INSTRUCTOR'S RESOURCE GUIDE AND SOLUTIONS MANUAL

to accompany

CALCULUS WITH APPLICATIONS

EIGHTH EDITION

AND

CALCULUS WITH APPLICATIONS
Brief Version

EIGHTH EDITION

Lial • Greenwell • Ritchey

PEARSON
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PREFACE

This book provides several resources for instructors using *Calculus with Applications,* Eighth Edition, by Margaret L. Lial, Raymond N. Greenwell, and Nathan P. Ritchey.

- Hints for teaching *Calculus with Applications* are provided as a resource for faculty.

- One open–response form and one multiple–choice form of a pretest are provided. These tests are an aid to instructors in identifying students who may need assistance.

- One open–response form and one multiple–choice form of a final examination are provided.

- Solutions for nearly all of the even–numbered exercises in the textbook are included. Solutions are usually not provided for exercises with open–response answers.

The following people have made valuable contributions to the production of this *Instructor's Resource Guide and Solutions Manual:* LaurelTech Integrated Publishing Services, editors; Judy Martinez and Sheri Minkner, typists; and Joe Vetere, Senior Author Support/Technology Specialist.

TEACHING HINTS

HINTS FOR TEACHING CALCULUS WITH APPLICATIONS

Algebra Reference

Some instructors obtain best results by going through this chapter carefully at the beginning of the semester. Others find it better to refer to it as needed throughout the course. Use whichever method works best for your students. As in the previous edition, we refer to the chapter as a "Reference" rather than a "Review," and the regular page numbers don't begin until Chapter 1. We hope this will make your students less anxious if you don't cover this material.

Chapter 1

Instructors sometimes go to either of two extremes in this chapter and the next. Some feel that their students have already covered enough precalculus in high school or in previous courses, and consequently begin with Chapter 3. Unfortunately, if they are wrong, their students may do poorly. Other instructors spend at least half a semester on Chapters 1 and 2 and the algebra reference chapter, and subsequently have little time for calculus. Such a course should not be labeled as calculus. We recommend trying to strike a balance, which may still not make all your students happy. A few may complain that the review of algebra, functions, and graphs is too quick; such students should be sent to a more basic course. Those students who are familiar with this material may become lazy and develop habits that will hurt them later in the course. You may wish to assign a few challenging exercises to keep these students on their toes.

Chapter 1 of *Calculus with Applications* is identical to Chapter 1 of *Finite Mathematics* and may be skipped by students who have already taken a course using that text. In this edition, we have streamlined the chapter from four to three sections, allowing instructors to reach the calculus material more quickly.

Section 1.1

This section and the next may seem fairly basic to students who covered linear functions in high school. Nevertheless, many students who have graphed hundreds of lines in their lifetime still lack a thorough understanding of slope, which hampers their understanding of the derivative. Such students could benefit from doing dozens of exercises similar to 39-42.

Perpendicular lines are not used in future chapters and could be skipped if you are in a hurry.

Section 1.2

Much recent research has been devoted to students' misunderstandings of the function concept. Such misunderstandings are among the major impediments to learning calculus. One way to help students is to study a simple class of functions first, as we do in this section. In this edition, even more of the general material on functions, including domain and range, is postponed until Chapter 2.

Supply and demand provides the students' first experience with a mathematical model. Spend time developing both the economics and the mathematics involved.

Stress that for cost, revenue, and profit functions, x represents the number of units. For supply and demand functions, we use the economists' notation of q to represent the number of units.

Emphasize the difference between the profit earned on 100 units sold as opposed to the number of units that must be sold to produce a profit of $100.

Section 1.3

The statistical functions on a calculator can greatly simplify these calculations, allowing more time for discussion and further examples. In this edition, we use "parallel presentation" to allow instructor choice on the extent technology is used. This section may be skipped if you are in a hurry, but your students can benefit from the realistic model and the additional work with equations of lines.

Section 2.1

After learning about linear functions in the previous chapter, students now learn about functions in general. This concept is critical for success in calculus. Unless sufficient time is devoted to this section, the results will become apparent later when students don't understand the derivative. One device that helps students distinguish $f(x+h)$ from $f(x)+h$ is to use a box in place of the letter x, as we do in this section after Example 4.

Section 2.2

This section combines the topics of quadratic functions and translation and reflection, with a minimal amount of material on completing the square. Our experience is that students graph quadratics most easily by first finding the y-intercept, then finding the x-intercepts when they exist (using factoring or the quadratic formula), and finding the vertex last by locating the point midway between the x-intercepts or, if the quadratic formula was used, by letting $x = -b/(2a)$.

Quadratics are among a small group of functions that can be analyzed completely with ease, so they are used throughout the text. On the other hand, the advent of graphing calculators has made ease of graphing less important, so we rely on quadratics less than in previous editions.

Some instructors pressed for time may choose to skip translations and reflections. But we have found that students who understand that the graph of $f(x) = 5 - \sqrt{4-x}$ is essentially the same as the graph of $f(x) = \sqrt{x}$, just shifted and reflected, will have an easier time when using the derivative to graph functions. Since students are familiar with very few classes of functions at this point, it helps to work with functions defined solely by their graphs, such as Exercises 25–28.

Exercises 33–40 cover stretching and shrinking of graphs in the vertical and horizontal directions. Covering these exercises carefully will not only give students a better grasp of functions, but will help them later to interpret the chain rule.

Section 2.3

Graphing calculators have made point plotting of functions less important than before. Plotting points by hand should not be entirely neglected, however, because a small amount is helpful when using the derivative to graph functions.

The two main goals of this section are to have an understanding of what an n-th degree polynomial looks like, and to be able to find the asymptotes of a rational function. Students who master these ideas will be better prepared for the chapter on curve sketching.

Exercise 59 is the first of several in this chapter asking students to find what type of function best fits a set of data. (See also Section 2.4, Exercises 49 and 50, and the Review Exercises 92 and 105.) The class can easily get bogged down in these exercises, particularly if you decide to explore the regression features in a calculator such as the TI-83/84 Plus. But there is a powerful payoff in terms of mastery of functions for the student who succeeds at these exercises.

Section 2.4

Some instructors may prefer at this point to continue with Chapter 3 and to postpone discussion of the exponential and logarithmic functions until later. The overwhelming preference of instructors we surveyed, however, was to cover exponential and logarithmic functions early and then to use these functions throughout the rest of the course. Instructors who wish to postpone this material will also need to omit for now those examples and exercises in Sections 3.1–4.3 that refer to exponential and logarithmic functions.

Students typically have no problem with $f(x) = 2^x$, but the number e often remains a mystery. Like π, the number e is a transcendental number, but students have had years of schooling to get used to π. Have your students approximate e with a calculator, as the textbook does before the definition of e. Notice how we use compound interest to help students get a handle on this number.

Section 2.5

Logarithms are a very difficult topic for many students. It's easy to say that a logarithm is just an exponent, but the fact that it is the exponent to which one must raise the base to get the number whose logarithm we are calculating is a rather obtuse concept. Therefore, spend lots of time going over examples that can be done without a calculator, such as $\log_2 8$. Students will also tend to come up with many incorrect pseudoproperties of logarithms, similar in form to the properties of logarithms given in this section. Take as much time and patience as necessary in gently correcting the many errors students inevitably will make at first.

Even after receiving a thorough treatment of logarithms, some students will still be stumped when solving a problem such as Example 7. Some of these students can get the correct answer using trial-and-error. The instructor should take consolation in the fact that at least such a student understands exponentials better than the one who uses logarithms incorrectly to solve Example 7 and comes up with the nonsensical answer $t = -7.51$ without questioning whether this makes sense. Be sure to teach your students to question the reasonableness of their answers; this will help them catch their errors.

Section 2.6

This section gives students much needed practice with exponentials and logarithms, and the applications keep students interested and motivated. Instructors should keep this in mind and not worry about having students memorize formulas. We have removed the formulas for present value in this edition, having decided that it's better for students to just solve the compound amount formula for P. This reduces by two the number of formulas that students need to remember.

There is a summary of graphs of basic functions in the end-of-chapter review.

Section 3.1

This is the first section on calculus, and perhaps the most important, since limits are what really distinguish calculus from algebra. Students will have the best understanding of limits if they have studied them graphically (as in Exercises 5–12), numerically (as in Exercises 15–20), and analytically (as in Exercises 31–52). The graphing calculator is a powerful tool for studying limits. Notice in Example 9 (c) and (d) that we have modified the method of finding limits at infinity by dividing by the highest power of x in the denominator, which avoid the problem of division by 0.

Section 3.2

The section on continuity should be straightforward if students have mastered limits from the previous section.

Section 3.3

This section introduces the derivative, even though that term doesn't appear until the next section. In a class full of business and social science majors, an instructor may wish to place less emphasis on velocity, an approach more suited to physics majors. But we have found velocity to be the manifestation of the derivative that is most intuitive to all students, regardless of their major.

Instructors in a hurry can skip the material on estimating the instantaneous rate of change from a set of data, but it helps solidify students' understanding of the derivative by giving them one more point of view.

Section 3.4

Students who have learned differentiation formulas in high school usually want (and deserve) some explanation of why they need to learn to take derivatives using the definition. You might try explaining to your students that getting the right formula is not the only goal; graphing calculators can give derivatives numerically. The most important thing for students to learn is the concept of the derivative, which they don't learn if they only memorize differentiation formulas.

Zooming in on a function with a graphing calculator until the graph appears to be a straight line gives students a very concrete image of what the derivative means.

After students have learned the differentiation formulas, they may forget about the definition of derivative. We have found that if we want them to use the definition on a test, it is important to say so clearly and emphatically, or they will simply use the shortcut formulas.

Section 3.5

One way to get students to focus on the concept of the derivative, rather than the mechanics, is to emphasize graphical differentiation. We have therefore devoted an entire section to this topic. Graphical differentiation is difficult for many students because there are no formulas to rely on. One must thoroughly understand what's going on to do anything. On the other hand, we have seen students who are weak in algebra but who possess a good intuitive grasp of geometry find this topic quite simple.

Section 4.1

Students tend to learn these differentiation formulas fairly quickly. These and the formulas in the next few sections are included in a summary at the end of the chapter.

Section 4.2

The product and quotient rules are more difficult for students to keep straight than those of the previous section. People seem to remember these rules better if they use an incantation such as "The first times the derivative of the second, plus the second times the derivative of the first." Some instructors have argued that this formulation of the product rule doesn't generalize well to products of three or more functions, but that's not important at this level. Some instructors allow their students to bring cards with formulas to the tests. This does not eliminate the need for students to understand the use of the formulas, but it does eliminate the anxiety students may have about forgetting a key formula under the pressure of an exam.

Section 4.3

No matter how many times an instructor cries out to his or her students, "Remember the chain rule!", many will still forget this rule at some time later in the course. But if a few more students remember the rule because the instructor reminds them so often, such reminders are worthwhile.

Section 4.4 and 4.5

In going through these sections, you may need to frequently refer to the rules of differentiation in the previous sections. You may also need to review the last three sections of Chapter 2.

Section 5.1 and 5.2

If students have understood Chapter 3, then the connection between the derivative, increasing and decreasing functions, and relative extrema should be obvious, and these sections should go quickly and smoothly.

Section 5.3

Students often confuse concave downward and upward with increasing and decreasing; carefully go over Figure 31 or the equivalent with your class.

Section 5.4

Graphing calculators have made curve sketching techniques less essential, but curve sketching is still one of the best ways to unify the various concepts introduced in this and the previous two chapters. Students should use graphing calculators to check their work.

Because this section is the culmination of many ideas, students often find it difficult and start to forget things they previously knew. For example, a student might state that a function is increasing on an interval and then draw it decreasing. The best solution seems to be lots of practice with immediate help and feedback from the instructor.

Students sometimes stumble over this topic because they use the rules for differentiation incorrectly, or because they make mistakes in algebra when simplifying. Exercises such as 35–39 are excellent for testing whether students really grasp the concepts.

Section 6.1

This section should not be conceptually difficult, but students need constant reminders to check the endpoints of an interval when finding the absolute maxima and minima.

Section 6.2

This section is one of the high points of the course. Some of the best applications of calculus involve maxima and minima. Notice that some exercises have a maximum or minimum at the endpoint of an interval, so students cannot ignore checking endpoints.

Almost everyone finds this material difficult because most people are not skilled at word problems. Remind your students that if they ever wonder whether mathematics is of any use, this section will show them.

Why are word problems so difficult? One theory is that word problems require the use of two different modes of thinking, which students are using simultaneously for the first time. People use words in daily life without difficulty, but when they study mathematics, they often turn off that part of their brain and begin thinking in a very formal, mechanical way. In word problems, both modes of thinking must be active. If and when the NCTM Standards become widely accepted in the schools, children will get more practice at an early age in such ways of thinking. Meanwhile, the steps for solving applied problems given in this book might make the process a little more straightforward, and hence achievable by the average student.

Section 6.3

This section continues the ideas of the previous one. The point of studying economic lot size should not be to apply Equation (3), but to learn how to apply calculus to solve such problems. We therefore urge you to cover Exercises 10–13, in which we vary the assumptions, so Equation (3) does not necessarily apply. In this edition, we have changed the presentation to be consistent with that of business texts.

The material on elasticity can be skipped, but it is an important application that should interest students who have studied even a little economics.

Section 6.4

There are two main reasons for covering implicit differentiation. First, it reinforces the chain rule. Second, it is needed for doing related rate problems. If you skip related rates due to lack of time, it is not essential to cover implicit differentiation, either.

Section 6.5

Related rate applications are less important than applied extrema problems, but they use some of the same skills in setting up word problems, and for that reason are worth covering. The best application exercises are under the heading "Physical Sciences," because those are the exercises in which no formula is given to the student; the student must construct a formula from the words. The geometrical formulas needed are kept to a minimum: the Pythagorean theorem, the area of a circle, the volume of a sphere, the volume of a cone, and the volume of a cylinder with a triangular cross section. Some instructors allow their students to use a card with such formulas on the exam. These formulas are summarized in a table in the back of the book.

Section 6.6

Differentials may be skipped by instructors in a hurry; you need not fear that this omission will hamper your students in the chapter on integration. The differentials used there are not the same as those used here, and the required techniques are easily picked up when integration by substitution is covered. The exception is for instructors who intend to cover Section 10.3 on Euler's Method, since differentials are used to motivate and derive that method.

As in the previous edition, our presentation of differentials emphasizes linear approximation, a topic of considerable importance in mathematics.

Section 7.1

Students sometimes start to get differentiation and antidifferentiation confused when they reach this section. Some believe the antiderivative of x^{-2} is $(-1/3)x^{-3}$; after all, if n is -2, isn't $n + 1 = -3$? Carefully clarify this point.

Section 7.2

The main difficulty here is teaching students what to choose as u. The advice given before Example 3 should be helpful.

Section 7.3

Some instructors who are pressed for time go lightly over the topic of the area as the limit of the sum of rectangular areas. This is possible, but care should be taken that students don't lose track of what the definite integral represents. Also, a light treatment here lessens the excitement of the Fundamental Theorem of Calculus.

We have continued the trend of the previous edition in covering three ways of approximating a definite integral with rectangles: the right endpoint, the left endpoint, and the midpoint. The trapezoidal rule is briefly introduced here as the average of the left sum and the right sum.

Section 7.4

The Fundamental Theorem of Calculus should be one of the high points of the course. Make a big deal about how the theorem unifies these two separate topics of area as a limit of sums and the antiderivative.

When using substitution on a definite integral, the text recommends changing the limits and the variable of integration. (See Example 4 and the Caution which follows the example.) Some instructors prefer instead to have their students solve an indefinite integral, and then to evaluate the integral using the limits on x. One advantage of this method is that students don't have to remember to change the limits. This method also has two disadvantages. The first is that it takes slightly longer, since it requires changing the integral to u and then back to x. Second, it prevents students at this stage from solving problems such as $\int_0^{1/2} x\sqrt{1-16x^4}\,dx$, which can be solved using the substitution $u = 4x^2$ and the fact that the integral $\int_0^1 \sqrt{1-u^2}\,du$ represents the area of a quarter circle. This is one section in which we deliberately did not use more than one method of presentation, because this would inevitably lead to confusion in the minds of some students.

Section 7.5

This section gives more motivation to the topic of integration. Consumers' and producers' surplus are important, realistic applications. We have downplayed sketching the curves that bound the area under consideration. Such sketches take time and are not necessary in solving these problems. But they clarify what is happening and make it possible to avoid memorizing formulas. Using a graphing calculator to sketch the curves can be helpful.

Section 7.6

The ubiquity of computers and graphing calculators has made numerical integration more important. Rather than computing a definite integral by an integration technique, one can just as easily enter the function into a calculator and press the integration key. Point out to students that this is valuable when the function to be integrated is complicated. On the other hand, using the antiderivative makes it easier to see the effect of changing the upper limit, say, from 2 to 3, or for working with the definite integral when one or both limits are parameters, such as a and b, rather than numbers.

Simpson's rule is the most accurate of the simple integration formulas. To achieve greater accuracy, a more complicated method must be used. This is why, unlike the trapezoidal rule, Simpson's rule is actually used by mathematicians and engineers.

You may wish to give your students the programs for the trapezoidal rule and Simpson's rule in *The Graphing Calculator Manual* that is available with the text.

Section 8.1

Students usually find column integration simpler than traditional integration by parts. We show both methods to give instructors a choice, and also to emphasize that the two methods are equivalent. Column integration makes organizing the details simpler, but is no more mechanical than the traditional method, as some have mistakenly claimed. At Hofstra University, students even use this method when neither the instructor nor the book discuss it. They find out about it from other students, and so it has become an underground method. Some instructors feel that students will lose any theoretical understanding of what they are doing if they use this method. Our experience is that almost no students at this level have a theoretical understanding of what integration by parts is about, but the better students can at least master the mechanics. With column integration, almost all of the students master the mechanics.

Section 8.2 and 8.3

These two sections give more applications of integration. Coverage of either section is optional.

Section 8.4

Improper integration is not really an application of integration, but it makes further applications of integration possible.

Many mathematicians use shorthand notation such as the following:
$\int_0^\infty e^{-x}\,dx = -e^{-x}|_0^\infty = 0-(-1) = 1$. For students at this level, it may be best to avoid the shorthand notation.

Section 9.1

The major difficulty students have with this section, and indeed with this entire chapter, is that they cannot visualize surfaces in 3-dimensional space, even though they live there. Fortunately, such visualization is not really necessary for doing the exercises in this chapter. A student who wants to explore what various surfaces look like can use any of the commercial or public domain computer programs available.

Section 9.2

Students who have mastered the differentiation techniques should have no difficulty with this section.

Section 9.3

This section corresponds to the section on applied extrema problems in the chapter on applications of the derivative, but with less emphasis on word problems. In Exercise 30, we revisit the topic of the least squares line, first covered in Section 1.3.

Section 9.4

Lagrange multipliers are an important application of calculus to economics. At some colleges, the business school is very insistent that the mathematics department cover this material.

Section 9.5

This section corresponds to the section on differentials in the chapter on applications of the derivative.

Section 9.6

Students who have trouble visualizing surfaces in 3-dimensional space are sometimes bothered by double integrals over variable regions. Instructors should assure such students that all they need to do is draw a good sketch of the region in the xy-plane, and not try to draw the volume in three dimensions.

Section 10.1

Differential equations of the form $dy/dx = f(x)$ are treated lightly in this section because they were already covered in the chapter on integration, although the terminology and notation were different then. Remind students that solving such differential equations is the same as antidifferentiation. The rest of the section is on separable differential equations. Students sometimes have trouble with this section because they have forgotten how to find an antiderivative, particularly one requiring substitution.

Section 10.2

If you get this far, your students have covered most of the techniques for solving first-order differential equations. You can find further techniques in differential equations texts, but most first-order equations that come up in real applications are either separable, linear, or not solvable by any exact method.

Section 10.3

This is one of the most calculator/computer-intensive sections of the book. In practice, more accurate methods than Euler's method are almost always used, but Euler's method introduces students to a way of solving problems that would otherwise be beyond their grasp. You may wish to give your students the program for Euler's method in *The Graphing Calculator Manual* that is available with the text.

Section 10.4

This is a fun section of assorted applications, showing students that the techniques they have learned were not in vain. You can pick and choose those applications of greatest interest to yourself and your students. You can also supplement the text with applications from other sources, such as those published by the Consortium for Mathematics and Its Applications Project (COMAP).

Chapter 11

Probability is one of the best applications of calculus around. In fact, statistics instructors sometimes feel the temptation to start discussing the definite integral even when their students know no calculus. This chapter is just a brief introduction, but it covers some of the most important concepts, such as mean, variance, standard deviation, expected value, and probability as the area under the curve. The third section covers three of the most important continuous probability distributions: uniform, exponential, and normal.

Chapter 12

Except for geometric series, most of the material in this chapter is of greater interest to the professional mathematician or engineer than to the student in business, management, economics, life sciences or social sciences, so many instructors may choose to skip this chapter.

Many students confuse sequences with series, and often have trouble with the various tests for convergence. To avoid these sources of confusion, we emphasize summation as a unifying concept, whether of a few terms of a sequence or of an infinite series. Except for convergence of geometric series and a few words on the interval of convergence, we devote little coverage to the topic of convergence. We believe this approach is appropriate for a course at this level.

Section 12.1

Geometric sequences are the most important type of sequences for applications. Even instructors who wish to skip most of this chapter may want to cover the first two sections.

Section 12.2

This material is similar to the material in the Mathematics of Finance chapter in our textbook *Finite Mathematics.* The section shows how geometric sequences are critical for an understanding of annuities, mortgages, and amortization. Don't be put off by the strange notation for the amount or the present value of an annuity; this notation is not at all strange in the world of finance.

Section 12.3

Taylor polynomials are introduced here as an approximation method, with no hint of infinite series.

Section 12.4

The emphasis here is on geometric series, the most important and simplest type of infinite series to understand.

Section 12.5

Some students find Taylor series a strange and abstract concept. To help make this concept more concrete, cover Example 4 on the normal curve, as well as the derivation of the rule of 70 and the rule of 72, introduced without proof in Chapter 2.

Section 12.6

Students with a “zero” feature on their calculator may lack motivation to learn Newton’s method unless you can interest them in how one might develop techniques for finding a zero. Newton’s method is not, however, the method typically used by calculators; calculator manufacturers are usually reluctant to discuss the actual algorithms.

Section 12.7

We have no applications for this sections, but students and instructors who enjoy symbol manipulation may still find this section satisfying.

Chapter 13

This chapter is a brief introduction to trigonometry and its uses in calculus. Students who need a more thorough treatment of this subject would be better served by a calculus book designed for mathematics majors. The presentation is brief, with a limited number of examples. As a result, students may find some of these exercises challenging. Therefore, tread carefully through this chapter.

PRETESTS AND ANSWERS

CALCULUS WITH APPLICATIONS

Name:

Pretest, Form A

1. Evaluate the expression $-x^2 + 3y - z^3$ when $x = -2$, $y = 3$, and $z = -1$. **1.** ______

Perform the indicated operations.

2. $\left(2x^2 - 3x + 7\right) - \left(5x^2 - 8x - 9\right)$ **2.** ______

3. $(y - 4)^2$ **3.** ______

4. $(2x - 1)\left(x^2 + 3x - 4\right)$ **4.** ______

5. $(5a + 2b)(5a - 2b)$ **5.** ______

Factor each polynomial completely.

6. $3x^2y^3 - 6x^3y^2 + 15x^2y^2$ **6.** ______

7. $2ac - 3ad + 8bc - 12bd$ **7.** ______

Solve each equation.

8. $6(x - 1) - 4(2x + 8) = 2 - 5(x + 2)$ **8.** ______

9. $\dfrac{1}{4}x - 7 = \dfrac{2x}{3} + \dfrac{1}{2}$ **9.** ______

10. $\dfrac{x - 1}{5} = \dfrac{3 - 2x}{4}$ **10.** ______

11. $(x - 2)(x + 1) = 4$ **11.** ______

12. Solve the inequality $6x - 8(x + 2) \leq 5 - (x + 14)$. Write your answer in interval notation. **12.** ______

13. Solve the equation $2x + 4y = ay - bx$ for y. **13.** ______

14. Solve the equation $\frac{5}{a} + \frac{c}{2b} = \frac{7}{3a}$ for a. **14.** ______

15. Solve the following system of equations.

$$\begin{aligned} 3x - 2y &= 12 \\ -4x + 5y &= -23 \end{aligned}$$

15. ______

Perform the indicated operation. Write answers in lowest terms.

16. $\dfrac{18a^2b^2}{27xy^3} \cdot \dfrac{10xy^2}{8ab^2}$ **16.** ____________

17. $\dfrac{2x+8}{3y-15} \div \dfrac{x^2+7x+12}{y^2-10y+25}$ **17.** ____________

18. $\dfrac{4}{x^2-1} + \dfrac{2}{x^2+3x+2}$ **18.** ____________

Simplify each expression. Write answers with only positive exponents.

19. $\left(2x^2y^{-3}\right)^{-3}$ **19.** ____________

20. $\dfrac{6x^{-2}y^5}{15x^4y^{-2}}$ **20.** ____________

Simplify each radical. Assume all variables represent nonnegative real numbers.

21. $\sqrt{72x^3y^4}$ **21.** ____________

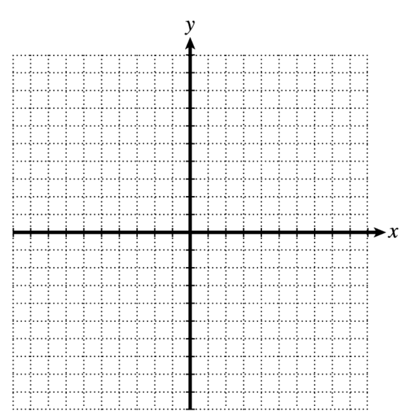
22. $\sqrt[3]{72x^3y^4}$ **22.** ____________

Write an equation in the form $ax + by = c$ ***for each line.***

23. The line through the points with coordinates $(-3, -14)$ and $(5, 2)$ **23.** ____________

24. The line through the points with coordinates $(3, -7)$ and $(3, 5)$ **24.** ____________

25. Graph the solution of the inequality $2x - 3y \leq 12$. **25.**



CALCULUS WITH APPLICATIONS

Name:

Pretest, Form B

Choose the best answer.

1. Evaluate the expression $\frac{y^2-z}{-x^2+6}$ when $x=-2$, $y=4$, and $z=-8$. **1.** ________

(a) $\frac{12}{5}$ (b) 4 (c) 12 (d) $\frac{4}{5}$

Perform the indicated operations.

2. $\left(x^2-3x+7\right)+\left(4x^3-6x^2-5x+4\right)$ **2.** ________

(a) $4x^3-6x^4-8x^2+11$ (b) $4x^3-5x^2-8x+11$
(c) $4x^3-5x^2-2x+11$ (d) $4x^3-7x^2-8x+11$

3. $(5x-4w)^2$ **3.** ________

(a) $25x^2-16w^2$ (b) $25x^2+16w^2$
(c) $25x^2-40wx+16w^2$ (d) $25x^2-20wx+16w^2$

4. $(x-3)\left(x^2+x-2\right)$ **4.** ________

(a) x^3-2x^2-5x+6 (b) x^3-3x^2-5x+6
(c) x^3-3x^2+6x+6 (d) x^3-2x^2+5x+6

5. $(4x+9)(4x-9)$ **5.** ________

(a) $16x^2+72x-81y^2$ (b) $16x^2-72xy-81y^2$
(c) $16x^2-81y^2$ (d) $16x^2-72xy+81y^2$

Factor each polynomial completely.

6. $14a^2b^3-7a^2b^2+35ab^4$ **6.** ________

(a) $7ab\left(2ab^2-ab+5b^3\right)$ (b) $a^2b^3\left(14-7b+35b^2\right)$
(c) $7a^3b\left(2b^2-7b+5b^3\right)$ (d) $7ab^2\left(2ab-a+5b^2\right)$

7. $6xt+8x-3yt-4y$ **7.** ________

(a) $(2x-y)(3t+4)$ (b) $(2x+y)(3t-4)$
(c) $(2x-y)(3t-4)$ (d) $(2x+y)(3t+4)$

Solve each equation.

8. $4m - (7m - 6) = -m$ **8.** ________

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) -3 (d) 3

9. $-\frac{5}{3} + \frac{2}{r-1} = \frac{1}{6}$ **9.** ________

(a) $-\frac{1}{3}$ (b) $\frac{23}{11}$ (c) 3 (d) -6

10. $\frac{3y+2}{5} = \frac{7-y}{4}$ **10.** ________

(a) $\frac{43}{7}$ (b) $\frac{27}{7}$ (c) $\frac{27}{17}$ (d) $\frac{43}{17}$

11. $2z^2 + z = 28$ **11.** ________

(a) $-4, 3$ (b) $4, -3$

(c) $-4, \frac{7}{2}$ (d) $4, -\frac{7}{2}$

12. Solve the inequality $-6y + 2 \geq 4y - 7$.
Write your answer in interval notation. **12.** ________

(a) $\left(-\infty, -\frac{9}{10}\right]$ (b) $\left[\frac{1}{2}, \infty\right)$

(c) $\left[\frac{9}{10}, \infty\right)$ (d) $\left(-\infty, \frac{9}{10}\right]$

13. Solve the equation $x = \frac{4(y-z)}{3k}$ for y. **13.** ________

(a) $y = \frac{4}{3kx+4z}$ (b) $y = \frac{4}{kx-z}$

(c) $y = -\frac{3kxz}{4}$ (d) $y = \frac{3x+4z}{4}$

14. Solve the equation $\frac{4}{x} + \frac{a}{y} = \frac{2}{3}$ for x. **14.** ________

(a) $x = \frac{2xy-12y}{3s}$ (b) $x = \frac{12y}{2y-3a}$

(c) $x = \frac{12y+3ax}{2y}$ (d) $x = \frac{3a-2y}{12y}$

15. Solve the following system of equations and then determine the value of $x+y$ for the solution.

$$\begin{aligned} 3x + 2y &= 0 \\ x - 5y &= 17 \end{aligned}$$

15. ________

(a) -1 (b) 1 (c) 2 (d) -3

Perform the indicated operation. Write answers in lowest terms.

16. $\dfrac{6yz^3}{30a^2b^4} \cdot \dfrac{15a^3b^3}{12y^2z^3}$

16. ________

(a) $\frac{3a^2}{12aby}$ (b) $\frac{4y^3z^6}{25a^5b^7}$

(c) $\frac{a}{4by}$ (d) $\frac{b}{4ay}$

17. $\dfrac{x^2-y^2}{4a^5b^7} \div \dfrac{x^2+3xy+2y^2}{12a^3b^{12}}$

17. ________

(a) $\frac{3b^5}{2a^2}$ (b) $\frac{3b^5(x+y)}{a^2(x-2y)}$

(c) $\frac{3b^5(x-y)}{a^2(x+2y)}$ (d) $\frac{3b^3(x+y)}{(x-2y)}$

18. $\dfrac{1}{2x} - \dfrac{2}{3y} + \dfrac{5}{6xy}$

18. ________

(a) $\frac{2}{3xy}$ (b) $\frac{3y-4x+5}{6xy}$

(c) $3y-4x+5$ (d) $\frac{4x-3y+5}{6xy}$

Simplify each expression. Write answers with only positive exponents.

19. $\left(3a^{-2}b^3\right)^{-4}$

19. ________

(a) $-\frac{12}{a^6b}$ (b) $-\frac{12a^8}{b^{12}}$

(c) $\frac{a^8}{12b^{12}}$ (d) $\frac{a^8}{81b^{12}}$

20. $\left(\dfrac{r^2m^{-1}}{r^3m^2}\right)^{-2}$

20. ________

(a) r^2m^2 (b) $\frac{1}{r^2m^2}$ (c) r^2m^6 (d) $\frac{1}{r^2m^6}$

Simplify each radical. Assume all variables represent nonnegative real numbers.

21. $\sqrt{108a^4b^3}$ **21.** ________

(a) $3a^2b\sqrt{12b}$ (b) $6a^2b\sqrt{3b}$

(c) $6a^2b\sqrt{3a}$ (d) $54a^2b\sqrt{b}$

22. $\sqrt[3]{108a^4b^3}$ **22.** ________

(a) $3ab\sqrt[3]{4a}$ (b) $6a^2b\sqrt[3]{3b}$

(c) $36ab\sqrt[3]{a}$ (d) $9a^2b\sqrt[3]{12a}$

Write an equation in the form $ax + by = c$ ***for each line.***

23. The line through the points with coordinates $(1, -1)$ and $(-1, -2)$ **23.** ________

(a) $x - 2y = 3$ (b) $x + 2y = 3$

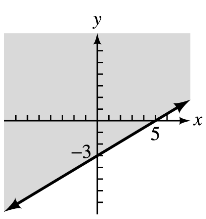
(c) $x + 2y = -3$ (d) $x - 2y = -3$

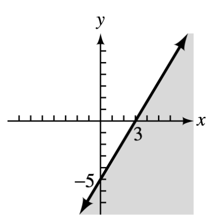
24. The line through the points with coordinates $(-5, -2)$ and $(-3, -2)$ **24.** ________

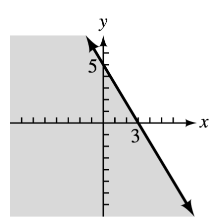
(a) $x + y = -2$ (b) $x = -2$

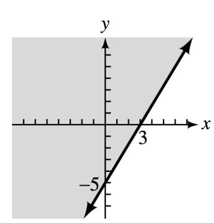
(c) $x - y = -2$ (d) $y = -2$

25. Graph the solution of the inequality $5x - 3y \geq 15$. **25.** ________

(a)


(b)


(c)


(d)


ANSWERS TO PRETESTS

PRETEST, FORM A

1. 6

2. $-3x^2+5x+16$

3. $y^2-8y+16$

4. $2x^3+5x^2-11x+4$

5. $25a^2-4b^2$

6. $3x^2y^2(y-2x+5)$

7. $(a+4b)(2c-3d)$

8. 10

9. -18

10. $\frac{19}{14}$

11. $-2,\ 3$

12. $[-7,\infty)$

13. $y=\frac{(b+2)x}{a-4}$

14. $a=-\frac{16b}{3c}$

15. $(2,-3)$

16. $\frac{5a}{6y}$

17. $\frac{2(y-5)}{3(x+3)}$

18. $\frac{6}{(x-1)(x+2)}$

19. $\frac{y^9}{8x^6}$

20. $\frac{2y^7}{5x^6}$

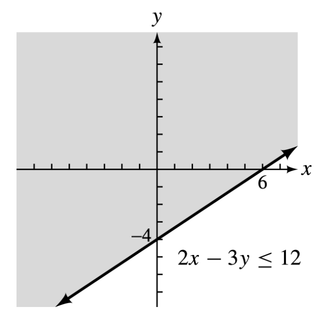
21. $6xy^2\sqrt{2x}$

22. $2xy\sqrt[3]{9y}$

23. $2x-y=8$

24. $x=3$

25.



PRETEST, FORM B

1. (c)

2. (b)

3. (c)

4. (a)

5. (c)

6. (d)

7. (a)

8. (d)

9. (b)

10. (c)

11. (c)

12. (d)

13. (d)

14. (b)

15. (a)

16. (c)

17. (c)

18. (b)

19. (d)

20. (c)

21. (b)

22. (a)

23. (a)

24. (d)

25. (b)

FINAL EXAMINATIONS AND ANSWERS

CALCULUS WITH APPLICATIONS

Name:

Final Examination, Form A

1. Find the coordinates of the vertex of the graph of $f(x) = -2x^2 + 30x + 45$.

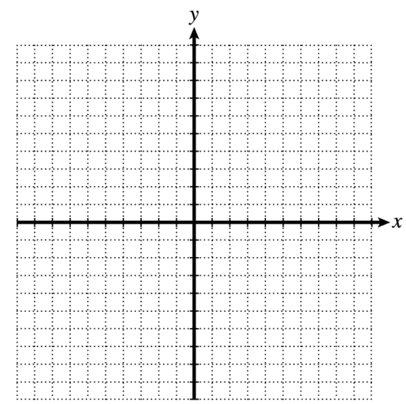
1. ______________________

2. Write the equation $2^a = d$ using logarithms.

2. ______________________

3. Graph the function $y = \log_2(x-3) + 4$.

3.



4. Evaluate $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$.

4. ______________________

5. Find all values of x at which $g(x) = \frac{x-2}{x-5}$ is not continuous.

5. ______________________

6. Find the average rate of change of $y = \sqrt{2x+1}$ between $x = 4$ and $x = 12$.

6. ______________________

7. Find the derivative of $y = 7x^2e^{3x}$.

7. ______________________

8. Find the derivative of $y = \frac{\ln(3x)}{x+1}$.

8. ______________________

9. Find the instantaneous rate of change of

$$s(t) = 3t^2 - \frac{8}{t}$$

at $t = 2$.

9. ______________________

10. Find an equation of the tangent line to the graph of $y = \left(x^2 + 3x + 3\right)^8$ at the point $(-1, 1)$.

10. ______________________

11. Find $h'(2)$ if $h(x) = \frac{6}{\sqrt{4x+17}}$.

11. ______________________

12. Find the largest open intervals where f is increasing or decreasing if

$$f(x) = 2x^3 - 24x^2 + 72x.$$

12. ______________________

13. Find the locations and values of all relative extrema of g if

$$g(x) = \frac{x^2 + 5x + 3}{x - 1}.$$

13. ______________________

14. For the function $f(x) = x^4 - 4x^3 - 5$, find

(a) all intervals where f is concave upward;

(b) all intervals where f is concave downward;

(c) all points of inflection.

14. (a) ______________________

(b) ______________________

(c) ______________________

15. Find the fourth derivative of $h(x) = xe^x$.

15. ______________________

16. Find the locations and values of all absolute extrema of

$$f(x) = x^3 - 6x^2$$

on the interval $[-1, 2]$.

16. ______________________

17. For a particular commodity, its price per unit in dollars is given by

$$p(x) = 120 - \frac{x^2}{10},$$

where x, measured in thousands, is the number of units sold. This function is valid on the interval $[0, 34]$. How many units must be sold to maximize revenue?

17. ______________________

18. Find $\frac{dy}{dx}$ if $3\sqrt{x} - 5y^3 = 7xy$.

18. ______________________

19. If $xy = y - 9$, find $\frac{dy}{dt}$ if $\frac{dx}{dt} = 12$, $x = -3$, and $y = 4$.

19. ______________________

20. If $y = -3\left(2 + x^2\right)^3$, find dy.

20. ______________________

21. Using differentials, approximate the volume of coating on a sphere of radius 5 inches, if the coating is .03 inch thick.

21. ______________________

22. Find $\int \left(3x^2 - 7x + 2\right)\, dx$.

22. ______________________

23. Find $\int \frac{8}{x+9}\, dx$.

23. ______________________

24. Evaluate $\int_0^2 x\sqrt{3x^2 + 4}\, dx$.

24. ______________________

25. Find the area of the region between the x-axis and the graph of $f(x) = xe^{x^2}$ on the interval $[0, 2]$. **25.** ______________

26. Find the area of the region enclosed by the graphs of $f(x) = x^2 - 4$ and $g(x) = 1 - x^2$. **26.** ______________

27. Find $\int x^2 e^x \, dx$. **27.** ______________

28. Evaluate $\int_1^{e^2} 3x \ln x \, dx$. **28.** ______________

29. Find the average value of $f(x) = x\left(x^2 + 1\right)^3$ over the interval $[0, 2]$. **29.** ______________

30. Find the volume of the solid of revolution formed by rotating the region bounded by the graphs of $f(x) = \sqrt{x + 4}$, $y = 0$, and $x = 12$ about the x-axis. **30.** ______________

31. Determine whether the improper integral $\int_{-\infty}^{-1} x^{-2} \, dx$ converges or diverges. If it converges, find its value. **31.** ______________

32. If $f(x, y) = 3x^2 - 4xy + y^3$, find $f_{xy}(1, -3)$. **32.** ______________

33. For $f(x, y) = 2x^2 - 2xy + 2y^2 + 4x + 4y + 5$, find all of the following:

(a) relative maxima; **33. (a)** ______________

(b) relative minima; **(b)** ______________

(c) saddle points. **(c)** ______________

34. Maximize $f(x, y) = x^2 y$, subject to the constraint $y + 4x = 84$. **34.** ______________

35. Find dz, given $z = 3x^2 - 4xy + y^2$, where $x = 0$, $y = 2$, $dx = .02$, and $dy = .01$. **35.** ______________

36. Evaluate $\int_1^2 \int_0^3 e^{2x} \, dy \, dx$. **36.** ______________

37. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+e^x}{2y}$. **37.** ______________

38. Find the particular solution of the differential equation $\frac{dy}{dx} = 3x^2 - 7x + 2$; $y = 4$ when $x = 0$. **38.** ____________

39. The marginal cost function for a particular commodity is $\frac{dy}{dx} = 8x - 9x^2$. The fixed cost is \$20. Find the cost function. **39.** ____________

40. The probability density function of a random variable is defined by $f(x) = \frac{1}{4}$ for x in the interval $[12, 16]$. Find $P(x \le 14)$. **40.** ____________

41. For the probability density function $f(x) = 3x^{-4}$ on $[1, \infty)$, find

(a) the expected value; **41. (a)** ____________

(b) the variance; **(b)** ____________

(c) the standard deviation. **(c)** ____________

42. The average height of a particular type of tree is 20 feet, with a standard deviation of 2 feet. Assuming a normal distribution, what is the probability that a tree of this kind will be shorter than 17 feet? **42.** ____________

43. For the function $f(x) = e^{-3x}$, find

(a) the Taylor polynomial of degree 4 at $x = 0$; **43. (a)** ____________

(b) an approximation, rounded to four decimal places, of $e^{-.03}$ using the Taylor polynomial from part (a). **(b)** ____________

44. Find the sum of the convergent geometric series

$$5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \cdots .$$

44. ____________

45. Use Newton's method to find a solution to the nearest hundredth of $3x^3 - 2x^2 + x - 1 = 0$ in the interval $[0, 1]$. **45.** ____________

46. Use l'Hospital's rule to find $\lim_{x \to 0} \frac{3e^{2x} - 3}{x}$. **46.** ____________

47. Find the derivative of $f(x) = \tan\left(x^2 + 1\right)$. **47.** ____________

48. Find the derivative of $y = x^3 \sin^2 x$. **48.** ____________

49. Find $\int \sec^2(3x + 1)\, dx$. **49.** ____________

50. Evaluate $\int_0^{\pi/6} \sin^3 x \cos x\, dx$. **50.** ____________

CALCULUS WITH APPLICATIONS

Name:

Final Examination, Form B

Choose the best answer.

1. Find the coordinates of the vertex of the graph of $f(x) = 2x^2 + 10x - 17$. **1.** ________

(a) $\left(-\frac{5}{2}, -\frac{59}{2}\right)$ (b) $\left(\frac{5}{2}, -\frac{59}{2}\right)$ (c) $(-5, -42)$ (d) $(5, -42)$

2. Write the equation $m^p = q$ using logarithms. **2.** ________

(a) $\log_m p = q$ (b) $\log_q p = m$
(c) $\log_p q = m$ (d) $\log_m q = p$

3. Solve $\left(\frac{1}{e}\right)^{2x} = 14$. (Round to nearest thousandth.) **3.** ________

(a) -1.320 (b) 2.639 (c) -2.639 (d) 1.320

4. Evaluate $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$. **4.** ________

(a) 0 (b) 3 (c) 6 (d) The limit does not exist.

5. Find all values of x at which

$$f(x) = \frac{x^2 - 5x + 6}{x^2 - 7x + 12}$$

is not continuous. **5.** ________

(a) 3 (b) 4 (c) 3 and 4
(d) The function is continuous everywhere.

6. Find the average rate of change of $y = x^2 + x$ between $x = 4$ and $x = 12$. **6.** ________

(a) 9 (b) 10 (c) 17 (d) 25

7. Find the derivative of $y = -3x^3e^{2x}$. **7.** ________

(a) $-27x^2e^{2x} - 6x^3e^{2x}$ (b) $-18x^2e^{3x}$
(c) $-9x^2e^{2x} - 6x^3e^{2x}$ (d) $-6x^2e^{2x} - 6x^3e^{2x}$

8. Find the derivative of $y = \frac{x^2}{\ln 3x}$. **8.** ________

(a) $\dfrac{2x^2}{(\ln 3x)^2}$ (b) $\dfrac{6x \ln 3x - x}{3(\ln 3x)^2}$ (c) $\dfrac{2x \ln 3x - x}{(\ln 3x)^2}$ (d) $\dfrac{6x \ln 3x + x}{3(\ln 3x)^2}$

9. Find the instantaneous rate of change of

$$s(t) = \frac{t^2}{2} - \frac{2}{t^2}$$

at $t = 2$. **9.** ________

(a) $\dfrac{3}{2}$ (b) $\dfrac{5}{2}$ (c) $\dfrac{7}{2}$ (d) $\dfrac{9}{2}$

10. Find an equation of the tangent line to the curve $f(x) = -\frac{1}{x+2}$ at the point $\left(2, -\frac{1}{4}\right)$. **10.** ________

(a) $x - 16y = 6$ (b) $x - 4y = 3$

(c) $f'(x) = \dfrac{1}{16}$ (d) $y = \dfrac{1}{(x+2)^2}$

11. Find $h'(3)$ if $h(x) = -\frac{400}{\sqrt{3x+7}}$. **11.** ________

(a) 100 (b) $\dfrac{25}{8}$ (c) $\dfrac{75}{8}$ (d) 200

12. Find the largest open interval(s) over which the function $g(x) = x^3 - 12x^2 + 36x + 1$ is increasing. **12.** ________

(a) $(2, 6)$ (b) $(-\infty, 2)$

(c) $(-\infty, 6)$ (d) $(-\infty, 2)$ and $(6, \infty)$

13. Find the location and value of all relative extrema for the function

$$f(x) = \frac{x^2 + 5x + 3}{x - 1}.$$

13. ________

(a) Relative minimum of 1 at -2; relative maximum of 13 at 4

(b) Relative maximum of 1 at -2; relative minimum of 13 at 4

(c) Relative minimum of 1 at 4; relative minimum of 13 at -2

(d) No relative extrema

14. Find the coordinates of all points of inflection of the function $g(x) = x^3 - 6x^2$. **14.** ________

(a) $(2, -16)$ (b) $(2, -12)$ (c) $(2, 0)$ (d) $(-2, 16)$

15. Find the third derivative of $f(x) = \sqrt{2x+1}$. **15.** ________

(a) $\frac{3}{8}(2x+1)^{-5/2}$ (b) $3(2x+1)^{-5/2}$

(c) $-(2x+1)^{-3/2}$ (d) $-\frac{1}{4}(2x+1)^{-3/2}$

16. Find the absolute minimum of $f(x) = x^4 - 4x^3 - 5$ on the interval $[-1, 2]$. **16.** ________

(a) -30 (b) 0 (c) -21 (d) No absolute minimum

17. Botanists, Inc., a consulting firm, monitors the monthly growth of an unusual plant. They determine that the growth (in inches) is given by

$$g(x) = 4x - x^2,$$

where x represents the average daily number of ounces of water the plant receives. Find the maximum monthly growth of the plant. **17.** ________

(a) 2 inches (b) 8 inches

(c) 4 inches (d) 6 inches

18. Find $\frac{dy}{dx}$, given $4x^2 - 7 = 3\sqrt{y} + \frac{2}{x^3}$. **18.** ________

(a) $\dfrac{4\sqrt{y}\left(4x^5 - 3\right)}{3x^4}$ (b) $\dfrac{4\sqrt{y}\left(4x^5 + 3\right)}{3x^4}$

(c) $\dfrac{4\sqrt{y}\left(4x^3 - 3\right)}{3x^2}$ (d) $\dfrac{4\sqrt{y}\left(4x^3 + 3\right)}{3x^2}$

19. If $xy = y + 6$, find $\frac{dy}{dt}$ if $\frac{dx}{dt} = -3$, $x = 4$, and $y = 2$. **19.** ________

(a) 2 (b) 0 (c) -2 (d) -4

20. Evaluate dy, given $y = 45 - 2x - 3x^2$, with $x = 3$ and $\triangle x = .02$. **20.** ________

(a) $-.4$ (b) $.4$ (c) $.32$ (d) $-.32$

21. Using differentials, approximate the volume of coating on a cube with 3-cm sides, if the coating is .02 cm thick. **21.** ________

(a) $.54 \text{ cm}^3$ (b) 27.54 cm^3
(c) 26.46 cm^3 (d) 81.54 cm^3

22. Find $\int \left(4x^2 + 3x - 5\right) dx$. **22.** ________

(a) $\frac{4}{3}x^3 + \frac{3}{2}x^2 - 5 + C$ (b) $\frac{4}{3}x^3 + \frac{3}{2}x^2 - 5x$
(c) $\frac{4}{3}x^3 + \frac{3}{2}x^2 - 5x + C$ (d) $8x + 3$

23. Find $\int \frac{-3}{(2x+5)}\, dx$. **23.** ________

(a) $-\frac{3}{2}\ln|2x+5| + C$ (b) $-3\ln|2x+5| + C$
(c) $-\frac{1}{6}\ln|2x+5| + C$ (d) $\frac{1}{2}\ln|2x+5| + C$

24. Evaluate $\int_0^1 2x\sqrt{4x^2+5}\, dx$. **24.** ________

(a) $\frac{3}{8}$ (b) $\frac{27 - 5\sqrt{5}}{6}$
(c) $\frac{27 - 5\sqrt{5}}{8}$ (d) $\frac{27 - 5\sqrt{5}}{4}$

25. Find the area of the region between the x-axis and the graph of $f(x) = \sqrt{x+1}$ on the interval $[0, 3]$. **25.** ________

(a) $\frac{14}{3}$ (b) 7 (c) $\frac{1}{4}$ (d) $\frac{21}{2}$

26. Find the area of the region enclosed by the graphs of $f(x) = 9 - x^2$ and $g(x) = x^2 - 9$. **26.** ________

(a) -36 (b) 36 (c) 0 (d) 72

27. Find $\int 3x^2 e^{2x}\, dx$. **27.** ________

(a) $3e^{x^2} + C$ (b) $\frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C$

(c) $x^3e^{2x} + C$ (d) $3x^2e^{2x} - 6xe^{2x} + 6e^{2x} + C$

28. Evaluate $\int_1^4 9x^2 \ln x\, dx$. **28.** ________

(a) $192 \ln 4 - 45$ (b) $192 \ln 4 - 65$

(c) $192 \ln 4 - 51$ (d) $192 \ln 4 - 63$

29. Find the average value of the function $f(x) = \sqrt{3x+1}$ over the interval $[0, 8]$. **29.** ________

(a) $\frac{31}{3}$ (b) $\frac{248}{9}$ (c) $\frac{31}{9}$ (d) $\frac{248}{3}$

30. Find the volume of the solid of revolution formed by rotating the region bounded by $f(x) = \sqrt{x-1}$, $y = 0$, and $x = 10$ about the x-axis. **30.** ________

(a) 40.5 (b) $\frac{81\pi}{2}$ (c) 18π (d) 40π

31. Evaluate $\int_{-\infty}^{-2} \frac{1}{x^3}\, dx$. **31.** ________

(a) $-\frac{1}{8}$ (b) Diverges (c) $\frac{1}{64}$ (d) $-\frac{1}{64}$

32. Given $z = f(x, y) = x^2 - 3xy + 2y^3$, find $f_{yx}(0, -2)$. **32.** ________

(a) -16 (b) 2 (c) -24 (d) -3

33. Which of the following applies to the function

$$f(x,y) = 2x^2 + 8y^3 - 12xy + 7?$$

33. ________

(a) f has a relative maximum at $\left(\frac{9}{2}, \frac{3}{2}\right)$
(b) f has a relative minimum at $(0, 0)$
(c) f has a relative minimum at $\left(\frac{9}{2}, \frac{3}{2}\right)$
(d) f has a saddle point at $\left(\frac{9}{2}, \frac{3}{2}\right)$

34. Maximize $f(x,y) = x^2y$, subject to the constraint $y + 4x = 84$. **34.** ________

(a) 0 (b) 14 (c) 5488 (d) 3642

35. Find dz, given $z = \sqrt{x^2 + y^2}$, with $x = 3,\ y = 4,\ dx = -.01$, and $dy = .02$. **35.** ________

(a) .022 (b) .01 (c) .001 (d) .05

36. Evaluate $\int_0^2 \int_2^4 x^2 y\, dy\, dx$. **36.** ________

(a) $\dfrac{112}{3}$ (b) 16 (c) 32 (d) 12

37. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{e^{2x} - 4}{y^2}$. **37.** ________

(a) $y = \left(\dfrac{3}{2}e^{2x} - 12x + C\right)^{1/3}$ (b) $y = \dfrac{1}{2}e^{2x} - 4x + C$

(c) $y = \left(\dfrac{1}{2}e^{2x} - 4x + C\right)^{1/3}$ (d) $y^2 = e^{2x} - 4$

38. Find the particular solution of the differential equation $\frac{dy}{dx} = 7 - 2x + 3x^2;\ y = -2$ when $x = 0$. **38.** ________

(a) $y = 7x - x^2 + x^3 + C$ (b) $y = 7x - x^2 + x^3 - 2$
(c) $y = -x^2 + x^3 + C$ (d) $y = 7x - x^2 + x^3 + 2 + C$

39. The marginal cost function for a particular commodity is $\frac{dy}{dx} = 10x - 7x^2$. The fixed cost is \$200. Find the cost function. **39.** ________

(a) $y = 200 + 5x^2 - 14x$ (b) $y = 5x^2 - 14x$

(c) $y = 5x^2 - \dfrac{7x^3}{3} + 200$ (d) $y = 10 - 14x$

40. The probability density function of a random variable is defined by $f(x) = \frac{1}{5}$ for $[20, 25]$. Find $P(x \leq 22)$. **40.** ________

(a) .2 (b) .8 (c) .4 (d) .6

41. Find the standard deviation for the probability density function $f(x) = 3x^{-4}$ on $[1, \infty)$. **41.** ________

(a) 1.5 (b) .866 (c) .75 (d) .67

42. The average height of a particular type of tree is 20 feet, with a standard deviation of 2 feet. Assuming a normal distribution, what is the probability that a tree of this kind will be taller than 17 feet? **42.** ________

(a) .0668 (b) .9332 (c) .8023 (d) .1977

43. Find the monthly house payment necessary to amortize a \$150,000 loan at 7.2% for 30 years. **43.** ________

(a) \$1018.18 (b) \$975.34 (c) \$1112.87 (d) \$5478.44

44. Find the sum of the convergent geometric series

$$2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$$

44. ________

(a) 2 (b) 8 (c) 4 (d) $\dfrac{8}{3}$

45. The nth term of the Taylor series expansion of $\ln\left(1+\frac{x}{2}\right)$ is given by which of the following? **45.** ________

(a) $\dfrac{(-1)^n x^{n+1}}{(n+1)\cdot 2^{n+1}}$ (b) $\dfrac{x^{n+1}}{n+1}$ (c) $\dfrac{x^{n+1}}{(n+1)\cdot 2^{n+1}}$ (d) $\dfrac{(-1)^n x^{n+1}}{n+1}$

46. Evaluate $\lim\limits_{x\to 0} \dfrac{-2e^{4x}+2}{x}$. **46.** ________

(a) -2 (b) -8 (c) 0 (d) The limit does not exist.

47. Find the derivative of $f(x) = \tan\sqrt{2x+1}$. **47.** ________

(a) $\dfrac{\sec^2\sqrt{2x+1}}{\sqrt{2x+1}}$ (b) $\dfrac{\sec^2\sqrt{2x+1}}{2}$

(c) $x\sec^2\sqrt{2x+1}$ (d) $2x\sec^2\sqrt{2x+1}$

48. Find the derivative of $y = x^2\cos^3 x$. **48.** ________

(a) $x\cos^2 x\,(2\cos x + 3x)$ (b) $x\cos^2 x\,(2\cos x - 3x\sin x)$

(c) $x\cos^2 x\,(2\cos x - 3x)$ (d) $x\cos^2 x\,(2\cos x + 3x\sin x)$

49. Find $\displaystyle\int \tan(2x)\sec^2(2x)\;dx$. **49.** ________

(a) $\dfrac{1}{4}\tan^2(2x) + C$ (b) $\tan^2(2x) + C$

(c) $\dfrac{1}{4}\sec(2x) + C$ (d) $\sec(2x) + C$

50. Evaluate $\displaystyle\int_0^{\pi/2} \left(10 - 10\sin^2 x\cos x\right)\,dx$. **50.** ________

(a) $\dfrac{20}{3}$ (b) $5\pi - 10$

(c) $\dfrac{30\pi - 10}{3}$ (d) $\dfrac{15\pi - 10}{3}$

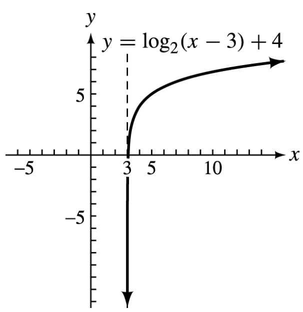
ANSWERS TO FINAL EXAMINATIONS

FINAL EXAMINATION, FORM A

1. $(7.5, 157.5)$

2. $\log_2 d = a$

3.



4. 4

5. 5

6. $\frac{1}{4}$

7. $y' = 7xe^{3x}(2+3x)$

8. $y' = \frac{x+1-x\ln(3x)}{x(x+1)^2}$

9. 14

10. $y = 8x + 9$

11. $-\frac{12}{125}$

12. Increasing on $(-\infty, 2)$ and $(6, \infty)$; decreasing on $(2, 6)$

13. Relative maximum of 1 at -2; relative minimum of 13 at 4

14. **(a)** $(-\infty, 0)$ and $(2, \infty)$ **(b)** $(0, 2)$ **(c)** $(0, -5)$ and $(2, -21)$

15. $(x+4)\,e^x$

16. Absolute maximum of 0 at 0; absolute minimum of -16 at 2

17. 20,000

18. $\frac{dy}{dx} = \frac{3-14x^{1/2}y}{14x^{3/2}+30x^{1/2}y^2}$

19. 12

20. $dy = -18x\left(2+x^2\right)^2\,dx$

21. 3π in^3

22. $x^3 - \frac{7}{2}x^2 + 2x + C$

23. $\frac{1}{8}\ln|x+9| + C$

24. $\frac{56}{9}$

25. $\frac{1}{2}\left(e^4 - 1\right)$

26. $\frac{10\sqrt{10}}{3} \approx 10.54$

27. $e^x\left(x^2 - 2x + 2\right) + C$

28. $\frac{3}{4}\left(3e^4 + 1\right) \approx 123.60$

29. 39

30. 120π

31. Converges; 1

32. -4

33. **(a)** None **(b)** -3 at $(-2,-2)$ **(c)** None

34. 5488 when $x = 14$ and $y = 28$

35. $-.12$

36. $\frac{3}{2}e^4 - \frac{3}{2}e^2 \approx 70.81$

37. $y^2 = x + e^x + C$

38. $y = x^3 - \frac{7}{2}x^2 + 2x + 4$

39. $y = 4x^2 - 3x^3 + 20$

40. .5

41. **(a)** $\frac{3}{2}$ **(b)** .75 **(c)** .866

42. .0668

43. **(a)** $1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8}$
(b) .9704

44. $\frac{15}{2}$

45. .78

46. 6

47. $2x\sec^2\left(x^2+1\right)$

48. $x^2 \sin x\,(3\sin x + 2x\cos x)$

49. $\frac{1}{3}\tan(3x+1) + C$

50. $\frac{1}{64}$

FINAL EXAMINATION, FORM B

1. (a)	**11.** (c)	**21.** (a)	**31.** (a)	**41.** (b)
2. (d)	**12.** (d)	**22.** (c)	**32.** (d)	**42.** (b)
3. (a)	**13.** (b)	**23.** (a)	**33.** (c)	**43.** (a)
4. (c)	**14.** (a)	**24.** (b)	**34.** (c)	**44.** (d)
5. (b)	**15.** (b)	**25.** (a)	**35.** (b)	**45.** (a)
6. (c)	**16.** (c)	**26.** (d)	**36.** (b)	**46.** (b)
7. (c)	**17.** (c)	**27.** (b)	**37.** (a)	**47.** (a)
8. (c)	**18.** (b)	**28.** (d)	**38.** (b)	**48.** (b)
9. (b)	**19.** (a)	**29.** (c)	**39.** (c)	**49.** (a)
10. (a)	**20.** (a)	**30.** (b)	**40.** (c)	**50.** (d)

SOLUTIONS
TO
EVEN–NUMBERED
EXERCISES

Chapter R

ALGEBRA REFERENCE

R.1 Polynomials

2. $(-4y^2 - 3y + 8) - (2y^2 - 6y - 2)$
$= (-4y^2 - 3y + 8) + (-2y^2 + 6y + 2)$
$= -4y^2 - 3y + 8 - 2y^2 + 6y + 2$
$= (-4y^2 - 2y^2) + (-3y + 6y)$
$\quad + (8 + 2)$
$= -6y^2 + 3y + 10$

4. $2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5)$
$= (6r^2 + 8r + 4) + (3r^2 - 12r + 15)$
$= (6r^2 + 3r^2) + (8r - 12r)$
$\quad + (4 + 15)$
$= 9r^2 - 4r + 19$

6. $.83(5r^2 - 2r + 7) - (7.12r^2 + 6.423r - 2)$
$= (4.15r^2 - 1.66r + 5.81)$
$\quad + (-7.12r^2 - 6.423r + 2)$
$= (4.15r^2 - 7.12r^2)$
$\quad + (-1.66r - 6.423r) + (5.81 + 2)$
$= -2.97r^2 - 8.083r + 7.81$

8. $(6k - 1)(2k - 3)$
$= (6k)(2k) + (6k)(-3) + (-1)(2k)$
$\quad + (-1)(-3)$
$= 12k^2 - 18k - 2k + 3$
$= 12k^2 - 20k + 3$

10. $(9k + q)(2k - q)$
$= (9k)(2k) + (9k)(-q) + (q)(2k)$
$\quad + (q)(-q)$
$= 18k^2 - 9kq + 2kq - q^2$
$= 18k^2 - 7kq - q^2$

12. $\left(\frac{3}{4}r - \frac{2}{3}s\right)\left(\frac{5}{4}r + \frac{1}{3}s\right)$
$= \left(\frac{3}{4}r\right)\left(\frac{5}{4}r\right) + \left(\frac{3}{4}r\right)\left(\frac{1}{3}s\right) + \left(-\frac{2}{3}s\right)\left(\frac{5}{4}r\right)$
$\quad + \left(-\frac{2}{3}s\right)\left(\frac{1}{3}s\right)$
$= \frac{15}{16}r^2 + \frac{1}{4}rs - \frac{5}{6}rs - \frac{2}{9}s^2$
$= \frac{15}{16}r^2 - \frac{7}{12}rs - \frac{2}{9}s^2$

14. $(6m + 5)(6m - 5)$
$= (6m)(6m) + (6m)(-5) + (5)(6m)$
$\quad + (5)(-5)$
$= 36m^2 - 30m + 30m - 25$
$= 36m^2 - 25$

16. $(2p - 1)(3p^2 - 4p + 5)$
$= (2p)(3p^2) + (2p)(-4p) + (2p)(5)$
$\quad + (-1)(3p^2) + (-1)(-4p) + (-1)(5)$
$= 6p^3 - 8p^2 + 10p - 3p^2 + 4p - 5$
$= 6p^3 - 11p^2 + 14p - 5$

18. $(k + 2)(12k^3 - 3k^2 + k + 1)$
$= k(12k^3) + k(-3k^2) + k(k) + k(1)$
$\quad + 2(12k^3) + 2(-3k^2) + 2(k) + 2(1)$
$= 12k^4 - 3k^3 + k^2 + k + 24k^3 - 6k^2$
$\quad + 2k + 2$
$= 12k^4 + 21k^3 - 5k^2 + 3k + 2$

20. $(r - 3s + t)(2r - s + t)$
$= r(2r) + r(-s) + r(t) - 3s(2r)$
$\quad - 3s(-s) - 3s(t) + t(2r) + t(-s)$
$\quad + t(t)$
$= 2r^2 - rs + rt - 6rs + 3s^2 - 3st$
$\quad + 2rt - st + t^2$
$= 2r^2 - 7rs + 3s^2 + 3rt - 4st + t^2$

22. $(x - 1)(x + 2)(x - 3)$
$= [x(x + 2) + (-1)(x + 2)](x - 3)$
$= (x^2 + 2x - x - 2)(x - 3)$
$= (x^2 + x - 2)(x - 3)$
$= x^2(x - 3) + x(x - 3) + (-2)(x - 3)$
$= x^3 - 3x^2 + x^2 - 3x - 2x + 6$
$= x^3 - 2x^2 - 5x + 6$

24. $(x - 2y)^3$
$= [(x - 2y)(x - 2y)](x - 2y)$
$= (x^2 - 2xy - 2xy + 4y^2)(x - 2y)$
$= (x^2 - 4xy + 4y^2)(x - 2y)$
$= (x^2 - 4xy + 4y^2)x + (x^2 - 4xy + 4y^2)(-2y)$
$= x^3 - 4x^2y + 4xy^2 - 2x^2y + 8xy^2 - 8y^3$
$= x^3 - 6x^2y + 12xy^2 - 8y^3$

R.2 Factoring

2. $3y^3 + 24y^2 + 9y$
$= 3y \cdot y^2 + 3y \cdot 8y + 3y \cdot 3$
$= 3y(y^2 + 8y + 3)$

4. $60m^4 - 120m^3n + 50m^2n^2$
$= 10m^2 \cdot 6m^2 - 10m^2 \cdot 12mn + 10m^2 \cdot 5n^2$
$= 10m^2(6m^2 - 12mn + 5n^2)$

6. $x^2 + 4x - 5 = (x+5)(x-1)$
since $5(-1) = -5$ and $-1 + 5 = 4$.

8. $b^2 - 8b + 7 = (b-7)(b-1)$
since $(-7)(-1) = 7$ and $-7 + (-1) = -8$.

10. $s^2 + 2st - 35t^2 = (s-5t)(s+7t)$
since $(-5t)(7t) = -35t^2$ and $7t + (-5t) = 2t$.

12. $6a^2 - 48a - 120 = 6(a^2 - 8a - 20)$
$= 6(a-10)(a+2)$

14. $2x^2 - 5x - 3$

The possible factors of $2x^2$ are $2x$ and x and the possible factors of -3 are -3 and 1, or 3 and -1. Try various combinations until one works.

$$2x^2 - 5x - 3 = (2x+1)(x-3)$$

16. $2a^2 - 17a + 30 = (2a-5)(a-6)$

18. $21m^2 + 13mn + 2n^2 = (7m+2n)(3m+n)$

20. $32z^5 - 20z^4a - 12z^3a^2$
$= 4z^3(8z^2 - 5za - 3a^2)$
$= 4z^3(8z+3a)(z-a)$

22. $9m^2 - 25 = (3m)^2 - (5)^2$
$= (3m+5)(3m-5)$

24. $9x^2 + 64$ is the *sum* of two perfect squares. It cannot be factored. It is prime.

26. $m^2 - 6mn + 9n^2$
$= m^2 - 2(3mn) + (3n)^2$
$= (m-3n)^2$

28. $a^3 - 216$
$= a^3 - 6^3$
$= (a-6)[(a)^2 + (a)(6) + (6)^2]$
$= (a-6)(a^2 + 6a + 36)$

30. $64m^3 + 125$
$= (4m)^3 + 5^3$
$= (4m+5)[(4m)^2 - (4m)(5) + (5)^2]$
$= (4m+5)(16m^2 - 20m + 25)$

32. $16a^4 - 81b^4$
$= (4a^2)^2 - (9b^2)^2$
$= (4a^2 + 9b^2)(4a^2 - 9b^2)$
$= (4a^2 + 9b^2)[(2a)^2 - (3b)^2]$
$= (4a^2 + 9b^2)(2a+3b)(2a-3b)$

R.3 Rational Expressions

2. $\dfrac{25p^3}{10p^2} = \dfrac{5 \cdot 5 \cdot p \cdot p \cdot p}{2 \cdot 5 \cdot p \cdot p} = \dfrac{5p}{2}$

4. $\dfrac{3(t+5)}{(t+5)(t-3)} = \dfrac{3}{t-3}$

6. $\dfrac{36y^2 + 72y}{9y} = \dfrac{36y(y+2)}{9y}$
$= \dfrac{9 \cdot 4 \cdot y(y+2)}{9 \cdot y}$
$= 4(y+2)$

8. $\dfrac{r^2 - r - 6}{r^2 + r - 12} = \dfrac{(r-3)(r+2)}{(r+4)(r-3)}$
$= \dfrac{r+2}{r+4}$

10. $\dfrac{z^2 - 5z + 6}{z^2 - 4} = \dfrac{(z-3)(z-2)}{(z+2)(z-2)}$
$= \dfrac{z-3}{z+2}$

12. $\dfrac{6y^2 + 11y + 4}{3y^2 + 7y + 4} = \dfrac{(3y+4)(2y+1)}{(3y+4)(y+1)}$
$= \dfrac{2y+1}{y+1}$

14. $\dfrac{15p^3}{9p^2} \div \dfrac{6p}{10p^2} = \dfrac{15p^3}{9p^2} \cdot \dfrac{10p^2}{6p}$
$= \dfrac{150p^5}{54p^3}$
$= \dfrac{25 \cdot 6p^5}{9 \cdot 6p^3}$
$= \dfrac{25p^2}{9}$

16. $\dfrac{a-3}{16} \div \dfrac{a-3}{32} = \dfrac{a-3}{16} \cdot \dfrac{32}{a-3}$

$$= \frac{a-3}{16} \cdot \frac{16 \cdot 2}{a-3}$$
$$= \frac{2}{1}$$
$$= 2$$

18. $\dfrac{9y-18}{6y+12} \cdot \dfrac{3y+6}{15y-30}$

$$= \frac{9(y-2)}{6(y+2)} \cdot \frac{3(y+2)}{15(y-2)}$$
$$= \frac{27}{90} = \frac{3 \cdot 3}{10 \cdot 3} = \frac{3}{10}$$

20. $\dfrac{6r-18}{9r^2+6r-24} \cdot \dfrac{12r-16}{4r-12}$

$$= \frac{6(r-3)}{3(3r^2+2r-8)} \cdot \frac{4(3r-4)}{4(r-3)}$$
$$= \frac{6(r-3)}{3(3r-4)(r+2)} \cdot \frac{4(3r-4)}{4(r-3)}$$
$$= \frac{6}{3(r+2)}$$
$$= \frac{2}{r+2}$$

22. $\dfrac{m^2+3m+2}{m^2+5m+4} \div \dfrac{m^2+5m+6}{m^2+10m+24}$

$$= \frac{m^2+3m+2}{m^2+5m+4} \cdot \frac{m^2+10m+24}{m^2+5m+6}$$
$$= \frac{(m+1)(m+2)}{(m+4)(m+1)} \cdot \frac{(m+6)(m+4)}{(m+3)(m+2)}$$
$$= \frac{m+6}{m+3}$$

24. $\dfrac{6n^2-5n-6}{6n^2+5n-6} \cdot \dfrac{12n^2-17n+6}{12n^2-n-6}$

$$= \frac{(2n-3)(3n+2)}{(2n+3)(3n-2)} \cdot \frac{(3n-2)(4n-3)}{(3n+2)(4n-3)}$$
$$= \frac{2n-3}{2n+3}$$

26. $\dfrac{3}{p} + \dfrac{1}{2}$

Multiply the first term by $\frac{2}{2}$ and the second by $\frac{p}{p}$.

$$\frac{2 \cdot 3}{2 \cdot p} + \frac{p \cdot 1}{p \cdot 2} = \frac{6}{2p} + \frac{p}{2p}$$
$$= \frac{6+p}{2p}$$

28. $\dfrac{1}{6m} + \dfrac{2}{5m} + \dfrac{4}{m}$

$$= \frac{5 \cdot 1}{5 \cdot 6m} + \frac{6 \cdot 2}{6 \cdot 5m} + \frac{30 \cdot 4}{30 \cdot m}$$
$$= \frac{5}{30m} + \frac{12}{30m} + \frac{120}{30m}$$
$$= \frac{5+12+120}{30m}$$
$$= \frac{137}{30m}$$

30. $\dfrac{6}{r} - \dfrac{5}{r-2}$

$$= \frac{6(r-2)}{r(r-2)} - \frac{5r}{r(r-2)}$$
$$= \frac{6(r-2)-5r}{r(r-2)}$$
$$= \frac{6r-12-5r}{r(r-2)}$$
$$= \frac{r-12}{r(r-2)}$$

32. $\dfrac{2}{5(k-2)} + \dfrac{3}{4(k-2)}$

$$= \frac{4 \cdot 2}{4 \cdot 5(k-2)} + \frac{5 \cdot 3}{5 \cdot 4(k-2)}$$
$$= \frac{8}{20(k-2)} + \frac{15}{20(k-2)}$$
$$= \frac{8+15}{20(k-2)}$$
$$= \frac{23}{20(k-2)}$$

34. $\dfrac{2y}{y^2+7y+12} - \dfrac{y}{y^2+5y+6}$

$$= \frac{2y}{(y+4)(y+3)} - \frac{y}{(y+3)(y+2)}$$
$$= \frac{2y(y+2)}{(y+4)(y+3)(y+2)}$$
$$- \frac{y(y+4)}{(y+3)(y+2)(y+4)}$$
$$= \frac{2y(y+2)-y(y+4)}{(y+4)(y+3)(y+2)}$$
$$= \frac{2y^2+4y-y^2-4y}{(y+4)(y+3)(y+2)}$$
$$= \frac{y^2}{(y+4)(y+3)(y+2)}$$

36. $\dfrac{4m}{3m^2+7m-6} - \dfrac{m}{3m^2-14m+8}$

$$= \frac{4m}{(3m-2)(m+3)} - \frac{m}{(3m-2)(m-4)}$$

$$= \frac{4m(m-4)}{(3m-2)(m+3)(m-4)}$$

$$- \frac{m(m+3)}{(3m-2)(m-4)(m+3)}$$

$$= \frac{4m(m-4)-m(m+3)}{(3m-2)(m-4)(m+3)}$$

$$= \frac{4m^2-16m-m^2-3m}{(3m-2)(m+3)(m-4)}$$

$$= \frac{3m^2-19m}{(3m-2)(m+3)(m-4)}$$

$$= \frac{m(3m-19)}{(3m-2)(m+3)(m-4)}$$

38. $\dfrac{5x+2}{x^2-1} + \dfrac{3}{x^2+x} - \dfrac{1}{x^2-x}$

$$= \frac{5x+2}{(x+1)(x-1)} + \frac{3}{x(x+1)} - \frac{1}{x(x-1)}$$

$$= \left(\frac{x}{x}\right)\left(\frac{5x+2}{(x+1)(x-1)}\right) + \left(\frac{x-1}{x-1}\right)\left(\frac{3}{x(x+1)}\right)$$

$$- \left(\frac{x+1}{x+1}\right)\left(\frac{1}{x(x-1)}\right)$$

$$= \frac{x(5x+2)+(x-1)(3)-(x+1)(1)}{x(x+1)(x-1)}$$

$$= \frac{5x^2+2x+3x-3-x-1}{x(x+1)(x-1)}$$

$$= \frac{5x^2+4x-4}{x(x+1)(x-1)}$$

R.4 Equations

2. $\dfrac{5}{6}k - 2k + \dfrac{1}{3} = \dfrac{2}{3}$

Multiply both sides of the equation by 6.

$$6\left(\frac{5}{6}k\right) - 6(2k) + 6\left(\frac{1}{3}\right) = 6\left(\frac{2}{3}\right)$$

$$\begin{aligned} 5k - 12k + 2 &= 4 \\ -7k + 2 &= 4 \\ -7k &= 2 \\ k &= -\frac{2}{7} \end{aligned}$$

The solution is $-\frac{2}{7}$.

4.
$$\begin{aligned} 5x + 2 &= 8 - 3x \\ 8x + 2 &= 8 \\ 8x &= 6 \\ x &= \frac{3}{4} \end{aligned}$$

The solution is $\frac{3}{4}$.

6.
$$\begin{aligned} 5(a+3) + 4a - 5 &= -(2a-4) \\ 5a + 15 + 4a - 5 &= -2a + 4 \\ 9a + 10 &= -2a + 4 \\ 11a + 10 &= 4 \\ 11a &= -6 \\ a &= -\frac{6}{11} \end{aligned}$$

The solution is $-\frac{6}{11}$.

8.
$$\begin{aligned} 4[2p - (3-p) + 5] &= -7p - 2 \\ 4[2p - 3 + p + 5] &= -7p - 2 \\ 4[3p + 2] &= -7p - 2 \\ 12p + 8 &= -7p - 2 \\ 19p + 8 &= -2 \\ 19p &= -10 \\ p &= -\frac{10}{19} \end{aligned}$$

The solution is $-\frac{10}{19}$.

10.
$$\begin{aligned} x^2 &= 3 + 2x \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x - 3 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = 3 \quad &\text{or} \quad x = -1 \end{aligned}$$

The solutions are 3 and −1.

12.
$$\begin{aligned} 2k^2 - k &= 10 \\ 2k^2 - k - 10 &= 0 \\ (2k-5)(k+2) &= 0 \\ 2k - 5 = 0 \quad &\text{or} \quad k + 2 = 0 \\ k = \frac{5}{2} \quad &\text{or} \quad k = -2 \end{aligned}$$

The solutions are $\frac{5}{2}$ and −2.

14.
$$\begin{aligned} m(m-7) &= -10 \\ m^2 - 7m + 10 &= 0 \\ (m-5)(m-2) &= 0 \\ m - 5 = 0 \quad &\text{or} \quad m - 2 = 0 \\ m = 5 \quad &\text{or} \quad m = 2 \end{aligned}$$

The solutions are 5 and 2.

16.
$$z(2z+7)=4$$
$$2z^2+7z-4=0$$
$$(2z-1)(z+4)=0$$
$$2z-1=0 \quad \text{or} \quad z+4=0$$
$$z=\frac{1}{2} \quad \text{or} \quad z=-4$$

The solutions are $\frac{1}{2}$ and -4.

18. $3x^2-5x+1=0$

Use the quadratic formula.

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(3)(1)}}{2(3)}=\frac{5\pm\sqrt{25-12}}{6}$$
$$x=\frac{5+\sqrt{13}}{6} \quad \text{or} \quad x=\frac{5-\sqrt{13}}{6}$$
$$\approx 1.434 \qquad\qquad \approx .232$$

The solutions are $\frac{5+\sqrt{13}}{6}\approx 1.434$ and $\frac{5-\sqrt{13}}{6}\approx .232$.

20. $p^2+p-1=0$

$$p=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)}=\frac{-1\pm\sqrt{5}}{2}$$

The solutions are $\frac{-1+\sqrt{5}}{2}\approx .618$ and $\frac{-1-\sqrt{5}}{2}\approx -1.618$.

22. $2x^2+12x+5=0$

$$x=\frac{-12\pm\sqrt{(12)^2-4(2)(5)}}{2(2)}$$
$$=\frac{-12\pm\sqrt{104}}{4}=\frac{-12\pm\sqrt{4\cdot 26}}{4}$$
$$=\frac{-12\pm\sqrt{4}\sqrt{26}}{4}=\frac{-12\pm 2\sqrt{26}}{4}$$
$$=\frac{2(-6\pm\sqrt{26})}{2\cdot 2}=\frac{-6\pm\sqrt{26}}{2}$$

The solutions are $\frac{-6+\sqrt{26}}{2}\approx -.450$ and $\frac{-6-\sqrt{26}}{2}\approx -5.550$.

24. $2x^2-7x+30=0$

$$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(2)(30)}}{2(2)}$$
$$x=\frac{7\pm\sqrt{49-240}}{4}$$
$$x=\frac{7\pm\sqrt{-191}}{4}$$

Since there is a negative number under the radical sign, $\sqrt{-191}$ is not a real number. Thus, there are no real number solutions.

26. $5m^2+5m=0$

$$5m(m+1)=0$$
$$5m=0 \quad \text{or} \quad m+1=0$$
$$m=0 \quad \text{or} \quad m=-1$$

The solutions are 0 and -1.

28. $\frac{x}{3}-7=6-\frac{3x}{4}$

Multiply both sides by 12, the least common denominator of 3 and 4.

$$12\left(\frac{x}{3}-7\right)=12\left(6-\frac{3x}{4}\right)$$
$$12\left(\frac{x}{3}\right)-(12)(7)=(12)(6)-(12)\left(\frac{3x}{4}\right)$$
$$4x-84=72-9x$$
$$13x-84=72$$
$$13x=156$$
$$x=12$$

The solution is 12.

30. $\frac{5}{2p+3}-\frac{3}{p-2}=\frac{4}{2p+3}$

Multiply both sides by $(2p+3)(p-2)$. Note that $p\neq-\frac{3}{2}$ and $p\neq 2$.

$$(2p+3)(p-2)\left(\frac{5}{2p+3}-\frac{3}{p-2}\right)$$
$$=(2p+3)(p-2)\left(\frac{4}{2p+3}\right)$$
$$(2p+3)(p-2)\left(\frac{5}{2p+3}\right)-(2p+3)(p-2)\left(\frac{3}{p-2}\right)$$
$$=(2p+3)(p-2)\left(\frac{4}{2p+3}\right)$$
$$(p-2)(5)-(2p+3)(3)=(p-2)(4)$$
$$5p-10-6p-9=4p-8$$
$$-p-19=4p-8$$
$$-5p-19=-8$$
$$-5p=11$$
$$p=-\frac{11}{5}$$

The solutions is $-\frac{11}{5}$.

32. $\dfrac{2y}{y-1} = \dfrac{5}{y} + \dfrac{10-8y}{y^2-y}$

$$\frac{2y}{y-1} = \frac{5}{y} + \frac{10-8y}{y(y-1)}$$

Multiply both sides by $y(y-1)$.
Note that $y \neq 0$ and $y \neq 1$.

$$\begin{aligned}
y(y-1)\left(\frac{2y}{y-1}\right) &= y(y-1)\left[\frac{5}{y} + \frac{10-8y}{y(y-1)}\right] \\
y(y-1)\left(\frac{2y}{y-1}\right) &= y(y-1)\left(\frac{5}{y}\right) \\
&\quad + y(y-1)\left[\frac{10-8y}{y(y-1)}\right] \\
y(2y) &= (y-1)(5) + (10-8y) \\
2y^2 &= 5y - 5 + 10 - 8y \\
2y^2 &= 5 - 3y \\
2y^2 + 3y - 5 &= 0 \\
(2y+5)(y-1) &= 0 \\
2y + 5 = 0 \quad &\text{or} \quad y - 1 = 0 \\
y = -\frac{5}{2} \quad &\text{or} \quad y = 1
\end{aligned}$$

Since $y \neq 1$, 1 is not a solution.
The solution is $-\frac{5}{2}$.

34. $\dfrac{5}{a} + \dfrac{-7}{a+1} = \dfrac{a^2-2a+4}{a^2+a}$

$$a(a+1)\left(\frac{5}{a} + \frac{-7}{a+1}\right) = a(a+1)\left(\frac{a^2-2a+4}{a^2+a}\right)$$

Note that $a \neq 0$ and $a \neq -1$.

$$\begin{aligned}
5(a+1) + (-7)(a) &= a^2 - 2a + 4 \\
5a + 5 - 7a &= a^2 - 2a + 4 \\
5 - 2a &= a^2 - 2a + 4 \\
5 &= a^2 + 4 \\
0 &= a^2 - 1 \\
0 &= (a+1)(a-1) \\
a + 1 = 0 \quad \text{or} \quad & a - 1 = 0 \\
a = -1 \quad \text{or} \quad & a = 1
\end{aligned}$$

Since -1 would make two denominators zero, 1 is the only solution.

36. $\dfrac{2}{x^2-2x-3} + \dfrac{5}{x^2-x-6} = \dfrac{1}{x^2+3x+2}$

$$\frac{2}{(x-3)(x+1)} + \frac{5}{(x-3)(x+2)} = \frac{1}{(x+2)(x+1)}$$

Multiply both sides by $(x-3)(x+1)(x+2)$.
Note that $x \neq 3$, $x \neq -1$, and $x \neq -2$.

$$\begin{aligned}
&(x-3)(x+1)(x+2)\left(\frac{2}{(x-3)(x+1)}\right) \\
&\quad + (x-3)(x+1)(x+2)\left(\frac{5}{(x-3)(x+2)}\right) \\
&\quad = (x-3)(x+1)(x+2)\left(\frac{1}{(x+2)(x+1)}\right)
\end{aligned}$$

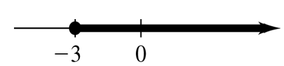
$$\begin{aligned}
2(x+2) + 5(x+1) &= x - 3 \\
2x + 4 + 5x + 5 &= x - 3 \\
7x + 9 &= x - 3 \\
6x + 9 &= -3 \\
6x &= -12 \\
x &= -2
\end{aligned}$$

However, $x \neq -2$. Therefore there is no solution.

R.5 Inequalities

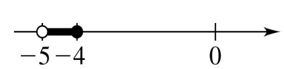
2. $x \geq -3$

Because the inequality sign means "greater than or equal to," the endpoint at -3 is included. This inequality is written in interval notation as $[-3, \infty)$. To graph this interval on a number line, place a solid circle at -3 and draw a heavy arrow pointing to the right.



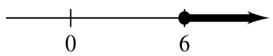
4. $-5 < x \leq -4$

The endpoint at -4 is included, but the endpoint at -5 is not. This inequality is written in interval notation as $(-5, -4]$. To graph this interval, place a an open circle at -5 and a closed circle at -4; then draw a heavy line segment between them.



6. $6 \le x$

This inequality may be written as $x \ge 6$, and is written in interval notation as $[6, \infty)$. Note that the endpoint at 6 is included. To graph this interval, place a closed circle at 6 and draw a heavy arrow pointing to the right.



8. $[2, 7)$

This represents all the numbers between 2 and 7, including 2 but not including 7. This interval can be written as the inequality $2 \le x < 7$.

10. $(3, \infty)$

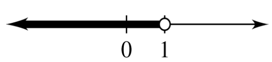
This represents all the numbers to the right of 3, and does not include the endpoint. This interval can be written as the inequality $x > 3$.

12. Notice that neither endpoint is included. The interval shown in the graph can be written as $0 < x < 8$.

14. Notice that the endpoint 0 is not included, but 3 is included. The interval shown in the graph can be written as $x < 0$ or $x \ge 3$.

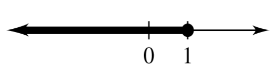
16.
$$\begin{aligned} 6k - 4 &< 3k - 1 \\ 6k &< 3k + 3 \\ 3k &< 3 \\ k &< 1 \end{aligned}$$

The solution in interval notation is $(-\infty, 1)$.



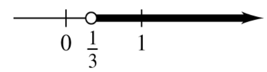
18.
$$\begin{aligned} -2(3y - 8) &\ge 5(4y - 2) \\ -6y + 16 &\ge 20y - 10 \\ -6y + 16 + (-16) &\ge 20y - 10 + (-16) \\ -6y &\ge 20y - 26 \\ -6y + (-20y) &\ge 20y + (-20y) - 26 \\ -26y &\ge -26 \\ -\frac{1}{26}(-26)y &\le -\frac{1}{26}(-26) \\ y &\le 1 \end{aligned}$$

The solution is $(-\infty, 1]$.



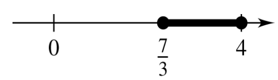
20.
$$\begin{aligned} x + 5(x + 1) &> 4(2 - x) + x \\ x + 5x + 5 &> 8 - 4x + x \\ 6x + 5 &> 8 - 3x \\ 6x &> 3 - 3x \\ 9x &> 3 \\ x &> \frac{1}{3} \end{aligned}$$

The solution is $\left(\frac{1}{3}, \infty\right)$.



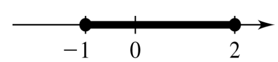
22.
$$\begin{aligned} 8 &\le 3r + 1 \le 13 \\ 8 + (-1) &\le 3r + 1 + (-1) \le 13 + (-1) \\ 7 &\le 3r \le 12 \\ \frac{1}{3}(7) &\le \frac{1}{3}(3r) \le \frac{1}{3}(12) \\ \frac{7}{3} &\le r \le 4 \end{aligned}$$

The solution is $\left[\frac{7}{3}, 4\right]$.



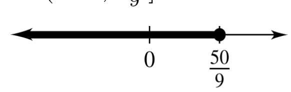
24.
$$\begin{aligned} -1 &\le \frac{5y + 2}{3} \le 4 \\ 3(-1) &\le 3\left(\frac{5y + 2}{3}\right) \le 3(4) \\ -3 &\le 5y + 2 \le 12 \\ -5 &\le 5y \le 10 \\ -1 &\le y \le 2 \end{aligned}$$

The solution is $[-1,\ 2]$.



26.
$$\begin{aligned} \frac{8}{3}(z - 4) &\le \frac{2}{9}(3z + 2) \\ (9)\frac{8}{3}(z - 4) &\le (9)\frac{2}{9}(3z + 2) \\ 24(z - 4) &\le 2(3z + 2) \\ 24z - 96 &\le 6z + 4 \\ 24z &\le 6z + 100 \\ 18z &\le 100 \\ z &\le \frac{100}{18} \\ z &\le \frac{50}{9} \end{aligned}$$

The solution is $(-\infty,\ \frac{50}{9}]$.



28. $(t+6)(t-1) \geq 0$

Solve $(t+6)(t-1) = 0$.

$$(t+6)(t-1) = 0$$
$$t = -6 \quad \text{or} \quad t = 1$$

Intervals: $(-\infty, -6),\ (-6, 1),\ (1, \infty)$

For $(-\infty, -6)$, choose -7 to test for t.

$$(-7+6)(-7-1) = (-1)(-8) = 8 \geq 0$$

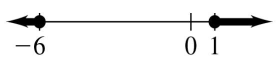
For $(-6,\ 1)$, choose 0.

$$(0+6)(0-1) = (6)(-1) = -6 \ngeq 0$$

For $(1,\ \infty)$, choose 2.

$$(2+6)(2-1) = (8)(1) = 8 \geq 0$$

Because the symbol $\geq$ is used, the endpoints -6 and 1 are included in the solution, $(-\infty, -6] \cup [1, \infty)$.



30. $2k^2 + 7k - 4 > 0$

Solve $2k^2 + 7k - 4 = 0$.

$$2k^2 + 7k - 4 = 0$$
$$(2k-1)(k+4) = 0$$
$$k = \frac{1}{2} \quad \text{or} \quad k = -4$$

Intervals: $(-\infty, -4), \left(-4, \frac{1}{2}\right) \left(\frac{1}{2}, \infty\right)$

For $(-\infty, -4)$, choose -5.

$$2(-5)^2 + 7(-5) - 4 = 11 > 0$$

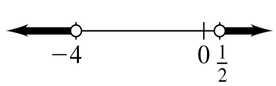
For $\left(-4, \frac{1}{2}\right)$, choose 0.

$$2(0)^2 + 7(0) - 4 = -4 \not> 0$$

For $\left(\frac{1}{2}, \infty\right)$, choose 1.

$$2(1)^2 + 7(1) - 4 = 5 > 0$$

The solution is $(-\infty, -4) \cup \left(\frac{1}{2}, \infty\right)$.



32. $2k^2 - 7k - 15 \leq 0$

Solve $2k^2 - 7k - 15 = 0$.

$$2k^2 - 7k - 15 = 0$$
$$(2k+3)(k-5) = 0$$
$$k = -\frac{3}{2} \quad \text{or} \quad k = 5$$

Intervals: $\left(-\infty, -\frac{3}{2}\right), \left(-\frac{3}{2}, 5\right), (5, \infty)$

For $\left(-\infty, -\frac{3}{2}\right)$, choose -2.

$$2(-2)^2 - 7(-2) - 15 = 7 \nleq 0$$

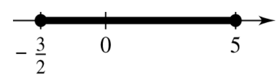
For $\left(-\frac{3}{2}, 5\right)$, choose 0.

$$2(0)^2 - 7(0) - 15 = -15 \leq 0$$

For $(5, \infty)$, choose 6.

$$2(6)^2 - 7(6) - 15 \nleq 0$$

The solution is $\left[-\frac{3}{2}, 5\right]$.



34. $10r^2 + r \leq 2$

Solve $10r^2 + r = 2$.

$$10r^2 + r = 2$$
$$10r^2 + r - 2 = 0$$
$$(5r-2)(2r+1) = 0$$
$$r = \frac{2}{5} \quad \text{or} \quad r = -\frac{1}{2}$$

Intervals: $\left(-\infty, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{2}{5}\right), \left(\frac{2}{5}, \infty\right)$

For $\left(-\infty, -\frac{1}{2}\right)$, choose -1.

$$10(-1)^2 + (-1) = 9 \nleq 2$$

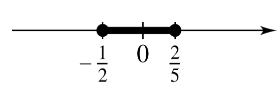
For $\left(-\frac{1}{2}, \frac{2}{5}\right)$, choose 0.

$$10(0)^2 + 0 = 0 \leq 2$$

For $\left(\frac{2}{5}, \infty\right)$, choose 1.

$$10(1)^2 + 1 = 11 \nleq 2$$

The solution is $\left[-\frac{1}{2}, \frac{2}{5}\right]$.



36. $3a^2 + a > 10$

Solve $3a^2 + a = 10$.

$$\begin{aligned} 3a^2 + a &= 10 \\ 3a^2 + a - 10 &= 0 \\ (3a-5)(a+2) &= 0 \end{aligned}$$

$$a = \frac{5}{3} \quad \text{or} \quad a = -2$$

Intervals: $(-\infty, -2), \left(-2, \frac{5}{3}\right), \left(\frac{5}{3}, \infty\right)$

For $(-\infty, -2)$, choose -3.

$$3(-3)^2 + (-3) = 24 > 10$$

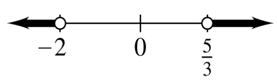
For $\left(-2, \frac{5}{3}\right)$, choose 0.

$$3(0)^2 + 0 = 0 \ngtr 10$$

For $\left(\frac{5}{3}, \infty\right)$, choose 2.

$$3(2)^2 + 2 = 14 > 10$$

The solution is $(-\infty, -2) \cup \left(\frac{5}{3}, \infty\right)$.



38. $p^2 - 16p > 0$

Solve $\text{p}^2 - 16p = 0$.

$$\begin{aligned} p^2 - 16p &= 0 \\ p(p-16) &= 0 \end{aligned}$$

$$p = 0 \quad \text{or} \quad p = 16$$

Intervals: $(-\infty, 0),\ (0, 16),\ (16, \infty)$

For $(-\infty, 0)$, choose -1.

$$(-1)^2 - 16(-1) = 17 > 0$$

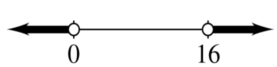
For $(0, 16)$, choose 1.

$$(1)^2 - 16(1) = -15 \ngtr 0$$

For $(16, \infty)$, choose 17.

$$(17)^2 - 16(17) = 17 > 0$$

The solution is $(-\infty, 0) \cup (16, \infty)$.



40. $\dfrac{r+1}{r-1} > 0$

Solve the equation $\dfrac{r+1}{r-1} = 0$.

$$\begin{aligned} \frac{r+1}{r-1} &= 0 \\ (r-1)\frac{r+1}{r-1} &= (r-1)(0) \\ r+1 &= 0 \\ r &= -1 \end{aligned}$$

Find the value for which the denominator equals zero.

$$\begin{aligned} r - 1 &= 0 \\ r &= 1 \end{aligned}$$

Intervals: $(-\infty, -1),\ (-1, 1),\ (1, \infty)$

For $(-\infty, -1)$, choose -2.

$$\frac{-2+1}{-2-1} = \frac{-1}{-3} = \frac{1}{3} > 0$$

For $(-1, 1)$, choose 0.

$$\frac{0+1}{0-1} = \frac{1}{-1} = -1 \ngtr 0$$

For $(1, \infty)$, choose 2.

$$\frac{2+1}{2-1} = \frac{3}{1} = 3 > 0$$

The solution is $(-\infty, -1) \cup (1, \infty)$.

42. $\dfrac{a-5}{a+2} < -1$

Solve the equation $\dfrac{a-5}{a+2} = -1$.

$$\begin{aligned} \frac{a-5}{a+2} &= -1 \\ a - 5 &= -1(a+2) \\ a - 5 &= -a - 2 \\ 2a &= 3 \\ a &= \frac{3}{2} \end{aligned}$$

Set the denominator equal to zero and solve for a.

$$\begin{aligned} a + 2 &= 0 \\ a &= -2 \end{aligned}$$

Intervals: $(-\infty, -2), \left(-2, \frac{3}{2}\right), \left(\frac{3}{2}, \infty\right)$

For $(-\infty, -2)$, choose -3.

$$\frac{-3-5}{-3+2} = \frac{-8}{-1} = 8 \not< -1$$

For $\left(-2, \frac{3}{2}\right)$, choose 0.

$$\frac{0-5}{0+2} = \frac{-5}{2} = -\frac{5}{2} < -1$$

For $\left(\frac{3}{2}, \infty\right)$, choose 2.

$$\frac{2-5}{2+2} = \frac{-3}{4} = -\frac{3}{4} \not< -1$$

The solution is $\left(-2, \frac{3}{2}\right)$.

44. $\dfrac{a+2}{3+2a} \le 5$

For the equation $\dfrac{a+2}{3+2a} = 5$.

$$\begin{aligned} \frac{a+2}{3+2a} &= 5 \\ a+2 &= 5(3+2a) \\ a+2 &= 15+10a \\ -9a &= 13 \\ a &= -\frac{13}{9} \end{aligned}$$

Set the denominator equal to zero and solve for a.

$$\begin{aligned} 3+2a &= 0 \\ 2a &= -3 \\ a &= -\frac{3}{2} \end{aligned}$$

Intervals: $\left(-\infty, -\frac{3}{2}\right), \left(-\frac{3}{2}, -\frac{13}{9}\right), \left(-\frac{13}{9}, \infty\right)$

For $\left(-\infty, -\frac{3}{2}\right)$, choose -2.

$$\frac{-2+2}{3+2(-2)} = \frac{0}{-1} = 0 \le 5$$

For $\left(-\frac{3}{2}, -\frac{13}{9}\right)$, choose -1.46.

$$\frac{-1.46+2}{3+2(-1.46)} = \frac{.54}{.08} = 6.75 \not\le 5$$

For $\left(-\frac{13}{9}, \infty\right)$, choose 0.

$$\frac{0+2}{3+2(0)} = \frac{2}{3} \le 5$$

The value $-\frac{3}{2}$ cannot be included in the solution since it would make the denominator zero. The solution is $\left(-\infty, -\frac{3}{2}\right) \cup \left[-\frac{13}{9}, \infty\right)$.

46. $\dfrac{5}{p+1} > \dfrac{12}{p+1}$

Solve the equation $\dfrac{5}{p+1} = \dfrac{12}{p+1}$.

$$\begin{aligned} \frac{5}{p+1} &= \frac{12}{p+1} \\ 5 &= 12 \end{aligned}$$

The equation has no solution.
Set the denominator equal to zero and solve for p.

$$\begin{aligned} p+1 &= 0 \\ p &= -1 \end{aligned}$$

Intervals: $(-\infty, -1),\ (-1, \infty)$
For $(-\infty, -1)$, choose -2.

$\dfrac{5}{-2+1} = -5$ and $\dfrac{12}{-2+1} = -12$, so

$$\frac{5}{-2+1} > \frac{12}{-2+1}.$$

For $(-1, \infty)$, choose 0.

$\dfrac{5}{0+1} = 5$ and $\dfrac{12}{0+1} = 12$, so

$$\frac{5}{0+1} \not> \frac{12}{0+1}.$$

The solution is $(-\infty, -1)$.

48. $\dfrac{8}{p^2+2p} > 1$

Solve the equation $\dfrac{8}{p^2+2p} = 1$.

$$\begin{aligned} \frac{8}{p^2+2p} &= 1 \\ 8 &= p^2+2p \\ 0 &= p^2+2p-8 \\ 0 &= (p+4)(p-2) \end{aligned}$$

$$p+4=0 \quad \text{or} \quad p-2=0$$
$$p=-4 \quad \text{or} \quad p=2$$

Set the denominator equal to zero and solve for p.

$$\begin{aligned} p^2+2p &= 0 \\ p(p+2) &= 0 \end{aligned}$$

$$p=0 \quad \text{or} \quad p+2=0$$
$$p=-2$$

Intervals: $(-\infty, -4),\ (-4, -2),\ (-2, 0),\ (0, 2),\ (2, \infty)$

For $(-\infty, -4)$, choose -5.

$$\frac{8}{(-5)^2 + 2(-5)} = \frac{8}{15} \not> 1$$

For $(-4, -2)$, choose -3.

$$\frac{8}{(-3)^2 + 2(-3)} = \frac{8}{9-6} = \frac{8}{3} > 1$$

For $(-2, 0)$, choose -1.

$$\frac{8}{(-1)^2 + 2(-1)} = \frac{8}{-1} = -8 \not> 1$$

For $(0, 2)$, choose 1.

$$\frac{8}{(1)^2 + 2(1)} = \frac{8}{3} > 1$$

For $(2, \infty)$, choose 3.

$$\frac{8}{(3)^2 + (2)(3)} = \frac{8}{15} \not> 1$$

The solution is $(-4, -2) \cup (0, 2)$.

50. $\dfrac{a^2 + 2a}{a^2 - 4} \le 2$

Solve the equation $\dfrac{a^2 + 2a}{a^2 - 4} = 2$.

$$\begin{aligned} \frac{a^2 + 2a}{a^2 - 4} &= 2 \\ a^2 + 2a &= 2(a^2 - 4) \\ a^2 + 2a &= 2a^2 - 8 \\ 0 &= a^2 - 2a - 8 \\ 0 &= (a-4)(a+2) \\ a - 4 = 0 \quad &\text{or} \quad a + 2 = 0 \\ a = 4 \quad &\text{or} \quad a = -2 \end{aligned}$$

But -2 is not a possible solution.
Set the denominator equal to zero and solve for a.

$$\begin{aligned} a^2 - 4 &= 0 \\ (a+2)(a-2) &= 0 \\ a + 2 = 0 \quad &\text{or} \quad a - 2 = 0 \\ a = -2 \quad &\text{or} \quad a = 2 \end{aligned}$$

Intervals: $(-\infty, -2)$, $(-2, 2)$, $(2, 4)$, $(4, \infty)$

For $(-\infty, -2)$, choose -3.

$$\frac{(-3)^2 + 2(-3)}{(-3)^2 - 4} = \frac{9-6}{9-4} = \frac{3}{5} \le 2$$

For $(-2, 2)$, choose 0.

$$\frac{(0)^2 + 2(0)}{0 - 4} = \frac{0}{-4} = 0 \le 2$$

For $(2, 4)$, choose 3.

$$\frac{(3)^2 + 2(3)}{(3)^2 - 4} = \frac{9+6}{9-5} = \frac{15}{4} \not\le 2$$

For $(4, \infty)$, choose 5.

$$\frac{(5)^2 + 2(5)}{(5)^2 - 4} = \frac{25+10}{25-4} = \frac{35}{21} \le 2$$

The value 4 will satisfy the original inequality, but the values -2 and 2 will not since they make the denominator zero. The solution is $(-\infty, -2) \cup (-2, 2) \cup [4, \infty)$.

R.6 Exponents

2. $3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$

4. $(-12)^0 = 1$, by definition.

6. $-(-3^{-2}) = -\left(-\dfrac{1}{3^2}\right) = -\left(-\dfrac{1}{9}\right) = \dfrac{1}{9}$

8. $\left(\dfrac{4}{3}\right)^{-3} = \dfrac{1}{\left(\frac{4}{3}\right)^3} = \dfrac{1}{\frac{64}{27}} = \dfrac{27}{64}$

10. $\dfrac{8^9 \cdot 8^{-7}}{8^{-3}} = 8^{9+(-7)-(-3)} = 8^{9-7+3} = 8^5$

12. $$\begin{aligned} \left(\frac{5^{-6} \cdot 5^3}{5^{-2}}\right)^{-1} &= (5^{-6+3-(-2)})^{-1} \\ &= (5^{-6+3+2})^{-1} = (5^{-1})^{-1} \\ &= 5^{(-1)(-1)} = 5^1 = 5 \end{aligned}$$

14. $\dfrac{y^9 y^7}{y^{13}} = y^{9+7-13} = y^3$

16. $$\begin{aligned} \frac{(3z^2)^{-1}}{z^5} &= \frac{3^{-1}(z^2)^{-1}}{z^5} = \frac{3^{-1}z^{2(-1)}}{z^5} \\ &= \frac{3^{-1}z^{-2}}{z^5} = 3^{-1}z^{-2-5} \\ &= 3^{-1}z^{-7} = \frac{1}{3} \cdot \frac{1}{z^7} = \frac{1}{3z^7} \end{aligned}$$

18. $\frac{5^{-2}m^2y^{-2}}{5^2m^{-1}y^{-2}} = \frac{5^{-2}}{5^2} \cdot \frac{m^2}{m^{-1}} \cdot \frac{y^{-2}}{y^{-2}}$

$= 5^{-2-2}m^{2-(-1)}y^{-2-(-2)}$

$= 5^{-2-2}m^{2+1}y^{-2+2}$

$= 5^{-4}m^3y^0 = \frac{1}{5^4} \cdot m^3 \cdot 1$

$= \frac{m^3}{5^4}$

20. $\left(\frac{2c^2}{d^3}\right)^{-2} = \frac{2^{-2}(c^2)^{-2}}{(d^3)^{-2}}$

$= \frac{2^{-2}c^{(2)(-2)}}{d^{(3)(-2)}} = \frac{2^{-2}c^{-4}}{d^{-6}}$

$= \frac{d^6}{2^2c^4}$

22. $\left(\frac{a^{-7}b^{-1}}{b^{-4}a^2}\right)^{1/3} = \left(a^{-7-2}b^{-1-(-4)}\right)^{1/3}$

$= \left(a^{-9}b^3\right)^{1/3}$

$= \left(a^{-9}\right)^{1/3}\left(b^3\right)^{1/3}$

$= a^{-3}b^1$

$= \frac{b}{a^3}$

24. $b^{-2} - a = \frac{1}{b^2} - a$

$= \frac{1}{b^2} - a\left(\frac{b^2}{b^2}\right)$

$= \frac{1}{b^2} - \frac{ab^2}{b^2}$

$= \frac{1-ab^2}{b^2}$

26. $\left(\frac{m}{3}\right)^{-1} + \left(\frac{n}{2}\right)^{-2} = \left(\frac{3}{m}\right)^1 + \left(\frac{2}{n}\right)^2$

$= \frac{3}{m} + \frac{4}{n^2}$

$= \left(\frac{3}{m}\right)\left(\frac{n^2}{n^2}\right) + \left(\frac{4}{n^2}\right)\left(\frac{m}{m}\right)$

$= \frac{3n^2}{mn^2} + \frac{4m}{mn^2}$

$= \frac{3n^2+4m}{mn^2}$

28. $\left(x^{-2}+y^{-2}\right)^{-2} = \left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-2}$

$= \left[\left(\frac{1}{x^2}\right)\left(\frac{y^2}{y^2}\right) + \left(\frac{x^2}{x^2}\right)\left(\frac{1}{y^2}\right)\right]^{-2}$

$= \left(\frac{y^2}{x^2y^2} + \frac{x^2}{x^2y^2}\right)^{-2}$

$= \left(\frac{y^2+x^2}{x^2y^2}\right)^{-2} = \left(\frac{x^2y^2}{y^2+x^2}\right)^2$

$= \frac{(x^2)^2(y^2)^2}{(x^2+y^2)^2} = \frac{x^4y^4}{(x^2+y^2)^2}$

30. $27^{1/3} = \sqrt[3]{27} = 3$

32. $-125^{2/3} = -(125^{1/3})^2 = -5^2 = -25$

34. $\left(\frac{64}{27}\right)^{1/3} = \frac{64^{1/3}}{27^{1/3}} = \frac{4}{3}$

36. $625^{-1/4} = \frac{1}{625^{1/4}} = \frac{1}{5}$

38. $\left(\frac{121}{100}\right)^{-3/2} = \frac{1}{\left(\frac{121}{100}\right)^{3/2}} = \frac{1}{\left[\left(\frac{121}{100}\right)^{1/2}\right]^3}$

$= \frac{1}{\left(\frac{11}{10}\right)^3} = \frac{1}{\frac{1331}{1000}} = \frac{1000}{1331}$

40. $27^{2/3} \cdot 27^{-1/3} = 27^{(2/3)+(-1/3)}$

$= 27^{2/3-1/3}$

$= 27^{1/3}$

42. $\frac{3^{-5/2} \cdot 3^{3/2}}{3^{7/2} \cdot 3^{-9/2}}$

$= 3^{(-5/2)+(3/2)-(7/2)-(-9/2)}$

$= 3^{-5/2+3/2-7/2+9/2}$

$= 3^0 = 1$

44. $\frac{12^{3/4} \cdot 12^{5/4} \cdot y^{-2}}{12^{-1} \cdot (y^{-3})^{-2}} = \frac{12^{3/4+5/4} \cdot y^{-2}}{12^{-1} \cdot y^{(-3)(-2)}}$

$= \frac{12^{8/4} \cdot y^{-2}}{12^{-1} \cdot y^6}$

$= \frac{12^2 \cdot y^{-2}}{12^{-1}y^6}$

$= 12^{2-(-1)} \cdot y^{-2-6}$

$= 12^3y^{-8}$

$= \frac{12^3}{y^8}$

46. $\dfrac{8p^{-3}(4p^2)^{-2}}{p^{-5}} = \dfrac{8p^{-3}\cdot 4^{-2}p^{(2)(-2)}}{p^{-5}}$

$= \dfrac{8p^{-3}4^{-2}p^{-4}}{p^{-5}}$

$= 8\cdot 4^{-2}p^{(-3)+(-4)-(-5)}$

$= 8\cdot 4^{-2}p^{-3-4+5}$

$= 8\cdot 4^{-2}p^{-2}$

$= 8\cdot \dfrac{1}{4^2}\cdot\dfrac{1}{p^2}$

$= 8\cdot \dfrac{1}{16}\cdot\dfrac{1}{p^2}$

$= \dfrac{8}{16p^2} = \dfrac{1}{2p^2}$

48. $\dfrac{x^{1/3}\cdot y^{2/3}\cdot z^{1/4}}{x^{5/3}\cdot y^{-1/3}\cdot z^{3/4}}$

$= x^{1/3-(5/3)}y^{(2/3)-(-1/3)}z^{1/4-(3/4)}$

$= x^{1/3-5/3}y^{2/3+1/3}z^{1/4-3/4}$

$= x^{-4/3}y^{3/3}z^{-2/4}$

$= \dfrac{y}{x^{4/3}z^{2/4}}$

$= \dfrac{y}{x^{4/3}z^{1/2}}$

50. $\dfrac{m^{7/3}\cdot n^{-2/5}\cdot p^{3/8}}{m^{-2/3}\cdot n^{3/5}\cdot p^{-5/8}}$

$= m^{7/3-(-2/3)}n^{-2/5-(3/5)}p^{3/8-(-5/8)}$

$= m^{7/3+2/3}n^{-2/5-3/5}p^{3/8+5/8}$

$= m^{9/3}n^{-5/5}p^{8/8}$

$= m^3n^{-1}p^1 = \dfrac{m^3p}{n}$

52. $6x(x^3+7)^2 - 6x^2(3x^2+5)(x^3+7)$

$= 6x(x^3+7)(x^3+7) - 6x(x)(3x^2+5)(x^3+7)$

$= 6x(x^3+7)[(x^3+7) - x(3x^2+5)]$

$= 6x(x^3+7)(x^3+7-3x^3-5x)$

$= 6x(x^3+7)(-2x^3-5x+7)$

54. $9(6x+2)^{1/2} + 3(9x-1)(6x+2)^{-1/2}$

$= 3\cdot 3(6x+2)^{-1/2}(6x+2)^1$

$\quad + 3(9x-1)(6x+2)^{-1/2}$

$= 3(6x+2)^{-1/2}[3(6x+2)+(9x-1)]$

$= 3(6x+2)^{-1/2}(18x+6+9x-1)$

$= 3(6x+2)^{-1/2}(27x+5)$

56. $(4x^2+1)^2(2x-1)^{-1/2} + 16x(4x^2+1)(2x-1)^{1/2}$

$= (4x^2+1)(4x^2+1)(2x-1)^{-1/2}$

$\quad + 16x(4x^2+1)(2x-1)^{-1/2}(2x-1)$

$= (4x^2+1)(2x-1)^{-1/2}$

$\quad \cdot [(4x^2+1) + 16x(2x-1)]$

$= (4x^2+1)(2x-1)^{-1/2}(4x^2+1+32x^2-16x)$

$= (4x^2+1)(2x-1)^{-1/2}(36x^2-16x+1)$

R.7 Radicals

2. $\sqrt[4]{1296} = \sqrt[4]{6^4} = 6$

4. $\sqrt{50} = \sqrt{25\cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

6. $\sqrt{32y^5} = \sqrt{(16y^4)(2y)}$

$= \sqrt{16y^4}\sqrt{2y}$

$= 4y^2\sqrt{2y}$

8. $4\sqrt{3} - 5\sqrt{12} + 3\sqrt{75}$

$= 4\sqrt{3} - 5(\sqrt{4}\sqrt{3}) + 3(\sqrt{25}\sqrt{3})$

$= 4\sqrt{3} - 5(2\sqrt{3}) + 3(5\sqrt{3})$

$= 4\sqrt{3} - 10\sqrt{3} + 15\sqrt{3}$

$= (4-10+15)\sqrt{3} = 9\sqrt{3}$

10. $3\sqrt{28} - 4\sqrt{63} + \sqrt{112}$

$= 3(\sqrt{4}\sqrt{7}) - 4(\sqrt{9}\sqrt{7}) + (\sqrt{16}\sqrt{7})$

$= 3(2\sqrt{7}) - 4(3\sqrt{7}) + (4\sqrt{7})$

$= 6\sqrt{7} - 12\sqrt{7} + 4\sqrt{7}$

$= (6-12+4)\sqrt{7}$

$= -2\sqrt{7}$

12. $2\sqrt[3]{3} + 4\sqrt[3]{24} - \sqrt[3]{81}$

$= 2\sqrt[3]{3} + 4\sqrt[3]{8\cdot 3} - \sqrt[3]{27\cdot 3}$

$= 2\sqrt[3]{3} + 4(2)\sqrt[3]{3} - 3\sqrt[3]{3}$

$= 2\sqrt[3]{3} + 8\sqrt[3]{3} - 3\sqrt[3]{3}$

$= 7\sqrt[3]{3}$

14. $\sqrt{2x^3y^2z^4} = \sqrt{x^2y^2z^4\cdot 2x}$

$= xyz^2\sqrt{2x}$

16. $\sqrt[3]{16x^8y^4z^5} = \sqrt[3]{8x^6y^3z^3\cdot 2x^2yz^2}$

$= 2x^2yz\sqrt[3]{2x^2yz^2}$

18. $\sqrt{a^3b^5} - 2\sqrt{a^7b^3} + \sqrt{a^3b^9}$

$= \sqrt{a^2b^4ab} - 2\sqrt{a^6b^2ab} + \sqrt{a^2b^8ab}$

$= ab^2\sqrt{ab} - 2a^3b\sqrt{ab} + ab^4\sqrt{ab}$

$= (ab^2 - 2a^3b + ab^4)\sqrt{ab}$

$= ab\sqrt{ab}(b - 2a^2 + b^3)$

20. $\dfrac{5}{\sqrt{7}} = \dfrac{5}{\sqrt{7}} \cdot \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{5\sqrt{7}}{7}$

22. $$\begin{aligned} \frac{-3}{\sqrt{12}} &= \frac{-3}{\sqrt{4\cdot 3}} \\ &= \frac{-3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{-3\sqrt{3}}{6} \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

24. $$\begin{aligned} \frac{3}{1-\sqrt{5}} &= \frac{3}{1-\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} \\ &= \frac{3(1+\sqrt{5})}{1-5} \\ &= \frac{-3(1+\sqrt{5})}{4} \end{aligned}$$

26. $$\begin{aligned} \frac{-2}{\sqrt{3}-\sqrt{2}} &= \frac{-2}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} \\ &= \frac{-2(\sqrt{3}+\sqrt{2})}{3-2} \\ &= \frac{-2(\sqrt{3}+\sqrt{2})}{1} \\ &= -2(\sqrt{3}+\sqrt{2}) \end{aligned}$$

28. $$\begin{aligned} \frac{1}{\sqrt{r}-\sqrt{3}} &= \frac{1}{\sqrt{r}-\sqrt{3}} \cdot \frac{\sqrt{r}+\sqrt{3}}{\sqrt{r}+\sqrt{3}} \\ &= \frac{\sqrt{r}+\sqrt{3}}{r-3} \end{aligned}$$

30. $$\begin{aligned} \frac{y-5}{\sqrt{y}-\sqrt{5}} &= \frac{y-5}{\sqrt{y}-\sqrt{5}} \cdot \frac{\sqrt{y}+\sqrt{5}}{\sqrt{y}+\sqrt{5}} \\ &= \frac{(y-5)(\sqrt{y}+\sqrt{5})}{y-5} \\ &= \sqrt{y}+\sqrt{5} \end{aligned}$$

32. $$\begin{aligned} \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} &= \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} \cdot \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}} \\ &= \frac{x+2\sqrt{x(x+1)}+(x+1)}{x-(x+1)} \\ &= \frac{2x+2\sqrt{x(x+1)}+1}{-1} \\ &= -2x-2\sqrt{x(x+1)}-1 \end{aligned}$$

34. $$\begin{aligned} \frac{1+\sqrt{2}}{2} &= \frac{(1+\sqrt{2})(1-\sqrt{2})}{2(1-\sqrt{2})} \\ &= \frac{1-2}{2(1-\sqrt{2})} \\ &= -\frac{1}{2(1-\sqrt{2})} \end{aligned}$$

36. $$\begin{aligned} &\frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} \\ &= \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} \cdot \frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} \\ &= \frac{x-(x+1)}{x-2\sqrt{x}\cdot\sqrt{x+1}+(x+1)} \\ &= \frac{-1}{2x-2\sqrt{x(x+1)}+1} \end{aligned}$$

38. $$\begin{aligned} &\sqrt{16-8x+x^2} \\ &= \sqrt{(4-x)^2} \\ &= |4-x| \end{aligned}$$

Since $\sqrt{}$ denotes the nonnegative root, we must have $4-x \geq 0$.

40. $\sqrt{4-25z^2} = \sqrt{(2+5z)(2-5z)}$

This factorization does not produce a perfect square, so the expression $\sqrt{4-25z^2}$ cannot be simplified.

Chapter 1

LINEAR FUNCTIONS

1.1 Slopes and Equations of Lines

2. Find the slope of the line through $(5,-4)$ and $(1,3)$.

$$\begin{aligned} m &= \frac{3-(-4)}{1-5} \\ &= \frac{3+4}{-4} = -\frac{7}{4} \end{aligned}$$

4. Find the slope of the line through $(1,5)$ and $(-2,5)$.

$$m = \frac{5-5}{-2-1} = \frac{0}{-3} = 0$$

6. $y = 3x - 2$

This equation is in slope-intercept form, $y = mx + b$. Thus, the coefficient of the x-term, 3, is the slope.

8. $4x + 7y = 1$

Rewrite the equation in slope-intercept form.

$$\begin{aligned} 7y &= 1 - 4x \\ \frac{1}{7}(7y) &= \frac{1}{7}(1) - \frac{1}{7}(4x) \\ y &= \frac{1}{7} - \frac{4}{7}x \\ y &= -\frac{4}{7}x + \frac{1}{7} \end{aligned}$$

The slope is $-\frac{4}{7}$.

10. The x-axis is the horizontal line $y = 0$. Horizontal lines have a slope of 0.

12. $y = -4$

By rewriting this equation in the slope-intercept form, $y = mx + b$, we get $y = 0x - 4$, with the slope, m, being 0.

14. Find the slope of a line perpendicular to $6x = y - 3$.
First, rewrite the given equation in slope-intercept form.

$$\begin{aligned} 6x &= y - 3 \\ 6x + 3 &= y \\ \text{or} \qquad y &= 6x + 3 \end{aligned}$$

The slope of this line is 6.
Let m be the slope of any line perpendicular to the given line. Then

$$\begin{aligned} 6m &= -1 \\ m &= -\frac{1}{6}. \end{aligned}$$

16. The line goes through $(2,4)$, with slope $m = -1$. Use point-slope form.

$$\begin{aligned} y - 4 &= -1(x-2) \\ y - 4 &= -x + 2 \\ y &= -x + 6 \end{aligned}$$

18. The line goes through $(-8,1)$, with undefined slope. Since the slope is undefined, the line is vertical. The equation of the vertical line passing through $(-8,1)$ is $x = -8$.

20. The line goes through $(8,-1)$ and $(4,3)$. Find the slope, then use point-slope form with either of the two given points.

$$\begin{aligned} m &= \frac{3-(-1)}{4-8} \\ &= \frac{3+1}{-4} \\ &= \frac{4}{-4} = -1 \end{aligned}$$

$$\begin{aligned} y - (-1) &= -1(x-8) \\ y + 1 &= -x + 8 \\ y &= -x + 7 \end{aligned}$$

22. The line goes through $\left(-2, \frac{3}{4}\right)$ and $\left(\frac{2}{3}, \frac{5}{2}\right)$.

$$\begin{aligned} m &= \frac{\frac{5}{2} - \frac{3}{4}}{\frac{2}{3} - (-2)} = \frac{\frac{10}{4} - \frac{3}{4}}{\frac{2}{3} + \frac{6}{3}} \\ &= \frac{\frac{7}{4}}{\frac{8}{3}} = \frac{21}{32} \end{aligned}$$

$$\begin{aligned} y - \frac{3}{4} &= \frac{21}{32}[x - (-2)] \\ y - \frac{3}{4} &= \frac{21}{32}x + \frac{42}{32} \\ y &= \frac{21}{32}x + \frac{42}{32} + \frac{3}{4} \\ y &= \frac{21}{32}x + \frac{21}{16} + \frac{12}{16} \\ y &= \frac{21}{32}x + \frac{33}{16} \end{aligned}$$

24. The line goes through $(-1, 3)$ and $(0, 3)$.

$$m = \frac{3 - 3}{-1 - 0} = \frac{0}{-1} = 0$$

This is a horizontal line; the value of y is always 3. The equation of this line is $y = 3$.

26. The line has x-intercept -2 and y-intercept 4. Two points on the line are $(-2, 0)$ and $(0, 4)$. Find the slope; then use slope-intercept form.

$$\begin{aligned} m &= \frac{4 - 0}{0 - (-2)} = \frac{4}{2} = 2 \\ y &= mx + b \\ y &= 2x + 4 \end{aligned}$$

28. The line is horizontal, through $(8, 7)$.
The line has an equation of the form $y = k$ where k is the y-coordinate of the point. In this case, $k = 7$, so the equation is $y = 7$.

30. Write the equation of the line through $(2, -5)$, parallel to $y - 4 = 2x$. Rewrite the equation in slope-intercept form.

$$\begin{aligned} y - 4 &= 2x \\ y &= 2x + 4 \end{aligned}$$

The slope of this line is 2.
Use $m = 2$ and the point $(2, -5)$ in the point-slope form.

$$\begin{aligned} y - (-5) &= 2(x - 2) \\ y + 5 &= 2x - 4 \\ y &= 2x - 9 \end{aligned}$$

32. Write the equation of the line through $(-2, 6)$, perpendicular to $2x - 3y = 5$.
Rewrite the equation in slope-intercept form.

$$\begin{aligned} 2x - 3y &= 5 \\ -3y &= -2x + 5 \\ y &= \frac{2}{3}x - \frac{5}{3} \end{aligned}$$

The slope of this line is $\frac{2}{3}$. To find the slope of a perpendicular line, solve

$$\begin{aligned} \frac{2}{3}m &= -1. \\ m &= -\frac{3}{2} \end{aligned}$$

Use $m = -\frac{3}{2}$ and $(-2, 6)$ in the point-slope form.

$$\begin{aligned} y - 6 &= -\frac{3}{2}[x - (-2)] \\ y - 6 &= -\frac{3}{2}(x + 2) \\ y - 6 &= -\frac{3}{2}x - 3 \\ y &= -\frac{3}{2}x + 3 \end{aligned}$$

34. Write the equation of the line with x-intercept $-\frac{2}{3}$, perpendicular to $2x - y = 4$.
Find the slope of the given line.

$$\begin{aligned} 2x - y &= 4 \\ 2x - 4 &= y \end{aligned}$$

The slope of this line is 2. Since the lines are perpendicular, the slope of the needed line is $-\frac{1}{2}$. The line also has an x-intercept of $-\frac{2}{3}$. Thus, it passes through the point $\left(-\frac{2}{3}, 0\right)$.
Using the point-slope form, we have

$$\begin{aligned} y - 0 &= -\frac{1}{2}\left[x - \left(-\frac{2}{3}\right)\right] \\ y &= -\frac{1}{2}\left(x + \frac{2}{3}\right) \\ y &= -\frac{1}{2}x - \frac{1}{3} \end{aligned}$$

36. (a) Write the given line in slope-intercept form.

$$\begin{aligned}2x+3y&=6\\3y&=-2x+6\\y&=-\frac{2}{3}x+2\end{aligned}$$

This line has a slope of $-\frac{2}{3}$. The desired line has a slope of $-\frac{2}{3}$ since it is parallel to the given line. Use the definition of slope.

$$\begin{aligned}m&=\frac{y_2-y_1}{x_2-x_1}\\-\frac{2}{3}&=\frac{2-(-1)}{k-4}\\-\frac{2}{3}&=\frac{3}{k-4}\\-2(k-4)&=(3)(3)\\-2k+8&=9\\-2k&=1\\k&=-\frac{1}{2}\end{aligned}$$

(b) Write the given line in slope-intercept form.

$$\begin{aligned}5x-2y&=-1\\2y&=5x+1\\y&=\frac{5}{2}x+\frac{1}{2}\end{aligned}$$

This line has a slope of $\frac{5}{2}$. The desired line has a slope of $-\frac{2}{5}$ since it is perpendicular to the given line. Use the definition of slope.

$$\begin{aligned}m&=\frac{y_2-y_1}{x_2-x_1}\\&=\frac{2-(-1)}{k-4}\\-\frac{2}{5}&=\frac{2+1}{k-4}\\\frac{-2}{5}&=\frac{3}{k-4}\\-2(k-4)&=(3)(5)\\-2k+8&=15\\-2k&=7\\k&=-\frac{7}{2}\end{aligned}$$

38. Two lines are perpendicular if the product of their slopes is -1.
The slope of the diagonal containing $(4,5)$ and $(-2,-1)$ is

$$m=\frac{5-(-1)}{4-(-2)}=\frac{6}{6}=1.$$

The slope of the diagonal containing $(-2,5)$ and $(4,-1)$ is

$$m=\frac{5-(-1)}{-2-4}=\frac{6}{-6}=-1.$$

The product of the slopes is $(1)(-1)=-1$, so the diagonals are perpendicular.

40. The line goes through $(1,3)$ and $(2,0)$.

$$m=\frac{3-0}{1-2}=\frac{3}{-1}=-3$$

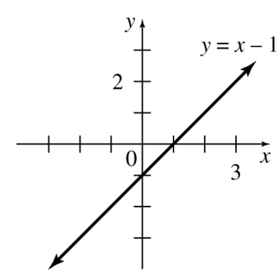
The correct choice is (f).

42. The line goes through $(-2,0)$ and $(0,1)$.

$$m=\frac{1-0}{0-(-2)}=\frac{1}{2}$$

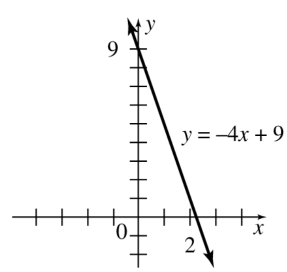
44. $y=x-1$

Three ordered pairs that satisfy this equation are $(0,-1)$, $(1,0)$, and $(4,3)$. Plot these points and draw a line through them.



46. $y=-4x+9$

Three ordered pairs that satisfy this equation are $(0,9)$, $(1,5)$, and $(2,1)$. Plot these points and draw a line through them.



48. $2x - 3y = 12$

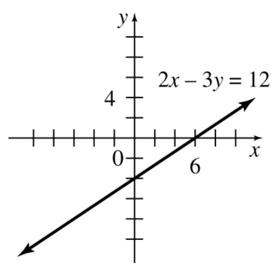
Find the intercepts.
If $y = 0$, then

$$\begin{aligned} 2x - 3(0) &= 12 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

so the x-intercept is 6.
If $x = 0$, then

$$\begin{aligned} 2(0) - 3y &= 12 \\ -3y &= 12 \\ y &= -4 \end{aligned}$$

so the y-intercept is -4.
Plot the ordered pairs $(6, 0)$ and $(0, -4)$ and draw a line through these points. (A third point may be used as a check.)



50. $3y + 4x = 12$

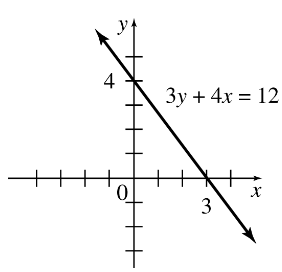
Find the intercepts.
If $y = 0$, then

$$\begin{aligned} 3(0) + 4x &= 12 \\ 4x &= 12 \\ x &= 3, \end{aligned}$$

so the x-intercept is 3.
If $x = 0$, then

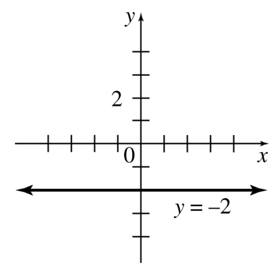
$$\begin{aligned} 3y + 4(0) &= 12 \\ 3y &= 12 \\ y &= 4, \end{aligned}$$

so the y-intercept is 4.
Plot the ordered pairs $(3, 0)$ and $(0, 4)$ and draw a line through these points. (A third point may be used as a check.)



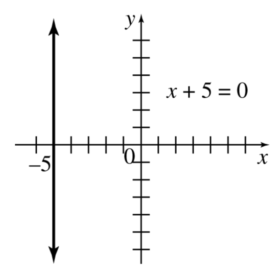
52. $y = -2$

The equation $y = -2$, or, equivalently, $y = 0x - 2$, always gives the same y-value, -2, for any value of x. The graph of this equation is the horizontal line with y-intercept -2.



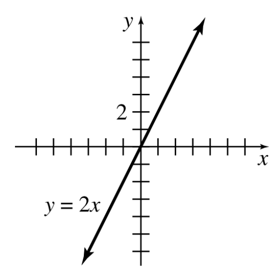
54. $x + 5 = 0$

This equation may be rewritten as $x = -5$. For any value of y, the x-value is -5. Because all ordered pairs that satisfy this equation have the same first number, this equation does not represent a function. The graph is the vertical line with x-intercept -5.



56. $y = 2x$

Three ordered pairs that satisfy this equation are $(0, 0)$, $(-2, -4)$, and $(2, 4)$. Use these points to draw the graph.

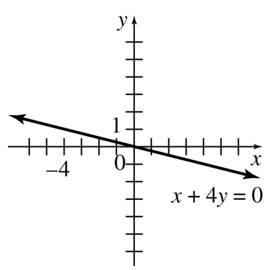


58. $x + 4y = 0$

If $y = 0$, then $x = 0$, so the x-intercept is 0. If $x = 0$, then $y = 0$, so the y-intercept is 0. Both intercepts give the same ordered pair, $(0, 0)$.
To get a second point, choose some other value of x (or y). For example if $x = 4$, then

$$\begin{aligned} x + 4y &= 0 \\ 4 + 4y &= 0 \\ 4y &= -4 \\ y &= -1, \end{aligned}$$

giving the ordered pair $(4, -1)$. Graph the line through $(0, 0)$ and $(4, -1)$.



60. **(a)** The line goes through (2, 27,000) and (5, 63,000).

$$m = \frac{63{,}000 - 27{,}000}{5 - 2} = 12{,}000$$

$$\begin{aligned} y - 27{,}000 &= 12{,}000(x - 2) \\ y - 27{,}000 &= 12{,}000x - 24{,}000 \\ y &= 12{,}000x + 3000 \end{aligned}$$

(b) Let y =100,000; find x.

$$\begin{aligned} 100{,}000 &= 12{,}000x + 3000 \\ 97{,}000 &= 12{,}000x \\ 8.08 &= x \end{aligned}$$

Sales would surpass $100,000 after 8 years, 1 month.

62. **(a)** Using the points $(.7, 1.4)$ and $(5.3, 10.9)$, we obtain

$$\begin{aligned} m &= \frac{10.9 - 1.4}{5.3 - .7} = \frac{9.5}{4.6} \\ &\approx 2.065. \end{aligned}$$

To avoid round-off error, keep all digits for the value of m in your calculator; then round the decimals in the final step.

Use the point-slope form.

$$\begin{aligned} y - 1.4 &= \frac{9.5}{4.6}(x - .7) \\ y - 1.4 &= \frac{9.5}{4.6}x - \frac{9.5}{4.6}(.7) \\ y &= \frac{9.5}{4.6}x - \frac{9.5}{4.6}(.7) + 1.4 \\ y &= 2.065x - .0456 \end{aligned}$$

(b) $y = 2.065(4.9) - .0456 \approx 10.1$ million passengers; this agrees favorably with the FAA predication of 10.3 million.

64. **(a)** Let $x =$ age.

$$\begin{aligned} u &= .85(220 - x) = 187 - .85x \\ l &= .7(220 - x) = 154 - .7x \end{aligned}$$

(b) $u = 187 - .85(20) = 170$
$l = 154 - .7(20) = 140$

The target heart rate zone is 140 to 170 beats per minute.

(c) $u = 187 - .85(40) = 153$
$l = 154 - .7(40) = 126$

The target heart rate zone is 126 to 153 beats per minute.

(d)
$$\begin{aligned} 154 - .7x &= 187 - .85(x + 36) \\ 154 - .7x &= 187 - .85x - 30.6 \\ 154 - .7x &= 156.4 - .85x \\ .15x &= 2.4 \\ x &= 16 \end{aligned}$$

The younger woman is 16; the older woman is $16 + 36 = 52$. $l = .7(220 - 16) \approx 143$ beats per minute.

66. Let $x = 0$ correspond to 1900. Then the "life expectancy from birth" line contains the points $(0, 46)$ and $(100, 76.9)$.

$$m = \frac{76.9 - 46}{100 - 0} = \frac{30.9}{100} = .309$$

Since $(0, 46)$ is one of the points, the line is given by the equation

$$y = .309x + 46.$$

The "life expectancy from age 65" line contains the points $(0, 76)$ and $(100, 82.9)$.

$$m = \frac{82.9 - 76}{100 - 0} = \frac{6.9}{100} = .069$$

Since $(0, 76)$ is one of the points, the line is given by the equation

$$y = .069x + 76.$$

Set the two equations equal to determine where the lines intersect. At this point, life expectancy should increase no further.

$$\begin{aligned} .309x + 46 &= .069x + 76 \\ .24x &= 30 \\ x &= 125 \end{aligned}$$

Determine the y-value when $x = 125$. Use the first equation.

$$\begin{aligned} y &= .309(125) + 46 \\ &= 38.625 + 46 \\ &= 84.625 \end{aligned}$$

Thus, the maximum life expectancy for humans is about 85 years.

68. **(a)** The line goes through the points $(0, 86{,}821)$ and (26, 217,753).

$$\begin{aligned} m &= \frac{217{,}753 - 86{,}821}{26 - 0} \\ &= \frac{130{,}932}{26} \\ &\approx 5035.85 \end{aligned}$$

Since one of the points is $(0, 86{,}821)$, the line is given by the equation

$$y = 5035.85x + 86{,}821.$$

(b) The year 2010 corresponds to $x = 36$.

$$\begin{aligned} y &= 5035.85(36) + 86{,}821 \\ y &\approx 268{,}112 \end{aligned}$$

We predict that the number of immigrants to California in 2010 will be about 268,112.

70. **(a)** Using the points $(0, 9.6)$ and $(31, 19.2)$,

$$\begin{aligned} m &= \frac{19.2 - 9.6}{31 - 0} \\ &= \frac{9.6}{31} \\ &\approx .31 \end{aligned}$$

Since $(0, 9.6)$ is on the line, the equation is given by

$$p = .31t + 9.6.$$

(b) The year 2010 corresponds to $t = 40$.

$$p = .31(40) + 9.6 = 22$$

If the trend continues, about 22% of college students will be 35 and older in 2010.

(c) Let $p = 31$ and solve the equation for t.

$$\begin{aligned} 31 &= .31t + 9.6 \\ 21.4 &= .31t \\ t &\approx 69 \end{aligned}$$

This corresponds to the year 1970 + 69 = 2039.

72. **(a)** If the temperature rises .3C° per decade, it rises .03C° per year.

$m = .03$
$b = 15$, since a point is $(0, 15)$.

$$T = .03t + 15$$

(b) Let $T = 19$; find t.

$$\begin{aligned} 19 &= .03t + 15 \\ 4 &= .03t \\ 133.3 &= t \\ 133 &\approx t \\ 1970 + 133 &= 2103 \end{aligned}$$

The temperature will rise to 19°C in about the year 2103.

74. **(a)** $m = \dfrac{13{,}150 - 2773}{2000 - 1950} = \dfrac{10{,}377}{50} = 207.54$

This means that each year there is an average increase of about 208 stations.

(b) Use the point-slope form with $(2000, 13{,}150)$.

$$\begin{aligned} y - 13{,}150 &= 207.54(x - 2000) \\ y - 13{,}150 &= 207.54x - 415{,}080 \\ y &= 207.54x - 401{,}930 \end{aligned}$$

(c) Let $y = 15{,}000$ and solve the equation for x.

$$\begin{aligned} 15{,}000 &= 207.54x - 401{,}930 \\ 416{,}930 &= 207.54x \\ x &\approx 2008.9 \end{aligned}$$

The estimated year when it is expected that the number of stations will first exceed 15,000 is 2009.

1.2 Linear Functions and Applications

2. This statement is false.

The graph of $f(x) = -3$ is a horizontal line.

4. This statement is true.

For any value of a,

$$f(0) = a \cdot 0 = 0,$$

so the point $(0, 0)$, which is the origin, lies on the line.

8. \$12 is the fixed cost and \$1 is the cost per hour.

$$\begin{aligned} \text{Let } x =& \text{ number of hours;} \\ C(x) =& \text{ cost of renting a saw for} \\ & x \text{ hours.} \end{aligned}$$

Thus,

$$\begin{aligned} C(x) =& \text{ fixed cost } + \text{ (cost per hour)} \\ & \cdot \text{ (number of hours)} \\ C(x) =& \ 12 + 1x \\ & 12 + x. \end{aligned}$$

10. 50¢ is the fixed cost and 35¢ is the cost per half-hour.

$$\begin{aligned} \text{Let } x =& \text{ the number of half-hours;} \\ C(x) =& \text{ the cost of parking a car for} \\ & x \text{ half-hours.} \end{aligned}$$

Thus,

$$\begin{aligned} C(x) &= 50 + 35x \\ &= 35x + 50. \end{aligned}$$

12. Fixed cost, \$100; 50 items cost \$1600 to produce.

$$\begin{aligned} \text{Let } C(x) =& \text{ cost of producing } x \text{ items.} \\ C(x) =& \ mx + b, \text{ where } b \text{ is the fixed} \\ & \text{cost.} \end{aligned}$$

$$C(x) = mx + 100$$

Now,

$C(x) = 1600$ when $x = 50$, so

$$\begin{aligned} 1600 &= m(50) + 100 \\ 1500 &= 50m \\ 30 &= m. \end{aligned}$$

Thus, $C(x) = 30x + 100$.

14. Marginal cost, \$90; 150 items cost \$16,000 to produce.

$$C(x) = 90x + b$$

Now, $C(x) = 16{,}000$ when $x = 150$.

$$\begin{aligned} 16{,}000 &= 90(150) + b \\ 16{,}000 &= 13{,}500 + b \\ 2500 &= b \end{aligned}$$

Thus, $C(x) = 90x + 2500$.

16. For a linear function, the average rate of change will be the same as the slope of the line. If the function is not linear, the average rate of change is the slope of the secant line connecting the beginning and ending points.

18. $D(q) = 16 - \dfrac{5}{4}q$

(a) $D(0) = 16 - \dfrac{5}{4}(0) = 16 - 0 = 16$

When 0 can openers are demanded, the price is \$16.

(b) $D(4) = 16 - \dfrac{5}{4}(4) = 16 - 5 = 11$

When 400 can openers are demanded, the price is \$11.

(c) $D(8) = 16 - \dfrac{5}{4}(8) = 16 - 10 = 6$

When 800 can openers are demanded, the price is \$6.

(d) Let $D(q) = 8$. Find q.

$$\begin{aligned} 8 &= 16 - \frac{5}{4}q \\ \frac{5}{4}q &= 8 \\ q &= 6.4 \end{aligned}$$

When the price is \$8, 640 can openers are demanded.

(e) Let $D(q) = 10$. Find q.

$$\begin{aligned} 10 &= 16 - \frac{5}{4}q \\ \frac{5}{4}q &= 6 \\ q &= 4.8 \end{aligned}$$

When the price is \$10, 480 can openers are demanded.

(f) Let $D(q) = 12$. Find q.

$$12 = 16 - \frac{5}{4}q$$
$$\frac{5}{4}q = 4$$
$$q = 3.2$$

When the price is $12, 320 can openers are demanded.

(g)

(h) $S(q) = \frac{3}{4}q$

Let $S(q) = 0$. Find q.

$$0 = \frac{3}{4}q$$
$$0 = q$$

When the price is $0, 0 can openers are supplied.

(i) Let $S(q) = 10$. Find q.

$$10 = \frac{3}{4}q$$
$$\frac{40}{3} = q$$
$$q = 13.\overline{3}$$

When the price is $10, about 1333 can openers are supplied.

(j) Let $S(q) = 20$. Find q.

$$20 = \frac{3}{4}q$$
$$\frac{80}{3} = q$$
$$q = 26.\overline{6}$$

When the price is $20, about 2667 can openers are demanded.

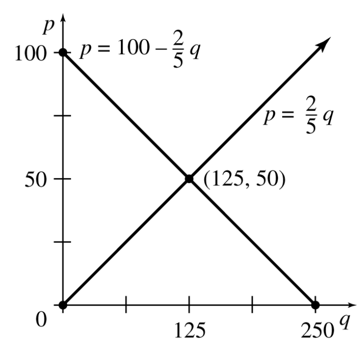
(k) .

(l)
$$D(q) = S(q)$$
$$16 - \frac{5}{4}q = \frac{3}{4}q$$
$$16 = 2q$$
$$8 = q$$
$$S(8) = \frac{3}{4}(8) = 6$$

The equilibrium quantity is 800, and the equilibrium price is $6.

20. $p = S(q) = \frac{2}{5}q$; $p = D(q) = 100 - \frac{2}{5}q$

(a)



(b)
$$S(q) = D(q)$$
$$\frac{2}{5}q = 100 - \frac{2}{5}q$$
$$\frac{4}{5}q = 100$$
$$q = 125$$
$$S(125) = \frac{2}{5}(125) = 50$$

The equilibrium quantity is 125, the equilibrium price is $50

22. **(a)** $C(x) = mx + b;\ m = 3.50;\ C(60) = 300$

$$C(x) = 3.50x + b$$

Find b.

$$\begin{aligned} 300 &= 3.50(60) + b \\ 300 &= 210 + b \\ 90 &= b \\ C(x) &= 3.50x + 90 \end{aligned}$$

(b) $R(x) = 9x$
$C(x) = R(x)$

$$\begin{aligned} 3.50x + 90 &= 9x \\ 90 &= 5.5x \\ 16.36 &= x \end{aligned}$$

Yoshi must produce and sell 17 shirts.

