of a rectangle does not have constant Laplacian.

If the rectangle has sides $2a$ and $2b$, and if we choose the orientation of the coordinate system as illustrated in Fig. 5.4, we require

$$\begin{aligned} \nabla^2 v + \frac{G}{\mu} &= 0 \;, \\ v = 0 \quad \text{on } x_1 &= \pm a \;, \\ v = 0 \quad \text{on } x_2 &= \pm b \;. \end{aligned} \tag{5.19}$$

If we set

$$v(x_1, x_2) = \left(\frac{G}{2\mu} \right) \left[b^2 - x_2^2 + u(x_1, x_2) \right] \;, \tag{5.20}$$

we require instead

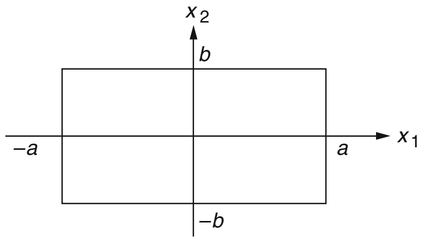


Fig. 5.4 Rectangular pipe

$$\nabla^2 u = 0 \, , \tag{5.21}$$

$$u = x_2^2 - b^2 \quad \text{on } x_1 = \pm a \, , \tag{5.22}$$

$$u = 0 \quad \text{on } x_2 = \pm b \, . \tag{5.23}$$

Since Laplace's equation (5.21) is linear, a sum of its solutions is also a solution. As we shall see, the required solution can be expressed as a sum of terms of the form $X(x_1).Y(x_2)$. This separation of variables solves (5.21) if and only if

$$\frac{X''}{X} = -\frac{Y''}{Y} \, . \tag{5.24}$$

Since the left side of (5.24) depends only on x_1 and the right side depends only on x_2, both sides must equal the same constant, say c_n^2.

If we choose

$$c_n = \left(n + \frac{1}{2}\right)\frac{\pi}{b} \, , \tag{5.25}$$

and let n be an integer, the function

$$Y_n(x_2) = \cos c_n x_2 \tag{5.26}$$

vanishes on $x_2 = \pm b$ and satisfies

$$-\frac{Y_n''}{Y_n} = c_n^2 \, . \tag{5.27}$$

The corresponding $X_n(x_1)$ is determined from

$$\frac{X_n''}{X_n} = c_n^2 \, , \tag{5.28}$$

i.e.,

$$X_n = A_n \cosh c_n x_1 + B_n \sinh c_n x_1 \, . \tag{5.29}$$

Because of the symmetry of the problem, we take $B_n = 0$. Thus we let

$$u(x_1, x_2) = \sum_{n=0}^{\infty} A_n \cosh c_n x_1 \cos c_n x_2 \ , \tag{5.30}$$

and set about determining the coefficients A_n so as to satisfy the boundary conditions (5.22). With the symmetry of the hyperbolic cosine, we need

$$\sum_{n=0}^{\infty} A_n^* \cos c_n x_2 = x_2^2 - b^2 \ , \tag{5.31}$$

where

$$\begin{aligned} A_n^* &= A_n \cosh c_n a \\ &= A_n \cosh \left[\left(n + \frac{1}{2} \right) \frac{\pi a}{b} \right] . \end{aligned} \tag{5.32}$$

We can now follow the usual procedures of Fourier analysis, observing that

$$\int_{-b}^{b} \cos c_n x_2 \cos c_m x_2 \, dx_2 = \begin{cases} 0, \text{ if } m \neq n \\ 1, \text{ if } m = n \end{cases} . \tag{5.33}$$

Thus, if we multiply both sides of (5.31) by $\cos c_m x_2$ and integrate, we obtain

$$\begin{aligned} A_m^* &= \frac{1}{b} \int_{-b}^{b} (x_2^2 - b^2) \cos c_m x_2 \, dx_2 \\ &= \frac{4(-1)^{m+1}}{b c_m^3} \\ &= \frac{32(-1)^{m+1} b^2}{(2m+1)^3 \pi^3} . \end{aligned} \tag{5.34}$$

We now obtain the solution for the fluid velocity by going back through the various substitutions:

$$v = \frac{G}{2\mu} \left[b^2 - x_2^2 + \frac{32 b^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cosh(2n+1)\pi x_1/2b \cos(2n+1)\pi x_2/2b}{(2n+1)^3 \cosh(2n+1)\pi a/2b} \right] . \tag{5.35}$$

5.4 Conformal Mapping Methods

Up to the present point this monograph has been self-contained, in the sense that the reader was not assumed to have prior knowledge of any mathematics beyond elementary differential equations and vector analysis. However, in order to present a very general method of solving pipe flow problems, we must bring in some techniques from the theory of functions of a complex variable. The reader who is not familiar with this subject, and who has not the time to learn it now, is invited to rejoin us at the beginning of Chap. 6. The remainder of the book is not contingent upon the present section.

As indicated in Sect. 5.1, the transformation

$$v = \frac{G}{2\mu}\left[u - \frac{(x_1^2 + x_2^2)}{2}\right] \tag{5.36}$$

reduces the problem of flow in a pipe to the Dirichlet problem

$$\nabla^2 u = 0\,,$$
$$u = \frac{(x_1^2 + x_2^2)}{2} \quad \text{on } \boldsymbol{\Gamma}\,. \tag{5.37}$$

Since u is harmonic, it can be represented as the real part of an analytic function of $z = x_1 + ix_2$; equivalently, it could be represented as the imaginary part of another analytic function of z. Moreover

$$x_1^2 + x_2^2 = z\bar{z}\,. \tag{5.38}$$

Hence the Dirichlet problem (5.37) is equivalent to the selection of a function, analytic inside $\boldsymbol{\Gamma}$, whose real part, or whose imaginary part, assumes the value $z\bar{z}/2$ on $\boldsymbol{\Gamma}$.

If the cross section of the pipe is simply-connected, the selection of the analytic function is effected by conformal mapping of the cross section onto the unit circle. Let the cross section occupy the region $\mathbf{R}$ in the z-plane and let the conformal transformation

$$z = m(w), \quad w = m^{-1}(z) \tag{5.39}$$

map the region $\mathbf{R}$ onto the unit disc $|w| \leq 1$. Let $F(z)$ be the function whose real part[1] assumes the value $z\bar{z}/2$ on the boundary of $\mathbf{R}$. If we set

[1] We could equally well deal with the imaginary part, as is done in Sokolnikoff (1983). Our treatment, based on the real part, is similar to that carried out in Muskhelishvili (1963).

$$f(w) = F[m(w)] , \tag{5.40}$$

then the real part of the $f(w)$ assumes the value $\frac{1}{2}m(w)\overline{m(w)}$ on **C**, the unit circle in the w-plane.

Since $F(z)$ is analytic inside Γ , $f(w)$ is analytic inside **C**. Hence by Cauchy's integral formula we have, for w inside **C**,

$$f(w) = \frac{1}{2\pi i} \oint_{\mathbf{C}} \frac{f(\zeta)d\zeta}{\zeta - w} . \tag{5.41}$$

By a somewhat less well-known result of the theory of functions (the proof is set out in Sokolnikoff (1983) and Muskhelishvili (1963)),

$$\overline{f(0)} = \frac{1}{2\pi i} \oint_{\mathbf{C}} \frac{\overline{f(\zeta)}d\zeta}{\zeta - w} \tag{5.42}$$

for any w inside **C**. Combining these results,

$$\begin{aligned} f(w) + \overline{f(0)} &= \frac{1}{2\pi i} \oint_{\mathbf{C}} \frac{f(\zeta) + \overline{f(\zeta)}}{\zeta - w} d\zeta \\ &= \frac{1}{\pi i} \oint_{\mathbf{C}} \frac{\Re f(\zeta)}{\zeta - w} d\zeta \\ &= \frac{1}{2\pi i} \oint_{\mathbf{C}} \frac{m(\zeta)\overline{m(\zeta)}}{\zeta - w} d\zeta . \end{aligned} \tag{5.43}$$

However

$$\overline{f(0)} = \Re f(0) - i\Im f(0) = \frac{1}{2}\left[f(0) + \overline{f(0)}\right] - i\Im f(0) . \tag{5.44}$$

If we set $w = 0$ in the formula (5.43) and then substitute the resulting expression for $f(0) + \overline{f(0)}$ into the right side of Eq. (5.44), we obtain

$$\overline{f(0)} = \frac{1}{4\pi i} \oint_{\mathbf{C}} \frac{m(\zeta)\overline{m(\zeta)}}{\zeta} d\zeta - i\Im f(0) . \tag{5.45}$$

Equation (5.43) therefore becomes

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \oint_{\mathbf{C}} \frac{m(\zeta)\overline{m(\zeta)}}{\zeta - w} d\zeta - \frac{1}{4\pi i} \oint_{\mathbf{C}} \frac{m(\zeta)\overline{m(\zeta)}}{\zeta} d\zeta + i\Im f(0) \\ &= \frac{1}{4\pi i} \oint_{\mathbf{C}} \frac{\zeta + w}{\zeta(\zeta - w)} m(\zeta)\overline{m(\zeta)} d\zeta + i\Im f(0) . \end{aligned} \tag{5.46}$$

Hence the fluid velocity through the pipe is determined. Retracing the various transformations, we obtain

$$\begin{aligned}
\left(\frac{2\mu}{G}\right) v(x_1, x_2) + \frac{(x_1^2 + x_2^2)}{2} &= u(x_1, x_2) \\
&= \Re F(z) \\
&= \Re f\left[m^{-1}(z)\right] \\
&= \Re f(w) \\
&= \Re\left[\frac{1}{4\pi i}\oint_{\mathbf{C}} \frac{\zeta + w}{\zeta(\zeta - w)} m(\zeta)\overline{m(\zeta)} d\zeta\right].
\end{aligned} \tag{5.47}$$

If the required conformal mapping can be found and the resulting integrations carried out, Eq. (5.47) will give a closed-form expression for the fluid velocity at any point in the pipe.

The conformal method also opens the way for approximation techniques. If the mapping function can be constructed, the integral in (5.47) can be evaluated by quadrature. Also it may be possible to find transformations which almost map the region **R** conformally onto the unit disc. The subject of approximate conformal mapping has been extensively studied in the literature of elasticity theory, to which the interested reader is referred. He will also discover several examples of Dirichlet problems solved exactly by conformal mapping methods.

Numerical solution of the Dirichlet problem by the conformal mapping method may or may not be easier than the direct solution of a finite difference approximation to Laplace's equation. It depends upon the shape of the region **R** and upon the computing equipment one has available.

5.4.1 Multiply-Connected Regions: Flow Between Eccentric Cylinders

If the cross section of the pipe is not simply-connected, it cannot be mapped conformally onto the unit disc. Consequently the method derived above breaks down. Sometimes, however, the cross section can be mapped onto a region simple enough that the solution to the resulting Dirichlet problem can be recognized. This procedure makes explicit use of the fact that Laplace's equation is invariant under conformal transformation.

By way of example, the bilinear transformation

$$z = \frac{w}{1 - aw}, \quad w = \frac{z}{1 + az}, \tag{5.48}$$

where a is a real positive constant, can be used to map the region between a pair of eccentric circles in the z-plane onto a concentric ring in the w-plane. The circle

$$|w| = \rho \, , \tag{5.49}$$

where $0 < \rho < 1/a$, transforms to a circle with center on the real axis at

$$z = \frac{a\rho^2}{1 - a^2\rho^2} \, , \tag{5.50}$$

and with radius

$$r = \frac{\rho}{1 - a^2\rho^2} \, . \tag{5.51}$$

Consequently the ring

$$0 < \rho_1 < |w| < \rho_2 < 1/a \tag{5.52}$$

maps onto the region between two circles of radii

$$r_1 = \frac{\rho_1}{1 - a^2\rho_1^2}, \quad r_2 = \frac{\rho_2}{1 - a^2\rho_2^2}, \tag{5.53}$$

with distance

$$l = \frac{a\rho_2^2}{1 - a^2\rho_2^2} - \frac{a\rho_1^2}{1 - a^2\rho_1^2} \tag{5.54}$$

between their centers. Conversely, if we wish to map the region between circles of radii r_1 and r_2, with distance l between their centers, onto a circular ring, we select ρ_1, ρ_2, and a by solving equations (5.53) and (5.54):

$$a = \frac{l}{\sqrt{(r_2^2 - r_1^2)^2 - 2l^2(r_1^2 + r_2^2) + l^4}} \, ,$$

$$\rho_1 = \frac{\sqrt{1 + 4r_1^2a^2} - 1}{2r_1a^2}, \quad \rho_2 = \frac{\sqrt{1 + 4r_2^2a^2} - 1}{2r_2a^2} \, . \tag{5.55}$$

As before, we let the fluid velocity in the eccentric circular pipe be given by

$$v = \frac{G}{2\mu}\left[u - \frac{(x_1^2 + x_2^2)}{2}\right] , \tag{5.56}$$

where u is harmonic and assumes the values $z\bar{z}/2$ on the pipe walls. If we employ the mapping (5.48) and let

$$w = We^{i\theta} \tag{5.57}$$

then, in the ring $\rho_1 < W < \rho_2$, u obeys Laplace's equation in polar coordinates:

$$\frac{\partial^2 u}{\partial W^2} + \frac{1}{W}\frac{\partial u}{\partial W} + \frac{1}{W^2}\frac{\partial^2 u}{\partial \theta^2} = 0\,. \tag{5.58}$$

Since we seek a single valued solution on a complete ring, the θ-dependence of $u(W, \theta)$ must have period 2π. Therefore, noting that

$$(a_+ W^n + a_- W^{-n}) \sin n\theta + (b_+ W^n + b_- W^{-n}) \cos n\theta \tag{5.59}$$

solves (5.58) for any integer n and any set of constants $a_\pm, b_\pm$, we set

$$u = b_0 + b_l \ln W + \sum_{n=1}^{\infty} [(a_n W^n + a_- W^{-n}) \sin n\theta + (b_n W^n + b_{-n} W^{-n}) \cos n\theta] \tag{5.60}$$

and determine the $a_n, a_{-n}, b_n, b_{-n}, b_0, b_l$ in terms of the Fourier coefficients of the transformed boundary conditions.

We have

$$\begin{aligned}
\frac{1}{2} z\bar{z} &= \frac{1}{2}\frac{w}{1-aw} \cdot \frac{\bar{w}}{1-a\bar{w}} \\
&= \frac{1}{2}\frac{W^2}{(1-aw)(1-a\bar{w})} \\
&= \frac{W^2}{2(1-a^2W^2)}\left\{1 + \frac{aw}{1-aw} + \frac{a\bar{w}}{1-a\bar{w}}\right\} \\
&= \frac{W^2}{2(1-a^2W^2)}\{1 + (aw + a^2w^2 + \ldots) + (a\bar{w} + a^2\bar{w}^2 + \ldots)\} \\
&= \frac{W^2}{(1-a^2W^2)}\{\frac{1}{2} + aW\cos\theta + a^2W^2\cos 2\theta + \ldots\} \\
&= \frac{W^2}{(1-a^2W^2)}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} a^n W^n \cos n\theta\right\}\,.
\end{aligned} \tag{5.61}$$

The series converges absolutely for $W < 1/a$. Hence the ring (5.52) lies within the circle of convergence.

Since $u(W, \theta)$, given by (5.60), must assume the values of $z\bar{z}/2$ on $W = \rho_1, \rho_2$, the coefficients a_n, a_{-n} must of necessity vanish, for there are no sine terms in (5.61). Notice that physically, the flow is symmetric about the line of centers. The $b's$ are determined pairwise by a system of algebraic equations:

$$
\begin{aligned}
b_0 + b_l \ln \rho_1 &= \frac{\rho_1^2}{2} \frac{\rho_1^2}{2(1 - a^2 \rho_1^2)}, \\
b_0 + b_l \ln \rho_2 &= \frac{\rho_2^2}{2} \frac{\rho_2^2}{2(1 - a^2 \rho_2^2)}, \\
b_n \rho_1^{2n} + b_{-n} &= \frac{a^n \rho_1^{2(n+1)}}{(1 - a^2 \rho_1^2)} \quad n = 1, 2, 3, \ldots, \\
b_n \rho_2^{2n} + b_{-n} &= \frac{a^n \rho_2^{2(n+1)}}{(1 - a^2 \rho_2^2)} \quad n = 1, 2, 3, \ldots.
\end{aligned}
\tag{5.62}
$$

The analogy between pipe flow and torsion is not quite complete in the case of multiply connected cross sections. In the torsion of a rod bounded by eccentric circular cylinders, for example, there is sufficient arbitrariness in the boundary conditions that the logarithm term—essential in the corresponding pipe flow problem treated above—need not be retained.

Chapter 6
Flow Past a Sphere

Abstract The classical approaches to the problem of flow past a sphere are presented, along with the more modern technique of matched inner and outer expansions. For the huge literature on flow past nonspherical obstacles, a list of references is provided. Stokeslets are introduced and related to Lighthill's compelling work on the propulsion of microorganisms.

One of the most deeply studied problems in viscous hydrodynamics deals with the steady-state flow past a sphere placed in an otherwise uniform stream. Despite the apparent simplicity of the geometry of this problem, a closed-form exact solution appears to be permanently out of reach. However the asymptotic solution of the problem for small Reynolds number has received considerable attention, beginning with the researches of G.G. Stokes in 1851. Stokes attacked the problem using the equations of creeping viscous flow. Approximate solutions to viscous flow problems obtained by using the creeping flow equations are sometimes called **Stokes solutions**, and the problem of flow past a sphere is sometimes called the **Stokes problem**. Subsequently certain conceptual difficulties were noted in the theory of creeping flow past obstacles, and modern research on the problem has centered around these difficulties.

ln this chapter we shall present a more-or-less historical development of the problem, principally because the modern approaches draw so heavily upon the classical treatments as developed e.g. in Lamb (1995).

6.1 The Equations of Creeping Viscous Flow

In Sect. 2.2, we pointed out that, in the absence of body forces, the creeping viscous flow of an incompressible fluid is governed by the linear equations

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_6,

$$\frac{\partial v_i}{\partial x_i} = 0 , \tag{6.1}$$

$$\frac{\partial p}{\partial x_i} = \mu \nabla^2 v_i , \tag{6.2}$$

in which Cartesian tensor notation is employed. Differentiating (6.2) and using (6.1) gives us

$$\nabla^2 p = 0 . \tag{6.3}$$

Thus, *for creeping flow, the pressure is harmonic*.

The introduction of a stream function is especially valuable in problems of creeping viscous flow. If v_3 is independent of x_3, we express the remaining velocity components in terms of a stream function ψ according to Eqs. (2.180):

$$v_1 = \frac{\partial \psi}{\partial x_2} , \quad v_2 = -\frac{\partial \psi}{\partial x_1} . \tag{6.4}$$

If the fluid motion is referred to cylindrical coordinates, we have, according to Eqs. (3.81),

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \; v_\theta = -\frac{\partial \psi}{\partial r} . \tag{6.5}$$

If we set the density of the fluid equal to zero, Eqs. (2.186) and (3.82) each reduce to the **biharmonic equation**

$$\nabla^4 \psi = 0 . \tag{6.6}$$

For Cartesian coordinates the operator ∇^4 denotes $(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)^2$; for cylindrical coordinates it signifies $[\partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2)\partial^2/\partial \theta^2]^2$. Note that for creeping flow the equation for ψ is not coupled to the equation for v_3.

For axially symmetric flow we introduce the Stokes stream function. In cylindrical coordinate we have, according to (3.84),

$$v_z = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad v_r = -\frac{\partial \Psi}{\partial z} . \tag{6.7}$$

In spherical coordinates we use (3.105):

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}, \; v_\theta = -\frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r} . \tag{6.8}$$

Setting the fluid density equal to zero in Eq. (3.86), or in (3.107), we obtain

$$E^4 \Psi = 0 . \tag{6.9}$$

For cylindrical coordinates E^4 denotes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 ; \tag{6.10}$$

for spherical coordinates

$$E^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}\right)^2 . \tag{6.11}$$

For creeping flow the motion in the meridional planes is not coupled to the swirl, which is governed by

$$E^2\Omega = 0 . \tag{6.12}$$

6.2 Creeping Flow Past a Sphere

Let us suppose that a solid sphere of radius a is held fixed in an otherwise uniform stream of speed U. As illustrated in Fig. 6.1, we refer the motion to a spherical coordinate system with origin at the center of the sphere. The $\theta = 0$ axis is chosen in the direction of free-stream flow. In view of the axial symmetry, the choice of $\phi = 0$ direction is unimportant.

With Eqs. (6.8), the no-slip condition requires that

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \theta} = 0 \quad \text{at } r = a. \tag{6.13}$$

At large r we expect that the motion will approach the free-stream flow. Thus

$$v_r \to U\cos\theta, \quad v_\theta \to U\sin\theta \quad \text{as } r \to \infty. \tag{6.14}$$

However the free-stream flow is derivable from the stream function $\frac{1}{2}Ur^2\sin^2\theta$ $+C$, where C is an arbitrary constant of no importance. Thus, with a tacit assumption of smooth behavior at infinity, (6.14) can be replaced by the condition

$$\Psi(r,\theta) \sim \Psi_\infty(r,\theta) \overset{def.}{=} \frac{1}{2}Ur^2\sin^2\theta \quad \text{as } r \to \infty. \tag{6.15}$$

Condition (6.15) suggests the trial solution

$$\Psi(r,\theta) = \frac{1}{2}Uf(r)\sin^2\theta , \tag{6.16}$$

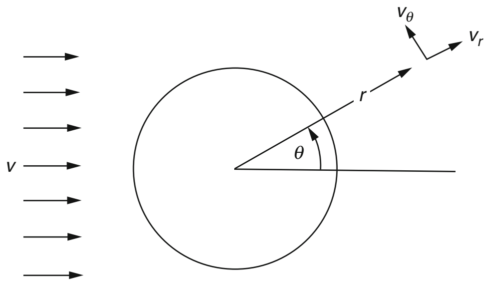


Fig. 6.1 Geometry and coordinate system for flow past a sphere

which, when substituted into Eq. (6.9), yields

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2} \right)^2 f(r) = 0 \, . \tag{6.17}$$

We can satisfy this equation by a sum of terms of the form cr^n, provided that each n satisfy the algebraic equation

$$[(n-2)(n-3)-2][n(n-1)-2] = 0 \, . \tag{6.18}$$

The roots of (6.18) are $n = -1, 1, 2, 4$, so that

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4 \, . \tag{6.19}$$

The limiting condition (6.15) requires that

$$C = 1, \quad D = 0 \, . \tag{6.20}$$

The no-slip condition (6.13) then provides simultaneous equations for A and B:

$$\left(\frac{1}{a^2} \right) A - B = 2a \, ,$$
$$\left(\frac{1}{a^2} \right) A + aB = -a^2 \, . \tag{6.21}$$

Thus $A = a^3/2, B = -3a/2$ and

$$\psi(r, \theta) = \frac{1}{2} U \left(\frac{a^3}{2r} - \frac{3ar}{2} + r^2 \right) \sin^2 \theta \, . \tag{6.22}$$

The lines of constant Ψ, i,e., the streamlines of the flow, are illustrated in Fig. 6.2.

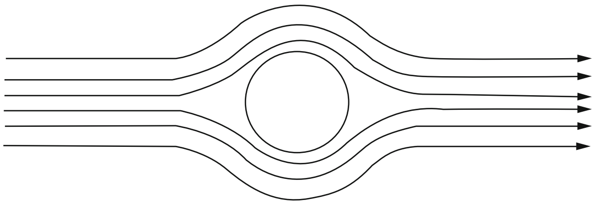


Fig. 6.2 Streamlines of the flow past a fixed sphere

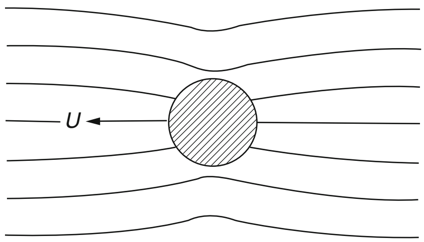


Fig. 6.3 Streamlines of the flow generated by a sphere translating through a fluid otherwise at rest

If we subtract off the contribution of the free-stream flow and plot the lines of constant Ψ^*, where

$$\Psi^* = \Psi - \Psi_\infty = \frac{1}{2}U\left(\frac{a^3}{2r} - \frac{3ar}{2}\right)\sin^2\theta \ , \tag{6.23}$$

we obtain the streamlines which result when a sphere is slowly and steadily translated through fluid otherwise at rest. These streamlines, illustrated in Fig. 6.3, seem peculiar, but we must remember that the frame of reference is fixed in the fluid rather than in the sphere. Hence the motion is unsteady and the streamlines do not coincide with the particle paths. Figure 6.3 is an instantaneous picture, taken as the sphere moves across the field of view from right to left. As it moves, it tends to push some of the fluid just ahead out of its way and to drag with it some of the fluid just behind.

If we substitute the stream function $\Psi(r, \theta)$ into Eqs. (6.8), we obtain the velocity components for the creeping flow past a sphere:

$$\begin{aligned} v_r &= U\left[\frac{1}{2}\left(\frac{a}{r}\right)^3 - \frac{3}{2}\left(\frac{a}{r}\right) + 1\right]\cos\theta \ , \\ v_\theta &= U\left[\frac{1}{4}\left(\frac{a}{r}\right)^3 + \frac{3}{4}\left(\frac{a}{r}\right) - 1\right]\sin\theta \ . \end{aligned} \tag{6.24}$$

In order to exhibit more clearly the disturbing effect of the sphere, we express this result in terms of the velocity components in a Cartesian coordinate system with origin at the center of the sphere and with positive x_3-axis in the direction of flow. Writing (6.24) in covariant or contravariant form and applying the rules of vector transformation set out in Chap. 3, we find

$$\begin{aligned} v_1 &= -\frac{3}{4}\frac{Uax_1x_3}{r^3}\left(1-\frac{a^2}{r^2}\right) , \\ v_2 &= -\frac{3}{4}\frac{Uax_2x_3}{r^3}\left(1-\frac{a^2}{r^2}\right) , \\ v_3 &= U\left[1-\frac{3}{4}\frac{ax_3^2}{r^3}\left(1-\frac{a^2}{r^2}\right)-\frac{1}{4}\frac{a}{r}\left(3+\frac{a^2}{r^2}\right)\right] . \end{aligned} \tag{6.25}$$

The pressure could now be determined by substituting the results (6.25) into Eq. (6.1) and integrating. However the manipulations are somewhat easier if we stay with spherical coordinates. For incompressible, axially symmetric creeping flow, Eqs. (3.102) and (3.103) reduce to

$$\begin{aligned} \frac{1}{\mu}\frac{\partial p}{\partial r} &= \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}-\frac{2}{r^2}\right)v_r \\ &\quad -\frac{2}{r^2}\left(\frac{\partial}{\partial\theta}+\cot\theta\right)v_\theta , \end{aligned} \tag{6.26}$$

$$\begin{aligned} \frac{1}{\mu r}\frac{\partial p}{\partial \theta} &= \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}-\frac{1}{r^2\sin^2\theta}\right)v_\theta \\ &\quad +\frac{2}{r^2}\frac{\partial v_r}{\partial\theta} . \end{aligned} \tag{6.27}$$

With Eqs. (6.24), these become

$$\frac{\partial p}{\partial r} = \left(\frac{3\mu aU}{r^3}\right)\cos\theta , \tag{6.28}$$

$$\frac{\partial p}{\partial \theta} = \left(\frac{3\mu aU}{2r^2}\right)\sin\theta , \tag{6.29}$$

so that

$$\begin{aligned} p &= p_\infty - \left(\frac{3\mu aU}{2r^2}\right)\cos\theta \\ &= p_\infty - \frac{3\mu aUx_3}{2r^3} , \end{aligned} \tag{6.30}$$

where p_∞ denotes the pressure far from the sphere.

It is of interest to calculate the net force acting on the sphere. First we note that by symmetry this force acts along the x_3-axis. Intuitively we feel that it acts in the increasing x_3-direction, so that the streaming fluid exerts a drag on the sphere. There are two methods of procedure: we can compute the x_3-component of the stress vector acting on a generic point of the sphere and integrate over the surface of the sphere; alternatively we can calculate the total rate at which mechanical energy is being dissipated in the flow field and equate it to the power required to pull the sphere at velocity U through a fluid otherwise at rest.

The first of these procedures is most easily carried out in Cartesian coordinates. Substituting the results (6.25) and (6.30) into the stress-deformation relation (2.125), we obtain

$$T_{33} = -p_\infty + \frac{3}{2}\frac{\mu a^3 U x_3}{r^5}\left[3 + \frac{x_3^2}{a^2}\left(3 - 5\frac{a^2}{r^2}\right)\right] ,$$
$$T_{13} = \frac{3}{2}\frac{\mu a^3 U x_1}{r^5}\left[1 + \frac{3x_3^2}{a^2} - \frac{5x_3^2}{r^2}\right] . \qquad (6.31)$$

Consider now a point on the intersection of the sphere with the $x_2 = 0$ plane. Specifically consider the point with coordinates

$$x_1 = a\sin\theta, \quad x_2 = 0, \quad x_3 = a\cos\theta . \qquad (6.32)$$

At this point the stress components given by (6.31) become

$$T_{33} = -p_\infty + \frac{3}{2}\frac{\mu U}{a}\left[3 - 2\cos^2\theta\right]\cos\theta ,$$
$$T_{13} = \frac{3}{2}\frac{\mu U}{a}\left[1 - 2\cos^2\theta\right]\sin\theta . \qquad (6.33)$$

Moreover the unit normal to the sphere at this point has components

$$n_1 = \sin\theta, \quad n_2 = 0, \quad n_3 = \cos\theta . \qquad (6.34)$$

Consequently

$$\begin{aligned} T_{3j}n_j &= T_{13}\sin\theta + T_{33}\cos\theta \\ &= -p_\infty\cos\theta + \frac{3\mu U}{2a} . \end{aligned} \qquad (6.35)$$

In view of the axial symmetry, the x_3-component of the stress vector has the value given by (6.35) at each point of the ring

$$x_1 = a\sin\theta\cos\phi, \quad x_2 = a\sin\theta\sin\phi, \quad x_3 = a\cos\theta, \quad 0 \le \phi \le 2\pi . \qquad (6.36)$$

In addition we note that $T_{3j}n_j$ is independent of θ, except for the term in p_∞. Thus, since $p_\infty\cos\theta$ gives no contribution when we integrate over the surface of the sphere, the only part of the stress vector which contributes to the drag is constant over the sphere. Therefore the net drag D is obtained by multiplying $3\mu U/2a$ by the area of the sphere. We thereby obtain the familiar **Stokes' law**

$$D = \left(\frac{3\mu U}{2a}\right)(4\pi a^2) \\ = 6\pi\mu a U\,. \tag{6.37}$$

Although the energy method leads to the same result, it is instructive to carry out the details. It is easiest to use spherical coordinates. Substituting the velocity components given by (6.24) into Eqs. (3.98), we find that the physical components of the rate of deformation tensor for the flow past a sphere are given by

$$-\frac{1}{2}e_{rr} = e_{\theta\theta} = e_{\phi\phi} = -\frac{3}{4}\frac{aU}{r^2}\left(1-\frac{a^2}{r^2}\right)\cos\theta\,, \\ e_{r\theta} = e_{\theta r} = -\frac{3a^3U}{4r^4}\sin\theta\,, \\ e_{r\phi} = e_{\phi r} = e_{\theta\phi} = e_{\phi\theta} = 0\,. \tag{6.38}$$

Since a rigid-body motion does not contribute, these same results apply when we consider that the sphere is pulled through fluid otherwise at rest.

For incompressible flow the dissipation term in the energy equation (3.99) is

$$2\mu \sum_{\alpha,\beta=r,\theta,\phi} e_{\alpha\beta}^2\,. \tag{6.39}$$

If we call this Φ, Eqs. (6.38) give us

$$\Phi = \frac{9\mu}{4}\left(\frac{aU}{r^2}\right)^2\left[3\left(1-\frac{a^2}{r^2}\right)^2\cos^2\theta + \left(\frac{a^2}{r^2}\right)^2\sin^2\theta\right]. \tag{6.40}$$

The net power expended in pulling the sphere through the fluid is the product of the drag and the speed. Equating this to the integral of the dissipation over the flow field, we obtain

$$DU = \iiint_{r\geq a} \Phi\, dV \\ = \frac{9\mu a^2U^2}{4}\int_a^\infty \frac{1}{r^4}\int_0^\pi \left[3\left(1-\frac{a^2}{r^2}\right)^2\cos^2\theta + \left(\frac{a^2}{r^2}\right)^2\sin^2\theta\right] \\ \times\,[2\pi r^2\sin\theta]\,d\theta\,dr = 6\pi\mu a U^2\,, \tag{6.41}$$

in agreement with (6.37).

6.3 Oseen's Criticism

The Stokes solution for flow past a sphere was constructed by assuming that the inertia term in the Navier-Stokes equation is negligible compared with the viscous term. We can now check to see if this assumption is justified a posteriori.

As pointed out in Sect. 2.8, we need not concern ourselves with that part of the inertia term arising from the dynamic head associated with the Stokes flow, for the pressure could be modified to take care of this. Rather we need only calculate the contribution arising from $\mathbf{v} \times \boldsymbol{\xi}$ and compare it with the viscous term. In general tensor notation $\mathbf{v} \times \boldsymbol{\xi}$ corresponds to the covariant vector ζ_i, where

$$\begin{aligned} \zeta_i &= v^j (v_{j,i} - v_{i,j}) \\ &= v^j \left(\frac{\partial v_j}{\partial x^i} - \Gamma^k_{ji} v_k - \frac{\partial v_i}{\partial x^j} + \Gamma^k_{ij} v_k \right) \\ &= v^j \left(\frac{\partial v_j}{\partial x^i} - \frac{\partial v_i}{\partial x^j} \right) . \end{aligned} \tag{6.42}$$

For Stokes flow, $v^3 = 0$. Consequently Eqs. (3.93) give us

$$\begin{aligned} \zeta_1 &= \frac{v_\theta}{r} \left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] , \\ \zeta_2 &= v_r \left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(r v_\theta) \right] , \\ \zeta_3 &= 0 . \end{aligned} \tag{6.43}$$

With v_r and v_θ given by Eqs. (6.24), the corresponding physical components are

$$\begin{aligned} \zeta_r &= \frac{3}{2} \frac{aU^2}{r^2} \left(1 - \frac{a}{r}\right) \left(1 + \frac{1}{4}\frac{a}{r} + \frac{1}{4}\frac{a^2}{r^2}\right) \sin^2\theta , \\ \zeta_\theta &= \frac{3}{2} \frac{aU^2}{r^2} \left(1 - \frac{a}{r}\right) \left(1 - \frac{1}{2}\frac{a}{r} - \frac{1}{2}\frac{a^2}{r^2}\right) \sin\theta \cos\theta , \\ \zeta_\phi &= 0 . \end{aligned} \tag{6.44}$$

Hence the magnitude ζ of the vector ζ_i is given by

$$\begin{aligned} \zeta &= \sqrt{\zeta_r^2 + \zeta_\theta^2 + \zeta_\phi^2} \qquad (6.45) \\ &= \frac{3}{2} \frac{aU^2 \sin\theta}{r^2} \left(1 - \frac{a}{r}\right) \sqrt{\left(1 + \frac{a}{4r} + \frac{a^2}{4r^2}\right)^2 \sin^2\theta + \left(1 - \frac{a}{2r} - \frac{a^2}{2r^2}\right)^2 \cos^2\theta} . \end{aligned}$$

Let υ_i denote the viscous term in the covariant Navier-Stokes equation (3.52). Thus

$$\upsilon_i = \nu g^{jk} v_{i,jk} \; . \tag{6.46}$$

The corresponding physical components can be obtained by inspection from Eqs. (3.102) through (3.104). With the velocity components given by Eqs. (6.24),

$$\begin{aligned}
\frac{\upsilon_r}{\nu} &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{2}{r^2} \right) v_r - \frac{2}{r^2}\left(\frac{\partial}{\partial \theta} + \cot\theta \right) v_\theta \\
&= \frac{3aU}{r^3}\cos\theta \\
\frac{\upsilon_\theta}{\nu} &= \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{1}{r^2\sin^2\theta} \right) v_\theta \\
&= \frac{3aU}{2r^3}\sin\theta \; , \\
\upsilon_\phi &= 0 \; .
\end{aligned} \tag{6.47}$$

The magnitude υ of the vector υ_i is then given by

$$\upsilon = \frac{3\nu a U}{2r^3}\sqrt{\sin^2\theta + 4\cos^2\theta} \; . \tag{6.48}$$

The relative importance of inertia and viscosity is measured by the ratio of ζ to υ. With Eqs. (6.45) and (6.48), we have

$$\begin{aligned}
\frac{\zeta}{\upsilon} &= \mathcal{R}e\left(\frac{r}{a} - 1\right)\sin\theta \\
&\quad \times \sqrt{\frac{(1 + a/4r + a^2/4r^2)^2\sin^2\theta + (1 - a/2r - a^2/2r^2)^2\cos^2\theta}{\sin^2\theta + 4\cos^2\theta}} \; ,
\end{aligned} \tag{6.49}$$

where $\mathcal{R}e$ is the Reynolds number based on the sphere radius as length scale,

$$\mathcal{R}e = \frac{aU}{\nu} \; . \tag{6.50}$$

Inertia is negligible compared with viscosity if ζ/υ is small compared with unity. From Eq. (6.49) we see that this is indeed the case in the neighborhood of the sphere, i.e., where r is not much larger than a, provided only that $\mathcal{R}e$ is small compared with unity.[1] Far from the sphere, however, conditions are somewhat distressing. Equation (6.49) reveals that

$$\frac{\zeta}{\upsilon} \sim \frac{\mathcal{R}e\, r}{a\sqrt{1 + 4\cot^2\theta}} \quad \text{as } r \to \infty\ .$$

Thus for any preassigned finite value of $\mathcal{R}e$ however small, and for any fixed value of θ other than 0 or π, the assumptions underlying the Stokes solution fail to hold at points far enough from the sphere so that r is comparable with $(a/\mathcal{R}e)\sqrt{1 + 4\cot^2\theta}$.

The analysis presented so far in this section represents the now-classical criticism of the Stokes solution by Carl Wilhelm Oseen (1879–1944), and it raises an even more troublesome point. In constructing the Stokes solution, one applies the boundary condition that the velocity field approaches uniform streaming as r approaches infinity. However, on its way out to infinity via any path not asymptotic to the axis of symmetry, r passes into a region where the neglect of inertia is not valid. Consequently if the equations of creeping flow are applied only where they are valid, the boundary condition at infinity never enters the picture and the problem is underdetermined, except for the degenerate and idealized case of a completely massless fluid. However it is possible to salvage the validity of the Stokes solution in the region close to the sphere for a fluid of finite density. From Eqs. (6.15) and (6.23), we observe that

$$\frac{\Psi_\infty - \Psi}{\Psi_\infty} \sim \frac{3a}{2r} \quad \text{as } r \to \infty\ . \tag{6.51}$$

Thus for any fixed value of the Reynolds number the Stokes flow approaches the uniform stream at the same rate as the viscous term ceases to be dominant. For sufficiently small Reynolds number, therefore, the Stokes flow approaches arbitrarily close to the uniform stream before the underlying assumptions break down. In a sense, then, the boundary condition can be moved in from infinity to a location where it can be validly applied to the equations of creeping flow. Hence for sufficiently low Reynolds number the Stokes solution is self-consistent in its prediction of the flow field in the neighborhood of the sphere; in particular, the expression (6.37) for the drag is reliable.[2]

Things are not always so fortunate. Attempts to find a Stokes solution for flow past a circular cylinder, with uniform flow at infinity, lead to frustration: the best one

---

[1]The component υ_r vanishes identically on the $\theta = \pi/2$ plane, where ζ_r remains finite. Hence it might be objected that the inertia term in Eq. (3.102) cannot be neglected in the neighborhood of this plane, no matter how small the Reynolds number. However the vanishing of υ_r is merely an accident resulting from the selection of coordinate system. It would not, in general, be observed in a coordinate system with origin, say, off the axis of symmetry. The Navier-Stokes equation is a vector equation and it is the relative vector magnitudes of ζ_i and υ_i that matter.

[2]Experimental evidence cited by Schlichting (1960), p. 16, indicates that (6.37) is accurate within the limits of experimental error for $\mathcal{R}e < 0.3$.

can do is to construct a solution with a logarithmic singularity at infinity, obviously an unsatisfactory result. Even for the flow past a sphere one encounters difficulties. Thirty-eight years after Stokes published his result, Alfred North Whitehead (1879–1944) attempted to improve upon it by obtaining higher-order approximations to the flow when the Reynolds number is not negligibly small. His method was the obvious one of using the Stokes result to calculate the inertia term in the Navier-Stokes equation and considering it to be the driving force of a perturbation flow field. Unfortunately his result behaved at infinity in a way which is incompatible with the uniform stream condition. This difficulty seems to pervade all problems of uniform streaming past finite bodies, and is sometimes called "Whitehead's paradox". The paradox was, of course, resolved by Oseen's criticism, which came 21 years later.

Oseen improved upon the Stokes solution by linearizing the Navier-Stokes equation in such a way as to account for inertia where it is important, but to neglect it in the region close to the sphere. We now turn our attention to his method of attack.

Let us examine the nonlinear inertia term in the covariant Navier-Stokes equation (3.52). If we denote the contravariant components of the uniform velocity far from the sphere by U^i, we may write

$$v^j v_{i,j} = U^j v_{i,j} - (U^j - v^j) v_{i,j} \ . \tag{6.52}$$

Far from the sphere, U^j and v^j are practically the same. Hence we approximate

$$v^j v_{i,j} \approx U^j v_{i,j} \ . \tag{6.53}$$

Close to the sphere, (6.53) does not apply. For small Reynolds number, however, *a posteriori* analysis of the Stokes solution revealed that, close to the sphere, the inertia term was, in fact, small compared with the viscous term. It seems not unreasonable to hope, subject to *a posteriori* confirmation, that near the sphere the inertia term modified according to Eq. (6.53) will also be negligible compared with the viscous term. If $v^j v_{i,j}$ and $U^j v_{i,j}$ are both negligibly small, the presence of either one in the equation presumably does no harm.

For steady motion in the absence of body forces the Oseen technique thus involves the replacement of the covariant Navier-Stokes equation (3.52) by the approximate equation

$$\rho U^j v_{i,j} = -\frac{\partial p}{\partial x^i} + \mu g^{jk} v_{i,jk} \tag{6.54}$$

on the grounds that: (1) far from the sphere, the left side approximates $\rho v^j v_{i,j}$ sufficiently well; (2) close to the sphere, for small Reynolds number, $\rho U^j v_{i,j}$ and $\rho v^j v_{i,j}$ are both negligibly small, so that it doesn't matter which is used.

For the spherical coordinate system used in Sect. 6.2,

$$U^1 = U \cos\theta, \quad U^2 = -\frac{U}{r} \sin\theta, \quad U^3 = 0 \ . \tag{6.55}$$

By substituting these components into (6.54) we obtain, as approximations to (3.102) and (3.103),

$$\rho\ U\left[\frac{\partial v_r}{\partial r}\cos\theta + \left(v_\theta - \frac{\partial v_r}{\partial\theta}\right)\frac{\sin\theta}{r}\right]$$
$$= -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2v_\theta\cot\theta}{r^2}\right), \tag{6.56}$$

$$\rho\ U\left[\frac{\partial v_\theta}{\partial r}\cos\theta - \left(v_r + \frac{\partial v_\theta}{\partial\theta}\right)\frac{\sin\theta}{r}\right]$$
$$= -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta}\right), \tag{6.57}$$

where ∇^2 is the axisymmetric Laplacian, i.e.,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\,. \tag{6.58}$$

If we express v_r and v_θ in terms of a Stokes stream function, according to (6.8), and eliminate the pressure from (6.56) and (6.57) by cross-differentiation, we obtain after a manipulation

$$E^4\Psi = \left(\frac{\mathcal{R}e}{a}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)E^2\Psi\,. \tag{6.59}$$

Equivalently we may write

$$\left(aE^2 - \mathcal{R}e\frac{\partial}{\partial x_3}\right)E^2\Psi = 0\,, \tag{6.60}$$

where, as before,

$$x_3 = r\cos\theta\ . \tag{6.61}$$

The construction of an exact solution of (6.59), subject to the boundary conditions (6.13) and (6.15), is a matter of some difficulty. Fortunately, however, it is easy to exhibit a solution which solves the problem to a degree of approximation no worse than that used in setting it up.[3] This solution, which is the one given by Oseen himself, takes the form

[3] The exact solution to the problem was given by Goldstein (1929). Investigations of this sort are motivated by the idea that Oseen's equation and the Navier-Stokes equation are qualitatively similar, so that solutions of the former might be expected to yield qualitative information about solutions of the latter for all Reynolds numbers.

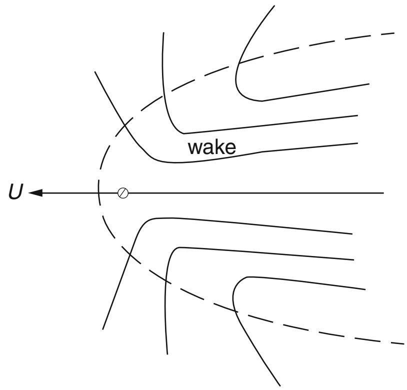


Fig. 6.4 Streamlines of the Oseen solution in absence of the uniform flow

$$\Psi = \frac{1}{2}a^2U\left[\left(\frac{a}{2r}+\frac{r^2}{a^2}\right)\sin^2\theta - \frac{3}{\mathcal{R}e}(1+\cos\theta)(1-e^{-(\mathcal{R}e\,r/2a)(1-\cos\theta)})\right]. \tag{6.62}$$

We can verify by direct substitution that this function satisfies the differential equation (6.59); that it satisfies (6.15), the boundary condition at infinity, is obvious. The remaining boundary conditions (6.13) are not satisfied, but expanding the exponential term in (6.62) as a Maclaurin series in $\mathcal{R}e\,r/a$, yields

$$\Psi = \frac{1}{2}a^2U\left(\frac{a}{2r}-\frac{3r}{2a}+\frac{r^2}{a^2}\right)\sin^2\theta + O\left(\frac{\mathcal{R}e\,r}{a}\right). \tag{6.63}$$

Thus for small Reynolds number the Oseen solution merges with the Stokes solution (6.22) in the region where the Stokes solution is valid. Consequently the stream function given by (6.62) satisfies the no-slip condition on the sphere, with neglect only of terms too small to be discussed within the framework of the Oseen approximation.

Since the Stokes and Oseen results coincide in the neighborhood of the sphere, both predict the result (6.37) for the net drag.

Far from the sphere, however, the picture of flow afforded by the Oseen solution is quite different from that given by Stokes' method. If we subtract off the uniform streaming and plot streamlines, we obtain the configuration shown in Fig. 6.4. The main feature which distinguishes this flow pattern from the Stokes pattern shown in Fig. 6.3 is the **laminar wake** behind the sphere.

Before passing on to modern refinements of the Stokes problem, we note that the Oseen equation (6.54) can be formally derived by considering the flow field to be a perturbation of the uniform stream. Thus, if we set

$$\begin{aligned} v_i &= U_i + \varepsilon u_i\,, \\ v^i &= U^i + \varepsilon u^i\,, \end{aligned} \tag{6.64}$$

where ε is a small parameter whose square is negligible, we obtain

$$\begin{aligned} v^j v_{i,j} &= U^j U_{i,j} + \varepsilon(U^j u_{i,j} + u^j U_{i,j}) + \varepsilon^2 u^j u_{i,j} \\ &\approx U^j v_{i,j} + \varepsilon u^j U_{i,j} \ . \end{aligned} \tag{6.65}$$

Since $U_{i,j}$ vanishes,[4] we recover (6.53). Conceptually, however, this view-point leaves something to be desired, for it requires the perturbation velocity u_i to become large of order ε^{-1} on the surface of the sphere. It is better to take the view that (6.53) approximates the inertia term where this term matters, and introduces an unimportant error in the neighborhood of the sphere.