(c) $P(x) = R(x) - C(x);\ P(x) = 500$

$$\begin{aligned} 500 &= 9x - (3.50x + 90) \\ 500 &= 5.5x - 90 \\ 590 &= 5.5x \\ 107.27 &= x \end{aligned}$$

To make a profit of \$500, Yoshi must produce and sell 108 shirts.

24. **(a)** Using the points $(100, 11.02)$ and $(400, 40.12)$,

$$m = \frac{40.12 - 11.02}{400 - 100} = \frac{29.1}{300} = .097.$$

$$\begin{aligned} y - 11.02 &= .097(x - 100) \\ y - 11.02 &= .097x - 9.7 \\ y &= .097x + 1.32 \\ C(x) &= .097x + 1.32 \end{aligned}$$

(b) The fixed cost is given by the constant in $C(x)$. It is \$1.32.

(c) $C(1000) = .097(1000) + 1.32 = 97 + 1.32$
$= 98.32$

The total cost of producing 1000 cups is \$98.32.

(d) $C(1001) = .097(1001) + 1.32 = 97.097 + 1.32$
$= 98.417$

The total cost of producing 1001 cups is \$98.417.

(e) Marginal cost $= 98.417 - 98.32$
$= \$.097$ or $9.7¢$

(f) The marginal cost for *any* cup is the slope, \$.097 or 9.7¢. This means the cost of producing one additional cup of coffee would be 9.7¢.

26. **(a)** $(100{,}000)(50) = 5{,}000{,}000$

Sales in 1996 would be 100,000 + 5,000,000 = 5,100,000.

(b) The ordered pairs are (1, 100,000) and (6, 5,100,000).

(c) $m = \dfrac{5{,}100{,}000 - 100{,}000}{6 - 1} = \dfrac{5{,}000{,}000}{5}$
$= 1{,}000{,}000$

$$\begin{aligned} y - 100{,}000 &= 1{,}000{,}000(x - 1) \\ y - 100{,}000 &= 1{,}000{,}000x - 1{,}000{,}000 \\ y &= 1{,}000{,}000x - 900{,}000 \\ S(x) &= 1{,}000{,}000x - 900{,}000 \end{aligned}$$

(d) Let $S(x) = 1{,}000{,}000{,}000$. Find x.

$$\begin{aligned} 1{,}000{,}000{,}000 &= 1{,}000{,}000x - 900{,}000 \\ 1{,}000{,}900{,}000 &= 1{,}000{,}000x \\ x &= 1000.9 \approx 1001 \end{aligned}$$

Sales would reach one billion dollars in about 1991 + 1001 = 2992.

(e) According to our linear model, in 2003, $x = 13$ and estimated sales would be

$$S(13) = 1{,}000{,}000(13) - 900{,}000 = 12{,}100{,}000$$

or about \$12,100,000. Sales are growing much faster than linearly if they reached one billion dollars in 2003.

28. $C(x) = 12x + 39;\ R(x) = 25x$

(a)
$$\begin{aligned} C(x) &= R(x) \\ 12x + 39 &= 25x \\ 39 &= 13x \\ 3 &= x \end{aligned}$$

The break-even quantity is 3 units.

(b)
$$\begin{aligned} P(x) &= R(x) - C(x) \\ P(x) &= 25x - (12x + 39) \\ P(x) &= 13x - 39 \\ P(250) &= 13(250) - 39 \\ &= 3250 - 39 \\ &= 3211 \end{aligned}$$

The profit from 250 units is \$3211.

(c) $P(x) = \$130$; find x.

$$\begin{aligned} 130 &= 13x - 39 \\ 169 &= 13x \\ 13 &= x \end{aligned}$$

For a profit of \$130, 13 units must be produced.

30. $C(x) = 105x + 6000$
$R(x) = 250x$

Set $C(x) = R(x)$ to find the break-even quantity.

$$\begin{aligned} 105x + 6000 &= 250x \\ 6000 &= 145x \\ 41.38 &\approx x \end{aligned}$$

The break-even quantity is about 41 units, so you should decide to produce.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 250x - (105x + 6000) \\ &= 145x - 6000 \end{aligned}$$

The profit function is $P(x) = 145x - 6000$.

32. $C(x) = 1000x + 5000$
$R(x) = 900x$

$$\begin{aligned} 900x &= 1000x + 5000 \\ -5000 &= 100x \\ -50 &= x \end{aligned}$$

It is impossible to make a profit when the break-even quantity is negative. Cost will always be greater than revenue.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 900x - (1000x + 5000) \\ &= -100x - 5000 \end{aligned}$$

The profit function is $P(x) = -100x - 500$ (always a loss).

34. Use the formula derived in Example 7 in this section of the textbook.

$$F = \frac{9}{5}C + 32$$
$$C = \frac{5}{9}(F - 32)$$

(a) $C = 37$; find F.

$$\begin{aligned} F &= \frac{9}{5}(37) + 32 \\ F &= \frac{333}{5} + 32 \\ F &= 98.6 \end{aligned}$$

The Fahrenheit equivalent of 37°C is 98.6°F.

(b) $C = 36.5$; find F.

$$\begin{aligned} F &= \frac{9}{5}(36.5) + 32 \\ F &= 65.7 + 32 \\ F &= 97.7 \end{aligned}$$

$C = 37.5$; find F.

$$\begin{aligned} F &= \frac{9}{5}(37.5) + 32 \\ &= 67.5 + 32 = 99.5 \end{aligned}$$

The range is between 97.7°F and 99.5°F.

1.3 The Least Squares Line

2. For the set of points $(1, 4)$, $(2, 5)$, and $(3, 6)$, $Y = x + 3$. For the set $(4, 1)$, $(5, 2)$, and $(6, 3)$, $Y = x - 3$.

4.
$$\begin{aligned} nb + (\textstyle\sum x)m &= \textstyle\sum y \\ (\textstyle\sum x)b + (\textstyle\sum x^2)m &= \textstyle\sum xy \\ nb + (\textstyle\sum x)m &= \textstyle\sum y \\ nb &= (\textstyle\sum y) - (\textstyle\sum x)m \\ b &= \frac{\sum y - m(\sum x)}{n} \\ (\textstyle\sum x)\left(\frac{\sum y - m(\sum x)}{n}\right) + (\textstyle\sum x^2)m &= \textstyle\sum xy \\ (\textstyle\sum x)[(\textstyle\sum y) - m(\textstyle\sum x)] + nm(\textstyle\sum x^2) &= n(\textstyle\sum xy) \\ (\textstyle\sum x)(\textstyle\sum y) - m(\textstyle\sum x)^2 + nm(\textstyle\sum x^2) &= n(\textstyle\sum xy) \\ nm(\textstyle\sum x^2) - m(\textstyle\sum x)^2 &= n(\textstyle\sum xy) - (\textstyle\sum x)(\textstyle\sum y) \\ m\left[n(\textstyle\sum x^2) - (\textstyle\sum x)^2\right] &= n(\textstyle\sum xy) - (\textstyle\sum x)(\textstyle\sum y) \\ m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \end{aligned}$$

6. (a)

$$10b + 965m = 95.3$$

$$\begin{aligned} 10b + 965m &= 95.3 \\ 10b &= 95.3 - 965m \\ b &= \frac{95.3 - 965m}{10} \\ b &= 95.3 - 96.5m \end{aligned}$$

$$\begin{aligned} 965b + 93{,}205m &= 9165.1 \\ 965(9.53 - 96.5m) + 93{,}205m &= 9165.1 \\ 9196.45 - 93{,}122.5m + 93,205m &= 9165.1 \\ 82.5m &= -31.35 \\ m &= -.38 \end{aligned}$$

$$b = 9.53 - 96.5(-.38) = 46.2$$

$$Y = -.38x + 46.2$$

(b) The year 2004 corresponds to $x = 104$.

$Y = -.38(104) + 46.2 = 6.68$ (in thousands)

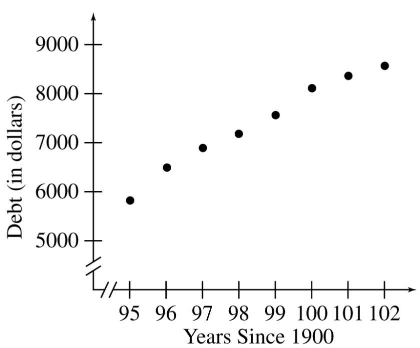
If the trend continues, there will be about 6680 banks in 2004.

(c) $$r = \frac{10(9165.1) - (965)(95.3)}{\sqrt{10(93{,}205) - 965^2} \cdot \sqrt{10(920.47) - 95.3^2}}$$

$$\approx -.986$$

This means that the least squares line fits the data points very well. The negative sign indicates that the number of banks is decreasing as the years increase.

8. (a)



Yes, the pattern is linear.

(b)

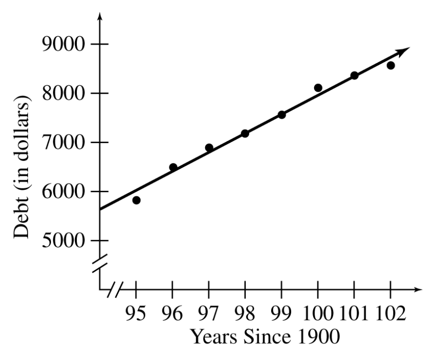
x	y	xy	x^2	y^2
95	5832	554,040	9025	34,012,224
96	6487	622,752	9216	42,081,169
97	6900	669,300	9409	47,610,000
98	7188	704,424	9604	51,667,344
99	7564	748,836	9801	57,214,096
100	8123	812,300	10,000	65,983,129
101	8367	845,067	10,201	70,006,689
102	8562	873,324	10,404	73,307,844
788	59,023	5,830,043	77,660	441,882,495

$$\begin{aligned} 8b + 788m &= 59{,}023 \\ 8b &= 59{,}023 - 788m \\ b &= \frac{59{,}023 - 788m}{8} \end{aligned}$$

$$\begin{aligned} 788b + 77{,}660m &= 5{,}830{,}043 \\ 788\left(\frac{59{,}023 - 788m}{8}\right) + 77{,}660m &= 5{,}830{,}043 \\ 788(59{,}023 - 788m) + 621{,}280m &= 46{,}640{,}344 \\ 46{,}510{,}124 - 620{,}944m + 621{,}280\text{m} &= 46{,}640{,}344 \\ 336m &= 130{,}220 \\ m &\approx 387.56 \end{aligned}$$

$$b = \frac{59{,}023 - 788\left(\frac{130{,}220}{336}\right)}{8} = -30{,}796.74$$

$$Y = 387.56x - 30{,}796.74$$



The line appears to be a good fit.

(c)

$$r = \frac{8(5{,}830{,}043) - (788)(59{,}023)}{\sqrt{8(77{,}660) - 788^2} \cdot \sqrt{8(441{,}882{,}495) - 59{,}023^2}}$$

$$\approx .991$$

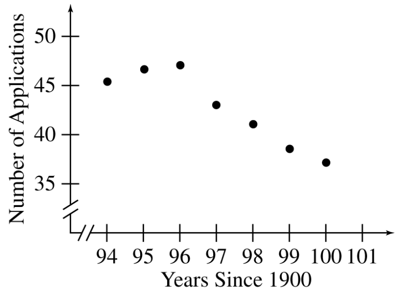
This indicates that the least squares line fits the data points very well.

(d) Let $Y = 10{,}000$ and solve for x.

$$\begin{aligned} 10,000 &= 387.56x - 30{,}796.74 \\ 40{,}796.74 &= 387.56x \\ x &\approx 105 \end{aligned}$$

If the trend continues, household debt will reach \$10,000 in 1900 + 105 = 2005.

10. (a)



If all points are included, the pattern is not linear.

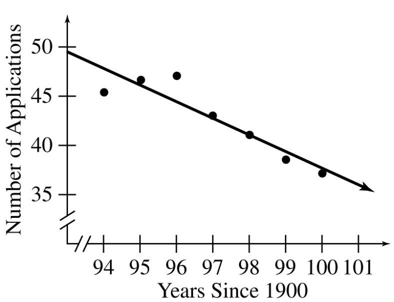
(b)

x	y	xy	x^2	y^2
94	45.4	4267.6	8836	2061.16
95	46.6	4427	9025	2171.56
96	47.0	4512	9216	2209
97	43.0	4171	9409	1849
98	41.0	4018	9604	1681
99	38.5	3811.5	9801	1482.25
100	37.1	3710	10,000	1376.41
679	298.6	28,917.1	65,891	12,830.38

$$7b + 679m = 298.6$$
$$7b = 298.6 - 679m$$
$$b = \frac{298.6 - 679m}{7}$$

$$679b + 65{,}891m = 28{,}917.1$$
$$679\left(\frac{298.6 - 679m}{7}\right) + 65{,}891m = 28{,}917.1$$
$$679(298.6 - 679m) + 461{,}237m = 202{,}419.7$$
$$202{,}749.4 - 461{,}041m + 461{,}237m = 202{,}419.7$$
$$196m = -329.7$$
$$m \approx -1.682$$

$$b = \frac{298.6 - 679\left(\frac{-329.7}{196}\right)}{7} = 205.825$$
$$Y = -1.682x + 205.825$$



The line fits the data reasonable well.

(c) $r = \dfrac{7(28{,}917.1) - (679)(298.6)}{\sqrt{7(65{,}891) - (679)^2} \cdot \sqrt{7(12{,}830.38) - 298.6^2}}$

$r \approx -.923$

This indicates a reasonably good fit, as concluded in part (b).

(d) The line lies near most of the points. The most distant two are still relatively close to the line.

12. (a)

x	y	xy	x^2	y^2
88.6	20.0	1772	7849.96	400.0
71.6	16.0	1145.6	5126.56	256.0
93.3	19.8	1847.34	8704.89	392.04
84.3	18.4	1551.12	7106.49	338.56
80.6	17.1	1378.26	6496.36	292.41
75.2	15.5	1165.6	5655.04	240.25
69.7	14.7	1024.59	4858.09	216.09
82.0	17.1	1402.2	6724	292.41
69.4	15.4	1068.76	4816.36	237.16
83.3	16.2	1349.46	6938.89	262.44
79.6	15.0	1194	6336.16	225
82.6	17.2	1420.72	6822.76	295.84
80.6	16.0	1289.6	6496.36	256.0
83.5	17.0	1419.5	6972.25	289.0
76.3	14.4	1098.72	5821.69	207.36
1200.6	249.8	20,127.47	96,725.86	4200.56

$$15b + 1200.6m = 249.8$$
$$1200.6b + 96{,}725.86m = 20{,}127.47$$

$$15b = 249.8 - 1200.6m$$
$$b = \frac{249.8 - 1200.6m}{15}$$

$$1200.6\left(\frac{249.8 - 1200.6m}{15}\right) + 96{,}725.86m = 20{,}127.47$$
$$1200.6(249.8 - 1200.6m) + 1{,}450{,}887.9m = 301{,}912.05$$
$$299{,}909.88 - 1{,}441{,}440.36m + 1{,}450{,}887.9m = 301{,}912.05$$
$$9447.54m = 2002.17$$
$$m \approx .212$$

$$b = \frac{249.8 - 1200.6(.212)}{15} = -.315$$
$$Y = .212x - .315$$

(b) Let $x = 73$; find Y.

$Y = .212(73) - .315$

≈ 15.2

If the temperature were 73°F, you would expect to hear 15.2 chirps per second.

(c) Let $Y = 18$; find x.

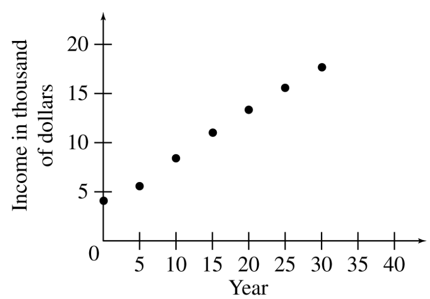
$$18 = .212x - .315$$
$$18.315 = .212x$$
$$86.4 \approx x$$

When the crickets are chirping 18 times per second, the temperature is 86.4°F.

(d)

$$r = \frac{15(20{,}127) - (1200.6)(249.8)}{\sqrt{15(96{,}725.86) - (1200.6)^2} \cdot \sqrt{15(4200.56) - (249.8)^2}}$$
$$= .835$$

14. (a)



Yes, the data appear to lie along a straight line.

(b)

$$r = \frac{7(1460.97) - (105)(75.402)}{\sqrt{7(2275) - 105^2} \cdot \sqrt{7(968.270792) - 75.402^2}}$$
$$r \approx .998$$

Yes, there is a strong positive linear correlation between the income and the year.

(c)
$$7b + 105m = 75.402$$
$$7b = 75.402 - 105m$$
$$b = \frac{75.402 - 105m}{7}$$
$$105b + 2275m = 1460.97$$
$$105\left(\frac{75.402 - 105m}{7}\right) + 2275m = 1460.97$$
$$105(75.402 - 105m) + 15{,}925m = 10{,}226.79$$
$$7917.21 - 11{,}025m + 15{,}925m = 10{,}226.79$$
$$4900m = 2309.58$$
$$m \approx .471$$

$$b = \frac{75.402 - 105\left(\frac{2309.58}{4900}\right)}{7} \approx 3.702$$
$$Y = .471x + 3.702$$

(d) The year 2015 corresponds to $x = 45$.

$$Y = .471(45) + 3.702$$
$$Y \approx 24.897$$

The predicted poverty level in the year 2015 is \$24,897.

16. (a)

x	y	xy	x^2	y^2
150	5000	750,000	22,500	25,000,000
175	5500	962,500	30,625	30,250,000
215	6000	1,290,000	46,225	36,000,000
250	6500	1,625,000	62,500	42,250,000
280	7000	1,960,000	78,400	49,000,000
310	7500	2,325,000	96,100	56,250,000
350	8000	2,800,000	122,500	64,000,000
370	8500	3,145,000	136,900	72,250,000
420	9000	3,780,000	176,400	81,000,000
450	9500	4,275,000	202,500	90,250,000
2970	72,500	22,912,500	974,650	546,250,000

$$10b + 2970m = 72{,}500$$
$$2970b + 974{,}650m = 22{,}912{,}500$$

$$10b = 72{,}500 - 2970m$$
$$b = 7250 - 297m$$

$$2970(7250 - 297m) + 974{,}650m = 22{,}912{,}500$$
$$21{,}532{,}500 - 882{,}090m + 974{,}650m = 22{,}912{,}500$$
$$92{,}560m = 1{,}380{,}000$$
$$m = 14.9$$

$$b = 7250 - 297(14.9) \approx 2820$$
$$Y = 14.9x + 2820$$

(b) Let $x = 150$; find Y.

$$Y = 14.9(150) + 2820$$
$$Y \approx 5060, \text{ compared to actual } 5000$$

Let $x = 280$; find Y.

$$Y = 14.9(280) + 2820$$
$$\approx 6990, \text{ compared to actual } 7000$$

Let $x = 420$; find Y.

$$Y = 14.9(420) + 2820$$
$$\approx 9080, \text{ compared to actual } 9000$$

(c) Let $x = 230$; find Y.

$$Y = 14.9(230) + 2820$$
$$\approx 6250$$

Adam would need to buy a 6500 BTU air conditioner.

18. (a)

x	y	xy	x^2	y^2
5	113.4	567	25	12,859.56
15	111.9	1678.5	225	12,521.61
25	111.9	2797.5	625	12,521.61
35	109.7	3839.5	1225	12,034.09
45	106.6	4797	2025	11,363.56
55	105.7	5813.5	3025	11,172.49
65	104.3	6779.5	4225	10,878.49
75	103.7	7777.5	5625	10,753.69
85	101.73	8647.05	7225	10,348.9929
95	101.11	9605.45	9025	10,223.2321
500	1070.04	52,302.5	33,250	114,677.325

$$\begin{aligned}
10b + 500m &= 1070.04 \\
500b + 33{,}250\text{m} &= 52{,}302.5 \\
10b &= 1070.04 - 500m \\
b &= 107.004 - 50m \\
500(107.004 - 50m) + 33{,}250m &= 52{,}302.5 \\
53{,}502 - 25{,}000m + 33{,}250m &= 52{,}302.5 \\
8250m &= -1199.5 \\
m &= -.1454
\end{aligned}$$

$b = 107.004 - 50(-.1454)$

$b \approx 114.27$

$Y = -.1454x + 114.27$

(b)

x	y	xy	x^2	y^2
25	144.0	3600.0	625	20, 736
35	135.6	4746	1225	18, 387.36
45	132.0	5940.0	2025	17, 424
55	125.0	6875.0	3025	15, 625
65	118.0	7670.0	4225	13, 924
75	117.48	8811	5625	13,801.5504
85	113.28	9628.8	7225	12, 832.3584
95	113.28	10,761.6	9025	12, 832.3584
480	998.64	58,032.4	33,000	125,562.6272

$$\begin{aligned}
8b + 480m &= 998.64 \\
480b + 33{,}000m &= 58{,}032.4 \\
8b &= 998.64 - 480m \\
b &= 124.83 - 60m \\
480(124.83 - 60m) + 33{,}000m &= 58{,}032.4 \\
59{,}918.4 - 28{,}800m + 33{,}000m &= 58{,}032.4 \\
4200m &= -1886 \\
m &= -.4490
\end{aligned}$$

$b = 124.83 - 60(-.4490)$

$b = 151.77$

$Y = -.4490x + 151.77$

(c) $$\begin{aligned} -.1454x + 114.27 &= -.4490x + 151.77 \\ .3036x &= 37.5 \\ x &\approx 123.52 \approx 124 \\ 1900 + 124 &= 2024 \end{aligned}$$

The women will catch up to the men in the year 2024.

(d) $$r_{\text{men}} = \frac{10(52{,}302.5) - (500)(1070.04)}{\sqrt{10(33{,}250) - 500^2} \cdot \sqrt{10(114{,}677.325) - (1070.04)^2}} = -.9877$$

$$r_{\text{women}} = \frac{8(58{,}032.4) - (480)(998.64)}{\sqrt{8(33{,}000) - 480^2} \cdot \sqrt{8(125{,}562.6272) - (998.64)^2}} = -9.688$$

Both sets of points closely fit a line with negative slope.

20. (a)

x	y	xy	x^2
1	33	33	1
2	34	68	4
3	36	108	9
4	35	140	16
5	40	200	25
6	44	264	36
7	48	336	49
8	45	360	64
9	46	414	81
10	48	480	100
11	49	539	121
12	49	588	144
13	48	624	169
14	54	756	196
15	57	855	225
120	666	5765	1240

$$\begin{aligned} 15b + 120m &= 666 \\ 120b + 1240m &= 5765 \\ 15b &= 666 - 120m \\ b &= \frac{666 - 120m}{15} \end{aligned}$$

$$120\left(\frac{666 - 120m}{15}\right) + 1240m = 5765$$

$$\begin{aligned} 8(666 - 120m) + 1240m &= 5765 \\ 5328 - 960m + 1240m &= 5765 \\ 280m &= 437 \\ m &\approx 1.5607 \end{aligned}$$

$$b = \frac{666 - 120(1.5607)}{15} \approx 31.914$$

$$Y = 1.5607x + 31.914$$

(b) Let $Y = 75$ (1 hour and 15 minutes beyond 2 hours); find x.

$$\begin{aligned} 75 &= 1.5607x + 31.914 \\ 43.086 &= 1.5607x \\ 27.61 &\approx x \end{aligned}$$

If the trend continues, the average completion time will be 3 hours and 15 minutes in the year $1980 + 27.61 \approx 2008$.

Chapter 1 Review Exercises

2. To complete the coefficient of correlation, you need to compute the following quantities: $\sum x$, $\sum y$, $\sum xy$, $\sum x^2$, $\sum y^2$, and n.

4. Through $(4, -1)$ and $(3, -3)$.

$$\begin{aligned} m &= \frac{-3-(-1)}{3-4} \\ &= \frac{-3+1}{-1} \\ &= \frac{-2}{-1} = 2 \end{aligned}$$

6. Through the origin and $(0, 7)$

$$m = \frac{7-0}{0-0} = \frac{7}{0}$$

The slope of the line is undefined.

8. $$\begin{aligned} 4x - y &= 7 \\ -y &= -4x + 7 \\ y &= 4x - 7 \\ m &= 4 \end{aligned}$$

10. $$\begin{aligned} 3y - 1 &= 14 \\ 3y &= 14 + 1 \\ 3y &= 15 \\ y &= 5 \end{aligned}$$

This is a horizontal line. The slope of a horizontal line is 0.

12. $$\begin{aligned} x &= 5y \\ \frac{1}{5}x &= y \\ m &= \frac{1}{5} \end{aligned}$$

14. Through $(8, 0)$, with slope $-\frac{1}{4}$

Use point-slope form.

$$\begin{aligned} y - 0 &= -\frac{1}{4}(x - 8) \\ y &= -\frac{1}{4}x + 2 \end{aligned}$$

16. Through $(2, -3)$ and $(-3, 4)$

$$m = \frac{4-(-3)}{-3-2} = -\frac{7}{5}$$

Use point-slope form.

$$\begin{aligned} y - (-3) &= -\frac{7}{5}(x - 2) \\ y + 3 &= -\frac{7}{5}x + \frac{14}{5} \\ y &= -\frac{7}{5}x + \frac{14}{5} - 3 \\ y &= -\frac{7}{5}x + \frac{14}{5} - \frac{15}{5} \\ y &= -\frac{7}{5}x - \frac{1}{5} \end{aligned}$$

18. Through $(-2, 5)$, with slope 0

Horizontal lines have 0 slope and an equation of the form $y = k$.
The line passes through $(-2, 5)$ so $k = 5$. An equation of the line is $y = 5$.

20. Through $(0, 5)$, perpendicular to $8x + 5y = 3$
Find the slope of the given line first.

$$\begin{aligned} 8x + 5y &= 3 \\ 5y &= -8x + 3 \\ y &= \frac{-8}{5}x + \frac{3}{5} \\ m &= -\frac{8}{5} \end{aligned}$$

The perpendicular line has $m = \frac{5}{8}$.
Use point-slope form.

$$\begin{aligned} y - 5 &= \frac{5}{8}(x - 0) \\ y &= \frac{5}{8}x + 5 \end{aligned}$$

22. Through $(3, -5)$, parallel to $y = 4$

Find the slope of the given line.
$y = 0x + 4$, so $m = 0$, and the required line will also have slope 0.
Use the point-slope form.

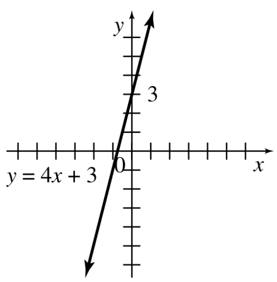
$$\begin{aligned} y - (-5) &= 0(x - 3) \\ y + 5 &= 0 \\ y &= -5 \end{aligned}$$

24. $y = 4x + 3$

Let $x = 0$. $\quad y = 4(0) + 3$
$\qquad\qquad\quad y = 3$

Let $y = 0$. $\quad 0 = 4x + 3$
$\qquad\qquad\quad -3 = 4x$
$\qquad\qquad\quad -\dfrac{3}{4} = x$

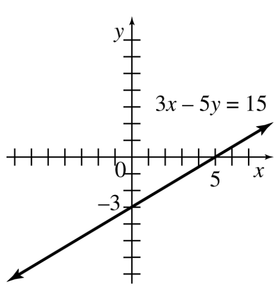
Draw the line through $(0, 3)$ and $\left(-\frac{3}{4}, 0\right)$.



26. $3x - 5y = 15$

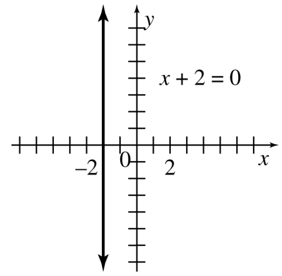
$$\begin{aligned} -5y &= -3x + 15 \\ y &= \frac{3}{5}x - 3 \end{aligned}$$

When $x = 0$, $y = -3$.
When $y = 0$, $x = 5$.
Draw the line through $(0, -3)$ and $(5, 0)$.



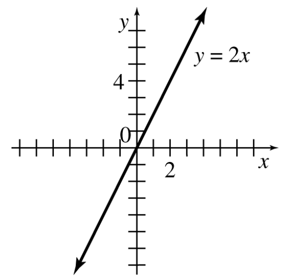
28. $x + 2 = 0$
$x = -2$

This is the vertical line through $(-2, 0)$.



30. $y = 2x$

When $x = 0$, $y = 0$.
When $x = 1$, $y = 2$.
Draw the line through $(0, 0)$ and $(1, 2)$.



32. **(a)** $E = 352 + 42x$ (where x is in thousands)

(b) $R = 130x$ (where x is in thousands)

(c)
$$\begin{aligned} R &> E \\ 130x &> 352 + 42x \\ 88x &> 352 \\ x &> 4 \end{aligned}$$

For a profit to be made, more than 4000 chips must be sold.

34. Using the points $(60, 40)$ and $(100, 60)$,

$$m = \frac{60 - 40}{100 - 60} = \frac{20}{40} = .5.$$

$$\begin{aligned} p - 40 &= .5(q - 60) \\ p - 40 &= .5q - 30 \\ p &= .5q + 10 \\ S(q) &= .5q + 10 \end{aligned}$$

36.
$$\begin{aligned} S(q) &= D(q) \\ .5q + 10 &= -.5q + 72.50 \\ q &= 62.5 \end{aligned}$$

$S(62.5) = .5(62.5) + 10 = 31.25 + 10 = 41.25$

The equilibrium price is \$41.25, and the equilibrium quantity is 62.5 diet pills.

38. Fixed cost is \$2000; 36 units cost \$8480.
Two points on the line are $(0, 2000)$ and $(36, 8480)$, so

$$m = \frac{8480 - 2000}{36 - 0} = \frac{6480}{36} = 180.$$

Use point-slope form.

$$\begin{aligned} y &= 180x + 2000 \\ C(x) &= 180x + 2000 \end{aligned}$$

40. Thirty units cost \$1500; 120 units cost \$5640.
Two points on the line are $(30, 1500)$, $(120, 5640)$, so

$$m = \frac{5640 - 1500}{120 - 30} = \frac{4140}{90} = 46.$$

Use point-slope form.

$$\begin{aligned} y - 1500 &= 46(x - 30) \\ y &= 46x - 1380 + 1500 \\ y &= 46x + 120 \\ C(x) &= 46x + 120 \end{aligned}$$

42. (a) $C(x) = 3x + 160$; $R(x) = 7x$

$$\begin{aligned} C(x) &= R(x) \\ 3x + 160 &= 7x \\ 160 &= 4x \\ 40 &= x \end{aligned}$$

The break-even quantity is 40 pounds.

(b) $R(40) = 7 \cdot 40 = \$280$

The revenue for 40 pounds is \$280.

44. Using the points (91, 6) and $(101, 19)$,

$$m = \frac{19 - 6}{101 - 01} = \frac{13}{10} = 1.3$$

$$\begin{aligned} y - 10 &= 1.3(x - 91) \\ y - 10 &= 1.3 - 118.3 \\ y &= 1.3 - 108.3 \end{aligned}$$

46. (a)

x	y	xy	x^2	y^2
75	6000	450,000	5625	36,000,000
80	7500	600,000	6400	56,250,000
85	12,000	1,020,000	7225	144,000,000
90	16,000	1,440,000	8100	256,000,000
95	20,400	1,938,000	9025	416,160,000
100	24,900	2,490,000	10,000	620,010,000
525	86,800	7,938,000	46,375	1,528,420,000

$$\begin{aligned} 6b + 525m &= 86{,}800 \\ 6b &= 86{,}800 - 525m \\ b &= \frac{86{,}800 - 525m}{6} \end{aligned}$$

$$\begin{aligned} 525b + 46{,}375m &= 7{,}938{,}000 \\ 525\left(\frac{86{,}800 - 525m}{6}\right) + 46{,}375m &= 7{,}938{,}000 \\ 525(86{,}800 - 525m) + 278{,}250m &= 47{,}628{,}000 \\ 45{,}570{,}000 - 275{,}625m + 278{,}250m &= 47{,}628{,}000 \\ 2625m &= 2{,}058{,}000 \\ m &= 784 \end{aligned}$$

$$\begin{aligned} b &= \frac{86{,}800 - 525(784)}{6} \approx -54{,}133.333 \\ Y &= 784x - 54{,}133.33 \end{aligned}$$

(b) $Y = 784(105) - 54{,}133.33$
$\approx 28{,}186.67$

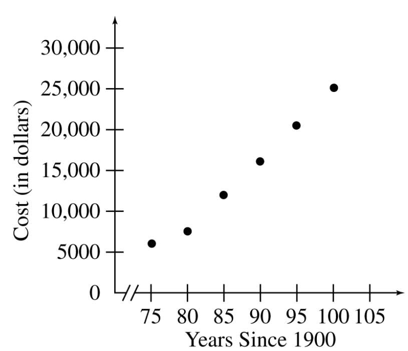
The average cost of a new car in the year 2005 is predicted to be about \$28,187.

(c)

$$r = \frac{6(7{,}938{,}000) - (525)(86{,}800)}{\sqrt{6(46{,}375) - 525^2} \cdot \sqrt{6(1{,}528{,}420{,}000) - 86{,}800^2}}$$
$$\approx .993$$

Yes, this indicates that the line fits the data points quite well.

(d)



No, the scatterplot suggests that the trend is linear.

48. **(a)**

x	y	xy	x^2	y^2
130	170	22,100	16,900	28,900
138	160	22,080	19,044	25,600
142	173	24,566	20,164	29,929
159	181	28,779	25,281	32,761
165	201	33,165	27,225	40,401
200	192	38,400	40,000	36,864
210	240	50,400	44,100	57,600
250	290	72,500	62,500	84,100
1394	1607	291,990	255,214	336,155

$$8b + 1394m = 1607$$
$$1394b + 255{,}214m = 291,990$$

$$8b = 1607 - 1394m$$
$$b = \frac{1607-1394m}{8}$$

$$1394\left(\frac{1607 - 1394m}{8}\right) + 255{,}214m = 291{,}990$$

$$1394(1607 - 1394m) + 2{,}041{,}712m = 2{,}335{,}920$$
$$2{,}240{,}158 - 1{,}943{,}236m + 2{,}041{,}712m = 2{,}335{,}920$$
$$98{,}476m = 95{,}762$$
$$m = .9724400$$
$$\approx .97$$

$$b = \frac{1607 - 1394(.97)}{8} \approx 31.85$$
$$Y = .97x + 31.85$$

(b) Let $x = 190$; find Y.

$$Y = .97(190) + 31.85$$
$$Y = 216.15 \approx 216$$

The cholesterol level for a person whose blood sugar level is 190 would be about 216.

(c)

$$r = \frac{8(291{,}990) - (1394)(1607)}{\sqrt{8(255{,}214) - 1394^2} \cdot \sqrt{8(336{,}155) - 1607^2}}$$
$$= .933814 \approx .93$$

50. Using the points $(90, 52.6)$ and $(100, 59.9)$,

$$m = \frac{59.9 - 52.6}{100 - 90} = \frac{7.3}{10} = .73$$

$$y - 59.9 = .73(x - 100)$$
$$y - 59.9 = .73x - 73$$
$$y = .73x - 13.1$$

Extended Application: Using Extrapolation to Predict Life Expectancy

1.

x	y	xy	x^2	y^2
1950	71.3	139,035	3,802,500	5083.69
1960	73.1	143,276	3,841,600	5343.61
1970	74.7	147,159	3,880,900	5580.09
1980	77.4	153,252	3,920,400	5990.76
1985	78.2	155,227	3,940,225	6115.24
1990	78.8	156,812	3,960,100	6209.44
1995	78.9	157,406	3,980,025	6225.21
13,830	532.4	1,052,167	27,325,750	40,548

$$7b + 13{,}830m = 532.4$$
$$13{,}830b + 27{,}325{,}750m = 1{,}052{,}167$$
$$7b = 532.4 - 13{,}830m$$
$$b = \frac{532.4 - 13{,}830m}{7}$$

$$13{,}830\left(\frac{532.4 - 13{,}830m}{7}\right) + 27{,}325{,}750m = 1{,}052{,}167$$

$$13{,}830(532.4 - 13{,}830m) + 191{,}280{,}250m = 7{,}365{,}169$$
$$7{,}363{,}092 - 191{,}268{,}900m + 191{,}280{,}250m = 7{,}365{,}169$$
$$11{,}350m = 2077$$
$$m \approx .183$$

$$b = \frac{532.4 - 13{,}830(.183)}{7} \approx -285$$

$$Y = -285 + .183x$$

2. Let $x = 1900$

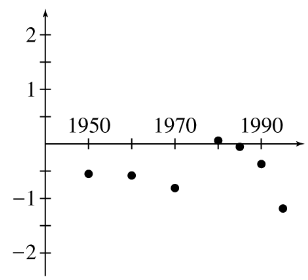
$$Y = -285 + .183(1900)$$
$$Y = 62.7$$

From the equation, the guess is the life expectancy of females born is 1900 is 62.7 years.

3. The poor prediction isn't surprising, since we were extrapolating far beyond the range of the original data.

4.

x	Predicted value	Residual
1950	71.850	−.550
1960	73.680	−.580
1970	75.510	−.810
1980	77.340	.060
1985	78.255	−.055
1990	79.170	−.370
1995	80.085	−1.185



5. It's not clear that any simple smooth function will fit this data–there seems to be a break in the pattern between 1970 and 1980. This will make it difficult to predict the life expectancy for females born in 2010.

6. You'll get 0 slope and 0 intercept, because you've already subtracted out the linear component of the data.

7. They used a regression equation of some kind to predict this value!

NONLINEAR FUNCTIONS

2.1 Properties of Functions

2. The x-value of 27 corresponds to two y-values, 69 and 50. In a function, each value of x must correspond to exactly one value of y.
The rule is not a function.

4. 9 corresponds to 3 and -3, 4 corresponds to 2 and -2, and 1 corresponds to -1 and 1.
The rule is not a function.

6. $y = \sqrt{x}$

Each x-value corresponds to exactly one y-value.
The rule is a function.

8. $x = y^4 - 1$

Solve the rule for y.

$$y^4 = 1 + x \quad \text{or} \quad y = \pm\sqrt[4]{1+x}$$

Each value of x (except -1) corresponds to two y-values

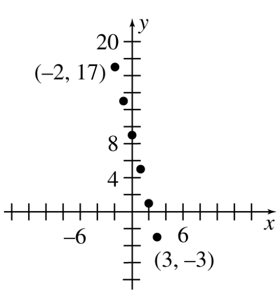
$y = \sqrt[4]{1+x}$ and $y = -\sqrt[4]{1+x}$.

The rule is not a function.

10. $y = -4x + 9$

x	-2	-1	0	1	2	3
y	17	13	9	5	1	-3

Pairs: $(-2, 17)$, $(-1, 13)$, $(0, 9)$, $(1, 5)$, $(2, 1)$, $(3, -3)$
Range: $\{-3, 1, 5, 9, 13, 17\}$

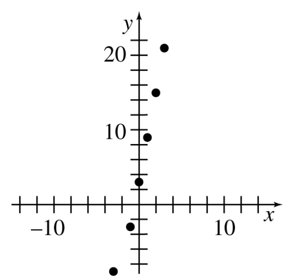


12. $6x - y = -3$

$$-y = -6x - 3$$
$$y = 6x + 3$$

x	-2	-1	0	1	2	3
y	-9	-3	3	9	15	21

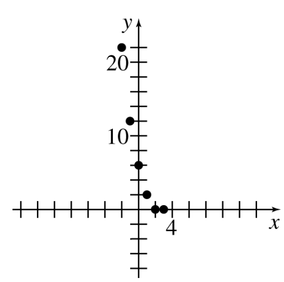
Pairs: $(-2, -9)$, $(-1, -3)$, $(0, 3)$, $(1, 9)$, $(2, 15)$, $(3, 21)$
Range: $\{-9, -3, 3, 9, 15, 21\}$



14. $y = (x - 2)(x - 3)$

x	-2	-1	0	1	2	3
y	20	12	6	2	0	0

Pairs: $(-2, 20)$, $(-1, 12)$, $(0, 6)$, $(1, 2)$, $(2, 0)$, $(3, 0)$
Range: $\{0, 2, 6, 12, 20\}$

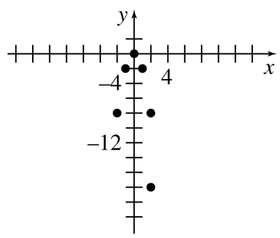


16. $y = -2x^2$

x	-2	-1	0	1	2	3
y	-8	-2	0	-2	-8	-18

Pairs: $(-2,-8),\ (-1,-2),\ (0,0),$
$(1,-2),\ (2,-8),\ (3,-18)$

Range: $\{-18,-8,-2,0\}$

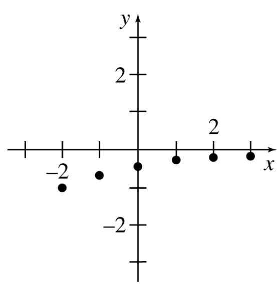


18. $y = \dfrac{-2}{x+4}$

x	-2	-1	0	1	2	3
y	-1	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{2}{5}$	$-\frac{1}{3}$	$-\frac{2}{7}$

Pairs: $(-2,-1),\ (-1,-\frac{2}{3}),\ (0,-\frac{1}{2}),$
$(1,-\frac{2}{5}),\ (2,-\frac{1}{3}),\ (3,-\frac{2}{7})$

Range: $\{-1,-\frac{2}{3},-\frac{1}{2},-\frac{2}{5},-\frac{1}{3},-\frac{2}{7}\}$

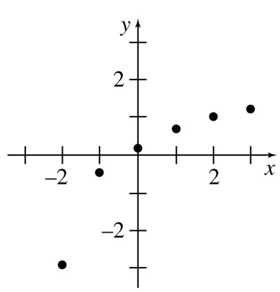


20. $y = \dfrac{2x+1}{x+3}$

x	-2	-1	0	1	2	3
y	-3	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{4}$	1	$\frac{7}{6}$

Pairs: $(-2,-3),\ (-1,-\frac{1}{2}),\ (0,\frac{1}{3}),$
$(1,\frac{3}{4}),\ (2,1),\ (3,\frac{7}{6})$

Range: $[-3,-\frac{1}{2},\frac{1}{3},\frac{3}{4},1,\frac{7}{6}\}$



22. $f(x) = x + 2$

x can take on any value, so the domain is the set of real numbers, which is written $(-\infty,\infty)$.

24. $f(x) = (x-2)^2$

x can take on any value, so the domain is the set of real numbers, $(-\infty,\infty)$.

26. $f(x) = |x-1|$

x can take on any value, so the domain is the set of real numbers, $(-\infty,\infty)$.

28. $f(x) = (3x+5)^{1/2} = \sqrt{3x+5}$

For $f(x)$ to be a real number,

$$\begin{aligned} 3x+5 &\ge 0 \\ 3x &\ge -5 \\ \frac{1}{3}(3x) &\ge \frac{1}{3}(-5) \\ x &\ge -\frac{5}{3}. \end{aligned}$$

In interval notation, the domain is $[-\frac{5}{3},\infty)$.

30. $f(x) = \dfrac{-8}{x^2-36}$

In order for $f(x)$ to be a real number, x^2-36 cannot be equal to 0.

When $x^2 - 36 = 0,$

$$\begin{aligned} x^2 &= 36 \\ x = 6 \quad &\text{or} \quad x = -6. \end{aligned}$$

Thus, the domain is any real number except 6 or -6. In interval notation, the domain is

$$(-\infty,-6)\cup(-6,6)\cup(6,\infty).$$

32. $f(x) = -\sqrt{\dfrac{5}{x^2+36}}$

x can take on any value. No choice for x will produce a zero in the denominator. Also, no choice for x will produce a negative number under the radical. The domain is $(-\infty,\infty)$.

34. $f(x) = \sqrt{15x^2 + x - 2}$

The expression under the radical must be nonnegative.

$$15x^2 + x - 2 \geq 0$$
$$(5x + 2)(3x - 1) \geq 0$$

Solve $(5x + 2)(3x - 1) = 0$.

$$5x + 2 = 0 \quad \text{or} \quad 3x - 1 = 0$$
$$5x = -2 \qquad 3x = 1$$
$$x = -\frac{2}{5} \quad \text{or} \quad x = \frac{1}{3}$$

Use these numbers to divide the number line into 3 intervals, $\left(-\infty, -\frac{2}{5}\right)$, $\left(-\frac{2}{5}, \frac{1}{3}\right)$, and $\left(\frac{1}{3}, \infty\right)$. Only values in the intervals $\left(-\infty, -\frac{2}{5}\right)$ and $\left(\frac{1}{3}, \infty\right)$ satisfy the inequality. The domain is $\left(-\infty, -\frac{2}{5}\right] \cup \left[\frac{1}{3}, \infty\right)$.

36. $f(x) = \sqrt{\dfrac{x+1}{x-1}}$

For $f(x)$ to be a real number,

$$\frac{x+1}{x-1} \geq 0.$$

Solve $\dfrac{x+1}{x-1} = 0$.

$$(x-1)\left(\frac{x+1}{x-1}\right) = (x-1)(0)$$
$$x + 1 = 0$$
$$x = -1$$

Also, $x \neq 1$, since this will cause the denominator to be zero.
Use the numbers -1 and 1 to divide the number line into 3 intervals, $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.
Only the values in the intervals $(-\infty, -1]$ and $(1, \infty)$ satisfy the inequality.
The value -1 is included, since the numerator may be zero, but the value 1 is not included since it would make the denominator zero.
The domain is $(-\infty, -1] \cup (1, \infty)$.

38. By reading the graph, the domain is all numbers greater than or equal to -5. The range is all numbers greater than or equal to 0.
Domain: $[-5, \infty)$ Range: $[0, \infty)$

40. By reading the graph, both x and y can take on any values.
Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$

42. $f(x) = (x + 3)(x - 4)$

(a) $f(4) = (4 + 3)(4 - 4) = (7)(0) = 0$

(b) $$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2} + 3\right)\left(-\frac{1}{2} - 4\right) = \left(\frac{5}{2}\right)\left(-\frac{9}{2}\right) = -\frac{45}{4}$$

(c) $f(a) = [(a) + 3][(a) - 4] = (a + 3)(a - 4)$

(d) $$f\left(\frac{2}{m}\right) = \left(\frac{2}{m} + 3\right)\left(\frac{2}{m} - 4\right) = \left(\frac{2 + 3m}{m}\right)\left(\frac{2 - 4m}{m}\right) = \frac{(2 + 3m)(2 - 4m)}{m^2}$$
$$\text{or} \quad \frac{2(2 + 3m)(1 - 2m)}{m^2}$$

(e)
$$f(x) = 1$$
$$(x + 3)(x - 4) = 1$$
$$x^2 - x - 12 = 1$$
$$x^2 - x - 13 = 0$$

$$x = \frac{1 \pm \sqrt{1 + 52}}{2}$$
$$x = \frac{1 \pm \sqrt{53}}{2}$$
$$x \approx -3.140 \text{ or } x \approx 4.140$$

44. $f(x) = \dfrac{3x - 5}{2x + 3}$

(a) $$f(4) = \frac{3(4) - 5}{2(4) + 3} = \frac{12 - 5}{8 + 3} = \frac{7}{11}$$

(b) $$f\left(-\frac{1}{2}\right) = \frac{3\left(-\frac{1}{2}\right) - 5}{2\left(-\frac{1}{2}\right) + 3} = \frac{-\frac{3}{2} - 5}{-1 + 3} = \frac{-\frac{3}{2} - \frac{10}{2}}{2} = \frac{-\frac{13}{2}}{2} = -\frac{13}{4}$$

(c) $$f(a) = \frac{3(a) - 5}{2(a) + 3} = \frac{3a - 5}{2a + 3}$$

(d) $$f\left(\frac{2}{m}\right) = \frac{3\left(\frac{2}{m}\right) - 5}{2\left(\frac{2}{m}\right) + 3} = \frac{\frac{6}{m} - \frac{5m}{m}}{\frac{4}{m} + \frac{3m}{m}} = \frac{\frac{6-5m}{m}}{\frac{4+3m}{m}} = \frac{6 - 5m}{m} \cdot \frac{m}{4 + 3m} = \frac{6 - 5m}{4 + 3m}$$

(e)
$$f(x) = 1$$
$$\frac{3x-5}{2x+3} = 1$$
$$3x - 5 = 2x + 3$$
$$x = 8$$

46. The domain is all real numbers between the end points of the curve, or $[-2, 4]$.
The range is all real numbers between the minimum and maximum values of the function or $[0, 5]$.

(a) $f(-2) = 5$ **(b)** $f(0) = 0$

(c) $f\left(\frac{1}{2}\right) = 1$

(d) From the graph, if $f(x) = 1,\ x = -.2,\ .5,\ 1.2,$ or 2.8.

48. The domain is all real numbers between the end points of the curve, or $[-2, 4]$.
The range is all real numbers between the minimum and maximum values of the function, or in this case, $\{3\}$.

(a) $f(-2) = 3$

(b) $f(0) = 3$

(c) $f\left(\frac{1}{2}\right) = 3$

(d) From the graph, $f(x)$ is 1 nowhere.

50.
$$\begin{aligned} f(x) &= 6x^2 - 2 \\ f(2r-1) &= 6(2r-1)^2 - 2 \\ &= 6(4r^2 - 4r + 1) - 2 \\ &= 24r^2 - 24r + 6 - 2 \\ &= 24r^2 - 24r + 4 \end{aligned}$$

52. $g(z - p)$
$$\begin{aligned} &= (z-p)^2 - 2(z-p) + 5 \\ &= z^2 - 2zp + p^2 - 2z + 2p + 5 \end{aligned}$$

54.
$$\begin{aligned} g\left(-\frac{5}{z}\right) &= \left(-\frac{5}{z}\right)^2 - 2\left(-\frac{5}{z}\right) + 5 \\ &= \frac{25}{z^2} + \frac{10}{z} + 5 \\ &= \frac{25}{z^2} + \frac{10z}{z^2} + \frac{5z^2}{z^2} \\ &= \frac{25 + 10z + 5z^2}{z^2} \end{aligned}$$

56. A vertical line drawn anywhere through the graph will intersect the graph in only one place. The graph represents a function.

58. A vertical line drawn through the graph will intersect the graph in two or more places. The graph does not represent a function.

60. A vertical line is not a function since the one x-value in the domain corresponds to more than one, in fact, infinitely many y-values. The graph does not represent a function.

62. $f(x) = 8 - 3x^2$

(a)
$$\begin{aligned} f(x+h) &= 8 - 3(x+h)^2 \\ &= 8 - 3(x^2 + 2xh + h^2) \\ &= 8 - 3x^2 - 6xh - 3h^2 \end{aligned}$$

(b) $f(x+h) - f(x)$
$$\begin{aligned} &= (8 - 3x^2 - 6xh - 3h^2) \\ &\quad - (8 - 3x^2) \\ &= 8 - 3x^2 - 6xh - 3h^2 - 8 + 3x^2 \\ &= -6xh - 3h^2 \end{aligned}$$

(c)
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{-6xh - 3h^2}{h} \\ &= \frac{h(-6x - 3h)}{h} \\ &= -6x - 3h \end{aligned}$$

64. $f(x) = -4x^2 + 3x + 2$

(a) $f(x+h)$
$$\begin{aligned} &= -4(x+h)^2 + 3(x+h) + 2 \\ &= -4(x^2 + 2hx + h^2) + 3x + 3h + 2 \\ &= -4x^2 - 8hx - 4h^2 + 3x + 3h + 2 \end{aligned}$$

(b) $f(x+h) - f(x)$
$$\begin{aligned} &= -4x^2 - 8hx - 4h^2 + 3x + 3h + 2 \\ &\quad - (-4x^2 + 3x + 2) \\ &= -4x^2 - 8hx - 4h^2 + 3x + 3h + 2 \\ &\quad + 4x^2 - 3x - 2 \\ &= -8hx - 4h^2 + 3h \end{aligned}$$

(c)
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{-8hx - 4h^2 + 3h}{h} \\ &= \frac{h(-8x - 4h + 3)}{h} \\ &= -8x - 4h + 3 \end{aligned}$$

66. $f(x) = -\dfrac{1}{x^2}$

(a) $$f(x+h) = -\frac{1}{(x+h)^2} = -\frac{1}{x^2+2xh+h^2}$$

(b) $$\begin{aligned} f(x+h) - f(x) &= -\frac{1}{x^2+2xh+h^2} - \left(-\frac{1}{x^2}\right) \\ &= -\frac{1}{x^2+2xh+h^2} + \frac{1}{x^2} \\ &= -\frac{x^2}{x^2(x^2+2xh+h^2)} + \frac{(x^2+2xh+h^2)}{x^2(x^2+2xh+h^2)} \\ &= \frac{-x^2+x^2+2xh+h^2}{x^2(x^2+2xh+h^2)} \\ &= \frac{2xh+h^2}{x^2(x^2+2xh+h^2)} \end{aligned}$$

(c) $$\frac{f(x+h)-f(x)}{h} = \frac{\frac{h(2x+h)}{x^2(x^2+2xh+h^2)}}{h} = \frac{2x+h}{x^2(x^2+2xh+h^2)}$$

68. If x is a whole number of days, the cost of renting a saw in dollars is $S(x) = 7x+4$. For x in whole days and a fraction of a day, substitute the next whole number for x in $7x+4$, because a fraction of a day is charged as a whole day.

(a) $S\left(\frac{1}{2}\right) = S(1) = 7(1) + 4 = 11$

The cost is \$11.

(b) $S(1) = 7(1) + 4 = 11$

The cost is \$11.

(c) $S\left(1\frac{1}{4}\right) = S(2) = 7(2) + 4 = 14 + 4 = 18$

The cost is \$18.

(d) $S\left(3\frac{1}{2}\right) = S(4) = 7(4) + 4 = 28 + 4 = 32$

The cost is \$32.

(e) $S(4) = 7(4) + 4 = 28 + 4 = 32$

The cost is \$32.

(f) $S\left(4\frac{1}{10}\right) = S(5) = 7(5) + 4 = 35 + 4 = 39$

The cost is \$39.

(g) $S\left(4\frac{9}{10}\right) = S(5) = 7(5) + 4 = 35 + 4 = 39$

The cost is \$39.

(h) To continue the graph, continue the horizontal bars up and to the right.

(i) The independent variable is x, the number of full and partial days.

(j) The dependent variable is S, the cost of renting a saw.

70. **(a)** The curve in the graph crosses the point with x-coordinate 17:37 and y-coordinate of approximately 140. So, at time 17 hours, 37 minutes the whale reaches a depth of about 140 m.

(b) The curve in the graph crosses the point with x-coordinate 17:39 and y-coordinate of approximately 240. So, at time 17 hours, 39 minutes the whale reaches a depth of about 240 m.

72. **(a)(i)** By the given function f, a muskrat weighing 800 g expends

$$\begin{aligned} f(800) &= .01(800)^{.88} \\ &\approx 3.6, \text{ or approximately} \end{aligned}$$

3.6 kcal/km when swimming at the surface of the water.

(ii) A sea otter weighing 20,000 g expends

$$\begin{aligned} f(20{,}000) &= .01(20{,}000)^{.88} \\ &\approx 61, \text{ or approximately} \end{aligned}$$

61 kcal/km when swimming at the surface of the water.

(b) If z is the number of kilogram of an amimal's weight, then $x = g(z) = 1000z$ is the number of grams since 1 kilogram equals 1000 grams.

74. **(a)** $P = 2L + 2W$

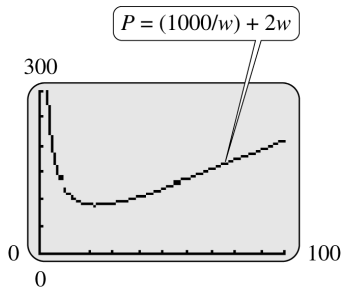
However, $LW = 500$, so $L = \dfrac{500}{W}$.

$$P(W) = 2\left(\frac{500}{W}\right) + 2W$$

$$P(W) = \frac{1000}{W} + 2W$$

(b) Since $L = \frac{500}{W}$, $W \neq 0$ but W could be any positive value. Therefore, the domain of P is $0 < W < \infty$.

(c)



2.2 Quadratic Functions; Translation and Reflection

4. The graph of $y = (x-3)^2 + 2$ is the graph of $y = x^2$ translated 3 units to the right and 2 units upward.
This is graph A.

6. The graph of $y = -(3-x)^2 + 2$ is the same as the graph of $y = -(x-3)^2 + 2$. This is the graph of $y = x^2$ reflected in the x-axis, translated 3 units to the right, and translated 2 units upward.
This is graph C.

8. $y = x^2 + 6x + 5$
$y = (x+5)(x+1)$

Set $y = 0$ to find the x-intercepts.

$$0 = (x+5)(x+1)$$
$$x = -5,\ x = -1$$

The x-intercepts are -5 and -1.
Set $x = 0$ to find the y-intercept.

$$y = 0^2 + 6(0) + 5$$
$$y = 5$$

The y-intercept is 5.
The x-coordinate of the vertex is

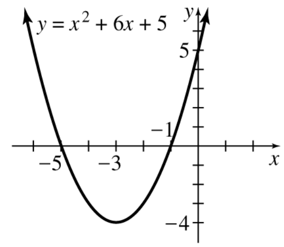
$$x = \frac{-b}{2a} = \frac{-6}{2} = -3.$$

Substitute to find the y-coordinate.

$$y = (-3)^2 + 6(-3) + 5 = 9 - 18 + 5 = -4$$

The vertex is $(-3, -4)$.

The axis is $x = -3$, the vertical line through the vertex.



10. $y = -2x^2 - 12x - 16$
$= -2(x^2 + 6x + 8)$
$= -2(x+4)(x+2)$

Let $y = 0$.

$$0 = -2(x+4)(x+2)$$
$$x = -4,\ x = -2$$

-4 and -2 are the x-intercepts.
Let $x = 0$.

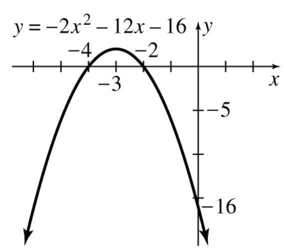
$$y = -2(0)^2 + 12(0) - 16$$

-16 is the y-intercept.

Vertex: $x = \dfrac{-b}{2a} = \dfrac{12}{-4} = -3$

$$y = -2(-3)^2 - 12(-3) - 16$$
$$= -18 + 36 - 16 = 2$$

The vertex is $(-3, 2)$.
The axis is $x = -3$, the vertical line through the vertex.



12. $y = 2x^2 + 12x - 16$

Let $y = 0$.

$$2x^2 + 12x - 16 = 0$$
$$x^2 + 6x - 8 = 0$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(-8)}}{2(1)}$$
$$= \frac{-6 \pm \sqrt{68}}{2} = \frac{-6 \pm 2\sqrt{17}}{2}$$
$$= -3 \pm \sqrt{17}$$

The x-intercepts are $-3 \pm \sqrt{17} \approx 1.12$ or -7.12.

Let $x = 0$.

$$y = 2(0)^2 + 12(0) - 16 = -16$$

The y-intercept is -16.

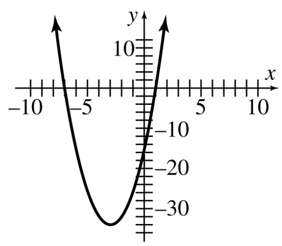
The x-coordinate of the vertex is

$$x = \frac{-b}{2a} = -\frac{12}{4} = -3.$$

If $x = -3$,

$$y = 2(-3)^2 + 12(-3) - 16 = 18 - 52 = -34.$$

The vertex is $(-3, -34)$.
The axis is $x = -3$.



$f(x) = 2x^2 + 12x - 16$

14. $f(x) = 2x^2 - 4x + 5$

Let $f(x) = 0$.

$0 = 2x^2 - 4x + 5$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(5)}}{2(2)}$$
$$= \frac{4 \pm \sqrt{16 - 40}}{4} = \frac{4 \pm \sqrt{-24}}{4}$$

Since the radicand is negative, there are no x-intercepts.

Let $x = 0$.

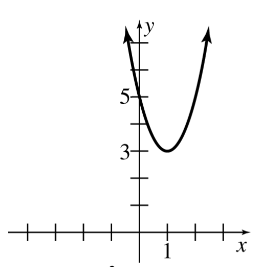
$$y = 2(0)^2 - 4(0) + 5$$
$$y = 5$$

5 is the y-intercept..

Vertex: $x = \frac{-b}{2a} = \frac{-(-4)}{2(2)} = \frac{4}{4} = 1$

$$y = 2(1)^2 - 4(1) + 5 = 2 - 4 + 5 = 3$$

The vertex is $(1, 3)$.
The axis is $x = 1$.



$f(x) = 2x^2 - 4x + 5$

16. $f(x) = -\frac{1}{3}x^2 + 2x + 4$

Let $f(x) = 0$.

$$0 = -\frac{1}{3}x^2 + 2x + 4$$

Multiply by -3 to clear fractions.

$$0 = x^2 - 6x - 12$$

Use the quadratic formula.

$$x = \frac{6 \pm \sqrt{36 - 4(1)(-12)}}{2}$$
$$= \frac{6 \pm \sqrt{84}}{2} = \frac{6 \pm 2\sqrt{21}}{2}$$
$$= \frac{2(3 \pm \sqrt{21})}{2} = 3 \pm \sqrt{21}$$

The x-intercepts are $3 + \sqrt{21} \approx 7.58$ and $3 - \sqrt{21} \approx -1.58$.

Let $x = 0$.

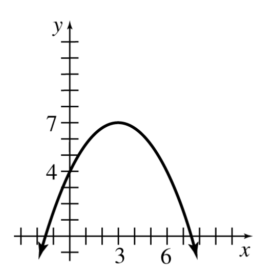
$$y = -\frac{1}{3}(0)^2 + 2(0) + 4$$
$$y = 4$$

4 is the y-intercept.

Vertex: $x = \frac{-b}{2a} = \frac{-2}{2\left(-\frac{1}{3}\right)} = \frac{-2}{-\frac{2}{3}} = 3$

$$y = -\frac{1}{3}(3)^2 + 2(3) + 4$$
$$= -3 + 6 + 4 = 7$$

The vertex is $(3, 7)$.
The axis is $x = 3$.



$f(x) = -(1/3)x^2 + 2x + 4$

18. $y = \frac{2}{3}x^2 - \frac{8}{3}x + \frac{5}{3}$

Let $y = 0$.

$$0 = \frac{2}{3}x^2 - \frac{8}{3}x + \frac{5}{3}$$

Multiply by 3.

$0 = 2x^2 - 8x + 5$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(5)}}{2(2)}$$
$$= \frac{8 \pm \sqrt{64 - 40}}{4} = \frac{8 \pm \sqrt{24}}{4}$$
$$= \frac{8 \pm 2\sqrt{6}}{4} = 2 \pm \frac{\sqrt{6}}{2}$$

The x-intercepts are $2 + \frac{\sqrt{6}}{2} \approx 3.22$ and $2 - \frac{\sqrt{6}}{2} \approx .78$.

Let $x = 0$.

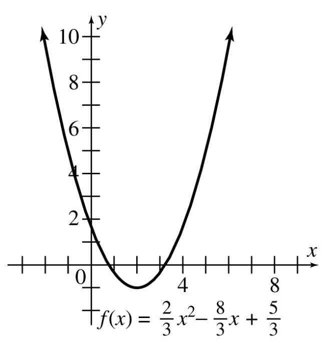
$$y = \frac{2}{3}(0)^2 - \frac{8}{3}(0) + \frac{5}{3}$$

$\frac{5}{3}$ is the y-intercept.

Vertex: $x = \frac{-b}{2a} = \frac{-\left(-\frac{8}{3}\right)}{2\left(\frac{2}{3}\right)} = \frac{\frac{8}{3}}{\frac{4}{3}} = 2$

$$y = \frac{2}{3}(2)^2 - \frac{8}{3}(2) + \frac{5}{3}$$
$$= \frac{8}{3} - \frac{16}{3} + \frac{5}{3} = -\frac{3}{3} = -1$$

The vertex is $(2, -1)$.
The axis is $x = 2$.

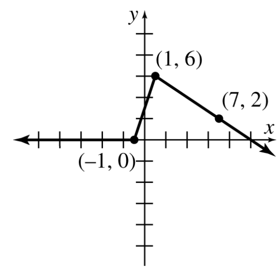


20. The graph of $y = \sqrt{x+2} - 4$ is the graph of $y = \sqrt{x}$ translated 2 units to the left and 4 units downward.
This is graph D.

22. The graph of $y = \sqrt{-x+2} - 4$ is the graph of $y = \sqrt{-(x-2)} - 4$, which is the graph of $y = \sqrt{x}$ reflected in the y-axis, translated 2 units to the right, and translated 4 units downward.
This is graph C.

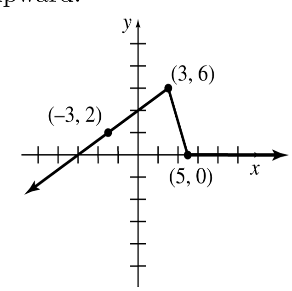
24. The graph of $y = -\sqrt{x+2} - 4$ is the graph of $y = \sqrt{x}$ reflected in the x-axis, translated 2 units to the left, and translated 4 units downward.
This is graph E.

26. The graph of $y = f(x-2) + 2$ is the graph of $y = f(x)$ translated 2 units to the right and 2 units upward.



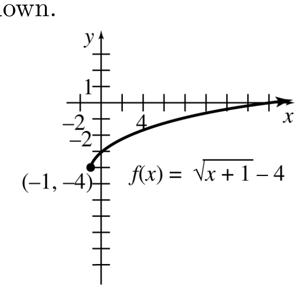
28. $y = f(2 - x) + 2$
$y = f[-(x-2)] + 2$

This is the graph of $y = f(x)$ reflected in the y-axis, translated 2 units to the right, and translated 2 units upward.



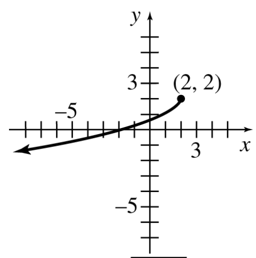
30. $f(x) = \sqrt{x+1} - 4$

Translate the graph of $f(x) = \sqrt{x}$ 1 unit left and 4 units down.



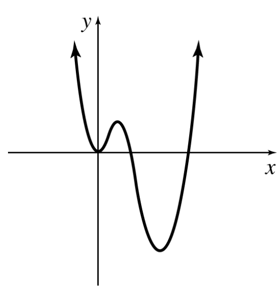
32. $f(x) = -\sqrt{2-x} + 2 = -\sqrt{-(x-2)} + 2$

Translate the graph of $f(x) = \sqrt{x}$ 2 units right and 2 units up.
Reflect vertically and horizontally.

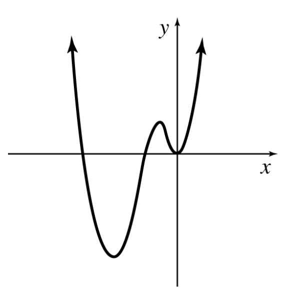


$f(x) = -\sqrt{2-x} + 2$

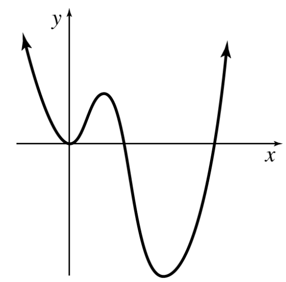
34. If $1 < a$, the graph of $f(ax)$ will be taller and thinner than the graph of $f(x)$. Multiplying x by a constant greater than 1 pairs x-values of smaller absolute value with y-values of points for which the x-values have larger absolute value.



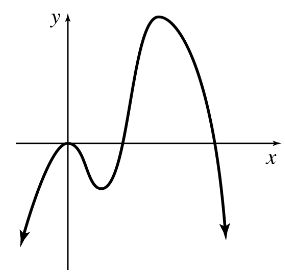
36. If $a < -1$, the graph of $f(ax)$ will be reflected horizontally, since a is negative. It will also be taller and thinner.



38. If $1 < a$ (or $a > 1$), the graph of $af(x)$ will be taller than the graph of $f(x)$. The absolute value of the y-value will be larger than the original y-values, while the x-values will remain the same.



40. If $a < -1$, the graph of $af(x)$ will be reflected vertically. It will also be taller than the graph of $f(x)$ since the absolute value of each y-value will be larger than the original y-values, while the x-values stay the same.

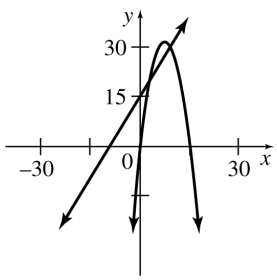


42. **(a)** Since the graph of $y = f(x)$ is reflected vertically to obtain the graph of $y = -f(x)$, the y-intercept of the graph of $y = -f(x)$ is $-b$.

(b) Since the graph of $y = f(x)$ is reflected horizontally to obtain the graph of $y = f(-x)$, the y-intercept is unchanged. The y-intercept of the graph of $y = f(-x)$ is b.

(c) Since the graph of $y = f(x)$ is reflected both horizontally and vertically to obtain the graph of $y = -f(-x)$, the y-intercept of the graph of $y = -f(-x)$ is $-b$.

44. **(a)**



(b) Break-even quantities are values of x = number of widgets for which revenue and cost are equal. Set $R(x) = C(x)$ and solve for x.

$$\begin{aligned}
-\frac{x^2}{2} + 8x &= \frac{3}{2}x + 15 \\
-x^2 + 16x &= 3x + 30 \\
x^2 - 13x + 30 &= 0 \\
(x-10)(x-3) &= 0 \\
x - 10 = 0 \quad &\text{or} \quad x - 3 = 0 \\
x = 10 \quad &\text{or} \quad x = 3
\end{aligned}$$

So, the break-even quantities are 3 and 10. The minimum break-even quantity is $x = 3$.

(c) The maximum revenue occurs at the vertex of R. Since $R(x) = -\frac{x^2}{2} + 8x$, then the x-coordinate of the vertex is

$$x = \frac{-b}{2a} = \frac{-8}{2\left(-\frac{1}{2}\right)} = 8.$$

So, the maximum revenue is

$$R(8) = -\frac{8^2}{2} + 8(8) = 32.$$

(d) The maximum profit is the maximum difference $R(x) - C(x)$. Since

$$\begin{aligned}
P(x) = R(x) - C(x) &= -\frac{x^2}{2} + 8x - \left(\frac{3}{2}x + 15\right) \\
&= -\frac{x^2}{2} + \frac{13}{2}x - 15
\end{aligned}$$

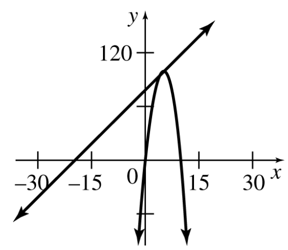
is a quadratic function, we can find the maximum profit by finding the vertex of P. This occurs at

$$x = -\frac{b}{2a} = \frac{-\frac{13}{2}}{2\left(-\frac{1}{2}\right)} = \frac{13}{2}.$$

Therefore, the maximum profit is

$$P\left(\frac{13}{2}\right) = \frac{-\left(\frac{13}{2}\right)^2}{2} + \frac{13}{2}\left(\frac{13}{2}\right) - 15 = 6.125.$$

46. **(a)**



(b) Break-even quantities are values of x for which revenue equals cost.
Sete $R(x) = C(x)$ and solve for x.

$$\begin{aligned}
-4x^2 + 40x &= 4x + 77 \\
4x^2 - 36x + 77 &= 0 \\
4x^2 - 14x - 22x + 77 &= 0 \\
2x(2x-7) - 11(2x-7) &= 0 \\
(2x-7)(2x-11) &= 0 \\
2x - 7 = 0 \quad &\text{or} \quad 2x - 11 = 0 \\
x = 3.5 \quad &\text{or} \quad x = 5.5
\end{aligned}$$

So, the break-even quantites are 3.5 and 5.5. The minimum break-even quantity is $x = 3.5$.

(c) The maximum revenue occurs at the vertex of R. Since $R(x) = -4x^2 + 40x$, then x-coordinate of the vertex is

$$x = \frac{-b}{2a} = \frac{-40}{2(-4)} = 5.$$

So, the maximum revenue is

$$R(5) = -4(5)^2 + 40(5) = 100.$$

(d) The maximum profit is the maximum difference $R(x) - C(x)$. Since

$$\begin{aligned}
P(x) &= R(x) - C(x) \\
&= -4x^2 + 40x - (4x + 77) \\
&= -4x^2 + 36x - 77
\end{aligned}$$

is a quadratic function, we can find the maximum profit by finding the vertex of P. This occurs at

$$x = -\frac{b}{2a} = -\frac{36}{2(-4)} = \frac{9}{2} = 4.5.$$

Therefore, the maximum profit is

$$P\left(\frac{9}{2}\right) = -4\left(\frac{9}{2}\right)^2 + 36\left(\frac{9}{2}\right) - 77 = 4.$$

48. **(a)** The revenue is

$$R(x) = (\text{Price per ticket}) \cdot (\text{Number of people flying}).$$

Number of people flying $= 100 - x$
Price per ticket $= 200 + 4x$

$$R(x) = (200 + 4x)(100 - x) = 20{,}000 + 200x - 4x^2$$

(b) $R(x) = -4x^2 + 200x + 20{,}000$

x-intercepts:

$$0 = (200 + 4x)(100 - x)$$
$$x = -50 \quad \text{or} \quad x = 100$$

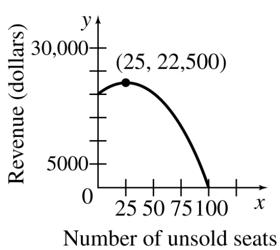
y-intercept:

$$y = -4(0)^2 + 200(0) + 20{,}000 = 20{,}000$$

Vertex: $x = \dfrac{-b}{2a} = \dfrac{-200}{-8} = 25$

$$y = -4(25)^2 + 200(25) + 20{,}000 = 22{,}500$$

This is a parabola which opens downward. The vertex is at (25, 22,500).



(c) The maximum revenue occurs at the vertex, (25, 22,500).
This will happen when $x = 25$, or there are 25 unsold seats.

(d) The maximum revenue is \$22,500, as seen from the graph.

50. Let $x =$ the number of weeks to wait.

(a) Income per pound (in cents):

$$40 - 2x$$

(b) Yield in pounds per tree:

$$100 + 5x$$

(c) Revenue per tree (in cents):

$$R(x) = (100 + 5x)(40 - 2x)$$
$$R(x) = 4000 - 10x^2$$

(d) Find the vertex.

$$x = \frac{-b}{2a} = \frac{0}{-20} = 0$$
$$y = 4000 - 10(0)^2 = 4000$$

The vertex is $(0, 4000)$.
To produce maximum revenue, wait 0 weeks. Pick the peaches now.

(e) $R(0) = 4000 - 10(0)^2 = 4000$

or the maximum revenue is 4000 cents per tree or \$40.00 per tree.

52. $S(x) = -\dfrac{1}{4}(x - 10)^2 + 40$ for $0 \le x \le 10$

(a) $S(0) = -\dfrac{1}{4}(0 - 10)^2 + 40 = -25 + 40 = 15$

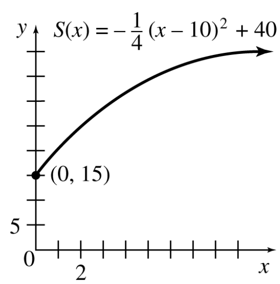
The increase in sales is \$15,000.

(b) For \$10,000, $x = 10$.

$$S(10) = -\frac{1}{4}(10 - 10)^2 + 40$$
$$S(10) = 40$$

The increase in sales is \$40,000.

(c) The graph of $S(x) = -\frac{1}{4}(x - 10)^2 + 40$ is the graph of $y = -\frac{1}{4}x^2$ translated 10 units to the right and 40 units upward.



54. $f(x) = -.2369x^2 + 1.425x + 6.905$

This function defines a parabola opening downward, so the maximum percent is at the vertex.
The x-coordinate of the vertex is

$$x = \frac{-b}{2a} = \frac{-1.425}{2(-.2369)} \approx 3.0076$$

Since $x = 0$ represents the year 1992, the percent of freshmen reached its maximum in 1992 + 3, or 1995.
The domain of $f(x)$ is $0 \le x \le 6$.

56. **(a)** The vertex of the quadratic function $y = .057x - .001x^2$ is at

$$x = -\frac{b}{2a} = -\frac{.057}{2(-.001)} = 28.5.$$

Since the coefficient of the leading term, $-.001$, is negative, then the graph of the function opens downward, so a maximum is reached at 28.5 weeks of gestation.

(b) The maximum splenic artery resistance reached at the vertex is

$$\begin{aligned} y &= .057(28.5) - .001(28.5)^2 \\ &\approx .81. \end{aligned}$$

(c) The splenic artery resistance equals 0, when $y = 0$.

$$\begin{aligned} .057x - .001x^2 &= 0 && \text{Substitute in the expression in } x \text{ for } y. \\ x(.057 - .001x) &= 0 && \text{Factor.} \\ x = 0 \text{ or } .057 - .001x &= 0 && \text{Set each factor equal to 0.} \\ x &= \frac{.057}{.001} = 57 \end{aligned}$$

So, the splenic artery resistance equals 0 at 0 weeks or 57 weeks of gestation.
No, this is not reasonable because at $x = 0$ or 57 weeks, the fetus does not exist.

58. **(a)** $f(x) = -19.321x^2 + 3608.7x - 168,310$

$$\frac{-b}{2a} = \frac{-3608.7}{2(-19.321)} \approx 93.388$$

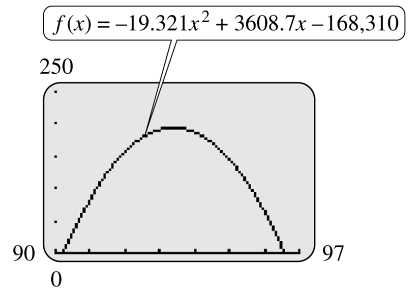
The value $x = 93.388$ corresponds to 1993.

(b) $f(93.388)$
$$\begin{aligned} &= -19.321(93.388)^2 + 3608.7(93.388) \\ &\quad - 168,310 \\ &\approx 194.68 \end{aligned}$$

The maximum amount spent is approximately \$195 million.

(c) Graph

$f(x) = -19.321x^2 + 3608.7x - 168{,}310.$



60. $h = 32t - 16t^2$
$= -16t^2 + 32t$

(a) Find the vertex.

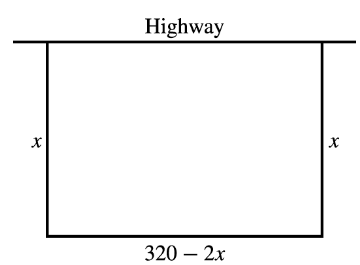
$$x = \frac{-b}{2a} = \frac{-32}{-32} = 1$$
$$\begin{aligned} y &= -16(1)^2 + 32(1) \\ &= 16 \end{aligned}$$

The vertex is $(1, 16)$, so the maximum height is 16 ft.
(b) When the object hits the ground, $h = 0$, so

$$\begin{aligned} 32t - 16t^2 &= 0 \\ 16t(2 - t) &= 0 \\ t = 0 \quad \text{or} \quad t &= 2. \end{aligned}$$

When $t = 0$, the object is thrown upward. When $t = 2$, the object hits the ground; that is, after 2 sec.

62. Let $x =$ the width.
Then $320 - 2x =$ the length.



$$\text{Area} = x(320 - 2x) = -2x^2 + 320x$$

Find the vertex:

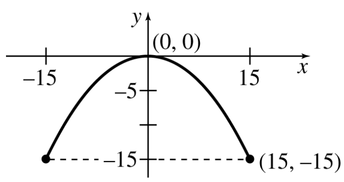
$$x = \frac{-b}{2a} = \frac{-320}{-4} = 80$$

$$y = -2(80)^2 + 320(80) = 12,800$$

The graph of the area function is a parabola with vertex (80, 12,800).
The maximum area of 12,800 sq ft occurs when the width is 80 ft and the length is

$$320 - 2x = 320 - 2(80) = 160 \text{ ft.}$$

64. Draw a sketch of the arch with the vertex at the origin.



Since the arch is a parabola that opens downward, the equation of the parabola is the form $y = a(x-h)^2 + k$, where the vertex $(h, k) = (0, 0)$ and $a < 0$. That is, the equation is of the form $y = ax^2$.
Since the arch is 30 meters wide at the base and 15 meters high, the points $(15, -15)$ and $(-15, -15)$ are on the parabola. Use $(15, -15)$ as one point on the parabola.

$$-15 = a(15)^2$$
$$a = \frac{-15}{15^2} = -\frac{1}{15}$$

So, the equation is

$$y = -\frac{1}{15}x^2.$$

Ten feet from the ground (the base) is at $y = -5$. Substitute -5 for y and solve for x.

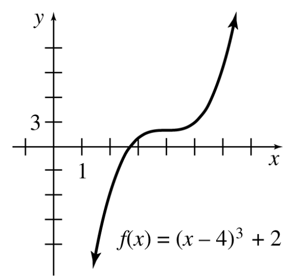
$$-5 = -\frac{1}{15}x^2$$
$$x^2 = -5(-15) = 75$$
$$x = \pm\sqrt{75} = \pm 5\sqrt{3}$$

The width of the arch ten feet from the ground is then

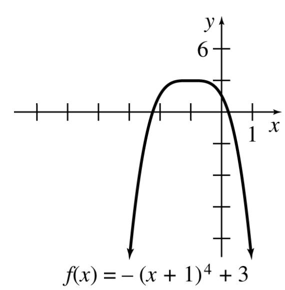
$$5\sqrt{3} - (-5\sqrt{3}) = 10\sqrt{3} \text{ meters}$$
$$\approx 17.32 \text{ meters.}$$

2.3 Polynomial and Rational Functions

4. The graph of $f(x) = (x-4)^3 + 2$ is the graph of $y = x^3$ translated 4 units to the right and 3 units upward.



6. The graph of $f(x) = -(x+1)^4 + 3$ is the graph of $y = x^4$ reflected in the x-axis, and translated 1 unit to the left and 3 units upward.



8. The graph of $y = -x^3 + 4x^2 + 3x - 8$ has the right end down, the left end up, at most two turning points, and a y-intercept of -8.
The is graph C.

10. The graph of $y = 2x^3 + 4x + 5$ has the right end up, the left end down, at most two turning points, and a y-intercept of 5.
This is graph B.

12. The graph of $y = x^4 + 4x^3 - 20$ has both ends up, at most three turning points, and a y-intercept of -20.
This is graph F.

14. The graph of $y = .7x^5 - 2.5x^4 - x^3 + 8x^2 + x + 2$ has the right end up, the left end down, at most four turning points, and a y-intercept of 2.
This is graph H.

16. The graph of $y = \frac{2x^2+3}{x^2-1}$ has the lines with equations $x = 1$ and $x = -1$ as vertical asymptotes, the line with equation $y = 2$ as a horizontal asymptote, and a y-intercept of -3.
This is graph B.

18. The graph of $y = \frac{-2x^2-3}{x^2-1}$ has the lines with equations $x = 1$ and $x = -1$ as vertical asymptotes, the line with equation $y = -2$ as a horizontal asymptote, and a y-intercept of 3.
This is graph A.

20. The graph of $y = \frac{2x^2+3}{x^3-1}$ has the line with equation $x = 1$ as a vertical asymptote, the x-axis as a horizontal asymptote, and a y-intercept of -3.
This is graph C.

22. The graph has the right end up, the left end down, and four turning points. The degree is an odd integer equal to 5 or more. The x^n term has a + sign.

24. Both ends are up, and there are five turning points. The degree is an even integer equal to 6 or more. The x^n term has a + sign.

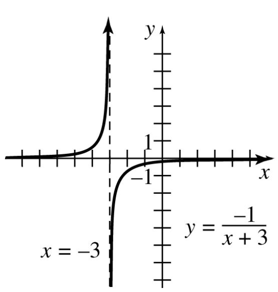
26. The right end is up, the left end is down, and there are six turning points. The degree is an odd integer equal to 7 or more. The x^n term has a + sign.

28. $y = \dfrac{-1}{x+3}$

A vertical asymptote occurs when $x + 3 = 0$ or when $x = -3$, since this value makes the denominator 0.

x	-6	-5	-4	-2	-1	0
$x+3$	-3	-2	-1	1	2	3
y	$\frac{1}{3}$	$\frac{1}{2}$	1	-1	$-\frac{1}{2}$	$-\frac{1}{3}$

As $|x|$ gets larger, $\frac{-1}{x+3}$ approaches 0, so $y = 0$ is a horizontal asymptote.
Asymptotes: $y = 0,\ x = -3$



x-intercept:
none, since the x-axis is an asymptote
y-intercept:
$-\frac{1}{3}$, the value when $x = 0$

30. $y = \dfrac{4}{5+3x}$

Undefined for

$$5 + 3x = 0$$
$$3x = -5$$
$$x = -\frac{5}{3}$$

Since $x = -\frac{5}{3}$ causes the denominator to equal 0, $x = -\frac{5}{3}$ is a vertical asymptote.

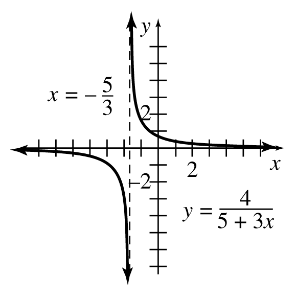
x	-4	-3	-2	-1	0	1
$5+3x$	-7	-4	-1	2	5	8
y	$-.571$	-1	-4	2	$.8$	$.5$

The graph approaches $y = 0$, so the line $y = 0$ is a horizontal asymptote.
Asymptotes: $y = 0,\ x = -\frac{5}{3}$

x-intercept:
none, since the x-axis is an asymptote

y-intercept:

$\frac{4}{5}$, the value when $x = 0$



32. $y = \dfrac{4x}{3-2x}$

Since $x = \frac{3}{2}$ causes the denominator to equal 0, $x = \frac{3}{2}$ is a vertical asymptote.

x	-3	-2	-1	0	1
$4x$	-12	-8	-4	0	4
$3-2x$	9	7	5	3	1
y	-1.33	-1.14	$-.8$	0	4

x	2	3	4
$4x$	8	12	16
$3-2x$	-1	-3	-5
y	-8	-4	-3.2

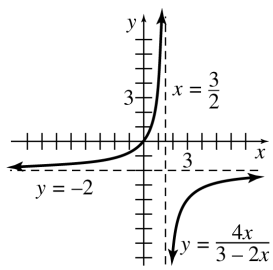
As x gets larger,

$$\frac{4x}{3-2x} \approx \frac{4x}{-2x} = -2.$$

Thus, the line $y = -2$ is a horizontal asymptote.
Asymptotes: $y = -2,\ x = \frac{3}{2}$

x-intercept:
0, the value when $y = 0$

y-intercept:
0, the value when $x = 0$



34. $y = \dfrac{x-3}{x+5}$

Since $x = -5$ causes the denominator to equal 0, $x = -5$ is a vertical asymptote.

x	-8	-7	-6	-4	-3
$x-3$	-11	-10	-9	-7	-6
$x+5$	-3	-2	-1	1	2
y	3.67	5	9	-7	-3

x	-2	-1	0
$x-3$	-5	-4	-3
$x+5$	3	4	5
y	-1.67	-1	$-.6$

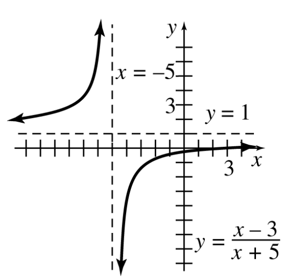
As x gets larger,

$$\frac{x-3}{x+5} \approx \frac{x}{x} = 1.$$

Thus, the line $y = 1$ is a horizontal asymptote.
Asymptotes: $y = 1, x = -5$
x-intercept: 3, the value when $y = 0$

y-intercept:

$-\frac{3}{5}$, the value when $x = 0$



36. $y = \dfrac{6-3x}{4x+12}$

$4x + 12 = 0$ when $4x = -12$ or $x = -3$, so $x = -3$ is a vertical asymptote.

x	-6	-5	-4	-2	-1	0
$6-3x$	24	21	18	12	9	6
$4x+12$	-12	-8	-4	4	8	12
y	-2	-2.625	-4.5	3	1.125	$.5$

As x gets larger,

$$\frac{6-3x}{4x+12} \approx \frac{-3x}{4x} = -\frac{3}{4}.$$

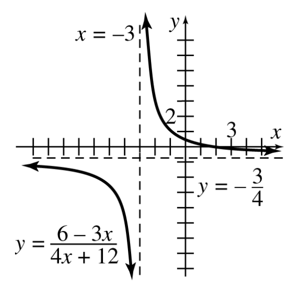
The line $y = -\frac{3}{4}$ is a horizontal asymptote.

Asymptotes: $y = -\frac{3}{4},\ x = -3$

x-intercept:
2, the value when $y = 0$

y-intercept:

$\frac{1}{2}$, the value when $x = 0$



38. $y = \dfrac{-x+8}{2x+5}$

$2x + 5 = 0$ when $2x = -5$ or $x = -\frac{5}{2}$, so $x = -\frac{5}{2}$ is a vertical asymptote.

x	-5	-4	-3	-2	-1	0
$-x+8$	13	12	11	10	9	8
$2x+5$	-5	-3	-1	1	3	5
y	-2.6	-4	-11	10	3	1.6

As x gets larger,

$$\frac{-x+8}{2x+5} \approx \frac{-x}{2x} = -\frac{1}{2}.$$

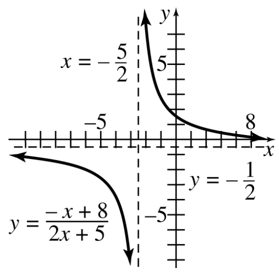
The line $y = -\frac{1}{2}$ is a horizontal asymptote.

Asymptotes: $y = -\frac{1}{2},\ x = -\frac{5}{2}$

x-intercept:
8, the value when $y = 0$

y-intercept:

$\frac{8}{5}$, the value when $x = 0$



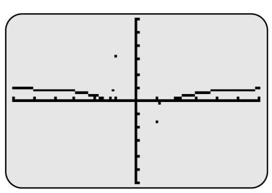
40. For a vertical asymptote at $x = -2$, put $x + 2$ in the denominator. For a horizontal asymptote at $y = 0$, the only condition is that the degree of the numerator is less than the degree of the denominator. If the degree of the denominator is 1, then put a constant in the numerator to make y approach 0 as x gets larger. So, one possible solution is $y = \frac{3}{x+2}$.

42. Graph

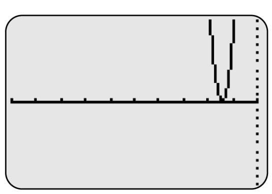
$$f(x) = \frac{x^7 - 4x^5 - 3x^4 + 4x^3 + 12x^2 - 12}{x^7}$$

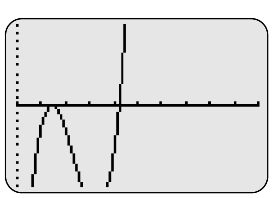
using a graphing calculator with the indicated viewing windows.

(a) There appear to be two x-intercepts, one at $x = -1.4$ and one at $x = 1.4$.



(b) There appear to be three x-intercepts, one at $x = -1.414$, one at $x = 1.414$, and one at $x = 1.442$.





44. $\overline{C}(x) = \dfrac{500}{x + 30}$

(a)
$$\overline{C}(10) = \frac{500}{10 + 30} = \frac{500}{40} = \$12.50$$

$$\overline{C}(20) = \frac{500}{20 + 30} = \frac{500}{50} = \$10$$

$$\overline{C}(50) = \frac{500}{50 + 30} = \frac{500}{80} = \$6.25$$

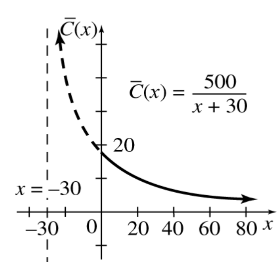
$$\overline{C}(75) = \frac{500}{75 + 30} = \frac{500}{105} \approx \$4.76$$

$$\overline{C}(100) = \frac{500}{100 + 30} = \frac{500}{130} = 3.8461538 \approx \$3.85$$

(b) $(0, \infty)$ would be a more reasonable domain for average cost than $[0, \infty)$. If zero were included in the domain, there would be no units produced. It is not reasonable to discuss the average cost per unit of zero units.

(c) The graph has a vertical asymptote at $x = -30$, a horizontal asymptote at $y = 0$ (the x-axis), and y-intercept $\frac{500}{30} \approx 16.7$.

(d)



46. $y = x(100 - x)(x^2 + 500)$,

$x =$ tax rate;
$y =$ tax revenue in hundreds of thousands of dollars.

(a) $x = 10$

$$\begin{aligned} y &= 10(100 - 10)(10^2 + 500) \\ &= 10(90)(600) \\ &= \$54 \text{ billion} \end{aligned}$$

(b) $x = 40$

$$\begin{aligned} y &= 40(100 - 40)(40^2 + 500) \\ &= 40(60)(2100) \\ &= \$504 \text{ billion} \end{aligned}$$

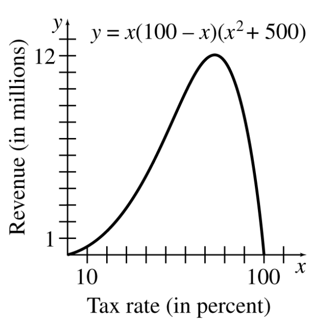
(c) $x = 50$

$$\begin{aligned} y &= 50(100 - 50)(50^2 + 500) \\ &= 50(50)(3000) \\ &= \$750 \text{ billion} \end{aligned}$$

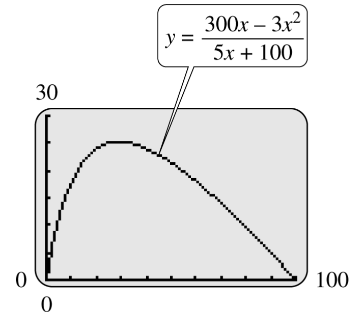
(d) $x = 80$

$$\begin{aligned} y &= 80(100 - 80)(80^2 + 500) \\ &= 80(20)(6900) \\ &= \$1104 \text{ billion} \end{aligned}$$

(e)



48. (a)



(b) The tax rate for maximum revenue is 29.0%. The maximum revenue is \$25.2 million.

50. $y = \dfrac{6.5x}{102 - x}$

$y =$ percent of pollutant;
$x =$ cost in thousands of dollars.

(a) $x = 0$

$$y = \frac{6.5(0)}{102 - 0} = \frac{0}{102} = \$0$$

$x = 50$

$$y = \frac{6.5(50)}{102 - 50} = \frac{325}{52} = 6.25$$

$6.25(1000) = \$6250$

$x = 80$

$$y = \frac{6.5(80)}{102 - 80} = \frac{520}{22} = 23.636$$

$$\begin{aligned} (23.636)(1000) &= \$23{,}636 \\ &\approx \$24{,}000 \end{aligned}$$

$x = 90$

$$y = \frac{6.5(90)}{102 - 90} = \frac{585}{12} = 48.75$$

$$\begin{aligned} (48.75)(1000) &= \$48{,}750 \\ &\approx \$48{,}800 \end{aligned}$$

$x = 95$

$$y = \frac{6.5(95)}{102 - 95} = \frac{617.5}{7} = 88.214$$

$$\begin{aligned} (88.214)(1000) &= 88{,}214 \\ &\approx \$88{,}000 \end{aligned}$$

$x = 99$

$$y = \frac{6.5(99)}{102 - 99} = \frac{643.5}{3} = 214.5$$

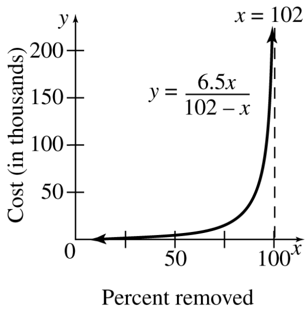
$$(214.500)(1000) = 214{,}500 \approx \$214{,}500$$

$x = 100$

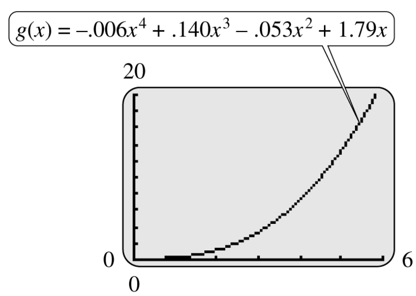
$$y = \frac{6.5(100)}{102 - 100} = \frac{650}{2} = 325$$

$(325)(1000) = \$325{,}000$

(b)



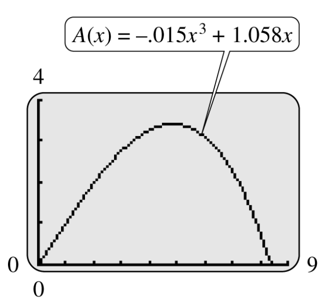
52. (a)



(b) Because the leading coefficient is negative and the degree of the polynomial is even, the graph will have right end down, so it cannot keep increasing forever.

54. $A(x) = -.015x^3 + 1.058x$

(a)



(b) Reading the graph, we find that concentration is maximum between 4 hours and 5 hours, but closer to 5 hours.

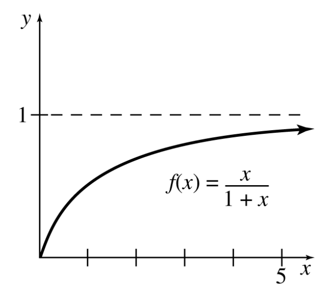
(c) Concentration exceeds .08% from less than 1 hr to about 8.4 hours.

56. $f(x) = \dfrac{\lambda x}{1 + (ax)^b}$

(a) A reasonable domain for the function is $[0, \infty)$. Populations are not measured using negative numbers and they may get extremely large.

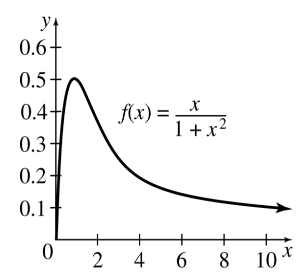
(b) If $\lambda = a = b = 1$, the function becomes

$$f(x) = \frac{x}{1 + x}.$$



(c) If $\lambda = a = 1$ and $b = 2$, the function becomes

$$f(x) = \frac{x}{1 + x^2}.$$

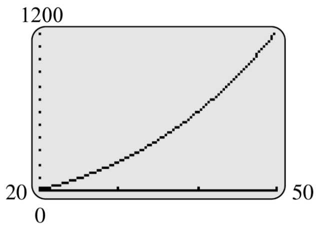


(d) As seen from the graphs, when b increases, the population of the next generation, $f(x)$, gets smaller when the current generation, x, is larger.

58. (a) When $c = 30, w = \frac{30^3}{100} - \frac{1500}{30} = 220$, so the brain weights 220 g when its circumference measures 30 cm. When $c = 40, w = \frac{40^3}{100} - \frac{1500}{40} = 602.5$, so the brain weighs 602.5 g when its circumference is 40 cm. When $c = 50, w = \frac{50^3}{100} - \frac{1500}{50} = 1220$, so the brain weighs 1220 g when its circumference is 50 cm.

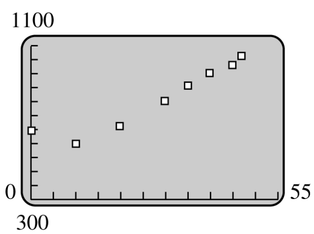
(b) Set the window of a graphing calculator so you can trace to the positive x-intercept of the function. Using a "root" or "zero" program, this x-intercept is found to be approximately 19.68. Notice in the graph that positive c values less than 19.68 correspond to negative w values. Therefore, the answer is $c < 19.68$.

(c)



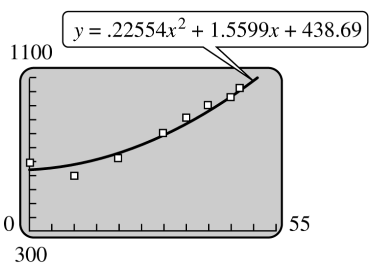
(d) One method is to graph the line $y = 700$ on the graph found in part (c) and use an "intercept" program to find the point of intersection of the two graphs. This point has the approximate coordinates $(41.9, 700)$. Therefore, an infant has a brain weighing 700 g when the circumference measures 41.9 cm.

60. **(a)**



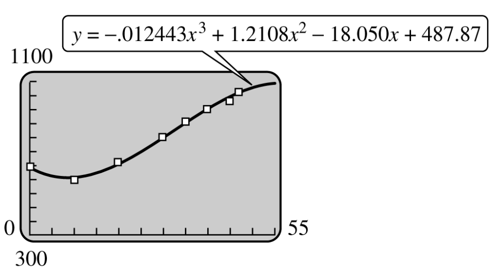
(b) $f(x) = .22554x^2 + 1.5599x + 438.69$

(c)



(d) $f(x) = -.012443x^3 + 1.2108x^2 - 18.050x + 487.87$

(e)



2.4 Exponential Functions

2. 500 sheets are 2 inches high

$$\frac{500}{2 \text{ in.}} = \frac{2^{50}}{x \text{ in.}}$$
$$x = \frac{2 \cdot 2^{50}}{500}$$
$$= 4.503599627 \times 10^{12} \text{ in.}$$
$$= 71{,}079{,}539.57 \text{ mi}$$

4. The graph of $y = 3^{-x}$ is the graph of $y = 3^x$ reflected in the y-axis.
This is graph D.

6. The graph of $y = 3^{x+1}$ is the graph of $y = 3^x$ translated 1 unit to the left.
This is graph F.

8. The graph of $y = \left(\frac{1}{3}\right)^x$ is the graph of $y = (3^{-1})^x = 3^{-x}$. This is the graph of $y = 3^x$ reflected in the y-axis.
This is graph D.

10. The graph of $y = -2 + 3^{-x}$ is the same as the graph of $y = 3^{-x} - 2$. This is the graph of $y = 3^x$ reflected in the y-axis and translated 2 units downward.
This is graph B.

14.
$$4^x = 64$$
$$4^x = 4^3$$
$$x = 3$$

16.
$$4^x = 8^{x+1}$$
$$(2^2)^x = (2^3)^{x+1}$$
$$2^{2x} = 2^{3x+3}$$
$$2x = 3x + 3$$
$$-x = 3$$
$$x = -3$$

18.
$$16^{-x+1} = 8^x$$
$$(2^4)^{-x+1} = (2^3)^x$$
$$2^{-4x+4} = 2^{3x}$$
$$-4x + 4 = 3x$$
$$4 = 7x$$
$$\frac{4}{7} = x$$

20. $$\begin{aligned}(e^4)^{-2x} &= e^{-x+1}\\ e^{-8x} &= e^{-x+1}\\ -8x &= -x+1\\ -7x &= 1\\ x &= -\frac{1}{7}\end{aligned}$$

22. $$\begin{aligned}2^{|x|} &= 16\\ 2^{|x|} &= 2^4\\ |x| &= 4\\ x = 4 \quad &\text{or} \quad x=-4\end{aligned}$$

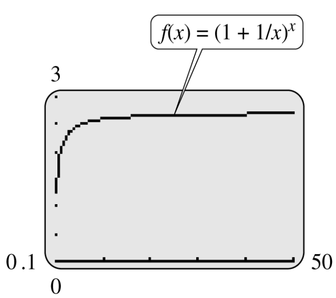
24. $$\begin{aligned}2^{x^2-4x} &= \left(\frac{1}{16}\right)^{x-4}\\ 2^{x^2-4x} &= (2^{-4})^{x-4}\\ 2^{x^2-4x} &= 2^{-4x+16}\\ x^2-4x &= -4x+16\\ x^2-16 &= 0\\ (x+4)(x-4) &= 0\\ x=-4 \quad &\text{or} \quad x=4\end{aligned}$$

26. $$\begin{aligned}8^{x^2} &= 2^{5x+2}\\ (2^3)^{x^2} &= 2^{5x+2}\\ 2^{3x^2} &= 2^{5x+2}\\ 3x^2 &= 5x+2\\ 3x^2-5x-2 &= 0\\ (3x+1)(x-2) &= 0\\ x = -\frac{1}{3} \quad &\text{or} \quad x=2\end{aligned}$$

28. $$\begin{aligned}e^{x^2-3x+2} &= 1\\ e^{x^2-3x+2} &= e^0\\ x^2-3x+2 &= 0\\ (x-1)(x-2) &= 0\\ x-1=0 \quad \text{or} \quad x-2 &= 0\\ x=1 \quad \text{or} \quad x &= 2\end{aligned}$$

30. 4 and 6 cannot be easily written as powers of the same base, so the equation $4^x = 6$ cannot be solved using this approach.

32.



$f(x)$ approaches $e \approx 2.71828$.

34. $A = P\left(1+\frac{r}{m}\right)^{tm}$, $P = 26{,}000$, $r = .12$, $t = 3$

(a) annually, $m = 1$

$$\begin{aligned}A &= 26{,}000\left(1+\frac{.12}{1}\right)^{3(1)}\\ &= 26{,}000(1.12)^3\\ &= 36{,}528.13\end{aligned}$$

$$\begin{aligned}\text{Interest} &= \$36{,}528.13 - \$26{,}000\\ &= \$10{,}528.13\end{aligned}$$

(b) semiannually, $m = 2$

$$\begin{aligned}A &= 26{,}000\left(1+\frac{.12}{2}\right)^{3(2)}\\ &= 26{,}000(1.06)^6\\ &= 36{,}881.50\end{aligned}$$

$$\begin{aligned}\text{Interest} &= \$36{,}881.50 - \$26{,}000\\ &= \$10{,}881.50\end{aligned}$$

(c) quarterly, $m = 4$

$$\begin{aligned}A &= 26{,}000\left(1+\frac{.12}{4}\right)^{3(4)}\\ &= 26{,}000(1.03)^{12}\\ &= 37{,}069.78\end{aligned}$$

$$\begin{aligned}\text{Interest} &= \$37{,}069.78 - \$26{,}000\\ &= \$11{,}069.78\end{aligned}$$

(d) monthly, $m = 12$

$$\begin{aligned}A &= 26{,}000\left(1+\frac{.12}{12}\right)^{3(12)}\\ &= 26{,}000(1.01)^{36}\\ &= 37{,}199.99\end{aligned}$$

$$\begin{aligned}\text{Interest} &= \$37{,}199.99 - \$26{,}000\\ &= \$11{,}199.99\end{aligned}$$

36. $A = P\left(1+\frac{r}{m}\right)^{tm}$, $P = 5000$, $A = 8000, t = 4$

(a) $m = 1$

$$\begin{aligned}8000 &= 5000\left(1+\frac{r}{1}\right)^{4(1)}\\ \frac{8}{5} &= (1+r)^4\\ \left(\frac{8}{5}\right)^{1/4} - 1 &= r\\ .125 &= r\end{aligned}$$

The interest rate is 12.5%.

(b) $m = 4$

$$8000 = 5000\left(1+\frac{r}{4}\right)^{4(4)}$$
$$\frac{8}{5} = \left(1+\frac{r}{4}\right)^{16}$$
$$\left(\frac{8}{5}\right)^{1/16} - 1 = \frac{r}{4}$$
$$4\left[\left(\frac{8}{5}\right)^{1/16} - 1\right] = r$$
$$.119 = r$$

The interest rate is 11.9%.

38. $P = \$25{,}000,\ r = 9\%$

Use the formula for continuous compounding,

$$A = Pe^{rt}.$$

(a) $t = 1$

$$A = 25,000e^{.09(1)}$$
$$= \$27,354.36$$

(b) $t = 5$

$$A = 25{,}000e^{.09(5)}$$
$$= \$39{,}207.80$$

(c) $t = 10$

$$A = 25{,}000e^{.09(10)}$$
$$= \$61{,}490.08$$

40. (a) $30{,}000 = 10{,}500\left(1+\frac{r}{4}\right)^{(4)(12)}$

$$\frac{300}{105} = \left(1+\frac{r}{4}\right)^{48}$$
$$1+\frac{r}{4} = \left(\frac{300}{105}\right)^{1/48}$$
$$4+r = 4\left(\frac{300}{105}\right)^{1/48}$$
$$r = 4\left(\frac{300}{105}\right)^{1/48} - 4$$
$$r \approx .0884$$

The required interest rate is 8.84%.

(b) $30{,}000 = 10{,}500e^{12r}$

$$\frac{300}{105} = e^{12r}$$
$$12r = \ln\left(\frac{300}{105}\right)$$
$$r = \frac{\ln\left(\frac{300}{105}\right)}{12}$$
$$r \approx .0875$$

The required interest rate is 8.75%.

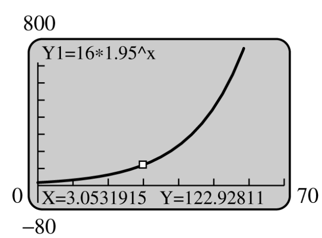
42. (a) $f(x) = f_0 a^x$

$$f_0 = 16 \text{ million}$$

Use the point $(5, 451.04)$ to find a.

$$451.04 = 16a^5$$
$$a^5 = \frac{451.04}{16}$$
$$a = \sqrt[5]{\frac{451.04}{16}}$$
$$\approx 1.950$$
$$f(x) = 16(1.950)^x$$

(b)



These were about 120 million users in December 1998. Yes, this is close to the estimate.

(c) 1995-1996:

$$\frac{f(1)-f(0)}{f(0)} \approx \frac{31.2-16}{16}$$
$$= .95 = 95\%$$

1996-1997:

$$\frac{f(2)-f(1)}{f(1)} \approx \frac{60.84-31.2}{31.2}$$
$$= .95 = 95\%$$

1997-1998:

$$\frac{f(3)-f(2)}{f(2)} \approx \frac{118.638-60.84}{60.84}$$
$$= .95 = 95\%$$

1998-1999:

$$\frac{f(4)-f(3)}{f(3)} \approx \frac{231.3441-118.638}{118.638} = .95 = 95\%$$

1999-2000:

$$\frac{f(5)-f(4)}{f(4)} \approx \frac{451.120995-231.344}{231.3441} = .95 = 95\%$$

The average yearly percent increase in users was about 95%.

(d) $f(6) \approx 879.69$

In 2001 there were approximately 879.69 million users according to the model. No, the model does not hold.

44. $A(t) = 2600e^{.017t}$

(a) 1970: $t = 20$

$$\begin{aligned} A(20) &= 2600e^{.017(20)} \\ &= 2600e^{.34} \\ &\approx 3650 \end{aligned}$$

The function gives a population of about 3650 million in 1970.
This is very close to the actual population of about 3700 million.

(b) 1990: $t = 40$

$$\begin{aligned} A(40) &= 2600e^{.017(40)} \\ &= 2600e^{.68} \\ &\approx 5130 \end{aligned}$$

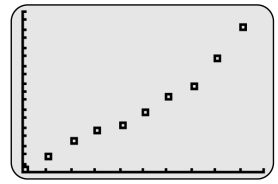
The function gives a population of 5130 million in 1990.

(c) 2010: $t = 60$

$$\begin{aligned} &= 2600e^{.017(60)} \\ &= 2600e^{1.02} \\ &= 7210 \end{aligned}$$

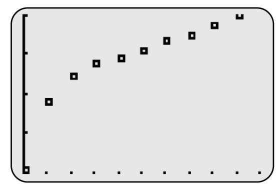
From the function, we estimate that the world population in 2010 will be 7210 million.

46. (a)



No, the data do not appear to lie along a straight line.

(b) Yes, the graph appears to be more linear, especially if the first point is eliminated.



(c) $y = .03421x + 9.191$

(d) $y = 9807(1.0348)^x$

(e)
$$\begin{aligned} y &= 9807(1.0348)^x \\ \ln y &= \ln[9807(1.0348)^x] \\ &= \ln 9807 + \ln(1.0348^x) \\ &= 9.191 + x \cdot \ln(1.0348) \\ &= 9.191 + .03421x \end{aligned}$$

48. $Q(t) = 1000(5^{-.3t})$

(a)
$$\begin{aligned} Q(6) &= 1000[5^{-.3(6)}] \\ &= 1000(5^{-1.8}) \\ &= 55 \end{aligned}$$

The amount present in 6 months will be 55 grams.

(b)
$$\begin{aligned} 8 &= 1000(5^{-.3t}) \\ \frac{1}{125} &= 5^{-.3t} \\ 5^{-3} &= 5^{-.3t} \\ -3 &= -.3t \\ 10 &= t \end{aligned}$$

It will take 10 months to reduce the substance to 8 grams.

50. (a) In 1971, $x = 0$ and $y = 2250$.
In 2000, $x = 29$ and $y = 42{,}000{,}000$

First, we find a function of the form $y = mx + b$. The two points are $(0, 2250)$ and $(29, 42{,}000{,}000)$.

$$m = \frac{42{,}000{,}000 - 2250}{29 - 0} \approx 1{,}448{,}198$$

$$b = 2250$$

Therefore, $y = 1{,}448{,}198x + 2250$.
Next, we find a function of the form $y = ax^2 + b$.
When $x = 0, y = 2250$.

$$\begin{aligned} 2250 &= a(0)^2 + b \\ b &= 2250 \\ y &= ax^2 + 2250 \end{aligned}$$

When $x = 29, y = 42{,}000{,}000$.

$$\begin{aligned} 42{,}000{,}000 &= a(29)^2 + 2250 \\ 41{,}997{,}750 &= 841a \\ a &\approx 49{,}938 \end{aligned}$$

$$y = 49{,}938x^2 + 2250$$

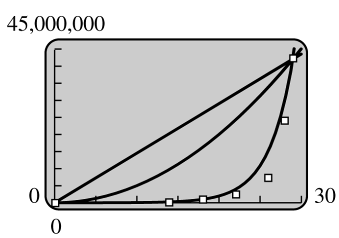
Finally, we find a function of the form $y = ab^x$.
When $x = 0, y = 2250$.

$$\begin{aligned} 2250 &= ab^0 \\ a &= 2250 \\ y &= 2250b^x \end{aligned}$$

When $x = 29, y = 42{,}000{,}000$.

$$\begin{aligned} 42{,}000{,}000 &= 2250b^{26} \\ b &= \sqrt[29]{\frac{42{,}000{,}000}{2250}} \approx 1.4037 \\ y &= 2250(1.4037)^x \end{aligned}$$

(b)



$y = 2250(1.4037)^x$ is the best fit.

(c) In 2008, $x = 37$

$$\begin{aligned} y &= 2250(1.4037)^x \\ &= 2500(1.4037)^{37} \\ &\approx 633{,}000{,}000 \end{aligned}$$

(d) $y = 2491(1.369)^x$

This is close to the function found in part b.

2.5 Logarithmic Functions

2. $5^2 = 25$

Since $a^y = x$ means $y = \log_a x$, the equation in logarithmic form is

$$\log_5 25 = 2.$$

4. $6^3 = 216$

Since $a^y = x$ means $y = \log_a x$, the equation in logarithmic form is

$$\log_6 216 = 3$$

6. $\left(\frac{4}{3}\right)^{-2} = \frac{9}{16}$

Since $a^y = x$ means $y = \log_a x$, the equation in logarithmic form is

$$\log_{4/3} \frac{9}{16} = -2.$$

8. $\log_3 81 = 4$

Since $y = \log_a x$ means $a^y = x$, the equation in exponential form is

$$3^4 = 81.$$

10. $\log_2 \frac{1}{8} = -3$

The equation in exponential form is

$$2^{-3} = \frac{1}{8}.$$

12.
$$\begin{aligned} \log .00001 &= -5 \\ \log_{10} .00001 &= -5 \\ 10^{-5} &= .00001 \end{aligned}$$

When no base is written, $\log_{10}$ is understood.

14. Let $\log_9 81 = x$.

$$\begin{aligned} \text{Then, } 9^x &= 81 \\ 9^x &= 9^2 \\ x &= 2. \end{aligned}$$

Thus, $\log_9 81 = 2$.

16. $\log_6 216 = x$

$$\begin{aligned} 6^x &= 216 \\ 6^x &= 6^3 \\ x &= 3 \end{aligned}$$

18. $\log_3 \frac{1}{27} = x$

$$3^x = \frac{1}{27}$$
$$3^x = 3^{-3}$$
$$x = -3$$

20. $\log_8 \sqrt[4]{\frac{1}{2}} = x$

$$8^x = \sqrt[4]{\frac{1}{2}} = \left(\frac{1}{2}\right)^{1/4}$$
$$(2^3)^x = 2^{-1/4}$$
$$3x = -\frac{1}{4}$$
$$x = -\frac{1}{12}$$

22. $\ln e^2 = x$

Recall that ln means $\log_e$.

$$e^x = e^2$$
$$x = 2$$

24. $\ln 1 = x$

$$e^x = 1$$
$$e^x = e^0$$
$$x = 0$$

28. $\log_5 (8p) = \log_5 8 + \log_5 p$

30. $\log_7 \frac{11p}{13y}$

$$= \log_7 11p - \log_7 13y$$
$$= (\log_7 11 + \log_7 p) - (\log_7 13 + \log_7 y)$$
$$= \log_7 11 + \log_7 p - \log_7 13 - \log_7 y$$

32. $\ln \frac{9\sqrt[3]{5}}{\sqrt[4]{3}}$

$$= \ln 9\sqrt[3]{5} - \ln \sqrt[4]{3}$$
$$= \ln 9 \cdot 5^{1/3} - \ln 3^{1/4}$$
$$= \ln 9 + \ln 5^{1/3} - \ln 3^{1/4}$$
$$= \ln 9 + \frac{1}{3} \ln 5 - \frac{1}{4} \ln 3$$

34. $\log_b 24$

$$= \log_b (8 \cdot 3)$$
$$= \log_b (2^3 \cdot 3)$$
$$= \log_b 2^3 + \log_b 3$$
$$= 3 \log_b 2 + \log_b 3$$
$$= 3a + c$$

36. $\log_b (4b^2) = \log_b 4 + \log_b b^2$

$$= \log_b 2^2 + \log_b b^2$$
$$= 2 \log_b 2 + 2 \log_b b$$
$$= 2a + 2(1)$$
$$= 2a + 2$$

38. $\log_{12} 170 = \frac{\ln 170}{\ln 12}$

$$\approx 2.07$$

40. $\log_{2.8} .12 = \frac{\ln .12}{\ln 2.8}$

$$\approx -2.06$$

42. $\log_9 27 = m$

$$9^m = 27$$
$$(3^2)^m = 3^3$$
$$3^{2m} = 3^3$$
$$2m = 3$$
$$m = \frac{3}{2}$$

44. $\log_y 8 = \frac{3}{4}$

$$y^{3/4} = 8$$
$$(y^{3/4})^{4/3} = 8^{4/3}$$
$$y = (8^{1/3})^4$$
$$= 2^4 = 16$$

46. $\log_3 (5x + 1) = 2$

$$3^2 = 5x + 1$$
$$9 = 5x + 1$$
$$5x = 8$$
$$x = \frac{8}{5}$$

48. $\log_4 x - \log_4 (x + 3) = -1$

$$\log_4 \frac{x}{x+3} = -1$$
$$4^{-1} = \frac{x}{x+3}$$
$$\frac{1}{4} = \frac{x}{x+3}$$
$$4x = x + 3$$
$$3x = 3$$
$$x = 1$$

50. $$\begin{aligned} \log(x+5)+\log(x+2) &= 1 \\ \log[(x+5)(x+2)] &= 1 \\ (x+5)(x+2) &= 10^1 \\ x^2+7x+10 &= 10 \\ x^2+7x &= 0 \\ x(x+7) &= 0 \\ x=0 \quad \text{or} \quad x &= -7 \end{aligned}$$

$x=-7$ is not a solution of the original equation because if $x=-7, x+5$ and $x+2$ would be negative, and the domain of $y=\log x$ is $(0,\infty)$. Therefore, $x=0$.

52. $\log_3(x^2+17)-\log_3(x+5)=1$

$$\begin{aligned} \log_3 \frac{x^2+17}{x+5} &= 1 \\ 3^1 &= \frac{x^2+17}{x+5} \\ 3x+15 &= x^2+17 \\ 0 &= x^2-3x+2 \\ 0 &= (x-1)(x-2) \end{aligned}$$

$x=1$ or $x=2$

54. $$\begin{aligned} 3^x &= 5 \\ \ln 3^x &= \ln 5 \\ x \ln 3 &= \ln 5 \\ x &= \frac{\ln 5}{\ln 3} \approx 1.46 \end{aligned}$$

56. $$\begin{aligned} e^{k-1} &= 4 \\ \ln e^{k-1} &= \ln 4 \\ (k-1) \ln e &= \ln 4 \\ k-1 &= \frac{\ln 4}{\ln e} \\ k-1 &= \frac{\ln 4}{1} \\ k &= 1+\ln 4 \\ &\approx 2.39 \end{aligned}$$

58. $$\begin{aligned} 2e^{5a+12} &= 8 \\ e^{5a+12} &= 4 \\ \ln e^{5a+12} &= \ln 4 \\ (5a+12) \ln e &= \ln 4 \\ 5a+12 &= \ln 4 \\ a &= \frac{\ln 4-12}{5} \\ a &\approx -2.12 \end{aligned}$$

60. $$\begin{aligned} 5(.10)^x &= 4(.12)^x \\ \ln[5(.10)^x] &= \ln[4(.12)^x] \\ \ln 5 + x\ln .10 &= \ln 4 + x \ln .12 \\ x(\ln .12 - \ln .10) &= \ln 5 - \ln 4 \\ x &= \frac{\ln 5-\ln 4}{\ln .12-\ln .10} \\ &\approx 1.22 \end{aligned}$$

62. $f(x)=\log(3-x)$

$$\begin{aligned} 3-x &> 0 \\ -x &> -3 \\ x &< 3 \end{aligned}$$

The domain of f is $x<3$.

64. $\log A-\log B=0$

$$\begin{aligned} \log \frac{A}{B} &= 0 \\ \frac{A}{B} &= 10^0 = 1 \\ A &= B \\ A-B &= 0 \end{aligned}$$

Thus, solving $\log A-\log B=0$ is equivalent to solving $A-B=0$.

66. Let $m=\log_a x^r$ and $n=\log_a x$.
Then, $a^m=x^r$ and $a^n=x$.
Substituting gives

$$a^m=x^r=(a^n)^r=a^{nr}.$$

Therefore, $m=nr$, or

$$\log_a x^r = r\log_a x.$$

68. **(a)** $$\begin{aligned} t &= \frac{\ln 2}{\ln(1+r)} \\ t &= \frac{\ln 2}{\ln(1+.06)} \\ t &\approx 11.90 \end{aligned}$$

It will take 12 years for the compound amount to be at least double.

(b) $$\begin{aligned} t &= \frac{\ln 3}{\ln(1+.06)} \\ t &\approx 18.85 \end{aligned}$$

It will take 19 years for the compound amount to at least triple.

(c) The rule of 72 gives

$$\frac{72}{100(.06)} = 12$$

years as the doubling time.

70.

r (sec)	.001	.02	.05	.08	.12
$\frac{\ln 2}{\ln(1+r)}$	693.5	35	14.2	9.01	6.12
$\frac{70}{100r}$	700	35	14	8.75	5.83
$\frac{72}{100r}$	720	36	14.4	9	6

For $.001 \le r < .05$, the Rule of 70 is more accurate. For $.05 < r \le .12$, the Rule of 72 is more accurate. At $r = .05$, the two are equally accurate.

72. If the number N is proportional to $m^{-.6}$, where m is the mass, then $N = km^{-.6}$, for some constant of proportionality k.
Taking the common log of both sides, we have

$$\begin{aligned}\log N &= \log(km^{-.6})\\ &= \log k + \log m^{-.6}\\ &= \log k - .6\log m.\end{aligned}$$

This is a linear equation in $\log m$. Its graph is a straight line with slope $-.6$ and vertical intercept $\log k$.

74. $H = -[P_1 \ \ln \ P_1 + P_2 \ \ln \ P_2$

$\qquad + P_3 \ \ln \ P_3 + P_4 \ \ln \ P_4]$

$H = -[.521 \ \ln \ .521 + .324 \ \ln \ .324$
$\qquad + .081 \ \ln \ .081 + \ .074 \ \ln .074]$
$H = 1.101$

76.
$$\begin{aligned} mX + N &= m \ \log_b \ x + \log_b \ n\\ &= \log_b \ x^m + \log_b \ n\\ &= \log_b \ nx^m\\ &= \log_b \ y\\ &= Y\end{aligned}$$

Thus, $Y = mX + N$.

78. **(a)** From the given graph, when $x = .3$ kg
$y \approx 4.3$ ml/ min, and when $x = .7$ kg
$y \approx 7.8$ ml/ min.

(b) If $y = ax^b$, then

$$\begin{aligned}\ln \ y &= \ln \ (ax^b)\\ &= \ln \ a + b \ \ln \ x.\end{aligned}$$

Thus, there is a linear relationship between $\ln \ y$ and $\ln \ x$.

(c)
$$\begin{aligned}4.3 &= a(.3)^b\\ 7.8 &= a(.7)^b\\ \frac{4.3}{7.8} &= \frac{a(.3)^b}{a(.7)^b}\\ \frac{4.3}{7.8} &= \left(\frac{.3}{.7}\right)^b\\ \ln\left(\frac{4.3}{7.8}\right) &= \ln\left(\frac{.3}{.7}\right)^b\\ \ln\left(\frac{4.3}{7.8}\right) &= b\ \ln\left(\frac{.3}{.7}\right)\\ b &= \frac{\ln\left(\frac{4.3}{7.8}\right)}{\ln\left(\frac{.3}{.7}\right)}\\ b &\approx .7028\end{aligned}$$

Substituting this value into $4.3 = a(.3)^b$,

$$\begin{aligned}4.3 &= a(.3)^{.7028}\\ a &= \frac{4.3}{(.3)^{.7028}} \approx 10.02.\end{aligned}$$

Therefore, $y = 10.02x^{.7028}$.

(d) If $x = .5$,

$$\begin{aligned}y &= 10.02(.5)^{.7028}\\ &\approx 6.16.\end{aligned}$$

We predict that the oxygen consumption for a guinea pig weighing .5 kg will be about 6.16 ml/min.

80. $N(r) = -5000 \ \ln \ r$

(a) $N(.9) = -5000 \ \ln \ (.9) \approx 530$

(b) $N(.5) = -5000 \ \ln \ (.5) \approx 3500$

(c) $N(.3) = -5000 \ \ln \ (.3) \approx 6000$

(d) $N(.7) = -5000 \ \ln \ (.7) \approx 1800$

(e) $-5000 \ \ln \ r = 1000$

$$\begin{aligned}\ln \ r &= \frac{1000}{-5000}\\ \ln \ r &= -\frac{1}{5}\\ r &= e^{-1/5}\\ r &\approx .8\end{aligned}$$

82. Decibel rating: $10 \log \dfrac{I}{I_0}$

(a) Intensity, $I = 115I_0$

$$10 \log \left(\frac{115I_0}{I_0}\right)$$
$$= 10 \cdot \log 115$$
$$\approx 21$$

(b) $I = 9{,}500{,}000I_0$

$$10 \log \left(\frac{9.5 \times 10^6 I_0}{I_0}\right) = 10 \log 9.5 \times 10^6 \approx 70$$

(c) $I = 1{,}200{,}000{,}000I_0$

$$10 \log \left(\frac{1.2 \times 10^9 I_0}{I_0}\right) = 10 \log 1.2 \times 10^9 \approx 91$$

(d) $I = 895{,}000{,}000{,}000I_0$

$$10 \log \left(\frac{8.95 \times 10^{11} I_0}{I_0}\right) = 10 \log 8.95 \times 10^{11} \approx 120$$

(e) $I = 109{,}000{,}000{,}000{,}000I_0$

$$10 \log \left(\frac{1.09 \times 10^{14} I_0}{I_0}\right) = 10 \log 1.09 \times 10^{14} \approx 140$$

(f) $I_0 = .0002$ microbars

$$1{,}200{,}000{,}000I_0 = 1{,}200{,}000{,}000(.0002) = 240{,}000 \text{ microbars}$$
$$895{,}000{,}000{,}000I_0 = 895{,}000{,}000{,}000(.0002) = 179{,}000{,}000 \text{ microbars}$$

84. $R(I) = \log \dfrac{I}{I_0}$

(a) $R(1{,}000{,}000\ I_0)$

$$= \log \frac{1{,}000{,}000\ I_0}{I_0}$$
$$= \log 1{,}000{,}000$$
$$= 6$$

(b) $R(100{,}000{,}000\ I_0)$

$$= \log \frac{100{,}000{,}000\ I_0}{I_0}$$
$$= \log 100{,}000{,}000$$
$$= 8$$

(c) $R(I) = \log \dfrac{I}{I_0}$

$$6.7 = \log \frac{I}{I_0}$$
$$10^{6.7} = \frac{I}{I_0}$$
$$I \approx 5{,}000{,}000I_0$$

(d) $R(I) = \log \dfrac{I}{I_0}$

$$8.1 = \log \frac{I}{I_0}$$
$$10^{8.1} = \frac{I}{I_0}$$
$$I \approx 126{,}000{,}000I_0$$

(e) $\dfrac{1985 \text{ quake}}{1999 \text{ quake}} = \dfrac{126{,}000{,}000I_0}{5{,}000{,}000I_0} \approx 25$

The 1985 earthquake had an amplitude more than 25 times that of the 1999 earthquake.

(f) $R(E) = \dfrac{2}{3} \log \dfrac{E}{E_0}$

For the 1999 earthquake

$$6.7 = \frac{2}{3} \log \frac{E}{E_0}$$
$$10.05 = \log \frac{E}{E_0}$$
$$\frac{E}{E_0} = 10^{10.05}$$
$$E = 10^{10.05} E_0$$

For the 1985 earthquake,

$$8.1 = \frac{2}{3} \log \frac{E}{E_0}$$
$$12.15 = \log \frac{E}{E_0}$$
$$\frac{E}{E_0} = 10^{12.15}$$
$$E = 10^{12.15} E_0$$

The ratio of their energies is

$$\frac{10^{12.15} E_0}{10^{10.05} E_0} = 10^{2.1} \approx 126$$

The 1985 earthquake had an energy about 126 times that of the 1999 earthquake.

(g) Find the energy of a magnitude 6.7 earthquake. Using the formula from part f,

$$\begin{aligned}6.7 &= \frac{2}{3}\log\frac{E}{E_0}\\ \log\frac{E}{E_0} &= 10.05\\ \frac{E}{E_0} &= 10^{10.05}\\ E &= E_0 10^{10.05}\end{aligned}$$

For an earthquake that releases 15 times this much energy, $E = E_0(15)10^{10.05}$.

$$\begin{aligned}R(E_0(15)10^{10.05}) &= \frac{2}{3}\log\left(\frac{E_0(15)10^{10.05}}{E_0}\right)\\ &= \frac{2}{3}\log(15\cdot 10^{10.05})\\ &\approx 7.5\end{aligned}$$

So, it's true that a magnitude 7.5 earthquake releases 15 times more energy than one of magnitude 6.7.

2.6 Applications: Growth and Decay; Mathematics of Finance

2. y_0 represents the initial quantity; k represents the rate of growth or decay.

4. The half-life of a quantity is the time period for the quantity to decay to one-half of the initial amount.

6. Assume that $y = y_0e^{kt}$ is the amount left of a radioactive substance decaying with a half-life of T. From Exercise 5, we know $k = \frac{-\ln 2}{T}$, so

$$\begin{aligned}y &= y_0e^{(-\ln 2/T)t} = y_0e^{-(t/T)\ln 2} = y_0e^{\ln(2^{-t/T})}\\ &= y_0 2^{-t/T} = y_0\left[\left(\frac{1}{2}\right)^{-1}\right]^{-t/T} = y_0\left(\frac{1}{2}\right)^{t/T}\end{aligned}$$

8. $r = 18\%$ compounded monthly, $m = 12$

$$\begin{aligned}r_E &= \left(1+\frac{.18}{12}\right)^{12} - 1\\ &\approx .1956\\ &= 19.56\%\end{aligned}$$

10. $r = 7\%$ compounded continuously

$$\begin{aligned}r_E &= e^r - 1\\ &= e^{.07} - 1\\ &\approx .0725\\ &= 7.25\%\end{aligned}$$

12. $A =$\$45,678.93, $r = 12.6\%$, $m = 12$, $t = 11$ months

$$\begin{aligned}P &= A\left(1+\frac{r}{m}\right)^{-tm}\\ &= 45{,}678.93\left(1+\frac{.126}{12}\right)^{-(11/12)(12)}\\ &\approx \$40{,}720.81\end{aligned}$$

14. $A =$ \$25,000, $r = 9\%$ compounded continuously, $t = 8$

$$\begin{aligned}A &= Pe^{rt}\\ P &= \frac{A}{e^{rt}}\\ &= \frac{25{,}000}{e^{.09(8)}}\\ &\approx \$12{,}168.81\end{aligned}$$

16. $r = 7.2\%$, $m = 4$

$$\begin{aligned}r_E &= \left(1+\frac{.072}{4}\right)^4 - 1\\ &\approx .0740\\ &= 7.40\%\end{aligned}$$

18. $A =$ \$20,000, $t = 4$, $r = 8\%$, $m = 1$

$$\begin{aligned}A &= P\left(1+\frac{r}{m}\right)^{mt}\\ 20{,}000 &= P\left(1+\frac{.08}{1}\right)^{1\cdot 4}\\ \frac{20{,}000}{(1.08)^4} &= P\\ \$14{,}700.60 &= P\end{aligned}$$

20. $A =$ \$20,000, $t = 5$

(a) $r = .08$, $m = 4$

$$\begin{aligned}A &= P\left(1+\frac{r}{m}\right)^{mt}\\ 20{,}000 &= P\left(1+\frac{.08}{1}\right)^{5(4)}\\ \frac{20{,}000}{(1.02)^{20}} &= P\\ \$13{,}459.43 &= P\end{aligned}$$

(b) $\text{Interest} = \$20{,}000 - \$13{,}459.43 = \$6540.57$

(c) $P = \$10{,}000$

$$A = 10{,}000\left(1 + \frac{.08}{4}\right)^{5(4)}$$
$$= \$14{,}859.47$$

The amount needed will be

$\$20{,}000 - \$14{,}859.47 = \$5140.53.$

22. **(a)**
$$11{,}000 = 5000\left(1 + \frac{.063}{4}\right)^{4t}$$
$$\frac{11}{5} = (1.01575)^{4t}$$
$$\ln\left(\frac{11}{5}\right) = 4t \ln 1.01575$$
$$t = \frac{\ln\left(\frac{11}{5}\right)}{4 \ln 1.01575}$$
$$t \approx 12.61$$

Since the interest is only added at the end of the quarter, it will take 12.75 years.

(b) $11{,}000 = 5000e^{.063t}$
$$\frac{11}{5} = e^{.063t}$$
$$.063t = \ln\left(\frac{11}{5}\right)$$
$$t = \frac{\ln\left(\frac{11}{5}\right)}{.063}$$
$$t \approx 12.52$$

It will take about 12.52 years.

24. The statement is not correct.

$$(1+r)^{17} = 4.78 + 1$$
$$1 + r = (5.78)^{1/17}$$
$$r = (5.78)^{1/17} - 1$$
$$r \approx .109$$

The required annual percent increase is 10.9%.

26. $S(x) = 5000 - 4000e^{-x}$

(a) $S(0) = 5000 - 4000e^0$
$$= 1000$$

Since $S(x)$ represents sales in thousands, in year 0 the sales are \$1,000,000.

(b) Let $S(x) = 4500.$
$$4500 = 5000 - 4000e^{-x}$$
$$-500 = -4000e^{-x}$$
$$.125 = e^{-x}$$
$$-\ln .125 = x$$
$$x \approx 2$$

It will take about 2 years for sales to reach \$45,500,000.

(c) Graphing the function $y = S(x)$ on a graphing calculator will show that there is a horizontal asymptote at $y = 5000$. Since this represents \$5000 thousand or \$5,000,000, the limit on sales is \$5,000,000.

28. **(a)** The function giving the number after t hours is

$$y = y_0 2^{t/12}$$

(b) For 10 days, $t = 10 \cdot 24$ or 240.
$$y = (1)2^{240/12}$$
$$= 2^{20}$$
$$= 1{,}048{,}576$$

For 15 days, $t = 15 \cdot 24$ or 360.
$$y = (1)2^{360/12}$$
$$= 2^{30}$$
$$= 1{,}073{,}741{,}824$$

30. $y = y_0e^{kt}$
$y = 40{,}000,\ y_0 = 25{,}000,\ t = 10$

(a) $40{,}000 = 25{,}000e^{k(10)}$
$$1.6 = e^{10k}$$
$$\ln 1.6 = 10k$$
$$.047 = k$$

The equation is

$$y = 25{,}000e^{.047t}.$$

(b)
$$y = 60{,}000$$
$$60{,}000 = 25{,}000e^{.047t}$$
$$2.4 = e^{.047t}$$
$$\ln 2.4 = .047t$$
$$18.6 = t$$

There will be 60,000 bacteria in about 18.6 hours.

32. $f(t) = 500e^{.1t}$

(a)
$$\begin{aligned} f(t) &= 3000 \\ 3000 &= 500e^{.1t} \\ 6 &= e^{.1t} \\ \ln 6 &= .1t \\ 17.9 &\approx t \end{aligned}$$

It will take 17.9 days.

(b) If $t = 0$ corresponds to January 1, the date January 17 should be placed on the product. January 18 would be more than 17.9 days.

34. **(a)** From the graph, the risks of chromosomal abnormality per 1000 at ages 20, 35, 42, and 49 are 2, 5, 29, and 125, respectively.

(Note: It is difficult to read the graph accurately. If you read different values from the graph, your answers to parts (b)-(e) may differ from those given here.)

(b) $y = Ce^{kt}$

When $t = 20$, $y = 2$, and when $t = 35$, $y = 5$.

$$\begin{aligned} 2 &= Ce^{20k} \\ 5 &= Ce^{35k} \\ \frac{5}{2} &= \frac{Ce^{35k}}{Ce^{20k}} \\ 2.5 &= e^{15k} \\ 15k &= \ln 2.5 \\ k &= \frac{\ln 2.5}{15} \\ k &\approx .061 \end{aligned}$$

(c) $y = Ce^{kt}$

When $t = 42$, $y = 29$, and when $t = 49$, $y = 125$.

$$\begin{aligned} 29 &= Ce^{42k} \\ 125 &= Ce^{49k} \\ \frac{125}{29} &= \frac{Ce^{49k}}{Ce^{42k}} \\ \frac{125}{29} &= e^{7k} \\ 7k &= \ln \left(\frac{125}{29}\right) \\ k &= \frac{\ln \left(\frac{125}{29}\right)}{7} \\ k &\approx .21 \end{aligned}$$

(d) Since the values of k are different, we cannot assume the graph is of the form $y = Ce^{kt}$.

(e) The results are summarized in the following table.

n	Value of k for [20, 35]	Value of k for [42, 49]
2	.0011	.0023
3	2.6×10^{-5}	3.4×10^{-5}
4	6.8×10^{-7}	5.5×10^{-7}

The value of n should be somewhere between 3 and 4.

36.
$$\begin{aligned} \frac{1}{2}A_0 &= A_0e^{-.053t} \\ \frac{1}{2} &= e^{-.053t} \\ \ln \frac{1}{2} &= -.053t \\ \ln 1 - \ln 2 &= -.053t \\ \frac{0 - \ln 2}{-.053} &= t \\ 13 &\approx t \end{aligned}$$

The half-life of plutonium 241 is about 13 years.

38. **(a)**
$$\begin{aligned} A(t) &= A_0\left(\frac{1}{2}\right)^{t/13} \\ A(100) &= 2.0\left(\frac{1}{2}\right)^{100/13} \\ A(100) &\approx .0097 \end{aligned}$$

After 100 years, about .0097 gram will remain.

(b)
$$\begin{aligned} .1 &= 2.0\left(\frac{1}{2}\right)^{t/13} \\ \frac{.1}{2.0} &= \left(\frac{1}{2}\right)^{t/13} \\ \ln .05 &= \frac{t}{13} \ln \left(\frac{1}{2}\right) \\ t &= \frac{13 \ln .05}{\ln \left(\frac{1}{2}\right)} \\ t &\approx 56.19 \end{aligned}$$

It will take 56 years.

40. **(a)** $y = y_0 e^{kt}$

When $t = 0,\ y = 500$, so $y_0 = 500$.
When $t = 3,\ y = 386$.

$$\begin{aligned} 386 &= 500e^{3k} \\ \frac{386}{500} &= e^{3k} \\ e^{3k} &= .772 \\ 3k &= \ln\ .772 \\ k &= \frac{\ln\ .772}{3} \\ k &\approx -.0863 \\ y &= 500e^{-.0863t} \end{aligned}$$

(b) $$\begin{aligned} \frac{1}{2}\,y_0 &= y_0 e^{-.0863t} \\ \ln\ \frac{1}{2} &= -.08863t \\ t &= \frac{\ln\ \left(\frac{1}{2}\right)}{-.0863} \\ t &\approx 8.0 \end{aligned}$$

The half-life is about 8.0 days.

42. $y = 40e^{-.004t}$

(a) $t = 180$

$$\begin{aligned} y &= 40e^{-.004(180)} = 40e^{-.72} \\ &\approx 19.5 \text{ watts} \end{aligned}$$

(b) $$\begin{aligned} 20 &= 40e^{-.004t} \\ \frac{1}{2} &= e^{-.004t} \\ \ln\ \frac{1}{2} &= -.004t \\ \frac{\ln\ 1 - \ln\ 2}{-.004} &= t \\ 173 &\approx t \end{aligned}$$

It will take about 173 days.

(c) The power will never be completely gone. The power will approach 0 watts but will never be exactly 0.

44. $P(t) = 100e^{-.1t}$

(a) $P(4) = 100e^{-.1(4)} \approx 67\%$

(b) $P(10) = 100e^{-.1(10)} \approx 37\%$

(c) $$\begin{aligned} 10 &= 100e^{-.1t} \\ .1 &= e^{-.1t} \\ \ln\ (.1) &= -.1t \\ \frac{-\ln\ (.1)}{.1} &= t \\ 23 &\approx t \end{aligned}$$

It would take about 23 days.

(d) $$\begin{aligned} 1 &= 100e^{-.1t} \\ .01 &= e^{-.1t} \\ \ln\ (.01) &= -.1t \\ \frac{-\ln\ (.01)}{.1} &= t \\ 46 &\approx t \end{aligned}$$

It would take about 46 days.

46. $t = 9,\ T_0 = 18,\ C = 5,\ k = .6$

$$f(t) = T_0 + Ce^{-kt}$$

$$\begin{aligned} f(t) &= 18 + 5e^{-.6(9)} \\ &= 18 + 5e^{-5.4} \\ &\approx 18.02 \end{aligned}$$

The temperature is about 18.02°.

48. $C = -14..6,\ k = .6,\ T_0 = 18°,$

$f(t) = 10°$
$f(t) = T_0 + Ce^{-kt}$

$$\begin{aligned} f(t) &= 18 + (-14.6)e^{-.6t} \\ -8 &= -14.6e^{-.6t} \\ .5479 &= e^{-.6t} \\ \ln\ .5479 &= -.6t \\ \frac{-\ln\ .5479}{.6} &= t \\ 1 &\approx t \end{aligned}$$

It would take about 1 hour for the pizza to thaw.

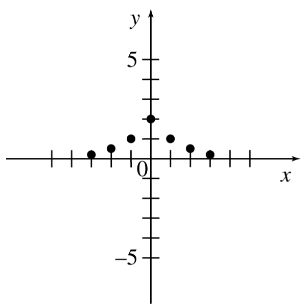
Chapter 2 Review Exercises

6. $y = \dfrac{2}{x^2+1}$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{5}$	$\frac{2}{5}$	1	2	1	$\frac{2}{5}$	$\frac{1}{5}$

Pairs: $\left(-3, \frac{1}{5}\right), \left(-2, \frac{2}{5}\right), (-1, 1),$
$(0, 2), (1, 1), \left(2, \frac{2}{5}\right), \left(3, \frac{1}{5}\right)$

Range: $\{\frac{1}{5}, \frac{2}{5}, 1, 2\}$



8. $f(x) = 8 - x - x^2$

(a) $f(6) = 8 - 6 - (6)^2$
$= 8 - 6 - 36$
$= -34$

(b) $f(-2) = 8 - (-2) - (-2)^2$
$= 8 + 2 - 4 = 6$

(c) $f(-4) = 8 - (-4) - (-4)^2$
$= 8 + 4 - 16 = -4$

(d) $f(r+1)$
$= 8 - (r+1) - (r+1)^2$
$= 8 - r - 1 - (r^2 + 2r + 1)$
$= 8 - r - 1 - r^2 - 2r - 1$
$= 6 - 3r - r^2$

10. $y = \dfrac{\sqrt{x-2}}{2x+3}$

$x - 2 \geq 0$ and $2x + 3 \neq 0$
$x \geq 2$ $\quad 2x \neq -3$
$x \neq -\dfrac{3}{2}$

Domain: $[2, \infty)$

12. $y = \dfrac{3x-4}{x}$

$x \neq 0$

Domain: $(-\infty, 0) \cup (0, \infty)$

14. $y = -\dfrac{1}{4}x^2 + x + 2$

The graph is a parabola.
Let $y = 0$.

$$0 = -\frac{1}{4}x^2 + x + 2$$

Multiply by 4.

$$0 = -x^2 + 4x + 8$$

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4(-1)(8)}}{2(-1)} \\ &= \frac{-4 \pm \sqrt{48}}{-2} \\ &= 2 \pm 2\sqrt{3} \end{aligned}$$

The x-intercepts are $2 + 2\sqrt{3} \approx 5.46$ and $2 - 2\sqrt{3} \approx -1.46$.
Let $x = 0$.

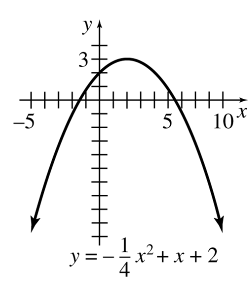
$$y = -\frac{1}{4}(0)^2 + 0 + 2$$

$y = 2$ is the y-intercept.

Vertex: $x = \dfrac{-b}{2a} = \dfrac{-1}{2\left(-\frac{1}{4}\right)} = 2$

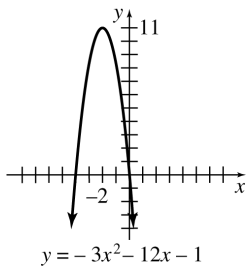
$$\begin{aligned} y &= -\frac{1}{4}(2)^2 + 2 + 2 \\ &= -1 + 4 \\ &= 3 \end{aligned}$$

The vertex is $(2, 3)$.



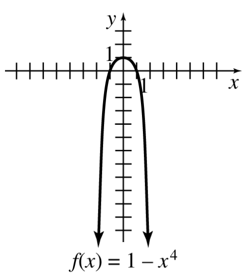
16. $y = -3x^2 - 12x - 1$

x-intercepts: -3.91 and $-.09$
y-intercept: -1
Vertex: $(-2, 11)$



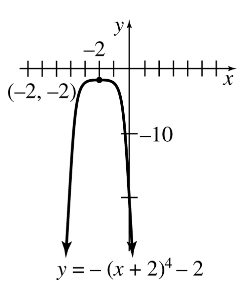
18. $f(x) = 1 - x^4$
$= -x^4 + 1$

Translate the graph of $f(x) = x^4$ 1 unit upward and reflect vertically.



20. $y = -(x + 2)^4 - 2$

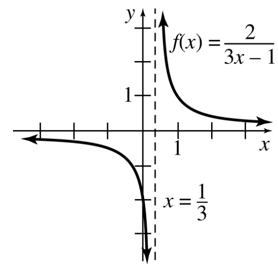
Translate the graph of $y = x^4$ 2 units to the left and 2 units downward.
Reflect vertically.



22. $f(x) = \dfrac{2}{3x - 1}$

Vertical asymptote: $3x - 1 = 0$ or $x = \frac{1}{3}$
Horizontal asymptote: $y = 0$, since $\frac{2}{3x-1}$ approaches zero as x gets larger.

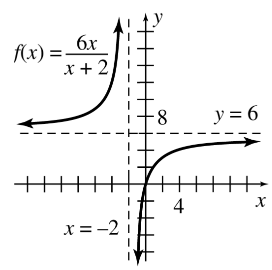
x	0	1	-1	2	-2
y	-2	1	$-\frac{1}{2}$	$\frac{2}{5}$	$-\frac{2}{7}$



24. $f(x) = \dfrac{6x}{x + 2}$

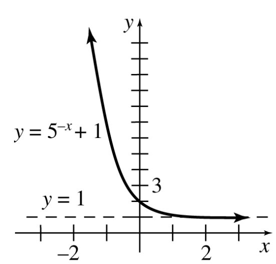
Vertical asymptote: $x = -2$
Horizontal asymptote: $y = 6$

x	-5	-4	-3	-1	0	1	2
y	10	12	18	-6	0	2	3



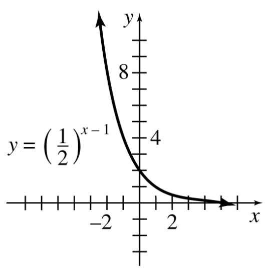
26. $y = 5^{-x} + 1$

x	-2	-1	0	1	2
y	26	6	2	$\frac{6}{5}$	$\frac{26}{5}$



28. $y = \left(\frac{1}{2}\right)^{x-1}$

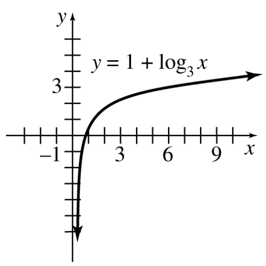
x	-2	-1	0	1	2
y	8	4	2	1	$\frac{1}{2}$



30.

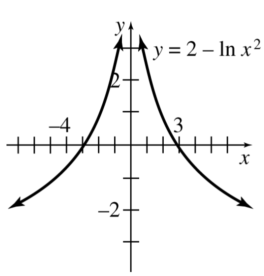
$$\begin{aligned} y &= 1 + \log_3\ x \\ y - 1 &= \log_3\ x \\ 3^{y-1} &= x \end{aligned}$$

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
y	-1	0	1	2	3



32. $y = 2 - \ln\ x^2$

x	-4	-3	-2	-1	1	2	3	4
y	$-.8$	$-.2$	$.6$	2	2	$.6$	$-.2$	$-.8$



34.

$$\begin{aligned} \left(\frac{9}{16}\right)^x &= \frac{3}{4} \\ \left(\frac{3}{4}\right)^{2x} &= \left(\frac{3}{4}\right)^1 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

36.

$$\begin{aligned} \frac{1}{2} &= \left(\frac{b}{4}\right)^{1/4} \\ \left(\frac{1}{2}\right)^4 &= \frac{b}{4} \\ 4\left(\frac{1}{2}\right)^4 &= b \\ 4\left(\frac{1}{16}\right) &= b \\ \frac{1}{4} &= b \end{aligned}$$

38. $3^{1/2} = \sqrt{3}$

The equation in logarithmic form is

$$\log_3\ \sqrt{3} = \frac{1}{2}.$$

40. $10^{1.07918} = 12$

The equation in logarithmic form is

$$\log_{10}\ 12 = 1.07918.$$

42. $\log_{10}\ 100 = 2$

The equation in exponential form is

$$10^2 = 100.$$

44. $\log\ 15.46 = 1.18921$

The equation in exponential form is

$$10^{1.18921} = 15.46.$$

Recall that $\log\ x$ means $\log_{10}\ x$.

46.

$$\begin{aligned} \log_{32}\ 16 &= x \\ 32^x &= 16 \\ 2^{5x} &= 2^4 \\ 5x &= 4 \\ x &= \frac{4}{5} \end{aligned}$$

48.

$$\begin{aligned} \log_{100} 1000 &= x \\ 100^x &= 1000 \\ (10^2)^x &= 10^3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

50. $\log_3\ 2y^3 - \log_3\ 8y^2$

$$\begin{aligned} &= \log_3\ \frac{2y^3}{8y^2} \\ &= \log_3\ \frac{y}{4} \end{aligned}$$

52. $5 \log_4 r - 3 \log_4 r^2$
$$= \log_4 r^5 - \log_4 (r^2)^3$$
$$= \log_4 \frac{r^5}{r^6}$$
$$= \log_4 \frac{1}{r}$$
$$= \log_4 r^{-1}$$
$$= -\log_4 r$$

54.
$$\begin{aligned} 3^{z-2} &= 11 \\ \ln 3^{z-2} &= \ln 11 \\ (z-2) \ln 3 &= \ln 11 \\ z-2 &= \frac{\ln 11}{\ln 3} \\ z &= \frac{\ln 11}{\ln 3} + 2 \\ &\approx 4.183 \end{aligned}$$

56.
$$\begin{aligned} 15^{-k} &= 9 \\ \ln 15^{-k} &= \ln 9 \\ -k \ln 15 &= \ln 9 \\ k &= -\frac{\ln 9}{\ln 15} \\ &\approx -.811 \end{aligned}$$

58.
$$\begin{aligned} e^{3x-1} &= 12 \\ \ln (e^{3x-1}) &= \ln 12 \\ 3x - 1 &= \ln 12 \\ 3x &= 1 + \ln 12 \\ x &= \frac{1 + \ln 12}{3} \\ &\approx 1.162 \end{aligned}$$

60.
$$\begin{aligned} \left(1 + \frac{2p}{5}\right)^2 &= 3 \\ 1 + \frac{2p}{5} &= \pm\sqrt{3} \\ 5 + 2p &= \pm 5\sqrt{3} \\ 2p &= -5 \pm 5\sqrt{3} \\ p &= \frac{-5 \pm 5\sqrt{3}}{2} \\ p &= \frac{-5 + 5\sqrt{3}}{2} \approx 1.830 \\ \text{or} \quad p &= \frac{-5 - 5\sqrt{3}}{2} \approx -6.830 \end{aligned}$$

62.
$$\begin{aligned} \log_3(2x+5) &= 2 \\ 3^2 &= 2x + 5 \\ 9 &= 2x + 5 \\ 4 &= 2x \\ x &= 2 \end{aligned}$$

64.
$$\begin{aligned} \log_2(5m-2) - \log_2(m+3) &= 2 \\ \log_2 \frac{5m-2}{m+3} &= 2 \\ \frac{5m-2}{m+3} &= 2^2 \\ 5m - 2 &= 4(m+3) \\ 5m - 2 &= 4m + 12 \\ m &= 14 \end{aligned}$$

66. $f(x) = \log_a x$; $a > 0$, $a \neq 1$

(a) The domain is $(0, \infty)$.

(b) The range is $(-\infty, \infty)$.

(c) The x-intercept is 1.

(d) There are no discontinues..

(e) The y-axis, $x = 0$, is a vertical asymptote.

(f) f is increasing if $a > 1$.

(g) f is decreasing if $0 < a < 1$.

68. **(a)** For x in the interval $0 < x \leq 1$, the renter is charged the fixed cost of \$40 and 1 day's rent of \$40 so
$$\begin{aligned} C\left(\tfrac{3}{4}\right) &= \$40 + \$40(1) \\ &= \$40 + \$40 \\ &= \$80. \end{aligned}$$

(b)
$$\begin{aligned} C\left(\tfrac{9}{10}\right) &= \$40 + \$40(1) \\ &= \$40 + \$40 \\ &= \$80. \end{aligned}$$

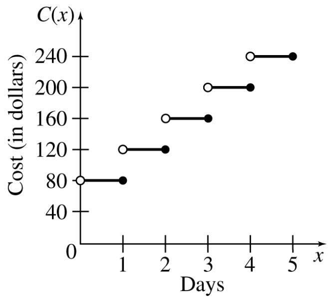
(c)
$$\begin{aligned} C(1) &= \$40 + \$40(1) \\ &= \$40 + \$40 \\ &= \$80. \end{aligned}$$

(d) For x in the interval $1 < x \leq 2$, the renter is charged the fixed cost of \$40 and 2 days rent of \$80, so
$$\begin{aligned} C\left(1\tfrac{5}{8}\right) &= \$40 + \$40(2) \\ &= \$40 + \$80 \\ &= \$120. \end{aligned}$$

(e) For x in the interval $2 < x \leq 3$ the renter is charged the fixed cost of \$40 and 3 days rent of \$120. So

$$\begin{aligned} C\left(2\tfrac{1}{9}\right) &= \$40 + \$40(3) \\ &= \$40 + \$120 \\ &= \$160. \end{aligned}$$

(f)



(g) The independent variable is the number of days, or x.

(h) The dependent variable is the cost, or $C(x)$.

70. $P = \$6902,\ r = 12\%,\ t = 8,\ m = 2$

$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$

$$\begin{aligned} A &= 6902\left(1 + \frac{.12}{2}\right)^{8(2)} \\ &= 6902(1.06)^{16} \\ &= \$17,533.51 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= A - P \\ &= \$17,533.51 - \$6902 \\ &= \$10,631.51 \end{aligned}$$

72. \$1000 deposited at 6% compounded semiannually.

$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$

To double:

$$\begin{aligned} 2(1000) &= 1000\left(1 + \frac{.06}{2}\right)^{t\cdot 2} \\ 2 &= 1.03^{2t} \\ \ln 2 &= 2t \ln 1.03 \\ t &= \frac{\ln 2}{2 \ln 1.03} \\ &\approx 12 \text{ years} \end{aligned}$$

To triple:

$$\begin{aligned} 3(1000) &= 1000\left(1 + \frac{.06}{2}\right)^{t\cdot 2} \\ 3 &= 1.03^{2t} \\ \ln 3 &= 2t \ln 1.03 \\ t &= \frac{\ln 3}{2 \ln 1.03} \\ &\approx 19 \text{ years} \end{aligned}$$

74. $P = \$12{,}104,\ r = 8\%,\ t = 2$

$$\begin{aligned} A &= Pe^{rt} \\ &= 12,104e^{.08(2)} \\ &= \$14,204.18 \end{aligned}$$

76. $A = \$1500,\ r = .10,\ t = 9$

$$\begin{aligned} A &= Pe^{rt} \\ &= 1500e^{.10(9)} \\ &= 1500e^{.9} \\ &= \$3689.40 \end{aligned}$$

78. $r = 7\%,\ m = 4$

$$\begin{aligned} r_E &= \left(1 + \frac{r}{m}\right)^m = 1 \\ &= \left(1 + \frac{.07}{4}\right)^4 - 1 \\ &= .0719 = 7.19\% \end{aligned}$$

80. $r = 9\%$ compounded continuously

$$\begin{aligned} r_E &= e^r - 1 \\ &= e^{.09} - 1 \\ &= .0942 = 9.42\% \end{aligned}$$

82. $A = \$10{,}000,\ r = 8\%,\ m = 2,\ t = 6$

$$\begin{aligned} P &= A\left(1 + \frac{r}{m}\right)^{-tm} \\ &= 10{,}000\left(1 + \frac{.08}{2}\right)^{-2(6)} \\ &= 10{,}000(1.04)^{-12} \\ &= \$6245.97 \end{aligned}$$

84. $P = \$1,\ r = .08$

$$\begin{aligned} A &= Pe^{rt},\ A = 3(1) \\ 3 &= 1e^{.08t} \\ \ln\ 3 &= .08t \\ \frac{\ln\ 3}{.08} &= t \\ 13.7 &= t \end{aligned}$$

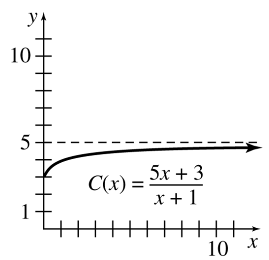
It would take about 13.7 years.

86. $P = A\left(1 + \frac{r}{m}\right)^{-tm}$

$$P = 25{,}000\left(1 + \frac{.06}{12}\right)^{-3(12)}$$
$$= 25{,}000(1.005)^{-36}$$
$$= \$20{,}891.12$$

88. $C(x) = \frac{5x+3}{x+1}$

(a)



(b) $C(x+1) - C(x)$

$$= \frac{5(x+1)+3}{(x+1)+1} - \frac{5x+3}{x+1}$$
$$= \frac{5x+8}{x+2} - \frac{5x+3}{x+1}$$
$$= \frac{(5x+8)(x+1) - (5x+3)(x+2)}{(x+2)(x+1)}$$
$$= \frac{5x^2+13x+8-5x^2-13x-6}{(x+2)(x+1)}$$
$$= \frac{2}{(x+2)(x+1)}$$

(c) $A(x) = \frac{C(x)}{x} = \frac{\frac{5x+3}{x+1}}{x}$

$$= \frac{5x+3}{x(x+1)}$$

(d) $A(x+1) - A(x)$

$$= \frac{5(x+1)+3}{(x+1)[(x+1)+1]} - \frac{5x+3}{x(x+1)}$$
$$= \frac{5x+8}{(x+1)(x+2)} - \frac{5x+3}{x(x+1)}$$
$$= \frac{x(5x+8) - (5x+3)(x+2)}{x(x+1)(x+2)}$$
$$= \frac{5x^2+8x-5x^2-13x-6}{x(x+1)(x+2)}$$
$$= \frac{-5x-6}{x(x+1)(x+2)}$$

90. $F(x) = -\frac{2}{3}x^2 + \frac{14}{3}x + 96$

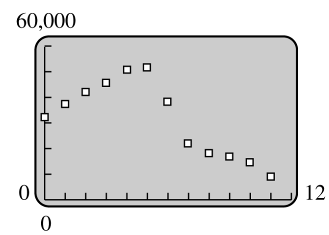
The maximum fever occurs at the vertex of the parabola.

$$x = \frac{-b}{2a} = \frac{-\frac{14}{3}}{-\frac{4}{3}} = \frac{7}{2}$$

$$y = -\frac{2}{3}\left(\frac{7}{2}\right)^2 + \frac{14}{3}\left(\frac{7}{2}\right) + 96$$
$$= -\frac{2}{3}\left(\frac{49}{4}\right) + \frac{49}{3} + 96$$
$$= -\frac{49}{6} + \frac{49}{3} + 96$$
$$= -\frac{49}{6} + \frac{98}{6} + \frac{576}{6} = \frac{625}{6} \approx 104.2$$

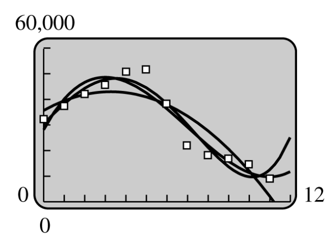
The maximum fever occurs on the third day. It is about 104.2°F.

92. (a)



(b) $y = -680.8x^2 + 4454x + 35{,}620$
$y = 151.7x^3 - 3184x^2 + 14{,}997x + 28{,}111$
$y = 20.59x^4 - 301.2x^3 - 63.25x^2 + 8074x + 30{,}441$

(c)



The cubic or quartic function best models the data.

(d) $x = 12$

quadratic:

$$y = -680.8(12)^2 + 4454(12) + 35{,}620$$
$$\approx -8967$$

cubic:

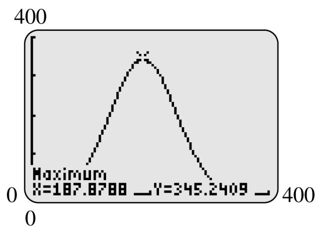
$$y = 151.7(12)^3 - 3184(12)^2 + 14,997(12) + 28,111$$
$$\approx 11{,}717$$

quartic:

$$\begin{aligned} y &= 20.59(12)^4 - 301.2(12)^3 - 63.25(12)^2 \\ &\quad + 8074(12) + 30{,}441 \\ &\approx 24{,}702 \end{aligned}$$

The first value is negative, which is impossible, and the last two are higher than the 2001 figure, yet the trend is clearly towards a drop in the number of deaths.

94.



This function has a maximum value at $x \approx 187.9$. At $x \approx 187.9, y \approx 345$. The largest girth for which this formula gives a reasonable answer is 187.9 cm. The predicted mass of a polar bear with this girth is 345 kg.

96. $p(t) = \dfrac{1.79 \cdot 10^{11}}{(2026.87 - t)^{.99}}$

(a) $p(1999) \approx 6.64$ billion

This is about 640 million more than the estimate of 5996 million.

(b) $p(2020) \approx 26.56$ billion
$p(2025) \approx 96.32$ billion

98. Graph

$$y = c(t) = e^{-t} - e^{-2t}$$

on a graphing calculator and locate the maximum point. A calculator shows that the x-coordinate of the maximum point is about .69, and the y-coordinate is exactly .25. Thus, the maximum concentration of .25 occurs at about .69 minutes.

100. $y = y_0 e^{-kt}$

(a)

$$\begin{aligned} 128{,}000 &= 100{,}000e^{-k(-5)} \\ 128{,}000 &= 100{,}000^{5k} \\ \frac{128}{100} &= e^{5k} \\ \ln\left(\frac{128}{100}\right) &= 5k \\ .05 &\approx k \end{aligned}$$

$y = 100{,}000e^{-.05t}$

(b)

$$\begin{aligned} 70{,}000 &= 100{,}000e^{-.05t} \\ \frac{7}{10} &= e^{-.05t} \\ \ln \frac{7}{10} &= -.05t \\ 7.1 &\approx t \end{aligned}$$

It will take about 7.1 years.

102. **(a)** Since the speed in one direction is $v + w$ and in the other direction is $v - w$, the time in one direction is $\frac{d}{v+w}$ and in the other direction is $\frac{d}{v-w}$. So the total time is $\frac{d}{v+w} + \frac{d}{v-w}$.

(b) The average speed is the total distance divided by the total time. So

$$v_{aver} = \frac{2d}{\frac{d}{v+w} + \frac{d}{v-w}}.$$

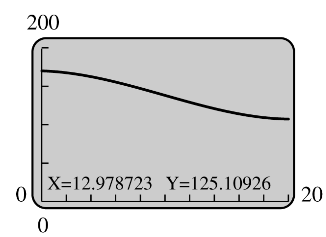
(c)

$$\begin{aligned} &\frac{2d}{\frac{d}{v+w} + \frac{d}{v-w}} \\ &= \frac{2d}{\frac{d}{v+w} + \frac{d}{v-w}} \cdot \frac{(v+w)(v-w)}{(v+w)(v-w)} \\ &= \frac{2d(v^2 - w^2)}{d(v-w) + d(v+w)} \\ &= \frac{2d(v^2 - w^2)}{dv - dw + dv + dw} \\ &= \frac{2d(v^2 - w^2)}{2dv} \\ &= \frac{v^2 - w^2}{v} = v - \frac{w^2}{v} \end{aligned}$$

(d) $v_{aver} = v - \dfrac{w^2}{v}$

v_{aver} will be greatest when $w = 0$.

104. **(a)**



Emissions have decreased the entire time, although in recent years the rate of decrease has slowed down.

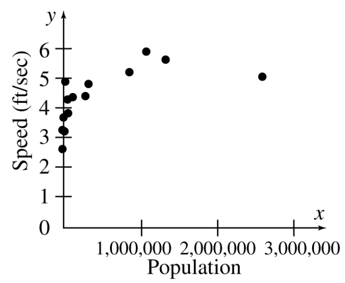
(b) Emissions were about 125 million short tons 13 years after 1983, or in 1996.

106. **(a)** From the graph, horsepower is maximum when the engine speed is 5750 rpm.

(b) The maximum horsepower is approximately 310.

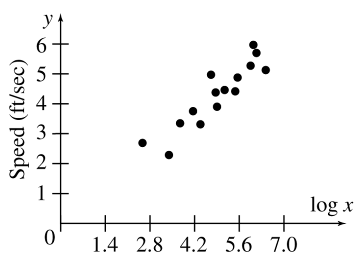
(c) When the horsepower is maximum, the torque is 280 foot-pounds.

108. **(a)**



(b) Using a graphing calculator, $r = .63$.

(c)



Yes, the data are now more linear than in part (a).

(d) Using a graphing calculator, $r = .91$. r is closer to 1.

(e) Using a graphing calculator,

$$Y = .873 \log x - .0255.$$

Extended Application: Characteristics of the Monkeyface Prickleback

1.
$$\begin{aligned} L_t &= L_x\,(1 - e^{-kt}) \\ L_t &= 71.5(1 - e^{-.1t}) \\ L_4 &= 71.5(1 - e^{-.4}) \approx 23.6 \\ L_{11} &= 71.5(1 - e^{-1.1}) \approx 47.7 \\ L_{17} &= 71.5(1 - e^{-1.7}) \approx 58.4 \end{aligned}$$

The estimates are low.

2. $W = aL^b$
$W = .01289L^{2.9}$

If $L = 25,\ W = .01289(25)^{2.9} \approx 146.0$
If $L = 40,\ W = .01289(40)^{2.9} \approx 570.5$
If $L = 60,\ W = .01289(60)^{2.9} \approx 1848.8$

Compared to the curve, the answers are reasonable.

Chapter 3

THE DERIVATIVE

3.1 Limits

2. Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = -1$,

$\lim_{x\to 2} f(x) = -1$. The answer is a.

4. Since $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = -\infty$,

$\lim_{x\to 1} f(x) = -\infty$. The answer is b.

6. By reading the graph, as x gets closer to 2 from the left or the right, $F(x)$ gets closer to 4.

$$\lim_{x\to 2} F(x) = 4$$

8. By reading the graph, as x gets closer to 3 from the left or the right, $g(x)$ gets closer to 2.

$$\lim_{x\to 3} g(x) = 2$$

10. **(a)** (i) By reading the graph, as x gets closer to 1 from the left, $f(x)$ gets closer to 1.

$$\lim_{x\to 1^-} f(x) = 1$$

(ii) By reading the graph, as x gets closer to 1 from the right, $f(x)$ gets closer to 1.

$$\lim_{x\to 1^+} f(x) = 1$$

(iii) Since $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = 1$,

$$\lim_{x\to 1} f(x) = 1.$$

(iv) $f(1) = 2$ since $(1, 2)$ is part of the graph.

(b) (i) By reading the graph, as x gets closer to 2 from the left, $f(x)$ gets closer to 0.

$$\lim_{x\to 2^-} f(x) = 0$$

(ii) By reading the graph, as x gets closer to 2 from the right, $f(x)$ gets closer to 0.

$$\lim_{x\to 2^+} f(x) = 0$$

(iii) Since $\lim_{x\to 2^-} f(x) = 0$ and $\lim_{x\to 2^+} f(x) = 0$,

$$\lim_{x\to 2} f(x) = 0.$$

(iv) $f(2) = 0$ since $(2, 0)$ is point of the graph.

12. By reading the graph, as x moves further to the left, $g(x)$ gets larger and larger. Therefore, $\lim_{x\to -\infty} g(x) = \infty$.

16. $f(x) = 2x^2 - 4x + 3$; find $\lim_{x\to 1} f(x)$.

Substitute .9 for x in the expression at the right to get $f(.9) = 1.02$.
Continue substituting to complete the table.

x	.9	.99	.999
$f(x)$	1.02	1.0002	1.000002

x	1.001	1.01	1.1
$f(x)$	1.000002	1.0002	1.02

As x approaches 1 from the left or the right, $f(x)$ approaches 1.

$$\lim_{x\to 1} f(x) = 1$$

18. $f(x) = \dfrac{2x^3 + 3x^2 - 4x - 5}{x + 1}$; find $\lim_{x\to -1} f(x)$.

x	-1.1	-1.01	-1.001
$f(x)$	-3.68	-3.969	-3.996

x	$-.999$	$-.99$	$-.9$
$f(x)$	-4.002	-4.02	-4.28

As x approaches -1 from the left or the right, $f(x)$ approaches -4.

$$\lim_{x\to -1} f(x) = -4$$

20. $f(x) = \dfrac{\sqrt{x}-3}{x-3}$; find $\lim\limits_{x\to 3} f(x)$.

x	2.9	2.99	2.999
$f(x)$	12.9706	127.0838	1268.237

x	3.001	3.01	3.1
$f(x)$	−1267.66	−125.506	−12.6795

$\lim\limits_{x\to 3^-} f(x) = \infty$

$\lim\limits_{x\to 3^+} f(x) = -\infty$

Thus, $\lim\limits_{x\to 3} f(x)$ does not exist.

22. $\lim\limits_{x\to 4} [(g(x)\cdot f(x)]$

$$= \left[\lim_{x\to 4} g(x)\right]\cdot\left[\lim_{x\to 4} f(x)\right]$$
$$= 8\cdot 16 = 128$$

24. $\lim\limits_{x\to 4} \log_2 f(x) = \log_2 \lim\limits_{x\to 4} f(x)$

$$= \log_2 16 = 4$$

26. $\lim\limits_{x\to 4} \sqrt[3]{g(x)} = \lim\limits_{x\to 4} [g(x)]^{1/3}$

$$= \left[\lim_{x\to 4} g(x)\right]^{1/3}$$
$$= 8^{1/3} = 2$$

28. $\lim\limits_{x\to 4} [1+f(x)]^2 = \left[\lim\limits_{x\to 4} (1+f(x))\right]^2$

$$= \left[\lim_{x\to 4} 1 + \lim_{x\to 4} f(x)\right]^2$$
$$= (1+16)^2 = 17^2$$
$$= 289$$

30. $\lim\limits_{x\to 4} \dfrac{5g(x)+2}{1-f(x)} = \dfrac{\lim\limits_{x\to 4} [5g(x)+2]}{\lim\limits_{x\to 4} [1-f(x)]}$

$$= \frac{5 \lim_{x\to 4} g(x) + \lim_{x\to 4} 2}{\lim_{x\to 4} 1 - \lim_{x\to 4} f(x)}$$
$$= \frac{5\cdot 8+2}{1-16}$$
$$= -\frac{42}{15} = -\frac{14}{5}$$

32. $\lim\limits_{x\to -2} \dfrac{x^2-4}{x+2} = \lim\limits_{x\to -2} \dfrac{(x+2)(x-2)}{(x+2)}$

$$= \lim_{x\to -2} (x-2)$$
$$= -2-2 = -4$$

34. $\lim\limits_{x\to -3} \dfrac{x^2-9}{x^2+x-6} = \lim\limits_{x\to -3} \dfrac{(x-3)(x+3)}{(x-2)(x+3)}$

$$= \lim_{x\to -3} \frac{x-3}{x-2}$$
$$= \frac{-3-3}{-3-2}$$
$$= \frac{-6}{-5}$$
$$= \frac{6}{5}$$

36. $\lim\limits_{x\to 5} \dfrac{x^2-3x-10}{x-5} = \lim\limits_{x\to 5} \dfrac{(x-5)(x+2)}{(x-5)}$

$$= \lim_{x\to 5} (x+2)$$
$$= 5+2 = 7$$

38. $\lim\limits_{x\to 0} \dfrac{\frac{-1}{x+2}-\frac{1}{2}}{x}$

$$= \lim_{x\to 0} \left(\frac{-1}{x+2}-\frac{1}{2}\right)\left(\frac{1}{x}\right)$$
$$= \lim_{x\to 0} \left[\frac{-2}{2(x+2)}+\frac{x+2}{2(x+2)}\right]\left(\frac{1}{x}\right)$$
$$= \lim_{x\to 0} \frac{-2+x+2}{2(x+2)(x)}$$
$$= \lim_{x\to 0} \frac{x}{2x(x+2)}$$
$$= \lim_{x\to 0} \frac{1}{2(x+2)}$$
$$= \frac{1}{2(0+2)}$$
$$= \frac{1}{4}$$

40. $\lim\limits_{x\to 36} \dfrac{\sqrt{x}-6}{x-36}$

$$= \lim_{x\to 36} \frac{\sqrt{x}-6}{x-36}\cdot\frac{\sqrt{x}+6}{\sqrt{x}+6}$$
$$= \lim_{x\to 36} \frac{(x-36)}{(x-36)(\sqrt{x}+6)}$$
$$= \lim_{x\to 36} \frac{1}{\sqrt{x}+6}$$
$$= \frac{1}{\sqrt{36}+6}$$
$$= \frac{1}{6+6}$$
$$= \frac{1}{12}$$

42. $\lim_{h\to 0} \dfrac{(x+h)^3 - x^3}{h}$

$= \lim_{h\to 0} \dfrac{(x^3 + 3x^2 + 3xh^2 + h^3) - x^3}{h}$

$= \lim_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3}{h}$

$= \lim_{h\to 0} \dfrac{h(3x^2 + 3xh + h^2)}{h}$

$= \lim_{h\to 0} (3x^2 + 3xh + h^2)$

$= 3x^2 + 3x(0) + (0)^2$

$= 3x^2$

44. $\lim_{x\to -\infty} \dfrac{8x+2}{2x-5} = \lim_{x\to -\infty} \dfrac{\frac{8x}{x} + \frac{2}{x}}{\frac{2x}{x} - \frac{5}{x}}$

$= \lim_{x\to -\infty} \dfrac{8 + \frac{2}{x}}{2 - \frac{5}{x}}$

$= \dfrac{8+0}{2-0}$

$= 4$

46. $\lim_{x\to\infty} \dfrac{x^2 + 2x - 5}{3x^2 + 2} = \lim_{x\to\infty} \dfrac{\frac{x^2}{x^2} + \frac{2x}{x^2} - \frac{5}{x^2}}{\frac{3x^2}{x^2} + \frac{2}{x^2}}$

$= \lim_{x\to\infty} \dfrac{1 + \frac{2}{x} - \frac{5}{x^2}}{3 + \frac{2}{x^2}}$

$= \dfrac{1+0-0}{3+0} = \dfrac{1}{3}$

48. $\lim_{x\to\infty} \dfrac{2x^2 - 1}{3x^4 + 2} = \lim_{x\to\infty} \dfrac{\frac{2x^2}{x^4} - \frac{1}{x^4}}{\frac{3x^4}{x^4} + \frac{2}{x^4}}$

$= \lim_{x\to\infty} \dfrac{\frac{2}{x^2} - \frac{1}{x^4}}{3 + \frac{2}{x^4}}$

$= \dfrac{0-0}{3+0} = \dfrac{0}{3} = 0$

50. $\lim_{x\to\infty} \dfrac{x^4 - x^3 - 3x}{7x^2 + 9} = \lim_{x\to\infty} \dfrac{\frac{x^4}{x^4} - \frac{x^3}{x^4} - \frac{3x}{x^4}}{\frac{7x^2}{x^4} + \frac{9}{x^4}}$

$= \lim_{x\to\infty} \dfrac{1 - \frac{1}{x} - \frac{3}{x^3}}{\frac{7}{x^2} + \frac{9}{x^4}}$

$= \dfrac{1-0-0}{0+0} = \dfrac{1}{0}$

Division by 0 is undefined, so this limit does not exist. By examining what happens as larger and larger values of x are put into the function, we see that

$$\lim_{x\to\infty} \frac{x^4 - x^3 - 3x}{7x^2 + 9} = \infty.$$

52. $\lim_{x\to\infty} \dfrac{-5x^3 - 4x^2 + 8}{6x^2 + 3x + 2}$

$= \lim_{x\to\infty} \dfrac{\frac{-5x^3}{x^2} - \frac{4x^2}{x^2} + \frac{8}{x^2}}{\frac{6x^2}{x^2} + \frac{3x}{x^2} + \frac{2}{x^2}}$

$= \lim_{x\to\infty} \dfrac{-5x - 4 + \frac{8}{x^2}}{6 + \frac{3}{x} + \frac{2}{x^2}}$

The denominator approaches 6, while the numerator becomes a negative number that is larger and larger in magnitude, so

$$\lim_{x\to\infty} \frac{-5x^3 - 4x^2 + 8}{6x^2 + 3x + 2} = -\infty.$$

54. $G(x) = \dfrac{-6}{(x-4)^2}$

(a) $\lim_{x\to 4} G(x) = -\infty$, since by looking at the graph of $G(x)$, we see that as x gets closer to 4 from either the right or the left, $g(x)$ gets smaller.

(b) Since $(x-4)^2 = 0$ when $x = 4$, $x = 4$ is the vertical asymptote of the graph of $G(x)$.

(c) The two answers are related. Since $x = 4$ is a vertical asymptote, we know the $\lim_{x\to 4} G(x)$ does not exist.

58. (a) $y = xe^{-x}$

From the graph, it appears that

$$\lim_{x\to\infty} xe^{-x} = 0.$$

x	1	10	50
y	.37	.00045	9.64×10^{-21}

(b) $y = x^2e^{-x}$

From the graph, it appears that

$$\lim_{x\to\infty} x^2e^{-x} = 0.$$

x	1	10	50
y	.37	.0045	4.82×10^{-19}

(c) $\lim_{x\to\infty} x^ne^{-x} = 0$. As n gets larger, y is still very small but larger than smaller values of n.

60. (a) $y = x \ln x$

From the graph, it appears that

$$\lim_{x\to 0^+} x \ln x = 0.$$

x	1	.5	.1	.01	.001
y	0	$-.347$	$-.230$	$-.0461$	$-.0069$

(b) $y = x(\ln\ x)^2$

From the graph it appears that

$$\lim_{x\to 0^+} x(\ln\ x)^2 = 0.$$

x	1	.5	.1	.01	.001
y	0	.240	.53	.212	.048

(c) $\lim_{x\to 0^+} x(\ln\ x)^n = 0$. As n gets closer to 0, $|y|$ is still very small but bigger than smaller values of n.

64. $\lim_{x\to 2} \dfrac{x^4 + x - 18}{x^2 - 4}$

(a)

x	2.01	2.001	2.0001	1.99	1.999	1.9999
$f(x)$	8.29	8.25	8.25	8.21	8.25	8.25

(b) Graph

$$y = \frac{x^4 + x - 18}{x^2 - 4}.$$

One suitable choice for the viewing window is $[-5, 5]$ by $[0, 20]$. Because $x^2 - 4 = 0$ when $x = -2$ or $x = 2$, we know that the function is undefined at these two x-values. The graph shows an asymptote at $x = -2$. There should be open circle to show a "hole" in the graph at $x = 2$. The graphing calculator doesn't show the hole, but if we try to find the value of the function at $x = 2$, we see that it is undefined. (Using the TABLE feature on a TI-83, we see that for $x = 2$, the y-value is listed as "ERROR.")

By viewing the function near $x = 2$ and using the ZOOM feature, we verify that the required limit is 8.25. (We may not be able to get this value exactly.)

66. $\lim_{x\to 4} \dfrac{x^{3/2} - 8}{x + x^{1/2} - 6}$

(a)

x	4.1	4.01	4.001	4.0001
$f(x)$	2.4179	2.4018	2.4002	2.4

x	3.9	3.99	3.999	3.9999
$f(x)$	2.3819	2.3982	2.3998	2.4

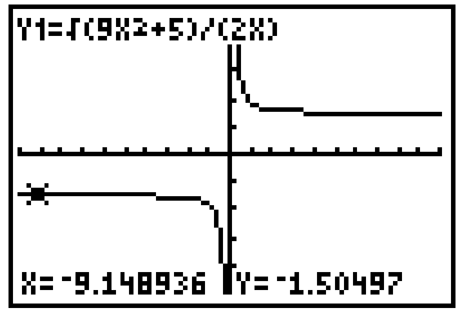
(b) Graph

$$y = \frac{x^{3/2} - 8}{x + x^{1/2} - 6}.$$

This function is undefined at $x = 4$ because this value would make the denominator equal to 0. However, by viewing the function near $x = 4$ and using the ZOOM feature, we verify that the required limit is 2.4.

68. $\lim_{x\to -\infty} \dfrac{\sqrt{9x^2 + 5}}{2x}$

Graph this function on a graphing calculator. A good choice for the viewing window is $[-10, 10]$ by $[-5, 5]$.



(a) The graph appears to have horizontal asymptotes at $y = \pm 1.5$. We see that as $x \to -\infty$, $y \to -1.5$, so we determine that

$$\lim_{x\to -\infty} \frac{\sqrt{9x^2 + 5}}{2x} = -1.5.$$

(b) As $x \to -\infty$,

$$\sqrt{9x^2 + 5} \to \sqrt{9x^2} = 3\,|x|\,,$$

and

$$\frac{\sqrt{9x^2 + 5}}{2x} \to \frac{3\,|x|}{2x}.$$

Since $x < 0$, $|x| = -x$, so

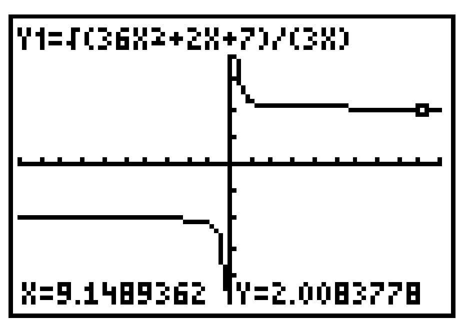
$$\frac{3\,|x|}{2x} = \frac{3(-x)}{2x} = -\frac{3}{2}.$$

Thus,

$$\lim_{x\to -\infty} \frac{\sqrt{9x^2 + 5}}{2x} = -\frac{3}{2} \text{ or } -1.5.$$

70. $\lim_{x\to \infty} \dfrac{\sqrt{36x^2 + 2x + 7}}{3x}$

Graph this function on a graphing calculator. A good choice for the viewing window is $[-10, 10]$ by $[-5, 5]$.



(a) The graph appears to have horizontal asymptotes at $y = \pm 2$. We see that as $x \to \infty$, $y \to 2$, so we determine that

$$\lim_{x\to\infty} \frac{\sqrt{36x^2+2x+7}}{3x} = 2.$$

(b) As $x \to \infty$,

$$\sqrt{36x^2+2x+7} \to \sqrt{36x^2} = 6\,|x|\,,$$

and

$$\frac{\sqrt{36x^2+2x+7}}{3x} \to \frac{6\,|x|}{3x}.$$

Since $x > 0$, $|x| = x$, so

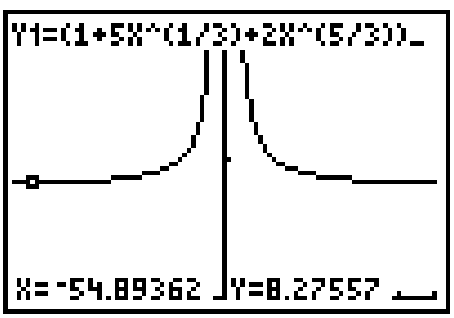
$$\frac{6\,|x|}{3x} = \frac{6x}{3x} = 2.$$

Thus,

$$\lim_{x\to\infty} \frac{\sqrt{36x^2+2x+7}}{3x} = 2.$$

72. $\displaystyle\lim_{x\to-\infty} \frac{\left(1+5x^{1/3}+2x^{5/3}\right)^3}{x^5}$

Graph this function on a graphing calculator. a good choice for the viewing window is $[-60, 60]$ by $[0, 20]$ with Xscl = 10, Yscl = 10.



(a) The graph appears to have a horizontal asymptote at $y = 8$. We see that as $x \to -\infty, y \to 8$, so we determine that

$$\lim_{x\to-\infty} \frac{\left(1+5x^{1/3}+2x^{5/3}\right)^3}{x^5} = 8.$$

(b) As $x \to -\infty$, the highest power term dominates in the numerator, so

$$\left(1+5x^{1/3}+2x^{5/3}\right)^3 \to (2x^{5/3}) = 2^3x^5 = 8x^5,$$

and

$$\frac{\left(1+5x^{1/3}+2x^{5/3}\right)^3}{x^5} \to \frac{8x^5}{x^5} = 8.$$

Thus,

$$\lim_{x\to-\infty} \frac{\left(1+5x^{1/3}+2x^{5/3}\right)^3}{x^5} = 8.$$

74. **(a)** $\displaystyle\lim_{x\to 12} G(t)$

As t approaches 12 from either direction, the value of $G(t)$ for the corresponding point on the graph approaches 3.
Thus, $\displaystyle\lim_{x\to 12} G(t) = 3$ which represents 3 million gallons.

(b) $\displaystyle\lim_{x\to 16^+} G(t) = 1.5$

$\displaystyle\lim_{x\to 16^-} G(t) = 2$

Since $\displaystyle\lim_{x\to 16^+} G(t) \neq \lim_{x\to 16^-} G(t)$, $\displaystyle\lim_{x\to 16} G(t)$ does not exist.

(c) $G(16)$ is the value of function $G(t)$ when $t = 16$. This value occurs at the solid dot on the graph
$G(16) = 2$ which represents 2 million gallons.

(d) The tipping point occurs at the break in the graph, when $t = 16$ months.

76. $C(x) = 15{,}000 + 6x$

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{15{,}000+6x}{x} = \frac{15{,}000}{x} + 6$$

$$\lim_{x\to\infty} \overline{C}(x) = \lim_{x\to\infty} \frac{15{,}000}{x} + 6 = 0 + 6 = 6$$

This means that the average cost approaches \$6 as the number of tapes produced becomes very large.

78. $$\lim_{n\to\infty} \left[R\left[\frac{1-(1+i)^{-n}}{i}\right]\right]$$

$$= \frac{R}{i}\lim_{n\to\infty}\left[1-(1+i)^{-n}\right]$$

$$= \frac{R}{i}\left[\lim_{n\to\infty} 1 - \lim_{n\to\infty}(1+i)^{-n}\right]$$

$$= \frac{R}{i}[1-0] = \frac{R}{i}$$

80. **(a)** $N(65) = 71.8e^{-8.96e^{(-.0685(65))}}$
≈ 64.68

To the nearest whole number, this species of alligator has approximately 65 teeth after 65 days of incubation by this formula.

(b) Since $\displaystyle\lim_{t\to\infty}(-8.96e^{-.0685t}) = -8.96 \cdot 0 = 0$, it follows that

$$\lim_{t\to\infty} 71.8e^{-8.96e^{(-.0685t)}} = 71.8e^0$$
$$= 71.8 \cdot 1$$
$$= 71.8$$

So, to the nearest whole number, $\lim_{t\to\infty} N(t) \approx 72$. Therefore, by this model a newborn alligator of this species will have about 72 teeth.

82. $A(h) = \dfrac{.17h}{h^2+2}$

$$\lim_{x\to\infty} A(h) = \lim_{x\to\infty} \frac{.17h}{h^2+2} = \lim_{x\to\infty} \frac{\frac{.17h}{h^2}}{\frac{h^2}{h^2}+\frac{2}{h^2}} = \lim_{x\to\infty} \frac{\frac{.17}{h}}{1+\frac{2}{h^2}} = \frac{0}{1+0} = 0$$

This means that the concentration of the drug in the bloodstream approaches 0 as the number of hours after injection increases.

3.2 Continuity

2. Discontinuous at $x = -1$

(a) $\lim_{x\to-1^-} f(x) = 2$

(b) $\lim_{x\to-1+} f(x) = 4$

(c) $\lim_{x\to-1} f(x)$ does not exist (since parts (a) and (b) have different answers).

(d) $f(-1) = 2$

4. Discontinuous at $x = -2$ and $x = 3$

(a) $\lim_{x\to-2^-} f(x) = -1$ $\quad\lim_{x\to3^-} f(x) = -1$

(b) $\lim_{x\to-2^+} f(x) = -1$ $\quad\lim_{x\to3+} f(x) = -1$

(c) $\lim_{x\to-2} f(x) = -1$ (since parts (a) and (b) have the same answer)
$\lim_{x\to3} f(x) = -1$ (since parts (a) and (b) have the same answer)

(d) $f(-2) = 1$ $\quad f(3) = 1$

6. Discontinuous at $x = 0$ and $x = 2$

(a) $\lim_{x\to0^-} f(x) = -\infty$ $\quad\lim_{x\to2^-} f(x) = -2$

(b) $\lim_{x\to0^+} f(x) = -\infty$ $\quad\lim_{x\to2+} f(x) = -2$

(c) $\lim_{x\to0} f(x) = -\infty$ (since parts (a) and (b) have the same answer)

$\lim_{x\to2} f(x) = -2$ (since parts (a) and (b) have the same answer)

(d) f(0) does not exist.

f(2) does not exist.

8. $f(x) = \dfrac{-2x}{(2x+1)(3x+6)}$

$f(x)$ is discontinuous at $x = -\frac{1}{2}$ and $x = -2$ since the denominator equals 0 at these two values.

$\lim_{x\to-2} f(x)$ does not exist since

$\lim_{x\to-2^-} = +\infty$ and $\lim_{x\to-2^+} f(x) = -\infty$.

$\lim_{x\to-\frac{1}{2}} f(x)$ does not exist since

$\lim_{x\to-\frac{1}{2}^-} f(x) = -\infty$ and $\lim_{x\to-\frac{1}{2}^+} f(x) = +\infty$.

10. $f(x) = \dfrac{x^2-25}{x+5}$

$f(x)$ is discontinuous at $x = -5$ since the denominator equals zero at that value.

Since

$$\frac{x^2-25}{x+5} = \frac{(x+5)(x-5)}{x+5} = x-5,$$

$$\lim_{x\to-5} f(x) = -5-5 = -10.$$

12. $q(x) = -3x^3 + 2x^2 - 4x + 1$

Since $q(x)$ is a polynomial function, it is continuous everywhere and thus discontinuous nowhere.

14. $r(x) = \dfrac{|5-x|}{x-5}$

$r(x)$ is discontinuous at $x = 5$ since the denominator is undefined at that value.
Since $\lim_{x\to5^+} r(x) = 1$ and $\lim_{x\to5-} r(x) = -1$, $\lim_{x\to5} r(x)$ does not exist.

16. $j(x) = e^{1/x}$

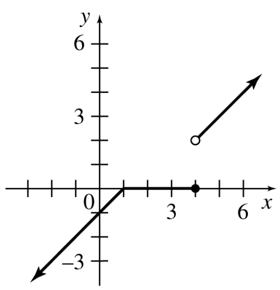
$j(x)$ is discontinuous at $x = 0$ since the function is undefined there.
$\lim_{x \to 0} j(x)$ does not exist since $\lim_{x \to 0} j(x) = 0$ and $\lim_{x \to 0} j(x) = \infty$.

18. $j(x) = \ln\left(\frac{x+2}{x-1}\right)$

The function is discontinuous for $-2 \leq x \leq 1$ since the function is undefined for these values. The limit of the function as x approaches any value a with $-2 \leq a \leq 1$ does not exist since the function is undefined for all values on this interval.

20. $f(x) = \begin{cases} x-1 & \text{if} \quad x < 1 \\ 0 & \text{if} \quad 1 \leq x \leq 4 \\ x-2 & \text{if} \quad x > 4 \end{cases}$

(a)

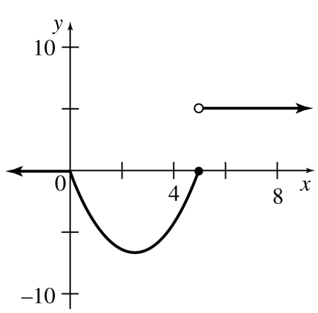


(b) $f(x)$ is discontinuous at $x = 4$.

(c) $\lim_{x \to 4^-} f(x) = 0, \lim_{x \to 4^+} f(x) = 2$

22. $g(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^2 - 5x & \text{if} \quad 0 \leq x \leq 5 \\ 5 & \text{if} \quad x > 5 \end{cases}$

(a)



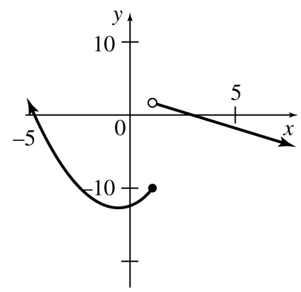
(b) $g(x)$ is discontinuous at $x = 5$.

(c) $\lim_{x \to 5^-} g(x) = 5^2 - 5(5)$
$= 0$

$\lim_{x \to 5^+} g(x) = 5$

24. $h(x) = \begin{cases} x^2 + x - 12 & \text{if} \quad x \leq 1 \\ 3 - x & \text{if} \quad x > 1 \end{cases}$

(a)



(b) $h(x)$ is discontinuous at $x = 1$.

(c) $\lim_{x \to 1^-} h(x) = 1^2 + 1 - 12$
$= -10$

$\lim_{x \to 1^+} h(x) = 3 - 1$
$= 2$

26. Find k so that $x^3 + k = kx - 5$ for $x = 3$.

$$\begin{aligned} 3^3 + k &= 3k - 5 \\ 27 + k &= 3k - 5 \\ 32 &= 2k \\ 16 &= k \end{aligned}$$

28. $\frac{3x^2 + 2x - 8}{x+2} = \frac{(3x-4)(x+2)}{x+2} = 3x - 4$
Find k so that $3x - 4 = 3x + k$ for $x = -2$.

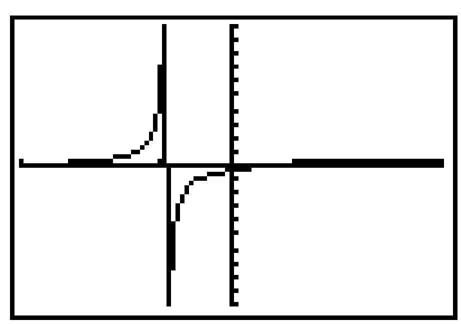
$$\begin{aligned} 3(-2) - 4 &= 3(-2) + k \\ -6 - 4 &= -6 + k \\ -10 &= -6 + k \\ -4 &= k \end{aligned}$$

32. $f(x) = \frac{x^2 + 3x - 2}{x^3 - .9x^2 + 4.14x + 5.4} = \frac{P(x)}{Q(x)}$

(a) Graph

$$Y_1 = \frac{P(x)}{Q(x)} = \frac{x^2 + 3x - 2}{x^3 - .9x^2 + 4.14x + 5.4}$$

on a graphing calculator. A good choice for the viewing window is $[-3, 3]$ by $[-10, 10]$.

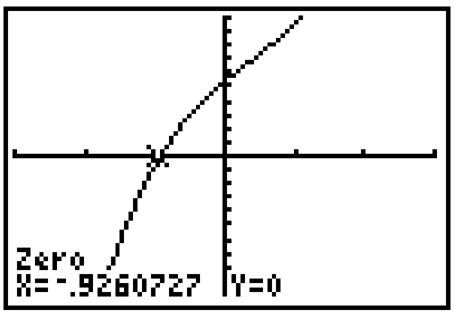


The graph has a vertical asymptote at $x \approx -.9$. It is difficult to read this value accurately from the graph.

(b) Graph

$$Y_2 = Q(x) = x^3 - .9x^2 + 4.14x + 5.4$$

using the same viewing window.



We see that this graph has one x-intercept, $\approx -.926$. This indicates that $-.926$ is the only real solution of the equation $Q(x) = 0$.

A rational function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

will be discontinuous wherever $Q(x) = 0$, so we see that f is discontinuous at $x \approx -.926$.

The result in part (b) is consistent with the result in part (a), but the result in part (b) is more accurate.

34. In dollars,

$C(x) = 4x$ if $0 < x \leq 150$
$C(x) = 3x$ if $150 < x \leq 400$
$C(x) = 2.5x$ if $400 < x$.

(a) $C(130) = 4(130) = \$520$

(b) $C(150) = 4(150) = \$600$

(c) $C(210) = 3(210) = \$630$

(d) $C(400) = 3(400) = \$1200$

(e) $C(500) = 2.5(500) = \$1250$

(f) C is discontinuous at $x = 150$ and $x = 400$ because those represent points of price change.

36. In dollars,

$C(t) = 30t$ if $0 < t \leq 5$
$C(t) = 30(5) = 150$ if $t = 6$ or $t = 7$
$C(t) = 150 + 30(t - 7)$ if $7 < t \leq 12$.

The average cost per day is

$$A(t) = \frac{C(t)}{t}.$$

(a) $A(4) = \frac{30(4)}{4} = \$30$

(b) $A(5) = \frac{30(5)}{5} = \$30$

(c) $A(6) = \frac{150}{6} = \$25$

(d) $A(7) = \frac{150}{7} \approx \$21.43$

(e) $A(8) = \frac{150 + 30(8 - 7)}{8}$

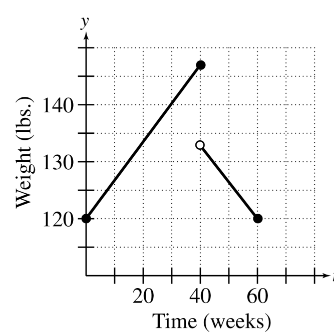
$= \frac{180}{8} = \$22.50$

(f) $\lim_{t \to 5^-} A(t) = 30$ because as t approaches 5 from the left, $A(t)$ approaches 30 (think of the graph for $t = 1, 2, ..., 5$).

(g) $\lim_{t \to 5^+} A(t) = 25$ because as t approaches 5 from the right, $A(t)$ approaches 25.

(h) A is discontinuous at $t = 5$, $t = 6$, $t = 7$, and so on, because the average cost will differ for each different rental length.

38. **(a)** Since $t = 0$ weeks the woman weighs 120 lbs. and at $t = 40$ weeks she weighs 147 lbs., graph the line beginning at coordinate $(0, 120)$ and ending at $(40, 147)$, with closed circles at these points. Since immediately after giving birth, she loses 14 lbs. and continues to lose 13 more lbs. over the following 20 weeks, graph the line between the points $(40, 133)$ and $(60, 120)$ with an open circle at $(40, 133)$ and a closed circle at $(60, 120)$.



(b) From the graph, we see that

$$\lim_{t\to 40^-} w(t) = 147 \neq 133 = \lim_{t\to 40^+} w(t),$$

where $w(t)$ is the weight in pounds t weeks after conception. Therefore, w is discontinuous at $t = 40$.

3.3 Rates of Change

2. $y = -4x^2 - 6 = f(x)$ between $x = 2$ and $x = 5$

Average rate of change

$$= \frac{f(5) - f(2)}{5 - 2} = \frac{(-106) - (-22)}{5 - 2} = \frac{-106 + 22}{3} = \frac{-84}{3} = -28$$

4. $y = -3x^3 + 2x^2 - 4x + 1 = f(x)$ between $x = 0$ and $x = 1$

$$\text{Average rate of change} = \frac{f(1) - f(0)}{1 - 0} = \frac{(-4) - (1)}{1 - 0} = \frac{-5}{1} = -5$$

6. $y = \sqrt{3x - 2} = f(x)$ between $x = 1$ and $x = 2$

$$\text{Average rate of change} = \frac{f(2) - f(1)}{2 - 1} = \frac{2 - 1}{2 - 1} = \frac{1}{1} = 1$$

8. $y = \dfrac{-5}{2x - 3} = f(x)$ between $x = 2$ and $x = 4$

$$\text{Average rate of change} = \frac{f(4) - f(2)}{4 - 2} = \frac{-1 - (-5)}{4 - 2} = \frac{-1 + 5}{2} = \frac{4}{2} = 2$$

10. $s(t) = t^2 + 5t + 2$

$$\lim_{h\to 0} \frac{s(1 + h) - s(1)}{h}$$
$$= \lim_{h\to 0} \frac{[(1 + h)^2 + 5(1 + h) + 2] - [(1)^2 + 5(1) + 2]}{h}$$
$$= \lim_{h\to 0} \frac{[1 + 2h + h^2 + 5 + 5h + 2] - [1 + 5 + 2]}{h}$$
$$= \lim_{h\to 0} \frac{8 + 7h + h^2 - 8}{h}$$
$$= \lim_{h\to 0} \frac{7h + h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(7 + h)}{h}$$
$$= \lim_{h\to 0} (7 + h) = 7$$

The instantaneous velocity at $t = 1$ is 7.

12. $s(t) = t^3 + 2t + 9$

$$\lim_{h\to 0} \frac{s(4 + h) - s(4)}{h}$$
$$= \lim_{h\to 0} \frac{(4+h)^3 + 2(4+h) + 9 - [4^3 + 2(4) + 9]}{h}$$
$$= \lim_{h\to 0} \frac{4^3 + 3(4^2)h + 3(4h^2) + h^3 + 8 + 2h + 9 - (4^3 + 8 + 9)}{h}$$
$$= \lim_{h\to 0} \frac{h^3 + 12h^2 + 48h + 2h}{h}$$
$$= \lim_{h\to 0} \frac{h^3 + 12h^2 + 50h}{h}$$
$$= \lim_{h\to 0} \frac{h(h^2 + 12h + 50)}{h}$$
$$= \lim_{h\to 0} (h^2 + 12h + 50) = 50$$

The instantaneous velocity at $t = 4$ is 50.

14. $s(t) = -4t^2 - 6$ at $t = 2$

$$\lim_{h\to 0} \frac{s(2 + h) - s(2)}{h}$$
$$= \lim_{h\to 0} \frac{-4(2 + h)^2 - 6 - [-4(2)^2 - 6]}{h}$$
$$= \lim_{h\to 0} \frac{-4(4 + 4h + h^2) - 6 + 16 + 6}{h}$$
$$= \lim_{h\to 0} \frac{-16h - 4h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(-16 - 4h)}{h}$$
$$= \lim_{h\to 0} (-16 - 4h)$$
$$= -16$$

The instantaneous rate of change at $t = 2$ is -16.

16. $F(x) = x^2 + 2$ at $x = 0$

$$\lim_{h\to 0} \frac{F(0+h) - F(0)}{h}$$
$$= \lim_{h\to 0} \frac{(0+h)^2 + 2 - [0^2 + 2]}{h}$$
$$= \lim_{h\to 0} \frac{h^2}{h}$$
$$= \lim_{h\to 0} h$$
$$= 0$$

The instantaneous rate of change at $x = 0$ is 0.

18. $f(x) = x^x$ at $x = 3$

h	
.01	$\frac{f(3+.01) - f(3)}{.01} = \frac{3.01^{3.01} - 3^3}{.01} = 57.3072$
.001	$\frac{f(3+.001) - f(3)}{.001} = \frac{3.001^{3.001} - 3^3}{.001} = 56.7265$
.00001	$\frac{f(3+.00001) - f(3)}{.00001} = \frac{3.00001^{3.00001} - 3^3}{.00001} = 56.6632$
.000001	$\frac{f(3+.000001) - f(3)}{.000001} = \frac{3.000001^{3.000001} - 3^3}{.000001} = 56.6626$
.0000001	$\frac{f(3+.0000001) - f(3)}{.0000001} = \frac{3.0000001^{3.0000001} - 3^3}{.0000001} = 56.6625$

The instantaneous rate of change at $x = 3$ is 56.6625.

20. $f(x) = x^{\ln x}$ at $x = 3$

h	
.01	$\frac{f(3+.01) - f(3)}{.01} = \frac{3.01^{\ln 3.01} - 3^{\ln 3}}{.01} = 2.4573$
.001	$\frac{f(3+.001) - f(3)}{.001} = \frac{3.001^{\ln 3.001} - 3^{\ln 3}}{.001} = 2.4495$
.00001	$\frac{f(3+.00001) - f(3)}{.00001} = \frac{3.00001^{\ln 3.00001} - 3^{\ln 3}}{.00001} = 2.4486$

The instantaneous rate of change at $x = 3$ is 2.4486.

22. If the instantaneous rate of change of $f(x)$ with respect to x is positive when $x = 1$, the function would be increasing.

24. **(a)** $p(1995) = 17.4$
$p(1997) = 16.4$

$$\text{Slope} = \frac{p(1997) - p(1995)}{1997 - 1995} = \frac{16.4 - 17.4}{2} = \frac{-1}{2} = -.5$$

The rate of change is $-.5$ which means that the percentage of sales consisting of imports decreased an average of .5% per year from 1995 to 1997.

(b) $p(1999) = 21.1$
$p(2001) = 25.0$

$$\text{Slope} = \frac{p(2001) - p(1999)}{2001 - 1999} = \frac{25.0 - 21.1}{2} = \frac{3.9}{2} = 1.95$$

The rate of change is 1.95 which means that the percentage of sales consisting ofimports increased an average of 1.95% per year from 1999 to 2001.

(c) No

26. **(a)** By looking at the graphs, we see that from 1991 to 1997 the curve representing the ratio of Internet messages to telephone calls is steeper than the curve representing the ratio of households with PCs to households without PCs; thus the curve representing the ratio of Internet messages to telephone calls has a greater rate of change over this time interval.

(b) The nonlinear curve, that is, the curve representing the ratio of Internet messages to telephone calls, has an increasing rate of change.

28. $R = 10x - .002x^2$

(a) Average rate of change $= \dfrac{R(1001) - R(1000)}{1001 - 1000} = \dfrac{8005.998 - 8000}{1} = 5.998$

The average rate of change is \$5998.

(b) $$\begin{aligned}\text{Marginal revenue} &= \lim_{h\to 0} \frac{R(1000+h) - R(1000)}{h}\\ &= \lim_{h\to 0} \frac{[10(1000+h) - .002(1000+h)^2] - [10(1000) - .002(1000)^2]}{h}\\ &= \lim_{h\to 0} \frac{[10{,}000\ + 10h - .002(1{,}000{,}000\ + 2000h + h^2] - 8000}{h}\\ &= \lim_{h\to 0} \frac{10{,}000\ + 10h - 2000 - 4h - .002h^2 - 8000}{h}\\ &= \lim_{h\to 0} \frac{6h - .002h^2}{h} = \lim_{h\to 0} \frac{h(6 - .002h)}{h} = \lim_{h\to 0} (6 - .002h) = 6\end{aligned}$$

The marginal revenue is \$6000.

(c) $$\begin{aligned}\text{Additional revenue} &= R(1001) - R(1000) = [10(1001) - .002(1001)^2] - [10(1000) - .002(1000)^2]\\ &= 8005.998 - 8000\end{aligned}$$

The additional revenue is \$5998.

(d) The answers to parts (a) and (c) are the same.

30. $p(t) = t^2 + t$

(a) $p(1) = 1^2 + 1 = 2$
$p(4) = 4^2 + 4 = 20$

Average rate of change $= \dfrac{p(4) - p(1)}{4 - 1} = \dfrac{20 - 2}{3} = 6$

The average rate of change is 6% per day.

(b) $$\begin{aligned}&\lim_{h\to 0} \frac{p(3+h) - p(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 + (3+h) - (3^2+3)}{h} = \lim_{h\to 0} \frac{9 + 6h + h^2 + 3 + h - 12}{h}\\ &= \lim_{h\to 0} \frac{h^2 + 7h}{h} = \lim_{h\to 0} \frac{h(h+7)}{h} = \lim_{h\to 0} (h+7) = 7\end{aligned}$$

The instantaneous rate of change is 7% per day.

32. **(a)** $P(2)=5,\ P(1)=3$

Average rate of change $=\dfrac{P(2)-P(1)}{2-1}=\dfrac{5-3}{2-1}=\dfrac{2}{1}=2$

From 1 min to 2 min, the population of bacteria increases, on the average, 2 million per min.

(b) $P(3)=4.2,\ P(2)=5$

Average rate of change $=\dfrac{P(3)-P(2)}{3-2}=\dfrac{4.2-5}{3-2}=\dfrac{-.8}{1}=-.8$

From 2 min to 3 min, the population of bacteria decreases, on the average, $-.8$ million or 800,000 per min.

(c) $P(3)=4.2,\ P(4)=2$

Average rate of change

$=\dfrac{P(4)-P(3)}{4-3}=\dfrac{2-4.2}{4-3}=\dfrac{-2.2}{1}=-2.2$

From 3 min to 4 min, the population of bacteria decreases, on the average, -2.2 million per min.

(d) $P(4)=2,\ P(5)=1$

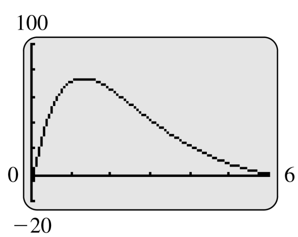
Average rate of change$=\dfrac{P(5)-P(4)}{5-4}=\dfrac{1-2}{5-4}=\dfrac{-1}{1}=-1$

From 4 min to 5 min, the population decreases, on the average, -1 million per min.

(e) The population increased up to 2 min after the bactericide was introduced, but decreased after 2 min.

(f) The rate of decrease of the population slows down at about 3 min.
The graph becomes less and less steep after that point.

34. **(a)**



(b) The average rate of change during the first hour is

$$\frac{F(1)-F(0)}{1-0}\approx 81.51$$

kilojoules per hour per hour.

(c) Store $F(t)$ in a function menus of a graphing calculator.
Store $\frac{Y_1(1+X)-Y_1(1)}{X}$ as Y_2 in the function menu, where Y_1 represents $F(t)$. Substitute small values for X in Y_2 perhaps with use of a table feature of the graphing calculator. As X is allowed to get smaller, Y_2 approaches 18.81 kilojoules per hour per hour.

(d) Through use of a MAX/feature program of a graphing calculator, the maximum point seen in part (a) is estimated to occur at approximately $t=1.3$ hours.

36. Let $C(t)$ be cannabis production (in metric tons) for year t and $O(t)$ be opium production (in metric tons) for year t.

(a) $C(1998) = 15{,}800$
$C(2000) = 14{,}500$
$C(2001) = 14{,}900$

Average rate of change from 1998 to 2000

$$= \frac{C(2000) - C(1998)}{2000 - 1998} = \frac{14{,}500 - 15{,}800}{2} = -650$$

On average, the production decreased about 650 metric tons per year.
Average rate of change from 2000 to 2001

$$= \frac{C(2001) - C(2000)}{2001 - 2000} = \frac{14{,}900 - 14{,}500}{1} = 400$$

On average, the production increased about 400 metric tons per year.

$$O(1998) = 4486$$
$$O(2000) = 5010$$
$$O(2001) = 1236$$

Average rate of change from 1998 to 2000

$$= \frac{O(2000) - O(1998)}{2000 - 1998} = \frac{5010 - 4486}{2} = 262$$

On average, the production increased about 260 metric tons per year.

Average rate of change from 2000 to 2001

$$= \frac{O(2001) - O(2000)}{2001 - 2000} = \frac{1236 - 5010}{1} = -3774$$

On average, the production decreased about 3800 metric tons per year.

(b) Opium had the greatest change in net production.

38. **(a)** $$\frac{T(3000) - T(1000)}{3000 - 1000} = \frac{23 - 15}{2000} = \frac{8}{2000} = \frac{4}{1000}$$

From 1000 to 3000 ft, the temperature changes about 4° per 1000 ft; the temperature rises (on the average).

(b) $$\frac{T(5000) - T(1000)}{5000 - 1000} = \frac{22 - 15}{4000} = \frac{7}{4000} = \frac{1.75}{1000}$$

From 1000 to 5000 ft, the temperature changes about 1.75° per 1000 ft; the temperature rises (on the average).

(c) $$\frac{T(9000) - T(3000)}{9000 - 3000} = \frac{15 - 23}{6000} = \frac{-8}{6000} = \frac{-\frac{4}{3}}{1000}$$

From 3000 to 9000 ft, the temperature changes about $-\frac{4}{3}°$ per 1000 ft; the temperature falls (on the average).

(d) $$\frac{T(9000) - T(1000)}{9000 - 1000} = \frac{15 - 15}{8000} = 0$$

From 1000 to 9000 ft, the temperature changes about 0° per 1000 ft; the temperature stays constant (on the average).

(e) The temperature is highest at 3000 ft and lowest at 1000 ft. If 7000 ft is changed to 10,000 ft, the lowest temperature would be at 10,000 ft.

(f) The temperature at 9000 ft is the same as 1000 ft.

40. **(a)** Average rate of change from .5 to 1:

$$\frac{f(1) - f(.5)}{1 - .5} = \frac{48 - 20}{.5} = 56 \text{ mph}$$

Average rate of change from 1 to 1.5:

$$\frac{f(1.5) - f(1)}{1.5 - 1} = \frac{80 - 48}{.5} = 64 \text{ mph}$$

Estimate of instantaneous velocity is

$$\frac{56 + 64}{2} = 60 \text{ mph.}$$

(b) Average rate of change from 1.5 to 2:

$$\frac{f(2) - f(1.5)}{2 - 1.5} = \frac{104 - 80}{.5} = 48 \text{ mph}$$

Average rate of change from 2 to 2.5

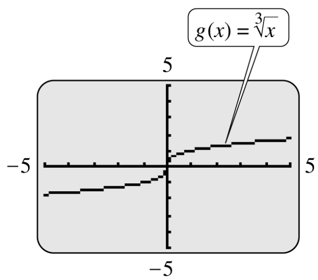
$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{126 - 104}{.5} = 44 \text{ mph}$$

Estimate of instantaneous velocity is

$$\frac{48 + 44}{2} = 46 \text{ mph.}$$

3.4 Definition of the Derivative

2. (a)



The line tangent to $g(x) = \sqrt[3]{x}$ at $x = 0$ is a vertical line. Since the slope of a vertical line is undefined, $g'(0)$ does not exist.

4. If the rate of change of $f(x)$ is zero when $x = a$, the tangent line at that point must have a slope of zero. Thus, the tangent line is horizontal at that point.

6. Using the points $(2, 2)$ and $(-2, 6)$, we have

$$m = \frac{6 - 2}{-2 - 2} = \frac{4}{-4} = -1.$$

8. Using the points $(3, -1)$ and $(-2, 3)$, we have

$$m = \frac{3 - (-1)}{-2 - 3} = \frac{4}{-5} = -\frac{4}{5}.$$

10. The line tangent to the curve at $(4, 2)$ is a vertical line. A vertical line has undefined slope.

12. $f(x) = 6x^2 - 4x$

Step 1 $f(x + h)$

$$\begin{aligned} &= 6(x + h)^2 - 4(x + h) \\ &= 6(x^2 + 2xh + h^2) - 4x - 4h \\ &= 6x^2 + 12xh + 6h^2 - 4x - 4h \end{aligned}$$

Step 2 $f(x + h) - f(x)$

$$\begin{aligned} &= 6x^2 + 12xh + 6h^2 - 4x - 4h \\ &\quad - 6x^2 + 4x \\ &= 6h^2 + 12xh - 4h \\ &= h(6h + 12x - 4) \end{aligned}$$

Step 3

$$\frac{f(x + h) - f(x)}{h} = \frac{h(6h + 12x - 4)}{h} = 6h + 12x - 4$$

Step 4

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} (6h + 12x - 4) \\ &= 12x - 4 \end{aligned}$$

$$\begin{aligned} f'(-2) &= 12(-2) - 4 = -28 \\ f'(0) &= 12(0) - 4 = -4 \\ f'(3) &= 12(3) - 4 = 32 \end{aligned}$$

14. $f(x) = \dfrac{6}{x}$

$$f(x + h) = \frac{6}{x + h}$$

$$\begin{aligned} f(x + h) - f(x) &= \frac{6}{x + h} - \frac{6}{x} \\ &= \frac{6x - 6(x + h)}{x(x + h)} \\ &= \frac{-6h}{x(x + h)} \end{aligned}$$

$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{-6h}{hx(x + h)} \\ &= \frac{-6}{x(x + h)} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{-6}{x(x + h)} \\ &= \frac{-6}{x^2} \end{aligned}$$

$$f'(-2) = \frac{-6}{(-2)^2} = \frac{-6}{4} = -\frac{3}{2}$$

$f'(0) = \dfrac{-6}{0^2}$ which is undefined so $f'(0)$ does not exist.

$$f'(3) = \frac{-6}{3^2} = \frac{-6}{9} = -\frac{2}{3}$$

16. $f(x) = -3\sqrt{x}$

Steps 1-3 are combined.

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{-3\sqrt{x+h}+3\sqrt{x}}{h}$$

Rationalize the numerator.

$$= \frac{-3\sqrt{x+h}+3\sqrt{x}}{h} \cdot \frac{-3\sqrt{x+h}-3\sqrt{x}}{-3\sqrt{x+h}-3\sqrt{x}}$$
$$= \frac{9(x+h)-9x}{h(-3\sqrt{x+h}-3\sqrt{x})}$$
$$= \frac{9x+9h-9x}{h(-3\sqrt{x+h}-3\sqrt{x})}$$
$$= \frac{9}{-3\sqrt{x+h}-3\sqrt{x}}$$
$$= \frac{3}{-\sqrt{x+h}-\sqrt{x}}$$
$$f'(x) = \lim_{h\to 0} \frac{3}{-\sqrt{x+h}-\sqrt{x}}$$
$$= \frac{3}{-\sqrt{x}-\sqrt{x}} = \frac{3}{-2\sqrt{x}}$$

$f'(-2) = \dfrac{3}{-2\sqrt{-2}}$ which is undefined so $f'(-2)$ does not exist.

$f'(0) = \dfrac{3}{-2\sqrt{0}} = \dfrac{3}{0}$ which is undefined so $f'(0)$ does not exist.

$$f'(3) = \frac{3}{-2\sqrt{3}} = -\frac{3}{2\sqrt{3}}$$

18. $f(x) = 6 - x^2;\ x = -1$

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{[6-(x+h)^2]-[6-(x)^2]}{h}$$
$$= \frac{[6-(x^2+2xh+h^2)]-[6-x^2]}{h}$$
$$= \frac{6-x^2-2xh-h^2-6+x^2}{h}$$
$$= \frac{-2xh-h^2}{h} = \frac{h(-2x-h)}{h} = -2x-h$$
$$f'(x) = \lim_{h\to 0}(-2x-h) = -2x$$

$f'(-1) = -2(-1) = 2$ is the slope of the tangent line at $x = -1$. Use $m = 2$ and $(-1, 5)$ in the point-slope form.

$$y - 5 = 2(x+1)$$
$$y - 5 = 2x + 2$$
$$y = 2x + 7$$

20. $f(x) = \dfrac{-3}{x+1};\ x = 1$

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{\frac{-3}{(x+h)+1} - \frac{-3}{x+1}}{h}$$
$$= \frac{\frac{-3(x+1)+3(x+h+1)}{(x+1)(x+h+1)}}{h}$$
$$= \frac{\frac{-3x-3+3x+3h+3}{(x+1)(x+h+1)}}{h}$$
$$= \frac{3h}{h(x+1)(x+h+1)}$$
$$f'(x) = \lim_{h\to 0}\frac{3}{(x+1)(x+h+1)}$$
$$= \frac{3}{(x+1)^2}$$

$f'(1) = \frac{3}{(1+1)^2} = \frac{3}{4}$ is the slope of the tangent line at $x = 1$. Use $m = \frac{3}{4}$ and $\left(1, -\frac{3}{2}\right)$ in the point-slope form.

$$y - \left(-\frac{3}{2}\right) = \frac{3}{4}(x-1)$$
$$y + \frac{3}{2} = \frac{3}{4}x - \frac{3}{4}$$
$$y = \frac{3}{4}x - \frac{9}{4}$$

22. $f(x) = \sqrt{x};\ x = 25$

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{\sqrt{x+h}-\sqrt{x}}{h}$$
$$= \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$f'(x) = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{2\cdot 5} = \frac{1}{10}$$

Use $m = \frac{1}{10}$ and $(25, 5)$ in the point-slope form.

$$y - 5 = \frac{1}{10}(x-25)$$
$$y - 5 = \frac{1}{10}x - \frac{25}{10}$$
$$y = \frac{1}{10}x + \frac{5}{2}$$

24. $f(x) = 6x^2 - 4x$

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{6(x+h)^2 - 4(x+h) - (6x^2-4x)}{h}$$
$$= \frac{12xh + 6h^2 - 4h}{h}$$
$$= 12x + 6h - 4$$

$$f'(x) = \lim_{h\to 0}(12x+6h-4)$$
$$= 12x - 4$$

$$f'(2) = 12(2) - 4 = 24 - 4 = 20$$
$$f'(16) = 12(16) - 4 = 192 - 4 = 188$$
$$f'(-3) = 12(-3) - 4 = -36 - 4 = -40$$

26. $f(x) = -9x - 5$

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{-9(x+h) - 5 - [-9x-5]}{h}$$
$$= \frac{-9x - 9h - 5 + 9x + 5}{h}$$
$$= \frac{-9h}{h} = -9$$

$$f'(x) = \lim_{h\to 0}(-9) = -9$$

$$f'(-2) = -9$$
$$f'(16) = -9$$
$$f'(-3) = -9$$

28. $f(x) = \dfrac{6}{x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{6}{x+h} - \frac{6}{x}}{h}$$
$$= \frac{6x - 6(x+h)}{hx(x+h)}$$
$$= \frac{-6}{x(x+h)}$$

$$f'(x) = \lim_{h\to 0}\frac{-6}{x(x+h)}$$
$$= \frac{-6}{x^2}$$

$$f'(2) = \frac{-6}{(2)^2} = \frac{-6}{4} = -\frac{3}{2}$$
$$f'(16) = \frac{-6}{16^2} = \frac{-6}{256} = -\frac{3}{128}$$
$$f'(-3) = \frac{-6}{(-3)^2} = \frac{-6}{9} = -\frac{2}{3}$$

30. $f(x) = -3\sqrt{x}$

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{-3\sqrt{x+h}+3\sqrt{x}}{h}$$
$$= \frac{-3\sqrt{x+h}+3\sqrt{x}}{h}\cdot\frac{-3\sqrt{x+h}-3\sqrt{x}}{-3\sqrt{x+h}-3\sqrt{x}}$$
$$= \frac{9(x+h)-9x}{h(-3\sqrt{x+h}-3\sqrt{x})}$$
$$= \frac{9}{-3\sqrt{x+h}-3\sqrt{x}}$$
$$= \frac{3}{-\sqrt{x+h}-\sqrt{x}}$$

$$f'(x) = \lim_{h\to 0}\frac{3}{-\sqrt{x+h}-\sqrt{x}}$$
$$= \frac{3}{-\sqrt{x}-\sqrt{x}} = -\frac{3}{2\sqrt{x}}$$

$$f'(2) = -\frac{3}{2\sqrt{2}}$$
$$f'(16) = \frac{-3}{2\sqrt{16}} = -\frac{3}{8}$$

$f'(-3) = \frac{-3}{2\sqrt{-3}}$, which is not a real number, so

$f'(-3)$ does not exist.

32. No derivative exists at $x = -6$ because the function is not defined at $x = -6$.

34. For $x = -5$ and $x = 0$, the function $f(x)$ is not defined. For $x = -3$ and $x = 2$ the graph of $f(x)$ has sharp points. For $x = 4$, the tangent to the graph is vertical. Therefore, no derivative exists for $x = -5$, $x = -3$, $x = 0$, $x = 2$, or $x = 4$.

36. The zeros of graph (b) correspond to the turning points of graph (a), the points where the derivative is zero. Graph (a) gives the distance, while graph (b) gives the velocity.

38. $f(x) = x^x, a = 2$

(a)

h	
.01	$\frac{f(2+.01)-f(2)}{.01} = \frac{2.01^{2.01}-2^2}{.01} = 6.84$
.001	$\frac{f(2+.001)-f(2)}{.001} = \frac{2.001^{2.001}-2^2}{.001} = 6.779$
.0001	$\frac{f(2+.0001)-f(2)}{.0001} = \frac{2.0001^{2.0001}-2^2}{.0001} = 6.773$
.00001	$\frac{f(2+.00001)-f(2)}{.00001} = \frac{2.00001^{2.00001}-2^2}{.00001} = 6.7727$
.000001	$\frac{f(2+.000001)-f(2)}{.000001} = \frac{2.000001^{2.000001}-2^2}{.000001} = 6.7726$

It appears that $f'(2) = 6.7726$.

(b) Graph the function on a graphing calculator and move the cursor to an x-value near $x = 2$. A good choice for the initial viewing window is $[0, 3]$ by $[0, 10]$.
Now zoom in on the function several times. Each time you zoom in, the graph will look less like a curve and more like a straight line. When the graph appears to be a straight line, use the TRACE feature to select two points on the graph, and record their coordinates. Use these two points to compute the slope. The result will be very close to the most accurate value found in part (a), which is 6.7726.

40. $f(x) = x^{1/x}, a = 2$

(a)

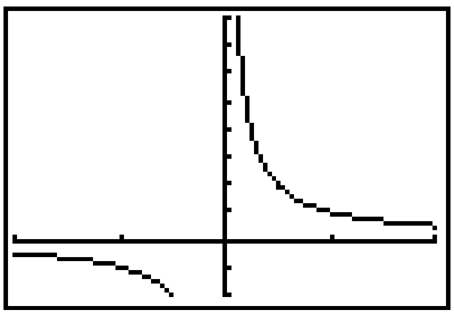
h	
.01	$\frac{f(2+.01)-f(2)}{.01} = \frac{2.01^{1/2.01}-2^{1/2}}{.01} = .1071$
.001	$\frac{f(2+.001)-f(2)}{.001} = \frac{2.001^{1/2.001}-2^{1/2}}{.001} = .1084$
.0001	$\frac{f(2+.0001)-f(2)}{.0001} = \frac{2.0001^{1/2.0001}-2^{1/2}}{.0001} = .1085$
.00001	$\frac{f(2+.00001)-f(2)}{.00001} = \frac{2.00001^{1/2.00001}-2^{1/2}}{.00001} = .1085$
.000001	$\frac{f(2+.000001)-f(2)}{.000001} = \frac{2.000001^{1/2.000001}-2^2}{.000001} = .1085$

It appears that $f'(2) = .1085$.

(b) Graph this function on a graphing calculator and move the cursor to an x-value near $x = 2$. A good choice for the initial viewing window is $[0, 5]$ by $[0, 3]$.
Follow the procedure outlined in the solution for Exercise 38, part (b). The final result will be close to the value found in part (a) of this exercise, which is .1085.

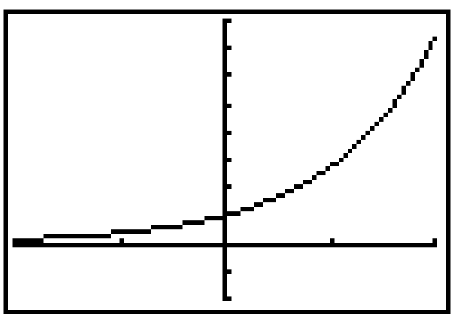
42. For Column A, let $h = .01$
$f(x) = \ln |x|$

Graph $y = \dfrac{\ln |x + .01| - \ln |x|}{.01}$.



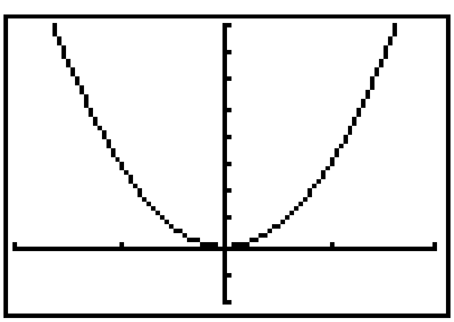
$f(x) = e^x$

Graph $y = \dfrac{e^{x+.01} - e^x}{.01}$.

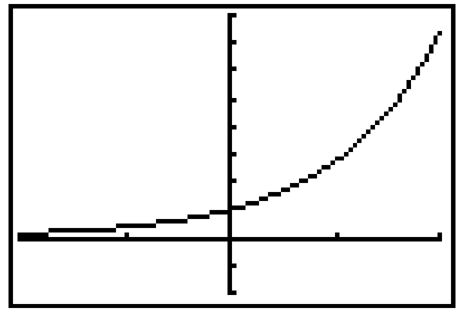


$f(x) = x^3$

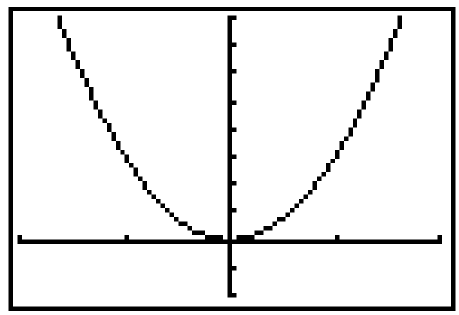
Graph $y = \dfrac{(x + .01)^3 - x^3}{.01}$.



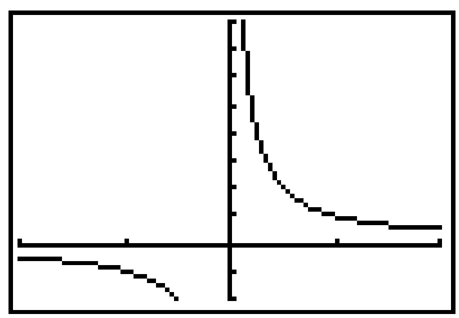
Column B
Graph $y = e^x$



Graph $y = 3x^2$



Graph $y = \dfrac{1}{x}$



We observe that the graph of

$$y = \frac{\ln |x + .01| - \ln |x|}{.01}$$

is very similar to the graph of

$$y = \frac{1}{x},$$

the graph of

$$y = \frac{e^{x+.01} - e^x}{.01}$$

is very similar to the graph of

$$y = e^x, \text{ and}$$

the graph of

$$y = \frac{(x+.01)^3 - x^3}{.01}$$

is very similar to the graph of

$$y = 3x^2.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, the derivative of e^x is e^x, and the derivative of x^3 is $3x^2$.

44. **(a)** $f(x) = -4x^2 + 11x$

$$\begin{aligned} f(x+h) &= -4(x+h)^2 + 11(x+h) \\ &= -4(x^2 + 2xh + h^2) + 11(x+h) \\ &= -4x^2 - 8xh - 4h^2 + 11x + 11h \end{aligned}$$

$$\begin{aligned} &f(x+h) - f(x) \\ &= -4x^2 - 8xh - 4h^2 + 11x + 11h + 4x^2 - 11x \\ &= -8xh - 4h^2 + 11h \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{-8xh - 4h^2 + 11h}{h} \\ &= -8x + 4h + 11 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} (-8x + 4h + 11) \\ &= -8x + 11 \end{aligned}$$

$$f'(3) = -8(3) + 11 = -13$$

$$\begin{aligned} &\frac{f(3+.1) - f(3)}{.1} \\ &= \frac{f(3.1) - f(3)}{.1} \\ &= \frac{-4(3.1)^2 + 11(3.1) - (-4(3)^2 + 11(3))}{.1} \\ &= \frac{-4(9.61) + 11(3.1) + 4(9) - 11(3)}{.1} \\ &= \frac{-38.44 + 34.1 + 36 - 33}{.1} \\ &= \frac{1.34}{.1} = -13.4 \end{aligned}$$

$$\begin{aligned} &\frac{f(3+.1) - f(3-.1)}{2(.1)} \\ &= \frac{f(3.1) - f(2.9)}{.2} \\ &= \frac{-4(3.1)^2 + 11(3.1) - (-4(2.9)^2 + 11(2.9))}{.2} \\ &= \frac{-4(9.61) + 11(3.1) + 4(8.41) - 11(2.9)}{.2} \\ &= \frac{-38.44 + 34.1 + 33.64 - 31.9}{.2} = \frac{-2.6}{.2} = -13 \end{aligned}$$

(b) $f'(3) = -8(3) + 11 = -13$

$$\begin{aligned} &\frac{f(3+.01) - f(3)}{.01} \\ &= \frac{f(3.01) - f(3)}{.01} \\ &= \frac{-4(3.01)^2 + 11(3.01) - (-4(3)^2 + 11(3))}{.01} \\ &= \frac{-4(9.0601) + 11(3.01) + 4(9) - 11(3)}{.01} \\ &= \frac{-36.2404 + 33.11 + 36 - 33}{.01} \\ &= \frac{-.1304}{.01} = -13.04 \end{aligned}$$

$$\begin{aligned} &\frac{f(3+.01) - f(3-.01)}{2(.01)} \\ &= \frac{f(3.01) - f(2.99)}{.02} \\ &= \frac{-4(3.01)^2 + 11(3.01) - (-4(2.99)^2 + 11(2.99))}{.02} \\ &= \frac{-4(9.0601) + 11(3.01) + 4(8.9401) - 11(2.99)}{.02} \\ &= \frac{-36.2404 + 33.11 + 35.7604 - 32.89}{.02} \\ &= \frac{-.26}{.02} = -13 \end{aligned}$$

(c)

$$\begin{aligned} f(x) &= \frac{-2}{x} \\ f(x+h) &= \frac{-2}{x+h} \\ f(x+h) - f(x) &= \frac{-2}{x+h} - \left(\frac{-2}{x}\right) \\ &= \frac{-2}{x+h} + \frac{2}{x} \\ &= \frac{-2x + 2(x+h)}{x(x+h)} \\ &= \frac{2h}{x(x+h)} \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\frac{2h}{x(x+h)}}{h} \\ &= \frac{2h}{x(x+h)} \cdot \frac{1}{h} \\ &= \frac{2}{x(x+h)} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{2}{x(x+h)} = \frac{2}{x^2} \end{aligned}$$

$$f'(3) = \frac{2}{3^2} = \frac{2}{9} \approx .222222$$

$$\frac{f(3+.1)-f(3)}{.1} = \frac{f(3.1)-f(3)}{.1} = \frac{\frac{-2}{3.1}-\frac{-2}{3}}{.1} \approx .215054$$

$$\frac{f(3+.1)-f(3-.1)}{2(.1)} = \frac{f(3.1)-f(2.9)}{.2} = \frac{\frac{-2}{3.1}-\frac{-2}{2.9}}{.2} \approx .222469$$

(d) $f'(3) = \frac{2}{3^2} = \frac{2}{9} \approx .222222$

$$\frac{f(3+.01)-f(3)}{.01} = \frac{f(3.01)-f(3)}{.01} = \frac{\frac{-2}{3.01}-\frac{-2}{3}}{.01} \approx .221484$$

$$\frac{f(3+.01)-f(3-.01)}{2(.01)} = \frac{f(3.01)-f(2.99)}{.02} = \frac{\frac{-2}{3.01}-\frac{-2}{2.99}}{.02} \approx .222225$$

(e) $f(x) = \sqrt{x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{h(\sqrt{x+h}+\sqrt{x})}$$

$$f'(x) = \lim_{h\to 0} \frac{1}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

$$f'(3) = \frac{1}{2\sqrt{3}} \approx .288675$$

$$\frac{f(3+.1)-f(3)}{.1} = \frac{f(3.1)-f(3)}{.1} = \frac{\sqrt{3.1}-\sqrt{3}}{.1} \approx .286309$$

$$\frac{f(3+.1)-f(3-.1)}{2(.1)} = \frac{f(3.1)-f(2.9)}{.2} = \frac{\sqrt{3.1}-\sqrt{2.9}}{.2} \approx .288715$$

(f) $f'(3) = \frac{1}{2\sqrt{3}} \approx .288675$

$$\frac{f(3+.01)-f(3)}{.01} = \frac{f(3.01)-f(3)}{.01} = \frac{\sqrt{3.01}-\sqrt{3}}{.01} \approx .288435$$

$$\frac{f(3+.01)-f(3-.01)}{2(.01)} = \frac{f(3.01)-f(2.99)}{.02} = \frac{\sqrt{3.01}-\sqrt{2.99}}{.02} \approx .288676$$

46. $P(x) = 1000 + 32x - 2x^2$

(a) \$8000 is 8 thousands, so $x = 8$.

$$P'(8) = 32 - 4(8) = 32 - 32 = 0$$

No, the firm should not increase production, since the marginal profit is 0.

(b) \$6000, $x = 6$

$$P'(6) = 32 - 4(6) = 32 - 24 = 8$$

Yes, the first should increase production, since the marginal profit is positive.

(c) \$12,000, $x = 12$

$$P'(12) = 32 - 4(12) = 32 - 48 = -16$$

No, because the marginal profit is negative.

(d) \$20,000, $x = 20$

$$P'(20) = 32 - 4(20) = 32 - 80 = -48$$

No, because the marginal profit is negative.

48. $C(x) = 1000 + .24x^2,\ 0 \le x \le 30{,}000$

(a) The marginal cost is given by $C'(x) = .48x$, $0 \le x \le 30{,}000$.

(b) $C'(100) = .48(100) = 48$

(c) This represents the fact that the cost of producing the next (101st) taco is approximately 48.

The exact cost to produce the 101st taco is

$$\begin{aligned} C(101) - C(100) &= [1000 + .24(101)^2] \\ &\quad - [1000 + .24(100)^2] \\ &= 1000 + 2448.24 - 1000 - 2400 \\ &= 48.24. \end{aligned}$$

(d) The exact cost of producing the 101st taco is .24 greater than the approximate cost. They are very close.

50. **(a)** From the graph, V_{mp} is just about at the turning point of the curve. Thus, the slope of the tangent line is approximately zero. The power expenditure is not changing.

(b) From the graph, the slope of the tangent line at V_{mr} is approximately .1. The power expended is increasing .1 unit per unit increase in speed.

(c) The slope of the tangent line at V_{opt} is a bit greater than that at V_{mr}, about .12. The power expended increases .12 units for each unit increase in speed.

(d) The power level first decreases to V_{mp}, then increases at greater rates.

(e) V_{mr} is the point which produces the smallest slope of a line.

52. $I(t) = 27 + 72t - 1.5t^2$

(a)

$$\begin{aligned} I'(t) &= \lim_{h\to 0} \frac{I(t+h) - I(t)}{h} \\ &= \lim_{h\to 0} \frac{27+72t+72h-1.5t^2-3th-1.5h^2-27-72t+1.5t^2}{h} \\ &= \lim_{h\to 0} \frac{72h - 3th - 1.5h^2}{h} \\ &= \lim_{h\to 0} 72 - 3t - 1.5h \\ &= 72 - 3t \end{aligned}$$

$$\begin{aligned} I'(5) &= 72 - 3(5) \\ &= 72 - 15 \\ &= 57 \end{aligned}$$

The rate of change of the intake of food 5 minutes into a meal is 57 grams per minute.

(b)

$$\begin{aligned} I'(24) &\stackrel{?}{=} 0 \\ 72 - 3(24) &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

24 minutes after the meal starts the rate of food consumption is 0.

(c) After 24 minutes the rate of food consumption is negative according to the function where a rate of zero is more accurate. A logical range for this function is

$$0 \le t \le 24.$$

54. The slope of the tangent line to the graph at the first point is found by finding two points on the tangent line.

$$\begin{aligned} (x_1, y_1) &= (1000, 13.5) \\ (x_2, y_2) &= (0, 18.5) \end{aligned}$$

$$m = \frac{18.5 - 13.5}{0 - 1000} = \frac{5}{-1000} = -.005$$

At the second point, we have

$$\begin{aligned} (x_1, y_1) &= (1000, 13.5) \\ (x_2, y_2) &= (2000, 21.5). \\ m &= \frac{21.5 - 13.5}{2000 - 1000} \\ &= \frac{8}{1000} \\ &= .008 \end{aligned}$$

At the third point, we have

$$\begin{aligned} (x_1, y_1) &= (5000, 20) \\ (x_2, y_2) &= (3000, 22.5). \\ m &= \frac{22.5 - 20}{3000 - 5000} \\ &= \frac{2.5}{-2000} \\ &= -.00125 \end{aligned}$$

At 500 ft, the temperature decreases .005° per foot. At about 1500 ft, the temperature increases .008° per foot. At 5000 ft, the temperature decreases .00125° per foot.

56. The slope of the graph at $x = 24$ can be estimated using the points $(24, 360)$ and $(33, 395)$.

$$\text{slope} = \frac{395 - 360}{33 - 24} = \frac{35}{9} = 3\tfrac{8}{9}$$

Thus, the derivative for a 24 ounce bat is about 4 ft per oz which means that the distance the ball travels is increasing 4 feet per ounce.

The slope of the graph at $x = 51$ can be estimated using the points $(60, 380)$ and $(51, 390)$.

$$\text{slope} = \frac{390 - 380}{51 - 60} = \frac{10}{-9} = -\frac{10}{9} \approx -1.1$$

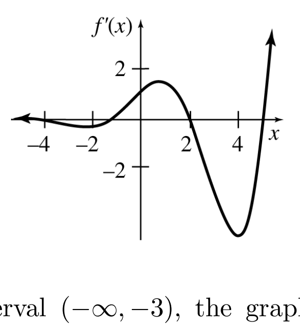
Thus, the derivative for a 51 ounce bat is about -1.1 ft per oz which means that the distance the ball travels is decreasing 1.1 feet per ounce.

3.5 Graphical Differentiation

4. Since the x-intercepts of the graph f' occur whenever the graph of f has a horizontal tangent line, Y_2 is the derivative of Y_1. Notice that Y_2 has 2 x-intercepts; each occurs at an x-value where the tangent line to Y_1 is horizontal.
Note also that Y_2 is negative whenever Y_1 is decreasing, and Y_2 is positive whenever Y_1 is increasing.

6. Since the x-intercepts of the graph f' occur whenever the graph of f has a horizontal tangent line, Y_1 is the derivative of Y_2. Notice that Y_1 has 4 x-intercepts; each occurs at an x-value where the tangent line to Y_2 is horizontal.
Note also that Y_1 is negative whenever Y_2 is decreasing and Y_1 is positive whenever Y_2 is increasing.

8. To graph f', observe the intervals where the slopes of lines are positive and where they are negative to determine where the derivative is positive and where it is negative. Also, whereever f has a horizontal tangent, f' will be 0.
Estimate the magnitude of the slope at several points by drawing tangents to the graph of f.



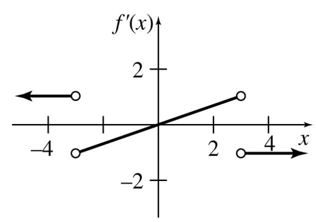
10. On the interval $(-\infty, -3)$, the graph of f is a straight line, so its slope is constant. To find this slope, use the points $(-6, -3)$ and $(-3, 0)$.

$$m = \frac{0 - (-3)}{-3 - (-6)} = \frac{3}{3} = 1$$

On the interval $(3, \infty)$, the slope of f is also constant. To find this slope, use the points $(3, 0)$ and $(6, -3)$.

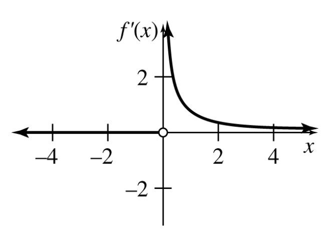
$$m = \frac{-3 - 0}{6 - 3} = \frac{-3}{3} = -1$$

Thus, we have $f'(x) = 1$ on $(-\infty, -3)$ and $f'(x) = -1$ on $(3, \infty)$. Because the graph of f has sharp points at $x = -3$ and $x = 3$, we know that $f'(-3)$ and $f'(3)$ do not exist. We show this on the graph of $f'(x)$ by using open circles.
We also observe that the slopes of tangent lines are negative on $(-3, 0)$, that the graph has a horizontal tangent at $x = 0$, and that the slopes of tangent lines are positive on $(0, 3)$. Thus, f' is negative on $(-\infty, 3)$, 0 at $x = 0$, and positive on $(3, \infty)$. Furthermore, by drawing tangents, we see that on $(-3, 3)$, the value of f' increases from -1 to 1.



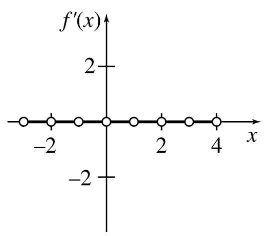
12. On the interval $(-\infty, 0)$, the graph of f is a horizontal line, so its slope is 0. Thus, the graph of f' is $y = 0$ (the x-axis) on $(-\infty, 0)$.

Since f is discontinuous at $x = 0$, $f'(0)$ does not exist. Thus, the graph of f' has an open circle at $x = 0$.
On the interval $(0, \infty)$, the slopes are positive but decreasing at a slower rate as x gets larger. Therefore, the value of f' will be positive but decreasing on this interval. This value approaches 0, but never becomes 0.

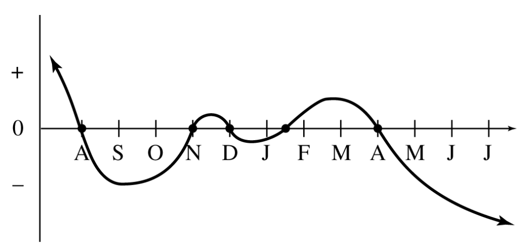


14. The graph of f is a step function. (This is the greatest integer function, $f(x) = [[x]]$.) The graph is made up of an infinite series of horizontal line segments. Thus, the derivative will be 0 everywhere it is defined. However, since f is discontinuous wherever x is an integer, $f'(x)$ does not exist at any integer.

The graph of f' is an infinite set of line segments on the x-axis (where $y = f'(x) = 0$) separated by open circles wherever x is an integer.



16. Notice how the slope of the graph changes with time. The slope is positive until early August, then it is negative until November, then it is positive until December, then it is negative until mid-January, then it is positive until April, then it is negative for the remainder of the data. Use this information to sketch the graph of the rate of change with respect to time.



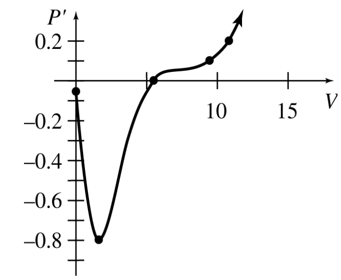
18. Let $P(V)$ represent the power corresponding to a given value of the tern's speed, V.

The rate of change of power as a function of time is given by the derivative of this function, $P'(V)$. We use the graph of P to sketch the graph of P'.

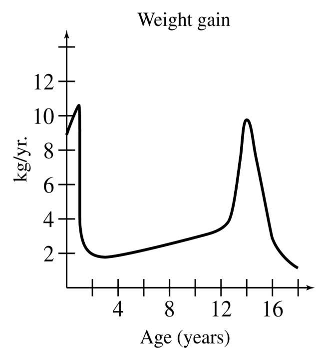
First, we observe that the graph of P has one turning point, at $V = V_{mp} \approx 5.5$. At this value of V, the graph has a horizontal tangent, so the graph of V' has an x-intercept at this value of V.

Since the slopes of tangent lines are negative on the interval $(0, V_{mp})$ and positive when $V > V_{mp}$, the value of V' is negative on $(0, V_{mp})$ and positive when $V > V_{mp}$.

Use the tangents drawn on the graph and additional tangent as needed to estimate the slope at several points on the graph of V to improve the accuracy of the graph of V'.



20.



Chapter 3 Review Exercises

4. The derivative can be used to find the instantaneous rate of change at a point on a function, and the slope of a tangent line at a point on a function.

6. **(a)** $\lim_{x\to -1^-} g(x) = -2$

(b) $\lim_{x\to -1^+} g(x) = 2$

(c) $\lim_{x\to -1} g(x)$ does not exist since parts (a) and (b) have different answers.

(d) $g(-1) = -2$, since $(-1, -2)$ is a point on the graph.

8. **(a)** $\lim_{x\to 2^-} h(x) = 1$

(b) $\lim_{x\to 2^+} h(x) = 1$

(c) $\lim_{x\to 2} h(x) = 1$

(d) $h(2)$ does not exists since the graph has no point with an x-value of 2.

10. $\lim_{x\to\infty} f(x) = -3$ since the line $y = -3$ is a horizontal asymptote for the graph.

12. Let $f(x) = \dfrac{2x+5}{x-3}$.

x	2.9	2.99	2.999
$f(x)$	−108	−1098	−10,998

x	3.1	3.01	3.001
$f(x)$	112	1102	11,002

As x approaches 3 from the left, $f(x)$ gets infinitely smaller. As x approaches 3 from the right, $f(x)$ gets infinitely larger. Therefore, $\lim_{x\to3} \frac{2x+5}{x-3}$ does not exist.

14. $$\lim_{x\to2} \frac{x^2+3x-10}{x-2} = \lim_{x\to2} \frac{(x+5)(x-2)}{x-2} = \lim_{x\to2}(x+5) = 2+5 = 7$$

16. $$\lim_{x\to3} \frac{3x^2-2x-21}{x-3} = \lim_{x\to3} \frac{(3x+7)(x-3)}{x-3} = \lim_{x\to3}(3x+7) = 9+7 = 16$$

18. $$\lim_{x\to16} \frac{\sqrt{x}-4}{x-16} = \lim_{x\to16} \frac{\sqrt{x}-4}{x-16}\cdot\frac{\sqrt{x}+4}{\sqrt{x}+4} = \lim_{x\to16} \frac{x-16}{(x-16)(\sqrt{x}+4)} = \lim_{x\to16} \frac{1}{\sqrt{x}+4} = \frac{1}{\sqrt{16}+4} = \frac{1}{4+4} = \frac{1}{8}$$

20. $$\lim_{x\to\infty} \frac{x^2+6x+8}{x^3+2x+1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^3}+\frac{6x}{x^3}+\frac{8}{x^3}}{\frac{x^3}{x^3}+\frac{2x}{x^3}+\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{6}{x^2}+\frac{8}{x^3}}{1+\frac{2}{x^2}+\frac{1}{x^3}} = \frac{0+0+0}{1+0+0} = 0$$

22. $$\lim_{x\to-\infty}\left(\frac{9}{x^4}+\frac{1}{x^2}-3\right) = \lim_{x\to-\infty}\frac{9}{x^4} + \lim_{x\to-\infty}\frac{1}{x^2} - \lim_{x\to-\infty}3 = 0+0-3 = -3$$

24. As shown on the graph, $f(x)$ is discontinuous at x_1 and x_4.

26. $f(x) = \dfrac{2-3x}{(1+x)(2-x)}$

The function is discontinuous at $x=-1$ and $x=2$ because those values make the denominator of the fraction equal to zero.
$\lim_{x\to-1} f(x)$ does not exist since $\lim_{x\to-1^-} f(x) = -\infty$ and $\lim_{x\to-1^+} f(x) = \infty$.
$\lim_{x\to2} f(x)$ does not exist since $\lim_{x\to2^-} f(x) = -\infty$ and $\lim_{x\to2^+} f(x) = \infty$.
$f(-1)$ and $f(2)$ do not exist since there is no point of the graph that has an x-value of −1 or 2.

28. $f(x) = \dfrac{x^2-9}{x+3}$

The function is discontinuous at $x=-3$ since this value makes the denominator of the fraction equal to zero.

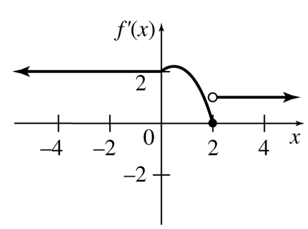
$$\lim_{x\to-3}\frac{x^2-9}{x+3} = \lim_{x\to-3}\frac{(x+3)(x-3)}{x+3} = \lim_{x\to-3}(x-3) = -3-3 = -6$$

$f(-3)$ does not exist since there is no point on the graph with an x-value of −3.

30. $f(x) = 2x^2-5x-3$ has no points of discontinuity since it is a polynomial function, which is continuous everywhere.

32. $$f(x) = \begin{cases} 2 & \text{if } x<0 \\ -x^2+x+2 & \text{if } 0\le x\le 2 \\ 1 & \text{if } x>2 \end{cases}$$

(a)



(b) The graph is discontinuous at $x=2$.

(c) $\lim_{x\to2^-} f(x) = -4+2+2 = 0$
$\lim_{x\to2^+} f(x) = 1$

34. $f(x) = \dfrac{x^4 + 3x^3 + 7x^2 + 11x + 2}{x^3 + 2x^2 - 3x - 6}$

(a) Find values of $f(x)$ when x is close to -2.

x	$f(x)$
-2.01	-12.62
-2.001	-12.96
-2.0001	-13
-1.99	-13.41
-1.999	-13.04
-1.9999	-13

It appears that $\lim\limits_{x\to -2} f(x) = -13.$

(b) Graph

$$y = \frac{x^4 + 3x^3 + 7x^2 + 11x + 2}{x^3 + 2x^2 - 3x - 6}$$

on a graphing calculator. One suitable choice for the viewing window is $[-5, 5]$ by $[-10, 10]$.
By viewing the function near $x = -2$, we see that as x gets close to -2 from the left on the right, y gets close to -13, suggesting that

$$\lim_{x\to -2} \frac{x^4 + 3x^3 + 7x^2 + 11x + 2}{x^3 + 2x^2 - 3x - 6} = -13.$$

36. $y = -2x^3 - x^2 + 5 = f(x)$

$$f(6) = -2(6)^3 - (6)^2 + 5 = -463$$
$$f(-2) = -2(-2)^3 - (-2)^2 + 5 = 17$$

Average rate of change:

$$= \frac{f(6) - f(-2)}{6 - (-2)}$$
$$= \frac{-463 - 17}{6 + 2} = \frac{-480}{8} = -60$$
$$y' = -6x^2 - 2x$$

Instantaneous rate of change at $x = -2$:

$$f'(-2) = -6(-2)^2 - 2(-2) = -6(4) + 4 = -20$$

38. $y = \dfrac{x + 4}{x - 1} = f(x)$

$$f(5) = \frac{5 + 4}{5 - 1} = \frac{9}{4}$$
$$f(2) = \frac{2 + 4}{2 - 1} = 6$$

Average rate of change:

$$= \frac{\frac{9}{4} - 6}{5 - 2} = \frac{\frac{-15}{4}}{3} = -\frac{5}{4}$$

$$y' = \frac{(x - 1)(1) - (x + 4)(1)}{(x - 1)^2}$$
$$= \frac{x - 1 - x - 4}{(x - 1)^2}$$
$$= \frac{-5}{(x - 1)^2}$$

Instantaneous rate of change at $x = 2$:

$$f'(2) = \frac{-5}{(2 - 1)^2} = \frac{-5}{1} = -5$$

40. $y = 5x^2 + 6x$

$$y' = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{[5(x + h)^2 + 6(x + h) - [5x^2 + 6x]}{h}$$
$$= \lim_{h\to 0} \frac{5(x^2 + 2xh + h^2) + 6x + 6h - 5x^2 - 6x}{h}$$
$$= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 + 6x + 6h - 5x^2 - 6x}{h}$$
$$= \lim_{h\to 0} \frac{10xh + 5h^2 + 6h}{h}$$
$$= \lim_{h\to 0} \frac{h(10x + 5h + 6)}{h}$$
$$= \lim_{h\to 0} (10x + 5h + 6)$$
$$= 10x + 6$$

42. $f(x) = x^{\ln x},\ x_0 = 3$

(a)

h	
.01	$\frac{f(3+.01)-f(3)}{.01} = \frac{3.01^{\ln 3.01} - 3^{\ln 3}}{.01} = 2.4573$
.001	$\frac{f(3+.001)-f(3)}{.001} = \frac{3.001^{\ln 3.001} - 3^{\ln 3}}{.001} = 2.4495$
.0001	$\frac{f(3+.0001)-f(3)}{.0001} = \frac{3.0001^{\ln 3.0001} - 3^{\ln 3}}{.0001} = 2.4487$
.00001	$\frac{f(3+.00001)-f(3)}{.00001} = \frac{3.00001^{\ln 3.00001} - 3^{\ln 3}}{.00001} = 2.4486$
.000001	$\frac{f(3+.000001)-f(3)}{.000001} = \frac{3.000001^{\ln 3.000001} - 3^{\ln 3}}{.000001} = 2.4486$

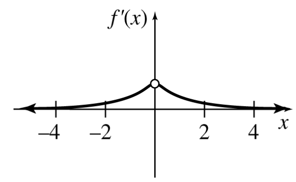
It appears that $\lim\limits_{x\to 3} f(x) = 2.4486$.

(b) Graph the function on a graphing calculator and move the cursor to an x-value near $x = 3$. A good choice for the viewing window is $[0, 10]$ by $[0, 10]$.

Zoom in on the function until the graph looks like a straight line. Use the TRACE feature to select two points on the graph, and use these points to compute the slope. The result will be close to the most accurate value found in part (a), which is 2.4486.

44. On the intervals $(-\infty, 0)$ and $(0, \infty)$, the slope of any tangent line will be positive, so the derivative will be positive. Thus, the graph of f' will lie above the y-axis. The slope of f and thus the value of f' approaches 0 when $x \to -\infty$ and $x \to \infty$ and approaches some particular but unknown positive value > 1 when $x \to 0^-$ and $x \to 0^+$.

Because f is discontinuous at $x = 0$, we know that $f'(0)$ does not exist, which we indicate with an open circle at $x = 0$ on the graph of f'.



46. $R(x) = 5000 + 16x - 3x^2$

(a) $R'(x) = 16 - 6x$

(b) Since x is in hundreds of dollars, \$1000 corresponds to $x = 10$.

$$R'(10) = 16 - 6(10) = 16 - 60 = -44$$

An increase of \$100 spent on advertising when advertising expenditures are \$1000 will result in the revenue decreasing by \$44.

48. $P(x) = 15x + 25x^2$

(a) $P(6) = 15(6) + 25(6)^2 = 90 + 900 = 990$

$P(7) = 15(7) + 25(7)^2 = 105 + 1225 = 1330$

Average rate of change:

$$= \frac{P(7) - P(6)}{7 - 6} = \frac{1330 - 990}{1}$$

$= 340$ cents or \$3.40

(b) $P(6) = 990$

$P(6.5) = 15(6.5) + 25(6.5)^2 = 97.5 + 1056.25 = 1153.75$

Average rate of change:

$$= \frac{P(6.5) - P(6)}{6.5 - 6} = \frac{1153.75 - 990}{.5}$$

$= 327.5$ cents or \$3.28

(c)
$$P(6) = 990$$
$$\begin{aligned} P(6.1) &= 15(6.1) + 25(6.1)^2 \\ &= 91.5 + 930.25 \\ &= 1021.75 \end{aligned}$$

Average rate of change:

$$\begin{aligned} &= \frac{P(6.1) - P(6)}{6.1 - 6} \\ &= \frac{1021.75 - 990}{.1} \\ &= 317.5 \text{ cents or } \$3.18 \end{aligned}$$

(d)
$$\begin{aligned} P'(x) &= 15 + 50x \\ P'(6) &= 15 + 50(6) \\ &= 15 + 300 \\ &= 315 \text{ cents or } \$3.15 \end{aligned}$$

(e)
$$\begin{aligned} P'(20) &= 15 + 50(20) \\ &= 1015 \text{ cents or } \$10.15 \end{aligned}$$

(f)
$$\begin{aligned} P'(30) &= 15 + 50(30) \\ &= 1515 \text{ cents or } \$15.15 \end{aligned}$$

(g) The domain of x is $[0, \infty)$ since pounds cannot be measured with negative numbers.

(h) Since $P'(x) = 15 + 50x$ gives the marginal profit, and $x \geq 0$, $P'(x)$ can never be negative.

(i)
$$\begin{aligned} \overline{P}(x) &= \frac{P(x)}{x} \\ &= \frac{15x + 25x^2}{x} \\ &= 15 + 25x \end{aligned}$$

(j) $\overline{P}'(x) = 25$

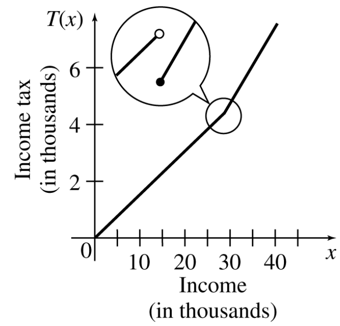
(k) The marginal average profit cannot change since $\overline{P}'(x)$ is constant. The profit per pound never changes, no matter now many pounds are sold.

50. (a)
$$\begin{aligned} \lim_{x \to 29{,}300^-} T(x) &= (29{,}300)(.15) \\ &= \$4395 \end{aligned}$$

(b)
$$\begin{aligned} \lim_{x \to 29{,}300^+} T(x) &= 4350 + (.27)(29{,}300 - 29{,}300) \\ &= \$4350 \end{aligned}$$

(c) $\lim_{x \to 29{,}300} T(x)$ does not exist since parts (a) and (b) have different answers.

(d)



(e) The graph is discontinuous at $x = 29{,}300$.

(f) For $0 \leq x \leq 29{,}300$,

$$A(x) = \frac{T(x)}{x} = \frac{.15x}{x} = .15.$$

For $x > 29{,}300$,

$$\begin{aligned} A(x) &= \frac{T(x)}{x} \\ &= \frac{4350 + (.27)(x - 29{,}300)}{x} \\ &= \frac{.27x - 3561}{x} \\ &= .27 - \frac{3561}{x}. \end{aligned}$$

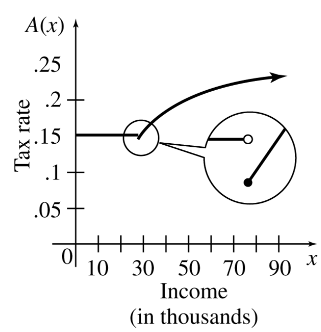
(g) $\lim_{x \to 29{,}300^-} A(x) = .15$

(h) $\lim_{x \to 29{,}300^+} A(x) = .27 - \frac{3561}{29{,}300} = .14846$

(i) $\lim_{x \to 29{,}300} A(x)$ does not exist since parts (g) and (h) have different answers.

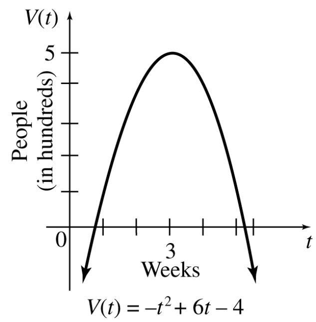
(j) $\lim_{x \to \infty} A(x) = .27 - 0 = .27$

(k)



52. $V(t) = -t^2 + 6t - 4$

(a)



(b) The x-intercepts of the parabola are .8 and 5.2, so a reasonable domain would be $[.8, 5.2]$, which represents the time period from .8 to 5.2 weeks.

(c) The number of cases reaches a maximum at the vertex;

$$x = \frac{-b}{2a} = \frac{-6}{-2} = 3$$

$$V(3) = -3^2 + 6(3) - 4 = 5$$

The vertex of the parabola is $(3, 5)$. This represents a maximum at 3 weeks of 500 cases.

(d) The rate of change function is

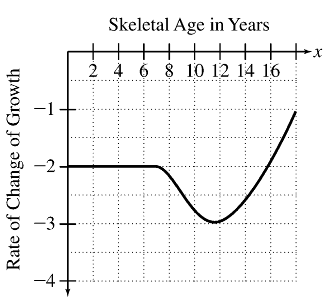
$$V'(t) = -2t + 6.$$

(e) The rate of change in the number of cases at the maximum is

$$V'(3) = -2(3) + 6 = 0.$$

(f) The sign of the rate of change up to the maximum is + because the function is increasing.. The sign of the rate of change after the maximum is − because the function is decreasing.

54.

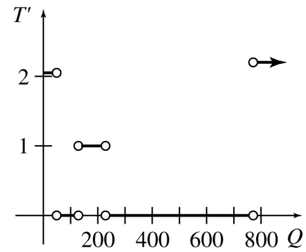


For a 10-year old girl, the remaining growth is about 14 cm and the rate of change is about −2.75 cm per year.

56. **(a)** The graph is discontinuous nowhere.

(b) The graph is not differentiable where the graph makes a sudden change, namely at $x = 50$, $x = 130, x = 230$, and $x = 770$.

(c)



Chapter 4

CALCULATING THE DERIVATIVE

4.1 Techniques for Finding Derivatives

2. $y = 3x^3 - x^2 - \frac{x}{12}$

$\frac{dy}{dx} = 3(3x^{3-1}) - 2x^{2-1} - \frac{1}{12}x^{1-1}$

$= 9x^2 - 2x - \frac{1}{12}$

4. $y = 3x^4 + 11x^3 + 2x^2 - 4x$

$\frac{dy}{dx} = 3(4x^{4-1}) + 11(3x^{3-1})$
$+ 2(2x^{2-1}) - 4x^{1-1}$
$= 12x^3 + 33x^2 + 4x - 4$

6. $f(x) = -2x^{2.5} + 8x^{.5}$
$f'(x) = -2(2.5x^{2.5-1}) + 8(.5x^{.5-1})$

$= -5x^{1.5} + 4x^{-.5}$ or $-5x^{1.5} + \frac{4}{x^{.5}}$

8. $y = -100\sqrt{x} - 11x^{2/3}$
$= -100x^{1/2} - 11x^{2/3}$

$\frac{dy}{dx} = -100\left(\frac{1}{2}x^{1/2-1}\right) - 11\left(\frac{2}{3}x^{2/3-1}\right)$

$= -50x^{-1/2} - \frac{22x^{-1/3}}{3}$ or $\frac{-50}{x^{1/2}} - \frac{22}{3x^{1/3}}$

10. $y = 10x^{-2} + 3x^{-4} - 6x$

$\frac{dy}{dx} = 10(-2x^{-2-1}) + 3(-4x^{-4-1}) - 6x^{1-1}$

$= -20x^{-3} - 12x^{-5} - 6$ or $\frac{-20}{x^3} - \frac{12}{x^5} - 6$

12. $f(t) = \frac{6}{t} - \frac{8}{t^2}$

$= 6t^{-1} - 8t^{-2}$
$f'(t) = 6(-1t^{-1-1}) - 8(-2t^{-2-1})$

$= -6t^{-2} + 16t^{-3}$ or $\frac{-6}{t^2} + \frac{16}{t^3}$

14. $y = \frac{9}{x^4} - \frac{8}{x^3} + \frac{2}{x}$
$= 9x^{-4} - 8x^{-3} + 2x^{-1}$

$\frac{dy}{dx} = 9(-4x^{-5}) - 8(-3x^{-4}) + 2(-x^{-2})$
$= -36x^{-5} + 24x^{-4} - 2x^{-2}$

or $\frac{-36}{x^5} + \frac{24}{x^4} - \frac{2}{x^2}$

16. $p(x) = -10x^{-1/2} + 8x^{-3/2}$

$p'(x) = -10\left(-\frac{1}{2}x^{-3/2}\right) + 8\left(-\frac{3}{2}x^{-5/2}\right)$

$= 5x^{-3/2} - 12x^{-5/2}$

or $\frac{5}{x^{3/2}} - \frac{12}{x^{5/2}}$

18. $y = \frac{6}{4\sqrt{x}} = 6x^{-1/4}$

$\frac{dy}{dx} = 6\left(-\frac{1}{4}\right)x^{-5/4}$

$= -\frac{3}{2}x^{-5/4}$

or $\frac{-3}{2x^{5/4}}$

20. $f(x) = \frac{x^2+2}{x}$

$= x + 2x^{-1}$
$f'(x) = x^{1-1} + 2(-1x^{-1-1})$
$= 1 - 2x^{-2}$

or $1 - \frac{2}{x^2}$

22. $g(x) = (8x^2 - 4x)^2$
$= 64x^4 - 64x^3 + 16x^2$
$g'(x) = 64(4x^{4-1}) - 64(3x^{3-1}) + 16(2x^{2-1})$
$= 256x^3 - 192x^2 + 32x$

24. A quadratic function has degree 2.
When the derivative is taken, the power will decrease by 1 and the derivative function will be linear, so the correct choice is (b).

26. $\frac{d}{dx}(4x^3 - 6x^{-2})$

$$\begin{aligned} &= 4(3x^2) - 6(-2x^{-3}) \\ &= 12x^2 + 12x^{-3} \\ &= 12x^2 + \frac{12}{x^3} \qquad \textit{choice (c)} \\ &= \frac{12x^2(x^3) + 12}{x^3} \\ &= \frac{12x^5 + 12}{x^3} \qquad \textit{choice (b)} \end{aligned}$$

Neither choice (a) nor choice (d) equals

$$\frac{d}{dx}(4x^3 - 6x^{-2}).$$

28. $D_x \left[\frac{8}{\sqrt[4]{x}} - \frac{3}{\sqrt{x^3}} \right]$

$$\begin{aligned} &= D_x \left[8x^{-1/4} - 3x^{-3/2}\right] \\ &= 8\left(-\frac{1}{4}x^{-5/4}\right) - 3\left(-\frac{3}{2}x^{-5/2}\right) \\ &= -2x^{-5/4} + \frac{9x^{-5/2}}{2} \end{aligned}$$

or $\frac{-2}{x^{5/4}} + \frac{9}{2x^{5/2}}$

30.

$$\begin{aligned} f(x) &= \frac{x^3}{9} - 8x^2 \\ &= \frac{1}{9}x^3 - 8x^2 \\ f'(x) &= \frac{1}{9}(3x^2) - 8(2x) \\ &= \frac{1}{3}x^2 - 16x \\ f'(3) &= \frac{1}{3}(3)^2 - 16(3) \\ &= 3 - 48 = -45 \end{aligned}$$

32.

$$\begin{aligned} y &= -2x^5 - 7x^3 + 8x^2;\ x = 1 \\ y' &= -2(5x^4) - 7(3x^2) + 8(2x) \\ &= -10x^4 - 21x^2 + 16x \\ y'(1) &= -10(1)^4 - 21(1)^2 + 16(1) \\ &= -10 - 21 + 16 \\ &= -15 \end{aligned}$$

The slope of the tangent line at $x = 1$ is -15. Use $m = -15$ and $(x_1, y_1) = (1, -1)$ to obtain the equation.

$$\begin{aligned} y - (-1) &= -15(x - 1) \\ y + 1 &= -15x + 15 \\ 15x + y &= 14 \end{aligned}$$

34.

$$\begin{aligned} y &= -x^{-3} + x^{-2};\ x = 1 \\ y' &= -3(-x^{-4}) + (-2x^{-3}) \\ &= 3x^{-4} - 2x^{-3} \\ &= \frac{3}{x^4} - \frac{2}{x^3} \\ y'(1) &= \frac{3}{(1)^4} - \frac{2}{(1)^3} = \frac{3}{1} - \frac{2}{1} \\ &= 3 - 2 \\ &= 1 \end{aligned}$$

The slope of the tangent line at $x = 1$ is 1.

36. $f(x) = 2x^3 + 9x^2 - 60x + 4$
$f'(x) = 6x^2 + 18x - 60$

If the tangent line is horizontal, then its slope is zero and $f'(x) = 0$.

$$\begin{aligned} 6x^2 + 18x - 60 &= 0 \\ 6(x^2 + 3x - 10) &= 0 \\ 6(x + 5)(x - 2) &= 0 \\ x = -5 \quad \text{or} \quad x &= 2 \end{aligned}$$

Thus, the tangent line is horizontal at $x = -5$ and $x = 2$.

38. $f(x) = x^3 - 4x^2 - 7x + 8$
$f'(x) = 3x^2 - 8x - 7$

If the tangent line is horizontal, then its slope is zero and $f'(x) = 0$.

$$3x^2 - 8x - 7 = 0$$

$$x = \frac{8 \pm \sqrt{64 + 84}}{6}$$

$$x = \frac{8 \pm \sqrt{148}}{6}$$

$$x = \frac{8 \pm 2\sqrt{37}}{6}$$

$$x = \frac{2(4 \pm \sqrt{37})}{6}$$

$$x = \frac{4 \pm \sqrt{37}}{3}$$

Thus, the tangent line is horizontal at $x = \frac{4 \pm \sqrt{37}}{3}$.

40. $f(x) = 6x^2 + 4x - 9$
$f'(x) = 12x + 4$

If the slope of the tangent line is -2, $f'(x) = -2$.

$$\begin{aligned} 12x + 4 &= -2 \\ 12x &= -6 \\ x &= -\frac{1}{2} \\ f\left(-\frac{1}{2}\right) &= -\frac{19}{2} \end{aligned}$$

The slope of the tangent line is -2 at $\left(-\frac{1}{2}, -\frac{19}{2}\right)$.

42. $f(x) = x^3 + 6x^2 + 21x + 2$
$f'(x) = 3x^2 + 12x + 21$

If the slope of the tangent line is 9, $f'(x) = 9$.

$$\begin{aligned} 3x^2 + 12x + 21 &= 9 \\ 3x^2 + 12x + 12 &= 0 \\ 3(x^2 + 4x + 4) &= 0 \\ 3(x + 2)^2 &= 0 \\ x &= -2 \\ f(-2) &= -24 \end{aligned}$$

The slope of the tangent line is 9 at $(-2, -24)$.

44.
$$\begin{aligned} f(x) &= \frac{1}{2}g(x) + \frac{1}{4}h(x) \\ f'(x) &= \frac{1}{2}g'(x) + \frac{1}{4}h'(x) \\ f'(2) &= \frac{1}{2}g'(2) + \frac{1}{4}h'(2) \\ &= \frac{1}{2}(3) + \frac{1}{4}(6) \\ &= \frac{3}{2} + \frac{3}{2} = 3 \end{aligned}$$

48. $\dfrac{f(x)}{k} = \dfrac{1}{k} \cdot f(x)$

Use the rule for the derivative of a constant times a function.

$$\begin{aligned} \frac{d}{dx}\left[\frac{f(x)}{k}\right] &= \frac{d}{dx}\left[\frac{1}{k} \cdot f(x)\right] \\ &= \frac{1}{k}f'(x) \\ &= \frac{f'(x)}{k} \end{aligned}$$

50. The demand is given by $x = 5000 - 100p$.

Solve for p.

$$\begin{aligned} p &= \frac{5000 - x}{100} \\ R(x) &= x\left(\frac{5000 - x}{100}\right) \\ &= \frac{5000x - x^2}{100} \\ R'(x) &= \frac{5000 - 2x}{100} \end{aligned}$$

(a) $R'(1000) = \dfrac{5000 - 2(1000)}{100}$
$= 30$

(b) $R'(2500) = \dfrac{5000 - 2(2500)}{100}$
$= 0$

(c) $R'(3000) = \dfrac{5000 - 2(3000)}{100}$
$= -10$

52.
$$\begin{aligned} S(t) &= 100 - 100t^{-1} \\ S'(t) &= -100(-1t^{-2}) \\ &= 100t^{-2} \\ &= \frac{100}{t^2} \end{aligned}$$

(a) $S'(1) = \dfrac{100}{(1)^2} = \dfrac{100}{1} = 100$

(b) $S'(10) = \dfrac{100}{(10)^2} = \dfrac{100}{100} = 1$

54. Profit = Revenue − Cost

$$\begin{aligned} P(x) &= xp(x) - C(x) \\ P(x) &= x\left(\frac{1000}{x^2} + 1000\right) - (.2x^2 + 6x + 50) \\ &= \frac{1000}{x} + 1000x - .2x^2 - 6x - 50 \\ &= 1000x^{-1} + 994x - .2x^2 - 50 \\ P'(x) &= -1000x^{-2} + 994 - .4x \\ &= 994 - .4x - \frac{1000}{x^2} \\ P'(10) &= 994 - .4(10) - \frac{1000}{(10)^2} \\ &= 994 - 4 - 10 \\ &= 980 \end{aligned}$$

The marginal profit is \$980.

56. **(a)** 1982:
$1982 - 1932 = 50$

$$\begin{aligned} C(50) &= .0102(50)^2 - .2164(50) + 3.23 \\ &= 17.91 \approx 18 \text{ cents} \end{aligned}$$

2002:
$2002 - 1932 = 70$

$$\begin{aligned} C(70) &= .0102(70)^2 - .2164(70) + 3.23 \\ &= 38.062 \approx 38 \text{ cents} \end{aligned}$$

(b) $$\begin{aligned} C'(x) &= .0102(2x) - .2164(1) \\ &= .0204x - .2164 \end{aligned}$$

1982:

$$\begin{aligned} C'(50) &= .0204(50) - .2164 \\ &= .8036 \text{ cents/year} \end{aligned}$$

2002:

$$\begin{aligned} C'(70) &= .0204(70) - .2164 \\ &= 1.2116 \text{ cents/year} \end{aligned}$$

58. $N(t) = .00437t^{3.2}$
$N'(t) = .013984t^{2.2}$

(a) $N'(5) \approx .4824$

(b) $N'(10) \approx 2.216$

60. $V(t) = -2159 + 1313t - 60.82t^2$

(a) $$\begin{aligned} V(3) &= -2159 + 1313(3) - 60.82(3)^2 \\ &= 1232.62 \text{ cm}^3 \end{aligned}$$

(b) $$\begin{aligned} V'(t) &= 1313 - 121.64t \\ V'(3) &= 1313 - 121.64(3) \\ &= 948.08 \text{ cm}^3/\text{yr} \end{aligned}$$

62. $v = 2.69l^{1.86}$

$$\frac{dv}{dl} = (1.86)2.69l^{1.86-1} \approx 5.00l^{.86}$$

64. $t = .0588s^{1.125}$

(a) When $s = 1609, t \approx 238.1$ seconds, or 3 minutes, 58.1 seconds.

(b) $$\begin{aligned} \frac{dt}{ds} &= .0588(1.125s^{1.125-1}) \\ &= .06615s^{.125} \end{aligned}$$

When $s = 100$, $\frac{dt}{ds} \approx .118$ sec/m. At 100 meters, the fastest possible time increases by .118 seconds for each additional meter.

(c) Yes, they have been surpassed. In 2000, the world record in the mile stood at 3:43.13. (Ref: www.runnersworld.com)

66. $\text{BMI} = \dfrac{703w}{h^2}$

(a) $6'2'' = 74$ in.

$$\text{BMI} = \frac{703(220)}{74^2} \approx 28$$

(b) $\text{BMI} = \dfrac{703w}{74^2} = 24.9$ implies

$$w = \frac{24.9(74)^2}{703} \approx 194.$$

A 220-lb person needs to lose 26 pounds to get down to 194 lbs.

(c) If $f(h) = \dfrac{703(125)}{h^2} = 87{,}875h^{-2}$, then

$$\begin{aligned} f'(h) &= 87{,}875(-2h^{-2-1}) \\ &= -175{,}750h^{-3} = -\frac{175{,}750}{h^3} \end{aligned}$$

(d) $f'(65) = -\dfrac{175{,}750}{65^3} \approx -.64$

For a 125-lb female with a height of 65 in. (5′5″), the BMI decreases by .64 for each additional inch of height.

(e) Sample Chart

ht/wt	140	160	180	200
60	27	31	35	39
65	23	27	30	33
70	20	23	26	29
75	17	20	22	25

68. $s(t) = 25t^2 - 9t + 8$

(a) $$\begin{aligned} v(t) = s'(t) &= 25(2t) - 9 + 0 \\ &= 50t - 9 \end{aligned}$$

(b) $$\begin{aligned} v(0) &= 50(0) - 9 = -9 \\ v(5) &= 50(5) - 9 = 241 \\ v(10) &= 50(10) - 9 = 491 \end{aligned}$$

70. $s(t) = -2t^3 + 4t^2 - 1$

(a) $$\begin{aligned} v(t) = s'(t) &= -2(3t^2) + 4(2t) - 0 \\ &= -6t^2 + 8t \end{aligned}$$

(b) $$\begin{aligned} v(0) &= -6(0)^2 + 8(0) = 0 \\ v(5) &= -6(5)^2 + 8(5) \\ &= -6(25) + 40 = -110 \\ v(10) &= -6(10)^2 + 8(10) \\ &= -6(100) + 80 = -520 \end{aligned}$$

72. $s(t) = -16t^2 + 64t$

(a) $v(t) = s'(t) = -16(2t) + 64$
$= -32t + 64$

$v(2) = -32(2) + 64 = -64 + 64 = 0$

$v(3) = -32(3) + 64 = -96 + 64 = -32$

The ball's velocity is 0 ft/sec after 2 seconds and −32 ft/sec after 3 seconds.

(b) As the ball travels upward, its speed decreases because of the force of gravity until, at maximum height, its speed is 0 ft/sec.
In part (a), we found that $v(2) = 0$.

It takes 2 seconds for the ball to reach its maximum height.

(c) $s(2) = -16(2)^2 + 64(2)$
$= -16(4) + 128$
$= -64 + 128$
$= 64$

It will go 64 ft high.

74. $y_1 = 4.13x + 14.63$
$y_2 = -.033x^2 + 4.647x + 13.347$

(a) When $x = 5, y_1 \approx 35$ and $y_2 \approx 36$.

(b) $\frac{dy_1}{dx} = 4.13$

$\frac{dy_2}{dx} = .033(2x) + 4.647$
$= -.066x + 4.647$

When $x = 5$, $\frac{dy_1}{dx} = 4.13$ and $\frac{dy_2}{dx} \approx 4.32$. These values are fairly close and represent the rate of change of four years for a dog for one year of a human, for a dog that is actually 5 years old.

(c) With the first three points eliminated, the dog age increases in 2-year steps and the human age increases in 8-year steps, for a slope of 4. The equation has the form $y = 4x + b$. A value of 16 for b makes the numbers come out right. $y = 4x + b$. For a dog of age $x = 5$ years or more, the equivalent human age is given by $y = 4x + 16$.

4.2 Derivatives of Products and Quotients

2. $y = (5x^2 - 1)(4x + 3)$

$$\begin{aligned}\frac{dy}{dx} &= (5x^2 - 1)(4) + (10x)(4x + 3)\\ &= 20x^2 - 4 + 40x^2 + 30x\\ &= 60x^2 + 30x - 4\end{aligned}$$

4. $y = (7x - 6)^2 = (7x - 6)(7x - 6)$

$$\begin{aligned}\frac{dy}{dx} &= (7x - 6)(7) + (7)(7x - 6)\\ &= 49x - 42 + 49x - 42\\ &= 98x - 84\end{aligned}$$

6.
$$\begin{aligned}g(t) &= (3t^2 + 2)^2\\ &= (3t^2 + 2)(3t^2 + 2)\\ g'(t) &= (3t^2 + 2)(6t) + (6t)(3t^2 + 2)\\ &= 18t^3 + 12t + 18t^3 + 12t\\ &= 36t^3 + 24t\end{aligned}$$

8.
$$\begin{aligned}y &= (2x - 3)(\sqrt{x} - 1)\\ &= (2x - 3)(x^{1/2} - 1)\\ \frac{dy}{dx} &= (2x - 3)\left(\frac{1}{2}x^{-1/2}\right) + 2(x^{1/2} - 1)\\ &= x^{1/2} - \frac{3}{2}x^{-1/2} + 2x^{1/2} - 2\\ &= 3x^{1/2} - \frac{3x^{-1/2}}{2} - 2\\ &\text{or}\quad 3x^{1/2} - \frac{3}{2x^{1/2}} - 2\end{aligned}$$

10.
$$\begin{aligned}q(x) &= (x^{-2} - x^{-3})(3x^{-1} + 4x^{-4})\\ q'(x) &= (x^{-2} - x^{-3})(-3x^{-2} - 16x^{-5})\\ &\quad + (-2x^{-3} + 3x^{-4})(3x^{-1} + 4x^{-4})\\ q'(x) &= -3x^{-4} - 16x^{-7} + 3x^{-5} + 16x^{-8} - 6x^{-4}\\ &\quad - 8x^{-7} + 9x^{-5} + 12x^{-8}\\ q'(x) &= -9x^{-4} + 12x^{-5} - 24x^{-7} + 28x^{-8}\end{aligned}$$

12. $f(x) = \frac{6x - 11}{8x + 1}$

$$\begin{aligned}f'(x) &= \frac{(8x + 1)(6) - (6x - 11)(8)}{(8x + 1)^2}\\ &= \frac{48x + 6 - 48x + 88}{(8x + 1)^2}\\ &= \frac{94}{(8x + 1)^2}\end{aligned}$$

14. $y = \dfrac{9-7t}{1-t}$

$$\frac{dy}{dx} = \frac{(1-t)(-7)-(9-7t)(-1)}{(1-t)^2} = \frac{-7+7t+9-7t}{(1-t)^2} = \frac{2}{(1-t)^2}$$

16. $y = \dfrac{x^2-4x}{x+3}$

$$\frac{dy}{dx} = \frac{(x+3)(2x-4)-(x^2-4x)(1)}{(x+3)^2} = \frac{2x^2+6x-4x-12-x^2+4x}{(x+3)^2} = \frac{x^2+6x-12}{(x+3)^2}$$

18. $y = \dfrac{-x^2+6x}{4x^2+1}$

$$\frac{dy}{dx} = \frac{(4x^2+1)(-2x+6)-(-x^2+6x)(8x)}{(4x^2+1)^2} = \frac{-8x^3+24x^2-2x+6+8x^3-48x^2}{(4x^2+1)^2} = \frac{-24x^2-2x+6}{(4x^2+1)^2}$$

20. $k(x) = \dfrac{x^2+7x-2}{x-2}$

$$k'(x) = \frac{(x-2)(2x+7)-(x^2+7x-2)(1)}{(x-2)^2} = \frac{2x^2+7x-4x-14-x^2-7x+2}{(x-2)^2} = \frac{x^2-4x-12}{(x-2)^2}$$

22. $r(t) = \dfrac{\sqrt{t}}{2t+3} = \dfrac{t^{1/2}}{2t+3}$

$$r'(t) = \frac{(2t+3)\left(\frac{1}{2}t^{-1/2}\right)-(t^{1/2})(2)}{(2t+3)^2} = \frac{t^{1/2}+\frac{3}{2}t^{-1/2}-2t^{1/2}}{(2t+3)^2} = \frac{-t^{1/2}+\frac{3}{2t^{1/2}}}{(2t+3)^2}$$

$$= \frac{-\sqrt{t}+\frac{3}{2\sqrt{t}}}{(2t+3)^2} \quad \text{or} \quad \frac{-2t+3}{2\sqrt{t}(2t+3)^2}$$

24. $h(z) = \dfrac{z^{2.2}}{z^{3.2}+5}$

$$h'(z) = \frac{(z^{3.2}+5)(2.2z^{1.2})-z^{2.2}(3.2z^{2.2})}{(z^{3.2}+5)^2} = \frac{2.2z^{4.4}+11z^{1.2}-3.2z^{4.4}}{(z^{3.2}+5)^2} = \frac{-z^{4.4}+11z^{1.2}}{(z^{3.2}+5)^2}$$

26. $f(x) = \dfrac{(3x^2+1)(2x-1)}{5x+4}$

$$f'(x) = \frac{(5x+4)[(3x^2+1)(2)+(6x)(2x-1)]-(3x^2+1)(2x-1)(5)}{(5x+4)^2}$$

$$= \frac{(5x+4)(18x^2-6x+2)-(3x^2+1)(10x-5)}{(5x+4)^2}$$

$$= \frac{90x^3-30x^2+10x+72x^2-24x+8-30x^3+15x^2-10x+5}{(5x+4)^2}$$

$$= \frac{60x^3+57x^2-24x+13}{(5x+4)^2}$$

28. $h(x) = \dfrac{f(x)}{g(x)}$

$$h'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

$$h'(3) = \frac{g(3)f'(3)-f(3)g'(3)}{[g(3)]^2} = \frac{4(6)-7(5)}{4^2} = -\frac{11}{16}$$

30. In the first step, the denominator, $(x^3)^2 = x^6$, was omitted. The correct work follows.

$$D_x\left(\frac{x^2-4}{x^3}\right) = \frac{x^3(2x)-(x^2-4)(3x^2)}{(x^3)^2} = \frac{2x^4-3x^4+12x^2}{x^6} = \frac{-x^4+12x^2}{x^6} = \frac{x^2(-x^2+12)}{x^2(x^4)} = \frac{-x^2+12}{x^4}$$

32. $f(x) = \dfrac{u(x)}{v(x)}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{u(x+h)}{v(x+h)}-\frac{u(x)}{v(x)}}{h} = \lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x+h)}{hv(x+h)v(x)}$$

$$= \lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x)+u(x)v(x)-u(x)v(x+h)}{hv(x+h)v(x)}$$

$$= \lim_{h\to 0}\frac{v(x)[u(x+h)-u(x)]-u(x)[v(x+h)-v(x)]}{hv(x+h)v(x)}$$

$$= \lim_{h\to 0}\frac{v(x)\frac{u(x+h)-u(x)}{h}-u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)} = \frac{v(x)\cdot u'(x)-u(x)v'(x)}{[v(x)]^2}$$

34. Graph the numerical derivative of $f(x) = (x^2-2)(x^2-\sqrt{2})$ for x ranging from -2 to 2. The derivative crosses the x-axis at 0 and at approximately -1.307 and 1.307.