6.4 Matching Techniques

It is not difficult to verify from Eq. (6.62) that the Oseen solution of Stokes problem is self-consistent at sufficiently low Reynolds number. Far from the sphere, (6.53) is a reasonable approximation; close to the sphere, the error doesn't matter. In principle, then, it should be possible to obtain higher approximations to the flow by Whitehead's iterative procedure, using Oseen's solution, rather than Stokes' solution, as a first approximation. In practice, however, the construction of Oseen-type solutions is a wearing task. Moreover the scheme entails a built-in drawback, already evident in the Oseen solution (6.62): the dependence of the solution upon the Reynolds number is somewhat obscured by the functional form. This is understandable, for the Oseen method seeks out a uniformly valid approximation which describes the flow both near the sphere, where inertia is unimportant, and far from the sphere, where inertia dominates. To the degree of approximation considered so far, relevant information about the flow can be extracted from Eq. (6.62), without inordinate difficulty, by straightforward asymptotic methods. In this manner we found, for example, that the Stokes and Oseen solutions predict the same net drag. When higher order approximations are undertaken, however, it is of obvious advantage to obtain the various flow quantities expressed as power series, or as simple extensions of power series, in the Reynolds number.

In order to obtain expressions of this sort, we develop and consider simultaneously two expansions—an inner expansion valid close to the sphere, an outer expansion valid far away. These are termed "Stokes" and "Oseen" expansions, respectively, since their leading terms are related to the corresponding approximate solutions. The Stokes expansion is constrained to obey the no-slip condition at the sphere; the Oseen expansion is generated so as to approach uniform streaming at infinity. It is almost self-evident that these boundary conditions are not enough. To render the problem determinate, it is necessary to use the fact that the Stokes and

[4]This is most easily seen by observing that it vanishes in a Cartesian system.

Oseen expansions are different forms of the same function; this leads to a matching of the expansions which makes it possible to derive alternately the successive terms in each expansion.

The Stokes expansion is of the sort envisaged by Whitehead: a perturbation expansion in the Reynolds number for fixed values of the spatial coordinates. To bring the Reynolds number explicitly into the analysis, we normalize the stream function and the spatial variables according to

$$\begin{aligned} \Psi &= a^2 U \Psi^{(s)} , \\ r &= aR , \\ \beta &= \cos\theta . \end{aligned} \tag{6.66}$$

The last of these relationships is introduced not for conceptual purposes but to simplify the forthcoming algebra. The quantities $\Psi^{(s)}, R, \beta$ are sometimes called the **Stokes variables**.

It is quickly verified, beginning with Eq. (3.107) with the inappropriate terms deleted, that the dimensionless stream function is governed by the equation

$$\frac{1}{R^2}\frac{\partial(\Psi^{(s)}, D^2\Psi^{(s)})}{\partial(R,\beta)} + \frac{2D^2\Psi^{(s)}}{R^2}\left(\frac{\beta}{1-\beta^2}\frac{\partial\Psi^{(s)}}{\partial R} + \frac{1}{R}\frac{\partial\Psi^{(s)}}{\partial\beta}\right) = \frac{1}{\mathcal{R}e}D^4\Psi^{(s)} , \tag{6.67}$$

where D^2 is the dimensionless operator corresponding to E^2, i.e.,

$$D^2 = \frac{\partial^2}{\partial R^2} + \frac{1-\beta^2}{R^2}\frac{\partial^2}{\partial\beta^2} . \tag{6.68}$$

We assume an expansion of the form

$$\Psi^{(s)} = \psi_0(R,\beta) + f_1(\mathcal{R}e)\psi_1(R,\beta) + f_2(\mathcal{R}e)\psi_2(R,\beta) + \dots , \tag{6.69}$$

where

$$f_1(\mathcal{R}e) \to 0, \quad \frac{f_{n+1}(\mathcal{R}e)}{f_n(\mathcal{R}e)} \to 0 \quad \text{as } \mathcal{R}e \to 0 . \tag{6.70}$$

The expansion (6.69) should be regarded as representing the exact solution $\Psi^{(s)}(R,\beta;\mathcal{R}e)$ for small values of $\mathcal{R}e$ at a fixed point in space. This involves the rather plausible assumption that there is no singular dependence on $\mathcal{R}e$ in the *finite* part of the flow field, such as gives rise to Whitehead's paradox by appearing at infinity.

The expansion (6.69) is required to satisfy the differential equation (6.67) and the no-slip condition on the sphere. Since the expansion is invalid at large values of R, the uniform stream condition at infinity cannot be applied; it is replaced by the requirement that the expansion be perfectly matched to an expansion which *is* valid in the outer region.

As Oseen pointed out, Whitehead's successive approximation scheme based on the Stokes solution breaks down because inertia and viscous effects become comparable at large values of R. The outer expansion should therefore be sought in terms of variables normalized in such a way that the Reynolds number does not appear in the governing equation, thereby reflecting explicitly the fact that all terms in the equation are of comparable magnitude. This can be done in many ways, but is simplest to use the **Oseen variables**

$$\lambda = \mathcal{R}e R = \frac{\mathcal{R}e\, r}{a} = \left(\frac{U}{\nu}\right) r \,,$$

$$\Psi^{(o)} = \mathcal{R}e^2 \Psi^{(s)} = \frac{\mathcal{R}e^2 \Psi}{a^2 U} = \left(\frac{U}{\nu^2}\right) \Psi \,, \tag{6.71}$$

in terms of which the governing equation (6.67) becomes

$$\frac{1}{\lambda^2} \frac{\partial(\Psi^{(o)}, L^2\Psi^{(o)})}{\partial(\lambda, \beta)} + \frac{2L^2\Psi^{(o)}}{\lambda^2} \left(\frac{\beta}{1-\beta^2} \frac{\partial \Psi^{(o)}}{\partial \lambda} + \frac{1}{\lambda} \frac{\partial \Psi^{(o)}}{\partial \beta} \right) = L^4 \Psi^{(o)} \,, \tag{6.72}$$

where

$$L^2 = \frac{\partial^2}{\partial \lambda^2} + \frac{1-\beta^2}{\lambda^2} \frac{\partial^2}{\partial \beta^2} \,. \tag{6.73}$$

The Oseen expansion is assumed to take the form

$$\Psi^{(o)} = \Psi_0(\lambda, \beta) + F_1(\mathcal{R}e)\Psi_1(\lambda, \beta) + F_2(\mathcal{R}e)\Psi_2(\lambda, \beta) + \ldots \,, \tag{6.74}$$

where

$$F_1(\mathcal{R}e) \to 0, \quad \frac{F_{n+1}(\mathcal{R}e)}{F_n(\mathcal{R}e)} \to 0 \quad \text{as } \mathcal{R}e \to 0 \,. \tag{6.75}$$

That the leading term in (6.74) should be independent of $\mathcal{R}e$ is implicit in the choice of the Oseen variables (6.71).

The expansion (6.74) is required to satisfy the differential equation (6.72) and the condition of uniform streaming at infinity. The no-slip condition on the sphere, which in the Oseen coordinates has shrunk to an infinitesimal sphere of radius $\mathcal{R}e$, is replaced by the condition that the Oseen expansion (6.74) should be matched to the Stokes expansion (6.69) at small values of λ. This procedure entails the assumption that the flow exhibits no singular dependence on λ and β as $\mathcal{R}e \to 0$, except possibly at $\lambda = 0$.

The Oseen expansion relates to the limiting process of $\mathcal{R}e$ approaching zero with λ and β fixed. For fixed values of U and ν, the first of Eqs. (6.71) reveals that

λ fixed corresponds to r fixed. Thus the limit process corresponds to considering the flow at a given point in space as the radius of the sphere shrinks to zero. As $\mathcal{R}e \to 0$, the Oseen coordinate λ becomes enormously stretched in comparison to the Stokes coordinate R. The matching procedure is therefore carried out by relating the behavior at zero in the stretched coordinate to the behavior at infinity in the unstretched coordinate. We require that the form of the Oseen expansion as $\lambda \to 0$ should be the same as the form of the Stokes expansion as $R \to \infty$.

If we make the fairly safe assumption that the disturbance caused by the sphere vanishes with its radius, we arrive at the conclusion that the leading term in the Oseen expansion corresponds to the uniform stream. Thus

$$\Psi_0 = \frac{1}{2}\lambda^2(1-\beta^2)\ . \tag{6.76}$$

We suspect that the leading term in the Stokes expansion is the classical Stokes solution itself, i.e., that

$$\psi_0 = \frac{1}{2}\left(\frac{1}{2R} - \frac{3R}{2} + R^2\right)(1-\beta^2)\ , \tag{6.77}$$

and this is easily verified. That (6.77) satisfies the no-slip condition at $R = 1$ is immediately evident; that it satisfies the differential equation (6.67), with neglect only of terms of order $\mathcal{R}e$, follows from the observation that $D^4\psi_0 = 0$. It remains only to show that (6.77) merges with (6.76) as R approaches infinity. To check this final point, we need only observe that

$$\psi_0 \sim \frac{1}{2}R^2(1-\beta^2) \quad \text{as } R \to \infty\ . \tag{6.78}$$

The second of Eqs. (6.71) stipulates that $\Psi^{(0)}$ and $\mathcal{R}e^2\Psi^{(s)}$ represent the same function. Since $\lambda = \mathcal{R}e\,R$, comparison of (6.76) and (6.78) reveals, as expected,

$$\psi_0 \sim \mathcal{R}e^{-2}\Psi_0 \quad \text{as } R \to \infty\ . \tag{6.79}$$

The second term in the outer expansion is closely related to the classical Oseen solution (6.62). If we express (6.62) in terms of the Oseen variables (6.71), we obtain

$$\Psi^{(o)}_{classical} = \frac{1}{2}\lambda^2(1-\beta^2) + \frac{1}{4}\mathcal{R}e^3\lambda^{-1}(1-\beta^2) - \frac{3}{2}\mathcal{R}e(1+\beta)(1-e^{-\frac{1}{2}\lambda(1-\beta)})\ . \tag{6.80}$$

The first term on the right side has already appeared as Ψ_0; the second term, involving as it does $\mathcal{R}e^3$, should not appear at this stage of the expansion. Thus, the evidence is strong that

$$F_1(\mathcal{R}e) = \mathcal{R}e, \quad \Psi_1(\lambda,\beta) = -\frac{3}{2}(1+\beta)(1-e^{-\frac{1}{2}\lambda(1-\beta)}) \,. \tag{6.81}$$

With $\Psi_1(\lambda,\beta)$ so defined, it is easily verified that $\Psi_0 + \mathcal{R}e\Psi_1$ satisfies the uniform stream condition at infinity and obeys the differential equation (6.72), with neglect only of terms of order $\mathcal{R}e^2$. Furthermore for small values of λ we have

$$\Psi_0 + \mathcal{R}e\Psi_1 = \frac{1}{2}\lambda^2(1-\beta^2) - \frac{3}{4}\mathcal{R}e[\lambda(1-\beta^2) + O(\lambda^2)] \,. \tag{6.82}$$

Expressing this result in terms of the Stokes variables gives

$$\mathcal{R}e^{-2}(\Psi_0 + \mathcal{R}e\Psi_1) = \frac{1}{2}(R^2 - \frac{3}{2}R)(1-\beta^2) + O(\mathcal{R}e) \,. \tag{6.83}$$

The term of order $\mathcal{R}e^0$ in this equation should be asymptotic to ψ_0 as $R \to \infty$. Examination of Eq. (6.77) reveals that this is the case. In fact not only do the R^2-terms have the same coefficient, but so do the R-terms; this coincidence is related to the fact that the assumed form of the classical Oseen solution reduces to the classical Stokes solution in the neighborhood of the sphere.

We are now in a position to proceed beyond the classical results by obtaining the second term in the Stokes expansion. With $\psi_0(R,\beta)$ given by Eq. (6.77), substitution of the expansion (6.69) into the differential equation (6.67) yields

$$f_1(\mathcal{R}e)D^4\psi_1 = \frac{9}{4}\mathcal{R}e\beta(\beta^2-1)\left(\frac{2}{R^2} - \frac{3}{R^3} + \frac{1}{R^5}\right) + O[\mathcal{R}e\, f_1(\mathcal{R}e)] \,. \tag{6.84}$$

The form of this equation tentatively suggests that we set

$$f_1(\mathcal{R}e) = \mathcal{R}e \,, \tag{6.85}$$

but we must allow for the possibility that the constants which arise in the integration of (6.84) may be functions of $\mathcal{R}e$. This possibility will not arise in the present calculation, but it does enter when higher order approximations are considered.

With (6.84) and (6.85), ψ_1 is required to satisfy the differential equation

$$D^4\psi_1 = \frac{9}{4}\beta(\beta^2-1)\left(\frac{2}{R^2} - \frac{3}{R^3} + \frac{1}{R^5}\right) \,, \tag{6.86}$$

subject to appropriate boundary conditions. If we measure the stream function from the axis of symmetry, then

$$\psi_1 = 0 \quad \text{for } \beta = \pm 1 \,. \tag{6.87}$$

On the sphere the no-slip condition applies. Thus

$$\frac{\partial \psi_1}{\partial R} = \frac{\partial \psi_1}{\partial \beta} = 0 \quad \text{on } R = 1\,. \tag{6.88}$$

The remaining condition on ψ_1 is the matching condition. As R approaches infinity, $\psi_0 + \mathcal{R}e\psi_1$ should be asymptotic to that part of the Oseen solution obtained so far, evaluated at small values of λ. Since terms in $\mathcal{R}e^0$ have already been matched, we must look one term beyond those explicitly set down in Eqs. (6.82) and (6.83). With (6.76) and (6.81), we have

$$\Psi_0+\mathcal{R}e\Psi_1 = \frac{1}{2}\lambda^2(1-\beta^2)-\frac{3}{4}\mathcal{R}e[\lambda(1-\beta^2)-\frac{1}{4}\lambda^2(1-\beta^2)(1-\beta)+O(\lambda^3)]\,. \tag{6.89}$$

Passing to the Stokes variables gives us

$$\mathcal{R}e^{-2}(\Psi_0+\mathcal{R}e\Psi_1) = \frac{1}{2}(R^2-\frac{3}{2}R)(1-\beta^2)+\frac{3}{16}\mathcal{R}e\,R^2(1-\beta^2)(1-\beta)+O(\mathcal{R}e^2)\,. \tag{6.90}$$

The matching condition on ψ_1 thus becomes

$$\psi_1(R,\beta) \sim \frac{3}{16}R^2(1-\beta^2)(1-\beta) \quad \text{as } R \to \infty\,. \tag{6.91}$$

A particular integral of (6.86) which satisfies all boundary conditions except the matching condition (6.91) is

$$\frac{3}{16}\beta(\beta^2-1)\left(\frac{1}{2R^2}-\frac{1}{2R}+\frac{1}{2}-\frac{3R}{2}+R^2\right)\,. \tag{6.92}$$

We must add to this a complementary function ψ_c which satisfies

$$\begin{aligned} D^4\psi_c &= 0\,,\\ \psi_c &= 0 \quad \text{for } \beta = \pm 1\,,\\ \frac{\partial \psi_c}{\partial R} = \frac{\partial \psi_c}{\partial \beta} &= 0 \quad \text{on } R = 1\,,\\ \psi_c &\sim \frac{3}{16}R^2(1-\beta^2) \quad \text{as } R \to \infty\,. \end{aligned} \tag{6.93}$$

If the factor $\frac{3}{16}$ in the last of these requirements were replaced by $\frac{1}{2}$, we would have the classical Stokes problem. Thus, there is no difficulty whatsoever in selecting the appropriate complementary function: we have

$$\psi_c = \frac{3}{8}\psi_0\,. \tag{6.94}$$

Adding this result to the particular integral (6.92) gives us

$$\psi_1 = \frac{3}{16}\left(\frac{1}{2R} - \frac{3R}{2} + R^2\right)(1-\beta^2) - \frac{3}{16}\left(\frac{1}{2R^2} - \frac{1}{2R} + \frac{1}{2} - \frac{3R}{2} + R^2\right)(1-\beta^2)\beta \,. \tag{6.95}$$

Because of a fortuitous combination of circumstances, it is quite easy to calculate the contribution of ψ_1 to the net drag on the sphere. First we note that the second term on the right side of (6.95) is an odd function of β. Consequently the flow field described by this term is a mirror image of itself with respect to the equatorial plane of the sphere. This part of the flow generates a drag-force in one hemisphere, a push-force in the other; the two precisely balance. Hence the only contribution of ψ_1 to the net drag arises from the first term on the right side of (6.95). But as we have already noted, this term is merely the classical Stokes result, multiplied by $\frac{3}{8}$. Since the net drag is in proportion, we have, to the present degree of approximation,

$$D = 6\pi\mu a U(1 + \frac{3}{8}\mathcal{R}e) \,. \tag{6.96}$$

The result (6.96) is also obtained if the Oseen equation (6.54), rather than the Navier-Stokes equation, is used as the starting point. That the correct result is thus obtained is often considered a remarkable stroke of luck, for the Oseen formulation neglects inertia in the neighborhood of the sphere,[5] and the appearance of the Reynolds number certainly indicates that the inertia contributes to (6.96). This "inverse paradox" is cleared away if we retrace the derivation of (6.96). The term in (6.95) which is odd in β is the particular integral (6.92), and hence represents the contribution of inertia in the neighborhood of the sphere, i.e., it results from the right side of (6.86). As we pointed out, this term does not contribute to the net drag. The term which does contribute is the complimentary function ψ_c. This term was calculated from the differential equation of creeping flow and was influenced by inertia only through the matching condition. Thus inertia enters the result (6.96) in a manner compatible with the assumptions used in deriving Oseen's equations.

The back-and-forth procedure we have been using can, in principle, be continued indefinitely. However we have now reached the point beyond which further calculations can be set out only through the extensive sacrifice of either brevity or lucidity. It does seem worthwhile, though, to quote the next order of approximation for the drag:

$$D = 6\pi\mu a U[1 + \frac{3}{8}\mathcal{R}e + \frac{9}{40}\mathcal{R}e^2 \ln \mathcal{R}e + O(\mathcal{R}e^2)] \,. \tag{6.97}$$

[5] In fact it does worse than neglect the inertia in this region: it misrepresents it. However the misrepresentation is also negligible.

The logarithmic term arises in the evaluation of constants of integration (note the reservations we expressed following Eq. (6.85)). For a quantitative outline of the analysis leading to (6.97), we refer the reader to Proudman and Pearson (1957). The logarithmic term does not appear if the Oseen equation is used instead of the Navier-Stokes equation. Goldstein's result (1960, p. 144) gives

$$D = 6\pi\mu a U[1 + \frac{3}{8}\mathcal{R}e - \frac{19}{320}\mathcal{R}e^2 + O(\mathcal{R}e^3)]\ .$$

As Goldstein himself pointed out, this approach gives valid results only as far as the term of order $\mathcal{R}e$.

A more accurate relation for the drag is proposed by Ockendon and Ockendon (1995):

$$D = 6\pi\mu a U[1 + \frac{3}{8}\mathcal{R}e + \frac{9}{40}\mathcal{R}e^2 \ln \mathcal{R}e + \frac{1}{40}(9\gamma + 15\ln 2 - \frac{323}{40})\mathcal{R}e^2 + \ldots]\ ,$$

where $\gamma = 0.5772156649$ is Euler's constant. After Chester et al. (1969), the next term in the drag is

$$\frac{27}{80}\mathcal{R}e^3 \ln \mathcal{R}e.$$

6.5 Flow Past Non-spherical Obstacles

Naturally enough, the researches of Stokes and Oseen on the flow past a sphere stimulated other work on the flow past obstacles. Research along this line has been given a more recent impetus with the emergence, in the late 1950s, of the matching technique described in Sect. 6.4. Unfortunately, when geometries less symmetric than the sphere are considered, the analysis becomes exceedingly complicated. To get involved with calculations of this sort would lead us far afield. Instead, we refer the interested reader to Rubinow and Keller (1961), Hasimoto (1956), Payne and Pell (1960), Pell and Payne (1960), Chester (1962), Brenner (1962), Carrier (1953), and Hill and Power (1956).

6.6 Stokeslets

In the theory of suspensions, it's useful to simplify the results for the steady motion of a sphere through a liquid at rest by letting the sphere become infinitesimal. See, for example, Brenner et al. (1990). If the sphere moves with velocity U_i, (6.37) indicates that the force it exerts on the fluid will be

$$F_i = -6\pi\mu a U_i\ . \tag{6.98}$$

With an appropriate translation and rotation of coordinates, (6.25) becomes

$$v_i = \left[\frac{1}{8\pi\mu}\left(\frac{F_i}{r} + \frac{x_i x_j F_j}{r^3}\right)\right] + \left[\frac{a^2}{8\pi\mu}\left(\frac{F_i}{3r^3} - \frac{x_i x_j F_j}{r^5}\right)\right], \tag{6.99}$$

where $r = (x_k x_k)^{1/2}$. Far from the sphere, the second term in brackets becomes insignificant compared with the first. If we disregard it, what remains is itself a solution to the Stokes equations, with a singularity at the origin.

This solution

$$v_i^S = \left[\frac{1}{8\pi\mu}\left(\frac{F_i}{r} + \frac{x_i x_j F_j}{r^3}\right)\right] \tag{6.100}$$

was termed a **Stokeslet** by Hancock (1953). It can be interpreted as the flow resulting from a delta-function point force of magnitude F^S acting at the origin:

$$-\frac{\partial p^S}{\partial x_i} + \mu \frac{\partial^2 v_i^S}{\partial x_j \partial x_j} = -F_i^S \delta(x_k), \tag{6.101}$$

where $\delta(\mathbf{x})$ is the Dirac function. The velocity (6.100) is induced by this forcing and may take the form

$$v_i^S = \frac{\mathcal{G}_{ij} F_j}{8\mu\pi}, \tag{6.102}$$

where the **Oseen-Burgers tensor**

$$\mathcal{G}_{ij} = \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}, \tag{6.103}$$

is in fact the Green's function for the Stokes equation. As such, since the Stokes equations are linear, it is the basis of boundary integral and singularity methods for building up solutions to boundary value problems. A detailed description, along with a copious bibliography, is provided in the book by Shankar (2007). Another source is provided by Guazzelli and Morris (2011).

6.6.1 Propulsion of Microorganisms

In the case of large animals, swimming and flying are accomplished almost entirely through inertial effects. In complete contrast, at the micrometer scale viscosity is the whole story. The whipping of the flagellum on a spermatozoon might be reminiscent of the flapping of a fish's tail, but there is no dynamical analogy, except that some kind of hydrodynamics is involved in both cases. The propulsion might best be described as swimming for the fish, squirming for the spermatozoon.

Many mechanisms exist for the propulsion of microorganisms by viscous effects. These include, but are not limited to: pseudopod extension achieved by internal cytoplasmic streaming; cilia beating, random in some cases, organized in others; flagellar motion, the most extensively studied case, sometimes involving a single flagellum, sometimes more.

For flagellar motion there are two important subcases. Sometimes, often in the case of bacteria, the flagellum essentially rotates as a rigid body, propelled by a biochemical motor at its base. In the case of eukaryotes (organisms whose cells incorporate membrane-bound organelles), waves propagate along the flagellum.

Early studies of motion propelled by flagella, for example Gray and Hancock (1955), have been termed *flagellar dynamics*. They achieved a certain degree of success, despite the use of local resistance coefficients, based on the oversimplified assumption that the fluid resists any local movement with a local force proportional to the velocity.

In a now classic paper, Lighthill (1976), Michael James Lighthill (1924–1998) initiated a true flagellar *hydrodynamics*. The dynamical problem involves simultaneous treatment of a pair of conditions:

1. With the insignificance of inertial effects, the translational motion of the microorganism, combined with whatever undulatory motions it might make, will generate zero resultant force on the body.
2. A system of forces with zero resultant might still have a net moment, making it necessary to determine both translation and rotation vectors for the organism such that the forces between the body and the fluid produce a system with zero resultant and zero moment.

Lighthill's treatment is far too detailed and extensive to be replicated here, but it hinges on a simple (a posteriori!) idea embodied in his "Theorem 6.1":

Theorem 6.1. *If $f(s)$ is the force per unit length with which a flagellum of radius a acts on a fluid, where the variable s signifies length measured along the centerline of the flagellum from some given cross section, then the resulting fluid motion can be represented by a distribution of Stokeslets along the centerline, of strength $f(s)$ per unit length, accompanied by dipoles of strength $-a^2 f_n(s)/(4\mu)$ per unit length; here $f_n(s)$ is the vector normal to the centerline obtained by resolving $f(s)$ onto planes normal to the centerline.*

The strengths of the Stokelets and dipoles are obtained by solving an integral equation representing the no-slip condition on the surface of the flagellum.

Subsequent developments can be found in a paper published by Lighthill (1996) shortly before his death[6] and in a subsequent paper by Purcell (1997).

[6]Lighthill was an accomplished open-water swimmer, believed to be the first to complete the 14 km swim around Sark. He first did it at age 49, and repeated it five times subsequently. He had nearly completed his seventh attempt, at age 74, when he incurred a fatal rupture of his mitral valve.

Chapter 7
Plane Flow

Abstract Complex variable methods for treating plane flow are discussed, including a treatment of the Stokes paradox. Approximation methods for the flow in a channel of varying width are presented. Hele-Shaw flow is treated in detail.

The neglect of inertia in continuum mechanics is sometimes criticized with surprising contentiousness. From one point of view the objection is understandable: we have already seen that the promiscuous neglect of inertia in the Stokes problem leads to inconsistent results. For some reason, however, other examples of this sort of thing are treated more tolerantly. It is often postulated that the material under consideration is incompressible—leading to the conclusion that disturbances propagate with infinite speed; one sometimes defines an "inviscid fluid"—which cannot, in general, satisfy the no-slip condition; it is not uncommon to assume that the flow field is isothermal—the heat generated by viscous dissipation being whisked off to hell by a Maxwellian demon. Yet postulates that the fluid is incompressible, or inviscid, or isothermal are seldom greeted with apoplectic revulsion. Rightly so: correctly handled, these postulates can lead to accurate and useful approximations of the way real materials behave under properly delineated circumstances, and do so through the development of analytical methods quite interesting in themselves. Moreover, when a postulate leads unexpectedly to anomalous results, a study of the paradox often deepens our understanding of the mathematical structure of fluid dynamics and of the underlying physics.

In like manner, to postulate that the fluid is massless does not automatically brand one as a hydrodynamical sophist. From an abstract standpoint, it is of no concern whether the hypothetical material exists in nature—the analysis provides its own justification. More pragmatically, we suspect that under some circumstances the flow of a real fluid can be approximated by the flow of a massless fluid. Needless to say, neither viewpoint justifies the use of creeping flow theory when inertia is obviously important, or when paradoxes appear; bad analysis remains bad analysis.

In the present chapter we shall study the general subject of creeping flow in the plane. This subject includes the well-known **Stokes paradox**—the non-existence of

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_7,

a Stokes solution for flow past a circular cylinder with uniform streaming at infinity. Therefore much of our treatment will be directed toward the task of deciding when creeping flow theory is more than a mathematical exercise.

The first four sections of this chapter deal with the use of complex variable methods in creeping flow. The results of these sections are not used elsewhere in the book; hence readers with no background in complex variable theory can pick up the thread at Sect. 7.5.

7.1 Description of Plane Creeping Flow in Terms of Complex Potentials

In Sect. 6.1, we recalled that if the velocity component v_3 is independent of x_3, then the velocity components v_1 and v_2 can be expressed in terms of a stream function. To reduce the number of subscripts, we set

$$x_1 = x, \quad x_2 = y, \quad v_1 = u, \quad v_2 = v\ . \tag{7.1}$$

We then have

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}\ , \tag{7.2}$$

where, for creeping flow, the stream function ψ satisfies the biharmonic equation

$$\nabla^4 \psi = 0\ . \tag{7.3}$$

In Eq. (7.3) and throughout this chapter, ∇^2 denotes the two-dimensional Laplacian, i.e.,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\ . \tag{7.4}$$

An immediate consequence of (7.3) is that the function $(\nabla^2 \psi)$ is harmonic. However

$$\begin{aligned} \nabla^2 \psi &= \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \\ &= -\xi\ , \end{aligned} \tag{7.5}$$

where ξ denotes the component of vorticity in the x_3 direction, i.e., the component normal to the plane of flow. Since $\nabla^2 \psi$ is harmonic, so is ξ.

We also saw in Sect. 6.1, that for creeping flow the pressure obeys Laplace's equation in three dimensions. When v_3 is independent of x_3, Eq. (6.2) reveals that $\partial^2 p / \partial x_3^2 = 0$. Hence,

$$\nabla^2 p = 0 \, , \tag{7.6}$$

where, as indicated above, ∇^2 is the two-dimensional Laplacian.

From (7.5) we observe that

$$\frac{\partial \xi}{\partial x} = \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} \, . \tag{7.7}$$

However, since $\partial v_3/\partial x_3 = 0$, the continuity equation (6.1) gives us

$$-\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \, , \tag{7.8}$$

so that

$$\frac{\partial \xi}{\partial x} = \nabla^2 v \, . \tag{7.9}$$

In a similar manner we find

$$\frac{\partial \xi}{\partial y} = -\nabla^2 u \, . \tag{7.10}$$

Introducing (6.2) into (7.9) and (7.10), we find that ξ and (p/μ) satisfy the Cauchy-Riemann equations

$$\frac{\partial \xi}{\partial x} = \frac{\partial (p/\mu)}{\partial y}, \quad \frac{\partial \xi}{\partial y} = -\frac{\partial (p/\mu)}{\partial x} \, . \tag{7.11}$$

Thus, ξ and p/μ are conjugate harmonic functions, so that they represent the real and imaginary parts of an analytic function of $z = x + iy$.

Let $4\phi(z) = 4g(x, y) + 4i\tilde{g}(x, y)$ be one of the infinitely many analytic functions whose derivative is $\xi + ip/\mu$. Further let

$$G(x, y) = xg(x, y) + y\tilde{g}(x, y) + \psi(x, y) \, . \tag{7.12}$$

We then have

$$\begin{aligned} \nabla^2 G &= x\nabla^2 g + y\nabla^2 \tilde{g} + 2\left(\frac{\partial g}{\partial x} + \frac{\partial \tilde{g}}{\partial y}\right) + \nabla^2 \psi \\ &= 4\frac{\partial g}{\partial x} + \nabla^2 \psi = 4\Re\phi'(z) + \nabla^2 \psi = \xi + \nabla^2 \psi = 0 \, , \end{aligned} \tag{7.13}$$

so that G is harmonic. If we let $\chi(z)$ be the analytic function whose real part is $-G(x, y)$ Eq. (7.12) gives us

$$\psi(x, y) = -\Re[\chi(z)] - [xg(x, y) + y\tilde{g}(x, y)] \, . \tag{7.14}$$

However $xg + y\tilde{g} = \Re[\bar{z}\phi(z)]$, so that

$$\psi(x, y) = -\Re[\bar{z}\phi(z) + \chi(z)] \,. \tag{7.15}$$

Thus we have $\psi(x, y)$ expressed in terms of a pair of **complex potentials** $\phi(z)$ and $\chi(z)$. Moreover this followed entirely from the fact that ψ is biharmonic; that $\nabla^2\psi$ and its harmonic conjugate have direct physical significance is purely incidental. Conversely, it is easily verified that any expression of the form (7.15) is biharmonic, provided only that ϕ and χ are analytic functions.

There is a certain degree of arbitrariness in the selection of the complex potentials. It is a straightforward matter to show that, if we *simultaneously* replace ϕ by $\phi + \tilde{\phi}$ and χ by $\chi + \tilde{\chi}$, where

$$\tilde{\phi} = iaz + c, \quad \tilde{\chi} = k - \bar{c}z, \tag{7.16}$$

in which a is a real constant and c, k are complex constants, then the value of ψ is changed only by the additive constant $-\Re k$. As usual, an additive constant in a stream function is irrelevant.

The velocity components can be expressed directly in terms of the complex potentials. With Eqs. (7.2),

$$i(u + iv) = \frac{\partial\psi}{\partial x} + i\frac{\partial\psi}{\partial y} \,. \tag{7.17}$$

Calculating the derivatives of ψ from Eq. (7.15) gives[1]

$$\begin{aligned} -i(u + iv) &= \phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)} \\ &\stackrel{def}{=} \Lambda(z, \bar{z}) \,. \end{aligned} \tag{7.18}$$

---

[1]The value of the derivative of an analytic function does not depend upon the direction of differentiation. If $f(z) = R(x, y) + iI(x, y)$, differentiating parallel to the x-axis gives

$$f'(z) = \frac{\partial R}{\partial x} + i\frac{\partial I}{\partial x}$$

and differentiating parallel to the y-axis gives

$$f'(z) = -i\left(\frac{\partial R}{\partial y} + i\frac{\partial I}{\partial y}\right) .$$

This, of course, is how the Cauchy-Riemann equations are established. In the matter at hand,

$$\frac{\partial R}{\partial x} + i\frac{\partial R}{\partial y} = \frac{\partial R}{\partial x} - i\frac{\partial I}{\partial x} = \overline{f'(z)} \,.$$

The $\overline{\chi'(z)}$ term in (7.18) follows immediately; the remaining terms are easily obtained, but a bit of care must be exercised, for $\bar{z}$ is not an analytic function of z.

In a similar way, the stress components can be expressed in terms of ϕ and χ. Since $p = 4\mu\Im[\phi'(z)]$, we have

$$\begin{aligned}
T_{11} &= -p + 2\mu\frac{\partial u}{\partial x} \\
&= -4\mu\Im[\phi'(z)] - 2\mu\frac{\partial}{\partial x}\Im[\phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)}] \\
&= -2\mu\Im[2\phi'(z) - \bar{z}\phi''(z) - \chi''(z)]\ , \\
T_{22} &= -p + 2\mu\frac{\partial v}{\partial y} = -p - 2\mu\frac{\partial u}{\partial x} \\
&= -2\mu\Im[2\phi'(z) + \bar{z}\phi''(z) + \chi''(z)]\ , \\
T_{12} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&= \mu\frac{\partial}{\partial x}\Re[\Lambda(z,\bar{z})] - \mu\frac{\partial}{\partial y}\Im[\Lambda(z,\bar{z})] \\
&= 2\mu\Re[\bar{z}\phi''(z) + \chi''(z)]\ .
\end{aligned} \tag{7.19}$$

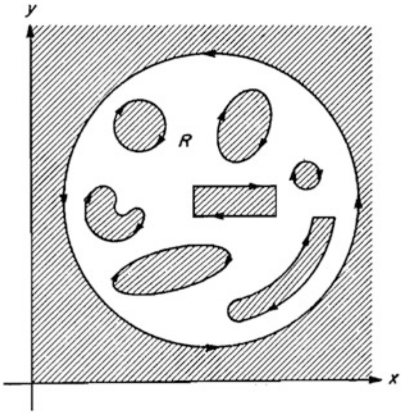
As we can see, the solution of a problem in creeping viscous flow reduces to the selection of two appropriate analytic functions to serve as the complex potentials. On solid boundaries the velocity components are specified; hence, Eq. (7.18) provides a boundary condition for ϕ and χ. If the velocity at infinity is specified, (7.18) governs the limiting behavior of the potentials. Even if there are free surfaces or fluid interfaces, the boundary conditions on the surface tractions can, by use of Eqs. (7.19) be converted to conditions on ϕ and χ; in this case, however, the location of the boundaries is not necessarily known a priori. The extension to free-surface flows was pointed out by Moisil (1955).

A sidelight is worth noting. The flow is irrotational, so that $\partial u/\partial y = \partial v/\partial x$, if and only if $\partial(\Re\bar{\Lambda})/\partial x = \partial(\Im\bar{\Lambda})/\partial y$, which is one of the Cauchy-Riemann equations. By (7.18), $\bar{\Lambda} = v + iu$, and the incompressibility of the fluid guarantees that the real and imaginary parts of $\bar{\Lambda}$ satisfy the second Cauchy-Riemann equation. Therefore when the flow is irrotational, and only when it is so, the complex velocity $v + iu$ is an analytic function of $z = x + iy$—a fact which finds widespread application in the theory of inviscid flow.

7.2 The Uniqueness Theorem for Creeping Flows in Bounded Regions

If a massless fluid is contained in a finite region bounded only by stationary solid walls, we expect, on purely physical grounds, that the fluid is at rest: no energy is supplied to balance any losses which may arise from viscous dissipation. This

Fig. 7.1 Geometrical configuration of region **R**



argument assumes that the flow field is free from sources, sinks, doublets, and other discontinuities. By considering the velocity potentials, we can confirm that this is indeed the case.

We shall need to use the complex variable form of **Stokes' theorem** relating contour integrals to double integrals. The theorem[2] states that if $f(z, \bar{z})$ is differentiable in the region **R**, which may be simply or multiply connected, and if the boundary of **R** is the piecewise smooth contour **C**, then

$$\oint_{\mathbf{C}} f(z, \bar{z}) dz = 2i \iint_{\mathbf{R}} \frac{\partial f}{\partial \bar{z}} \mathbf{dR}; \quad \oint_{\mathbf{C}} f(z, \bar{z}) d\bar{z} = -2i \iint_{\mathbf{R}} \frac{\partial f}{\partial z} \mathbf{dR} . \tag{7.20}$$

The cyclic integrals over various parts of **C** are taken in the direction which keeps the region **R** on the left.

Referring to Fig. 7.1, let us now suppose that a region **R** is bounded externally by a closed contour smooth enough for Stokes' theorem to apply. We also permit **R** to be bounded internally by a collection of piecewise-smooth, closed contours. We assume that these external and internal closed contours complete the boundary **C**, so that Stokes' theorem applies to the region **R**.

Let

$$\begin{aligned} f_1(z, \bar{z}) &= \phi'(z)\overline{\Lambda(z, \bar{z})} \\ &= \phi'(z)[\overline{\phi(z)} + \bar{z}\phi'(z) + \chi'(z)] . \end{aligned} \tag{7.21}$$

Since

$$\frac{\partial f_1}{\partial \bar{z}} = \phi'(z)[\phi'(z) + \overline{\phi'(z)}] = 2\phi'(z)\Re\phi'(z) , \tag{7.22}$$

[2]For a proof see Milne-Thomson (1996), p. 130.

the first half of Stokes' theorem gives

$$4i \iint\limits_{\mathbf{R}} \phi'(z)\Re\phi'(z)\mathbf{dR} = \oint_{\mathbf{C}} \phi'(z)\overline{\Lambda(z,\bar{z})}dz\,. \tag{7.23}$$

If all portions of the boundary are at rest, Eq. (7.18) implies that $\Lambda(z,\bar{z})$ vanishes on **C**. Equation (7.23) then reduces to

$$\iint\limits_{\mathbf{R}} \phi'(z)\Re\phi'(z)\mathbf{dR} = 0\,. \tag{7.24}$$

In like manner, if we let

$$f_2(z,\bar{z}) = \overline{\phi'(z)}\Lambda(z,\bar{z})\;, \tag{7.25}$$

so that

$$\frac{\partial f_2}{\partial z} = 2\overline{\phi'(z)}\Re\phi'(z)\;, \tag{7.26}$$

the second half of Stokes theorem gives, if all boundaries are at rest,

$$\iint\limits_{\mathbf{R}} \overline{\phi'(z)}\Re\phi'(z)\mathbf{dR} = 0\,. \tag{7.27}$$

Adding (7.24) and (7.27) yields

$$2\iint\limits_{\mathbf{R}} [\Re\phi'(z)]^2\mathbf{dR} = 0\,. \tag{7.28}$$

Since $[\Re\phi'(z)]^2$ is everywhere non-negative, (7.28) implies that $[\Re\phi'(z)] = 0$. Thus, since ϕ(z) is analytic,

$$\phi(z) = aiz + c\;, \tag{7.29}$$

where a is a real constant and c is a complex constant.

Substituting (7.29) into (7.18) gives

$$\Lambda(z,\bar{z}) = c + \overline{\chi'(z)}\,. \tag{7.30}$$

As already noted, Λ vanishes on **C**. Hence $\chi'(z) = -\bar{c}$ on **C**. However χ is analytic in **R** and hence so is χ'. By the identity theorem for analytic functions, an analytic function which is constant on the periphery of a region is constant throughout the region. Hence

$$\chi(z) = k - \bar{c}z \tag{7.31}$$

throughout **R**. Substituting (7.29) and (7.31) into (7.18) yields

$$u + iv = 0\,. \tag{7.32}$$

We thus see that if all portions of the boundary **C** are at rest, the fluid everywhere in **R** is at rest. Moreover time considerations never enter the picture. For time-dependent motion we have quite generally: the fluid motion vanishes throughout **R** at any instant when all portions of **C** are at rest. Physically, motion in a real fluid persists after the boundaries stop only because of fluid inertia. For extremely viscous fluids, the residual motion is almost nonexistent. The problems of a dog with a big piece of caramel are not unfamiliar.

Since the equations governing creeping viscous flow are linear, the result derived above immediately generalizes to a uniqueness theorem for creeping viscous flow in a bounded region. Let us suppose that at some instant of time the boundary **C** is not completely at rest. If ψ and $\hat{\psi}$ are two functions, biharmonic in **R**, which satisfy the instantaneous boundary conditions on **C**, their difference is biharmonic in **R** and has vanishing derivatives on **C**. Thus $\psi - \hat{\psi}$ is the stream function for creeping flow in a bounded region with fixed boundaries; as we have seen, the state of rest is all that can exist in such a region. Hence ψ and $\hat{\psi}$ differ by at most an additive constant.

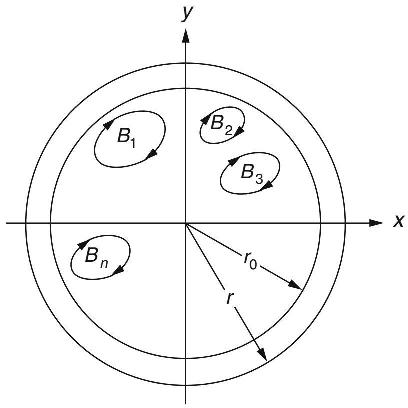
Ladyzhenskaya (1969) is a useful source of information on existence and uniqueness theorems for viscous flow at finite Reynolds number.

7.3 The Stokes Paradox

The approach used in Sect. 6.2, to calculate the creeping flow past a sphere, cannot be modified to treat successfully the flow past a circular cylinder: it leads to a velocity field with a logarithmic singularity at infinity.

The nonexistence of a solution for the creeping flow past a cylinder is not too easy to accept. One might argue that the neglected inertia terms provide a mechanism for momentum exchange far from the sphere which makes possible a solution with uniform streaming at infinity: we did observe that inertia becomes dominant at large distances in the flow past the sphere. This argument is valid at small but finite Reynolds number. However it begs a more persistent question: What happens at *zero* Reynolds number? If we are willing to postulate a massless fluid, we somehow feel that this hypothetical material should behave decently for us. However, this is pushing our intuition beyond its bounds. Without the mitigating effect of at least a slight bit of inertia, we find that dragging a cylinder through a fluid entails dragging

Fig. 7.2 Array of obstacles



all the fluid with it. Stokes himself gave the following physical picture of motion starting from rest:

The pressure of a cylinder on the fluid continually tends to increase the quantity of fluid which it carries with it, while the friction of the fluid at a distance from the cylinder continually tends to diminish it. In the case of a sphere, these two causes eventually counteract each other, and the motion becomes uniform. But in the case of a cylinder, the increase in the quantity of fluid carried continually gains on the decrease due to the friction of the surrounding fluid, and the quantity carried increases indefinitely as the cylinder moves on.

Although basically reasonable, this traditional explanation is not completely precise. As we pointed out in Sect. 7.2, time plays no role in the equations of creeping viscous flow. If we set the density of the fluid equal to zero, the $\partial/\partial t$ term in the Navier-Stokes equation disappears along with the non-linear term. Thus creeping viscous flow is *quasi-static*: time enters only through the boundary conditions; at any specific instant of time, only spatial variation of the flow variables matters in a boundary value problem. Thus in Stokes' explanation of his paradox the reference to what the flow does "eventually" is really irrelevant. The paradox sets in as soon as the cylinder begins to move.

Our inability to produce a bounded solution for the flow past a cylinder does not itself prove that a paradox really exists. One might conjecture that there is a bounded solution which has so far eluded discovery. We shall presently see that this is not the case.

One feels intuitively that, if the paradox exists, it should not be confined to the special case of a circular cylinder. Let us consider in Fig. 7.2 the general problem of creeping flow in the presence of an array of stationary obstacles $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$, all of which lie in the bounded region $|z| < r_0$ and all of whose boundaries are smooth enough for Stokes' theorem to hold.

We shall see that the state of rest is the only solution to the creeping flow equations which does not have a logarithmic (or stronger) singularity at infinity.

The investigation proceeds in somewhat the same way as it did for the case of bounded flow. The principal difference, of course, is that we now have no outer boundary upon which the velocity components are specified. Instead we enclose the obstacles in a circle of radius r, centered at the origin, and investigate asymptotic behavior as $r \to \infty$.

The complex potentials are analytic throughout the flow field, but inside the obstacles there may be singularities. In particular there may be branch points. Because of this possibility, $\phi(z)$ and $\chi(z)$ may be multiple-valued functions.[3]

Let us assume that, along any path circling the obstacle $\mathbf{B}_k$ precisely once in the counterclockwise sense, and enclosing no other obstacle, the function $\phi(z)$ increases by $2\pi i A_k$, where A_k is a complex constant. We can represent this jump by including in $\phi(z)$ a term of the form $A_k \ln(z - z_k)$, where z_k lies within $\mathbf{B}_k$. Thus

$$\phi(z) = \phi_1(z) + \sum_{k=1}^{n} A_k \ln(z - z_k) \; , \tag{7.33}$$

where $\phi_1(z)$ is single-valued in the region of flow.

Although $\phi(z)$ and $\chi(z)$ may be multiple-valued, the functions $\phi'(z)$ and $\Lambda(z, \bar{z})$ cannot be, for they represent, respectively, $\frac{1}{4}(\xi + ip/\mu)$ and $-i(u + iv)$—quantities with physical meaning. Consequently introducing (7.33) into (7.18), and noting that $\overline{\ln(z - z_k)}$ increases by $-2\pi i$ as we circle $\mathbf{B}_k$, yields

$$\chi'(z) = \Phi(z) + \sum_{k=1}^{n} \bar{A}_k \ln(z - z_k) \; , \tag{7.34}$$

where $\Phi(z)$ is single-valued in the region of flow. Integrating (7.34) gives

$$\chi(z) = \chi_1(z) + \sum_{k=1}^{n} \bar{A}_k (z - z_k) \ln(z - z_k) \; , \tag{7.35}$$

where $\chi_1(z)$ is a single-valued function related to $\Phi(z)$ according to

$$\Phi(z) = \chi_1'(z) + \sum_{k=1}^{n} \bar{A}_k \; . \tag{7.36}$$

Let us now examine the behavior of the complex potentials far from the obstacles. We may write

$$\ln(z - z_k) = \ln z + \ln\left(1 - \frac{z_k}{z}\right) \; . \tag{7.37}$$

[3] The term "multiple-valued function" is used as in Knopp (1945). Those who find the term distasteful should have no difficulty in recasting the forthcoming arguments in terms of Riemann sheets, except that the phrasing gets awkward in spots.