36. $C(x) = \dfrac{3x+2}{x+4}$

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{3x+2}{x^2+4x}$$

(a) $\overline{C}(10) = \dfrac{3(10)+2}{10^2+4(10)} = \dfrac{32}{140} \approx .2286$ hundreds of dollars or \$22.86 per unit

(b) $\overline{C}(20) = \dfrac{3(20)+2}{(20)^2+4(20)} = \dfrac{62}{480} \approx .1292$ hundreds of dollars or \$12.92 per unit

(c) $\overline{C}(x) = \dfrac{3x+2}{x^2+4x}$ per unit

(d) $\overline{C}'(x) = \dfrac{(x^2+4x)(3)-(3x+2)(2x+4)}{(x^2+4x)^2} = \dfrac{3x^2+12x-6x^2-12x-4x-8}{(x^2+4x)^2} = \dfrac{-3x^2-4x-8}{(x^2+4x)^2}$

38. $M(d) = \dfrac{100d^2}{3d^2+10}$

(a) $M'(d) = \dfrac{(3d^2+10)(200d)-(100d^2)(6d)}{(3d^2+10)^2} = \dfrac{600d^3+2000d-600d^3}{(3d^2+10)^2} = \dfrac{2000d}{(3d^2+10)^2}$

(b) $M'(2) = \dfrac{2000(2)}{[3(2)^2+10]^2} = \dfrac{4000}{484} \approx 8.3$

This means the new employee can assemble about 8.3 additional bicycles per day after 2 days of training.

$$M'(5) = \frac{2000(5)}{[3(5)^2+10]^2} = \frac{10{,}000}{7225} \approx 1.4$$

This means the new employee can assemble about 1.4 additional bicycles per day after 5 days of training.

40. $\overline{C}(x) = \dfrac{C(x)}{x}$

Let $u(x) = C(x)$, with $u'(x) = C'(x)$.
Let $v(x) = x$ with $v'(x) = 1$. Then, by the quotient rule,

$$\begin{aligned}\overline{C}(x) &= \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2}\\ &= \frac{x\cdot C'(x) - C(x)\cdot 1}{x^2}\\ &= \frac{xC'(x) - C(x)}{x^2}\end{aligned}$$

42. $f(x) = \dfrac{Kx}{A+x}$

(a) $f'(x) = \dfrac{(A+x)K - Kx(1)}{(A+x)^2}$

$f'(x) = \dfrac{AK}{(A+x)^2}$

(b) $$\begin{aligned}f'(A) &= \frac{AK}{(A+A)^2}\\ &= \frac{AK}{4A^2}\\ &= \frac{K}{4A}\end{aligned}$$

44. $R(w) = \dfrac{30(w-4)}{w-1.5}$

(a) $$\begin{aligned}R(5) &= \frac{30(5-4)}{5-1.5}\\ &\approx 8.57 \text{ min}\end{aligned}$$

(b) $$\begin{aligned}R(7) &= \frac{30(7-4)}{7-1.5}\\ &\approx 16.36 \text{ min}\end{aligned}$$

(c) $$\begin{aligned}R'(w) &= \frac{(w-1.5)(30) - 30(w-4)(1)}{(w-1.5)^2}\\ &= \frac{30w - 45 - 30w + 120}{(w-1.5)^2}\\ &= \frac{75}{(w-1.5)^2}\end{aligned}$$

$$\begin{aligned}R'(5) &= \frac{75}{(5-1.5)^2}\\ &\approx 6.12\frac{\text{min}^2}{\text{kcal}}\\ R'(7) &= \frac{75}{(7-1.5)^2}\\ &\approx 2.48\frac{\text{min}^2}{\text{kcal}}\end{aligned}$$

46. $$\begin{aligned}f(t) &= \frac{90t}{99t-90}\\ f'(t) &= \frac{(99t-90)(90) - (90t)(99)}{(99t-90)^2}\\ &= \frac{-8100}{(99t-90)^2}\end{aligned}$$

(a) $$\begin{aligned}f'(1) &= \frac{-8100}{(99-90)^2}\\ &= \frac{-8100}{9^2}\\ &= \frac{-8100}{81} = -100\end{aligned}$$

(b) $$\begin{aligned}f'(10) &= \frac{-8100}{[99(10)-90]^2}\\ &= \frac{-8100}{(900)^2}\\ &= \frac{-8100}{810,000}\\ &= -\frac{1}{100} \text{ or } -.01\end{aligned}$$

4.3 The Chain Rule

In Exercises 2-6, $f(x) = 4x^2 - 2x$ and $g(x) = 8x + 1$.

2. $$\begin{aligned}g(-5) &= 8(-5) + 1\\ &= -40 + 1 = -39\end{aligned}$$

$$\begin{aligned}f[g(-5)] &= f[-39]\\ &= 4(-39)^2 - 2(-39)\\ &= 6084 + 78 = 6162\end{aligned}$$

4. $$\begin{aligned}f(-5) &= 4(-5)^2 - 2(-5)\\ &= 100 + 10 = 110\end{aligned}$$

$$\begin{aligned}g[f(-5)] &= g[110]\\ &= 8(110) + 1\\ &= 880 + 1 = 881\end{aligned}$$

6. $$\begin{aligned}f(5z) &= 4(5z)^2 - 2(5z)\\ &= 4(25z^2) - 10z\\ &= 100z^2 - 10z\end{aligned}$$

$$\begin{aligned}g[f(5z)] &= 8(100z^2 - 10z) + 1\\ &= 800z^2 - 80z + 1\end{aligned}$$

8. $f(x) = -6x + 9;\ g(x) = \dfrac{x}{5} + 7$

$$\begin{aligned} f[g(x)] &= -6\left[\frac{x}{5} + 7\right] + 9 \\ &= \frac{-6x}{5} - 42 + 9 \\ &= \frac{-6x}{5} - 33 \\ &= \frac{-6x - 165}{5} \\ g[f(x)] &= \frac{-6x + 9}{5} + 7 \\ &= \frac{-6x + 9}{5} + \frac{35}{5} \\ &= \frac{-6x + 44}{5} \end{aligned}$$

10. $f(x) = \dfrac{2}{x^4};\ g(x) = 2 - x$

$$f[g(x)] = \frac{2}{(2 - x)^4}$$

$$g[f(x)] = 2 - \left(\frac{2}{x^4}\right) = 2 - \frac{2}{x^4}$$

12. $f(x) = 9x^2 - 11x;\ g(x) = 2\sqrt{x + 2}$

$$\begin{aligned} f[g(x)] &= 9(2\sqrt{x + 2})^2 - 11(2\sqrt{x + 2}) \\ &= 9[4(x + 2)] - 22\sqrt{x + 2} \\ &= 36(x + 2) - 22\sqrt{x + 2} \\ &= 36x + 72 - 22\sqrt{x + 2} \\ g[f(x)] &= 2\sqrt{(9x^2 - 11x) + 2} \\ &= 2\sqrt{9x^2 - 11x + 2} \end{aligned}$$

14. $f(x) = \dfrac{8}{x};\ g(x) = \sqrt{3 - x}$

$$\begin{aligned} f[g(x)] &= \frac{8}{\sqrt{3 - x}} \\ &= \frac{8}{\sqrt{3 - x}} \cdot \frac{\sqrt{3 - x}}{\sqrt{3 - x}} \\ &= \frac{8\sqrt{3 - x}}{3 - x} \end{aligned}$$

$$\begin{aligned} g[f(x)] &= \sqrt{3 - \frac{8}{x}} = \sqrt{\frac{3x - 8}{x}} \\ &= \frac{\sqrt{3x - 8}}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} \\ &= \frac{\sqrt{3x^2 - 8x}}{x} \end{aligned}$$

16. $y = (3x - 7)^{1/3}$

If $f(x) = x^{1/3}$ and $g(x) = 3x - 7$, then

$$y = f[g(x)] = (3x - 7)^{1/3}.$$

18. $y = \sqrt{9 - 4x}$

If $f(x) = \sqrt{x}$ and $g(x) = 9 - 4x$, then

$$y = f[g(x)] = \sqrt{9 - 4x}.$$

20. $y = (x^{1/2} - 3)^2 + (x^{1/2} - 3) + 5$

If $f(x) = x^2 + x + 5$ and $g(x) = x^{1/2} - 3$, then

$$\begin{aligned} y &= f[g(x)] \\ &= (x^{1/2} - 3)^2 + (x^{1/2} - 3) + 5. \end{aligned}$$

22. $y = (2x^3 + 9x)^5$

Let $f(x) = x^5$ and $g(x) = 2x^3 + 9x$. Then $(2x^3 + 9x)^5 = f[g(x)]$.

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x)$$

$$\begin{aligned} f'(x) &= 5x^4 \\ f'[g(x)] &= 5[g(x)]^4 \\ &= 5(2x^3 + 9x)^4 \\ g'(x) &= 6x^2 + 9 \end{aligned}$$

$$\frac{dy}{dx} = 5(2x^3 + 9x)^4(6x^2 + 9)$$

24. $f(x) = -8(3x^4 + 2)^3$

Use the generalized power rule with $u = 3x^4 + 2$, $n = 3$ and $u' = 12x^3$.

$$\begin{aligned} f'(x) &= -8[3(3x^4 + 2)^{3-1} \cdot 12x^3] \\ &= -8[36x^3(3x^4 + 2)^2] \\ &= -288x^3(3x^4 + 2)^2 \end{aligned}$$

26. $s(t) = 12(2t^4 + 5)^{3/2}$

Use generalized power rule with $u = 2t^4 + 5$, $n = \frac{3}{2}$, and $u' = 8t^3$.

$$\begin{aligned} s'(t) &= 12\left[\frac{3}{2}(2t^4 + 5)^{1/2} \cdot 8t^3\right] \\ &= 12[12t^3(2t^4 + 5)^{1/2}] \\ &= 144t^3(2t^4 + 5)^{1/2} \end{aligned}$$

28. $f(t) = 8\sqrt{4t^2+7}$
$= 8(4t^2+7)^{1/2}$

Use generalized power rule with $u = 4t^2+7$, $n = \frac{1}{2}$, and $u' = 8t$.

$$\begin{aligned} f'(t) &= 8\left[\frac{1}{2}(4t^2+7)^{-1/2}\cdot 8t\right] \\ &= 8[4t(4t^2+7)^{-1/2}] \\ &= 32t(4t^2+7)^{-1/2} \\ &= \frac{32t}{(4t^2+7)^{1/2}} \\ &= \frac{32t}{\sqrt{4t^2+7}} \end{aligned}$$

30. $r(t) = 4t(2t^5+3)^2$

Use the product rule and the power rule.

$$\begin{aligned} r'(t) &= 4t[2(2t^5+3)\cdot 10t^4] \\ &\quad + (2t^5+3)^2\cdot 4 \\ &= 80t^5(2t^5+3) + 4(2t^5+3)^2 \\ &= 4(2t^5+3)[20t^5+(2t^5+3)] \\ &= 4(2t^5+3)(22t^5+3) \end{aligned}$$

32. $y = (x^3+2)(x^2-1)^2$

Use the product rule and the power rule.

$$\begin{aligned} \frac{dy}{dx} &= (x^3+2)[2(x^2-1)\cdot 2x] \\ &\quad + (x^2-1)^2(3x^2) \\ &= (x^3+2)[4x(x^2-1)] \\ &\quad + 3x^2(x^2-1)^2 \\ &= (x^2-1)[4x(x^3+2)+3x^2(x^2-1)] \\ &= (x^2-1)(4x^4+8x+3x^4-3x^2) \\ &= (x^2-1)(7x^4-3x^2+8x) \end{aligned}$$

34. $p(z) = z(6z+1)^{4/3}$

Use the product rule and the power rule.

$$\begin{aligned} p'(z) &= z\cdot\frac{4}{3}(6z+1)^{1/3}\cdot 6 \\ &\quad + 1\cdot(6z+1)^{4/3} \\ &= 8z(6z+1)^{1/3} + (6z+1)^{4/3} \\ &= (6z+1)^{1/3}[8z+(6z+1)^1] \\ &= (6z+1)^{1/3}(14z+1) \end{aligned}$$

36. $y = \frac{1}{(3x^2-4)^5} = (3x^2-4)^{-5}$

$$\begin{aligned} \frac{dy}{dx} &= -5(3x^2-4)^{-6}\cdot 6x \\ &= -30x(3x^2-4)^{-6} \\ &= \frac{-30x}{(3x^2-4)^6} \end{aligned}$$

38. $p(t) = \frac{(2t+3)^3}{4t^2-1}$

$$\begin{aligned} p'(t) &= \frac{(4t^2-1)[3(2t+3)^2\cdot 2]-(2t+3)^3(8t)}{(4t^2-1)^2} \\ &= \frac{6(4t^2-1)(2t+3)^2-8t(2t+3)^3}{(4t^2-1)^2} \\ &= \frac{(2t+3)^2[6(4t^2-1)-8t(2t+3)]}{(4t^2-1)^2} \\ &= \frac{(2t+3)^2[24t^2-6-16t^2-24t]}{(4t^2-1)^2} \\ &= \frac{(2t+3)^2[8t^2-24t-6]}{(4t^2-1)^2} \\ &= \frac{2(2t+3)^2(4t^2-12t-3)}{(4t^2-1)^2} \end{aligned}$$

40. $y = \frac{x^2+4x}{(3x^3+2)^4}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(3x^3+2)^4(2x+4)-(x^2+4x)[4(3x^3+2)^3\cdot 9x^2]}{[(3x^3+2)^4]^2} \\ &= \frac{(3x^3+2)^4(2x+4)-36x^2(x^2+4x)(3x^3+2)^3}{(3x^3+2)^8} \\ &= \frac{2(3x^3+2)^3[(3x^3+2)(x+2)-18x^2(x^2+4x)]}{(3x^3+2)^8} \\ &= \frac{2(3x^4+6x^3+2x+4-18x^4-72x^3)}{(3x^3+2)^5} \\ &= \frac{-30x^4-132x^3+4x+8}{(3x^3+2)^5} \end{aligned}$$

42. Let $f(x) = x^n$. Then $y = f(g(x)) = [g(x)]^n$. Using the chain rule,

$$\frac{dy}{dx} = f'[g(x)]\cdot g'(x).$$

Then, using the power rule, $f'(x) = nx^{n-1}$.

$$\frac{dy}{dx} = n[g(x)]^{n-1}\cdot g'(x).$$

44. **(a)** $D_x\ (g[f(x)])$ at $x = 1$

$$\begin{aligned} &= g'[f(1)]\cdot f'(1) \\ &= g'(2)\cdot(-6) \\ &= \frac{3}{7}(-6) = -\frac{18}{7} \end{aligned}$$

(b) $D_x\ (g[f(x)])$ at $x = 2$

$$\begin{aligned} &= g'[f(2)]\cdot f'(2) \\ &= g'(4)\cdot(-7) \\ &= \frac{5}{7}(-7) = -5 \end{aligned}$$

46. $f(x) = (x^3+7)^{2/3}; x = 1$

$$f'(x) = \frac{2}{3}(x^3+7)^{-1/3}(3x^2)$$

$$f'(x) = \frac{2x^2}{(x^3+7)^{1/3}}$$

$$f'(1) = \frac{2}{2} = 1$$

$$f'(1) = 8^{2/3} = 4$$

We use $m = 1$ and the point $P(1, 4)$.

$$y - 4 = 1(x-1)$$
$$y = x + 3$$

48. $f(x) = x^2\sqrt{x^4-12}; x = 2$

$$f(x) = x^2(x^4-12)^{1/2}$$

$$f'(x) = x^2 \cdot \frac{1}{2}(x^4-12)^{-1/2}(4x^3) + 2x(x^4-12)^{1/2}$$

$$f'(x) = \frac{2x^5}{(x^4-12)^{1/2}} + 2x(x^4-12)^{1/2}$$

$$f'(2) = \frac{64}{4^{1/2}} + 4(4)^{1/2}$$

$$f'(2) = 32 + 8 = 40$$

$$f(2) = 4\sqrt{4} = 8$$

We use $m = 40$ and the point $P(2, 8)$.

$$y - 8 = 40(x-2)$$
$$y = 40x - 72$$

50. $f(x) = \dfrac{x}{(x^2+4)^4}$

$$f'(x) = \frac{(x^2+4)^4 \cdot 1 - x \cdot 4(x^2+4)^3(2x)}{(x^2+4)^8}$$

$$= \frac{(x^2+4)^3[(x^2+4) - 8x^2]}{(x^2+4)^8} = \frac{4-7x^2}{(x^2+4)^5}$$

If the tangent line is horizontal, its slope is zero and $f'(x) = 0$.

$$\frac{4-7x^2}{(x^2+4)^5} = 0$$

$$4 - 7x^2 = 0$$

$$x^2 = \frac{4}{7}$$

$$x = \pm\frac{2}{\sqrt{7}}$$

The tangent line is horizontal at $x = \pm\frac{2}{\sqrt{7}}$.

54. $R(x) = 1000\left(1 - \dfrac{x}{500}\right)^2$

$$R'(x) = 1000\left[2\left(1-\frac{x}{500}\right)\left(-\frac{1}{500}\right)\right]$$

$$= 1000\left[\frac{-2}{500}\left(1-\frac{x}{500}\right)\right]$$

$$= -4\left(1-\frac{x}{500}\right)$$

(a) $R'(400) = -4\left(1 - \dfrac{400}{500}\right)$

$$= -4(1-.8)$$
$$= -4(.2) = -.8$$

(b) $R'(500) = -4\left(1 - \dfrac{500}{500}\right)$

$$= -4(1-1)$$
$$= -4(0) = 0$$

(c) $R'(600) = -4\left(1 - \dfrac{600}{500}\right)$

$$= -4(1-1.2)$$
$$= -4(-.2)$$
$$= .8$$

(d) $\overline{R}(x) = \dfrac{R(x)}{x}$

$$= \frac{1000}{x}\left(1-\frac{x}{500}\right)^2$$

(e) $\overline{R}'(x) = \dfrac{1000}{x}\left[2\left(1-\dfrac{x}{500}\right)\left(-\dfrac{1}{500}\right)\right] + \left(1-\dfrac{x}{500}\right)^2\left(-\dfrac{1000}{x^2}\right)$

$$= \frac{1000}{x}\left(1-\frac{x}{500}\right)\cdot\left[-\frac{1}{250} - \frac{1}{x}\left(1-\frac{x}{500}\right)\right]$$

$$= \frac{1000}{x}\left(1-\frac{x}{500}\right)\cdot\left(-\frac{1}{250} - \frac{1}{x} + \frac{1}{500}\right)$$

$$= \frac{1000}{x}\left(1-\frac{x}{500}\right)\left(-\frac{1}{500} - \frac{1}{x}\right)$$

$$= \left(1-\frac{x}{500}\right)\left(-\frac{2}{x} - \frac{1000}{x^2}\right)$$

56. $q = D(p)$

$$= 30\left(5 - \frac{p}{\sqrt{p^2+1}}\right)$$

$$= 150 - \frac{30p}{\sqrt{p^2+1}}$$

$$= 150 - \frac{30p}{(p^2+1)^{1/2}}$$

$$\frac{dq}{dp} = 0 - \left[\frac{(p^2+1)^{1/2}D_p(30p) - (30p)(D_p(p^2+1)^{1/2})}{[(p^2+1)^{1/2}]^2}\right]$$

$$= -\left[\frac{(p^2+1)^{1/2}(30) - (30p)\left(\frac{1}{2}\right)(p^2+1)^{-1/2}(2p)}{(p^2+1)}\right]$$

$$= -\left[\frac{(p^2+1)^{1/2}(30) - (30p)(p)(p^2+1)^{-1/2}}{(p^2+1)}\right]$$

$$= -\left[\frac{(30)(p^2+1)^{-1/2}([p^2+1]-p^2)}{(p^2+1)}\right]$$

$$= \frac{-30(p^2+1)^{-1/2}(1)}{(p^2+1)}$$

$$= -\frac{30}{(p^2+1)^{3/2}}$$

58. $C = 2000x + 3500$
$x = \sqrt{15,000 - 1.5p}$

Solve for p.

$$x = \sqrt{15{,}000 - 1.5p}$$
$$x^2 = 15,000 - 1.5p$$
$$\frac{x^2 - 15{,}000}{-1.5} = p$$
$$\frac{x^2}{-1.5} + \frac{15{,}000}{1.5} = p$$
$$\frac{-2x^2}{3} + 10{,}000 = p$$

(a) $R(x) = xp = x\left(\frac{-2x^2}{3} + 10{,}000\right)$

$$= \frac{-2x^3}{3} + 10{,}000x$$

$$= \frac{-2x^3 + 30{,}000x}{3}$$

$$= \frac{30{,}000x - 2x^3}{3}$$

(b) $P(x) = R(x) - C(x)$

$$= \frac{30{,}000x - 2x^3}{3} - (2000x + 3500)$$

$$= \left(\frac{-2x^3}{3} + 10{,}000x\right) - (2000x + 3500)$$

$$= \frac{-2x^3}{3} + 8000x - 3500$$

$$= 8000x - \frac{2x^3}{3} - 3500$$

$P(x)$ is the profit function.

(c) $\frac{dP}{dx} = D_x\left(8000x - \frac{2x^3}{3} - 3500\right)$

$$= 8000 - 2x^2$$

$\frac{dP}{dx}$ or $P'(x)$ gives the marginal profit.

(d) If $p = \$5000$ and

$x = \sqrt{15{,}000 - 1.5p},$

then $x = \sqrt{15{,}000 - 1.5(5000)}$

$= \sqrt{15{,}000 - 7500}$

$= \sqrt{7500}.$

Thus, $x^2 = 7500$.

$$P'(x) = 8000 - 2x^2$$
$$= 8000 - 2(7500)$$
$$= 8000 - 15{,}000$$
$$= -7000$$

When the price is \$5000, the marginal profit is −\$7000.

60. **(a)**

$$A(r) = \pi r^2$$
$$r(t) = t^2$$
$$A[r(t)] = \pi[t^2]^2$$
$$= \pi(t^4)$$
$$= \pi t^4$$

This function represents the area of the oil slick as a function of time t after the beginning of the leak.

(b)
$$\begin{aligned} D_tA[r(t)] &= 4\pi t^3 \\ D_tA[r(100)] &= 4\pi(100)^3 \\ &= 4{,}000{,}000\pi \end{aligned}$$

At 100 minutes the area of the spill is changing at the rate of $4{,}000{,}000\pi$ ft^2/min.

62. $N(t) = 2t(5t+9)^{1/2} + 12$

$$\begin{aligned} N'(t) &= (2t)\left[\frac{1}{2}(5t+9)^{-1/2}(5)\right] \\ &\quad + 2(5t+9)^{1/2} + 0 \\ &= 5t(5t+9)^{-1/2} + 2(5t+9)^{1/2} \\ &= (5t+9)^{-1/2}[5t + 2(5t+9)] \\ &= (5t+9)^{-1/2}(15t+18) \\ &= \frac{15t+18}{(5t+9)^{1/2}} \end{aligned}$$

(a)
$$\begin{aligned} N'(0) &= \frac{15(0)+18}{[5(0)+9]^{1/2}} \\ &= \frac{18}{9^{1/2}} = 6 \end{aligned}$$

(b)
$$\begin{aligned} N'\left(\frac{7}{5}\right) &= \frac{15\left(\frac{7}{5}\right)+18}{\left[5\left(\frac{7}{5}\right)+9\right]^{1/2}} \\ &= \frac{21+8}{(7+9)^{1/2}} \\ &= \frac{39}{(16)^{1/2}} \\ &= \frac{39}{4} = 9.75 \end{aligned}$$

(c)
$$\begin{aligned} N'(8) &= \frac{15(8)+18}{[5(8)+9]^{1/2}} \\ &= \frac{120+18}{(49)^{1/2}} \\ &= \frac{138}{7} \approx 19.71 \end{aligned}$$

64. (a) $R(Q) = Q\left(C - \frac{Q}{3}\right)^{1/2}$

$$\begin{aligned} R'(Q) &= Q\left[\frac{1}{2}\left(C - \frac{Q}{3}\right)^{-1/2}\left(-\frac{1}{3}\right)\right] \\ &\quad + \left(C - \frac{Q}{3}\right)^{1/2}(1) \\ &= -\frac{1}{6}Q\left(C - \frac{Q}{3}\right)^{-1/2} + \left(C - \frac{Q}{3}\right)^{1/2} \\ &= -\frac{Q}{6\left(C - \frac{Q}{3}\right)^{1/2}} + \left(C - \frac{Q}{3}\right)^{1/2} \end{aligned}$$

(b) $R'(Q) = -\frac{Q}{6\left(C - \frac{Q}{3}\right)^{1/2}} + \left(C - \frac{Q}{3}\right)^{1/2}$

If $Q = 87$ and $C = 59$, then

$$\begin{aligned} R'(Q) &= \left(59 - \frac{87}{3}\right)^{1/2} - \frac{87}{6\left(59 - \frac{87}{3}\right)^{1/2}} \\ &= (30)^{1/2} - \frac{87}{6(30)^{1/2}} \\ &= 5.48 - \frac{87}{32.88} \\ &= 5.48 - 2.65 \\ &= 2.83. \end{aligned}$$

(c) Because $R'(Q)$ is positive, the patient's sensitivity to the drug is increasing.

4.4 Derivatives of Exponential Functions

2.
$$\begin{aligned} y &= e^{-2x} \\ \text{Let} \quad g(x) &= -2x, \\ \text{with} \quad g'(x) &= -2. \\ \frac{dy}{dx} &= -2e^{-2x} \end{aligned}$$

4. $y = .2e^{5x}$

$$\begin{aligned} \frac{dy}{dx} &= .2(5e^{5x}) \\ &= e^{5x} \end{aligned}$$

6. $y = -4e^{-.1x}$

$$\begin{aligned} \frac{dy}{dx} &= -4(-.1e^{-.1x}) \\ &= .4e^{-.1x} \end{aligned}$$

8.
$$\begin{aligned} y &= e^{-x^2} \\ g(x) &= -x^2 \\ g'(x) &= -2x \\ \frac{dy}{dx} &= -2xe^{-x^2} \end{aligned}$$

10.
$$\begin{aligned} y &= -5e^{4x^3} \\ g(x) &= 4x^3 \\ g'(x) &= 12x^2 \\ \frac{dy}{dx} &= (-5)(12x^2)e^{4x^3} \\ &= -60x^2e^{4x^3} \end{aligned}$$

12.
$$\begin{aligned} y &= -3e^{3x^2+5} \\ g(x) &= 3x^2+5 \\ g'(x) &= 6x \\ y' &= (-3)(6x)e^{3x^2+5} \\ &= -18xe^{3x^2+5} \end{aligned}$$

14. $y = x^2e^{-2x}$

Use the product rule.

$$\begin{aligned} \frac{dy}{dx} &= x^2(-2e^{-2x}) + 2xe^{-2x} \\ &= -2x^2e^{-2x} + 2xe^{-2x} \\ &= 2x(1-x)e^{-2x} \end{aligned}$$

16. $y = (3x^2-4x)e^{-3x}$

Use the product rule.

$$\begin{aligned} \frac{dy}{dx} &= (3x^2-4x)(-3)(e^{-3x}) \\ &\quad + (6x-4)e^{-3x} \\ &= (-9x^2+12x)(e^{-3x}) \\ &\quad + (6x-4)e^{-3x} \\ &= (6x-4-9x^2+12x)e^{-3x} \\ &= (-9x^2+18x-4)e^{-3x} \end{aligned}$$

18.
$$\begin{aligned} y &= \frac{e^x}{2x+1} \\ \frac{dy}{dx} &= \frac{e^x(2x+1)-(2)(e^x)}{(2x+1)^2} \\ &= \frac{e^x(2x+1-2)}{(2x+1)^2} \\ &= \frac{e^x(2x-1)}{(2x+1)^2} \end{aligned}$$

20.
$$\begin{aligned} y &= \frac{e^x-e^{-x}}{x} \\ \frac{dy}{dx} &= \frac{x(e^x-(-1)e^{-x})-(e^x-e^{-x})(1)}{x^2} \\ &= \frac{xe^x+xe^{-x}-e^x+e^{-x}}{x^2} \\ &= \frac{e^x(x-1)+e^{-x}(x+1)}{x^2} \end{aligned}$$

22.
$$\begin{aligned} p &= \frac{500}{12+5e^{-.5t}} \\ \frac{dp}{dt} &= \frac{(12+5e^{-.5t})\cdot 0 - 500[0+5(-.5)e^{-.5t}]}{(12+5e^{-.5t})^2} \\ &= \frac{1250e^{-.5t}}{(12+5e^{-.5t})^2} \end{aligned}$$

24. $y = 8^{5x}$

Let $g(x) = 5x$, with $g'(x) = 5$. Then

$$\begin{aligned} \frac{dy}{dx} &= (\ln\ 8)8^{5x}\cdot 5 \\ &= 5(\ln\ 8)8^{5x} \end{aligned}$$

26. $y = 3\cdot 4^{x^2+2}$

Let $g(x) = x^2+2$, with $g'(x) = 2x$. Then

$$\begin{aligned} \frac{dy}{dx} &= 3(\ln\ 4)4^{x^2+2}\cdot 2x \\ &= 6x(\ln\ 4)4^{x^2+2} \end{aligned}$$

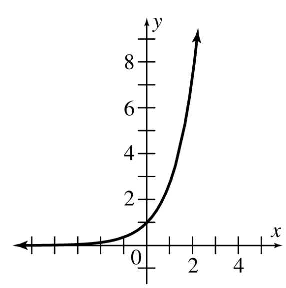
28. $s = 2\cdot 3^{\sqrt{t}}$

Let $g(t) = \sqrt{t}$, with $g'(t) = \dfrac{1}{2\sqrt{t}}$. Then

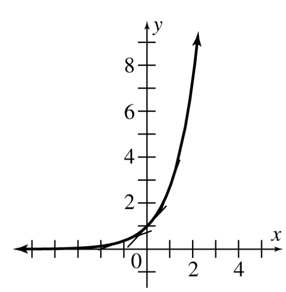
$$\begin{aligned} \frac{ds}{dt} &= 2(\ln\ 3)3^{\sqrt{t}}\cdot\frac{1}{2\sqrt{t}} \\ &= \frac{(\ln\ 3)3^{\sqrt{t}}}{\sqrt{t}} \end{aligned}$$

30.
$$\begin{aligned} y &= y_oe^{kt} \\ \frac{dy}{dt} &= \frac{d}{dt}[y_oe^{kt}] \\ &= y_oke^{kt} \\ &= k(y_oe^{kt}) \\ &= ky \end{aligned}$$

32. Graph the function $y = e^x$.



Sketch the lines tangent to the graph at $x = -1$, 0, 1, 2.



Estimate the slopes of the tangent lines at these points.

At $x = -1$ the slope is a little steeper than $\frac{1}{3}$ or approximately $0.\bar{3}$.
At $x = 0$ the slope is 1.
At $x = 1$ the slope is a little steeper than $\frac{5}{2}$ or 2.5.
At $x = 2$ the slope is a little steeper than $7\frac{1}{3}$ or $7.\bar{3}$.
Note that $e^{-1} \approx .36787944, e^0 = 1,$
$e^1 = e \approx 2.7182812$, and $e^2 \approx 7.3890561$. The values are close enough to the slopes of the tangent lines to convince us that $\frac{de^x}{dx} = e^x$.

34. $C(x) = \sqrt{900 - 800 \cdot 1.1^{-x}}$

$$C(x) = [900 - 800(1.1^{-x})]^{1/2}$$

$$= \frac{1}{2}[900 - 800(1.1^{-x})]^{-1/2} \cdot [-800(\ln\ 1.1)(1.1^{-x})(-1)]$$

$$C'(x) = \frac{(400\ \ln\ 1.1)(1.1^{-x})}{\sqrt{900 - 800(1.1^{-x})}}$$

(a) $C'(0) = \dfrac{400\ \ln\ 1.1}{\sqrt{100}} \approx 3.81$
The marginal cost is \$3.81.

(b) $C'(20) = \dfrac{(400\ \ln\ 1.1)(1.1^{-20})}{\sqrt{900 - 800(1.1^{-20})}} \approx .20$

The marginal cost is \$.20.

(c) As x becomes larger and larger, $C'(x)$ approaches zero.

36. $f(x) = 16(1.950)^x$
$f'(x) = 16(\ln\ 1.950)1.950^x$

(a) For 1998, $x = 3$.

$$f'(3) = 16(\ln\ 1.950)1.950^3 \approx 79$$

The instantaneous rate of change is 79,000,000 users per year in 1998.

(b) For 2000, $x = 5$.

$$f'(5) = 16(\ln\ 1.950)1.950^5 \approx 301$$

The instantaneous rate of change is 301,000,000 users per year in 2000.

38. $p(t) = 14.0(1.017)^t$
$p'(t) = 14.0(\ln\ 1.017)(1.017)^t$

(a) For July, 1998, $t = 4$.

$$p'(4) = 14.0(\ln\ 1.017)(1.017)^4 \approx .252$$

The instantaneous rate of change is 252,000 people.

(b) For January, 2005, $t = 10.5$.

$$p'(10.5) = 14.0(\ln\ 1.017)(1.017)^{10.5} \approx .282$$

The instantaneous rate of change is 282,000 people.

40. $G(t) = \dfrac{mG_o}{G_o + (m - G_o)e^{-kmt}}$, where $G_o = 400$; $m = 5200$; and $k = .0001$.

(a) $G(t) = \dfrac{(5200)(400)}{400 + (5200 - 400)e^{(-.0001)(5200)t}}$

$$= \frac{(400)(5200)}{400 + 4800e^{-.52t}}$$

$$= \frac{5200}{1 + 12e^{-.52t}}$$

(b) $G(t) = 5200(1 + 12e^{-.52t})^{-1}$
$G'(t) = -5200(1 + 12e^{-.52t})^{-2}(-6.24e^{-.52t})$

$$= \frac{32{,}448e^{-.52t}}{(1 + 12e^{-.52t})^2}$$

$$G(1) = \frac{5200}{1 + 12e^{-.52}} \approx 639$$

$$G'(1) = \frac{32{,}448e^{-.52}}{(1 + 12e^{-.52})^2} \approx 292$$

(c) $G(4) = \dfrac{5200}{1 + 12e^{-2.08}} \approx 2081$

$$G'(4) = \frac{32{,}448e^{-2.08}}{(1 + 12e^{-2.08})^2} \approx 649$$

(d) $G(10) = \dfrac{5200}{1 + 12e^{-5.2}} \approx 4877$

$$G'(10) = \frac{34,448e^{-5.2}}{(1 + 12e^{-5.2})^2} \approx 167$$

(e) It increases for a while and then gradually decreases to 0.

42. $V(t) = 1100[1023e^{-.02415t} + 1]^{-4}$

(a) $V(240) = 1100[1023e^{-.02415(240)} + 1]^{-4}$
$\approx 3.857\text{ cm}^3$

(b) $V = \frac{4}{3}\pi r^3$, so $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$

$$r(3.857) = \sqrt[3]{\frac{3(3.857)}{4\pi}} \approx .973 \text{ cm}$$

(c) $V(t) = 1100[1023e^{-.02415t} + 1]^{-4} = .5$

$$[1023e^{-.02415t} + 1]^{-4} = \frac{1}{2200}$$
$$(1023e^{-.02415t} + 1)^4 = 2200$$
$$1023e^{-.02415t} + 1 = 2200^{1/4}$$
$$1023e^{-.02415t} = 2200^{1/4} - 1$$
$$e^{-.02415t} = \frac{2200^{1/4} - 1}{1023}$$
$$-.02415t = \ln\left(\frac{2200^{1/4} - 1}{1023}\right)$$
$$t = \frac{1}{-.02415}\ln\left(\frac{2200^{1/4} - 1}{1023}\right) \approx 214 \text{ months}$$

The tumor has been growing for almost 18 years.

(d) As t goes to infinity, $e^{-.02415t}$ goes to zero, and $V(t) = 1100[1023e^{-.02415t}+1]^{-4}$ goes to 1100 cm^3, which corresponds to a sphere with a radius of $\sqrt[3]{\frac{3(1100)}{4\pi}} \approx 6.4$ cm. It makes sense that a tumor growing in a person's body reaches a maximum volume of this size.

(e) By the chain rule,

$$\frac{dV}{dt} = 1100(-4)[1023e^{-.02415t} + 1]^{-5} \cdot (1023)(e^{-.02415t})(-.02415)$$
$$= 108{,}703.98[1023e^{-.02415t} + 1]^{-5}e^{-.02415t}$$

At $t = 240$, $\frac{dV}{dt} \approx .282$.

At 240 months old, the tumor is increasing in volume at the instantaneous rate of .282 cm^3/month.

44. $URR = 1 - \left\{(.96)^{.14t-1} + \frac{8t}{126t+900}[1 - (.96)^{.14t-1}]\right\}$

(a) When $t = 180$, $URR \approx .589$. The patient has not received adequate dialysis.

(b) When $t = 240$, $URR \approx .690$. The patient has received adequate dialysis.

(c) $D_t URR$

$$= -\Big\{(\ln .96)(.96)^{.14t-1}(.14) + \frac{8t}{126t+900}(-\ln .96)(.96)^{.14t-1}(.14) + \frac{(126t+900)(8) - 8t(126)}{(126t+900)^2}[1 - (.96)^{.14t-1}]\Big\}$$

When $t = 240, D_t URR \approx .001$. The URR is increasing instantaneously by .001 units per minute when $t = 240$ minutes.

The rate of increase is low, and it will take a significant increase in time on dialysis to increase URR significantly.

46. $M(t) = 3102e^{-e^{-.022(t-56)}}$

(a) $M(200) = 3102e^{-e^{-.022(200-56)}} \approx 2974.15$ grams, or about 3 kilograms.

(b) As t gets very large, $-e^{-.022(t-56)}$ goes to zero, $e^{-e^{-.022(t-56)}}$ goes to 1, and $M(t)$ approaches 3102 grams or about 3.1 kilograms.

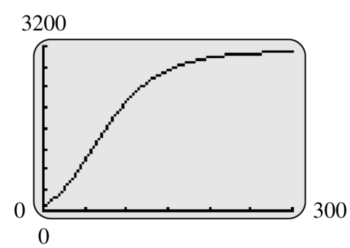
(c) 80% of 3102 is 2481.6.

$$2481.6 = 3102e^{-e^{-.022(t-56)}}$$
$$-\ln\frac{2481.6}{3102} = e^{-.022(t-56)}$$
$$\ln\left(\ln\frac{3102}{2481.6}\right) = -.022(t-56)$$
$$t = -\frac{1}{.022}\ln\left(\ln\frac{3102}{2481.6}\right) + 56$$
$$\approx 124 \text{ days}$$

(d)
$$D_t M(t) = 3102e^{-e^{-.022(t-56)}} D_t(-e^{-.022(t-56)})$$
$$= 3102e^{-e^{-.022(t-56)}}(-e^{-.022(t-56)})(-.022)$$
$$= 68.244e^{-e^{-.022(t-56)}}e^{-.022(t-56)}$$

When $t = 200$, $D_t M(t) \approx 2.75$ g/day.

(e)



Growth is initially rapid, then tapers off.

(f)

Day	*Weight*	*Rate*
50	991	24.88
100	2122	17.73
150	2734	7.60
200	2974	2.75
250	3059	.94
300	3088	.32

48. $W_1(t) = 509.7(1 - .941e^{-.00181t})$
$W_2(t) = 498.4(1 - .889e^{-.00219t})^{1.25}$

(a) Both W_1 and W_2 are strictly increasing functions, so they approach their maximum values as t approaches ∞.

$$\lim_{t\to\infty} W_1(t) = \lim_{t\to\infty} 509.7(1 - .941e^{-.00181t}) = 509.7(1-0) = 509.7$$

$$\lim_{t\to\infty} W_2(t) = \lim_{t\to\infty} 498.4(1 - .889e^{-.00219t})^{1.25} = 498.4(1-0)^{1.25} = 498.4$$

So, the maximum values of W_1 and W_2 are 509.7 kg and 498.4 kg respectively.

(b)
$$.9(509.7) = 509.7(1 - .941e^{-.00181t})$$
$$.9 = 1 - .941e^{-.00181t}$$
$$\frac{.1}{.941} = e^{-.00181t}$$
$$1239 \approx t$$

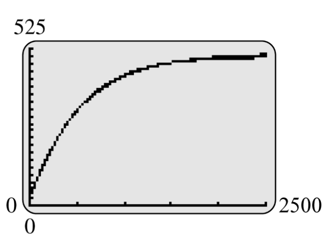
$$.9(498.4) = 498.4(1 - .889e^{-.00219t})^{1.25}$$
$$.9 = (1 - .889e^{-.00219t})^{1.25}$$
$$\frac{1 - .9^{.8}}{.889} = e^{-.00219t}$$
$$1095 \approx t$$

Respectively, it will take the average beef cow about 1239 days or 1095 days to reach 90% of its maximum.

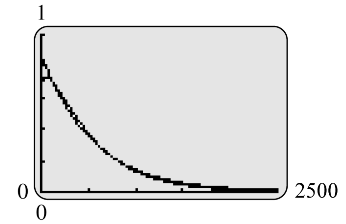
(c)
$$W_1'(t) = (509.7)(-.941)(-.00181)e^{-.00181t} \approx .868126e^{-.00181t}$$
$$W_1'(750) \approx .868126e^{-.00181(750)} \approx .22 \text{ kg/day}$$
$$W_2'(t) = (498.4)(1.25)(1 - .889e^{-.00219t})^{.25} \cdot (-.889)(-.00219)e^{-.00219t} \approx 1.21292e^{-.00219t}(1 - .889e^{-.00219t})^{.25}$$
$$W_2'(750) \approx 1.12192e^{-.00219(750)} \cdot (1 - .889e^{-.00219(750)})^{.25} \approx .22 \text{ kg/day}$$

Both functions yield a rate of change of about .22 kg per day.

(d) Looking at the graph, the growth patterns of the two functions are very similar.



(e) The graphs of the rates of change of the two functions are also very similar.



50. $P(t) = 26.7e^{.023t}$

(a) $P(5) = 26.7e^{.023(5)} \approx 29.95$

so, the U.S. Latino-American population in 2000 was approximately 29,950,000.

(b)
$$P'(t) = 26.7e^{.023t}(.023) = .6141e^{.023t}$$
$$P'(5) = .6141e^{.023(5)} \approx .689$$

The Latino-American population was increasing at the rate of .689 million/year at the end of the year 2000.

52. $Q(t) = CV(1 - e^{-t/RC})$

(a)
$$I_c = \frac{dQ}{dt} = CV\left[0 - e^{-t/RC}\left(-\frac{1}{RC}\right)\right] = CV\left(\frac{1}{RC}\right)e^{-t/RC} = \frac{V}{R}e^{-t/RC}$$

(b) When $C = 10^{-5}$ farads, $R = 10^7$ ohms, and $V = 10$ volts, after 200 seconds

$I_c = \frac{10}{10^7}e^{-200/(10^7 \cdot 10^{-5})} \approx 1.35 \times 10^{-7}$ amps

4.5 Derivatives of Logarithmic Functions

2. $y = \ln(-4x)$

$$\frac{dy}{dx} = \frac{d}{dx}[\ln(-4x)] = \frac{d}{dx}[\ln 4 + \ln(-x)] = \frac{d}{dx}\ln 4 + \frac{d}{dx}\ln(-x) = 0 + \frac{-1}{-x} = \frac{1}{x}$$

4. $y = \ln(1 + x^2)$
$g(x) = 1 + x^2$
$g'(x) = 2x$

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{2x}{1+x^2}$$

6. $$y = \ln \left|-8x^2 + 6x\right|$$
$$g(x) = -8x^2 + 6x$$
$$g'(x) = -16x + 6$$
$$\frac{dy}{dx} = \frac{-16x+6}{-8x^2+6x} = \frac{2(-8x+3)}{2(-4x^2+3x)} = \frac{-8x+3}{-4x^2+3x}$$

8. $$y = \ln \sqrt{2x+1} = \ln (2x+1)^{1/2}$$
$$g(x) = (2x+1)^{1/2}$$
$$g'(x) = \frac{1}{2}(2x+1)^{-1/2}(2) = (2x+1)^{-1/2}$$
$$\frac{dy}{dx} = \frac{(2x+1)^{-1/2}}{(2x+1)^{1/2}} = \frac{1}{2x+1}$$

10. $$y = \ln (5x^3 - 2x)^{3/2} = \frac{3}{2} \ln (5x^3 - 2x)$$
$$\frac{dy}{dx} = \frac{3}{2} D_x \ln (5x^3 - 2x)$$
$$g(x) = 5x^3 - 2x$$
$$g'(x) = 15x^2 - 2$$
$$\frac{dy}{dx} = \frac{3}{2}\left(\frac{15x^2-2}{5x^3-2x}\right) = \frac{3(15x^2-2)}{2(5x^3-2x)}$$

12. $$y = (3x+1) \ln (x-1)$$

Use the product rule.

$$\frac{dy}{dx} = (3x+1)\frac{1}{x-1} + 3 \ln (x-1) = \frac{3x+1}{x-1} + 3 \ln (x-1)$$

14. $$y = x \ln \left|2 - x^2\right|$$

Use the product rule.

$$\frac{dy}{dx} = x\left(\frac{1}{2-x^2}\right)(-2x) + \ln \left|2-x^2\right| = -\frac{2x^2}{2-x^2} + \ln \left|2-x^2\right|$$

16. $$v = \frac{\ln u}{u^3}$$

Use the quotient rule.

$$\frac{dv}{du} = \frac{u^3\left(\frac{1}{u}\right) - (\ln u)(3u^2)}{(u^3)^2} = \frac{u^2 - 3u^2 \ln u}{u^6} = \frac{u^2(1 - 3 \ln u)}{u^6} = \frac{1 - 3 \ln u}{u^4}$$

18. $$y = \frac{-2 \ln x}{3x-1}$$

Use the quotient rule.

$$\frac{dy}{dx} = \frac{(3x-1)(-2)\left(\frac{1}{x}\right) - (-2 \ln x)(3)}{(3x-1)^2} = \frac{\frac{-2(3x-1)}{x} + 6 \ln x}{(3x-1)^2} = \frac{-2(3x-1) + 6x \ln x}{x(3x-1)^2} = \frac{-2(3x - 1 - 3x \ln x)}{x(3x-1)^2}$$

20. $$y = \frac{x^3-1}{2 \ln x}$$
$$\frac{dy}{dx} = \frac{(2 \ln x)(3x^2) - (x^3-1)(2)\left(\frac{1}{x}\right)}{(2 \ln x)^2} = \frac{6x^2 \ln x - \frac{2}{x}(x^3-1)}{(2 \ln x)^2} \cdot \frac{x}{x} = \frac{6x^3 \ln x - 2(x^3-1)}{4x (\ln x)^2} = \frac{3x^3 \ln x - (x^3-1)}{2x (\ln x)^2}$$

22. $$y = \sqrt{\ln |x-3|} = (\ln |x-3|)^{1/2}$$
$$\frac{dy}{dx} = \frac{1}{2}(\ln |x-3|)^{-1/2} \cdot \frac{d}{dx}(\ln |x-3|) = \frac{1}{2(\ln |x-3|)^{1/2}}\left(\frac{1}{x-3}\right) = \frac{1}{2(x-3)(\ln |x-3|)^{1/2}} = \frac{1}{2(x-3)\sqrt{\ln |x-3|}}$$

24. $y = (\ln\ 4)(\ln\ |3x|)$

$$\frac{dy}{dx} = (\ln\ 4)\left(\frac{1}{3x}\right)(3) = \frac{3\ \ln\ 4}{3x} = \frac{\ln\ 4}{x}$$

(Recall that ln 4 is a constant.)

26. $y = e^{2x-1}\ \ln\,(2x-1)$

$$\frac{dy}{dx} = (2)e^{2x-1}\ \ln\,(2x-1) + (e^{2x-1})\left(\frac{1}{2x-1}\right)(2) = 2e^{2x-1}\ \ln\,(2x-1) + \frac{2e^{2x-1}}{2x-1}$$

28. $p(y) = \dfrac{\ln\ y}{e^y}$

$$p'(y) = \frac{e^y \cdot \frac{1}{y} - (\ln\ y)e^y}{(e^y)^2} = \frac{e^y - y(\ln\ y)e^y}{ye^{2y}} = \frac{e^y(1 - y\ \ln\ y)}{ye^{2y}} = \frac{1 - y\ \ln\ y}{ye^y}$$

30. $y = \log\,(6x)$

$g(x) = 6x$ and $g'(x) = 6$.

$$\frac{dy}{dx} = \frac{1}{\ln\ 10}\left(\frac{6}{6x}\right) = \frac{1}{x\ \ln\ 10}$$

32. $y = \log\ |1 - x|$

$g(x) = 1 - x$ and $g'(x) = -1$.

$$\frac{dy}{dx} = \frac{1}{\ln\ 10} \cdot \frac{-1}{1-x} = -\frac{1}{(\ln\ 10)(1-x)} \quad \text{or} \quad \frac{1}{(\ln\ 10)(x-1)}$$

34. $y = \log_5\ \sqrt{5x+2}$

$g(x) = \sqrt{5x+2}$ and $g'(x) = \dfrac{5}{2\sqrt{5x+2}}$.

$$\frac{dy}{dx} = \frac{1}{\ln\ 5} \cdot \frac{\frac{5}{2\sqrt{5x+2}}}{\sqrt{5x+2}} = \frac{5}{2\ \ln\ 5\ (5x+2)}$$

36. $y = \log_3\,(x^2+2x)^{3/2}$

$g(x) = (x^2+2x)^{3/2}$ and

$$g'(x) = \frac{3}{2}(x^2+2x)^{1/2} \cdot (2x+2) = 3(x+1)(x^2+2x)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{\ln\ 3} \cdot \frac{3(x+1)(x^2+2x)^{1/2}}{(x^2+2x)^{3/2}} = \frac{3(x+1)}{(\ln\ 3)(x^2+2x)}$$

38. $w = \log_8\,(6^p - 1)$

$g(p) = 6^p - 1$ and $g'(p) = (\ln 6)6^p$.

$$\frac{dw}{dp} = \frac{1}{\ln\ 8} \cdot \frac{(\ln\ 6)\ 6^p}{6^p - 1} = \frac{(\ln 6)6^p}{(\ln\ 8)(6^p - 1)}$$

44. Use the derivative of $\ln x$.

$$\frac{d\ln[u(x)v(x)]}{dx} = \frac{1}{u(x)v(x)} \cdot \frac{d[u(x)v(x)]}{dx}$$

$$\frac{d\ln u(x)}{dx} = \frac{1}{u(x)} \cdot \frac{d[u(x)]}{dx}$$

$$\frac{d\ln v(x)}{dx} = \frac{1}{v(x)} \cdot \frac{d[v(x)]}{dx}$$

Then since $\ln[u(x)v(x)] = \ln u(x) + \ln v(x)$,

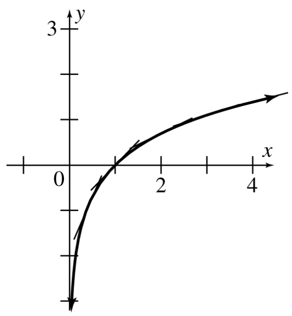
$$\frac{1}{u(x)v(x)} \cdot \frac{d[u(x)v(x)]}{dx} = \frac{1}{u(x)} \cdot \frac{d[u(x)]}{dx} + \frac{1}{v(x)} \cdot \frac{d[v(x)]}{dx}.$$

Multiply both sides of this equation by $u(x)v(x)$. Then

$$\frac{d[u(x)v(x)]}{dx} = v(x)\frac{d[u(x)]}{dx} + u(x)\frac{d[v(x)]}{dx}.$$

This is the product rule.

46. Graph the function $y = \ln x$. Sketch lines tangent to the graphat $x = \frac{1}{2}$, 1, 2, 3, 4.



Estimate the slopes of the tangent lines at these points

x	slope of tangent
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
4	$\frac{1}{4}$

The values of the slopes at x are $\frac{1}{x}$.

Thus we see that $\frac{d \ln x}{dx} = \frac{1}{x}$.

48. $R(x) = 30 \ \ln (2x+1)$

$$C(x) = \frac{x}{2}$$

$$P(x) = R(x) - C(x) = 30 \ln (2x+1) - \frac{x}{2}$$

The profit will be a maximum when the derivative of the profit function is equal to 0.

$$P'(x) = 30\left(\frac{1}{2x+1}\right)(2) - \frac{1}{2} = \frac{60}{2x+1} - \frac{1}{2}$$

Now, $P'(x) = \dfrac{60}{2x+1} - \dfrac{1}{2} = 0$

when

$$\frac{60}{2x+1} = \frac{1}{2}$$
$$120 = 2x + 1$$
$$\frac{119}{2} = x.$$

Thus, a maximum profit occurs when $x = \frac{119}{2}$ or, in a practical sense, when 59 or 60 items are manufactured. (Both 59 and 60 give the same profit.)

50. $C(x) = 100x + 100$

(a) The marginal cost is given by $C'(x)$.

$$C'(x) = 100$$

(b)
$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= x\left(100 + \frac{50}{\ln \ x}\right) \\ &\quad - (100x + 100) \\ &= 100x + \frac{50x}{\ln \ x} - 100x - 100 \\ &= \frac{50x}{\ln \ x} - 100 \end{aligned}$$

(c) The profit from one more unit is is $\frac{dP}{dx}$ for $x = 8$.

$$\frac{dP}{dx} = \frac{(\ln \ x)(50) - 50x\left(\frac{1}{x}\right)}{(\ln \ x)^2}$$
$$= \frac{50 \ \ln \ x - 50}{(\ln \ x)^2} = \frac{50(\ln \ x - 10)}{(\ln \ x)^2}$$

When $x = 8$, the profit from one more unit is

$$\frac{50(\ln \ 8 - 1)}{(\ln \ 8)^2} = \$12.48.$$

(d) The manager can use the information from part (c) to decide whether it is profitable to make and sell additional items.

52. $A(w) = 4.688w^{.8168 - .0154 \log_{10} w}$

(a)
$$\begin{aligned} A(4000) &= 4.688(4000)^{.8168 - .0154 \log_{10} 4000} \\ &\approx 2590 \text{ cm}^2 \end{aligned}$$

(b) $\dfrac{A(w)}{4.688} = w^{.8166 - .0154 \log_{10} w}$

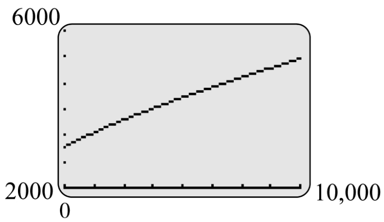
$$\ln A(w) - \ln 4.688 = (\ln w)\left(.8168 - .0154 \frac{\ln w}{\ln 10}\right)$$

$$\begin{aligned} \frac{A'(w)}{A(w)} &= \frac{1}{w}\left(.8168 - .0154 \frac{\ln w}{\ln 10}\right) \\ &\quad + \ln w \left(\frac{-.0154}{\ln 10}\right)\frac{1}{w} \\ &= \frac{.8168}{w} - \frac{.0308}{\ln 10}\frac{\ln w}{w} \\ &= \frac{1}{w}\left(.8168 - \frac{.0308}{\ln 10}\ln w\right) \\ A'(w) &= \frac{1}{w}\left(.8168 - \frac{.0308}{\ln 10}\ln w\right) \\ &\quad \cdot (4.688 w^{.8168 - .0154 \log_{10} w}) \end{aligned}$$

$$A'(4000) \approx .4571 \approx .46 \text{ g/cm}^2$$

When the infant weighs 4000 g, it is gaining .46 square contimeters per gram of weight increase.

(c)



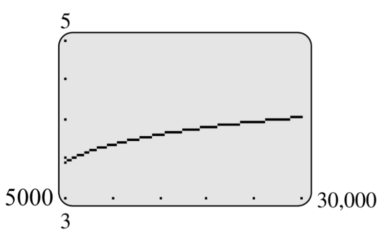
54. $F(x) = .774 + .727 \log(x)$

(a) $$\begin{aligned} F(25{,}000) &= .774 + .727 \log(25{,}000) \\ &= 3.9713\ldots \\ &\approx 4 \text{ kJ/day} \end{aligned}$$

(b) $$\begin{aligned} F'(x) &= .727 \frac{1}{x \ln 10} \\ &= \frac{.727}{\ln 10} x^{-1} \\ F'(25{,}000) &= \frac{.727}{\ln 10} 25{,}000^{-1} \\ &\approx .000012629\ldots \\ &\approx 1.3 \times 10^{-5} \end{aligned}$$

When a fawn is 25 kg in size, the rate of change of the energy expenditure of the fawn is about 1.3×10^{-5} kJ/day per gram.

(c)



56. $M(t) = (.1t + 1) \ln \sqrt{t}$

(a) $$\begin{aligned} M(15) &= [.1(15) + 1] \ln \sqrt{15} \\ &\approx 3.385 \end{aligned}$$

When the temperature is 15°C, the number of matings is about 3.

(b) $$\begin{aligned} M(25) &= [.1(25) + 1] \ln \sqrt{25} \\ &\approx 5.663 \end{aligned}$$

When the temperature is 25°C, the number of matings is about 6.

(c) $$\begin{aligned} M(t) &= (.1t + 1) \ln \sqrt{t} \\ &= (.1t + 1) \ln t^{1/2} \\ M'(t) &= (.1t + 1)\left(\frac{1}{2} \cdot \frac{1}{t}\right) \\ &\quad + (\ln t^{1/2})(.1) \\ &= .1 \ln \sqrt{t} + \frac{1}{2t}(.1t + 1) \\ M'(15) &= .1 \ln \sqrt{15} + \frac{1}{2 \cdot 15}[(.1)(15) + 1] \\ &\approx .22 \end{aligned}$$

When the temperature is 15°C, the rate of change of the number of matings is about .22.

58. $M = \frac{2}{3} \log \frac{E}{.007}$

(a) $$\begin{aligned} 8.9 &= \frac{2}{3} \log \frac{E}{.007} \\ 13.35 &= \log \frac{E}{.007} \\ 10^{13.35} &= \frac{E}{.007} \\ E &= .007(10^{13.35}) \\ &\approx 1.567 \times 10^{11} \text{ kWh} \end{aligned}$$

(b) $$\begin{aligned} &10{,}000{,}000 \times 247 \text{ kWh/month} \\ &= 2{,}470{,}000{,}000 \text{ kWh/month} \end{aligned}$$

$$\frac{1.567 \times 10^{11} \text{ kWh}}{2{,}470{,}000{,}000 \text{ kWh/month}} \approx 63.4 \text{ months}$$

(c) $$\begin{aligned} M &= \frac{2}{3} \log E - \frac{2}{3} \log .007 \\ \frac{dM}{dE} &= \frac{2}{3}\left(\frac{1}{(\ln 10)E}\right) \\ &= \frac{2}{(3 \ln 10)E} \end{aligned}$$

When $E = 70{,}000$,

$$\begin{aligned} \frac{dM}{dE} &= \frac{2}{(3 \ln 10)70{,}000} \\ &\approx 4.14 \times 10^{-6} \end{aligned}$$

(d) $\frac{dM}{dE}$ varies inversely with E, so as E increases, $\frac{dM}{dE}$ decreases and approaches zero.

Chapter 4 Review Exercises

2. $y = x^3 - 4x^2$

$$\frac{dy}{dx} = 3x^2 - 4(2x) = 3x^2 - 8x$$

4. $y = -3x^{-2}$

$$\frac{dy}{dx} = (-3)(-2)x^{-3}$$
$$= 6x^{-3} \quad \text{or} \quad \frac{6}{x^3}$$

6. $f(x) = 6x^{-1} - 2\sqrt{x} = 6x^{-1} - 2(x)^{1/2}$

$$f'(x) = -6(x^{-2}) - 2\left(\frac{1}{2}x^{-1/2}\right)$$
$$= -6x^{-2} - x^{-1/2}$$
$$\text{or} \quad \frac{-6}{x^2} - \frac{1}{x^{1/2}}$$

8. $r(x) = \dfrac{-8}{2x+1} = -8(2x+1)^{-1}$

$$r'(x) = -8[(-1)(2x+1)^{-2}(2)]$$
$$= 16(2x+1)^{-2}$$
$$= \frac{16}{(2x+1)^2}$$

10. $y = \dfrac{2x^3 - 5x^2}{x+2}$

$$\frac{dy}{dx} = \frac{(x+2)(6x^2 - 10x) - (2x^3 - 5x^2)(1)}{(x+2)^2}$$
$$= \frac{6x^3 + 12x^2 - 10x^2 - 20x - 2x^3 + 5x^2}{(x+2)^2}$$
$$= \frac{4x^3 + 7x^2 - 20x}{(x+2)^2}$$

12. $k(x) = (5x-1)^6$

$$k'(x) = 6(5x-1)^5(5)$$
$$= 30(5x-1)^5$$

14. $y = -3\sqrt{8t-1} = -3(8t-1)^{1/2}$

$$\frac{dy}{dx} = -3\left[\frac{1}{2}(8t-1)^{-1/2}(8)\right]$$
$$= -12(8t-1)^{-1/2}$$
$$\text{or} \quad \frac{-12}{(8t-1)^{1/2}}$$

16. $y = 4x^2(3x-2)^5$

$$\frac{dy}{dx} = (4x^2)[5(3x-2)^4(3)] + (3x-2)^5(8x)$$
$$= 60x^2(3x-2)^4 + 8x(3x-2)^5$$
$$= 4x(3x-2)^4[15x + 2(3x-2)]$$
$$= 4x(3x-2)^4(15x + 6x - 4)$$
$$= 4x(3x-2)^4(21x - 4)$$

18. $s(t) = \dfrac{t^3 - 2t}{(4t-3)^4}$

$$s'(t) = \frac{(4t-3)^4(3t^2-2) - (t^3-2t)(4)(4t-3)^3(4)}{[(4t-3)^4]^2}$$
$$= \frac{(4t-3)^4(3t^2-2) - 16(t^3-2t)(4t-3)^3}{(4t-3)^8}$$
$$= \frac{(4t-3)^3[(4t-3)(3t^2-2) - 16(t^3-2t)]}{(4t-3)^8}$$
$$= \frac{(4t-3)^3(12t^3 - 9t^2 - 8t + 6 - 16t^3 + 32t)}{(4t-3)^8}$$
$$= \frac{-4t^3 - 9t^2 + 24t + 6}{(4t-3)^5}$$

20. $g(t) = t^3(t^4+5)^{7/2}$

$$g'(t) = t^3 \cdot \frac{7}{2}(t^4+5)^{5/2}(4t^3) + 3t^2 \cdot (t^4+5)^{7/2}$$
$$= 14t^6(t^4+5)^{5/2} + 3t^2(t^4+5)^{7/2}$$
$$= t^2(t^4+5)^{5/2}[14t^4 + 3(t^4+5)]$$
$$= t^2(t^4+5)^{5/2}(17t^4 + 15)$$

22. $y = 8e^{.5x}$

$$\frac{dy}{dx} = 8(.5e^{.5x}) = 4e^{.5x}$$

24. $y = -4e^{x^2}$

$$g(x) = x^2$$
$$g'(x) = 2x$$
$$\frac{dy}{dx} = (2x)(-4e^{x^2})$$
$$= -8xe^{x^2}$$

26. $y = -7x^2e^{-3x}$

Use the product rule.

$$\frac{dy}{dx} = (-7x^2)(-3e^{-3x}) + e^{-3x}(-14x)$$
$$= 21x^2e^{-3x} - 14xe^{-3x}$$
$$\text{or } 7xe^{-3x}(3x-2)$$

28. $y = \ln(5x+3)$
$g(x) = 5x+3$
$g'(x) = 5$

$$\frac{dy}{dx} = \frac{5}{5x+3}$$

30. $y = \frac{\ln|2x-1|}{x+3}$

$$\frac{dy}{dx} = \frac{(x+3)\left(\frac{2}{2x-1}\right) - (\ln|2x-1|)(1)}{(x+3)^2} \cdot \frac{2x-1}{2x-1}$$
$$= \frac{2(x+3) - (2x-1)\ln|2x-1|}{(2x-1)(x+3)^2}$$

32. $y = \frac{(x^2+1)e^{2x}}{\ln x}$

$$\frac{dy}{dx} = \frac{\ln x[(x^2+1)(2e^{2x}) + (e^{2x})(2x)] - (x^2+1)e^{2x}\left(\frac{1}{x}\right)}{(\ln x)^2} \cdot \frac{x}{x}$$
$$= \frac{x\ln x[2e^{2x}(x^2+1) + 2xe^{2x}] - (x^2+1)e^{2x}}{x(\ln x)^2}$$
$$= \frac{e^{2x}[2x(\ln x)(x^2+1+x) - (x^2+1)]}{x(\ln x)^2}$$

34. $q = (e^{2p+1} - 2)^4$

Use the chain rule.

$$q'\frac{dq}{dp} = 4(e^{2p+1}-2)^3[2e^{2p+1}]$$
$$= 8e^{2p+1}(e^{2p+1}-2)^3$$

36. $y = 10 \cdot 2^{\sqrt{x}}$

$$\frac{dy}{dx} = 10 \cdot (\ln 2) \cdot 2^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2}$$
$$= \frac{5(\ln 2)2^{\sqrt{x}}}{x^{1/2}}$$

38. $h(z) = \log(1+e^z)$

$$h'(z) = \frac{1}{\ln 10} \cdot \frac{e^z}{1+e^z}$$
$$= \frac{e^z}{(\ln 10)(1+e^z)}$$

40. $y = x^2 - 6x$; tangent at $x = 2$

$$\frac{dy}{dx} = 2x - 6$$

Slope $= y'(2) = 2(2) - 6 = -2$

Use $(2, -8)$ and point-slope form.

$$y - (-8) = -2(x-2)$$
$$y + 8 = -2x + 4$$
$$y + 2x = -4$$
$$y = -2x - 4$$

42. $y = \frac{3}{x-1}$; tangent at $x = -1$

$$y = \frac{3}{x-1} = 3(x-1)^{-1}$$
$$\frac{dy}{dx} = 3(-1)(x-1)^{-2}(1)$$
$$= -3(x-1)^{-2}$$

Slope $= y'(-1) = -3(-1-1)^{-2} = -\frac{3}{4}$

Use $\left(-1, -\frac{3}{2}\right)$ and point-slope form.

$$y - \left(-\frac{3}{2}\right) = -\frac{3}{4}[x - (-1)]$$
$$y + \frac{3}{2} = -\frac{3}{4}(x+1)$$
$$y + \frac{6}{4} = -\frac{3}{4}x - \frac{3}{4}$$
$$y = -\frac{3}{4}x - \frac{9}{4}$$

44. $y = \sqrt{6x-2}$; tangent at $x = 3$
$y = \sqrt{6x-2} = (6x-2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(6x-2)^{-1/2}(6)$$
$$= 3(6x-2)^{-1/2}$$

$$\text{slope} = y'(3) = 3(6 \cdot 3 - 2)^{-1/2}$$
$$= 3(16)^{-1/2}$$
$$= \frac{3}{16^{1/2}}$$
$$= \frac{3}{4}$$

Use $(3, 4)$ and point-slope form.

$$y - 4 = \frac{3}{4}(x-3)$$
$$y - \frac{16}{4} = \frac{3}{4}x - \frac{9}{4}$$
$$y = \frac{3}{4}x + \frac{7}{4}$$

46. $y = e^x$; $x = 0$

$$\frac{dy}{dx} = e^x$$

The value of $\frac{dy}{dx}$ when $x = 0$ is the slope $m = e^0 = 1$.
When $x = 0$, $y = e^0 = 1$. Use $m = 1$ with $P(0, 1)$.

$$y - 1 = 1(x - 0)$$
$$y = x + 1$$

48. $y = \ln\, x;\; x = 1$

$$\frac{dy}{dx} = \frac{1}{x}$$

The value of $\frac{dy}{dx}$ when $x = 1$ is the slope $m = \frac{1}{1} = 1$.
When $x = 1,\; y = \ln\; 1 = 0$. Use $m = 1$ with $P(1,\; 0)$.

$$y - 0 = 1(x - 1)$$
$$y = x - 1$$

50. The slope of the graph of $y = x + k$ is 1. First, we find the point on the graph of $f(x) = \sqrt{2x-1}$ at which the slope is also 1.

$$f(x) = (2x - 1)^{1/2}$$
$$f'(x) = \frac{1}{2}(2x - 1)^{-1/2}(2)$$
$$f'(x) = \frac{1}{\sqrt{2x-1}}$$

The slope is 1 when

$$\frac{1}{\sqrt{2x-1}} = 1$$
$$1 = \sqrt{2x-1}$$
$$1 = 2x - 1$$
$$2x = 2$$
$$x = 1,$$

and

$$f(1) = 1.$$

Therefore, at $P(1, 1)$ on the graph of $f(x) = \sqrt{2x-1}$, the slope is 1. An equation of the tangent line is

$$y - 1 = 1(x - 1)$$
$$y - 1 = x - 1$$
$$y = x + 0.$$

Any tangent line intersects the curve in exactly one point.
From this we see that if $k = 0$, there is one point of intersection.

The graph of f is below the line $y = x + 0$. Therefore, if $k > 0$, the graph of $y = x + k$ will not intersect the graph.

Consider the point $Q(\frac{1}{2}, 0)$ on the graph. We find an equation of the line through Q with slope 1.

$$y - 0 = 1\left(x - \frac{1}{2}\right)$$
$$y = x - \frac{1}{2}$$

The line with a slope of 1 through $Q(\frac{1}{2}, 0)$ will intersect the graph in two points. One is Q and the other is some point on the graph to the right of P.
The graph of $y = x + 0$ intersects the graph in one point, while the graph of $y = x - \frac{1}{2}$ intersects it in two points. If we use a value of k in $y = x + k$ with $-\frac{1}{2} < k < 0$, we will have a line with a y-intercept between $-\frac{1}{2}$ and a 0 and a slope of 1 which will intersect the graph in two points.
If k, the y-intercept, is less than $-\frac{1}{2}$, the graph of $y = x + k$ will be below point Q and will intersect the graph of f in exactly one point.

To summarize, the graph of $y = x + k$ will intersect the graph of $f(x) = \sqrt{2x-1}$ in

(1) no points if $k > 0$;

(2) exactly one point if $k = 0$ or if $k < -\frac{1}{2}$;

(3) exactly two points if $-\frac{1}{2} \leq k < 0$.

52. Using the result $\hat{fg} = \hat{f} + \hat{g}$, the total amount of tuition collected goes up by approximately $2\% + 3\% = 5\%$.
Let T = tuition per person before the increase and S = number of students before the increase. Then the new tuition is $1.03T$ and the new numbers of students is $1.02S$, so the total amount of tuition collected is $(1.03T)(1.02S) = 1.0506TS$, which is an increase of 5.06%.

54. $C(x) = \sqrt{3x+2}$

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{\sqrt{3x+2}}{x} = \frac{(3x+2)^{1/2}}{x}$$

$$\overline{C}'(x) = \frac{x\left[\frac{1}{2}(3x+2)^{-1/2}(3)\right] - (3x+2)^{1/2}(1)}{x^2}$$
$$= \frac{\frac{3}{2}x(3x+2)^{-1/2} - (3x+2)^{1/2}}{x^2}$$
$$= \frac{3x(3x+2)^{-1/2} - 2(3x+2)^{1/2}}{2x^2}$$
$$= \frac{(3x+2)^{-1/2}[3x - 2(3x+2)]}{2x^2}$$
$$= \frac{3x - 6x - 4}{2x^2(3x+2)^{1/2}} = \frac{-3x - 4}{2x^2(3x+2)^{1/2}}$$

56. $C(x) = (4x+3)^4$

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{(4x+3)^4}{x}$$

$$\overline{C}'(x) = \frac{x[4(4x+3)^3(4)] - (4x+3)^4(1)}{x^2}$$
$$= \frac{16x(4x+3)^3 - (4x+3)^4}{x^2}$$
$$= \frac{(4x+3)^3[16x - (4x+3)]}{x^2}$$
$$= \frac{(4x+3)^3(12x-3)}{x^2}$$

58. $C(x) = \ln(x+5)$

$$\overline{C}(x) = \frac{\ln(x+5)}{x}$$

$$\overline{C}'(x) = \frac{x\cdot\frac{1}{x+5} - \ln(x+5)\cdot 1}{x^2}$$
$$= \frac{x - (x+5)\ \ln(x+5)}{x^2(x+5)}$$

60. $P(x) = \dfrac{x^2}{x-1}$, where $x > 1$

$$P'(x) = \frac{(x-1)(2x) - (x^2)(1)}{(x-1)^2}$$
$$= \frac{2x^2 - 2x - x^2}{(x-1)^2}$$
$$= \frac{x^2 - 2x}{(x-1)^2}$$

(a) $P'(4) = \dfrac{(4)^2 - 2(4)}{(4-1)^2} = \dfrac{16-8}{9} = \dfrac{8}{9}$

In dollars, this is $\frac{8}{9}(100) = \$88.89$, which represents the approximate increase in profit from selling the fifth unit.

(b) $P'(12) = \dfrac{(12)^2 - 2(12)}{(12-1)^2} = \dfrac{144-24}{121} = \dfrac{120}{121}$

In dollars, this is $\frac{120}{121}(100) = \$99.17$, which represents the approximate increase in profit from selling the thirteenth unit.

(c) $P'(20) = \dfrac{(20)^2 - 2(20)}{(20-1)^2} = \dfrac{400-40}{361} = \dfrac{360}{361}$

In dollars, this is $\frac{360}{361}(100) = \$99.72$, which represents the approximate increase in profit from selling the twenty-first unit.

(d) As the number of units sold increases, the marginal profit increases.

(e) The average profit is defined by

$$\overline{P}(x) = \frac{P(x)}{x} = \frac{\frac{x^2}{x-1}}{x} = \frac{x}{x-1}.$$

The marginal average profit is given by

$$\frac{d}{dx}(\overline{P}(x)) = \frac{(x-1)(1) - (x)(1)}{(x-1)^2}$$
$$= \frac{x-1-x}{(x-1)^2}$$
$$= \frac{-1}{(x-1)^2}.$$

The marginal average profit when 4 units are sold is

$$\frac{d}{dx}(\overline{P}(4)) = \frac{-1}{(4-1)^2} = -\frac{1}{9}.$$

In dollars, this is $-\frac{1}{9}(100) = -\frac{100}{9}$. This indicates that the average profit is going down at a rate of $\frac{100}{9}$, or \$11.11, per unit when 4 units are sold.

62. $A(r) = 1000\left(1 + \dfrac{r}{400}\right)^{48}$

$$A'(r) = 1000\cdot 48\left(1+\frac{r}{400}\right)^{47}\cdot\frac{1}{400}$$
$$= 120\left(1+\frac{r}{400}\right)^{47}$$

$$A'(5) = 120\left(1+\frac{5}{400}\right)^{47} \approx 215.15$$

The balance increases by approximately \$215.15 for every 1% increase in the interest rate when the rate is 5%.

64. $T(r) = \dfrac{\ln\ 2}{\ln\ \left(1+\frac{r}{100}\right)}$

$$T(r) = \ln\ 2\ \left[\ln\ \left(1+\frac{r}{100}\right)\right]^{-1}$$

$$T'(r) = (\ln\ 2)(-1)\left[\ln\ \left(1+\frac{r}{100}\right)\right]^{-2}\cdot\frac{\frac{1}{100}}{1+\frac{r}{100}}$$

$$T'(r) = \frac{-\ln\ 2}{(100+r)\left[\ln\ \left(1+\frac{r}{100}\right)\right]^2}$$

$$T'(5) = -\frac{\ln\ 2}{105\ (\ln\ 1.05)^2} \approx -2.77$$

The doubling time decreases by approximately 2.77 years for every 1% increase in the interest rate when the interest rate is 5%.

66. $G(t) = \dfrac{m\ G_o}{G_o+(m-G_o)e^{-kmt}}$, where $m = 30{,}000$, $G_o = 2000$, and $k = 5\cdot 10^{-6}$.

(a) $$G(t) = \frac{(30{,}000)(2000)}{2000+(30{,}000-2000)e^{-5\cdot 10^{-6}(30{,}000)t}} = \frac{30,000}{1+14e^{-.15t}}$$

(b) $$G(t) = 30,000(1+14e^{-.15t})^{-1}$$
$$G'(t) = -30,000(1+14e^{-.15t})^{-2}(-2.1e^{-.15t}) = \frac{63,000e^{-.15t}}{(1+14e^{-.15t})^2}$$
$$G'(6) = \frac{63,000e^{-.90}}{(1+14e^{-.90})^2} \approx 572$$
$$G(6) = 30,000(1+14e^{-.90})^{-1} \approx 4483$$

The population is 4483, and the rate of growth is 572.

68. $M(t) = 3583e^{-e^{-.020(t-66)}}$

(a) $M(250) = 3583e^{-e^{-.020(250-66)}}$
≈ 3493.76 grams,
or about 3.5 kilograms

(b) As $t\to\infty, -e^{-.020(t-66)}\to 0, e^{-e^{-.020(t-66)}}\to 1$, and $M(t)\to 3583$ grams or about 3.6 kilograms.

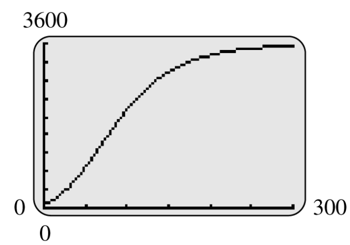
(c) 50% of 3583 is 1791.5.

$$1791.5 = 3583e^{-e^{-.020(t-66)}}$$
$$\ln\left(\frac{1791.5}{3583}\right) = -e^{-.020(t-66)}$$
$$\ln\left(\ln\frac{3583}{1791.5}\right) = -.020(t-66)$$
$$t = -\frac{1}{.020}\ln\left(\ln\frac{3583}{1791.5}\right)+66$$
$$\approx 84 \text{ days}$$

(d) $$D_tM(t) = 3583e^{-e^{-.020(t-66)}}D_t(-e^{-.020(t-66)})$$
$$= 3583e^{-e^{-.020(t-66)}}(-e^{-.020(t-66)})(-.020)$$
$$= 71.66e^{-e^{-.020(t-66)}}(e^{-.020(t-66)})$$

when $t = 250$, $D_tM(t) \approx \$1.76$ g/day.

(e)



Growth is initially rapid, then tapers off.

(f)

Day	Weight	Rate
50	904	24.90
100	2159	21.87
150	2974	11.08
200	3346	4.59
250	3494	1.76
300	3550	.66

70. $f(t) = \dfrac{8}{t+1}+\dfrac{20}{t^2+1}$

(a) The average velocity from $t = 1$ to $t = 3$ is given by

$$\text{average velocity} = \frac{f(3)-f(1)}{3-1} = \frac{\left(\frac{8}{4}+\frac{20}{10}\right)-\left(\frac{8}{2}+\frac{20}{2}\right)}{2} = \frac{4-14}{2} = -5$$

Belmar's average velocity between 1 sec and 3 sec is -5 ft/sec.

(b)
$$\begin{aligned}
f(t) &= 8(t+1)^{-1} + 20(t^2+1)^{-1}\\
f'(t) &= -8(t+1)^{-2}\cdot 1 - 20(t^2+1)^{-2}\cdot 2t\\
&= -\frac{8}{(t+1)^2} - \frac{40t}{(t^2+1)^2}\\
f'(3) &= -\frac{8}{16} - \frac{120}{100}\\
&= -.5 - 1.2\\
&= -1.7
\end{aligned}$$

Belmar's instantaneous velocity at 3 sec is -1.7 ft/sec.

72. **(a)** $N(t) = N_0 e^{-.217t}$, where $t = 1$ and $N_0 = 210$

$$\begin{aligned}
N(1) &= 210e^{-.217(1)}\\
&\approx 169
\end{aligned}$$

The number of words predicted to be in use in 1950 is 169, and the actual number in use was 167.

(b)
$$\begin{aligned}
N(2) &= 210e^{-.217(2)}\\
&\approx 136
\end{aligned}$$

In 2050 the will be about 136 words still being used.

(c)
$$\begin{aligned}
N(t) &= 210e^{-.217t}\\
N'(t) &= 210e^{-.217t}\cdot(-.217)\\
&= -45.57e^{-.217t}\\
N'(2) &= -45.57e^{-.217(2)}\\
&\approx -30
\end{aligned}$$

In the year 2050 the number of words in use will be decreasing by 30 words per millenium.

Chapter 5

GRAPHS AND THE DERIVATIVE

5.1 Increasing and Decreasing Functions

2. By reading the graph, f is

(a) increasing on $(-\infty, 4)$ and

(b) decreasing on $(4, \infty)$.

4. By reading the graph, g is

(a) increasing on $(3, \infty)$ and

(b) decreasing on $(-\infty, 3)$.

6. By reading the graph, h is

(a) increasing on $(1, 5)$ and

(b) decreasing on $(-\infty, 1)$ and $(5, \infty)$.

8. By reading the graph, f is

(a) increasing on $(-3, 0)$ and $(3, \infty)$ and

(b) decreasing on $(-\infty, -3)$ and $(0, 3)$.

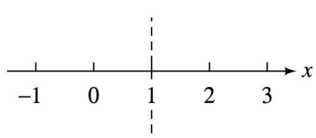
10. $y = .3 + .4x - .2x^2$

(a) $y' = .4 - .4x$

y' is zero when

$$\begin{aligned} .4 - .4x &= 0 \\ x &= 1, \end{aligned}$$

and there are no values of x where y' does not exist, so the only critical number is $x = 1$.



Test a point in each interval.

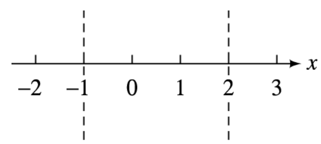
(b) When $x = 0$, $y' = .4 > 0$, so the function is increasing on $(-\infty, 1)$.

(c) When $x = 2$, $y' = -.4 < 0$, so the function is decreasing on $(1, \infty)$.

12. $f(x) = \dfrac{2}{3}x^3 - x^2 - 4x + 2$

(a)
$$\begin{aligned} f'(x) &= 2x^2 - 2x - 4 \\ &= 2(x^2 - x - 2) \\ &= 2(x + 1)(x - 2) \end{aligned}$$

$f'(x)$ is zero when $x = -1$ or $x = 2$, so the critical numbers are -1 and 2.



Test a point in each interval.

$$\begin{aligned} f'(-2) &= 8 > 0 \\ f'(0) &= -4 < 0 \\ f'(3) &= 8 > 0 \end{aligned}$$

(b) f is increasing on $(-\infty, -1)$ and $(2, \infty)$.

(c) f is decreasing on $(-1, 2)$.

14. $f(x) = 4x^3 - 9x^2 - 30x + 6$

(a)
$$\begin{aligned} f'(x) &= 12x^2 - 18x - 30 \\ &= 6(2x^2 - 3x - 5) \\ &= 6(2x - 5)(x + 1) \end{aligned}$$

$f'(x)$ is zero when $x = \frac{5}{2}$ or $x = -1$, so the critical numbers are $\frac{5}{2}$ and -1.

Test a point in each interval.

$$\begin{aligned} f(-2) &= 54 > 0 \\ f(0) &= -30 < 0 \\ f(3) &= 24 > 0 \end{aligned}$$

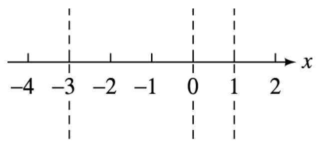
(b) f is increasing on $(-\infty, -1)$ and $\left(\frac{5}{2}, \infty\right)$.

(c) f is decreasing on $\left(-1, \frac{5}{2}\right)$.

16. $f(x) = 3x^4 + 8x^3 - 18x^2 + 5$

(a) $$\begin{aligned} f'(x) &= 12x^3 + 24x^2 - 36x \\ &= 12x(x^2 + 2x - 3) \\ &= 12x(x+3)(x-1) \end{aligned}$$

$f'(x)$ is zero when $x = 0$, $x = -3$, or $x = 1$, so the critical numbers are $0, -3$, and 1.



Test a point in each interval.

$$\begin{aligned} f'(-4) &= -240 < 0 \\ f'(-1) &= 48 > 0 \\ f'\left(\frac{1}{2}\right) &= -\frac{21}{2} < 0 \\ f'(2) &= 120 > 0 \end{aligned}$$

(b) f is increasing on $(-3, 0)$ and $(1, \infty)$.

(c) f is decreasing on $(-\infty, -3)$ and $(0, 1)$.

18. $y = 6x - 9$

(a) $y' = 6 > 0$

There are no critical numbers since y' can never be 0 and always exists.

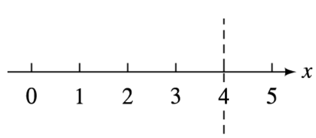
(b) Since y' is always positive, the function is increasing everywhere, or on the interval $(-\infty, \infty)$.

(c) y' *is* never negative, so the function is decreasing on no interval.

20. $f(x) = \dfrac{x+3}{x-4}$

(a) $$\begin{aligned} f'(x) &= \frac{(1)(x-4) - (1)(x+3)}{(x-4)^2} \\ &= \frac{x - 4 - x - 3}{(x-4)^2} = \frac{-7}{(x-4)^2} \end{aligned}$$

f' is never 0, but it fails to exist when $x = 4$. Since 4 is not in the domain of f, 4 is not a critical number. Thus, there are no critical numbers. However, the line $x = 4$ is an asymptote of the graph, so the function might change direction from one side of the asymptote to the other.



$$\begin{aligned} f'(0) &= -\frac{7}{16} < 0 \\ f'(5) &= -7 < 0 \end{aligned}$$

(b) $f'(x)$ is always negative, so $f(x)$ is increasing on no interval.

(c) $f'(x)$ is always negative, so $f(x)$ is decreasing everywhere that is defined. Since $f(x)$ is not defined at $x = 4$, these intervals are $(-\infty, 4)$ and $(4, \infty)$.

22. $y = x\sqrt{9 - x^2} = x(9 - x^2)^{1/2}$

(a) Use the product rule.

$$\begin{aligned} y' &= (1)(9 - x^2)^{1/2} \\ &\quad + \frac{1}{2}(9 - x^2)^{-1/2}(-2x)(x) \\ &= (9 - x^2)^{1/2} - x^2(9 - x^2)^{-1/2} \\ &= (9 - x^2)^{-1/2}(9 - x^2 - x^2) \\ &= (9 - x^2)^{-1/2}(9 - 2x^2) \\ &= \frac{9 - 2x^2}{\sqrt{9 - x^2}} \end{aligned}$$

Critical numbers occur when $y' = 0$ or when y' fails to exist.
$y' = 0$ when

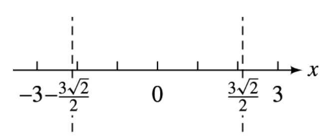
$$\begin{aligned} 9 - 2x^2 &= 0 \\ x &= \frac{\pm 3}{\sqrt{2}} = \frac{\pm 3\sqrt{2}}{2}. \end{aligned}$$

y' fails to exist when

$$\begin{aligned} 9 - x^2 &= 0 \\ x &= \pm 3. \end{aligned}$$

Thus, the critical numbers are $\frac{\pm 3\sqrt{2}}{2}$ and ± 3.

These four values determine three intervals since $f(x)$ is defined only on $[-3, 3]$. Note that $\pm\frac{3\sqrt{2}}{2} \approx \pm 2.12$.



$$\begin{aligned} f'(-2.5) &= -2.11 < 0 \\ f'(0) &= 3 > 0 \\ f'(2.5) &= -2.11 < 0 \end{aligned}$$

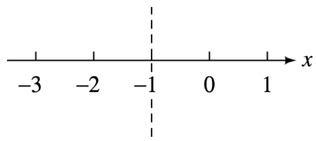
(b) f is increasing on $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

(c) f is decreasing on $\left(-3, -\frac{3\sqrt{2}}{2}\right)$ and $\left(\frac{3\sqrt{2}}{2}, 3\right)$.

24. $f(x) = (x+1)^{4/5}$

(a) $f'(x) = \frac{4}{5}(x+1)^{-1/5} = \frac{4}{5(x+1)^{1/5}}$

$f'(x)$ is never zero, but fails to exist when $x = -1$, so the critical number is -1.



$$f'(-2) = -\frac{4}{5} < 0$$

$$f'(0) = \frac{4}{5} > 0$$

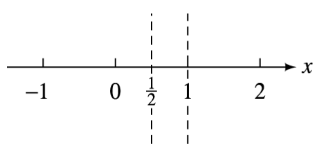
(b) f is increasing on $(-1, \infty)$.

(c) f is decreasing on $(-\infty, -1)$.

26. $y = xe^{x^2-3x}$

(a)
$$\begin{aligned} y' &= xe^{x^2-3x}(2x-3) + e^{x^2-3x}(1) \\ &= e^{x^2-3x}[x(2x-3)+1] \\ &= e^{x^2-3x}(2x^2-3x+1) \\ &= e^{x^2-3x}(2x-1)(x-1) \end{aligned}$$

y' is zero when $x = \frac{1}{2}$ or $x = 1$, so the critical numbers are $\frac{1}{2}$ and 1.



Test a point in each interval.

$$f'(0) = 1(-1)(-1) = 1 > 0$$

$$f'(.6) = e^{-1.44}(.2)(-.4) = \frac{-.08}{e^{1.44}} < 0$$

$$f'(2) = e^{-2}(3)(1) = \frac{3}{e^2} > 0$$

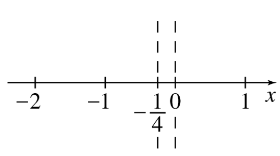
(b) The function is increasing on $\left(-\infty, \frac{1}{2}\right)$ and $(1, \infty)$.

(c) The function is decreasing on $\left(\frac{1}{2}, 1\right)$.

28. $y = x^{1/3} + x^{4/3}$

(a) $y' = \frac{1}{3}x^{-2/3} + \frac{4}{3}x^{1/3} = \frac{1+4x}{3x^{2/3}}$

$y' = 0$ when $x = -\frac{1}{4}$. The derivative does not exist at $x = 0$. So the critical numbers are $-\frac{1}{4}$ and 0.



Test a number in each interval.

$$y'(-1) = -1 < 0$$

$$y'\left(-\frac{1}{8}\right) = \frac{2}{3} > 0$$

$$y'(1) = \frac{5}{3} > 0$$

(b) The derivative is positive on $\left(-\frac{1}{4}, 0\right)$ and $(0, \infty)$ and the function is defined at $x = 0$. Thus, y is increasing on $\left(-\frac{1}{4}, \infty\right)$.

(c) y is decreasing on $\left(-\infty, -\frac{1}{4}\right)$.

30. $f(x) = ax^2 + bx + c,\ a > 0$
$f'(x) = 2ax + b$

Let $f'(x) = 0$ to find the critical number.

$$\begin{aligned} 2ax + b &= 0 \\ 2ax &= -b \\ x &= \frac{-b}{2a} \end{aligned}$$

Choose a value in the interval $\left(-\infty, \frac{-b}{2a}\right)$. Since $a > 0$,

$$\frac{-b}{2a} - \frac{1}{2a} = \frac{-b-1}{2a} < \frac{-b}{2a}.$$

$$\begin{aligned} f'\left(\frac{-b-1}{2a}\right) &= 2a\left(\frac{-b-1}{2a}\right) + b \\ &= -1 < 0 \end{aligned}$$

Choose a value in the interval $\left(\frac{-b}{2a}, \infty\right)$. Since $a > 0$,

$$\frac{-b}{2a} + \frac{1}{2a} = \frac{-b+1}{2a} > \frac{-b}{2a}.$$

$$f'\left(\frac{-b+1}{2a}\right) = 1 > 0.$$

$f(x)$ is increasing on $\left(\frac{-b}{2a}, \infty\right)$ and decreasing on $\left(-\infty, \frac{-b}{2a}\right)$.
This tells us that the curve opens upward and $x = \frac{-b}{2a}$ is the x-coordinate of the vertex.

$$\begin{aligned} f\left(\frac{-b}{2a}\right) &= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c \\ &= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c \\ &= \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a} \\ &= \frac{4ac-b^2}{4a} \end{aligned}$$

The vertex is $\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$ or $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

32. $f(x) = e^x$
$f'(x) = e^x > 0$

$f(x) = e^x$ is increasing on $(-\infty, \infty)$.
$f(x) = e^x$ is decreasing nowhere.
Since $f'(x)$ is always positive, it is never equal zero. Therefore, the tangent line is horizontal nowhere.

34. $H(r) = \dfrac{300}{1 + .03r^2} = 300(1 + .03r^2)^{-1}$

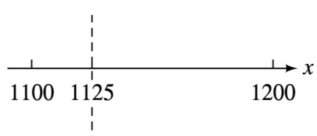
$$H'(r) = 300[-1(1 + .03r^2)^{-2}(.06r)]$$
$$= \frac{-18r}{(1 + .03r^2)^2}$$

Since r is a mortgage rate (in percent), it is always positive. Thus, $H'(r)$ is always negative.

(a) H is increasing on nowhere.

(b) H is decreasing on $(0, \infty)$.

36. $C(x) = 4.8x - .0004x^2,\ 0 \le x \le 2250$
$R(x) = 8.4x - .002x^2,\ 0 \le x \le 2250$
$$P(x) = R(x) - C(x)$$
$$= 8.4x - .002x^2 - (4.8x - .0004x^2)$$
$$= 3.6x - .0016x^2$$
$$P'(x) = 3.6 - .0032x$$
$P'(x) = 0$ when
$$x = \frac{3.6}{.0032}$$
$$= 1125.$$



$P'(0) = 3.6 > 0$
$P'(1200) = -.24 < 0$

P is increasing on $(0, 1125)$.

38. **(a)** These curves are graphs of functions since they all pass the vertical line test.

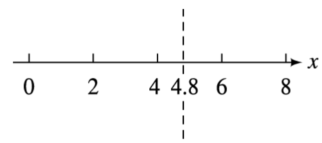
(b) The graph for particulates increases from April to July; it decreases from July to November; it is constant from January to April and November to December.

(c) All graphs are constant from January to April and November to December. When the temperature is low, as it is during these months, air pollution is greatly reduced.

40. $A(x) = -.015x^3 + 1.058x$
$A'(x) = -.045x^2 + 1.058$
$A'(x) = 0$ when

$$-.045x^2 + 1.058 = 0$$
$$-.045x^2 = -1.058$$
$$x^2 \approx 23.5$$
$$x \approx \pm 4.8.$$

The function only applies for the interval $[0, 8]$, so we disregard the solution -4.8.
Then, 4.8 divides $[0, 8]$ into two intervals.



$A'(4) = .338 > 0$
$A'(5) = -.067 < 0$

(a) A is increasing on the interval $(0, 4.8)$.

(b) A is decreasing on the interval $(4.8, 8)$.

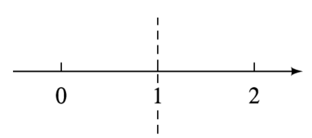
42. $K(t) = \dfrac{5t}{t^2 + 1}$

$$K'(t) = \frac{5(t^2 + 1) - 2t(5t)}{(t^2 + 1)^2}$$
$$= \frac{5t^2 + 5 - 10t^2}{(t^2 + 1)^2}$$
$$= \frac{5 - 5t^2}{(t^2 + 1)^2}$$

$K'(t) = 0$ when

$$\frac{5 - 5t^2}{(t^2 + 1)^2} = 0$$
$$5 - 5t^2 = 0$$
$$5t^2 = 5$$
$$t = \pm 1.$$

Since t is the time after a drug is administered, the function applies only for $[0, \infty)$, so we discard $t = -1$. Then 1 divides the interval into two intervals.



$K'(.5) = 2.4 > 0$
$K'(2) = -.6 < 0$

(a) K is increasing on $(0, 1)$.

(b) K is decreasing on $(1, \infty)$.

44. **(a)** $F(t) = -10.28 + 175.9te^{-t/1.3}$

$$\begin{aligned} F'(t) &= (175.9)(e^{-t/1.3}) \\ &\quad + (175.9.9t)\left(-\frac{1}{1.3}e^{-t/1.3}\right) \\ &= (175.9)(e^{-t/1.3})\left(1 - \frac{t}{1.3}\right) \\ &\approx 175.9e^{-t/1.3}(1 - .769t) \end{aligned}$$

(b) $F'(t)$ is equal to 0 at $t = 1.3$. Therefore, 1.3 is a critical number. Since the domain is $(0, \infty)$, test values in the intervals from $(0, 1.3)$ and $(1.3, \infty)$.

$$F'(1) \approx 18.83 > 0 \text{ and } F'(2) \approx -20.32 < 0$$

$F'(t)$ is increasing on $(0, 1.3)$ and decreasing on $(1.3, \infty)$.

46. $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{2\pi}}e^{-x^2/2}(-x) \\ &= \frac{-x}{\sqrt{2\pi}}e^{-x^2/2} \end{aligned}$$

$f'(x) = 0$ when $x = 0$.
Choose a value from each of the intervals $(-\infty, 0)$ and $(0, \infty)$.

$$\begin{aligned} f'(-1) &= \frac{1}{\sqrt{2\pi}}e^{-1/2} > 0 \\ f'(1) &= \frac{-1}{\sqrt{2\pi}}e^{-1/2} < 0 \end{aligned}$$

The function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

48. As shown on the graph,

(a) horsepower increases with engine speed on $(1000, 6100)$;

(b) horsepower decreases with engine speed on $(6100, 6500)$;

(c) torque increases with engine speed on $(1000, 3000)$ and $(3600, 4200)$;

(d) torque decreases with engine speed on $(3000, 3600)$ and $(4200, 6500)$.

5.2 Relative Extrema

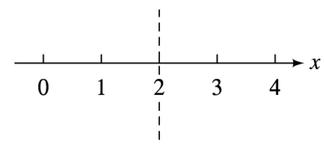
2. As shown on the graph, the relative maximum of 1 occurs when $x = 4$.

4. As shown on the graph, the relative maximum of -4 occurs when $x = 3$.

6. As shown on the graph, the relative minimum of -6 occurs when $x = 1$ and the relative maximum of 2 occurs when $x = 5$.

8. As shown on the graph, the relative maximum of 4 occurs when $x = 0$; the relative minimum of 0 occurs when $x = -3$ and $x = 3$.

10. $f(x) = x^2 - 4x + 6$
$f'(x) = 2x - 4 = 2(x - 2)$
$f'(x)$ is zero when $x = 2$.



$f'(0) = 2(0) - 4 = -4 < 0$
$f'(3) = 2(3) - 4 = 2 > 0$

Thus, $f(x)$ is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$, so $f(2)$ is a relative minimum.

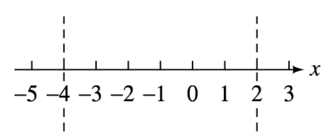
$$f(2) = (2)^2 - 4(2) + 6 = 4 - 8 + 6 = 2$$

Relative minimum of 2 at 2

12. $f(x) = x^3 + 3x^2 - 24x + 2$

$$\begin{aligned} f'(x) &= 3x^2 + 6x - 24 \\ &= 3(x^2 + 2x - 8) \\ &= 3(x + 4)(x - 2) \end{aligned}$$

$f'(x)$ is zero when $x = -4$ and $x = 2$.



$f'(-5) = 21 > 0$
$f'(0) = -24 < 0$
$f'(3) = 21 > 0$

f is increasing on $(-\infty, -4)$ and decreasing on $(-4, 2)$. Thus, a relative maximum occurs at $x = -4$. $f(x)$ is decreasing on $(-4, 2)$ and increasing on $(2, \infty)$. Thus, a relative minimum occurs at $x = 2$.

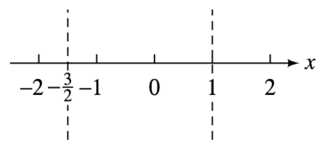
$$\begin{aligned} f(-4) &= (-4)^3 + 3(-4)^2 - 24(-4) + 2 \\ &= 82 \\ f(2) &= (2)^3 + 3(2)^2 - 24(2) + 2 \\ &= -26 \end{aligned}$$

Relative maximum of 82 at -4; relative minimum of -26 at 2

14. $f(x) = -\frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x - 4$

$$\begin{aligned} f'(x) &= -2x^2 - x + 3 \\ &= -(2x^2 + x - 3) \\ &= -(2x + 3)(x - 1) \end{aligned}$$

$f'(x)$ is zero when $x = -\frac{3}{2}$ or $x = 1$.



$$\begin{aligned} f'(-2) &= -3 < 0 \\ f'(0) &= 3 > 0 \\ f'(2) &= -7 < 0 \end{aligned}$$

f is decreasing on $\left(-\infty, -\frac{3}{2}\right)$ and increasing on $\left(-\frac{3}{2}, 1\right)$. Thus, a relative minimum occurs at $x = -\frac{3}{2}$. $f(x)$ is increasing on $\left(-\frac{3}{2}, 1\right)$ and decreasing on $(1, \infty)$. Thus, a relative maximum occurs at $x = 1$.

$$f\left(-\frac{3}{2}\right) = -\frac{59}{8}$$

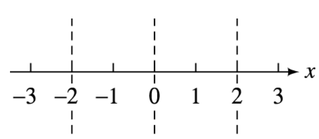
$$f(1) = -\frac{13}{6}$$

Relative maximum of $-\frac{13}{6}$ at 1; relative minimum of $-\frac{59}{8}$ at $-\frac{3}{2}$

16. $f(x) = x^4 - 8x^2 + 9$

$$\begin{aligned} f'(x) &= 4x^3 - 16x \\ &= 4x(x^2 - 4) \\ &= 4x(x + 2)(x - 2) \end{aligned}$$

$f'(x)$ is zero when $x = 0$ or $x = -2$ or $x = 2$.



$$\begin{aligned} f'(-3) &= -60 < 0 \\ f'(-1) &= 12 > 0 \\ f'(1) &= -12 < 0 \\ f(3) &= 60 > 0 \end{aligned}$$

f is increasing on $(-2, 0)$ and $(2, \infty)$; f is decreasing on $(-\infty, -2)$ and $(0, 2)$.

$$\begin{aligned} f(-2) &= -7 \\ f(0) &= 9 \\ f(2) &= -7 \end{aligned}$$

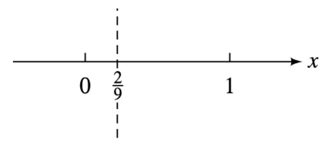
Relative maximum of 9 at 0; relative minimum of -7 at -2 and 2

18. $f(x) = (2 - 9x)^{2/3}$

$$\begin{aligned} f'(x) &= \frac{2}{3}(2 - 9x)^{-1/3}(-9) \\ &= \frac{-6}{(2 - 9x)^{1/3}} \end{aligned}$$

Critical number:

$$\begin{aligned} 2 - 9x &= 0 \\ x &= \frac{2}{9} \end{aligned}$$



$$\begin{aligned} f'(0) &= -4.76 < 0 \\ f'(1) &= 3.14 > 0 \end{aligned}$$

f is decreasing on $\left(-\infty, \frac{2}{9}\right)$ and increasing on $\left(\frac{2}{9}, \infty\right)$.

$$f\left(\frac{2}{9}\right) = 0$$

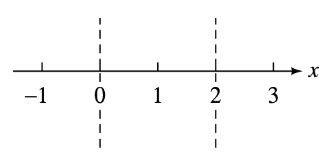
Relative minimum of 0 at $\frac{2}{9}$

20. $f(x) = 3x^{5/3} - 15x^{2/3}$

$$\begin{aligned} f'(x) &= 5x^{2/3} - 10x^{-1/3} \\ &= 5x^{-1/3}(x - 2) \\ &= \frac{5(x - 2)}{x^{1/3}} \end{aligned}$$

Find the critical numbers:

$$\begin{aligned} 5(x - 2) &= 0 \qquad & x^{1/3} &= 0 \\ x &= 2 & x &= 0 \end{aligned}$$



$$\begin{aligned} f'(-1) &= 15 > 0 \\ f'(1) &= -5 < 0 \\ f'(3) &= 3.47 > 0 \end{aligned}$$

f is increasing on $(-\infty, 0)$ and $(2, \infty)$.
f is decreasing on $(0, 2)$.

$$\begin{aligned} f(x) &= 3x^{2/3}(x - 5) \\ f(0) &= 3 \cdot 0(0 - 5) = 0 \\ f(2) &= 3 \cdot 2^{2/3}(2 - 5) \\ &= -9 \cdot 2^{2/3} \approx -14.287 \end{aligned}$$

Relative maximum of 0 at 0; relative minimum of $-9 \cdot 2^{2/3} \approx -14.287$ at 2

22. $f(x) = x^2 + \dfrac{1}{x}$

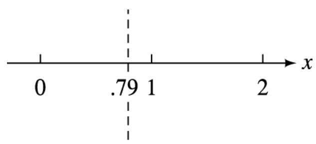
$$f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2}$$

Find the critical number:

$$2x^3 - 1 = 0$$

$$x = \frac{1}{\sqrt[3]{2}} = \frac{\sqrt[3]{4}}{2} \approx .79$$

Note that both $f(x)$ and $f'(x)$ do not exist at $x = 0$, so 0 is not a critical number.



$$f'(-1) = -3 < 0$$
$$f'(1) = 1 > 0$$

$f(x)$ is decreasing on $\left(-\infty, \frac{\sqrt[3]{4}}{2}\right)$ and increasing on $\left(\frac{\sqrt[3]{4}}{2}, \infty\right)$.

$$f\left(\frac{\sqrt[3]{4}}{2}\right) = \left(\frac{\sqrt[3]{4}}{2}\right)^2 + \left(\frac{\sqrt[3]{4}}{2}\right)^{-1}$$
$$= \left(\frac{\sqrt[3]{4}}{2}\right)^{-1}\left[\left(\frac{\sqrt[3]{4}}{2}\right)^3 + 1\right]$$
$$= \frac{2}{\sqrt[3]{4}}\left(\frac{4}{8} + 1\right) = \frac{2}{\sqrt[3]{4}}\left(\frac{3}{2}\right) \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}}$$
$$= \frac{3\sqrt[3]{2}}{2} \approx 1.890$$

Relative minimum of $\frac{3\sqrt[3]{2}}{2} \approx 1.890$ at $\frac{\sqrt[3]{4}}{2}$

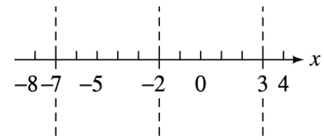
24. $f(x) = \dfrac{x^2 - 6x + 9}{x + 2}$

$$f'(x) = \frac{(2x - 6)(x + 2) - (1)(x^2 - 6x + 9)}{(x + 2)^2}$$
$$= \frac{x^2 + 4x - 21}{(x + 2)^2}$$
$$= \frac{(x + 7)(x - 3)}{(x + 2)^2}$$

Find the critical numbers:

$$(x + 7)(x - 3) = 0$$
$$x = -7 \quad \text{or} \quad x = 3$$

Note that $f(x)$ and $f'(x)$ do not exist at $x = -2$, so the only critical numbers are -7 and 3.



$$f'(-8) = \frac{11}{36} > 0$$
$$f'(-3) = -24 < 0$$
$$f'(0) = -\frac{21}{4} < 0$$
$$f'(4) = \frac{11}{36} > 0$$

f is increasing on $(-\infty, -7)$ and $(3, \infty)$.
f is decreasing on $(-7, -2)$ and $(-2, 3)$.

$$f(-7) = -20$$
$$f(3) = 0$$

Relative maximum of -20 at -7; relative minimum of 0 at 3

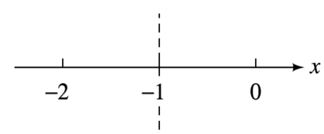
26. $f(x) = 3xe^x + 2$

$$f'(x) = 3xe^x + 3e^x = 3e^x(x + 1)$$

Find the critical numbers:

$$3e^x = 0 \quad \text{or} \quad x + 1 = 0$$
$$e^x = 0 \quad \text{or} \quad x = -1$$

e^x is always positive, so the only critical number is -1.



$$f'(-2) = 3e^{-2}(-2 + 1) = -3e^{-2} < 0$$
$$f'(0) = 3e^0(0 + 1) = 3 > 0$$

f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

$$f(-1) = 3(-1)e^{-1} + 2 = \frac{-3}{e} + 2 \approx .9$$

Relative minimum of .90 at -1

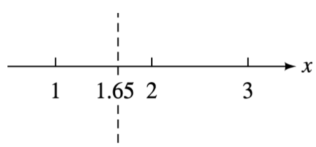
28. $f(x) = \dfrac{x^2}{\ln x}$

$$f'(x) = \frac{(\ln x)\,2x - x^2\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{2x \ln x - x}{(\ln x)^2}$$
$$= \frac{x(2 \ln x - 1)}{(\ln x)^2}$$

Find the critical numbers:

$$x = 0 \quad \text{or} \quad 2 \ln x - 1 = 0 \qquad \text{or} \quad \ln x = 0$$
$$2 \ln x = 1 \qquad\qquad x = 1$$
$$\ln x = \frac{1}{2}$$
$$x = e^{1/2}$$
$$= \sqrt{e}$$
$$\approx 1.65$$

Since the domain of $y = \ln x$ is $(0, \infty)$, 0 is not a critical number. Since $\ln 1 = 0$, we see that 1 is also not in the domain of f. Thus $\sqrt{e} \approx 1.65$ is the only critical number.



$$f'(1.5) = \frac{1.5(2 \ln 1.5 - 1)}{(\ln 1.5)^2} < 0$$

$$f'(2) = \frac{2(2 \ln 2 - 1)}{(\ln 2)^2} > 0$$

$f(x)$ is increasing on $(1.65, \infty)$ and decreasing on $(1, 1.65)$.

$$f(1.65) = \frac{(1.65)^2}{\ln (1.65)} \approx 5.44$$

Relative minimum of 5.44 at 1.65

30. $y = ax^2 + bx + c$
$y' = 2ax + b$

The vertex occurs when $y' = 0$.

$$2ax + b = 0$$
$$x = \frac{-b}{2a}$$

$$a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c = \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c$$
$$= \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}$$
$$= \frac{4ac - b^2}{4a}.$$

The vertex is at $\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$.

32. $f(x) = -x^5 - x^4 + 2x^3 - 25x^2 + 9x + 12$

$f'(x) = -5x^4 - 4x^3 + 6x^2 - 50x + 9$

Graph f' on a graphing calculator. A suitable choice for the viewing window is $[-4, 4]$ by $[-100, 100]$, Yscl = 20.
Use the calculator to estimate the x-intercepts of this graph. These numbers are the solutions of the equation $f'(x) = 0$ and thus the critical numbers for f. Rounded to three decimal places, these x-values are .183 and -2.703.
Examine the graph of f' near $x = -2.703$ and $x = .183$. Observe that $f'(x) < 0$ to the left of $x = -2.703$ and $f'(x) > 0$ to the right of $x = 2.703$. Also observe that $f'(x) > 0$ to the left of $x = .183$ and $f'(x) < 0$ to the right of $x = .183$. The first derivative test allows us to conclude that f has a relative minimum at $x = -2.703$ and a relative maximum at $x = .183$.

$$f(.183) \approx 12.821$$
$$f(-2.703) \approx -143.572$$

Relative maximum of 12.821 at .183; relative minimum of -143.572 at -2.703

34. **(a)** When graphing $g(x)$ in the standard window, no graph seems to appear.

(b) $g(x) = \dfrac{1}{x^{12}} - 2\left(\dfrac{1000}{x}\right)^6$

$$g'(x) = \frac{-12}{x^{13}} - 12\left(\frac{1000}{x}\right)^5\left(\frac{-1000}{x^2}\right)$$
$$= \frac{-12}{x^{13}} + \frac{1.2 \times 10^{19}}{x^7}$$
$$= \frac{-12}{x^{13}} + \frac{(1.2 \times 10^{19})x^6}{x^{13}}$$
$$= \frac{-12 + (1.2 \times 10^{19})x^6}{x^{13}}$$

Find the critical numbers:

$$-12 + (1.2 \times 10^{19})x^6 = 0 \qquad \text{or} \quad x^{13} = 0$$
$$(1.2 \times 10^{19})x^6 = 12 \qquad\qquad x = 0$$
$$x^6 = \frac{12}{1.2 \times 10^{19}}$$
$$x^6 = \frac{10}{10^{19}} = 10^{-18}$$
$$x = \sqrt[6]{10^{-18}}$$
$$= \pm .001 \qquad \text{or} \qquad x = 0$$

$$g'(-1) = \frac{-12 + (1.2 \times 10^{19})(-1)^6}{(-1)^{13}} < 0$$
$$g'(-.0001) = \frac{-12 + (1.2 \times 10^{19})(-.0001)^6}{(-.0001)^{13}} > 0$$
$$g'(.0001) = \frac{-12 + (1.2 \times 10^{19})(.0001)^6}{(.0001)^{13}} < 0$$
$$g'(1) = \frac{-12 + (1.2 \times 10^{19})1^6}{1^{13}} > 0$$

$g(.001) = -10^{-36}$; $g(-.001) = -10^{36}$

Minimum of -10^{36} at $x = \pm.001$

36. $C(x) = 25x + 5000$; $p = 80 - .01x$

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= px - C(x) \\ &= (80 - .01x)x - (25x + 5000) \\ &= 55x - .01x^2 - 5000 \end{aligned}$$

(a) Since the graph of P is a parabola that opens downward, we know that its vertex will be a maximum point. To find the x-value of this point, we find the critical number.

$$P'(x) = 55 - .02x$$

$P'(x) = 0$ when

$$\begin{aligned} 55 - .02x &= 0 \\ -.02x &= -55 \\ x &= 2750. \end{aligned}$$

The number of units that produces maximum profit is 2750.

(b) If $x = 2750$,

$$\begin{aligned} p &= 80 - .01(2750) \\ &= 52.50. \end{aligned}$$

The price that produces maximum profit is \$52.50.

(c)
$$\begin{aligned} P(x) &= 55x - .01x^2 - 5000 \\ P(2750) &= 55(2750) - .01(2750)^2 - 5000 \\ &= 70{,}625 \end{aligned}$$

The maximum profit is \$70,625.

38. $P(x) = \ln(-x^3 + 3x^2 + 72x + 1)$, for x in $[0, 10]$.

(a)
$$\begin{aligned} P'(x) &= \frac{-3x^2 + 6x + 72}{-x^3 + 3x^2 + 72x + 1} \\ &= \frac{-3(x-6)(x+4)}{-x^3 + 3x^2 + 72x + 1} \end{aligned}$$

$P'(x) = 0$ when $x = 6$ or $x = -4$, but 4 is not in the domain of $[0, 10]$. $P'(x)$ fails to exist for $x \approx -0.14$ and $x \approx 10.123$, neither of which are in the domain of $[0, 10]$. Thus, the critical number is 6.

Use the first derivative test to verify that $x = 6$ gives a maximum profit.

$$P'(5) = \frac{27}{311} > 0$$
$$P'(7) = -\frac{11}{103} < 0$$

The maximum profit results when 6 units are sold.

(b) $P(6) \approx 5.784$

The maximum profit is about \$5784.

40.
$$\begin{aligned} p &= D(q) = 500qe^{-.0016q^2} \\ R(q) &= pq = (500qe^{-.0016q^2})q \\ &= 500q^2e^{-.0016q^2} \\ R'(q) &= 500e^{-.0016q^2}(2q) + 500q^2e^{-.0016q^2}(-.0016 \cdot 2q) \\ &= (1000q - 1.6q^3)e^{-.0016q^2} \end{aligned}$$

Since $e^{-.0016q^2} > 0$ for all values of q,

$$\begin{aligned} R'(q) &= 0 \\ (1000q - 1.6q^3)e^{-.0016q^2} &= 0 \\ 1000q - 1.6q^3 &= 0 \\ 1.6q^3 &= 1000q \\ q^2 &= 625 \\ q &= 25 \end{aligned}$$

If $q < 25$, $R'(q) > 0$; if $q > 25$, $R'(q) < 0$. So p has a maximum value at $q = 25$.

$$p = D(25) = 500(25)e^{-.0016(25)^2} \approx 4600$$

Maximum revenue occurs when the price is about \$4600 and 25 computer systems are sold.

42.
$$\begin{aligned} a(t) &= .008t^3 - .288t^2 + 2.304t + 7 \\ a'(t) &= .024t^2 - .576t + 2.304 \end{aligned}$$

Set $a' = 0$ and use the quadratic formula to solve for t.

$$\begin{aligned} .024t^2 - .576t + 2.304 &= 0 \\ .024(t^2 - 24t + 96) &= 0 \\ t^2 - 24t + 96 &= 0 \\ t \approx 5.07 \text{ or } t &\approx 18.93 \end{aligned}$$

$t = 5.07 = 5$ hours $+ .07 \cdot 60$ minutes corresponds to 5:04 P.M.
$t = 18.93 = 18$ hours $+ .93 \cdot 60$ minutes corresponds to 6:56 A.M.

44. $M(t) = 369(.93)^t(t)^{.36}$

$$M'(t) = (369)(.93)^t \ln(.93)(t^{.36}) + 369(.93)^t(.36)(t)^{-.64}$$
$$= (369t^{.36})(.93^t \ln .93) + \frac{132.84(.93)^t}{t^{.64}}$$

$M'(t) = 0$ when $t \approx 4.96$.
Verify that $t \approx 4.96$ gives a maximum.

$$M'(4) > 0$$
$$M'(5) < 0$$

Find $M(4.96)$

$$M(4.96) = 369(.93)^{4.96}(4.96)^{.36} \approx 458.22$$

The female moose reaches a maximum weight of about 458.22 kilograms at about 4.96 years.

46. $f(x) = .198x^3 - 1.56x^2 + 2.03x + 14.8$
$f'(x) = .594x^2 - 3.12x + 2.03$

$f'(x) = 0$ when $x \approx .76$ and $x \approx 4.49$.
Since the function is accurate only on the finite domain, $\{0, 1, 2, 3, 4, 5\}$, the critical numbers are $x = 1$ and $x = 4$, the closest numbers in the domain to the real critical numbers.

$f(1) = 15.468$ $\qquad f(4) = 10.632$

Therefore, there is a maximum of about 1550 casualties in 1991 ($x = 1$) and a minimum of about 1060 casualties in 1994 ($x = 4$).

48. $R(t) = \dfrac{20t}{t^2 + 100}$

$$R'(t) = \frac{20(t^2 + 100) - 20t(2t)}{(t^2 + 100)^2} = \frac{2000 - 20t^2}{(t^2 + 100)^2}$$

$R'(t) = 0$ when

$$2000 - 20t^2 = 0$$
$$-20t^2 = -2000$$
$$t^2 = 100$$
$$t = \pm 10.$$

Disregard the negative value.
Use the first derivative test to verify that $t = 10$ gives a maximum rating.

$$R'(9) = .0116 > 0$$
$$R'(11) = -.0086 < 0$$

The film should be 10 minutes long.

5.3 Higher Derivatives, Concavity, and the Second Derivative Test

2. $f(x) = x^3 + 4x^2 + 2$
$f'(x) = 3x^2 + 8x$
$f''(x) = 6x + 8$
$f''(0) = 6(0) + 8 = 8$
$f''(2) = 6(2) + 8 = 20$

4. $f(x) = -x^4 + 2x^3 - x^2$
$f'(x) = -4x^3 + 6x^2 - 2x$
$f''(x) = -12x^2 + 12x - 2$
$f''(0) = -12(0)^2 + 12(0) - 2 = -2$
$f''(2) = -12(2)^2 + 12(2) - 2 = -26$

6. $f(x) = 8x^2 + 6x + 5$
$f'(x) = 16x + 6$
$f''(x) = 16$
$f''(0) = 16$
$f''(2) = 16$

8. $f(x) = \dfrac{-x}{1 - x^2}$

$$f'(x) = \frac{(1 - x^2)(-1) - (-x)(-2x)}{(1 - x^2)^2}$$
$$= \frac{-1 + x^2 - 2x^2}{(1 - x^2)^2}$$
$$= \frac{-1 - x^2}{(1 - x^2)^2}$$
$$f''(x) = \frac{(1-x^2)^2(-2x) - (-1-x^2)(2)(1-x^2)(-2x)}{(1 - x^2)^4}$$
$$= \frac{(1 - x^2)[-2x(1 - x^2) + 4x(-1 - x^2)]}{(1 - x^2)^4}$$
$$= \frac{-2x^3 - 6x}{(1 - x^2)^3}$$
$$= \frac{-2x(x^2 + 3)}{(1 - x^2)^3}$$
$$f''(0) = \frac{-2(0)(0^2 + 3)}{(1 - 0^2)^3}$$
$$= 0$$
$$f''(2) = \frac{-2(2)(4 + 3)}{(1 - 2^2)^3}$$
$$= \frac{-28}{-27}$$
$$= \frac{28}{27}$$

10. $f(x) = \sqrt{2x+9}$
$= (2x+9)^{1/2}$
$f'(x) = \frac{1}{2}(2x+9)^{-1/2} \cdot 2$
$= (2x+9)^{-1/2}$
$f''(x) = -\frac{1}{2}(2x+9)^{-3/2} \cdot 2$
$= -(2x+9)^{-3/2}$
$= \frac{-1}{(2x+9)^{3/2}}$
$f''(0) = \frac{-1}{9^{3/2}}$
$= -\frac{1}{27}$
$f''(2) = -\frac{1}{13^{3/2}}$
$\approx -.0213$

12. $f(x) = -2x^{2/3}$
$f'(x) = -\frac{4}{3}x^{-1/3}$
$f''(x) = \frac{4}{9}x^{-4/3}$ or $\frac{4}{9x^{4/3}}$
$f''(0)$ does not exist.
$f''(2) = \frac{4}{9 \cdot 2^{4/3}} \approx .1764$

14. $f(x) = .5e^{x^2}$
$f'(x) = (.5)(2x)e^{x^2} = xe^{x^2}$
$f''(x) = xe^{x^2}(2x) + e^{x^2}(1) = 2x^2e^{x^2} + e^{x^2}$
or $e^{x^2}(2x^2+1)$
$f''(0) = e^{0^2}[2(0)^2+1] = 1(1) = 1$
$f''(2) = e^{2^2}[2(2)^2+1] = e^4(9) = 9e^4$
≈ 491.4

16. $f(x) = \ln\ x + \frac{1}{x}$
$f'(x) = \frac{1}{x} - \frac{1}{x^2}$
$f''(x) = \frac{-1}{x^2} + \frac{2}{x^3}$
$= \frac{-x}{x^3} + \frac{2}{x^3} = \frac{2-x}{x^3}$
$f''(0)$ does not exist since the denominator is 0.
$f''(2) = \frac{2-2}{2^3} = \frac{0}{8} = 0$

18. $f(x) = 2x^4 - 3x^3 + x^2$
$f'(x) = 8x^3 - 9x^2 + 2x$
$f''(x) = 24x^2 - 18x + 2$
$f'''(x) = 48x - 18$
$f^{(4)}(x) = 48$

20. $f(x) = 3x^5 - x^4 + 2x^3 - 7x$
$f'(x) = 15x^4 - 4x^3 + 6x^2 - 7$
$f''(x) = 60x^3 - 12x^2 + 12x$
$f'''(x) = 180x^2 - 24x + 12$
$f^{(4)}(x) = 360x - 24$

22. $f(x) = \frac{x+1}{x}$
$f'(x) = \frac{(1)(x) - 1(x+1)}{x^2}$
$= -\frac{1}{x^2} = -x^{-2}$
$f''(x) = 2x^{-3}$ or $\frac{2}{x^3}$
$f'''(x) = -6x^{-4}$ or $-\frac{6}{x^4}$
$f^{(4)}(x) = 24x^{-5}$ or $\frac{24}{x^5}$

24. $f(x) = \frac{x}{2x+1}$
$f'(x) = \frac{(1)(2x+1) - (2)(x)}{(2x+1)^2}$
$= \frac{1}{(2x+1)^2} = (2x+1)^{-2}$
$f''(x) = -2(2x+1)^{-3}(2)$
$= -4(2x+1)^{-3}$
$f'''(x) = 12(2x+1)^{-4}(2)$
$= 24(2x+1)^{-4}$
or $\frac{24}{(2x+1)^4}$
$f^{(4)}(x) = -96(2x+1)^{-5}(2)$
$= -192(2x+1)^{-5}$
or $\frac{-192}{(2x+1)^5}$

26. $f(x) = e^x$
$f'(x) = e^x$
$f''(x) = e^x$
$f'''(x) = e^x$
$f^{(n)}(x) = e^x$

28. Concave upward on $(-\infty, 3)$
Concave downward on $(3, \infty)$
Point of inflection at $(3, 7)$

30. Concave upward on $(-2, 6)$
Concave downward on $(-\infty, -2)$ and $(6, \infty)$
Point of inflection at $(-2, -4)$ and $(6, -1)$

32. Concave upward on $(-\infty, 0)$
Concave downward on $(0, \infty)$
No points of inflection

34. $f(x) = 8 - 6x - x^2$
$f'(x) = -6 - 2x$
$f''(x) = -2 < 0$ for all x.

Always concave downward
No points of inflection

36. $f(x) = -x^3 - 12x^2 - 45x + 2$
$f'(x) = -3x^2 - 24x - 45$
$f''(x) = -6x - 24$

$$\begin{aligned} f''(x) = -6x - 24 &> 0 \text{ when} \\ -6(x+4) &> 0 \\ x + 4 &< 0 \\ x &< -4. \end{aligned}$$

Concave upward on $(-\infty, -4)$

$$\begin{aligned} f''(x) = -6x - 24 &< 0 \text{ when} \\ -6(x+4) &< 0 \\ x + 4 &> 0 \\ x &> -4. \end{aligned}$$

Concave downward on $(-4, \infty)$

$$\begin{aligned} f''(x) = -6x - 24 &= 0 \text{ when} \\ -6(x+4) &= 0 \\ x &= -4. \end{aligned}$$

$f(-4) = 54$

Point of inflection at $(-4, 54)$

38. $f(x) = \dfrac{-2}{x+1}$
$= -2(x+1)^{-1}$

$f'(x) = 2(x+1)^{-2}$

$f''(x) = -4(x+1)^{-3} = \dfrac{-4}{(x+1)^3}$

$$\begin{aligned} f''(x) = \frac{-4}{(x+1)^3} &> 0 \text{ when} \\ x + 1 &< 0 \\ x &< -1. \end{aligned}$$

Concave upward on $(-\infty, -1)$

$$\begin{aligned} f''(x) = \frac{-4}{(x+1)^3} &< 0 \text{ when} \\ x + 1 &> 0 \\ x &> -1. \end{aligned}$$

Concave downward on $(-1, \infty)$

$f''(x) \neq 0$ for any value of x; it does not exist when $x = -1$. There is a change of concavity there, but no point of inflection since $f(-1)$ does not exist.

40. $f(x) = -x(x-3)^2$

$$\begin{aligned} f'(x) &= -1(x-3)^2 + 2(x-3)(-x) \\ &= -(x-3)^2 - 2x^2 + 6x \\ f''(x) &= -2(x-3) - 4x + 6 \\ &= -2x + 6 - 4x + 6 \\ &= -6x + 12 \end{aligned}$$

$$\begin{aligned} f''(x) = -6x + 12 &> 0 \text{ when} \\ -6(x-2) &> 0 \\ x - 2 &< 0 \\ x &< 2. \end{aligned}$$

Concave upward on $(-\infty, 2)$

$$\begin{aligned} f''(x) = -6x + 12 &< 0 \text{ when} \\ -6(x-2) &< 0 \\ x - 2 &> 0 \\ x &> 2. \end{aligned}$$

Concave downward on $(2, \infty)$

$f''(x) = -6x + 12 = 0$ when $x = 2$.
$f(2) = -2$

Point of inflection at $(2, -2)$

42. $f(x) = 2e^{-x^2}$

$$\begin{aligned} f'(x) &= 2e^{-x^2}(-2x) = -4xe^{-x^2} \\ f''(x) &= -4xe^{-x^2}(-2x) + e^{-x^2}(-4) \\ &= -4e^{-x^2}(-2x^2 + 1) \end{aligned}$$

$$\begin{aligned} f''(x) = 0 \text{ when } \quad -2x^2 + 1 &= 0 \\ 1 &= 2x^2 \\ \frac{1}{2} &= x^2 \\ \pm\frac{\sqrt{2}}{2} &= x \end{aligned}$$

Check the sign of $f''(x)$ in each of the intervals determined by $x = \frac{\sqrt{2}}{2}$ and $x = -\frac{\sqrt{2}}{2}$ using test points.

$$\begin{aligned} f''(-1) &= -4e^{-(-1)^2}[-2(-1)^2 + 1] \\ &= \frac{-4}{e}(-1) = \frac{4}{e} > 0 \\ f''(0) &= -4e^{-0^2}[-2(0)^2 + 1] \\ &= -4(1) = -4 < 0 \\ f''(1) &= -4e^{-1^2}[-2(1)^2 + 1] \\ &= \frac{-4}{e}(-1) = \frac{4}{e} > 0 \end{aligned}$$

Concave upward on $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, \infty\right)$; concave downward on $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$$f\left(\pm\tfrac{\sqrt{2}}{2}\right) = 2e^{-1/2} = \frac{2}{\sqrt{e}}$$

Points of inflection at $\left(-\frac{\sqrt{2}}{2}, \frac{2}{\sqrt{e}}\right)$ and $\left(\frac{\sqrt{2}}{2}, \frac{2}{\sqrt{e}}\right)$

44. $f(x) = x^{7/3} + 56x^{4/3}$

$$f'(x) = \frac{7}{3}x^{4/3} + \frac{224}{3}x^{1/3}$$

$$f''(x) = \frac{28}{9}x^{1/3} + \frac{224}{9}x^{-2/3} = \frac{28(x+8)}{9x^{2/3}}$$

$f''(x) = 0$ when $x = -8$
$f''(x)$ fails to exist when $x = 0$
Note that both $f(x)$ and $f'(x)$ exist at $x = 0$.
Check the sign of $f''(x)$ in three intervals determined by $x = -8$ and $x = 0$ using test points.

$$f''(-27) = \frac{28(-19)}{9(9)} = -\frac{532}{81} < 0$$

$$f''(-1) = \frac{28(7)}{9(1)} = \frac{196}{9} > 0$$

$$f''(1) = \frac{28(9)}{9(1)} = 28 > 0$$

Concave upward on $(-8, \infty)$; concave downward on $(-\infty, -8)$

$$f(-8) = (-8)^{7/3} + 56(-8)^{4/3} = -128 + 896 = 768$$

Point of inflection at $(-8, 768)$

46. **(a)** The slope of the tangent line to $f(x) = e^x$ as $x \to -\infty$ is close to 0 since the tangent line is almost horizontal, and a horizontal line has a slope of 0.

(b) The slope of the tangent line to $f(x) = e^x$ as $x \to 0$ is close to 1 since the first derivative represents the slope of the tangent line, $f'(x) = e^x$, and $e^0 = 1$.

48. $f(x) = -x^2 - 10x - 25$
$f'(x) = -2x - 10$
$= -2(x+5) = 0$

Critical number: -5
$f''(x) = -2 < 0$ for all x.

The curve is concave downward, which means a relative maximum occurs at $x = -5$.

50. $f(x) = 3x^3 - 3x^2 + 1$
$f'(x) = 9x^2 - 6x$
$= 3x(3x-2) = 0$

Critical numbers: 0 and $\frac{2}{3}$

$f''(x) = 18x - 6$
$f''(0) = -6 < 0$, which means that a relative maximum occurs at $x = 0$.

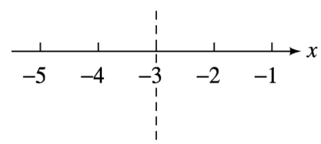
$f''\left(\frac{2}{3}\right) = 6 > 0$, which means that a relative minimum occurs at $x = \frac{2}{3}$.

52. $f(x) = (x+3)^4$
$f'(x) = 4(x+3)^3 = 0$

Critical number: $x = -3$

$f''(x) = 12(x+3)^2$
$f''(-3) = 12(-3+3)^2 = 0$

The second derivative test fails.
Use the first derivative test.



$$f'(-4) = 4(-4+3)^2 = 4(-1)^3 = -4 < 0$$

This indicates that f is decreasing on $(-\infty, -3)$.

$$f'(0) = 4(0+3)^3 = 4(3)^3 = 108 > 0$$

This indicates that f is increasing on $(-3, \infty)$.
A relative minimum occurs at -3.

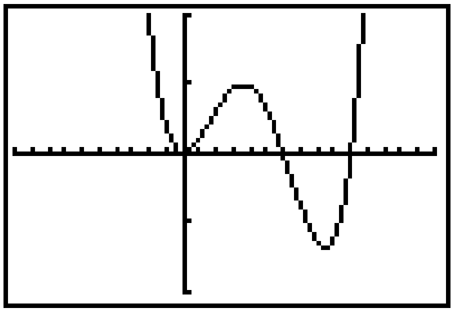
54. There are many examples. The easiest is $f(x) = \sqrt{x}$. This graph is increasing and concave downward.

$$f'(x) = \tfrac{1}{2}x^{-1/2} = \tfrac{1}{2\sqrt{x}}$$

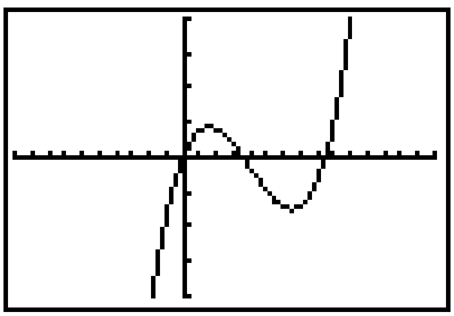
$f'(0)$ does not exist, while $f'(x) > 0$ for all $x > 0$. (Note that the domain of f is $[0, \infty)$.)
As x increases, the value of $f'(x)$ decreases, but remains positive. It approaches zero, but never becomes zero or negative.

56. $f'(x) = 10x^2(x-1)(5x-3)$
$= 10x^2(5x^2 - 8x + 3)$
$= 50x^4 - 80x^3 + 30x^2$
$f''(x) = 200x^3 - 240x^2 + 60x$
$= 20x(10x^2 - 12x + 3)$

Graph f' in the window $[-1, 1.5]$ by $[-2, 2]$, Xscl = .1.



This window does not give a good view of the graph of f'', so we graph f'' in the window $[-1, 1.5]$ by $[-20, 20]$, Xscl = .1. Yscl = 5.



(a) The critical numbers of f are the x-intercepts of the graph of f'. (Note that there are no values where $f'(x)$ does not exist.) From the graph or by examining the factored expression for f', we see that the critical numbers of f are 0, .6, and 1. By either looking at the graph of f' and applying the first derivative test or by looking at the graph of f'' and applying the second derivative test, we see that f has a relative minimum at 1 and a relative maximum at .6.
(At $x = 0$, the second derivative test fails since $f''(0) = 0$, and the first derivative does not change sign, so there is no relative extremum at 0.)

(b) Examine the graph of f' to determine the intervals where the graph lies above and below the x-axis. We see that $f'(x) \geq 0$ on $(-\infty, .6)$, $f'(x) < 0$ on $(.6, 1)$, and $f'(x) > 0$ on $(1, \infty)$. Therefore, f is increasing on $(-\infty, .6)$ and $(1, \infty)$ and decreasing on $(.6, 1)$.

(c) Examine the graph of f''. We see that this graph has three x-intercepts, so there are three values where $f''(x) = 0$. These x-value are 0, about .36, and about .85. Because the sign of f'' and thus the concavity of f changes at these three values, we see that the x-values of the inflection points of the graph of f are 0, about .36, and about .85.

(d) We observe from the graph of f'' that $f''(x) > 0$ on $(0, .36)$ and $(.85, \infty)$, so f is concave upward on the same intervals. Likewise, $f''(x) < 0$ on $(-\infty, 0)$ and $(.36, .85)$, so f is concave downward on the same intervals.

58. **(a)** The left side of the graph changes from concave upward to concave downward at the point of inflection between cellular phones and digital video disc players. The rate of growth of sales begins to decline at the point of inflection.

(b) Food processors are closest to the right-hand point of inflection. This inflection point indicates that the rate of decline of sales is beginning to slow.

60. $R(x) = \frac{4}{27}(-x^3 + 66x^2 + 1050x - 400)$

$0 \leq x \leq 25$

$$R'(x) = \frac{4}{27}(-3x^2 + 132x + 1050)$$
$$R''(x) = \frac{4}{27}(-6x + 132)$$

A point of diminishing returns occurs at a point of inflection, or where $R''(x) = 0$.

$$\frac{4}{27}(-6x + 132) = 0$$
$$-6x + 132 = 0$$
$$6x = 132$$
$$x = 22$$

Test $R''(x)$ to determine whether concavity changes at $x = 22$.

$$R''(20) = \frac{4}{27}(-6 \cdot 20 + 132) = \frac{16}{9} > 0$$
$$R''(24) = \frac{4}{27}(-6 \cdot 24 + 132) = -\frac{16}{9} < 0$$

$R(x)$ is concave upward on $(0, 22)$ and concave downward on $(22, 25)$.

$$R(22) = \frac{4}{27}[-(22)^3+66(22)^2+1060(22)-400] \approx 6517.9$$

The point of diminishing returns is $(22, 6517.9)$.

62. $R(x) = -.6x^3 + 3.7x^2 + 5x,\ 0 \le x \le 6$
$R'(x) = -1.8x^2 + 7.4x + 5$
$R''(x) = -3.6x + 7.4$

A point of diminishing returns occurs at a point of inflection or where $R''(x) = 0$.

$$-3.6x + 7.4 = 0$$
$$-3.6x = -7.4$$
$$x = \frac{-7.4}{-3.6} \approx 2.06$$

Test $R''(x)$ to determine whether concavity changes at $x = 2.05$.

$$R''(2) = -3.6(2) + 7.4 = -7.2 + 7.4 = .2 > 0$$
$$R''(3) = -3.6(3) + 7.4 = -10.8 + 7.4 = -3.4 < 0$$

$R(x)$ is concave upward on $(0, 2.06)$ and concave downward on $(2.06, 6)$.

$$R(2.06) = -.6(2.06)^3 + 3.7(2.06)^2 + 5(2.06) \approx 20.8$$

The point of diminishing returns is $(2.06, 20.8)$.

64. Let $D(q)$ represent the demand function.
The revenue function, $R(q)$, is $R(q) = qD(q)$.
The marginal revenue is given by

$$R'(q) = qD'(q) + D(q)(1) = qD'(q) + D(q).$$
$$R''(q) = qD''(q) + D'(q)(1) + D'(q) = qD''(q) + 2D'(q)$$

gives the rate of decline of marginal revenue.
$D'(q)$ gives the rate of decline of price.
If marginal revenue declines more quickly than price,

$$qD''(q) + 2D'(q) - D'(q) < 0 \quad \text{or} \quad qD''(q) + D'(q) < 0.$$

66. **(a)** $R(t) = t^2(t-18) + 96t + 1000;\ 0 < t < 8$
$= t^3 - 18t^2 + 96t + 1000$
$R'(t) = 3t^2 - 36t + 96$

Set $R'(t) = 0$.

$$3t^2 - 36t + 96 = 0$$
$$t^2 - 12t + 32 = 0$$
$$(t-8)(t-4) = 0$$
$$t = 8 \quad \text{or} \quad t = 4$$

8 is not in the domain of $R(t)$.
$R''(t) = 6t - 36$
$R''(4) = -12 < 0$ implies that $R(t)$ is maximized at $t = 4$, so the population is maximized at 4 hours.

(b) $R(4) = 16(-14) + 96(4) + 1000$
$= -224 + 384 + 1000$
$= 1160$

The maximum population is 1160 million.

68. $K(x) = \dfrac{3x}{x^2+4}$

(a) $K'(x) = \dfrac{3(x^2+4) - (2x)(3x)}{(x^2+4)^2} = \dfrac{-3x^2+12}{(x^2+4)^2} = 0$

$$-3x^2 + 12 = 0$$
$$x^2 = 4$$
$$x = 2 \quad \text{or} \quad x = -2$$

For this application, the domain of K is $[0, \infty)$, so the only critical number is 2.

$$K''(x) = \frac{(x^2+4)^2(-6x) - (-3x^2+12)(2)(x^2+4)(2x)}{(x^2+4)^4} = \frac{-6x(x^2+4) - 4x(-3x^2+12)}{(x^2+4)^3} = \frac{6x^3 - 72x}{(x^2+4)^3}$$

$K''(2) = \frac{-96}{512} = -\frac{3}{16} < 0$ implies that $K(x)$ is maximized at $x = 2$.
Thus, the concentration is a maximum after 2 hours.

(b) $K(2) = \dfrac{3(2)}{(2)^2+4} = \dfrac{3}{4}$

The maximum concentration is $\frac{3}{4}\%$.

70. $G(t) = \dfrac{10{,}000}{1 + 49e^{-.1t}}$

$$G'(t) = \frac{(1 + 49e^{-.1t})(0) - (10{,}000)(-4.9e^{-.1t})}{(1 + 49e^{-.1t})^2} = \frac{49{,}000e^{-.1t}}{(1 + 49e^{-.1t})^2}$$

To find $G''(t)$, apply the quotient rule to find the derivative of $G'(t)$.
The numerator of $G''(t)$ will be

$$\begin{aligned}&(1 + 49e^{-.1t})^2(-4900e^{-.1t})\\ &\quad - (49{,}000e^{-.1t})(2)(1 + 49e^{-.1t})(-4.9e^{-.1t})\\ &\quad = (1 + 49e^{-.1t})(-4900e^{-.1t})\\ &\quad\quad \cdot [(1 + 49e^{-.1t}) - 20(4.9e^{-.1t})]\\ &\quad = (-4900e^{-.1t})[1 + 49e^{-.1t} - 98e^{-.1t}]\\ &\quad = (-4900e^{-.1t})(1 - 49e^{-.1t}).\end{aligned}$$

Thus,

$$G''(t) = \frac{(-4900e^{-.1t})(1 - 49e^{-.1t})}{(1 + 49e^{-.1t})^4}.$$

$G''(t) = 0$ when $-4900e^{-.1t} = 0$ or $1 - 49e^{-.1t} = 0$.
$-4900e^{-.1t} < 0$, and thus never equals zero.

$$\begin{aligned}1 - 49e^{-.1t} &= 0\\ 1 &= 49e^{-.1t}\\ \frac{1}{49} &= e^{-.1t}\\ \ln\left(\frac{1}{49}\right) &= -.1t\end{aligned}$$

$$\begin{aligned}\ln 1 - \ln 49 &= -.1t\\ -\ln 49 &= -.1t\\ \ln 49 &= .1t\\ \ln 7^2 &= .1t\\ 2\ln 7 &= .1t\\ 20\ln 7 &= t\\ 38.9182 &\approx t\end{aligned}$$

The point of inflection is $(38.9182, 5000)$.