Since $\ln(1 - z_k/z)$ is analytic and single-valued everywhere outside $|z| = r_0$, Eqs. (7.33) through (7.35) can be rewritten

$$\phi(z) = \phi_0(z) + A \ln z \,, \tag{7.38}$$

$$\chi'(z) = \Phi_0(z) + \bar{A} \ln z \,, \tag{7.39}$$

$$\chi(z) = \chi_0(z) + (\bar{A}z - B) \ln z \,, \tag{7.40}$$

where $\phi_0(z)$, $\Phi_0(z)$ and $\chi_0(z)$ are analytic and single-valued for $|z| > r_0$, and the complex constants A, B are defined by

$$A = \sum_{k=1}^{n} A_k, \quad B = \sum_{k=1}^{n} \bar{A}_k z_k \,. \tag{7.41}$$

The functions $\Phi_0(z)$ and $\chi_0(z)$ are related by

$$\Phi_0(z) = \chi_0'(z) + \bar{A} - \frac{B}{z} \,. \tag{7.42}$$

Substituting (7.38) and (7.39) into (7.18) yields

$$-i(u + iv) = \phi_0(z) + z\overline{\phi_0'(z)} + \overline{\Phi_0(z)} + \frac{\bar{A}z}{\bar{z}} + 2A \ln |z| \,, \tag{7.43}$$

which holds for $|z| > r_0$. We now introduce the key assumption

$$u + iv = o(\ln |z|) \quad \text{as } z \to \infty \,. \tag{7.44}$$

Since $\phi_0(z)$ and $\Phi_0(z)$ are analytic and single-valued outside the circle $|z| = r_0$, neither of these functions has a branch point at infinity. Hence the grouping $[\phi_0(z) + z\overline{\phi_0'(z)} + \overline{\Phi_0(z)} + Az/\bar{z}]$ in Eq. (7.43) has no logarithmic infinity to cancel $2\bar{A} \ln |z|$. Therefore the assumption (7.44) entails

$$A = 0. \tag{7.45}$$

Equations (7.38) and (7.39) then give

$$\phi(z) = \phi_0(z), \quad \chi'(z) = \Phi_0(z) \,. \tag{7.46}$$

If we express $\phi_0(z)$ and $\Phi_0(z)$ as the Laurent series

$$\phi_0(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \quad \Phi_0(z) = \sum_{j=-\infty}^{\infty} b_j z^j \,, \tag{7.47}$$

substitution into Eq. (7.43), with $A = 0$, yields

$$-i(u + iv) = \sum_{j=-\infty}^{\infty} [(a_j z^{j-1} + j\bar{a}_j \bar{z}^{j-1})z + \bar{b}_j \bar{z}^j] . \tag{7.48}$$

The assumption (7.44) then requires that

$$\begin{aligned} b_j &= 0, \quad j \geq 1, \\ a_j &= 0, \quad j \geq 2, \\ a_1 + \bar{a}_1 &= 0. \end{aligned} \tag{7.49}$$

The last of these conditions can be rephrased as

$$a_1 = i\alpha , \tag{7.50}$$

where α is real.

Since $\phi(z) = \phi_0(z),\ \chi'(z) = \Phi_0(z)$ outside the circle of radius r_0, we may write

$$\phi(z) = i\alpha z + a_0 + \sum_{j=1}^{\infty} a_{-j} z^{-j} , \quad |z| > r_0 , \tag{7.51}$$

$$\chi(z) = b + b_0 z + b_{-1} \ln z - \sum_{j=2}^{\infty} \frac{b_{-j}}{j-1} z^{-j+1} , \quad |z| > r_0 , \tag{7.52}$$

where b is a complex constant of integration. Comparing with (7.40), we see that

$$b_{-1} = -B . \tag{7.53}$$

To get the proper asymptotic behavior, we now introduce modified complex potentials $\hat{\phi}(z)$ and $\hat{\chi}(z)$, defined by

$$\hat{\phi}(z) = \phi(z) - i\alpha z - a_0 , \tag{7.54}$$

$$\hat{\chi}(z) = \chi(z) + \bar{a}_0 z . \tag{7.55}$$

As indicated in Sect. 7.1, the potential pairs (ϕ, χ) and $(\hat{\phi}, \hat{\chi})$ describe the same flow field. Thus,

$$\begin{aligned} -i(u + iv) &= \hat{\phi}(z) + z\overline{\hat{\phi}'(z)} + \overline{\hat{\chi}'(z)} \\ &\stackrel{\text{def}}{=} \hat{\Lambda}(z, \bar{z}) . \end{aligned} \tag{7.56}$$

Equations (7.51) and (7.52) reveal that, for $|z| \to \infty$,

$$\begin{aligned} \hat{\phi}(z) &= O(z^{-1}) \,, \\ \hat{\phi}'(z) &= O(z^{-2}) \,, \\ \hat{\chi}'(z) &= \bar{a}_0 + b_0 + O(z^{-1}) \,, \\ \hat{\Lambda}(z, \bar{z}) &= a_0 + \bar{b}_0 + O(z^{-1}) \,. \end{aligned} \tag{7.57}$$

Consider now the region of flow bounded internally by the obstacles and bounded externally by the circle $|z| = r > r_0$. If we let

$$f_1(z, \bar{z}) = \hat{\phi}'(z)\overline{\hat{\Lambda}(z, \bar{z})}, \quad f_2(z, \bar{z}) = \overline{\hat{\phi}'(z)}\hat{\Lambda}(z, \bar{z}) \,, \tag{7.58}$$

then Eqs. (7.57) tell us that $f_1(z, \bar{z})$ and $f_2(z, \bar{z})$ are both $O(z^{-2})$ as $z \to \infty$. Consequently their integrals around $|z| = r$ are $O(r^{-1})$ as $r \to \infty$.

The arguments of Sect. 7.2 can now be applied *mutatis mutandis*. The functions $f_1(z, \bar{z})$, $f_2(z, \bar{z})$ vanish identically on the internal boundaries, and their integrals around the ficticious outer boundary can be made arbitrarily small in magnitude by choosing r large enough. Stokes' theorem can then be used to show that the magnitude of the complex velocity is less than any preassigned positive quantity, i.e., the fluid is at rest.

Thus, if the fluid is bounded internally by stationary obstacles and is unbounded externally, it is at rest unless the assumption (7.44) is violated. Consequently any streaming past a configuration of obstacles is, of necessity, accompanied by a logarithmic (or stronger) singularity at infinity.

The establishment of Stokes' paradox can also be taken as a uniqueness proof for flow generated by motion of the obstacles. If we find one solution to the equations of creeping flow which produces the given velocity components on all the solid boundaries, then any other solution obeying these same boundary conditions is logarithmically singular at infinity.

Stokes' paradox does not necessarily obtain if the flow field is only partially bounded at infinity. For example, it is possible to calculate a physically reasonable solution for the creeping flow past a cylinder situated between a pair of infinite, parallel walls, see Bairstow et al. (1922). In this problem, and in other problems involving what might be termed **internal flows**, it is physically meaningful to specify a finite pressure gradient in the distant portions of the flow field, cf. Fig. 7.3. Since $\phi'(z) = \frac{1}{4}(\xi + ip/\mu)$, the key arguments concerning the asymptotic behavior of ϕ' do not apply.

The uniqueness proofs set out in Sects. 7.2 and 7.3 are taken from Krakowski and Charnes (1951), which extends the results to a wider class of obstacles. The boundary is no longer required to consist of piecewise-smooth, closed contours; it is sufficient that, in a properly defined sense, they can be uniformly approximated by a family of such contours. For example, a line-segment can be considered as the limit of a family of nested ellipses, major axis fixed, minor axis shrinking to zero.

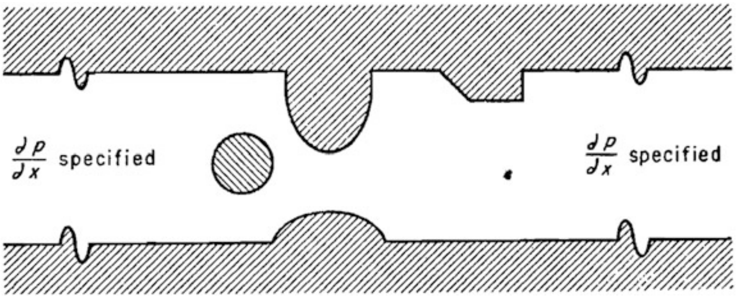


Fig. 7.3 Internal flow with specified pressure gradient

Expressions for the forces and moments on an obstacle in terms of the complex potentials are also included.

An alternate method of establishing the Stokes paradox is presented in Finn and Noll (1957). Complex variable methods are not used. Instead a series of lemmas dealing with the asymptotic behavior of harmonic functions is proved; Stokes' paradox follows from applying these lemmas to the vorticity. A uniqueness theorem for three-dimensional flow is also proved. It is shown that for any piecewise smooth obstacle there is at most one three-dimensional flow past the obstacle which has a prescribed uniform velocity at infinity.

Kaplun (1957) applied the matching procedure illustrated in Sect. 6.4 to the more difficult problem of flow past a circular cylinder.

In a significant paper, Chang (1961) treated the general problem of two-dimensional flow at large distance from an object of finite size moving through a viscous fluid, assuming neither large nor small Reynolds number. The paper develops asymptotic expansions of the velocity and pressure, valid at large distances from the body, for a fixed but arbitrary value of the Reynolds number. The expansion parameter is a dimensionless distance from the obstacle. Terms of logarithmic order appear in the expansion, a phenomenon which the author named **switchback** and discussed extensively. The flow field far from an object gives rise to **Filon's paradox**: the second approximation predicts infinite torque on unsymmetrical bodies. The paper provides a systematic way of showing that higher order terms in the expansion are necessary and sufficient to keep the torque finite.

7.4 Conformal Mapping and Biharmonic Flow

If the boundary configuration is such that the Stokes paradox results, there is little more to be said about creeping flow; we must resort to some technique, such as Oseen's, which takes at least partial account of inertia. For internal flows, however, creeping flow may give a good physical approximation, so that it is of interest to determine the complex potentials.

Except for the most degenerate geometries, the selection of the potentials is an exceedingly difficult task. Since the problem has been cast in terms of analytic functions of a complex variable, we might look to conformal mapping methods for some help. These do indeed simplify the problem, but one should be forewarned that, unlike the case of the Dirichlet problem, finding a conformal mapping only carries us part way to the solution. Fundamentally the difference is rooted in the fact that the Laplace's equation is preserved under conformal mapping, whereas the biharmonic equation is not.

Let us suppose that the conformal transformation

$$z = m(w), \quad w = m^{-1}(z) \tag{7.59}$$

maps the flow region **R** into a (presumably simpler) region **S** in the $w = X + iY$ plane. Since the mapping is conformal, the functions m and m^{-1} are analytic. Hence X and Y, considered as functions of x and y, satisfy the Cauchy-Riemann equations

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x} . \tag{7.60}$$

It then follows by a straightforward calculation that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left[\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2\right]\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) . \tag{7.61}$$

Thus, if ∇^2_{xy} and ∇^2_{XY} denote the Laplacian before and after transformation, we have

$$\begin{aligned} \nabla^2_{xy} &= \left|\frac{dw}{dz}\right|^2 \nabla^2_{XY} \\ &= \left|\frac{dz}{dw}\right|^{-2} \nabla^2_{XY} . \end{aligned} \tag{7.62}$$

Since neither dw/dz nor dz/dw can vanish in a conformal transformation, a function which is harmonic with respect to one of the sets of variables is also harmonic with respect to the other.

If we apply formula (7.61) twice, however, we find

$$\nabla^4_{xy} = \left|\frac{dw}{dz}\right|^2 \nabla^2_{XY}\left(\left|\frac{dw}{dz}\right|^2 \nabla^2_{XY}\right) . \tag{7.63}$$

Thus a function which is biharmonic with respect to the $X - Y$ variables will not, in general, be biharmonic with respect to the $x - y$ variables. If the transformation (7.59) is linear, so that $w = \alpha z + \beta$ (α, β constant), then dw/dz is constant, and the biharmonic character of a function is preserved.

In solving boundary value problems, linear transformations are almost useless. If we can't solve a problem for a small region of given shape, it's unlikely that we'd be able to solve the problem for a big region of the same shape. Hence we must resign ourselves to the expectation that functions which are biharmonic in the region of flow will not be biharmonic in the transformed region.

As expected, the power of conformal mapping methods for biharmonic problems lies with the expression of a biharmonic function in terms of complex potentials. These potentials are analytic functions, both in the flow plane and in the transformed plane. Thus, if we represent the stream function according to Eq. (7.15), so that

$$\psi(x, y) = -\Re[\bar{z}\phi(z) + \chi(z)] , \tag{7.64}$$

and we let

$$\psi(x, y) = \boldsymbol{\Psi}(X, Y) , \tag{7.65}$$

then

$$\boldsymbol{\Psi}(X, Y) = -\Re[\overline{m(w)}\boldsymbol{\Phi}(w) + \boldsymbol{\chi}(w)] , \tag{7.66}$$

where

$$\boldsymbol{\Phi}(w) = \phi[m(w)], \quad \boldsymbol{\chi}(w) = \chi[m(w)] . \tag{7.67}$$

On solid boundaries the representation (7.18) provides a boundary condition on the complex potentials. Let us suppose that on a solid boundary **C**, the complex velocity $v - iu$ is a specified function of z, say $f(z)$. Equation (7.18) then gives

$$\phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)} = f(z) \quad \text{on } \mathbf{C} . \tag{7.68}$$

It is an easy matter to calculate the corresponding boundary condition in the w-plane. Since

$$\begin{aligned} \phi'(z) &= \frac{dw}{dz}\boldsymbol{\Phi}'(w) \\ &= \frac{1}{m'(w)}\boldsymbol{\Phi}'(w) \end{aligned} \tag{7.69}$$

we have, with (7.67),

$$\boldsymbol{\Phi}(w) + \frac{m(w)}{\overline{m'(w)}}\overline{\boldsymbol{\Phi}'(w)} + \overline{\Upsilon(w)} = F(w) \quad \text{on } \boldsymbol{\Gamma} , \tag{7.70}$$

where

$$\Upsilon(w) = \frac{1}{m'(w)}\chi'(w) , \quad (7.71)$$

$$F(w) = f[m(w)] , \quad (7.72)$$

and $\boldsymbol{\Gamma}$ is the curve in the w-plane into which $\mathbf{C}$ transforms under (7.59).

If the fluid motion is generated solely by the motion of rigid boundaries, all boundaries conditions in the flow plane are of the form (7.68); consequently all boundary conditions in the transformed plane are of the form (7.70). The problem of selecting the potentials under such circumstances is called the **first fundamental problem for biharmonic functions**. It has been extensively studied in the literature of plane elasticity, where it deals with the state of stress in a plate subjected to a prescribed distribution of normal forces on its boundaries.

In view of the complete analogy between the fluid-flow problem and the elastic stress problem established by Goodier (1934), one might wonder at first why the first fundamental problem for biharmonic functions is virtually always couched in the language of elasticity theory rather than in the language of hydrodynamics. The answer appears when we reflect on the sort of regions primarily of interest in the two disciplines. To consider the stress field in a simply-connected plate subjected to edge loading is certainly a meaningful undertaking. It is precisely for regions of this sort that the first fundamental problem is most amenable to solution. For the various methods of attack, the reader is referred to Kantorovich and Krylov (1964) and to Muskhelishvili (1963), a well-known treatise. On the other hand creeping viscous flow in a simply-connected region is a rather artificial sort of problem.[4] To the hydrodynamicist, multiply connected regions are of genuine interest, but in studying them he finds himself working in an area where it seems every problem is either trivial or impossible. Some meaningful results, however, do exist; the best place to start looking is probably Dean (1958).

Another class of problems can be reduced to the first fundamental problem. Let us consider the problem of streaming through a channel which is uniform except for a bounded array of obstacles, as illustrated, for example, in Fig. 7.2. We set

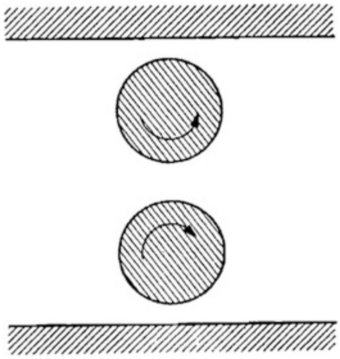
$$\psi = \psi_0 + \psi_1 , \quad (7.73)$$

where ψ_0 is a stream-function for parabolic flow through the channel. Thus, if the walls of the channel are at $y = \pm h$, we can take

$$\psi_0 = \frac{Q_0}{4h^3}(3h^2y - y^3) , \quad (7.74)$$

[4] One might pose, as a counterexample, flow in a rectangle, one side of which is moving. However, this problem, and most others like it, involves discontinuous boundary conditions (in the example cited, at two corners)—this can make the first fundamental problem most difficult.

Fig. 7.4 Two cylinders rotating in opposite directions in a plane channel



where Q_0 is constant. We cannot expect ψ_0 to satisfy the no-slip condition at the obstruction, but this gives us boundary conditions for ψ_1. If the surfaces of the obstacles move with velocity components U, V,

$$\frac{\partial \psi_1}{\partial x} = -V - \frac{\partial \psi_0}{\partial x}, \quad \frac{\partial \psi_1}{\partial y} = U - \frac{\partial \psi_0}{\partial y}. \tag{7.75}$$

Since ψ_0 satisfies no-slip on the channel walls,

$$\frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_1}{\partial y} = 0, \quad \text{on } y = \pm h \tag{7.76}$$

if the walls are stationary; if one or both of the walls is in motion, (7.76) undergoes an obvious modification.

Following the usual procedure, we can convert (7.75) and (7.76) to the boundary conditions of the first fundamental problem. If we can solve this problem for ψ_1, we are almost finished, but there is a detail to clear up. The flow represented by ψ_1 may contribute a flux Q_1 through the channel. One feels that this might be the case if the obstruction consists, for example, of two cylinders, arranged across the channel, which rotate in opposite directions as is shown in Fig. 7.4. However, for a given set of obstacles and boundary velocities, use of the boundary conditions (7.75) gives Q_1, as a function of Q_0. Thus the total flux $Q = Q_0 + Q_1$ through the channel is given by an expression of the form

$$Q = Q_0 + f(Q_0). \tag{7.77}$$

In a typical problem, the total flux Q will be specified a priori. If (7.77) can be solved uniquely for Q_0 in terms of Q, the stream function $\psi = \psi_0 + \psi_1$ is the required solution.

The success of this technique hinges upon the linearity of creeping flow, which permits no coupling between ψ_0 and ψ_1 in the governing equations.

7.5 Pressure Flow Through a Channel of Varying Width

The solution of a problem in plane creeping flow need not always be attempted in terms of the complex potentials. To do so for an extremely simple problem, such as Couette flow for example, would obviously be quite foolish. In very complicated problems approximation methods based upon the conformal mapping technique may provide the best line of attack, but we should not overlook other possibilities.

One might expect, for instance, that the flow at a given point is most strongly influenced by the local geometry, and is only slightly modified by the presence of boundaries far away. By considering various regions of the flow field, it is often possible to patch together a fairly good picture of the flow field. Moreover the patching is not necessarily a mortar-and-trowel numerical matter: in this section we treat a class of problems in which it is carried out gradually, and in analytic terms, from one part of the flow field to another.

The reader will find additional details on the problem of flow through non-uniform channels in Langlois (1958).

We consider creeping flow through an infinitely long channel of smoothly varying gap, cf. Fig. 7.5. For simplicity we assume that the channel is symmetric with respect to an axis stretching along the channel, which we choose for the x-axis. Thus, the walls of the channel follow the curves

$$y = \pm h(x) , \tag{7.78}$$

where $h(x)$ is smooth and positive for all values of x.

We take the walls to be stationary, so that

$$u[x, \pm h(x)] = v[x, \pm h(x)] = 0 \tag{7.79}$$

for all x. There may be a net flux Q through the channel. Incompressibility requires that this be the same at all cross sections. Therefore

$$\int_{-h(x)}^{h(x)} u \, dy = Q \quad (\text{all } x) . \tag{7.80}$$

As usual the differential equations governing the motion are the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{7.81}$$

and the force-balance equations

$$\frac{\partial p}{\partial x} = \mu \nabla^2 u , \tag{7.82}$$

$$\frac{\partial p}{\partial y} = \mu \nabla^2 v . \tag{7.83}$$

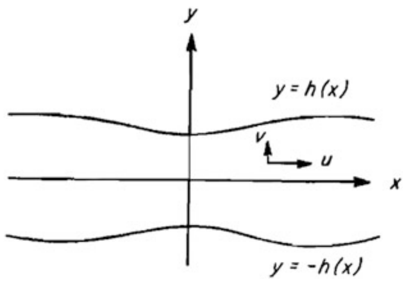


Fig. 7.5 Channel with curvy walls

When suitable restrictions are placed upon $h(x)$, the boundary value problem represented by Eqs. (7.79) through (7.83) is precisely of the sort in which it is helpful to look at the local picture of flow.

7.5.1 Wall Slope Everywhere Negligible

If $h'(x)$ is everywhere small compared with unity, it is reasonable to assume that, at each value of x, the components of velocity and pressure gradient are approximately equal to those obtaining in a channel of uniform width. This gives the familiar parabolic velocity profile with pressure gradient parallel to the x-axis, i.e.,

$$u = \frac{3Q}{4h^3}(h^2 - y^2), \quad v = 0\,, \tag{7.84}$$

$$\frac{\partial p}{\partial x} = -\frac{3Q\mu}{2h^3}, \quad \frac{\partial p}{\partial y} = 0\,. \tag{7.85}$$

Since $h(x)$ is nearly constant, it is tempting to infer from (7.85) that $p = \text{const.} - 3Q\mu x/2h^3$, but this is incorrect. The components of pressure gradient are approximated *locally* by the uniform-channel values, but the *integrated* value of p can be quite different. The correct result is

$$p = \left(\frac{3Q\mu}{2}\right)\int_x^c \frac{dx}{h^3}\,, \tag{7.86}$$

where c is a constant of integration.

Direct substitution of (7.84) into (7.79) and (7.80) reveals that the boundary conditions are exactly satisfied. Since the assumed values of v and $\partial p/\partial y$ are both zero, Eq. (7.83) is trivially satisfied.

The remaining Eqs. (7.81) and (7.82), are satisfied by (7.84) and (7.85), provided we neglect terms in $h'(x)$ and $h(x)h''(x)$. The validity of this approach therefore requires

$$|h'(x)| \ll 1, \tag{7.87}$$

$$|h(x)h''(x)| \ll 1\,. \tag{7.88}$$

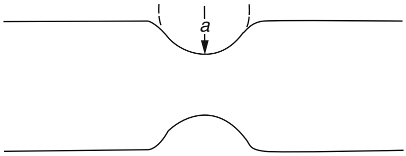


Fig. 7.6 Channel with smooth bumps

The meaning of (7.87) is clear, but (7.88) requires a bit of explanation, best provided by an example. Suppose, for simplicity, that the channel is uniform, except for a slight constriction caused by a pair of smooth bumps as in Fig. 7.6. If a, the nominal radius of the bumps, is small compared with the channel width, so that (7.88) is violated, the flow near one of the bumps is precisely that—the flow near a bump; uniform-channel flow would be a bad local approximation.

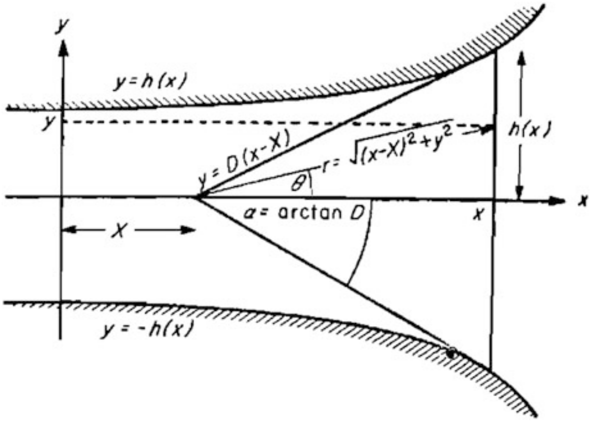


Fig. 7.7 Channel with a divergent wedge

7.5.2 *Wall Curvature Everywhere Negligible*

Under suitable conditions the restriction (7.87) can be removed by assuming the flow locally to be as if $h(x)$ were a linear function of x. Where $h'(x)$ is positive, the channel is approximated by a divergent wedge with a source of flux Q at its vertex; where $h'(x)$ is negative, the wedge is convergent with a sink at its vertex. We shall carry out the analysis explicitly for $h'(x)$ positive as Fig. 7.7 shows, but we shall see *a posteriori* that the expressions for the flow variables have the same functional form, regardless of the sign of $h'(x)$; the discontinuities of these expressions at $h'(x) = 0$ are removable, the results reducing to those of Sect. 7.5.1.

The solution to the full Navier-Stokes equation for flow in a wedge-shaped region was set out in Chap. 4. As we saw, this solution is obtained implicitly, in terms of elliptical integrals. For creeping flow, however, the solution is much simpler and lends itself readily to the method we have in mind.

In plane polar coordinates, the equations of creeping flow become

$$\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_\theta}{\partial \theta} = 0\ ,$$

$$\frac{\partial p}{\partial r} = \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)\ , \tag{7.89}$$

$$\frac{\partial p}{\partial \theta} = \mu r\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right)\ , \tag{7.90}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\ . \tag{7.91}$$

For wedge flow the boundary conditions are

$$v_r(r, \pm\alpha) = v_\theta(r, \pm\alpha) = 0,$$

$$\int_{-\alpha}^{\alpha} r v_r d\theta = Q\ . \tag{7.92}$$

It can be verified by direct substitution that the differential equations (7.89) and the boundary conditions (7.92) are all exactly satisfied by

$$v_r = \frac{Q}{r}\frac{\sin^2\alpha - \sin^2\theta}{\sin\alpha\cos\alpha - \alpha + 2\alpha\sin^2\alpha}\ ,$$

$$v_\theta = 0\ , \tag{7.93}$$

$$p = \frac{\mu Q}{r^2}\frac{\cos^2\theta - \sin^2\theta}{\sin\alpha\cos\alpha - \alpha + 2\alpha\sin^2\alpha} + \text{const.}$$

In order to use these results in the varying-channel problem, we shall convert them to Cartesian coordinates. With the notation indicated in Fig. 7.7,

$$u = v_r\cos\theta, \quad v = v_r\sin\theta\ ,$$

$$\frac{\partial p}{\partial x} = \frac{x - X}{r}\frac{\partial p}{\partial r} - \frac{y}{r^2}\frac{\partial p}{\partial \theta}\ ,$$

$$\frac{\partial p}{\partial y} = \frac{y}{r}\frac{\partial p}{\partial r} + \frac{x - X}{r^2}\frac{\partial p}{\partial \theta}\ ,$$

$$\cos\theta = \frac{(x - X)}{\sqrt{(x - X)^2 + y^2}}\ , \tag{7.94}$$

$$\sin\theta = \frac{y}{\sqrt{(x - X)^2 + y^2}}\ ,$$

$$r = \sqrt{(x - X)^2 + y^2} \,,$$
$$\sin\alpha = \frac{D}{\sqrt{1 + D^2}}, \quad \cos\alpha = \frac{1}{\sqrt{1 + D^2}} \,.$$

Equations (7.93) then yield

$$\begin{aligned}
u &= \frac{Q(x - X)}{E} \frac{D^2(x - X)^2 - y^2}{[(x - X)^2 + y^2]^2} \,, \\
v &= \frac{Qy}{E} \frac{D^2(x - X)^2 - y^2}{[(x - X)^2 + y^2]^2} \,, \\
\frac{\partial p}{\partial x} &= -\frac{2\mu Q(1 + D^2)(x - X)}{E} \frac{(x - X)^2 - 3y^2}{[(x - X)^2 + y^2]^3} \,, \\
\frac{\partial p}{\partial y} &= -\frac{2\mu Q(1 + D^2)y}{E} \frac{3(x - X)^2 - y^2}{[(x - X)^2 + y^2]^3} \,,
\end{aligned} \tag{7.95}$$

where

$$\begin{aligned}
E &= (\sin\alpha\cos\alpha - \alpha + 2\alpha\sin^2\alpha)(1 + \tan^2\alpha) \\
&= D - (1 - D^2)\arctan D \,.
\end{aligned} \tag{7.96}$$

The somewhat fictitious length X can be removed from Eqs. (7.95) by noting that

$$h = D(x - X) \,. \tag{7.97}$$

Thus

$$\begin{aligned}
u &= \frac{QD^3h}{E} \frac{h^2 - y^2}{(h^2 + D^2y^2)^2} \,, \\
v &= \frac{QD^4y}{E} \frac{h^2 - y^2}{(h^2 + D^2y^2)^2} \,,
\end{aligned} \tag{7.98}$$

$$\begin{aligned}
\frac{\partial p}{\partial x} &= -\frac{2\mu Q(1 + D^2)D^3h}{E} \frac{h^2 - 3D^2y^2}{(h^2 + D^2y^2)^3} \,, \\
\frac{\partial p}{\partial y} &= -\frac{2\mu Q(1 + D^2)D^4y}{E} \frac{3h^2 - D^2y^2}{(h^2 + D^2y^2)^3} \,.
\end{aligned} \tag{7.99}$$

If the curvature of the walls is everywhere small (in the sense that $h(x)h''(x)$ is small compared with unity), it might be hoped that, at each value of x, the flow in a non-uniform channel can be approximated by the flow in a wedge with vertex at $[X(x), 0]$ and vertex angle $[2\arctan D(x)]$, where

$$D(x) = h'(x),$$
$$X(x) = x - \frac{h(x)}{D(x)} \; . \tag{7.100}$$

Upon checking by direct substitution, we find that the velocity components given by (7.98) satisfy exactly the boundary conditions (7.79) and (7.80). We also find that (7.98) and (7.99) satisfy approximately the differential equations (7.81)–(7.83), provided

$$|h(x)h''(x)| \ll 1 \; , \tag{7.101}$$
$$|h(x)^2 h'''(x)| \ll 1 \; . \tag{7.102}$$

If (7.101) is satisfied, it is easy to verify that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

approximates an exact differential when $\partial p/\partial x$ and $\partial p/\partial y$ are given by Eqs. (7.99). Therefore,

$$\begin{aligned} p &= \int_{C,0}^{x,y} \left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy \right) \\ &= 2\mu Q \int_x^C \frac{(1+D^2)D^3}{Eh^3}dx \\ &- \frac{2\mu Q D^4(1+D^2)}{E} \int_0^y \frac{3h^2 y - D^2 y^3}{(h^2+D^2y^2)^3}dy \\ &= 2\mu Q \int_x^C \frac{(1+D^2)D^3}{Eh^3}dx - \frac{\mu Q D^4(1+D^2)}{h^2 E} \frac{3(y/h)^2 + D^2(y/h)^4}{[1+D^2(y/h)^2]^2} \; , \end{aligned} \tag{7.103}$$

where C is a constant of integration.

If we set

$$\psi(x, y) = \frac{Q}{2E} \left[\frac{Dh(1+D^2)y}{h^2+D^2y^2} - (1-D^2)\arctan(Dy/h) \right] \; , \tag{7.104}$$

we see that

$$u = \frac{\partial \psi}{\partial y} \; ,$$
$$v = -\frac{\partial \psi}{\partial x} + \text{terms in } h(x)D'(x) \; . \tag{7.105}$$

Thus, within the limits of validity of the approximation, $\psi(x, y)$ is a stream function for the flow.

7.5.3 Power Series Expansion in the Wall Slope

The method of Sect. 7.5.2 leads to rather cumbersome results, even for analytically simple forms of $h(x)$. Consequently it is desirable to compromise between the methods of Sects. 7.5.1 and 7.5.2 by developing a method wherein condition (7.87) is not completely removed, but is merely relaxed.

It may happen that a function $h(x)$ satisfying (7.101) and (7.102) is such that $h'(x)$ is small but not negligible, in the sense that the modified condition

$$|D^n| = |h'(x)|^n \ll 1 \tag{7.106}$$

is satisfied for some positive integer n. The case $n = 1$ corresponds to Sect. 7.5.1. For $n > 1$, the results of Sect. 7.5.2 can be expanded in a power series in D and terms of the nth degree or higher in D neglected. The treatment given here will be restricted to the case $n = 3$, for with $n > 3$ the resulting expressions are so complicated that no advantage is gained over a direct use of Sect. 7.5.2.

Since

$$\arctan D = D - \frac{1}{3}D^3 + \frac{1}{5}D^5 + O(D^7)\ , \tag{7.107}$$

Eq. (7.96), which defines the function E, gives us

$$E = \frac{4}{3}D^3[1 - \frac{2}{5}D^2 + O(D^4)]\ , \tag{7.108}$$

so that

$$\frac{D^3}{E} = \frac{3}{4}[1 + \frac{2}{5}D^2 + O(D^4)]\ . \tag{7.109}$$

If terms of the third degree or higher in D are neglected in the expansions of Eqs. (7.98) and (7.99), it follows that

$$u = \left(\frac{3Q}{4h}\right)\left[1 - \left(\frac{y}{h}\right)^2\right]\left[1 - 2D^2\left(\frac{y}{h}\right)^2 + \left(\frac{2}{5}\right)D^2\right]\ ,$$

$$v = \left(\frac{3QD}{4h}\right)\left(\frac{y}{h}\right)\left[1 - \left(\frac{y}{h}\right)^2\right]\ , \tag{7.110}$$

$$\frac{\partial p}{\partial x} = -\left(\frac{3\mu Q}{2h^3}\right)\left[1 - 6D^2\left(\frac{y}{h}\right)^2 + \left(\frac{7}{5}\right)D^2\right]\ ,$$

$$\frac{\partial p}{\partial y} = -\left(\frac{9\mu QD}{h^3}\right)\left(\frac{y}{h}\right)\ . \tag{7.111}$$

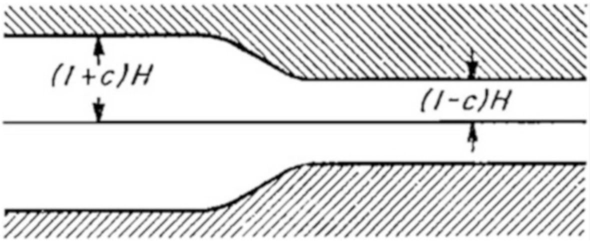


Fig. 7.8 Channel with a smooth constriction

Equation (7.103), the integrated equation for the pressure, becomes

$$p = \frac{3\mu Q}{4}\left(2\int_x^C \frac{1+\frac{7}{5}D^2}{h^3}dx - \frac{3Dy^2}{h^4}\right) \tag{7.112}$$

and Eq. (7.104) for the stream function reduces to

$$\psi(x,y) = \frac{3Q}{4}\frac{y}{h}\left\{\left[1-\frac{1}{3}\left(\frac{y}{h}\right)^2\right] + \frac{2}{5}D^2\left[1-\left(\frac{y}{h}\right)^2\right]\right\}^2 . \tag{7.113}$$

7.5.4 The Flow Through a Smooth Constriction

With reference to Fig. 7.8, let us suppose that the function $h(x)$ is given by

$$h(x) = H\left[1 - c\tanh\left(\frac{x}{L}\right)\right] , \tag{7.114}$$

where H and L are constant lengths and c is a positive, dimensionless constant less than unity. Application of the general results developed in the preceding sections give an illustration of the type of flow to be expected when a very viscous fluid is forced through a smooth constriction connecting two uniform channels of different widths.

From (7.114),

$$\begin{aligned} h'(x) &= D(x) = -\left(\frac{ch}{L}\right)\left\{\operatorname{sech}^2\left(\frac{x}{L}\right)\right\} , \\ h''(x) &= \left(\frac{cH}{L^2}\right)\left\{2\operatorname{sech}^2\left(\frac{x}{L}\right)\tanh\left(\frac{x}{L}\right)\right\} , \\ h'''(x) &= \left(\frac{cH}{L^3}\right)\left\{2\operatorname{sech}^2\left(\frac{x}{L}\right)\left[\operatorname{sech}^2\left(\frac{x}{L}\right) - 2\tanh^2\left(\frac{x}{L}\right)\right]\right\} . \end{aligned} \tag{7.115}$$

Elementary methods can be used to verify that the factors enclosed in braces all have magnitude less than or equal to unity. Hence (7.87) is satisfied if

$$\left|\frac{cH}{L}\right| \ll 1\,, \tag{7.116}$$

condition (7.88), which is the same as (7.101), is satisfied if

$$\left|c(1+c)\left(\frac{H}{L}\right)^2\right| \ll 1\,, \tag{7.117}$$

condition (7.102) is satisfied if

$$\left|c(1+c)^2\left(\frac{H}{L}\right)^3\right| \ll 1\,, \tag{7.118}$$

and condition (7.106), with $n = 3$, is satisfied if

$$\left|\frac{cH}{L}\right|^3 \ll 1\,. \tag{7.119}$$

If we assume that conditions (7.117), (7.118), and (7.119) are satisfied, and permit (7.116) to be violated, then the method of Sect. 7.5.3 is applicable, but the method of Sect. 7.5.1 may not be.

Let us take

$$\frac{cH}{L} = \delta$$

and substitute (7.114) into Eq. (7.110). We obtain

$$\begin{aligned} u &= \frac{3Q}{4H[1-c\tanh(x/L)]}\left[1-\frac{y^2}{H^2[1-c\tanh(x/L)]^2}\right] \\ &\quad\times\left[1-2\delta^2\frac{y^2\mathrm{sech}^4(x/L)}{H^2[1-c\tanh(x/L)]^2}+\frac{2}{5}\delta^2\mathrm{sech}^4(x/L)\right] \\ v &= -\frac{3Qy\delta\,\mathrm{sech}^2(x/L)}{4H^2[1-c\tanh(x/L)]^2}\left[1-\frac{y^2}{H^2[1-c\tanh(x/L)]^2}\right]. \end{aligned} \tag{7.120}$$

Equation (7.112) for the pressure gives

$$p = \frac{3\mu Q}{4}\left[2\int_x^C \frac{1+\frac{7}{5}\delta^2\mathrm{sech}^4(x/L)}{H^3[1-c\tanh(x/L)]^3}dx + \frac{3\delta y^2\mathrm{sech}^2(x/L)}{H^4[1-c\tanh(x/L)]^4}\right], \tag{7.121}$$

and Eq. (7.113) for the stream function becomes

$$\psi = \frac{3Qy}{4H[1 - c\tanh(x/L)]}\left\{\left[1 - \frac{y^2}{3H^2[1 - c\tanh(x/L)]^2}\right] + \frac{2}{5}\delta^2\mathrm{sech}^2(x/L)\left[1 - \frac{y^2}{H^2[1 - c\tanh(x/L)]^2}\right]\right\} . \tag{7.122}$$

7.6 Hele-Shaw Flow

The Hele-Shaw flow (after Henry Selby Hele-Shaw, 1854–1941) and its theory are quite useful in modern engineering applications: to compute for example the flow in injection molding, the viscous fingering instability related to oil recovery, flow through porous media.

The theory of Hele-Shaw cells may be found in Lamb (1995) and Batchelor (1992). Let us consider the creeping flow between two fixed parallel plates, cf. Fig. 4.1, of a very viscous fluid in a layer of thickness $2h$. Inside the gap is placed an obstacle of cylindrical shape with its generators orthogonal to the plates, and of characteristic length L. The geometrical aspect ratio, defined as

$$\varepsilon = \frac{h}{L} , \tag{7.123}$$

is such that $\varepsilon \ll 1$. With the incompressibility constraint (2.194), an order of magnitude estimate of the various terms yields

$$|v_3| \approx \varepsilon v_1 \simeq \varepsilon v_2 \ll |v_1| \simeq |v_2| . \tag{7.124}$$

As the length scales in the directions parallel and orthogonal to the plates are quite different, we write

$$\left|\frac{\partial^2 v_1}{\partial x_1^2}\right| \ll \left|\frac{\partial^2 v_1}{\partial x_3^2}\right| , \quad \left|\frac{\partial^2 v_2}{\partial x_1^2}\right| \ll \left|\frac{\partial^2 v_2}{\partial x_3^2}\right| . \tag{7.125}$$

The Stokes equations (6.1) and (6.2) reduce to the set

$$\frac{\partial v_i}{\partial x_i} = 0 , \tag{7.126}$$

$$\frac{\partial p}{\partial x_i} = \mu \nabla^2 v_i , \tag{7.127}$$

for $v_i = v_i(x_1, x_2), i = 1, 2$. Since v_3 is of order ε, the $i = 3$ component of (6.2) indicates that the pressure is essentially constant across the gap, i.e, $p = p(x_1, x_2)$.This set of equations is meaningful in the symmetry plane parallel to the plates containing the origin of the coordinate axes. It is possible with the help of Eqs. (7.123) and (7.124) to write

$$v_i(x_1, x_2, x_3) = v_i(x_1, x_2, 0) f(x_3) , \tag{7.128}$$

as the variation of v_i with respect to x_1, x_2 is slower than that of f with respect to x_3. With pressure variation across the gap ignored, the Poiseuille flow solution provides the relationship

$$f(x_3) = \left(1 - \frac{x_3^2}{h^2}\right) . \tag{7.129}$$

The momentum equations (7.127) become

$$\mu \frac{\partial^2 v_1}{\partial x_3^2} = \frac{\partial p}{\partial x_1} , \tag{7.130}$$

$$\mu \frac{\partial^2 v_2}{\partial x_3^2} = \frac{\partial p}{\partial x_2} . \tag{7.131}$$

Taking Eqs. (7.128) and (7.129) into account, one obtains

$$v_i(x_1, x_2, 0) = -\frac{h^2}{2\mu} \frac{\partial p}{\partial x_i}, \quad \text{i=1,2} . \tag{7.132}$$

Thus the velocity is a gradient field (i.e., irrotational). Therefore it can be derived from a velocity potential:

$$v_i = \frac{\partial \varphi}{\partial x_i} , \tag{7.133}$$

and Eq. (7.132) yields

$$\varphi = -\frac{h^2 p}{2\mu} . \tag{7.134}$$

The streamline configuration will be the same in planes $x_3 = cst$. Furthermore they will be similar to those of a two-dimensional potential flow of an inviscid fluid around obstacles of the same shape. Near the obstacles the viscous fluid sticks to the walls, but this influence will be limited to a zone of thickness h. This discussion explains why the Hele-Shaw cell is used in many experiments to provide the observer with the geometrical pattern resulting from the presence of bodies inside an internal flow.

Chapter 8
Rotary Flow

Abstract Axially symmetric creeping flow generated by rotary motion of solid boundaries is treated. If the Reynolds number is not zero, there will be a secondary flow, due to centrifugal force, in the meridional planes. For the case of flow between concentric spheres, this is calculated to first order using the Reynolds number as a perturbation parameter. Rotlets are introduced.

ln this chapter we focus our attention on axially symmetric flows generated entirely by rotary motion of solid boundaries. Flows of this sort are often used to measure viscosity.

Two special cases of rotary flow were considered in Chap. 4. The axisymmetric Couette flow between cylinders, calculated in Sect. 4.9, is particularly simple: the streamlines are all circles about the axis of rotation. In Sect. 4.12 we treated von Karman's problem of flow generated by a rotating disc, and found that the non-uniform centrifugal force generated a pumping effect, so that a secondary flow in the meridional planes was superimposed upon the circular motion. This centrifugal pumping is characteristic of rotary flow when fluid inertia is taken into account; the axisymmetric Couette flow is the only known counterexample. Even in this case secondary flow enters the picture. Taylor (1922) showed that at moderate Reynolds number the Couette flow passes over into a different mode. When this happens, there is a flow in the meridional planes which presents the appearance of a system of vortices contained in rectangular compartments, so that the axial variation of flow is periodic rather than uniform. For massless fluids there is no centrifugal pumping, so that it is reasonable to seek solutions without secondary flow.

Taylor vortices are the first sign of flow instability. This topic is beyond the scope of the present monograph and the reader is referred to, e.g., Drazin and Reid (2004), Koschmieder (1993), and Schmid and Henningson (2001).

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_8,

8.1 The Equations Governing Creeping Rotary Flow

Let us assume that a massless fluid occupies an axially symmetric region of space, which may be either finite or infinite in extent. The boundaries of the region will, in general, consist partly of rigid walls, partly of free surfaces. The only motion we permit the rigid walls to have is a rotation about the axis of symmetry, and we specify that flow in the region is generated only by this boundary rotation.

The general problem of axisymmetric creeping flow was treated in Sect. 6.1. We showed that the Stokes stream function Ψ and the swirl Ω are governed by

$$E^4\Psi = 0 , \tag{8.1}$$

$$E^2\Omega = 0 , \tag{8.2}$$

where E^2 is a second-order differential operator, defined in cylindrical coordinates by

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{8.3}$$

and in spherical coordinates by

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} . \tag{8.4}$$

For the class of problems under consideration the velocity components in the meridional planes vanish on the solid boundaries. The no-slip condition on these boundaries and the differential equation (8.1) are both satisfied by taking

$$\Psi = \text{const} . \tag{8.5}$$

Since purely rotary flow results, the kinematic boundary condition that the free surfaces always consist of the same particles is automatically satisfied.

In Chap. 3 the swirl was defined as the velocity-moment about the axis of symmetry. Thus in cylindrical coordinates we have, according to Eq. (3.85),

$$\Omega = r v_\theta . \tag{8.6}$$

For spherical coordinates, Eq. (3.106) gives

$$\Omega = r \sin\theta \, v_\phi . \tag{8.7}$$

In terms of the velocity, Eq. (8.2) becomes

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}\right)v_\theta=0 \quad \text{(cylindrical coordinates)} \quad , \tag{8.8}$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}-\frac{1}{r^2\sin^2\theta}\right)v_\phi=0 \quad \text{(spherical coordinates)} \quad . \tag{8.9}$$

It is easily checked from the Navier-Stokes equations for a massless incompressible fluid, referred to the appropriate coordinate system, that the pressure is uniform throughout a region of creeping rotary flow.

8.2 Flow Between Parallel Discs

In the parallel disc viscometer the fluid to be tested is contained in the cylindrical region between two discs, which rotate with different angular velocities about their common axis. In practice one of the plates is held fixed, and there is no loss of generality in assuming that this is always the case, cf. Fig. 8.1. If both discs rotate, we can let the coordinate system rotate with one of them. For a massless fluid the acceleration of the reference frame does not matter. If the stationary disc lies in the $z = 0$ plane and the moving disc lies in the $z = h$ plane, the no-slip condition requires that

$$v_\theta(r,0)=0\ , \tag{8.10}$$

$$v_\theta(r,h)=\omega r\ , \tag{8.11}$$

where ω is the angular velocity of the moving disc.

The cylindrical surface $r = a, 0 < z < h$ is, ideally, force-free, except for an isotropic and uniform ambient pressure. Since this cannot be achieved in practice, the problem of edge effects has received considerable attention, see, for example, Kestin and Persen (1955). Since the outward normal to the free surface is in the direction of increasing r, the stress boundary conditions for the ideal case are

$$T_{rr}=-p_a, \quad T_{r\theta}=T_{rz}=0 \quad \text{on } r=a\ , \tag{8.12}$$

where p_a is the ambient pressure. Setting $v_r = v_z = \partial v_\theta/\partial\theta = 0$ in Eqs. (3.74) and (3.76) then gives

$$p(a,z)=p_a\ , \tag{8.13}$$

$$a\left[\frac{\partial}{\partial r}v_\theta(r,z)\right]_{r=a}-v_\theta(a,z)=0\ . \tag{8.14}$$

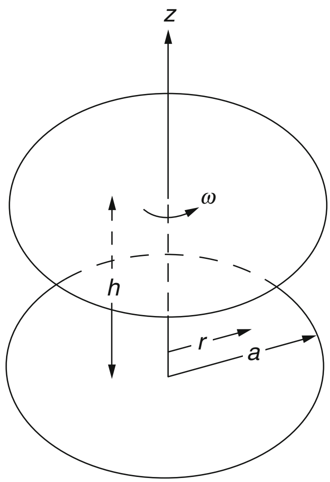


Fig. 8.1 Coordinate system for the parallel discs

In view of the remarks ending Sect. 8.1, the condition (8.13) requires that

$$p(r, z) = p_a \,. \tag{8.15}$$

The solution of (8.8) subject to (8.10) and (8.14) is trivial. We find by inspection that

$$v_\theta(r, z) = \omega \frac{rz}{h} \,. \tag{8.16}$$

Throughout the fluid,

$$T_{\theta z} = 2\mu e_{\theta z} = \mu \frac{\partial v_\theta}{\partial z} = \mu\omega \frac{r}{h} \,. \tag{8.17}$$

Consequently the moment M required to rotate the top disc, or to hold the bottom one still, is given by

$$M = \int_0^a (2\pi r)(r T_{\theta z}) dr = \frac{2\pi\mu\omega}{h} \int_0^a r^3 dr = \frac{\pi\mu\omega a^4}{2h} \,. \tag{8.18}$$

Since the quantities M, ω, a, h can presumably be measured, (8.18) can be used to determine the viscosity of the fluid, provided the underlying assumptions of creeping flow and idealized geometry are sufficiently well approximated.

8.3 Flow Between Coaxial Cones

Another widely used type of viscometer contains the fluid between a pair of rotating coaxial cones as depicted in Fig. 8.2. The rate of deformation tensor is almost constant throughout the flow field generated in this instrument. For this reason it is often the viscometer which is chosen when the experimenter wishes to measure the dynamic behavior of the fluid under uniform kinematic conditions. Basically it is a compromise between a coaxial cylinder (Couette) viscometer, in which uniform conditions are quite well approximated (there is a small variation across the gap), and a parallel disc viscometer, which is easily loaded even when extremely viscous liquids are measured.

The problem is most easily studied in spherical coordinates. If we denote the semivertical angles of the cones by α and $\tilde{\alpha}$ and the corresponding angular velocities by ω and $\tilde{\omega}$, the no-slip condition requires

$$v_\phi(r,\alpha) = r\omega \sin\alpha, \quad v_\phi(r,\tilde{\alpha}) = r\tilde{\omega}\sin\tilde{\alpha} . \tag{8.19}$$

Since Eq. (8.9) is homogeneous in r, it is reasonable to seek a solution of the form

$$v_\phi(r,\theta) = r \sin\theta\, \xi(\theta) . \tag{8.20}$$

Substituting (8.20) into (8.9) gives an ordinary differential equation for $\xi(\theta)$:

$$\frac{d^2}{d\theta^2}(\xi \sin\theta) + \cot\theta \frac{d}{d\theta}(\xi \sin\theta) + \left(\frac{2\sin^2\theta - 1}{\sin\theta}\right)\xi = 0 . \tag{8.21}$$

Comparing (8.19) and (8.20), we obtain the boundary conditions

$$\xi(\alpha) = \omega, \quad \xi(\tilde{\alpha}) = \tilde{\omega} . \tag{8.22}$$

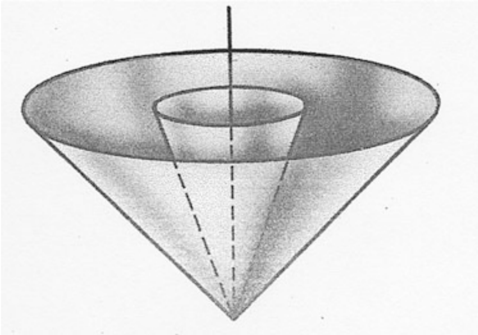
One family of solutions to (8.21) is $\xi = \text{const}$. Physically, this corresponds to the rigid body motion which results when $\omega = \tilde{\omega}$. The second solution follows easily when we rewrite (8.21) as

$$\left(\sin\theta \frac{d}{d\theta} + 3\cos\theta\right)\frac{d\xi}{d\theta} = 0 , \tag{8.23}$$

from which we obtain

$$\frac{d\xi}{d\theta} = \frac{\text{const.}}{\sin^3\theta} . \tag{8.24}$$

Fig. 8.2 Coaxial cones



Integrating (8.24) then gives the general solution to (8.21):

$$\xi(\theta) = A + B\left(\ln\tan\frac{1}{2}\theta - \frac{\cos\theta}{\sin^2\theta}\right) . \tag{8.25}$$

The constants of integration A, B are determined by the conditions (8.22); we obtain the untidy but symmetric result

$$A = \frac{\omega\left(\ln\tan\frac{1}{2}\tilde{\alpha} - \frac{\cos\tilde{\alpha}}{\sin^2\tilde{\alpha}}\right) - \tilde{\omega}\left(\ln\tan\frac{1}{2}\alpha - \frac{\cos\alpha}{\sin^2\alpha}\right)}{\ln\left(\frac{\tan\frac{1}{2}\tilde{\alpha}}{\tan\frac{1}{2}\alpha}\right) - \frac{\cos\tilde{\alpha}}{\sin^2\tilde{\alpha}} + \frac{\cos\alpha}{\sin^2\alpha}} , \tag{8.26}$$

$$B = \frac{(\tilde{\omega} - \omega)}{\ln\left(\frac{\tan\frac{1}{2}\tilde{\alpha}}{\tan\frac{1}{2}\alpha}\right) - \frac{\cos\tilde{\alpha}}{\sin^2\tilde{\alpha}} + \frac{\cos\alpha}{\sin^2\alpha}} .$$

Substituting (8.25) into (8.20) yields

$$v_\phi(r,\theta) = Ar\sin\theta + Br(\sin\theta\ln\tan\frac{1}{2}\theta - \cot\theta) . \tag{8.27}$$

The physical components of the rate of deformation tensor can now be calculated from Eqs. (3.98). All components vanish except

$$e_{\theta\phi} = e_{\phi\theta} = \frac{B}{\sin^2\theta} . \tag{8.28}$$

As indicated earlier, $e_{\theta\phi}$ is nominally constant throughout the flow field: it is entirely independent of r and, if $\varepsilon = (\tilde{\alpha}-\alpha)/\alpha$ is small compared with unity, the percentage variation of $\sin^2\theta$ across the gap is proportional to ε.

So far in this section we have ignored the question of free-surface conditions. If the region of flow extends to infinity, the question never arises. However in most laboratories the viscometer is filled only to a finite level. Since the only non-zero component of the rate of deformation tensor is given by (8.28), the stress components are given by

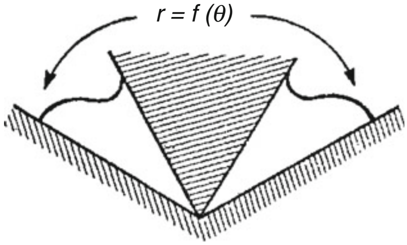


Fig. 8.3 Free surface between the coaxial cones

$$
\begin{aligned}
T_{rr} &= T_{\theta\theta} = T_{\phi\phi} = -p \ , \\
T_{\theta\phi} &= T_{\phi\theta} = \frac{2\mu B}{\sin^2\theta} \ , \\
T_{r\theta} &= T_{\theta r} = T_{r\phi} = T_{\phi r} = 0;
\end{aligned}
\tag{8.29}
$$

as indicated at the end of Sect. 8.1, the pressure is constant. Let us suppose that as shown in Fig. 8.3 the fluid fills the viscometer only up to the surface

$$r = f(\theta) \ . \tag{8.30}$$

Above this surface is an inviscid atmosphere, so that the stress is isotropic and uniform. The unit normal to the surface (8.30) has components

$$n_r = \frac{r}{\sqrt{r^2 + [f'(\theta)]^2}}, \quad n_\theta = -\frac{f'(\theta)}{\sqrt{r^2 + [f'(\theta)]^2}} \tag{8.31}$$

in the r- and θ-directions, respectively.

For the configuration described, the stress boundary conditions take the form

$$T_{rr}n_r + T_{r\theta}n_\theta = -p_a n_r \ , \tag{8.32}$$

$$T_{\theta r}n_r + T_{\theta\theta}n_\theta = -p_a n_\theta \ , \tag{8.33}$$

$$T_{\phi r}n_r + T_{\theta\phi}n_\theta = 0 \ , \tag{8.34}$$

where p_a is the ambient pressure. With the stresses given by (8.29), conditions (8.32) and (8.33) are satisfied by taking $p = p_a$. However (8.34) cannot be satisfied unless either B or n_θ vanishes. The case of B $= 0$ corresponds to rigid body motion; if we ignore this case, rotary flow is an acceptable solution only if $n_\theta = 0$. From (8.30) and (8.31) we see that this is the case only if the surface is a ring cut from the surface of a sphere with center at $r = 0$.

If the fluid surface is not spherical, one might conjecture that the flow is almost purely rotary except for a churning motion near the surface; it is conceivable that the fluid would ultimately take up the spherical shape as the only stable possibility. In practice, however, the discussion is academic, for the presence of even a slightly positive fluid density destroys the rotary pattern, especially near the surface, where gravity will not permit the spherical shape. Thus, the use of the bi-conical viscometer is predicated on the hope that, for very viscous fluids at least, the secondary motions arising from fluid inertia and edge effects do not couple strongly with the main flow. If this is the case, the moment acting on the inner cone is given by

$$\begin{aligned} M &= \int_0^R (2\pi r \sin\alpha)(r \sin\alpha)[T_{\phi\theta}]_{\theta=\alpha}\, dr \\ &= 4\pi\mu B \int_0^R r^2 dr = \frac{4}{3}\pi\mu B R^3 \,, \end{aligned} \tag{8.35}$$

where it is assumed that the fluid extends to the surface $r = R$; the moment on the outer cone has the same magnitude.

8.4 Flow Between Concentric Spheres

The question of edge effects never enters when we consider the flow produced in a fluid contained between two concentric spheres which rotate slowly about a common axis of symmetry with different angular velocities as in Fig. 8.4.

We make the obvious choice of a spherical coordinate system with origin at the common center of the spheres and $\theta = 0$ axis along the axis of rotation. The radius and angular velocity of the inner sphere are denoted by R and ω respectively and those of the outer sphere by $\tilde{R}$ and $\tilde{\omega}$. Thus we seek a solution to (8.9) which satisfies the boundary conditions

$$\begin{aligned} v_\phi(R,\theta) &= \omega R \sin\theta \,, \\ v_\phi(\tilde{R},\theta) &= \tilde{\omega}\tilde{R} \sin\theta \,. \end{aligned} \tag{8.36}$$

A trial solution of the form

$$v_\phi(r,\theta) = \eta(r)\sin\theta \tag{8.37}$$

does the job, for substitution in (8.9) yields the ordinary differential equation

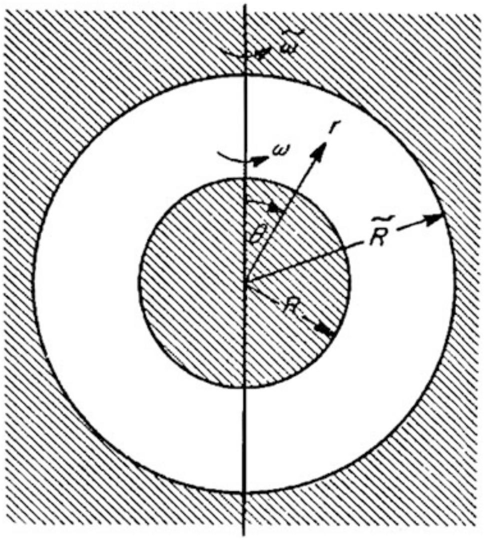
$$r^2\eta''(r) + 2r\eta'(r) - 2\eta(r) = 0. \tag{8.38}$$

The general solution is

$$\eta(r) = \omega\left(Kr + \frac{LR^3}{r^2}\right) \tag{8.39}$$

and the dimensionless constants of integration K, L are easily obtained from the boundary conditions (8.36):

Fig. 8.4 Concentric spheres



$$K = \frac{\tilde{R}^3 \tilde{\omega}/\omega - R^3}{\tilde{R}^3 - R^3} \;, \tag{8.40}$$

$$L = \frac{\tilde{R}^3 (1 - \tilde{\omega}/\omega)}{\tilde{R}^3 - R^3} \;. \tag{8.41}$$

If the inner sphere is absent, a condition of finite velocity at $r = 0$ requires that L vanishes. Equation (8.37) then represents the solid body rotation obtained when a revolving sphere is completely filled with fluid. The coefficient L also vanishes, so that solid body rotation is again obtained, if $\omega = \tilde{\omega}$.

If the outer sphere is absent, it is K that must vanish. In this case L is unity and Eq. (8.37) describes the flow generated in an infinite volume of massless fluid by a rotating sphere.

Combining (8.37) and (8.39) yields

$$v_\phi(r, \theta) = \omega \left(Kr + \frac{LR^3}{r^2} \right) \sin\theta \;, \tag{8.42}$$

from which the stresses can be calculated via Eqs. (3.98) and (3.100). We obtain

$$\begin{aligned} T_{rr} &= T_{\theta\theta} = T_{\phi\phi} = -p \;, \\ T_{r\theta} &= T_{\theta r} = T_{\theta\phi} = T_{\phi\theta} = 0 \;, \\ T_{r\phi} &= = T_{\phi r} = -3\mu\omega LR^3 \frac{\sin\theta}{r^3} ; \end{aligned} \tag{8.43}$$

the pressure p may be assigned any constant value p_0.

The moment on the outer sphere, which has the same magnitude as the moment on the inner sphere, is given by

$$M = \int_0^\pi (2\pi \tilde{R} \sin\theta)(\tilde{R} \sin\theta)[T_{r\phi}]_{r=\tilde{R}}\, d(\tilde{R}\theta) \tag{8.44}$$

$$= -6\pi\mu\omega LR^3 \int_0^\pi \sin^3\theta d\theta = -8\pi\mu\omega LR^3 = \frac{8\pi\mu R^3 \tilde{R}^3(\tilde{\omega}-\omega)}{\tilde{R}^3 - R^3}\,.$$

Kanwal (1961) treated the general problem of creeping rotary flow generated by an axially symmetric body rotating in an unbounded fluid. Several special bodies, including the lens and the torus, were treated explicitly.

8.4.1 Secondary Flow

As we mentioned in Chap. 6, Whitehead attempted to improve upon Stokes' solution for the flow past a sphere by treating the inertia associated with Stokes' result as the driving force of a perturbation flow field. The attempt was doomed from the start, since the Stokes solution is not self-consistent if the fluid has any density whatsoever: as Oseen pointed out, inertia dominates over viscosity in region sufficiently far from the sphere. For the rotary flow between spheres, however, Whitehead's procedure works very well.

Since the pressure is constant and the inertia is neglected in creeping rotary flow, the only term that remains in the Navier-Stokes equation is the viscous term. Consequently solving the flow problem involves seeing to it that this term vanishes; hence it is pointless to examine its magnitude. A measure of the relative importance of viscosity and inertia can, however, be obtained by comparing a typical term in (8.9) with a typical inertia term in the full Navier-Stokes equation. We need only consider the case of unbounded flow, for this is where Whitehead's paradox might be encountered. If we set $K = 0$ in Eq. (8.42), we see that a typical term in (8.9) has the form $\omega R^3 r^{-4} f(\theta)$. A typical neglected term in the Navier-Stokes equation is $(\rho/\mu r)v_\phi^2 = (\rho\omega^2 R^6/\mu)r^{-5}\sin^2\theta$. If we ignore the θ-dependence and take the ratio δ, we obtain a measure of the relative importance of inertia and viscosity:

$$\delta = \frac{\rho R^3 \omega}{\mu r}\,. \tag{8.45}$$

The worst case occurs not at infinity but on the sphere, where δ equals the Reynolds number

$$\mathcal{R}e = \frac{\rho R^2 \omega}{\mu}\,. \tag{8.46}$$

Consequently, for sufficiently small $\mathcal{R}e$, inertia is everywhere negligible compared with viscosity.

The flow field

$$\begin{aligned} v_r &= v_\theta = 0 \,, \\ v_\phi &= \omega\left(Kr + \frac{LR^3}{r^2}\right)\sin\theta \,, \\ p &= p_0 \,, \end{aligned} \tag{8.47}$$

represents the zero-order term in a perturbation expansion with $\mathcal{R}e$ as the parameter. In calculating higher order terms, it is best to use a dimensionless formulation. This can be done several ways, but it is easiest to use R as length scale and $R\omega$ as velocity scale, for the problem of a single sphere in unbounded flow then appears as a special case. Thus we set

$$\begin{aligned} r &= R\lambda \,, \\ v_r &= R\omega[\mathcal{R}e\, u_\lambda(\lambda,\theta) + O(\mathcal{R}e^2)] \,, \\ v_\theta &= R\omega[\mathcal{R}e\, u_\theta(\lambda,\theta) + O(\mathcal{R}e^2)] \,, \\ v_\phi &= R\omega\left[\left(K\lambda + \frac{L}{\lambda^2}\right)\sin\theta + \mathcal{R}e\, u_\phi(\lambda,\theta) + O(\mathcal{R}e^2)\right] \,, \\ p &= p_0 + \mu\omega[\mathcal{R}e\, \varpi(\lambda,\theta) + O(\mathcal{R}e^2)] \,. \end{aligned} \tag{8.48}$$

We now substitute these results into the incompressible flow Eqs. (3.101) through (3.104) with time-dependent terms and body force terms deleted. Terms in $\mathcal{R}e^0$ cancel out and we ignore terms in $\mathcal{R}e^2$. The terms in $\mathcal{R}e^1$ give us the equations governing the perturbation flow; with the axial symmetry,

$$\frac{\partial u_\lambda}{\partial\lambda} + \frac{2u_\lambda}{\lambda} + \frac{1}{\lambda}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\theta\cot\theta}{\lambda} = 0 \,, \tag{8.49}$$

$$\tilde{\nabla}^2 u_\lambda - \frac{2u_\lambda}{\lambda^2} - \frac{2}{\lambda^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2u_\theta\cot\theta}{\lambda^2} = \frac{\partial\varpi}{\partial\lambda} - \lambda\left(K + \frac{L}{\lambda^3}\right)^2\sin^2\theta \,, \tag{8.50}$$

$$\tilde{\nabla}^2 u_\theta + \frac{2}{\lambda^2}\frac{\partial u_\lambda}{\partial\theta} - \frac{u_\theta}{\lambda^2\sin^2\theta} = \frac{1}{\lambda}\frac{\partial\varpi}{\partial\theta} - \lambda\left(K + \frac{L}{\lambda^3}\right)^2\cos\theta\sin\theta \,, \tag{8.51}$$

$$\tilde{\nabla}^2 u_\phi - \frac{u_\phi}{\lambda^2\sin^2\theta} = 0 \,, \tag{8.52}$$

where $\tilde{\nabla}^2$ is the dimensionless Laplacian, i.e.,

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial\lambda^2} + \frac{2}{\lambda}\frac{\partial}{\partial\lambda} + \frac{1}{\lambda^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{\lambda^2}\frac{\partial}{\partial\theta} \,. \tag{8.53}$$

Since the basic flow (8.48) satisfies the no-slip conditions, all velocity components of the perturbation solution must vanish on both spheres. Thus,

$$u_\lambda(1,\theta) = u_\lambda(\alpha,\theta) = 0\,, \tag{8.54}$$

$$u_\theta(1,\theta) = u_\theta(\alpha,\theta) = 0\,, \tag{8.55}$$

$$u_\phi(1,\theta) = u_\phi(\alpha,\theta) = 0\,, \tag{8.56}$$

in which $\alpha = \tilde{R}/R$.

With (8.52) and (8.56), we have immediately

$$u_\phi(\lambda,\theta) = 0\,. \tag{8.57}$$

Thus, to the present order of approximation, the perturbation flow consists entirely of a circulation in the meridional planes. To determine the nature of this circulation, we introduce a dimensionless Stokes stream function $\Upsilon(\lambda,\theta)$ such that

$$u_\lambda = \frac{1}{\lambda^2 \sin\theta}\frac{\partial \Upsilon}{\partial \theta}, \quad u_\theta = -\frac{1}{\lambda \sin\theta}\frac{\partial \Upsilon}{\partial \lambda}\,. \tag{8.58}$$

Cross-differentiating (8.50) and (8.51) so as to eliminate ϖ yields

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\lambda^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{\lambda^2}\frac{\partial}{\partial\theta}\right)^2 \Upsilon \\
&= \sin\theta\left\{\frac{\partial}{\partial\lambda}\left[\left(K\lambda + \frac{L}{\lambda^2}\right)^2 \cos\theta\sin\theta\right] - \frac{\partial}{\partial\theta}\left[\lambda\left(K + \frac{L}{\lambda^3}\right)^2 \sin^2\theta\right]\right\} \\
&= -6L\left(\frac{K}{\lambda^2} + \frac{L}{\lambda^5}\right)\sin^2\theta\cos\theta\,.
\end{aligned} \tag{8.59}$$

The boundary conditions on Υ can be deduced by substituting (8.58) into (8.54) and (8.55). From (8.55) we see immediately that $\partial\Upsilon/\partial\lambda$ must vanish on both spheres. Condition (8.54) tells us that $\partial\Upsilon/\partial\theta$ vanishes on the spheres, or equivalently, that Υ is constant on each sphere. If we assume $\Upsilon(1,\theta) = c$ and $\Upsilon(\alpha,\theta) = \tilde{c}$, where c and $\tilde{c}$ are different, we make nonsense of (8.58): for u_θ to remain finite on the axis of symmetry, $\partial\Upsilon/\partial\lambda$ must vanish along $\theta = 0$, which is not the case if Υ varies continuously from the value c to the different value $\tilde{c}$. Consequently $\tilde{c} = c$. Moreover we may take $c = \tilde{c} = 0$ without affecting the meaning of the stream function. We thus obtain the boundary conditions

$$\Upsilon(1,\theta) = \Upsilon(\alpha,\theta) = \frac{\partial}{\partial\lambda}\Upsilon(1,\theta) = \frac{\partial}{\partial\lambda}\Upsilon(\alpha,\theta) = 0\,. \tag{8.60}$$

The solution of (8.59) subject to (8.60) is easily carried out with the help of the identity

$$\left(\frac{\partial^2}{\partial\lambda^2}+\frac{1}{\lambda^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{\lambda^2}\frac{\partial}{\partial\theta}\right)^2(\lambda^n\sin^2\theta\cos\theta)=\sin^2\theta\cos\theta\left(\frac{\partial^2}{\partial\lambda^2}-\frac{6}{\lambda^2}\right)^2\lambda^n$$
$$=\sin^2\theta\cos\theta[(n-2)(n-3)-6][\;n\;\;(n-1)-6]\lambda^{n-4}$$
$$=n(n-5)(n-3)(n+2)\lambda^{n-4}\qquad\sin^2\theta\cos\theta\;. \tag{8.61}$$

A solution to (8.59) which retains sufficient arbitrariness to satisfy (8.60) is provided by

$$\Upsilon(\lambda,\theta)=-\frac{1}{4}L\left(A+\frac{B}{\lambda^2}+C\lambda^3+D\lambda^5+K\lambda^2-\frac{L}{\lambda}\right)\sin^2\theta\cos\theta\;, \tag{8.62}$$

where A, B, C, D are constants of integration. With (8.60),

$$\begin{aligned}A+B+C+D&=L-K\;,\\ \alpha^2A+B+\alpha^5C+\alpha^7D&=\alpha(L-\alpha^3K)\;,\\ -2B+3C+5D&=-(L+2K)\;,\\ -2B+3\alpha^5C+5\alpha^7D&=-\alpha(L+2\alpha^3K)\;.\end{aligned} \tag{8.63}$$

Expanding Δ, the determinant of coefficients, we find

$$\Delta=\begin{vmatrix}1&1&1&1\\ \alpha^2&1&\alpha^5&\alpha^7\\ 0&-2&3&5\\ 0&-2&3\alpha^5&5\alpha^7\end{vmatrix}=\begin{vmatrix}(1-\alpha^2)&(\alpha^5-\alpha^2)&(\alpha^7-\alpha^2)\\ -2&3&5\\ -2&3\alpha^5&5\alpha^7\end{vmatrix}$$
$$=4\alpha^{12}-25\alpha^9+42\alpha^7-25\alpha^5+4\alpha^2 \tag{8.64}$$
$$=\alpha^2(\alpha-1)^4(4\alpha^6+16\alpha^5+40\alpha^4+55\alpha^3+40\alpha^2+16\alpha+4)\;.$$

By Descartes' rule of signs, the only positive real root of $\Delta=0$ is the quadruple root at unity. Since $\alpha=\tilde{R}/R>1$, the coefficients A, B, C, D are uniquely determined by (8.63).

There is, of course, no conceptual difficulty in carrying out the explicit solution of the system (8.63). This was in fact the approach used in a note by Haberman (1962). Haberman's notation is somewhat different from that used here. In particular his stream function is chosen with opposite sign.

However it is instructive to use a somewhat different approach, which involves working backwards: the boundary conditions are met first, *then* the coefficients are adjusted to satisfy the differential equation. Since $\Upsilon(r,\theta)$ has a double zero on both spheres, we set

$$\Upsilon(\lambda,\theta)=-\frac{1}{4}L\frac{(1-\lambda)^2(\alpha-\lambda)^2}{\lambda^2}(a+b\lambda+c\lambda^2+d\lambda^3)\sin^2\theta\cos\theta\;, \tag{8.65}$$

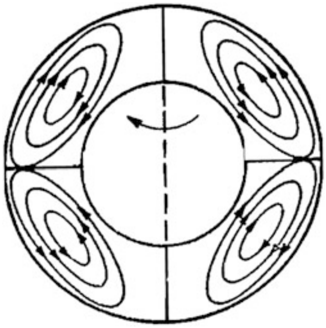


Fig. 8.5 Streamlines in a meridional plane

and adjust the coefficients a, b, c, and d so that the expansion of (8.65) gives a result consistent with (8.62). This procedure leads to the result

$$\begin{aligned}
&-\frac{4(4\alpha^6+16\alpha^5+40\alpha^4+55\alpha^3+40\alpha^2+16\alpha+4)\lambda^2}{(1-\lambda)^2(\alpha-\lambda)^2\sin^2\theta\cos\theta}\Upsilon(\lambda,\theta)\\
&= KL[\alpha^2(4\alpha^2+7\alpha+4)+2\alpha(\alpha+1)(4\alpha^2+7\alpha+4)\lambda\\
&+4(\alpha+1)^2(\alpha^2+3\alpha+1)\lambda^2+2(\alpha+1)(\alpha^2+3\alpha+1)\lambda^3]\\
&+\frac{L^2}{\alpha}[2(\alpha+1)(\alpha^4+3\alpha^3+7\alpha^2+3\alpha+1)\\
&+(4\alpha^4+16\alpha^3+25\alpha^2+16\alpha+4)\lambda\\
&\quad +2(\alpha+1)(3\alpha^2+4\alpha+3)\lambda^2+(3\alpha^2+4\alpha+3)\lambda^3]\,. \qquad (8.66)
\end{aligned}$$

In a typical case, the projections of the streamlines upon a meridional plane form four separate vortices, one in each quadrant, as illustrated in Fig. 8.5. If K and L have different signs, however, there may be other dividing streamlines, viz., the circles $\lambda = \lambda_\nu$, where λ_ν represents the zeros of the right side of (8.66).

If the outer sphere is absent, so that $K = 0$ and $L = 1$, the calculations become quite a bit easier. To avoid infinite velocities, the coefficients C and D in Eq. (8.62) are set equal to zero. The remaining coefficients are determined by the boundary conditions at $\lambda = 1$:

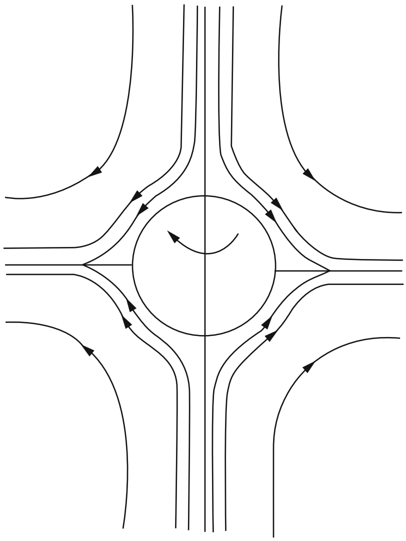
$$\begin{aligned}
A + B &= 1\,,\\
-2B &= -1\,. \qquad (8.67)
\end{aligned}$$

Hence, $A = B = 1/2$, and (8.62) becomes

$$\Upsilon(\lambda,\theta) = -\frac{1}{8}\left(1-\frac{2}{\lambda}+\frac{1}{\lambda^2}\right)\sin^2\theta\cos\theta \qquad (8.68)$$

$$= -\frac{1}{8}(\lambda-1)^2\lambda^{-2}\sin^2\theta\cos\theta\,. \qquad (8.69)$$

Fig. 8.6 Streamlines around one sphere



For this case the streamline projections do not close upon themselves: if the stream function and the colatitude are assigned specific values $\Upsilon_0, \theta_0, (\theta_0 \neq 0, \pi/2, \pi)$, Eq. (8.68) takes the form

$$k^2\lambda^2 = (\lambda - 1)^2 \ , \tag{8.70}$$

where

$$k^2 = -\frac{8\Upsilon_0}{\sin^2\theta_0 \cos\theta_0} < \lim_{\lambda\to\infty} \frac{(\lambda-1)^2}{\lambda^2} = 1; \tag{8.71}$$

Eq. (8.70) has only one positive solution, viz., $\lambda = (1-k)^{-1}$. Thus, if we proceed along the ray $\theta = \theta_0$, we encounter the streamline $\Upsilon = \Upsilon_0$ only once. We obtain the configuration shown in Fig. 8.6; since $\Upsilon(r, \theta)$ approaches a finite limit as r tends to infinity, θ fixed, the streamlines become asymptotically radial.

8.5 Rotlets

As discussed in Sect. 6.6, the flow generated at great distance from a concentrated force applied to the fluid is that of a Stokeslet, a fact that is often exploited in the solution of boundary value problems. If the fluid is acted upon by a concentrated force couple, a similar result is found.

When the outer sphere is absent, (8.37) becomes

$$v_\phi = \omega(R^3/r^2)\sin\theta \ , \tag{8.72}$$

and the expression (8.44) for the moment which the sphere exerts on the fluid reduces to

$$M = 8\pi\mu R^3\omega \; . \tag{8.73}$$

If the fluid rotates about an arbitrary axis, so that the rotation is described by a rotation vector $\boldsymbol{\omega}$, the corresponding moment vector $\mathbf{M}$ exerted by the sphere on the fluid is given by

$$\mathbf{M} = 8\pi\mu R^3\boldsymbol{\omega}. \tag{8.74}$$

With an appropriate rotation of coordinates, (8.72) can be expressed in terms of $\mathbf{M}$. The resulting velocity field

$$v_i^R = (1/8\pi\mu r^3)\epsilon_{ijk}M_j x_k \tag{8.75}$$

is termed a **rotlet**. Just as a Stokeslet can be interpreted as the flow resulting from a delta-function force acting at the origin, a rotlet, as discussed by Guazzelli and Morris (2011), can be interpreted as the flow created by a delta function force couple acting at the origin.

Chapter 9
Lubrication Theory

Abstract Lubrication theory is the hydrodynamical analog of shell theory, capitalizing on the fact that the physical domain is thin in one direction compared with the others. A stretched coordinate, akin to that used in boundary layer theory, is used to derive the general Reynolds equation. If the lubricant is incompressible, this is a linear equation for the pressure in terms of time and the transverse space variables. In the important case where the lubricant is an isothermal gas, the Reynolds equation is nonlinear. For slider bearings, externally pressurized bearings, and journal bearings, the pressure is determined by the steady-state Reynolds equations, unless transients are of interest. Squeeze bearings are governed by the time-dependent Reynolds equation.

It has long been known that the presence of a fluid film greatly reduces the sliding friction between solid objects. The enormous practical importance of this effect has, quite naturally, stimulated a great deal of research, both theoretical and experimental. Since much of the work has been geared directly to application, there is some tendency among hydrodynamicists to regard lubrication theory as a prosaic subject, complicated but not fundamentally difficult, useful but not intellectually satisfying. There is at least a subconscious inclination to consider it not really part of hydrodynamics at all, but a separate subject which, like hydraulics and acoustics, happens to use a certain part of the hydrodynamics vocabulary. This is quite unfortunate, for a considerable amount of "real hydrodynamics" is involved in the foundations of lubrication theory.

Lubrication theory is the hydrodynamical analog of shell theory. In most lubricating films the thickness of the film is extremely small compared with its lateral dimensions. Properly handled, this observation can be used to eliminate from the hydrodynamic equations the dependence upon one of the three spatial variables. Roughly speaking, the continuity equation is integrated across the film and the Navier-Stokes equation is used to evaluate the quantities appearing as integrands.

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_9,

We shall consider only the fundamentals of lubrication theory, plus a few illustrative examples. Development and application are extensively covered in Pinkus and Sternlicht (1961), Gross (1962), and Tipei (1962). A more recent monograph is Hori (2006).

9.1 Physical Origins of Fluid-Film Lubrication

From a very superficial consideration of the matter one might expect that the main problem of lubrication theory is to predict the friction which results from a given bearing configuration. However a little more reflection reveals that the real problem is quite different. Lubricating films are usually found between two solid objects which are acted upon by forces tending to push them together. To carry this load, the film must develop normal stresses. We shall see in the next section that in a lubricating film the significant portion of the stress tensor is represented by the pressure term. Thus the first task is to predict the pressure distribution and from it the **load-carrying capacity**.

Many different types of film-lubricated bearings are in use, but the relevant features are represented by four basic types:

1. **In the *slider bearing* the lubricating pressure is generated by the lateral motion of two surfaces which are not quite parallel.**
2. **In the *externally pressurized bearing* lubricant is forced into the film at a pressure high enough to sustain the load.**
3. **The *squeeze-bearing*, which gets its load-carrying capacity from relative normal motion of the surfaces, was first developed[1] as the first edition of this book was being written, although it had been known for some time that lubricating pressure can be generated by such motion.**
4. **The *journal bearing* is a bearing wrapped around a cylinder. This could be any one of the three previous types, or a combination of them. Boundary conditions applied on the bearing periphery in the other cases are replaced here by periodicity conditions.**

Thus the four basic bearings represent three different principles of lubrication: lateral motion; external pressurization; relative normal motion. All fluid-film bearings generate their load-carrying capacity through application of these principles, singly or in combination. The journal bearing is included as a basic type because its boundary conditions are distinctive.

Again from a superficial viewpoint, one might obtain an erroneous concept of the physical origin of the lubricating pressure. The slider-bearing configuration illustrated in Fig. 9.1a brings to mind such concepts as hydrofoil, lifting surface, Bernoulli principle. However the principal effects in fluid-film lubrication have nothing to do with Bernoulli's equation, which assumes inviscid flow; in most lubrication problems the relevant Reynolds number is so small that viscosity dominates completely. The misconception arises since Fig. 9.1a is not drawn to

[1] For an engineering description, Salbu (1964).

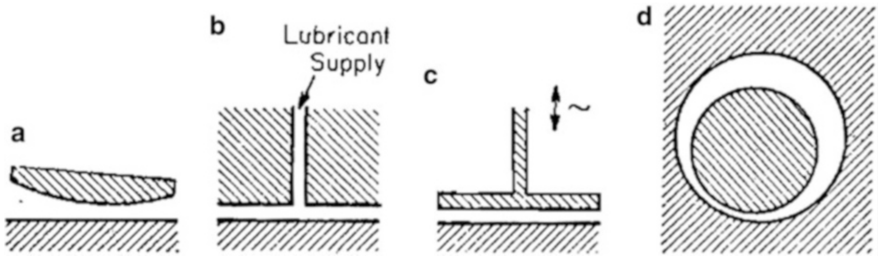


Fig. 9.1 (**a**) Slider bearing. (**b**) Externally pressurized bearing. (**c**) Squeeze bearing. (**d**) Journal bearing

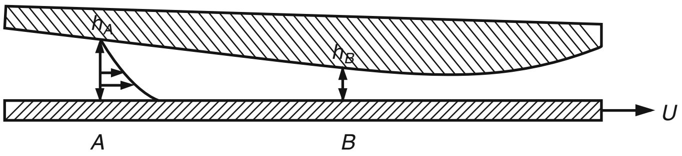


Fig. 9.2 Detailed slider bearing

scale; if it were, the gap would not be visible. The illustration sometimes given is that a slider bearing, drawn to scale, would correspond to two surfaces the size of football pitches, 1 cm apart at one end, 2 at the other.

The true physical origin of lubricating pressure is more easily grasped by considering first the externally-pressurized bearing. Lubricant supplied at the orifice streams toward the periphery, where ambient pressure obtains. This streaming, through an extremely narrow passage, is resisted by the viscosity of the fluid. To overcome this resistance, a pressure gradient must be set up. Thus the portion of the film away from the periphery is at a pressure higher than ambient,[2] so that there is a pressure difference across the upper surface of the bearing, i.e., there is a load-carrying capacity.

In the slider bearing the origin of the lubricating pressure is more subtle, but again it arises from the fluid viscosity rather than from inertia. To illustrate the point, let us consider an extremely simple case, which is nevertheless often approximated in practice: we assume that the lubricant is an incompressible Newtonian fluid and that the local velocity profile approximates the exact solution (4.5), with the transverse velocity v set equal to zero. By referring to Fig. 9.2, we see that the pressure gradient cannot vanish throughout the film: if it did, the resulting shear flow would produce a lubricant flux of $\frac{1}{2}Uh_A$ through the section at **A** and a smaller flux $\frac{1}{2}Uh_B$ through the section at **B**, violating continuity. To achieve the proper mass balance, the shear flow

[2]This explanation is somewhat oversimplified. In practice inertia dominates the flow in the immediate neighborhood of the orifice, giving rise to a **Bernoulli's region**, in which the pressure is subambient.

must be reduced by a component of back-flow, stronger at **A** than at **B**. Associated with this backflow is an adverse pressure gradient. Hence, as we proceed from **A** farther into the region of lubrication, the pressure increases. The rate of increase at **B** is slower, and eventually the backflow reverses, reinforcing the shear flow, so that the pressure returns to ambient at the trailing edge of the bearing.

Although the fluid motion in a lubricating film is “slow”, in the sense that the appropriate Reynolds number is quite small, it does not necessarily follow that compressibility can be neglected. In recent years gas bearings have come into widespread use. In a gas, significant variations in pressure are usually accompanied by significant variations in density.

Fortunately, however, there is seldom any need to bring in the energy equation. Gas lubrication films are extremely thin—from 0.5 to 25 μm, and the bearing surfaces are usually metals—excellent conductors of heat. Hence, unless we consider *refinements* of lubrication theory, we may consider the film to be isothermal, so that the density of the gas is proportional to its pressure.

The heuristic discussion presented above is of necessity somewhat vague. It should, however, provide at least a qualitative feeling for the way an analytical development should proceed. Such a development is considered next.

9.2 The Mathematical Foundations of Lubrication Theory

Assume that a thin, continuous film of fluid is contained between the surfaces

$$x_3 = \mathfrak{H}(x_1, x_2, t) \ , \tag{9.1}$$

$$x_3 = \mathfrak{H}'(x_1, x_2, t) \ , \tag{9.2}$$

where x_1, x_2, x_3 are right-handed Cartesian coordinates fixed in the ambient fluid as exhibited in Fig. 9.3. The film thickness h, defined by

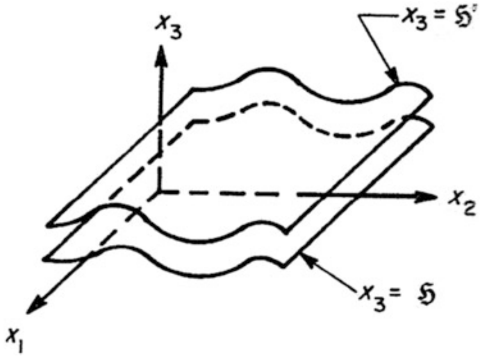
$$h(x_1, x_2, t) = \mathfrak{H}'(x_1, x_2, t) - \mathfrak{H}(x_1, x_2, t) \tag{9.3}$$

is positive for all values of x_1, x_2, t; the surfaces move relative to the ambient fluid with velocity components $V_\alpha, V_\alpha^{'}$.

The surfaces bounding the film may be either rigid or flexible, but are assumed continuous. At each point of each surface three components of velocity provide one degree of freedom too many and must therefore be related through a kinematic constraint analogous to the kinematic boundary condition on a free surface (see Sect. 2.6.4). With the convention that Latin indices extend over the values 1, 2, and Greek indices extend over 1, 2, 3, the constraints are

$$V_3 = \frac{\partial \mathfrak{H}}{\partial t} + V_i \frac{\partial \mathfrak{H}}{\partial x_i} \ ,$$

Fig. 9.3 General configuration of lubrication flow



$$V_3' = \frac{\partial \mathfrak{H}'}{\partial t} + V_i' \frac{\partial \mathfrak{H}'}{\partial x_i} , \tag{9.4}$$

repeated indices denoting summation.

We consider only cases in which the viscosity coefficients μ and λ can be assumed constant. We thus exclude many problems, occurring mostly for oil-film lubrication, in which temperature variations cause μ and λ to vary significantly. The appropriate form of the Navier-Stokes equation is therefore (2.130). Neglecting the body force term and converting to Greek indices gives us

$$\rho \frac{D v_\kappa}{Dt} = \frac{\partial}{\partial x_\kappa}[-p + (\lambda + \mu)\Delta] + \mu \frac{\partial^2 v_\kappa}{\partial x_\alpha \partial x_\alpha} , \tag{9.5}$$

where the dilation Δ is given by[3]

$$\Delta = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{\partial v_\alpha}{\partial x_\alpha} ; \tag{9.6}$$

the second part of (9.6) follows from the continuity equation (2.35).

Lubrication theory is founded on the assumption that the film thickness is small compared with the lateral dimensions of the bounding surfaces. Thus, if h_0 denotes a typical value of $h(x_1, x_2, t)$ and B denotes a typical lateral dimension, we introduce

$$\varepsilon = \frac{h_0}{B} \tag{9.7}$$

as a perturbation parameter.

[3]In Langlois (1962), Δ is defined with opposite sign. The form (9.6) is used here for consistency with Eqs. (3.71) and (3.95). With (9.6), Δ is positive when the fluid is expanding.

We shall want to compare the magnitudes of the various terms in the Navier-Stokes equation. However in the x_i coordinate system the terms have a misleading appearance, for the velocity components vary much more rapidly in the x_3-direction than in the lateral directions. In order to indicate the magnitude of each term by its coefficient, we introduce a dimensionless coordinate system which stretches the coordinate normal to the film:

$$X_i = \frac{x_i}{B} \quad (i = 1, 2), \tag{9.8}$$

$$z = \frac{x_3}{\varepsilon^m B} = \left(\frac{B}{h_0}\right)^m \left(\frac{x_3}{B}\right) , \tag{9.9}$$

where the value of the positive exponent m will be determined later.

To complete the normalization, we must select, in a meaningful way, a dimensionless time variable, dimensionless pressure and density, and dimensionless components of velocity.

For the time scale, we choose the reciprocal of a typical frequency ω of the squeeze component of surface motion. Thus we assume

$$V_3 = h_0\omega W, \quad V_3' = h_0\omega W' , \tag{9.10}$$

where W and W' are dimensionless velocities of order unity, and introduce a dimensionless time T defined by

$$T = \omega t . \tag{9.11}$$

To be consistent with (9.10), we let

$$v_3 = h_0\omega w . \tag{9.12}$$

We can now select the appropriate value of the exponent m in Eq. (9.9). The variation of v_3 across the film is of order $h_0\omega$, so that $\partial v_3/\partial x_3$ is of order ω. With (9.9) and (9.12), however,

$$\frac{\partial v_3}{\partial x_3} = \frac{h_0\omega}{\varepsilon^m B}\frac{\partial w}{\partial z} = \varepsilon^{1-m}\omega\frac{\partial w}{\partial z} . \tag{9.13}$$

If $\partial w/\partial z$ is to be of order unity, we must take $m = 1$. Equation (9.9) therefore becomes

$$z = \frac{x_3}{\varepsilon B} = \frac{x_3}{h_0} . \tag{9.14}$$

The scale of the lateral velocity components is not necessarily related to the scale of the squeeze component. While it is true that the squeeze motion forces fluid outward or sucks it inward, at a characteristic velocity ωB, there is also a contribution to the lateral velocity arising from the lateral motion of the bearing surfaces. We need not require that the lateral components of surface motion remain constant, but it is convenient to assume that their variation is not too great; this allows us to introduce a reference velocity V such that

$$V_i = VU_i, \quad V_i' = VU_i' , \tag{9.15}$$

where the dimensionless velocities U_i, U_i' are of order unity. To account for both contributions to the lateral velocity, we let

$$v_i = (\omega B + V)u_i , \tag{9.16}$$

and expect the u_i to be of order unity.

In terms of our dimensionless quantities, the constraints (9.4) becomes

$$\omega W = \omega \frac{\partial \tilde{\mathfrak{H}}}{\partial T} + \frac{V}{B} U_i \frac{\partial \tilde{\mathfrak{H}}}{\partial X_i} ,$$
$$\omega W' = \omega \frac{\partial \tilde{\mathfrak{H}}'}{\partial T} + \frac{V}{B} U_i' \frac{\partial \tilde{\mathfrak{H}}'}{\partial X_i} , \tag{9.17}$$

in which

$$\tilde{\mathfrak{H}} = \frac{\mathfrak{H}}{h_0}, \quad \tilde{\mathfrak{H}}' = \frac{\mathfrak{H}'}{h_0} . \tag{9.18}$$

The factor $V/\omega B$ implicit in Eq. (9.17) provides a measure of the relative magnitude of the two reciprocal times characteristic of the bearing kinematics: V/B represents a shear rate characteristic of the lateral motion; ω, as defined above, is a typical frequency of the squeeze motion.

We now turn our attention to the definition of a dimensionless pressure. As pointed out in Sect. 9.1, viscous effects dominate. Consequently it is a mistake to normalize with respect to the dynamic head. The typical pressure is proportional to the coefficient of viscosity, multiplied by an appropriate measure of the rate of deformation tensor. The parameter ε may enter into the coefficient of proportionality; to allow for this possibility, we set

$$p = \mu \left(\omega + \frac{V}{B} \right) \varepsilon^{-n} \varpi , \tag{9.19}$$

where the exponent n will be selected later. It is a priori conceivable that the form (9.19) might not be sufficiently general; if this were the case (it is not), the fact would become evident in the analysis which follows.

For the normalized density, we use

$$q = \frac{\rho}{\rho_a} \,, \tag{9.20}$$

where ρ_a denotes the ambient density.

With Eqs. (9.6), (9.8), (9.14), (9.12), and (9.16), the dilatational stress $(\lambda + \mu)\Delta$ can be expressed

$$(\lambda + \mu)\Delta = \mu \left(\omega\theta_S + \frac{V}{B}\theta_L \right) \,, \tag{9.21}$$

where θ_S and θ_L are dimensionless quantities, defined by

$$\theta_S = \left(1 + \frac{\lambda}{\mu}\right)\left(\frac{\partial u_i}{\partial X_i} + \frac{\partial w}{\partial z}\right) \,, \tag{9.22}$$

$$\theta_L = \left(1 + \frac{\lambda}{\mu}\right)\frac{\partial u_i}{\partial X_i} \,. \tag{9.23}$$

Near the bearing periphery, where steep gradients of pressure obtain, θ_S and θ_L may be quite large. In the interior of the film, however, they are normally of order unity. As indicated at the end of Sect. 9.1, we consider only films with continuous bounding surfaces. However bearings with steps and grooves are sometimes used in practical applications. In such bearings θ_S and θ_L may be large in the neighborhood of the discontinuity.