72.
$$\begin{aligned}L(t) &= Be^{-ce^{-kt}}\\ L'(t) &= Be^{-ce^{-kt}}(-ce^{-kt})'\\ &= Be^{-ce^{-kt}}[-ce^{-kt}(-kt)']\\ &= Bcke^{-ce^{-kt}-kt}\\ L''(t) &= Bcke^{-ce^{-kt}-kt}(-ce^{-kt} - kt)'\\ &= Bcke^{-ce^{-kt}-kt}[-ce^{-kt}(-kt)' - k]\\ &= Bcke^{-ce^{-kt}-kt}(cke^{-kt} - k)\\ &= Bck^2e^{-ce^{-kt}-kt}(ce^{-kt} - 1)\end{aligned}$$

$L''(t) = 0$ when $ce^{-kt} - 1 = 0$

$$\begin{aligned}ce^{-kt} - 1 &= 0\\ \frac{c}{e^{kt}} &= 1\\ e^{kt} &= c\\ kt &= \ln c\\ t &= \frac{\ln c}{k}\end{aligned}$$

Letting $c = 7.267963$ and $k = .670840$

$$t = \frac{\ln 7.267963}{.670840} \approx 2.96 \text{ years}$$

Verify that there is a point of inflection at $t = \frac{\ln c}{k} \approx 2.96$. For

$$L''(t) = Bck^2e^{-ce^{-kt}-kt}(ce^{-kt} - 1),$$

we only need to test the factor $ce^{-kt} - 1$ on the intervals determined by $t \approx 2.96$ since the other factors are always positive.
$L''(1)$ has the same sign as

$$7.267963e^{-.670840(1)} - 1 \approx 2.72 > 0.$$

$L''(3)$ has the same sign as

$$7.267963e^{-.670840(3)} - 1 \approx -.029 < 0.$$

Therefore L, is concave up on $\left(0, \frac{\ln c}{k} \approx 2.96\right)$ and concave down on $\left(\frac{\ln c}{k}, \infty\right)$, so there is a point of inflection at $t = \frac{\ln c}{k} \approx 2.96$ years.
This signifies the time when the rate of growth begins to slowdown since L changes from concave up to concave down at this inflection point.

74.
$$\begin{aligned}v(x) &= -35.98 + 12.09x - .4450x^2\\ v'(x) &= 12.09 - .89x\\ v''(x) &= -.89\end{aligned}$$

Since $-.89 < 0$, the function is always concave down.

76. Since the rate of violent crimes is decreasing but at a slower rate than in previous years, we know that $f'(t) < 0$ but $f''(t) > 0$. Note that since $f'(t) < 0$, f is decreasing, and since $f''(t) > 0$, the graph of f is concave upward.

78. $s(t) = -16t^2$
$v(t) = s'(t) = -32t$

(a) $v(3) = -32(3) = -96$ ft/sec

(b) $v(5) = -32(5) = -160$ ft/sec

(c) $v(8) = -32(8) = -256$ ft/sec

(d) $a(t) = v'(t) = s''(t)$
$= -32$ ft/sec^2

80. $s(t) = 256t - 16t^2$
$v(t) = s'(t) = 256 - 32t$
$a(t) = v'(t) = s''(t) = -32$

To find when the maximum height occurs, set $s'(t) = 0$.

$$256 - 32t = 0$$
$$t = 8$$

Find the maximum height.

$$s(8) = 256(8) - 16(8^2) = 1024$$

The maximum height of the ball is 1024 ft.
The ball hits the ground when $s = 0$.

$$256t - 16t^2 = 0$$
$$16t(16 - t) = 0$$
$$t = 0 \quad \text{(initial moment)}$$
$$t = 16 \text{ (final moment)}$$

The ball hits the ground 16 seconds after being thrown.

5.4 Curve Sketching

4. $f(x) = x^3 - \frac{15}{2}x^2 - 18x - 1$

$f'(x) = 3x^2 - 15x - 18$
$= 3(x^2 - 5x - 6)$
$= 3(x - 6)(x + 1) = 0$

Critical numbers: 6 and -1
Critical points: $(6, -163)$ and $(-1, 8.5)$

$f''(x) = 6x - 15$
$f''(6) = 21 > 0$
$f''(-1) = -21 < 0$

Relative maximum at $x = -1$, relative minimum at $x = 6$
Increasing on $(-\infty, -1)$ and $(6, \infty)$
Decreasing on $(-1, 6)$

$$f''(x) = 6x - 15 = 0$$
$$x = \frac{5}{2}$$

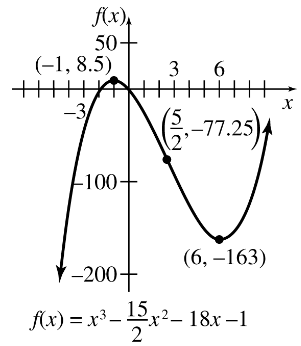
Point of inflection at $\left(\frac{5}{2}, -77.25\right)$

Concave upward on $\left(\frac{5}{2}, \infty\right)$

Concave downward on $\left(-\infty, \frac{5}{2}\right)$

y-intercept:

$$y = (0)^3 - \frac{15}{2}(0)^2 - 18(0) - 1 = -1$$



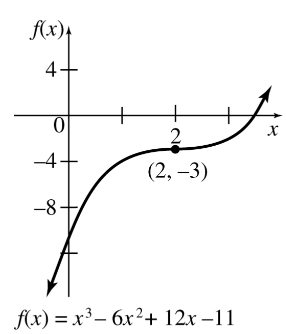
6. $f(x) = x^3 - 6x^2 + 12x - 11$
$f'(x) = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4)$
$= 3(x - 2)^2$

Critical number: 2
Critical point: $(2, -3)$

$f''(x) = 6x - 12$
$f''(0) = 6(0) - 12 = -12 < 0$
$f''(3) = 6(3) - 12 = 6 > 0$

No relative extrema
Increasing on $(-\infty, \infty)$
Concave upward on $(2, \infty)$; concave downward on $(-\infty, 2)$
Point of inflection: $(2, -3)$
y-intercept:

$$y = 0^3 - 6(0)^2 + 12(0) - 11 = -11$$



8. $f(x) = x^4 - 8x^2$

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2)$$

Critical numbers: 0, −2, and 2
Critical points: $(0,0)$, $(-2,-16)$ and $(2,-16)$

$$f''(x) = 12x^2 - 16$$
$$f''(0) = -16 < 0$$
$$f''(-2) = 32 > 0$$
$$f''(2) = 32 > 0$$

Relative maximum at 0, relative minima at −2 and 2
Increasing on $(-2, 0)$ and $(2, \infty)$
Decreasing on $(-\infty, -2)$ and $(0, 2)$

$$f''(x) = 12x^2 - 16 = 0$$
$$4(3x^2 - 4) = 0$$
$$x = \pm\sqrt{\frac{4}{3}}$$

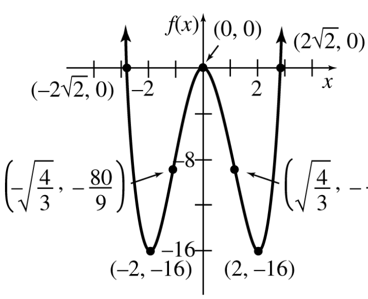
Points of inflection $\left(\sqrt{\frac{4}{3}}, -\frac{80}{9}\right)$ and $\left(-\sqrt{\frac{4}{3}}, -\frac{80}{9}\right)$

Concave upward on $\left(-\infty, -\sqrt{\frac{4}{3}}\right)$ and $\left(\sqrt{\frac{4}{3}}, \infty\right)$

Concave downward on $\left(-\sqrt{\frac{4}{3}}, \sqrt{\frac{4}{3}}\right)$

x-intercepts: $0 = x^4 - 8x^2$
$0 = x^2(x^2 - 8)$
$x = 0$ or $x = \pm 2\sqrt{2}$

y-intercept: $y = 0^4 - 8(0)^2 = 0$



$f(x) = x^4 - 8x^2$

10. $f(x) = x^5 - 15x^3$

$$f'(x) = 5x^4 - 45x^2 = 0$$
$$5x^2(x^2 - 9) = 0$$
$$5x^2(x+3)(x-3) = 0$$

Critical numbers: 0, −3, and 3
Critical points: $(0,0)$, $(-3,162)$, and $(3,-162)$

$$f''(x) = 20x^3 - 90x$$
$$f''(0) = 0$$
$$f''(-3) = -270 < 0$$
$$f''(3) = 270 > 0$$

Relative maximum at −3
Relative minimum at 3
No relative extremum at 0
Increasing on $(-\infty, -3)$ and $(3, \infty)$
Decreasing on $(-3, 3)$

$$f''(x) = 20x^3 - 90x = 0$$
$$10x(2x^2 - 9) = 0$$
$$x = 0 \quad \text{or} \quad x = \pm\frac{3}{\sqrt{2}}$$

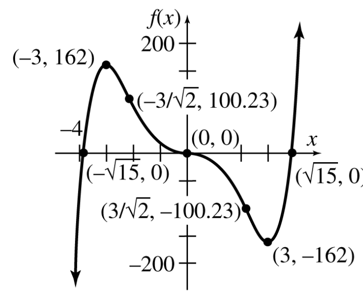
Points of inflection at $(0,0)$, $\left(-\frac{3}{\sqrt{2}}, 100.23\right)$, and $\left(\frac{3}{\sqrt{2}}, -100.23\right)$

Concave upward on $\left(-\frac{3}{\sqrt{2}}, 0\right)$ and $\left(\frac{3}{\sqrt{2}}, \infty\right)$

Concave downward on $\left(-\infty, -\frac{3}{\sqrt{2}}\right)$ and $\left(0, \frac{3}{\sqrt{2}}\right)$

x-intercepts: $0 = x^5 - 15x^3$
$0 = x^3(x^2 - 15)$
$x = 0$, $x = \pm\sqrt{15}$

y-intercept: $y = 0^5 - 15(0)^3 = 0$



$f(x) = x^5 - 15x^3$

12. $f(x) = 2x + \dfrac{8}{x}$

Vertical asymptote at $x = 0$

$$f'(x) = 2 - \frac{8}{x^2} = 0$$
$$\frac{2x^2 - 8}{x^2} = 0$$
$$2x^2 - 8 = 0$$
$$2(x^2 - 4) = 0$$

Critical numbers: −2 and 2
Critical points: $(-2,-8)$ and $(2,8)$

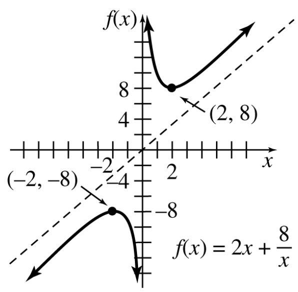
$$f''(x) = \frac{16}{x^3}$$
$$f''(-2) = -2 < 0$$
$$f''(2) = 2 > 0$$

Relative maximum at −2

Relative minimum at 2
Increasing on $(-\infty, -2)$ and $(2, \infty)$
Decreasing on $(-2, 0)$ and $(0, 2)$
(Recall that $f(x)$ does not exist at $x = 0$.)

$f''(x) = \frac{16}{x^3}$ is never zero.

There are no points of inflection.
Concave upward on $(0, \infty)$
Concave downward on $(-\infty, 0)$
$f(x)$ is never zero, so there are no x-intercepts.
$f(x)$ does not exist at $x = 0$, so there is no y-intercept.
$y = x$ is an oblique asymptote.



14. $f(x) = \dfrac{x}{1+x}$

Vertical asymptote at $x = -1$
Horizontal asymptote at $y = 1$

$$f'(x) = \frac{(1)(1+x) - (1)(x)}{(1+x^2)} = \frac{1}{(1+x)^2}$$

$f'(x)$ is never zero.
$f'(x)$ fails to exist for $x = -1$.

$$f''(x) = \frac{(1+x)^2(0) - 1(2)(1+x)}{(1+x)^4}$$
$$= \frac{-2(1+x)}{(1+x)^4} = \frac{-2}{(x+1)^3}$$

$f''(x)$ fails to exist for $x = -1$.
No critical numbers, so no maxima or minima
Increasing on $(-\infty, -1)$ and $(-1, \infty)$
(Recall that $f(x)$ does not exist at $x = -1$.)
No points of inflection

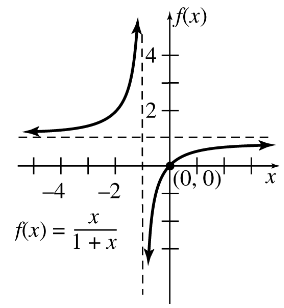
$$f''(-2) = 2 > 0$$
$$f''(0) = -2 < 0$$

Concave upward on $(-\infty, -1)$
Concave downward on $(-1, \infty)$

x-intercept: $0 = \dfrac{x}{1+x}$

$$0 = x$$

y-intercept: $y = \dfrac{0}{1+0} = 0$



16. $f(x) = \dfrac{-2}{x^2 - x - 6}$

Vertical asymptote when

$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = 3 \quad \text{or} \quad x = -2.$$

Horizontal asymptote at $y = 0$

$$f'(x) = \frac{(x^2 - x - 6)(0) - (-2)(2x-1)}{(x^2 - x - 6)^2}$$
$$= \frac{2(2x-1)}{(x^2 - x - 6)^2}$$

$f'(x) = 0$ when $2x - 1 = 0$

$$x = \frac{1}{2}$$

$$f'(0) = \frac{2(2 \cdot 0 - 1)}{(0^2 - 0 - 6)^2} = \frac{-2}{36} < 0$$

$$f'(1) = \frac{2(2 \cdot 1 - 1)}{(1^2 - 1 - 6)^2} = \frac{2}{36} > 0$$

$$f\left(\frac{1}{2}\right) = \frac{-2}{\left(\frac{1}{2}\right)^2 - \frac{1}{2} - 6} = \frac{-2}{-\frac{25}{4}} = \frac{8}{25}$$

Relative minimum at $\left(\frac{1}{2}, .32\right)$

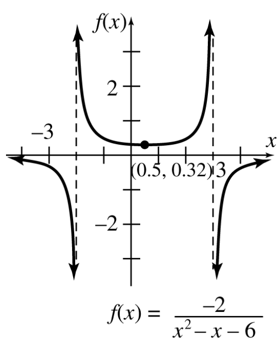
$$f''(x) = \frac{(x^2 - x - 6)^2(4) - 2(2x-1)(2)(x^2 - x - 6)(2x-1)}{(x^2 - x - 6)^4}$$
$$= \frac{4(x^2 - x - 6)[(x^2 - x - 6) - (2x-1)^2]}{(x^2 - x - 6)^4}$$
$$= \frac{4(x^2 - x - 6 - 4x^2 + 4x - 1)}{(x^2 - x - 6)^3}$$
$$= \frac{4(-3x^2 + 3x - 7)}{(x^2 - x - 6)^3}$$

$f''(x)$ is undefined when $x = 3$ and $x = -2$.
$f''(x) \neq 0$ since $-3x^2 + 3x - 7 < 0$ for all x.

$$f''(-3) = \frac{4[-3(-3)^2 + 3(-3) - 7]}{((-3)^2 - (-3) - 6)^3}$$
$$= -.7963 < 0$$
$$f''(0) = \frac{4[-3(0^2) + 3(0) - 7]}{(0^2 - 0 - 6)^3}$$
$$= .1296 > 0$$
$$f''(4) = \frac{4[-3(4^2) + 3(4) - 7]}{(4^2 - 4 - 6)^3}$$
$$= -7963 < 0$$

Concave downward on $(-\infty, -2)$ and $(3, \infty)$
Concave upward on $(-2, 3)$
There are no x-intercepts since $f(x)$ can never be 0.
No points of inflection

y-intercept: $y = \frac{-2}{0^2 - 0 - 6} = \frac{1}{3}$



18. $f(x) = \frac{1}{x^2 + 1}$

Horizontal asymptote at $y = 0$

$$f'(x) = \frac{(x^2 + 1)(0) - 1(2x)}{(x^2 + 1)^2}$$
$$= \frac{-2x}{(x^2 + 1)^2} = 0$$

Critical number: $x = 0$
Critical point: $(0, 1)$

$$f''(x) = \frac{(x^2+1)^2(-2)-(-2x)(2)(x^2+1)(2x)}{(x^2 + 1)^4}$$
$$= \frac{6x^2 - 2}{(x^2 + 1)^3}$$
$$f''(0) = -2 < 0$$

Relative maximum at 0
Increasing on $(-\infty, 0)$
Decreasing on $(0, \infty)$

$$f''(x) = \frac{6x^2 - 2}{(x^2 + 1)^3} = 0$$
$$6x^2 - 2 = 0$$
$$2(3x^2 - 1) = 0$$
$$x = \pm\frac{1}{\sqrt{3}}$$

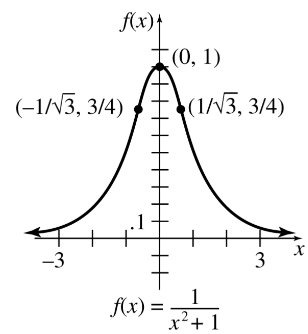
Points of inflection at $\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$

Concave upward on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$

Concave downward on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$f(x)$ is never zero, so there is no x-intercept.

y-intercept: $y = \frac{1}{0^2 + 1} = 1$



20. $f(x) = \frac{x}{x^2 - 1}$

Vertical asymptotes at $x = 1$ and $x = -1$
Horizontal asymptote at $y = 0$

$$f'(x) = \frac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2}$$
$$= \frac{-x^2 - 1}{(x^2 - 1)^2}$$

$f'(x)$ is never zero.
No critical values, no maxima nor minima
Decreasing on $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$
(Recall that $f(x)$ does not exist at $x = -1$ and $x = 1$.)

$$f''(x) = \frac{(x^2-1)^2(-2x)-(-x^2-1)(2)(x^2-1)(2x)}{(x^2 - 1)^4}$$
$$= \frac{2x^3 + 6x}{(x^2 - 1)^3} = 0$$
$$2x^3 + 6x = 0$$
$$2x(x^2+3) = 0$$
$$x = 0$$

Point of inflection: $(0, 0)$

$$f''(-3) = -\frac{9}{64} < 0$$

$$f''\left(-\frac{1}{2}\right) \approx 7.704 > 0$$

$$f''\left(\frac{1}{2}\right) \approx -7.704 < 0$$

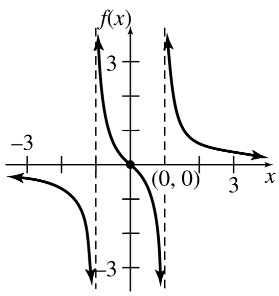
$$f''(3) = \frac{9}{64} > 0$$

Concave upward on $(-1, 0)$ and $(1, \infty)$
Concave downward on $(-\infty, -1)$ and $(0, 1)$

x-intercept: $0 = \dfrac{x}{x^2 - 1}$

$0 = x$

y-intercept: $y = \dfrac{0}{0-1} = 0$



$f(x) = \dfrac{x}{x^2 - 1}$

22. $y = x - \ln|x|$

Vertical asymptote at $x = 0$

$y' = 1 - \dfrac{1}{x} = \dfrac{x-1}{x}$

Critical number: $x = 1$
(Recall that y does not exist at $x = 0$.)

$$y'(-1) = \frac{-1-1}{-1} = 2 > 0$$

$$y'\left(\frac{1}{2}\right) = \frac{\frac{1}{2} - 1}{\frac{1}{2}} = -1 < 0$$

$$y'(2) = \frac{2-1}{2} = \frac{1}{2} > 0$$

$$y(1) = 1 - \ln|1| = 1$$

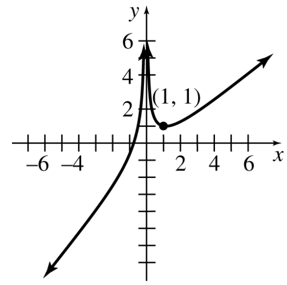
Relative minimum at $(1, 1)$
Increasing on $(-\infty, 0)$ and $(1, \infty)$
Decreasing on $(0, 1)$

$$y'' = \frac{x(1) - (x-1)(1)}{x^2} = \frac{1}{x^2}$$

No point of inflection
y'' is positive everywhere it is defined,

Concave upward on $(-\infty, 0)$ and $(0, \infty)$
There is no y-intercept
x-intercept: $x - \ln|x| = 0$

$$x = \ln|x|$$
$$x \approx -.567$$



$y = x - \ln|x|$

24. $y = \dfrac{\ln x^2}{x^2}$

There is a vertical asymptote at $x = 0$.
There is a no y-intercept.
There are x-intercepts when $y = 0$:

$$\ln x^2 = 0$$
$$x^2 = 1$$
$$x = \pm 1.$$

$$y' = \frac{x^2 \cdot \frac{1}{x^2} \cdot 2x - (\ln x^2)\, 2x}{x^4}$$
$$= \frac{2x(1 - \ln x^2)}{x^4} = \frac{2(1 - \ln x^2)}{x^3}$$

$y' = 0$ when $1 - \ln x^2 = 0$

$$\ln x^2 = 1$$
$$x^2 = e$$
$$x = \pm\sqrt{e}.$$

$$y'' = \frac{x^3\left(-2 \cdot \frac{1}{x^2} \cdot 2x\right) - 2(1 - \ln x^2)3x^2}{x^6}$$
$$= \frac{x^2[-4 - 6(1 - \ln x^2)]}{x^6}$$
$$= \frac{-10 + 6 \ln x^2}{x^4}$$

$$y''(\pm\sqrt{e}) = \frac{-10 + 6 \ln e}{e^2} = \frac{-4}{e^2} < 0$$

There are relative maxima at $\left(\pm\sqrt{e}, \frac{1}{e}\right)$.

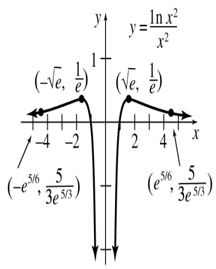
$y''(x) = 0$ when $-10 + 6 \ln x^2 = 0$

$$6 \ln x^2 = 10$$
$$\ln x^2 = \frac{5}{3}$$
$$x^2 = e^{5/3}$$
$$x = \pm\sqrt{e^{5/3}} = \pm e^{5/6}$$

$$y''(3) = \frac{-10 + 6\ \ln\ 9}{3^4} = .0393 > 0$$
$$y''(1) = \frac{-10 + 6\ \ln\ 1^2}{1^4} = -10 < 0$$
$$y''(-1) = \frac{-10 + 6\ \ln\ (-1)^2}{(-1)^4} = -10 < 0$$
$$y''(-3) = \frac{-10 + 6\ \ln\ (-3)^2}{(-3)^4} = .0393 > 0$$

Concave upward on $(-\infty, -e^{5/6})$ and $(e^{5/6}, \infty)$
Concave downward on $(-e^{5/6}, 0)$ and $(0, e^{5/6})$

Points of inflection at $\left(\pm e^{5/6}, \frac{5}{3e^{5/3}}\right)$



26. $y = x^2e^{-x}$

y-intercept: $y = 0^2e^{-0} = 0$

0 is the only intercept.

$$\begin{aligned} y' &= x^2e^{-x}(-1) + e^{-x}(2x) \\ &= xe^{-x}(-x + 2) \end{aligned}$$

Critical numbers $x = 0$ or $-x + 2 = 0$
$x = 2.$

$$f'(-1) = (-1)e^{-(-1)}[-(-1) + 2] = -3e < 0$$
$$f'(1) = (1)e^{-1}(-1 + 2) = \frac{1}{e} > 0$$
$$f'(3) = 3e^{-3}(-3 + 2) = \frac{-3}{e^3} < 0$$

The function is decreasing on $(-\infty, 0)$ and $(2, \infty)$ and increasing on $(0, 2)$.

$$f(0) = 0^2e^{-0} = 0$$
$$f(2) = 2^2e^{-2} = \frac{4}{e^2}$$

Relative minimum at $(0, 0)$; relative maximum at $\left(2, \frac{4}{e^2}\right)$

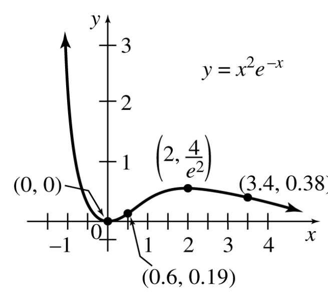
$$\begin{aligned} y'' &= xe^{-x}(-1) + (-x + 2)(-xe^{-x} + e^{-x}) \\ &= -xe^{-x} + x^2e^{-x} - xe^{-x} - 2xe^{-x} + 2e^{-x} \\ &= x^2e^{-x} - 4xe^{-x} + 2e^{-x} \\ &= e^{-x}(x^2 - 4x + 2) \end{aligned}$$

$y'' = 0$ when

$$\begin{aligned} x^2 - 4x + 2 &= 0 \\ x &= \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2} \\ &= \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}. \end{aligned}$$
$$x = 2 + \sqrt{2} \approx 3.4 \quad \text{or} \quad 2 - 2\sqrt{2} \approx .6$$

$$f''(0) = e^{-0}[0^2 - 4(0) + 2] = 2 > 0$$
$$f''(1) = e^{-1}(1^2 - 4(1) + 2) = -\frac{1}{e} < 0$$
$$f''(4) = e^{-4}[4^2 - 4(4) + 2] = \frac{2}{e^4} > 0$$

Concave upward on $(-\infty, .6)$ and $(3.4, \infty)$; downward on $(.6, 3.4)$
Points of inflection at $(.6, .19)$ and $(3.4, .38)$



28. $y = e^x + e^{-x}$

y-intercept: $y = e^0 + e^{-0} = 1 + 1 = 2$

The y-intercept is 2.

$y' = e^x - e^{-x}$

Critical numbers:

$$\begin{aligned} e^x - e^{-x} &= 0 \\ e^x - \frac{1}{e^x} &= 0 \\ \frac{e^{2x} - 1}{e^x} &= 0 \\ e^{2x} - 1 &= 0 \\ e^{2x} &= 1 \\ 2x &= \ln\ 1 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

The only critical number is 0.

$$f'(1) = e^{-1} - e^{-(-1)} = \frac{1}{e} - e \approx -2.35 < 0$$
$$f'(1) = e^1 - e^{-1} \approx 2.35 > 0$$

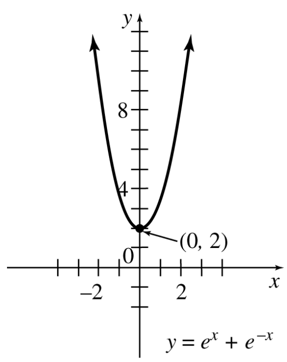
Decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$
Relative minimum at $(0, 2)$

$$y'' = e^x + e^{-x} = \frac{e^{2x}+1}{e^x}$$

Since $y'' \neq 0$ ($e^{2x}+1$ is always positive), there is no point of inflection.

$$f''(0) = e^0 + e^{-0} = 2 > 0$$

The entire graph is concave upward on $(-\infty, \infty)$.



30. $y = x^{1/3} + x^{4/3}$

$$y' = \frac{1}{3}x^{-2/3} + \frac{4}{3}x^{1/3} = \frac{1+4x}{3x^{2/3}}$$

Critical points: $x = -\frac{1}{4}$ and $x = 0$

$$y'' = \frac{3x^{2/3}(4) - (1+4x)3\left(\frac{2}{3}\right)x^{-1/3}}{(3x^{2/3})^2} = \frac{12x^{2/3} - 2(1+4x)x^{-1/3}}{9x^{4/3}} = \frac{12x - 2(1+4x)}{9x^{5/3}} = \frac{4x-2}{9x^{5/3}}$$

y-intercept: $y = 0^{1/3} + 0^{4/3} = 0$
x-intercept: $0 = x^{1/3} + x^{4/3}$
$= x^{1/3}(1+x)$
$x = 0$ or $x = -1$

$y' = 0$ when $1 + 4x = 0$

$$x = -\frac{1}{4}$$

$$y''\left(-\frac{1}{4}\right) = \frac{4\left(-\frac{1}{4}\right) - 2}{9\left(-\frac{1}{4}\right)^{5/3}} = \frac{4^{5/3}}{3} \approx 3.360 > 0$$

$$y\left(-\frac{1}{4}\right) = \left(-\frac{1}{4}\right)^{1/3} + \left(-\frac{1}{4}\right)^{4/3} = -\frac{3}{4^{4/3}} \approx -.472$$

Relative minimum at $\left(-\frac{1}{4}, -\frac{3}{4^{4/3}}\right) \approx (-.25, -.472)$

$y' = \frac{1+4x}{3x^{2/3}}$ undefined at $x = 0$.

Test sign of y' on intervals defined by $x = -\frac{1}{4}$, $x = 0$.

$$y'(-1) = \frac{-3}{3} = -1 < 0$$

$$y'\left(-\frac{1}{8}\right) = \frac{1-\frac{1}{2}}{\frac{3}{4}} > 0$$

$$y'(1) = \frac{5}{3} > 0$$

y increases on $\left(-\frac{1}{4}, \infty\right)$, y decreases on $\left(-\infty, -\frac{1}{4}\right)$
No extreme point at $(0, y(0) = 0^{1/3} + 0^{4/3} = 0)$
$y'' = 0$ when $4x - 2 = 0$

$$x = \frac{1}{2}$$

y'' undefined when $9x^{5/3} = 0$
$x = 0$

$$y''(-1) = \frac{4(-1)-2}{9(-1)^{5/3}} = \frac{2}{3} > 0$$

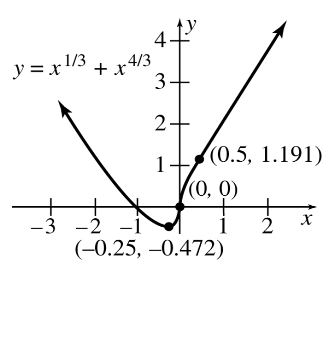
$$y''\left(\frac{1}{8}\right) = \frac{4\left(\frac{1}{8}\right) - 2}{9\left(\frac{1}{8}\right)^{5/3}} = -\frac{16}{3} < 0$$

$$y''(1) = \frac{4(1)-2}{9(1)^{5/3}} = \frac{2}{9} > 0$$

Concave upward on $(-\infty, 0)$ and $\left(\frac{1}{2}, \infty\right)$

Concave downward on $\left(0, \frac{1}{2}\right)$

Points of inflection at $(0, 0)$ and $\left(\frac{1}{2}, \frac{3}{2^{4/3}}\right) \approx (.5, 1.191)$



32. In exercise 4, 8, and 10, either the relative maximum or relative minimum is outside of the normal window of $-10 \le y \le 10$. In exercise 6, the y-intercept is outside the window.

34. In exercise 20, the values of $f(x)$ near the vertical asymptotes are so small and the asymptotes are so close together, it is difficult to determine the behavior of the function in that region. In exercise 24, the relative maxima and points of inflection are so small, they are hard to distinguish from the x-axis.

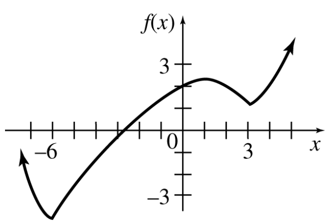
For Exercises 36 and 38, other graphs are possible.

36. **(a)** indicates that the curve may not contain breaks.

(b) and **(c)** indicate relative minima at -6 and 3 and a relative maximum at 1.

(d) and **(e)**, when combined with (b) and (c) show that concavity does not change between relative extrema.

(f) gives the y-intercept.



38. **(a)** indicates that the curve may not contain breaks.

(b) and **(c)** indicate relative maxima at -2 and 3 and a relative minimum at 0.

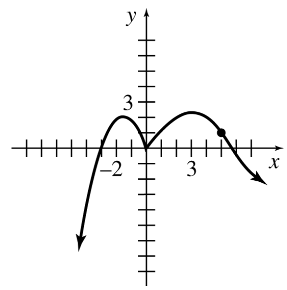
(d) shows that concavity does not change at 0.

(d) and **(e)** are consistent with (i).

(f) shows critical values.

(g) and **(h)** indicate that the function is not differentiable at 0, and is differentiable everywhere else.
Thus, a sharp corner must exist at 0.

(i) indicates that concavity changes just once, at $(5, 1)$.



Chapter 5 Review Exercises

6. $f(x) = -2x^2 - 3x + 4$
$f'(x) = -4x - 3$

$f'(x) = 0$ when $x = -\frac{3}{4}$ and f' exists everywhere, so the only critical number is $-\frac{3}{4}$.
By testing an x-value in the interval $(-\infty - \frac{3}{4})$ and an x-value in the interval $(-\frac{3}{4}, \infty)$, we see that $f'(x) > 0$ on $(-\infty, \frac{3}{4})$ and $f'(x) < 0$ on $(-\frac{3}{4}, \infty)$. These results tell us that f is increasing on $(-\infty, -\frac{3}{4})$ and decreasing on $(-\frac{3}{4}, \infty)$.

8. $f(x) = 4x^3 + 3x^2 - 18x + 1$
$$\begin{aligned} f'(x) &= 12x^2 + 6x - 18 \\ &= 6(2x^2 + x - 3) \\ &= 6(2x + 3)(x - 1) \end{aligned}$$

$f'(x) = 0$ when

$$2x + 3 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -\frac{3}{2} \quad \text{or} \quad x = 1.$$

f' exists everywhere, so the critical numbers are $-\frac{3}{2}$ and 1.
Test one x-value in each of the following intervals: $(\infty, -\frac{3}{2}), (-\frac{3}{2}, 1), (1, \infty)$. We find that $f'(x) > 0$ on $(-\infty, -\frac{3}{2})$ and $(1, \infty)$, and that $f'(x) < 0$ on $(-\frac{3}{2}, 1)$. From this information, we know that f is increasing on $(-\infty, -\frac{3}{2})$ and $(1, \infty)$ and decreasing on $(-\frac{3}{2}, 1)$.

10. $f(x) = \dfrac{5}{2x + 1}$

$f'(x) = \frac{-10}{(2x+1)^2} < 0$ for all x, but f is not defined for $x = -\frac{1}{2}$.
f is never increasing; it is decreasing on $\left(-\infty, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \infty\right)$.

12. $f(x) = 3xe^{2x}$

$$f'(x) = 3x(e^{2x})(2) + 3e^{2x} = 3e^{2x}(2x+1)$$

$f'(x) = 0$ when $2x + 1 = 0$

$$x = -\frac{1}{2}.$$

$$f'(-1) = 3e^{2(-1)}[2(-1)+1] = 3e^{-2}(-1) = \frac{-3}{e^2} < 0$$

$$f'(0) = 3e^{2(0)}(2 \cdot 0 + 1) = 3(1) = 3 > 0$$

Decreasing on $\left(-\infty, -\frac{1}{2}\right)$; increasing on $\left(\frac{1}{2}, \infty\right)$

14. $f(x) = x^2 - 6x + 4$
$f'(x) = 2x - 6$

$f'(x) = 0$ when $x = 3$.
Critical number: 3
$f''(x) = 2 > 0$ for all x, so $f(3)$ is a relative minimum.

$$f(3) = -5$$

Relative minimum of -5 at 3

16. $f(x) = -3x^2 + 2x - 5$
$f'(x) = -6x + 2$

$f'(x) = 0$ when $x = \frac{1}{3}$.

Critical number: $\frac{1}{3}$

Since $f''(x) = -6 < 0$ for all x, $f\left(\frac{1}{3}\right)$ is a relative maximum.

$$f\left(\frac{1}{3}\right) = -\frac{14}{3}$$

Relative maximum of $-\frac{14}{3}$ at $\frac{1}{3}$

18. $f(x) = 2x^3 + 3x^2 - 12x + 5$

$$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x+2)(x-1)$$

$f'(x) = 0$ when $x = -2$ or $x = 1$.
Critical numbers: $-2, 1$

$$f''(x) = 12x + 6$$

$f''(-2) = -18 < 0$, so a maximum occurs at $x = -2$.
$f''(1) = 18 > 0$, so a minimum occurs at $x = 1$.

$$f(-2) = 25$$
$$f(1) = -2$$

Relative maximum of 25 at -2
Relative minimum of -2 at 1

20. $y = \dfrac{\ln(3x)}{2x^2}$

$$y' = \frac{2x^2 \cdot \frac{1}{x} - (\ln 3x) \cdot 4x}{4x^4} = \frac{2x(1 - 2\ln 3x)}{4x^4} = \frac{1 - 2\ln 3x}{2x^3}$$

$y' = 0$ when

$$1 - 2\ln 3x = 0$$
$$1 = 2\ln 3x$$
$$\frac{1}{2} = \ln 3x$$
$$e^{1/2} = 3x$$
$$\frac{e^{1/2}}{3} = x$$
$$\frac{\sqrt{e}}{3} \approx .55$$

$$f'(1) = \frac{1 - 2\ln 3}{2} \approx -.6 < 0$$

$$f'\left(\frac{1}{3}\right) = \frac{1 - 2\ln 1}{2\left(\frac{1}{3}\right)^3} = \frac{1}{2 \cdot \frac{1}{27}} = \frac{27}{2} > 0$$

$$f\left(\frac{\sqrt{e}}{3}\right) \approx .83$$

Relative maximum at $\left(\frac{\sqrt{e}}{3}, .83\right)$ or $(.55, .83)$

22. $f(x) = 9x^3 + \dfrac{1}{x}$

$$= 9x^3 + x^{-1}$$

$$f'(x) = 27x^2 - x^{-2}$$

$$f''(x) = 54x + 2x^{-3} = 54x + \frac{2}{x^3}$$

$$f''(1) = 54(1) + \frac{2}{(1)^3} = 56$$

$$f''(-3) = 54(-3) + \frac{2}{(-3)^3} = -162 - \frac{2}{27} = -\frac{4376}{27}$$

24. $f(x) = \dfrac{4-3x}{x+1}$

$$f'(x) = \frac{-3(x+1)-(1)(4-3x)}{(x+1)^2}$$
$$= \frac{-7}{(x+1)^2} = -7(x+1)^{-2}$$
$$f''(x) = 14(x+1)^{-3} = \frac{14}{(x+1)^3}$$
$$f''(1) = \frac{14}{(1+1)^3} = \frac{7}{4}$$
$$f''(-3) = \frac{14}{(-3+1)^3} = -\frac{7}{4}$$

26. $f(t) = -\sqrt{5-t^2} = -(5-t^2)^{1/2}$

$$f'(t) = -\frac{1}{2}(5-t^2)^{-1/2}(-2t) = t(5-t^2)^{-1/2}$$
$$f''(t) = (1)(5-t^2)^{-1/2} + t\left[-\frac{1}{2}(5-t^2)^{-3/2}(-2t)\right]$$
$$= (5-t^2)^{-1/2} + t[t(5-t^2)^{-3/2}]$$
$$= (5-t^2)^{-3/2}[5-t^2+t^2] = \frac{5}{(5-t^2)^{3/2}}$$
$$f''(1) = \frac{5}{(5-1)^{3/2}} = \frac{5}{8}$$
$$f''(-3) = \frac{5}{(5-9)^{3/2}}$$

This value does not exist since $(-4)^{3/2}$ does not exist.

28. $f(x) = -\dfrac{4}{3}x^3 + x^2 + 30x - 7$

$$f'(x) = -4x^2 + 2x + 30$$
$$= -2(2x^2 - x - 15)$$
$$= -2(2x+5)(x-3) = 0$$

Critical numbers: $-\frac{5}{2}$ and 3
Critical points: $\left(-\frac{5}{2}, -54.91\right)$ and $(3, 56)$

$$f''(x) = -8x + 2$$
$$f''\left(-\frac{5}{2}\right) = 22 > 0$$
$$f''(3) = -22 < 0$$

Relative maximum at 3

Relative minimum at $-\frac{5}{2}$

Increasing on $\left(-\frac{5}{2}, 3\right)$

Decreasing on $\left(-\infty, -\frac{5}{2}\right)$ and $(3, \infty)$

$$f''(x) = -8x + 2 = 0$$
$$x = \frac{1}{4}$$

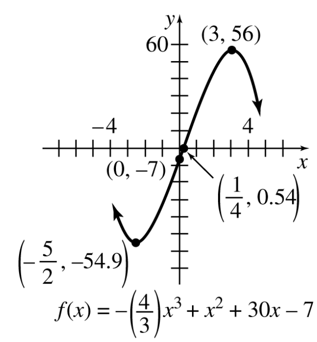
Point of inflection at $\left(\frac{1}{4}, .54\right)$

Concave upward on $\left(-\infty, \frac{1}{4}\right)$

Concave downward on $\left(\frac{1}{4}, \infty\right)$

y-intercept:

$$y = -\frac{4}{3}(0)^3 + (0)^3 + 30(0)^2 - 7$$
$$= -7$$



30. $f(x) = -\dfrac{2}{3}x^3 + \dfrac{9}{2}x^2 + 5x + 1$

$$f'(x) = -2x^2 + 9x + 5 = 0$$
$$(-2x-1)(x-5) = 0$$

Critical numbers: $-\frac{1}{2}$ and 5

Critical points: $\left(-\frac{1}{2}, -.29\right)$ and $(5, 55.17)$

$$f''(x) = -4x + 9$$
$$f''\left(-\frac{1}{2}\right) = 11 > 0$$
$$f''(5) = -11 < 0$$

Relative maximum at 5

Relative minimum at $-\frac{1}{2}$

Increasing on $\left(-\frac{1}{2}, 5\right)$

Decreasing on $\left(-\infty, -\frac{1}{2}\right)$ and $(5, \infty)$

$$f''(x) = -4x + 9 = 0$$
$$x = \frac{9}{4}$$

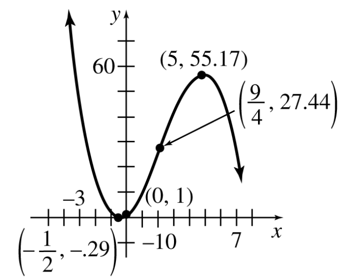
Point of inflection at $\left(\frac{9}{4}, 27.44\right)$

Concave upward on $\left(-\infty, \frac{9}{4}\right)$

Concave downward on $\left(\frac{9}{4}, \infty\right)$

y-intercept:

$$y = -\frac{2}{3}(0)^3 + \frac{9}{2}(0)^2 + 5(0) + 1 = 1$$



$f(x) = -\left(\frac{2}{3}\right)x^3 + \left(\frac{9}{2}\right)x^2 + 5x + 1$

32. $f(x) = \dfrac{2x-5}{x+3}$

Vertical asymptote at $x = -3$
Horizontal asymptote at $y = 2$

$$f'(x) = \frac{2(x+3) - (2x-5)}{(x+3)^2} = \frac{11}{(x+3)^2}$$

f' is never zero.
$f(x)$ has no extrema.

$$f''(x) = \frac{-22}{(x+3)^2}$$

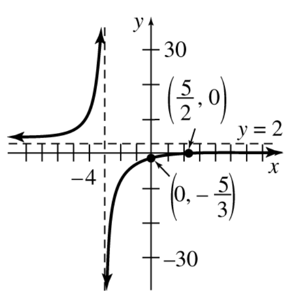
$f''(-4) = 22 > 0$
$f''(-2) = -22 < 0$

Concave upward on $(-\infty, -3)$
Concave downward on $(-3, \infty)$

x-intercept: $\dfrac{2x-5}{x+3} = 0$

$$x = \frac{5}{2}$$

y-intercept: $\dfrac{2(0)-5}{0+3} = -\dfrac{5}{3}$



$f(x) = \dfrac{2x-5}{x+3}$

34. $f(x) = x^3 + \frac{5}{2}x^2 - 2x - 3$

$f'(x) = 3x^2 + 5x - 2 = (3x-1)(x+2) = 0$

Critical numbers: $\frac{1}{3}$ and -2

Critical points: $\left(\frac{1}{3}, -3.35\right)$ and $(-2, 3)$

$f''(x) = 6x + 5$

$f''\left(\dfrac{1}{3}\right) = 7 > 0$

$f''(-2) = -7 < 0$

Relative maximum at -2

Relative minimum at $\frac{1}{3}$

Increasing on $(-\infty, -2)$ and $\left(\frac{1}{3}, \infty\right)$

Decreasing on $\left(-2, \frac{1}{3}\right)$

$f'(x) = 6x + 5 = 0$

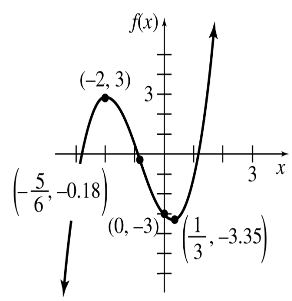
$$x = -\frac{5}{6}$$

Point of inflection at $\left(-\frac{5}{6} - .18\right)$

Concave upward on $\left(-\frac{5}{6}, \infty\right)$

Concave downward on $\left(-\infty, -\frac{5}{6}\right)$

We are unable to compute the x-intercept because this would require solving a cubic equation where the cubic polynomial cannot be factored. The graph will show that there are 3 x-intercepts.
y-intercept: -3



$f(x) = x^3 + \frac{5}{2}x^2 - 2x - 3$

36. $f(x) = 6x^3 - x^4$

$f'(x) = 18x^2 - 4x^3 = 2x^2(9 - 2x) = 0$

Critical numbers: 0 and $\frac{9}{2}$

Critical points: $(0, 0)$ and $\left(\frac{9}{2}, 136.7\right)$

$f''(x) = 36x - 12x^2 = 12x(3 - x)$
$f''(0) = 0$

$f''\left(\dfrac{9}{2}\right) = -81 < 0$

Relative maximum at $\frac{9}{2}$

No relative extrema at 0

Increasing on $\left(-\infty, \frac{9}{2}\right)$

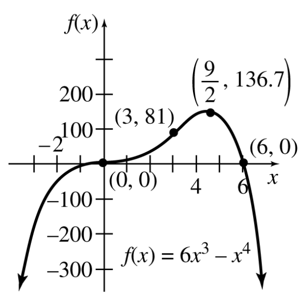
Decreasing on $\left(\frac{9}{2}, \infty\right)$

$$f''(x) = 12x(3-x) = 0$$
$$x = 0 \text{ or } x = 3$$

Points of inflection at $(0, 0)$ and $(3, 81)$
Concave upward on $(0, 3)$
Concave downward on $(-\infty, 0)$ and $(3, \infty)$

x-intercepts: $6x^3 - x^4 = 0$
$$x^3(6-x) = 0$$
$$x = 0,\ x = 6$$

The x-intercepts are 0 and 6.
y-intercept: 0



38. $f(x) = x + \dfrac{8}{x}$

Vertical asymptote at $x = 0$
Oblique asymptote at $y = x$

$$f'(x) = 1 - \frac{8}{x^2}$$
$$= \frac{x^2 - 8}{x^2} = 0$$

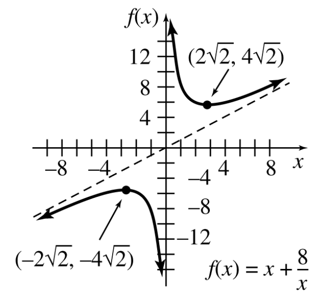
Critical numbers: $x = \pm 2\sqrt{2}$
Critical points: $(2\sqrt{2}, 4\sqrt{2}), (-2\sqrt{2}, -4\sqrt{2})$

$$f''(x) = \frac{16}{x^3}$$
$$f''(-2\sqrt{2}) = -\frac{\sqrt{2}}{2} < 0$$
$$f''(2\sqrt{2}) = \frac{\sqrt{2}}{2} > 0$$

Relative maximum at $-2\sqrt{2}$
Relative minimum at $2\sqrt{2}$
Increasing on $(-\infty, -2\sqrt{2})$ and $(2\sqrt{2}, \infty)$
Decreasing on $(-2\sqrt{2}, 0)$ and $(0, 2\sqrt{2})$

$f''(x) = \frac{16}{x^3} > 0$ for all x.

No inflection points
Concave upward on $(0, \infty)$
Concave downward on $(-\infty, 0)$



40. $f(x) = \dfrac{-4x}{1+2x}$

Vertical asymptote at $x = -\frac{1}{2}$

Horizontal asymptote at $y = -2$

$$f'(x) = \frac{-4(1+2x) - 2(-4x)}{(1+2x)^2}$$
$$= \frac{-4 - 8x + 8x}{(1+2x)^2}$$
$$= \frac{-4}{(1+2x)^2}$$

$f'(x)$ is never zero.
No critical values; no relative extrema

$$f'(0) = -4 < 0$$
$$f'(-1) = -4 < 0$$

Decreasing on $\left(-\infty, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \infty\right)$

$$f''(x) = \frac{16}{(1+2x)^3}$$

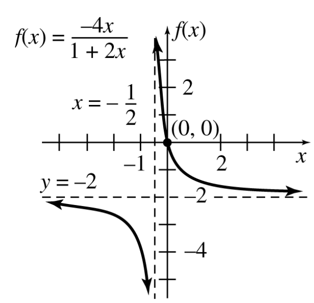
$f''(x)$ is never zero; no points of inflection.

$$f''(0) = 16 > 0$$
$$f''(-1) = -16 < 0$$

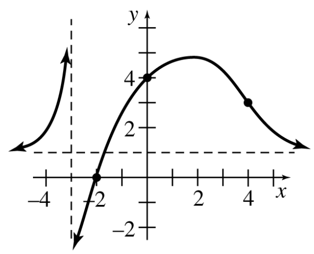
Concave upward on $\left(-\frac{1}{2}, \infty\right)$

Concave downward on $\left(-\infty, -\frac{1}{2}\right)$

x-intercept: 0; y-intercept: 0



42.



Other graphs are possible.

44. (a)-(b) When a stock reaches its highest price of the day, $P(t)$ is at a maximum. Maxima and minima occur when $P'(t) = 0$. Since this is a maximum, the graph would be concave down. Therefore, $P''(t) < 0$.

46. (a) Since the second derivative has many sign changes, the graph continually changes from concave upward to concave downward. Since there is a nonlinear decline, the graph must be one that declines, levels off, declines, levels off, etc. Therefore, the first derivative has many critical numbers where the first derivative is zero.

(b) The curve is always decreasing except at frequent points of inflection.

48. Sketch the curve for $l_1(v) = .08e^{.33v}$

$$l_1'(v) = .0264e^{.33v}$$
$$e^{.33v} \neq 0$$

$l_1'(v)$ has no critical points.

$$l_1''(v) = .008712e^{.33v}$$
$$e^{.33v} \neq 0$$

$l_1''(v)$ has no inflection points.

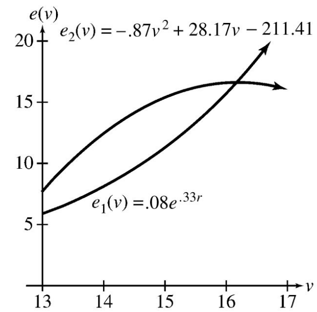
Sketch the curve for $l_2 = -.87v^2 + 28.17v - 211.41$

$$l_2'(v) = -1.74v + 28.17$$
$$-1.74v + 28.17 = 0$$
$$v \approx 16.19$$

Critical point: $(16.19, 16.62)$

$$l_2''(v) = -1.74$$

$l'(v)$ has no inflection points.
$l_2'(v)$ has a relative maximum at $(16.19, 16.62)$.



50. $y = 34.7(1.0186)^{-x}(x^{-.658})$

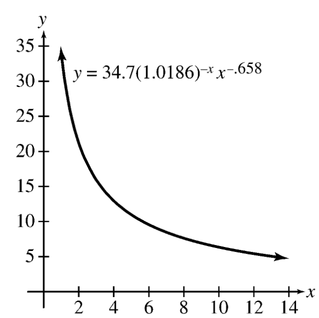
In chapter 4, the function was originally defined as

$$\log y = 1.54 - .008x - .658 \log x$$

so, $0 < x < \infty$, and $0 < y < \infty$.
The function will have a vertical asymptote at $x = 0$ and a horizontal assymptote at $y = 0$.

$$\frac{dy}{dx} = -34.7(1.0186)^{-x}x^{-.658}\left(\ln(1.0186) + \frac{.658}{x}\right)$$

For every value in the domain, $\frac{dy}{dx} < 0$, so y has no critical points and is decreasing on $(0, \infty)$.



52. (a)
$$f(x) = .817x^3 - 15.4x^2 + 49.4x + 874$$
$$f'(x) = 2.451x^2 - 30.8x + 49.4$$
$$f'(x) = 0 \text{ when } x \approx 1.89 \text{ and when } x \approx 10.68.$$
$$f'(1) = 21.051 > 0$$
$$f'(5) = -43.325 < 0$$
$$f'(12) = 32.744 > 0$$

There is a maximum at $x \approx 1.89$ and a minimum at $x \approx 10.68$. At year 2, which is 1992, there is a relative maximum of about 918. At year 11, which is 2001, there is a relative minimum of about 640.

(b) $f''(x) = 4.902x - 30.8$
$f''(x) = 0$ when $x \approx 6.28$

There is a point of inflection at about $x \approx 6.28$. In year 6, which is 1996, the rate of increase started to slow down.

54. (a) The U.S. stockpile was at a relative maximum between 1965 and 1967, at 1974, 1980, 1984, and at 1987.

(b) The U.S. stockpile was at its largest relative maximum from 1965 to 1967. During this period, the Soviet stockpile was concave upward. This means that the stockpile was increasing at an increasingly rapid rate.

Chapter 6

APPLICATIONS OF THE DERIVATIVE

6.1 Absolute Extrema

2. As shown on the graph, the absolute minimum occurs at x_1; there is no absolute maximum. (There is no functional value that is greater than all others.)

4. As shown on the graph, there are no absolute extrema.

6. As shown on the graph, the absolute maximum occurs at x_1; there is no absolute minimum.

8. As shown on the graph, the absolute maximum occurs at x_2; the absolute minimum occurs at x_1.

10. $f(x) = x^3 - 3x^2 - 24x + 5;\ [-3, 6]$

Find critical numbers:

$$\begin{aligned} f'(x) = 3x^2 - 6x - 24 &= 0 \\ 3(x^2 - 2x - 8) &= 0 \\ 3(x+2)(x-4) &= 0 \\ x = -2 \quad \text{or} \quad x &= 4 \end{aligned}$$

x	$f(x)$	
-3	23	
-2	33	Absolute maximum
4	-75	Absolute minimum
6	-31	

12. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 3;\ [-4, 4]$

Find critical numbers:

$$\begin{aligned} f'(x) = x^2 - x - 6 &= 0 \\ (x+2)(x-3) &= 0 \\ x = -2 \quad \text{or} \quad x &= 3 \end{aligned}$$

x	$f(x)$	
-4	$-\frac{7}{3} \approx -2.3$	
-2	$\frac{31}{3} \approx 10.3$	Absolute maximum
3	$-\frac{21}{2} \approx -10.5$	Absolute minimum
4	$-\frac{23}{3} \approx -7.7$	

14. $f(x) = x^4 - 32x^2 - 7;\ [-5, 6]$

$$\begin{aligned} f'(x) = 4x^3 - 64x &= 0 \\ 4x(x^2 - 16) &= 0 \\ 4x(x-4)(x+4) &= 0 \\ x = 0 \text{ or } x = 4 \text{ or } x &= -4 \end{aligned}$$

x	$f(x)$	
-5	-182	
-4	-263	Absolute minimum
0	-7	
4	-263	Absolute minimum
6	137	Absolute maximum

16. $f(x) = \frac{8+x}{8-x};\ [4, 6]$

$$\begin{aligned} f'(x) &= \frac{(8-x)(1) - (8+x)(-1)}{(8-x)^2} \\ &= \frac{16}{(8-x)^2} \end{aligned}$$

$f'(x)$ is never zero. Although $f'(x)$ fails to exist if $x = 8$, 8 is not in the given interval.

x	$f(x)$	
4	3	Absolute minimum
6	7	Absolute maximum

18. $f(x) = \frac{x}{x^2+2};\ [0, 4]$

$$\begin{aligned} f'(x) &= \frac{(x^2+2)1 - x(2x)}{(x^2+2)^2} \\ &= \frac{-x^2+2}{(x^2+2)^2} = 0 \\ -x^2 + 2 &= 0 \\ x^2 &= 2 \end{aligned}$$

$x = \sqrt{2}$ or $x = -\sqrt{2}$, but $-\sqrt{2}$ is not in $[0, 4]$. $f'(x)$ is defined for all x.

x	$f(x)$	
0	0	Absolute minimum
$\sqrt{2}$	$\frac{\sqrt{2}}{4} \approx .35$	Absolute maximum
4	$\frac{2}{9} \approx .22$	

20. $f(x) = (x^2 + 18)^{2/3};\ [-3, 3]$

$$f'(x) = \frac{2}{3}(x^2 + 18)^{-1/3}(2x) = \frac{4x}{3(x^2 + 18)^{1/3}}$$

The derivative always exists, and is 0 when

$$\frac{4x}{3(x^2 + 18)^{1/3}} = 0$$
$$4x = 0$$
$$x = 0.$$

x	$f(x)$	
0	$18^{2/3} \approx 6.87$	Absolute minimum
-3	9	Absolute maximum
3	9	Absolute maximum

22. $f(x) = (x + 1)(x + 2)^2;\ [-4, 0]$

$$\begin{aligned} f'(x) &= (x + 1)(2)(x + 2)(1) + (x + 2)^2(1) \\ &= 2(x + 1)(x + 2) + (x + 2)^2 \\ &= [2(x + 1) + x + 2](x + 2) = 0 \\ & (3x + 4)(x + 2) = 0 \end{aligned}$$

$$x = -\frac{4}{3} \quad \text{or} \quad x = -2$$

x	$f(x)$	
-4	-12	Absolute minimum
-2	0	
$-\frac{4}{3}$	$-\frac{4}{27}$	
0	4	Absolute maximum

24. $f(x) = \dfrac{x^3 + 2x + 5}{x^4 + 3x^3 + 10};\ [-3, 0]$

The indicated domain tells us the x-values to use for the viewing window, but we must experiment to find a suitable range for the y-values. In order to show the absolute extrema on $[-3, 0]$, we find that a suitable window is $[-3, 0]$ by $[-9, 1]$.

From the graph, we see that on $[-3, 0]$, f has an absolute maximum at 0 and an absolute minimum at about -2.35.

26. $f(x) = 12 - x - \dfrac{9}{x},\ x > 0$

$$f'(x) = -1 + \frac{9}{x^2} = \frac{9 - x^2}{x^2} = \frac{(3 + x)(3 - x)}{x^2}$$

$f'(x) = 0$ when $x = -3$ or $x = 3$, and $f'(x)$ does not exist when $x = 0$. However, the specified domain for f is $(0, \infty)$. Since -3 and 0 are not in the domain of f, the only critical number is 3.

x	$f(x)$
3	6

There is an absolute maximum at $x = 3$. There is no absolute minimum, as can be seen by looking at the graph of f.

28. $f(x) = x^4 - 4x^3 + 4x^2 + 1$

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 + 8x \\ &= 4x(x^2 - 3x + 2) \\ &= 4x(x - 2)(x - 1) \end{aligned}$$

The critical numbers are 0, 1, and 2.

x	$f(x)$
0	1
1	2
2	1

There is no absolute maximum, as can be seen by looking at the graph of f. There is an absolute minimum at $x = 0$ and $x = 2$.

30. $f(x) = \dfrac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{(1 + x)(1 - x)}{(x^2 + 1)^2}$$

The critical numbers are -1 and 1.

x	$f(x)$
-1	$-.5$
1	.5

There is an absolute maximum of .5 at $x = 1$ and an absolute minimum of $-.5$ at $x = -1$. This can be verified by looking at the graph of f.

32. $f(x) = 2x - 3x^{2/3}$

$$\begin{aligned} f'(x) &= 2 - 2x^{-1/3} \\ &= 2 - \frac{2}{\sqrt[3]{x}} \\ &= \frac{2\sqrt[3]{x} - 2}{\sqrt[3]{x}} \end{aligned}$$

$f'(x) = 0$ when
$$\begin{aligned} 2\sqrt[3]{x} - 2 &= 0 \\ 2\sqrt[3]{x} &= 2 \\ \sqrt[3]{x} &= 1 \\ x &= 1. \end{aligned}$$

$f'(x)$ is undefined at $x = 0$, but $f(x)$ is defined at $x = 0$. So the critical numbers are 0 and 1.

(a) On $[-1, .5]$

x	$f(x)$
-1	-5
0	0
1	-1
.5	$-.88988$

Absolute minimum of -5 at $x = -1$; absolute maximum of 0 at $x = 0$

(b) On $[.5, 2]$

x	$f(x)$
.5	$-.88988$
1	-1
2	$-.7622$

Absolute maximum of about $-.76$ at $x = 2$; absolute minimum of -1 at $x = 1$

34. (a) By looking at the graph, there are relative maxima of 226 in 1986, 264 in 1992, and 146 in 2002. There are relative minima of 206 in 1984, 222 in 1988, and 145 in 2000.

(b) The rate (per 100,000 inhabitants) at which automobiles were stolen reached an absolute maximum of 264 in 1992 and an absolute minimum of 145 in 2000.

36. $P(x) = -x^3 + 9x^2 + 120x - 400,\ x \geq 5$

$$\begin{aligned} P'(x) &= -3x^2 + 18x + 120 \\ &= -3(x^2 - 6x - 40) \\ &= -3(x - 10)(x + 4) = 0 \\ x &= 10 \quad \text{or} \quad x = -4 \end{aligned}$$

-4 is not relevant since $x \geq 5$, so the only critical number is 10.

The graph of $P'(x)$ is a parabola that opens downward, so $P'(x) > 0$ on the interval $[5, 10)$ and $P'(x) < 0$ on the interval $(10, \infty)$. Thus, $P(x)$ is a maximum at $x = 10$.
Since x is measured in hundred thousands, 10 hundred thousand or 1,000,000 tires must be sold to maximize profit.
Also,

$$\begin{aligned} P(10) &= -(10)^3 + 9(10)^2 + 120(10) - 400 \\ &= 700. \end{aligned}$$

The maximum profit is \$700 thousand or \$700,000.

38. $C(x) = x^3 + 37x + 250$

(a) $1 \leq x \leq 10$

$$\begin{aligned} \overline{C}(x) &= \frac{C(x)}{x} = \frac{x^3 + 37x + 250}{x} \\ &= x^2 + 37 + \frac{250}{x} \end{aligned}$$

$$\begin{aligned} \overline{C}'(x) &= 2x - \frac{250}{x^2} \\ &= \frac{2x^3 - 250}{x^2} = 0 \text{ when} \\ 2x^3 &= 250 \\ x^3 &= 125 \\ x &= 5. \end{aligned}$$

Test for relative minimum.

$$\begin{aligned} \overline{C}'(4) &= -7.625 < 0 \\ \overline{C}'(6) &\approx 5.0556 > 0 \\ \overline{C}(5) &= 112 \\ \overline{C}(1) &= 1 + 37 + 250 = 288 \\ \overline{C}(10) &= 100 + 37 + 25 = 162 \end{aligned}$$

The minimum on the interval $1 \leq x \leq 10$ is 112.

(b) $10 \leq x \leq 20$

There are no critical values in this interval. Check the endpoints.

$$\begin{aligned} \overline{C}(10) &= 162 \\ \overline{C}(20) &= 400 + 37 + 12.5 = 449.5 \end{aligned}$$

The minimum on the interval $10 \leq x \leq 20$ is 162.

40. The value $x = 11$ minimizes $\frac{f(x)}{x}$ because this is the point where the line from the origin to the curve is tangent to the curve.
A production level of 11 units results in the minimum cost per unit.

42. The value $x = 100$ maximizes $\frac{f(x)}{x}$ because this is the point where the line from the origin to the curve is tangent to the curve.
A production level of 100 units results in the maximum profit per item produced.

44. $S(x) = -x^3 + 3x^2 + 360x + 5000;\ 6 \le x \le 20$

$$\begin{aligned} S'(x) &= -3x^2 + 6x + 360 \\ &= -3(x^2 - 2x - 120) \\ S'(x) &= -3(x-12)(x+10) = 0 \end{aligned}$$

$x = 12$ or $x = -10$ (not in the interval)

x	$f(x)$
6	7052
12	8024
10	7900

12° is the temperature that produces the maximum number of salmon.

46. The function is defined on the interval $[15, 46]$. We look first for critical numbers in the interval. We find

$$R'(T) = -.00021T^2 + .0802T - 1.6572$$

Using our graphing calculator, we find one critical number in the interval at about 21.92

T	$R(T)$
15	81.01
21.92	79.29
46	98.89

The relative humidity is minimized at about 21.92°C.

48. $M(x) = -.018x^2 + 1.24x + 6.2,\ 30 \le x \le 60$

$$M'(x) = -.036x + 1.24 = 0$$
$$x = 34.4$$

x	$M(x)$
30	27.2
34.4	27.6
60	15.8

The absolute minimum of 15.8 mpg occurs at 60 mph.
The absolute maximum of 27.6 mpg occurs at 34.4 mph.

50. Total area $= A(x)$

$$= \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{12-x}{4}\right)^2$$
$$= \frac{x^2}{4\pi} + \frac{(12-x)^2}{16}$$

$$A'(x) = \frac{x}{2\pi} - \frac{12-x}{8} = 0$$
$$\frac{4x - \pi(12-x)}{8\pi} = 0$$
$$x = \frac{12\pi}{4+\pi} \approx 5.28$$

x	Area
0	9
5.28	5.04
12	11.46

The total area is maximized when all 12 feet of wire are used to form the circle.

6.2 Applications of Extrema

2. $x + y = 200$

Minimize $x^2 + y^2$.

(a) $y = 200 - x$

(b) Let
$$\begin{aligned} P = x^2 + y^2 &= x^2 + (200-x)^2 \\ &= x^2 + 40{,}000 - 400x + x^2 \\ &= 2x^2 - 400x + 40{,}000. \end{aligned}$$

(c) Since $y = 200 - x$ and x and y are nonnegative numbers, the domain of P is $[0, 200]$.

(d) $P' = 4x - 400$

$$\begin{aligned} 4x - 400 &= 0 \\ 4x &= 400 \\ x &= 100 \end{aligned}$$

(e)

x	P
0	40,000
100	20,000
200	40,000

(f) The minimum value of $x^2 + y^2$ occurs when $x = 100$ and $y = 200 - x = 200 - 100 = 100$. The minimum value is 20,000.

4. $x + y = 45$

Maximize xy^2.

(a) $y = 45 - x$

(b) Let
$$\begin{aligned} P = xy^2 &= x(45-x)^2 \\ &= x(2025 - 90x + x^2) \\ &= 2025x - 90x^2 + x^3. \end{aligned}$$

(c) Since $y = 45 - x$ and x and y are nonnegative numbers, the domain of P is $[0, 45]$.

(d) $P' = 2025 - 180x + 3x^2$

$$\begin{aligned} 3x^2 - 180x + 2025 &= 0 \\ 3(x^2 - 60x + 675) &= 0 \\ 3(x-15)(x-45) &= 0 \\ x = 15 \quad \text{or} \quad x &= 45 \end{aligned}$$

(e)

x	P
0	0
15	13,500
45	0

(f) The maximum value of xy^2 occurs when $x = 15$ and $y = 30$. The maximum value is 13,500.

6. $C(x) = 10 + 20x^{1/2} + 16x^{3/2}$

The average cost function is

$$\begin{aligned} A(x) = \overline{C}(x) &= \frac{C(x)}{x} \\ &= \frac{10 + 20x^{1/2} + 16x^{3/2}}{x} \\ &= \frac{10}{x} + 20x^{-1/2} + 16x^{1/2} \\ &\text{or} \quad 10x^{-1} + 20x^{-1/2} + 16x^{1/2}. \end{aligned}$$

Then

$$A'(x) = -10x^{-2} - 10x^{-3/2} + 8x^{-1/2}.$$

Graph $y = A'(x)$ on a graphing calculator. A suitable choice for the viewing window is $[0, 10]$ by $[-10, 10]$. (Negative values of x are not meaningful in this application.) We see that this graph has one x-intercept or "zero." Using the calculator, we find that this x-value is about 2.110, which shows that 2.110 is the only critical number of A.

Now graph $y = A(x)$ and use this graph to confirm that a minimum occurs at $x \approx 2.110$. Thus, the average cost is smallest at $x \approx 2.110$.

8. $p(x) = 6 - \frac{x}{8}$

(a) Revenue from x thousand compact discs:

$$\begin{aligned} R(x) &= 1000xp \\ &= 1000x\left(6 - \frac{x}{8}\right) \\ &= 6000x - 125x^2 \end{aligned}$$

(b) $R'(x) = 6000 - 250x$

$$\begin{aligned} 6000 - 250x &= 0 \\ 6000 &= 250x \\ 24 &= x \end{aligned}$$

The maximum revenue occurs when 24 thousand compact discs are sold.

(c)
$$\begin{aligned} R(24) &= 6000(24) - 125(24)^2 \\ &= 72{,}000 \end{aligned}$$

The maximum revenue is \$72,000.

10. Let $x =$ length of field;
$y =$ width of field.

Perimeter:

$$\begin{aligned} P = 2x + 2y &= 200 \\ x + y &= 100 \\ y &= 100 - x \end{aligned}$$

Area:

$$\begin{aligned} A &= xy \\ &= x(100 - x) \\ &= 100x - x^2 \end{aligned}$$

Thus,

$$\begin{aligned} A(x) &= 100x - x^2 \\ A'(x) &= 100 - 2x. \end{aligned}$$

$A'(x) = 0$ when

$$\begin{aligned} 100 - 2x &= 0 \\ x &= 50. \end{aligned}$$

$A''(x) = -2$, so $A''(50) = -2 < 0$, which confirms that a maximum value occurs at $x = 50$.
If $x = 50$,

$$y = 100 - x = 100 - 50 = 50.$$

A maximum area occurs when the length is 50 m and the width is 50 m.

12. Let $x =$ the length at \$3 per foot
$y =$ the width at \$6 per foot

$$xy = 20{,}000$$
$$y = \frac{20{,}000}{x}$$

$$\text{Perimeter} = 2x + 2y = 2x + \frac{40{,}000}{x}$$

$$\text{Cost} = 2x(3) + \frac{40{,}000}{x}(6) = 6x + \frac{240{,}000}{x}$$

Minimize cost.

$$C'(x) = 6 - \frac{240{,}000}{x^2}$$

$$6 - \frac{240{,}000}{x^2} = 0$$
$$6 = \frac{240{,}000}{x^2}$$
$$6x^2 = 240{,}000$$
$$x^2 = 40{,}000$$
$$x = 200$$
$$y = \frac{20{,}000}{200} = 100$$

400 ft at \$3 per foot will cost \$1200. 200 ft at \$6 per foot will cost \$1200. The entire cost will be \$2400.

14. Let $x =$ the number of seats.

Profit is 5 dollars per seat for $60 \le x \le 80$.

Profit (in dollars) is $5 - .05(x - 80)$per seat for $x > 80$.
We expect that the number of seats which makes the total profit a maximum will be greater than 80 because after 80 the profit is still increasing, though at a slower rate. (Thus we know the function is concave down and its one extremum will be a maximum.)

(a) The total profit for x seats is

$$\begin{aligned} P(x) &= [5 - .05(x - 80)]x \\ &= (5 - .05x + 4)x \\ &= (9 - .05x)x \\ &= 9x - .05x^2. \end{aligned}$$

$$P'(x) = 9 - .10x$$

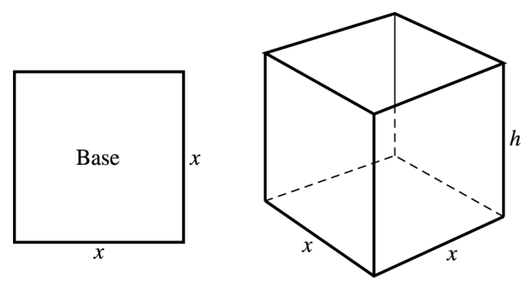
$$9 - .10x = 0$$
$$09 = .10x$$
$$x = 90$$

90 seats will produce maximum profit.

(b) $$\begin{aligned} P(90) &= 9(90) - .05(90)^2 \\ &= 810 - .05(8100)^2 \\ &= 405 \end{aligned}$$

The maximum profit is \$405.

16. Let $x =$ a side of the base
$h =$ the height of the box.



An equation for the volume of the box is

$$V = x^2 h,$$
$$\text{so} \quad 32 = x^2 h$$
$$h = \frac{32}{x^2}.$$

The box is open at the top so the area of the surface material $m(x)$ in square inches is the area of the base plus the area of the four sides.

$$\begin{aligned} m(x) &= x^2 + 4xh \\ &= x^2 + 4x\left(\frac{32}{x^2}\right) \\ &= x^2 + \frac{128}{x} \\ m'(x) &= 2x - \frac{128}{x^2} \end{aligned}$$

$$\frac{2x^3 - 128}{x^2} = 0$$
$$2x^3 - 128 = 0$$
$$2(x^3 - 64) = 0$$
$$x = 4$$

$m'(x) = 2 + \frac{256}{x^3} > 0$ since $x > 0$.
So, $x = 4$ minimizes the surface material.

If $x = 4$,

$$h = \frac{32}{x^2} = \frac{32}{16} = 2.$$

The dimensions that will minimize the surface material are 4 in by 4 in by 2 in.

18. Let $x =$ the width.
Then $2x =$ the length
and $h =$ the height.

An equation for volume is

$$\begin{aligned} V &= (2x)(x)h = 2x^2h \\ 36 &= 2x^2h. \end{aligned}$$

So, $h = \frac{18}{x^2}$.

The surface area $S(x)$ is the sum of the areas of the base and the four sides.

$$\begin{aligned} S(x) &= (2x)(x) + 2xh + 2(2x)h \\ &= 2x^2 + 6xh \\ &= 2x^2 + 6x\left(\frac{18}{x^2}\right) \\ &= 2x^2 + \frac{108}{x} \\ S'(x) &= 4x - \frac{108}{x^2} \end{aligned}$$

$$\begin{aligned} \frac{4x^3 - 108}{x^2} &= 0 \\ 4(x^3 - 27) &= 0 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} S''(x) &= 4 + \frac{108(2)}{x^3} \\ &= 4 + \frac{216}{x^3} > 0 \text{ since } x > 0. \end{aligned}$$

So $x = 3$ minimizes the surface material.

If $x = 3$,

$$h = \frac{18}{x^2} = \frac{18}{9} = 2.$$

The dimensions are 3 ft by 6 ft by 2 ft.

20. 120 centimeters of ribbon are available; it will cover 4 heights and 8 radii.

$$\begin{aligned} 4h + 8r &= 120 \\ h + 2r &= 30 \\ h &= 30 - 2r \end{aligned}$$

$$\begin{aligned} V &= \pi r^2 h \\ V &= \pi r^2(30 - 2r) \\ &= 30\pi r^2 - 2\pi r^3 \end{aligned}$$

Maximize volume.

$$\begin{aligned} V' &= 60\pi r - 6\pi r^2 \\ 60\pi r - 6\pi r^2 &= 0 \\ 6\pi r(10 - r) &= 0 \\ r = 0 \quad \text{or} \quad r &= 10 \end{aligned}$$

If $r = 0$, there is no box, so we discard this value. $V'' = 6\pi - 12\pi r < 0$ for $r = 10$, which implies that $r = 10$ gives maximum volume.
When $r = 10$, $h = 30 - 2(10) = 10$.
The volume is maximized when the radius and height are both 10 cm.

22. $V = \pi r^2 h = 16$

$$h = \frac{16}{\pi r^2}$$

The total cost is the sum of the cost of the top and bottom and the cost of the sides or

$$\begin{aligned} C &= 2(2)(\pi r^2) + 1(2\pi r h) \\ &= 4(\pi r^2) + 1(2\pi r)\left(\frac{16}{\pi r^2}\right) \\ &= 4\pi r^2 + \frac{32}{r}. \end{aligned}$$

Minimize cost.

$$C' = 8\pi r - \frac{32}{r^2}$$

$$\begin{aligned} 8\pi r - \frac{32}{r^2} &= 0 \\ 8\pi r^3 &= 32 \\ \pi r^3 &= 4 \\ r &= \sqrt[3]{\frac{4}{\pi}} \\ &\approx 1.08 \end{aligned}$$

$$h = \frac{16}{\pi(1.08)^2} \approx 4.34$$

The radius should be 1.08 ft and the height should be 4.34 ft. If these rounded values for the height and radius are used, the cost is

$$\begin{aligned} &\$2(2)(\pi r^2) + \$1(2\pi r h) \\ &= 4\pi(1.08)^2 + 2\pi(1.08)(4.34) \\ &= \$44.11. \end{aligned}$$

24. **(a)** From Example 3, the area of the base is $(12 - 2x)(12 - 2x) = 4x^2 - 48x + 144$ and the total area of all four walls is $4x(12 - 2x) = -8x^2 + 48x$. Since the box has maximum volume when $x = 2$, the area of the base is $4(2)^2 - 48(2) + 144 = 64$ square inches and the total area of all four walls is $-8(2)^2 + 48(2) = 64$ square inches. So, both are 64 square inches.

(b) From Exercise 23, the area of the base is $(3-2x)(8-2x)=4x^2-22x+24$ and the total area of all four walls is $2x(3-2x)+2x(8-2x)=-8x^2+22x$. Since the box has maximum volume when $x=\frac{2}{3}$, the area of the base is $4\left(\frac{2}{3}\right)^2-22\left(\frac{2}{3}\right)+24=\frac{100}{9}$ square feet and the total area of all four walls is $-8\left(\frac{2}{3}\right)^2+22\left(\frac{2}{3}\right)=\frac{100}{9}$ square feet. So, both are $\frac{100}{9}$ square feet.

(c) Based on the results from parts (a) and (b), it appears that the area of the base and the total area of the walls for the box with maximum volume are equal. (This conjecture is true.)

26. Distance on shore: $9-x$ miles
Cost on shore: \$400 per mile
Distance underwater: $\sqrt{x^2+36}$
Cost underwater: \$500 per mile
Find the distance from A, that is, $(9-x)$, to minimize cost, $C(x)$.

$$\begin{aligned}C(x)&=(9-x)(400)+(\sqrt{x^2+36})(500)\\&=3600-400x+500(x^2+36)^{1/2}\end{aligned}$$

$$\begin{aligned}C'(x)&=-400+500\left(\frac{1}{2}\right)(x^2+36)^{-1/2}(2x)\\&=-400+\frac{500x}{\sqrt{x^2+36}}\end{aligned}$$

If $C'(x)=0$,

$$\begin{aligned}\frac{500x}{\sqrt{x^2+36}}&=400\\\frac{5x}{4}&=\sqrt{x^2+36}\\\frac{25}{16}x^2&=x^2+36\\\frac{9}{16}x^2&=36\\x^2&=\frac{36\cdot 16}{9}\\x&=\frac{6\cdot 4}{3}=8.\end{aligned}$$

(Discard the negative solution.)
Then the distance should be

$$\begin{aligned}9-x&=9-8\\&=1\text{ mile from point A.}\end{aligned}$$

28. Let $x=$ the number of additional tables.
Then $90-.25x=$ the cost per table and
and $300+x=$ the number of tables ordered.

$$\begin{aligned}R&=(90-.25x)(300+x)\\&=27,000+15x-.25x^2\\R'&=15-.5x=0\\x&=30\end{aligned}$$

$R''=-.5<0$, so when $300+30=330$ tables are ordered, revenue is maximum.

Thus, the maximum revenue is

$$\begin{aligned}R(30)&=27{,}000+15(30)-.25(30)^2\\&=\$27{,}225\end{aligned}$$

The maximum revenue is \$27,225.

Minimum revenue is found by letting $R=0$.

$$\begin{aligned}&(90-.25x)(300+x)=0\\&90-.25x=0\quad\text{or}\quad 300+x=0\\&x=360\quad\text{or}\quad x=-300\\&\qquad\qquad\qquad\text{(impossible)}\end{aligned}$$

So when $300+360=660$ tables are ordered, revenue is 0, that is, each table is free.
I would fire the assistant.

30. In Exercise 29, we found that the cost of the aluminum to make the can is

$$.03\left(2\pi r^2+\frac{2000}{r}\right)=.06\pi r^2+\frac{60}{r}.$$

The cost for the vertical seam is $.01h$. From Example 4, we see that h and r are related by the equation

$$h=\frac{1000}{\pi r^2},$$

so the sealing cost is

$$\begin{aligned}.01h&=.01\left(\frac{1000}{\pi r^2}\right)\\&=\frac{10}{\pi r^2}.\end{aligned}$$

Thus, the total cost is given by the function

$$\begin{aligned}C(r)&=.06\pi r^2+\frac{60}{r}+\frac{10}{\pi r^2}\\&\text{or}\quad .06\pi r^2+60r^{-1}+\frac{10}{\pi}r^{-2}.\end{aligned}$$

Then

$$C'(x) = .12\pi r - 60r^{-2} - \frac{20}{\pi}r^{-3}$$

$$\text{or} \quad .12\pi r - \frac{60}{r^2} - \frac{20}{\pi r^3}.$$

Graph

$$y = .12\pi r - \frac{60}{r^2} - \frac{20}{\pi r^3}$$

on a graphing calculator. Since r must be positive, our window should not include negative values of x. A suitable choice for the viewing window is $[0, 10]$ by $[-10, 10]$. From the graph, we find that $C'(x) = 0$ when $x \approx 5.454$.

Thus, the cost is minimized when the radius is about 5.454 cm.

We can find the corresponding height by using the equation

$$h = \frac{1000}{\pi r^2}$$

from Example 4.

If $r = 5.454$,

$$h = \frac{1000}{\pi(5.454)^2} \approx 10.70.$$

To minimize cost, the can should have radius 5.454 cm and height 10.70 cm.

32. $p(t) = \dfrac{20t^3 - t^4}{1000}$, $[0, 20]$

(a) $p'(t) = \dfrac{3}{50}t^2 - \dfrac{1}{250}t^3$

$$= \frac{1}{50}t^2\left[3 - \frac{1}{5}t\right]$$

Critical numbers:

$$\frac{1}{50}t^2 = 0 \quad \text{or} \quad 3 - \frac{1}{5}t = 0$$

$$t = 0 \quad \text{or} \quad t = 15$$

t	$p(t)$
0	0
15	16.875
20	0

The number of people infected reaches a maximum in 15 days.

(b) $P(15) = 16.875\%$

34. **(a)** $p(t) = 10te^{-t/8}$, $[0, 40]$

$$p'(t) = 10te^{-t/8}\left(-\frac{1}{8}\right) + e^{-t/8}(10)$$

$$= 10e^{-t/8}\left(-\frac{t}{8} + 1\right)$$

Critical numbers:

$p'(t) = 0$ when

$$-\frac{t}{8} + 1 = 0$$

$$t = 8.$$

t	$p(t)$
0	0
8	29.43
40	2.6952

The percent of the population infected reaches a maximum in 8 days.

(b) $P(8) = 29.43\%$

36. $H(S) = f(S) - S$

$$f(S) = \frac{25S}{S+2}$$

$$H'(S) = \frac{(S+2)(25) - 25S}{(S+2)^2} - 1$$

$$= \frac{25S + 50 - 25S - (S+2)^2}{(S+2)^2}$$

$$= \frac{50 - (S^2 + 4S + 4)}{(S+2)^2}$$

$$= \frac{-S^2 - 4S + 46}{(S+2)^2}$$

$H'(S) = 0$ when

$$S^2 + 4S - 46 = 0$$

$$S = \frac{-4 \pm \sqrt{16 + 184}}{2}$$

$$= 5.071.$$

(Discard the negative solution.)

The number of creatures needed to sustain the population is $S_0 = 5.071$ thousand.

$$H'' = \frac{(S+2)^2(-2S-4) - (S^2 - 4S + 46)(2S+4)}{(S+2)^4} < 0,$$

so H is a maximum at $S_0 = 5.071$.

$$H(S_0) = \frac{25(5.071)}{7.071} - 5.071$$

$$\approx 12.86$$

The maximum sustainable harvest is 12.86 thousand.

38. $r = .1,\ P = 100$

$$f(S) = Se^{r(1-S/P)}$$
$$f'(S) = -\frac{1}{1000} \cdot Se^{.1(1-S/100)} + e^{.1(1-S/100)}$$
$$f'(S_0) = -.001S_0e^{.1(1-S_0/100)} + e^{.1(1-S_0/100)}$$

Graph

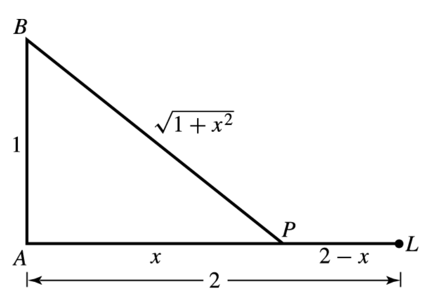
$$Y_1 = -.001xe^{.1(1-x/100)} + e^{.1(1-x/100)}$$

and

$$Y_2 = 1$$

on the same screen. A suitable choice for the viewing window is $[0, 60]$ by $[.5, 1.5]$ with Xscl $= 10$ and Yscl $= .5$. By zooming or using the "intersect" option, we find the graphs intersect when $x \approx 49.37$. The maximum sustainable harvest is 49.37.

40. Let $x =$ distance from P to A.



Energy used over land: 1 unit per mile

Energy used over water: $\frac{4}{3}$ units per mile

Distance over land: $(2 - x)$ mi
Distance over water: $\sqrt{1 + x^2}$ mi
Find the location of P to minimize energy used.

$$E(x) = 1(2 - x) + \frac{4}{3}\sqrt{1 + x^2}, \text{ where } 0 \le x \le 2.$$
$$E'(x) = -1 + \frac{4}{3}\left(\frac{1}{2}\right)(1 + x^2)^{-1/2}(2x)$$

If $E'(x) = 0$,

$$\frac{4}{3}x(1 + x^2)^{-1/2} = 1$$
$$\frac{4x}{3(1 + x^2)^{1/2}} = 1$$
$$\frac{4}{3}x = (1 + x^2)^{1/2}$$
$$\frac{16}{9}x^2 = 1 + x^2$$
$$\frac{7}{9}x^2 = 1$$
$$x^2 = \frac{9}{7}$$
$$x = \frac{3}{\sqrt{7}}$$
$$= \frac{3\sqrt{7}}{7}.$$

x	$E(x)$
0	3.3333
1.134	2.8819
2	2.9814

The absolute minimum occurs at $x \approx 1.134$.
Point P is $\frac{3\sqrt{7}}{7} \approx 1.134$ mi from Point A.

42. (a) $f(S) = aSe^{-bS}$

$$f(S) = Se^{r(1-S/p)}$$
$$= Se^{r - rS/p}$$
$$= Se^re^{-rS/p}$$
$$= e^rSe^{-(r/p)S}$$

Comparing the two terms, replace a with e^r and b with r/p.

(b) Shepherd:

$$f(S) = \frac{aS}{1 + (S/b)^c}$$
$$f'(S) = \frac{[1 + (S/b)^c](a) - (aS)[c(S/b)^{c-1}(1/b)]}{[1 + (S/b)^c]^2}$$
$$= \frac{a + a(S/b)^c - (acS/b)(S/b)^{c-1}}{[1 + (S/b)^c]^2}$$
$$= \frac{a + a(S/b)^c - ac(S/b)^c}{[1 + (S/b)^c]^2}$$
$$= \frac{a[1 + (1 - c)(S/b)^c]}{[1 + (S/b)^c]^2}$$

Ricker:

$$\begin{aligned} f(S) &= aSe^{-bS} \\ f'(S) &= ae^{-bS} + aSe^{-bS}(-b) \\ &= ae^{-bS}(1 - bS) \end{aligned}$$

Berverton-Holt:

$$\begin{aligned} f(S) &= \frac{aS}{1 + (S/b)} \\ f'(S) &= \frac{[1 + (S/b)](a) - aS(1/b)}{[1 + (S/b)]^2} \\ &= \frac{a + a(S/b) - a(S/b)}{[1 + (S/b)]^2} \\ &= \frac{a}{[1 + (S/b)]^2} \end{aligned}$$

(c) Shepherd:

$$f'(0) = \frac{a[1 + (1 - c)(0/b)^c]}{[1 + (0/b)^c]^2} = a$$

Ricker:

$$f'(0) = ae^{-b(0)}[1 - b(0)] = a$$

Beverton-Holt:

$$f'(0) = \frac{a}{[1 + (0/b)]^2} = a$$

The constant a represents the slope of the graph of $f(S)$ at $S = 0$.

(d) First find the critical numbers by solving $f'(S) = 0$.

Shepherd:

$$\begin{aligned} f'(0) &= 0 \\ a[1 + (1 - c)(S/b)^c] &= 0 \\ (1 - c)(S/b)^c &= -1 \\ (c - 1)(S/b)^c &= 1 \end{aligned}$$

Substitute $b = 248.72$ and $c = 3.24$ and solve for S.

$$(3.24 - 1)(S/248.72)^{3.24} = 1$$

$$\left(\frac{S}{248.72}\right)^{3.24} = \frac{1}{2.24}$$

$$\frac{S}{248.72} = \left(\frac{1}{2.24}\right)^{1/3.24}$$

$$S = 248.72\left(\frac{1}{2.24}\right)^{1/3.24}$$

$$S \approx 193.914$$

Using the Shepherd model, next year's population is maximized when this year's population is about 194,000 tons. This can be verified by examing the graph of $f(S)$.

(e) First find the critical numbers by solving $f'(S) = 0$.

Ricker:

$$\begin{aligned} f'(S) &= 0 \\ ae^{-bS}(1 - bS) &= 0 \\ 1 - bS &= 0 \\ bS &= 1 \\ S &= \frac{1}{b} \end{aligned}$$

Substitute $b = .0039$ and solve for S.

$$S = \frac{1}{.0039}$$

$$S \approx 256.410$$

Using the Ricker model, next year's population is maximized when this year's population is about 256,000 tons. This can be verified by examining the graph of $f(S)$.

(g) We solve the equation $f'(S) = 1$ using the Shepherd model.

$$\frac{a[1 + (1 - c)(S/b)^c]}{[1 + (S/b)^c]^2} = 1$$

$$a\left[1 + (1 - c)\left(\frac{S}{b}\right)^c\right] = \left[1 + \left(\frac{S}{b}\right)^c\right]^2$$

$$a + a(1 - c)\left(\frac{S}{b}\right)^c = 1 + 2\left(\frac{S}{b}\right)^c + \left(\frac{S}{b}\right)^{2c}$$

$$\left(\frac{S}{b}\right)^{2c} + [2 - a(1 - c)]\left(\frac{S}{b}\right)^c + 1 - a = 0$$

Substitute 3.026, 248.72, and 3.24 for a, b, and c, respectively, simplify, and solve the resulting equation in S using a graphing calculator.

$$\left(\frac{S}{248.72}\right)^{2(3.24)} + [2 - 3.026(1 - 3.24)]\left(\frac{S}{248.72}\right)^{3.24} + 1 - 3.026 = 0$$

$$\frac{S^{6.48}}{248.72^{6.48}} + 8.77824\frac{S^{3.24}}{248.72^{3.24}} - 2.026 = 0$$

$$S \approx 156.958$$

Using the Shepherd model, the maximum sustainable harvest occurrs when this year's population is about 157,000 tons.

44. Let $x =$ width.
Then $x =$ height
and $108 - 4x =$ length.

(since length plus girth = 108)

$$\begin{aligned} V(x) &= l \cdot w \cdot h \\ &= (108 - 4x)x \cdot x \\ &= 108x^2 - 4x^3 \\ V'(x) &= 216x - 12x^2 \end{aligned}$$

Set $V'(x) = 0$, and solve for x.

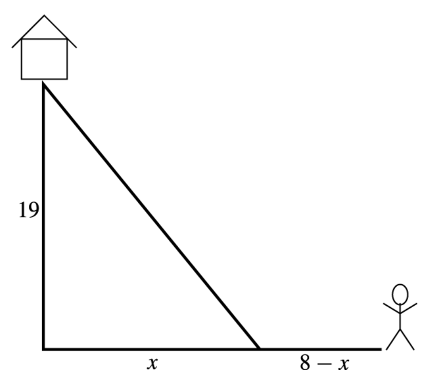
$$\begin{aligned} 216x - 12x^2 &= 0 \\ 12x(18 - x) &= 0 \\ x = 0 \quad \text{or} \quad x &= 18 \end{aligned}$$

0 is not in the domain, so the only critical number is 18.

Width = 18
Height = 18
Length = 108 − 4(18) = 36

The dimensions of the box with maximum volume are 36 inches by 18 inches by 18 inches.

46. Let $8 - x =$ the distance the hunter will travel on the river.



Then $\sqrt{19^2 + x^2} =$ the distance he will travel on land.

Since the rate on the river is 5 mph, the rate on land is 2 mph, and $t = \frac{d}{r}$,

$$\frac{8 - x}{5} = \text{the time on the river}$$

$$\frac{\sqrt{361 + x^2}}{2} = \text{the time on the land.}$$

The total time is

$$\begin{aligned} T(x) &= \frac{8-x}{5} + \frac{\sqrt{361+x^2}}{2} \\ &= \frac{8}{5} - \frac{1}{5}x + \frac{1}{2}(361+x^2)^{1/2}. \\ T'(x) &= -\frac{1}{5} + \frac{1}{4}\cdot 2x(361+x^2)^{-1/2} \end{aligned}$$

$$-\frac{1}{5} + \frac{x}{2(361+x^2)^{1/2}} = 0$$

$$\frac{1}{5} = \frac{x}{2(361+x^2)^{1/2}}$$

$$\begin{aligned} 2(361+x^2)^{1/2} &= 5x \\ 4(361+x^2) &= 25x^2 \\ 1444+4x^2 &= 25x^2 \\ 1444 &= 21x^2 \\ \frac{38}{\sqrt{21}} &= x \\ 8.29 &= x \end{aligned}$$

8.29 is not possible, since the cabin is only 8 miles west. Check the endpoints.

x	$T(x)$
0	11.1
8	10.3

$T(x)$ is minimized when $x = 8$.

The distance along the river is given by $8 - x$, so the hunter should travel $8 - 8 = 0$ miles along the river. He should complete the entire trip on land.

6.3 Further Business Applications: Economic Lot Size; Economic Order Quantity; Elasticity of Demand

4. Use equation (3) with $k = 1, M = 100{,}000$, and $f = 500$.

$$\begin{aligned} q &= \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(500)(100{,}000)}{1}} \\ &= \sqrt{100{,}000{,}000} = 10{,}000 \end{aligned}$$

10,000 lamps should be made in each batch to minimize production costs.

6. From Exercise 4, $M = 100{,}000$, and $q = 10{,}000$. The number of batches per year is $\frac{M}{q} = \frac{100{,}000}{10{,}000} = 10$.

8. Here $k = .50$, $M = 100{,}000$, and $f = 60$. We have

$$\begin{aligned} q &= \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(60)(100{,}000)}{.50}} \\ &= \sqrt{24{,}000{,}000} \approx 4898.98 \end{aligned}$$

$T(4898) = 2449.489792$ and $T(4899) = 2449.489743$, so ordering 4899 copies per order minimizes the annual costs.

10. Using maximum inventory size,

$$\begin{aligned} T(q) &= \frac{fM}{q} + gM + kq;\ (0,\infty) \\ T'(q) &= \frac{-fM}{q^2} + k \end{aligned}$$

Set the derivative equal to 0.

$$\begin{aligned} \frac{-fM}{q^2} + k &= 0 \\ k &= \frac{fM}{q^2} \\ q^2k &= fM \\ q^2 &= \frac{fM}{k} \\ q &= \sqrt{\frac{fM}{k}} \end{aligned}$$

Since $\lim\limits_{q\to 0} T(q) = \infty$, $\lim\limits_{q\to\infty} T(q) = \infty$, and

$q = \sqrt{\frac{fM}{k}}$ is the only critical value in $(0,\infty)$, $q = \sqrt{\frac{fM}{k}}$ is the number of units that should be ordered or manufactured to minimize total costs.

12. Assuming an annual cost, k_1, for storing a single unit, plus an annual cost per unit, k_2, that must be paid for each unit up to the maximum number of units stored, we have

$$\begin{aligned} T(q) &= \frac{fM}{q} + gM + \frac{k_1 q}{2} + k_2 q;\ (0,\infty) \\ T'(q) &= \frac{-fM}{q^2} + \frac{k_1}{2} + k_2 \end{aligned}$$

Set this derivative equal to 0.

$$\frac{-fM}{q^2} + \frac{k_1}{2} + k_2 = 0$$
$$\frac{k_1}{2} + k_2 = \frac{fM}{q^2}$$
$$\frac{k_1 + 2k_2}{2} = \frac{fM}{q^2}$$
$$\frac{q^2(k_1 + 2k_2)}{2} = fM$$
$$q^2 = \frac{2fM}{k_1 + 2k_2}$$
$$q = \sqrt{\frac{2fM}{k_1 + 2k_2}}$$

Since $\lim_{q\to 0} T(q) = \infty$, $\lim_{q\to\infty} T(q) = \infty$, and

$q = \sqrt{\frac{2fM}{k_1+2k_2}}$ is the only critical value in $(0, \infty)$,

$q = \sqrt{\frac{2fM}{k_1+2k_2}}$ is the number of units that should be ordered or manufactured to minimize the total cost in this case.

14. $q = 25{,}000 - 50p$

(a) $\frac{dq}{dp} = -50$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$
$$= -\frac{p}{25{,}000 - 50p}(-50)$$
$$= \frac{p}{500 - p}$$

(b) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$.
Since $q = 0$ means no revenue, set $1 - E = 0$.

$$E = 1$$

From part (a),

$$\frac{p}{500 - p} = 1$$
$$p = 500 - p$$
$$p = 250.$$
$$q = 25{,}000 - 50p$$
$$= 25{,}000 - 50(250)$$
$$= 12{,}500$$

Total revenue is maximized if $q = 12{,}500$.

16. $q = 48{,}000 - 10p^2$

(a) $\frac{dq}{dp} = -20p$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$
$$= \frac{-p}{48{,}000 - 10p^2}(-20p)$$
$$= \frac{20p^2}{48{,}000 - 10p^2}$$
$$= \frac{2p^2}{4800 - p^2}$$

(b) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$. Since $q = 0$ means no revenue, set $1 - E = 0$.

$$E = 1$$

From part (a),

$$\frac{2p^2}{4800 - p^2} = 1$$
$$2p^2 = 4800 - p^2$$
$$3p^2 = 4800$$
$$p^2 = 1600$$
$$p = \pm 40.$$

Since p must be positive, $p = 40$.

$$q = 48{,}000 - 10p^2$$
$$= 48{,}000 - 10(40^2)$$
$$= 48{,}000 - 10(1600)$$
$$= 48{,}000 - 16{,}000$$
$$= 32{,}000$$

18. $q = 10 - \ln p$

(a) $\frac{dq}{dp} = -\frac{1}{p}$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$
$$= \frac{-p}{10 - \ln p}\left(-\frac{1}{p}\right)$$
$$= \frac{1}{10 - \ln p}$$

(b) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$. Since $q = 0$ means no revenue, set $1 - E = 0$.

$$E = 1$$

From part (a),

$$\frac{1}{10 - \ln p} = 1$$

$$\begin{aligned} 1 &= 10 - \ln p \\ \ln p &= 9 \\ p &= e^9 \\ q &= 10 - \ln p \\ &= 10 - \ln e^9 \\ &= 10 - 9 \\ &= 1 \end{aligned}$$

Note that $E = \frac{1}{q}$, thus we would expect E to be maximum when $q = 1$.

20. $q = 300 - 2p$

$$\frac{dq}{dp} = -2$$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$\begin{aligned} E &= -\frac{p(-2)}{300 - 2p} \\ &= \frac{2p}{300 - 2p} \end{aligned}$$

(a) When $p = \$100$,

$$\begin{aligned} E &= \frac{200}{300 - 200} \\ &= 2. \end{aligned}$$

Since $E > 1$, demand is elastic. This indicates that a percentage increase in price will result in a greater percentage decrease in demand.

(b) When $p = \$50$,

$$\begin{aligned} E &= \frac{100}{300 - 100} \\ &= \frac{1}{2} < 1. \end{aligned}$$

Since $E < 1$, supply is inelastic. This indicates that a percentage change in price will result in a smaller percentage change in demand.

22. $q = 100p^{-.17}$

(a) $\frac{dq}{dp} = -17p^{-1.17}$

$$\begin{aligned} E &= -\frac{p}{q} \cdot \frac{dq}{dp} \\ &= \frac{-p}{100p^{-.17}}(-17p^{-1.17}) \\ &= \frac{17p^{-.17}}{100p^{-.17}} = .17 \end{aligned}$$

(b) Since $E = .17 < 1$, the demand is inelastic.

24. $q = -.225p + 3.74$

(a) $\frac{dq}{dp} = -.225$

$$\begin{aligned} E &= -\frac{p}{q} \cdot \frac{dq}{dp} \\ &= \frac{-p}{-.225p + 3.74}(-.225) \\ &= \frac{.225p}{-.225p + 3.74} \end{aligned}$$

(b) When $p = 10$,

$$\begin{aligned} E &= \frac{.225(10)}{-.225(10) + 3.74} \\ &= \frac{2.25}{-2.25 + 3.74} \\ &= 1.51. \end{aligned}$$

(c) $\frac{.225p}{-.225p + 3.74} > 1$

$$\begin{aligned} .225p &> -.225p + 3.74 \\ .450p &> 3.74 \\ p &> \frac{3.74}{.450} \\ p &> 8.31 \end{aligned}$$

At \$8.31, the demand for marijuana is unit elastic.

26. $E = -\frac{p}{q} \cdot \frac{dq}{dp}$

Since $p \neq 0$, $E = 0$ when $\frac{dq}{dp} = 0$. The derivative is zero, which implies that the demand function has a horizontal tangent line at the value of p where $E = 0$.

6.4 Implicit Differentiation

2. $2x^2 - 5y^2 = 4$

$$\begin{aligned}
\frac{d}{dx}(2x^2 - 5y^2) &= \frac{d}{dx}(4) \\
\frac{d}{dx}(2x^2) - \frac{d}{dx}(5y^2) &= \frac{d}{dx}(4) \\
4x - 10y\frac{dy}{dx} &= 0 \\
10y\frac{dy}{dx} &= 4x \\
\frac{dy}{dx} &= \frac{2x}{5y}
\end{aligned}$$

4. $8x^2 = 6y^2 + 2xy$

$$\begin{aligned}
\frac{d}{dx}(8x^2) &= \frac{d}{dx}(6y^2 + 2xy) \\
16x &= \frac{d}{dx}(6y^2 + 2xy) \\
16x &= 12y\frac{dy}{dx} + 2x\frac{d}{dx}(y) + y\frac{d}{dx}(2x) \\
16x &= 12y\frac{dy}{dx} + 2x\frac{dy}{dx}(1) + 2y \\
16x - 2y &= \frac{dy}{dx}(12y + 2x) \\
\frac{16x - 2y}{12y + 2x} &= \frac{dy}{dx} \\
\frac{8x - y}{6y + x} &= \frac{dy}{dx}
\end{aligned}$$

6. $x^3 - 6y^2 = 10$

$$\begin{aligned}
\frac{d}{dx}(x^3 - 6y^2) &= \frac{d}{dx}(10) \\
\frac{d}{dx}(x^3) - \frac{d}{dx}(6y^2) &= 0 \\
3x^2 - 12y\frac{dy}{dx} &= 0 \\
-12y\frac{dy}{dx} &= -3x^2 \\
\frac{dy}{dx} &= \frac{x^2}{4y}
\end{aligned}$$

8. $2y^2 = \dfrac{5 + x}{5 - x}$

$$\begin{aligned}
\frac{d}{dx}(2y^2) &= \frac{d}{dx}\left(\frac{5 + x}{5 - x}\right) \\
4y\frac{dy}{dx} &= \frac{(1)(5 - x) - (-1)(5 + x)}{(5 - x)^2} \\
4y\frac{dy}{dx} &= \frac{5 - x + 5 + x}{(5 - x)^2} = \frac{10}{(5 - x)^2} \\
\frac{dy}{dx} &= \frac{10}{4y(5 - x)^2} = \frac{5}{2y(5 - x)^2}
\end{aligned}$$

10. $2\sqrt{x} - \sqrt{y} = 1$

$$\begin{aligned}
\frac{d}{dx}(2x^{1/2} - y^{1/2}) &= \frac{d}{dx}(1) \\
x^{-1/2} - \frac{1}{2}y^{-1/2}\frac{dy}{dx} &= 0 \\
-\frac{1}{2}y^{-1/2}\frac{dy}{dx} &= -x^{-1/2} \\
\frac{dy}{dx} &= -2y^{1/2}(-x^{-1/2}) \\
&= \frac{2y^{1/2}}{x^{1/2}}
\end{aligned}$$

12. $(xy)^{4/3} + x^{1/3} = y^6 + 1$

$$\begin{aligned}
\frac{d}{dx}[(xy)^{4/3} + x^{1/3}] &= \frac{d}{dx}(y^6 + 1) \\
\frac{d}{dx}(x^{4/3}y^{4/3}) + \frac{d}{dx}(x^{1/3}) &= \frac{d}{dx}(y^6) + \frac{d}{dx}(1) \\
x^{4/3} \cdot \frac{4}{3}y^{1/3}\frac{dy}{dx} + \frac{4}{3}x^{1/3}y^{4/3} + \frac{1}{3}x^{-2/3} & \\
&= 6y^5\frac{dy}{dx} + 0 \\
\frac{4}{3}x^{1/3}y^{4/3} + \frac{1}{3}x^{-2/3} &= 6y^5\frac{dy}{dx} - \frac{4}{3}x^{4/3}y^{1/3}\frac{dy}{dx} \\
4x^{1/3}y^{4/3} + x^{-2/3} &= 18y^5\frac{dy}{dx} - 4x^{4/3}y^{1/3}\frac{dy}{dx} \\
4x^{1/3}y^{4/3} + x^{-2/3} &= (18y^5 - 4x^{4/3}y^{1/3}) \cdot \frac{dy}{dx} \\
\frac{4x^{1/3}y^{4/3} + x^{-2/3}}{18y^5 - 4x^{4/3}y^{1/3}} &= \frac{dy}{dx} \\
\frac{x^{2/3}}{x^{2/3}} \cdot \frac{4x^{1/3}y^{4/3} + x^{-2/3}}{18y^5 - 4x^{4/3}y^{1/3}} &= \frac{dy}{dx} \\
\frac{4xy^{4/3} + 1}{18x^{2/3}y^5 - 4x^2y^{1/3}} &= \frac{dy}{dx}
\end{aligned}$$

14. $x^2e^y + y = x^3$

$$\begin{aligned}
\frac{d}{dx}(x^2e^y + y) &= \frac{d}{dx}(x^3)\\
\frac{d}{dx}(x^2e^y) + \frac{d}{dx}(y) &= 3x^2\\
2xe^y + x^2e^y\frac{dy}{dx} + \frac{dy}{dx} &= 3x^2\\
x^2e^y\frac{dy}{dx} + \frac{dy}{dx} &= 3x^2 - 2xe^y\\
(x^2e^y + 1)\frac{dy}{dx} &= 3x^2 - 2xe^y\\
\frac{dy}{dx} &= \frac{3x^2 - 2xe^y}{x^2e^y + 1}
\end{aligned}$$

16. $y\ln x + 2 = x^{3/2}y^{5/2}$

$$\begin{aligned}
\frac{d}{dx}(y\ln x + 2) &= \frac{d}{dx}(x^{3/2}y^{5/2})\\
\ln x\frac{dy}{dx} + \frac{y}{x} + 0 &= \frac{3}{2}x^{1/2}y^{5/2} + \frac{5}{2}x^{3/2}y^{3/2}\frac{dy}{dx}\\
\ln x\frac{dy}{dx} - \frac{5}{2}x^{3/2}y^{3/2}\frac{dy}{dx} &= \frac{3}{2}x^{1/2}y^{5/2} - \frac{y}{x}\\
\frac{dy}{dx}\left(\ln x - \frac{5}{2}x^{3/2}y^{3/2}\right) &= \frac{3}{2}x^{1/2}y^{5/2} - \frac{y}{x}\\
\frac{dy}{dx} &= \frac{\frac{3}{2}x^{1/2}y^{5/2} - \frac{y}{x}}{\ln x - \frac{5}{2}x^{3/2}y^{3/2}}\\
&= \frac{3x^{3/2}y^{5/2} - 2y}{x(2\ln x - 5x^{3/2}y^{3/2})}
\end{aligned}$$

18. $x^2 + y^2 = 100$; tangent at $(8, -6)$

$$\begin{aligned}
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(100)\\
2x + 2y\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}$$

$$\begin{aligned}
m = -\frac{x}{y} &= -\frac{8}{-6} = \frac{4}{3}\\
y - y_1 &= m(x - x_1)\\
y + 6 &= \frac{4}{3}(x - 8)\\
3y + 18 &= 4x - 32\\
3y &= 4x - 50
\end{aligned}$$

20. $x^2y^3 = 8$; tangent at $(-1, 2)$

$$\begin{aligned}
\frac{d}{dx}(x^2y^3) &= \frac{d}{dx}(8)\\
2xy^3 + 3x^2y^2\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= -\frac{2xy^3}{3x^2y^2}
\end{aligned}$$

$$\begin{aligned}
m = -\frac{2xy^3}{3x^2y^2} &= -\frac{2(-1)(2)^3}{3(-1)^2(2)^2}\\
&= \frac{16}{12} = \frac{4}{3}\\
y - 2 &= \frac{4}{3}(x + 1)\\
3y - 6 &= 4x + 4\\
3y &= 4x + 10
\end{aligned}$$

22. $y + \dfrac{\sqrt{x}}{y} = 3$; tangent at $(4, 2)$

$$\begin{aligned}
\frac{d}{dx}\left(y + \frac{\sqrt{x}}{y}\right) &= \frac{d}{dx}(3)\\
\frac{dy}{dx} + \frac{d}{dx}\left(\frac{\sqrt{x}}{y}\right) &= 0\\
\frac{dy}{dx} + \frac{y\left(\frac{1}{2}\right)x^{-1/2} - \sqrt{x}\,\frac{dy}{dx}}{y^2} &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-\frac{1}{2}yx^{-1/2} + \sqrt{x}\,\frac{dy}{dx}}{y^2}\\
y^2\frac{dy}{dx} &= -\frac{1}{2}yx^{-1/2} + \sqrt{x}\,\frac{dy}{dx}\\
(y^2 - \sqrt{x})\frac{dy}{dx} &= -\frac{1}{2}yx^{-1/2}\\
\frac{dy}{dx} &= \frac{-y}{2x^{1/2}(y^2 - \sqrt{x})}
\end{aligned}$$

$$\begin{aligned}
m &= \frac{-y}{2x^{1/2}(y^2 - \sqrt{x})}\\
&= \frac{-2}{2(2)(4 - 2)}\\
&= -\frac{1}{4}\\
y - 2 &= -\frac{1}{4}(x - 4)\\
y &= -\frac{1}{4}x + 3\\
x + 4y &= 12
\end{aligned}$$

24. $$y^3 + 2x^2y - 8y = x^3 + 19,\ x = 2$$

$$3y^2\frac{dy}{dx} + 2x^2\frac{dy}{dx} + 4xy - 8\frac{dy}{dx} = 3x^2$$
$$(3y^2 + 2x^2 - 8)\frac{dy}{dx} = 3x^2 - 4xy$$
$$\frac{dy}{dx} = \frac{3x^2 - 4xy}{3y^2 + 2x^2 - 8}$$

Find y when $x = 2$.

$$y^3 + 8y - 8y = 8 + 19$$
$$y^3 = 27$$
$$y = 3$$

$$\frac{dy}{dx} = \frac{12 - 24}{27 + 8 - 8} = \frac{-12}{27} = -\frac{4}{9}$$

$$y - 3 = -\frac{4}{9}(x - 2)$$
$$y - 3 = -\frac{4}{9}x + \frac{8}{9}$$
$$y = -\frac{4}{9}x + \frac{35}{9}$$

26. $y^4(1 - x) + xy = 2,\ x = 1$

Find the y-value.

$$y^4(1 - 1) + y = 2$$
$$y = 2$$

The point is $(1, 2)$.

Find $\frac{dy}{dx}$.

$$y^4(-1) + 4y^3(1 - x)\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$
$$-y^4 + 4y^3\frac{dy}{dx} - 4xy^3\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$
$$(4y^3 - 4xy^3 + x)\frac{dy}{dx} = y^4 - y$$
$$\frac{dy}{dx} = \frac{y^4 - y}{4y^3 - 4xy^3 + x}$$

$\frac{dy}{dx}$ at $(1, 2)$ is

$$\frac{2^4 - 2}{4 \cdot 2^3 - 4(1)2^3 + 1} = \frac{16 - 2}{32 - 32 + 1} = 14.$$

$$y - 2 = 14(x - 1)$$
$$y - 2 = 14x - 14$$
$$y = 14x - 12$$

28. $\frac{y}{18}(x^2 - 64) + x^{2/3}y^{1/3} = 12,\ x = 8$

Find the y-value of the point.

$$\frac{y}{18}(64 - 64) + 8^{2/3}y^{1/3} = 12$$
$$4y^{1/3} = 12$$
$$y^{1/3} = 3$$
$$y = 27$$

The point is $(8, 27)$.

Find $\frac{dy}{dx}$.

$$\frac{y}{18}(2x) + \frac{1}{18}(x^2 - 64)\frac{dy}{dx} + \frac{1}{3}x^{2/3}y^{-2/3}\frac{dy}{dx}$$
$$+ \frac{2}{3}x^{-1/3}y^{1/3} = 0$$
$$\left(\frac{x^2 - 64}{18} + \frac{x^{2/3}y^{-2/3}}{3}\right)\frac{dy}{dx} = \frac{-2xy}{18} - \frac{2x^{-1/3}y^{1/3}}{3}$$
$$\frac{dy}{dx} = \frac{\frac{-2xy}{18} - \frac{2x^{-1/3}y^{1/3}}{3}}{\frac{x^2-64}{18} + \frac{x^{2/3}y^{-2/3}}{3}}$$
$$= \frac{-2xy - 12x^{-1/3}y^{1/3}}{x^2 - 64 + 6x^{2/3}y^{-2/3}}$$

$\frac{dy}{dx}$ at $(8, 27)$ is

$$\frac{-2(8)(27) - 12(8)^{-1/3}(27)^{1/3}}{64 - 64 + 6(8)^{2/3}(27)^{-2/3}} = \frac{-432 - 18}{\frac{24}{9}}$$
$$= (-450)\left(\frac{9}{24}\right)$$
$$= \frac{-675}{4}$$

$$y - 27 = -\frac{675}{4}(x - 8)$$
$$y - 27 = -\frac{675}{4}x + 1350$$
$$y = -\frac{675}{4}x + 1377$$

30. $x^{2/3} + y^{2/3} = 2;\ (1, 1)$

Find $\frac{dy}{dx}$.

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$
$$\frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$$

$$\frac{dy}{dx} = \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}}$$
$$= -\frac{y^{1/3}}{x^{1/3}}$$

At $(1,1)$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1^{1/3}}{1^{1/3}} = -1 \\ y-1 &= -1(x-1) \\ y-1 &= -x+1 \\ y &= -x+2 \end{aligned}$$

32. $y^2(x^2+y^2) = 20x^2$; $(1,2)$

Find $\frac{dy}{dx}$.

$$\begin{aligned} 2y(x^2+y^2)\frac{dy}{dx} + y^2\left(2x + 2y\frac{dy}{dx}\right) &= 40x \\ 2x^2y\frac{dy}{dx} + 2y^3\frac{dy}{dx} + 2xy^2 + 2y^3\frac{dy}{dx} &= 40x \\ 2x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} &= -2xy^2 + 40x \\ (2x^2y + 4y^3)\left(\frac{dy}{dx}\right) &= -2xy^2 + 40x \\ \frac{dy}{dx} &= \frac{-2xy^2 + 40x}{2x^2y + 4y^3} \end{aligned}$$

At $(1,2)$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{-2(1)(2)^2 + 40(1)}{2(1)^2(2) + 4(2)^3} \\ &= \frac{32}{36} = \frac{8}{9} \\ y-2 &= \frac{8}{9}(x-1) \\ y-2 &= \frac{8}{9}x - \frac{8}{9} \\ y &= \frac{8}{9}x + \frac{10}{9} \end{aligned}$$

34. $x^2+y^2+1=0$

$$\begin{aligned} \frac{d}{dx}(x^2+y^2) &= \frac{d}{dx}(-1) \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y} \end{aligned}$$

If x and y are real numbers, x^2 and y^2 are nonnegative. 1 plus a nonnegative number cannot equal zero, so there is no function $y = f(x)$ that satisfies $x^2+y^2+1=0$.

36. $\sqrt{u} + \sqrt{2v+1} = 5$

$$\begin{aligned} \frac{dv}{du}(\sqrt{u} + \sqrt{2v+1}) &= \frac{dv}{du}(5) \\ \frac{1}{2}u^{-1/2} + \frac{1}{2}(2v+1)^{-1/2}(2)\frac{dv}{du} &= 0 \\ (2v+1)^{-1/2}\frac{dv}{du} &= -\frac{1}{2}u^{-1/2} \\ \frac{dv}{du} &= -\frac{(2v+1)^{1/2}}{2u^{1/2}} \end{aligned}$$

38. $C^2 = x^2 + 100\sqrt{x} + 50$

(a)

$$\begin{aligned} 2C\frac{dC}{dx} &= 2x + \frac{1}{2}(100)x^{-1/2} \\ \frac{dC}{dx} &= \frac{2x + 50x^{-1/2}}{2C} \\ \frac{dC}{dx} &= \frac{x + 25x^{-1/2}}{C} \cdot \frac{x^{1/2}}{x^{1/2}} \\ \frac{dC}{dx} &= \frac{x^{3/2} + 25}{Cx^{1/2}} \end{aligned}$$

When $x = 5$, the approximate increase in cost of an additional unit is

$$\begin{aligned} \frac{(5)^{3/2} + 25}{(5^2 + 100\sqrt{5} + 50)^{1/2}(5)^{1/2}} &= \frac{36.18}{(17.28)\sqrt{5}} \\ &\approx .94. \end{aligned}$$

(b) $900(x-5)^2 + 25R^2 = 22,500$

$$\begin{aligned} R^2 &= 900 - 36(x-5)^2 \\ 2R\frac{dR}{dx} &= -72(x-5) \\ \frac{dR}{dx} &= \frac{-36(x-5)}{R} = \frac{180 - 36x}{R} \end{aligned}$$

When $x = 5$, the approximate change in revenue for a unit increase in sales is

$$\frac{180 - 36(5)}{R} = \frac{0}{R} = 0.$$

40. $b-a=(b+a)^3$

$$\begin{aligned}
\frac{d}{db}(b-a) &= \frac{d}{db}[(b+a)^3] \\
1-\frac{da}{db} &= 3(b+a)^2\frac{d}{db}(b+a) \\
1-\frac{da}{db} &= 3(b+a)^2\left(1+\frac{da}{db}\right) \\
1-\frac{da}{db} &= 3(b+a)^2+3(b+a)^2\frac{da}{db} \\
-\frac{da}{db}-3(b+a)^2\frac{da}{db} &= 3(b+a)^2-1 \\
[-1-3(b+a)^2]\frac{da}{db} &= 3(b+a)^2-1 \\
\frac{da}{db} &= \frac{3(b+a)^2-1}{-1-3(b+a)^2} \\
\frac{da}{db} &= 0 \\
3(b+a)^2-1 &= 0 \\
(b+a)^2 &= \frac{1}{3} \\
b+a &= \frac{1}{\sqrt{3}}
\end{aligned}$$

Since $b-a=(b+a)^3=\left(\frac{1}{\sqrt{3}}\right)^3=\frac{1}{3\sqrt{3}}$.

$$\begin{aligned}
b+a &= \frac{1}{\sqrt{3}} \\
-(b-a) &= -\frac{1}{3\sqrt{3}} \\
\hline
2a &= \frac{2}{3\sqrt{3}} \\
a &= \frac{1}{3\sqrt{3}}
\end{aligned}$$

42. $s^3-4st+2t^3-5t=0$

$$\begin{aligned}
3s^2\frac{ds}{dt}-\left(4t\frac{ds}{dt}+4s\right)+6t^2-5 &= 0 \\
3s^2\frac{ds}{dt}-4t\frac{ds}{dt}-4s+6t^2-5 &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{ds}{dt}(3s^2-4t) &= 4s-6t^2+5 \\
\frac{ds}{dt} &= \frac{4s-6t^2+5}{3s^2-4t}
\end{aligned}$$

6.5 Related Rates

2. $8y^3+x^2=1;\ \frac{dx}{dt}=2,\ x=3,\ y=-1$

$$\begin{aligned}
24y^2\frac{dy}{dt}+2x\frac{dx}{dt} &= 0 \\
\frac{dy}{dt} &= \frac{-2x\,\frac{dx}{dt}}{24y^2} = -\frac{x\,\frac{dx}{dt}}{12y^2} \\
&= -\frac{(3)(2)}{12(-1)^2} \\
&= -\frac{1}{2}
\end{aligned}$$

4. $4x^3-9xy^2+y=-80;\ \frac{dx}{dt}=4,\ x=-3,\ y=1$

$$\begin{aligned}
12x^2\frac{dx}{dt}-\left(9y^2\frac{dx}{dt}+18xy\frac{dy}{dt}\right)+\frac{dy}{dt} &= 0 \\
12x^2\frac{dx}{dt}-9y^2\frac{dx}{dt}-18xy\frac{dy}{dt}+\frac{dy}{dt} &= 0 \\
(1-18xy)\frac{dy}{dt} &= (9y^2-12x^2)\frac{dx}{dt}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dt} &= \frac{3(3y^2-4x^2)\,\frac{dx}{dt}}{1-18xy} = \frac{3(3\cdot1-4\cdot9)(4)}{1-18(-3)(1)} \\
&= \frac{-396}{55} = -\frac{36}{5}
\end{aligned}$$

6. $\frac{y^3-x^2}{x+2y}=\frac{17}{7};\ \frac{dx}{dt}=1,\ x=-3,\ y=-2$

$$\begin{aligned}
7(y^3-x^2) &= 17(x+2y) \\
7y^3-7x^2 &= 17x+34y \\
21y^2\frac{dy}{dt}-14x\frac{dx}{dt} &= 17\frac{dx}{dt}+34\frac{dy}{dt} \\
(21y^2-34)\frac{dy}{dt} &= (17+14x)\frac{dx}{dt} \\
\frac{dy}{dt} &= \frac{(17+14x)\,\frac{dx}{dt}}{21y^2-34} \\
&= \frac{[17+14(-3)]\cdot 1}{21\cdot 4-34} \\
&= \frac{-25}{50} = -\frac{1}{2}
\end{aligned}$$

8. $y\ln x + xe^y = 6;\ \dfrac{dx}{dt} = 5, x = 1, y = 0$

$$\ln x\frac{dy}{dt} + \frac{y}{x}\frac{dx}{dt} + e^y\frac{dx}{dt} + xe^y\frac{dy}{dt} = 0$$
$$(\ln x + xe^y)\frac{dy}{dt} + \left(\frac{y}{x} + e^y\right)\frac{dx}{dt} = 0$$
$$(\ln x + xe^y)\frac{dy}{dt} = -\left(\frac{y}{x} + e^y\right)\frac{dx}{dt}$$
$$\frac{dy}{dt} = -\frac{(\frac{y}{x} + e^y)\frac{dx}{dt}}{\ln x + xe^y}$$
$$= -\frac{(y + xe^y)\frac{dx}{dt}}{x\ln x + x^2e^y}$$
$$= -\frac{[0 + (1)e^0](5)}{(1)\ln 1 + 1^2e^0} = -5$$

10. $C = \dfrac{R^2}{400{,}000} + 10{,}000;\ \dfrac{dC}{dx} = 10$

$R = 20{,}000$

$$\frac{dC}{dx} = \frac{R}{200{,}000}\cdot\frac{dR}{dx}$$
$$10 = \frac{20{,}000}{200{,}000}\cdot\frac{dR}{dx}$$
$$10 = .1\frac{dR}{dx}$$
$$100 = \frac{dR}{dx}$$

Revenue is changing at a rate of \$100 per unit.

12. $R = 50x - .4x^2,\ C = 5x + 15;\ x = 200,\ \dfrac{dx}{dt} = 50$

(a) $$\frac{dR}{dt} = 50\frac{dx}{dt} - .8x\frac{dx}{dt}$$
$$= 50(50) - .8(200)(50)$$
$$= 2500 - 8000$$
$$= -5500$$

Revenue is decreasing at a rate of \$5500 per day.

(b) $$\frac{dC}{dt} = (5)\frac{dx}{dt}$$
$$= (5)(50)$$
$$= 250$$

Cost is increasing at a rate of \$250 per day.

(c) $$P = R - C$$
$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt}$$
$$= -5500 - 250$$
$$= -5750$$

Profit is decreasing at a rate of \$5750 per day.

14. $R = pq;\ \dfrac{dq}{dt} = 25$

Find the relationship between p and q by finding the equation of the line through $(0, 70)$, and $(100, 60)$.

$$m = \frac{70 - 60}{6 - 100} = \frac{10}{-100} = -\frac{1}{10}$$
$$p - 70 = -\frac{1}{10}(q - 0)$$
$$p - 70 = -\frac{1}{10}q$$
$$p = -\frac{1}{10}q + 70$$

$$R = \left(-\frac{1}{10}q + 70\right)q$$
$$= -\frac{1}{10}q^2 + 70q$$
$$\frac{dR}{dt} = -\frac{1}{5}q\frac{dq}{dt} + 70\frac{dq}{dt}$$
$$= -\frac{1}{5}(20)(25) + 70(25)$$
$$= -100 + 1750$$
$$= \$1650$$

Revenue is increasing at a rate of \$1650 per day.

16. $y = nx^m$

Note that n is a constant.

$$\ln\ y = \ln\ (nx^m)$$
$$\ln\ y = \ln\ n + \ln\ x^m$$
$$\ln\ y = \ln\ n + m\ \ln\ x$$

Now take the derivative of both sides with respect to t.

$$\frac{1}{y}\frac{dy}{dt} = 0 + m\frac{1}{x}\frac{dx}{dt}$$
$$\frac{1}{y}\frac{dy}{dt} = m\frac{1}{x}\frac{dx}{dt}$$

18. $E = 429w^{-.35}$

$$\begin{aligned}\frac{dE}{dt} &= 429(-.35)w^{-1.35}\frac{dw}{dt}\\ &= -150.15w^{-1.35}\frac{dw}{dt}\\ &= -150.15(10)^{-1.35}(.001)\\ &\approx -.0067\end{aligned}$$

The rate of change of the energy expenditure is about $-.0067$ cal/g/hr^2.

20. $E = 26.5w^{-.34}$

$$\begin{aligned}\frac{dE}{dt} &= 26.5(-.34)w^{-1.34}\frac{dw}{dt}\\ &= -9.01w^{-1.34}\frac{dw}{dt}\\ &= -9.01(5)^{-1.34}(0.05)\\ &\approx -.0521\end{aligned}$$

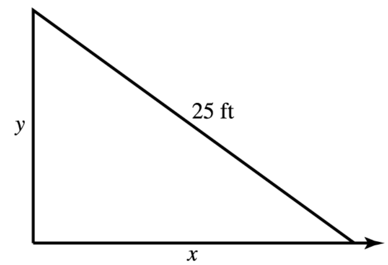
The rate of change of the energy expenditure is about $-.0521$ kcal/kg/km/day.

22. $T(x) = \frac{2+x}{2+x^2},\ x = 4,\ \frac{dx}{dt} = 4$

Find $\frac{dT}{dt}$.

$$\begin{aligned}\frac{dT}{dt} &= \frac{(2+x^2)\frac{d}{dt}(2+x) - (2+x)\frac{d}{dt}(2+x^2)}{(2+x^2)^2}\\ &= \frac{(2+x^2)\frac{dx}{dt} - (2+x)(2x)\frac{dx}{dt}}{(2+x^2)^2}\\ &= \frac{(2+x^2)\frac{dx}{dt} - (4x+2x^2)\frac{dx}{dt}}{(2+x^2)^2}\\ &= \frac{(2-4x-x^2)\frac{dx}{dt}}{(2+x^2)^2}\\ &= \frac{(2-16-16)4}{(2+16)^2}\\ &= -\frac{120}{324}\\ &\approx -.370\end{aligned}$$

24. Let $x =$ the distance of the base of the ladder from the base of the building;
$y =$ the distance up the side of the building to the top of the ladder.



Find $\frac{dy}{dt}$ when $x = 7$ ft and $\frac{dx}{dt} = 4$ ft/min.

Since $y = \sqrt{25^2 - x^2}$, when $x = 7$,

$$y = 24.$$

By the Pythagorean theorem,

$$x^2 + y^2 = 25^2.$$

$$\begin{aligned}\frac{d}{dt}(x^2+y^2) &= \frac{d}{dt}(25)^2\\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0\\ 2y\frac{dy}{dt} &= -2x\frac{dx}{dt}\\ \frac{dy}{dt} &= \frac{-2x}{2y}\frac{dx}{dt} = -\frac{x}{y}\frac{dx}{dt}\\ &= -\frac{7}{24}(4)\\ &= -\frac{7}{6}\end{aligned}$$

The ladder is sliding down the building at the rate of $\frac{7}{6}$ ft/min.

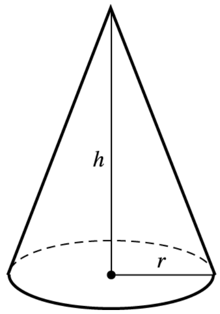
26. Let $r =$ the radius of the circle formed by the ripple.

Find $\frac{dA}{dt}$ when $r = 4$ ft and $\frac{dr}{dt} = 2$ ft/min.

$$\begin{aligned}A &= \pi r^2\\ \frac{dA}{dt} &= 2\pi r\frac{dr}{dt}\\ &= 2\pi(4)(2)\\ &= 16\pi\end{aligned}$$

The area is changing at the rate of 16π ft^2/min.

28. Let $r =$ the radius of the base of the conical pile.



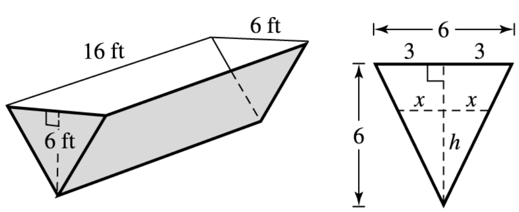
Find $\frac{dV}{dt}$ when $r = 5$ in, and $\frac{dr}{dt} = 1$ in/min.
$h = 2r$ for all t.

$$V = \frac{\pi}{3}r^2 h$$
$$V = \frac{\pi}{3}r^2(2r)$$
$$= \frac{2\pi}{3}r^3$$
$$\frac{dV}{dt} = \frac{3 \cdot 2\pi r^2}{3}\,\frac{dr}{dt}$$
$$\frac{dV}{dt} = 2\pi(5^2)(1)$$
$$= 50\pi$$

The volume in changing at the rate of 50π in^3/min.

30. Let $x =$ one-half the width of the triangular cross section;
$h =$ the height of the water;
$V =$ the volume of the water.

$$\frac{dV}{dt} = 4 \text{ cu ft per min}$$



Find $\frac{dh}{dt}$ when $h = 4$.

$$V = \begin{pmatrix} \text{Area of} \\ \text{triangular} \\ \text{cross section} \end{pmatrix} \cdot (\text{length})$$

Area of triangular cross section

$$= \frac{1}{2}(\text{base})(\text{height})$$
$$= \frac{1}{2}(2x)(h) = xh$$

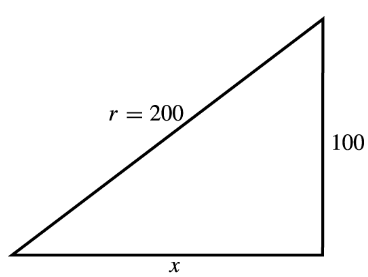
By similar triangles,

$$\frac{6}{2x} = \frac{6}{h},$$
$$\text{so} \quad x = \frac{h}{2}.$$

$$V = (xh)(16)$$
$$= \left(\frac{h}{2}h\right) 16$$
$$= 8h^2$$
$$\frac{dV}{dt} = 16h\,\frac{dh}{dt}$$
$$\frac{1}{16h}\,\frac{dV}{dt} = \frac{dh}{dt}$$
$$\frac{1}{16(4)}(4) = \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{1}{16}$$

The height of the water is increasing at a rate of $\frac{1}{16}$ ft/min.

32. Let $x =$ the horizontal length;
$r =$ the rope length.



$$\frac{dx}{dt} = 50 \text{ ft/min}$$

By the Pythagorean theorem,

$$\begin{aligned} x^2 + 100^2 &= 200^2 \\ x &= \sqrt{30{,}000} = 100\sqrt{3}. \\ r^2 &= x^2 + 100^2 \\ 2r\frac{dr}{dt} &= 2x\frac{dx}{dt} + 0 \\ r\frac{dr}{dt} &= x\frac{dx}{dt} \\ \frac{dr}{dt} &= \frac{x\frac{dx}{dt}}{r} \\ \frac{dr}{dt} &= \frac{100\sqrt{3}(50)}{200} = 25\sqrt{3} \\ &\approx 43.3 \end{aligned}$$

She must let out the string at a rate of $25\sqrt{3} \approx 43.3$ ft/min.

6.6 Differentials: Linear Approximation

2. $y = x^2 - 3x;\ x = 3,\ \Delta x = .1$

$$\begin{aligned} dy &= (2x - 3)\,dx \\ &\approx (2x - 3x)\,\Delta x \\ &= [2(3) - 3](.1) \\ &= (3)(.1) = .3 \end{aligned}$$

4. $y = 2x^3 + x^2 - 4x;\ x = 2,\ \Delta x = -.2$

$$\begin{aligned} dy &= (6x^2 + 2x - 4)\,dx \\ &\approx (6x^2 + 2x - 4)\,\Delta x \\ &= [6(2)^2 + 2(2) - 4](-.2) \\ &= (24 + 4 - 4)(-.2) = -4.8 \end{aligned}$$

6. $y = \sqrt{4x - 1};\ x = 5,\ \Delta x = .08$

$$\begin{aligned} dy &= \frac{1}{2}(4x - 1)^{-1/2}(4)\,dx \\ &= 2(4x - 1)^{-1/2}\,dx \\ &\approx 2(4x - 1)^{-1/2}\,\Delta x \\ &= 2[4(5) - 1]^{-1/2}(.08) \\ &= 2(19)^{-1/2}(.08) \\ &= \frac{2(.08)}{(19)^{1/2}} = .037 \end{aligned}$$

8. $y = \dfrac{6x - 3}{2x + 1};\ x = 3,\ \Delta x = -.04$

$$\begin{aligned} dy &= \frac{6(2x + 1) - 2(6x - 3)}{(2x + 1)^2}\,dx \\ &= \frac{12}{(2x + 1)^2}\,dx \\ &\approx \frac{12}{(2x + 1)^2}\,\Delta x \\ &= \frac{12}{[2(3) + 1]^2}(-.04) \\ &= \frac{-.48}{49} = -.010 \end{aligned}$$

10. $\sqrt{23}$

We know $\sqrt{25} = 5,\ f(x) = \sqrt{x},\ x = 25$, and $dx = -2$.

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

$$dy = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2\sqrt{25}}(-2) = -\frac{1}{5} = -.2.$$

$$\sqrt{23} \approx f(x) + dy = 5 - .2 = 4.8$$

By calculator, $\sqrt{23} \approx 4.7958$.
The difference is $|4.8 - 4.7958| = .0042$.

12. $\sqrt{17.02}$

We know $\sqrt{16} = 4,\ f(x) = \sqrt{x},\ x = 16$, and $dx = 1.02$.

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

$$\begin{aligned} dy &= \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2\sqrt{16}}(1.02) \\ &= \frac{1}{8}(1.02) = .1275 \end{aligned}$$

$$\sqrt{17.02} \approx f(x) + dy = 4 + .1275 = 4.1275$$

By calculator, $\sqrt{17.02} \approx 4.1255$.
The difference is $|4.1275 - 4.1255| = .0020$.

14. $e^{-.002}$

We know $e^0 = 1,\ f(x) = e^x,\ x = 0$, and $dx = -.002$.

$$\frac{dy}{dx} = e^x$$

$$dy = e^x\,dx = e^0(-.002) = -.002$$

$$e^{-.002} \approx f(x) + dy = 1 - .002 = .998$$

By calculator, $e^{-.002} \approx .9980$.
The difference is $|.9980 - .998| = 0$.

16. ln .98

We know $\ln 1 = 0$, $f(x) = \ln x$, $x = 1$, and $dx = -.02$.

$$\frac{dy}{dx} = \frac{1}{x}$$
$$dy = \frac{1}{x}\,dx = \frac{1}{1}(-.02) = -.02$$
$$\ln .98 \approx f(x) + dy = 0 - .02 = -.02$$

By calculator, $\ln .98 \approx -.0202$.
The difference is $|-.02 - (-.0202)| = .0002$.

18. $A(x) = .04x^3 + .1x^2 + .5x + 6$

(a) $x = 3,\ \Delta x = 1$

$$\begin{aligned} dA &= (.12x^2 + .2x + .5)\,dx \\ &= (.12x^2 + .2x + .5)\,\Delta x \\ &= [(.12)(3)^2 + (.2)(3) + (.5)](1) \\ &= 2.18 \end{aligned}$$

(b) $x = 5,\ \Delta x = 1$

$$\begin{aligned} dA &= [(.12)(5)^2 + (.2)(5) + (.5)](1) \\ &= 4.5 \end{aligned}$$

20. $P(x) = -390 + 24x + 5x^2 - \frac{1}{3}x^3$

$x = 1000,\ \Delta x = 1$

$$\begin{aligned} dP &= (24 + 10x - x^2)\,dx \\ &\approx (24 + 10x - x^2)\,\Delta x \\ &= (24 + 10{,}000 - 100{,}000) \cdot 1 \\ &\approx -990{,}000 \end{aligned}$$

The change in profit is about $-\$990,000$.

22. Let $x =$ the number of beach balls;
$V =$ the volume of x beach balls.
Then $\frac{dV}{dr} \approx$ the volume of material in beach balls since they are hollow.

$$V = \frac{4}{3}\pi r^3 x$$
$$r = 6 \text{ in},\ x = 5000,\ \Delta r = .03 \text{ in}$$
$$\begin{aligned} dV &= \frac{4}{3}\pi(3r^2 x + r^3)\Delta r \\ &= \frac{4}{3}\pi(3 \cdot 36 \cdot 5000 + 216)(.03) \\ &= 21{,}608.64\pi \end{aligned}$$

$21{,}608\pi$ in^3 of material would be needed.

24. $C = \frac{5x}{9 + x^2}$

$$\begin{aligned} dC &= \frac{5(9 + x^2) - 2x(5x)}{(9 + x^2)^2}\,dx \\ &= \frac{45 + 5x^2 - 10x^2}{(9 + x^2)^2}\,dx \\ &= \frac{45 - 5x^2}{(9 + x^2)^2}\,dx \\ &\approx \frac{45 - 5x^2}{(9 + x^2)^2}\,\Delta x \end{aligned}$$

(a) $x = 1,\ \Delta x = .5$

$$\begin{aligned} dC &\approx \frac{45 - 5(1)^2}{(9 + 1)^2}(.5) \\ &= \frac{40}{100}(.5) \\ &= .2 \end{aligned}$$

(b) $x = 2,\ \Delta x = .25$

$$\begin{aligned} dC &\approx \frac{45 - 5(2)^2}{(9 + 4)^2}(.25) \\ &= .037 \end{aligned}$$

26. $A = \pi r^2,\ r = 1.7 \text{ mm},\ \Delta r = -.1 \text{ mm}$

$$\begin{aligned} dA &= 2\pi r\,dr \\ \Delta A &\approx 2\pi r\,\Delta r \\ &= 2\pi(1.7)(-.1) \\ &= -.34\pi \text{ mm}^2 \end{aligned}$$

28. $A = \pi r^2,\ r = 1.2 \text{ mi},\ \Delta r = .2 \text{ mi}$

$$\begin{aligned} dA &= 2\pi r\,dr \\ \Delta A &\approx 2\pi r\,\Delta r \\ &= 2\pi(1.2)(.2) \\ &= .48\pi \text{ mi}^2 \end{aligned}$$

30. $A(p) = \frac{1.181p}{94.359 - p}$

(a) Since values for p must be non-negative and the denominator can't be zero, a sensible domain would be from 0 to about 94.

(b)
$$\begin{aligned} dA &= \frac{(94.359 - p)(1.181) - 1.181p(-1)}{(94.359 - p)^2}dp \\ &= \frac{111.437979 - 1.181p + 1.181p}{(94.359 - p)^2}dp \\ &= \frac{111.437979}{(94.359 - p)^2}dp \end{aligned}$$

We are given $p = 60$ and $dp = 65 - 60 = 5$.

$$dA \approx \frac{111.437979}{(94.359 - 60)^2}(5) \approx .472$$

It will take about .47 years.
The actual value is

$$A(65) - A(60) \approx 2.615 - 2.062 = .553$$

or about 0.55 years.

32. $V = \frac{4}{3}\pi r^3,\ r = 4 \text{ cm},\ \Delta r = .2 \text{ cm}$

$$\begin{aligned} dV &= 4\pi r^2\, dr \\ \Delta V &\approx 4\pi r^2\, \Delta r \\ &= 4\pi(4)^2(.2) \\ &= 12.8\pi \text{ cm}^3 \end{aligned}$$

34. $A = x^2;\ x = 3.45,\ \Delta x = \pm.002$

$$\begin{aligned} dA &= 2x\, dx \\ \Delta A &\approx 2x\, \Delta x \\ &= 2(3.45)(\pm.002) \\ &= \pm.0138 \text{ sq in}^2 \end{aligned}$$

36. $V = \frac{4}{3}\pi r^3;\ r = 5.81,\ \Delta r = \pm.003$

$$\begin{aligned} dV &= \frac{4}{3}\pi(3r^2)\, dr \\ \Delta V &\approx \frac{4}{3}\pi(3r^2)\, \Delta r \\ &= 4\pi(5.81)^2(\pm.003) \\ &= \pm.405\pi \approx \pm1.273 \text{ in}^3 \end{aligned}$$

Chapter 6 Review Exercises

2. $f(x) = 4x^2 - 8x - 3;\ [-1, 2]$
$f'(x) = 8x - 8 = 0$ when $x = 1$.

$$\begin{aligned} f(-1) &= 9 \\ f(1) &= -7 \\ f(2) &= -3 \end{aligned}$$

Absolute maximum of 9 at -1; absolute minimum of -7 at 1

4. $f(x) = -2x^3 - 2x^2 + 2x - 1;\ [-3, 1]$
$f'(x) = -6x^2 - 4x + 2$

$f'(x) = 0$ when

$$\begin{aligned} 3x^2 + 2x - 1 &= 0 \\ (3x - 1)(x + 1) &= 0 \end{aligned}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -1.$$

$$\begin{aligned} f(-3) &= 29 \\ f(-1) &= -3 \\ f\left(\frac{1}{3}\right) &= -\frac{17}{27} \\ f(1) &= -3 \end{aligned}$$

Absolute maximum of 29 at -3; absolute minimum of -3 at -1 and 1.