In terms of the dimensionless quantities introduced above, the $\kappa = 1, 2$ components of the equation of motion (9.5) become

$$\begin{aligned} \varepsilon^{2-n}\frac{\partial \varpi}{\partial X_i} &= \frac{\partial^2 u_i}{\partial z^2} - \mathcal{R}e_S q\left(\frac{\partial u_i}{\partial T} + w\frac{\partial u_i}{\partial z}\right) - (\mathcal{R}e_S + \mathcal{R}e_L)qu_j\frac{\partial u_i}{\partial X_j} \\ &+ \varepsilon^2\left[\frac{\partial\theta_S/\partial X_i}{(1 + V/\omega B)} + \frac{\partial\theta_L/\partial X_i}{(1 + \omega B/V)} + \frac{\partial^2 u_i}{\partial X_j \partial X_j}\right] \,, \end{aligned} \tag{9.24}$$

where $\mathcal{R}e_S$ and$\mathcal{R}e_L$ are modified Reynolds numbers corresponding, respectively, to the squeeze motion and to the lateral motion:

$$\mathcal{R}e_S = \frac{\omega\rho_a h_0^2}{\mu}, \quad \mathcal{R}e_L = \frac{V\rho_a h_0^2}{B\mu} \,. \tag{9.25}$$

For Eq. (9.24) to be meaningful, we must set $n = 2$, so that the pressure normalization (9.19) becomes

$$p = \mu\left(\omega + \frac{V}{B}\right)\frac{\varpi}{\varepsilon^2} \,. \tag{9.26}$$

The $\kappa = 3$ component of Eq. (9.5) becomes

$$\frac{\partial \varpi}{\partial z} = \frac{\varepsilon^2}{(1+V/\omega B)}[\frac{\partial^2 w}{\partial z^2} + \frac{\partial \theta_S}{\partial z} - \mathcal{R}e_S q\left(\frac{\partial w}{\partial T} + w\frac{\partial w}{\partial z}\right)$$
$$- (\mathcal{R}e_S + \mathcal{R}e_L) q u_i \frac{\partial w}{\partial X_i}]$$
$$+ \frac{\varepsilon^2}{(1+\omega B/V)}\frac{\partial \theta_L}{\partial z} + \frac{\varepsilon^4}{(1+V/\omega B)}\frac{\partial^2 w}{\partial X_i \partial X_i}\,. \tag{9.27}$$

Equation (9.27) implies that, with neglect only of terms of the second degree or higher in ε, the pressure is constant across the film. This conclusion does not necessarily apply near the periphery of the bearing nor in regions where one or both of the dimensionless dilational stresses θ_S, θ_L become large (of order ε^{-2}). Moreover either $q\mathcal{R}e_S$ or $q\mathcal{R}e_L$ could conceivably be of order ε^{-2}. In most cases of interest, however, it is correct to infer from Eq. (9.27) that $\partial\varpi/\partial z$ vanishes throughout the interior of the film, and we shall proceed on the assumption that this is the case. Consistent with this assumption is the reduction of Eq. (9.24) to

$$\frac{\partial \varpi}{\partial X_i} = \frac{\partial^2 u_i}{\partial z^2} - \mathcal{R}e_S q\left(\frac{\partial u_i}{\partial T} + w\frac{\partial u_i}{\partial z}\right) - (\mathcal{R}e_S + \mathcal{R}e_L) q u_j \frac{\partial u_i}{\partial X_j}\,. \tag{9.28}$$

In most bearing applications, the Reynolds numbers are negligibly small, so that (9.28) becomes

$$\frac{\partial \varpi}{\partial X_i} = \frac{\partial^2 u_i}{\partial z^2}\,. \tag{9.29}$$

In view of Eqs. (9.15), (9.16), and (9.18), the dimensionless velocity components satisfy the boundary conditions

$$[u_i]_{z=\tilde{\mathfrak{H}}} = \frac{U_i}{(1+\omega B/V)}\,,$$
$$[u_i]_{z=\tilde{\mathfrak{H}}'} = \frac{U_i'}{(1+\omega B/V)}\,. \tag{9.30}$$

Since ϖ is assumed constant across the film, integration of Eq. (9.29) subject to the boundary conditions (9.30) yields

$$u_i = \frac{1}{2}\frac{\partial \varpi}{\partial X_i}(z-\tilde{\mathfrak{H}})(z-\tilde{\mathfrak{H}}') + \frac{U_i\tilde{\mathfrak{H}}' - U_i'\tilde{\mathfrak{H}} + (U_i' - U_i)z}{(1+\omega B/V)H}\,, \tag{9.31}$$

in which

$$H = \frac{h}{h_0} = \tilde{\mathfrak{H}}' - \tilde{\mathfrak{H}}\,. \tag{9.32}$$

Equation (9.31) represents a preliminary result to which we shall return presently. For the moment, however, let us consider the continuity equation. In terms of the normalized variables

$$\frac{\partial q}{\partial T} + \left(1 + \frac{V}{\omega B}\right) \frac{\partial}{\partial X_i}(q u_i) + \frac{\partial}{\partial z}(q w) = 0\,. \tag{9.33}$$

As indicated previously, we neglect the z-variation of pressure. It was pointed out in Sect. 9.1 that temperature variations across gas films are also negligible. Hence, the density, and the normalized density, can be assumed constant across the film, even if the lubricant is a gas. Integration of Eq. (9.33) across the film therefore yields

$$H\frac{\partial q}{\partial T} + \left(1 + \frac{V}{\omega B}\right) \int_{\tilde{\mathfrak{H}}}^{\tilde{\mathfrak{H}}'} \frac{\partial (q u_i)}{\partial X_i} dz + q(W' - W) = 0\,. \tag{9.34}$$

However with the boundary conditions (9.30),

$$\begin{aligned} &\left(1 + \frac{V}{\omega B}\right) \int_{\tilde{\mathfrak{H}}}^{\tilde{\mathfrak{H}}'} \frac{\partial (q u_i)}{\partial X_i} dz \\ &= \left(1 + \frac{V}{\omega B}\right) \frac{\partial}{\partial X_i} \int_{\tilde{\mathfrak{H}}}^{\tilde{\mathfrak{H}}'} q u_i\, dz + \frac{qV}{\omega B}\left(U_i \frac{\partial \tilde{\mathfrak{H}}}{\partial X_i} - U_i' \frac{\partial \tilde{\mathfrak{H}}'}{\partial X_i}\right)\,, \end{aligned} \tag{9.35}$$

so that, if we again neglect density variation across the film,

$$\begin{aligned} &H\frac{\partial q}{\partial T} + q(W' - W) + \frac{qV}{\omega B}\left(U_i \frac{\partial \tilde{\mathfrak{H}}}{\partial X_i} - U_i' \frac{\partial \tilde{\mathfrak{H}}'}{\partial X_i}\right) \\ &+ \left(1 + \frac{V}{\omega B}\right) \frac{\partial}{\partial X_i}\left(q \int_{\tilde{\mathfrak{H}}}^{\tilde{\mathfrak{H}}'} u_i\, dz\right) = 0\,. \end{aligned} \tag{9.36}$$

In view of the kinematic constraints (9.17), Eq. (9.36) becomes

$$\frac{1}{1 + V/\omega B} \frac{\partial (qH)}{\partial T} + \frac{\partial}{\partial X_i}\left(q \int_{\tilde{\mathfrak{H}}}^{\tilde{\mathfrak{H}}'} u_i\, dz\right) = 0\,. \tag{9.37}$$

With the dimensionless velocity components u_i given by Eq. (9.31), Eq. (9.37) becomes

$$\begin{aligned} &\frac{\partial}{\partial X_i}\left(H^3 q \frac{\partial \varpi}{\partial X_i}\right) \\ &= \frac{6}{(1 + \omega B/V)} \frac{\partial}{\partial X_i}\left[qH(U_i + U_i')\right] + \frac{12}{(1 + V/\omega B)} \frac{\partial (qH)}{\partial T}\,. \end{aligned} \tag{9.38}$$

In terms of the original variables (9.38) becomes the **Reynolds equation**

$$\frac{\partial}{\partial x_i}\left(h^3\rho\frac{\partial p}{\partial x_i}\right) = 6\mu\left\{2\frac{\partial(\rho h)}{\partial t} + \frac{\partial}{\partial x_i}\left[\rho h(V_i + V_i')\right]\right\} , \tag{9.39}$$

which is the fundamental equation of lubrication theory.

If the lubricant is incompressible, the density cancels out of (9.39), leaving a linear equation for the pressure:

$$\frac{\partial}{\partial x_i}\left(h^3\frac{\partial p}{\partial x_i}\right) = 6\mu\left\{2\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_i}\left[h(V_i + V_i')\right]\right\} , \tag{9.40}$$

The other important case is that of an isothermal gas, for which the density is proportional to the pressure. Equation (9.39) then becomes

$$\frac{\partial}{\partial x_i}\left(h^3 p\frac{\partial p}{\partial x_i}\right) = 6\mu\left\{2\frac{\partial(ph)}{\partial t} + \frac{\partial}{\partial x_i}\left[ph(V_i + V_i')\right]\right\} , \tag{9.41}$$

If compressibility and temperature variation are both important, the density cannot be eliminated from the Reynolds equation, which is therefore coupled to an energy equation for the film. Sometimes, however, it is possible to get around this by using the **polytropic approximation**

$$p = c\rho^n , \tag{9.42}$$

where c and n are constants. In an isothermal film the **polytropic index** n takes the value unity. At the other extreme, seldom encountered in practice, when the lubricant is an adiabatic gas, $n = \gamma$, where γ is the specific-heat ratio. In general

$$1 \leq n \leq \gamma . \tag{9.43}$$

As we have indicated several times already, however, gas films are almost always isothermal. Thus, in most practical applications, either the incompressible form (9.40) or the isothermal form (9.41) is used, the former for liquid films, the latter for gas films.

Each boundary value problem considered in earlier chapters was treated by deriving an equation for a velocity component or for the stream function, and by solving this equation subject to appropriate boundary conditions. Now, in the lubrication problem, we derive instead an equation for the *pressure*. It is not yet clear that this can lead anywhere: by itself the pressure has no physical significance, since force boundary conditions involve the components of stress—of which the pressure is only a part. However, introducing the normalizations (9.8), (9.14), (9.12), (9.16), (9.26), (9.21) into the constitutive equation (2.125), we obtain

$$\frac{\varepsilon^2 T_{\alpha\beta}}{\mu(\omega + V/B)} = -\varpi\delta_{\alpha\beta} + \frac{\varepsilon^2\lambda}{(\lambda+\mu)}\frac{\omega\theta_S + V\theta_L/B}{(\omega + V/B)}\delta_{\alpha\beta} + \varepsilon E_{\alpha\beta} , \tag{9.44}$$

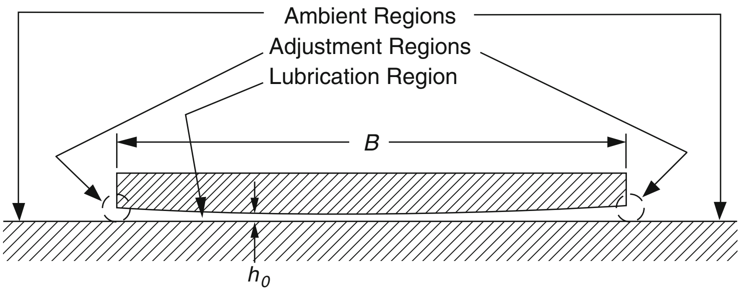


Fig. 9.4 Sketch of the different regions in the bearing

where the components of $E_{\alpha\beta}$ are given by

$$\begin{aligned} E_{ij} &= \varepsilon \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) , \\ E_{i3} = E_{3i} &= \frac{\partial u_i}{\partial z} + \varepsilon^2 \left(1 + \frac{V}{\omega B} \right)^{-1} \frac{\partial w}{\partial X_i} , \\ E_{33} &= 2\varepsilon \left(1 + \frac{V}{\omega B} \right)^{-1} \frac{\partial w}{\partial z} . \end{aligned} \tag{9.45}$$

If we drop terms involving ε from the right side of Eq. (9.44) and return to the physical variables, we obtain

$$T_{\alpha\beta} = -p\delta_{\alpha\beta} , \tag{9.46}$$

so that force boundary conditions can be expressed in terms of the pressure.

At the periphery of the bearing the lubricant is no longer constrained to flow through a narrow gap, so that a rapid adjustment to ambient conditions takes place, cf. Fig. 9.4. The extent of the adjustment region is measured by h_0. Since the typical lateral dimension of the film is $B = h_0/\varepsilon$, it is consistent with the approximations already introduced to neglect the breadth of the adjustment region. Thus for a boundary condition on the Reynolds equation, we require that p equal the ambient pressure all along the bearing periphery.

The four basic problems of lubrication theory are discussed in the sections which follow. These sections give only a brief introduction to a big subject; for the manifold extensions, nuances, and ramifications, Pinkus and Sternlicht (1961), Gross (1962), Tipei (1962), and Hori (2006) can be consulted.

9.3 Slider Bearings

If there is no relative normal motion of the bearing surfaces, the time-derivative term drops out of the Reynolds equation (9.39) and we obtain

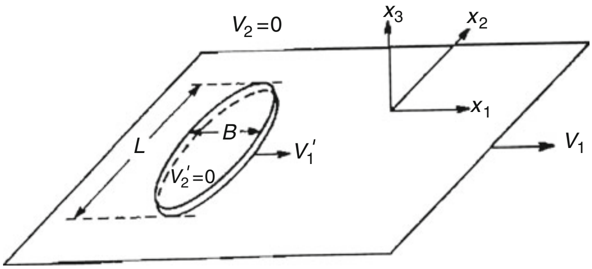


Fig. 9.5 The slider bearing problem

$$\frac{\partial}{\partial x_i}\left(h^3\rho\frac{\partial p}{\partial x_i}\right) = 6\mu\frac{\partial}{\partial x_i}\left[\rho h(V_i + V_i')\right] , \tag{9.47}$$

In many slider bearing problems, we can orient the coordinate system so that $V_2 = V_2' = 0$, see Fig. 9.5. In this case, Eq. (9.47) can be written

$$\frac{\partial}{\partial x_1}\left(h^3\rho\frac{\partial p}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(h^3\rho\frac{\partial p}{\partial x_2}\right) = 6\mu\frac{\partial}{\partial x_1}\left[\rho h(V_1 + V_1')\right] , \tag{9.48}$$

The extent of the bearing in the direction of motion is called the **breadth**; the transverse extent is called the **length**. This infuriating terminology is carried over from the study of journal bearings, where it corresponds to the definitions of length and breadth in their usual (non-technical) sense.

If the length-to-breadth ratio is very large, and if the leading and trailing edges of the bearing are straight, the $\partial/\partial x_2$ term in Eq. (9.48) may be unimportant, i.e., the **side leakage** may be negligible. In this case, (9.48) becomes a second order ordinary differential equation; dropping the subscripts, we obtain

$$\frac{d}{dx}\left(h^3\rho\frac{dp}{dx}\right) = 6\mu\frac{d}{dx}\left[\rho h(V + V')\right] . \tag{9.49}$$

For boundary conditions we require that the pressure be ambient at the leading and trailing edges.

A first integral of (9.49) is obtained immediately:

$$\frac{dp}{dx} = \frac{6\mu(V + V')}{h^2} - \frac{C}{\rho h^3} , \tag{9.50}$$

where C is a constant of integration. For incompressible films p is obtained by one more integration, plus use of the boundary conditions. For isothermal films ρ is proportional to p and (9.50) represents a first order differential equation, which may or may not be soluble in closed form.

In the more general case of arbitrary bearing plan-form, the mathematical character of Eq. (9.47) is best studied by rewriting it in normalized form. We use the dimensionless variables introduced in Sect. 9.2, except now we normalize the pressure with respect to the ambient pressure p_a, i.e., we set

$$P = \frac{p}{p_a} \, . \tag{9.51}$$

Equation (9.47) then becomes, in terms of the normalized density $q = \rho/\rho_a$,

$$\frac{\partial}{\partial X_i}\left(H^3 q \frac{\partial P}{\partial X_i}\right) = \Lambda \frac{\partial}{\partial X_i}\left[qH(U_i + U_i')\right] \, , \tag{9.52}$$

where the dimensionless parameter Λ is the **bearing number**, defined by

$$\Lambda = \frac{6\mu B V}{p_a h_0^2} \, . \tag{9.53}$$

If the lubricant is incompressible, there is little more to be said at the conceptual level. The normalized density cancels out of (9.52), which then becomes a linear equation of the elliptic type. If the bearing geometry and the functional forms of H, U_i, U_i' are simple enough, analytic solution can be achieved by well-known methods. If not, computer solution can be carried out by equally well-known numerical methods.

The isothermal case is less trivial. With q proportional to P, (9.52) becomes the nonlinear equation

$$\frac{\partial}{\partial X_i}\left(H^3 P \frac{\partial P}{\partial X_i}\right) = \Lambda \frac{\partial}{\partial X_i}\left[PH(U_i + U_i')\right] \, . \tag{9.54}$$

Except for the most trivial problems the search for exact solutions now appears futile. Numerical methods remain available. We should not, however, overlook the possibility of *asymptotic* solutions, especially since (9.54) contains a parameter (the bearing number).

If the bearing number is precisely zero, the pressure is ambient throughout the film. Consequently, for small Λ, we might expect the pressure to depart only slightly from ambient. We therefore set

$$P = 1 + \sum_{n=1}^{\infty} \Lambda^n P^{(n)} \, , \tag{9.55}$$

whence (9.54) becomes

$$\Lambda \frac{\partial}{\partial X_i}\left(H^3 \frac{\partial P^{(1)}}{\partial X_i}\right) + \sum_{n=2}^{\infty} \Lambda^n \frac{\partial}{\partial X_i}\left[H^3\left(\frac{\partial P^{(n)}}{\partial X_i} + \sum_{m=1}^{n-1} P^{(m)} \frac{\partial P^{(n-m)}}{\partial X_i}\right)\right]$$
$$= \Lambda \frac{\partial}{\partial X_i}\left[H(U_i + U_i')\right] + \sum_{n=2}^{\infty} \Lambda^n \frac{\partial}{\partial X_i}\left[P^{(n-1)} H(U_i + U_i')\right] \, . \tag{9.56}$$

Collecting equal powers of Λ yields

$$\frac{\partial}{\partial X_i}\left(H^3\frac{\partial P^{(1)}}{\partial X_i}\right)=\frac{\partial}{\partial X_i}\left[H(U_i+U_i')\right]\ , \tag{9.57}$$

$$\frac{\partial}{\partial X_i}\left[H^3\left(\frac{\partial P^{(n)}}{\partial X_i}+\sum_{m=1}^{n-1}P^{(m)}\frac{\partial P^{(n-m)}}{\partial X_i}\right)\right]=\frac{\partial}{\partial X_i}\left[P^{(n-1)}H(U_i+U_i')\right]$$
$$(n=2,3,\ldots)\ . \tag{9.58}$$

Since the pressure is ambient at the bearing periphery, all the $P^{(n)}$ must vanish there.
It is of interest to set

$$1+\Lambda P^{(1)}=\Pi_\Lambda\ , \tag{9.59}$$

so that, with the expansion (9.55),

$$P=\Pi_\Lambda+O(\Lambda^2)\ . \tag{9.60}$$

The boundary condition requires that Π_Λ be unity on the bearing periphery, and Eq. (9.57) implies that

$$\frac{\partial}{\partial X_i}\left(H^3\frac{\partial \Pi_\Lambda}{\partial X_i}\right)=\Lambda\frac{\partial}{\partial X_i}\left[H(U_i+U_i')\right]\ , \tag{9.61}$$

Thus Π_Λ satisfies the equation for an incompressible lubricant. Hence, with neglect only of terms of the second degree and higher in the bearing number, an isothermal gas can be considered incompressible. Physically, at low bearing number nothing very violent happens to the gas as it passes between the bearing surfaces.

9.4 Externally Pressurized Bearings

If there is neither lateral motion nor relative normal motion, the entire right side of the Reynolds equation vanishes, so that

$$\frac{\partial}{\partial x_i}\left(h^3\rho\frac{\partial p}{\partial x_i}\right)=0\ . \tag{9.62}$$

Whatever the pressure-density relationship, this homogeneous equation yields non-trivial solutions only if non-homogeneous boundary conditions are applied, i.e., only if somewhere on the periphery the pressure differs from ambient. In the usual construction of such bearings, as typified by Fig. 9.1b, ambient pressure obtains on the external boundary, but there are one or more internal boundaries across which lubricant is supplied at pressure p_s higher than ambient.

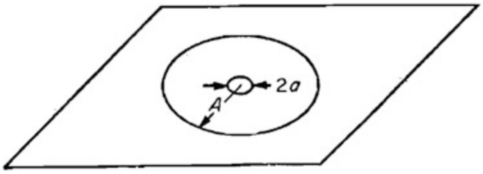


Fig. 9.6 Simplified pressurized bearing

For externally pressurized bearings there are two cases which yield linear equations. One of these, of course, is the case of incompressible lubricant. The other comes about when h is constant, so that (9.62) reduces (in the case of an isothermal gas) to

$$\nabla^2 p^2 = 0\,, \tag{9.63}$$

where ∇^2 is the two-dimensional Laplacian. By treating p^2 as the dependent variable, the methods of potential theory become available.

One of the simplest problems is important in practice. Let us consider that a flat, circular disc, with a hole in its center, is placed parallel to a flat back-up plate as in Fig. 9.6. Through the hole lubricant is supplied at pressure p_s. Thus, if the inner and outer radius of the disc are denoted, respectively, by a and A, the boundary conditions on (9.63) become[4]

$$\begin{aligned} p^2 &= p_s^2 \quad \text{on } x_1^2 + x_2^2 = a^2\,, \\ p^2 &= p_a^2 \quad \text{on } x_1^2 + x_2^2 = A^2\,. \end{aligned} \tag{9.64}$$

In view of the axial symmetry we seek a solution to (9.63), subject to (9.64), in the form

$$p = \sqrt{f(r)}\,, \tag{9.65}$$

where

$$r = \sqrt{x_1^2 + x_2^2}\,. \tag{9.66}$$

Substituting into (9.63) yields the ordinary differential equation

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) = 0\,. \tag{9.67}$$

[4] In practice the first of these idealized boundary conditions may or may not a good approximation. If the film thickness is too large or the supply pressure too high, it may be necessary to account for the Bernoulli region—or even turbulence and shock waves—in the neighborhood of the inner boundary.

Since $f(a) = p_s^2$ and $f(A) = p_a^2$, integration of (9.67) yields

$$f(r) = p_s^2 \left[1 - \frac{p_s^2 - p_a^2}{p_s^2} \frac{\ln(r/a)}{\ln(A/a)} \right] . \tag{9.68}$$

Using (9.65) then gives the pressure profile

$$\frac{p}{p_s} = \sqrt{1 - -\frac{p_s^2 - p_a^2}{p_s^2} \frac{\ln(r/a)}{\ln(A/a)}} . \tag{9.69}$$

9.5 Squeeze Films

If there is no lateral motion of the bearing surfaces, (9.39) reduces to the **squeeze-film equation**

$$\frac{\partial}{\partial x_i} \left(h^3 \rho \frac{\partial p}{\partial x_i} \right) = 12\mu \frac{\partial(\rho h)}{\partial t} . \tag{9.70}$$

For squeeze films the difference between incompressible and isothermal lubricants involves much more than the difference between linear and non-linear differential equations. If ρ is constant, (9.70) becomes

$$\frac{\partial}{\partial x_i} \left(h^3 \frac{\partial p}{\partial x_i} \right) = 12\mu \frac{\partial h}{\partial t} , \tag{9.71}$$

in which the time-derivatives of the dependent variable do not appear. Thus the incompressible squeeze-film problem is quasi-static: time enters only through the *coefficients* in the differential equation.

For isothermal films, however, (9.70) becomes the non-linear parabolic equation

$$\frac{\partial}{\partial x_i} \left(h^3 p \frac{\partial p}{\partial x_i} \right) = 12\mu \frac{\partial(ph)}{\partial t} . \tag{9.72}$$

This is a tough one, even for the simplest possible forms for the dependence of h on x_i and t. For asymptotic investigations, we use the dimensionless variables of Sect. 9.2, excepting again the pressure, which is normalized according to (9.48). Equation (9.72) then becomes

$$\frac{\partial}{\partial X_i} \left(H^3 P \frac{\partial P}{\partial X_i} \right) = \sigma \frac{\partial(PH)}{\partial t} , \tag{9.73}$$

where the dimensionless parameter σ is the **squeeze number**, defined by

$$\sigma = 12\mu \frac{B^2\omega}{p_a h_0^2} . \tag{9.74}$$

We proceed by analogy with the slider bearing problem. When the squeeze number is small, we expect that the pressure will not differ much from ambient: small squeeze number corresponds to low-frequency normal motion, so that the bearing has time to leak. Thus we use σ as a perturbation parameter, setting

$$P = 1 + \sum_{n=1}^{\infty} \sigma^n P^{(n)} . \tag{9.75}$$

The isothermal squeeze-film equation (9.72) then becomes

$$\sigma \frac{\partial}{\partial X_i}\left(H^3 \frac{\partial P^{(1)}}{\partial X_i}\right) + \sum_{n=2}^{\infty} \sigma^n \frac{\partial}{\partial X_i}\left[H^3\left(\frac{\partial P^{(n)}}{\partial X_i} + \sum_{m=1}^{n-1} P^{(m)} \frac{\partial P^{(n-m)}}{\partial X_i}\right)\right]$$
$$= \sigma \frac{\partial H}{\partial T} + \sum_{n=2}^{\infty} \sigma^n \frac{\partial (P^{(n-1)} H)}{\partial T} , \tag{9.76}$$

and collecting equal powers of σ yields

$$\frac{\partial}{\partial X_i}\left(H^3 \frac{\partial P^{(1)}}{\partial X_i}\right) = \frac{\partial H}{\partial T} , \tag{9.77}$$

$$\frac{\partial}{\partial X_i}\left[H^3\left(\frac{\partial P^{(n)}}{\partial X_i} + \sum_{m=1}^{n-1} P^{(m)} \frac{\partial P^{(n-m)}}{\partial X_i}\right)\right] = \frac{\partial (P^{(n-1)} H)}{\partial T} \qquad (n = 2, 3, \ldots) . \tag{9.78}$$

All the $P^{(n)}$ vanish on the bearing periphery.

If we set

$$1 + \sigma P^{(1)} = \Pi_\sigma , \tag{9.79}$$

so that, with (9.75),

$$P = \Pi_\sigma + O(\sigma^2) , \tag{9.80}$$

the boundary conditions require that Π be unity on the periphery, and (9.77) requires that Π_σ satisfy the incompressible squeeze-film equation. Thus, with neglect only of terms of the second degree and higher in the squeeze number, an isothermal gas film can be considered incompressible. Physically, at low squeeze number the gas leaks out before it is significantly compressed.

Squeeze-film effects are not confined to lubrication phenomena. For example, the tackiness of liquid adhesives is a squeeze-film effect. For a discussion of this subject, see Bikerman (1958).

9.6 Journal Bearings

As mentioned in Sect. 9.1, a journal bearing is nothing more than a bearing wrapped around a cylinder. Since there is nothing new except the periodicity condition, we shall consider only the simplest possible case, viz., that of an infinitely long bearing with incompressible lubricant and neither squeeze motion nor external pressurization.

Figure 9.7 is not drawn to scale. In practice the radius R of the rotating shaft (the **journal**) is more than 1,000 times the **clearance** c. Thus, although the derivation of the Reynolds equation presented in Sect. 9.1 does not hold for curved films, we can ignore the curvature in most journal bearing problems. Curvature corrections for the journal bearing have been calculated by Elrod (1960). We simply let $x = R\theta$ in the infinitely-long slider bearing Eq. (9.49). Since $V = \Omega R$ and $V' = 0$, and since ρ is assumed constant, we obtain

$$\frac{d}{d\theta}\left(h^3 \frac{dp}{d\theta}\right) = 6\mu\Omega R^2 \frac{dh}{d\theta} , \tag{9.81}$$

or the equivalent first integral

$$\frac{dp}{d\theta} = 6\mu\Omega R^2 \left(\frac{1}{h^2} - \frac{ck}{h^3}\right) , \tag{9.82}$$

where k is a dimensionless constant of integration.

The film thickness h depends both upon the clearance c and upon the **eccentricity** e. For $c/R \ll 1$, we may write

$$h = c(1 + \varepsilon \cos\theta) , \tag{9.83}$$

where ε is the **eccentricity ratio** e/c.

Substituting (9.83) into (9.82) and carrying out the rather lengthy integration yields

$$\begin{aligned} p &= p_0 \\ &\quad + \frac{6\mu\Omega R^2}{c^2(1-\varepsilon^2)^2}\left\{\frac{\varepsilon \sin\theta}{(1+\varepsilon\cos\theta)^2}\left[\frac{k}{2}(4-\varepsilon^2+3\varepsilon\cos\theta) - (1-\varepsilon^2)(1+\varepsilon\cos\theta)\right]\right. \\ &\quad \left. + \frac{1}{\sqrt{1-\varepsilon^2}}\left[2(1-\varepsilon^2) - (2+\varepsilon^2)k\right] \arctan\left[\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan\frac{\theta}{2}\right]\right\} . \end{aligned} \tag{9.84}$$

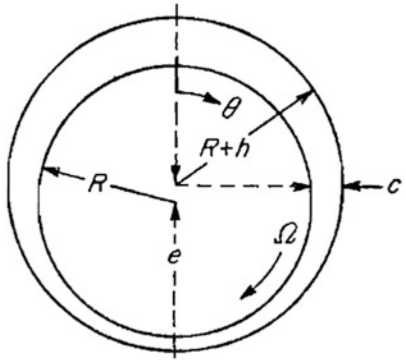


Fig. 9.7 Journal bearing

Since the fluid film is not open to the atmosphere, the constant of integration p_0 can be assigned arbitrarily. However the pressure must be single-valued function of θ, so that the coefficient of the arctangent term in (9.84) must be made to vanish. Therefore

$$k = 2\left(\frac{1-\varepsilon^2}{2+\varepsilon^2}\right) \tag{9.85}$$

and

$$p = p_0 + \frac{6\mu\Omega R^2\varepsilon(2+\varepsilon\cos\theta)\sin\theta}{c^2(2+\varepsilon^2)(1+\varepsilon\cos\theta)^2}\,. \tag{9.86}$$

In practice, μ, Ω, R, and c are known a priori, but ε is not. Under stable operation the journal assumes whatever eccentricity is required for Eq. (9.86) to produce the proper load-carrying capacity.

9.6.1 The Wannier Flow

Wannier (1950) solves the cylindrical bearing problem in Cartesian coordinates. Solutions of the biharmonic equation (7.3) are expressed as linear combinations of harmonic functions denoted by the generic variable φ. Therefore $x_1\varphi$ and $x_2\varphi$ are biharmonic as is the expression $(x_1^2 + x_2^2)\varphi$. Wannier uses electrostatics theory, especially the treatment of cylindrical geometries, to obtain the lubrication solution. The reader may consult the original paper for the full relationship.

An interesting limit case is obtained when the radius of the outer cylinder goes to infinity. The flow occurs in the upper half plane and is set into motion by the lower wall moving with a constant velocity U in the x_1 direction. A fixed cylinder of radius R is located above the wall with its center at a distance d. The Wannier solution for the velocity components is

$$v_1 = U - \frac{2(A+Fx_2)}{K_1}[(s+x_2) + \frac{K_1}{K_2}(s-x_2)] - \frac{B}{K_1}[(s+2x_2) - \frac{2x_2(s+x_2)^2}{K_2}]$$
$$- \frac{C}{K_2}[(s-2x_2) - \frac{2x_2(s-x_2)^2}{K_2}] - F\,ln(\frac{K_1}{K_2}), \tag{9.87}$$

$$v_2 = -\frac{2x_1(A+Fx_2)}{K_1K_2}(K_2-K_1) - \frac{2Bx_1x_2(s+x_2)}{K_1^2} - \frac{2Cx_1x_2(s-x_2)}{K_2^2}, \tag{9.88}$$

where

$$A = -\frac{Ud}{ln(\Gamma)},\ B = \frac{2U(d+s)}{ln(\Gamma)},\ C = \frac{2U(d-s)}{ln(\Gamma)},\ F = \frac{U}{ln(\Gamma)},\ \Gamma = \frac{d+s}{d-s},$$
$$K_1 = x_1^2 + (s+x_2)^2,\ K_2 = x_1^2 + (s-x_2)^2,\ s^2 = d^2 - R^2. \tag{9.89}$$

This solution is very useful for checking the numerical accuracy of Navier-Stokes solvers on irregular geometries.

Chapter 10
Introduction to the Finite Element Method

Abstract The necessary steps for setting up the finite element method are described, beginning with the weak formulation, which is the cornerstone of Galerkin's method. Various schemes for constructing the finite elements to implement this method are developed.

The numerical treatment of the fluid mechanics equations is classically performed by a method discretizing the space. The numerical process starts by the construction of a mesh, i.e. a set of grid points which the discrete variables are attached to. Then a methodology is built to replace the continuous partial derivatives with discrete approximations or operators that will eventually lead to the creation of matrix equations and the use of associated solvers.

Today three major methods constitute the state-of-the-art of the scientific domain called computational fluid dynamics (CFD): the finite difference method (FDM), the finite volume method (FVM) and the finite element method (FEM). These acronyms are subtly designated by the generic name FXM, the choice for the user being reduced to X.

FDM is the simplest tool to understand and to set up. The derivatives are approximated by discrete operators generated by Taylor series of the problem variables. For the velocity-pressure formulation of the Navier-Stokes equation, FDM led in the sixties of the last century to the well-known MAC (Marker and Cell) method Harlow and Welch (1965) and the projection methods due to Chorin (1968) and Témam (1969). FDM presents weaknesses in coping with complicated industrial geometries and in discretizating accurate boundary conditions.

FVM avoids some drawbacks of FDM by using the conservative form of the equations based on the presence of the divergence operator defined on the global computational domain. Finite volumes are generated by hexahedra or tetrahedra; then the divergence theorem is locally applied on each volume. Therefore the volume integrals are replaced by surface integrals for three-dimensional problems and by curvilinear integrals for two-dimensional cases. The variables are expressed

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_10,

by fluxes. This is one of the reasons why numericists who favor the physical interpretation of computational results are keen FVM practioners. One of the major FVM advantages rests upon the ability of using unstructured meshes that allow dealing with complex geometries. The reader is referred to Eymard et al. (2000), Hirsch (1991), and Versteeg and Malalasekera (2007) for further developments.

The finite element method is based on the weak formulation, the Galerkin method, and finite element theory. In this chapter we will describe all the necessary steps to set up the finite element method for diffusion problems in one and two space dimensions. However in order to keep the developments on a pedestrian track, we will skip some heavy mathematical details that the reader may get acquainted with from references given in due course. For general introductions without being exhaustive we may cite Carey and Oden (1983), Ciarlet (1978), Girault and Raviart (1986), Gresho and Sani (2000), Johnson (1990), Quarteroni and Valli (1997), and Strang and Fix (1973). The present description of the finite element method is inspired by the monograph of Azaïez et al. (2011).

The recent monograph by Sengupta (2013) is highly recommended as it covers the full world of discretization methods with the view of achieving high quality and high accuracy numerics.

10.1 Weak Formulation

For the sake of simplicity let us examine first a very simple boundary value problem given by the relationship

$$Lu = f, \quad \text{on} \quad \Omega\,, \tag{10.1}$$

where L is a linear operator described by partial derivatives, $u = u(\mathbf{x})$ the dependent variable and $f = f(\mathbf{x})$ a given source term. The domain Ω is bounded by a smooth and continuous boundary $\partial\Omega$. For the sake of facility we assume that (10.1) is subjected to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. The problem (10.1) is called the **strong formulation** because if, for example, $L = -\Delta$, the Laplace operator, it involves second order partial derivatives that from a mathematical standpoint should at least exist for the problem to be well posed. The weak formulation lowers the order of the derivatives to enlarge the function space where solutions are sought. Typically the weak formulation will use integration by parts to reduce the degree of the derivatives and consequently will require less continuity in the solution.

In a formal setting, the weak formulation is based on the relation

$$(Lu, w) = (f, w) \quad \text{for every } w\,, \tag{10.2}$$

where w is a well-chosen, sufficiently smooth function having first order continuous derivatives on Ω. The notation $(.,.)$ is a scalar product based on the integral definition

$$(f, g) = \int_\Omega f\, g\, d\Omega\ . \tag{10.3}$$

Note that Eq. (10.2) may be recast as

$$(Lu - f, w) = \int_\Omega (Lu - f)\, w d\Omega\ , \tag{10.4}$$

where the quantity $Lu - f$ is called the residual. If u were the solution of problem (10.1) then the residual would vanish. In the context of a numerical approach this non zero residual is projected to minimize the error. This theory is known as the **weighted residual method**.

The key question is: which space is involved in the selection of the test function w? If the choice is appropriate it is possible to prove that weak formulation and minimization of a variational principle are equivalent procedures. This is no trivial matter and we refer the reader to the following references where these topics are treated in full detail and with great care: Deville et al. (2002), Strang and Fix (1973), and Strang (1986).

Let us treat the Poisson equation

$$(-\Delta u, w) = (f, w) \quad \text{for every}\ w\ , \tag{10.5}$$

with the boundary condition $u = 0$ on $\partial\Omega$. The weak form is obtained by integration by parts of (10.5) with the requirement that the test function also satisfies the homogeneous boundary condition $w = 0$ on $\partial\Omega$. One obtains

$$\int_\Omega \left(\frac{\partial u}{\partial x_1}\frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial w}{\partial x_2} + \frac{\partial u}{\partial x_3}\frac{\partial w}{\partial x_3} \right) d\Omega = \int_\Omega f\, w\, d\Omega\ . \tag{10.6}$$

This is the weak form of the Poisson equation with homogeneous Dirichlet boundary condition. Equation (10.6) is an instance of the **Green's formula** for a scalar variable u derived from the divergence theorem

$$\int_\Omega \left(w\, \frac{\partial^2 u}{\partial x_j \partial x_j} + \frac{\partial u}{\partial x_j}\frac{\partial w}{\partial x_j} \right) d\Omega = \int_{\partial\Omega} w\, \frac{\partial u}{\partial x_j} n_j\, dS\ , \tag{10.7}$$

with $\mathbf{n}$ the outward unit vector normal to the boundary. In the context of the weak formulation, the Dirichlet condition is called an **essential boundary condition**.

Suppose now that a Neumann condition is imposed on $\partial\Omega$, i.e. the normal derivative is specified by

$$\frac{\partial u}{\partial x_j} n_j = g\ , \tag{10.8}$$

where g is a given function. The weak form becomes in this case

$$\int_{\Omega} \left(\frac{\partial u}{\partial x_j} \frac{\partial w}{\partial x_j} - fw \right) d\Omega = \int_{\partial\Omega} w \frac{\partial u}{\partial x_j} n_j \, dS = \int_{\partial\Omega} w \, g \, dS \,. \qquad (10.9)$$

The Neumann condition is called a **natural condition** as it is incorporated automatically in the weak formulation.

The weak formulation is the cornerstone for building up Galerkin's method. The first step consists in choosing a finite set of approximation functions like, e.g., Fourier polynomials (spectral methods) or piecewise Lagrange interpolating polynomials (finite elements). Then the variable(s) of the problem is (are) expressed as finite series of the approximation functions, which are inserted in the weak form. The coefficients of the approximation constitute the unknowns of the problem and their computation is performed with the use of the test functions w needed in the weak form. From symmetry considerations, the **Galerkin method** requires the test functions to be the same as the approximation functions. This process leads most of the time to matrices that are symmetric.

As already mentioned, the Galerkin method is a weighted residual method. When the series of approximating functions is substituted into the weak form a residual is produced. If the series were infinite and involved, for example, eigenfunctions of the operator as approximating functions, the residual might be zero. The actual finite series produces a non-vanishing residual that is a projection obtained via the test function. This projection in the Galerkin framework is an orthogonal projection that minimizes the error in the least square sense.

10.2 The Finite Elements

In order to construct the finite element (FE) theory, four basic ingredients are needed

1. A mesh $\mathcal{M}$, sometimes called a grid,
2. A reference domain $\hat{\Omega}$,
3. A set of parameters $u_j, j = 1, \ldots, N$, named the degrees of freedom or the problem variables,
4. A set of J associated functions $\hat{\varphi}_j, j = 1, \ldots, N$ defined in $\hat{\Omega}$.

The mesh $\mathcal{M}$ is built by a decomposition of the computational domain in E subdomains $\left\{\Omega^i\right\}_{i=1}^{E}$ such that

$$\overline{\Omega} = \cup_{i=1}^{E} \Omega^i \quad \text{and} \quad \Omega^k \cap \Omega^\ell = \emptyset, \quad \forall k \neq \ell \,, \qquad (10.10)$$

where $\overline{\Omega}$ is the closure of Ω and $\emptyset$ is the empty set. Mesh or grid generation is based on Delaunay triangulation which produces unstructured meshes able to cope with complex geometries. An excellent introduction to this topic is the monograph by Frey and George (2000). The subdomains are the finite elements and they are composed of geometrical building blocks of various shapes. These physical elements are matched onto a **reference** or **parent element** defined for the one-dimensional (1D) case as

$$\hat{\Omega} = [-1, +1] \ . \tag{10.11}$$

In two or three dimensions, these reference elements generalize easily to triangles, rectangles, parallelepipeds and tetrahedra.

In the 1D case, let us choose $\Omega = (a, b)$. The mesh is built between the end points $x_1 = a$ and $x_{E+1} = b$. Then each element is such that $\Omega^i = (x_i, x_{i+1})$. FE meshes are generally irregular (unlike the finite difference method where regular meshes are mostly used) and each subdomain has a size $h_i = x_{i+1} - x_i$. Each point $P \in \overline{\Omega}^i$ of coordinate x such that $x_i \leq x \leq x_{i+1}$ has an image $\hat{P} \in \hat{\Omega}$ provided by the affine mapping

$$\xi = \frac{2}{h_i}\left(x - \frac{x_i + x_{i+1}}{2}\right) , \qquad -1 \leq \xi \leq +1, \tag{10.12}$$

where ξ is the **local coordinate** of $\hat{P}$. This linear relation is invertible. Each point $\hat{P}$ of local coordinate ξ yields one and only one image P in $\overline{\Omega}^i$, characterized by the physical coordinate x

$$x = \frac{h_i}{2}\left(\xi + \frac{x_i + x_{i+1}}{h_i}\right) , \qquad x_i \leq x \leq x_{i+1} \ . \tag{10.13}$$

Note that every node of coordinate x_{i+1} belongs to element $\overline{\Omega}^i$ as the image of $\xi = +1$ and to element $\overline{\Omega}^{i+1}$ as the image of $\xi = -1$.

10.3 One-Dimensional Q_1 Lagrange Element

The 1D Q_1 Lagrange finite element uses Lagrangian linear interpolation. Then, we have two degrees of freedom as the interpolator is a linear polynomial. On the parent element the interpolation nodes are the end points of $\hat{\Omega}$ corresponding to $\xi_1 = -1$ and $\xi_2 = +1$, respectively. As a consequence every function $w(\xi)$ is approximated by the linear interpolation

$$w(\xi) \approx w(-1)\hat{\varphi}_1(\xi) + w(+1)\hat{\varphi}_2(\xi), \qquad -1 \leq \xi \leq +1, \tag{10.14}$$

where

$$\hat{\varphi}_1(\xi) = \frac{1-\xi}{2} \quad \text{and} \quad \hat{\varphi}_2(\xi) = \frac{1+\xi}{2} \ . \tag{10.15}$$

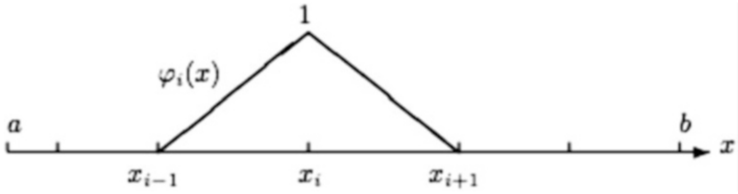


Fig. 10.1 Hat function of the Q_1 element

It is easily checked that $\hat{\varphi}_i(\xi_j) = \delta_{ij}$, which is a feature of Lagrange interpolation known as the **cardinality condition**. The unknowns of the Q_1 finite element will be the nodal values at the end points of the parent element.

The application of the mapping (10.13) is carried out over all elements $\overline{\Omega}^i$. Taking into account the fact that x_i belongs by construction to two adjacent elements, we will approximate a function $u(x) \in H^1(\Omega)$ by $u_{\mathcal{I}}(x)$, the Q_1 interpolation of u,

$$u(x) \approx u_{\mathcal{I}}(x) = \sum_{i=1}^{E+1} u_i \varphi_i(x), \qquad a \le x \le b\,, \tag{10.16}$$

which is a global approximation of the variable u in the domain $\overline{\Omega} = (a, b)$. The space $H^1(\Omega)$ is the Sobolev space, where the function and its first-order derivatives (hence the superscript being 1) are each square integrable, i.e. in L^2. We recall that a square integrable function f is such that

$$\int_\Omega |f|^2 \, d\Omega < \infty$$

meaning that the function contains a finite energy. The space of square integrable functions is usually denoted as the L^2 space. In (10.16), the quantity $u_i = u(x_i)$ is the nodal value at the physical mesh node x_i. The interpolating polynomial is defined as

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases}, \qquad 2 \le i \le E. \tag{10.17}$$

The function displayed in Fig. 10.1 is called a **hat function** and results from the assembly of adjacent basis functions given by (10.15). Note that for the Q_1 element, the number of grid points is $N = E + 1$.

The theory of interpolation Strang and Fix (1973) shows that if the mesh size h, defined as the maximum of h_i, goes to zero, the L^2 norm of the interpolation error varies as $O(h^2)$. More precisely

$$\|u - u_{\mathcal{I}}\|_{L^2} = \left(\int_a^b (u - u_{\mathcal{I}})^2 dx \right)^{1/2} \approx Ch^2 \,, \tag{10.18}$$

where the constant C depends on the interpolated function and is independent of the grid size.

10.4 One-Dimensional Q_2 Lagrange Element

The 1D Q_2 Lagrange finite element uses Lagrangian quadratic interpolation. Then, we have three degrees of freedom as the interpolator is a polynomial of degree two. On the parent element the interpolation nodes are the end points of $\hat{\Omega}$, $\xi_1 = -1$ and $\xi_3 = +1$ and the middle point $\xi_2 = 0$. As a consequence every function $w(\xi)$ is approximated by the quadratic interpolation

$$w(\xi) \approx w(-1)\hat{\varphi}_1(\xi) + w(0)\hat{\varphi}_2(\xi) + w(+1)\hat{\varphi}_3(\xi), \qquad -1 \le \xi \le +1 \,, \tag{10.19}$$

where

$$\hat{\varphi}_1(\xi) = \frac{1}{2}\xi(\xi - 1), \quad \hat{\varphi}_2(\xi) = 1 - \xi^2, \quad \hat{\varphi}_3(\xi) = \frac{1}{2}\xi(\xi + 1) \,, \tag{10.20}$$

as shown in Fig. 10.2.

Approximating again $u(x)$ by its global interpolation $u_{\mathcal{I}}(x)$, we can write

$$u_{\mathcal{I}}(x) = \sum_{i=1}^{E+1} u_i \, \varphi_i(x) \; + \; \sum_{i=1}^{E} u_{i+1/2} \, \varphi_{i+1/2}(x), \quad a \le x \le b, \tag{10.21}$$

with $u_i = u(x_i)$ and the global basis functions

$$\varphi_i(x) = \begin{cases} \frac{1}{h_{i-1}^2}(2x - x_{i-1} - x_i)(x - x_{i-1}), & x \in [x_{i-1}, x_i] \,, \\ \frac{1}{h_i^2}(2x - x_i - x_{i+1})(x - x_{i+1}), & x \in [x_i, x_{i+1}] \,, \\ 0, & x \notin [x_{i-1}, x_{i+1}] \,, \end{cases} \tag{10.22}$$

$$\varphi_{i+1/2}(x) = \begin{cases} \frac{2}{h_i^2}(x_{i+1} - x)(x - x_i), & x \in [x_i, x_{i+1}] \,, \\ 0, & x \notin [x_i, x_{i+1}] \,, \end{cases} \tag{10.23}$$

respectively for $i = 2, \ldots, E$ and for $i = 1, \ldots, E$, with the basis functions corresponding to the end points of Ω being excluded.

For the Q_2 element the number of interpolation nodes is $2E + 1$. For the sake of facility we may write (10.21) similarly to (10.16) as

$$u(x) \approx u_{\mathcal{I}}(x) = \sum_{i=1}^{2E+1} u_i \varphi_i(x) \,, \qquad a \le x \le b \,, \tag{10.24}$$

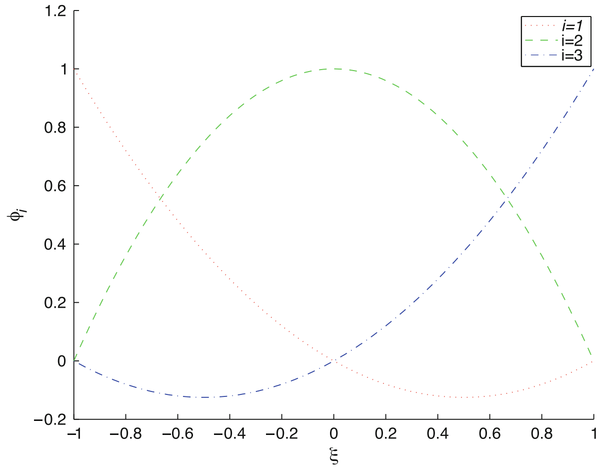


Fig. 10.2 Basis functions of the Q_2 element

where the odd nodes are the elemental boundary nodes and the even nodes the element interior nodes. For the Q_2 element, the L^2 norm of the interpolation error is proven to be $O(h^3)$.

10.5 Implementation of the Galerkin Method

Considering the 1D version of Eq. (10.6) with ordinary differentials replacing partial derivatives, the problem is stated as follows, with the convention that the (continuous) space variable x_1 is denoted by x:

Find $u(x) \in H_e^1(\Omega)$ *such that*

$$\int_\Omega \frac{du}{dx}\frac{dw}{dx}dx = \int_\Omega f\,w\,dx\,, \quad \forall w(x) \in H_e^1(\Omega)\,, \tag{10.25}$$

where $H_e^1(\Omega)$ is a subspace of $H^1(\Omega)$ defined as

$$H_e^1(\Omega) = \{w(x)|w(x) \in H^1(\Omega), w(a) = w(b) = 0\}\,, \tag{10.26}$$

the lower index indicating that the test functions satisfy the essential boundary conditions. Covering Ω with a mesh $\mathcal{M}$ comprising E elements $\{\Omega^i\}_{i=1}^E$, we then construct V_N, a subspace of dimension N of $H^1(\Omega)$, with Q_1 finite elements. This procedure yields global basis functions $\{\varphi_k(x)\}_{k=1}^N$. We build up the interpolation functions $u_N(x) \in V_{N-2}$ and $f_N \in V_N$ such that

$$u_N(x) = \sum_{k=2}^{N-1} u_k\varphi_k(x), \quad f_N(x) = \sum_{k=1}^{N} f_k\varphi_k(x)\,. \tag{10.27}$$

The coefficients $\{f_k\}_{k=1}^{N}$ are the problem data while the $\{u_k\}_{k=2}^{N-1}$ constitute the problem unknowns inasmuch the essential boundary conditions impose $u_1 = u_N = 0$.

The discrete weak formulation becomes: *Find* $u_N(x) \in V_{N-2}$ *such that*

$$\int_\Omega \frac{du_N}{dx}\frac{dw_N}{dx}dx = \int_\Omega f_N w_N\, dx\,, \quad \forall w_N \in V_{N-2}\,. \tag{10.28}$$

We take full advantage of a property of the Galerkin method: the test functions w_N are the same as the basis functions. Recall that for Q_1 elements, $N = E + 1$. Introducing (10.27) into (10.28) we generate a linear system of algebraic equations of order $N - 2$

$$\sum_{k=2}^{N-1} K_{i,k}\, u_k = \sum_{k=1}^{N} M_{i,k}\, f_k, \quad 2 \le i \le N-1\,. \tag{10.29}$$

In the left side of (10.29) the square matrix $[K]$ is of the order $N - 2$ and has components

$$K_{i,k} = \int_\Omega \frac{d\varphi_i}{dx}\frac{d\varphi_k}{dx}dx, \quad 2 \le i,k \le N-1\,. \tag{10.30}$$

The matrix $[M]$ on the right side is rectangular and has $N-2$ rows and N columns

$$M_{i,k} = \int_\Omega \varphi_i\varphi_k dx, \quad 2 \le i \le N-1,\ 1 \le k \le N\,. \tag{10.31}$$

The **stiffness matrix** (or rigidity matrix) $[K]$ is symmetric and positive definite. The matrix $[M]$ is the **mass matrix**. Adding the relations $u_1 = 0$ and $u_N = 0$ to the linear system (10.29) yields the algebraic equations

$$[K]\underline{u} = [M]\underline{f}\,. \tag{10.32}$$

The vector $\underline{u} = [0\, u_2 \ldots u_{N-1}\, 0]$ is formed with all the unknowns and the boundary conditions. Standard direct methods like the Cholesky algorithm may be used to solve (10.32).

The classical description of the finite element method uses Gauss-Legendre quadrature rules to approximate the integrals appearing in the stiffness (10.30) and mass (10.31) matrices. However it is possible to compute these integrals in closed form. As an example we carry out the evaluation of the left side of (10.29). We first fix i to a chosen value. Then due to the fact that the local basis extends over two adjacent elements, the integral contains non vanishing contributions only for $k = i-1, i, i+1$. Therefore one obtains taking (10.13) into account

$$
\begin{aligned}
\sum_{k=2}^{N-1} K_{i,k}\, u_k &= \sum_{k=2}^{N-1} u_k \int_\Omega \frac{d\varphi_i}{dx}\frac{d\varphi_k}{dx}dx \\
&= u_{i-1}\int_\Omega \frac{d\varphi_i}{dx}\frac{d\varphi_{i-1}}{dx}dx + u_i\int_\Omega \frac{d\varphi_i}{dx}\frac{d\varphi_i}{dx}dx + u_{i+1}\int_\Omega \frac{d\varphi_i}{dx}\frac{d\varphi_{i+1}}{dx}dx \\
&= u_{i-1}\int_{x_{i-1}}^{x_i} \frac{d\varphi_i}{dx}\frac{d\varphi_{i-1}}{dx}dx + u_i\int_{x_{i-1}}^{x_{i+1}} \left(\frac{d\varphi_i}{dx}\right)^2 dx + u_{i+1}\int_{x_i}^{x_{i+1}} \frac{d\varphi_i}{dx}\frac{d\varphi_{i+1}}{dx}dx \\
&= \frac{2u_{i-1}}{h_{i-1}}\int_{-1}^{+1} \frac{d\hat\varphi_2}{d\xi}\frac{d\hat\varphi_1}{d\xi}d\xi + \frac{2u_i}{h_{i-1}}\int_{-1}^{+1}\left(\frac{d\hat\varphi_2}{d\xi}\right)^2 d\xi \\
&+\frac{2u_i}{h_i}\int_{-1}^{+1}\left(\frac{d\hat\varphi_1}{d\xi}\right)^2 d\xi + \frac{2u_{i+1}}{h_i}\int_{-1}^{+1} \frac{d\hat\varphi_2}{d\xi}\frac{d\hat\varphi_1}{d\xi}d\xi \\
&= -\frac{u_{i-1}}{h_{i-1}} + \left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right)u_i - \frac{u_{i+1}}{h_i}\,.
\end{aligned}
\tag{10.33}
$$

A similar development for the right side of (10.29) yields

$$
\sum_{k=1}^{N} M_{i,k}\, f_k = \frac{h_{i-1}}{6} f_{i-1} + (h_{i-1}+h_i)\frac{f_i}{3} + \frac{h_i}{6} f_{i+1}\,. \tag{10.34}
$$

The discrete equations are

$$
-\frac{u_{i-1}}{h_{i-1}} + \left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right)u_i - \frac{u_{i+1}}{h_i} = \frac{h_{i-1}}{6} f_{i-1} + (h_{i-1}+h_i)\frac{f_i}{3} + \frac{h_i}{6} f_{i+1}\,, \quad 2 \le i \le N-1\,. \tag{10.35}
$$

The $[K]$ matrix is tridiagonal. For a regular mesh with $h_i = \Delta x, i = 2\ldots, N$ Eq. (10.35) becomes

$$
-\frac{1}{\Delta x}(u_{i-1} - 2u_i + u_{i+1}) = \Delta x\left(\frac{1}{6} f_{i-1} + \frac{2f_i}{3} + \frac{1}{6} f_{i+1}\right)\,, \tag{10.36}
$$

where the left side is exactly the second order centered finite difference of the second derivative. If we lump the mass matrix, which consists in replacing the diagonal coefficient by the sum of all coefficients of the row, then the condensed or **lumped mass matrix** is the identity matrix and the right side of the discrete equation is f_i, the finite difference approximation. We may conclude at this point that the finite element methodology is able to produce finite difference approximations, if need be. This is especially interesting for irregular meshes.

10.6 Natural Boundary Conditions

The 1D continuous problem is stated as

$$-\frac{d^2u}{dx^2} = f, \quad \text{on} \quad \Omega = (a, b) \,, \tag{10.37}$$

with the Dirichlet boundary condition $u(x = a) = 0$ and the Neumann condition $du/dx(x = b) = 0$. Because $g = 0$ in Eq. (10.9) the weak form is the same as (10.25), with the space H^1_e now defined as

$$H^1_e(\Omega) = \{w(x)|w(x) \in H^1(\Omega), w(a) = 0\} \,. \tag{10.38}$$

The global basis functions are still $\{\varphi_k(x)\}_{k=1}^N$, while the interpolation functions $u_N(x) \in V_{N-1}$ and $f_N \in V_N$ are given by

$$u_N(x) = \sum_{k=2}^{N} u_k \varphi_k(x), \quad f_N(x) = \sum_{k=1}^{N} f_k \varphi_k(x) \,. \tag{10.39}$$

The $\{u_k\}_{k=2}^N$ constitute the problem unknowns as the essential boundary condition imposes $u_1 = 0$. Therefore the computation involves the last grid point x_N. The global functions for the two extreme nodes are given as

$$\varphi_1 = \begin{cases} \frac{x_2 - x}{h_1} \,, & x \in [x_1, x_2] \\ 0 \,, & x \in [x_2, x_{E+1}] \end{cases}, \quad \varphi_{E+1} = \begin{cases} 0 \,, & x \in [x_1, x_E] \\ \frac{x - x_E}{h_E} \,, & x \in [x_E, x_{E+1}] \end{cases} . \tag{10.40}$$

The discrete interior equations are still given by (10.35). For $i = N$, the equation reads

$$\frac{u_N - u_{N-1}}{h_E} = h_E\left(\frac{f_N}{3} + \frac{f_{N-1}}{6}\right) . \tag{10.41}$$

With $h_E = \Delta x$, this last relation yields

$$\frac{1}{\Delta x}(u_N - u_{N-1}) = O(\Delta x) \tag{10.42}$$

or

$$\frac{du}{dx}(x = b) = 0 + O(\Delta x) \,. \tag{10.43}$$

Unlike the finite difference approximation, we observe that the FE method generates the appropriate discrete approximation of the Neumann boundary condition. The procedure is self-contained and suffices to close the linear system.

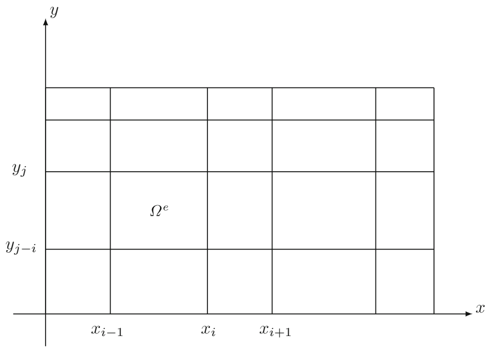


Fig. 10.3 Two-dimensional mesh

10.7 Multidimensional Finite Elements

We will introduce here the two-dimensional (2D) FE, the extension for the three-dimensional (3D) case being straightforward. The integration domain Ω is split into E elements $\Omega^e, e = 1, \ldots, E$ which for the sake of simplicity we assume to be quadrilaterals. The mesh size h is defined as the diameter of the circumscribed circle of the largest quadrilateral

$$h = \max_{e=1,\cdots,E} \text{diam}(\Omega^e) \; . \tag{10.44}$$

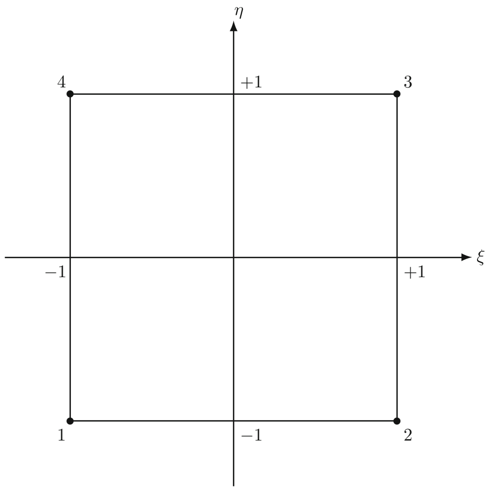


Fig. 10.4 Two-dimensional Q_1 parent element

Each element is such that $\Omega^e = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$ with mesh sizes $h_i^x = x_i - x_{i-1}, h_j^y = y_j - y_{j-1}$, as shown in Fig. 10.3.

The parent element is defined as the square $\hat{\Omega} = [-1, +1]^2$ with local coordinates $\boldsymbol{\xi} = (\xi, \eta)$, as it is exhibited in Fig. 10.4.

10.7.1 Two-Dimensional Q_1 Element

The 2D Q_1 FE is built as a tensor product of the 1D linear interpolators; the basis is therefore linear in x for every y and vice versa. The degrees of freedom in the parent element are the nodal values attached to the vertices $\hat{v}_j$, $j = 1, \ldots 4$, if we go around the parent element in the trigonometric sense, cf. Fig. 10.4. Based on the Lagrange interpolation properties, each function $w(\boldsymbol{\xi})$ may be written in $\hat{\Omega}$ as

$$w(\boldsymbol{\xi}) = \sum_{j=1}^{4} w(\hat{\mathbf{v}}_j)\hat{\varphi}_j(\boldsymbol{\xi}) , \tag{10.45}$$

where $\hat{\mathbf{v}}_j$ represent the nodal values at the element vertices. The canonical basis is made up of tensor products of the 1D bases (10.15) such that

$$\begin{aligned}
\tilde{\varphi}_1(\boldsymbol{\xi}) &= \hat{\varphi}_1(\xi)\,\hat{\varphi}_1(\eta) = (1-\xi)\,(1-\eta)/4 ,\\
\tilde{\varphi}_2(\boldsymbol{\xi}) &= \hat{\varphi}_2(\xi)\,\hat{\varphi}_1(\eta) = (1+\xi)\,(1-\eta)/4 ,\\
\tilde{\varphi}_3(\boldsymbol{\xi}) &= \hat{\varphi}_2(\xi)\,\hat{\varphi}_2(\eta) = (1+\xi)\,(1+\eta)/4 ,\\
\tilde{\varphi}_4(\boldsymbol{\xi}) &= \hat{\varphi}_1(\xi)\,\hat{\varphi}_2(\eta) = (1-\xi)\,(1+\eta)/4 .
\end{aligned} \tag{10.46}$$

These functions satisfy the cardinality condition

$$\tilde{\varphi}_j(\boldsymbol{\xi}_i) = \delta_{ij}, \quad i, j = 1, \ldots, 4 . \tag{10.47}$$

Usually the grid points in the mesh are labeled by two integer indices i and j that increase in the positive direction of the axes. This generates a **natural ordering** of the unknowns that are taken from the lower left corner of the physical domain up to the upper right end by sweeping each horizontal line from left to right and then visiting each horizontal grid line from bottom to top. By this procedure it is possible to obtain easily the global number of the element whose lower left corner is given by i, j as $e = i + (j-1)(N_x - 1)$ if N_x denotes the number of nodes in the x direction. The global approximation of the discrete variable denoted by u_h is given by

$$u_h(\mathbf{x}) = \sum_{k=1}^{N} u_k\,\varphi_k(\mathbf{x}), \qquad \mathbf{x} \in \Omega , \tag{10.48}$$

with $N = N_x N_y$, the total number of unknowns in the mesh. The global basis $\{\varphi_k\}_{k=1}^{N}$ is the canonical basis in Ω satisfying (10.47) and $u_k = u_h(\mathbf{x}_k)$.

10.7.2 Implementation of the 2D Galerkin Method

We consider Eq. (10.6) with the convention $x \equiv x_1, y \equiv x_2$. The physical domain is split in $E_x = N_x - 1$ elements in the x direction and $E_y = N_y - 1$ in the y direction such that $E = E_x E_y$. The space V_N is a subspace of dimension N of $H^1_e(\Omega)$ built with the FE basis $\left\{\varphi_k(\mathbf{x})\right\}_{k=1}^N$. The discrete space $V^1_{0,N}$, a subspace of V_h, is made of Q_1 FE in each Ω^e; however as we have homogeneous boundary conditions on $\partial\Omega$, we require that the bases corresponding to nodes belonging to the boundary vanish. The weak formulation is stated as: *Find* $u_h(\mathbf{x}) \in V^1_{0,N}$ *such that*

$$\int_\Omega \nabla u_h \cdot \nabla w_h \, d\mathbf{x} = \int_\Omega f(\mathbf{x}) v_h(\mathbf{x}) \, d\mathbf{x}, \quad \forall \, w_h(\mathbf{x}) \in V_N \tag{10.49}$$

We next introduce the geometrical cobble block $\Omega_{cb} = \Omega^e \cup \Omega^{e+1} \cup \Omega^{e+N_x-1} \cup \Omega^{e+N_x}$ made of the four contiguous elements contributing to the computation of the unknown $u_{ij} = u(x_i, y_j)$. We insert (10.48) into (10.49) and we choose as test functions $w_h = \varphi_i(\mathbf{x}), i = 1, \ldots, N_{int}$, with the notation $N_{int} = (N_x - 2)(N_y - 2)$, the number of interior nodes. Therefore, the locality of the FE bases leads to the relation

$$\sum_{m=1}^{N_{int}} u_m \int_{\Omega_{cb}} \frac{\partial\varphi_m}{\partial x}\frac{\partial\varphi_n}{\partial x} + \frac{\partial\varphi_m}{\partial y}\frac{\partial\varphi_n}{\partial y} \, d\mathbf{x} = \int_{\Omega_{cb}} f(\mathbf{x})\varphi_n(\mathbf{x}) d\mathbf{x}, \quad n = 1, \ldots, N_{int} \, . \tag{10.50}$$

We wish to generate the discrete equation related to node (i, j). We set $\varphi_n(\mathbf{x}) = \varphi_i(x)\varphi_j(y)$, where n designates the global number of node (i, j) inside the element Ω^e. To proceed, we replace the global approximation with one index by a global approximation with the two indices of the natural ordering

$$u_h(\mathbf{x}) = \sum_{k=2}^{N_x-1} \sum_{\ell=2}^{N_y-1} u_{k\,\ell} \, \varphi_k(x) \, \varphi_\ell(y) \, . \tag{10.51}$$

With (10.51) and a similar expression for the test functions, Eq. (10.50) yields

$$\sum_{k=2}^{N_x-1} \sum_{\ell=2}^{N_y-1} u_{k\,\ell} \int_{\Omega_{cb}} \left(\frac{\partial\varphi_k}{\partial x}\varphi_\ell \frac{\partial\varphi_i}{\partial x}\varphi_j + \varphi_k \frac{\partial\varphi_\ell}{\partial y}\varphi_i \frac{\partial\varphi_j}{\partial y}\right) d\mathbf{x} = \int_{\Omega_{cb}} f(\mathbf{x})\varphi_i(x)\varphi_j(y) d\,\mathbf{x} \, . \tag{10.52}$$

Fixing the indices i, j, the non vanishing contributions to the integrals of (10.52) are obtained for the indices $k = i - 1, i, i + 1$ and $\ell = j - 1, j, j + 1$.

Let us examine in detail the relevant contribution to $u_{i-1,j-1}$ in (10.52). The associated integral in the left side is carried out on the lower left of Ω_{cb}, as $k = i-1$ and $\ell = j-1$. One has

$$\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\left(\frac{\partial\varphi_{i-1}}{\partial x}\varphi_{j-1}\frac{\partial\varphi_i}{\partial x}\varphi_j + \varphi_{i-1}\frac{\partial\varphi_{j-1}}{\partial y}\varphi_i\frac{\partial\varphi_j}{\partial y}\right)d\mathbf{x}\,. \tag{10.53}$$

The double integrals are separable as a consequence of the tensorization of the bases. Therefore the first term of (10.53) may be written as

$$\int_{x_{i-1}}^{x_i}\frac{\partial\varphi_{i-1}(x)}{\partial x}\frac{\partial\varphi_i(x)}{\partial x}\,dx\int_{y_{j-1}}^{y_j}\varphi_{j-1}(y)\varphi_j(y)\,dy\,. \tag{10.54}$$

Due to the mapping of the physical element onto the parent element (in case of Fig. 10.3 the mapping is affine and is very easy to handle), the integral in x is the 1D rigidity matrix (10.30) and the integral in y is the mass matrix (10.31). The evaluation of the product (10.54) gives $-(1/6)\,(h^y_{j-1}/h^x_{i-1})$. Taking all k and ℓ values into account, the first integral of (10.52) yields

$$\begin{aligned}
&\sum_{k=1}^{N_x-2}\sum_{\ell=1}^{N_y-2}u_{k\,\ell}\int_{\Omega_{bb}}\frac{\partial\varphi_k(x)}{\partial x}\varphi_\ell(y)\frac{\partial\varphi_i(x)}{\partial x}\varphi_j(y)d\mathbf{x} = \\
&\quad -\frac{u_{i-1,j-1}}{6}\frac{h^y_{j-1}}{h^x_{i-1}} + \frac{u_{i-1,j}}{3}\left(\frac{h^y_j + h^y_{j-1}}{h^x_{i-1}}\right) - \frac{u_{i-1,j+1}}{6}\frac{h^y_j}{h^x_{i-1}} \\
&\quad -\left(\frac{u_{i,j-1}}{3}h^y_{j-1} - \frac{u_{i,j}}{3}\left(h^y_j + h^y_{j-1}\right) + \frac{u_{i,j+1}}{3}h^y_j\right)\left(\frac{1}{h^x_{i-1}} + \frac{1}{h^x_i}\right) \\
&\quad -\frac{u_{i+1,j-1}}{6}\frac{h^y_{j-1}}{h^x_i} + \frac{u_{i+1,j}}{3}\left(\frac{h^y_j + h^y_{j-1}}{h^x_i}\right) - \frac{u_{i+1,j+1}}{6}\frac{h^y_j}{h^x_i}\,.
\end{aligned} \tag{10.55}$$

This shows that the 2D Q_1 element involves 9 nodal values. The corresponding stiffness matrix is sparse and symmetric positive definite with a bandwidth of order N_x, i.e. the number of unknowns in the x direction.

10.7.3 Three-Dimensional Q1 Element

The mesh $\mathcal{M}$ is decomposed into parallelepipeds. The parent element is the unit cube $\hat{\Omega} = [-1, +1]^3$. The 3D element involves eight degrees of freedom corresponding to the vertices of the parent element, as the polynomial basis is made of Lagrange polynomials of degree one. Each node i, j, k is connected to 26 adjacent neighbors as a result of the presence of the adjacent 8 elements. The stiffness matrix has now a bandwidth of order N_xN_y corresponding to the number of points inside a plane of the mesh.

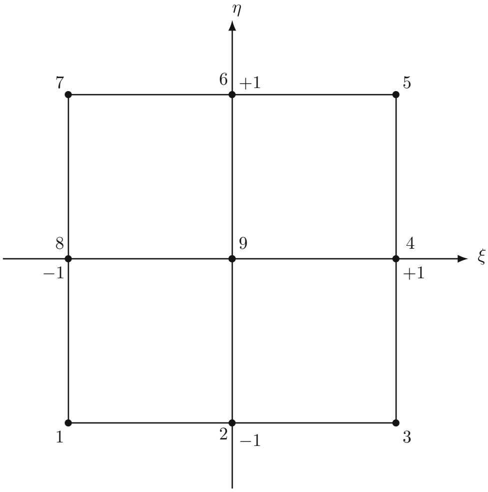


Fig. 10.5 Two-dimensional Q_2 parent element

10.8 Two-Dimensional Q_2 Element

The 2D Q_2 parent element is the unit square $\hat{\Omega} = [-1, +1]^2$. The element possesses nine nodes as shown in Fig. 10.5 with nodal values $\hat{v}_j$, $j = 1, \ldots, 9$ attached to the vertices, the mid-side points, and the central node. The polynomial basis is a tensor product of interpolation polynomials of degree two in each space direction. Each function $w(\boldsymbol{\xi})$ is written in $\hat{\Omega}$ as

$$w(\boldsymbol{\xi}) = \sum_{j=1}^{9} w(\hat{\mathbf{v}}_j)\hat{\varphi}_j(\boldsymbol{\xi}) \,. \tag{10.56}$$

The canonical basis is made of tensor products of the 1D bases (10.20) such that

$$\begin{aligned}
\tilde{\boldsymbol{\varphi}}_1(\boldsymbol{\xi}) &= \hat{\varphi}_1(\xi)\hat{\varphi}_1(\eta) = \xi\eta(1-\xi)(1-\eta)/4 \,, \\
\tilde{\boldsymbol{\varphi}}_2(\boldsymbol{\xi}) &= \hat{\varphi}_2(\xi)\hat{\varphi}_1(\eta) = -\eta(1+\xi)(1-\xi)(1-\eta)/2 \,, \\
\tilde{\boldsymbol{\varphi}}_3(\boldsymbol{\xi}) &= \hat{\varphi}_3(\xi)\hat{\varphi}_1(\eta) = -\xi\eta(1+\xi)(1-\eta)/4 \,, \\
\tilde{\boldsymbol{\varphi}}_4(\boldsymbol{\xi}) &= \hat{\varphi}_3(\xi)\hat{\varphi}_2(\eta) = \xi(1+\xi)(1+\eta)(1-\eta)/2 \,, \\
\tilde{\boldsymbol{\varphi}}_5(\boldsymbol{\xi}) &= \hat{\varphi}_3(\xi)\hat{\varphi}_3(\eta) = \xi\eta(1+\xi)(1+\eta)/4 \,, \\
\tilde{\boldsymbol{\varphi}}_6(\boldsymbol{\xi}) &= \hat{\varphi}_2(\xi)\hat{\varphi}_3(\eta) = \eta(1+\xi)(1-\xi)(1+\eta)/2 \,, \\
\tilde{\boldsymbol{\varphi}}_7(\boldsymbol{\xi}) &= \hat{\varphi}_1(\xi)\hat{\varphi}_3(\eta) = -\xi\eta(1-\xi)(1+\eta)/4 \,, \\
\tilde{\boldsymbol{\varphi}}_8(\boldsymbol{\xi}) &= \hat{\varphi}_1(\xi)\hat{\varphi}_2(\eta) = -\xi(1-\xi)(1+\xi)(1+\eta)(\eta)/2 \,, \\
\tilde{\boldsymbol{\varphi}}_9(\boldsymbol{\xi}) &= \hat{\varphi}_2(\xi)\hat{\varphi}_2(\eta) = (1+\xi)(1-\xi)(1+\eta)(1-\eta) \,.
\end{aligned} \tag{10.57}$$

For a domain Ω covered with $E_x \times E_y$ Q_2 elements and with Dirichlet homogeneous boundary condition, the total number of unknowns is $(2E_x - 1)(2E_y - 1)$. Each discrete equation involves 25 nodal values. For the 3D Q_2 element, the connection between neighbors extends over 125 nodes.

For a direct factorization solver, the Q_2 computational cost increases by a factor of 8 with respect to that of the Q_1 cost. This is one of the reason for choosing a preconditioned conjugate gradient (PCG) method instead of the Cholesky algorithm for three-dimensional problems.

10.9 Triangular Elements

For complex geometries the use of 2D triangular elements and 3D tetrahedra is practically mandatory as they are the most practical way to build up meshes that can cope with curvy boundaries, interior obstacles, and other complicated shapes. The triangle FEs of degree one and two are known as P_1 and P_2, respectively.

10.9.1 P_1 Finite Element

In the physical space, the triangular element Ω^e has the vertices $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$. The P_1 FE is related to the parent element $\hat{\Omega}$, a triangle with three degrees of freedom attached to the nodes $\hat{\mathbf{s}}_1 = (0,0), \hat{\mathbf{s}}_2 = (1,0), \hat{\mathbf{s}}_3 = (0,1)$ as it is shown in Fig. 10.6. Each regular function $w(\mathbf{x})$ may be interpolated in Ω^e as

$$w(\mathbf{x}) \approx \sum_{j=1}^{3} w(\mathbf{s}_j)\varphi_j^e(\mathbf{x}) , \qquad (10.58)$$

where φ_j^e belongs to the space of polynomial functions of total degree less than or equal to one given by the relationship

$$\varphi(\mathbf{x}) = a + bx + cy, \quad a, b, c \in \mathbb{R} . \qquad (10.59)$$

The canonical basis in the parent element is written as

$$\begin{aligned} \hat{\varphi}_1(\boldsymbol{\xi}) &= 1 - \xi - \eta, \\ \hat{\varphi}_2(\boldsymbol{\xi}) &= \xi, \\ \hat{\varphi}_3(\boldsymbol{\xi}) &= \eta. \end{aligned} \qquad (10.60)$$

We observe that this basis satisfies the cardinality property. The correspondence between the physical and parent elements is obtained through an affine

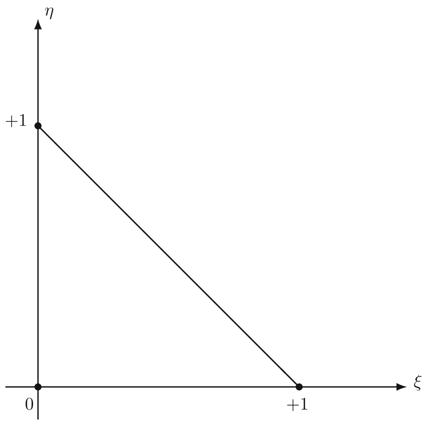


Fig. 10.6 Triangular P_1 parent element

transformation which maps $\hat{\Omega}$ onto Ω^e such that $\hat{\mathbf{s}}_i \longmapsto \mathbf{s}_i$, $i = 1, 2, 3$ with $\mathbf{s}_i = (x_i, y_i)$. We skip the details related to the construction of this transformation as many of the technicalities are intricate. The interested reader may consult Rappaz et al. (2003). The resulting stiffness matrix is sparse and symmetric. A PCG method is recommended for solving the linear system.

10.9.2 P_2 Finite Element

The triangular element Ω^e has the vertices $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ and also the mid-points $\mathbf{s}_{12}, \mathbf{s}_{13}, \mathbf{s}_{23}$ of the three edges $(\mathbf{s}_1, \mathbf{s}_2), (\mathbf{s}_1, \mathbf{s}_3), (\mathbf{s}_2, \mathbf{s}_3)$. Each regular function $w(\mathbf{x})$ may be interpolated in Ω^e as

$$w(\mathbf{x}) \approx \sum_{j=1}^{6} w(\mathbf{s}_j)\varphi_j^e(\mathbf{x}) \,, \tag{10.61}$$

where φ_j^e belongs to the space of polynomial functions of total degree less than or equal to two given by the relationship

$$\varphi(\mathbf{x}) = a + bx + cy + dx^2 + exy + fy^2, \quad a, b, c, d, e, f \in \mathbb{R} \,. \tag{10.62}$$

The canonical bases on the element Ω^e are given by

$$\varphi_j(\mathbf{x}) = \lambda_j\,(2\,\lambda_j - 1), \qquad 1 \le j \le 3 \,, \tag{10.63}$$

and φ_{12}, φ_{13} and φ_{23} are the functions

$$\varphi_{12}(\mathbf{x}) = 4\,\lambda_1\,\lambda_2, \quad \varphi_{13}(\mathbf{x}) = 4\,\lambda_1\,\lambda_3, \quad \varphi_{23}(\mathbf{x}) = 4\,\lambda_2\,\lambda_3 \,, \tag{10.64}$$

where the parameters λ_j $(1 \leq j \leq 3)$ are the **barycentric coordinates** of triangle Ω^e. Let Ω^e be the triangle with vertices $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ and P any point inside the triangle. The barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$ of P are defined by the expression

$$\lambda_i = \frac{\text{area } \Delta_{P\mathbf{s}_j\mathbf{s}_k}}{\text{area } \Delta_{\mathbf{s}_i\mathbf{s}_j\mathbf{s}_k}}, \qquad i = 1, 2, 3, \qquad i \neq j \neq k, \tag{10.65}$$

where, for example, $\Delta_{P\mathbf{s}_j\mathbf{s}_k}$ denotes the triangle $P\mathbf{s}_j\mathbf{s}_k$. The barycentric coordinates satisfy the following property

$$\lambda_1 + \lambda_2 + \lambda_3 = 1. \tag{10.66}$$

The 3D extension generates the P1 and P2 tetrahedra.

10.10 Spectral and Mortar Element Method

The spectral element method (SEM) constitutes a generalization of the FE method based on high-order Lagrange-Legendre interpolants used in conjunction with Gauss-Legendre quadrature rules. Beyond degree two, standard Lagrange interpolation with equally spaced nodes is prone to Runge instability that produces violent oscillations, destroying the accuracy. Therefore a cure consists in using the nodes as the roots of orthogonal polynomials to stabilize the interpolation process. Among the several possible choices the Legendre polynomials are best, as the associated weight in the weak formulation is unity. Furthermore the relevant Gauss-Legendre rules are based on quadrature nodes that match exactly the interpolation nodes. The Gauss-Lobatto-Legendre (GLL) rule implies nodes on the elemental edges, ensuring C^0 continuity between the elements.

To give more precise statements, the 1D spectral element uses the set of $N + 1$ GLL quadrature nodes on $\hat{\Omega}$. These nodes are the roots of the equation

$$(1 - \xi^2) L'_N(\xi) = 0, \quad \xi \in \hat{\Omega}\,, \tag{10.67}$$

where L'_N is the first derivative of the Legendre polynomial of degree N. The Lagrange interpolation polynomial of degree N of the regular function u at the GLL quadrature nodes is

$$u_{\mathcal{I}} = \sum_{j=0}^{N} u(\xi_j)\varphi_j(\xi)\,, \quad \xi \in \hat{\Omega}\,, \tag{10.68}$$

with $\varphi_j(\xi), j = 1, \ldots, N$ the associated interpolation basis of degree N. One may show that the elements of this basis are given by

$$\varphi_j = \frac{-1}{N(N+1)} \frac{(1-\xi^2)L'_N(\xi)}{(\xi-\xi_j)L_N(\xi_j)} , \quad 0 \le j \le N , \quad \xi \in \hat{\Omega} . \tag{10.69}$$

SEM exhibits several favorable computational properties, such as the use of tensor products, naturally diagonal mass matrices, adequacy to concurrent implementations. It rests on firm theoretical ground, e.g. Deville et al. (2002). The spectral element method aims at combining the high-order precision of the spectral methods Quarteroni and Valli (1997) and the geometrical flexibility of the finite element methods. Due to these advantages, the spectral element method exhibits the so-called spectral accuracy, where the error decreases exponentially, if smooth solutions of regular problems are sought.

The requirement of C^0 continuity across the element interfaces demands that any field be interpolated identically on both sides of a common interface. Accordingly any local increase of the polynomial-expansion degree propagates to the rest of the mesh, so that zones where the solution field undergoes little variation end up being meshed as finely as the zones of actual interest. An appropriate remedy was devised: the mortar element method, cf. Anagnostou et al. (1990) and Bernardi et al. (1994). Mortars consist in variational patches of the discontinuous field along the element interfaces. Mortars relax the C^0 continuity condition and thus remove the need for uniform polynomial expansion in every direction all over the mesh.

Chapter 11
Variational Principle, Weak Formulation and Finite Elements

Abstract Finite element treatment of the steady and unsteady Stokes equations is developed in detail. This is then extended to the advection-diffusion problem through the non-linear Burgers equation and finally to the full Navier-Stokes equation.

In this chapter we will focus our attention on the spatial finite element approximation of the steady Stokes problem. A main topic consists in describing stable elements which do not contain spurious pressure modes. Theoretical analyses go beyond the scope of an introductory text such as this monograph. The reader will find complementary reading in Azaïez et al. (2011), Brezzi and Fortin (1991), Girault and Raviart (1986), and Gresho and Sani (2000). Then we will consider time dependent problems that will need time discretizations. Finally, the full Navier-Stokes equation will challenge the talents of numericists: the nonlinear advection term is most difficult to treat correctly without inducing too much numerical dispersion and dissipation.

11.1 Variational Principle

In this section we follow a line of reasoning proposed by Rieutord (1997). With body forces neglected, the Stokes equation (6.2) for a Newtonian viscous incompressible fluid may be written as

$$\frac{\partial T_{ij}}{\partial x_j} = 0\,, \tag{11.1}$$

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_11,

with the Cauchy stress tensor given by

$$T_{ij} = -p\,\delta_{ij} + 2\mu e_{ij}\ . \tag{11.2}$$

The Stokes momentum equation (11.1) may be obtained from a **variational principle** in which a convex functional will be minimized. This functional is related to energy considerations and more specifically to viscous dissipation.

To this end consider a body of fluid in a given volume Ω bounded by the surface $\partial\Omega$. The global dissipation is the integral

$$D = \int_\Omega 2\mu e_{ij} e_{ij}\, d\Omega\ , \tag{11.3}$$

where the integrand is the stress power of Eq. (2.162). Indeed

$$T_{ij} e_{ij} = (-p\delta_{ij} + 2\mu\, e_{ij}) e_{ij} = -p e_{ii} + 2\mu\, e_{ij} e_{ij} = 2\mu\, e_{ij} e_{ij}\ . \tag{11.4}$$

The first-order variation δv_k of the velocity field satisfies a homogeneous boundary condition and implies a variation of D such that

$$\delta D = \int_\Omega 4\mu e_{ij} \delta e_{ij}\, d\Omega\ . \tag{11.5}$$

With (11.2) this becomes

$$\delta D = 2\int_\Omega T_{ij}\, \delta e_{ij}\, d\Omega\ . \tag{11.6}$$

With the symmetry of the stress tensor and the vanishing of δe_{ii}, Eq. (11.6) becomes

$$\delta D = 4\int_\Omega T_{ij}\, \frac{\partial\, \delta v_j}{\partial x_i}\, d\Omega\ . \tag{11.7}$$

With the help of (11.1) and the divergence theorem, one may write

$$\delta D = 4\int_\Omega \frac{\partial (T_{ij} \delta v_j)}{\partial x_i}\, d\Omega = 4\int_{\partial\Omega} T_{ij} \delta v_j n_i\, dS\ . \tag{11.8}$$

Since the homogeneous boundary condition $\delta v_j = 0$ applies on $\partial\Omega$, one obtains

$$\delta D = 0\ . \tag{11.9}$$

From (11.9) we conclude that viscous dissipation reaches an extremum when the velocity field is a solution of the Stokes problem. This extremum may easily be shown to be a minimum, as the functional (11.3) is a convex quadratic form. The associated Stokes solution is unique.

11.2 Weak Form of the Stokes Problem

Coming back to the Stokes problem, we write

$$\frac{\partial T_{ij}}{\partial x_j} + \rho f_i = 0 , \tag{11.10}$$

$$\frac{\partial v_j}{\partial x_j} = 0 , \tag{11.11}$$

where f_i is a given volume force. Insertion of (11.2) into (11.10) yields another form of the Stokes problem (where p is normalized by the density, i.e. $p = p/\rho$)

$$-\nu \Delta v_i + \frac{\partial p}{\partial x_i} = f_i , \tag{11.12}$$

$$\frac{\partial v_j}{\partial x_j} = 0 . \tag{11.13}$$

For the sake of simplicity, we assume that the Stokes equation (11.12) has homogeneous Dirichlet boundary conditions $v_i = 0$ on $\partial\Omega$.

The generalized **Green's theorem** for the tensorial case reads

$$\int_\Omega \frac{\partial T_{ij}}{\partial x_j} w_i \, d\Omega = -\int_\Omega T_{ij} \frac{\partial w_i}{\partial x_j} \, d\Omega + \int_{\partial\Omega} w_i T_{ij} n_j \, dS . \tag{11.14}$$

Note that we can easily incorporate stress boundary conditions, as the surface integral involves the stress vector $T_{ij} n_j$.

In order to design the weak formulation of the Stokes problem, let us choose the test functions w_i such that they vanish on the boundaries. As the Laplacian $\Delta = \text{div (grad)}$, the application of the Green's formula (11.14) to the viscous term of (11.12) leads to

$$\begin{aligned} -\nu \int_\Omega \frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j}\right) w_i \, d\Omega &= -\nu \int_{\partial\Omega} w_i \frac{\partial v_i}{\partial x_j} n_j \, dS + \nu \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\Omega \\ &= \nu \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\Omega . \end{aligned} \tag{11.15}$$

Equation (11.15) demands that first order derivatives of w_i should at least exist. It is customary to choose w_i in the Sobolev space H^1.

The weak form for the pressure gradient is

$$\begin{aligned}\int_\Omega \frac{\partial p}{\partial x_i} w_i \, d\Omega &= \int_\Omega \frac{\partial}{\partial x_i} (p w_i) \, d\Omega - \int_\Omega p \frac{\partial w_i}{\partial x_i} \, d\Omega \\ &= \int_{\partial\Omega} p \, n_i \, w_i \, dS - \int_\Omega p \frac{\partial w_i}{\partial x_i} \, d\Omega \\ &= - \int_\Omega p \frac{\partial w_i}{\partial x_i} \, d\Omega \, .\end{aligned} \tag{11.16}$$

The test function for the pressure variable is a scalar square integrable function q. More precisely, pressure will be sought in the L_0^2 space, a subspace of L^2 with vanishing average functions, i.e.

$$L_0^2 = \{\varphi \in L^2 | \int_\Omega \varphi \, d\Omega = 0\} \, . \tag{11.17}$$

Therefore the reference pressure is an average over the full domain; this is in contrast with finite difference or finite volume methods where the reference pressure is fixed at a predefined grid point. Recall that fixing the pressure at one particular point impedes the control at that point of the evolution of the numerical divergence of the velocity field.

Therefore the weak formulation of the Stokes problem (11.12) and (11.13) is given as

$$\nu \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\Omega - \int_\Omega p \frac{\partial w_i}{\partial x_i} \, d\Omega = \int_\Omega f_i \, w_i \, d\Omega \, , \tag{11.18}$$

$$- \int_\Omega q \, \frac{\partial v_j}{\partial x_j} \, d\Omega = 0 \, . \tag{11.19}$$

The first integral on the left side of (11.18) may be considered as a dissipation term of the form (11.3). Let us define the following notation

$$e_{ij}(\mathbf{v}) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad e_{ij}(\mathbf{w}) = \frac{1}{2} \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) \, . \tag{11.20}$$

Then using Eq. (2.76) and the symmetry of e_{ij}, we have

$$\int_\Omega e_{ij}(\mathbf{v}) e_{ij}(\mathbf{w}) \, d\Omega = \int_\Omega e_{ij}(\mathbf{v}) \frac{\partial w_i}{\partial x_j} \, d\Omega \, . \tag{11.21}$$

Moreover by integration by parts

$$\int_{\Omega} \frac{\partial v_j}{\partial x_i} \frac{\partial w_i}{\partial x_j} \, d\Omega = - \int_{\Omega} \frac{\partial}{\partial x_i} \left(\frac{\partial v_j}{\partial x_j} \right) w_i \, d\Omega = 0 \,. \tag{11.22}$$

Combining (11.21) and (11.22) we obtain the identity

$$2 \int_{\Omega} e_{ij}(\mathbf{v}) e_{ij}(\mathbf{w}) \, d\Omega = \int_{\Omega} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\Omega \,, \tag{11.23}$$

which shows the deep link between the variational principle and the weak formulation. We observe that the second term on the left side of (11.18) acts on the pressure variable. In fact the pressure is multiplied by the divergence of the test function. We conclude that the pressure is the Lagrange multiplier that imposes the incompressibility constraint. This is one of the major benefits of the weak formulation. It should also be emphasized that Eq. (11.19) is enforcing a weak divergence free condition. This also means that if one computes the strong formulation of the divergence of the velocity, it will never be zero.

The Stokes problem (11.18) and (11.19) corresponds to a **saddle point problem** as the energy functional with the incorporation of the pressure term is minimized for the velocity solution and maximized for the pressure. The name saddle point was given by taking into account the geometric features of the functional that is convex for the velocity minimization and concave for the pressure maximization. This results in an object shaped like a saddle with double curvature. The solution of the saddle point problem presents adequate stability and convergence properties if it satisfies the well-known **inf-sup condition**, which imposes a condition of compatibility of the functional spaces chosen for the velocity and the pressure. This theory is beyond the scope of the present monograph and interested readers are referred to the original papers by Brezzi (1974) and Babuška (1973).

11.3 Finite Element Discretization of the Stokes Equation

The domain Ω is covered with E polyhedral elements Ω^e such that $\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}^e$. On each element Ω^e, the velocity and pressure fields are approximated by

$$\mathbf{v}(\mathbf{x})_{|\Omega^e} \approx \mathbf{v}_h(\mathbf{x})_{|\Omega^e} = \mathbf{v}^e(\mathbf{x}) = \sum_{j=1}^{N_v} \mathbf{v}_j^e \boldsymbol{\varphi}_{v,j} \,, \tag{11.24}$$

$$p(\mathbf{x})_{|\Omega^e} \approx p_h(\mathbf{x})_{|\Omega^e} = p^e(\mathbf{x}) = \sum_{j=1}^{N_p} p_j^e \varphi_{p,j} \,, \tag{11.25}$$

where the subscript h refers to the FE mesh discretization. The velocity approximation uses N_v vector basis functions $\boldsymbol{\varphi}_{v,j}$ while the pressure involves N_p scalar basis functions $\varphi_{p,j}$, with $N_v \neq N_p$. The variational problem on Ω^e reads

$$\nu \sum_{i=1}^{d} \int_{\Omega^e} \nabla v_i^e \cdot \nabla w_i^e \, d\mathbf{x} - \int_{\Omega^e} \text{div } \mathbf{w} \, p^e \, d\mathbf{x} =$$

$$\int_{\Omega^e} \mathbf{f}^e \cdot \mathbf{w} \, d\mathbf{x} \,, \tag{11.26}$$

$$-\int_{\Omega^e} \text{div } \mathbf{v}^e \, q \, d\mathbf{x} = 0 \,, \tag{11.27}$$

where d is the space dimension and the dot denotes the scalar product. For the 2D case we suppose that the velocity components of $\mathbf{v}$ are subject to the same boundary conditions. They have an identical number of degrees of freedom (d.o.f.) which we will denote n_v and therefore $N_v = d \times n_v$.

Choosing the basis functions $\boldsymbol{\varphi}_{v,i}$ as test functions in (11.26) and $\varphi_{p,i}$ in (11.27), the variational problem at the element level is written

$$\begin{pmatrix} [K^e] & [0] & [D_1^e]^T \\ [0] & [K^e] & [D_2^e]^T \\ [D_1^e] & [D_2^e] & [0] \end{pmatrix} \begin{pmatrix} \underline{v}_1^e \\ \underline{v}_2^e \\ \underline{p}^e \end{pmatrix} = \begin{pmatrix} \underline{f}_1^e \\ \underline{f}_2^e \\ \underline{0} \end{pmatrix} . \tag{11.28}$$

The various elementary matrices are defined as follows:

1. The square matrix $[K^e]$ in $\mathbb{R}^{n_v \times n_v}$ is given by

$$[K^e]_{ij} == \nu \int_{\Omega^e} \nabla \varphi_{v,j}^e \cdot \nabla \varphi_{v,i}^e \, d\mathbf{x} \,;$$

2. The matrices $[D_i^e]$ are rectangular matrices of dimension n_p by n_v

$$[D_1^e]_{ij} = -\int_{\Omega^e} \varphi_{p,j}^e \frac{\partial \varphi_{v,i}^e}{\partial x} d\mathbf{x} \,,$$

$$[D_2^e]_{ij} = -\int_{\Omega^e} \varphi_{p,j}^e \frac{\partial \varphi_{v,i}^e}{\partial y} d\mathbf{x} \,;$$

3. The data are vectors of dimension n_v

$$(\underline{f}_1^e)_i = \int_{\Omega^e} f_1^e \varphi_{v,i}^e \, d\mathbf{x} \quad \text{and} \quad (\underline{f}_2^e)_i = \int_{\Omega^e} f_2^e \varphi_{v,i}^e \, d\mathbf{x} \,.$$

We notice that the matrices composing the discrete gradient are transposes of the matrices involved in the discrete divergence.