10. $x^2y^3 + 4xy = 2$

$$\begin{aligned} \frac{d}{dx}(x^2y^3 + 4xy) &= \frac{d}{dx}(2) \\ 2xy^3 + 3y^2\left(\frac{dy}{dx}\right)x^2 + 4y + 4x\frac{dy}{dx} &= 0 \\ (3x^2y^2 + 4x)\frac{dy}{dx} &= -2xy^3 - 4y \\ \frac{dy}{dx} &= \frac{-2xy^2 - 4y}{3x^2y^2 + 4x} \end{aligned}$$

12. $9\sqrt{x} + 4y^3 = \frac{2}{x}$

$$\begin{aligned} \frac{d}{dx}\left(9\sqrt{x} + 4y^3\right) &= \frac{d}{dx}\left(\frac{2}{x}\right) \\ \frac{9}{2}x^{-1/2} + 12y^2\frac{dy}{dx} &= \frac{-2}{x^2} \\ 12y^2\frac{dy}{dx} &= \frac{-2}{x^2} - \frac{9x^{-1/2}}{2} \\ 12y^2\frac{dy}{dx} &= \frac{-4 - 9x^{3/2}}{2x^2} \\ \frac{dy}{dx} &= \frac{-4 - 9x^{3/2}}{24x^2y^2} \end{aligned}$$

14.
$$\frac{x+2y}{x-3y} = y^{1/2}$$
$$x + 2y = y^{1/2}(x-3y)$$
$$\frac{d}{dx}(x+2y) = \frac{d}{dx}\left[y^{1/2}(x-3y)\right]$$
$$1 + 2\frac{dy}{dx} = y^{1/2}\left(1 - 3\frac{dy}{dx}\right) + \frac{1}{2}(x-3y)y^{-1/2}\frac{dy}{dx}$$
$$1 + 2\frac{dy}{dx} = y^{1/2} - 3y^{1/2}\frac{dy}{dx} + \frac{1}{2}xy^{-1/2}\frac{dy}{dx} - \frac{3}{2}y^{1/2}\frac{dy}{dx}$$
$$\left(2 + 3y^{1/2} - \frac{1}{2}xy^{-1/2} + \frac{3}{2}y^{1/2}\right)\frac{dy}{dx} = y^{1/2} - 1$$
$$\frac{2y^{1/2}\left(2 + \frac{9}{2}y^{1/2} - \frac{1}{2}xy^{-1/2}\right)}{2y^{1/2}}\frac{dy}{dx} = y^{1/2} - 1$$
$$\left(\frac{4y^{1/2} + 9y - x}{2y^{1/2}}\right)\frac{dy}{dx} = y^{1/2} - 1$$
$$\frac{dy}{dx} = \frac{2y - 2y^{1/2}}{4y^{1/2} + 9y - x}$$

16. $\sqrt{2x} - 4yx = -22$, tangent line at $(2,3)$
$$\frac{d}{dx}(\sqrt{2x} - 4yx) = \frac{d}{dx}(-22)$$
$$\frac{1}{2}(2x)^{-1/2}(2) - 4y - 4x\frac{dy}{dx} = 0$$
$$(2x)^{-1/2} - 4y = 4x\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{(2x)^{-1/2} - 4y}{4x} = \frac{\frac{1}{\sqrt{2x}} - 4y}{4x}$$

At $(2,3)$,
$$m = \frac{\frac{1}{\sqrt{2\cdot 2}} - 4\cdot 3}{4\cdot 2} = \frac{\frac{1}{2} - 12}{8} = -\frac{23}{16}.$$

The equation of the tangent line is
$$y - y_1 = m(x - x_1)$$
$$y - 3 = -\frac{23}{16}(x-2)$$
$$23x + 16y = 94.$$

20. $y = \dfrac{9-4x}{3+2x};\ \dfrac{dx}{dt} = -1,\ x = -3$
$$\frac{dy}{dt} = \frac{(-4)(3+2x) - (2)(9-4x)}{(3+2x)^2}\frac{dx}{dt} = \frac{-30}{(3+2x)^2}\frac{dx}{dt} = \frac{-30}{[3+2(-3)]^2}(-1) = \frac{30}{9} = \frac{10}{3}$$

22. $\dfrac{x^2+5y}{x-2y} = 2;\ \dfrac{dx}{dt} = 1,\ x = 2,\ y = 0$
$$x^2 + 5y = 2(x-2y)$$
$$x^2 + 5y = 2x - 4y$$
$$9y = -x^2 + 2x$$
$$y = \frac{1}{9}(-x^2 + 2x) = -\frac{1}{9}x^2 + \frac{2}{9}x$$
$$\frac{dy}{dt} = \left(-\frac{2}{9}x + \frac{2}{9}\right)\frac{dx}{dt} = \left[\left(-\frac{2}{9}\right)(2) + \frac{2}{9}\right](1) = -\frac{4}{9} + \frac{2}{9} = -\frac{2}{9}$$

24. $y = 8 - x^2 + x^3,\ x = -1,\ \Delta x = .02$
$$dy = (-2x + 3x^2)\,dx \approx (-2x+3x^2)\,\Delta x = [-2(-1) + 3(-1)^2](.02) = .1$$

26. $-12x + x^3 + y + y^2 = 4$
$$\frac{dy}{dx}(-12x + x^3 + y + y^2) = \frac{d}{dx}(4)$$
$$-12 + 3x^2 + \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$(1+2y)\frac{dy}{dx} = 12 - 3x^2$$
$$\frac{dy}{dx} = \frac{12 - 3x^2}{1+2y}$$

(a) If $\frac{dy}{dx} = 0$,

$$\begin{aligned} 12 - 3x^2 &= 0 \\ 12 &= 3x^2 \\ \pm 2 &= x. \end{aligned}$$

$x = 2$:

$$\begin{aligned} -24 + 8 + y + y^2 &= 4 \\ y + y^2 &= 20 \\ y^2 + y - 20 &= 0 \\ (y+5)(y-4) &= 0 \\ y = -5 \text{ or } y &= 4 \end{aligned}$$

$(2, -5)$ and $(2, 4)$ are critical points.
$x = -2$:

$$\begin{aligned} 24 - 8 + y + y^2 &= 4 \\ y + y^2 &= -12 \\ y^2 + y + 12 &= 0 \\ y &= \frac{-1 \pm \sqrt{1^2 - 48}}{2} \end{aligned}$$

This leads to imaginary roots.
$x = -2$ does not produce critical points.

(b)

x	y_1	y_2
1.9	−4.99	3.99
2	−5	4
2.1	−4.99	3.99

The point $(2, -5)$ is a relative minimum.
The point $(2, 4)$ is a relative maximum.

(c) There is no absolute maximum or minimum for x or y.

28. (a)

$$\begin{aligned} P(x) &= -x^3 + 10x^2 - 12x \\ P'(x) &= -3x^2 + 20x - 12 = 0 \\ 3x^2 - 20x + 12 &= 0 \\ (3x - 2)(x - 6) &= 0 \end{aligned}$$

$$3x - 2 = 0 \quad \text{or} \quad x - 6 = 0$$

$$x = \frac{2}{3} \quad \text{or} \quad x = 6$$

$$P''(x) = -6x + 20$$

$$P''\left(\frac{2}{3}\right) = 16,$$

which implies that $x = \frac{2}{3}$ is location of minimum.

$$P''(6) = -16,$$

which implies that $x = 6$ is location of maximum.
Thus, 600 boxes will produce a maximum profit.

(b) Maximum profit $= P(6)$

$$\begin{aligned} &= -(6)^3 + 10(6)^2 - 12(6) \\ &= 72 \end{aligned}$$

The maximum profit is \$720.

30. $V = \pi r^2 h = 40$, so $h = \frac{40}{\pi r^2}$.

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \left(\frac{40}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{80}{r} \end{aligned}$$

$$\begin{aligned} \text{Cost} = C(r) &= 4(2\pi r^2) + 3\left(\frac{80}{r}\right) \\ &= 8\pi r^2 + \frac{240}{r} \end{aligned}$$

$$\begin{aligned} C'(r) &= 16\pi r - \frac{240}{r^2} \\ 16\pi r - \frac{240}{r^2} &= 0 \\ 16\pi r^3 &= 240 \\ r^3 &= \frac{15}{\pi} \\ r &\approx 1.684 \end{aligned}$$

$C''(r) = 16\pi + \frac{480}{r^3} > 0$, so $r = 1.684$ minimizes cost.

$$h = \frac{40}{\pi r^2} = \frac{40}{\pi(1.684)^2} = 4.490$$

The radius should be 1.684 in and the height should be 4.490 in.

32. $M =$ 980,000 cases sold per year
$k = \$2$, cost to store 1 case for 1 yr
$f = \$20$, fixed cost for order
$x =$ the number of cases per order

$$\begin{aligned} x &= \sqrt{\frac{2fM}{k}} \\ &= \sqrt{\frac{2(20)(980{,}000)}{2}} \\ &= \sqrt{19{,}600{,}000} \\ &= 1400\sqrt{10} \\ &\approx 4427 \end{aligned}$$

The store should order 4427 cases each time.

34. Use equation (3) from Section 6.3 with $k = 2$, $M =$240,000, and $f = 15$.

$$q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(15)(240{,}000)}{2}}$$
$$= \sqrt{3{,}600{,}000} \approx 1897.4$$

$T(1897) \approx 3794.7333$ and $T(1898) \approx 3794.7334$. Since $T(1897) < T(1898)$, then the number of batches per year should be

$$\frac{M}{q} = \frac{240{,}000}{1897} \approx 127.$$

36. $q = \dfrac{A}{p^k}$

$$\frac{dq}{dp} = -k\,\frac{A}{p^{k+1}}$$
$$E = -\frac{p}{q}\cdot\frac{dq}{dp}$$
$$= \left(-\frac{p}{\frac{A}{p^k}}\right)(-k)\left(\frac{A}{p^{k+1}}\right)$$
$$= \left(-\frac{p^{k+1}}{A}\right)\left(-k\,\frac{A}{p^{k+1}}\right)$$
$$= k$$

The demand is elastic when $k > 1$ and inelastic when $k < 1$.

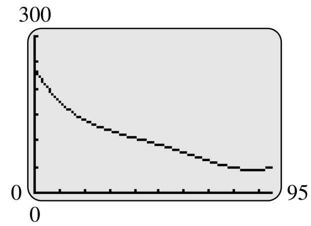
38. $\dfrac{dx}{dt} = rx(N - x)$

$$= rxN - rx^2$$
$$\frac{d^2x}{dt^2} = rN\,\frac{dx}{dt} - 2rx\,\frac{dx}{dt}$$
$$= r\,\frac{dx}{dt}\,[N - 2x]$$
$$= r[rx(N - x)][N - 2x]$$
$$= r^2x(N - x)(N - 2x)$$

$\dfrac{d^2x}{dt^2} = 0$ when $x = 0,\ x = N$, or $x = \frac{N}{2}$.

On $\left(0, \frac{N}{2}\right)$, $\frac{d^2x}{dt^2} > 0$; therefore, the curve is concave upward.
On $\left(\frac{N}{2}, N\right)$, $\frac{d^2x}{dt^2} < 0$; therefore, the curve is concave downward.
Hence $x = \frac{N}{2}$ is a point of inflection.

40. (a)



(b) To find where the maximum and minimum numbers occur, use a graphing calculator to locate any extreme points on the graph. One critical number is formed at about 87.78.

t	$P(t)$
0	237.09
87.78	43.56
95	48.66

The maximum number of polygons is about 237 at birth. The minimum number is about 44.

42. $\dfrac{dV}{dt} = 1.2\ \text{ft}^3/\text{min}$

Find $\frac{dr}{dt}$ when $r = 1.2$ ft.

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = \frac{4}{3}\pi(3)r^2\,\frac{dr}{dt}$$
$$= 4\pi r^2\,\frac{dr}{dt}$$
$$1.2 = 4\pi(1.2)^2\,\frac{dr}{dt}$$
$$\frac{1.2}{4\pi(1.2)^2} = \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{4.8\pi} \approx .0663$$

The radius is changing at the rate of

$\frac{1}{4.8\pi} \approx .0663$ ft/min.

44. $V = \dfrac{4}{3}\pi r^3,\ r = 4$ in, $\Delta r = .02$ in

$$dV = 4\pi r^2\,dr$$
$$\Delta V \approx 4\pi r^2\,\Delta r$$
$$= 4\pi(4)^2(.02)$$
$$= 1.28\pi \text{ or about } 4.021$$

The volume of the coating is 1.28π in^3 or about 4.021 in^3.

46.
$$\begin{aligned} V &= l \cdot w \cdot h \\ w &= 4 + h \\ l + g &= 130;\ g = 2(w+h) \\ l + 2(w+h) &= 130 \\ l + 2w + 2h &= 130 \end{aligned}$$

$$\begin{aligned} l &= 130 - 2w - 2h \\ &= 130 - 2(4+h) - 2h \\ &= 122 - 4h \end{aligned}$$

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= (122-4h)(4+h)h \\ &= 488h + 106h^2 - 4h^3 \end{aligned}$$

$$\frac{dV}{dh} = 488 + 212h - 12h^2$$

Set $\frac{dV}{dh} = 0$.

$$\begin{aligned} 488 + 212h - 12h^2 &= 0 \\ 12h^2 - 212h - 488 &= 0 \\ 3h^2 - 53h - 122 &= 0 \\ h &= \frac{53 \pm \sqrt{2809 + 1464}}{6} \\ &\approx -2.06 \quad \text{or} \quad h = 19.73 \end{aligned}$$

h can't be negative, so

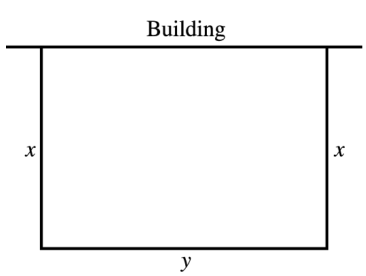
$$h \approx 19.73.$$

Thus,

$$\begin{aligned} l &= 122 - 4h \\ &\approx 43.1. \end{aligned}$$

The length that produces the maximum volume is about 43.1 inches.

48. Let $x =$ width of play area;
$y =$ length of play area.



An equation describing the amount of fencing is

$$\begin{aligned} 900 &= 2x + y \\ y &= 900 - 2x. \end{aligned}$$

Then,

$$\begin{aligned} A &= xy \\ A(x) &= x(900 - 2x) \\ &= 900x - 2x^2. \end{aligned}$$

If $A'(x) = 900 - 4x = 0,$
$x = 225.$

Then $y = 900 - 2(225) = 450.$

$A''(x) = -4 < 0$, so the area is maximized if the dimensions are 225 m by 450 m.

Extended Application: A Total Cost Model for a Training Program

1. $Z(m) = \frac{C_1}{m} + DtC_2 + DC_3\left(\frac{m-1}{2}\right)$

$$\begin{aligned} Z'(m) &= -\frac{C_1}{m^2} + 0 + \frac{DC_3}{2} \\ &= -\frac{C_1}{m^2} + \frac{DC_3}{2} \end{aligned}$$

2. $Z'(m) = 0$ when

$$\begin{aligned} \frac{DC_3}{2} &= \frac{C_1}{m^2} \\ m^2 &= \frac{2C_1}{DC_3} \\ m &= \sqrt{\frac{2C_1}{DC_3}}. \end{aligned}$$

3. $D = 3,\ t = 12,\ C_1 = 15{,}000,\ C_3 = 900$

$$\begin{aligned} m &= \sqrt{\frac{2(15{,}000)}{3(900)}} \\ &= \sqrt{\frac{30{,}000}{2700}} \\ &= \sqrt{\frac{100}{9}} \\ &\approx 3.33 \end{aligned}$$

4. $3 < 3.33 < 4$
$m^+ = 4$ and $m^- = 3$

5. $$Z(m) = \frac{C_1 + mDtC_2}{m} + \frac{DC_3\left[\frac{m(m-1)}{2}\right]}{m}$$
$$= \frac{C_1}{m} + DtC_2 + DC_3\left(\frac{m-1}{2}\right)$$

$C_1 =$ 15,000; $D = 3,\ t = 12,$
$C_2 = 100;\ C_3 = 900$

$$Z(m^+) = Z(4)$$
$$= \frac{15{,}000}{4} + 3(12)(100) + 3(900)\left(\frac{4-1}{2}\right)$$
$$= 3750 + 3600 + 4050$$
$$= \$11{,}400$$

$$Z(m^-) = Z(3)$$
$$= \frac{15,000}{3} + 3600 + 3\frac{(900)(3-1)}{2}$$
$$= 5000 + 3600 + 2700$$
$$= \$11{,}300$$

6. Since $Z(3) < Z(4)$, the optimal time interval is 3 months.

$$N = mD$$
$$= 3 \cdot 3$$
$$= 9$$

There should be 9 trainees per batch.

Chapter 7

INTEGRATION

7.1 Antiderivatives

6. $\int 2\,dy = 2\int 1\,dy = 2\int y^0\,dy$

$= \frac{2}{1}y^{0+1} + C$

$= 2y + C$

8. $\int (3x - 5)\,dx$

$= 3\int x\,dx - 5\int x^0\,dx$

$= 3 \cdot \frac{1}{2}x^2 - 5 \cdot \frac{1}{1}x + C$

$= \frac{3x^2}{2} - 5x + C$

10. $\int (5x^2 - 6x + 3)\,dx$

$= 5\int x^2\,dx - 6\int x\,dx + 3\int x^0\,dx$

$= \frac{5x^3}{3} - \frac{6x^2}{2} + 3x + C$

$= \frac{5x^3}{3} - 3x^2 + 3x + C$

12. $\int (12y^3 + 6y^2 - 8y + 5)\,dy$

$= 12\int y^3\,dy + 6\int y^2\,dy - 8\int y\,dy$

$+ 5\int dy$

$= \frac{12y^4}{4} + \frac{6y^3}{3} - \frac{8y^2}{2} + 5y + C$

$= 3y^4 + 2y^3 - 4y^2 + 5y + C$

14. $\int t^{1/4}\,dt = \frac{t^{1/4+1}}{\frac{1}{4}+1} + C$

$= \frac{t^{5/4}}{\frac{5}{4}} + C$

$= \frac{4t^{5/4}}{5} + C$

16. $\int x^2(x^4 + 4x + 3)dx = \int (x^6 + 4x^3 + 3x^2)dx$

$= \frac{x^7}{7} + \frac{4x^4}{4} + \frac{3x^3}{3} + C$

$= \frac{x^7}{7} + x^4 + x^3 + C$

18. $\int (15x\sqrt{x} + 2\sqrt{x})\,dx$

$= 15\int x(x^{1/2})\,dx + 2\int x^{1/2}\,dx$

$= 15\int x^{3/2}\,dx + 2\int x^{1/2}\,dx$

$= \frac{15x^{5/2}}{\frac{5}{2}} + \frac{2x^{3/2}}{\frac{3}{2}} + C$

$= 15\left(\frac{2}{5}\right)x^{5/2} + 2\left(\frac{2}{3}\right)x^{3/2} + C$

$= 6x^{5/2} + \frac{4x^{3/2}}{3} + C$

20. $\int (56t^{5/2} + 18t^{7/2})\,dt$

$= 56\int t^{5/2}\,dt + 18\int t^{7/2}\,dt$

$= \frac{56t^{7/2}}{\frac{7}{2}} + \frac{18t^{9/2}}{\frac{9}{2}} + C$

$= 16t^{7/2} + 4t^{9/2} + C$

22. $\int \left(\frac{4}{x^3}\right)\,dx = \int 4x^{-3}\,dx$

$= 4\int x^{-3}\,dx$

$= \frac{4x^{-2}}{-2} + C$

$= -2x^{-2} + C$

$= \frac{-2}{x^2} + C$

24. $\int \left(\sqrt{u} + \frac{1}{u^2}\right)\,du = \int u^{1/2}\,du + \int u^{-2}\,du$

$= \frac{u^{3/2}}{\frac{3}{2}} + \frac{u^{-1}}{-1} + C$

$= \frac{2u^{3/2}}{3} - \frac{1}{u} + C$

26. $\int (8x^{-3}+4x^{-1})\,dx = 8\int x^{-3}\,dx + 4\int x^{-1}\,dx$

$$= \frac{8x^{-2}}{-2} + 4\ \ln\ |x| + C$$
$$= -4x^{-2} + 4\ \ln\ |x| + C$$
$$= \frac{-4}{x^2} + 4\ \ln\ |x| + C$$

28. $\int \frac{2}{3x^4}\,dx = \int \frac{2}{3}x^{-4}dx$

$$= \frac{2}{3}\int x^{-4}dx$$
$$= \frac{2}{3}\left(\frac{x^{-3}}{-3}\right) + C$$
$$= -\frac{2}{9}x^{-3} + C$$
$$= -\frac{2}{9x^3} + C$$

30. $\int -4e^{.2v}\,dv = -4\int e^{.2v}\,dv$

$$= (-4)\frac{1}{.2}e^{.2v} + C$$
$$= -20e^{.2v} + C$$

32. $\int \left(\frac{9}{x} - 3e^{-.4x}\right)dx$

$$= \int \frac{9}{x}\,dx - 3\int e^{-.4x}\,dx$$
$$= 9\ \ln\ |x| - 3\left(-\frac{1}{.4}\right)e^{-.4x} + C$$
$$= 9\ \ln\ |x| + \frac{15e^{-.4x}}{2} + C$$

34. $\int \frac{2y^{1/2} - 3y^2}{y}\,dy$

$$= \int \frac{2y^{1/2}}{y}\,dy - \int \frac{3y^2\,dy}{y}$$
$$= 2\int y^{-1/2}\,dy - 3\int y\,dy$$
$$= 2\frac{y^{1/2}}{\frac{1}{2}} - \frac{3y^2}{2} + C$$
$$= 4y^{1/2} - \frac{3y^2}{2} + C$$

36. $\int (v^2 - e^{3v})\,dv = \int v^2\,dv - \int e^{3v}\,dv$

$$= \frac{v^3}{3} - \frac{e^{3v}}{3} + C$$
$$= \frac{v^3 - e^{3v}}{3} + C$$

38. $\int (2y-1)^2\,dy = \int (4y^2 - 4y + 1)\,dy$

$$= \frac{4y^3}{3} - \frac{4y^2}{2} + y + C$$
$$= \frac{4y^3}{3} - 2y^2 + y + C$$

40. $\int \frac{1 - 2\sqrt[3]{z}}{\sqrt[3]{z}}\,dz = \int \left(\frac{1}{\sqrt[3]{z}} + \frac{2\sqrt[3]{z}}{\sqrt[3]{z}}\right)dz$

$$= \int (z^{-1/3} - 2)\,dz$$
$$= \frac{z^{2/3}}{\frac{2}{3}} - 2z + C$$
$$= \frac{3z^{2/3}}{2} - 2z + C$$

42. $\int 3^{2x}dx = \frac{3^{2x}}{2(\ln 3)} + C$

44. Find $f(x)$ such that $f'(x) = 6x^2 - 4x + 3$, and $(0,1)$ is on the curve.

$$f(x) = \int (6x^2 - 4x + 3)\,dx$$
$$= \frac{6x^3}{3} - \frac{4x^2}{2} + 3x + C$$
$$= 2x^3 - 2x^2 + 3x + C$$

Since $(0,1)$ is on the curve, then $f(0) = 1$.

$$f(0) = 2(0)^3 - 2(0)^2 + 3(0) + C = 1$$
$$C = 1$$

Thus,

$$f(x) = 2x^3 - 2x^2 + 3x + 1.$$

46. $C'(x) = .2x^2 + 5x$; fixed cost is \$10.

$$C(x) = \int (.2x^2 + 5x)\,dx$$
$$= \frac{.2x^3}{3} + \frac{5x^2}{2} + k$$
$$C(0) = \frac{.2(0)^3}{3} + \frac{5(0)^2}{2} + k = k$$

Since $C(0) = 10,\ k = 10$.

Thus,

$$C(x) = \frac{.2x^3}{3} + \frac{5x^2}{2} + 10.$$

48. $C'(x) = x^{1/2}$, 16 units cost \$45.

$$C(x) = \int x^{1/2}\,dx = \frac{x^{3/2}}{\frac{3}{2}} + k = \frac{2}{3}x^{3/2} + k$$

$$C(16) = \frac{2}{3}(16)^{3/2} + k = \frac{2}{3}(64) + k = \frac{128}{3} + k$$

Since $C(16) = 45$,

$$\frac{128}{3} + k = 45$$
$$k = \frac{7}{3}.$$

Thus,

$$C(x) = \frac{2}{3}x^{3/2} + \frac{7}{3}.$$

50. $C'(x) = x + \frac{1}{x^2}$; 2 units cost \$5.50, so

$$C(2) = 5.50.$$

$$\begin{aligned} C(x) &= \int \left(x + \frac{1}{x^2}\right) dx \\ &= \int (x + x^{-2})\,dx \\ &= \frac{x^2}{2} + \frac{x^{-1}}{-1} + k \\ C(x) &= \frac{x^2}{2} - \frac{1}{x} + k \\ C(2) &= \frac{(2)^2}{2} - \frac{1}{2} + k \\ &= 2 - \frac{1}{2} + k \end{aligned}$$

Since $C(2) = 5.50$,

$$5.50 - 1.5 = k$$
$$4 = k.$$

Thus,

$$C(x) = \frac{x^2}{2} - \frac{1}{x} + 4.$$

52. $C'(x) = 1.2^x(\ln\ 1.2)$; 2 units cost \$9.44
(Hint: Recall that $a^x = e^{x\ \ln\ a}$.)

$$\begin{aligned} C(x) &= \int 1.2^x(\ln\ 1.2)\,dx \\ &= \ln\ 1.2 \int 1.2^x\,dx \\ &= \ln\ 1.2 \int e^{x\ \ln\ 1.2}\,dx \\ &= \ln\ 1.2\left(\frac{1}{\ln\ 1.2}e^{x\ \ln 1.2}\right) + k \\ &= e^{x\ \ln\ 1.2} + k \\ C(2) &= e^{2\ \ln\ 1.2} + k = 1.44 + k \end{aligned}$$

Since $C(2) = 9.44$,

$$144 + k = 9.44$$
$$k = 8.$$

Thus,

$$\begin{aligned} C(x) &= e^{x\ \ln\ 1.2} + 8 \\ &= 1.2^x + 8. \end{aligned}$$

54. $R'(x) = 50 - x$

$$\begin{aligned} R &= \int (50 - x)dx \\ &= 50x - \frac{x^2}{2} + C. \end{aligned}$$

If $x = 0$, then $R = 0$ (no items sold means no revenue), and

$$\begin{aligned} 0 &= 50(0) - \frac{(0)^2}{2} + C \\ 0 &= C. \end{aligned}$$

Thus, $R = 50x - \frac{x^2}{2}$

gives the revenue function. Now, recall that $R = xp$, where p is the demand function. Then

$$50x - \frac{x^2}{2} = xp$$
$$50 - \frac{x}{2} = p,$$

which gives the demand function.

56. $R'(x) = 600 - 5e^{.0002x}$

$$\begin{aligned} R &= \int (600 - 25{,}000e^{.0002x})dx \\ &= 600x - 25{,}000e^{.0002x} + C. \end{aligned}$$

If $x = 0$, then $R = 0$ (no items sold means no revenue), and

$$\begin{aligned} 0 &= 600(0) - 25{,}000e^{.0002(0)} + C \\ 0 &= -25{,}000 + C \\ 0 &= 25{,}000. \end{aligned}$$

Thus, $R = 600x - 25{,}000e^{.0002x} + 25{,}000$
$= 600x + 25{,}000(1 - e^{.0002x})$

gives the revenue function. Now, recall that $R = xp$, where p is the demand function. Then

$$600x + 25{,}000(1 - e^{.0002x}) = xp$$
$$600 + \frac{25{,}000}{x}(1 - e^{.0002x}) = p,$$

which gives the demand function.

58. $P'(x) = 2x + 20$; profit is -50 when 0 hamburgers are sold.

$$\begin{aligned} P(x) &= \int (2x+20)\,dx \\ &= \frac{2x^2}{2} + 20x + k \\ &= x^2 + 20x + k \\ P(0) &= 0^2 + 20(0) + k \end{aligned}$$

Since $P(0) = -50,\ k = -50.$

$$P(x) = x^2 + 20x - 50$$

60. **(a)** $f'(t) = .01e^{-.01t}$

$$\begin{aligned} f(t) &= \int .01e^{-.01t}\,dt \\ &= -\frac{.01e^{-.01t}}{.01} + k \\ &= -e^{-.01t} + k \end{aligned}$$

(b) $$\begin{aligned} f(0) &= -e^{-.01(0)} + k \\ &= -e^0 + k \\ &= -1 + k \end{aligned}$$

Since $f(0) = 0,$

$$\begin{aligned} 0 &= -1 + k \\ k &= 1. \\ f(t) &= -e^{-.01t} + 1 \end{aligned}$$

$$\begin{aligned} f(10) &= -e^{-.01(10)} + 1 \\ &= -e^{-.1} + 1 \\ &= -.905 + 1 \\ &= .095 \end{aligned}$$

.095 unit is excreted in 10 min.

62. **(a)** $c(t) = (c_0 - C)e^{-kAt/V} + M$

$$\begin{aligned} c'(t) &= (c_0 - C)\left(\frac{-kA}{V}\right)e^{-kAt/V} \\ &= \frac{-kA}{V}(c_0 - C)e^{-kAt/V} \end{aligned}$$

(b) Since equation (1) states

$$c'(t) = \frac{kA}{V}[C - c(t)],$$

then from (a) and by substituting from equation (2), we obtain

$$\begin{aligned} &(c_0 - C)\left(\frac{-kA}{V}\right)e^{-kAt/V} \\ &\quad = \frac{kA}{V}C - \frac{kA}{V}[(c_0 - C)e^{-kAt/V} + M] \\ &\frac{-kA}{V}(c_0 - C)e^{-kAt/V} + M \\ &\quad = \frac{-kA}{V}(c_0 - C)e^{-kAt/V} + \frac{kA}{V}C - \frac{kA}{V}M. \end{aligned}$$

If $t = 0,\ c(t) = c_0$, so

$$\begin{aligned} c_0 &= (c_0 - C)e^0 + M \quad \textit{Equation (2)} \\ c_0 &= c_0 - C + M \\ \text{or} \quad C &= M. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{-kA}{V}(c_0 - C)e^{-kAt/V} \\ &\quad = \frac{-kA}{V}(c_0 - C)e^{-kAt/V} + \frac{kA}{V}M - \frac{kA}{V}M \end{aligned}$$

or

$$\frac{-kA}{V}(c_0 - C)e^{-kAt/V} = \frac{-kA}{V}(c_0 - C)e^{-kAt/V}.$$

64. $g'(x) = -.00097x^3 + .022x^2 - .137x + .0989$

(a) $$\begin{aligned} g(x) &= \int (-.00097x^3 + .022x^2 - .137x + .0989)dx \\ &= -.000243x^4 + .00733x^3 - .0685x^2 \\ &\quad + .0989x + C \end{aligned}$$

In 1985 ($x = 0$), $g(x) = 2.4$, and

$$\begin{aligned} 2.4 &= -.000243(0)^4 + .00733x^3 - .0685(0)^2 \\ &\quad + .0989(0) + C \\ 2.4 &= C. \end{aligned}$$

Thus,

$$\begin{aligned} g(x) &= -.000243x^4 + .00733x^3 - .0685x^2 \\ &\quad + .0989x + 2.4. \end{aligned}$$

(b) In 2000, $x = 15$, and

$$\begin{aligned} g(15) &= -.000243(15)^4 + .00733(15)^3 - .0685(15)^2 \\ &\quad + .0989(15) + 2.4. \\ &\approx -12.3019 + 24.7388 - 15.4125 + 1.4835 + 2.4 \\ &\approx .91. \end{aligned}$$

There were .91 deaths per 100 million miles driven in 2000.

66. $v(t) = 6t^2 - \dfrac{2}{t^2}$

$$\begin{aligned} s &= \int v(t)\,dt \\ &= \int (6t^2 - 2t^{-2})\,dt \\ &= 2t^3 + 2t^{-1} + C \\ s &= 2t^3 + \frac{2}{t} + C \end{aligned}$$

Since $s(1) = 8$,

$$\begin{aligned} 8 &= 2(1)^3 + \frac{2}{1} + C \\ 8 &= 4 + C \\ 4 &= C. \end{aligned}$$

Thus,

$$s(t) = 2t^3 + \frac{2}{t} + 4.$$

68. $a(t) = 18t + 8$

$$\begin{aligned} v(t) &= \int (18t + 8)\,dt \\ &= 9t^2 + 8t + C_1 \\ v(1) &= 9(1)^2 + 8(1) + C_1 = 17 + C_1 \end{aligned}$$

Since $v(1) = 15,\ C_1 = -2.$

$$\begin{aligned} v(t) &= 9t^2 + 8t - 2 \\ s(t) &= \int (9t^2 + 8t - 2)\,dt \\ &= 3t^3 + 4t^2 - 2t + C_2 \\ s(1) &= 3(1)^3 + 4(1)^2 - 2(1) + C_2 \\ &= 5 + C_2 \end{aligned}$$

Since $s(1) = 19,\ C_2 = 14.$
Thus,

$$s(t) = 3t^3 + 4t^2 - 2t + 14.$$

70. (a) First find $v(t)$ by integrating $a(t)$:

$$v(t) = \int (-32)dt = -32t + k.$$

When $t = 0, v(t) = v_0$:

$$\begin{aligned} v_0 &= -32(0) + k \\ v_0 &= k \end{aligned}$$

and $v(t) = -32t + v_0$.
Now integrate $v(t)$ to find $s(t)$.

$$\begin{aligned} s(t) &= \int (-32t + v_0)dt \\ &= -16t^2 + v_0 t + C \end{aligned}$$

Since $s(t) = 0$ when $t = 0$, we can substitute these values into the equation for $s(t)$ to get $C = 0$ and

$$s(t) = -16t^2 + v_0 t.$$

(b) When $t = 14, s(t) = 0, so$

$$\begin{aligned} 0 &= -16(14)^2 + v_0(14) \\ v_0 &= 224 \end{aligned}$$

The velocity was 224 feet per second at time $t = 0$.

(c) $v_0 t = 224(14) = 3136$

The distance the rocket would travel horizontally would be 3136 feet.

7.2 Substitution

2. (a) $\int (3x^2 - 5)^4\, 2x\,dx$

Let $u = 3x^2 - 5$; then $du = 6x\,dx$.

(b) $\int \sqrt{1 - x}\,dx$

Let $u = 1 - x$; then $du = -dx$.

(c) $\int \dfrac{x^2}{2x^3 + 1}\,dx$

Let $u = 2x^3 + 1$; then $du = 6x^2\,dx$.

(d) $\int 4x^3 e^{x^4}\,dx$

Let $u = x^4$; then $du = 4x^3\,dx$.

4. $\int (-4t + 1)^3\,dt$

$$= -\frac{1}{4}\int -4(-4t + 1)^3\,dt$$

Let $u = -4t + 1,$ so that $du = -4\,dt.$

$$\begin{aligned} &= -\frac{1}{4}\int u^3\,du \\ &= -\frac{1}{4}\cdot\frac{u^4}{4} + C \\ &= \frac{-u^4}{16} + C \\ &= \frac{-(-4t + 1)^4}{16} + C \end{aligned}$$

6. $\int \frac{3\,du}{\sqrt{3u-5}} = \int 3(3u-5)^{-1/2}\,du$

Let $w = 3u-5$, so that $dw = 3\,du$.

$$= \int w^{-1/2}\,dw$$

$$= \frac{w^{1/2}}{\frac{1}{2}} + C$$

$$= 2w^{1/2} + C$$
$$= 2(3u-5)^{1/2} + C$$

8. $\int \frac{6x^2\,dx}{(2x^3+7)^{3/2}}$

$$= \int 6x^2(2x^3+7)^{-3/2}\,dx$$

Let $u = 2x^3+7$, so that $du = 6x^2\,dx$.

$$= \int u^{-3/2}\,du$$

$$= \frac{u^{-1/2}}{-\frac{1}{2}} + C$$

$$= -2u^{-1/2} + C$$

$$= \frac{-2}{u^{1/2}} + C$$

$$= \frac{-2}{(2x^3+7)^{1/2}} + C$$

10. $\int r\sqrt{r^2+2}\,dr = \int r(r^2+2)^{1/2}\,dr$

Let $u = r^2+2$, so that $du = 2r\,dr$.

$$= \frac{1}{2}\int 2r(r^2+2)^{1/2}\,dr$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\cdot\frac{u^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{u^{3/2}}{3} + C$$

$$= \frac{(r^2+2)^{3/2}}{3} + C$$

12. $\int 5e^{-.3g}\,dg = 5\int e^{-.3g}\,dg$

Let $u = -.3g$, so that $du = -.3\,dg$.

$$= \frac{5}{-.3}\int(-.3)e^{-.3g}\,dg$$

$$= \frac{-50}{3}\int e^u\,du$$

$$= \frac{-50e^u}{3} + C$$

$$= \frac{-50e^{-.3g}}{3} + C$$

14. $\int re^{-r^2}\,dr$

Let $u = -r^2$, so that $du = -2r\,dr$.

$$\int re^{-r^2}\,dr = -\frac{1}{2}\int -2re^{-r^2}\,dr$$

$$= -\frac{1}{2}\int e^u\,du$$

$$= \frac{-e^u}{2} + C$$

$$= \frac{-e^{-r^2}}{2} + C$$

16. $\int (x^2-1)e^{x^3-3x}\,dx$

Let $u = x^3-3x$, so that

$du = (3x^2-3)\,dx = 3(x^2-1)\,dx$.

$$\int (x^2-1)e^{x^3-3x}\,dx$$

$$= \frac{1}{3}\int 3(x^2-1)e^{x^3-3x}\,dx$$

$$= \frac{1}{3}\int e^u\,du = \frac{e^u}{3} + C$$

$$= \frac{e^{x^3-3x}}{3} + C$$

18. $\int \frac{e^{\sqrt{y}}}{2\sqrt{y}}\,dy = \int \frac{e^{y^{1/2}}}{2y^{1/2}}\,dy$

$$= \int \frac{1}{2}y^{-1/2}e^{y^{1/2}}\,dy$$

Let $u = y^{1/2}$, so that $du = \frac{1}{2}y^{-1/2}\,dy$.

$$= \int e^u\,du = e^u + C$$

$$= e^{y^{1/2}} + C = e^{\sqrt{y}} + C$$

20. $\displaystyle\int \frac{t^2+2}{t^3+6t+3}\,dx$

Let $u = t^3+6t+3$, so that $du = (3t^2+6)dt$.

$$\int \frac{t^2+2}{t^3+6t+3}\,dx = \frac{1}{3}\int \frac{(3t^2+6)dt}{t^3+6t+3}$$
$$= \frac{1}{3}\int \frac{du}{u}$$
$$= \frac{1}{3}\ln|u| + C$$
$$= \frac{1}{3}\ln\left|t^3+6t+3\right| + C$$

22. $\displaystyle\int \frac{B^3-1}{(2B^4-8B)^{3/2}}\,dB$

$$= \int (B^3-1)(2B^4-8B)^{-3/2}\,dB$$

Let $u = 2B^4-8B$, so that

$du = (8B^3-8)dB = 8(B^3-1)\,dB$.

$$= \frac{1}{8}\int 8(B^3-1)(2B^4-8B)^{-3/2}\,dB$$
$$= \frac{1}{8}\int u^{-3/2}\,du = \frac{1}{8}\cdot\frac{u^{-1/2}}{-\frac{1}{2}} + C$$
$$= -\frac{1}{4}u^{-1/2} + C = \frac{-1}{4u^{1/2}} + C$$
$$= \frac{-1}{4(2B^4-8B)^{1/2}} + C$$

24. $\displaystyle\int 4r\sqrt{8-r}\,dr$

$$= \int 4r(8-r)^{1/2}\,dr$$

Let $u = 8-r$, so that

$du = -dr$; also, $r = 8-u$.

$$= -4\int -r(8-r)^{1/2}\,dr$$
$$= -4\int (8-u)u^{1/2}\,du$$
$$= -4\int (8u^{1/2}-u^{3/2})\,du$$
$$= -4\left(\frac{8u^{3/2}}{\frac{3}{2}} - \frac{u^{5/2}}{\frac{5}{2}}\right) + C$$
$$= \frac{8(8-r)^{5/2}}{5} - \frac{64(8-r)^{3/2}}{3} + C$$

26. $\displaystyle\int \frac{2x}{(x+5)^6}\,dx$

$$= \int 2x(x+5)^{-6}\,dx = 2\int x(x+5)^{-6}\,dx$$

Let $u = x+5$, so that

$du = dx$; also, $u-5 = x$.

$$= 2\int (u-5)u^{-6}\,du = 2\int (u^{-5}-5u^{-6})\,du$$
$$= 2\left(\frac{u^{-4}}{-4}\right) - 10\left(\frac{u^{-5}}{-5}\right) + C = -\frac{u^{-4}}{2} + 2u^{-5} + C$$
$$= \frac{-1}{2(x+5)^4} + \frac{2}{(x+5)^5} + C$$

28. $\displaystyle\int (\sqrt{x^2-6x})(x-3)\,dx$

$$= \int (x^2-6x)^{1/2}(x-3)\,dx$$

Let $u = x^2-6x$, so that

$du = (2x-6)\,dx = 2(x-3)\,dx$.

$$= \frac{1}{2}\int (x^2-6x)^{1/2}2(x-3)\,dx$$
$$= \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + C$$
$$= \frac{u^{3/2}}{3} + C = \frac{(x^2-6x)^{3/2}}{3} + C$$

30. $\displaystyle\int \frac{-4x}{x^2+3}\,dx$

Let $u = x^2+3$, so that $du = 2x\,dx$.

$$\int \frac{-4x}{x^2+3}\,dx = -2\int \frac{2x\,dx}{x^2+3} = -2\int \frac{du}{u}$$
$$= -2\,\ln\,|u| + C$$
$$= -2\,\ln\,(x^2+3) + C$$

32. $\displaystyle\int \frac{\sqrt{2+\ln\,x}}{x}\,dx$

Let $u = 2+\ln\,x$, so that

$$du = \frac{1}{x}\,dx.$$

$$\int \frac{\sqrt{2+\ln\,x}}{x}\,dx = \int \sqrt{u}\,du$$
$$= \int u^{1/2}\,du$$
$$= \frac{u^{3/2}}{3/2} + C$$
$$= \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{3}(2+\ln\,x)^{3/2} + C$$

34. $\int \frac{1}{x(\ln\ x)}\,dx$

Let $u = \ln\ x$, so that

$$du = \frac{1}{x}\,dx.$$

$$\int \frac{1}{x(\ln\ x)}\,dx = \int \frac{1}{u}\,du$$
$$= \ln\ |u| + C$$
$$= \ln\ |\ln\ x| + C$$

38. (a) $M'(x) = \sqrt{x^2 + 12x}(2x + 12)$
$$= (x^2 + 12x)^{1/2}(2x + 12)$$
$$M(x) = \int M'(x)\,dx$$
$$= \int (x^2 + 12x)^{1/2}(2x + 12)\,dx$$

Let $u = x^2 + 12x$, so that

$$du = (2x + 12)\,dx.$$

$$M(x) = \int u^{1/2}\,du = \frac{u^{3/2}}{\frac{3}{2}} + C$$
$$= \frac{2u^{3/2}}{3} + C$$
$$M(x) = \frac{2(x^2 + 12x)^{3/2}}{3} + C$$

Now, when $x = 4$, $M = 612$.

$$612 = \frac{2[(4)^2 + 12(4)]^{3/2}}{3} + C$$
$$612 = \frac{2(16 + 48)^{3/2}}{3} + C$$
$$612 = \frac{2(64)^{3/2}}{3} + C$$
$$270.67 = C$$

Thus,

$$M(x) = \frac{2}{3}(x^2 + 12x)^{3/2} + 270.67.$$

(b) $M(x) = \frac{2}{3}(x^2 + 12x)^{3/2} + 270.67 = 2000$
$$\frac{2}{3}(x^2 + 12x)^{3/2} = 1729.33$$
$$(x^2 + 12x)^{3/2} = 2593.995$$
$$x^2 + 12x = 188.79$$
$$x^2 + 12x - 188.79 = 0$$

$$x = \frac{-12 \pm \sqrt{12^2 - 4(1)(-188.79)}}{2(1)}$$
$$x = \frac{-12 \pm 29.99}{2}$$
$$x = 8.99 \quad \text{or} \quad x = -20.99$$

9 yr must pass.

40. (a) $P'(x) = xe^{-x^2}$

Let $-x^2 = u$, so that

$$-2x\,dx = du$$
$$x\,dx = -\frac{du}{2}.$$
$$p = -\frac{1}{2}\int e^u\,du$$
$$= -\frac{1}{2}e^u$$
$$= -\frac{e^{-x^2}}{2} + C$$
$$P(3) = -\frac{e^{-9}}{2} + C$$

Since 10,000 = .01 million and $p(3) = .01$,

$$-\frac{e^{-9}}{2} + C = .01$$
$$C = .01 + \frac{e^{-9}}{2}$$
$$= .01006$$
$$\approx .01.$$

$$P(x) = \frac{-e^{-x^2}}{2} + .01$$

(b) $\lim_{x\to\infty} (x) = \lim_{x\to\infty}\left(\frac{-e^{x^2}}{2} + .01\right)$
$$= \lim_{x\to\infty}\left(-\frac{1}{2e^{x^2}} + .01\right)$$
$$= .01$$

Since profit is expressed in millions of dollars, the profit approaches .01(1,000,000) = \$10,000.

42.
$$\begin{aligned} V'(t) &= -kP(t) \\ P(t) &= P_0e^{-mt} \\ V'(t) &= -k\,P_0e^{-mt} \\ V(t) &= \frac{k}{m}\,P_0e^{-mt} + C \\ V(0) &= \frac{k}{m}\,P_0e^{0} + C \\ V_0 - \frac{k}{m}\,P_0 &= C \end{aligned}$$

Therefore,

$$\begin{aligned} V(t) &= \frac{k}{m}\,P_0e^{-mt} + V_0 - \frac{k}{m}\,P_0 \\ &= \frac{k\,P_0}{m}e^{-mt} + V_0 - \frac{k\,P_0}{m}. \end{aligned}$$

7.3 Area and the Definite Integral

2. $\int_0^3 (x^2+2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}({x_i}^2+2)\Delta x$, where $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and x_i is any value of x in the ith interval.

4. $f(x) = \frac{1}{x}$ and $x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$, and $\Delta x = \frac{1}{2}$

(a) $\sum_{i=1}^{4} f(x_i)\Delta x$

$$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$f(x_1) = f\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{2}} = 2$$

$$f(x_2) = f(1) = \frac{1}{1} = 1$$

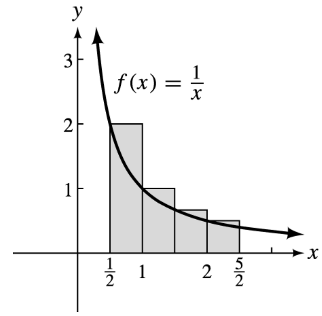
$$f(x_3) = f\left(\frac{3}{2}\right) = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$$f(x_4) = f(2) = \frac{1}{2}$$

Thus,

$$\begin{aligned} &\sum_{i=1}^{4} f(x_i)\Delta x \\ &= (2)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ &= \frac{12+6+4+3}{12} \\ &= \frac{25}{12}. \end{aligned}$$

(b)



The sum is approximated by the integral.

$$\int_{1/2}^{5/2} \frac{1}{x}\,dx$$

6. $f(x) = 3x + 2$ from $x = 1$ to $x = 5$

For $n = 4$ rectangles:

$$\Delta x = \frac{5-1}{4} = 2$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	1	5
2	2	8
3	3	11
4	4	14

$$\begin{aligned} A &= \sum_{i=1}^{4} f(x_i)\Delta x \\ &= 5(1) + 8(1) + 11(1) + 14(1) \\ &= 38 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	2	8
2	3	11
3	4	14
4	5	17

$$\begin{aligned} A &= 8(1) + 11(1) + 14(1) + 17(1) \\ &= 50 \end{aligned}$$

(c) Average $= \frac{38+50}{2} = \frac{88}{2} = 44$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	$\frac{3}{2}$	$\frac{13}{2}$
2	$\frac{5}{2}$	$\frac{19}{2}$
3	$\frac{7}{2}$	$\frac{25}{2}$
4	$\frac{9}{2}$	$\frac{31}{2}$

$$\begin{aligned} A &= \sum_{i=1}^{4} f(x_i)\Delta x \\ &= \frac{13}{2}(1) + \frac{19}{2}(1) + \frac{25}{2}(1) + \frac{31}{2}(1) \\ &= 44 \end{aligned}$$

8. $f(x) = x^2$ from $x = 1$ to 5

For $n = 4$ rectangles:

$$\Delta x = \frac{5-1}{4} = 1$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	1	1
2	2	4
3	3	9
4	4	16

$$\begin{aligned} A &= \sum_{i=1}^{4} f(x_i)\Delta x \\ &= 1(1) + 4(1) + 9(1) + 16(1) \\ &= 30 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	2	4
2	3	9
3	4	16
4	5	25

$$\begin{aligned} A &= 4(1) + 9(1) + 16(1) + 25(1) \\ &= 54 \end{aligned}$$

(c) Average $= \dfrac{30+54}{2} = \dfrac{84}{2} = 42$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	$\frac{3}{2}$	$\frac{9}{4}$
2	$\frac{5}{2}$	$\frac{25}{4}$
3	$\frac{7}{2}$	$\frac{49}{4}$
4	$\frac{9}{2}$	$\frac{81}{4}$

$$\begin{aligned} A &= \sum_{i=1}^{4} f(x_i)\Delta x \\ &= \frac{9}{4}(1) + \frac{25}{4}(1) + \frac{49}{4}(1) + \frac{81}{4}(1) \\ &= 41 \end{aligned}$$

10. $f(x) = e^x - 1$ from $x = 0$ to $x = 4$

For $n = 4$ rectangles:

$$\Delta x = \frac{4-0}{4} = 1$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	0	0
2	1	1.718
3	2	6.389
4	3	19.086

$$\begin{aligned} A &= \sum_{i=1}^{4} f(x_i)\Delta x \\ &= 0(1) + 1.718(1) + 6.389(1) + 19.086(1) \\ &\approx 27.19 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	1	1.718
2	2	6.389
3	3	19.086
4	4	53.598

$$\begin{aligned} A &= 1.718(1) + 6.389(1) + 19.086(1) \\ &\quad + 53.598(1) \\ &\approx 80.79 \end{aligned}$$

(c) Average $= \dfrac{27.19 + 80.79}{2} = 53.99$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	$\frac{1}{2}$	.649
2	$\frac{3}{2}$	3.482
3	$\frac{5}{2}$	11.182
4	$\frac{7}{2}$	32.115

$$A = \sum_{i=1}^{4} f(x_i)\Delta x$$
$$= .649(1) + 3.482(1) + 11.182(1) + 32.115(1)$$
$$\approx 47.43$$

12. $f(x) = \frac{1}{x}$ from $x = 1$ to 5

For $n = 4$ rectangles:

$$\Delta x = \frac{5-1}{4} = 1$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	1	1
2	2	.5
3	3	.33
4	4	.25

$$A = \sum_{i=1}^{4} f(x_i)\Delta x = 1 + .5 + .33 + .25 = 2.08$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	2	.5
2	3	.33
3	4	.25
4	5	.2

$$A = .5(1) + .33(1) + .25(1) + .2(1) = 1.28$$

(c) Average $= \frac{2.08 + 1.28}{2} = 1.68$

(d) Using the midpoints:

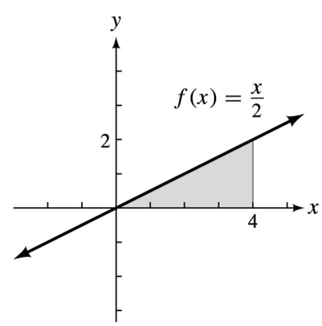
i	x_i	$f(x_i)$
1	$\frac{3}{2}$	$\frac{2}{3}$
2	$\frac{5}{2}$	$\frac{2}{5}$
3	$\frac{7}{2}$	$\frac{2}{7}$
4	$\frac{9}{2}$	$\frac{2}{9}$

$$A = \sum_{i=1}^{4} f(x_i)\Delta x$$
$$= \frac{2}{3}(1) + \frac{2}{5}(1) + \frac{2}{7}(1) + \frac{2}{9}(1)$$
$$\approx 1.57$$

14. **(a)** Width $= \frac{4-0}{4} = 1;\ f(x) = \frac{x}{2}$

$$\text{Area} = 1 \cdot f\left(\frac{1}{2}\right) + 1 \cdot f\left(\frac{3}{2}\right) + 1 \cdot f\left(\frac{5}{2}\right) + 1 \cdot f\left(\frac{7}{2}\right)$$
$$= \frac{1}{4} + \frac{3}{4} + \frac{5}{4} + \frac{7}{4} = \frac{16}{4} = 4$$

(b)



$$\int_0^4 f(x)\,dx = \int_0^4 \frac{x}{2}\,dx = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(4)(2) = 4$$

16. **(a)** Area of triangle is $\frac{1}{2} \cdot$ base $\cdot$ height.

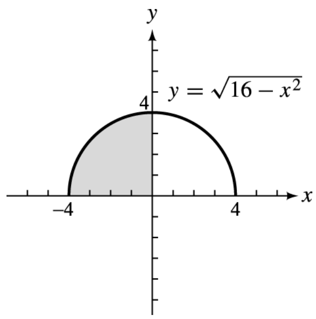
The base is 4; the height is 2.

$$\int_0^4 f(x)\,dx = \frac{1}{2} \cdot 4 \cdot 2 = 4$$

(b) The larger triangle has an area of $\frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$. The smaller triangle has an area of $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. The sum is $\frac{9}{2} + \frac{1}{2} = \frac{10}{2} = 5$.

18. $\int_{-4}^{0} \sqrt{16 - x^2}\, dx$

Graph $y = \sqrt{16 - x^2}$.

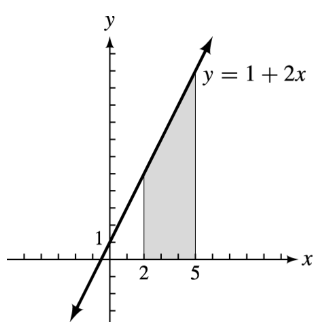


$\int_{-4}^{0} \sqrt{16 - x^2}\, dx$ is the area of the portion of the circle in the second quadrant, which is one-fourth of a circle. The circle has radius 4.

$$\text{Area} = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(4)^2 = 4\pi$$

20. $\int_{2}^{5} (1 + 2x)\, dx$

Graph $y = 1 + 2x$.



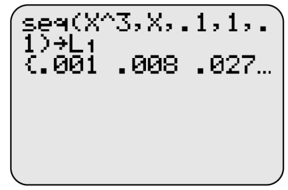
$\int_{2}^{5} (1 + 2x)\, dx$ is the area of the trapezoid with $B = 11$, $b = 5$, and $h = 3$. The formula for the area is

$$A = \frac{1}{2}(B + b)h,$$

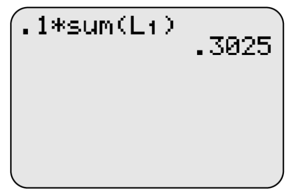
so we have

$$A = \frac{1}{2}(11 + 5)(3)$$
$$= 24.$$

22. **(a)** With $n = 10$, $\Delta x = \frac{1-0}{10} = .1$, and $x_1 = 0 + .1 = .1$, use the command seq(X^3,X, .1, .1, .1) →L1. The resulting screen is:

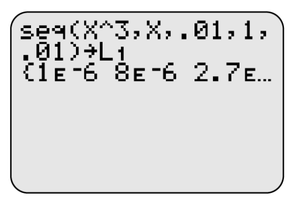


(b) Since $\sum_{i=1}^{n} f(x_i)\Delta x = \Delta x \left(\sum_{i=1}^{n} f(x_i)\right)$, use the command .1∗sum(L1) to approximate $\int_0^1 x^3 dx$. The resulting screen is:

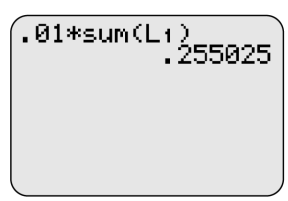


$$\int_0^1 x^3 dx \approx .3025$$

(c) With $n = 100, \Delta x = \frac{1-0}{100} = .01$, and $x_1 = 0 + .01 = .01$, use the command seq(X^3,X, .01, .1, .01) →L1. The resulting screen is:

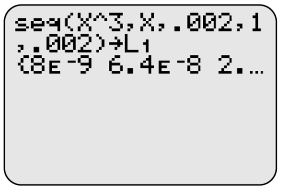


Use the command .01∗sum(L1) to approximate $\int_0^1 x^3 dx$. The resulting screen is:

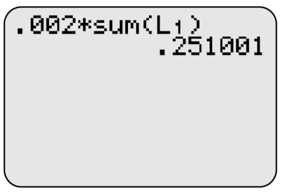


$$\int_0^1 x^3 dx \approx .255025$$

(d) With $n = 500, \Delta x = \frac{1-0}{500} = .002$, and $x_1 = 0 + .002 = .002$, use the command seq (X^3,X, .002, 1, .002) →L1. The resulting screen is:



Use the command .002∗sum(L1) to approximate $\int_0^1 x^3 dx$. The resulting screen is:



$$\int_0^1 x^3 dx \approx .251001$$

(e) As n gets larger the approximation for $\int_0^1 x^3 dx$ seems to be approaching .25 or $\frac{1}{4}$. We estimate $\int_0^1 x^3 dx = \frac{1}{4}$.

For Exercises 24-32, readings on the graphs and answers may vary.

24. Left endpoints:

Read values of the function from the graph for every 2 hours from midnight to 10 P.M. These values give the heights of 12 rectangles. The width of each rectangle is $\Delta x = 2$. We estimate the area under the curve as

$$\begin{aligned} A &= \sum_{i=1}^{12} f(x_i)\Delta x \\ &= 3.0(2) + 3.2(2) + 3.5(2) + 4.2(2) + 5.2(2) \\ &\quad + 6.2(2) + 8.0(2) + 11.0(2) + 11.8(2) \\ &\quad + 10.0(2) + 6.0(2) + 4.4(2) \\ &= 153.0 \end{aligned}$$

Right endpoints:

Read values of the function from the graph for every 2 hours from 2 A.M. to midnight. Now we estimate the area under the curve as

$$\begin{aligned} A &= \sum_{i=1}^{12} f(x_i)\Delta x \\ &= 3.2(2) + 3.5(2) + 4.2(2) + 5.2(2) + 6.2(2) \\ &\quad + 8.0(2) + 11.0(2) + 11.8(2) + 10.0(2) \\ &\quad + 6.0(2) + 4.4(2) + 3.8(2) \\ &= 154.6 \end{aligned}$$

Average:

$$\frac{153.0 + 154.6}{2} = \frac{307.6}{2} = 153.8$$

The area under the curve represents the total electricity usage. We estimate this usage as about 154 million kilowatt hours.

26. Left endpoints:

Read values of the function on the graph every hour from 0 to 7. These values give us the heights of 8 rectangles. The width of each rectangle is $\Delta x = 1$. We estimate the area under the curve as

$$\begin{aligned} A &= \sum_{i=1}^{8} f(x_i)\Delta x \\ &= 0(1) + 1.2(1) + 2.1(1) + 2.9(1) + 3.5(1) \\ &\quad + 3.7(1) + 3.3(1) + 2.4(1) \\ &= 19.1. \end{aligned}$$

Right endpoints:

Read values of the function on the graph every hour from 1 to 8. Now we estimate the area under the curve as

$$\begin{aligned} A &= \sum_{i=1}^{8} f(x_i)\Delta x \\ &= 1.2(1) + 2.1(1) + 2.9(1) + 3.5(1) \\ &\quad + 3.7(1) + 3.3(1) + 2.4(1) + 1.0(1) \\ &= 20.1. \end{aligned}$$

Average:

$$\frac{19.1 + 20.1}{2} = 19.6 \approx 20$$

The area under the curve represents the total alcohol concentration. We estimate that this concentration is about 20 units.

28. **(a)** Left endpoints:

Read values of the function from the graph for every 14 days from 18 Feb. through 30 Apr. The values give the heights of 6 rectangles. The width of each rectangle is $\Delta x = 14$. We estimate the area under the curve as

$$\begin{aligned} A &= \sum_{i=1}^{6} f(x_i)\Delta x \\ &= 0(14) + 15(14) + 33(14) + 40(14) \\ &\quad + 16(14) + 5(14) \\ &= 1526. \end{aligned}$$

Right endpoints:

Read values of the function from the graph for every 14 days from 4 Mar. through 13 May. Now we estimate the area under the curve as

$$\begin{aligned} A &= \sum_{i=1}^{6} f(x_i)\Delta x \\ &= 15(14) + 33(14) + 40(14) + 16(14) \\ &\quad + 5(14) + 1(14) \\ &= 1540. \end{aligned}$$

Average:

$$\frac{1526 + 1540}{2} = 1533$$

There were about 1533 cases of the disease.

(b) Left endpoints:

Read values of the function from the graph for every 14 days from 18 Feb. through 30 Apr. The values give the heights of 6 rectangles. The width of each rectangle is $\Delta x = 14$. We estimate the area under the curve as

$$\begin{aligned} A &= \sum_{i=1}^{6} f(x_i)\Delta x \\ &= 0(14) + 10(14) + 15(14) + 10(14) \\ &\quad + 3(14) + 1(14) \\ &= 546. \end{aligned}$$

Right endpoints:

Read values of the function from the graph for every 14 days from 4 Mar. through 13 May. Now we estimate the area under the curve as

$$\begin{aligned} A &= \sum_{i=1}^{6} f(x_i)\Delta x \\ &= 10(14) + 15(14) + 10(14) + 3(14) \\ &\quad + 1(14) + 1(14) \\ &= 560. \end{aligned}$$

Average:

$$\frac{546 + 560}{2} = 553$$

There would have been about 553 cases of the disease.

30. Read the value for the speed every 5 sec from $x = 2.5$ to $x = 22.5$. These are the midpoints of rectangles with width $\Delta x = 5$. Then read the speed for $x = 26.5$, which is the midpoint of a rectangle with width $\Delta x = 3$.

$$\begin{aligned} \sum_{i=1}^{6} f(x_i)\Delta x &\approx 28(5) + 54(5) + 72(5) \\ &\quad + 82(5) + 92(5) + 98(3) \\ &= 1934 \end{aligned}$$

$$\frac{1934}{3600}(5280) \approx 2800$$

The BMW 733i traveled about 2800 ft.

32. **(a)** Read the value for a plain glass window facing south for every 2 hr from 6 to 6. These are the heights, at the midpoints, of rectangles with width $\Delta x = 2$.

$$\begin{aligned} \sum &= 10(2) + 30(2) + 80(2) + 107(2) \\ &\quad + 79(2) + 29(2) + 10(2) \\ &\approx 690 \end{aligned}$$

The heat gain is about 690 BTUs per square foot.

(b) Read the value for a window with Shadescreen facing south for every 2 hr from 6 to 6. These are the heights, at the midpoints, of rectangles with width $\Delta x = 2$.

$$\begin{aligned} \sum &= 4(2) + 10(2) + 20(2) + 22(2) \\ &\quad + 20(2) + 10(2) + 4(2) \\ &\approx 180 \end{aligned}$$

The heat gain is about 180 BTUs per square foot.

34. Using the left endpoints:

$$\begin{aligned} \text{Distance} &= v_0(1) + v_1(1) + v_2(1) + v_3(1) \\ &= 0 + 8 + 13 + 17 \\ &= 38 \text{ feet} \end{aligned}$$

Using the right endpoints:

$$\begin{aligned} \text{Distance} &= v_1(1) + v_2(1) + v_3(1) + v_4(1) \\ &= 8 + 13 + 17 + 18 \\ &= 56 \text{ feet} \end{aligned}$$

36. **(a)** Using the left endpoints:

$$\text{Distance} = \sum_{i=1}^{n} f(x_i)\Delta x_i$$
$$= 0(1.84) + 12.9(1.96) + 23.8(2.58) + 26.3(.85) + 26.3(1.73) + 26.0(.87)$$
$$= 0 + 25.284 + 61.404 + 22.355 + 45.499 + 22.62$$
$$= 177.162$$

Since we multiplied the units of seconds by miles per hour, we need to divide by 3600 (the number of seconds in an hour) to get a distance in miles.

$$\frac{177.162}{3600} \approx .0492$$

The estimate of the distance is .0492 miles.

(b) Using the right endpoints:

$$\text{Distance} = \sum_{i=1}^{n} f(x_i)\Delta x_i$$
$$= 12.9(1.84) + 23.8(1.96) + 26.3(2.58) + 26.3(.85) + 26.0(1.73) + 25.7(.87)$$
$$= 23.736 + 46.648 + 67.854 + 22.355 + 44.98 + 22.359$$
$$= 227.932$$

Divide by 3600 (the number of seconds in an hour) to get a distance in miles.

$$\frac{227.932}{3600} \approx .0633$$

The estimate of the distance is .0633 miles.

(c) $\frac{100}{1609} \approx .0622$

Johnson actually ran .0622 miles. The answer to part b is closer.

7.4 The Fundamental Theorem of Calculus

2. $\int_{-4}^{1} 6x\,dx = 6\int_{-4}^{1} x\,dx = 3\cdot x^2\Big|_{-4}^{1}$

$$= 3[1^2 - (-4)^2] = 3(1-16) = -45$$

4. $\int_{-2}^{2} (4z+3)\,dz = 4\int_{-2}^{2} z\,dz + 3\int_{-2}^{2} dz$

$$= 2z^2\Big|_{-2}^{2} + 3z\Big|_{-2}^{2}$$
$$= 2[2^2 - (-2)^2] + 3[2-(-2)]$$
$$= 2(4-4) + 3(4) = 12$$

6. $\int_{-2}^{3} (-x^2 - 3x + 5)\,dx$

$$= -\int_{-2}^{3} x^2\,dx - 3\int_{-2}^{3} x\,dx + 5\int_{-2}^{3} dx$$
$$= -\frac{1}{3}x^3\Big|_{-2}^{3} - \frac{3}{2}x^2\Big|_{-2}^{3} + 5x\Big|_{-2}^{3}$$
$$= -\frac{1}{3}[3^3-(-2)^3] - \frac{3}{2}[3^2-(-2)^2] + 5[3-(-2)]$$
$$= -\frac{1}{3}(27+8) - \frac{3}{2}(9-4) + 5(5)$$
$$= -\frac{35}{3} - \frac{15}{2} + 25 = \frac{35}{6}$$

8. $\int_{3}^{9} \sqrt{2r-2}\,dr = \int_{3}^{9} (2r-2)^{1/2}\,dr$

Let $u = 2r-2$, so that $du = 2\,dr$.

If $r = 9,\ u = 2\cdot 9 - 2 = 16$.
If $r = 3,\ u = 2\cdot 3 - 2 = 4$.

$$\int_{3}^{9} (2r-2)^{1/2}\,dr = \frac{1}{2}\int_{3}^{9} (2r-2)^{1/2}\,2dr$$
$$= \frac{1}{2}\int_{4}^{16} u^{1/2}\,du$$
$$= \frac{1}{2}\cdot\frac{u^{3/2}}{\frac{3}{2}}\Big|_{4}^{16}$$
$$= \frac{1}{3}\cdot u^{3/2}\Big|_{4}^{16}$$
$$= \frac{1}{3}(16^{3/2} - 4^{3/2})$$
$$= \frac{1}{3}(64-8) = \frac{56}{3}$$

10. $\int_{0}^{4} -(3x^{3/2} + x^{1/2})\,dx$

$$= -3\int_{0}^{4} x^{3/2}\,dx - \int_{0}^{4} x^{1/2}\,dx$$
$$= -3\frac{x^{5/2}}{\frac{5}{2}}\Big|_{0}^{4} - \frac{x^{3/2}}{\frac{3}{2}}\Big|_{0}^{4}$$
$$= -\frac{6}{5}(32) - \frac{2}{3}(8)$$
$$= -\frac{192}{5} - \frac{16}{3} = -\frac{656}{15}$$

12. $\int_4^9 (4\sqrt{r} - 3r\sqrt{r})\,dr$

$$= 4\int_4^9 r^{1/2}\,dr - 3\int_4^9 r^{3/2}\,dr$$

$$= 4\frac{r^{3/2}}{\frac{3}{2}}\Big|_4^9 - 3\frac{r^{5/2}}{\frac{5}{2}}\Big|_4^9$$

$$= \frac{8}{3}r^{3/2}\Big|_4^9 - \frac{6}{5}r^{5/2}\Big|_4^9$$

$$= \frac{8}{3}(27-8) - \frac{6}{5}(243-32)$$

$$= \frac{8}{3}\cdot 19 - \frac{6}{5}(211)$$

$$= \frac{760}{15} - \frac{3798}{15} = -\frac{3038}{15}$$

14. $\int_1^4 \frac{-3}{(2p+1)^2}\,dp$

$$= -3\int_1^4 (2p+1)^{-2}\,dp$$

Let $u = 2p+1$, so that $du = 2\,dp$.

If $p = 4,\ u = 2\cdot 4 + 1 = 9$.
If $p = 1,\ u = 2\cdot 1 + 1 = 3$.

$$-3\int_1^4 (2p+1)^{-2}\,dp$$

$$= -\frac{3}{2}\int_3^9 u^{-2}\,du$$

$$= -\frac{3}{2}\cdot\frac{u^{-1}}{-1}\Big|_3^9$$

$$= \frac{3}{2u}\Big|_3^9$$

$$= \frac{3}{18} - \frac{3}{6} = -\frac{1}{3}$$

16. $\int_2^3 (3x^{-2} - x^{-4})\,dx = 3\int_2^3 x^{-2}\,dx - \int_2^3 x^{-4}\,dx$

$$= 3\frac{x^{-1}}{-1}\Big|_2^3 - \frac{x^{-3}}{-3}\Big|_2^3$$

$$= -\frac{3}{x}\Big|_2^3 + \frac{1}{3x^3}\Big|_2^3$$

$$= -3\left(\frac{1}{3} - \frac{1}{2}\right) + \frac{1}{81} - \frac{1}{24}$$

$$= -1 + \frac{3}{2} + \frac{1}{81} - \frac{1}{24} \approx .471$$

18. $\int_1^2 \left(\frac{-1}{B} + 3e^{.2B}\right) dB = \int_1^2 -\frac{1}{B}\,dB + \frac{3}{.2}\int_1^2 .2e^{.2B}\,dB$

$$= -\ln\ B\Big|_1^2 + \frac{3}{.2}e^{.2B}\Big|_1^2$$

$$= -\ln\ 2 + 15e^{.4} - 15e^{.2}$$

$$\approx 3.363$$

20. $\int_{.5}^1 (p^3 - e^{4p})\,dp = \int_{.5}^1 p^3\,dp - \int_{.5}^1 e^{4p}\,dp$

$$= \frac{p^4}{4}\Big|_{.5}^1 - \frac{e^{4p}}{4}\Big|_{.5}^1$$

$$= \frac{1}{4} - \frac{1}{64} - \left(\frac{e^4}{4} - \frac{e^2}{4}\right)$$

$$= \frac{15}{64} - \frac{e^4}{4} + \frac{e^2}{4} \approx -11.568$$

22. $\int_0^3 m^2(4m^3+2)^3\,dm$

Let $u = 4m^3 + 2$, so that

$$du = 12m^2\,dm \text{ and } \tfrac{1}{12}\,du = m^2\,dm.$$

Also, when $m = 3$,

$$u = 4(3^3) + 2 = 110,$$

and when $m = 0$,

$$u = 4(0^3) + 2 = 2.$$

$$\frac{1}{12}\int_2^{110} u^3\,du = \frac{1}{12}\cdot\frac{u^4}{4}\Big|_2^{110} = \frac{1}{48}u^4\ \Big|_2^{110}$$

$$= \frac{146{,}410{,}000}{48} - \frac{16}{48}$$

$$= \frac{146{,}409{,}984}{48}$$

$$= \frac{9{,}150{,}624}{3}$$

$$\approx\ 3{,}050{,}208$$

24. $\int_1^8 \frac{3-y^{1/3}}{y^{2/3}}\,dy$

$$= \int_1^8 (3y^{-2/3} - y^{-1/3})\,dy$$
$$= \int_1^8 3y^{-2/3}\,dy - \int_1^8 y^{-1/3}\,dy$$
$$= \frac{3y^{1/3}}{\frac{1}{3}}\Big|_1^8 - \frac{y^{2/3}}{\frac{2}{3}}\Big|_1^8$$
$$= 9y^{1/3}\Big|_1^8 - \frac{3y^{2/3}}{2}\Big|_1^8$$
$$= 9(2-1) - \frac{3}{2}(4-1)$$
$$= 9 - \frac{9}{2} = \frac{9}{2}$$

26. $\int_1^3 \frac{\sqrt{\ln x}}{x}\,dx$

Let $u = \ln x$, so that

$$du = \frac{1}{x}\,dx.$$

When $x = 3,\ u = \ln 3$, and
when $x = 1,\ u = \ln 1 = 0$.

$$\int_0^{\ln 3} \sqrt{u}\,du = \int_0^{\ln 3} u^{1/2}\,du$$
$$= \frac{u^{3/2}}{\frac{3}{2}}\Big|_0^{\ln 3}$$
$$= \frac{2}{3}u^{3/2}\Big|_0^{\ln 3}$$
$$= \frac{2}{3}(\ln 3)^{3/2} - \frac{2}{3}(0)^{3/2}$$
$$= \frac{2}{3}(\ln 3)^{3/2}$$
$$\approx .76767$$

28. $\int_1^2 \frac{3}{x(1+\ln x)}\,dx$

Let $u = 1 + \ln x$, so that

$$du = \frac{1}{x}\,dx.$$

When $x = 2,\ u = 1 + \ln 2$, and
when $x = 1,\ u = 1 + \ln 1 = 1$.

$$\int_1^{1+\ln 2} \frac{3}{u}\,du$$
$$= 3\ln |u|\Big|_1^{1+\ln 2}$$
$$= 3\ \ln(1+\ln 2) - 3\ \ln 1$$
$$= 3\ln(1+\ln 2)$$
$$\approx 1.5798$$

30. $\int_0^1 \frac{e^{2z}}{\sqrt{1+e^{2z}}}\,dz$

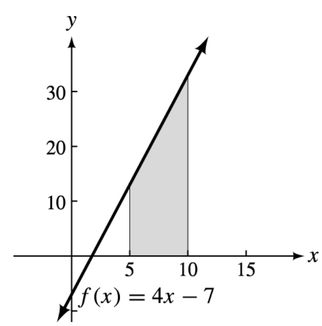
Let $u = 1 + e^{2z}$, so that

$$du = 2e^{2z}\,dz \text{ and } \frac{1}{2}\,du = e^{2z}\,dz.$$

When $z = 1,\ u = 1 + e^2$, and
when $z = 0,\ u = 1 + e^0 = 2$.

$$\frac{1}{2}\int_2^{1+e^2} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_2^{1+e^2} u^{-1/2}\,du$$
$$= \frac{1}{2}\cdot\frac{u^{1/2}}{\frac{1}{2}}\Big|_2^{1+e^2}$$
$$= u^{1/2}\Big|_2^{1+e^2}$$
$$= \sqrt{1+e^2} - \sqrt{2}$$
$$\approx 1.4822$$

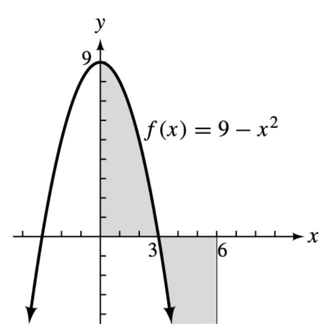
32. $f(x) = 4x - 7;\ [5, 10]$



The area of the region is

$$\int_5^{10} (4x-7)\,dx = (2x^2 - 7x)\Big|_5^{10}$$
$$= (200 - 70) - (50 - 35)$$
$$= 115.$$

34. $f(x) = 9 - x^2;\ [0, 6]$



The graph crosses the x-axis at

$$0 = 9 - x^2$$
$$x^2 = 9$$
$$x = \pm 3.$$

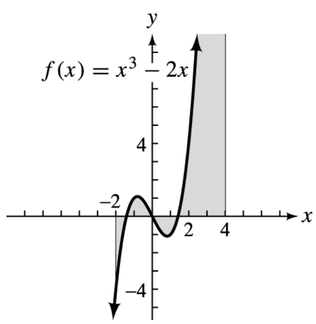
In the interval, the graph crosses at $x = 3$.
The area of the region is

$$\int_0^3 (9 - x^2)\,dx + \left|\int_3^6 (9 - x^2)\,dx\right|$$

$$= \left(9x - \frac{x^3}{3}\right)\Big|_0^3 + \left|\left(9x - \frac{x^3}{3}\right)\Big|_3^6\right|$$

$$= (27 - 9) + |(54 - 72) - (27 - 9)|$$
$$= 18 + |-36|$$
$$= 18 + 36 = 54.$$

36. $f(x) = x^3 - 2x;\ [-2, 4]$



The graph crosses the x-axis at

$$0 = x^3 - 2x$$
$$= x(x^2 - 2)$$

$x = 0,\ x = \sqrt{2}$, and $x = -\sqrt{2}$.
These locations are all in the interval.
The area of the region is

$$\left|\int_{-2}^{-\sqrt{2}} (x^3 - 2x)\,dx\right| + \left|\int_{-\sqrt{2}}^{0} (x^3 - 2x)\,dx\right|$$

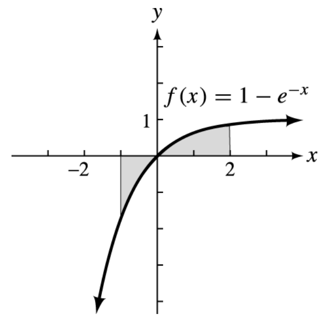
$$+ \left|\int_0^{-\sqrt{2}} (x^3 - 2x)\,dx\right| + \left|\int_{-\sqrt{2}}^{4} (x^3 - 2x)\,dx\right|$$

$$= \left|\left(\frac{x^4}{4} - x^2\right)\Big|_{-2}^{-\sqrt{2}}\right| + \left|\left(\frac{x^4}{4} - x^2\right)\Big|_{-\sqrt{2}}^{0}\right|$$

$$+ \left|\left(\frac{x^4}{4} - x^2\right)\Big|_0^{\sqrt{2}}\right| + \left|\left(\frac{x^4}{4} - x^2\right)\Big|_{\sqrt{2}}^{4}\right|$$

$$= |(1 - 2) - (4 - 4)| + |0 - (1 - 2)|$$
$$+ |(1 - 2) - 0 + (64 - 16) - 1 - 2|$$
$$= |-1| + |1| + |-1| + |49|$$
$$= 1 + 1 + 1 + 49$$
$$= 52.$$

38. $f(x) = 1 - e^{-x};\ [-1, 2]$



The graph crosses the x-axis at

$$0 = 1 - e^{-x}$$
$$e^{-x} = 1$$
$$-x \ln e = \ln 1$$
$$-x = 0$$
$$x = 0.$$

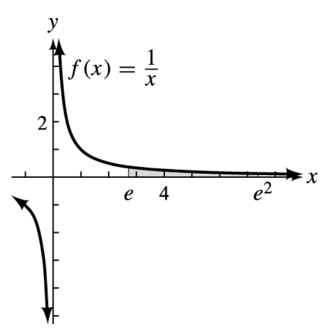
The area of the region is

$$\left|\int_{-1}^{0} (1 - e^{-x})\,dx\right| + \int_0^2 (1 - e^{-x})\,dx$$

$$= \left|(x + e^{-x})\Big|_{-1}^{0}\right| + \left|(x + e^{-x})\right|_0^2$$

$$= \left|(1) - (-1 + e^1)\right| + (2 + e^{-2}) - (e^0)$$
$$= |2 - e| + 2 + e^{-2} - 1$$
$$= |-.718| + 1 + e^{-2}$$
$$= .718 + 1.135$$
$$= 1.854.$$

40. $f(x) = \frac{1}{x};\ [e, e^2]$



The graph does not cross the x-axis.

$$\int_e^{e^2} \frac{1}{x}\,dx = \ln x\Big|_e^{e^2}$$
$$= \ln e^2 - \ln e$$
$$= 2 - 1 = 1$$

42. $f(x) = x^2 - 2x;\ [-1, 2]$

From the graph, we see that the total area is

$$\int_{-1}^{0} (x^2 - 2x)\,dx + \left|\int_{0}^{2} (x^2 - 2x)\,dx\right|$$
$$= \left(\frac{x^3}{3} - x^2\right)\Bigg|_{-1}^{0} + \left|\left(\frac{x^3}{3} - x^2\right)\Bigg|_{0}^{2}\right|$$
$$= -\left(-\frac{1}{3} - 1\right) + \left|\frac{8}{3} - 4\right|$$
$$= \frac{4}{3} + \left|-\frac{4}{3}\right| = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}.$$

44. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

46. $$\int_0^{16} f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx + \int_5^8 f(x)\,dx + \int_8^{16} f(x)\,dx$$
$$= \frac{1}{2}\cdot 2(1+3) + \frac{\pi(3^2)}{4} - \frac{\pi(3^2)}{4} - \frac{1}{2}(3)(8)$$
$$= 4 + \frac{9}{4}\pi - \frac{9}{4}\pi - 12$$
$$= -8$$

48. Prove: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

Let $F(x)$ be an antiderivative of $f(x)$.

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
$$= F(x)\Big|_a^c + F(x)\Big|_c^b$$
$$= [F(c) - F(a)] + [F(b) - F(c)]$$
$$= F(c) - F(a) + F(b) - F(c)$$
$$= F(b) - F(a)$$
$$= \int_a^b f(x)\,dx$$

50. $\int_{-1}^{4} f(x)\,dx$

$$= \int_{-1}^{0} (2x+3)\,dx \int_0^4 \left(-\frac{x}{4} - 3\right)\,dx$$
$$= (x^2 + 3x)\Big|_{-1}^{0} + \left(-\frac{x^2}{8} - 3x\right)\Big|_0^4$$
$$= -(1-3) + (-2-12)$$
$$= 2 - 14$$
$$= -12$$

52. **(a)** $g(t) = t^4$ and $c = 1$, use substitution.

$$f(x) = \int_c^x g(t)dt$$
$$= \int_1^x t^4 dt$$
$$= \frac{t^5}{5}\Big|_1^x$$
$$= \frac{x^5}{5} - \frac{(1)^5}{5}$$
$$= \frac{x^5}{5} - \frac{1}{5}$$

(b)

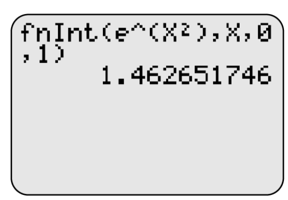
$$f'(x) = \frac{d}{dx}(f(x))$$
$$= \frac{d}{dx}\left(\frac{x^5}{5} - \frac{1}{5}\right)$$
$$= \frac{1}{5}\cdot\frac{d}{dx}(x^5) - \frac{d}{dx}\left(\frac{1}{5}\right)$$
$$= \frac{1}{5}\cdot 5x^4 - 0$$
$$= x^4$$

Since $g(t) = t^4$, then $g(x) = x^4$ and we see $f'(x) = g(x)$.

(c) Let $g(t) = e^{t^2}$ and $c = 0$, then $f(x) = \int_0^x e^{t^2}\,dt$.

$f(1) = \int_0^1 e^{t^2}\,dt$ and $f(1.01) = \int_0^{1.01} e^{t^2}\,dt$.

Use the fnInt command in the Math menu of your calculator to find $\int_0^1 e^{x^2}\,dx$ and $\int_0^{1.01} e^{x^2}\,dx$. The resulting screens are:



$f(1) \approx 1.46265$
$f(1.01) \approx 1.49011$

Use $\frac{f(1+h)-f(1)}{h}$ to approximate $f'(1)$ with $h = .01$

$$\begin{aligned}\frac{f(1+h)-f(1)}{h} &= \frac{f(1.01)-f(1)}{.01}\\ &\approx \frac{1.49011-1.46265}{.01}\\ &= 2.746\end{aligned}$$

So $f'(1) \approx 2.746$, and
$g(1) = e^{1^2} = e \approx 2.718.$

54. $H'(x) = 20 - 2x$ is the rate of change of the number of hours it takes a worker to produce the xth item.

(a) The total number of hours required to produce the first 5 items is

$$\begin{aligned}\int_0^5 (20-2x)\,dx &= (20x - x^2)\Big|_0^5\\ &= 100 - 25 = 75.\end{aligned}$$

It would take 75 hr to produce 5 items.

(b) The total number of hours required to produce the first 10 items is

$$\begin{aligned}\int_0^{10} (20-2x)\,dx &= (20x - x^2)\Big|_0^{10}\\ &= (200-100)-(0) = 100.\end{aligned}$$

It would take 100 hr to produce the first 10 items.

56. The tanker is leaking oil at a rate in barrels per hour of

$$L'(t) = \frac{80\ \ln\,(t+1)}{t+1}.$$

(a) $\int_0^{24} \frac{80\ \ln\,(t+1)}{t+1}\,dt$

Let $u = \ln\,(t+1)$, so that $du = \frac{1}{t+1}\,dt.$

When $t = 24,\ u = \ln\ 25.$
When $t = 0,\ u = \ln\ 1 = 0.$

$$\begin{aligned}80\int_0^{\ln\ 25} u\,du &= 80\frac{u^2}{2}\Big|_0^{\ln\ 25}\\ &= 40u^2\Big|_0^{\ln\ 25}\\ &= 40(\ln\ 25)^2 - 40(0)^2\\ &\approx 414\end{aligned}$$

About 414 barrels will leak on the first day.

(b) $\int_{24}^{48} \frac{80\ \ln\,(t+1)}{t+1}\,dt$

Let $u = \ln\,(t+1)$, so that the limits of integration with respect to u are $\ln\ 25$ and $\ln\ 49$.

$$\begin{aligned}80\int_{\ln\ 25}^{\ln\ 49} u\,du &= 40u^2\Big|_{\ln\ 25}^{\ln\ 49}\\ &= 40(\ln\ 49)^2 - 40(\ln\ 25)^2\\ &\approx 191\end{aligned}$$

About 191 barrels will leak on the second day.

(c) $\lim_{t\to\infty} L'(t) = \lim_{t\to\infty} \frac{80\ \ln\,(t+1)}{t+1} = 0$

The number of barrels of oil leaking per day is decreasing to 0.

58. Total growth after 2.5 days is

$$\begin{aligned}\int_0^{2.5} R'(x)\,dx &= \int_0^{2.5} 200e^{.2x}\,dx\\ &= 200\frac{e^{.2x}}{.2}\Big|_0^{2.5}\\ &= 1000e^{.2x}\Big|_0^{2.5}\\ &= 1000e^{.5} - 1000e^0\\ &\approx 648.72.\end{aligned}$$

60.
$$\begin{aligned}F(T) &= \int_0^T f(x)\,dx\\ &= \int_0^T kb^x\,dx\\ &= \int_0^T ke^{(\ln\ b)x}\,dx\\ &= k\int_0^T e^{(\ln\ b)x}\,dx\\ &= \frac{k}{\ln\ b}\cdot e^{(\ln\ b)x}\Big|_0^T\\ &= \frac{k}{\ln\ b}\left[e^{(\ln\ b)T} - 1\right]\\ &= \frac{k}{\ln\ b}\left[b^T - 1\right]\end{aligned}$$

62. $$w'(t) = (4t+1)^{1/3}$$

$$\begin{aligned} w(t) = \int_0^3 (4t+1)^{1/3}\,dt &= \frac{1}{4}\cdot\frac{(4t+1)^{4/3}}{\frac{4}{3}}\Big|_0^3 \\ &= \frac{3(4t+1)^{4/3}}{16}\Big|_0^3 \\ &= \frac{3}{16}\left(13^{4/3} - 1^{4/3}\right) \\ &= \frac{3}{16}\left(13^{4/3} - 1\right) \\ &= 5.5439 \end{aligned}$$

The change in weight is 5.5439 mg.

64. $$\int_3^9 (.1762x^2 - 3.986x + 22.68)dx$$

$$\begin{aligned} &= \left(\frac{.1762}{3}x^3 - \frac{3.986}{2}x^2 + 22.68x\right)\Big|_3^9 \\ &= 85.5036 - 51.6888 \\ &= 33.8148 \end{aligned}$$

The total increase in the length of a ram's horn during the period is about 33.8 cm.

66. $$\int_0^{100} .85e^{.0133x}dx$$

Let $u = .0133x$, so that $du = .0133dx$, or $dx = \frac{1}{.0133}du$

When $x = 100, u = 1.33$.

When $x = 0, u = 0$.

$$\begin{aligned} \int_0^{1.33} .85e^u\left(\frac{1}{.0133}\right)du &= \frac{.85}{.0133}e^u\Big|_0^{1.33} \\ &= \frac{.85}{.0133}(e^{1.33} - e^0) \\ &\approx 177.736 \end{aligned}$$

The total mass of the column is about 178 g.

68. $$\int_{2.5}^5 (.058x^3 - 1.08x^2 + 4.81x + 6.26)dx$$

$$\begin{aligned} &= \left(\frac{.058}{4}x^4 - \frac{1.08}{3}x^3 + \frac{4.81}{2}x^2 + 6.26x\right)\Big|_{2.5}^5 \\ &= (.0145x^4 - .36x^3 + 2.405x^2 + 6.26x)\Big|_{2.5}^5 \\ &\approx 55.4875 - 25.6227 \\ &= 29.8648 \end{aligned}$$

In 2000, approximately 30% of the population had an income between \$25,000 and \$50,000.

70. $c'(t) = 1.2e^{.04t}$

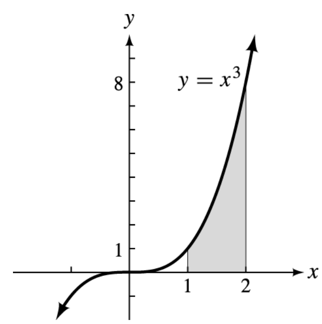
$$\begin{aligned} c(T) &= \int_0^T 1.2e^{.04t} \\ &= \frac{1.2}{.04}e^{.04t}\Big|_0^T \\ &= 30(e^{.04T} - e^0) \\ &= 30\left(e^{.04T} - 1\right) \end{aligned}$$

In 5 yr,

$$\begin{aligned} c(5) &= 30(e^{.04(5)} - 1) \\ &= 30(e^{.2} - 1) \\ &\approx 6.64 \text{ billion barrels.} \end{aligned}$$

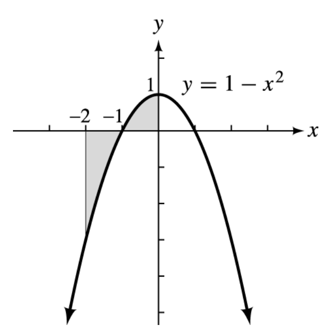
7.5 The Area Between Two Curves

2. $x = 1,\ x = 2,\ y = x^3,\ y = 0$



$$\begin{aligned} \int_1^2 (x^3 - 0)\,dx &= \frac{x^4}{4}\Big|_1^2 \\ &= 4 - \frac{1}{4} \\ &= \frac{15}{4} \end{aligned}$$

4. $x = -2,\ x = 0,\ y = 1 - x^2,\ y = 0$



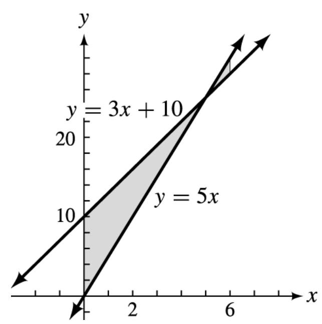
The region is composed of two separate regions in $[-2, 0]$ because $y = 1 - x^2$ intersects $y = 0$ at $x = -1$.

Let $f(x) = 1 - x^2,\ g(x) = 0$.

In the interval $[-2, -1]$, $g(x) \geq f(x)$.
In the interval $[-1, 0]$, $f(x) \geq g(x)$.

$$\int_{-2}^{-1} [0 - (1 - x^2)]\,dx + \int_{-1}^{0} [(1 - x^2) - 0]\,dx$$
$$= \left(-x + \frac{x^3}{3}\right)\Big|_{-2}^{-1} + \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{0}$$
$$= \left(1 - \frac{1}{3}\right) - \left(2 - \frac{8}{3}\right) + 0 - \left(-1 + \frac{1}{3}\right)$$
$$= 1 - \frac{1}{3} - 2 + \frac{8}{3} + 1 - \frac{1}{3}$$
$$= 2$$

6. $x = 0,\ x = 6,\ y = 5x,\ y = 3x + 10$



To find the intersection of $y = 5x$ and $y = 3x + 10$, substitute for y.

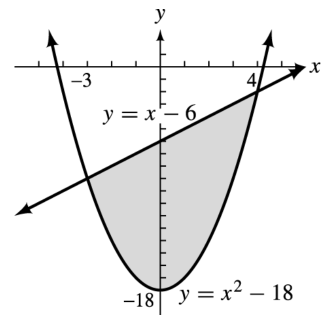
$$5x = 3x + 10$$
$$2x = 10$$
$$x = 5$$

If $x = 5$, $y = 5(5) = 25$.
The region is composed of two separate regions because $y = 5x$ and $y = 3x + 10$ intersect at $x = 5$ (that is, $(5, 25)$).
Let $f(x) = 3x + 10$, $g(x) = 5x$.

In the interval $[0, 5]$, $f(x) \geq g(x)$.
In the interval $[5, 6]$, $g(x) \geq f(x)$.

$$\int_0^5 (3x + 10 - 5x)\,dx + \int_5^6 [5x - (3x + 10)]\,dx$$
$$= \int_0^5 (-2x + 10)\,dx + \int_5^6 (2x - 10x)\Big|_5^6$$
$$= \left(\frac{-2x^2}{2} + 10x\right)\Big|_0^5 + \left(\frac{2x^2}{2} - 10x\right)\Big|_5^6$$
$$= (-x^2 + 10x)\Big|_0^5 + (x^2 - 10x)\Big|_5^6$$
$$= -25 + 50 + (36 - 60) - (25 - 50)$$
$$= 26$$

8. $y = x^2 - 18,\ y = x - 6$



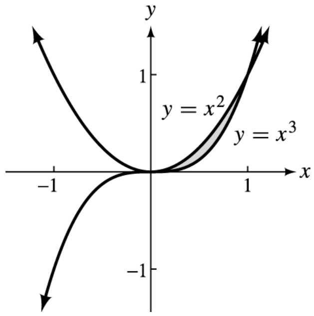
Find the intersection points.

$$x^2 - 18 = x - 6$$
$$x^2 - x - 12 = 0$$
$$(x - 4)(x + 3) = 0$$

The curves intersect at $x = -3$ and $x = 4$.

$$\int_{-3}^{4} [(x - 6) - (x^2 - 18)]\,dx$$
$$= \int_{-3}^{4} (x - 6 - x^2 + 18)\,dx$$
$$= \int_{-3}^{4} (-x^2 + x + 12)\,dx$$
$$= \left(\frac{-x^3}{3} + \frac{x^2}{2} + 12x\right)\Big|_{-3}^{4}$$
$$= \left(\frac{-64}{3} + 8 + 48\right) - \left(9 + \frac{9}{2} - 36\right)$$
$$= \left(\frac{-64}{3} + 56\right) - \left(-27 + \frac{9}{2}\right)$$
$$= \frac{-64}{3} + 83 - \frac{9}{2}$$
$$= \frac{343}{6}$$
$$\approx 57.167$$

10. $y = x^2,\ y = x^3$



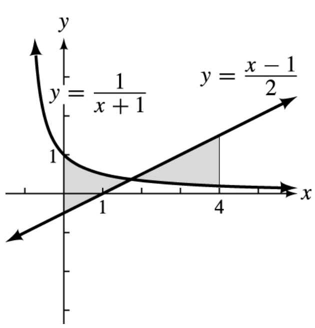
Find the intersection points.

$$\begin{aligned} x^2 &= x^3 \\ x^2 - x^3 &= 0 \\ x^2(1-x) &= 0 \end{aligned}$$

The curves intersect at $x = 0$ and $x = 1$. In the interval $[0, 1]$, $x^2 > x^3$.

$$\begin{aligned} \int_0^1 (x^2 - x^3)\,dx &= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 \\ &= \frac{1}{3} - \frac{1}{4} \\ &= \frac{1}{12} \end{aligned}$$

12. $x = 0,\ x = 4,\ y = \dfrac{1}{x+1},\ y = \dfrac{x-1}{2}$



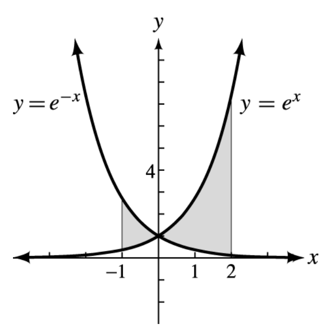
Find the intersection points.

$$\begin{aligned} \frac{1}{x+1} &= \frac{x-1}{2} \\ x^2 - 1 &= 2 \\ x^2 - 3 &= 0 \end{aligned}$$

In the interval $[0, 4]$, the only intersection point is at $x = \sqrt{3}$.

$$\begin{aligned} &\int_0^{\sqrt{3}} \left(\frac{1}{x+1} - \frac{x-1}{2}\right) dx + \int_{\sqrt{3}}^4 \left(\frac{x-1}{2} - \frac{1}{x+1}\right) dx \\ &= \left(\ln|x+1| - \frac{x^2}{4} + \frac{x}{2}\right)\Big|_0^{\sqrt{3}} \\ &\quad + \left(\frac{x^2}{4} - \frac{x}{2} - \ln\ |x+1|\right)\Big|_{\sqrt{3}}^4 \\ &= \ln(\sqrt{3}+1) - \frac{3}{4} + \frac{\sqrt{3}}{2} \\ &\quad + \left\{(4 - 2 - \ln\ 5) - \left[\frac{3}{4} - \frac{\sqrt{3}}{2} - \ln(\sqrt{3}+1)\right]\right\} \\ &= \ln(\sqrt{3}+1) - \frac{3}{4} + \frac{\sqrt{3}}{2} + 2 \\ &\quad - \ln\ 5 - \frac{3}{4} + \frac{\sqrt{3}}{2} + \ln(\sqrt{3}+1) \\ &= \ln(\sqrt{3}+1) + \ln(\sqrt{3}+1) \\ &\quad - \ln\ 5 + \frac{1}{2} + \sqrt{3} \\ &= \ln\frac{(\sqrt{3}+1)^2}{5} + \frac{1}{2} + \sqrt{3} \\ &\approx 2.633 \end{aligned}$$

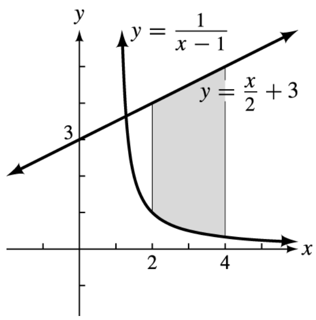
14. $x = -1,\ x = 2,\ y = e^{-x},\ y = e^x$



The total area between the curves from $x = -1$ to $x = 2$ is

$$\begin{aligned} &\int_{-1}^0 (e^{-x} - e^x)\,dx + \int_0^2 (e^x - e^{-x})\,dx \\ &= (-e^{-x} - e^x)\Big|_{-1}^0 + (e^x + e^{-x})\Big|_0^2 \\ &= [(-1-1) - (-e - e^{-1})] \\ &\quad + [(e^2 + e^{-2}) - (1+1)] \\ &= e^2 + e^{-2} + e + e^{-1} - 4 \\ &\approx 6.6106. \end{aligned}$$

16. $x = 2,\ x = 4,\ y = \dfrac{x}{2} + 3,\ y = \dfrac{1}{x-1}$



Find the intersection points in $[2, 4]$.

$$\begin{aligned} \frac{x}{2} + 3 &= \frac{1}{x-1} \\ \frac{x+6}{2} &= \frac{1}{x-1} \\ x^2 + 5x - 6 &= 2 \\ x^2 + 5x - 8 &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{25 + 4(8)}}{2} \\ &= \frac{-5 \pm \sqrt{57}}{2} \\ x &= 1.27 \quad \text{or} \quad x = -6.27 \end{aligned}$$

There are no intersection points in the interval.

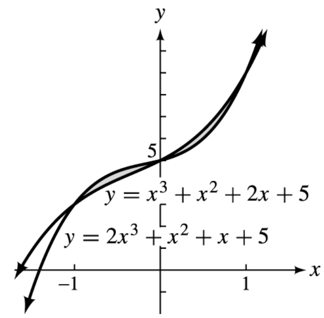
$$\frac{x}{2} + 3 \geq \frac{1}{x-1} \text{ in } [2,4].$$

$$\int_2^4 \left[\left(\frac{x}{2} + 3 \right) - \left(\frac{1}{x-1} \right) \right] dx$$

$$= \left(\frac{x^2}{4} + 3x - \ln |x-1| \right) \Big|_2^4$$

$$\begin{aligned} &= (4 + 12 - \ln 3) - (1 + 6 - 0) \\ &= 16 - \ln 3 - 7 \\ &= 9 - \ln 3 \\ &\approx 7.901 \end{aligned}$$

18. $y = 2x^3 + x^2 + x + 5,$
$y = x^3 + x^2 + 2x + 5$



To find the points of intersection, substitute for y.

$$\begin{aligned} 2x^3 + x^2 + x + 5 &= x^3 + x^2 + 2x + 5 \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \end{aligned}$$

The points of intersection are at $x = 0,\ x = -1,$ and $x = 1$.
The area of the region between the curves is

$$\int_{-1}^{0} [(2x^3 + x^2 + x + 5) - (x^3 + x^2 + 2x + 5)]\, dx$$

$$+ \int_0^1 [(x^3 + x^2 + 2x + 5) - (2x^3 + x^2 + x + 5)]\, dx$$

$$\begin{aligned} &= \int_{-1}^{0} (x^3 - x)\, dx + \int_0^1 (-x^3 + x)\, dx \\ &= \left(\frac{x^4}{4} - \frac{x^2}{2} \right) \Big|_{-1}^{0} + \left(-\frac{x^4}{4} + \frac{x^2}{2} \right) \Big|_0^1 \\ &= \left[0 - \left(\frac{1}{4} - \frac{1}{2} \right) \right] + \left[\left(-\frac{1}{4} + \frac{1}{2} \right) - 0 \right] \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

20. $y = x^5 - 2\ \ln (x + 5),$
$y = x^3 - 2\ \ln (x + 5)$

To find the points of intersection, substitute for y.

$$\begin{aligned} x^5 - 2\ \ln (x + 5) &= x^3 - 2\ \ln (x + 5) \\ x^5 - x^3 &= 0 \\ x^3(x^2 - 1) &= 0 \end{aligned}$$

The points of intersection are at $x = 0$ and $x = 1$ and $x = -1$.

In the interval $[-1, 0]$,

$$x^5 - 2\ \ln (x + 5) > x^3 - 2 \ln (x + 5).$$

In the interval $[0, 1]$,

$$x^5 - 2\ \ln (x + 5) < x^3 - 2 \ln (x + 5).$$

The area between the curves is

$$\int_{-1}^{0}[(x^5 - 2\ \ln\,(x+5)) - (x^3 - 2\ \ln\,(x+5))]\,dx$$
$$+\int_{0}^{1}[(x^3 - 2\ \ln\,(x+5)) - (x^5 - 2\ \ln\,(x+5))]\,dx$$

$$= \int_{-1}^{0}(x^5 - x^3)\,dx + \int_{0}^{1}(x^3 - x^5)\,dx$$

$$= \left(\frac{x^6}{6} - \frac{x^4}{4}\right)\Bigg|_{-1}^{0} + \left(\frac{x^4}{4} - \frac{x^6}{6}\right)\Bigg|_{0}^{1}$$

$$= \left[0 - \left(\frac{1}{6} - \frac{1}{4}\right)\right] + \left[\left(\frac{1}{4} - \frac{1}{6}\right) - 0\right]$$

$$= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}.$$

22. $y = \sqrt{x},\ y = x\sqrt{x}$

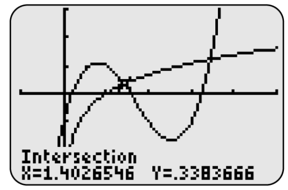
To find the points of intersection, substitute for y.

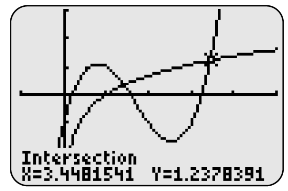
$$\begin{aligned} \sqrt{x} &= x\sqrt{x} \\ x\sqrt{x} - \sqrt{x} &= 0 \\ \sqrt{x}(x-1) &= 0 \end{aligned}$$

The points of intersection are at $x = 0$ and $x = 1$.
In $[0, 1]$, $\sqrt{x} > x\sqrt{x}$.
The area between the curves is

$$\begin{aligned} \int_0^1 (\sqrt{x} - x\sqrt{x})\,dx &= \int_0^1 (x^{1/2} - x^{3/2})\,dx \\ &= \left(\frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\right)\Bigg|_0^1 \\ &= \left(\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right)\Bigg|_0^1 \\ &= \left[\frac{2}{3}(1) - \frac{2}{5}(1)\right] - 0 \\ &= \frac{4}{15}. \end{aligned}$$

24. Graph $y_1 = \ln x$ and $y_2 = x^3 - 5x^2 + 6x - 1$ on your graphing calculator. Use the intersect command to find the two intersection points. The resulting screens are:





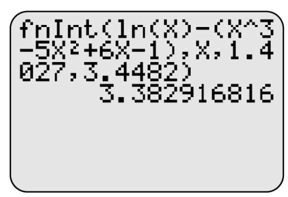
These screens show that $\ln x = x^3 - 5x^2 + 6x - 1$ when $x \approx 1.4027$ and $x \approx 3.4482$.
In the interval $[1.4027, 3.4482]$,

$$\ln x > x^3 - 5x^2 + 6x - 1.$$

The area between the curves is given by

$$\int_{1.4027}^{3.4482} [\ln x - (x^3 - 5x^2 + 6x - 1)]dx.$$

Use the fnInt command to approximate this definite integral. The resulting screen is:



The last screen shows that the area is approximately 3.3829.

26. **(a)** $S(x) = -x^2 + 4x + 8, C(x) = \dfrac{3}{25}x^2$

$$\begin{aligned} S(x) &= C(x) \\ -x^2 + 4x + 8 &= \frac{3}{25}x^2 \\ -25x^2 + 100x + 200 &= 3x^2 \end{aligned}$$

$$\begin{aligned} 0 &= 28x^2 - 100x - 200 \\ 0 &= 7x^2 - 25x - 50 \\ 0 &= (7x + 10)(x - 5) \end{aligned}$$

$$x = -\frac{10}{7} \quad \text{or} \quad x = 5$$

Since time would not be negative, 5 is the only solution.

It will pay to use the device for 5 yr.

(b) The total savings over 5 yr is given by

$$\int_0^5 (-x^2 + 4x + 8)\,dx$$
$$= \left(\frac{-x^3}{3} + 2x^2 + 8x\right)\Big|_0^5$$
$$= \frac{-125}{3} + 90$$
$$= 48.33.$$

The total cost over 5 yr is given by

$$\int_0^5 \frac{3}{25}x^2\,dx = \frac{x^3}{25}\Big|_0^5 = 5.$$

$$\text{Net savings} = \$48.33 \text{ million} - \$5 \text{ million}$$
$$= \$43.33 \text{ million}$$

28. (a) $R(t) = 104 - .4e^{t/2}; C(t) = .3e^{t/2}$

It will no longer be profitable when $C(t) > R(t)$. Find t when $C(t) > R(t)$.

$$.3e^{t/2} > 104 - .4e^{t/2}$$
$$.7e^{t/2} > 104$$
$$e^{t/2} > \frac{104}{.7}$$
$$\ln\, e^{t/2} > \ln\left(\frac{104}{.7}\right)$$
$$t > 2\,\ln\left(\frac{104}{.7}\right)$$
$$t > 10$$

It will no longer be profitable to use the process after 10 yr.

(b) The total net savings is

$$\int_0^{10} [(104 - .4e^{t/2}) - .3e^{t/2}]\,dt$$
$$= \int_0^{10} (104 - .7e^{t/2})\,dt$$
$$= \left(104t - \frac{.7e^{t/2}}{1/2}\right)\Big|_0^{10}$$
$$= (104t - 1.4e^{t/2})\Big|_0^{10}$$
$$= [(104t - 1.4e^{t/2}) - (0 - 1.4)]$$
$$= 1041.4 - 1.4e^5$$
$$\approx 834.$$

The net total savings will be $834,000.

30. $S(q) = 100 + 3q^{3/2} + q^{5/2}$; equilibrium quantity is $q = 9$.

$$\text{Producers' surplus} = \int_0^{q_0} [p_0 - S(q)]\,dq$$
$$p_0 = S(9) = 424$$
$$\int_0^9 [424 - (100 + 3q^{3/2} + q^{5/2})]\,dq$$
$$= \int_0^9 (324 - 3q^{3/2} - q^{5/2})]\,dq$$
$$= \left(324q - \frac{6}{5}q^{5/2} - \frac{2}{7}q^{7/2}\right)\Big|_0^9$$
$$= \left[\left(324(9) - \frac{6}{5}(9)^{5/2} - \frac{2}{7}(9)^{7/2}\right) - 0\right]$$
$$= 2916 - \frac{1458}{5} - \frac{4374}{7}$$
$$= 1999.54$$

The producers' surplus is 1999.54.

32. $D(q) = \dfrac{16,000}{(2q+8)^3}$; equilibrium quantity is $q = 6$.

$$\text{Consumers' surplus} = \int_0^{q_0} [D(q) - p_0]\,dq$$
$$p_0 = D(6) = \frac{16{,}000}{20^3} = 2$$
$$\int_0^6 \left[\frac{16{,}000}{(2q+8)^3} - 2\right]\,dq$$
$$= \int_0^6 \frac{16{,}000}{(2q+8)^3}\,dq - \int_0^6 2\,dq$$

Let $u = 2q + 8$, so that

$$du = 2\,dq \text{ and } \frac{1}{2}\,du = dq.$$

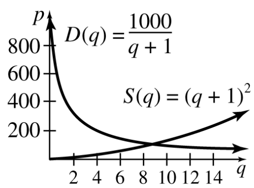
$$= \frac{1}{2}\int_8^{20} \frac{16{,}000}{u^3}\,du - \int_0^6 2\,dq$$
$$= 8000\int_8^{20} u^{-3}\,du - \int_0^6 2\,dq$$
$$= 8000 \cdot \frac{u^{-2}}{-2}\Big|_8^{20} - 2q\Big|_0^6$$
$$= \frac{-4000}{u^2}\Big|_8^{20} - 2q\Big|_0^6$$
$$= \left(\frac{-4000}{400} + \frac{4000}{64}\right) - 12$$
$$= -10 + 62.5 - 12$$
$$= 40.5$$

The consumers' surplus is 40.50.

34. $S(q) = (q+1)^2$

$$D(q) = \frac{1000}{q+1}$$

(a) The graph of the supply function is a parabola with vertex at $(-1, 0)$. The graph of the demand function is the graph of a rational function with vertical asymptote of $x = -1$ and horizontal asymptote of $y = 0$.



(b) Find the equilibrium point by setting the two functions equal.

$$\begin{aligned}(q+1)^2 &= \frac{1000}{q+1}\\ (q+1)^3 &= 1000\\ q^3 + 3q^2 + 3q + 1 &= 1000\\ q^3 + 3q^2 + 3q - 999 &= 0\\ (q-9)(q^2 + 12q + 111) &= 0\end{aligned}$$

Since $q^2 + 12q + 111$ has no real roots, $q = 9$ is the only root. At the equilibrium point where the supply and demand are both 9 items, the price is

$$S(9) = (9+1)^2 = 100.$$

The equilibrium point is $(9, 100)$.

(c) The consumers' surplus is given by

$$\begin{aligned}\int_0^9 \left(\frac{1000}{q+1} - 100\right) dq &= (1000 \ln|q+1| - 100q)\Big|_0^9\\ &= 1000\ln(9+1) - 100(9) - 0\\ &\approx 1402.59\end{aligned}$$

Here the consumers' surplus is 1402.59.

(d) The producers' surplus is given by

$$\begin{aligned}\int_0^9 [100 - (q+1)^2]dq &= \int_0^9 (99 - q^2 - 2q)dq\\ &= (99q - \frac{1}{3}q^3 - q^2)\Big|_0^9\\ &= 99(9) - \frac{1}{3}(9)^3 - (9)^2 - 0\\ &= 567\end{aligned}$$

Here the producers' surplus is 567.

38. (a) The pollution level in the lake is changing at the rate $f(t) - g(t)$ at any time t. We find the amount of pollution by integrating.

$$\begin{aligned}&\int_0^{12} [f(t) - g(t)]dt\\ &= \int_0^{12} [15(1 - e^{-.05t}) - .3t]dt\\ &= \left(15t - 15\frac{1}{-.05}e^{-.05t} - .3\frac{1}{2}t^2\right)\Big|_0^{12}\\ &= (300e^{-.05t} + 15t - .15t^2)\Big|_0^{12}\\ &= [300e^{-.05(12)} + 15(12) - .15(12)^2]\\ &\quad - [300e^{-.05(0)} + 15(0) - .15(0)^2]\\ &= (300e^{-.6} + 158.4) - (300)\\ &= 300e^{-.6} - 141.6\\ &\approx 23.04\end{aligned}$$

After 12 hours, there are about 23.04 gallons.

(b) The graphs of the functions intersect at about 44.63. So the rate that pollution enters the lake equals the rate the pollution is removed at about 44.63 hours.

(c) $\int_0^{44.63} [f(t) - g(t)]dt$

$$\begin{aligned}&= (300e^{-.05t} + 15t - .15t^2)\Big|_0^{44.63}\\ &= [300e^{-.05(44.63)} + 15(44.63) - .15(44.63)^2]\\ &\quad - 300\\ &= (300e^{-2.2315} + 370.674465) - 300\\ &= 300e^{-2.2315} + 70.674465\\ &\approx 102.88\end{aligned}$$

After 44.63 hours, there are about 102.88 gallons.

(d) For $t > 44.63, g(t) > f(t)$, and pollution is being removed at the rate $g(t) - f(t)$. So, we want to solve for c, where

$$\int_0^c [f(t) - g(t)]dt = 0$$

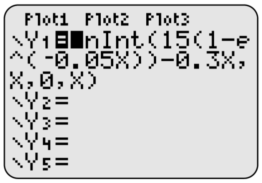
(Alternatively, we could solve for c in

$$\int_{44.63}^c [g(t) - f(t)]dt = 102.88.)$$

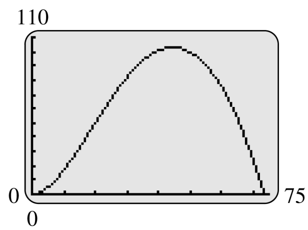
One way to do this with a graphing calculator is to graph the function

$$y = \int_0^x [f(t) - g(t)]dt$$

and determine the values of x for which $y = 0$. The first window shows how the function can be defined.

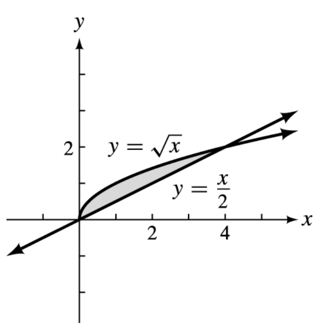


A suitable window for the graph is [.75] by $[0, 110]$.



Use the calculator's features to approximate where the graph intersects the x-axis. These are at 0 and abaout 73.47. Therefore, the pollution will be removed from the lake after about 73.47 hours.

40. $y = \sqrt{x},\ y = \dfrac{x}{2}$



To find the points of intersection, substitute for y.

$$\sqrt{x} = \frac{x}{2}$$
$$\frac{x}{2} - \sqrt{x} = 0$$
$$x - 2\sqrt{x} = 0$$
$$\sqrt{x}(\sqrt{x} - 2) = 0$$
$$x = 0 \quad \text{or} \quad x = 4$$

$$\text{Area} = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \int_0^4 \left(x^{1/2} - \frac{x}{2}\right) dx$$
$$= \left(\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right)\Big|_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$$

7.6 Numerical Integration

2. $\displaystyle\int_0^2 (2x+1)\,dx$

$n = 4,\ b = 2,\ a = 0,\ f(x) = 2x + 1$

i	x_i	$f(x_i)$
0	0	1
1	$\frac{1}{2}$	2
2	1	3
3	$\frac{3}{2}$	4
4	2	5

(a) Trapezoidal rule:

$$\int_0^2 (2x+1)\,dx \approx \frac{2-0}{4}\left[\frac{1}{2}(1) + 2 + 3 + 4 + \frac{1}{2}(5)\right]$$
$$= \frac{1}{2}\left(\frac{1}{2} + 2 + 3 + 4 + \frac{5}{2}\right) = 6$$

(b) Simpson's rule:

$$\int_0^2 (2x+1)\,dx$$
$$\approx \frac{2-0}{3(4)}[1 + 4(2) + 2(3) + 4(4) + 5]$$
$$\frac{1}{6}(36) = 6$$

(c) Exact value:

$$\int_0^2 (2x+1)\,dx = (x^2 + x)\Big|_0^2$$
$$= 4 + 2$$
$$= 6$$

4. $\displaystyle\int_1^5 \frac{1}{x+1}\,dx$

$n = 4,\ b = 5,\ a = 1,\ f(x) = \frac{1}{x+1}$

i	x_i	$f(x_i)$
0	1	$\frac{1}{2}$
1	2	$\frac{1}{3}$
2	3	$\frac{1}{4}$
3	4	$\frac{1}{5}$
4	5	$\frac{1}{6}$

(a) Trapezoidal rule:

$$\int_1^5 \frac{dx}{x+1} \approx \frac{5-1}{4}\left[\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{2}\left(\frac{1}{6}\right)\right]$$
$$= \frac{1}{4}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{12}$$
$$\approx 1.1167$$

(b) Simpson's rule:

$$\int_1^5 \frac{dx}{x+1}$$
$$\approx \frac{5-1}{3(4)}\left[\frac{1}{2}+4\left(\frac{1}{3}\right)+2\left(\frac{1}{4}\right)+4\left(\frac{1}{5}\right)+\frac{1}{6}\right]$$
$$= \frac{1}{3}\left(\frac{1}{2}+\frac{4}{3}+\frac{1}{2}+\frac{4}{5}+\frac{1}{6}\right) = 1.1000$$

(c) Exact value:

$$\int_1^5 \frac{dx}{x+1} = \ln\,|x+1|\Big|_1^5$$
$$= \ln\,6 - \ln\,2 = \ln\,\frac{6}{2}$$
$$= \ln\,3 \approx 1.0986$$

6. $\int_0^3 (2x^2+1)\,dx$

$n=4,\ b=3,\ a=0,\ f(x)=2x^2+1$

i	x	$f(x)$
0	0	1
1	.75	2.125
2	1.5	5.5
3	2.25	11.125
4	3	19

(a) Trapezoidal rule:

$$\int_0^3 (2x^2+1)\,dx$$
$$= \frac{3-0}{4}\left[\frac{1}{2}(1)+2.125+5.5+11.125+\frac{1}{2}(19)\right]$$
$$= \frac{3}{4}(28.75)$$
$$= 21.5625$$

(b) Simpson's rule:

$$\int_0^3 (2x^2+1)\,dx$$
$$= \frac{3-0}{3(4)}[1+4(2.125)+2(5.5)$$
$$+4(11.125)+19]$$
$$= \frac{1}{4}(1+8.5+11+44.5+19)$$
$$= 21.0000$$

(c) Exact value:

$$\int_9^3 (2x^2+1)\,dx = \left(\frac{2x^3}{3}+x\right)\Big|_0^3$$
$$= 18+3 = 21$$

8. $\int_2^4 \frac{1}{x^3}\,dx$

$n=4,\ b=4,\ a=2,\ f(x)=\frac{1}{x^3}$

i	x_i	$f(x_i)$
0	2	.125
1	2.5	.064
2	3	.03703
3	3.5	.02332
4	4	.015625

(a) Trapezoidal rule:

$$\int_2^4 \frac{dx}{x^3} \approx \frac{4-2}{4}\left[\frac{1}{2}(.125)+.064+.03703\right.$$
$$\left.+\ .02332+\frac{1}{2}(.015625)\right]$$
$$\approx \frac{1}{2}(.19466) \approx .0973$$

(b) Simpson's rule:

$$\int_2^4 \frac{dx}{x^3} \approx \frac{4-2}{3(4)}[.125+4(.064)+2(.03703)$$
$$+4(.02332)+.015625]$$
$$\approx \frac{1}{6}(.56397)$$
$$\approx .0940$$

(c) Exact value:

$$\int_2^4 \frac{dx}{x^3} = \int_2^4 x^{-3}\,dx = \frac{x^{-2}}{-2}\Big|_2^4 = \frac{-1}{2x^2}\Big|_2^4$$
$$= \frac{-1}{32}+\frac{1}{8} = \frac{3}{32} = .09375$$

10. $\int_0^4 \sqrt{2x+1}\,dx$

$n = 4,\ b = 4,\ a = 0,\ f(x) = \sqrt{2x+1}$

i	x_i	$f(x_i)$
0	0	1
1	1	1.7321
2	2	2.2361
3	3	2.6458
4	4	3

(a) Trapezoidal rule:

$$\int_0^4 \sqrt{2x+1}\,dx$$

$$\approx \frac{4-0}{4}\left[\frac{1}{2}(1)+1.7321+2.2361+2.6458+\frac{1}{2}(3)\right]$$

$$\approx 8.614$$

(b) Simpson's rule:

$$\int_0^4 \sqrt{2x+1}\,dx$$

$$\approx \frac{4-0}{3(4)}[1+4(1.7321)+2(2.2361)+4(2.6458)+3]$$

$$\approx \frac{1}{3}[1+6.9284+4.4722+10.5832+3]$$

$$\approx \frac{1}{3}[25.9838] \approx 8.661$$

(c) Exact value:

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\frac{(2x+1)^{3/2}}{\frac{3}{2}}\Big|_0^4$$

$$= \frac{1}{3}(2x+1)^{3/2}\Big|_0^4$$

$$= 9 - \frac{1}{3} = \frac{26}{3}$$

$$\approx 8.6667$$

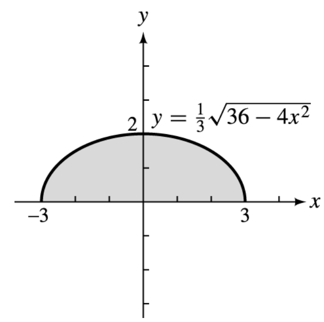
12. $4x^2 + 9y^2 = 36$

$$y^2 = \frac{36-4x^2}{9}$$

$$y = \pm\frac{1}{3}\sqrt{36-4x^2}$$

An equation of the semiellipse is

$$y = \frac{1}{3}\sqrt{36-4x^2}.$$



$n = 12,\ b = -3,\ a = 3$

i	x_i	y
0	−3	0
1	−2.5	1.1055
2	−2	1.4907
3	−1.5	1.7321
4	−1	1.8856
5	−.5	1.9720
6	0	2
7	.5	1.9720
8	1	1.8856
9	1.5	1.7321
10	2	1.4907
11	2.5	1.1055
12	3	0

(a) Trapezoidal rule:

$$\frac{A}{2} = \frac{6}{12}\left[\,\frac{1}{2}(0) + 1.1055 + 1.4907 + 1.7321 + 1.8856 + 1.972 + 2 + 1.972 + 1.8856 + 1.7321 + 1.4907 + 1.1055 + 2(0)\right]$$

$$= 9.1859$$

(b) Simpson's rule:

$$\frac{A}{2} = \frac{6}{3(12)}[0 + 4(1.1055) + 2(1.4907) + 4(1.7321) + 2(1.8856) + 4(1.972) + 2(2) + 4(1.972) + 2(1.8856) + 4(1.7321) + 2(1.4907) + 4(1.1055) + 0]$$

$$= \frac{1}{6}(55.982)$$

$$\approx 9.3304$$

(c) The trapezoidal rule gives the area of the region as 9.1859. Simpson's rule gives the area of the region as 9.3304. The actual area is $3\pi \approx 9.4248$. Simpson's rule is a better approximation.

14. **(a)** $f(x) = x^2;\ [0, 3]$

$$T > \int_a^b f(x)\,dx$$

By looking at the graph of $y = x^2$ and dividing the area between 0 and 3 into an even number of trapezoids, you can see that each trapezoid has an area greater than the actual area.

(b) $f(x) = \sqrt{x};\ [0, 9]$

$$T < \int_a^b f(x)\,dx$$

By looking at the graph of $y = \sqrt{x}$ and dividing area between 0 and 9 into an even number of trapezoids, you can see that each trapezoid has an area less than the actual area.

(c) You can't say which is larger because some trapezoids are greater than the given area and some are less than the given area.

16. As n changes from 4 to 8, for example, the error changes from .020703 to .005200.

$$.020703a = .005200$$
$$a \approx \frac{1}{4}$$

Similar results would be obtained using other values for n.
The error is multiplied by $\frac{1}{4}$.

18. As n changes form 4 to 8, the error changes from .0005208 to .0000326.

$$.0005208a = .0000326$$
$$a \approx \frac{1}{16}$$

Similar results would be obtained using other values for n.
The error is multiplied by $\frac{1}{16}$.

20. Midpoint rule: $n = 4, b = 4, a = 2, f(x) = \dfrac{1}{x^3}$,

$\Delta x = \dfrac{1}{2}$

i	x_i	$f(x_i)$
1	$\frac{9}{4}$	$\frac{64}{729}$
2	$\frac{11}{4}$	$\frac{64}{1331}$
3	$\frac{13}{4}$	$\frac{64}{2197}$
4	$\frac{15}{4}$	$\frac{64}{3375}$

$$\int_2^4 \frac{1}{x^3}dx$$
$$\approx \sum_{i=1}^{4} f(x_i)\Delta x$$
$$= \frac{64}{729}\left(\frac{1}{2}\right) + \frac{64}{1331}\left(\frac{1}{2}\right) + \frac{64}{2197}\left(\frac{1}{2}\right) + \frac{64}{3375}\left(\frac{1}{2}\right)$$
$$\approx .09198$$

Simpson's rule:

$n = 8, b = 4, a = 2, f(x) = \dfrac{1}{x^3}$

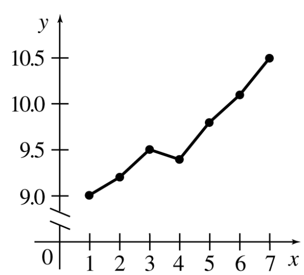
i	x_i	$f(x_i)$
0	2	$\frac{1}{8}$
1	$\frac{9}{4}$	$\frac{64}{729}$
2	$\frac{5}{2}$	$\frac{8}{125}$
3	$\frac{11}{4}$	$\frac{64}{1331}$
4	3	$\frac{1}{27}$
5	$\frac{13}{4}$	$\frac{64}{2197}$
6	$\frac{7}{2}$	$\frac{8}{343}$
7	$\frac{15}{4}$	$\frac{64}{3375}$
8	4	$\frac{1}{64}$

$$\int_2^4 \frac{1}{x^3}dx$$
$$\approx \frac{4-2}{3(8)}\left[\frac{1}{8}+4\left(\frac{64}{729}\right)+2\left(\frac{8}{125}\right)+4\left(\frac{64}{1331}\right)\right.$$
$$+2\left(\frac{1}{27}\right)+4\left(\frac{64}{2197}\right)+2\left(\frac{8}{343}\right)$$
$$\left.+4\left(\frac{64}{3375}\right)+\frac{1}{64}\right]$$
$$\approx \frac{1}{12}(1.125223) \approx .09377$$

From #8 part a, $T \approx .0973$, when $n = 4$. To verify the formula evaluate $\frac{2M+T}{3}$.

$$\frac{2M+T}{3} \approx \frac{2(.09198)+.0973}{3}$$
$$\approx .09377$$

22. (a)



(b) $A = \frac{7-1}{6}\left[\frac{1}{2}(9)+9.2+9.5+9.4\right.$
$$\left.+9.8+10.1+\frac{1}{2}(10.5)\right]$$
$$= 57.75$$

(c) $A = \frac{7-1}{3(6)}[9.0+4(9.2)+2(9.5)$
$$+4(9.4)+2(9.8)+4(10.1)+10.5]$$
$$= \frac{1}{3}(172.9)$$
$$= 57.63$$

24. $y = \int_1^7 \left(\frac{2}{t}+e^{-t^2/2}\right)dt$

$n = 12,\ b = 7,\ a = 1$

i	x_i	$f(x_i)$
0	1	2.607
1	1.5	1.658
2	2	1.135
3	2.5	.8439
4	3	.6778
5	3.5	.5736
6	4	.5003
7	4.5	.4445
8	5	.4000
9	5.5	.3636
10	6	.3333
11	6.5	.3077
12	7	.2857

(a) Total growth
$$= \frac{7-1}{12}\left[\frac{1}{2}(2.607)+1.658+1.135+.8439+.6778\right.$$
$$+.5736+.5003+.4445+.4000+.3636+.3333$$
$$\left.+.3077+\frac{1}{2}(.2857)\right]$$
$$\approx 4.3421 \text{ ft}$$

(b) Total growth
$$= \frac{7-1}{3(12)}[2.607+4(1.658)+2(1.135)+4(.8439)$$
$$+2(.6778)+4(.5736)+2(.5003)+4(.4445)$$
$$+2(.4000)+4(.3636)+2(.3333)+4(.3077)$$
$$+.2857]$$
$$\approx 4.2919 \text{ ft}$$

26. $n = 10,\ b = 20,\ a = 0$

i	x_i	y
0	0	0
1	2	2.0
2	4	2.9
3	6	3.0
4	8	2.5
5	10	2.0
6	12	1.75
7	14	1.0
8	16	.75
9	18	.50
10	20	.25

$$A = \frac{20-0}{10}\left[\frac{1}{2}(0) + 2 + 2.9 + 3 + 2.5 + 2 + 1.75 + 1.0 + .75 + .5 + \frac{1}{2}(.25)\right]$$

$= 33.05$ (This answer may vary depending upon readings from the graph.)

The area under the curve, about 33 mcg/ml, represents the total amount of drug available to the patient.

28. The area both under the curve for Formulation B and above the minimum effective concentration line is on the interval $(2, 10)$.
$n = 8,\ b = 10,\ a = 2$

i	x_i	y
0	2	2.0
1	3	2.4
2	4	2.9
3	5	2.8
4	6	3.0
5	7	2.6
6	8	2.5
7	9	2.2
8	10	2.0

Let A_B = area under Formulation B curve between $t = 2$ and $t = 10$.

$$A_B = \frac{10-2}{8}\left[\frac{1}{2}(2) + 2.4 + 2.9 + 2.8 + 3 + 2.6 + 2.5 + 2.2 + \frac{1}{2}(2)\right]$$

$$A_B = 20.4$$

Let A_{ME} = area under minimum effective concentration curve between $t = 2$ and $t = 10$.

$$A_{ME} = (10-2)(2) = 16$$

So the area between A_B and A_{ME} between $t = 2$ and $t = 10$ is $20.4 - 16 = 4.4$.
This area, about 4.4 mcg/ml, represents the total effective amount of the drug available to the patient.

Notice that between $t = 0$ and $t = 12$, the graph for Formulation B is below the line.
Thus, no area exists under the curve for Formulation B and above the minimum effective concentration line in the intervals $(0, 2)$ and $(10, 12)$.

30. For the period Feb. 18 through May 13, there are six 14-day intervals or 84 days.

$n = 6, b = 84, a = 0, f(t)$ as listed

(a)

i	t_i	$f(t_i)$
0	Feb. 18	0
1	Mar. 4	12
2	Mar. 18	30
3	Apr. 1	40
4	Apr. 15	18
5	Apr. 29	8
6	May 13	3

Simpson's rule:

$$\int_a^b f(t)dt$$

$$\approx \frac{84-0}{3(6)}[0 + 4(12) + 2(30) + 4(40) + 2(18) + 4(8) + 3]$$

$$= \frac{14}{3}(339)$$

$$= 1{,}582$$

There were about 1,582 cases.

(b)

i	t_i	$f(t_i)$
0	Feb. 18	0
1	Mar. 4	10
2	Mar. 18	14
3	Apr. 1	11
4	Apr. 15	2
5	Apr. 29	1
6	May 13	1

Simpson's rule:

$$\int_a^b f(t)dt$$

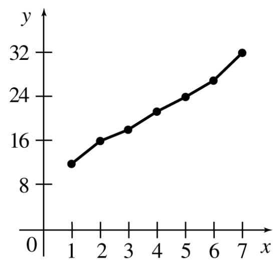
$$\approx \frac{84-0}{3(6)}[0 + 4(10) + 2(14) + 4(11) + 2(2) + 4(1) + 1]$$

$$\approx \frac{14}{3}(121)$$

$$\approx 564.67$$

There were about 565 cases.

32. **(a)**



(b) $A = \frac{7-1}{6}\left[\frac{1}{2}(12) + 16 + 18 + 21 + 24 + 27 + \frac{1}{2}(32)\right]$

$A = 1(128)$
$= 128$

(c) $A = \frac{7-1}{3(6)}[12 + 4(16) + 2(18) + 4(21) + 2(24) + 4(27) + 32]$
$A = 128$

34. We need to evaluate

$$\int_{.1}^{1.1} (.2x^5 - .68x^4 + .8x^3 - .39x^2 + .005x + 100)\,dx.$$

Using a calculator program for Simpson's rule with $n = 100$, we obtain 99.9929 as the value of this integral, which represents the area under the curve from .1 sec to 1.1 sec.

Chapter 7 Review Exercises

6. $\int (5x-1)\,dx = \frac{5x^2}{2} - x + C$

8. $\int (6-x^2)\,dx = 6x - \frac{x^3}{3} + C$

10. $\int \frac{\sqrt{x}}{2}\,dx = \int \frac{1}{2}x^{1/2}\,dx$

$= \frac{\frac{1}{2}x^{3/2}}{\frac{3}{2}} + C$

$= \frac{x^{3/2}}{3} + C$

12. $\int (2x^{4/3} + x^{-1/2})\,dx$

$= \frac{2x^{7/3}}{\frac{7}{3}} + \frac{x^{1/2}}{\frac{1}{2}} + C$

$= \frac{6x^{7/3}}{7} + 2x^{1/2} + C$

14. $\int \frac{5}{x^4}\,dx = \int 5x^{-4}\,dx$

$= \frac{5x^{-3}}{-3} + C$

$= -\frac{5}{3x^3} + C$

16. $\int 5e^{-x}\,dx = -5e^{-x} + C$

18. $\int 2xe^{x^2}\,dx = e^{x^2} + C$

20. $\int \frac{-x}{2-x^2}\,dx = -\frac{1}{2}\int \frac{-2x\,dx}{2-x^2}$

Let $u = 2 - x^2$, so that

$du = -2x\,dx.$

$= \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ \ln\ |u| + C$

$= \frac{1}{2}\ \ln\ \left|2-x^2\right| + C$

22. $\int (x^2-5x)^4(2x-5)\,dx$

Let $u = x^2 - 5x$, so that

$du = (2x-5)\,dx.$

$\int (x^2-5x)^4(2x-5)\,dx$

$= \int u^4\,du$

$= \frac{u^5}{5} + C$

$= \frac{(x^2-5x)^5}{5} + C$

24. $\int e^{3x^2+4}\,x\,dx$

Let $u = 3x^2 + 4$ so that

$dy = 6x\,dx.$

$\int e^{3x^2+4}\,x\,dx = \frac{1}{6}\int (6x)(e^{3x^2})\,dx$

$= \frac{1}{6}\int e^u\,du$

$= \frac{1}{6}e^u + C$

$= \frac{e^{3x^2+4}}{6} + C$

26. **(a)** $\int_0^4 f(x)\,dx = 0$, since the area above the x-axis from 0 to 2 is identical to the area below the x-axis from 2 to 4.

(b) $\int_0^4 f(x)\,dx$ can be computed by calculating the area of the rectangle and triangle that make up the region shown in graph.

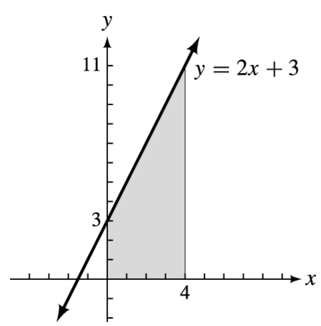
$$\begin{aligned}\text{Area of rectangle} &= (\text{length})(\text{width})\\ &= (3)(1) = 3\end{aligned}$$

$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2}(\text{base})(\text{height})\\ &= \frac{1}{2}(1)(3) = \frac{3}{2}\end{aligned}$$

$$\int_0^4 f(x)\,dx = 3 + \frac{3}{2} = \frac{9}{2} = 4.5$$

28. $\int_0^4 (2x+3)\,dx$

Graph $y = 2x + 3$.



$\int_0^4 (2x+3)\,dx$ is the area of a trapezoid with $B = 11,\ b = 3,\ h = 4$. The formula for the area is

$$A = \frac{1}{2}(B+b)h.$$

$$A = \frac{1}{2}(11+3)(4)$$

$$A = 28,$$

so

$$\int_0^4 (2x+3)\,dx = 28.$$

30. The Fundamental Theorem of Calculus states that

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where f is continuous on $[a, b]$ and F is any antiderivative of f.

32. $\int_1^6 (2x^2 + x)\,dx$

$$\begin{aligned}&= \left(\frac{2x^3}{3} + \frac{x^2}{2}\right)\Big|_1^6\\ &= \left[\frac{2(6)^3}{3} + \frac{(6)^2}{2}\right] - \left[\frac{2(1)^3}{3} + \frac{(1)^2}{2}\right]\\ &= 144 + 18 - \frac{2}{3} - \frac{1}{2}\\ &= 162 - \frac{2}{3} - \frac{1}{2}\\ &= \frac{965}{6}\\ &\approx 160.83\end{aligned}$$

34. $\int_0^1 x\sqrt{5x^2+4}\,dx$

Let $u = 5x^2 + 4$, so that

$$du = 10x\,dx \text{ and } \frac{1}{10}\,du = x\,dx.$$

When $x = 0,\ u = 5(0^2) + 4 = 4$.
When $x = 1,\ u = 5(1^2) + 4 = 9$.

$$\begin{aligned}&= \frac{1}{10}\int_4^9 \sqrt{u}\,du = \frac{1}{10}\int_4^9 u^{1/2}\,du\\ &= \frac{1}{10}\cdot\frac{u^{3/2}}{3/2}\Big|_4^9 = \frac{1}{15}u^{3/2}\Big|_4^9\\ &= \frac{1}{15}(9)^{3/2} - \frac{1}{15}(4)^{3/2}\\ &= \frac{27}{15} - \frac{8}{15}\\ &= \frac{19}{15}\end{aligned}$$

36. $\int_1^6 8x^{-1}\,dx = \int_1^6 \frac{8}{x}\,dx$

$$\begin{aligned}&= 8(\ln\ x)\Big|_1^6\\ &= 8(\ln\ 6 - \ln\ 1)\\ &= 8\ \ln\ 6\\ &\approx 14.334\end{aligned}$$

38. $\int_1^6 \frac{5}{2}e^{4x}\,dx = \frac{1}{4}\cdot\frac{5}{2}\int_1^6 4e^{4x}\,dx$

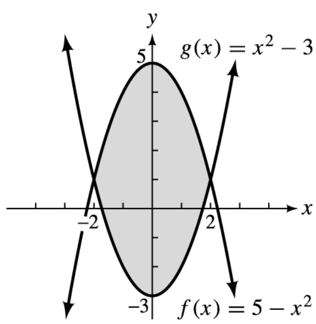
$$\begin{aligned}&= \frac{5e^{4x}}{8}\Big|_1^6\\ &= \frac{5(e^{24} - e^4)}{8}\\ &\approx 1.656 \times 10^{10}\end{aligned}$$

40. $f(x)=\sqrt{x-1};\ [1,10]$

$$\begin{aligned}\text{Area} &= \int_1^{10} \sqrt{x-1}\,dx\\ &= \int_1^{10} (x-1)^{1/2}\,dx\\ &= \frac{2}{3}(x-1)^{3/2}\Big|_1^{10}\\ &= \frac{2}{3}(9)^{3/2}-\frac{2}{3}(0)^{3/2}\\ &= \frac{2}{3}(27)\\ &= 18\end{aligned}$$

42. $\displaystyle\int_0^2 e^x\,dx = e^x\Big|_0^2 = e^2-e^0 = e^2-1 \approx 6.3891$

44. $f(x)=5-x^2,\ g(x)=x^2-3$



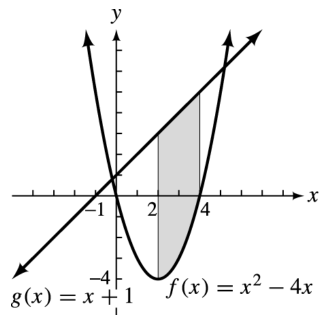
Points of intersection:

$$\begin{aligned}5-x^2 &= x^2-3\\ 2x^2-8 &= 0\\ 2(x^2-4) &= 0\\ x &= \pm 2\end{aligned}$$

Since $f(x)\ge g(x)$ in $[-2,2]$, the area between the graphs is

$$\begin{aligned}\int_{-2}^{2}[f(x)-g(x)]\,dx &= \int_{-2}^{2}[(5-x^2)-(x^2-3)]\,dx\\ &= \int_{-2}^{2}(-2x^2+8)\,dx\\ &= \left(\frac{-2x^3}{3}+8x\right)\Big|_{-2}^{2}\\ &= -\frac{2}{3}(8)+16+\frac{2}{3}(-8)-8(-2)\\ &= \frac{-32}{3}+32\\ &= \frac{64}{3}.\end{aligned}$$

46. $f(x)=x^2-4x,\ g(x)=x+1,\ x=2,\ x=4$



$g(x)>f(x)$ in the interval $[2,4]$.

$$\begin{aligned}&\int_2^4 [(x+1)-(x^2-4x)]\,dx\\ &= \int_2^4 (x+1-x^2+4x)\,dx\\ &= \int_2^4 (5x+1-x^2)\,dx\\ &= \left(\frac{5x^2}{2}+x-\frac{x^3}{3}\right)\Big|_2^4\\ &= \left(\frac{5}{2}(4)^2+4-\frac{(4)^3}{3}\right)\\ &\quad -\left(\frac{5}{2}(2)^2+2-\frac{(2)^3}{3}\right)\\ &= \left(40+4-\frac{64}{3}\right)-\left(10+2-\frac{8}{3}\right)\\ &= \frac{40}{3}\end{aligned}$$

48. $\displaystyle\int_1^3 \frac{\ln\ x}{x}\,dx$

Trapezoidal Rule:

$n=4,\ b=3,\ a=1,\ f(x)=\frac{\ln\ x}{x}$

i	x_1	$f(x_i)$
0	1	0
1	1.5	.27031
2	2	.34657
3	2.5	.36652
4	3	.3662

$$\begin{aligned}\int_1^3 \frac{\ln\ x}{x}\,dx &\approx \frac{3-1}{4}\left[\frac{1}{2}(0)+.27031+.34657\right.\\ &\qquad\left.+.36652+\frac{1}{2}(.3662)\right]\\ &= .58325\end{aligned}$$

Exact value:

$$\int_1^3 \frac{\ln\ x}{x}\, dx$$

$$= \frac{1}{2}(\ln\ x)^2 \Big|_1^3$$

$$= \frac{1}{2}(\ln\ 3)^2 - \frac{1}{2}(\ln\ 1)^2$$

$$\approx .60347$$

50. $\int_0^1 e^x\sqrt{e^x+1}\, dx$

Trapezoidal Rule:
$n = 4,\ b = 1,\ a = 0,\ f(x) = e^x\sqrt{e^x+1}$

i	x_i	$f(x_i)$
0	0	1.4142
1	.25	1.9405
2	.5	2.6833
3	.75	3.7376
4	1	5.2416

$$\int_0^1 e^x\sqrt{e^x+1}\, dx$$

$$= \frac{1-0}{4}\left[\frac{1}{2}(1.4142) + 1.905 + 2.6833\right.$$

$$\left. + 3.7376 + \frac{1}{2}(5.2416)\right]$$

$$\approx 2.9223$$

Exact Value:

$$\int_0^1 e^x\sqrt{e^x+1}\, dx$$

$$= \int_0^1 e^x(e^x+1)^{1/2}\, dx$$

$$= \frac{2}{3}(e^x+1)^{3/2}\Big|_0^1$$

$$= \frac{2}{3}(e+1)^{3/2} - \frac{2}{3}(2)^{3/2}$$

$$\approx 2.8943$$

52. $\int_2^{10} \frac{x\, dx}{x-1}$

Simpson's Rule:

i	x_i	$f(x_1)$
0	2	2
1	4	$\frac{4}{3}$
2	6	$\frac{6}{5}$
3	8	$\frac{8}{7}$
4	10	$\frac{10}{9}$

$$\int_2^{10} \frac{x}{x-1}\, dx$$

$$\approx \frac{10-2}{3(4)}\left[2 + 4\left(\frac{4}{3}\right) + 2\left(\frac{6}{5}\right) + 4\left(\frac{8}{7}\right) + \frac{10}{9}\right]$$

$$\approx 10.28$$

This answer is close to the answer of 10.197 obtained from the exact integral in Exercise 49.