Now let us discuss the possible choice of the elements and the local shape functions $\varphi_{v,i}^e$ and $\varphi_{p,i}^e$.

11.4 Stable Finite Elements for Viscous Incompressible Fluids

The finite element theory analyzes at large the velocity-pressure discretizations. Even if we are tempted to choose the same polynomial degree for both fields, this is a very bad idea. Indeed the saddle point problem related to the Stokes equation requires that the discrete fields satisfy the inf-sup compatibility condition. If this condition is not enforced the computational results show the presence of **spurious pressure modes** which do not affect the computation of the velocities as those modes produce vanishing pressure gradients. However they should be avoided, especially if the stress has to be evaluated on the boundaries as is required in **fluid-structure interaction**.

The functions $\varphi^e_{v,i}$ and $\varphi^e_{p,i}$ cannot be taken arbitrarily. In practice the pressure approximation is one unit lower than the polynomial degree of the velocity. Figure 11.1 shows the $Q_1 - Q_0$ element where Q_1 refers to the bilinear approximation of the velocity and Q_0 to the constant approximation of pressure. The crosses are the velocity nodes with components v_1, v_2 and the central node is the pressure.

Unfortunately this element is also unstable and presents one spurious pressure mode, namely the **checkerboard mode**. This means that the pressure oscillates in sign from one element to the next in both space directions. Even though this mode may be filtered (see e.g. Sani et al. (1981)) it is safer to rely on stable elements like the one introduced by Taylor and Hood (1973). This element is the $Q_2 - Q_1$ nine node Lagrangian element where the velocity is locally approximated by quadratics and the pressure by bilinear functions as shown in Fig. 11.2.

Note that the 2D $Q_2 - Q_1$ element has 18 velocity d.o.f. and 4 pressures. In 3D this leads to 81 velocities and 8 pressures. The associated bandwidth of the linear system and consequently the solution cost increase. This is one of the reason why practioners still prefer to use the 3D $Q_1 - Q_0$ element with filtering as it is less expensive. The triangular counterpart of the $Q_2 - Q_1$ element is the $P_2 - P_1$ element. Both elements offer second order accuracy in space. More specifically, if the velocity is approximated by polynomials of degree k and the pressure by polynomials of degree $k - 1$, then the following error estimate can be proved:

$$\|\mathbf{v} - \mathbf{v}_h\|_{H^1} + \|p - p_h\|_{L^2} \leq C h^k \left(\|\mathbf{v}\|_{H^{k+1}} + \|p\|_{H^k}\right),$$

where C is a constant independent of h and the various norms are evaluated in the appropriate spaces. The method is accurate to order k, i.e. the convergence is $O(h^k)$. Consequently the $Q_2 - Q_1$ element is $O(h^2)$.

If we carry out the direct stiffness procedure which performs the assembling of the local contributions (11.28) to construct the global matrix system, we obtain

$$[\mathbf{K}]\,\underline{\mathbf{v}} + [\mathbf{D}^T]\,\underline{p} = [\mathbf{M}]\,\underline{\mathbf{f}}\,, \tag{11.29}$$

$$[D]\,\underline{\mathbf{v}} = \underline{0}\,. \tag{11.30}$$

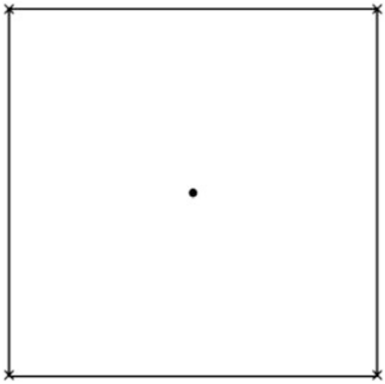


Fig. 11.1 2D $Q_1 - Q_0$ element

Note that the matrices yielding vector quantities as a result are written in boldface.

The stiffness and mass matrices, $[\mathbf{K}]$ and $[\mathbf{M}]$ are square in $\mathbb{R}^{N,N}$, the gradient matrix $[\mathbf{D}]^T$ is a rectangular matrix of $\mathbb{R}^{N,M}$ and the transposed matrix $[D]$ is the discrete divergence. The data $\underline{\mathbf{f}}$ is a vector of $\mathbb{R}^N$ and the unknowns are the vectors $\underline{\mathbf{v}}$ of $\mathbb{R}^N$ and $\underline{p}$ of $\mathbb{R}^M$ with N and M, both integers. One can show that the system (11.29) and (11.30) has a unique solution that is computed by decoupling the velocity and the pressure by first solving the relation

$$[S]\,\underline{p} = \underline{b}\,, \tag{11.31}$$

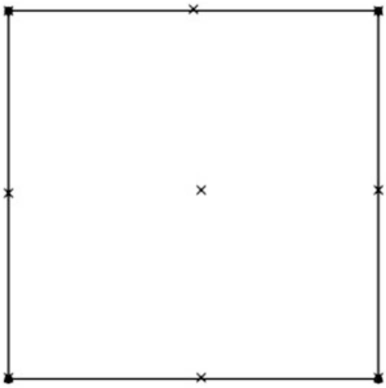


Fig. 11.2 2D $Q_2 - Q_1$ element

with

$$[S] = [D][\mathbf{K}]^{-1}[\mathbf{D}^T]\,, \tag{11.32}$$

$$\underline{b} = [D][\mathbf{K}]^{-1}[\mathbf{M}]\,\underline{\mathbf{f}}\,, \tag{11.33}$$

and afterwards

$$[\mathbf{K}]\,\underline{\mathbf{v}} = [\mathbf{M}]\underline{\mathbf{f}} - [\mathbf{D}]^T\,\underline{p}\,. \tag{11.34}$$

The $[S]$ matrix is called the **Uzawa matrix** and has the following properties

1. $[S]^T = [S]$,
2. $([S]\,\underline{p}, \underline{p}) \geq 0$,
3. $([S]\,\underline{p}, \underline{p}) = 0$ if and only if $p = 0$.

The solution of Eq. (11.31) is performed iteratively so we can take full advantage of the symmetric and positive definite characteristic features of $[S]$. A preconditioned conjugate gradient method is an excellent choice.

11.5 Unsteady Stokes Equation

With the given body force **f**, the transient Stokes equation governs the evolution of the velocity and pressure fields

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} \qquad \text{in} \quad \Omega\times]0,T]\,, \tag{11.35}$$

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in} \quad \Omega\times]0,T]\,, \tag{11.36}$$

$$\mathbf{v} = \mathbf{0} \qquad \text{on} \quad \partial\Omega\times]0,T]\,, \tag{11.37}$$

$$\mathbf{v}(t=0,\mathbf{x}) = \mathbf{v}^0(\mathbf{x}) \qquad \text{in} \quad \Omega\,. \tag{11.38}$$

The symbol T is the given final time and $\mathbf{v}^0(\mathbf{x})$ is the initial incompressible velocity. Let us denote the time step by Δt.

Now, on each element Ω^e, the velocity and pressure fields are approximated by

$$\mathbf{v}(\mathbf{x},t)_{|\Omega^e} \approx \mathbf{v}_h(\mathbf{x},t)_{|\Omega^e} = \mathbf{v}^e(\mathbf{x},t) = \sum_{j=1}^{N_v} \mathbf{v}_j^e(t)\boldsymbol{\varphi}_{v,j}\;, \tag{11.39}$$

$$p(\mathbf{x},t)_{|\Omega^e} \approx p_h(\mathbf{x},t)_{|\Omega^e} = p^e(\mathbf{x},t) = \sum_{j=1}^{N_p} p_j^e(t)\varphi_{p,j}\;. \tag{11.40}$$

Inserting (11.39) and (11.40) into (11.35) and (11.36) yields

$$[\mathbf{M}]\,\underline{\dot{\mathbf{v}}} + \nu[\mathbf{K}]\,\underline{\mathbf{v}} + [\mathbf{D}]^T\,\underline{\mathbf{p}} = [\mathbf{M}]\,\underline{\mathbf{f}}, \tag{11.41}$$

$$[D]\,\underline{\mathbf{v}} = \underline{0}\,. \tag{11.42}$$

Equations (11.41) and (11.42) form a **differential-algebraic system** where the ordinary differential equations (11.41) are constrained by the set of algebraic equations (11.42). Let us observe that even if we choose an explicit time scheme, we will have to solve a matrix system to get the evolution of the velocity, unless we lump the mass matrix. Furthermore the time scheme should be stable. This means that for a highly viscous fluid, a stability restriction like the von Neumann criterion $\nu \Delta t / h^2 \leq 1/2$ reduces dramatically the time step. Therefore most numericists today use an implicit scheme for the Stokes problem.

For ease of description, the time scheme is chosen as the first order Euler implicit scheme

$$[\mathbf{M}]\,\dot{\underline{\mathbf{v}}} \approx \frac{[\mathbf{M}]}{\Delta t}\left(\underline{\mathbf{v}}^{n+1} - \underline{\mathbf{v}}^{n}\right) , \tag{11.43}$$

where the superscript indicates the time level. Combining (11.43) with (11.41) and (11.42) yields the matrix system

$$\begin{pmatrix} [\mathbf{H}] & [\mathbf{D}]^T \\ [D] & [0] \end{pmatrix} \begin{pmatrix} \underline{\mathbf{v}}^{n+1} \\ \underline{p}^{n+1} \end{pmatrix} = \begin{pmatrix} [\mathbf{M}]\underline{\mathbf{f}} \\ \underline{0} \end{pmatrix} + \frac{1}{\Delta t} \begin{pmatrix} [\mathbf{M}]\underline{\mathbf{v}}^n \\ \underline{0} \end{pmatrix}. \tag{11.44}$$

The symbol $[\mathbf{H}]$ is the discrete Helmholtz operator

$$[\mathbf{H}] = \frac{1}{\Delta t}[\mathbf{M}] + \nu[\mathbf{K}], \tag{11.45}$$

composed of d block matrices $[H] = 1/(\Delta t)[M] + \nu[K]$. The same applies to the stiffness matrix $[\mathbf{K}]$. The system (11.44) is solved by the **Uzawa algorithm**. The pressure is computed by solving

$$[S_t]\,\underline{p}^{n+1} = \underline{b}^n , \tag{11.46}$$

with

$$[S_t] = [D][\mathbf{H}]^{-1}[\mathbf{D}]^T , \tag{11.47}$$

and

$$\underline{b}^n = [D][\mathbf{H}]^{-1}\left([\mathbf{M}]\underline{\mathbf{f}} + \frac{1}{\Delta t}[\mathbf{M}]\underline{\mathbf{v}}^n\right) , \tag{11.48}$$

and then the velocity is obtained from

$$[\mathbf{H}]\,\underline{\mathbf{v}}^{n+1} = [\mathbf{M}](\underline{\mathbf{f}} + \frac{1}{\Delta t}\underline{\mathbf{v}}^n) - [\mathbf{D}]^T\,\underline{p}^{n+1}. \tag{11.49}$$

As the Uzawa method is reputed to converge (too) slowly, projection methods are widely used to undertake the velocity-pressure decoupling.

To this end, let us LU decompose the matrix (11.44)

$$\begin{pmatrix} [\mathbf{H}] & [\mathbf{D}]^T \\ [D] & [0] \end{pmatrix} = \begin{pmatrix} [\mathbf{H}] & [\mathbf{0}] \\ [D] & -[D][\mathbf{H}]^{-1}[\mathbf{D}]^T \end{pmatrix} \begin{pmatrix} [\mathbf{I}] & [\mathbf{H}]^{-1}[\mathbf{D}]^T \\ [\mathbf{0}] & [I] \end{pmatrix} . \tag{11.50}$$

Then the L **step** consists in computing a temporary velocity $\underline{\tilde{\mathbf{v}}}^{n+1}$ and pressure $\underline{\tilde{p}}^{n+1}$, solutions of

$$\begin{aligned} [\mathbf{H}]\,\underline{\tilde{\mathbf{v}}}^{n+1} &= \underline{\mathbf{b}}^{n+1}, \\ [D]\underline{\tilde{\mathbf{v}}}^{n+1} - [D][\mathbf{H}]^{-1}[\mathbf{D}]^T\,\underline{\tilde{p}}^{n+1} &= \underline{0}, \end{aligned} \tag{11.51}$$

and afterwards the U **step** computes the new time level velocity and pressure by solving

$$\begin{aligned} \underline{\mathbf{v}}^{n+1} + [\mathbf{H}]^{-1}[\mathbf{D}]^T\,\underline{p}^{n+1} &= \underline{\tilde{\mathbf{v}}}^{n+1}, \\ \underline{p}^{n+1} &= \underline{\tilde{p}}^{n+1}. \end{aligned}$$

If we wish to alleviate the burden of the L step, we can proceed with an approximate factorization where the inverse of $[\mathbf{H}]$ is obtained by a truncated Taylor series

$$\begin{aligned} [\mathbf{H}]^{-1} &= \left(\frac{1}{\Delta t}[\mathbf{M}] + \nu[\mathbf{K}]\right)^{-1} \\ &= \Delta t\left([\mathbf{I}] + \nu\Delta t[\mathbf{M}]^{-1}[\mathbf{K}]\right)^{-1}[\mathbf{M}]^{-1} \\ &= \Delta t\sum_{j=0}^{\infty}(-1)^j\left(\nu\Delta t[\mathbf{M}]^{-1}[\mathbf{K}]\right)^j[\mathbf{M}]^{-1} \\ &= \Delta t([\mathbf{I}] - \nu\Delta t[\mathbf{M}]^{-1}[\mathbf{K}] + \cdots)[\mathbf{M}]^{-1}. \end{aligned} \tag{11.52}$$

The first and simplest choice is to take the first term in (11.52)

$$[\mathbf{H}]^{-1} \approx \Delta t[\mathbf{M}]^{-1} . \tag{11.53}$$

The algorithm computes first the temporary velocity $\underline{\tilde{\mathbf{v}}}^{n+1}$ through Eq. (11.51); then the final pressure p^{n+1} is obtained from

$$-\Delta t[D][\mathbf{M}]^{-1}[\mathbf{D}]^T\,\underline{p}^{n+1} = [D]\underline{\tilde{\mathbf{v}}}^{n+1}. \tag{11.54}$$

Finally the velocity results from

$$\underline{\mathbf{v}}^{n+1} = \underline{\tilde{\mathbf{v}}}^{n+1} - \Delta t[\mathbf{M}]^{-1}[\mathbf{D}]^T\,\underline{p}^{n+1} .$$

If the mass matrix is lumped and therefore diagonal, the matrix in (11.54) is $[D][\mathbf{D}]^T$, i.e. a divergence times a gradient. This corresponds to a discrete Laplacian and this relation is a Poisson pressure equation. This approximate method was proposed independently by Chorin (1968) and Témam (1969). The projection method, while easing the computation, induces a **splitting error** resulting from the velocity-pressure decoupling. Here the splitting error is $O(\Delta t)$.

11.6 Advection-Diffusion Equation

The beauty and the difficulty of fluid flow problems come from the presence of the nonlinear term in the Navier-Stokes equation. As this monograph treats mainly slow viscous flow the problem of tackling turbulence or instabilities is put aside. However the advent of the computer era has allowed entry into this nonlinear world and discover what it brings to flow understanding. In this section we will describe briefly the way finite elements discretize the nonlinear terms and how this spatial approximation may be incorporated into a full time scheme. Our mathematical model is based on the advection-diffusion equation, where the advective part is nonlinear.

11.6.1 One Dimensional Burgers Equation

The Burgers equation is a one dimensional Navier-Stokes equation where the pressure influence is discarded:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}\,, \quad -1 < x < 1, \quad t \in]0, T]\,, \tag{11.55}$$

$$u(x)^0 = -\sin \pi x\,, \tag{11.56}$$

$$u(-1,t) = u(+1,t) = 0\,. \tag{11.57}$$

Using the change of variable

$$u = -\frac{2\nu}{\theta}\left(\frac{\partial \theta}{\partial x}\right) \tag{11.58}$$

in (11.55) yields a linear diffusion equation for the θ variable

$$\frac{\partial \theta}{\partial t} = \nu\frac{\partial^2 \theta}{\partial x^2}\,. \tag{11.59}$$

The analytical solution of (11.55) obtained by Cole and compiled by Benton and Platzmann (1972) is

$$u(t,x) \;=\; 4\pi\nu \,\frac{\sum_{n=1}^{\infty} n\, a_n e^{-\nu n^2\pi^2 t}\sin(n\pi x)}{a_0 + 2\sum_{n=1}^{\infty} n\, a_n e^{-\nu n^2\pi^2 t}\cos(n\pi x)}. \tag{11.60}$$

In this expression, the a_n coefficients are given by

$$a_n \;=\; (-1)^n I_n(1/2\pi\nu), \tag{11.61}$$

where $I_n(z)$ is the modified Bessel function of first kind and of order n. This analytical solution is very difficult to evaluate for small values of t $(0 \le t \le 2/\pi)$ and of ν as $I_n(z)$, when z goes to ∞, behaves asymptotically as $e^z(2\pi z)^{-1/2}$, independently of the n value. High-order numerical solutions are found in Basdevant et al. (1986). An interesting quantity for measuring the scheme accuracy is the time evolution of the slope at the origin. For $\nu = 1/100\pi$ one has $|\partial u/\partial x|_{max} = 152.00516$ for $t_{max} = 0.5105$.

As far as the dynamics is concerned, the peaks of the initial profile start moving with a velocity close to unity toward the origin. As the viscosity is very low, the initial times are advection dominated and therefore the profile deforms and takes the shape of a sawtooth. We get an internal shear layer. Note that the profile is not discontinuous. In the long run however viscosity enters into play and damps the profile to zero.

11.6.1.1 The Galerkin Method

Let us apply the Q1 FE method to (11.55). The weak formulation reads: *Find* $u(x,t) \in H_e^1(\Omega)$ *such that*

$$\frac{\partial}{\partial t}\int_\Omega u(x,t)w(x)dx + \nu\int_\Omega \frac{\partial u}{\partial x}\frac{\partial w}{\partial x}dx + \int_\Omega (u\frac{\partial u}{\partial x})w\,dx = 0\,, \quad \forall w(x) \in H_e^1(\Omega)\,. \tag{11.62}$$

Choosing the approximation u_N as in (10.27), with the u_k being time dependent, and the test function w_N in $H_e^1(\Omega)$, one obtains

$$\int_\Omega \left(\frac{\partial u_N}{\partial t}w_N + \nu\frac{\partial u_N}{\partial x}\frac{\partial w_N}{\partial x} + w_N\, u_N\frac{\partial u_N}{\partial x}\right)dx = 0\,. \tag{11.63}$$

The semi-discrete equation for the mesh node i is given by

$$\frac{h_{i-1}}{6}\frac{du_{i-1}}{dt} + \frac{h_{i-1}+h_i}{3}\frac{du_i}{dt} + \frac{h_i}{6}\frac{du_{i+1}}{dt} + \frac{u_{i-1}+u_i+u_{i+1}}{3}\frac{u_{i+1}-u_{i-1}}{2} \\ -\nu\left(\frac{u_{i-1}}{h_{i-1}} - \left(\frac{1}{h_{i-1}}+\frac{1}{h_i}\right)u_i + \frac{u_{i+1}}{h_i}\right) = 0\,. \tag{11.64}$$

As usual we observe the presence of the stiffness and mass matrices. In the nonlinear term we have a centered scheme for the first order derivative with the contribution of the local velocity taken as the mean over three successive points. This is in contrast with the value u_i that would be produced by a FD discretization.

In compact form one gets

$$[M]\,\frac{d\underline{u}(t)}{dt} + [K]\,\underline{u}(t) + [C(\underline{u}(t))]\,\underline{u}(t) = 0\,, \tag{11.65}$$

where the operator $[C(\underline{u}(t))]$ acts on the vector $\underline{u}$. Note that $[C]$ is not a matrix. A simple time scheme consists of the explicit forward Euler scheme

$$\frac{[M]}{\Delta t}\,\left(\underline{u}^{n+1} - \underline{u}^{n}\right) + [K]\,\underline{u}^{n} + [C(\underline{u}^{n})]\,\underline{u}^{n} = 0. \tag{11.66}$$

Note that as the time scheme is explicit it is subject to stability conditions. If we lump the mass matrix and therefore get back to FD, the stability is restricted by two conditions, namely:

$$\frac{\nu \Delta t}{h^2} \leq \frac{1}{2}, \tag{11.67}$$

and

$$\frac{u_i^n\,\Delta t}{h} < 1,\ \forall i \in [2, N]. \tag{11.68}$$

Condition (11.67) is the **von Neumann condition** imposed on the viscous part of the operator. If we treat a very viscous fluid this condition is quite stringent and this is one of the reasons why viscous terms are very often treated implicitly. Condition (11.68) is the **CFL** (Courant-Friedrichs-Lewy) **condition** imposing that within one time step, the numerical information should not travel a distance longer than one mesh spacing. For high Reynolds number flows where advection is the dominating physical phenomenon this CFL condition constitutes a severe restriction and reduces drastically the time step of the time integration. If one tries to avoid the CFL condition, the other possible time scheme would be implicit. However as the non linearity leads to quadratic terms, a Newton linearization is needed. The resulting matrix system is no longer symmetric and must be solved using the GMRES method Saad (2003).

The scheme proposed in (11.66) is only first order time accurate. To reach second order accuracy, many numericists resort to **time splitting**. This means that different time integrators are used for the linear viscous part and the nonlinear term. A state-of-the-art scheme is given by the BE2/AB2 formula meaning that the linear term is treated implicitly by the second order backward Euler scheme (hence BE2) and the non linearity is handled explicitly by the second order Adams-Bashforth (hence AB2) scheme

$$\frac{[M]}{2\Delta t}\left(3\underline{u}^{n+1} - 4\underline{u}^{n} + \underline{u}^{n-1}\right) + [K]\,\underline{u}^{n+1} + \frac{3}{2}\,[C(\underline{u}^{n})]\,\underline{u}^{n} - \frac{1}{2}\,[C(\underline{u}^{n-1})]\,\underline{u}^{n-1} = 0\,. \qquad (11.69)$$

However even though each time integrator is second order accurate the full splitting error is only first order, a very deceptive result. Second order accuracy is achieved if the nonlinear term is evaluated by a second order time extrapolation denoted by EXT2

$$[C(\underline{u}^{n+1})]\,\underline{u}^{n+1} \approx 2\,[C(\underline{u}^{n})]\,\underline{u}^{n} - [C(\underline{u}^{n-1})]\,\underline{u}^{n-1}\,, \qquad (11.70)$$

leading to the full scheme BE2/EXT2.

11.6.1.2 Petrov-Galerkin Method

The Galerkin method generates centered schemes for first-order derivatives. For the nonlinear term this seems a bit odd because if the local velocity is positive this means that to compute the derivative at node i we need to go downstream to $i+1$ to pick up the relevant discrete values. Should we not use some kind of upwind schemes that would be more appropriate from the physical point of view? The **Petrov-Galerkin** (PG) **method** allows one to build such an approach. Here the test functions are different from the basis functions. For the Q_1 case, the test functions in $\hat{\Omega}$ are basis functions modified by the addition of a parabolic contribution that gives more weight to the upstream quantities:

$$\{\hat{\psi}_1(\xi), \hat{\psi}_2(\xi)\} = \{\frac{1-\xi}{2} - 3\alpha(1-\xi^2), \frac{1+\xi}{2} + 3\alpha(1-\xi^2)\} \qquad (11.71)$$

where the parameter α will manage the amount of upwinding needed. The PG method does not affect the discrete scheme of the viscous term; it will modify the nonlinear term which becomes

$$\frac{u_{i-1} + u_i + u_{i+1}}{3}\left(\frac{u_{i+1} - u_{i-1}}{2h}\right) - \frac{\alpha h}{4}\,\frac{u_{i-1}^2 - 2u_i^2 + u_{i+1}^2}{h^2}\,, \qquad (11.72)$$

where the additional PG term is a diffusive term expressed as a function of u^2 instead of u. This will improve the stability of the scheme.

11.6.1.3 Numerical Results

Solving (11.55)–(11.57) by BE2/EXT2 with $E = 1{,}000$ elements for $\nu = 1/(100\pi)$ yields the results shown in Fig. 11.3.

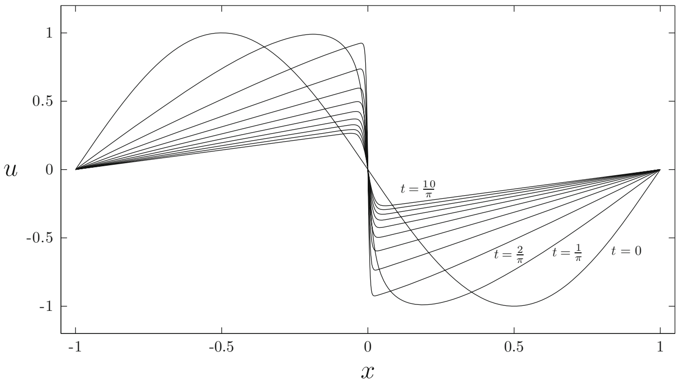


Fig. 11.3 Burgers equation, $\nu = 1/100\pi$, $t \in]0, 10/\pi]$, $E = 1{,}000$ (Reprinted with permission from Azaïez et al. (2011))

If the number of elements is not sufficient, e.g. $E = 100$, oscillations are induced near the origin because of the steep gradient. The maximum slope at the origin is equal to 155.7 and occurs at time $t_{max} = 0.5093$, indicating that the numerical solution leads the analytical solution by a bit.

11.6.2 Multidimensional Burgers Equation

The multidimensional Burgers equation is described through the problem unknown $\mathbf{v}(\mathbf{x}, t)$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \Delta \mathbf{v}, \qquad \text{in } \Omega \times]0, T] , \tag{11.73}$$

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{0} \qquad \text{on } \partial\Omega \times]0, T] , \tag{11.74}$$

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}^0(\mathbf{x}) \qquad \text{in } \Omega . \tag{11.75}$$

Each velocity component of $\mathbf{v}$ satisfies a nonlinear advection-diffusion equation where a coupling occurs via the transport term. For the sake of simplicity we present the Galerkin method applied to Eq. (11.73).

Let us build $\mathbf{V}_h$, a subspace of dimension N of $(H_e^1(\Omega))^d$, with the help of the Q_1 FE basis $\{\boldsymbol{\varphi}_j(\mathbf{x})\}_{j=1}^N$. The global polynomial approximation of $\mathbf{v}$ is given by

$$\mathbf{v}_h = \sum_{j=1}^{N} \mathbf{v}_j(t) \boldsymbol{\varphi}_j(\mathbf{x}) . \tag{11.76}$$

The test functions are chosen in $\mathbf{V}_h$ such that $\mathbf{w}_h = \boldsymbol{\varphi}_k$, $k = 1, \ldots, N$. The weak formulation of the problem integrates the diffusive term by parts, the nonlinear term remaining unchanged. One then has the following problem:

Find $\mathbf{v}_h(\mathbf{x}, t) \in \mathbf{V}_h$ *such that* $\forall\ \mathbf{w}_h \in \mathbf{V}_h$

$$\int_\Omega \mathbf{w}_h \left(\frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla \mathbf{v}_h \right) d\mathbf{x} = -\nu \int_\Omega \nabla \mathbf{v}_h \nabla \mathbf{w}_h \, d\mathbf{x} \, . \tag{11.77}$$

For each component of $\mathbf{v}_h$, one performs the discretization of (11.77), obtaining the mass [**M**] and the stiffness [**K**] matrices for the linear terms. In the 2D case, the nonlinear term generates the following contributions at the node i, j in a structured mesh

$$\begin{aligned} N_{1,ij} v_{1,j} &= \left(v_{1,k} \int_\Omega \varphi_i \varphi_k \frac{\partial \varphi_j}{\partial x} d\mathbf{x} + v_{2,k} \int_\Omega \varphi_i \varphi_k \frac{\partial \varphi_j}{\partial y} d\mathbf{x} \right) v_{1,j} \ , \\ N_{2,ij} v_{2,j} &= \left(v_{1,k} \int_\Omega \varphi_i \varphi_k \frac{\partial \varphi_j}{\partial x} d\mathbf{x} + v_{2,k} \int_\Omega \varphi_i \varphi_k \frac{\partial \varphi_j}{\partial y} d\mathbf{x} \right) v_{2,j} \ , \end{aligned}$$

with the summation convention on repeated indices j and k. We are led to the compact form

$$[\mathbf{M}] \frac{d \underline{\mathbf{v}}}{dt} + [\mathbf{K}] \underline{\mathbf{v}} + [\mathbf{N}(\underline{\mathbf{v}})] \underline{\mathbf{v}} = \mathbf{0} \, , \tag{11.78}$$

where the vector $\underline{\mathbf{v}} = (\underline{v}_1, \underline{v}_2)^T$. The matrices [**M**] and [**K**] are diagonal matrices with d blocks, equal respectively to $[M]$ and $[K]$. This system of nonlinear ordinary differential equations is integrated by time schemes similar to those proposed in Sect. 11.6.1.

As an example, the Crank-Nicolson/AB2 (CN/AB2) method yields

$$[\mathbf{M}] \frac{\underline{\mathbf{v}}^{n+1} - \underline{\mathbf{v}}^n}{\Delta t} + \frac{[\mathbf{K}]}{2} \left(\underline{\mathbf{v}}^{n+1} + \underline{\mathbf{v}}^n \right) + \frac{3}{2} [\mathbf{N}(\underline{\mathbf{v}}^n)] \underline{\mathbf{v}}^n - \frac{1}{2} [\mathbf{N}(\underline{\mathbf{v}}^{n-1})] \underline{\mathbf{v}}^{n-1} = \mathbf{0} \, .$$

11.7 Navier-Stokes Equation

The velocity-pressure formulation of the Navier-Stokes equation is summarized as follows

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \Delta \mathbf{v} - \nabla p + \mathbf{f} \, , \tag{11.79}$$

$$\operatorname{div} \mathbf{v} = 0 \, , \tag{11.80}$$

$$\mathbf{v}(\mathbf{x}, t = 0) = \mathbf{v}^0 \, , \tag{11.81}$$

where $\mathbf{v}$ is the velocity and p the pressure normalized by the density. The finite element discretization in space produces the semi-discrete problem: *Find the velocity field* $\underline{\mathbf{v}}(t)$ *and the scalar field* $\underline{p}(t)$ *which are solutions of*

$$[\mathbf{M}]\,\dot{\underline{\mathbf{v}}} + ([\mathbf{K}] + \mathbf{N}(\underline{\mathbf{v}}))\,\underline{\mathbf{v}} + [\mathbf{D}]^T\,\underline{p} = [\mathbf{M}]\,\underline{\mathbf{f}}\,, \tag{11.82}$$

$$[D]\,\underline{\mathbf{v}} = \underline{0}\,. \tag{11.83}$$

A second order accurate time scheme with an adaptive time stepping is proposed by Gresho et al. (1980). The time integrator is a predictor-corrector scheme where the prediction step is AB2 and the correction step CN. The choice of the time step is controlled by the time error during the evolution of the dynamics. The detailed description of this excellent and sophisticated method is beyond the scope of the present text and the reader is referred to the original paper. This method is very useful for problems where the time evolution may become wild or chaotic.

A good example is blood flow simulation in the aorta, where the systole peak may induce strong pressure gradients and steep accelerations Tu et al. (1992).

Another application that needs time accuracy is the von Kármán street behind a circular cylinder. This is composed of vortices that are shed alternately from the top and bottom parts of the rear boundary layer. Accurate capture of the shedding frequency remains a real challenge for most of the numerical methods undertaking this difficult problem.

An easier time treatment of (11.82) and (11.83) consists in using split schemes with an implicit part for the Stokes problem and an explicit treatment for the nonlinearity. One might carry out a BE2/AB2 scheme.

11.8 Spectral Elements for the Navier-Stokes Equation

Let us define by $\mathbb{P}_N$ the set of polynomials of degree $\leq N$ with respect to each of the space variables. Then the spectral element discretization for the Navier-Stokes primitive variables $\mathbf{v}$ and p is based on the $\mathbb{P}_N - \mathbb{P}_{N-2}$ element which is free of spurious pressure modes. (The choice $\mathbb{P}_N - \mathbb{P}_{N-1}$ still contains spurious modes.) One can show that this element is optimal. The underlying quadrature rules involve the Gauss–Lobatto–Legendre nodes for the velocity field and the Gauss–Legendre grid for the pressure. Figure 11.4 displays the staggered spectral element for the polynomial degree $N = 6$ (left) and $N = 7$ (right). The velocities are continuous along the element interfaces while the pressure is not necessarily continuous. We observe in Fig. 11.4 that the dashed lines represent the pressure grid and the solid lines the velocity grid. The two grids are entwined and in the right element there is no pressure node in the central strips of the element. For smooth problems with regular solutions, the velocity error decays as N^{1-N} while the 3D pressure decays like N^{2-N} ($N^{3/2-N}$ for 2D problems).

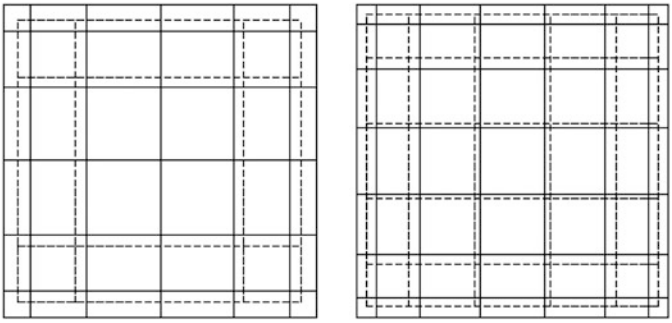


Fig. 11.4 2D staggered spectral elements for $N = 6$ (*left*) and $N = 7$ (*right*) (Reprinted with permission from Deville et al. (2002))

To illustrate the ability of SEM to solve challenging problems we will consider the flow that occurs behind a horizontal circular cylinder placed between two lateral vertical walls. The solution we show is due to Latt (2013, private communication) and is based on the software described in Bosshard et al. (2011). The physical phenomena and experimental data are summarized in Williamson (1996).

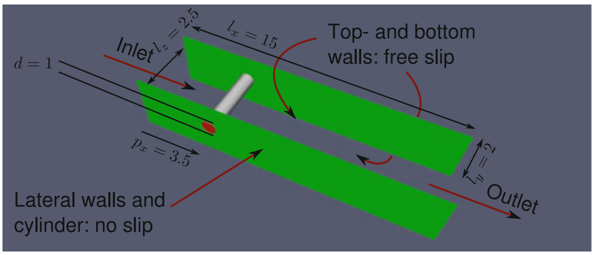


Fig. 11.5 Horizontal cylinder between vertical walls

Let us refer to Fig. 11.5 for the problem description. The horizontal cylinder has a diameter $d = 1$ and span $l_z = 2.5$. The two vertical walls have length $l_x = 15$ and height $l_y = 2$. The symmetry axis of the cylinder is located at a distance $p_x = 3.5$ from the inlet section where a uniform velocity profile is prescribed. On the vertical walls and on the cylinder, no-slip boundary conditions are applied. On the top and bottom walls of the computational box free-slip conditions are imposed. At the outlet we let the flow leave the domain with a stress free boundary condition.

Figure 11.6 shows the SEM grid in the symmetry plane of the computational box. This grid is extruded in the spanwise direction. We note that most elements are uniform quadrilaterals (or more precisely uniform hexahedra). However curvy elements are needed close to the cylinder boundaries to resolve the boundary layers. They are taken into account by the method of **isoparametric elements** where the geometry is described by the same polynomial approximation as the one used for the velocity variables. This consistent approach yields the same accuracy for the geometry as for the problem unknowns.

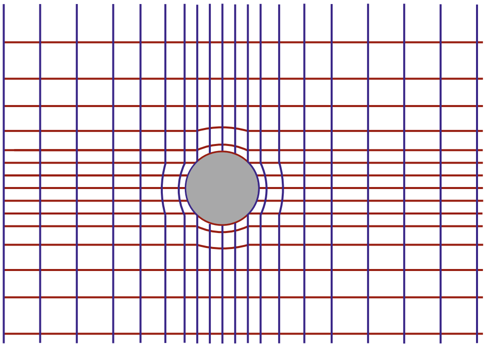


Fig. 11.6 The spectral element mesh

All the computations were carried out with $E = 4{,}096$ elements with polynomial degree $N = 8$ in each space direction. The parallel toolbox Speculoos (http://sourceforge.net/projects/openspeculoos) was used on 4,096 cores of the EPFL IBM BlueGene.

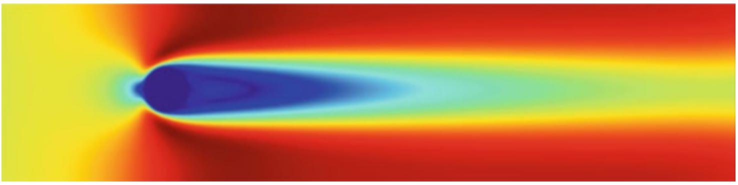


Fig. 11.7 Stationary flow at $\mathcal{R}e = 80$

Because the lateral walls induce viscous effects on the main flow, the transition to a time periodic flow is slowed down and delayed. Figure 11.7 shows the velocity magnitude for $\mathcal{R}e = 80$ in the symmetry plane. Obviously the flow is laminar, steady state, and presents a symmetric recirculation zone in the wake of the cylinder.

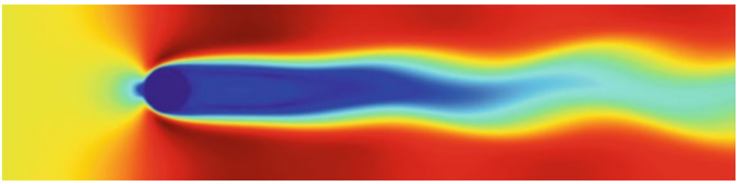


Fig. 11.8 Time-periodic von Karman vortex structure at $\mathcal{R}e = 160$

At Reynolds number $\mathcal{R}e = 160$, Fig. 11.8 displays the first time-periodic feature of the Hopf bifurcation and a von Kármán vortex street starts evolving. The Hopf bifurcation has a single characteristic frequency corresponding to an elliptical limit cycle in the phase plane.

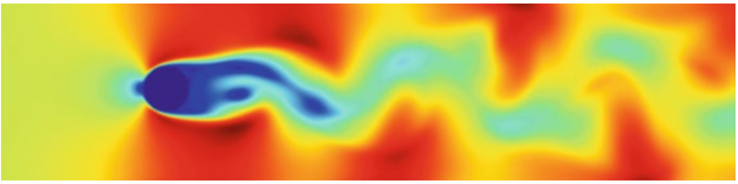


Fig. 11.9 Quasi-periodic flow at $\mathcal{R}e = 320$

Increasing the Reynolds number to $\mathcal{R}e = 320$ brings more complexity to the flow in terms of spatial structures. Figure 11.9 exhibits the various geometrical patterns of a quasi-periodic flow. Here the frequency analysis of the flow dynamics reveals that the fundamental frequency is present with higher harmonics.

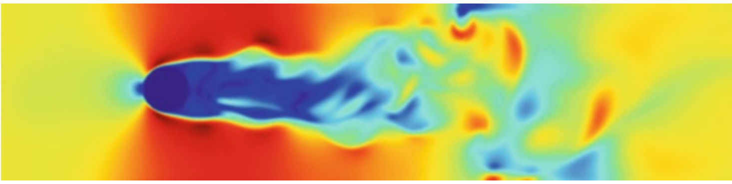


Fig. 11.10 Weak turbulent flow at $\mathcal{R}e = 640$

Finally, in Fig. 11.10 for $\mathcal{R}e = 640$, a turbulent flow is generated wherein more eddies are created. The temporal behavior becomes chaotic and weak turbulence constitutes the dominant phenomenon as a consequence of the nonlinear dynamics.

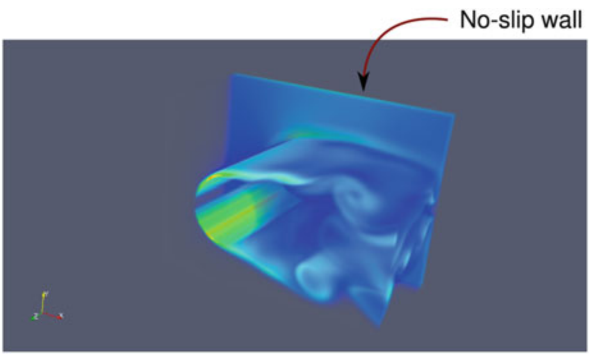


Fig. 11.11 Vorticity at $\mathcal{R}e = 640$

Figure 11.11 represents the norm of the vorticity field at $\mathcal{R}e = 640$. The two vortices shed by the flow instability are clearly identified. Note also that vorticity develops on the lateral wall.

To summarize the chapter, we may conclude that direct numerical simulation is feasible only for small or moderate Reynolds numbers. To compute developed turbulent flows ($\mathcal{R}e = 10^6 \ldots 10^7$) additional modeling is needed. Numericists integrate the **Reynolds-averaged Navier-Stokes equation** that is derived by (Reynolds) decomposing the velocity field into a mean contribution based on a time averaging and a fluctuating part. Another route to tackle turbulent flows uses **large eddy simulation** where the dynamics of gross structures is resolved by integrating a filtered Navier-Stokes equation while small structures are modeled by a subgrid scale approach. Obviously both theoretical developments are beyond the scope of the present monograph. For further reading the reader is referred to Deville and Gatski (2012).

Chapter 12
Stokes Flow and Corner Eddies

Abstract Creeping flow in two- and three-dimensional corners is investigated. A solution to the paint scraper problem is presented. The Stokes operator is numerically analyzed and the eigenspectrum with the eigenvalues and the eigenmodes is calculated. A three-dimensional solution for the steady Stokes equations, based on harmonic solutions of the Laplace equation, is presented.

This chapter is devoted to the creeping flow occurring in two- and three-dimensional corners. Analytical solutions are available and constitute a set of benchmark references for numerical solutions of the Navier-Stokes equation. Even in this century where the computational power is growing at a rapid pace, the need for closed form solutions is still imperative as a tool for deeper insight into and understanding of fluid flow phenomena. The reader who wants to go further than this introductory text should consult Barthès-Biesel (2012) and Shankar (2007).

As the Stokes operator is the cornerstone of many numerical methods for integrating the Navier-Stokes equation, we will survey the present knowledge of the associated eigenspectrum.

12.1 Two-Dimensional Corners

With the assumption of plane creeping flow, the governing equation for the stream function ψ is the biharmonic relation (6.6) with the velocity components given by (6.5). For the sequel it is useful to remember that the operator ∇^4 is defined in polar coordinates by

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2 . \tag{12.1}$$

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3_12,

We look for separable solutions of (6.6) expressed by

$$\psi = r^{\lambda} f(\theta) \,, \tag{12.2}$$

where λ is a real or complex number. Carrying through the algebra, the biharmonic equation yields

$$\nabla^4 \psi = r^{\lambda-4} \left(\frac{d^2}{d\theta^2} + \lambda^2 \right) \left(\frac{d^2}{d\theta^2} + (\lambda - 2)^2 \right) f = 0 \,. \tag{12.3}$$

If $\lambda \neq 0, 1, 2$, the differential equation (12.3) has solutions behaving like $f(\theta) \sim e^{i\lambda\theta}$ or $e^{i(\lambda-2)\theta}$. Therefore, according to Jeffrey and Sherwood (1980), we may list the solutions

- For $\lambda \neq 0, 1, 2$,

$$\psi = r^{\lambda} \left(A e^{i\lambda\theta} + B e^{i(\lambda-2)\theta} \right) \,, \tag{12.4}$$

  or, as is explained below,

$$\psi = \left(A r^{\lambda} + B r^{\lambda+2} \right) e^{i\lambda\theta} \,. \tag{12.5}$$

- For $\lambda = 0$,

$$\psi = A e^{2i\theta} + C\theta + D \,. \tag{12.6}$$

- For $\lambda = 1$,

$$\psi = r \left(A e^{i\theta} + B\theta e^{i\theta} \right) \,, \tag{12.7}$$

  or

$$\psi = (Ar + Br \ln r) \, e^{i\theta} \,. \tag{12.8}$$

- For $\lambda = 2$,

$$\psi = r^2 \left(A e^{2i\theta} + C\theta + D \right) \,. \tag{12.9}$$

- Solution independent of θ

$$\psi = A r^2 \ln r + B \ln r + C r^2 + D \,. \tag{12.10}$$

Note that (12.4) and (12.5) are equivalent because $r^{\lambda} e^{i\lambda\theta}$ is a harmonic function, so that $r^{\lambda+2} e^{i\lambda\theta}$ is a solution of the biharmonic equation. The parameters λ, A, B, C, D are, in principle, complex numbers.

12.2 The Paint-Scraper Problem

This problem was first solved by Taylor (1962). Let us consider the flow generated in a corner as is displayed in Fig. 12.1. The lower wall is fixed while the wall inclined with angle θ_0 is in a uniform translation at constant speed U. This is an idealized situation of a plate scraping paint over a fixed horizontal wall. Near the origin, the velocity gradients are very high; however, it is expected that the viscous forces will be preponderant in the neighborhood of the origin. To perform a steady state formulation of the problem, the coordinate axes are chosen with the origin at the intersection of both walls and moving with the upper wall. In this case, the boundary conditions are

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U, \qquad \frac{\partial \psi}{\partial r} = 0 \quad \text{on} \quad \theta = 0\,, \tag{12.11}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0, \qquad \frac{\partial \psi}{\partial r} = 0 \quad \text{on} \quad \theta = \theta_0\,. \tag{12.12}$$

The expression of the boundary conditions suggests that ψ be given by

$$\psi = r\, f(\theta)\,. \tag{12.13}$$

This situation corresponds to the special 2D corner flow with $\lambda = 1$ and the solution given by (12.7) is such that

$$f(\theta) = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta\,. \tag{12.14}$$

The boundary conditions (12.11) and (12.12) impose $f(0) = 0, f'(0) = -U, f(\theta_0) = 0, f'(\theta_0) = 0$. One finds

$$A, B, C, D = \left(-\theta_0^2, 0, \theta_0 - \sin\theta_0\cos\theta_0, \sin^2\theta_0\right)\frac{U}{\theta_0^2 - \sin^2\theta_0}\,. \tag{12.15}$$

For the particular case of a rectangular corner

$$\psi = \frac{rU}{(\frac{\pi}{2})^2 - 1}\left(-(\frac{\pi}{2})^2\sin\theta + \frac{\pi}{2}\theta\sin\theta + \theta\cos\theta\right)\,, \tag{12.16}$$

which leads easily to the velocity components

$$v_r = \frac{U}{(\frac{\pi}{2})^2 - 1}\left((1 - \frac{\pi^2}{4})\cos\theta + \frac{\pi}{2}\sin\theta + \frac{\pi}{2}\theta\cos\theta - \theta\cos\theta\right), \tag{12.17}$$

$$v_\theta = -\frac{U}{(\frac{\pi}{2})^2 - 1}\left(-(\frac{\pi}{2})^2\sin\theta + \frac{\pi}{2}\theta\sin\theta + \theta\cos\theta\right). \tag{12.18}$$

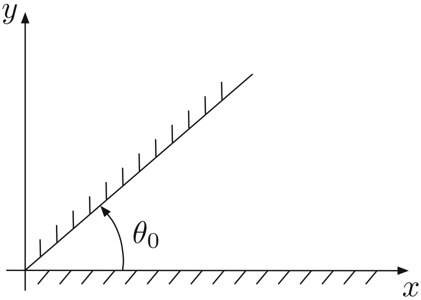


Fig. 12.1 Geometry of the paint-scraper

We can examine *a posteriori* the validity of the creeping flow hypothesis. Indeed the acceleration components given in Eqs. (3.78) and (3.79) are evaluated with (12.17) and (12.18) and are proportional to U^2/r with a factor depending on θ which is of order of unity. The viscous effects are of the order $\mu U/r^2$. Consequently the assumption of creeping flow is satisfied if $\rho r U/\mu \ll 1$ is enforced. This is certainly true in a region close enough to the origin such that $r \ll \nu U$. Farther away the solution is no longer correct as the inertia forces become quickly of the same order of magnitude as the viscous forces.