54. (a) $\int_1^5 \left[\sqrt{x-1} - \left(\frac{x-1}{2}\right)\right]\, dx$

$$= \int_1^5 \left(\sqrt{x-1} - \frac{x}{2} + \frac{1}{2}\, dx\right)$$

$$= \left(\frac{2}{3}(x-1)^{3/2} - \frac{x^2}{4} + \frac{x}{2}\right)\Big|_1^5$$

$$= \left(\frac{16}{3} - \frac{25}{4} + \frac{5}{2}\right) - \left(0 - \frac{1}{4} + \frac{1}{2}\right)$$

$$= \frac{16}{3} - 6 + 2 = \frac{4}{3}$$

(b) $n = 4,\ b = 5,\ a = 1,\ f(x) = \sqrt{x-1} - \frac{x}{2} + \frac{1}{2}$

i	x_i	$f(x_i)$
0	1	0
1	2	.5
2	3	.41421
3	4	.23205
4	5	0

$$\int_1^5 \left(\sqrt{x-1} - \frac{x}{2} + \frac{1}{2}\right)\, dx$$

$$= \left(\frac{5-1}{4}\right)\left[\frac{1}{2}(0) + .5 + .41421\right.$$

$$\left. + .23205 + \frac{1}{2}(0)\right]$$

$$= 1.14626$$

(c) $\int_1^5 \left(\sqrt{x-1} - \frac{x}{2} + \frac{1}{2}\right) dx$

$$= \left(\frac{5-1}{3(4)}\right)[0 + 4(.5) + 2(.41421) + 4(.23205) + 0]$$
$$= \left(\frac{1}{3}\right)(3.75662)$$
$$= 1.2522$$

56. $C'(x) = 3\sqrt{2x-1}$; 13 units cost \$270.

$$C(x) = \int 3(2x-1)^{1/2}\, dx$$
$$= \frac{3}{2}\int 2(2x-1)^{1/2}\, dx$$

Let $u = 2x - 1$, so that

$$du = 2\, dx.$$
$$= \frac{3}{2}\int u^{1/2}\, du$$
$$= \frac{3}{2}\left(\frac{u^{3/2}}{3/2}\right) + C$$
$$= (2x-1)^{3/2} + C$$
$$C(13) = [2(13) - 1]^{3/2} + C$$

Since $C(13) = 270$,

$$270 = 25^{3/2} + C$$
$$270 = 125 + C$$
$$C = 145.$$

Thus,

$$C(x) = (2x-1)^{3/2} + 145.$$

58. Read values for the rate of investment income accumulation for every 2 years from year 1 to year 9. These are the heights of rectangles with width $\Delta x = 2$.

Total accumulated income
$= 11,000(2) + 9000(2) + 12,000(2) + 10,000(2) + 6000(2) \approx \$96,000$

60. $S'(x) = \sqrt{x} + 2$

$$S(x) = \int_0^9 (x^{1/2} + 2)\, dx$$
$$= \left(\frac{x^{3/2}}{3/2} + 2x\right)\Bigg|_0^9$$
$$= \frac{2}{3}(9)^{3/2} + 18 = 36$$

Total sales $= 36{,}000$

62. $S(q) = q^2 + 5q + 100$
$D(q) = 350 - q^2$
$S(q) = D(q)$ at the equilibrium point.

$$q^2 + 5q + 100 = 350 - q^2$$
$$2q^2 + 5q - 250 = 0$$
$$(-2q + 25)(q - 10) = 0$$
$$q = -\frac{25}{2} \quad \text{or} \quad q = 10$$

Since the number of units produced would not be negative, the equilibrium point occurs when $q = 10$.

Equilibrium supply
$= (10)^2 + 5(10) + 100 = 250$

Equilibrium demand
$= 350 - (10)^2 = 250$

(a) Producers' surplus

$$= \int_0^{10} [250 - (q^2 + 5q + 100)]\, dx$$
$$= \int_0^{10} (-q^2 - 5q + 150)\, dx$$
$$= \left(\frac{-q^3}{3} - \frac{5q^2}{2} + 150q\right)\Bigg|_0^{10}$$
$$= \frac{-1000}{3} - \frac{500}{2} + 1500$$
$$= \frac{\$2750}{3} \approx \$916.67$$

(b) Consumers' surplus

$$= \int_0^{10} [(350 - q^2) - 250]\, dx$$
$$= \int_0^{10} (100 - q^2)\, dx$$
$$= \left(100q - \frac{q^3}{3}\right)\Bigg|_0^{10}$$
$$= 1000 - \frac{1000}{3}$$
$$= \frac{\$2000}{3}$$
$$\approx \$666.67$$

64. $f(t) = 100 - \sqrt{2.4t + 1}$

The total number of additional spiders in the first ten months is

$$\int_0^{10} (100 - \sqrt{2.4t + 1})\, dt,$$

where t is the time in months.

$$= \int_0^{10} 100\, dt - \int_0^{10} (2.4t + 1)^{1/2}\, dt$$

Let $u = 2.4t + 1$, so that

$du = 2.4t\, dt$ and $\frac{1}{2.4}\, du = dt$.

When $t = 10,\ u = 25$. When $t = 0,\ u = 1$.

$$\begin{aligned}
&= \int_0^{10} 100\, dt - \frac{1}{2.4} \int_1^{25} u^{1/2}\, du \\
&= 100t \Big|_0^{10} - \frac{5}{12} \cdot \frac{u^{3/2}}{3/2} \Big|_1^{25} \\
&= 1000 - \frac{5}{18} u^{3/2} \Big|_1^{25} \\
&= 1000 - \frac{5}{18}(125) + \frac{5}{18} \\
&= 1000 - \frac{310}{9} \\
&\approx 965.56 \approx 966
\end{aligned}$$

The total number of additional spiders in the first 10 months is about 966.

66. **(a)** The total area is the area of the triangle on $[0, 12]$ with height .024 plus the area of the rectangle on $[12, 17.6]$ with height .024.

$$\begin{aligned}
A &= \frac{1}{2}(12 - 0)(.024) + (17.6 - 12)(.024) \\
&= .144 + .1344 \\
&= .2784
\end{aligned}$$

(b) On $[0, 12]$ we defined the function $f(x)$ with slope $\frac{.024 - 0}{12 - 0} = .002$ and y-intercept 0.

$$f(x) = .002x$$

On $[12, 17.6]$, define $g(x)$ as the constant value.

$$g(x) = .024.$$

The area is the sum of the integrals of these two functions.

$$\begin{aligned}
A &= \int_0^{12} .002x dx + \int_{12}^{17.6} .024 dx \\
&= .001x^2 \Big|_0^{12} + .024x \Big|_{12}^{17.6} \\
&= .001(12^2 - 0^2) + .024(17.6 - 12) \\
&= .144 + .1344 \\
&= .2784
\end{aligned}$$

70. $v(t) = t^2 - 2t$

$$s(t) = \int_0^t (t^2 - 2t)\, dt$$

$$s(t) = \frac{t^3}{3} - t^2 + s_0$$

If $t = 3,\ s = 8$.

$$\begin{aligned}
8 &= 9 - 9 + s_0 \\
8 &= s_0
\end{aligned}$$

Thus,

$$s(t) = \frac{t^3}{3} - t^2 + 8.$$

Extended Application: Estimating Depletion Dates for Minerals

1. $2{,}300{,}000 \div 17{,}100 \approx 135$

The reserves would last about 135 yr.

2.

$$\begin{aligned}
2{,}300{,}000 &= \frac{17{,}100}{.02}(e^{.02T_1} - 1) \\
\frac{2{,}300{,}000(.02)}{17{,}100} &= e^{.02T_1} - 1 \\
2.6901 + 1 &= e^{.02T_1} \\
3.6901 &= e^{.02T_1} \\
\ln\ 3.6901 &= .02T_1 \\
T_1 &= \frac{\ln\ 3.6901}{.02} \\
T_1 &\approx 65.3
\end{aligned}$$

The reserves would last about 65.3 yr.

3.

$$\begin{aligned}
15{,}000{,}000 &= \frac{63{,}000}{.06}(e^{.06T_1} - 1) \\
\frac{15{,}000{,}000(.06)}{63{,}000} + 1 &= e^{.06T_1} \\
15.286 &\approx e^{.06T_1} \\
\ln\ 15.286 &\approx .06T_1 \\
T_1 &\approx \frac{\ln\ 15.286}{.06} \\
&= 45.4
\end{aligned}$$

The depletion time for bauxite is about 45.4 yr.

4.
$$2{,}000{,}000 = \frac{2200}{.04}(e^{.04T_1} - 1)$$
$$\frac{2{,}000{,}000(.04)}{2200} + 1 = e^{.04T_1}$$
$$37.36 = e^{.04T_1}$$
$$\ln\ 37.36 = .04T_1$$
$$T_1 \approx 90.5$$

The depletion time for bituminous coal is about 90.5 yr.

5. $k(t) = \dfrac{.5}{t+25}$

(a) For $t = 0$,

$$k(t) = \frac{.5}{0+25} = .02.$$

This gives a growth rate of 2% for 1970.
For $t = 25$,

$$k(t) = \frac{.5}{25+25} = .01.$$

This gives a growth rate of 1% for 1996.

(b) Use the form of the function $k(t) = \dfrac{a}{t+b}$, where a and b are both constants. Since $k(0) = .03, k(t) = \dfrac{a}{t+b}$, where

$$.03 = \frac{a}{0+b} = \frac{a}{b}. \text{ Or } a = .03b.$$

Also, since $k(25) = .02$,

$$.02 = \frac{a}{25+b}. \text{ Or } a = .02(25+b).$$

Solve:

$$.03b = .02(25+b)$$
$$.03b = .5 + .02b$$
$$.01b = .5$$
$$b = 50$$

Find a using substitution.

$$a = .03b$$
$$a = .03(50)$$
$$a = 1.5$$

The function that satisfies these conditions is

$$k(t) = \frac{1.5}{t+50}.$$

6. **(a)** Total consumption $= 17{,}100 \displaystyle\int_0^T e^{k(t)\cdot t}dt$

$$= 17{,}100 \int_0^T e^{1.5t/(t+50)}dt$$

(b) Use the fnInt command on a graphing calculator to evaluate

$$17{,}100 \int_0^T \mathrm{e}^{1.5t/(t+50)}\mathrm{dt}$$

for different values of T.

For $T = 70$ the integral is about 2,158,000.
For $T = 71$ the integral is about 2,199,000.
For $T = 72$ the integral is about 2,240,000.
For $T = 73$ the integral is about 2,282,000.
For $T = 74$ the integral is about 2,324,000.

We would estimate that starting in 1970 the petroleum reserves would last for about 73 years, that is, until 2043.

Chapter 8

FURTHER TECHNIQUES AND APPLICATIONS OF INTEGRATION

8.1 Integration by Parts

2. $\int (x+1)e^x\,dx$

Let $dv = e^x\,dx$ and $u = x+1$.

Then $v = \int e^x\,dx$ and $du = dx$.

$v = e^x + C$

Use the formula

$$\int u\,dv = uv - \int v\,du.$$

$$\int (x+1)e^x\,dx = (x+1)e^x - \int e^x\,dx$$
$$= xe^x + e^x - e^x + C = xe^x + C$$

4. $\int (6x+3)e^{-2x}\,dx$

Let $dv = e^{-2x}\,dx$ and $u = 6x+3$

Then $v = \int e^{-2x}\,dx$ and $du = 6\,dx$.

$v = \frac{e^{-2x}}{-2} + C$

$$\int (6x+3)e^{-2x}\,dx$$
$$= \frac{(6x+3)e^{-2x}}{-2} - \int \frac{6e^{-2x}}{-2}\,dx$$
$$= -\frac{1}{2}(6x+3)e^{-2x} + \frac{3e^{-2x}}{-2} + C$$
$$= -\frac{1}{2}(6x+3)e^{-2x} - \frac{3}{2}e^{-2x} + C$$

6. $\int_0^1 \frac{1-x}{3e^x}\,dx = \frac{1}{3}\int_0^1 (1-x)e^{-x}\,dx$

Let $dv = e^{-x}\,dx$ and $u = 1-x$.
Then $v = -e^{-x}$ and $du = -dx$.

$$\frac{1}{3}\int (1-x)e^{-x}\,dx$$
$$= \frac{1}{3}\left[-(1-x)e^{-x} - \int e^{-x}\,dx\right]$$
$$= \frac{1}{3}[-(1-x)e^{-x} + e^{-x}\,dx] = \frac{1}{3}xe^{-x}$$

$$\int_0^1 (1-x)e^{-x}\,dx = \frac{1}{3}xe^{-x}\Big|_0^1$$
$$= \frac{1}{3e} \approx .123$$

8. $\int_1^2 \ln\ 5x\,dx$

Let $dv = dx$ and $u = \ln\ 5x$.

Then $v = x$ and $du = \frac{1}{x}\,dx$.

$$\int \ln\ 5x\,dx = x\ \ln\ 5x - \int x\left(\frac{1}{x}\,dx\right) = x\ \ln\ 5x - x$$

$$\int_1^2 \ln\ 5x\,dx = (x\ \ln\ 5x - x)\Big|_1^2$$
$$= 2\ \ln\ 10 - 2 - \ln\ 5 + 1$$
$$= \ln\ \frac{10^2}{5} - 1 = \ln\ 20 - 1 \approx 1.996$$

10. $\int x^2\ \ln\ x\,dx,\ x > 0$

Let $dv = x^2\,dv$ and $u = \ln\ x$.

Then $v = \frac{x^3}{3}$ and $du = \frac{1}{x}\,dx$.

$$\int x^2\ \ln\ x\,dx = \frac{x^3\ \ln\ x}{3} - \int \frac{x^3}{3}\left(\frac{1}{x}\right)dx$$
$$= \frac{x^3\ \ln\ x}{3} - \frac{1}{3}\int x^2\,dx$$
$$= \frac{x^3\ \ln\ x}{3} - \frac{x^3}{9} + C$$

12. $A = \int_0^1 xe^x\,dx$

Let $dv = e^x\,dx$ and $u = x$.
Then $v = e^x$ and $du = dx$.

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$
$$= xe^x - e^x = e^x(x-1)$$
$$A = e^x(x-1)\Big|_0^1$$
$$= e(0) - 1(-1)$$
$$= 1$$

14. $\displaystyle\int_1^2 (1-x^2)e^{2x}\,dx$

Let $u = 1 - x^2$ and $dv = e^{2x}\,dx$.
Use column integration.

D		I
$1-x^2$	+	e^{2x}
$-2x$	−	$\frac{e^{2x}}{2}$
-2	+	$\frac{e^{2x}}{4}$
0		$\frac{e^{2x}}{8}$

$$\int (1-x^2)e^{2x}\,dx$$

$$= \frac{(1-x^2)e^{2x}}{2} - \frac{(-2x)e^{2x}}{4} + \frac{(-2)e^{2x}}{8}$$

$$= \frac{(1-x^2)e^{2x}}{2} + \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}$$

$$= \frac{e^{2x}}{2}\left(1 - x^2 + x - \frac{1}{2}\right) = \frac{e^{2x}}{2}\left(\frac{1}{2} - x^2 + x\right)$$

$$\int_1^2 (1-x^2)e^{2x}\,dx = \frac{e^{2x}}{2}\left(\frac{1}{2} - x^2 + x\right)\Bigg|_1^2$$

$$= \frac{e^4}{2}\left(-\frac{3}{2}\right) - \frac{e^2}{2}\left(\frac{1}{2}\right)$$

$$= -\frac{e^2}{4}(3e^2 + 1) \approx -42.80$$

16. $\displaystyle\int (2x-1)\ln(3x)\,dx$

Let $dv = (2x-1)\,dx$ and $u = \ln 3x$.

Then $v = x^2 - x$ and $du = \frac{1}{x}\,dx$.

$$\int (2x-1)\ln(3x)\,dx$$

$$= (x^2 - x)\ln 3x - \int (x^2 - x)\left(\frac{1}{x}\,dx\right)$$

$$= (x^2 - x)\ln 3x - \int (x-1)\,dx$$

$$= (x^2 - x)\ln 3x - \frac{x^2}{2} + x + C$$

18. $\displaystyle\int xe^{x^2}\,dx = \frac{1}{2}\int 2xe^{x^2}\,dx$

Let $u = x^2$ and $du = 2x\,dx$.

$$= \frac{1}{2}e^{x^2} + C$$

$$= \frac{e^{x^2}}{2} + C$$

20. $\displaystyle\int_0^1 \frac{x^2\,dx}{2x^3+1}$

Let $u = 2x^3 + 1$.
Then $du = 6x^2\,dx$.

$$\int_0^1 \frac{x^2\,dx}{2x^3+1} = \frac{1}{6}\int_0^1 \frac{6x^2\,dx}{2x^3+1}$$

$$= \frac{1}{6}\left(\ln\left|2x^3+1\right|\Big|_0^1\right)$$

$$= \frac{1}{6}(\ln 3)$$

$$\approx .183$$

22. $\displaystyle\int \frac{x^2\,dx}{2x^3+1} = \frac{1}{6}\int \frac{6x^2\,dx}{2x^3+1}$

Let $u = 2x^3 + 1$.
Then $du = 6x^2\,dx$.

$$= \frac{1}{6}\ln\left|2x^3+1\right| + C$$

24. $\displaystyle\int \frac{6}{x^2-9}\,dx$

$$= 6\int \frac{1}{x^2-9}\,dx$$

Use entry 8 from table with $a = 3$.

$$= 6\left[\frac{1}{2(3)}\ln\left|\frac{x-3}{x+3}\right|\right] + C$$

$$= \ln\left|\frac{x-3}{x+3}\right| + C$$

26. $\displaystyle\int \frac{2}{3x(3x-5)}\,dx$

$$= \frac{2}{3}\int \frac{1}{x(3x-5)}\,dx$$

Use entry 13 from table with $a = 3,\ b = -5$.

$$= \frac{2}{3}\left(-\frac{1}{5}\ln\left|\frac{x}{3x-5}\right|\right) + C$$

$$= -\frac{2}{15}\ln\left|\frac{x}{3x-5}\right| + C$$

28. $\displaystyle\int \sqrt{x^2+10}\,dx$

Use entry 15 from table with $a = \sqrt{10}$.

$$\int \sqrt{x^2+10}\,dx$$

$$= \frac{x}{2}\sqrt{x^2 + (\sqrt{10})^2}$$

$$+ \frac{(\sqrt{10})^2}{2}\ln\left|x + \sqrt{x^2 + (\sqrt{10})^2}\right| + C$$

$$= \frac{x}{2}\sqrt{x^2+10} + 5\ln\left|x + \sqrt{x^2+10}\right| + C$$

32. $\int x^n e^{ax}\,dx, a \neq 0$

Let $dv = e^{ax}\,dx$ and $u = x^n$;

then $v = \frac{1}{a}e^{ax}$ and $du = nx^{n-1}\,dx$.

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \int \left(\frac{1}{a}e^{ax} \cdot nx^{n-1}\right) dx$$
$$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx + C$$

34. The integration constant is missing.

36. $r(x) = \int_1^6 2x^2 e^{-x}\,dx$

Let $dv = e^{-x}\,dx$ and $u = 2x^2$.

Then $v = -e^{-x}$ and $du = 4x\,dx$.

Use column integration.

D		I
$2x^2$	+	e^{-x}
$4x$	−	$-e^{-x}$
4	+	e^{-x}
0		$-e^{-x}$

$$\int 2x^2 e^{-x}\,dx$$

$$= 2x^2(-e^{-x}) - 4x(e^{-x}) + 4(-e^{-x})$$
$$= -2x^2e^{-x} - 4xe^{-x} - 4e^{-x}$$
$$= -2e^{-x}(x^2 + 2x + 2)$$

$$\int_1^6 2x^2 e^{-x}\,dx = -2e^{-x}(x^2+2x+2)\Big|_1^6$$
$$= -100e^{-6} + 10e^{-1} \approx 3.431$$

The total reaction to the drug from $x = 1$ to $x = 6$ is about 3.431.

38. $A = \int_4^9 \sqrt{t}\ln t\,dt$

Let $u = \ln t$ and $dv = \sqrt{t}\,dt = t^{1/2}dt$.

Then $du = \frac{1}{t}dt$ and $v = \frac{2}{3}t^{3/2}$.

$$\int \sqrt{t}\ln t\,dt = \frac{2}{3}t^{3/2}\ln t - \int \left(\frac{2}{3}t^{3/2} \cdot \frac{1}{t}\right) dt$$
$$= \frac{2}{3}t^{3/2}\ln t - \int \frac{2}{3}t^{1/2}dt$$
$$= \frac{2}{3}t^{3/2}\ln t - \frac{4}{9}t^{3/2} + C$$
$$\int_4^9 \sqrt{t}\ln t\,dt = 18\ln 9 - \frac{16}{3}\ln 4 - \frac{76}{9}$$
$$\approx 23.71 \text{ sq cm}$$

40. $\int_0^1 ke^{-kt}(1-t)dt$

Let $u = 1 - t$ and $dv = e^{-kt}dt$.

Then $du = -dt$ and $v = -\frac{1}{k}e^{-kt}$

$$\int_0^1 ke^{-kt}(1-t)dt$$

$$= k\left[(1-t)\left(-\frac{1}{k}e^{-kt}\right) - \int\left(-\frac{1}{k}e^{-kt}\right)(-dt)\right]\Bigg|_0^1$$

$$= k\left[-\frac{1}{k}e^{-kt}(1-t) - \frac{1}{k}\int e^{-kt}dt\right]\Bigg|_0^1$$

$$= -e^{-kt}(1-t) + \frac{1}{k}e^{-kt}\Big|_0^1$$

$$= -e^{-kt}\left(1 - t - \frac{1}{k}\right)\Big|_0^1$$

$$= \left[-e^{-k}\left(-\frac{1}{k}\right)\right] - \left[-e^0\left(1 - \frac{1}{k}\right)\right]$$

$$= 1 - \frac{1}{k} + \frac{1}{k}e^{-k}$$

(a) $k = \dfrac{1}{12}$:

$$\int_0^1 \frac{1}{12}e^{-t/12}(1-t)dt$$
$$= 1 - \frac{1}{1/12} + \frac{1}{1/12}e^{-1/12}$$
$$= 12e^{-1/12} - 11$$
$$\approx .0405$$

$k = \dfrac{1}{24}$:

$$\int_0^1 \frac{1}{24}e^{-t/24}(1-t)dt$$
$$= 1 - \frac{1}{1/24} + \frac{1}{1/24}e^{-1/24}$$
$$= 24e^{-1/24} - 23$$
$$\approx .0205$$

$k = \dfrac{1}{48}$:

$$\int_0^1 \frac{1}{48}e^{-t/48}(1-t)dt$$
$$= 1 - \frac{1}{1/48} + \frac{1}{1/48}e^{-1/48}$$
$$= 48e^{-1/48} - 47$$
$$\approx .0103$$

(b) $\displaystyle\int_1^6 ke^{-kt}\frac{6-t}{5}dt$

The integral, easily found by comparing it to the integral in part (a), is

$$-e^{-kt}\left(\frac{6}{5} - \frac{t}{5} - \frac{1}{5k}\right)\Bigg|_1^6$$
$$= \left[-e^{-6k}\left(\frac{6}{5} - \frac{6}{5} - \frac{1}{5k}\right)\right] - \left[-e^{-k}\left(\frac{6}{5} - \frac{1}{5} - \frac{1}{5k}\right)\right]$$
$$= \frac{1}{5k}e^{-6k} + \left(1 - \frac{1}{5k}\right)e^{-k}$$

$k = \dfrac{1}{12}$:

$$\int_1^6 \frac{1}{12}e^{-t/12}\frac{6-t}{5}dt$$
$$= \frac{1}{5(1/12)}e^{-6(1/12)} + \left[1 - \frac{1}{5(1/12)}\right]e^{-1/12}$$
$$= \frac{12}{5}e^{-1/2} - \frac{7}{5}e^{-1/12}$$
$$\approx .1676$$

$k = \dfrac{1}{24}$:

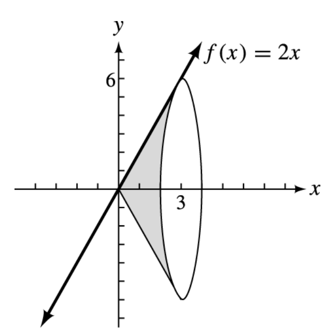
$$\int_1^6 \frac{1}{24}e^{-t/24}\frac{6-t}{5}dt$$
$$= \frac{1}{5(1/24)}e^{-6(1/24)} + \left[1 - \frac{1}{5(1/24)}\right]e^{-1/24}$$
$$= \frac{24}{5}e^{-1/4} - \frac{19}{5}e^{-1/24}$$
$$\approx .0933$$

$k = \dfrac{1}{48}$:

$$\int_1^6 \frac{1}{48}e^{-t/48}\frac{6-t}{5}dt$$
$$= \frac{1}{5(1/48)}e^{-6(1/48)} + \left[1 - \frac{1}{5(1/48)}\right]e^{-1/48}$$
$$= \frac{48}{5}e^{-1/8} - \frac{43}{5}e^{-1/48}$$
$$\approx .0493$$

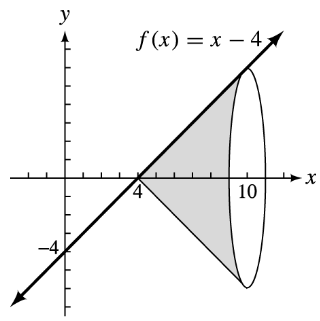
8.2 Volume and Average Value

2. $f(x) = 2x,\ y = 0,\ x = 0,\ x = 3$
Graph $f(x) = 2x$. Then show the solid of revolution formed by rotating about the x-axis the region bounded by $f(x)$, $x = 0$, and $x = 3$.



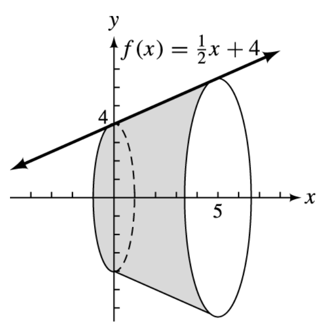
$$V = \pi\int_0^3 (2x)^2\,dx$$
$$= \pi\int_0^3 4x^2\,dx$$
$$= \frac{4\pi x^3}{3}\Bigg|_0^3$$
$$= 36\pi$$

4. $f(x) = x - 4,\ y = 0,\ x = 4,\ x = 10$



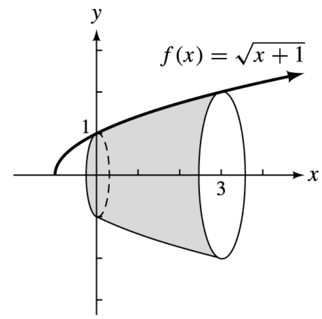
$$V = \pi \int_4^{10} (x-4)^2\,dx$$
$$= \left.\frac{\pi(x-4)^3}{3}\right|_4^{10}$$
$$= \frac{\pi}{3}[(6)^3 - 0]$$
$$= 72\pi$$

6. $f(x) = \frac{1}{2}x + 4,\ y = 0,\ x = 0,\ x = 5$



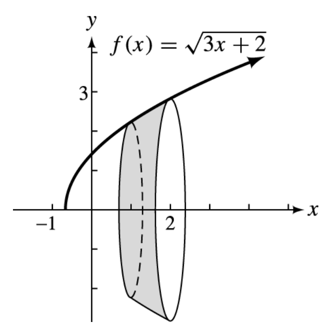
$$V = \pi \int_0^5 \left(\frac{1}{2}x + 4\right)^2 dx$$
$$= 2\pi \int_0^5 \frac{1}{2}\left(\frac{1}{2}x+4\right)^2 dx$$
$$= 2\pi \left.\frac{\left(\frac{1}{2}x+4\right)^3}{3}\right|_0^5$$
$$= \frac{2\pi}{3}\left[\left(\frac{5}{2}+4\right)^3 - (4)^3\right]$$
$$= \frac{2\pi}{3}\left[\left(\frac{13}{2}\right)^3 - 64\right]$$
$$= \frac{2\pi}{3}\left(\frac{2197}{8} - \frac{512}{8}\right)$$
$$= \frac{1685\pi}{12}$$

8. $f(x) = \sqrt{x+1},\ y = 0,\ x = 0,\ x = 3$



$$V = \pi \int_0^3 (\sqrt{x+1})^2 dx$$
$$= \pi \int_0^3 (x+1)\,dx$$
$$= \pi\left.\left(\frac{x^2}{2} + x\right)\right|_0^3 = \pi\left(\frac{9}{2}+3\right)$$
$$= \frac{15\pi}{2}$$

10. $f(x) = \sqrt{3x+2},\ y = 0,\ x = 1,\ x = 2$



$$V = \pi \int_1^2 (\sqrt{3x+2})^2\,dx$$
$$= \pi \int_1^2 (3x+2)\,dx$$
$$= \pi\left.\left(\frac{3x^2}{2} + 2x\right)\right|_1^2$$
$$= \pi\left[\left(\frac{3}{2}(2)^2 + 2(2)\right) - \left(\frac{3}{2}(1) + 2(1)\right)\right]$$
$$= \pi\left(10 - \frac{7}{2}\right)$$
$$= \frac{13\pi}{2}$$

12. $f(x) = 2e^x,\ y = 0,\ x = -2,\ x = 1$

$$\begin{aligned} V &= \pi \int_{-2}^{1} (2e^x)^2\,dx \\ &= \pi \int_{-2}^{1} 4x^{2x}\,dx \\ &= \frac{4\pi}{2}(e^{2x})\Big|_{-2}^{1} \\ &= 2\pi(e^2 - e^{-4}) \approx 46.3 \end{aligned}$$

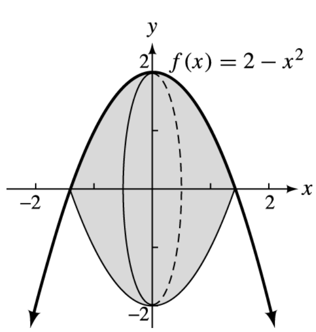
14. $f(x) = \dfrac{1}{\sqrt{x+1}},\ y = 0,\ x = 0,\ x = 2$

$$\begin{aligned} V &= \pi \int_0^2 \left(\frac{1}{\sqrt{x+1}}\right)^2 dx \\ &= \pi \int_0^2 \frac{dx}{x+1} \\ &= \pi(\ln|x+1|)\Big|_0^2 \\ &= \pi \ln 3 \\ &= \pi \ln 3 \approx 3.45 \end{aligned}$$

16. $f(x) = \dfrac{x^2}{2},\ y = 0,\ x = 0,\ x = 4$

$$\begin{aligned} V &= \pi \int_0^4 \left(\frac{x^2}{2}\right)^2 dx \\ &= \pi \int_0^4 \frac{x^4}{4}\,dx \\ &= \frac{\pi}{4}\left(\frac{x^5}{5}\right)\Big|_0^4 = \frac{\pi}{20}(4^5) \\ &= \frac{256\pi}{5} \end{aligned}$$

18. $f(x) = 2 - x^2,\ y = 0$



Since $f(x) = 2 - x^2$ intersects $y = 0$ where

$$\begin{aligned} 2 - x^2 &= 0 \\ x &= \pm\sqrt{2}, \end{aligned}$$

$$a = -\sqrt{2} \quad \text{and} \quad b = \sqrt{2}.$$

$$\begin{aligned} V &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)^2\,dx \\ &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 4x^2 + x^4)\,dx \\ &= \pi\left(4x - \frac{4x^3}{3} + \frac{x^5}{5}\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} \\ &= \pi\left[\left(4\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{4}{5}\sqrt{2}\right) - \left(-4\sqrt{2} + \frac{8}{3}\sqrt{2} - \frac{4}{5}\sqrt{2}\right)\right] \\ &= \pi\left(\frac{32}{15}\sqrt{2} + \frac{32}{15}\sqrt{2}\right) \\ &= \frac{64\pi\sqrt{2}}{15} \end{aligned}$$

20. $f(x) = \sqrt{16 - x^2}$
$r = \sqrt{16} = 4$

$$\begin{aligned} V &= \pi \int_{-4}^{4} (\sqrt{16 - x^2})^2\,dx = \pi \int_{-4}^{4} (16 - x^2)\,dx \\ &= \pi\left(16x - \frac{x^3}{3}\right)\Big|_{-4}^{4} \\ &= \pi\left[\left(64 - \frac{64}{3}\right) - \left(-64 + \frac{64}{3}\right)\right] \\ &= \pi\left(128 - \frac{128}{3}\right) = \frac{256\pi}{3} \end{aligned}$$

22.
$$\begin{aligned} V &= \pi \int_{-a}^{a} \left[\frac{b}{a}\sqrt{a^2 - x^2}\right]^2 dx \\ &= \pi \int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2)\,dx \\ &= \frac{\pi b^2}{a^2}\left(a^2 x - \frac{x^3}{3}\right)\Big|_{-a}^{a} \\ &= \frac{\pi b^2}{a^2}\left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right] \\ &= \frac{\pi b^2}{a^2}\left(2a^3 - \frac{2a^3}{3}\right) = \frac{\pi b^2}{a^2}\left(\frac{4a^3}{3}\right) = \frac{4ab^2\pi}{3} \end{aligned}$$

24. $f(x) = 3 - 2x^2;\ [1, 9]$

Average value

$$\begin{aligned} &= \frac{1}{9-1}\int_1^9 (3 - 2x^2)\,dx = \frac{1}{8}\left(3x - \frac{2x^3}{3}\right)\Big|_1^9 \\ &= \frac{1}{8}(27 - 486) - \frac{1}{8}\left(3 - \frac{2}{3}\right) \approx -57.67 \end{aligned}$$

26. $f(x) = (2x-1)^{1/2};\ [1, 13]$

Average value

$$= \frac{1}{13-1}\int_1^{13}(2x-1)^{1/2}dx$$
$$= \frac{1}{12}\left(\frac{1}{2}\right)\int_1^{13} 2(2x-1)^{1/2}\,dx$$
$$= \frac{1}{24}\cdot\frac{2}{3}(2x-1)^{3/2}\Big|_1^{13}$$
$$= \frac{1}{36}(25^{3/2}-1)$$
$$= \frac{1}{36}(125) - \frac{1}{36}$$
$$= \frac{124}{36} = \frac{31}{9} \approx 3.44$$

28. $f(x) = e^{.1x};\ [0, 10]$

$$\text{Average value} = \frac{1}{10-0}\int_0^{10} e^{.1x}\,dx$$
$$= \frac{10}{10}(e^{.1x})\Big|_0^{10}$$
$$= e^1 - 1 = e - 1 \approx 1.718$$

30. $f(x) = x\ln x;\ [1, e]$

$$\text{Average value} = \frac{1}{e-1}\int_1^e x\ln x\,dx$$

Let $u = \ln x$ and $dv = x\,dx$.
Use column integration.

D		I
$\ln x$	$+$	x
$\frac{1}{x}$	$-$	$\frac{1}{2}x^2$

$$\int x\ln x\,dx$$
$$= \frac{1}{2}x^2\ln x - \int\left(\frac{1}{x}\cdot\frac{1}{2}x^2\right)dx$$
$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$$
$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

$$\frac{1}{e-1}\int_1^e x\ln x\,dx$$
$$= \frac{1}{e-1}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_1^e$$
$$= \frac{1}{e-1}\left(\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}\right)$$
$$= \frac{1}{e-1}\left(\frac{e^2+1}{4}\right)$$
$$= \frac{e^2+1}{4(e-1)} \approx 1.221$$

32. $f(x) = \frac{1}{1+x^2}$, $y = 0, x = -1, x = 1$

$$V = \pi\int_{-1}^1\left(\frac{1}{1+x^2}\right)^2 dx$$
$$= \pi\int_{-1}^1 (1+x^2)^{-2}dx$$

Using a graphing utility with the $fnInt$ feature to evaluate the integral, we get $4.038197427 \approx 4.0382$.

34. Use the formula for average value with $a = 0$ and $b = 5$. The average price is

$$\frac{1}{5-0}\int_0^5 [t(25-5t)+18]dt$$
$$= \frac{1}{5}\int_0^5 (25t - 5t^2 + 18)dt$$
$$= \frac{1}{5}\left(\frac{25t^2}{2} - \frac{5t^3}{3} + 18t\right)\Big|_0^5$$
$$= \frac{1}{5}\left(\frac{625}{2} - \frac{625}{3} + 90\right)$$
$$= \frac{125}{2} - \frac{125}{3} + 18 \approx \$38.83$$

36. At the end of any given business day, CFFC has $400 - 80t$ cases of perfume on hand, where $t = 1$ represents Monday, $t = 2$ represents Tuesday, and so on.
The average daily number of cases in inventory is

$$\frac{1}{5-0}\int_0^5 (400-80t)dt = \frac{1}{5}\left(400t - \frac{80t^2}{2}\right)\Big|_0^5$$
$$= \frac{1}{5}(400t - 40t^2)\Big|_0^5$$
$$= \frac{1}{5}(2000 - 1000)$$
$$= 200 \text{ cases}$$

38. **(a)** $\int_0^R 2\pi r k(R^2 - r^2)\,dr$

$$= 2\pi k \int_0^R r(R^2 - r^2)\,dr$$

(b)
$$\begin{aligned} &= 2\pi k \int_0^R (rR^2 - r^3)\,dr \\ &= 2\pi k \left[\frac{r^2R^2}{2} - \frac{r^4}{4}\right]\Bigg|_0^R \\ &= 2\pi k \left[\frac{R^4}{2} - \frac{R^4}{4} - 0\right] \\ &= 2\pi k \left(\frac{R^4}{4}\right) = \frac{\pi k R^4}{2} \end{aligned}$$

40. From problem 22, the volume of an ellipsoid with major axis $2a$ and minor axis $2b$ is

$$V = \frac{4}{3}ab^2\pi.$$

(a) i. Canada goose: $a = \frac{8.6}{2} = 4.3, b = \frac{5.8}{2} = 2.9$

$$V = \frac{4}{3}(4.3)(2.9)^2\pi \approx 151.48$$

The volume is about 151 cubic cm.

ii. Robin: $a = .95, b = .75$

$$V = \frac{4}{3}(.95)(.75)^2\pi \approx 2.238$$

The volume is about 2.24 cubic cm.

iii. Turtledove: $a = 1.55, b = 1.15$

$$V = \frac{4}{3}(1.55)(1.15)^2\pi \approx 8.586$$

The volume is about 8.59 cubic cm.

iv. Hummingbird: $a = .5, b = .5$

$$V = \frac{4}{3}(.5)(.5)^2\pi \approx .5236$$

The volume is about .524 cubic cm.

v. Raven: $a = 2.5, b = 1.65$

$$V = \frac{4}{3}(2.5)(1.65)^2\pi \approx 28.51$$

The volume is about 28.5 cubic cm.

(b) Using the formula

$$V = \frac{4}{3}ab^2\pi$$

a is half the length, l, and b half the midth, w, of a bird egg.
Substituting these expressions and writing l in terms of w, we have

$$\begin{aligned} V &= \frac{4}{3}\left(\frac{l}{2}\right)\left(\frac{w}{2}\right)^2\pi \\ &= \frac{4}{3}\left(\frac{1.585w - .487}{2}\right)\left(\frac{w^2}{4}\right)\pi \\ &= \frac{\pi(1.585w^3 - .487w^2)}{6}. \end{aligned}$$

i Canada goose: $w = 5.8$

$$\begin{aligned} V &= \frac{\pi[1.585(5.8)^3 - .487(5.8)^2]}{6} \\ &\approx 153.3 \end{aligned}$$

The volume is about 153 cubic cm.

ii Robin: $w = 1.5$

$$\begin{aligned} V &= \frac{\pi[1.585(1.5)^3 - .487(1.5)^2]}{6} \\ &\approx 2.227 \end{aligned}$$

The volume is about 2.23 cubic cm.

iii Turtledove: $w = 2.3$

$$\begin{aligned} V &= \frac{\pi[1.585(2.3)^3 - .487(2.3)^2]}{6} \\ &\approx 8.749 \end{aligned}$$

The volume is about 8.75 cubic cm.

iv Hummingbird: $w = 1.0$

$$\begin{aligned} V &= \frac{\pi[1.585(1.0)^3 - .487(1.0)^2]}{6} \\ &\approx .5749 \end{aligned}$$

The volume is about .575 cubic cm.

v Raven: $w = 3.3$

$$\begin{aligned} V &= \frac{\pi[1.585(3.3)^3 - .487(3.3)^2]}{6} \\ &\approx 27.047 \end{aligned}$$

The volume is about 27.0 cubic cm.

42. $W(t) = -3.75t^2 + 30t + 40$

(a) $W(0) = -3.75(0)^2 + 30(0) + 40$
$W(0) = 40$ words/minute

(b)
$$W'(t) = -7.50t + 30$$
$$-7.50t + 30 = 0$$
$$t = 4$$

If $0 \le t < 4$, $W'(t) > 0$.
If $4 < t \le 5$, $W'(t) < 0$.
Therefore, a maximum occurs when $t = 4$.

$$W(4) = -3.75(4)^2 + 30(4) + 40$$
$$W(4) = 100$$

A maximum speed of 100 words per minute occurs when $t = 4$ minutes.

(c) The average value of W over $[0, 5]$ is given by

$$\frac{1}{5-0}\int_0^5 (-3.75t^2 + 30t + 40)\,dt$$
$$= \frac{1}{5}(-1.25t^3 + 15t^2 + 40t)\Big|_0^5$$
$$= \frac{1}{5}[(-156.25 + 375 + 200) - 0]$$
$$= 83.75.$$

The average value is 83.75 words per minute.

8.3 Continuous Money Flow

2. $f(x) = 300$

(a)
$$P = \int_0^{10} (300)e^{-.12x}\,dx$$
$$= \frac{300}{-.12}e^{-.12x}\Big|_0^{10}$$
$$= -2500(e^{-1.2} - e^0)$$
$$= \$1747.01$$

Store the value for P without rounding in your calculator.

(b)
$$A = e^{.12(10)}\int_0^{10} 300e^{-.12x}\,dx$$
$$= e^{1.2}P$$
$$= \$5800.29$$

4. $f(x) = 2000$

(a)
$$P = \int_0^{10} 2000e^{-.12x}\,dx$$
$$= \frac{2000}{-.12}e^{-.12x}\Big|_0^{10}$$
$$= -16{,}666.67(e^{-1.2} - e^0)$$
$$= \$11{,}646.76$$

(b)
$$A = e^{.12(10)}\int_0^{10} 2000e^{-.12x}\,dx$$
$$= e^{1.2}P$$
$$= \$38{,}668.62$$

6. $f(x) = 800e^{.05x}$

(a)
$$P = \int_0^{10} 800e^{.05x}e^{-.12x}\,dx = 800\int_0^{10} e^{-.07x}\,dx$$
$$= \frac{800}{-.07}e^{-.07x}\Big|_0^{10} = -11{,}428.57(e^{-.7} - 1)$$
$$= \$5753.31$$

(b)
$$A = e^{.12(10)}\int_0^{10} 800e^{.05x}e^{-.12x}\,dx$$
$$= e^{1.2}\int_0^{10} 800e^{-.07x}\,dx$$
$$= e^{1.2}P = \$19{,}101.66$$

8. $f(x) = 1000e^{-.02x}$

(a)
$$P = 1000\int_0^{10} e^{-.02x}e^{-.12x}\,dx = 1000\int_0^{10} e^{-.14x}\,dx$$
$$= \frac{1000}{-.14}e^{-.14x}\Big|_0^{10} = -7142.86(e^{-1.4} - 1)$$
$$= \$5381.45$$

(b)
$$A = e^{.12(10)}\int_0^{10} e^{-.02x}e^{-.12x}\,dx$$
$$= e^{1.2}\int_0^{10} e^{-.14x}\,dx$$
$$= e^{1.2}P = \$17{,}867.04$$

10. $f(x) = .5x$

(a) $P = \int_0^{10} .5xe^{-.12x}\,dx = .5\int_0^{10} xe^{-.12x}\,dx$

$.5\int xe^{-.12x}\,dx$

$$= .5\left(\frac{xe^{-.12x}}{-.12}\right) - \frac{.5}{-.12}\int e^{-.12x}\,dx$$

$$P = \left[.5\left(\frac{xe^{-.12x}}{-.12}\right) + \frac{.5}{.12}\left(\frac{e^{-.12x}}{-.12}\right)\right]\Bigg|_0^{10}$$

$$= -4.167(10e^{-1.2})$$
$$+ (-34.72)(e^{-1.2} - 1)$$
$$= \$11.71$$

(b) $A = e^{.12(10)}\int_0^{10} .5xe^{-.12x}\,dx$

$$= e^{1.2}P$$
$$= \$38.89$$

12. $f(x) = .05x + 500$

(a) $P = \int_0^{10} (.05x + 500)e^{-.12x}\,dx$

$$= .05\int_0^{10} xe^{-.12x}\,dx + 500\int_0^{10} e^{-.12x}\,dx$$

$.05\int xe^{-.12x}\,dx$

$$= .05\left(\frac{xe^{-.12x}}{-.12} - \frac{1}{-.12}\int e^{-.12x}\,dx\right)$$

$$= .05\left(\frac{xe^{-.12x}}{-.12} + \frac{1}{.12}\cdot\frac{e^{-.12x}}{-.12}\right)$$

$$P = \left[.05\left(\frac{xe^{-.12x}}{-.12} - \frac{e^{-.12x}}{(.12)^2}\right) + \frac{500}{-.12}(e^{-.12x})\right]\Bigg|_0^{10}$$

$$= -.4167(10e^{-1.2}) - 3.472(e^{-1.2} - 1)$$
$$- 4166.67(e^{-1.2} - 1)$$
$$= \$2912.86$$

(b) $A = e^{.12(10)}\int_0^{10} (.05x + 500)e^{-.12x}\,dx$

$$= e^{1.2}P = \$9671.04$$

14. $f(x) = 2000x - 150x^2$

(a) $P = \int_0^{10} (2000x - 150x^2)e^{-.12x}\,dx$

$$= \int_0^{10} 2000xe^{-.12x}\,dx - \int_0^{10} 150x^2e^{-.12x}\,dx$$

For $\int_0^{10} 2000xe^{-.12x}\,dx$, let $u = 2000x$ and $dv = e^{-.12x}\,dx$; then $du = 2000\,dx$ and

$$v = \left(-\frac{1}{.12}\right)e^{-.12x}.$$

$\int 2000xe^{-.12x}\,dx$

$$= \left(-\frac{2000x}{.12}\right)e^{-.12x} - \int \frac{2000}{-.12}e^{-.12x}\,dx$$

$$= \frac{2000e^{-.12x}}{-.12} - \frac{2000}{(-.12)^2}e^{-.12x} + C$$

For $\int 150x^2e^{-.12x}\,dx$, let $u = 150x^2$ and $dv = e^{-.12x}\,dx$.

Use column integration.

D		I
$150x^2$	+	$e^{-.12x}$
$300x$	−	$\left(\frac{1}{-.12}\right)e^{-.12x}$
300	+	$\left(\frac{1}{-.12}\right)^2e^{-.12x}$
0		$\left(\frac{1}{-.12^3}\right)e^{-.12x}$

$\int 150x^2e^{-.12x}\,dx$

$$= \frac{150x^2e^{-.12x}}{-.12} - \frac{300xe^{-.12x}}{(-.12)^2}$$
$$+ \frac{300e^{-.12x}}{(-.12)^3} + C$$

$\int_0^{10} 2000xe^{-.12x}\,dx - \int_0^{10} 150x^2e^{-.12x}\,dx$

$$= \left[\frac{2000xe^{-.12x}}{-.12} - \frac{2000e^{-.12x}}{(-.12)^2}\right.$$
$$- \frac{150x^2e^{-.12x}}{-.12} + \frac{300xe^{-.12x}}{(-.12)^2}$$
$$\left.- \frac{300e^{-.12x}}{(-.12)^3}\right]$$

$$= \frac{20{,}000e^{-1.2}}{-.12} - \frac{2000e^{-1.2}}{(-.12)^2}$$
$$- \frac{15{,}000e^{-1.2}}{-.12} + \frac{3000e^{-1.2}}{(-.12)^2}$$
$$- \frac{300e^{-1.2}}{(-.12)^3} + \frac{2000}{(-.12)^2} + \frac{300}{(-.12)^3}$$
$$= \$25{,}934.95$$

(b) $e^{.12(10)} \int_0^{10} (2000x - 150x^2)e^{-.12x}\,dx$

$= e^{1.2}\,P$
$= \$86{,}107.05$

16. (a) Present value

$$= \int_0^6 8000e^{-.12x}\,dx$$
$$= \frac{8000}{-.12}(e^{-.12x})\Big|_0^6$$
$$= -66{,}666.67(e^{-.72} - 1)$$
$$= \$34{,}216.52$$

(b) Present value

$$= \int_0^6 8000e^{-.10x}\,dx$$
$$= \frac{8000}{-.10}(e^{-.10x})\Big|_0^6$$
$$= -80{,}000(e^{-.6} - 1)$$
$$= \$36{,}095.07$$

(c) Present value

$$= \int_0^6 8000e^{-.15x}\,dx$$
$$= \frac{8000}{-.15}(e^{-.15x})\Big|_0^6$$
$$= -53{,}333.33(e^{-.9} - 1)$$
$$= \$31{,}649.62$$

18. (a) Present value

$$= \int_0^4 1000e^{.05x}e^{-.11x}\,dx$$
$$= 1000\int_0^4 e^{-.06x}\,dx$$
$$= \frac{1000}{-.06}(e^{-.06x})\Big|_0^4$$
$$= -16{,}666.67(e^{-.24} - 1)$$
$$= \$3556.20$$

(b) Final amount

$$= e^{.11(4)} \int_0^4 1000e^{.05x}e^{-.11x}\,dx$$
$$= e^{.44}(3556.20)$$
$$= \$5521.74$$

20. $A = e^{.10(3)} \int_0^3 (1000 - x^2)e^{-.10x}\,dx$

$$= e^{.3}\int_0^3 1000e^{-.1x}\,dx - e^{.3}\int_0^3 x^2e^{-.1x}\,dx$$
$$= \frac{1000e^{.3}e^{-.1x}}{-.1}\Big|_0^3 - e^{.3}(7.2)$$
$$= -13{,}498.59(e^{-.3} - 1) - 9.72$$
$$= 3498.59 - 9.72$$
$$= \$3488.87$$

8.4 Improper Integrals

2. $\displaystyle\int_5^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_5^b \frac{dx}{x^2} = \lim_{b\to\infty}\int_5^b x^{-2}\,dx$

$$= \lim_{b\to\infty}\left(\frac{x^{-1}}{-1}\right)\Big|_5^b = \lim_{b\to\infty}\left(-\frac{1}{x}\right)\Big|_5^b$$
$$= \lim_{b\to\infty}\left(-\frac{1}{b} + \frac{1}{5}\right)$$

As $b\to\infty$, $-\frac{1}{b}\to 0$. The integral is convergent.

$$\int_5^\infty \frac{1}{x^2}\,dx = 0 + \frac{1}{5} = \frac{1}{5}$$

4. $\displaystyle\int_{16}^\infty \frac{-3}{\sqrt{x}}\,dx = \lim_{b\to\infty}\int_{16}^b -3x^{-1/2}\,dx$

$$= \lim_{b\to\infty}\left(\frac{-3x^{1/2}}{\frac{1}{2}}\right)\Big|_{16}^b$$
$$= \lim_{b\to\infty}(-6\sqrt{b} - 6\sqrt{16})$$
$$= \lim_{b\to\infty}(-6\sqrt{b} - 24)$$

As $b\to\infty$, $(-6\sqrt{b} - 24) \to -\infty$, so the integral is divergent.

6. $$\int_{-\infty}^{-4} \frac{3}{x^4}\,dx = \lim_{a\to-\infty} \int_a^{-4} 3x^{-4}\,dx$$
$$= \lim_{a\to-\infty} \left(\frac{3x^{-3}}{-3}\right)\Big|_a^{-4}$$
$$= \lim_{a\to-\infty} \left(-\frac{1}{x^3}\right)\Big|_a^{-4}$$
$$= \lim_{a\to-\infty} \left(\frac{1}{64} + \frac{1}{a^3}\right)$$

As $a \to \infty$, $\frac{1}{a^3} \to 0$. The integral is convergent.

$$\int_{-\infty}^{-4} \frac{3}{x^4}\,dx = \frac{1}{64} + 0 = \frac{1}{64}$$

8. $$\int_1^{\infty} \frac{1}{x^{.999}}\,dx = \lim_{b\to\infty} \int_1^b x^{-.999}\,dx$$
$$= \lim_{b\to\infty} \left(\frac{x^{.001}}{.001}\right)\Big|_1^b$$
$$= \lim_{b\to\infty} (1000x^{.001})\Big|_1^b$$
$$= \lim_{b\to\infty} (1000b^{.001} - 1000)$$

As $b \to \infty$, $(1000b^{.001} - 1000) \to \infty$.
The intergral is divergent.

10. $$\int_{-\infty}^{-4} x^{-2}\,dx = \lim_{a\to-\infty} \int_a^{-4} x^{-2}\,dx$$
$$= \lim_{a\to-\infty} \left(\frac{x^{-1}}{-1}\right)\Big|_a^{-4}$$
$$= \lim_{a\to-\infty} \left(\frac{-1}{x}\right)\Big|_a^{-4}$$
$$= \lim_{a\to-\infty} \left(\frac{-1}{-4} + \frac{1}{a}\right)$$

As $a \to -\infty$, $\frac{1}{a} \to 0$. The integral is convergent.

$$\int_{-\infty}^{-4} x^{-2}\,dx = \frac{1}{4} + 0 = \frac{1}{4}$$

12. $$\int_{-\infty}^{-27} x^{-5/3}\,dx = \lim_{a\to-\infty} \int_a^{-27} x^{-5/3}\,dx$$
$$= \lim_{a\to-\infty} \left(\frac{x^{-2/3}}{-\frac{2}{3}}\right)\Big|_a^{-27}$$
$$= \lim_{a\to-\infty} \left(-\frac{3}{2}x^{-2/3}\right)\Big|_a^{-27}$$
$$= \lim_{a\to-\infty} \left[-\frac{3}{2}(-27)^{-2/3} + \frac{3}{2}(a)^{-2/3}\right]$$
$$= \lim_{a\to-\infty} \left(-\frac{1}{6} + \frac{3}{a^{2/3}}\right)$$

As $a \to -\infty$, $\frac{3}{a^{2/3}} \to 0$. The integral is convergent.

$$\int_{-\infty}^{27} x^{-5/3}\,dx = -\frac{1}{6} + 0 = -\frac{1}{6}$$

14. $$\int_0^{\infty} 10e^{-10x}\,dx = \lim_{b\to\infty} \int_0^b 10e^{-10x}\,dx$$
$$= \lim_{b\to\infty} (-e^{-10x})\Big|_0^b$$
$$= \lim_{b\to\infty} (-e^{-10b} + e^0)$$
$$= \lim_{b\to\infty} (-e^{-10b} + 1)$$

As $b \to \infty$, $\frac{-1}{e^{10b}} \to 0$. The integral is convergent.

$$\int_0^{\infty} 10e^{-10x}\,dx = 0 + 1 = 1$$

16. $$\int_{-\infty}^{0} 3e^{4x}\,dx = \lim_{a\to-\infty} \int_a^0 3e^{4x}\,dx$$
$$= \lim_{a\to-\infty} \left(\frac{3e^{4x}}{4}\right)\Big|_a^0$$
$$= \lim_{a\to-\infty} \left(\frac{3}{4} - \frac{3}{4}e^{4a}\right)$$

As $a \to -\infty$, e^{4a} is in the denominator of the fraction. So, $-\frac{3}{4}e^{4a} \to 0$. The integral is convergent.

$$\int_{-\infty}^{0} 3e^{4x}\,dx = \frac{3}{4} + 0 = \frac{3}{4}$$

18. $$\int_1^{\infty} \ln|x|\,dx = \lim_{b\to\infty} \int_1^b \ln|x|\,dx$$

Let $u = \ln|x|$ and $dv = dx$.

Then $du = \frac{1}{x}\,dx$ and $v = x$.

$$\int \ln|x|\,dx = x\ln|x| - \int \frac{x}{x}\,dx$$
$$= x\ln|x| - x + C$$

$$\int_1^{\infty} \ln|x|\,dx$$
$$= \lim_{b\to\infty} (x\ln|x| - x)\Big|_1^b$$
$$= \lim_{b\to\infty} [(b\ln b - b) - (-1)]$$
$$= \lim_{b\to\infty} [b\ln b - b + 1]$$

As $b \to \infty$, $(b\ln b - b + 1) \to \infty$. The integral is divergent.

20. $\displaystyle\int_0^{\infty} \frac{dx}{(2x+1)^3}$

$$= \lim_{b\to\infty} \int_0^b \frac{dx}{(2x+1)^3}$$

$$= \lim_{b\to\infty} \left(\frac{1}{2}\int_0^b 2(2x+1)^{-3}\,dx\right)$$

$$= \lim_{b\to\infty} \left[\frac{1}{2}\cdot\frac{(2x+1)^{-2}}{-2}\right]\Big|_0^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{4}(2x+1)^{-2}\right]\Big|_0^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{4}(2b+1)^{-2} + \frac{1}{4}(1)^{-2}\right]$$

As $b\to\infty$, $-\frac{1}{4}(2b+1)^{-2}\to 0$. The integral is convergent.

$$\int_0^{\infty} \frac{dx}{(2x+1)^3} = 0 + \frac{1}{4} = \frac{1}{4}$$

22. $\displaystyle\int_1^{\infty} \frac{2x+3}{x^2+3x}\,dx$

$$= \lim_{b\to\infty} \int_1^b \frac{2x+3}{x^2+3x}\,dx$$

$$= \lim_{b\to\infty} \left[\ln\left|x^2+3x\right|\Big|_1^b\right]$$

$$= \lim_{b\to\infty} \left[\ln(b^2+3b) - \ln 1\right]$$

As $b\to\infty$, $[\ln(b^2+3b) - 1\,0]\to\infty$.
The integral is divergent.

24. $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$

$$= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx \quad \textit{Use substitution}$$

$$= \lim_{b\to\infty} \left(-\frac{1}{\ln x}\Big|_2^b\right)$$

$$= \lim_{b\to\infty} \left(\frac{-1}{\ln b} + \frac{1}{\ln 2}\right)$$

As $b\to\infty$, $-\dfrac{1}{\ln b}\to 0$. The integral is convergent.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

26. $\displaystyle\int_{-\infty}^{0} xe^{3x}\,dx = \lim_{a\to-\infty}\int_a^0 xe^{3x}\,dx$

Let $dv = e^{3x}\,dx$ and $u = x$.

Then $v = \frac{1}{3}e^{3x}$ and $du = dx$.

$$\int xe^{3x}\,dx = \frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,dx = \frac{xe^{3x}}{3} - \frac{1}{9}e^{3x} + C$$

$$\int_{-\infty}^{0} xe^{3x}\,dx = \lim_{a\to-\infty}\left(\frac{xe^{3x}}{3} - \frac{1}{9}e^{3x}\right)\Big|_a^0 = \lim_{a\to-\infty}\left(-\frac{1}{9} - \frac{ae^{3a}}{3} + \frac{1}{9}e^{3a}\right)$$

As $a\to-\infty$, e^{3a} is in the denominator of a fraction. The integral is convergent.

$$\int_{-\infty}^{0} xe^{3x}\,dx = -\frac{1}{9} - 0 + 0 = -\frac{1}{9}$$

28. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}dx = \int_{-\infty}^{0} e^{-|x|}dx + \int_0^{\infty} e^{-|x|}dx$

We evaluate each of the two improper integrals on the right.

$$\int_{-\infty}^{0} e^{-|x|}dx = \int_{-\infty}^{0} e^{x}dx = \lim_{b\to-\infty}\int_b^0 e^x dx = \lim_{b\to-\infty}\left[e^x\Big|_b^0\right] = \lim_{b\to-\infty}(1-e^b)$$

As $b\to-\infty, e^b\to 0$. The integral is convergent.

$$\int_{-\infty}^{0} e^{-|x|}dx = 1 - 0 = 1$$

$$\int_0^{\infty} e^{-|x|}dx = \int_0^{\infty} e^{-x}dx = \lim_{b\to\infty}\int_0^b e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\Big|_0^b\right] = \lim_{b\to\infty}\left[-e^{-b}+1\right]$$

As $b\to\infty, e^{-b}\to 0$. The integral is convergent.

$$\int_0^{\infty} e^{-|x|}dx = -0 + 1 = 1$$

Since each of the improper integrals converges, the original improper integral converges.

$$\int_{-\infty}^{\infty} e^{-|x|}dx = 1 + 1 = 2$$

30. $\displaystyle\int_{-\infty}^{\infty} \frac{2x+1}{x^2+x+4}\,dx$

$$= \int_{-\infty}^{0} \frac{2x+1}{x^2+x+4}\,dx + \int_{0}^{\infty} \frac{2x+1}{x^2+x+4}\,dx$$

We evaluate the first proper integral on the right.

$$\int_{-\infty}^{0} \frac{2x+1}{x^2+x+4}\,dx$$

$$= \lim_{b\to-\infty} \int_{b}^{0} \frac{2x+1}{x^2+x+4}\,dx \quad \textit{Use substitution}$$

$$= \lim_{b\to-\infty} [\ln(x^2+x+4)\Big|_b^0$$

$$= \lim_{b\to-\infty} [\ln 4 - \ln(b^2+b+4)]$$

As $b \to -\infty$, $\ln(b^2+b+4) \to \infty$. The integral is divergent. Since one of the two improper integrals on the right diverges, the original improper integral diverges.

32. $f(x) = e^{-x}$ for $(-\infty, e]$

$$\int_{-\infty}^{e} e^{-x} = \lim_{a\to-\infty} \int_{a}^{e} e^{-x}\,dx$$

$$= \lim_{a\to-\infty} (-e^{-x})\Big|_a^e$$

$$= \lim_{a\to-\infty} (-e^{-e} + e^{-a})$$

As $a \to -\infty$, $e^{-a} \to \infty$, and $(-e^{-e} + e^{-a}) \to \infty$. The integral is divergent, so the area cannot be found.

34. $f(x) = \dfrac{1}{(x-1)^3}$ for $(-\infty, 0]$

$$\int_{-\infty}^{0} \frac{1}{(x-1)^3}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{(x-1)^3}$$

$$= \lim_{a\to-\infty} \left(\frac{(x-1)^{-2}}{-2}\right)\Big|_a^0$$

$$= \lim_{a\to-\infty} \left[-\frac{1}{2(x-1)^2}\right]\Big|_a^0$$

$$= \lim_{a\to-\infty} \left[-\frac{1}{2} + \frac{1}{2(a-1)^2}\right]$$

$$= -\frac{1}{2}$$

Since area is positive, the area is $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

36. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\,dx$

$$= \int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx + \int_{0}^{\infty} \frac{x}{(1+x^2)^2}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{x\,dx}{(1+x^2)^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{x\,dx}{(1+x^2)^2}$$

$$= \lim_{a\to-\infty} \left(\frac{-1}{2(1+x^2)}\right)\Big|_a^0 + \lim_{b\to\infty} \left(\frac{-1}{2(1+x^2)}\right)\Big|_0^b$$

$$= \lim_{a\to-\infty} \left[\frac{-1}{2(1+0^2)} + \frac{1}{2(1+a^2)}\right]$$

$$+ \lim_{b\to\infty} \left[\frac{-1}{2(1+b^2)} + \frac{1}{2(1+0^2)}\right]$$

$$= -\frac{1}{2} + 0 + 0 + \frac{1}{2} = 0$$

40. **(a)** Use the $fnInt$ feature on a graphing utility to obtain

$$\int_0^1 e^{-x^2}\,dx \approx .7468; \int_0^5 e^{-x^2}\,dx \approx .8862;$$

$$\int_0^{10} e^{-x^2}\,dx \approx .8862; \int_0^{20} e^{-x^2}\,dx \approx .8862.$$

(b) Since the approximate values of the last three integrals in part a are all .8862, it appears that the integral $\int_0^\infty e^{-x^2}\,dx$ converges with an approximate value of .8862.

(c) For $x > 1, e^{-x^2} < e^{-x}$. Thus, the area between the graph of $y = e^{-x^2}$ and the x-axis on the interval $[1, \infty)$ is less than the area between $y = e^{-x}$ and the x-axis. Since

$$\int_1^\infty e^{-x}dx = \lim_{b\to\infty} \int_1^b e^{-x}dx$$

$$= \lim_{b\to\infty} [-e^{-x}\Big|_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{e^b} + \frac{1}{e}\right]$$

$$= 0 + \frac{1}{e} = \frac{1}{e},$$

the area between $y = e^{-x}$ and the x-axis on the interval $[1, \infty)$ is finite. It follows that the area between $y = e^{-x^2}$ and the x-axis, being smaller, must also be finite, so the integral $\int_1^\infty e^{-x^2}\,dx$ converges. Since the area between $y = e^{-x^2}$ and the x-axis on the interval $[0, 1]$ is finite as well, the integral $\int_0^\infty e^{-x^2}\,dx$ converges.

42. Capital value is

$$\int_0^\infty 60{,}000e^{-.08t}\,dt$$

$$= \lim_{b\to\infty} \int_0^b 60{,}000e^{-.08t}\,dt$$

$$= \lim_{b\to\infty} \left(\frac{60{,}000}{-.08}e^{-.08t}\right)\Big|_0^b$$

$$= -750{,}000 \lim_{b\to\infty} (e^{-.08b} - 1)$$

$$= -750{,}000 \lim_{b\to\infty} \left(\frac{1}{e^{.08b}} - 1\right).$$

As $b\to\infty$, $\frac{1}{e^{.08b}}\to 0$.

$$= -750{,}000(-1)$$
$$= \$750{,}000$$

44. (a) $$\int_0^\infty 6000e^{-.08t}\,dt = \lim_{b\to\infty}\int_0^b 6000e^{-.08t}\,dt$$

$$= \lim_{b\to\infty} \frac{6000e^{-.08t}}{-.08}\Big|_0^b$$

$$= \lim_{b\to\infty}\left(\frac{6000e^{-.08b}}{-.08} - \frac{6000}{-.08}\right)$$

$$= 0 + \; 75{,}000 = \$75{,}000$$

(b) $$\int_0^\infty 6000e^{-.1t}\,dt = \lim_{b\to\infty}\int_0^b 6000e^{-.1t}\,dt$$

$$= \lim_{b\to\infty} \frac{6000e^{-.1t}}{-.1}\Big|_0^b$$

$$= \lim_{b\to\infty}\left(\frac{6000e^{-.1b}}{-.1} - \frac{6000}{-.1}\right)$$

$$= 0 + 60{,}000 = \$60{,}000$$

46. $R(t) = 6000 + 200t$, $r = .05$

The capital value is given by

$$\int_0^\infty (6000 + 200t)e^{-.05t}dt$$

$$= \lim_{b\to\infty}\int_0^b (6000 + 200t)e^{-.05t}dt.$$

Let $u = 6000 + 200t$ and $dv = e^{-.05t}dt$.

Then $du = 200dt$ and $v = \dfrac{1}{-.05}e^{-.05t} = -20e^{-.05t}$.

$$\int (6000 + 200t)e^{-.05t}dt$$

$$= -20e^{-.05t}(6000 + 200t) + \int 4000e^{-.05t}dt$$
$$= -20e^{-.05t}(6000 + 200t) - 80{,}000e^{-.05t} + C$$

$$\int_0^\infty (6000 + 200t)e^{-.05t}dt$$

$$= \lim_{b\to\infty}\int_0^b (6000 + 200t)e^{-.05t}dt$$

$$= \lim_{b\to\infty} [-20e^{-.05t}(6000 + 200t)$$
$$- 80{,}000e^{-.05t})]\Big|_0^b$$

$$= \lim_{b\to\infty} [e^{-.05t}((-20)(6000 + 200t) - 80{,}000)]\Big|_0^b$$

$$= \lim_{b\to\infty} [e^{-.05b}(-120{,}000 - 4000b - 80{,}000)$$
$$+ 200{,}000]$$
$$= 0 + 200{,}000 = \$200{,}000$$

48. $r'(x) = 2x^2e^{-x}$

$$r(x) = \int_0^\infty 2x^2e^{-x}\,dx = \lim_{b\to\infty}\int_0^b 2x^2e^{-x}\,dx$$

Let $u = 2x^2$ and $dv = e^{-x}\,dx$.
Use column integration to obtain

$$\lim_{b\to\infty}\int_0^b 2x^2e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[(-2x^2e^{-x} - 4xe^{-x} - 4e^{-x})\Big|_0^b\right]$$

$$= \lim_{b\to\infty} [(-2b^2e^{-b} - 4be^{-b} - 4e^{-b}) - (0 - 4\cdot 1)]$$

$$= -(-4) \quad \textit{Use hint to evaluate limits}$$
$$= 4.$$

50. $$P = \int_0^\infty e^{-rt}(at + b)K\,dt = K\lim_{c\to\infty}\int_o^c (at + b)e^{-rt}dt$$

Evaluate $\int_o^c (at + b)e^{-rt}dt$ using integration by parts.

Let $u = at + b$ and $dv = e^{-rt}dt$.
Then $du = a\,dt$ and $v = -\dfrac{1}{r}e^{-rt}$.

$$\int_0^c (at + b)e^{-rt}\,dt$$

$$= \left[(at + b)\left(-\frac{1}{r}e^{-rt}\right) - \int\left(-\frac{1}{r}e^{-rt}\right)a\,dt\right]\Big|_0^c$$

$$= \left[-\frac{at + b}{r}e^{-rt} + \frac{a}{r}\int e^{-rt}dt\right]\Big|_0^c$$

$$= \left[-\frac{at + b}{r}e^{-rt} - \frac{a}{r^2}e^{-rt}\right]\Big|_0^c$$

$$= \left(-\frac{ac + b}{r}e^{-rc} - \frac{a}{r^2}e^{-rc}\right) - \left(-\frac{b}{r}e^0 - \frac{a}{r^2}e^0\right)$$

Therefore,

$$K \lim_{c\to\infty} \int_0^c (at+b)e^{-rt}dt$$

$$= K \lim_{c\to\infty} \left(-\frac{ac+b}{r}e^{-rc} - \frac{a}{r^2}e^{-rc} + \frac{b}{r} + \frac{a}{r^2}\right)$$

$$= K\left(0 - 0 + \frac{b}{r} + \frac{a}{r^2}\right)$$

$$= \frac{K(a+br)}{r^2}$$

52. $\int_0^\infty 50e^{-.04t}\,dt$

$$= 50 \lim_{b\to\infty} \int_0^b e^{-.04t}\,dt$$

$$= 50 \lim_{b\to\infty} \left.\frac{e^{-.04t}}{-.04}\right|_0^b$$

$$= \frac{50}{-.04} \lim_{b\to\infty} \left(\frac{1}{e^{.04b}} - 1\right)$$

As $b \to \infty$, $\frac{1}{e^{.04b}} \to 0$.

$$= -1250(-1)$$
$$= 1250$$

Chapter 8 Review Exercises

4. $\int x(8-x)^{3/2}\,dx$

Let $u = x$ and $dv = (8-x)^{3/2}$.

Then $du = dx$ and $v = -\frac{2}{5}(8-x)^{5/2}$.

$$\int x(8-x)^{3/2}\,dx$$

$$= -\frac{2}{5}x(8-x)^{5/2} + \int \frac{2}{5}(8-x)^{5/2}\,dx$$

$$= -\frac{2}{5}x(8-x)^{5/2} - \frac{2}{5}\left(\frac{2}{7}\right)(8-x)^{7/2} + C$$

$$= -\frac{2x}{5}(8-x)^{5/2} - \frac{4}{35}(8-x)^{7/2} + C$$

6. $\int xe^x\,dx$

Let $u = x$ and $dv = e^x\,dx$.

Then $du = dx$ and $v = e^x$.

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

8. $\int \ln|2x+3|\,dx$

First, use substitution.

Let $a = 2x+3$. Then $da = 2\,dx$.

$$\int \ln|2x+3|\,dx = \frac{1}{2}\int \ln|a|\,da$$

Second, integrate by parts.

Let $u = \ln|a|$ and $dv = da$.

Then $du = \frac{1}{a}\,da$ and $v = a$.

$$\frac{1}{2}\int \ln|a|\,da$$

$$= \frac{1}{2}(a\ln|a| - \int da)$$

$$= \frac{1}{2}(a\ln|a| - a) + C$$

Finally, substitute $2x+3$ for a.

$$\int \ln|2x+3|\,dx$$

$$= \frac{1}{2}[(2x+3)\ln|2x+3| - (2x+3) + C]$$

$$= \frac{1}{2}(2x+3)[\ln|2x+3| - 1] + C$$

10. $\int \frac{x}{9-4x^2}\,dx$

Use substitution.

Let $u = 9-4x^2$. Then $du = -8x\,dx$.

$$\int \frac{x}{9-4x^2}\,dx = -\frac{1}{8}\int \frac{-8x\,dx}{9-4x^2}$$

$$= -\frac{1}{8}\int \frac{du}{u}$$

$$= -\frac{1}{8}\ln|u| + C$$

$$= -\frac{1}{8}\ln\left|9-4x^2\right| + C$$

12. $\int_1^e x^3 \ln x \, dx$

Let $u = \ln x$ and $dv = x^3 \, dx$.
Use column integration.

D		I
$\ln x$	+	x^3
$\frac{1}{x}$	−	$\frac{1}{4}x^4$

$$\int x^3 \ln x \, dx$$
$$= \frac{1}{4}x^4 \ln x - \int \left(\frac{1}{4}x^4 \cdot \frac{1}{x}\right) dx$$
$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx$$
$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

$$\int_1^e x^3 \ln x \, dx$$
$$= \left(\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right)\Big|_1^e$$
$$= \left(\frac{e^4}{4}\right)(1) - \frac{e^4}{16} - 0 + \frac{1}{16}$$
$$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16}$$
$$= \frac{3e^4+1}{16}$$
$$\approx 10.300$$

14. $A = \int_0^1 (3+x^2)e^{2x} \, dx$

$$= \int_0^1 3e^{2x} \, dx + \int_0^1 x^2 e^{2x} \, dx$$

$$\int 3e^{2x} \, dx = \frac{3}{2}e^{2x} + C$$

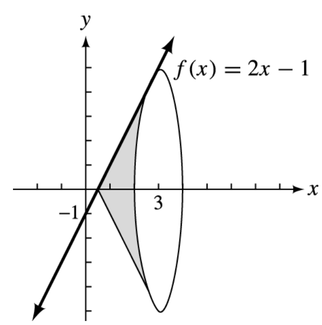
For the second integral, $\int x^2 e^{2x} \, dx$, let $u = x^2$ and $dv = e^{2x} \, dx$.
Use column integration.

D		I
x^2	+	e^{2x}
$2x$	−	$\frac{e^{2x}}{2}$
2	+	$\frac{e^{2x}}{4}$
0		$\frac{e^{2x}}{8}$

$$\int x^2 e^{2x} \, dx = x^2\frac{e^{2x}}{2} - 2x\frac{e^{2x}}{4} + 2\frac{e^{2x}}{8}$$
$$= \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4}$$

$$A = \left(\frac{3}{2}e^{2x} + \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4}\right)\Big|_0^1$$
$$= \frac{3}{2}e^2 + \frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{4} - \left(\frac{3}{2} + \frac{1}{4}\right)$$
$$= \left(\frac{6+2-2+1}{4}\right)e^2 - \left(\frac{7}{4}\right)$$
$$= \frac{7}{4}(e^2-1) \approx 11.181$$

16. $f(x) = 2x - 1, \; y = 0, \; x = 3$



Since $f(x) = 2x - 1$ intersects $y = 0$ at $x = \frac{1}{2}$, the integral has a lower bound $a = \frac{1}{2}$.

$$V = \pi \int_{1/2}^3 (2x-1)^2 \, dx$$
$$= \pi \int_{1/2}^3 (4x^2 - 4x + 1) \, dx$$
$$= \pi \left(\frac{4x^3}{3} - \frac{4x^2}{2} + x\right)\Big|_{1/2}^3$$
$$= \pi \left(36 - 18 + 3 - \frac{1}{6} + \frac{1}{2} - \frac{1}{2}\right)$$
$$= \pi \left(21 - \frac{1}{6}\right) = \frac{125\pi}{6}\pi \approx 65.45$$

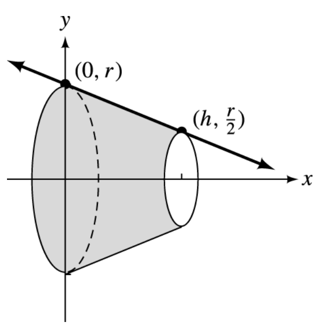
18. $f(x) = e^{-x}, \; y = 0, \; x = -2, \; x = 1$

$$V = \pi \int_{-2}^1 e^{-2x} \, dx = \frac{\pi e^{-2x}}{-2}\Big|_{-2}^1$$
$$= \frac{\pi e^{-2}}{-2} + \frac{\pi e^4}{2} = \frac{\pi(e^4 - e^{-2})}{2}$$
$$\approx 85.55$$

20. $f(x) = 4 - x^2,\ y = 0,\ x = -1,\ x = 1$

$$\begin{aligned} V &= \pi \int_{-1}^{1} (4 - x^2)^2\,dx \\ &= \pi \int_{-1}^{1} (16 - 8x^2 + x^4)\,dx \\ &= \pi \left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\Bigg|_{-1}^{1} \\ &= \pi \left(16 - \frac{8}{3} + \frac{1}{5} + 16 - \frac{8}{3} + \frac{1}{5}\right) \\ &= \pi \left(32 - \frac{16}{3} + \frac{2}{5}\right) \\ &= \frac{406\pi}{15} \\ &\approx 85.03 \end{aligned}$$

22. The frustum may be shown as follows.



Use the two points given to find

$$f(x) = -\frac{r}{2h}x + r.$$

$$\begin{aligned} V &= \pi \int_0^h \left(-\frac{r}{2h}x + r\right)^2\,dx \\ &= -\frac{2\pi h}{3r}\left(-\frac{r}{2h}x + r\right)^3\Bigg|_0^h \\ &= -\frac{2\pi h}{3r}\left[\left(-\frac{r}{2} + r\right)^3 - (0 + r)^3\right] \\ &= -\frac{2\pi h}{3r}\left[\left(\frac{r}{2}\right)^3 - r^3\right] \\ &= -\frac{2\pi h}{3r}\left(\frac{r^3}{8} - r^3\right) \\ &= -\frac{2\pi h}{3r}\left(-\frac{7r^3}{8}\right) \\ &= \frac{7\pi r^2 h}{12} \end{aligned}$$

24. $f(x) = \sqrt{x+1}$

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{8-0}\int_0^8 \sqrt{x+1}\,dx \\ &= \frac{1}{8}\int_0^8 (x+1)^{1/2}\,dx \\ &= \frac{1}{8}\left(\frac{2}{3}\right)(x+1)^{3/2}\Bigg|_0^8 \\ &= \frac{1}{12}(9)^{3/2} - \frac{1}{12}(1) \\ &= \frac{27}{12} - \frac{1}{12} = \frac{26}{12} \\ &= \frac{13}{6} \end{aligned}$$

26.
$$\begin{aligned} \int_1^\infty x^{-1}\,dx &= \lim_{b\to\infty}\int_1^b \frac{dx}{x} \\ &= \lim_{b\to\infty} \ln x\,\Big|_1^b \\ &= \lim_{b\to\infty} \ln b \end{aligned}$$

As $b\to\infty$, $\ln b\to\infty$. The integral is divergent.

28. $\displaystyle\int_0^\infty \frac{dx}{(5x+2)^2}$

$$\begin{aligned} &= \lim_{b\to\infty}\int_0^b (5x+2)^{-2}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{1}{5}(5x+2)^{-1}\right]\Bigg|_0^b \\ &= \lim_{b\to\infty}\left[\frac{-1}{5(5x+2)}\right]\Bigg|_0^b \\ &= \lim_{b\to\infty}\left[\frac{-1}{5(5b+2)} + \frac{1}{10}\right] \\ &= \frac{1}{10} \end{aligned}$$

30. $\displaystyle\int_{-\infty}^0 \frac{x}{x^2+3}\,dx$

$$\begin{aligned} &= \lim_{a\to-\infty}\frac{1}{2}\int_a^0 \frac{2x\,dx}{x^2+3} \\ &= \lim_{a\to-\infty}\frac{1}{2}(\ln\left|x^2+3\right|)\Big|_a^0 \\ &= \lim_{a\to-\infty}\frac{1}{2}(\ln 3 - \ln\left|a^2+3\right|) \end{aligned}$$

As $a\to-\infty$, $\frac{1}{2}(\ln 3 - \ln\left|a^2+3\right|)\to\infty$.
The integral is divergent.

32. $A = \int_{-\infty}^{1} \frac{3}{(x-2)^2}\,dx$

$$= \lim_{a\to-\infty} \int_a^1 3(x-2)^{-2}\,dx$$

$$= \lim_{a\to-\infty} \left.\frac{3(x-2)^{-1}}{-1}\right|_a^1$$

$$= \lim_{a\to-\infty} \left.\left(-\frac{3}{x-2}\right)\right|_a^1$$

$$= \lim_{a\to-\infty} \left(-\frac{3}{-1} + \frac{3}{a-2}\right)$$

$$= 3$$

36. $f(x) = 5000;\ 8$ yr; 9%

$$P = \int_0^8 5000e^{-.09x}\,dx = \left.\frac{5000}{-.09}e^{-.09x}\right|_0^8$$

$$= \frac{-5000}{.09}e^{-.72} + \frac{5000}{.09} \approx \$28{,}513.76$$

38. $f(x) = 100e^{.02x}$, 5 yr; 11%

$$P = \int_0^5 100e^{.02x} \cdot e^{-.11x}\,dx = \int_0^5 100e^{-.09x}\,dx$$

$$= \left.\frac{100}{-.09}e^{-.09x}\right|_0^5 = \frac{-100}{.09}e^{-.45} + \frac{100}{.09}$$

$$\approx \$402.64$$

40. $f(x) = 2000;\ 5$ yr, 12% per year

$$e^{.12(5)} \int_0^5 2000e^{-.12t}\,dt = e^{.6}\left.\left(\frac{2000}{-.12}e^{-.12t}\right)\right|_0^5$$

$$= e^{.6}\left(\frac{2000}{-.12}(e^{-.6}-1)\right)$$

$$\approx \$13{,}701.98$$

42. $f(x) = 20x;\ 6$ yr, 12% per year

$$e^{.12(6)} \int_0^6 20xe^{-.12x}\,dx = 20e^{.72}\int_0^6 xe^{-.12x}\,dx$$

Let $u = x\,dx$ and $dv = e^{-.12x}$.

Then $du = dx$ and $v = \frac{e^{-.12x}}{-.12}$.

$$\int xe^{-.12x}\,dx = \frac{xe^{-.12x}}{-.12} - \int \frac{e^{-.12x}}{-.12}\,dx$$

$$= -\frac{xe^{-.12x}}{.12} - \frac{e^{-.12x}}{(.12)^2}$$

$$20e^{.72}\int_0^6 xe^{-.12x}\,dx$$

$$= 20e^{.72}\left.\left[\frac{-xe^{-.12x}}{.12} - \frac{e^{-.12x}}{(.12)^2}\right]\right|_0^6$$

$$= 20e^{.72}\left[\frac{-6e^{-.72}}{.12} - \frac{e^{-.72}}{(.12)^2} + 0 + \frac{1}{(.12)^2}\right]$$

$$\approx \$464.49$$

44. $f(x) = Ce^{kx}$ where $C = 1000,\ k = .05$
$f(x) = 1000e^{.05x}$

Use $P = \int_0^t f(x)e^{-rx}\,dx$ with $r = .11$ and $t = 7$.

$$P = \int_0^7 1000e^{.05x} \cdot e^{-.11x}\,dx$$

$$= 1000\int_0^7 e^{-.06x}\,dx$$

$$= 1000\left.\left(\frac{1}{-.06}e^{-.06x}\right)\right|_0^7$$

$$= \frac{1000}{-.06}(e^{-.42}-1) = \$5715.89$$

46. $\int_0^b Re^{-kt}\,dt$

$$= \int_0^\infty 50{,}000e^{-.09t}\,dt$$

$$= \lim_{b\to\infty}\left.\left(\frac{50{,}000e^{-.09t}}{-.09}\right)\right|_0^b$$

$$= \lim_{b\to\infty}\left[\frac{50{,}000}{-.09}e^{-.09(b)} + \frac{50{,}000}{.09}(1)\right]$$

$$= \lim_{b\to\infty}\left[\frac{-50{,}000}{.09e^{.09b}} + \frac{50{,}000}{.09}\right]$$

$$= \$555{,}555.56$$

As $b\to\infty$, $e^{.09b}\to\infty$, so $\frac{-50{,}000}{.09e^{.09b}}\to 0$.

48. $f(x) = 100e^{-.05x}$

The total amount of oil is

$$\int_0^\infty 100e^{-.05x}\,dx = \lim_{b\to\infty}\int_0^b 100e^{-.05x}\,dx$$

$$= \lim_{b\to\infty}\left.\left(\frac{100e^{-.05x}}{-.05}\right)\right|_0^b$$

$$= \lim_{b\to\infty}\left(\frac{-2000}{e^{.05b}} + 2000\right)$$

$$= 0 + 2000 = 2000 \text{ gal.}$$

50. (a) $\overline{T} = \frac{1}{10-0}\int_0^{10}(400-.25x^2)\,dx$

$$= \frac{1}{10}\left(400x - \frac{.25x^3}{3}\right)\Big|_0^{10}$$
$$= \frac{1}{10}\left[400(10) - \frac{.25}{3}(10)^3\right]$$
$$= \frac{1}{10}(3916.7)$$
$$= 391.7$$

(b) $\overline{T} = \frac{1}{40-10}\int_{10}^{40}(400-.25x^2)\,dx$

$$= \frac{1}{30}\left(400x - \frac{.25x^3}{3}\right)\Big|_{10}^{40}$$
$$= \frac{1}{30}\left[\left(400(40) - \frac{.25(40)^3}{3}\right) - \left(400(10) - \frac{.25(10)^3}{3}\right)\right]$$
$$= \frac{1}{30}(10,666.7 - 3916.67)$$
$$= 225$$

(c) $\overline{T} = \frac{1}{40-0}\int_0^{40}(400-.25x^2)\,dx$

$$= \frac{1}{40}\left(400x - \frac{.25x^3}{3}\right)\Big|_0^{40}$$
$$= \frac{1}{40}\left[(400)(40) - \frac{(.25)(40)^3}{3}\right]$$
$$= \frac{1}{40}[10{,}666.71]$$
$$= 266.7$$

Extended Application: Estimating Learning Curves in Manufacturing with Integrals

1. $C(280) \approx C(1)\cdot 280^{-.152} \approx 284{,}000 \cdot 280^{-.152}$
$\approx \$121{,}000$

2. No; using the Change of Base formula for logarithms, we have

$$b = \frac{\ln r}{\ln 2} = \log_2 r = \frac{\log r}{\log 2}.$$

3. If $C(x) = ax^b$ is a solution to the function equation $C(2n) = r\cdot C(n)$, then

$$a(2n)^b = r\cdot an^b$$
$$a\cdot 2^b\cdot n^b = a\cdot r\cdot n^b$$
$$2^b = r,\ \ a\neq 0, n\neq 0$$
$$b\ln 2 = \ln r$$
$$b = \frac{\ln r}{\ln 2}$$

Thus, choose $b = \frac{\ln r}{\ln 2}$. The only condition on a is that it be nonzero. Thus, choose $a = 1$.

4. (a) The sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ is the area of the region comprised of the four rectangles in the picture. The integral $\int_1^5 \frac{1}{x}dx$ is the area of the part of the region lying below the graph of $y = \frac{1}{x}$. The graph at the right shows the parts of the region above the graph of $y = \frac{1}{x}$ stacked vertically inside a rectangle of height 1. The lowest point in the shaded region is at a height $\frac{1}{5}$; thus, the height of the shaded region is $1 - \frac{1}{5}$. Since the width of the shaded region is 1 at its widest points, the shaded regions lies in a rectangle of area $1 - \frac{1}{5}$. By observation, we see that slightly more than half of that rectangle is shaded, so $\frac{1-\frac{1}{5}}{2}$ is a slight underestimate of the area of the shaded region.
Thus, $\int_1^5 \frac{1}{x}dx + \frac{1-\frac{1}{5}}{2}$ is a slight underestimate of the area of the rectangles, which is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$.

(b) $\int_1^5 \frac{1}{x}\,dx + \frac{1-\frac{1}{5}}{2} = \ln x\ \Big|_1^5 + \frac{2}{5}$

$$= \ln 5 + \frac{2}{5} \approx 2.009$$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.083$

The percent error is approximately

$$\frac{2.009 - 2.083}{2.083} \approx -.036 \approx -3.6\%.$$

Chapter 9

MULTIVARIABLE CALCULUS

9.1 Functions of Several Variables

2. $g(x, y) = -x^2 - 4xy + y^3$

(a) $$g(-2, 4) = -(-2)^2 - 4(-2)(4) + (4)^3 \\ = 92$$

(b) $$g(-1, -2) = -(-1)^2 - 4(-1)(-2) + (-2)^3 \\ = -17$$

(c) $$g(-2, 3) = -(-2)^2 - 4(-2)(3) + (3)^3 \\ = 47$$

(d) $$g(5, 1) = -(5)^2 - 4(5)(1) + (1)^3 \\ = -44$$

4. $f(x, y) = \dfrac{\sqrt{9x + 5y}}{\log x}$

(a) $$f(10, 2) = \frac{\sqrt{9(10) + 5(2)}}{\log 10} = \frac{\sqrt{100}}{1} = 10$$

(b) $$f(100, 1) = \frac{\sqrt{9(100) + 5(1)}}{\log 100} = \frac{\sqrt{905}}{2}$$

(c) $$f(1000, 0) = \frac{\sqrt{9(1000) + 5(0)}}{\log 1000} = \frac{\sqrt{9000}}{3} = 10\sqrt{10}$$

(d) $$f\left(\frac{1}{10}, 5\right) = \frac{\sqrt{9\left(\frac{1}{10}\right) + 5(5)}}{\log \frac{1}{10}} = \frac{\sqrt{25.9}}{-1} = -\sqrt{25.9}$$

6. $x + y + z = 12$

To find x-intercept, let $y = 0,\ z = 0.$

$$x + 0 + 0 = 12 \\ x = 12$$

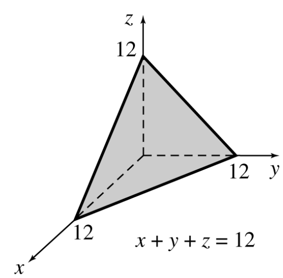
To find y-intercept, let $x = 0,\ z = 0.$

$$y = 12$$

To find z-intercept, let $x = 0,\ y = 0.$

$$z = 12$$

Sketch the portion of the plane in the first octant that contains these intercepts.



8. $4x + 2y + 3z = 24$

x-intercept: $y = 0,\ z = 0$

$$4x = 24 \\ x = 6$$

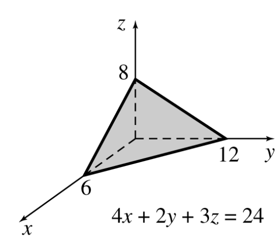
y-intercept: $x = 0,\ z = 0$

$$2y = 24 \\ y = 12$$

z-intercept: $x = 0,\ y = 0$

$$3z = 24 \\ z = 8$$

Sketch the portion of the plane in the first octant that contains these intercepts.



10. $y + z = 5$

x-intercept: $y = 0,\ z = 0$

$$0 = 5$$

Impossible, so no x-intercept

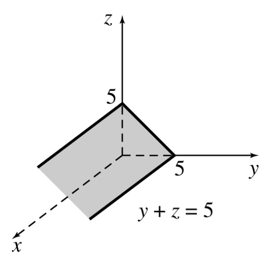
y-intercept: $x = 0,\ z = 0$

$$y = 5$$

z-intercept: $x = 0,\ y = 0$

$$z = 5$$

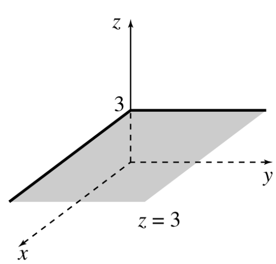
Sketch the portion of the plane in the first octant that contains these intercepts and its parallel to the x-axis.



12. $z = 3$

No x-intercept, no y-intercept

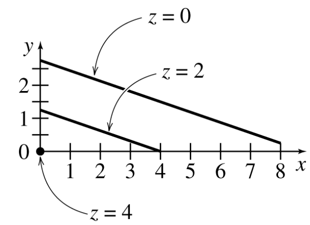
Sketch the portion of the plane in the first octant that passes through $(0, 0, 3)$ parallel to the xy-plane.



14. $x + 3y + 2z = 8$

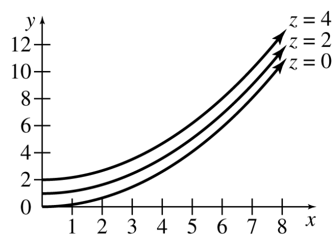
If $z = 0$, we have $x + 3y = 8$. The level curve is a line in the xy-plane with an x-intercept of 8 and a y-intercept of $\frac{8}{3}$.
If $z = 2$, we have $x + 3y = 4$. The level curve is a line in the plane $z = 2$ passing through the points $(4, 0, 2)$ and $\left(0, \frac{4}{3}, 2\right)$.
If $z = 4$, we have $x + 3y = 0$. The level curve is a line in the plane $z = 4$ passing through the point $(0, 0, 4)$ on the z-axis.

Sketch segments of these lines in the first octant.



16. $2y - \dfrac{x^2}{3} = z$

If $z = 0$, we have $y = \frac{1}{6}x^2$. The level curve is a parabola in the xy-plane with vertex at the origin.
If $z = 2$, we have $y = \frac{1}{6}x^2 + 1$. The level curve is a parabola in the plane $z = 2$ with vertex at the point $(0, 1, 2)$.
If $z = 4$, we have $y = \frac{1}{6}x^2 + 2$. The level curve is a parabola in the plane $z = 4$ with vertex at the point $(0, 2, 4)$.
Sketch portions of these curves in the first octant.



18. x-intercept: $y = 0, z = 0 \quad (a \neq 0)$

$$ax = d$$
$$x = \frac{d}{a}$$

y-intercept: $x = 0, z = 0 \quad (b \neq 0)$

$$by = d$$
$$y = \frac{d}{b}$$

z-intercept: $x = 0, y = 0 \quad (c \neq 0)$

$$cz = d$$
$$z = \frac{d}{c}$$

If $d \neq 0$, then for the plane to have a portion in the first octant, one of $\frac{d}{a}$, $\frac{d}{b}$, or $\frac{d}{c}$ must be positive, so at least one of a, b, or c must have the same sign as d.

If $d = 0$, then the trace in the xy-plane is the line $x = -\frac{b}{a}y$, the trace in the yz-plane is $y = -\frac{c}{b}z$, and the trace in the xz-plane is $x = -\frac{c}{a}z$. $-\frac{c}{a}$, $-\frac{c}{b}$, or $-\frac{b}{a}$ must be positive, so a, b, and c cannot all have the same sign.

22. $z^2 - y^2 - x^2 = 1$

If $z = 0$,

$$-(y^2 + x^2) = 1.$$

This is impossible so there is no xy-trace.
If $x = 0$,

$$z^2 - y^2 = 1.$$

yz-trace: hyperbola
If $y = 0$,

$$z^2 - x^2 = 1.$$

xz-trace: hyperbola
The equation is represented by a hyperboloid of two sheets, as shown in (f).

24. $z = y^2 - x^2$

If $z = 0$,

$$x^2 = y^2$$
$$x = \pm y.$$

xy-trace: two intersecting lines.
If $x = 0$,

$$z = y^2.$$

yz-trace: parabola, opening upward
If $y = 0$,

$$z = -x^2.$$

xz-trace: parabola, opening downward
Hyperbolic paraboloid

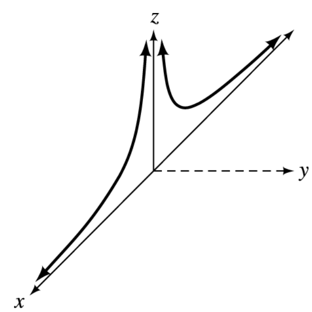
Both (a) and (e) are hyperbolic paraboloids, but only (a) has traces described by this function.

26. $z = 5(x^2 + y^2)^{-1/2} = \dfrac{5}{\sqrt{x^2 + y^2}}$

Note that $z > 0$ for all values of x and y.
xz-trace: $y = 0$

$$z = \frac{5}{\sqrt{x^2}} = \frac{5}{|x|}$$

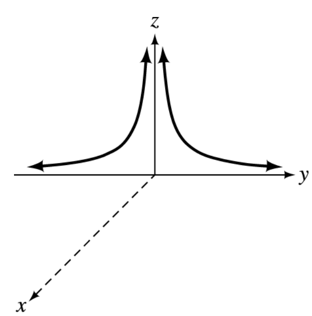
This gives one branch of the hyperbola $xz = 5$, where $z > 0$ and $x > 0$, and one branch of the hyperbola $xz = -5$, where $z > 0$ and $x < 0$.



yz-trace: $x = 0$

$$z = \frac{5}{\sqrt{y^2}} = \frac{5}{|y|}$$

This gives one branch of the hyperbola $yz = 5$, where $z > 0$ and $y > 0$, and one branch of the hyperbola $yz = -5$, where $z > 0$ and $y < 0$.



Level curves on planes $z = k$, where $k > 0$, are

$$k = \frac{5}{\sqrt{x^2 + y^2}}$$
$$x^2 + y^2 = \frac{25}{k^2}.$$

These are circles with centers $(0, 0, k)$ and radii $\frac{5}{k}$. The graph of $z = 5(x^2 + y^2)^{-1/2}$ is (d).

28. $f(x, y) = 7x^3 + 8y^2$

(a) $\dfrac{f(x+h, y) - f(x, y)}{h}$

$$= \frac{[7(x+h)^3 + 8y^2] - (7x^3 + 8y^2)}{h}$$
$$= \frac{7x^3 + 21x^2h + 21xh^2 + 7h^3 - 7x^3}{h}$$
$$= 21x^2 + 21xh + 7h^2$$

(b) $\dfrac{f(x,y+h)-f(x,y)}{h}$

$$= \frac{[7x^3+8(y+h)^2]-(7x^3+8y^2)}{h}$$
$$= \frac{8y^2+16yh+8h^2-8y^2}{h}$$
$$= 16y+8h$$

(c) $\displaystyle\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$

$$= \lim_{h\to 0}(21x^2+21xh+7h^2)$$
$$= 21x^2+21x(0)+7(0)^2$$
$$= 21x^2$$

(d) $\displaystyle\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$

$$= \lim_{h\to 0}(16y+8h)$$
$$= 16y+8(0)$$
$$= 16y$$

30. A linear regression on the $y = 0$ column gives $f_0(x) = 4.019 + 1.989x$.
The $y = 1$ column gives

$$f_1(x) = 7.045 + 1.98x.$$

The $y = 2$ column gives

$$f_2(x) = 9.978 + 2.013x.$$

The $y = 3$ column gives

$$f_3(x) = 12.989 + 2.019x.$$

Use the nearest integer values.

$$\begin{aligned} f_0(x) &= f(x,0) = 4+2x \\ &= a+bx+c(0) \\ &= a+bx \end{aligned}$$

$$\begin{aligned} f_1(x) &= f(x,1) = 7+2x \\ &= a+bx+c(1) \\ &= a+bx+c \end{aligned}$$

$$\begin{aligned} f_2(x) &= f(x,2) = 10+2x \\ &= a+bx+c(2) \\ &= a+bx+2c \end{aligned}$$

$$\begin{aligned} f_3(x) &= f(x,3) = 13+2x \\ &= a+bx+c(3) \\ &= a+bx+3c \end{aligned}$$

Thus, $b = 2$ and we have

$$\begin{aligned} 4 &= a \\ 7 &= a+c \\ 10 &= a+2c \\ 13 &= a+3c \end{aligned}$$

so $a = 4$ and $c = 3$.

$$f(x,y) = 4+2x+3y$$

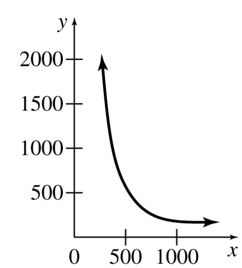
32. $M = f(25, .05, .28)$

$$= \frac{(1+.05)^{25}(1-.28)+.28}{[1+(1-.28)(.05)]^{25}}$$
$$\approx 1.12$$

Since $M > 1$, the IRA account grows faster.

34. $z = x^{.7}y^{.3}$ and $z = 500$, so $500 = x^{.7}y^{.3}$.

$$y^{.3} = \frac{500}{x^{.7}} \quad \text{or} \quad y^{3/10} = \frac{500}{x^{7/10}}$$
$$y = \left(\frac{500}{x^{7/10}}\right)^{10/3} \approx \frac{9.9\cdot 10^8}{x^{7/3}} \approx \frac{10^9}{x^{7/3}}$$



36. $z = x^{.7}y^{.3}$

If x is doubled, z is multiplied by $2^{.7}$, or approximately 1.6.
If y is doubled, z is multiplied by $2^{.3}$, or approximately 1.2.
If both are doubled, z is multiplied by $2^{.7}\cdot 2^{.3} = 2^{1.0} = 2$. Thus, z is doubled.

38. $m = \dfrac{2.5(T-F)}{w^{.67}}$

(a) $m = \dfrac{2.5(38-6)}{(32)^{.67}} \approx 7.85$

(b) $m = \dfrac{2.5(40-20)}{(43)^{.67}} \approx 4.02$

40. $A = .202W^{.425}H^{.725}$

(a) $A = .202(72)^{.425}(1.78)^{.725} \approx 1.89 \text{ m}^2$

(b) $A = .202(65)^{.425}(1.40)^{.725} \approx 1.52 \text{ m}^2$

(c) $A = .202(70)^{.425}(1.60)^{.725} \approx 1.73 \text{ m}^2$

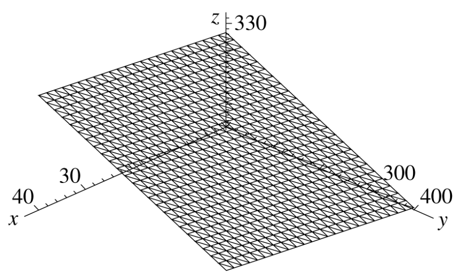
(d) Answer will vary for each person.

42. $N(R, C) = 329.32 + .0377R - .0171C$

(a) $N(141{,}319, 37{,}960)$
$= 329.32 + .0377(141{,}319)$
$- .0171(37{,}960)$
≈ 5008

Approximately 5008 deer were harvested in Tuscarawas County.

(b)



44. The girth is $2H + 2W$. Thus,

$$f(L, W, H) = L + 2H + 2W.$$

46. $f(H, D) = \sqrt{H^2 + D^2}$ with $D = 3.75$ in

(a) $\frac{H}{D} = \frac{3}{4}$

$H = \frac{3}{4}D$

$H = \frac{3}{4}(3.75)$

$H = 2.8125$
$f(2.8125, 3.75)$
$= \sqrt{(2.8125)^2 + (3.75)^2}$
≈ 4.69

The length of the ellipse is approximately 4.69 in, and its width is 3.75 inches.

(b) $\frac{H}{D} = \frac{2}{5}$

$H = \frac{2}{5}D$

$H = \frac{2}{5}(3.75)$

$H = 1.5$
$f(1.5, 3.75)$
$= \sqrt{(1.5)^2 + (3.75)^2}$
≈ 4.04

The length of the ellipse is approximately 4.04 inches, and its width is 3.75 inches.

9.2 Partial Derivatives

2. $z = g(x, y) = 5x + 9x^2y + y^2$

(a) $\frac{\partial g}{\partial x} = 5 + 18xy$

(b) $\frac{\partial g}{\partial y} = 9x^2 + 2y$

(c) $\frac{\partial z}{\partial y}(-3, 0) = \frac{\partial g}{\partial y}(-3, 0) = 9(-3)^2 + 2(0)$
$= 81$

(d) $g_x(2, 1) = \frac{\partial g}{\partial x}(2, 1) = 5 + 18(2)(1) = 41$

4. $f(x, y) = 4x^2y - 9y^2$

$f_x(x, y) = 8xy$
$f_y(x, y) = 4x^2 - 18y$
$f_x(2, -1) = 8(2)(-1) = -16$
$f_y(-4, 3) = 4(-4)^2 - 18(3) = 10$

6. $f(x, y) = -2x^2y^4$

$f_x(x, y) = -4xy^4$
$f_y(x, y) = -8x^2y^3$
$f_x(2, -1) = -4(2)(-1)^4 = -8$
$f_y(-4, 3) = -8(-4)^2(3)^3 = -3456$

8. $f(x, y) = 3e^{2x+y}$
$f_x(x, y) = 6e^{2x+y}$
$f_y(x, y) = 3e^{2x+y}$
$f_x(2, -1) = 6e^{2(2)-1} = 6e^3$
$f_y(-4, 3) = 3e^{2(-4)+3} = 3e^{-5}$

10. $f(x, y) = 8e^{7x-y}$
$f_x(x, y) = 56e^{7x-y}$
$f_y(x, y) = -8e^{7x-y}$
$f_x(2, -1) = 56e^{7(2)-(-1)} = 56e^{15}$
$f_y(-4, 3) = -8e^{7(-4)-3} = -8e^{-31}$

12. $f(x,y) = \dfrac{3x^2y^3}{x^2+y^2}$

$$f_x(x,y) = \frac{(x^2+y^2)\cdot 6xy^3 - 3x^2y^3\cdot 2x}{(x^2+y^2)^2} = \frac{6xy^5}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{(x^2+y^2)\cdot 9x^2y^2 - 3x^2y^3\cdot 2y}{(x^2+y^2)^2} = \frac{9x^4y^2+3x^2y^4}{(x^2+y^2)^2}$$

$$f_x(2,-1) = \frac{6(2)(-1)^5}{[(2)^2+(-1)^2]^2} = -\frac{12}{25}$$

$$f_y(-4,3) = \frac{9(-4)^4(3)^2+3(-4)^2(3)^4}{[(-4)^2+(3)^2]^2} = \frac{24{,}624}{625}$$

14. $f(x,y) = \ln\left|2x^5 - xy^4\right|$

$$f_x(x,y) = \frac{1}{2x^5-xy^4}\cdot(10x^4-y^4) = \frac{10x^4-y^4}{2x^5-xy^4}$$

$$f_y(x,y) = \frac{1}{2x^5-xy^4}\cdot(-4xy^3) = \frac{-4xy^3}{2x^5-xy^4} = \frac{4y^3}{y^4-2x^4}$$

$$f_x(2,-1) = \frac{10(2)^4-(-1)^4}{2(2^5)-2(-1)^4} = \frac{159}{62}$$

$$f_y(-4,3) = \frac{4(3)^3}{(3)^4-2(-4)^4} = -\frac{108}{431}$$

16. $f(x,y) = y^2e^{x+3y}$

$$\begin{aligned} f_x(x,y) &= y^2e^{x+3y} \\ f_y(x,y) &= y^2\cdot 3e^{x+3y} + e^{x+3y}\cdot 2y \\ &= ye^{x+3y}(3y+2) \\ f_x(2,-1) &= (-1)^2e^{2+3(-1)} = e^{-1} \\ f_y(-4,3) &= 3e^{-4+3(3)}[3(3)+2] \\ &= 33e^5 \end{aligned}$$

18. $f(x,y) = (7x^2+18xy^2+y^3)^{1/3}$

$$f_x(x,y) = \frac{14x+18y^2}{3(7x^2+18xy^2+y^3)^{2/3}}$$

$$f_y(x,y) = \frac{36xy+3y^2}{3(7x^2+18xy^2+y^3)^{2/3}}$$

$$f_x(2,-1) = \frac{14(2)+18(-1)^2}{3(7(2)^2+18(2)(-1)^2+(-1)^3)^{2/3}} = \frac{46}{3(63)^{2/3}}$$

$$f_y(-4,3) = \frac{36(-4)(3)+3(3)^2}{3(7(-4)^2+18(-4)(3)^2+(3)^3)^{2/3}} = -\frac{135}{(509)^{2/3}}$$

20. $f(x,y) = (7e^{x+2y}+4)(e^{x^2}+y^2+2)$

$$\begin{aligned} f_x(x,y) &= 7e^{x+2y}(e^{x^2}+y^2+2) \\ &\quad + 2xe^{x^2}(7e^{x+2y}+4) \\ f_y(x,y) &= 14e^{x+2y}(e^{x^2}+y^2+2) \\ &\quad + 2y(7e^{x+2y}+4) \\ f_x(2,-1) &= 7e^{2+2(-1)}(e^{2^2}+(-1)^2+2) \\ &\quad + 2(2)e^{2^2}(7e^{2+2(-1)}+4) \\ &= 7e^0(e^4+3)+4e^4(7e^0+4) \\ &= 51e^4+21 \\ f_y(-4,3) &= 14e^{-4+2(3)}(e^{(-4)^2}+3^2+2) \\ &\quad + 2(3)(7e^{-4+2(3)}+4) \\ &= 14e^2(e^{16}+11)+6(7e^2+4) \end{aligned}$$

22. $g(x,y) = 5xy^4+8x^3-3y$

$$\begin{aligned} g_x(x,y) &= 5y^4+24x^2 \\ g_{xx}(x,y) &= 48x \\ g_{xy}(x,y) &= 20y^3 \\ g_y(x,y) &= 20xy^3-3 \\ g_{yy}(x,y) &= 60xy^2 \\ g_{yx}(x,y) &= 20y^3 \end{aligned}$$

24. $h(x,y) = 30y+5x^2y+12xy^2$

$$\begin{aligned} h_x(x,y) &= 10xy+12y^2 \\ h_{xx}(x,y) &= 10y \\ h_{xy}(x,y) &= 10x+24y \\ h_y(x,y) &= 30+5x^2+24xy \\ h_{yy}(x,y) &= 24x \\ h_{yx}(x,y) &= 10x+24y \end{aligned}$$

26. $k(x,y) = \dfrac{-5y}{x+2y} = -5y(x+2y)^{-1}$

$$k_x(x,y) = 5y(x+2y)^{-2} = \frac{5y}{(x+2y)^2}$$

$$k_y(x,y) = \frac{(x+2y)(-5) - (-5y)\cdot 2}{(x+2y)^2} = \frac{-5x}{(x+2y)^2} = -5x(x+2y)^{-2}$$

$$k_{xx}(x,y) = -10y(x+2y)^{-3} = \frac{-10y}{(x+2y)^3}$$

$$k_{yy}(x,y) = 10x(x+2y)^{-3}\cdot 2 = \frac{20x}{(x+2y)^3}$$

$$k_{xy}(x,y) = \frac{(x+2y)^2\cdot 5 - 5y\cdot 2(x+2y)\cdot 2}{(x+2y)^4} = \frac{5x-10y}{(x+2y)^3}$$

$$k_{yx}(x,y) = \frac{(x+2y)^2\cdot(-5) - (-5x)\cdot 2(x+2y)}{(x+2y)^4} = \frac{5x-10y}{(x+2y)^3}$$

28. $z = -3ye^x$

$$\begin{aligned} z_x(x,y) &= -3ye^x \\ z_{xx}(x,y) &= -3ye^x \\ z_{xy}(x,y) &= -3e^x \\ z_y(x,y) &= -3e^x \\ z_{yy}(x,y) &= 0 \\ z_{yx}(x,y) &= -3e^x \end{aligned}$$

30. $k = \ln|5x-7y|$

$$k_x(x,y) = \frac{5}{5x-7y} = 5(5x-7y)^{-1}$$

$$k_y(x,y) = \frac{-7}{5x-7y} = -7(5x-7y)^{-1}$$

$$k_{xx}(x,y) = -5(5x-7)^{-2}\cdot 5 = -25(5x-7y)^{-2} \quad\text{or}\quad \frac{-25}{(5x-7y)^2}$$

$$k_{yy}(x,y) = 7(5x-7y)^{-2}\cdot(-7) = -49(5x-7y)^{-2} \quad\text{or}\quad \frac{-49}{(5x-7y)^2}$$

$$k_{xy}(x,y) = -5(5x-7)^{-2}\cdot(-7) = 35(5x-7y)^{-2} \quad\text{or}\quad \frac{35}{(5x-7y)^2}$$

$$k_{yx}(x,y) = 7(5x-7y)^{-2}\cdot(5) = 35(5x-7y)^{-2} \quad\text{or}\quad \frac{35}{(5x-7y)^2}$$

32.

$$\begin{aligned} z &= (y+1)\ln\left|x^3y\right| \\ &= (y+1)(3\ln|x| + \ln|y|) \end{aligned}$$

$$z_x(x,y) = (y+1)\cdot\left(3\cdot\frac{1}{x}\right) = \frac{3(y+1)}{x} = 3x^{-1}(y+1)$$

$$z_y(x,y) = (y+1)\cdot\left(\frac{1}{y}\right) + (3\ln x + \ln y)\cdot 1 = \frac{y+1}{y} + 3\ln|x| + \ln|y|$$

$$z_{xx}(x,y) = -3x^{-2}(y+1) = \frac{-3(y+1)}{x^2}$$

$$z_{yy}(x,y) = \frac{y\cdot 1 - (y+1)\cdot 1}{y^2} + \frac{1}{y} = -\frac{1}{y^2} + \frac{1}{y}$$

$$z_{xy}(x,y) = 3x^{-1} = \frac{3}{x}$$

$$z_{yx}(x,y) = 3\cdot\frac{1}{x} = \frac{3}{x}$$

34. $f(x,y) = 50 + 4x - 5y + x^2 + y^2 + xy$
$f_x(x,y) = 4 + 2x + y,\ f_y(x,y) = -5 + 2y + x$

Solve the system

$$\begin{aligned} 4 + 2x + y &= 0 \\ -5 + x + 2y &= 0. \end{aligned}$$

Multiply the second equation by -2 and add.

$$\begin{array}{r} 4 + 2x + \ \ y = 0 \\ \underline{10 - 2x - 4y = 0} \\ 14 \qquad - 3y = 0 \\ y = \dfrac{14}{3} \end{array}$$

Substitute into the second equation to get $x = -\frac{13}{3}$. The solution is $x = -\frac{13}{3}, y = \frac{14}{3}$.

36. $f(x,y) = 2200 + 27x^3 + 72xy + 8y^2$
$f_x(x,y) = 81x^2 + 72y,\ f_y(x,y) = 72x + 16y$

Solve the system

$$\begin{aligned} 81x^2 + 72y &= 0 \quad (1) \\ 72x + 16y &= 0. \quad (2) \end{aligned}$$

Divide equation (1) by 9 and equation (2) by 8.

$$\begin{aligned} 9x^2 + 8y &= 0 \quad (3) \\ 9x + 2y &= 0 \quad (4) \end{aligned}$$

From equation (4), $y = -\frac{9}{2}x$.
Substitute into equation (3).

$$\begin{aligned} 9x^2 - 36x &= 0 \\ 9x(x-4) &= 0 \\ x = 0 \quad \text{or} \quad x &= 4 \end{aligned}$$

Since $y = -\frac{9}{2}x$,

if $x = 0,\ y = 0$;

if $x = 4,\ y = -\frac{9}{2}(4) = -18$.

The solutions are $x = 0, y = 0$, and $x = 4$, $y = -18$.

38. $f(x,y,z) = 3x^5 - x^2 + y^5$
$f_x(x,y,z) = 15x^4 - 2x;\ f_y(x,y,z) = 5y^4;$
$f_z(x,y,z) = 0; f_{yz}(x,y,z) = 0$

40. $$\begin{aligned} f(x,y,z) &= \frac{2x^2 + xy}{yz - 2} \\ &= \frac{1}{yz-2}(2x^2 + xy) \\ &= (2x^2 + xy)(yz-2)^{-1} \end{aligned}$$

$$\begin{aligned} f_x(x,y,z) &= \frac{1}{yz-2}(4x+y) \\ &= \frac{4x+y}{yz-2} \\ f_y(x,y,z) &= \frac{(yz-2)\cdot x - (2x^2+xy)\cdot z}{(yz-2)^2} \\ &= \frac{-2x - 2x^2 z}{(yz-2)^2} \\ f_z(x,y,z) &= -(2x^2+xy)(yz-2)^{-2}\cdot y \\ &= -\frac{(2x^2y + xy^2)}{(yz-2)^2} \end{aligned}$$

$f_{yz}(x,y,z)$

$$= \frac{(yz-2)^2(-2x^2) - (-2x - 2x^2 z)\cdot 2(yz-2)\cdot y}{(yz-2)^4}$$

$$= \frac{4x^2 + 4xy + 2x^2yz}{(yz-2)^3}$$

42. $f(x,y,z) = \ln\left|8xy + 5yz - x^3\right|$

$$\begin{aligned} f_x(x,y,z) &= \frac{1}{8xy+5yz-x^3}\cdot(8y - 3x^2) \\ &= \frac{8y-3x^2}{8xy+5yz-x^3} \\ f_y(x,y,z) &= \frac{1}{8xy+5yz-x^3}\cdot(8x+5z) \\ &= \frac{8x+5z}{8xy+5yz-x^3} \\ f_z(x,y,z) &= \frac{1}{8xy+5yz-x^3}\cdot 5y \\ &= \frac{5y}{8xy+5yz-x^3} \\ f_{yz}(x,y,z) &= \frac{(8xy+5yz-x^3)\cdot 5 - (8x+5z)\cdot 5y}{(8xy+5yz-x^3)^2} \\ &= \frac{-5x^3}{(8xy+5yz-x^3)^2} \end{aligned}$$

44. $f(x,y) = (x+y^2)^{2x+y}$

(a) $f_x(2,1) = \lim\limits_{h\to 0}\dfrac{f(2+h,1) - f(2,1)}{h}$

We will use a small value for h. Let $h = .00001$.

$f_x(2,1)$

$$\begin{aligned} &\approx \frac{f(2.00001,1) - f(2,1)}{.00001} \\ &\approx \frac{(2.00001+1^2)^{2(2.00001)+1} - (2+1^2)^{2(2)+1}}{.00001} \\ &\approx \frac{(3.00001)^{5.00002} - 3^5}{.00001} \\ &\approx 938.9 \end{aligned}$$

(b) $f_y(2,1) = \lim\limits_{h\to 0}\dfrac{f(2,1+h) - f(2,1)}{h}$

Again, let $h = .00001$.

$f_y(2,1)$

$$\begin{aligned} &\approx \frac{f(2,1.00001) - f(2,1)}{.00001} \\ &\approx \frac{[2+(1.00001)^2]^{2(2)+1.00001} - (2+1^2)^{2(2)+1}}{.00001} \\ &\approx \frac{[2+(1.00001)^2]^{5.00001} - 3^5}{.00001} \\ &\approx 1077 \end{aligned}$$

46. $R(x, y) = 5x^2 + 9y^2 - 4xy$

(a) $R(9, 5) = 5(9)^2 + 9(5)^2 - 4(9)(5)$
$= 450$

$$R_x(x, y) = 10x - 4y$$
$$R_x(9, 5) = 10(9) - 4(5) = 70$$

R would increase by \$70.

(b) $R_y(x, y) = 18y - 4x$
$R_y(9, 5) = 18(5) - 4(9) = 54$

R would increase by \$54.

48. $P(x, y) = 100\sqrt{x^2 + y^2}$
$= 100(x^2 + y^2)^{1/2}$,

where x is labor, y is capital.

(a) $\dfrac{\partial P}{\partial x} = 50(x^2 + y^2)^{-1/2} \cdot 2x = \dfrac{100x}{\sqrt{x^2 + y^2}}$

$$\frac{\partial P}{\partial x}(4, 3) = \frac{100(4)}{\sqrt{(4)^2 + (3)^2}} = 80 \text{ units}$$

(b) $\dfrac{\partial P}{\partial y} = 50(x^2 + y^2)^{-1/2} \cdot 2y = \dfrac{100y}{\sqrt{x^2 + y^2}}$

$$\frac{\partial P}{\partial x}(4, 3) = \frac{100(3)}{\sqrt{(4)^2 + (3)^2}} = 60 \text{ units}$$

50. $z = x^{.7}y^{.3}$, where x is labor, y is capital.

Marginal productivity of labor is

$$\frac{\partial z}{\partial x} = .7x^{-.3}y^{.3}.$$

Marginal productivity of capital is

$$\frac{\partial z}{\partial y} = .3x^{.7}y^{-.7}.$$

52. $f(x, y) = 3x^{1/3}y^{2/3}$, where x is labor, y is capital.

(a) $f_x(x, y) = x^{-2/3}y^{2/3} = \dfrac{y^{2/3}}{x^{2/3}} = \left(\dfrac{y}{x}\right)^{2/3}$

$$f_x(64, 125) = \left(\frac{125}{64}\right)^{2/3} = \frac{25}{16} = 1.5625,$$

which is the approximate change in production (in thousands of units) for a 1 unit change in labor.

$$f_y(x, y) = 2x^{1/3}y^{-1/3}$$
$$= 2\frac{x^{1/3}}{y^{1/3}} = 2\left(\frac{x}{y}\right)^{1/3}$$
$$f_y(64, 125) = 2\left(\frac{64}{125}\right)^{1/3} = \frac{8}{5} = 1.6,$$

which is the approximate change in production (in thousands of units) for a 1 unit change in capital.

(b) Increasing to 65 units of labor would result in an increase of approximately

$$\frac{25}{16}(1000) \approx 1563 \text{ batteries.}$$

(c) An increase of approximately

$$\frac{16}{5}(1000) = 3200 \text{ batteries}$$

would be the effect of increasing capital to 126 while holding labor at 64. Increasing capital is the better option.

54. $m(T, F, w) = 2.5(T - F)w^{-.67}$
$= 2.5Tw^{-.67} - 2.5Fw^{-.67}$

(a) Increasing T from 38°C to 39°C while F remains at 12°C and w remains at 30 kg results in a change in oxygen consumption of

$$m_T(38, 12, 30) = 2.5w^{-.67}$$
$$= 2.5(30)^{-.67}$$
$$\approx .256.$$

(b) Increasing F from 14°C to 15°C while T remains at 36°C and w remains at 25 kg results in a change of oxygen consumption of

$$m_F(36, 14, 25) = -2.5w^{-.67}$$
$$= -2.5(25^{-.67})$$
$$\approx -.289.$$

56. $C(a, b, v) = \dfrac{b}{a - v} = b(a - v)^{-1}$

(a) $C(160, 200, 125) = \dfrac{200}{160 - 125}$
$= \dfrac{200}{35}$
≈ 5.71

(b) $C_a(a, b, v) = -b(a - v)^{-2} \cdot 1$
$= -\dfrac{b}{(a - v)^2}$

$$C_a(160, 200, 125) = -\frac{200}{(160 - 125)^2}$$
$$= -\frac{200}{35^2}$$
$$\approx -.163$$

(c) $C_b(a,b,v) = (a-v)^{-1}$

$$C_b(160,200,125) = (160-125)^{-1} = \frac{1}{35} \approx .0286$$

(d) $C_v(a,b,v) = -b(a-v)^{-2}\cdot(-1) = \dfrac{b}{(a-v)^2}$

$$C_v(160,200,125) = \frac{200}{(160-125)^2} = \frac{200}{35^2} \approx .163$$

(e) Changing a by 1 unit produces the greatest decrease in the liters of blood pumped, while changing v by 1 unit produces the same amount of increase in the liters of blood pumped.

58. $B(w,h) = \dfrac{703w}{h^2}$

(a) $B(220,74) = \dfrac{703(220)}{74^2} \approx 28$

(b) $\dfrac{\partial B}{\partial w} = \dfrac{703}{h^2}$

$$\frac{\partial B}{\partial h} = \frac{-2(703)w}{h^3} = -\frac{1406w}{h^3}$$

Since $\frac{\partial B}{\partial w}$ is positive, as w increases, so does B. Since $\frac{\partial B}{\partial h}$ is negative, B decreases as h increases.

(c) If w_m and h_m represent a person's weight and height, in kilograms and meters, respectively, then $w_m = .4555w$, where w is weight in pounds, and $h_m = .0254h$, where h is height in inches. To transform the formula $B = \frac{703w}{h^2}$ to handle metric inputs, make the substitutions $w = \frac{w_m}{.4555}$ and $h = \frac{h_m}{.0254}$.

$$B_m = \frac{703\frac{w_m}{.4555}}{\left(\frac{h_m}{.0254}\right)^2} = \frac{703(.0254)^2}{.4555}\cdot\frac{w_m}{h_m^2}.$$

Since $\dfrac{703(.0254)^2}{.4555} \approx .996 \approx 1$, use $B_m = \dfrac{w_m}{h_m^2}$.

60. $p(l,t) = 1 + \dfrac{l}{33}(1-2^{-t/5})$

(a)
$$p(33,10) = 1 + \frac{33}{33}(1-2^{-10/5}) = 1 + (1)(1-2^{-2}) = 1 + \frac{3}{4} = 1.75$$

The pressure at 33 feet for a 10-minute dive is 1.75 atmospheres.

(b)
$$p_l(l,t) = \frac{1}{33}(1-2^{-t/5})$$

$$p_t(l,t) = \frac{l}{33}(-(\ln 2)\left(-\frac{1}{5}2^{-t/5}\right)) = \frac{l}{165}(\ln 2)(2^{-t/5})$$

$$p_l(33,10) = \frac{1}{33}(1-2^{-10/5}) = \frac{1}{33}\left(\frac{3}{4}\right) \approx .023 \text{ atm/ft}$$

Increasing the depth of the dive by 1 foot (to 34 feet), while keeping the dive length at 10 minutes, increases the pressure by approximately .023 atmospheres.

$$p_t(33,10) = \frac{33}{165}(\ln 2)(2^{-10/5}) = \frac{1}{20}\ln 2 \approx .035 \text{ atm/min}$$

Increasing the dive time by 1 minute (to 11 minutes), while keeping the depth of the dive to 33 feet increases the pressure by approximately .035 atmospheres.

(c) Solve $p(66,t) = 2.15$

$$1 + \frac{66}{33}(1-2^{-t/5}) = 2.15$$
$$2(1-2^{-t/5}) = 1.15$$
$$1-2^{-t/5} = .575$$
$$2^{-t/5} = .425$$
$$-\frac{t}{5}\ln 2 = \ln .425$$
$$t = -\frac{5\ln .425}{\ln 2}$$
$$t \approx 6.17$$

The maximum dive length is 6.17 minutes.

62. **(a)** $f(90, 30) \approx 90$

(b) $f(90, 75) \approx 109$

(c) $f(80, 75) \approx 86$

(d) $$f_T(90, 30) \approx \frac{f(95, 30) - f(90, 30)}{5} \approx \frac{95 - 90}{5} = 1$$

(e) $$f_H(90, 30) \approx \frac{f(90, 35) - f(90, 30)}{5} \approx \frac{92 - 90}{5} = .4$$

(f) $$f_T(90, 75) \approx \frac{f(95, 75) - f(90, 75)}{5} \approx \frac{130 - 109}{5} = 4.2$$

(g) $$f_H(90, 75) \approx \frac{f(90, 80) - f(90, 75)}{5} \approx \frac{114 - 109}{5} = 1$$

64. $p = f(s, n, a) = .003a + .1(sn)^{1/2}$

(a) $$\begin{aligned} f(8, 6, 450) &= .003(450) + .1[(8)(6)]^{1/2} \\ &= 1.35 + .1(48)^{1/2} \\ &= 2.0428 \\ p &\approx 2.04\% \end{aligned}$$

(b) $$\begin{aligned} f(3, 3, 320) &= .003(320) + .1[(3)(3)]^{1/2} \\ &= .96 + .1(9)^{1/2} = 1.26 \\ p &= 1.26\% \end{aligned}$$

(c) $$f_n(s, n, a) = .003(0) + .1\left[\frac{1}{2}(sn)^{-1/2}(s)\right]$$

$$\begin{aligned} f_n(3, 3, 320) &= .1\left(\frac{1}{2}\right)[(3)(3)]^{-1/2}(3) \\ &= (.1)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)(3) = .05 \\ p_n &= .05\% \end{aligned}$$

$f_a(s, n, a) = .003$ for all ordered triples (s, n, a). Therefore, $p_a = .003\%$. $p_n = .05\%$ is the rate of change of the probability for an additional semester of high school math.
$p_a = .003\%$ is the rate of change of the probability per unit of change in an SAT score.

66. $$w(x, y) = \frac{x + y}{1 + \frac{xy}{c^2}}$$

(a) $w(50{,}000,\ 150{,}000)$

$$= \frac{50{,}000 + 150{,}000}{1 + \frac{(50{,}000)(150{,}000)}{(186{,}282)^2}} \approx 164{,}456$$

The rocket is traveling 164,456 m/sec relative to the stationary observer.

(b) $$\begin{aligned} w_x &= \frac{\left(1 + \frac{xy}{c^2}\right) - \frac{y}{c^2}(x + y)}{\left(1 + \frac{xy}{c^2}\right)^2} \\ &= \frac{1 + \frac{xy}{c^2} - \frac{xy}{c^2} - \frac{y^2}{c^2}}{\left(1 + \frac{xy}{c^2}\right)^2} \\ &= \frac{1 - \left(\frac{y}{c}\right)^2}{\left(1 + \frac{xy}{c^2}\right)^2} \end{aligned}$$

$w_x(50{,}000,\ 150{,}000)$

$$= \frac{1 - \left(\frac{150{,}000}{186{,}282}\right)^2}{\left(1 + \frac{(50{,}000)(150{,}000)}{(186{,}282)^2}\right)^2} \approx .238$$

The instantaneous rate of change is .238 m/sec per m/sec.

(c) $$w(c, c) = \frac{c + c}{1 + \frac{(c)(c)}{c^2}} = \frac{2c}{2} = c$$

The speed is the speed of light, c.

9.3 Maxima and Minima

2. $f(x, y) = 4xy + 8x - 9y$

$f_x(x, y) = 4y + 8,\ f_y(x, y) = 4x - 9,\ f_x(x, y) = 0$ and $f_y(x, y) = 0$ when

$$\begin{aligned} 4y + 8 &= 0 \\ y &= -2 \\ \text{and} \quad 4x - 9 &= 0 \\ x &= \frac{9}{4}. \end{aligned}$$

$f_{xx}(x, y) = 0,\ f_{yy}(x, y) = 0,\ f_{xy}(x, y) = 4$

Since

$$\begin{aligned} D &= f_{xx}\left(\frac{9}{4}, -2\right) \cdot f_{yy}\left(\frac{9}{4}, -2\right) - f_{xy}\left(\frac{9}{4}, -2\right) \\ &= 0 \cdot 0 - 4^2 = -16 < 0, \end{aligned}$$

$\left(\frac{9}{4}, -2\right)$ is a saddle point.

4. $f(x,y) = x^2 + xy + y^2 - 6x - 3$

$f_x(x,y) = 2x + y - 6,\ f_y(x,y) = x + 2y$

$$\begin{aligned} 2x + y - 6 &= 0 \\ x + 2y \quad &= 0 \end{aligned}$$

$$\begin{aligned} 2x + y - 6 &= 0 \\ -2x - 4y \quad &= 0 \\ \hline -3y - 6 &= 0 \\ y &= -2 \end{aligned}$$

$$\begin{aligned} x + 2(-2) &= 0 \\ x &= 4 \end{aligned}$$

$f_{xx}(x,y) = 2,\ f_{yy}(x,y) = 2,\ f_{xy}(x,y) = 1$

$D = 2 \cdot 2 - 1^2 = 3 > 0$ and $f_{xx}(x,y) > 0$

Relative minimum at $(4, -2)$

6. $f(x,y) = x^2 + xy + y^2 + 3x - 3y$

$f_x(x,y) = 2x + y + 3,\ f_y(x,y) = x + 2y - 3$

$$\begin{aligned} 2x + y + 3 &= 0 \\ x + 2y - 3 &= 0 \end{aligned}$$

$$\begin{aligned} 2x + y + 3 &= 0 \\ -2x - 4y + 6 &= 0 \\ \hline -3y + 9 &= 0 \\ y &= 3 \end{aligned}$$

$$\begin{aligned} x + 2(3) - 3 &= 0 \\ x &= -3 \end{aligned}$$

$f_{xx}(x,y) = 2,\ f_{yy}(x,y) = 2,\ f_{xy}(x,y) = 1$

$D = 2 \cdot 2 - 1^2 = 3 > 0$ and $f_{xx}(x,y) > 0$

Relative minimum at $(-3, 3)$

8. $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y - 4$

$f_x(x,y) = 5y - 14x + 3,\ f_y(x,y) = 5x - 2y - 6$

$$\begin{aligned} 5y - 14x + 3 &= 0 \\ -2y + 5x - 6 &= 0 \end{aligned}$$

$$\begin{aligned} 10y - 28x + 6 &= 0 \\ -10y + 25x - 30 &= 0 \\ \hline -3x - 24 &= 0 \\ x &= -8 \end{aligned}$$

$$\begin{aligned} -2y + 5(-8) - 6 &= 0 \\ -2y &= 46 \\ y &= -23 \end{aligned}$$

$f_{xx}(x,y) = -14,\ f_{yy}(x,y) = -2,\ f_{xy}(x,y) = 5$

$D = (-14)(-2) - 5^2 = 3 > 0$ and $f_{xx}(x,y) < 0$

Relative maximum at $(-8, -23)$

10. $f(x,y) = x^2 + xy + 3x + 2y - 6$

$f_x(x,y) = 2x + y + 3,\ f_y(x,y) = x + 2$

$$\begin{aligned} 2x + y + 3 &= 0 \\ x + 2 &= 0 \\ x &= -2 \end{aligned}$$

$$\begin{aligned} 2(-2) + y + 3 &= 0 \\ y &= 1 \end{aligned}$$

$f_{xx}(x,y) = 2,\ f_{yy}(x,y) = 0,\ f_{xy}(x,y) = 1$

$D = 2 \cdot 0 - 1^2 = -1 < 0$

Saddle point at $(-2, 1)$

12. $f(x,y) = x^2 + xy + y^2 - 3x - 5$

$f_x(x,y) = 2x + y - 3,\ f_y(x,y) = x + 2y$

$$\begin{aligned} 2x + y - 3 &= 0 \\ x + 2y \quad &= 0 \end{aligned}$$

$$\begin{aligned} 2x + y - 3 &= 0 \\ -2x - 4y \quad &= 0 \\ \hline -3y - 3 &= 0 \\ y &= -1 \end{aligned}$$

$$\begin{aligned} x + 2(-1) &= 0 \\ x &= 2 \end{aligned}$$

$f_{xx}(x,y) = 2,\ f_{yy}(x,y) = 2,\ f_{xy}(x,y) = 1$

$D = 2 \cdot 2 - 1^2 = 3 > 0$ and $f_{xx}(x,y) > 0$

Relative minimum at $(2, -1)$

14. $f(x,y) = 5x^3 + 2y^2 - 60xy - 3$

$f_x(x,y) = 15x^2 - 60y,\ f_y(x,y) = 4y - 60x$

$$\begin{aligned} 15x^2 - 60y &= 0 \quad (1) \\ 4y - 60x &= 0 \quad (2) \end{aligned}$$

Divide equation (1) by 15 and equation (2) by 4.

$$\begin{aligned} x^2 - 4y &= 0 \quad (3) \\ y - 15x &= 0 \quad (4) \end{aligned}$$

Solve equation (4) for y.

$$y = 15x$$

Substituting in equation (3), we have

$$\begin{aligned} x^2 - 4(15x) &= 0 \\ x^2 - 60x &= 0 \\ x = 0 \quad \text{or} \quad x &= 60 \\ y = 0 \quad \text{or} \quad y &= 900. \end{aligned}$$

$f_{xx}(x,y) = 30x,\ f_{yy}(x,y) = 4,\ f_{xy}(x,y) = -60$

At $(0,0)$,

$D = 0 \cdot 4 - (-60)^2 = -3600 < 0.$

Saddle point at $(0,0)$
At $(60,900)$,

$$\begin{aligned} D &= 1800 \cdot 4 - (-60)^2 \\ &= 3600 > 0 \text{ and } f_{xx}(x,y) > 0. \end{aligned}$$

Relative minimum at $(60,900)$

16. $f(x,y) = 3x^2 + 7y^3 - 42xy + 5$

$f_x(x,y) = 6x - 42y,\ f_y(x,y) = 21y^2 - 42x$

$$\begin{aligned} 6x - 42x &= 0 \quad (1) \\ 21y^2 - 42x &= 0 \quad (2) \end{aligned}$$

Divide equation (1) by 6 and equation (2) by 21.

$$\begin{aligned} x - 7y &= 0 \quad (3) \\ y^2 - 2x &= 0 \quad (4) \end{aligned}$$

Solve equation (3) for x.

$$x = 7y$$

Substituting in equation (4), we have

$$\begin{aligned} y^2 - 2(7y) &= 0 \\ y^2 - 14y &= 0 \\ y = 0 \quad &\text{or} \quad y = 14 \\ x = 0 \quad &\text{or} \quad x = 98. \end{aligned}$$

$f_{xx}(x,y) = 6,\ f_{yy}(x,y) = 42y,\ f_{xy}(x,y) = -42$

At $(0,0)$,

$D = 6 \cdot 0 - (-42)^2 = -1764 < 0.$

Saddle point at $(0,0)$
At $(98,14)$,

$$\begin{aligned} D &= 6 \cdot 588 - (-42)^2 \\ &= 1764 > 0 \text{ and } f_{xx}(x,y) > 0. \end{aligned}$$

Relative minimum at $(98,14)$

18. $f(x,y) = x^2 + e^y$

$f_x(x,y) = 2x,\ f_y(x,y) = e^y$

$$\begin{aligned} 2x &= 0 \\ e^y &= 0 \end{aligned}$$

The latter equation has no solutions. There are no extrema and no saddle points.

22. $z = \frac{3}{2}y - \frac{1}{2}y^3 - x^2y + \frac{1}{16}$

$\frac{\partial z}{\partial x} = -2xy, \quad \frac{\partial z}{\partial y} = \frac{3}{2} - \frac{3}{2}y^2 - x^2$

$$\begin{aligned} -2xy &= 0 \\ x = 0 \quad &\text{or} \quad y = 0 \end{aligned}$$

$$\frac{3}{2} - \frac{3}{2}y^2 - x^2 = 0$$

If $x = 0$,

$$\begin{aligned} \frac{3}{2} - \frac{3}{2}y^2 &= 0 \\ y &= \pm 1. \end{aligned}$$

If $y = 0$,

$$\begin{aligned} \frac{3}{2} - x^2 &= 0 \\ x &= \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}. \end{aligned}$$

$\frac{\partial^2 z}{\partial x^2} = -2y,\ \frac{\partial^2 z}{\partial y^2} = -3y,\ \frac{\partial^2 z}{\partial y \partial x} = -2x$

$D = (-2y)(-3y) - (-2x)^2 = 6y^2 - 4x^2$

At $(0,1)$,

$D = 6 > 0$ and $\frac{\partial^2 z}{\partial x^2} = -2 < 0.$

$z = \frac{3}{2} - \frac{1}{2} + \frac{1}{16} = 1\frac{1}{16}$

Relative maximum of $1\frac{1}{16}$ at $(0,1)$

At $(0,-1)$,

$D = 6 > 0$ and $\frac{\partial^2 z}{\partial x^2} = 2 > 0.$

$z = -\frac{3}{2} + \frac{1}{2} + \frac{1}{16} = -\frac{15}{16}$

Relative minimum of $-\frac{15}{16}$ at $(0,-1)$

At $\left(\pm\frac{\sqrt{6}}{2}, 0\right)$, $D = -6 < 0.$

Saddle points at $\left(\frac{\sqrt{6}}{2}, 0\right)$ and $\left(-\frac{\sqrt{6}}{2}, 0\right)$

The correct graph is (d).

24. $z = -2x^3 - 3y^4 + 6xy^2 + \frac{1}{16}$

$\frac{\partial z}{\partial x} = -6x^2 + 6y^2,\ \frac{\partial z}{\partial y} = -12y^3 + 12xy$

$$\begin{aligned} -6x^2 + \quad 6y^2 &= 0 \quad (1) \\ -12y^3 + 12xy &= 0 \quad (2) \end{aligned}$$

Divide equation (1) by 6 and equation (2) by 12.

$$\begin{aligned} -x^2 + y^2 &= 0 \quad (3) \\ -y^3 + xy &= 0 \quad (4) \end{aligned}$$

Solve equation (3) for y. Substitute $y = \pm x$ in equation (4).

If $y = x$,

$$\begin{aligned} -x^3 + x^2 &= 0 \\ -x^2(-x + 1) &= 0 \\ x = 0 \quad \text{or} \quad x &= 1. \end{aligned}$$

If $x = 0,\ y = 0$.
If $x = 1,\ y = 1$.

If $y = -x$,

$$\begin{aligned} x^3 - x^2 &= 0 \\ x^2(x - 1) &= 0 \\ x = 0 \quad \text{or} \quad x &= 1. \end{aligned}$$

If $x = 0,\ y = 0$
If $x = 1,\ y = -1$.

$\frac{\partial^2 z}{\partial x^2} = -12x,\ \frac{\partial^2 z}{\partial y^2} = -36y^2 + 12x,$

$\frac{\partial^2 z}{\partial y \partial x} = 12y$

$D = (-12x)(-36y^2 + 12x) - (12y^2)$

At $(0, 0),\ D = 0$, which gives no information.
At $(1, 1)$,

$$\begin{aligned} D &= -12(-36 + 12) - 144 \\ &= 144 > 0 \text{ and } \tfrac{\partial^2 z}{\partial x^2} = -12 < 0. \end{aligned}$$

$z = -2 - 3 + 6 + \frac{1}{16} = 1\frac{1}{16}$

so there is a relative maximum of $1\frac{1}{16}$ at $(1, 1)$.
At $(1, -1)$,

$$\begin{aligned} D &= -12(-36 + 12) - 144 \\ &= 144 > 0 \text{ and } \tfrac{\partial^2 z}{\partial x^2} = -12 < 0, \end{aligned}$$

so there is a relative maximum of $1\frac{1}{16}$ at $(1, -1)$.
The correct graph is (c).