It is interesting to compute the pressure from Eq. (3.79):

$$\frac{\partial p}{\partial \theta} = r\mu \left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} \right) = r\mu \left(-\nabla^2 \frac{\partial \psi}{\partial r} + \frac{2}{r^3}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial \psi}{\partial r} \right) , \tag{12.19}$$

which yields

$$p = \frac{2\mu}{r(\theta_0^2 - \sin^2\theta_0)} \left((\frac{1}{2}\sin 2\theta_0 - \theta_0)\sin\theta - \sin^2\theta_0 \cos\theta \right) . \tag{12.20}$$

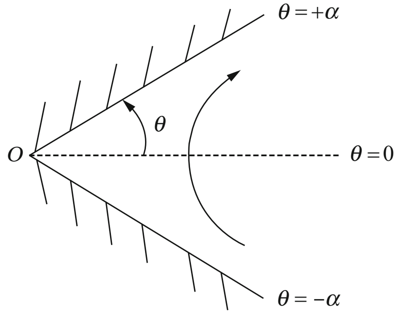
Observe that the pressure varying like r^{-1} becomes unbounded when we approach the corner. This dismal performance comes from the fact that at the corner the boundary conditions are not consistent with the real problem, which has always a tiny gap so that the forces remain finite.

12.3 Two-Dimensional Corner Eddies

Plane Stokes flows occur in engineering or physical problems in the neighborhood of slots or cracks in a wall. This situation is modeled by analyzing the creeping flow close to the vertex of a sharp wedge with an aperture angle 2α formed by the walls $\theta = \pm\alpha$ as shown in Fig. 12.2.

The forcing mechanism generating the corner flow is "far" from the wedge vertex. For example, in the lid-driven square cavity problem, corner vortices are generated by the influence of the main primary vortex.

Fig. 12.2 Geometry of the corner flow



For the flow in a wedge, two geometrical configurations are possible: the flow is asymmetric with respect to the symmetry axis $\theta = 0$; the flow is symmetric in θ and the symmetry axis plays the role of a mirror.

Here we will concentrate on the asymmetric case shown in Fig. 12.2 with the assumption of 2D creeping flow described by the stream function. Following Moffatt (1964), the 2D streamfunction is expanded in a series of basic solutions (12.2)

$$\psi = \Re \sum_{n=1}^{\infty} A_n r^{\lambda_n} f_{\lambda_n}(\theta) \;, \tag{12.21}$$

where the A_n are complex numbers and the λ_n satisfy the condition

$$1 < \Re\lambda_1 < \Re\lambda_2 < \ldots \tag{12.22}$$

The first inequality imposes that the flow vanish at the origin, which is located at the corner. The remaining inequalities indicate that the first term in (12.21) dominates over the others and then

$$\psi \approx A_1 r^{\lambda_1} f_{\lambda_1}(\theta) \equiv A r^{\lambda} f_{\lambda}(\theta) \;, \tag{12.23}$$

the last equivalence sign being an abuse of notation. As for the asymmetric solution $v_r(r, -\theta) = -v_r(r, \theta)$ and $v_\theta(r, -\theta) = v_\theta(r, \theta)$, the function f_λ has to be even in θ. The solution (12.4), rewritten as

$$A \sin \lambda\theta + B \cos \lambda\theta + C \cos(\lambda - 2)\theta + D \sin(\lambda - 2)\theta \;, \tag{12.24}$$

is such that the constants A and D vanish:

$$f_\lambda = B \cos \lambda\theta + C \cos(\lambda - 2)\theta \;. \tag{12.25}$$

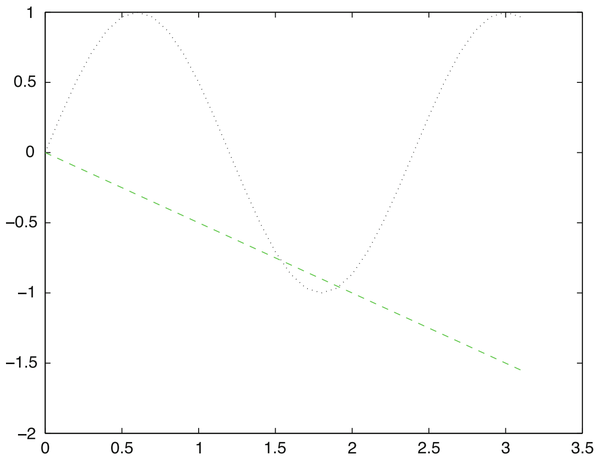


Fig. 12.3 Real solutions for $\alpha = 75°$

The no slip boundary conditions on the two walls $f_\lambda(\pm\alpha) = f'_\lambda(\pm\alpha) = 0$ yield the system

$$B\cos\lambda\alpha + C\cos(\lambda-2)\alpha = 0\,, \tag{12.26}$$

$$B\lambda\sin\lambda\alpha + C(\lambda-2)\sin(\lambda-2)\alpha = 0\,. \tag{12.27}$$

For a nonzero solution the determinant of this system must vanish, i.e.

$$\sin 2(\lambda-1)\alpha + (\lambda-1)\sin 2\alpha = 0\,. \tag{12.28}$$

12.3.1 Real Solutions for λ ($\alpha > 73.15°$)

The nonlinear equation (12.28) gives real solutions for an angle $\alpha > 73.15°$. Figure 12.3 shows the real solutions obtained as the intersections of the sine function $\sin 2(\lambda-1)\alpha$ in black and the straight line $-(\lambda-1)\sin 2\alpha$ in green with respect to the variable $(\lambda-1)\alpha$. The smallest value is the relevant one when we approach the corner $r \to 0$ as the solution goes like r^λ.

12.3.2 Complex Solutions for λ ($\alpha < 73.15°$)

Let us write $\lambda = p + 1 + iq$. The azimuthal velocity component is

$$v_\theta(r,\theta) = -\frac{\partial\psi}{\partial r} = \Re\left(-\lambda r^{\lambda-1} f(\theta)\right)\,. \tag{12.29}$$

Table 12.1 Main eigenvalue λ with respect to the corner half angle

α (°)	λ
2	$61.34043791 + i\,32.2266675$
10	$13.0794799 + i\,6.3843883$
20	$7.0578309 + i\,3.0953659$
30	$5.0593290 + i\,1.9520499$
40	$4.0674345 + i\,1.3395862$
50	$3.4792155 + i\,0.9303733$
60	$3.0941391 + i\,0.6045850$
70	$2.8268686 + i\,0.2616953$
73.155	$2.7634862 + i\,0$

On the symmetry axis of the corner, $\theta = 0$, and therefore

$$v_\theta(r,0) = \Re\left(-\lambda r^{\lambda-1} f(0)\right) = \Re\left(r^{\lambda-1} C\right) , \tag{12.30}$$

where $C = |C|e^{i\beta} \equiv -\lambda f(0)$. Equation (12.30) yields

$$v_\theta(r,0) = \Re\left(r^p |C| e^{iq\ln r} e^{i\beta}\right) = r^p |C| \cos(q \ln r + \beta) . \tag{12.31}$$

When $r \to 0$, $\ln r \to -\infty$ and the velocity $v_\theta(r,0)$ changes sign infinitely often. This behavior means that a string of counter-rotating vortices is present in the corner. The center of the nth corner eddy denoted by r_n is the distance of this center to the origin. It is given by the relation $v_\theta(r,0) = 0$ leading to

$$q \ln r_n + \beta = -(2n+1)\frac{\pi}{2},\ n = 0, 1, 2, \ldots, \quad \text{or} \quad r_n = e^{-(2n+1)\frac{\pi}{2q}} e^{-\frac{\beta}{q}} . \tag{12.32}$$

A simple calculation yields

$$\frac{r_n}{r_{n+1}} = \frac{r_n - r_{n+1}}{r_{n+1} - r_{n+2}} = e^{\frac{\pi}{q}} , \tag{12.33}$$

which shows that the sizes of the vortices fall off in geometrical progression with a common ratio $e^{\pi/q}$, depending on the aperture angle of the corner. If we now inspect the velocity maxima, we find $v_{\theta,max} = r^p |C|$ at points $r_{n+\frac{1}{2}} = e^{-n\frac{\pi}{2q}} e^{-\frac{\beta}{q}}$. The maximum velocity will be called the intensity of the eddy. The corner vortices have their intensities falling off in geometrical progression with the common ratio

$$\left|\frac{v_n}{v_{n+1}}\right| = e^{\pi p/q} , \tag{12.34}$$

which also depends on α.

The numerical solution of Eq. (12.28) for the angle α is given in Table 12.1. We observe that λ decreases when α increases. Furthermore, the imaginary part $\Im\lambda$ goes to 0 when α reaches the value 73.15°, meaning that λ then becomes real.

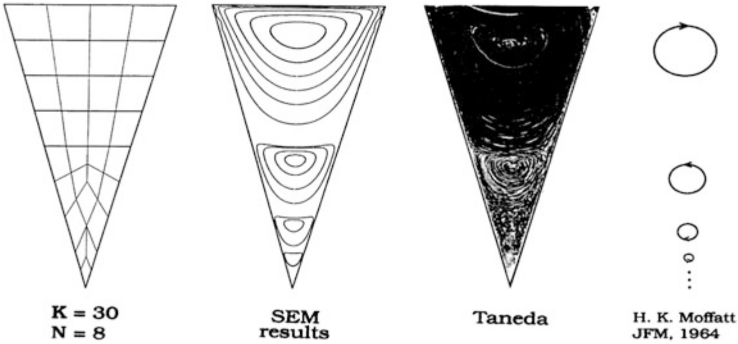


Fig. 12.4 Corner flow in a wedge of aperture $\alpha = 28.5°$ (Courtesy of E. Rønquist. The picture is reprinted with permission from Taneda (1979))

Figure 12.4 shows a spectral element solution computed by Rønquist (1991) for the Stokes flow in a wedge. As the top lid moves at unit velocity, a series of Moffat corner eddies is generated. These eddies are stacked from top to bottom in an infinite cascade to the tip. The wedge shown has an aperture angle of 28.5°. The asymptotic ratio of successive eddy intensities is 405. With the discretization shown in the figure ($K = E = 30, N = 8$), one obtains four eddies. The ratio of the strength of two successive eddies is from top to bottom 386, 406, and 411. We observe also that the computed results are very close to the experimental data provided by Taneda (1979) and Van Dyke (1983).

12.4 Stokes Eigenmodes and Corner Eddies

Most state-of-the-art numerical methods dealing with the Navier-Stokes equation rely on an implicit treatment of the Stokes operator and an explicit scheme for the nonlinearity (rejecting it as a source term). Hence it is essential to understand the structure of the Stokes operator. Furthermore if we can write the eigenmodes in closed form, or if we can compute them accurately, then we are able to use those modes as the basis for the approximation of the Stokes equation, see for example Batcho and Karniadakis (1994).

The Stokes eigenproblem is defined by setting $\frac{\partial \mathbf{v}}{\partial t} = \lambda \mathbf{v}$ in Eq. (11.35) and assuming that $\mathbf{f} = \mathbf{0}$. The eigenvalue λ provides the growth or decay rate of the velocity field. Now the eigensystem becomes

$$\lambda \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{0} \quad \text{in} \quad \Omega\,, \tag{12.35}$$

$$\text{div}\ \mathbf{v} = 0 \quad \text{in} \quad \Omega\,, \tag{12.36}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial\Omega\,. \tag{12.37}$$

12.4.1 Periodic Stokes Eigenmodes

Let us consider the fully periodic solutions of the transient Stokes problem Eqs. (11.35) in the open square domain $\Omega\ =]-1,+1[^2$ with the Fourier approximation

$$\mathbf{v}(\mathbf{x},t) = \sum_{\|\mathbf{k}\|<\infty} \hat{\mathbf{v}}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)}, \quad p = \sum_{\|\mathbf{k}\|<\infty} \hat{p}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)},$$
$$\mathbf{f}(\mathbf{x},t) = \sum_{\|\mathbf{k}\|<\infty} \hat{\mathbf{f}}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)}, \tag{12.38}$$

where $\hat{\mathbf{v}}$, $\hat{p}$, $\hat{\mathbf{f}}$ are the complex Fourier coefficients, $\mathbf{k}$ the wavevector and ω a complex frequency. The notation $\|\mathbf{k}\| < \infty$ is defined to mean $-\infty < k_i < +\infty$ for $i = 1, 2$.

Let us denote by $\mathbf{e_k}$ the unit vector in the direction of $\mathbf{k}$. Then $\mathbf{k} = k\,\mathbf{e_k}$. The solution is easily obtained

$$\hat{p} = -i\frac{\mathbf{e_k}\cdot\hat{\mathbf{f}}}{k}, \quad (i\omega + k^2)\hat{\mathbf{v}} = \left(\hat{\mathbf{f}} - \mathbf{e_k}(\mathbf{e_k}\cdot\hat{\mathbf{f}}\right). \tag{12.39}$$

The resulting periodic eigenmode corresponding to $\lambda = i\,\omega$ is

$$\hat{p} = 0, \quad (\lambda + k^2)\hat{\mathbf{v}} = \mathbf{0}, \quad \mathbf{k}\cdot\hat{\mathbf{v}} = 0\,. \tag{12.40}$$

The periodic Stokes modes are constant pressure modes driven only by diffusion as $\lambda = -k^2$. The incompressibility constraint div $\mathbf{v} = 0$ does not influence the space configuration, except that the wavevector $\mathbf{k}$ must be orthogonal to the velocity. Geometrically speaking the velocity is contained in a plane perpendicular to $\mathbf{k}$, while the pressure is aligned with the wavevector.

12.4.2 Channel Flow Stokes Eigenmodes

The problem is based on the plane channel flow between horizontal plates as treated in Orszag et al. (1986). The flow is assumed periodic in the x_2 direction while it is confined by rigid walls in $x_1 = \pm 1$. We seek a solution of the 2D Stokes equation in the form

$$\mathbf{v}(\mathbf{x},t) = \left(u(x_1)e^{ikx_2+\lambda t}, v(x_1)e^{ikx_2+\lambda t}\right), \quad p = p(x_1)e^{ikx_2+\lambda t}, \tag{12.41}$$

where k is a chosen wavenumber and u, v are complex functions. The Stokes equations satisfied by u, v, p are

$$\lambda u = -\frac{dp}{dx_1} + \nu(\frac{d^2u}{dx_1^2} - k^2 u) ,$$

$$\lambda v = -ikp + \nu(\frac{d^2v}{dx_1^2} - k^2 v) , \tag{12.42}$$

$$\frac{du}{dx_1} + ikv = 0 ,$$

for $-1 \leq x_1 \leq +1$. The boundary conditions are

$$\mathbf{v}(\pm 1, x_2, t) = \mathbf{0} , \tag{12.43}$$

for the no-slip walls. The elimination of v and p in Eqs. (12.42) yields

$$\lambda \left(D^2 - k^2\right) u = \nu \left(D^2 - k^2\right)^2 u , \tag{12.44}$$

with $u(\pm 1) = Du(\pm 1) = 0$, where $D = d/dx_1$. The solutions of Eq. (12.44) are either symmetric in x_1

$$u(x_1) = \cos\mu \cosh kx_1 - \cosh k \cos\mu x_1 , \tag{12.45}$$

or antisymmetric

$$u(x_1) = \sin\mu \sinh kx_1 - \sinh k \sin\mu x_1 . \tag{12.46}$$

The eigenvalues are

$$\lambda = -\nu(\mu^2 + k^2) , \tag{12.47}$$

satisfying the relations

$$k \tanh k = -\mu \tan\mu , \tag{12.48}$$

for (12.45) and

$$k \coth k = \mu \cot\mu , \tag{12.49}$$

for (12.46). For $k = 1$ and $k = 10$, the first symmetric eigenmode decays with $\lambda/\nu = -9.3137$ and $\lambda/\nu = -103.0394$, respectively, while, for $k = 1$ and $k = 10$, the antisymmetric eigenmode decays with $\lambda/\nu = -20.5706$ and $\lambda/\nu = -112.0836$. All the eigenvalues λ are real and negative, indicating strong damping by the viscous forces.

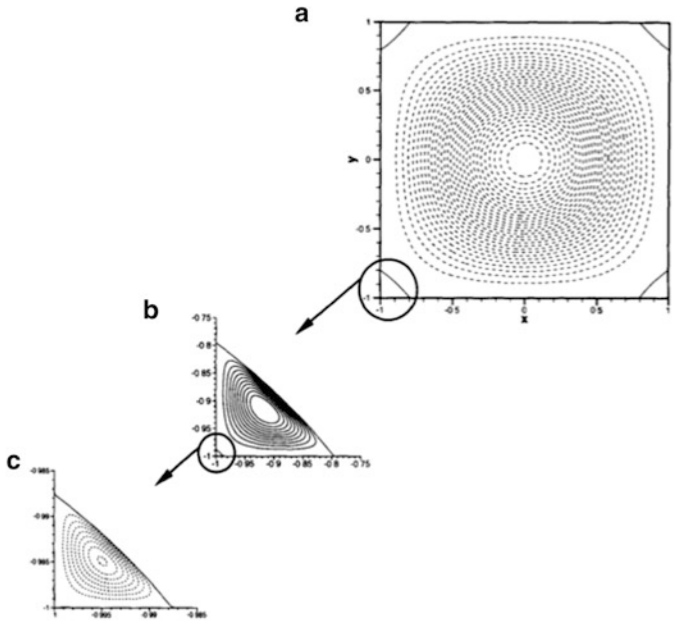


Fig. 12.5 Streamlines of the fundamental eigenmode. (**a**) Core vortex; (**b**) zoom of the primary corner eddy; (**c**) zoom of the secondary corner eddy (Reprinted with kind permission from Springer Science: Leriche and Labrosse (2005))

12.4.3 Stokes Eigenmodes in the Square Domain

Here, the Stokes problem is solved using a Chebyshev pseudo-spectral method. A split scheme of projection-diffusion type is set up, the details of which are found inLeriche and Labrosse (2004). The eigenmodes are investigated taking the symmetry properties into account, mainly rotations and reflections. As the projection method is only asymptotically divergence free when the polynomial degree of the Chebyshev approximation goes to infinity, corner eddies show up in the Stokes eigenspace computation. The presence of these eddies ensures that discrete incompressibility will be numerically achieved.

Among the various geometrical isometries, three of them are independent, namely rotation by a multiple of $\pi/2$ and reflections about the coordinate axes. This leads to accurate computation of the Stokes eigenmodes in the reference square for each of the six corresponding symmetry families. Figure 12.5 displays the streamline contours of the first eigenmode computed with $N = 96$ Chebyshev polynomials. The main core vortex has values in the range $[0, -1]$. The first corner eddy is in between $[0, 10^{-4}]$ and the second corner eddy is in $[0, -4.\,10^{-10}]$

The numerical results for the eigenvalues confirm the theoretical predictions by Constantin and Foias (1988), given by the next theorem:

Theorem 12.1. *Let $\Omega \subset \mathbb{R}^n, n = 2$ or 3. Then there exists a scale invariant constant C_0 such that $\forall\, k = 1, 2, \ldots$ the eigenvalues λ_k of the Stokes operator satisfies*

$$\lambda_k \geqslant C_0 k^{2/n} \lambda_1 \,. \tag{12.50}$$

In the periodic case, $\lambda_1 = \frac{4\pi^2}{L^2}$ where L is the length of the domain.

The computed spectra λ_k are indeed linear in k with a second term proportional to $\sqrt{k}$, Leriche and Labrosse (2004, 2005).

12.4.4 Corner Modes in the Cubic Domain

The extension of the Moffatt corner eddies to the three-dimensional case made of orthogonal planes has resisted analytical treatment till now. As Shankar (2007) writes in Sect. 11.3: "Nothing is known about such flows… There is still much to be done about three-dimensional corner flows". However recent studies based on numerical methods bring new insights into this challenging problem, e.g. Leriche and Labrosse (2011), Labrosse et al. (2014), and Scott (2013).

12.5 Three-Dimensional Stokes Solution

In this section we present a three-dimensional solution for the steady Stokes equations (6.1) and (6.2). This procedure is due to Tran-Cong and Blake (1982) and is based on harmonic solutions of the Laplace equation, assuming that separable solutions are relevant.

Suppose that **A** and B are vector and scalar fields satisfying Laplace's equations

$$A_{i,jj} = 0, \quad B_{,jj} = 0 \,. \tag{12.51}$$

Then the velocity v_i and pressure p are given by the relationships

$$v_i = \frac{\partial}{\partial x_i}\big(r_j A_j + B\big) - 2A_i \,, \tag{12.52}$$

$$\frac{p}{\mu} = 2\, A_{j,j} \,, \tag{12.53}$$

where r_j are the components of the position vector. The proofs given in Tran-Cong and Blake (1982) are based on theoretical developments coming from elasticity theory. For the sake of simplicity we will skip them. However let us examine how the methodology of solving both Laplace equations (12.51) and then combining the two harmonic solutions through (12.52)–(12.53) yields the Stokes solution. In Cartesian coordinates with $r_j = x_j$, Eqs. (12.52) and (12.53) give

$$v_i = \frac{\partial}{\partial x_i}(x_j A_j + B) - 2A_i = x_j A_{j,i} - A_i + B_{,i} \ , \tag{12.54}$$

$$\frac{p}{\mu} = 2A_{j,j} \ . \tag{12.55}$$

The incompressibility constraint is ensured:

$$v_{i,i} = x_j A_{j,ii} + A_{i,i} - A_{i,i} + B_{,ii} = 0 \ . \tag{12.56}$$

We next employ the equilibrium equation (6.2) and use (12.51) to eliminate some terms

$$\begin{aligned} v_{i,kk} &= \frac{\partial}{\partial x_k}\left(A_{k,i} + x_j A_{j,ik} - A_{i,k} + B_{,ik}\right) \\ &= A_{k,ik} + A_{k,ik} + x_j A_{j,ikk} - A_{i,kk} + B_{,ikk} = 2A_{k,ik} \\ &= \frac{1}{\mu}\frac{\partial p}{\partial x_i} \ . \end{aligned} \tag{12.57}$$

Let us apply the previous solution technique to the Stokes flow of a sphere of radius a moving at constant speed U in an infinite fluid, as reported in Sect. 6.2. We refer the problem to a Cartesian coordinate system with origin at the center of the sphere and with positive x_3-axis in the flow direction. The harmonic solutions are

$$A_1 = A_2 = 0, \quad A_3 = -U + \frac{3Ua}{4r}, \qquad B = -\frac{Ux_3a^2}{4r^3} \ . \tag{12.58}$$

Using (12.58) in (12.54) we easily obtain the Eqs. (6.25).

Appendix
Comments on Some Bibliographical Entries

The present biographical scheme is quite different from that of the first edition. There, each chapter ended with a short bibliography suggesting further reading. Each entry was accompanied by a short review.

Now, in keeping with the publishers current format, the entire bibliography is placed at the end, without reviews.

In the belief that some of the reviews retain value, we offer this appendix.

It has been wisely said that a trick used twice becomes a method. Some of the reviews in the first edition reported techniques which were novel at the time, but which now are widespread in the literature. Such reviews are omitted here.

Aris R This monograph goes quite deeply into the use of tensor methods in hydrodynamics.

Berker R An extensive collection of exact and approximate solutions to the Navier-Stokes equations.

Brenner H Presents a general theory for the motion of a particle when the fluid is not unbounded. Creeping flow is assumed and the particle is assumed to be small in comparison with its distance from the boundary. If the force on the particle in an unbounded stream is a pure drag—no side thrust—the wall-correction can be calculated from a knowledge of the drag on the particle in an unbounded stream and the wall-correction for a spherical particle.

Carrier GF Offers an alternate approach to the problem of flow past obstacles. The suggestion is made that the inertia term in the Navier Stokes equation be replaced by $cU^j v_{i,j}$ where c is a constant whose value lies between zero and unity. In the Stokes approximation $c = 0$; in the Oseen approximation, $c = 1$. Thus either of the classical approaches might be interpreted as replacing the factor v^j in $v^j v_{i,j}$ by a weighted average, and this report suggests that a better weighting might be found. From a study of sharp-edged obstacles, $c = 0.43$ appears to be the most appropriate choice; moreover, the corresponding formulas for drag fit the experimental data for spheres, cylinders, and flat plates for Reynolds numbers up to about 20. The approach might be termed "semi-empirical", but in view of the

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3,

physical insight this particular author showed on so many occasions, it would be unwise to dismiss the matter that briefly.

Ericksen JL Nominally an appendix to Truesdell and Toupin (1960), this contribution is of significant interest in its own right. Many advanced results in general tensor theory are derived and the main emphasis is on those aspects of the subject which are of direct importance in continuum physics.

Goldstein S. 1938 This well-known work covers many matters in the general area of viscous hydrodynamics. In particular the Hamel and von Kármán exact solutions are discussed in detail with flow curves provided.

Gross WA Although no explicit division of the book into two parts is indicated, that is the impression conveyed to the reader. Roughly speaking the first half of the book introduces the subject to a reader presumed to have no prior knowledge of lubrication theory. The remainder of the volume is, in essence, a handbook, cataloguing many solutions to lubrication problems and giving references rather than details. The bibliography, which extends over fifteen pages, is quite impressive.

Jeffreys H The standard reference on Cartesian tensor analysis is now somewhat dated, but still quite useful. Much of its material is also included in Chap. 3 of *Methods of Mathematical Physics* by H. and B. S. Jeffreys (Cambridge, 1951).

Kantorovich LV, Krylov VI This 700-page volume surveys many topics in higher analysis. Included is a lucid discussion of the fundamental biharmonic problems and of the conformal mapping methods for solving them. Although the connexion with plane elasticity is developed, it is not assumed that the reader is already familiar with elementary elasticity theory.

Lamb SH For a classical look at classical problems, one can hardly do better. The treatment of the Stokes problem is particularly instructive.

Lighthill J, 1976 The reader with little or no background in microbiology will appreciate that the author provides it, lucidly and in sufficient depth to convey what biological problems await treatment. After reviewing the successes and shortcomings of prior work, i.e., the use of local resistance coefficients, he sets the stage for flagellar hydrodynamics. This he then pursues in depth, covering several important cases. One ironic result: despite his obvious distaste for local resistance coefficients, he uses the results of his more sophisticated analysis to offer ways to improve them.

McConnell AJ A republication under a new title of *Applications of the Absolute Differential Calculus*. For many years the standard reference on the subject, this book offers a detailed and quite lucid treatment of the geometrical aspects of tensor analysis. Several applications to mathematical physics are also extensively treated, but this part of the book is now somewhat dated.

Muskhelishvili NI Treats extensively the use of complex variable techniques in elasticity theory.

Proudman I, Pearson JRA The matching procedure set out in Sect. 6.4 was taken from this paper, which also treats the corresponding, but more difficult, problem of flow past a circular cylinder.

Rivlin RS In preparing Chap. 1, we drew heavily upon Chap. 1 of Rivlin's notes. The treatment is not completely parallel, for Rivlin's treatment is intended to provide a background appropriate for elasticity theory, rather than hydrodynamics. Nevertheless we are grateful to Professor Rivlin for demonstrating that the use of Cartesian tensors in mechanics can be simply and succinctly presented.

Serrin J This article presents an accurate and thorough development of the basic principles underlying several aspects of hydrodynamics.

Sokolnikoff IS, 1951 Written by one of the most prominent workers in elasticity theory, this reference treats extensively the application of tensor methods to classical continuum mechanics.

Sokolnikoff IS, 1983 Chap. 4 includes the solution of a variety of torsion problems. By the analogy noted at the beginning of Chap. 5, these can be converted to pipe-flow problems.

Tipei N This is the first English edition of a book previously available only in Rumanian. The viewpoint taken appears somewhat strange to American and British workers in lubrication theory, but unorthodox viewpoints are often enlightening. Both liquid films and gas films are discussed.

Truesdell C, 1960 Not intended for beginners, this article presents a wealth of information on the basic principles underlying continuum physics. Much of the material is directly related to the subject matter of Chap. 2.

References

Abramowitz M, Stegun IA (1972) Handbook of mathematical functions with formulas, graphs, and mathematical tables, 10th edn. Wiley, New York

Adam NK (1968) The physics and chemistry of surfaces. Dover, New York

Anagnostou G, Maday Y, Mavriplis C, Patera A (1990) On the mortar element method: generalizations and implementation. In: Chan T, Glowinski R, Périaux J, Widlund O (eds) Third international symposium on domain decomposition methods for partial differential equations, Philadelphia. SIAM, pp 157–173

Aris R (1962) Vectors, tensors, and the basic equations of fluid mechanics. Dover, New York

Azaïez M, Deville M, Mund E (2011) Éléments finis pour les fluides incompressibles. Presses Polytechniques et Universitaires Romandes, Lausanne

Babuška I (1973) The finite element method with Lagrangian multipliers. Numer Math 20:179–192

Bairstow L, Cave BM, Lang ED (1922) The two-dimensional slow motion of viscous fluids. Proc R Soc Lond A 100:394–413

Barenblatt GI (2003) Scaling. Cambridge University Press, Cambridge

Barlow EJ, Langlois WE (1962) Diffusion of gas from a liquid into an expanding bubble. IBM J Res Dev 6:329–337

Barthès-Biesel D (2012) Microhydrodynamics and complex fluids. CRC/Taylor and Francis, Boca Raton

Basdevant C, Deville M, Haldenwang P, Lacroix JM, Ouazzani J, Peyret R, Orlandi P, Patera AT (1986) Spectral and finite difference solutions of the Burgers equation. Comput Fluids 14:23–41

Batchelor GK (1953) The theory of homogeneous turbulence. Cambridge University Press, Cambridge

Batchelor GK (1992) An introduction to fluid dynamics. Cambridge University Press, Cambridge

Batcho P, Karniadakis G (1994) Generalized Stokes eigenfunctions: a new trial basis for the solution of incompressible Navier-Stokes equations. J Comput Phys 115:121–146

Benton ER, Platzmann GN (1972) A table of solutions of one-dimensional Burgers equation. Q Appl Math 29:195–212

Berker R (1963) Intégration des équations du mouvement d'un fluide visqueux incompressible. In: Flügge S (ed) Encyclopedia of physics, vol VIII/2. Springer, Berlin, pp 1–384

Bernardi C, Maday Y, Patera A (1994) A new nonconforming approach to domain decomposition: the mortar element method. In: Brezis H, Lions JL (eds) Collège de France Seminar, vol XI. Pitman, Boston, pp 13–51

Bikerman JJ (1958) Surface chemistry. Academic, New York

Boger DV, Walters K (1993) Rheological phenomena in focus. Elsevier, Amsterdam

Bosshard C, Bouffanais R, Deville M, Gruber R, Latt J (2011) Computational performance of a parallelized three-dimensional high-order spectral element toolbox. Comput Fluids 44:1–8

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3,

Botsis J, Deville M (2006) Mécanique des milieux continus: une Introduction. Presses Polytechniques et Universitaires Romandes, Lausanne

Brenner H (1962) Effect of finite boundaries on the Stokes resistance of an arbitrary particle. J Fluid Mech 12:35–48

Brenner H, Graham AL, Abbott JR, Mondy LA (1990) Theoretical basis for falling-body rheometry in suspensions of neutrally buoyant spheres. Int J Multiph Flow 16:579–596

Brezzi F (1974) On the existence, uniqueness and approximation of saddle point problems arising from Lagrangian multipliers. RAIRO Anal Numer 8:129–151

Brezzi F, Fortin M (1991) Mixed and hybrid finite element methods. Springer, Berlin

Carey FG, Oden JT (1983) Finite elements: a second course. Prentice Hall, Englewood Cliffs

Carrier GF (1953) On slow viscous flow. Final report, Office of Naval Research Contract nonr-653(00), Brown University

Chang ID (1961) Navier-Stokes solutions at large distances from a finite body. Indiana Univ Math J 10:811–876

Chester W (1962) On Oseen's approximation. J Fluid Mech 13:557–569

Chester W, Breach DR, Proudman I (1969) On the flow past a sphere at low Reynolds number. J Fluid Mech 37:751–760

Chorin AJ (1968) Numerical solution of the Navier-Stokes equations. Math Comput 22:745–762

Ciarlet PG (1978) The finite element method for elliptic problems. North-Holland, Amsterdam

Cochran WG (1934) The flow due to a rotating disc. Math Proc Camb Philos Soc 30:365–375

Constantin P, Foias C (1988) Navier-Stokes equations. Chicago lectures in mathematics. University of Chicago Press, Chicago

Dean WR (1958) An application in hydrodynamics of the Green's function of an elastic plate. Mathematika 5:85–92

Deville MO, Gatski TB (2012) Mathematical modeling for complex fluids and flows. Springer, Berlin

Deville MO, Fischer PF, Mund EH (2002) High-order methods for incompressible fluid flow. Cambridge monographs on applied and computational mathematics. Cambridge University Press, Cambridge

Dowty EL (1963) A solution to the oscillating plate problem with discontinuous initial conditions. J Basic Eng 85D:477–478

Drazin PG, Reid WH (2004) Hydrodynamic stability, 2nd edn. Cambridge University Press, Cambridge

Dussan V EB (1976) On the difference between a bounding surface and a material surface. J Fluid Mech 75:609–623

Elrod HG (1960) A derivation of the basic equations for hydrodynamic lubrication with a fluid having constant properties. Q Appl Math 27:349–385

Ericksen JL (1960) Tensor fields. In: Flügge S (ed) Encyclopedia of physics, vol lll/l. Springer, Berlin, pp 794–858

Eringen AC (1962) Nonlinear theory of continuous media. McGraw-Hill, New York

Eymard R, Herbin R, Gallouet T (2000) Finite volume methods. In: Ciarlet P, Lions J (eds) Handbook of numerical analysis, vol VII. North Holland, Amsterdam, pp 713–1020

Finn R, Noll W (1957) On the uniqueness and nonexistence of Stokes flows. Arch Ration Mech Anal 1:97–106

Frey PJ, George PL (2000) Mesh generation: application to finite elements. Hermes Science, Paris

Fung YC (1984) Biodynamics: circulation. Springer, New York

Girault V, Raviart PA (1986) Finite element methods for Navier-Stokes equations. Springer, Berlin

Goldstein S (1929) The steady flow of a viscous fluid past a fixed spherical obstacle at small Reynolds numbers. Proc R Soc Lond A 123:225–235

Goldstein S (1938) Modern development in fluid dynamics. Clarendon Press, Philadelphia

Goldstein S (1960) Lectures on fluid mechanics. Lectures in applied mathematics, vol 2. Interscience, London

Goodier JN (1934) An analogy between the slow motions of a viscous fluid in two dimensions, and systems of plane stress. Philos Mag 17:554–576

Gray J, Hancock GJ (1955) The propagation of sea-urchin spermatozoa. J Exp Biol 32:802–804
Gresho PM, Sani RL (2000) Incompressible flow and the finite element method: Part I and II. Wiley, Chichester
Gresho PM, Lee RL, Sani RL (1980) On the time-dependent solution of the incompressible Navier-Stokes equations in two and three dimensions. In: Taylor C, Morgan K (eds) Recent advances in numerical methods in fluids, vol 1. Pineridge Press, Swansea, pp 27–79
Gross WA (1962) Gas film lubrication. Wiley, Chichester
Guazzelli E, Morris JF (2011) A physical introduction to suspension dynamics. Cambridge University Press, Cambridge
Guyon E, Hulin JP, Petit L (2012) Hydrodynamique physique, 3rd edn. CNRS Éditions & EDP Sciences, Paris
Haberman WL (1962) Secondary flow about a sphere rotating in a viscous liquid inside a coaxially rotating spherical container. Phys Fluids 5:625–626
Hamel G (1916) Spiralförmige Bewegungen zäher Flussigkeiten. Jahresber d Deutschen Mathematiker-Vereinigung 25:34–60
Hancock GJ (1953) The self-propulsion of microscopic organisms through liquids. Proc R Soc Lond A 217:96–121
Happel JR, Brenner H (2013) Low Reynolds number hydrodynamics: with special applications to particulate media. Springer, Berlin
Harlow FH, Welch JE (1965) Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. Phys Fluids 8:2181–2189
Hasimoto H (1956) A sphere theorem on the Stokes equation for axisymmetric viscous flow. J Phys Soc Jpn 11:793–797
Hayes WD, Probstein RF (2004) Hypersonic inviscid flow. Dover, Mineola
Hill R, Power G (1956) Extremum principles for slow viscous flow and the approximate calculation of drag. Q J Mech Appl Math 9:313–319
Hirsch C (1991) Numerical computation of internal and external flows: fundamentals of numerical discretization. Wiley, Chichester
Hori Y (2006) Hydrodynamic lubrication. Springer, Berlin
Jeffrey DJ, Sherwood JD (1980) Streamline patterns and eddies in low-Reynolds-number flow. J Fluid Mech 96:315–334
Jeffreys H (1931) Cartesian tensors. Cambridge University Press, Cambridge
Johnson C (1990) Numerical solution of partial differential equations by the finite element method. Cambridge University Press, Cambridge
Kantorovich LV, Krylov VI (1964) Approximate methods of higher analysis. Interscience, New York
Kanwal RP (1961) Slow steady rotation of axially symmetric bodies in a viscous fluid. J Fluid Mech 10:17–24
Kaplun S (1957) Low Reynolds number flow past a circular cylinder. J Math Mech 6:595–603
Kestin J, Persen LN (1955) The lnfluence of edges on shearing stresses in viscous laminar flow. Technical report no. 3, Brown University
Knopp K (1945) Theory of functions. Dover, New York
Koschmieder EL (1993) Bénard cells and Taylor vortices. Cambridge University Press, Cambridge
Kovasznay LIG (1948) Laminar flow behind a two-dimensional grid. Math Proc Camb Philos Soc 44:58–62
Krakowski M, Charnes A (1951) Stokes' paradox and biharmonic flows. Technical report no. 37, Carnegie Institute of Technology
Labrosse G, Leriche E, Lallemand P (2014) Stokes eigenmodes in cubic domain: their symmetry properties. Theor Comput Fluid Dyn. DOI: 10.1007/s00162-014-0318-5
Ladyzhenskaya OA (1969) The mathematical theory of viscous incompressible flow. Mathematics and its applications, vol 2, 2nd edn. Gordon and Breach, New York
Lamb SH (1995) Hydrodynamics. Cambridge University Press, Cambridge
Landau L, Lifshitz EM (1997) Fluid mechanics. Butterworth Heinemann, Oxford

Langlois WE (1958) Creeping viscous flow through a two-dimensional channel of varying gap. In: Proceedings of the third U.S. National Congress of applied mechanics, Rhode Island, p 777

Langlois WE (1962) Isothermal squeeze films. Q Appl Math 20:131–150

Langlois WE (1963) Similarity rules for isothermal bubble growth. J Fluid Mech 15:111–118

Langlois WE (1971) An elementary proof that the undetermined stress in an incompressible fluid is of the form $-p\mathbf{I}$. Am J Phys 39:641–642

Langlois WE (1981) Conservative differencing procedures for rotationally symmetric flows with swirl. Comput Methods Appl Mech Eng 25:315–333

Langlois WE (1985) Buoyancy-driven flows in crystal growth melts. Annu Rev Fluid Mech 17:191–215

Leal LG (2007) Advanced transport phenomena, fluid mechanics and convective transport processes. Cambridge University Press, Cambridge

Leriche E, Labrosse G (2004) Stokes eigenmodes in square domain and the stream function–vorticity correlation. J Comput Phys 200:489–511

Leriche E, Labrosse G (2005) Fundamental Stokes eigenmodes in the square: which expansion is more accurate, Chebyshev or Reid-Harris? Numer Algorithms 38:111–131

Leriche E, Labrosse G (2011) Are there localized eddies in the trihedral corners of the Stokes eigenmodes in cubical cavity? Comput Fluids 43:98–101

Lighthill J (1976) Flagellar hydrodynamics. SIAM Rev 18:161–230

Lighthill J (1996) Helical distributions of stokeslets. J Eng Math 30:35–78

McConnell AJ (1957) Applications of tensor analysis. Dover, New York

Milne-Thomson LM (1996) Theoretical hydrodynamics. Dover, New York

Moffatt HK (1964) Viscous and resistive eddies near a sharp corner. J Fluid Mech 18:1–18

Moisil GC (1955) Metoda funcţiilor analitice in hidrodinamica lichidelor viscoase. Comunicările Academiei R P Romîne 5:1411

Muskhelishvili NI (1963) Some basic problems of the mathematical theory of elasticity. Noordhoff, Groningen

Ockendon H, Ockendon JR (1995) Viscous flow. Cambridge University Press, Cambridge

Orszag S, Israeli M, Deville M (1986) Boundary conditions for incompressible flows. J Sci Comput 1:75–111

Panton RL (1984) Incompressible flow. Wiley, New York

Payne LE, Pell WH (1960) The Stokes problem for a class of axially symmetric bodies. J Fluid Mech 7:529–549

Pell WH, Payne LE (1960) On Stokes flow about a torus. Mathematika 7:78–92

Peyret R, Taylor TD (1983) Computational methods for fluid flow. Springer, Berlin

Pinkus O, Sternlicht B (1961) Theory of hydrodynamic lubrication. McGraw-Hill, New York

Plesset MS, Zwick SA (1954) The growth of vapor bubbles in superheated liquids. J Appl Phys 25:493–500

Pólya G (1948) Torsional rigidity, electrostatic capacity, and symmetrization. Q Appl Math 6:267–277

Pozrikidis C (2009) Fluid dynamics: theory, computation, and numerical simulation. Springer, New York

Proudman I, Pearson JRA (1957) Expansions at small Reynolds number for the flow past a sphere and a circular cylinder. J Fluid Mech 2:237–262

Purcell EM (1997) The efficiency of propulsion by a rotating flagellum. Proc Natl Acad Sci USA 94:11307–11311

Quarteroni A, Valli A (1997) Numerical approximation of partial differential equations. Springer, Berlin

Rappaz M, Bellet M, Deville M (2003) Numerical modeling in materials science and engineering. Springer, Berlin

Rieutord M (1997) Une introduction à la dynamique des fluides. Masson, Paris

Rivlin RS (1957) Preliminary notes to a course of lectures on solid mechanics. Technical report, summer seminar in applied mathematics, Boulder

Rivlin RS, Ericksen JL (1955) Stress-deformation relations for isotropic materials. Arch Ration Mech Anal 4:323–425

Rivlin RS, Smith GF (1957) The anisotropic tensors. Q Appl Math 15:308–314

Rønquist E (1991) Spectral element methods for the unsteady Navier-Stokes equations. In: Von Karman Institute for Fluid Dynamics (eds) Computational fluid dynamics. Lecture series 1991-01. von Karman Institute for Fluid Dynamics, Rhode St-Genèse, Belgium

Rubinow SI, Keller JB (1961) The transverse force on a spinning sphere moving in a viscous fluid. J Fluid Mech 11:447–459

Saad Y (2003) Iterative methods for sparse linear systems. SIAM, Philadelphia

Salbu EOJ (1964) Compressible squeeze films and squeeze bearings. J Basic Eng 86:355–366

Sani R, Gresho P, Lee R, Griffiths D, Engleman M (1981) The cause and cure (?) of the spurious pressures generated by certain FEM solutions of the incompressible Navier-Stokes equations. Int J Numer Meth Fluids 1:17–43; 171–204

Schlichting H (1960) Boundary layer theory. McGraw Hill, New York

Schmid PJ, Henningson DS (2001) Stability and transition in shear flows. Springer, New York

Scott JF (2013) Moffatt-type flows in a trihedral cone. J Fluid Mech 725:446–461

Sengupta TK (2013) High accuracy computing methods: fluid flows and wave phenomena. Cambridge University Press, Cambridge

Serrin J (1959) Mathematical principles of classical fluid mechanics. In: Flügge S (ed) Encyclopedia of physics, vol Vlll/l. Springer, Berlin, pp 125–263

Shankar PN (2007) Slow viscous flows: qualitative features and quantitative analysis using complex eigenfunction expansions. Imperial College Press, London

Snoeijer JH, Andreotti B (2013) Moving contact lines: scales, regimes, and dynamical transitions. Annu Rev Fluid Mech 45:269–292

Sokolnikoff IS (1951) Tensor analysis. Wiley, New York

Sokolnikoff IS (1983) Mathematical theory of elasticity. Krieger, Malabar

Spencer AJM (2004) Continuum mechanics. Dover, Mineola

Strang G (1986) Introduction to applied mathematics. Wellesley-Cambridge Press, Wellesley

Strang G, Fix G (1973) An analysis of the finite element method. Prentice Hall, Englewood Cliffs

Taneda S (1979) Visualization of separating Stokes flows. J Phys Soc Jpn 46:1935–1942

Taylor GI (1922) Stability of a viscous liquid contained between two rotating cylinders. Philos Trans R Soc A 73:289–343

Taylor GI (1962) On scraping viscous fluid from a plane surface. In: Batchelor GK (ed) The scientific papers of Sir Geoffrey Ingram Taylor, vol IV. Cambridge University Press, Cambridge, pp 410–413

Taylor C, Hood P (1973) A numerical solution of the Navier-Stokes equations using the finite element technique. Comput Fluids 1:73–100

Témam R (1969) Sur l'approximation des équations de Navier-Stokes par la méthode des pas fractionnaires ii. Arch Ration Mech Anal 33:377–385

Tipei N (1962) Theory of lubrication. Stanford University Press, Stanford

Tran-Cong T, Blake JR (1982) General solutions of the Stokes' flow equations. J Math Anal Appl 90:72–84

Tritton DJ (1980) Physical fluid dynamics. Van Nostrand Reinhold, New York

Truesdell C (1952) The mechanical foundations of elasticity and fluid dynamics. J Ration Mech Anal 1:125–300

Truesdell C (1966) The elements of continuum mechanics. Springer, New York

Truesdell C, Toupin RA (1960) The classical field theories. In: Flügge S (ed) Encyclopedia of physics, vol lll/l. Springer, Berlin, pp 226–793

Tu C, Deville M, Dheur L, Vanderschuren L (1992) Finite element simulation of pulsatile flow through arterial stenosis. J Biomech 25:1141–1152

Van Dyke M (1983) An album of fluid motion. Parabolic Press, Stanford

Versteeg HK, Malalasekera W (2007) An introduction to computational fluid dynamics: the finite volume method. Pearson, Harlow

von Kármán T (1921) Über laminare und turbulente Reibung. Zeit Ang Math Mech 1:233–252

von Mises R (2004) Mathematical theory of compressible flow. Dover, Mineola
Walsh O (1992) Eddy solutions of the Navier-Stokes equations. In: Heywood J, Masuda K, Rautmann R, Solonnikov S (eds) The Navier-Stokes equations II – theory and numerical methods. Lecture notes in mathematics, vol 1530. Springer, Berlin, pp 306–309
Wannier GH (1950) A contribution to the hydrodynamics of lubrication. Q Appl Math 8:1–32
Weinstein SJ, Ruschak KJ (2004) Coating flows. Annu Rev Fluid Mech 36:29–53
Widder DV (1989) Advanced calculus, 2nd edn. Dover, New York
Williamson CHK (1996) Vortex dynamics in cylinder wake. Annu Rev Fluid Mech 28:477–539
Zamir M (2000) The physics of pulsatile flow. Springer, New York

Index

W.E. Langlois and M.O. Deville, *Slow Viscous Flow*,
DOI 10.1007/978-3-319-03835-3,