26. $z = -y^4 + 4xy - 2x^2 + \frac{1}{16}$

$\frac{\partial z}{\partial x} = 4y - 4x,\ \frac{\partial z}{\partial y} = -4y^3 + 4x$

$$\begin{aligned} 4y - 4x &= 0 \quad (1) \\ -4y^3 + 4x &= 0 \quad (2) \end{aligned}$$

Divide equation (1) by 4 and equation (2) by -4.

$$\begin{aligned} y - x &= 0 \quad (3) \\ -y^3 + x &= 0 \quad (4) \end{aligned}$$

Solve equation (3) for y.

$$y = x$$

Substituting, we have

$$\begin{aligned} -x^3 + x &= 0 \\ x(-x^2 + 1) &= 0 \\ x = 0 \quad \text{or} \quad x = 1 \quad &\text{or} \quad x = -1. \end{aligned}$$

If $x = 0,\ y = 0$.
If $x = 1,\ y = 1$.
If $x = -1,\ y = -1$.

$\frac{\partial^2 z}{\partial x^2} = -4,\ \frac{\partial^2 z}{\partial y^2} = -12y^2,\ \frac{\partial^2 z}{\partial y \partial x} = 4$

$D = -4(-12y^2) - 4^2 = 48y^2 - 16$

At $(0, 0),\ D = -16 < 0$.
Saddle point at $(0, 0)$
At $(1, 1)$ and $(-1, -1)$,

$D = 48 - 16 = 32 > 0$ and $\frac{\partial^2 z}{\partial x^2} = -4 < 0$.

Relative maximum of $1\frac{1}{16}$ at $(1, 1)$ and at $(-1, -1)$.
The correct graph is (f).

28. $f(x,y) = x^3 + (x-y)^2$

$f_x(x,y) = 3x^2 + 2(x-y),\ f_y(x,y) = -2(x-y)$

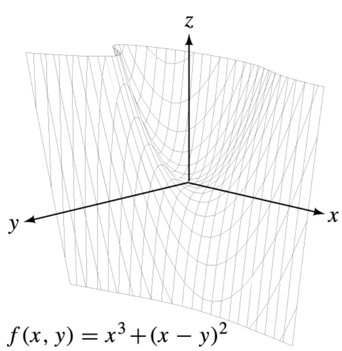
$$\begin{aligned} 3x^2 + 2x - 2y &= 0 \\ -2x + 2y &= 0 \\ \hline 3x^2 &= 0 \\ x &= 0 \end{aligned}$$

$$\begin{aligned} 0 + 2y &= 0 \\ y &= 0 \end{aligned}$$

$f_{xx}(x,y) = 6x + 2,\ f_{yy}(x,y) = 2,\ f_{xy}(x,y) = -2$

$D = (6x+2)(2) - (-2)^2 = 12x$

At $(0,0)$, $D = 0$, which gives no information. Examine the graph of $z = x^3 + (x-y)^2$: in the yz-plane, the trace is $z = y^2$, which has a minimum at $(0,0,0)$; in the xz-plane, the trace is $z = x^3 + x^2$, which has a minimum at $(0,0,0)$.
But in the plane $y = x$, the trace is $z = x^3$, which has neither a maximum nor a minimum at $(0,0,0)$. So the function has no relative extrema. Notice the orientation of axes in the following figure: This is a back view.



$f(x,y) = x^3 + (x-y)^2$

30.
$$\begin{aligned} S(m,b) &= \sum (mx + b - y)^2 \\ S_m(m,b) &= \sum 2(mx + b - y)^1(x) \\ &= 2\sum (mx^2 + bx - xy) \\ S_b(m,b) &= \sum 2(mx + b - y)(1) \\ &= 2\sum (mx + b - y) \end{aligned}$$

If $S_m(m,b) = 0$, then

$$\begin{aligned} 2\sum (mx^2 + bx - xy) &= 0 \\ \sum (mx^2) + \sum (bx) &= \sum (xy) \\ (\sum x^2)m + \sum (x)b &= \sum (xy). \quad (1) \end{aligned}$$

If $S_b(m,b) = 0$, then

$$\begin{aligned} 2\sum (mx + b - y) &= 0 \\ \sum (mx) + \sum (b) - \sum y &= 0 \\ (\sum x)m + nb &= \sum y. \quad (2) \end{aligned}$$

Equations (1) and (2) give us the following system of equations.

$$\begin{aligned} (\sum x)b + (\sum x^2)m &= \sum xy \\ nb + (\sum x)m &= \sum y \end{aligned}$$

32. $P(x,y) = 1000 + 24x - x^2 + 80y - y^2$

$$\begin{aligned} P_x(x,y) &= 24 - 2x \\ P_y(x,y) &= 80 - 2y \\ 24 - 2x &= 0 \\ 80 - 2y &= 0 \\ 12 &= x \\ 40 &= y \end{aligned}$$

A critical point is $(12, 40)$.

$P_{xx}(x,y) = -2,\ P_{yy}(x,y) = -2,\ P_{xy}(x,y) = 0$

For $(12, 40)$,

$$D = -2(-2) - 0^2 = 4 > 0.$$

Since $P_{xx}(x,y) < 0$,

$$\begin{aligned} P(12,40) &= 1000 + 24(12) - (12)^2 \\ &\quad + 80(40) - (40)^2 \\ &= 2744. \end{aligned}$$

The maximum profit is \$274,400.

34. $C(x,y) = 2x^2 + 3y^2 - 2xy + 2x - 126y + 3800$

$$\begin{aligned} C_x &= 4x - 2y + 2 \\ C_y &= 6y - 2x - 126 \end{aligned}$$

$$\begin{aligned} 0 &= 4x - 2y + 2 \\ 0 &= 6y - 2x - 126 \end{aligned}$$

$$\begin{aligned} 0 &= 2x - y + 1 \\ 0 &= -2x + 6y - 126 \\ \hline 0 &= 5y - 125 \\ y &= 25 \\ 0 &= 2x - 25 + 1 \end{aligned}$$

If $y = 25,\ x = 12$.
$(12, 25)$ is a critical point.

$$\begin{aligned} C_{xx}(x,y) &= 4 \\ C_{yy}(x,y) &= 6 \\ C_{xy}(x,y) &= -2 \end{aligned}$$

For $(12, 25)$,

$$D = 4 \cdot 6 - 4 = 20 > 0.$$

Since $C_{xx}(x, y) > 0$, 12 units of electrical tape and 25 units of packing tape should be produced to yield a minimum cost.

$$\begin{aligned} &C(12, 25) \\ &\quad = 2(12)^2 + 3(25)^2 - 2(12 \cdot 25) \\ &\qquad + 2(12) - 126(25) + 3800 \\ &\quad = 2237 \end{aligned}$$

The minimum cost is \$2237.

36. $P(x, y) = 36xy - x^3 - 8y^3$
$P_x(x, y) = 36y - 3x^2$
$P_y(x, y) = 36x - 24y^2$

$$\begin{aligned} P_x(x, y) &= 0 \\ 36y - 3x^2 &= 0 \\ 36y &= 3x^2 \\ y &= \frac{1}{12}x^2 \end{aligned}$$

$$\begin{aligned} P_x(x, y) &= 0 \\ 36x - 24y^2 &= 0 \\ 36x &= 24y^2 \\ x &= \frac{2}{3}y^2 \end{aligned}$$

Use substitution to solve the system of equations

$$y = \frac{1}{12}x^2$$
$$x = \frac{2}{3}y^2.$$

$$y = \frac{1}{12}\left(\frac{2}{3}y^2\right)^2$$
$$y = \frac{1}{12}\left(\frac{4}{9}\right)y^4$$
$$y = \frac{1}{27}y^4$$
$$\frac{1}{27}y^4 - y = 0$$
$$\left(\frac{1}{27}y^3 - 1\right)y = 0$$
$$\frac{1}{27}y^3 - 1 = 0 \quad \text{or} \quad y = 0$$
$$\frac{1}{27}y^3 = 1 \quad \text{or} \quad y = 0$$
$$y^3 = 27 \quad \text{or} \quad y = 0$$
$$y = 3 \quad \text{or} \quad y = 0$$

If $y = 3, x = \frac{2}{3}(3)^2 = 6$.

If $y = 0$, $x = \frac{2}{3}(0)^2 = 0$.

The critical points are $(6, 3)$ and $(0, 0)$.

$P_{xx}(x, y) = -6x$
$P_{yy}(x, y) = -48y$
$P_{xy}(x, y) = 36$
$P_{xx}(6, 3) = -36$
$P_{yy}(6, 3) = -144$
$P_{xy}(6, 3) = 36$

$D = (-36)(-144) - (36)^2 = 3888$

Here $D > 0$ and $P_{xx} < 0$, so there is a relative maximum at $(6, 3)$.

$P_{xx}(0, 0) = 0$
$P_{yy}(0, 0) = 0$
$P_{xy}(0, 0) = 36$

$D = 0 \cdot 0 - 36^2 = -1296$

Since $D < 0$, there is a saddle point at $(0, 0)$.

$$\begin{aligned} P(6, 3) &= 36(6)(3) - (6)^3 - 8(3)^3 \\ &= 648 - 216 - 216 \\ &= 216 \end{aligned}$$

So 6 tons of steel and 3 tons of aluminum produce a maximum profit of \$216,000.

38. $P(\alpha, r, s) = \alpha(3r^2(1 - r) + r^3) + (1 - \alpha)$
$(3s^2(1 - s) + s^3)$

(a)
$$\begin{aligned} P(.9, .5, .6) &= .9(3(.5)^2(1 - .5) + (.5)^3) \\ &\quad + (1 - .9)(3(.6)^2(1 - .6) + (.6)^3) \\ &= .5148 \\ P(.1, .8, .4) &= .1(3(.8)^2(1 - .8) + (.8)^3) \\ &\quad + (1 - .1)(3(.4)^2(1 - .4) + (.4)^3) \\ &= .4064 \end{aligned}$$

The jury is less likely to make the correct decision in the second situation.

(b) If $r = s = 1$ then $P(\alpha, 1, 1) = 1$, so the jury always makes a correct decision. These values do not depend on α, but in a real-life situation α is likely to influence r and s.

(c) When P reaches a maximum, P_α, P_r, and P_s equal 0.

$$\begin{aligned} P_\alpha(\alpha, r, s) &= 3r^2(1-r) + r^3 \\ &\quad - (3s^2(1-s) + s^3) \\ &= 3r^2 - 2r^3 - (3s^2 - 2s^3) \end{aligned}$$

$$\begin{aligned} P_r(\alpha, r, s) &= \alpha(6r(1-r) - 3r^2 + 3r^2) \\ &= 6\alpha r(1-r) \\ P_s(\alpha, r, s) &= (1-\alpha)(6s(1-s) - 3s^2 + 3s^2) \\ &= 6s(1-\alpha)(1-s) \\ P_\alpha(\alpha, r, s) &= 0 \quad \text{when} \quad r = s. \end{aligned}$$

Since $P_r(\alpha, r, s) = 6\alpha r(1-r)$, and $P_s(\alpha, r, s) = 6(1-\alpha)s(1-s)$, then P_α, P_r, and P_s are simultaneously 0 at the points $(\alpha, 1, 1)$ and $(\alpha, 0, 0)$. So $(\alpha, 1, 1)$ and $(\alpha, 0, 0)$ are critical points.
$P(\alpha, 0, 0) = 0$ while $P(\alpha, 1, 1) = 1$
Since $P(\alpha, r, s)$ represents a probability, $0 \le P(\alpha, r, s) \le 1$. Thus, $P(\alpha, 1, 1) = 1$ is a maximum value of the function.

9.4 Lagrange Multipliers

2. Maximize $f(x, y) = 4xy + 2$, subject to $x + y = 24$.

1. $g(x, y) = x + y - 24$

2. $F(x, y, \lambda) = 4xy + 2 - \lambda(x + y - 24)$

3. $F_x(x, y, \lambda) = 4y - \lambda$
$F_y(x, y, \lambda) = 4x - \lambda$
$F_\lambda(x, y, \lambda) = -(x + y - 24)$

4.
$$\begin{aligned} 4y - \lambda &= 0 \\ 4x - \lambda &= 0 \\ x + y - 24 &= 0 \end{aligned}$$

5. $\lambda = 4y$ and $\lambda = 4x$

$$\begin{aligned} 4y &= 4x \\ y &= x \end{aligned}$$

Since $x+y-24 = 0$ and $y = x$, we have $2x-24 = 0$, so $x = 12$ and $y = 12$.
The maximum is $f(12, 12) = 4(12)(12) + 2 = 578$.

4. Maximize $f(x, y) = 4xy^2$, subject to $3x - 2y = 5$.

1. $g(x, y) = 3x - 2y - 5$

2. $F(x, y, \lambda) = 4xy^2 - \lambda(3x - 2y - 5)$

3. $F_x(x, y, \lambda) = 4y^2 - 3\lambda$
$F_y(x, y, \lambda) = 8xy + 2\lambda$
$F_\lambda(x, y, \lambda) = -(3x - 2y - 5)$

4.
$$\begin{aligned} 4y^2 - 3\lambda &= 0 \\ 8xy + 2\lambda &= 0 \\ 3x - 2y - 5 &= 0 \end{aligned}$$

5. $\lambda = \dfrac{4y^2}{3}$ and $\lambda = -4xy$

$$\frac{4y^2}{3} = -4xy$$

$$0 = \frac{4}{3}y^2 + 4xy = 4y\left(\frac{1}{3}y + x\right)$$

$$y = 0 \quad \text{or} \quad y = -3x$$

Substituting into $3x - 2y - 5 = 0$, we have for $y = 0$, $3x - 5 = 0$, so $x = \frac{5}{3}$, and for $y = -3x$, $3x - 2(-3x) - 5 = 0$, so $x = \frac{5}{9}$ and $y = -\frac{5}{3}$.

$$f\left(\frac{5}{3}, 0\right) = 4\left(\frac{5}{3}\right)(0)^2 = 0$$

$$f\left(\frac{5}{9}, -\frac{5}{3}\right) = 4\left(\frac{5}{9}\right)\left(-\frac{5}{3}\right)^2 = \frac{500}{81}$$

The maximum is $f\left(\frac{5}{9}, -\frac{5}{3}\right) = \frac{500}{81} \approx 6.2$.

6. Minimize $f(x, y) = 3x^2 + 4y^2 - xy - 2$, subject to $2x + y = 21$.

1. $g(x, y) = 2x + y - 21$

2. $F(x, y, \lambda)$
$= 3x^2 + 4y^2 - xy - 2 - \lambda(2x + y - 21)$

3. $F_x(x, y, \lambda) = 6x - y - 2\lambda$
$F_y(x, y, \lambda) = 8y - x - \lambda$
$F_\lambda(x, y, \lambda) = -(2x + y - 21)$

4.
$$\begin{aligned} 6x - y - 2\lambda &= 0 \\ 8y - x - \lambda &= 0 \\ 2x + y - 21 &= 0 \end{aligned}$$

5. $\lambda = \dfrac{6x - y}{2}$ and $\lambda = 8y - x$

$$\frac{6x - y}{2} = 8y - x$$

$$y = \frac{8}{17}x$$

Substituting into $2x + y - 21 = 0$, we have

$$2x + \frac{8}{17}x - 21 = 0,$$

so $x = \frac{17}{2}$ and $y = 4$.

$$\text{Minimum is } f\left(\tfrac{17}{2}, 4\right) = 3\left(\frac{17}{2}\right)^2 + 4(4)^2 - \frac{17}{2}(4) - 2$$

$$= \frac{979}{4} = 244.75.$$

8. Maximize $f(x, y) = 12xy - x^2 - 3y^2$, subject to $x + y = 16$.

1. $g(x, y) = x + y - 16$

2. $F(x, y, \lambda)$
$= 12xy - x^2 - 3y^2 - \lambda(x + y - 16)$

3. $F_x(x, y, \lambda) = 12y - 2x - \lambda$
$F_y(x, y, \lambda) = 12x - 6y - \lambda$
$F_\lambda(x, y, \lambda) = -(x + y - 16)$

4. $12y - 2x - \lambda = 0$
$12x - 6y - \lambda = 0$
$x + y - 16 = 0$

5. $\lambda = 12y - 2x$ and $\lambda = 12x - 6y$

$$12y - 2x = 12x - 6y$$
$$y = \frac{7}{9}x$$

Substituting into $x + y - 16 = 0$, we have

$$x + \frac{7}{9}x - 16 = 0,$$

so $x = 9$ and $y = 7$.

Maximum is

$f(9, 7) = 12(9)(7) - (9)^2 - 3(7)^2 = 528.$

10. Maximize $f(x, y, z) = xy + 2xz + 2yz$, subject to $xyz = 32$.

1. $g(x, y, z) = xyz - 32$

2. $F(x, y, z, \lambda) = xy + 2xz + 2yz - \lambda(xyz - 32)$

3. $F_x(x, y, z, \lambda) = y + 2z - \lambda yz$
$F_y(x, y, z, \lambda) = x + 2z - \lambda xz$
$F_z(x, y, z, \lambda) = 2x + 2y - \lambda xy$
$F_\lambda(x, y, z, \lambda) = -(xyz - 32)$

4. $y + 2z - \lambda yz = 0$
$x + 2z - \lambda xz = 0$
$2x + 2y - \lambda xy = 0$
$xyz - 32 = 0$

5. $\lambda = \dfrac{y + 2z}{yz}$, $\lambda = \dfrac{x + 2z}{xz}$, and $\lambda = \dfrac{2x + 2y}{xy}$

$$\frac{y + 2z}{yz} = \frac{x + 2z}{xz}$$
$$xyz + 2xz^2 = xyz + 2yz^2$$
$$2yz^2 - 2xz^2 = 0$$
$$2z^2(y - x) = 0$$
$$z = 0 \quad \text{or} \quad y = x$$

and

$$\frac{y + 2z}{yz} = \frac{2x + 2y}{xy}$$
$$xy^2 + 2xyz = 2xyz + 2y^2z$$
$$2y^2z - xy^2 = 0$$
$$y^2(2z - x) = 0$$
$$y = 0 \quad \text{or} \quad z = \frac{x}{2}$$

Since $xyz = 32$, $z = 0$ and $y = 0$ are impossible. Substituting $y = x$ and $z = \frac{x}{2}$ into $xyz - 32 = 0$, we have

$x(x)\left(\dfrac{x}{2}\right) = 32$, so $x = 4$, $y = 4$, and $z = 2$.

Maximum is

$f(4, 4, 2) = 4(4) + 2(4)(2) + 2(4)(2) = 48.$

12. Let x and y be the two numbers such that $x + y = 36$.
Maximize $f(x, y) = x^2y$, subject to $x + y = 36$.

1. $g(x, y) = x + y - 36$

2. $F(x, y, \lambda) = x^2y - \lambda(x + y - 36)$

3. $F_x(x, y, \lambda) = 2xy - \lambda$
$F_y(x, y, \lambda) = x^2 - \lambda$
$F_\lambda(x, y, \lambda) = -(x + y - 36)$

4. $2xy - \lambda = 0$
$x^2 - \lambda = 0$
$x + y - 36 = 0$

5. $\lambda = 2xy$ and $\lambda = x^2$

$$2xy = x^2$$
$$x(x - 2y) = 0$$

Since $x = 0$ gives a smaller value of $f(x, y) = x^2y$ than any positive values of x and y, we can assume $x \neq 0$, so $y = \frac{x}{2}$.
Substituting into $x + y - 36 = 0$, we have

$$x + \frac{x}{2} - 36 = 0,$$

so $x = 24$ and $y = 12$.

14. Let x, y, and z be three positive numbers such that $x + y + z = 240$. Maximize $f(x, y, z) = xyz$, subject to $x + y + z = 240$.

1. $g(x, y) = x + y + z - 240$

2. $F(x, y, z, \lambda)$
$= xyz - \lambda(x + y + z - 240)$

3. $F_x(x, y, z, \lambda) = yz - \lambda$
$F_y(x, y, z, \lambda) = xz - \lambda$
$F_z(x, y, z, \lambda) = xy - \lambda$
$F_\lambda(x, y, z, \lambda) = -(x + y + z - 240)$

4. $yz - \lambda = 0$
$xz - \lambda = 0$
$xy - \lambda = 0$
$x + y + z - 240 = 0$

5. $\lambda = yz,\ \lambda = xz,\ \lambda = xy$
$yz = xz$
$z = 0$ (impossible) or $x = y$
$xz = xy$
$x = 0$ (impossible) or $z = y$

Thus, $x = y = z$.

$$\begin{aligned} x + x + x - 240 &= 0 \\ x &= 80 \end{aligned}$$

Thus,
$$x = y = z = 80.$$

The three numbers are 80, 80, and 80.

18. Let x be the width and y be the length of a field such that the cost in dollars to enclose the field is

$$\begin{aligned} 6x + 6y + 4x + 4y &= 1200 \\ 10x + 10y &= 1200. \end{aligned}$$

The area is
$$f(x, y) = xy.$$

1. $g(x, y) = 10x + 10y - 1200$

2. $F(x, y) = xy - \lambda(10x + 10y - 1200)$

3. $F_x(x, y, \lambda) = y - 10\lambda$
$F_y(x, y, \lambda) = x - 10\lambda$
$F_\lambda(x, y, \lambda) = -(10x + 10y - 1200)$

4.
$$\begin{aligned} y - 10\lambda &= 0 \\ x - 10\lambda &= 0 \\ 10x + 10y - 1200 &= 0 \end{aligned}$$

5. $10\lambda = y$ and $10\lambda = x$

$$y = x$$

Substituting into the third equation gives

$$\begin{aligned} 10x + 10x - 1200 &= 0 \\ 20x - 1200 &= 0 \\ x &= 60 \\ y &= 60. \end{aligned}$$

These dimensions, 60 feet by 60 feet, will maximize the area.

20. $C(x, y) = 2x^2 + 6y^2 + 4xy + 10$, subject to $x + y = 10$

1. $g(x, y) = x + y - 10$

2. $F(x, y)$
$= 2x^2 + 6y^2 + 4xy + 10$
$- \lambda(x + y - 10)$

3. $F_x(x, y, \lambda) = 4x + 4y - \lambda$
$F_y(x, y, \lambda) = 12y + 4x - \lambda$
$F_\lambda(x, y, \lambda) = -(x + y - 10)$

4. $4x + 4y - \lambda = 0$
$12y + 4x - \lambda = 0$
$x + y - 10 = 0$

5. $\lambda = 4x + 4y$ and $\lambda = 12y + 4x$.

$$\begin{aligned} 4x + 4y &= 12y + 4x \\ 8y &= 0 \\ y &= 0 \end{aligned}$$

Since $x + y = 10,\ x = 10$.
10 large kits and no small kits will maximize the cost.

22. $f(x, y) = 3x^{1/3}y^{2/3}$, subject to $80x + 150y = 40{,}000$

1. $g(x, y) = 80x + 150y - 40{,}000$

2. $F(x, y)$
$= 3x^{1/3}y^{2/3} - \lambda(80x + 150y - 40{,}000)$

3. and **4.**

$F_x(x, y, \lambda) = x^{-2/3}y^{2/3} - 80\lambda = 0$
$F_y(x, y, \lambda) = 2x^{1/3}y^{-1/3} - 150\lambda = 0$
$F_\lambda(x, y, \lambda) = -(80x + 150y - 40{,}000) = 0$

5. $\dfrac{x^{-2/3}y^{2/3}}{80} = \dfrac{2x^{1/3}y^{-1/3}}{150}$

$$\frac{15y}{16} = x$$

Substitute into the third equation.

$$\begin{aligned} 80\left(\frac{15y}{16}\right) + 150y - 40{,}000 &= 0 \\ y &= 178 \text{ (rounded)} \\ x &= \frac{15(178)}{16} \\ &\approx 167 \end{aligned}$$

Use about 167 units of labor and 178 units of capital to maximize production.

24. Let x and y be the dimensions of the field such that $2x + 2y = 200$, and the area is $f(x, y) = xy$.

1. $g(x, y) = 2x + 2y - 200$

2. $F(x, y)$
$= xy - \lambda(2x + 2y - 200)$

3. $F_x(x, y, \lambda) = y - 2\lambda$
$F_y(x, y, \lambda) = x - 2\lambda$
$F_\lambda(x, y, \lambda) = -(2x + 2y - 200)$

4. $y - 2\lambda = 0$
$x - 2\lambda = 0$
$2x + 2y - 200 = 0$

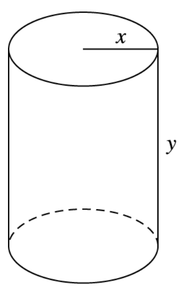
5. $2\lambda = y$ and $2\lambda = x$, so $x = y$.

$$2x + 2x - 200 = 0$$
$$4x - 200 = 0$$
$$x = 50$$

Thus, $y = 50.$

Dimensions of 50 m by 50 m will maximize the area.

26. Let x be the radius r of the circular base and y the height h of the can, such that the volume is $\pi x^2 y = 250\pi$.



The surface area is

$$f(x, y) = 2\pi xy + 2\pi x^2.$$

1. $g(x, y) = \pi x^2 y - 250\pi$

2. $F(x, y) = 2\pi xy + 2\pi x^2 - \lambda(\pi x^2 y - 250\pi)$

3. $F_x(x, y, \lambda) = 2\pi y + 4\pi x - \lambda(2\pi xy)$
$F_y(x, y, \lambda) = 2\pi x - \lambda(\pi x^2)$
$F_\lambda(x, y, \lambda) = -(\pi x^2 y - 250\pi)$

4. $2\pi y + 4\pi x - \lambda(2\pi xy) = 0$
$2\pi x - \lambda \pi x^2 = 0$
$\pi x^2 y - 250\pi = 0$

Simplifying these equations gives

$$y + 2x - 1\lambda xy = 0$$
$$2x - 1\lambda x^2 = 0$$
$$x^2 y - 250 = 0.$$

5. From the second equation,

$$x(2 - \lambda x) = 0$$
$$x = 0 \quad \text{or} \quad \lambda = \frac{2}{x}.$$

If $x = 0$, the volume will be 0, which is not possible.

Substituting $x = \frac{2}{\lambda}$ into the first equation gives

$$y + 2\left(\frac{2}{\lambda}\right) - \lambda\left(\frac{2}{\lambda}\right) y = 0$$
$$y + \frac{4}{\lambda} - 2y = 0$$
$$\frac{4}{\lambda} = y$$
$$\lambda = \frac{4}{y}.$$

Since $\lambda = \frac{2}{x}$, $y = 2x$.

Substituting into third equation gives

$$x^2(2x) - 250 = 0$$
$$2x^3 - 250 = 0$$
$$x = 5$$
$$y = 10.$$

Since $g(1, 250) = 0$ and

$$f(1, 250) = 502\pi > f(5, 10) = 150\pi,$$

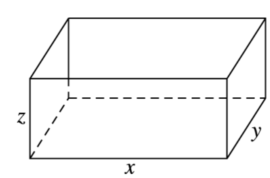
a can with radius of 5 inches and height of 10 inches will have a minimum surface area.

28. Let x, y, and z be the dimensions of the box such that the surface area is

$$xy + 2yz + 2xz = 500$$

and the volume is

$$f(x, y, z) = xyz.$$



1. $g(x, y, z) - 500 = 0$

2.$F(x, y, z) = xyz - \lambda(xy + 2yz + 2xz - 500)$

3. and **4.**

$$F_x(x, y, z, \lambda) = yz - \lambda(y + 2z) = 0 \quad (1)$$
$$F_y(x, y, z, \lambda) = xz - \lambda(x + 2z) = 0 \quad (2)$$
$$F_z(x, y, z, \lambda) = xy - \lambda(2y + 2x) = 0 \quad (3)$$
$$F_\lambda(x, y, z, \lambda) = -(xy + 2xz + 2yz - 500) = 0 \quad (4)$$

Multiplying equation (1) by x, equation (2) by y, and equation (3) by z gives

$$\begin{aligned} xyz - \lambda x(y+2z) &= 0 \\ xyz - \lambda y(x+2z) &= 0 \\ xyz - \lambda z(2y+2z) &= 0. \end{aligned}$$

5. Subtracting the first equation from the second equation gives

$$\begin{aligned} \lambda x(y+2z) - \lambda y(x+2z) &= 0 \\ 2\lambda xz - 2\lambda yz &= 0 \\ \lambda z(x-y) &= 0, \\ \text{so} \qquad x &= y. \end{aligned}$$

Subtracting the third equation from the second equation gives

$$\begin{aligned} \lambda z(2y+2x) - \lambda y(x+2z) &= 0 \\ 2\lambda xz - \lambda xy &= 0 \\ \lambda x(2z-y) &= 0, \\ \text{so} \qquad z &= \frac{y}{2}. \end{aligned}$$

Substituting into the fourth equation gives

$$\begin{aligned} y^2 + 2y\left(\frac{y}{2}\right) + 2y\left(\frac{y}{2}\right) - 500 &= 0 \\ 3y^2 &= 500 \\ y &= \sqrt{\frac{500}{3}} \\ &\approx 12.9099 \\ x &\approx 12.9099 \\ z &\approx \frac{12.9099}{2} \\ &\approx 6.4649. \end{aligned}$$

The dimensions are 12.91 m by 12.91 m by 6.45 m.

30. Let x, y, and z be the dimensions of the box. The surface area is

$$2xy + 2xz + 2yz.$$

We must minimize

$$f(x,y,z) = 2xy + 2xz + 2yz$$

subject to $xyz = 27$.

1. $g(x,y,z) = xyz - 27$

2. $F(x,y,z) = 2xy + 2xz + 2yz - \lambda(xyz - 27)$

3. $F_x(x,y,z,\lambda) = 2y + 2z - \lambda yz$
$F_y(x,y,z,\lambda) = 2x + 2z - \lambda xz$
$F_z(x,y,z,\lambda) = 2x + 2y - \lambda xy$
$F_\lambda(x,y,z,\lambda) = -(xyz - 27)$

4. $2y + 2z - \lambda yz = 0 \quad (1)$
$2x + 2z - \lambda xz = 0 \quad (2)$
$2x + 2y - \lambda xy = 0 \quad (3)$
$xyz - 27 = 0 \quad (4)$

5. Equations (1) and (2) give

$$\frac{2y+2z}{yz} = \lambda \text{ and } \frac{2x+2z}{xz} = \lambda.$$

Thus,

$$\begin{aligned} \frac{2y+2z}{yz} &= \frac{2x+2z}{xz} \\ 2xyz + 2xz^2 &= 2xyz + 2yz^2 \\ 2xz^2 - 2yz^2 &= 0 \\ 2z^2 = 0 \qquad & \text{or} \quad x - y = 0 \\ z = 0 \quad \text{(impossible)} \qquad & \text{or} \quad x = y. \end{aligned}$$

Equations (2) and (3) give

$$\frac{2x+2z}{xz} = \lambda \text{ and } \frac{2x+2y}{xy} = \lambda.$$

$$\frac{2x+2z}{xz} = \frac{2x+2y}{xy}$$

Thus,

$$\begin{aligned} 2x^2y + 2xyz &= 2x^2z + 2xyz \\ 2x^2y - 2x^2z &= 0 \\ 2x^2(y-z) &= 0 \\ 2x^2 = 0 \qquad & \text{or} \quad y - z = 0 \\ x = 0 \text{ (impossible)} \qquad & \text{or} \quad y = z. \end{aligned}$$

Therefore, $x = y = z$. Substituting into equation (4) gives

$$\begin{aligned} x^3 - 27 &= 0 \\ x^3 &= 27 \\ x &= 3. \end{aligned}$$

Thus,

$$y = 3 \text{ and } z = 3.$$

The dimensions that will minimize the surface area are 3 m by 3 m by 3 m.

32. **(a)** The surface area of the box is

$$SA = xy + 2xz + 2yz.$$

Let the sides with area xz be those made from the free material (along with the bottom). This gives the constraint $2xz + xy = 4$. The volume of the box is xyz, thus it will take $\frac{400}{xyz}$ trips to transport the material at a cost of $\frac{400}{xyz}(.10) = \dfrac{40}{xyz}$. The total cost also includes the cost of the material for the ends of the box: $(2yz)(20) = 40yz$.
Thus, the total cost is

$$f(x, y, z) = \frac{40}{xyz} + 40yz.$$

(b) Using a spread sheet, $x = 2$ yards, $y = 1$ yard, $z = \frac{1}{2}$ yard.

9.5 Total Differentials and Approximations

2. Let $z = f(x, y) = \sqrt{x^2 + y^2}$.

Then

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy.$$
$$dz = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x)\,dx + \frac{1}{2}(x^2 + y^2)^{-1/2}(2y)\,dx$$
$$= \frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy$$

To approximate $\sqrt{6.07^2 + 7.95^2}$, we let $x = 6$, $dx = .07$, $y = 8$, and $dy = -.05$.

$$dz = \frac{6}{\sqrt{6^2 + 8^2}}(.07) + \frac{8}{\sqrt{6^2 + 8^2}}(-.05)$$
$$= \frac{6}{10}(.07) + \frac{8}{10}(-.05)$$
$$= .002$$

$$\begin{aligned} f(6.07, 7.95) &= f(6, 8) + \Delta z \\ &\approx f(6.8) + dz \\ &= \sqrt{6^2 + 8^2 + .002} \\ f(6.07, 7.95) &\approx 10.002 \end{aligned}$$

Using a calculator, $\sqrt{6.07^2 + 7.95^2} \approx 10.0024$.
The absolute value of the difference of the two results is $|10.002 - 10.0024| = .0004$.

4. Let $z = f(x, y) = (x^2 - y^2)^{1/3}$.

Then

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy.$$
$$dz = \frac{1}{3}(x^2 - y^2)^{-2/3}(2x)\,dx + \frac{1}{3}(x^2 - y^2)^{-2/3}(-2y)\,dy$$
$$dz = \frac{2x}{3(x^2 - y^2)^{2/3}}\,dx - \frac{2y}{3(x^2 - y^2)^{2/3}}\,dy$$

To approximate $(2.93^2 - .94^2)^{1/3}$, we let $x = 3$, $dx = -.07$, $y = 1$, and $dy = -.06$.

$$dz = \frac{2(3)}{3[(3)^2 - (1)^2]^{2/3}}(-.07) - \frac{2(1)}{3[(3)^2 - (1)^2]^{2/3}}(-.06)$$
$$dz = \frac{1}{2}(-.07) - \frac{1}{6}(-.06)$$
$$= -.025$$

$$\begin{aligned} f(2.93, .94) &= f(3, 1) + \Delta z \\ &\approx f(3, 1) + dz \\ &= 2 + (-.025) \\ f(2.93, .94) &\approx 1.975 \end{aligned}$$

Using a calculator, $(2.93^2 - .94^2)^{1/3} \approx 1.9748$.
The absolute value of the difference of the two results is $|1.975 - 1.9748| = .0002$.

6. Let $z = f(x, y) = xe^y$.

Then

$$\begin{aligned} dz &= f_x(x, y)\,dx + f_y(x, y)\,dy \\ &= e^y\,dx + xe^y\,dy. \end{aligned}$$

To approximate $1.02e^{-.03}$ we let $x = 1$, $dx = .02$, $y = 0$, and $dy = -.03$.

$$\begin{aligned} dz &= e^0(.02) + (1)(e^0)(-.03) \\ &= -.01 \end{aligned}$$

$$\begin{aligned} f(1.02, -.03) &= f(1, 0) + \Delta z \\ &\approx f(1, 0) + dz \\ &= (1)e^0 + (-.01) \\ f(1.02, -.03) &\approx .99 \end{aligned}$$

Using a calculator, $1.02e^{-.03} \approx .9899$.
The absolute value of the difference of the two results is $|.99 - .9899| = .0001$.

8. Let $z = x \ln y$.

Then

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$$
$$= (\ln y)\,dx + \frac{x}{y}\,dy.$$

To approximate .95 ln 1.04, we let $x = 1$, $dx = -.05$, $y = 1$, and $dy = .04$.

$$dz = (\ln 1)(-.05) + \frac{1}{1}(.04)$$
$$= .04$$

$$f(.95, 1.04) = f(1, 1) + \Delta z$$
$$\approx f(1, 1) + dz$$
$$= (1)(\ln 1) + (.04)$$
$$f(.95, 1.04) \approx .04$$

Using a calculator, $.95 \ln 1.04 \approx .0373$.
The absolute value of the difference of the two results is $|.04 - .0373| = .0027$.

10. $z = 8x^3 + 2x^2y - y$
$x = 1,\ y = 3,$
$dx = .01,\ dy = .02$

$$dz = (24x^2 + 4xy)\,dx + (2x^2 - 1)\,dy$$
$$= [24(1)^2 + 4(1)(3)](.01)$$
$$+ (2(1)^2 - 1)(.02)$$
$$= .38$$

12. $z = \ln(x^2 + y^2)$
$x = 2,\ y = 3,\ dx = .02,\ dy = -.03$

$$dz = \frac{2x\,dx}{x^2 + y^2} + \frac{2y\,dy}{x^2 + y^2}$$

Substitute the given information.

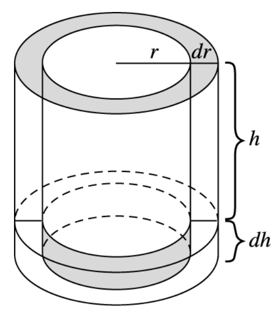
$$dz = \frac{2(2)(.02)}{2^2 + 3^2} + \frac{2(3)(-.03)}{2^2 + 3^2}$$
$$= -.00769$$

14. $w = x \ln(yz) - y \ln\left(\frac{x}{z}\right)$
$= x(\ln y + \ln z) - y(\ln x - \ln z),$

$x = 2,\ y = 1,\ z = 4,\ dx = .03,\ dy = .02,$
$dz = -.01$

$$dw = \left(\ln y + \ln z - \frac{y}{x}\right) dx + \left(\frac{x}{y} - \ln x + \ln z\right) dy$$
$$+ \left(\frac{x}{z} + \frac{y}{z}\right) dz$$
$$= \left(\ln 1 + \ln 4 - \frac{1}{2}\right)(.03)$$
$$+ \left(\frac{2}{1} - \ln 2 + \ln 4\right)(.02)$$
$$+ \left(\frac{2}{4} + \frac{1}{4}\right)(-.01)$$
$$\approx .0730$$

16. Let r be the radius inside the tumbler and h be the height inside.



$V = \pi r^2 h,\ r = 1.5,\ h = 9,\ dr = dh = .2$

$$dV = 2\pi rh\,dr + \pi r^2\,dh$$
$$= 2\pi(1.5)(9)(.2) + \pi(1.5)^2(.2)$$
$$\approx 18.4$$

Approximately 18.4 cm^3 of material is needed.

18. $M(x, y) = 40x^2 + 30y^2 - 10xy + 30$
$x = 4,\ y = 7,$
$dx = 5 - 4 = 1,\ dy = 6.50 - 7 = -.50$

$$dM = (80x - 10y)\,dx + (60y - 10x)\,dy$$
$$= [80(4) - 10(7)](1)$$
$$+ [60(7) - 10(4)](-.50)$$
$$= 60$$

Expect costs to increase by approximately \$60.

20. $z = x^{.8}y^{.2},\ x = 20,\ y = 18,$
$dx = 21 - 20 = 1,\ dy = 16 - 18 = -2$

$$\begin{aligned} dz &= .8x^{-.2}y^{.2}\,dx + .2x^{.8}y^{-.8}\,dy \\ &= .8\left(\frac{y^{.2}}{x^{.2}}\right)dx + .2\left(\frac{x^{.8}}{y^{.8}}\right)dy \\ &= .8\left(\frac{y}{x}\right)^{.2}dx + .2\left(\frac{x}{y}\right)^{.8}dy \\ &= .8\left(\frac{18}{20}\right)^{.2}(1) + .2\left(\frac{20}{18}\right)^{.8}(-2) \\ &\approx .348 \end{aligned}$$

The change in production is .348 units.

22. Assume that blood vessels are cylindrical.

$$V = \pi r^2 h,\ r = .8,\ h = 7.9,\ dr = dh = \pm .15$$

$$\begin{aligned} dV &= 2\pi rh\,dr + \pi r^2\,dh \\ &= 2\pi(.8)(7.9)(\pm .15) + \pi(.8)^2(\pm .15) \\ &\approx \pm 6.26 \text{ cm}^3 \end{aligned}$$

The maximum possible error is 6.26 cm^3.

24. $m = \dfrac{2.5(T-F)}{w^{.67}}$

$T = 38°,\ F = 12°,\ w = 30$ kg,
$dT = (36 - 38) = -2°,$
$dF = (13 - 12) = 1°,$
$dw = (31 - 30) = 1$ kg

$$\begin{aligned} m_T(T,F,w) &= \frac{w^{.67}(2.5)(1) - 2.5(T-F)(0)}{(w^{.67})^2} \\ &= \frac{2.5}{w^{.67}} \\ m_F(T,F,w) &= \frac{w^{.67}(2.5)(-1) - 2.5(T-F)(0)}{(w^{.67})^2} \\ &= \frac{-2.5}{w^{.67}} \\ m_w(T,F,w) &= \frac{w^{.67}(0) - 2.5(T-F)(.67)w^{-.33}}{(w^{.67})^2} \\ &= \frac{-1.675(T-F)}{w^{1.67}} \\ dm &= \frac{2.5}{w^{.67}}\,dT + \frac{-2.5}{w^{.67}}\,dF \\ &\quad + \frac{-1.675(T-F)}{w^{1.67}}\,dw \end{aligned}$$

$$\begin{aligned} &dm(38,12,30) \\ &= \frac{2.5}{30^{.67}}(-2) + \frac{-2.5}{30^{.67}}(1) \\ &\quad + \frac{-1.675(38-12)}{30^{1.67}}(1) \\ &= -.5120 - .2560 - .1490 \\ &= -.917 \text{ units} \end{aligned}$$

26. $V = \dfrac{h\pi}{3}(r_1^2 + r_1r_2 + r_2^2)$

(a) $V = \dfrac{40\pi}{3}(5^2 + 5\cdot 3 + 3^2) \approx 2052.51$

The volume is about 2052.5 cm^3.

(b)
$$\begin{aligned} dV &= V_h dh + V_{r_1} d_{r_1} + V_{r_2} d_{r_2} \\ &= \frac{\pi}{3}(r_1^2 + r_1r_2 + r_2^2)dh + \frac{h\pi}{3}(2r_1 + r_2)dr_1 \\ &\quad + \frac{h\pi}{3}(r_1 + 2r_2)dr_2 \end{aligned}$$

$$\begin{aligned} dh &= 42 - 40 = 2 \\ dr_1 &= 5.1 - 5 = .1 \\ dr_2 &= 2.9 - 3 = -.1 \end{aligned}$$

$$\begin{aligned} dV(40,5,3) &= \frac{\pi}{3}(5^2 + 5\cdot 3 + 3^2)(2) \\ &\quad + \frac{40\pi}{3}(2\cdot 5 + 3)(.1) \\ &\quad + \frac{40\pi}{3}(5 + 2\cdot 3)(-.1) \\ &\approx 111.00 \end{aligned}$$

$$\begin{aligned} V(42,5.1,2.9) &\approx V(40,5,3) + dV \\ &\approx 2052.51 + 111.00 \\ &\approx 2163.51 \end{aligned}$$

Using differentials, the volume is about 2163.5 cm^3.

$$\begin{aligned} V(42,5.1,2.9) &= \frac{42\pi}{3}(5.1^2 + 5.1\cdot 2.9 + 2.9^2) \\ &\approx 2164.37 \end{aligned}$$

The volume found by using the original formula is about 2164.4 cm^3.

28. The area is
$$A = \frac{1}{2}bh$$
with $h = 42.6$ cm, $b = 23.4$ cm, $dh = 1.2$ cm, $db = .9$ cm.
$$dA = \frac{1}{2}h\,db + \frac{1}{2}b\,dh$$
Substitute.
$$dA = \frac{1}{2}(42.6)(.9) + \frac{1}{2}(23.4)(1.2) = 33.2 \text{ cm}^2$$

30. Let $z = f(L, W, H) = LWH$

Then

$$dz = f_L(L,W,H)\,dL + f_W(L,W,H)\,dW + f_H(L,W,H)\,dH = WH dL + LH dW + LW dH.$$

A maximum 1% error in each measurement means that the maximum values of dL, dW, and dH are given by $dL = .01L$, $dW = .01W$, and $dH = .01H$.
Therefore,

$$dz = WH(.01L) + LH(.01W) + LW(.01H) = .01LWH + .01LWH + .01LWH = .03LWH.$$

Thus, an estimate of the maximum error in calculating the volume is 3%.

9.6 Double Integrals

2. $\int_1^4 (xy^2 - x)\,dy = \left(\frac{xy^3}{3} - xy\right)\Big|_1^4$

$= \left(\frac{64x}{3} - 4x\right) - \left(\frac{x}{3} - x\right)$

$= 18x$

4. $\int_3^7 (x + 5y)^{1/2}\,dy$

$= \frac{2}{15}(x + 5y)^{3/2}\Big|_3^7$

$= \frac{2}{15}[(x + 35)^{3/2} - (x + 15)^{3/2}]$

6. $\int_3^6 x\sqrt{x^2 + 3y}\,dx$

Let $u = x^2 + 3y$. Then $du = 2x\,dx$
When $x = 6,\ u = 36 + 3y$.
When $x = 3,\ u = 9 + 3y$.

$= \frac{1}{2}\int_{9+3y}^{36+y} u^{1/2}\,du$

$= \frac{1}{2}\cdot\frac{2}{3}(u^{3/2})\Big|_{9+3y}^{36+3y}$

$= \frac{1}{3}[(36 + 3y)^{3/2} - (9 + 3y)^{3/2}]$

8. $\int_2^7 \frac{3 + 5y}{\sqrt{x}}\,dy$

$= \left(\frac{3y}{\sqrt{x}} + \frac{5y^2}{2\sqrt{x}}\right)\Big|_2^7$

$= \frac{1}{\sqrt{x}}\left[\left(21 + \frac{245}{2}\right) - \left(6 + \frac{20}{2}\right)\right]$

$= \frac{255}{2\sqrt{x}}$

10. $\int_2^6 e^{x+4y}\,dx = e^{x+4y}\Big|_2^6$

$= e^{6+4y} - e^{2+4y}$

12. $\int_1^6 xe^{x^2+9y}\,dy = \frac{x}{y}e^{x^2+9y}\Big|_1^6$

$= \frac{x}{9}(e^{x^2+54} - e^{x^2+9})$

14. (See Exercise 2.)

$\int_0^3 \left[\int_1^4 (xy^2 - x)\,dy\right] dx$

$= \int_0^3 (18x)\,dx = 9x^2\Big|_0^3$

$= 81$

16. (See Exercise 5.)

$\int_0^3 \left[\int_4^5 x\sqrt{x^2 + 3y}\,dy\right] dx$

$= \int_0^3 \frac{2x}{9}[(x^2 + 15)^{3/2} - (x^2 + 12)^{3/2}]\,dx$

$= \frac{2}{45}[(x^2 + 15)^{5/2} - (x^2 + 12)^{5/2}]\Big|_0^3$

$= \frac{2}{45}(24^{5/2} - 21^{5/2} - 15^{5/2} + 12^{5/2})$

18. (See Exercise 8.)

$\int_{16}^{25} \left[\int_2^7 \frac{3 + 5y}{\sqrt{x}}\,dy\right] dx$

$= \int_{16}^{25} \frac{255}{2\sqrt{x}}\,dx$

$= \int_{16}^{25} \frac{255}{2}x^{-1/2}\,dx$

$= 255x^{1/2}\Big|_{16}^{25} = 255(5 - 4)$

$= 255$

20. $\displaystyle\int_1^4 \int_2^5 \frac{dy\,dx}{x} = \int_1^4 \frac{y}{x}\Big|_2^5 dx = \int_1^4 \left(\frac{5}{x} - \frac{2}{x}\right) dx$

$\displaystyle = \int_1^4 \frac{3}{x}\,dx = 3\ln|x|\Big|_1^4$

$= 3\ln 4$

22. $\displaystyle\int_3^4 \int_1^2 \left(\frac{6x}{5} + \frac{y}{x}\right) dx\,dy$

$\displaystyle = \int_3^4 \left(\frac{3x^2}{5} + y\ln x\right)\Big|_1^2 dy$

$\displaystyle = \int_3^4 \left(\frac{12}{5} + y\ln 2 - \frac{3}{5}\right) dy$

$\displaystyle = \int_3^4 \left(\frac{9}{5} + y\ln 2\right) dy$

$\displaystyle = \left(\frac{9y}{5} + \frac{y^2}{2}\ln 2\right)\Big|_3^4$

$\displaystyle = \frac{36}{5} + 8\ln 2 - \frac{27}{5} - \frac{9}{2}\ln 2$

$\displaystyle = \frac{9}{5} + \frac{7}{2}\ln 2$

24. $\displaystyle\iint_R (4x^3 + y^2)\,dx\,dy; 1 \le x \le 4, 0 \le y \le 2$

$\displaystyle\iint_R (4x^3 + y^2)\,dx\,dy$

$\displaystyle = \int_0^2 \int_1^4 (4x^3 + y^2)\,dx\,dy$

$\displaystyle = \int_0^2 (x^4 + xy^2)\Big|_1^4 dy$

$\displaystyle = \int_0^2 (256 + 4y^2 - 1 - y^2)\,dy$

$\displaystyle = \int_0^2 (255 + 3y^2)\,dy$

$\displaystyle = (255y + y^3)\Big|_0^2$

$= 510 + 8 = 518$

26. $\displaystyle\iint_R x^2\sqrt{x^3 + 2y}\,dx\,dy; 0 \le x \le 2, 0 \le y \le 3$

$\displaystyle\iint_R x^2\sqrt{x^3 + 2y}\,dx\,dy$

$\displaystyle = \int_0^3 \int_0^2 x^2(x^3 + 2y)^{1/2}\,dx\,dy$

$\displaystyle = \int_0^3 \frac{2}{9}(x^3 + 2y)^{3/2}\Big|_0^2 dy$

$\displaystyle = \int_0^3 \frac{2}{9}[(8 + 2y)^{3/2} - (2y)^{3/2}]\,dy$

$\displaystyle = \frac{2}{45}[(8 + 2y)^{5/2} - (2y)^{5/2}]\Big|_0^3$

$\displaystyle = \frac{2}{45}(14^{5/2} - 6^{5/2} - 8^{5/2})$

28. $\displaystyle\iint_R \frac{y}{\sqrt{6x + 5y^2}}\,dx\,dy; 0 \le x \le 3, 1 \le y \le 2$

$\displaystyle\iint_R \frac{y}{\sqrt{6x + 5y^2}}\,dx\,dy$

$\displaystyle = \int_1^2 \int_0^3 \frac{y}{\sqrt{6x + 5y^2}}\,dx\,dy$

$\displaystyle = \int_1^2 \int_0^3 y(6x + 5y^2)^{-1/2}\,dx\,dy$

$\displaystyle = \int_1^2 \frac{y}{3}(6x + 5y^2)^{1/2}\Big|_0^3 dy$

$\displaystyle = \int_1^2 \frac{y}{3}[(18 + 5y^2)^{1/2} - (5y^2)^{1/2}]\,dy$

$\displaystyle = \int_1^2 \frac{1}{3}[y(18 + 5y^2)^{1/2} - y(5y^2)^{1/2}]\,dy$

$\displaystyle = \frac{1}{3}\left[\frac{1}{15}(18 + 5y^2)^{3/2} - \frac{(5y^2)^{3/2}}{15}\right]\Big|_1^2$

$\displaystyle = \frac{1}{45}(38^{3/2} - 20^{3/2} - 23^{3/2} + 5^{3/2})$

or $\displaystyle\frac{1}{45}(38^{3/2} - 23^{3/2} - 35\sqrt{5})$

30. $\iint_R x^2 e^{x^3+2y}\, dx\, dy; 1 \le x \le 2, 1 \le y \le 3$

$$\begin{aligned}\iint_R x^2 e^{x^3+2y}\, dx\, dy \\ &= \int_1^3 \int_1^2 x^2 e^{x^3+2y}\, dx\, dy \\ &= \int_1^3 \frac{1}{3} e^{x^3+2y} \Big|_1^2 \, dy \\ &= \int_1^3 \frac{1}{3}(e^{8+2y} - e^{1+2y})\, dy \\ &= \frac{1}{6}(e^{8+2y} - e^{1+2y}) \Big|_1^3 \\ &= \frac{1}{6}(e^{14} - e^7 - e^{10} + e^3)\end{aligned}$$

32. $z = 9x + 5y + 12; 0 \le x \le 3, -2 \le y \le 1$

$$\begin{aligned}V &= \int_{-2}^1 \int_0^3 (9x + 5y + 12)\, dx\, dy \\ &= \int_{-2}^1 \left(\frac{9x^2}{2} + 5xy + 12x\right)\Big|_0^3 \, dy \\ &= \int_{-2}^1 \left(\frac{81}{2} + 15y + 36\right) dy \\ &= \int_{-2}^1 \left(\frac{153}{2} + 15y\right) dy \\ &= \left(\frac{153y}{2} + \frac{15y^2}{2}\right)\Big|_{-2}^1 \\ &= \frac{1}{2}(153 + 15 + 306 - 60) \\ &= 207\end{aligned}$$

34. $z = \sqrt{y}; 0 \le x \le 4, 0 \le y \le 9$

$$\begin{aligned}V &= \int_0^4 \int_0^9 y^{1/2}\, dy\, dx \\ &= \int_0^4 \frac{2}{3} y^{3/2}\Big|_0^9 \, dx = \int_0^4 \frac{2}{3}(27)\, dx \\ &= \int_0^4 18\, dx = 18x\Big|_0^4 \\ &= 72\end{aligned}$$

36. $z = yx\sqrt{x^2 + y^2}; 0 \le x \le 4, 0 \le y \le 1$

$$\begin{aligned}V &= \int_0^1 \int_0^4 yx(x^2 + y^2)^{1/2}\, dx\, dy \\ &= \int_0^1 \frac{y}{3}(x^2 + y^2)^{3/2}\Big|_0^4 \, dy \\ &= \int_0^1 \frac{y}{3}[(16 + y^2)^{3/2} - (y^2)^{3/2}]\, dy \\ &= \int_0^1 \frac{1}{3}[y(16 + y^2)^{3/2} - y^4]\, dy \\ &= \frac{1}{3}\left[\frac{1}{5}(16 + y^2)^{5/2} - \frac{y^5}{5}\right]\Big|_0^1 \\ &= \frac{1}{15}(17^{5/2} - 1 - 16^{5/2}) \\ &= \frac{1}{15}(17^{5/2} - 1 - 1024) \\ &= \frac{1}{15}(17^{5/2} - 1025)\end{aligned}$$

38. $z = e^{x+y}; 0 \le x \le 1, 0 \le y \le 1$

$$\begin{aligned}V &= \int_0^1 \int_0^1 e^{x+y}\, dx\, dy \\ &= \int_0^1 e^{x+y}\Big|_0^1 \, dy \\ &= \int_0^1 (e^{1+y} - e^y)\, dy \\ &= (e^{1+y} - e^y)\Big|_0^1 \\ &= e^2 - e - e + 1 \\ &= e^2 - 2e + 1\end{aligned}$$

40. $\iint_R 2x^3 e^{x^2 y}\, dx\, dy; 0 \le x \le 1, 0 \le y \le 1$

$$\begin{aligned}\iint_R 2x^3 e^{x^2 y}\, dx\, dy &= \int_0^1 \int_0^1 2x^3 e^{x^2 y}\, dy\, dx \\ &= \int_0^1 \left(2x^3 \cdot \frac{1}{x^2} e^{x^2 y}\right)\Big|_0^1 \, dx \\ &= \int_0^1 (2xe^{x^2} - 2xe^0)\, dx \\ &= (e^{x^2} - x^2)\Big|_0^1 \\ &= (e^1 - 1) - (e^0 - 0) \\ &= e - 2\end{aligned}$$

42. $\int_0^5 \int_0^{2y} (x^2+y)\,dx\,dy$

$$= \int_0^5 \left(\frac{x^3}{3} + xy\right)\bigg|_0^{2y} dy$$
$$= \int_0^5 \left(\frac{8y^3}{3} + 2y^2\right)\bigg|_0^5 dy$$
$$= \left(\frac{2y^4}{3} + \frac{2y^3}{3}\right)\bigg|_0^5$$
$$= \frac{2}{3}[625+125]$$
$$= 500$$

44. $\int_1^4 \int_0^x \sqrt{x+y}\,dy\,dx$

$$= \int_1^4 \int_0^x (x+y)^{1/2}\,dy\,dx$$
$$= \int_1^4 \frac{2}{3}(x+y)^{3/2}\bigg|_0^x dx$$
$$= \int_1^4 \frac{2}{3}[(2x)^{3/2} - x^{3/2}]\,dx$$
$$= \frac{2}{3}\left[\frac{1}{5}(2x)^{5/2} - \frac{2}{5}x^{5/2}\right]\bigg|_1^4$$
$$= \frac{2}{15}(8^{5/2} - 2(4)^{5/2} - 2^{5/2} + 2)$$
$$= \frac{2}{15}(8^{5/2} - 64 - 2^{5/2} + 2)$$
$$= \frac{2}{15}(8^{5/2} - 62 - 2^{5/2})$$

46. $\int_1^4 \int_x^{x^2} \frac{1}{y}\,dy\,dx = \int_1^4 \ln y\bigg|_x^{x^2} dx$

$$= \int_1^4 (\ln x^2 - \ln x)\,dx$$
$$= \int_1^4 (2\ln x - \ln x)\,dx$$
$$= \int_1^4 \ln x\,dx$$
$$= \left(x\ln x - \int dx\right)\bigg|_1^4$$

Integration by parts
$u = \ln x,\ dv = dx$
$du = \frac{1}{x}\,dx,\ v = x$

$$= (x\ln x - x)\bigg|_1^4$$
$$= 4\ln 4 - 4 + 1$$
$$= 4\ln 4 - 3$$

Note: We can say $\ln y$ instead $\ln|y|$ since x is in $[1,4]$ and y is in $[x, x^2]$, so $y > 0$.

48. $\int_0^1 \int_{2x}^{4x} e^{x+y}\,dy\,dx$

$$= \int_0^1 e^{x+y}\bigg|_{2x}^{4x} dx = \int_0^1 (e^{5x} - e^{3x})\,dx$$
$$= \left(\frac{1}{5}e^{5x} - \frac{1}{3}e^{3x}\right)\bigg|_0^1 = \frac{1}{5}e^5 - \frac{1}{3}e^3 - \frac{1}{5} + \frac{1}{3}$$
$$= \frac{e^5}{5} - \frac{e^3}{3} + \frac{2}{15}$$

50. $\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$

Change the order of integration.

$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx$$
$$= \int_0^1 e^{x^2} y\bigg|_0^{2x} dx$$
$$= \int_0^1 (2xe^{x^2} - 0)\,dx$$
$$= e^{x^2}\bigg|_0^1 = e^1 - e^0$$
$$= e - 1$$

52. $\iint_R (3x+9y)\,dy\,dx;\ 2 \le x \le 4, 2 \le y \le 3x$

$$\iint_R (3x+9y)\,dy\,dx$$
$$= \int_2^4 \int_2^{3x} (3x+9y)\,dy\,dx$$
$$= \int_2^4 \left[3xy + \frac{9y^2}{2}\right]\bigg|_2^{3x} dx$$
$$= \int_2^4 \left[9x^2 + \frac{81x^2}{2} - 6x - 18\right] dx$$
$$= \int_2^4 \left[\frac{99x^2}{2} - 6x - 18\right] dx$$
$$= \left(\frac{33x^3}{2} - 3x^2 - 18x\right)\bigg|_2^4$$
$$= 1056 - 48 - 72 - 132 + 12 + 36 = 852$$

54. $\iint\limits_R \frac{dy\,dx}{x}; 1 \le x \le 2, 0 \le y \le x-1$

$$\iint\limits_R \frac{dy\,dx}{x}$$

$$= \int_1^2 \int_0^{x-1} \frac{dy\,dx}{x}$$

$$= \int_1^2 \frac{y}{x}\Big|_0^{x-1} \, dx$$

$$= \int_1^2 \frac{x-1}{x} \, dx$$

$$= \int_1^2 \left[1 - \frac{1}{x}\right] dx$$

$$= (x - \ln x)\Big|_1^2$$

$$= 2 - \ln 2 - 1$$
$$= 1 - \ln 2$$

56. $\iint\limits_R (x^2 - y)\,dy\,dx; -1 \le x \le 1, -x^2 \le y \le x^2$

$$\iint\limits_R (x^2 - y)\,dy\,dx$$

$$= \int_{-1}^1 \int_{-x^2}^{x^2} (x^2 - y)\,dy\,dx$$

$$= \int_{-1}^1 \left[x^2 y - \frac{y^2}{2}\right]\Bigg|_{-x^2}^{x^2} dx$$

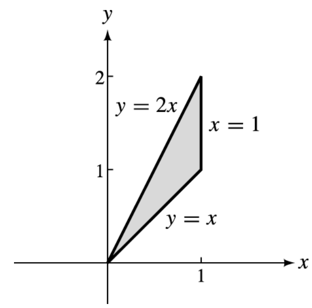
$$= \int_{-1}^1 \left[x^4 - \frac{x^4}{2} + x^4 + \frac{x^4}{2}\right] dx$$

$$= \int_{-1}^1 2x^4\,dx = \frac{2x^5}{5}\Big|_{-1}^1$$

$$= \frac{2}{5} + \frac{2}{5}$$

$$= \frac{4}{5}$$

58. $\iint\limits_R x^2y^2\,dy\,dx$; R bounded by $y = x$, $y = 2x$, and $x = 1$



$$\int_0^1 \int_x^{2x} x^2y^2\,dy\,dx$$

$$= \int_0^1 \frac{x^2y^3}{3}\Big|_x^{2x} dx$$

$$= \int_0^1 \left(\frac{8x^5}{3} - \frac{x^5}{3}\right) dx$$

$$= \int_0^1 \frac{7x^5}{3}\,dx$$

$$= \frac{7x^6}{18}\Big|_0^1$$

$$= \frac{7}{18}$$

62. $f(x, y) = 5xy + 2y;\ 1 \le x \le 4,\ 1 \le y \le 2$

The area of region R is

$$A = (4-1)(2-1) = 3.$$

The average value of f over R is

$$\frac{1}{3}\int_1^4 \int_1^2 (5xy + 2y)\,dy\,dx$$

$$= \frac{1}{3}\int_1^4 \left[\frac{5xy^2}{2} + y^2\right]\Bigg|_1^2 dx$$

$$= \frac{1}{3}\int_1^4 \left[10x + 4 - \frac{5x}{2} - 1\right] dx$$

$$= \frac{1}{3}\int_1^4 \left[\frac{15x}{2} + 3\right] dx$$

$$= \frac{1}{3}\left[\frac{15x^2}{4} + 3x\right]\Bigg|_1^4$$

$$= \frac{1}{3}\left[60 + 12 - \frac{15}{4} - 3\right]$$

$$= \frac{87}{4}.$$

64. $f(x,y) = e^{-5y+3x};\ 0 \le x \le 2,\ 0 \le y \le 2$

The area of region R is

$$(2-0)(2-0) = 4.$$

The average value of f over R is

$$\frac{1}{4}\int_0^2 \int_0^2 e^{-5y+3x}\,dy\,dx$$

$$= \frac{1}{4}\int_0^2 -\frac{1}{5}e^{-5y+3x}\Big|_0^2\,dx$$

$$= \frac{1}{4}\int_0^2 -\frac{1}{5}[e^{3x-10} - e^{3x}]\,dx$$

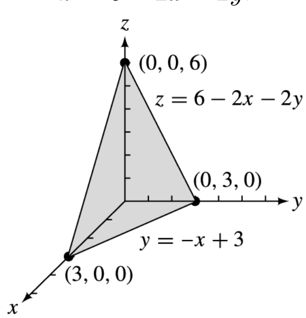
$$= -\frac{1}{20}\left[\frac{1}{3}e^{3x-10} - \frac{1}{3}e^{3x}\right]\Big|_0^2$$

$$= -\frac{1}{60}[e^{-4} - e^{6} - e^{-10} + 1]$$

$$= \frac{e^6 + e^{-10} - e^{-4} - 1}{60}.$$

66. The plane that intersects the axes has the equation

$$z = 6 - 2x - 2y.$$



$$V = \iint_R f(x,y)\,dA$$

$$= \int_0^3 \int_0^{-x+3} (6 - 2x - 2y)\,dy\,dx$$

$$= \int_0^3 (6y - 2xy - y^2)\Big|_0^{-x+3}\,dx$$

$$= \int_0^3 [-6x + 18 - 2x(-x+3)$$

$$- (3-x)^2]\,dx$$

$$= \int_0^3 (-6x + 18 + 2x^2 - 6x - 9 + 6x - x^2)\,dx$$

$$= \int_0^3 (x^2 - 6x + 9)\,dx = \left(\frac{x^3}{3} - 3x^2 + 9x\right)\Big|_0^3$$

$$= (9 - 27 + 27) - 0 = 9$$

The volume is 9 in^3.

68. $P(x,y) = 500x^{.2}y^{.8},\ 10 \le x \le 50,$
$20 \le y \le 40$

$A = 40 \cdot 20 = 800$

Average production:

$$\frac{1}{800}\int_{10}^{50}\int_{20}^{40} 500x^{.2}y^{.8}\,dy\,dx$$

$$= \frac{5}{8}\int_{10}^{50} \frac{x^{.2}y^{1.8}}{1.8}\Big|_{20}^{40}\,dx$$

$$= \frac{25}{72}\int_{10}^{50} x^{.2}(40^{1.8} - 20^{1.8})\,dx$$

$$= \frac{25(40^{1.8} - 20^{1.8})}{72}\cdot\frac{x^{1.2}}{1.2}\Big|_{10}^{50}$$

$$= \frac{125}{432}(40^{1.8} - 20^{1.8})(50^{1.2} - 10^{1.2})$$

$\approx 14{,}750$ units

70. $R = q_1p_1 + q_2p_2$ where $q_1 = 300 - 2p_1,$
$q_2 = 500 - 1.2p_2,\ 25 \le p_1 \le 50,$ and
$50 \le p_2 \le 75.$

$A = 25 \cdot 25 = 625$
$R = (300 - 2p_1)p_1 + (500 - 1.2p_2)p_2$
$R = 300p_1 - 2p_1^2 + 500p_2 - 1.2p_2^2$

Average Revenue:

$$\frac{1}{625}\int_{25}^{50}\int_{50}^{75} (300p_1 - 2p_1^2 + 500p_2 - 1.2p_2^2)\,dp_2dp_1$$

$$= \frac{1}{625}\int_{25}^{50} (300p_1p_2 - 2p_1^2p_2 + 250p_2^2 - .4p_2^3)\Big|_{50}^{75}\,dp_1$$

$$= \frac{1}{625}\int_{25}^{50} (22{,}500p_1 - 150p_1^2 + 1{,}406{,}250 + 50{,}000)dp$$

$$-168{,}750 - 15{,}000p_1 + 100p_1^2 - 625{,}000$$

$$= \frac{1}{625}\int_{25}^{50} (662{,}500 + 7500p_1 - 50p_1^2)\,dp_1$$

$$= \frac{1}{625}\left(662{,}500p_1 + 3750p_1^2 - \frac{50p_1^3}{3}\right)\Big|_{25}^{50}$$

$$= \frac{1}{625}\Bigg(33{,}125{,}000 + 9{,}375{,}000 - \frac{6{,}250{,}000}{3}$$

$$- 16{,}562{,}500 - 2{,}343{,}750 + \frac{781{,}250}{3}\Bigg)$$

$\approx \$34{,}833$

72. $P(x,y) = 36xy - x^3 - 8y^3$

$$\text{Areas} = (8-0)(4-0) = 32$$

The average profit is

$$\frac{1}{32}\iint_R (36xy - x^3 - 8y^3)dy\,dx$$

$$= \frac{1}{32}\int_0^8 \int_0^4 (36xy - x^3 - 8y^3)dy\,dx$$

$$= \frac{1}{32}\int_0^8 \left(\frac{36xy^2}{2} - x^3y - \frac{8y^4}{4}\right)\Bigg|_0^4 dx$$

$$= \frac{1}{32}\int_0^8 \left[\frac{36x(4-0)^2}{2} - x^3(4-0) - \frac{8(4-0)^4}{4}\right] dx$$

$$= \frac{1}{32}\int_0^8 (288x - 4x^3 - 512)dx$$

$$= \frac{1}{32}\left(\frac{288x^2}{2} - \frac{4x^4}{4} - 512x\right)\Bigg|_0^8 dx$$

$$= \frac{1}{32}\left[\frac{288(8-0)^2}{2} - \frac{4(8-0)^4}{4} - 512(8-0)\right]$$

$$= \frac{1}{32}(9216 - 4096 - 4096)$$

$= 32$ thousand dollars

Chapter 9 Review Exercises

4. $f(x,y) = -4x^2 + 6xy - 3$

$$f(-1,2) = -4(-1)^2 + 6(-1)(2) - 3 = -19$$

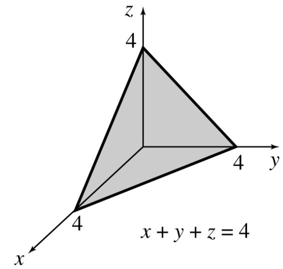
$$f(6,-3) = -4(6)^2 + 6(6)(-3) - 3 = -4(36) + (-108) - 3 = -255$$

6. $f(x,y) = \dfrac{x-3y}{x+4y}$

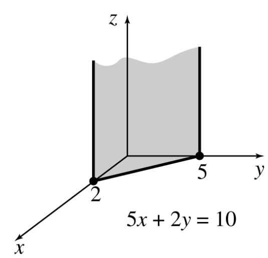
$$f(-1,2) = \frac{-1-6}{-1+8} = \frac{-7}{7} = -1$$

$$f(6,-3) = \frac{6-3(-3)}{6+4(-3)} = \frac{6+9}{6-12} = \frac{15}{-6} = -\frac{5}{2}$$

8. The plane $x + y + z = 4$ intersects the axes at $(4,0,0)$, $(0,4,0)$, and $(0,0,4)$.

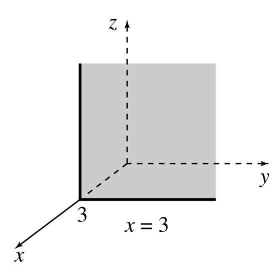


10. The plane $5x + 2y = 10$ intersects the x- and y-axes at $(2,0,0)$ and $(0,5,0)$. Note that there is no z-intercept since $x = y = 0$ is not a solution of the equation of the plane.



12. $x = 3$

The plane is parallel to the yz-plane. It intersects the x-axis at $(3,0,0)$.



14. Let $z = f(x,y) = -5x^2 + 7xy - y^2$

(a) $\dfrac{\partial z}{\partial x} = -10x + 7y$

(b) $\dfrac{\partial z}{\partial y} = 7x - 2y$

$$\left(\frac{\partial z}{\partial y}\right)(-1,4) = 7(-1) - 2(4) = -15$$

(c) $f_{xy}(x,y) = 7$

$f_{xy}(2,-1) = 7$

16. $f(x,y) = 9x^3y^2 - 5x$

$f_x(x,y) = 27x^2y^2 - 5$

$f_y(x,y) = 18x^3y$

18. $f(x,y) = \sqrt{4x^2+y^2}$

$$f_x(x,y) = \frac{1}{2}(4x^2+y^2)^{-1/2}(8x) = \frac{4x}{(4x^2+y^2)^{1/2}}$$

$$f_y(x,y) = \frac{1}{2}(4x^2+y^2)^{-1/2}(2y) = \frac{y}{(4x^2+y^2)^{1/2}}$$

20. $f(x,y) = x^2 \cdot e^{2y}$

$$f_x(x,y) = 2xe^{2y}$$
$$f_y(x,y) = 2x^2e^{2y}$$

22. $f(x,y) = \ln\left|2x^2+y^2\right|$

$$f_x(x,y) = \frac{1}{2x^2+y^2}\cdot 4x = \frac{4x}{2x^2+y^2}$$

$$f_y(x,y) = \frac{1}{2x^2+y^2}\cdot 2y = \frac{2y}{2x^2+y^2}$$

24. $f(x,y) = 4x^3y^2 - 8xy$

$$f_x(x,y) = 12x^2y^2 - 8y$$
$$f_{xx}(x,y) = 24xy^2$$
$$f_{xy}(x,y) = 24x^2y - 8$$

26. $f(x,y) = \dfrac{2x}{x-2y}$

$$f_x(x,y) = \frac{2(x-2y)-(2x)}{(x-2y)^2} = \frac{-4y}{(x-2y)^2} = -4y(x-2y)^{-2}$$

$$f_{xx}(x,y) = -4y[-2(x-2y)^{-3}] = 8y(x-2y)^{-3} = \frac{8y}{(x-2y)^3}$$

Use the quotient rule on

$$f_x(x,y) = \frac{-4y}{(x-2y)^2}$$

to get

$$f_{xy}(x,y) = \frac{-4(x-2y)^2+4y[2(x-2y)(-2)]}{(x-2y)^4} = \frac{-4(x-2y)[(x-2y)+4y]}{(x-2y)^4} = \frac{-4(x+2y)}{(x-2y)^3} = \frac{-4x-8y}{(x-2y)^3}.$$

28. $f(x,y) = x^2e^y$

$$f_x(x,y) = 2xe^y$$
$$f_{xx}(x,y) = 2e^y$$
$$f_{xy}(x,y) = 2xe^y$$

30. $f(x,y) = \ln\left|2-x^2y\right|$

$$f_x(x,y) = \frac{1}{2-x^2y}\cdot(-2xy) = \frac{2xy}{x^2y-2}$$

$$f_{xx}(x,y) = \frac{(x^2y-2)2y-2xy(2xy)}{(x^2y-2)^2} = \frac{2y[(x^2y-2)-2x^2y]}{(x^2y-2)^2} = \frac{2y(-x^2y-2)}{(x^2y-2)^2} = \frac{-2x^2y^2-4y}{(2-x^2y)^2}$$

$$f_{xy}(x,y) = \frac{2x(x^2y-2)-x^2(2xy)}{(x^2y-2)^2} = \frac{2x[(x^2y-2)-x^2y]}{(x^2y-2)^2} = \frac{2x(-2)}{(x^2y-2)^2} = \frac{-4x}{(2-x^2y)^2}$$

32. $z = x^2 + 2y^2 - 4y$

$$\begin{aligned} z_x(x,y) &= 2x \\ z_y(x,y) &= 4y - 4 \end{aligned}$$

Setting z_x and z_y equal to zero simultaneously implies $x = 0$ and $y = 1$.

$$z_{xx}(x,y) = 2,\ z_{yy}(x,y) = 4,\ z_{xy}(x,y) = 0$$

For $(0,1)$,

$$D = 2 \cdot 4 - 0 = 8 > 0.$$

Since

$$z_{xx}(x,y)(0,1) = 2 > 0,$$

z has a relative minimum at $(0,1)$.

34. $f(x,y) = x^2 + 5xy - 10x + 3y^2 - 12y$

$$\begin{aligned} f_x(x,y) &= 2x + 5y - 10 \\ f_y(x,y) &= 5x + 6y - 12 \end{aligned}$$

Setting f_x and f_y equal to zero and solving yields

$$\begin{aligned} 2x + 5y &= 10 \\ 5x + 6y &= 12. \end{aligned}$$

$$\begin{aligned} -10x - 25y &= -50 \\ 10x + 12y &= 24 \\ \hline -13y &= -26 \\ y &= 2 \end{aligned}$$

$$\begin{aligned} 2x + 10 &= 10 \\ x &= 0 \end{aligned}$$

$$f_{xx}(x,y) = 2,\ f_{yy}(x,y) = 6,\ f_{xy}(x,y) = 5.$$

For $(0,2)$,

$$D = 2 \cdot 6 - 5^2 = -13 < 0.$$

Therefore, f has a saddle point at $(0,2)$.

36. $z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 2xy - 5x - 7y + 10$

$$\begin{aligned} z_x(x,y) &= x + 2y - 5 \\ z_y(x,y) &= y + 2x - 7 \end{aligned}$$

Setting $z_x = z_y = 0$ and solving yields

$$\begin{aligned} x + 2y &= 5 \\ 2x + \ \ y &= 7. \\ -2x - 4y &= -10 \\ 2x + y &= 7 \\ \hline -3y &= -3 \\ y = 1,&\ x = 3. \end{aligned}$$

$$z_{xx}(x,y) = 1,\ z_{yy}(x,y) = 1,\ z_{xy}(x,y) = 2$$

For $(3,1)$,

$$D = 1 \cdot 1 - 4 = -3 < 0.$$

Therefore, z has a saddle point at $(3,1)$.

38. $z = x^3 + y^2 + 2xy - 4x - 3y - 2$

$$\begin{aligned} z_x(x,y) &= 3x^2 + 2y - 4 \\ z_y(x,y) &= 2y + 2x - 3 \end{aligned}$$

Setting $z_x(x,y) = z_y(x,y) = 0$ yields

$$\begin{aligned} 3x^2 + 2y - 4 &= 0 \quad (1) \\ 2y \ + 2x - 3 &= 0. \quad (2) \end{aligned}$$

Solving for $2y$ in equation (2) gives $2y = -2x + 3$. Substitute into equation (1).

$$\begin{aligned} 3x^2 + (-2x) + 3 - 4 &= 0 \\ 3x^2 - 2x - 1 &= 0 \\ (3x+1)(x-1) &= 0 \end{aligned}$$

$$x = -\frac{1}{3} \quad \text{or} \quad x = 1$$

$$y = \frac{11}{6} \quad \text{or} \quad y = \frac{1}{2}$$

$$z_{xx}(x,y) = 6x,\ z_{yy}(x,y) = 2,\ z_{xy}(x,y) = 2$$

For $\left(-\frac{1}{3}, \frac{11}{6}\right)$,

$$\begin{aligned} D &= 6\left(-\frac{1}{3}\right)(2) - 4 \\ &= -4 - 4 = -8 < 0, \end{aligned}$$

so z has a saddle point at $\left(-\frac{1}{3}, \frac{11}{6}\right)$.

$$D = 6(1)(2) - 4 = 8 > 0.$$

$z_{xx}\left(1, \frac{1}{2}\right) = 6 > 0$, so z has a relative minimum at $\left(1, \frac{1}{2}\right)$.

42. $f(x,y) = x^2 + y^2$, subject to $x = y + 2$.

1. $g(x,y) = x - y - 2$

2. $F(x,y,\lambda) = x^2 + y^2 - \lambda(x - y - 2)$

3. $F_x(x,y,\lambda) = 2x - \lambda$
$F_y(x,y,\lambda) = 2y + \lambda$
$F_\lambda(x,y,\lambda) = -(x - y - 2)$

4.
$$\begin{aligned} 2x - \lambda &= 0 \\ 2y + \lambda &= 0 \\ x - y - 2 &= 0 \end{aligned}$$

5. $\lambda = 2x,\ \lambda = -2y$

$$\begin{aligned} 2x &= -2y \\ y &= -x \end{aligned}$$

Substituting into $x - y - 2 = 0$, we have

$$\begin{aligned} x + x - 2 &= 0 \\ x = 1,\ y &= -1, \end{aligned}$$

so the extremum is $f(1,-1) = 2$ at $(1,-1)$.

$F_{xx} = 2,\ F_{yy} = 2,\ F_{xy} = 0$

$D = 2 \cdot 2 - (0)^2 = 4$ and $F_{xx} > 0$.

$F(1,-1) = f(1,-1) = 2$ is a relative minimum.

44. Maximize $f(x,y) = xy^2$, subject to $x + y = 50$.

1. $g(x,y) = x + y - 50$

2. $F(x,y,\lambda) = xy^2 - \lambda(x + y - 50)$

3. $F_x(x,y,\lambda) = y^2 - \lambda$
$F_y(x,y,\lambda) = 2xy - \lambda$
$F_\lambda(x,y,\lambda) = -(x + y - 50)$

4.
$$\begin{aligned} y^2 - \lambda &= 0 \\ 2xy - \lambda &= 0 \\ x + y - 50 &= 0 \end{aligned}$$

5. $\lambda = y^2,\ \lambda = 2xy$

$$\begin{aligned} y^2 &= 2xy \\ y^2 - 2xy &= 0 \\ y(y - 2x) &= 0 \end{aligned}$$

$y \neq 0$ since f is larger for positive x and y than for $y = 0$. So $y = 2x$. Substituting into $x + y - 50 = 0$, we have

$$x + 2x - 50 = 0,$$

so $x = \frac{50}{3},\ y = \frac{100}{3}$.

46. $z = 2x^2 - 4y^2 + 6xy;\ x = 2,\ y = -3,$
$dx = .01,\ dy = .05$

$$\begin{aligned} f_x(x,y) &= 4x + 6y \\ f_y(x,y) &= -8y + 6x \\ dz &= (4x + 6y)\,dx + (-8y + 6x)\,dy \end{aligned}$$

Substitute.

$$\begin{aligned} dz &= [4(2) + 6(-3)](.01) \\ &\quad + [-8(-3) + 6(2)](.05) \\ &= -.1 + 1.8 = 1.7 \end{aligned}$$

48. Let $z = f(x,y) = \sqrt{x^2 + y^2}$.

Then

$$\begin{aligned} dz &= f_x(x,y)\,dx + f_y(x,y)\,dy. \\ dz &= \frac{1}{2}(x^2 + y^2)^{-1/2}(2x)\,dx \\ &\quad + \frac{1}{2}(x^2 + y^2)(2y)\,dx \\ &= \frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy \end{aligned}$$

To approximate $\sqrt{5.1^2 + 12.05^2}$, we let $x = 5$, $dx = .1,\ y = 12$, and $dy = .05$.

Then,

$$\begin{aligned} dz &= \frac{5}{\sqrt{5^2 + 12^2}}(.1) + \frac{12}{\sqrt{5^2 + 12^2}}(.05) \\ &= \frac{5}{13}(.1) + \frac{12}{13}(.05) \\ &\approx .0846. \end{aligned}$$

Therefore,

$$\begin{aligned} f(5.1, 12.05) &= f(5,12) + \Delta z \\ &\approx f(5,12) + dz \\ &= \sqrt{5^2 + 12^2} + .0846 \\ f(5.1, 12.05) &\approx 13.0846 \end{aligned}$$

Using a calculator, $\sqrt{5.1^2 + 12.05^2} \approx 13.0848$. The absolute value of the difference of the two results is $|13.0846 - 13.0848| = .0002$.

50. $\displaystyle\int_4^9 \frac{6y - 8}{\sqrt{x}}\,dx$

$$\begin{aligned} &= (6y - 8)(2\sqrt{x})\Big|_4^9 \\ &= 2(3y - 4)(2)(\sqrt{9} - \sqrt{4}) \\ &= 4(3y - 4) \\ &= 12y - 16 \end{aligned}$$

52. $\displaystyle\int_0^5 \frac{6x}{\sqrt{4x^2 + 2y^2}}\,dx$

Let $u = 4x^2 + 2y^2$; then $du = 8x\,dx$.
When $x = 0,\ u = 2y^2$.
When $x = 5,\ u = 100 + 2y^2$.

$$\begin{aligned} &= \frac{3}{4}\int_{2y^2}^{100+2y^2} u^{-1/2}\,du \\ &= \frac{3}{4}(2u^{1/2})\Big|_{2y^2}^{100+2y^2} \\ &= \frac{3}{4} \cdot 2[(100 + 2y^2)^{1/2} - (2y^2)^{1/2}] \\ &= \frac{3}{2}[(100 + 2y^2)^{1/2} - (2y^2)^{1/2}] \end{aligned}$$

54. $\displaystyle\int_0^2 \left[\int_0^4 (x^2y^2+5x)\,dx\right] dy$

$$= \int_0^2 \left(\frac{1}{3}x^3y^2 + \frac{5}{2}x^2\right)\Big|_0^4 dx$$

$$= \int_0^2 \left(\frac{64}{3}y^2 + 40\right) dy$$

$$= \left(\frac{64y^3}{9} + 40y\right)\Big|_0^2$$

$$= \frac{64}{9}(8) + 40(2)$$

$$= \frac{512}{9} + \frac{720}{9}$$

$$= \frac{1232}{9}$$

56. $\displaystyle\int_3^4 \left[\int_2^5 \sqrt{6x+3y}\,dx\right] dy$

$$= \int_3^4 \frac{1}{9}(6x+3y)^{3/2}\Big|_2^5 dx$$

$$= \int_3^4 \frac{1}{9}[(30+3y)^{3/2} - (12+3y)^{3/2}]\,dy$$

$$= \frac{1}{3}\cdot\frac{1}{9}\cdot\frac{2}{5}$$

$$\cdot\,[(30+3y)^{5/2} - (12+3y)^{5/2}]\,\Big|_3^4$$

$$= \frac{2}{135}[(42)^{5/2} - (24)^{5/2} - (39)^{5/2} + (21)^{5/2}]$$

58. $\displaystyle\int_2^4 \int_2^4 \frac{dx\,dy}{y}$

$$= \int_2^4 \left(\frac{1}{y}x\right)\Big|_2^4 dy$$

$$= \int_2^4 \left[\frac{1}{y}(4-2)\right] dy$$

$$= 2\ \ln\,|y|\,\Big|_2^4$$

$$= 2\ \ln\left|\frac{4}{2}\right|$$

$$= 2\ \ln\ 2 \text{ or } \ln\ 4$$

60. $\displaystyle\iint\limits_R (x^2+y^2)\,dx\,dy;\ 0\le x\le 2,\ 0\le y\le 3$

$$\iint\limits_R (x^2+y^2)\,dx\,dy$$

$$= \int_0^3 \int_0^2 (x^2+y^2)\,dx\,dy$$

$$= \int_0^3 \left(\frac{x^3}{3} + y^2x\right)\Big|_0^2 dy$$

$$= \int_0^3 \left(\frac{8}{3} + 2y^2\right) dy$$

$$= \left(\frac{8}{3}y + \frac{2}{3}y^3\right)\Big|_0^3$$

$$= 8 + 18 = 26$$

62. $\displaystyle\iint\limits_R \sqrt{y+x}\,dx\,dy;\ 0\le x\le 7,\ 1\le y\le 9$

$$\iint\limits_R \sqrt{y+x}\,dx\,dy$$

$$= \int_0^7 \int_1^9 \sqrt{y+x}\,dy\,dx$$

$$= \int_0^7 \left[\frac{2}{3}(y+x)^{3/2}\right]\Big|_1^9 dx$$

$$= \int_0^7 \frac{2}{3}[(9+x)^{3/2} - (1+x)^{3/2}]\,dx$$

$$= \frac{2}{3}\cdot\frac{2}{5}[(9+x)^{5/2} - (1+x)^{5/2}]\,\Big|_0^7$$

$$= \frac{4}{15}[(16)^{5/2} - (8)^{5/2} - (9)^{5/2} + (1)^{5/2}]$$

$$= \frac{4}{15}[4^5 - (2\sqrt{2})^5 - 3^5 + 1]$$

$$= \frac{4}{15}(1024 - 32(4\sqrt{2}) - 243 + 1)$$

$$= \frac{4}{15}(782 - 128\sqrt{2})$$

$$= \frac{4}{15}(782 - 8^{5/2})$$

64. $z = x + 9y + 8;\ 1 \le x \le 6,\ 0 \le y \le 8$

$$V = \iint_R (x + 9y + 8)\,dx\,dy$$
$$= \int_0^8 \int_1^6 (x + 9y + 8)\,dx\,dy$$
$$= \int_0^8 \left[\frac{x^2}{2} + (9y + 8)x\,dx\right]\Bigg|_1^6 dy$$
$$= \int_0^8 \left[\frac{36 - 1}{2} + (9y + 8)(6 - 1)\right] dy$$
$$= \left[\frac{35}{2}y + 5\left(\frac{9}{2}y^2 + 8y\right)\right]\Bigg|_0^8$$
$$= \frac{35}{2}(8) + 5\left[\left(\frac{9}{2}\right)(64) + 8 \cdot 8\right)\right]$$
$$= \frac{35(8)}{2} + 5\left(\frac{9(64)}{2} + 8 \cdot 8\right)$$
$$= 140 + 5(352)$$
$$= 1900$$

66. $\int_0^1 \int_0^{2x} xy\,dy\,dx$

$$= \int_0^1 \left(\frac{xy^2}{2}\right)\Bigg|_0^{2x} dx$$
$$= \int_0^1 \frac{x}{2}(4x^2 - 0)\,dx$$
$$= \int_0^1 2x^3\,dx$$
$$= \left(\frac{1}{2}x^4\right)\Bigg|_0^1 = \frac{1}{2}$$

68. $\int_0^1 \int_{x^2}^{x} x^3y\,dy\,dx$

$$\int_0^1 \left(\frac{x^3}{2}y^2\right)\Bigg|_{x^2}^{x} dx$$
$$= \int_0^1 \frac{x^3}{2}(x^2 - x^4)\,dx$$
$$= \frac{1}{2}\int_0^1 (x^5 - x^7)\,dx$$
$$= \frac{1}{2}\left(\frac{x^6}{6} - \frac{x^8}{8}\right)\Bigg|_0^1$$
$$= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{8}\right) = \frac{1}{2} \cdot \frac{1}{24} = \frac{1}{48}$$

70. $\int_0^2 \int_{x/2}^{1} \frac{1}{y^2 + 1}\,dy\,dx$

Change the order of integration.

$$\int_0^2 \int_{x/2}^{1} \frac{1}{y^2 + 1}\,dy\,dx$$
$$= \int_0^1 \int_0^{2y} \frac{1}{y^2 + 1}\,dx\,dy$$
$$= \int_0^1 \frac{x}{y^2 + 1}\Bigg|_0^{2y} dy$$
$$= \int_0^1 \left[\frac{1}{y^2 + 1}(2y) - \frac{1}{y^2 + 1}(0)\right] dy$$
$$= \int_0^1 \frac{2y}{y^2 + 1}\,dy$$
$$= \ln(y^2 + 1)\Big|_0^1$$
$$= \ln 2 - \ln 1$$
$$= \ln 2 - 0 = \ln 2$$

72. $\iint_R (2x + 3y)\,dx\,dy;\ 0 \le y \le 1,$

$y \le x \le 2 - y$

$$\int_0^1 \int_y^{2-y} (2x + 3y)\,dx\,dy$$
$$= \int_0^1 (x^2 + 3xy)\Bigg|_y^{2-y} dy$$
$$= \int_0^1 [(2 - y)^2 - y^2 + 3y(2 - y - y)]\,dy$$
$$= \int_0^1 (4 - 4y + y^2 - y^2 + 6y - 6y^2)\,dy$$
$$= \int_0^1 (4 + 2y - 6y^2)\,dy$$
$$= (4y + y^2 - 2y^3)\Big|_0^1$$
$$= 4 + 1 - 2 = 3$$

74. $C(x, y) = 2x^2 + 4y^2 - 3xy + \sqrt{x}$

(a) $C(10, 5)$
$$= 2(10)^2 + 4(5)^2 - 3(10)(5) + \sqrt{10}$$
$$= 200 + 100 - 150 + \sqrt{10}$$
$$= \$(150 + \sqrt{10})$$

(b) $C(15, 10)$
$$= 2(15)^2 + 4(10)^2 - 3(15)(10) + \sqrt{15}$$
$$= 450 + 400 - 450 + \sqrt{15}$$
$$\$(400 + \sqrt{15})$$

(c) $C(20, 20)$
$= 2(20)^2 + 4(20)^2 - 3(20)(20) + \sqrt{20}$
$= 800 + 1600 - 1200 + \sqrt{20}$
$= \$(1200 + 2\sqrt{5})$

76. $z = x^{.6}y^{.4}$

(a) The marginal productivity of labor is

$$\frac{\partial z}{\partial x} = .6x^{-.4}y^{.4} = \frac{.6y^{.4}}{x^{.4}}.$$

(b) The marginal productivity of capital is

$$\frac{\partial z}{\partial y} = .4x^{.6}y^{-.6} = \frac{.4x^{.6}}{y^{.6}}.$$

78. $C(x, y) = \ln(x^2 + y) + e^{xy/20}$
$x = 15,\ y = 9,\ dx = 1,\ dy = -1$

$$dC = \left(\frac{2x}{x^2 + y} + \frac{y}{20}e^{xy/20}\right) dx + \left(\frac{1}{x^2 + y} + \frac{x}{20}e^{xy/20}\right) dy$$

$dC(15, 9)$

$$= \left(\frac{2(15)}{15^2 + 9} + \frac{9}{20}e^{(15)(9)/20}\right)(1) + \left(\frac{1}{15^2 + 9} + \frac{15}{20}e^{(9)(15)/20}\right)(-1)$$

$$= \frac{29}{234} - \frac{3}{10}e^{27/4}$$

$$= -256.10$$

Costs decrease by \$256.10.

80. $V = \frac{4}{3}\pi r^3,\ r = 2$ ft,

$dr = 1 \text{ in} = \frac{1}{12}$ ft

$$dV = 4\pi r^2\, dr = 4\pi(2)^2\left(\frac{1}{12}\right) \approx 4.19 \text{ ft}^3$$

82. $P(x, y) = .01(-x^2 + 3xy + 160x - 5y^2 + 200y + 2600)$

with $x + y = 280$.

(a) $y = 280 - x$

$$\begin{aligned} P(x) &= .01[-x^2 + 3x(280 - x) + 160x - 5(280 - x)^2 + 200(280 - x) + 2600] \\ &= .01(-x^2 + 840x - 3x^2 + 160x - 392{,}000 + 2800x - 5x^2 + 56{,}000 - 200x + 2600) \\ P(x) &= .01(-9x^2 + 3600x - 333{,}400) \end{aligned}$$

$$\begin{aligned} P'(x) &= .01(-18x + 3600) \\ .01(-18x + 3600) &= 0 \\ -18x &= -3600 \\ x &= 200 \end{aligned}$$

If $x < 200,\ P'(x) > 0$, and if $x > 200,\ P'(x) < 0$. Therefore, P is maximum when $x = 200$. If $x = 200,\ y = 80$.

$P(200, 80)$
$= .01[-200^2 + 3(200)(80) + 160(200) - 5(80)^2 + 200(80) + 2600]$
$= .01(26{,}600)$
$= 266$

Thus, \$200 spent on fertilizer and \$80 spent on seed will produce a maximum profit of \$266 per acre.

(b)
$$\begin{aligned} P(x, y) &= .01(-x^2 + 3xy + 160x - 5y^2 + 200y + 2600) \\ P_x(x, y) &= .01(-2x + 3y + 160) \\ P_y(x, y) &= .01(3x - 10y + 200) \\ .01(-2x + 3y + 160) &= 0 \\ .01(3x - 10y + 200) &= 0 \end{aligned}$$

These equations simplify to

$$\begin{aligned} -2x + 3y &= -160 \\ 3x - 10y &= -200. \end{aligned}$$

Solve this system.

$$\begin{aligned} -6x + 9y &= -480 \\ 6x - 20y &= -400 \\ \hline -11y &= -880 \\ y &= 80 \end{aligned}$$

If $y = 80$,

$$\begin{aligned} 3x - 10(80) &= -200 \\ 3x &= 600 \\ x &= 200. \end{aligned}$$

$$P_{xx}(x,y) = .01(-2) = -.02$$
$$P_{yy}(x,y) = .01(-10) = -.1$$
$$P_{xy}(x,y) = 0$$

For $(200, 80)$, $D = (-.02)(-.1) - 0^2 = .002 > 0$, and $P_{xx} < 0$, so there is a relative maximum at $(200, 80)$.

$P(200, 80) = 266$, as in part (a). Thus, \$200 spent on fertilizer and \$80 spent on seed will produce a maximum profit of \$266 per acre.

(c) Maximize $P(x, y)$
$= .01(-x^2 + 3xy + 160x - 5y^2 + 200y + 2600)$
subject to $x + y = 280$.

1. $g(x, y) = x + y - 280$

2. $F(x, y, \lambda)$
$= .01(-x^2 + 3xy + 160x - 5y^2 + 200y + 2600) - \lambda(x + y - 280)$

3. $F_x = .01(-2x + 3y + 160) - \lambda$
$F_y = .01(3x - 10y + 200) - \lambda$
$F_\lambda = -(x + y - 280)$

4. $.01(-2x + 3y + 160) - \lambda = 0$ *(1)*
$.01(3x - 10y + 200) - \lambda = 0$ *(2)*
$x + y - 280 = 0$ *(3)*

5. Equations (1) and (2) give

$$.01(-2x + 3y + 160) = .01(3x - 10y + 200)$$
$$-2x + 3y + 160 = 3x - 10y + 200$$
$$-5x + 13y = 40.$$

Multiplying equation (3) by 5 gives

$$5x + 5y - 1400 = 0.$$

$$\begin{array}{r} -5x + 13y = 40 \\ \underline{5x + 5y = 1400} \\ 18y = 1440 \\ y = 80 \end{array}$$

If $y = 80$,

$$5x + 5(80) = 1400$$
$$5x = 1000$$
$$x = 200.$$

Thus, $P(200, 80)$ is a maximum. As before, $P(200, 80) = 266$.
Thus, \$200 spent on fertilizer and \$80 spent on seed will produce a maximum profit of \$266 per acre.

84. $T(A, W, S) = -18.37 - .09A + .34W + .25S$

(a) $T(65, 85, 180) = -18.37 - .09(65) + .34(85) + .25(180)$
$= 49.68$

The total body water is 49.68 liters.

(b) $T_A(A, W, S) = -.09$

The approximate change in total body water if age is increased by 1 yr and weight and height are held constant is $-.09l$.

$T_W(A, W, S) = .34$

The approximate change in total body water if height is increased by 1 kg and age and height are held constant is $.34l$.

$T_S(A, W, S) = .25$

The approximate change in total body water if height is increased by 1 cm and age and weight are held constant is $.25l$.

86. (a) $f(60, 1900) \approx 50$

In 1900, 50% of those born 60 years earlier are still alive.

(b) $f(70, 1995) \approx 80$

In 1995, 80% of those born 70 years earlier are still alive.

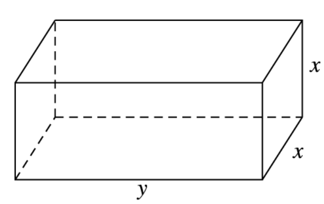
(c) $f_x(60, 1900) \approx -1.25$

In 1900, 5the percent of those born 60 years earlier who are still alive was dropping at a rate of 1.25 percent per additional year of life.

(d) $f_x(70, 1995) \approx -1.45$

In 1995, the percent of those born 70 years earlier who are still alive was dropping at at rate of 1.45 percent per additional year of life.

88. Let x be the length of each of the square faces of the box and y be the length of the box.



Since the volume must be 125, the constraint is $125 = x^2y$.

$f(x, y) = 2x^2 + 4xy$ is the surface area of the box.

1. $g(x) = x^2y - 125$

2. $F(x, y, \lambda) = 2x^2 + 4xy - \lambda(x^2y - 125)$

3. $F_x(x, y, \lambda) = 4x + 4y - 2xy\lambda$
$F_y(x, y, \lambda) = 4x - x^2\lambda$
$F_\lambda(x, y, \lambda) = -(x^2y - 125)$

4. $4x + 4y - 2xy\lambda = 0 \quad (1)$
$4x - x^2\lambda = 0 \quad (2)$
$x^2y - 125 = 0 \quad (3)$

5. Factoring equation (2) gives

$$x(4 - x\lambda) = 0$$
$$x = 0 \quad \text{or} \quad 4 - x\lambda = 0.$$

Since $x = 0$ is not a solution of equation (3), then

$$4 - x\lambda = 0$$
$$\lambda = \frac{4}{x}.$$

Substituting into equation (1) gives

$$4x + 4y - 2xy\left(\frac{4}{x}\right) = 0$$

or

$$4x + 4y - 8y = 0$$
$$x = y.$$

Substituting $x = y$ into equation (3) gives

$$x^2y - 125 = 0$$
$$y^3 = 125$$
$$y = 5.$$

Therefore, $x = y = 5$. The dimensions are 5 inches by 5 inches by 5 inches.

Extended Application: Using Multivariable Fitting to Create a Response Surface Design

1. The general cubic function of two variables has the form

$$G(x, y) = Ax^3 + By^3 + Cx^2y + Dxy^2 + Ex^2 + Fy^2 + Gxy + Hx + Ty + J$$

which has 10 terms.

2. The maximum appears to be close to orange = 56, banana = 48.

3. $G(x, y) = -.00202x^2 - .00163y^2 + .000194xy + .21380x + .14768y - 2.36204$
$G_x(x, y) = -.00404x + .000194y + .21380$
$G_y(x, y) = -.00326y + .000194x + .14768$
$-.00404x + .000194y = -.21380$
$-.00326y + .000194x = -.14768$

The solution to the system is $x \approx 55.254, y \approx 48.589$.

4. $G_{xx}(x, y) = -.00404$
$G_{yy}(x, y) = -.00326$
$G_{xy}(x, y) = .000194$

$$D = (-.00404)(-.00326) - (.000194)^2 \approx .0000131$$

Since $D > 0$ and $G_{xx} = -.00404 < 0$, $G(55.254, 48.589)$ is a relative maximum of $G(x, y)$.

5. The test results include random errors or "noise," which may explain the 7.2 rating. Also, the function that best fits all of the data points is not necessarily above all of the points.

6. The two flavor surfaces have saddle points near the middle of the domain. For overall flavor, the maxima are at the edges of the domain, either 400 minutes at about 130°C or 20 minutes at about 160°C.

7. With this more stringent requirements, the allowable regions in the three contour plots do not overlap. The blanching efficiency would need to be allowed to be less than 93%.

8. The lower area of overlap suggests a processing temperature between 145°C and 155°C with treatment times from 50 to 70 seconds. This region represents processing nuts at higher temperatures but for a shorter time.

Chapter 10

DIFFERENTIAL EQUATIONS

10.1 Solutions of Elementary and Separable Differential Equations

2. $\dfrac{dy}{dx} = 3e^{-2x}$

$$y = \int 3e^{-2x}\,dx = \frac{-3e^{-2x}}{2} + C$$

4. $3x^2 - 3\,\dfrac{dy}{dx} = 2$

$$\frac{dy}{dx} = \frac{3x^2-2}{3} = x^2 - \frac{2}{3}$$

$$y = \int \left(x^2 - \frac{2}{3}\right)dx = \frac{x^3}{3} - \frac{2x}{3} + C$$

6. $y\,\dfrac{dy}{dx} = x^2 - 1$

$$y\,dy = (x^2-1)\,dx$$

$$\int y\,dy = \int (x^2-1)\,dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} - x + C$$

$$y^2 = \frac{2x^3}{3} - 2x + C$$

8. $\dfrac{dy}{dx} = x^2 y$

$$\frac{1}{y}\,dy = x^2\,dx$$

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

$$|y| = e^{x^3/3+C} = e^{x^3/3}\cdot e^C$$

$$y = \pm e^C e^{x^3/3}$$

$$y = Me^{x^3/3}$$

10. $(y^2 - y)\,\dfrac{dy}{dx} = x$

$$(y^2-y)\,dy = x\,dx$$

$$\int (y^2-y)\,dy = \int x\,dx$$

$$\frac{y^3}{3} - \frac{y^2}{2} = \frac{x^2}{2} + C$$

$$2y^3 - 3y^2 = 3x^2 + C$$

12. $\dfrac{dy}{dx} = \dfrac{y}{x^2}$

$$\frac{1}{y}\,dy = \frac{1}{x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \int x^{-2}\,dx$$

$$\ln|y| = \frac{x^{-1}}{-1} + C$$

$$|y| = e^{-1/x+C} = e^{-1/x}\cdot e^C$$

$$y = \pm e^C e^{-1/x}$$

$$y = Me^{-1/x}$$

14. $\dfrac{dy}{dx} = 3 - y$

$$\frac{1}{3-y}\,dy = dx$$

$$\int \frac{1}{3-y}\,dy = \int dx$$

$$-\ln|3-y| = x + C$$

$$\ln|3-y| = -x + C$$

$$|3-y| = e^{-x+C} = e^{-x}e^C$$

$$3 - y = \pm e^C e^{-x} = Me^{-x}$$

$$y = 3 - Me^{-x}$$

16. $\dfrac{dy}{dx} = \dfrac{e^x}{e^y}$

$$e^y\,dy = e^x\,dx$$

$$\int e^y\,dy = \int e^x\,dx$$

$$e^y = e^x + C$$

$$y = \ln(e^x + C)$$

18. $x\,\dfrac{dy}{dx} = x^2e^{3x};\ y = \dfrac{8}{9}$ when $x = 0$.

$$\frac{dy}{dx} = xe^{3x}$$
$$y = \int xe^{3x}\,dx$$

Let $u = x \qquad dv = e^{3x}\,dx$

$du = dx \qquad v = \dfrac{e^{3x}}{3}.$

$$y = \frac{x}{3}e^{3x} - \int \frac{e^{3x}}{3}\,dx$$
$$y = \frac{x}{3}e^{3x} - \frac{e^{3x}}{9} + C$$
$$\frac{8}{9} = 0 - \frac{1}{9} + C$$
$$C = 1$$
$$y = \frac{x}{3}e^{3x} - \frac{e^{3x}}{9} + 1$$

20. $x\,\dfrac{dy}{dx} - y\sqrt{x} = 0;\ y = 1$ when $x = 0$.

$$\frac{1}{y}\,dy = \frac{\sqrt{x}}{x}\,dx$$
$$\int \frac{1}{y}\,dy = \int x^{-1/2}\,dx$$
$$\ln|x| = 2x^{1/2} + C$$
$$|x| = e^{2x^{1/2}+C}$$
$$y = Me^{2x^{1/2}}$$
$$1 = Me^0$$
$$M = 1$$
$$y = e^{2x^{1/2}}$$

22. $\dfrac{dy}{dx} = \dfrac{x^2+5}{2y-1};\ y = 11$ when $x = 0$.

$$(2y-1)\,dy = (x^2+5)\,dx$$
$$\int (2y-1)\,dy = \int (x^2+5)\,dx$$
$$y^2 - y = \frac{x^3}{3} + 5x + C$$
$$121 - 11 = C$$
$$C = 110$$
$$y^2 - y = \frac{x^3}{3} + 5x + 110$$

24. $\dfrac{dy}{dx} = \dfrac{2x+1}{y-3};\ y = 4$ when $x = 0$.

$$\int (y-3)\,dy = \int (2x+1)dx$$
$$\frac{y^2}{2} - 3y = \frac{2x^2}{2} + x + C$$

Since $y = 4$ when $x = 0$,

$$\frac{16}{2} - 12 = 0 + 0 + C$$
$$C = -4.$$

So,

$$\frac{y^2}{2} - 3y = x^2 + x - 4.$$

26. $x^2\,\dfrac{dy}{dx} = y;\ y = -1$ when $x = 1$.

$$\frac{1}{y}\,dy = \frac{1}{x^2}\,dx$$

See Exercise 12.

$$y = Me^{-1/x}$$
$$-1 = Me^{-1}$$
$$M = -e$$
$$y = -e^{1-1/x}$$
$$= -e^{(-1/x)+1}$$

28. $\dfrac{dy}{dx} = x^{1/2}y^2;\ y = 12$ when $x = 4$.

$$\frac{1}{y^2}\,dy = x^{1/2}\,dx$$
$$\int y^{-2}\,dy = \int x^{1/2}\,dx$$
$$-y^{-1} = \frac{2}{3}x^{3/2} + C$$
$$-\frac{1}{12} = \frac{2}{3}(4)^{3/2} + C$$
$$C = -\frac{65}{12}$$
$$\frac{-1}{y} = \frac{2}{3}x^{3/2} - \frac{65}{12}$$
$$\frac{-1}{y} = \frac{8x^{3/2} - 65}{12}$$
$$y = \frac{-12}{8x^{3/2} - 65}$$

30. $\dfrac{dy}{dx} = (x+2)^2 e^y;\ y = 0$ when $x = 1$.

$$\begin{aligned} e^{-y}\,dy &= (x+2)^2\,dx \\ \int e^{-y}\,dy &= \int (x+2)^2\,dx \\ -e^{-y} &= \frac{(x+2)^3}{3} + C \\ -1 &= 9 + C \\ C &= -10 \\ -e^{-y} &= \frac{(x+2)^3}{3} - 10 \\ e^{-y} &= 10 - \frac{(x+2)^3}{3} \\ -y &= \ln\left[10 - \frac{(x+2)^3}{3}\right] \\ y &= -\ln\left[10 - \frac{(x+2)^3}{3}\right] \end{aligned}$$

32. $\dfrac{dz}{dx} = k(1-z)z,\ 0 < z < 1$

Note $1 - z > 0$ also.

$$\frac{1}{(1-z)z}\,dz = k\,dx$$

Observe that

$$\begin{aligned} \frac{1}{1-z} + \frac{1}{z} &= \frac{1}{(1-z)z} \\ \left[\frac{1}{1-z} + \frac{1}{z}\right] dz &= k\,dx \\ \int \left[\frac{1}{1-z} + \frac{1}{z}\right] dz &= \int k\,dx \\ -\ln(1-z) + \ln z &= kx + C \\ \ln\left(\frac{z}{1-z}\right) &= kx + C \\ \frac{z}{1-z} &= e^{kx+C} \\ &= Me^{kx} \text{ where } M = e^C. \end{aligned}$$

$z = \dfrac{1}{2}$ when $x = x_0$. Let $b = e^{kx_0}$.

$$\begin{aligned} \frac{\frac{1}{2}}{1-\frac{1}{2}} &= Me^{kx_0}. \\ 1 &= Mb \\ M &= \frac{1}{b}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{x}{1-z} &= \frac{1}{b}e^{kx} \\ bz &= e^{kx} - ze^{kx} \\ (e^{kx} + b)z &= e^{kx} \\ z &= \frac{e^{kx}}{e^{kx} + b} \\ &= \frac{1}{1 + be^{-kx}}. \end{aligned}$$

34. $y = \dfrac{N}{1 - be^{-kx}}$ for all $x \neq \dfrac{\ln b}{k}$, where $0 < N < y_0$ and

$$b = \frac{y_0 - N}{y_0}.$$

(a) Since $0 < N < y_0$, $0 < y_0 - N$ and $0 < y_0$, so $b > 0$. Also, $y_0 - N < y_0$, so

$$\frac{y_0 - N}{y_0} < 1 \text{ and } b < 1.$$

(b) $\lim\limits_{x\to\infty} e^{-kx} = 0$ (assuming $k > 0$), so

$$\lim_{x\to\infty} \frac{N}{1 - be^{-kx}} = \frac{N}{1-0} = N.$$

Thus, $y = N$ is a horizontal asymptote to the curve as $x \to \infty$.

$$\lim_{x\to-\infty} e^{-kx} = \infty, \text{ so}$$

$$\lim_{x\to-\infty} \frac{N}{1 - be^{-kx}} = 0.$$

Thus, $y = 0$ is a horizontal asymptote to the curve as $x \to -\infty$.

(c) The graph has a vertical asymptote when

$$\begin{aligned} 1 - be^{-kx} &= 0 \\ 1 &= be^{-kx} \\ \frac{1}{b} &= e^{-kx} \\ b^{-1} &= e^{-kx} \\ \ln b^{-1} &= \ln e^{-kx} \\ -\ln b &= -kx \\ \frac{\ln b}{k} &= x. \end{aligned}$$

(d) $y = N(1 - be^{-kx})^{-1},\ x \neq \dfrac{\ln b}{k}$

We assume that $k > 0$.

$$\begin{aligned}\frac{dy}{dx} &= -N(1 - be^{-kx})^{-2}(-be^{-kx} \cdot (-k)) \\ &= \frac{-kbNe^{-kx}}{(1 - be^{-kx})^2}\end{aligned}$$

$e^{-kx} > 0$ for all x, $(1 - be^{-kx})^2 > 0$ for all

$x \neq \dfrac{\ln b}{k}$, $k > 0$, $b > 0$, and $N > 0$.

Therefore, $\frac{dy}{dx} < 0$, and y is decreasing on $\left(\frac{\ln b}{k}, \infty\right)$ and on $\left(-\infty, \frac{\ln b}{k}\right)$.

(e) To find $\frac{d^2y}{dx^2}$, apply the quotient rule to find the derivative of $\frac{dy}{dx}$. The numerator of $\frac{d^2y}{dx^2}$ is

$$\begin{aligned}\frac{d^2y}{dx^2} &= -kbN(1-be^{-kx})^2 \cdot e^{-kx}(-k) \\ &\quad + kbNe^{-kx} \cdot 2(1-be^{-kx}) \cdot (kbe^{-kx}) \\ &= kbN\left[-ke^{-kx}\{(1 - be^{-kx}) + 2be^{-kx}\}\right] \\ &= k^2bNe^{-kx}(1 + be^{-kx}),\end{aligned}$$

and the denominator is

$$[(1 - be^{-kx})^2]^2 = (1 - be^{-kx})^4.$$

Thus,

$$\frac{d^2y}{dx^2} = \frac{k^2bNe^{-kx}(1 + be^{-kx})}{(1 - be^{-kx})^3}.$$

$e^{-kx} > 0$ and $b > 0$, so $1 + be^{-kx} > 0$ for all x. $k^2 > 0$ and $N > 0$, so the numerator is always positive.

$(1 - be^{-kx})^3 > 0$, if and only if

$$\begin{aligned}1 - be^{-kx} &> 0 \\ 1 &> be^{-kx} \\ b^{-1} &> e^{-kx} \\ \ln b^{-1} &> \ln e^{-kx} \\ -\ln b &> -kx \\ \frac{\ln b}{k} &> x.\end{aligned}$$

Thus, y is concave upward on $\left(\frac{\ln b}{x}, \infty\right)$ and concave downward on $\left(-\infty, \frac{\ln b}{x}\right)$.

36. Let y = sales, t = time, and $k = -.25$.

(a) $\dfrac{dy}{dt} = -.25y$

(b) From Example 4, $y = Me^{-.25t}$.

(c) If $t = 0$, $y = .3M$.

$$\begin{aligned}.3M &= Me^{-.25t} \\ .3 &= e^{-.25t} \\ -.25t &= \ln .3 \\ t &= \frac{\ln .3}{-.25} \approx 4.8\end{aligned}$$

Sales will become 30% of their original value in approximately 4.8 yr.

38. $E = -\dfrac{p}{q} \cdot \dfrac{dq}{dp}$

If $E = \frac{4p^2}{q^2}$,

$$\begin{aligned}\frac{4p^2}{q^2} &= -\frac{p}{q} \cdot \frac{dq}{dp} \\ 4p\,dp &= -q\,dq. \\ \int 4p\,dp &= -\int q\,dq \\ 2p^2 &= -\frac{1}{2}q^2 + C_1\end{aligned}$$

Multiplying by 2, we have

$$\begin{aligned}4p^2 &= -q^2 + 2C_2 \\ q^2 &= -4p^2 + 2C_2.\end{aligned}$$

Let $C = 2C_2$. Then

$$\begin{aligned}q^2 &= -4p^2 + C \\ q &= \pm\sqrt{-4p^2 + C}.\end{aligned}$$

Since q cannot be negative,

$$q = \sqrt{-4p^2 + C}.$$

40. $\dfrac{dy}{dt} = -.03y$

$$\begin{aligned}\int \frac{dy}{y} &= -.03y \int dt \\ \ln|y| &= -.03t + C \\ e^{\ln|y|} &= e^{-.03t+C} \\ y &= Me^{-.03t}\end{aligned}$$

Since $y = 6$ when $t = 0$,

$$6 = Me^0$$
$$M = 6.$$

So $y = 6e^{-.03t}$.
If $t = 10$,

$$y = 6e^{-.03(10)}$$
$$\approx 4.4$$

After 10 minutes, about 4.4 cc of dye will be present.

42. $$\frac{dy}{dx} = .01(5000 - y)$$
$$\int \frac{dy}{5000 - y} = \int .01\,dx$$
$$-\ln|5000 - y| = .01x + C$$
$$\ln|5000 - y| = -.01x + C$$
$$|5000 - y| = e^{-.01x+C}$$
$$|5000 - y| = e^{-.01x} \cdot e^C$$
$$5000 - y = Me^{-.01x}$$
$$y = 5000 - Me^{-.01x}$$

Since $y = 150$ when $x = 0$,

$$150 = 5000 - Me^0$$
$$M = 4850.$$

So $y = 5000 - 4850e^{-.01x}$.
If $x = 5$,

$$y = 5000 - 4850^{-.01(5)}$$
$$\approx 387.$$

At the end of 5 years, there will be about 387 fish.

44. **(a)** $\frac{dw}{dt} = \frac{1}{3500}(2500 - 17.5w);\ wt = 180$ when $t = 0$.

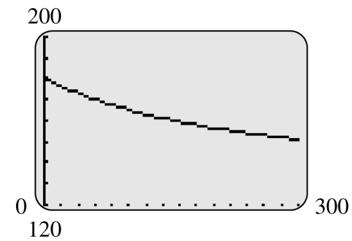
$$\frac{3500}{2500 - 17.5w}\,dw = dt$$
$$\int \frac{3500}{2500 - 17.5w}\,dw = \int dt$$
$$\frac{3500}{-17.5}\int \frac{-17.5}{2500 - 17.5w}\,dw = \int dt$$
$$-200\ln|2500 - 17.5w| = t + C_1$$
$$\ln|2500 - 17.5w| = -.005t + C_2$$
$$|2500 - 17.5w| = e^{-.005t+C_2}$$
$$|2500 - 17.5w| = e^{C_2}e^{-.005t}$$
$$2500 - 17.5w = \pm e^{C_2}e^{-.005t}$$
$$-17.5w = -2500 + C_3e^{-.005t}$$
$$w = 143 + C_4e^{-.005t}$$

Since $w = 180$ when $t = 0$,

$$180 = 143 + C_4(1)$$
$$C_4 = 37$$
$$w = 143 + 37e^{-.005t}.$$

(b) $$\lim_{t\to\infty} w = \lim_{t\to\infty}(143 + 37e^{-.005t})$$
$$= 143 + 37(0)$$
$$= 143$$

The asymptote is $w = 143$.

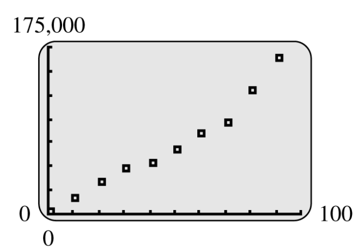


According to the model, the value 143 will never be attained.

(c) $$145 = 143 + 37e^{-.005t}$$
$$2 = 37e^{-.005t}$$
$$e^{-.005t} = \frac{2}{37}$$
$$-.005t = \ln\left(\frac{2}{37}\right)$$
$$t = \frac{\ln\left(\frac{2}{37}\right)}{-.005}$$
$$t \approx 583.55$$

About 584 days (about 19.5 months) would be required to reach a weight of 145.

46. **(a)**



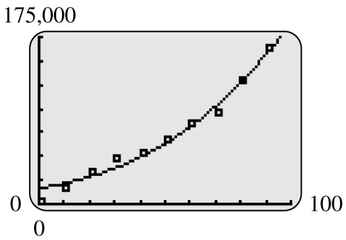
The points suggest that a logistic function is appropriate.

(b) A calculator with a logistic regression function determines that

$$y = \frac{484{,}900}{1 + 28.32e^{-.02893x}}$$

best fits the data.

(c)



The model does produce appropriate y-values for the given x-values, particularly for the more recent years.

(d) As x gets larger and larger, $e^{-.02893x}$ approaches 0, so y approaches

$$\frac{484{,}900}{1 + 28.32 \cdot 0} = \frac{484{,}900}{1} = 484{,}900.$$

Therefore, according to this model, the limiting size of the Jewish population in Toronto is 484,900.

48. $\dfrac{dy}{dt} = ky$

First separate the variables and integrate.

$$\frac{dy}{y} = k\,dt$$
$$\int \frac{dy}{y} = \int k\,dt$$
$$\ln|y| = kt + C_1$$

Solve for y.

$$|y| = e^{kt+C_1} = e^{C_1}e^{kt}$$
$$y = Ce^{kt}, \text{ where } C = \pm e^{C_1}.$$
$$y(0) = 35.5, \text{ so } 35.5 = Ce^0 = C, \text{ and}$$
$$y = 35.5e^{kt}.$$

Solve for k. Since $y(50) = 60.6$, then

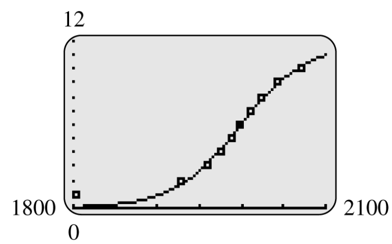
$$60.6 = 35.5e^{50k}$$
$$e^{50k} = \frac{60.6}{35.5}$$
$$50k = \ln\left(\frac{60.6}{35.5}\right)$$
$$k = \frac{\ln\left(\frac{60.6}{35.5}\right)}{50} \approx .01070,$$
$$\text{so} \quad y = 35.5e^{.01070t}.$$

50. (a) A calculator with a logistic regression function determines that

$$y = \frac{11.74}{1 + (1.423 \times 10^{22})e^{-.02554x}}$$

best fits the data.

(b) From the graph, the function from part (a) seems to fit the data from 1927 on very well. For the year 1804, the function does not fit the data very well.

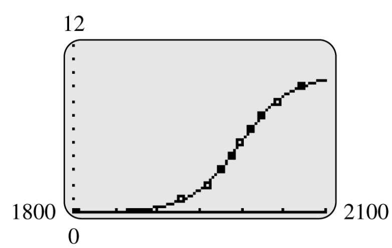


(c) After subtracting .99 from the y-values in the list, a calculator with a logistic regression function determines that

$$y = \frac{9.803}{1 + (2.612 \times 10^{29})e^{-.03391x}}$$

best fits the data.

(d)



From the graph, the function in part (c) does seem to fit the data better than the graph found in part (b).

(e) As x gets larger and larger, $e^{-.03391x}$ approaches 0 so that y approaches

$$\frac{9.803}{1+0} = 9.803.$$

If you add back the .99 that was subtracted from the y-values, the result is approximately 10.79 billion.

(f) For the function found in part (c), as x gets smaller and approaches negative infinity, the denominator of this logistic function approaches infinity so that y approaches 0. After adding back the .99 that was subtracted earlier, this would imply that the limiting value for the world population as you go further and further back in time is .99 billion. This does not seem reasonable though because the world population was not always more than 990 million.

52. (a) $\frac{dy}{dt} = -.05y$

(b) $\int \frac{dy}{y} = -\int .05\, dt$

$$\begin{aligned} \ln|y| &= -.05t + C \\ e^{\ln|y|} &= e^{-.05+C} \\ y &= \pm e^{-.05t} \cdot e^C \\ y &= Me^{-.05t} \end{aligned}$$

(c) Since $y = 90$ when $t = 0$.

$$\begin{aligned} 90 &= Me^0 \\ M &= 90. \end{aligned}$$

So $y = 90e^{-.05t}$.

(d) At $t = 10$,

$$\begin{aligned} y &= 90e^{-.05(10)} \\ &\approx 55. \end{aligned}$$

After 10 months, about 55 g are left.

10.2 Linear First-Order Differential Equations

2. $\frac{dy}{dx} + 4y = 10$

$$\begin{aligned} I(x) &= e^{\int 4\, dx} = e^{4x} \\ e^{4x}\frac{dy}{dx} + 4e^{4x}y &= 10e^{4x} \\ D_x(e^{4x}y) &= 10e^{4x} \\ e^{4x}y &= \int 10e^{4x}\, dx \\ &= \frac{5}{2}e^{4x} + C \\ y &= \frac{5}{2} + Ce^{-4x} \end{aligned}$$

4. $\frac{dy}{dx} + 2xy = x$

$$\begin{aligned} I(x) &= e^{\int 2x\, dx} = e^{x^2} \\ e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y &= xe^{x^2} \\ D_x(e^{x^2}y) &= xe^{x^2} \end{aligned}$$

$$\begin{aligned} e^{x^2}y &= \int xe^{x^2}\, dx \\ &= \frac{1}{2}e^{x^2} + C \\ y &= \frac{1}{2} + Ce^{-x^2} \end{aligned}$$

6. $x\frac{dy}{dx} + 2xy - x^2 = 0;\ x > 0$

$$\begin{aligned} \frac{dy}{dx} + 2y &= x \\ I(x) &= e^{\int 2\, dx} = e^{2x} \\ e^{2x}\frac{dy}{dx} + 2e^{2x}y &= xe^{2x} \\ D_x(e^{2x}y) &= xe^{2x} \end{aligned}$$

$$e^{2x}y = \int xe^{2x}\, dx$$

Integration by parts:

Let $u = x \qquad dv = e^{2x}\, dx$

$du = dx \qquad v = \frac{e^{2x}}{2}.$

$$\begin{aligned} e^{2x}y &= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\, dx \\ &= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C \\ y &= \frac{x}{2} - \frac{1}{4} + Ce^{-2x} \end{aligned}$$

8. $3\dfrac{dy}{dx} + 6xy + x = 0$

$$\frac{dy}{dx} + 2xy = -\frac{x}{3}$$
$$I(x) = e^{\int 2x\,dx} = e^{x^2}$$

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = -\frac{x}{3}e^{x^2}$$
$$D_x(e^{x^2}y) = -\frac{x}{3}e^{x^2}$$
$$e^{x^2}y = \int -\frac{x}{3}e^{x^2}\,dx$$
$$= -\frac{1}{6}e^{x^2} + C$$
$$y = -\frac{1}{6} + Ce^{-x^2}$$

10. $x^2\dfrac{dy}{dx} + xy = x^3 - 2x^2;\ x > 0$

$$\frac{dy}{dx} + \frac{1}{x}y = x - 2$$

$$I(x) = e^{\int 1/x\,dx} = e^{\ln x} = x$$

$$x\frac{dy}{dx} + y = x^2 - 2x$$
$$D_x(xy) = x^2 - 2x$$
$$xy = \int (x^2 - 2x)\,dx$$
$$= \frac{x^3}{3} - x^2 + C$$
$$y = \frac{x^2}{3} - x + \frac{C}{x}$$

12. $2xy + x^3 = x\dfrac{dy}{dx}$

$$x\frac{dy}{dx} - 2xy = x^3$$
$$\frac{dy}{dx} - 2y = x^2$$

$$I(x) = e^{\int -2\,dx} = e^{-2x}$$
$$e^{-2x}\frac{dy}{dx} - 2e^{-2x}y = x^2e^{-2x}$$
$$D_x(e^{-2x}y) = x^2e^{-2x}$$
$$e^{-2x}y = \int x^2e^{-2x}\,dx$$

Integration by parts:

Let $u = x^2$ $\quad dv = e^{-2x}\,dx$

$du = 2x\,dx$ $\quad v = \dfrac{e^{-2x}}{-2}.$

$$e^{-2x}y = \frac{-x^2e^{-2x}}{2} + \int xe^{-2x}\,dx$$

Let $u = x$ $\quad dv = e^{-2x}\,dx$

$du = dx$ $\quad v = \dfrac{e^{-2x}}{-2}.$

$$e^{-2x}y = \frac{-x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \int \frac{e^{-2x}}{2}\,dx$$
$$= \frac{-x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$$
$$y = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} + Ce^{2x}$$

14. $\dfrac{dy}{dx} + 2y = e^{3x};\ y = 50$ when $x = 0$.

$$I(x) = e^{\int 2\,dx} = e^{2x}$$
$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}e^{3x}$$
$$D_x(2e^{2x}y) = e^{5x}$$

$$e^{2x}y = \int e^{5x}\,dx + C$$
$$= \frac{e^{5x}}{5} + C$$
$$y = \frac{e^{3x}}{5} + Ce^{-2x}$$
$$50 = \frac{1}{5} + C$$
$$C = \frac{249}{5}$$
$$y = \frac{e^{3x}}{5} + \frac{249e^{-2x}}{5}$$

16. $x\,\dfrac{dy}{dx} - 3y + 2 = 0;\ y = 8$ when $x = 1$.

$$\frac{dy}{dx} - \frac{3}{x}y = -\frac{2}{x}$$

$$\begin{aligned} I(x) &= e^{\int -3/x\,dx} = e^{-3\ \ln\ x} \\ &= e^{\ln\ x^{-3}} = x^{-3} \end{aligned}$$

$$\begin{aligned} \frac{1}{x^3}\,\frac{dy}{dx} - \frac{3}{x^4}y &= -\frac{2}{x^4} \\ D_x\left(\frac{1}{x^3}y\right) &= -\frac{2}{x^4} \\ \frac{1}{x^3}y &= \int -2x^{-4}\,dx \\ &= \frac{2x^{-3}}{3} + C \\ y &= \frac{2}{3} + Cx^3 \\ 8 &= \frac{2}{3} + C \\ C &= \frac{22}{3} \\ y &= \frac{2}{3} + \frac{22x^3}{3} \end{aligned}$$

18. $2\,\dfrac{dy}{dx} - 4xy = 5x;\ y = 10$ when $x = 1$.

$$\frac{dy}{dx} - 2xy = \frac{5x}{2}$$

$$\begin{aligned} I(x) &= e^{\int -2x\,dx} = e^{-x^2} \\ e^{-x^2}\,\frac{dy}{dx} - 2xe^{-x^2}y &= \frac{5x}{2}e^{-x^2} \\ D_x(e^{-x^2}y) &= \frac{5x}{2}e^{-x^2} \\ e^{-x^2}y &= \int \frac{5x}{2}e^{-x^2}\,dx \\ &= -\frac{5}{4}e^{-x^2} + C \end{aligned}$$

$$\begin{aligned} y &= -\frac{5}{4} + Ce^{x^2} \\ 10 &= -\frac{5}{4} + Ce \\ C &= \frac{45}{4e} \\ y &= -\frac{5}{4} + \frac{45}{4e}e^{x^2} \\ \text{or} &\quad -\frac{5}{4} + \frac{45}{4}e^{x^2-1} \end{aligned}$$

20. $\dfrac{dy}{dx} + 2xy - e^{-x^2};\ y = 100$ when $x = 0$.

$$\begin{aligned} I(x) &= e^{\int 2x\,dx} = e^{x^2} \\ e^{x^2}\,\frac{dy}{dx} + 2xe^{x^2}y &= e^{x^2}\cdot e^{-x^2} \\ D_x(e^{x^2}y) &= 1 \\ e^{x^2}y &= \int 1\,dx \\ &= x + C \\ y &= xe^{-x^2} + Ce^{-x^2} \\ 100 &= 0 + C \\ y &= xe^{-x^2} + 100e^{-x^2} \end{aligned}$$

22.

$$\begin{aligned} \frac{dG}{dt} &= a - KG \\ dG &= (a - KG)\ dt \\ \frac{1}{a - KG}\,dG &= dt \\ \int \frac{1}{a - KG}\,dG &= \int dt \\ -\frac{1}{K}\int \frac{-K}{a - KG}\,dG &= \int dt \\ -\frac{1}{K}\ \ln|a - KG| &= t + C_1 \\ \ln|a - KG| &= -Kt + C_2 \\ |a - KG| &= e^{-Kt+C_2} \\ &= e^{C_2}e^{-Kt} \\ a - KG &= ke^{-Kt} \\ -KG &= -a + ke^{-Kt} \end{aligned}$$

Dividing by $-K$, we get

$G = \dfrac{a}{K} + Ce^{-Kt}$ where $C = \dfrac{k}{-K}$.

24. $$\frac{dN}{dt} = rN - \frac{\alpha r(\alpha+b+v)}{\beta\left[\alpha - r\left(1+\frac{v}{b+\gamma}\right)\right]}$$

$$\frac{dN}{dt} - rN = -\frac{\alpha r(\alpha+b+v)}{\beta\left[\alpha - r\left(1+\frac{v}{b+\gamma}\right)\right]}$$

Integrating factor: $I = e^{\int -r\,dt} = e^{-rt}$.

$$e^{-rt}\frac{dN}{dt} - e^{-rt}rN = e^{-rt}\left(-\frac{\alpha r(\alpha+b+v)}{\beta\left[\alpha - r\left(1+\frac{v}{b+\gamma}\right)\right]}\right)$$

$$D_t(Ne^{-rt}) = -\frac{\alpha r(\alpha+b+v)}{\beta\left[\alpha - r\left(1+\frac{v}{b+\gamma}\right)\right]}e^{-rt}$$

$$Ne^{-rt} = -\frac{\alpha r(\alpha+b+v)}{\beta\left[\alpha - r\left(1+\frac{v}{b+\gamma}\right)\right]}\int e^{-rt}dt$$

$$= -\frac{\alpha r(\alpha+b+v)}{\beta\left[\alpha - r\left(1+\frac{v}{b+\gamma}\right)\right]}\left(-\frac{1}{r}\right)e^{-rt} + C$$

$$= \frac{\alpha(\alpha+b+v)}{\beta\left[\alpha - r\left(1+\frac{v}{b+\gamma}\right)\right]}e^{-rt} + C$$

$$N(t) = \frac{\alpha+b+v}{\beta}\cdot\frac{\alpha}{\left[\alpha - r\left(1+\frac{v}{b+\gamma}\right)\right]} + Ce^{rt}$$

Use the initial condition $N(0) = (\alpha+b+\gamma)/\beta$.

$$N(0) = \frac{\alpha+b+v}{\beta}\cdot\frac{\alpha}{\alpha - r\left(1+\frac{v}{b+\gamma}\right)} + Ce^{r(0)}$$

$$\frac{\alpha+b+v}{\beta} = \frac{\alpha+b+v}{\beta}\cdot\frac{\alpha}{\alpha - r\left(1+\frac{v}{b+\gamma}\right)} + C$$

$$C = \frac{\alpha+b+v}{\beta} - \frac{\alpha+b+v}{\beta}\cdot\frac{\alpha}{\alpha - r\left(1+\frac{v}{b+\gamma}\right)}$$

$$= \frac{\alpha+b+v}{\beta}\left(1 - \frac{\alpha}{\alpha - r\left(1+\frac{v}{b+\gamma}\right)}\right)$$

Replace this in the antiderivative previously found.

$$N(t) = \frac{\alpha+b+v}{\beta}\cdot\frac{\alpha}{\alpha - r\left(1+\frac{v}{b+\gamma}\right)} + \frac{\alpha+b+v}{\beta}\left(1 - \frac{\alpha}{\alpha - r\left(1+\frac{v}{b+\gamma}\right)}\right)e^{rt}$$

Now use the substitution

$$R = \alpha - r\left(1+\frac{v}{b+\gamma}\right).$$

$$N(t) = \frac{\alpha+b+v}{\beta}\cdot\frac{\alpha}{\alpha - r\left(1+\frac{v}{b+\gamma}\right)} + \frac{\alpha+b+v}{\beta}\cdot\left(1 - \frac{\alpha}{\alpha - r\left(1+\frac{v}{b+\gamma}\right)}\right)e^{rt}$$

$$= \frac{\alpha+b+v}{\beta}\cdot\frac{\alpha}{R} + \frac{\alpha+b+v}{\beta}\left(1-\frac{\alpha}{R}\right)e^{rt}$$

$$= \frac{\alpha+b+v}{\beta}[\alpha + (R-\alpha)e^{rt}]$$

$$= \frac{\alpha+b+v}{\beta}[(R-\alpha)e^{rt} + \alpha]$$

26. $f(t) = e^{-t}$

$$\frac{dy}{dt} = ky + f(t),\ k = .02$$

$$\frac{dy}{dt} = .02y + e^{-t}$$

$$\frac{dy}{dt} - .02y = e^{-t}$$

$$I(t) = e^{\int -.02\,dt} = e^{-.02t}$$

$$D_t(y\cdot e^{-.02t}) = e^{-.02t}e^{-t}$$

$$ye^{-.02t} = \int e^{-1.02t}\,dt$$

$$= -.98e^{-1.02t} + C$$

$$y = -.98e^{-t} + Ce^{-.02t}$$

Since $y = 10{,}000$ when $t = 0$,

$$10{,}000 = -.98e^0 + Ce^0 = -.98 + C$$

$$C = 10{,}000.98.$$

Therefore,

$$y = -.98e^{-t} + 10{,}000.98e^{.02t}.$$

28. $f(t) = t$

$$\frac{dy}{dt} = ky + f(t)$$

$$\begin{aligned}\frac{dy}{dt} &= ky + t\\ \frac{dy}{dt} - ky &= t\\ I(t) = e^{-\int k\,dt} &= e^{-kt}\\ e^{-kt}\frac{dy}{dt} - ke^{-kt}y &= te^{-kt}\\ D_t(e^{-kt}y) &= te^{-kt}\\ e^{-kt}y &= \int te^{-kt}\,dt\end{aligned}$$

Use the table or integration by parts.

$$\begin{aligned}e^{-kt}y &= -\left(\frac{kt+1}{k^2}\right)e^{-kt} + C\\ y &= -\left(\frac{kt+1}{k^2}\right) + Ce^{kt}\end{aligned}$$

At $k = .02,\ y = 10{,}000,\ t = 0,$

$$\begin{aligned}10{,}000 &= -\left(\frac{0+1}{(.02)^2}\right) + Ce^0\\ &= -\left(\frac{1}{.0004}\right) + C(1)\\ 12{,}500 &= C.\end{aligned}$$

Therefore,

$$\begin{aligned}y &= -\left(\frac{.02t+1}{.0004}\right) + 12{,}500e^{.02t}\\ &= -50t - 2500 + 12{,}500e^{.02t}.\end{aligned}$$

30. $$\frac{dT}{dt} = -k(T - T_0) \text{ with } T - T_0 > 0$$

$$\begin{aligned}\frac{1}{T-T_0}\,dT &= -k\,dt\\ \int \frac{1}{T-T_0}\,dT &= -\int k\,dt\\ \ln(T - T_0) &= -kt + C_1\\ T - T_0 &= e^{-kt+C_1}\\ T &= T_0 + e^{C_1-kb}\\ T &= ce^{-kt} + T_0, \text{ where } c = e^{C_1}.\end{aligned}$$

32. Refer to Exercise 27.

$$T = Ce^{-kt} + T_0$$

In this problem, $T_0 = 68°$F; when $t = 0$, $T = 98.6°$F; and when $t = 1$, $T = 90°$F.

$$\begin{aligned}98.6 &= C(1) + 68\\ C &= 30.6\\ T &= 30.6e^{-kt} + 68\\ 90 &= 30.6e^{-k(1)} + 68\\ 30.6e^{-k} &= 22\\ e^{-k} &= \frac{22}{30.6}\\ -k &= \ln\left(\frac{22}{30.6}\right)\\ k &= -\ln\left(\frac{22}{30.6}\right)\\ &\approx .33\end{aligned}$$

Therefore, $T = 30.6e^{-.33t} + 68$.

(a) When $t = 2$,

$$\begin{aligned}T &= 30.6e^{-.33(2)} + 68\\ &= 83.8.\end{aligned}$$

After 2 hours, the temperature of the body will be 83.8°F.

(b) $$\begin{aligned}75 &= 30.6e^{-.33t} + 68\\ 7 &= 30.6e^{-.33t}\\ e^{-.33t} &= \frac{7}{30.6}\\ -.33t &= \ln\left(\frac{7}{30.6}\right)\\ t &= \frac{\ln\left(\frac{7}{30.6}\right)}{-.33}\\ &\approx 4.5\end{aligned}$$

The temperature of the body will be 75°F in approximately 4.5 hours.

(c) $$\begin{aligned}68.01 &= 30.6e^{-.33t} + 68\\ .01 &= 30.6e^{-.33t}\\ e^{-.33t} &= \frac{.01}{30.6}\\ -.33t &= \ln\left(\frac{.01}{30.6}\right)\\ t &= \frac{\ln\left(\frac{.01}{30.6}\right)}{-.33}\\ &\approx 24.3\end{aligned}$$

The temperature of the body will be within .01° of the surrounding air in approximately 24.3 hours.

10.3 Euler's Method

2. $\dfrac{dy}{dx} = xy + 2;\ y(0) = 0,\ h = .1;$ find $y(.5)$

$g(x, y) = xy + 2$
$x_0 = 0;\ y_0 = 0$
$g(x_0, y_0) = 0 + 2 = 2$
$x_1 = .1$

By Euler's method,

$y_{i+1} = y_i + g(x_i, y_i)h.$

Thus, $y_1 = 0 + 2(.1) = .2.$

$g(x_1, y_1) = .1(.2) + 2 = 2.02$

$x_2 = .2$
$y_2 = .2 + 2.02(.1) = .402$

$g(x_2, y_2) = .2(.402) + 2 = 2.0804$

$x_3 = .3$
$y_3 = .402 + 2.0804(.1) = .61004$
$g(x_3, y_3) = .3(.61004) + 2$
$= 2.183012$
$x_4 = .4$
$y_4 = .61004 + 2.183012(.1)$
$= .8283412$
$g(x_4, y_4) = .4(.8283412) + 2$
$= 2.33133648$
$x_5 = .5$
$y_5 = .8283412 + 2.33133648(.1)$
$= 1.061474848$

Tabulate the results as follows.

x_i	y_i
0	0
.1	.2
.2	.402
.3	.61004
.4	.8283412
.5	1.0614748

Therefore, $y(.5) \approx 1.061$.

Use Euler's method as outlined as in the solution for Exercise 2 in the following exercises. The results are tabulated.

4. $\dfrac{dy}{dx} = x + y^2; y(0) = 0, h = .1;$ find $y(.6)$.

x_i	y_i
0	0
.1	.0
.2	.01
.3	.03001
.4	6.01007×10^{-2}
.5	.1004613
.6	.1514705

Therefore, $y(.6) \approx .151$.

6. $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}; y(1) = 0, h = .1;$ find $y(1.4)$.

x_i	y_i
1	0
1.1	.1
1.2	.2090909
1.3	.3265152
1.4	.4516317

Therefore, $y(1.4) \approx .452$.

8. $\dfrac{dy}{dx} = e^{-y} + x; y(0) = 0; h = .1;$ find $y(.5)$.

x_i	y_i
0	0
.1	.1
.2	.2004838
.3	.3023172
.4	.4062276
.5	.5128435

Therefore, $y(.5) \approx .513$.

10. $\dfrac{dy}{dx} = 4x + 3;\ y(1) = 0,\ h = .1;$ find $y(1.5)$.

x_i	y_i
1	0
1.1	.7
1.2	1.44
1.3	2.22
1.4	3.04
1.5	3.9

Therefore, $y(1.5) \approx 3.900$.
Exact solution:

$$y = 2x^2 + 3x + C$$
$$0 = 2(1)^2 + 3(1) + C$$
$$C = -5$$
$$y = 2x^2 + 3x - 5$$

Therefore,

$$y(1.5) = 2(1.5)^2 + 3(1.5) - 5$$
$$= 4.000.$$

12. $\frac{dy}{dx} = \frac{1}{x};\ y(1) = 1,\ h = .1$; find $y(1.4)$.

x_i	y_i
1	1
1.1	1.1
1.2	1.190909
1.3	1.274243
1.4	1.351166

Therefore, $y(1.4) \approx 1.351$.

Exact solution:

$$\begin{aligned} y &= \ln x + C \\ 1 &= \ln 1 + C = C \\ y &= \ln x + 1 \\ y(1.4) &= \ln 1.4 + 1 \approx 1.336 \end{aligned}$$

14. $\frac{dy}{dx} = x^2y;\ y(0) = 1,\ h = .1$; find $y(.6)$.

x_i	y_i
0	1
.1	1
.2	1.001
.3	1.005004
.4	1.014049
.5	1.030274
.6	1.056031

Therefore, $y(.6) \approx 1.056$.

Exact solution:

$$\begin{aligned} \frac{\frac{dy}{dx}}{y} &= x^2 \\ \ln y &= \frac{x^3}{3} + C \\ \ln 1 &= 0 + C \\ 0 &= C \\ \ln y &= \frac{x^3}{3} \\ y = y(x) &= e^{x^3/3} \\ y(.6) &= e^{(.6)^3/3} \approx 1.075 \end{aligned}$$

16. $\frac{dy}{dx} = \frac{x}{y};\ y(0) = 2,\ h = .1$; find $y(.3)$.

x_i	y_i
0	2
.1	2
.2	2.2005
.3	2.014975

Therefore, $y(.3) \approx 2.015$.

Exact solution:

$$\begin{aligned} y\frac{dy}{dx} &= x \\ \frac{y^2}{2} &= \frac{x^2}{2} + C \\ 2 &= 0 + C = C \\ \frac{y^2}{2} &= \frac{x^2}{2} + 2 \\ y^2 &= x^2 + 4 \\ y &= \sqrt{x^2 + 4}, \text{ assuming } y > 0. \\ y(.3) &= \sqrt{(.3)^2 + 4} \approx 2.022 \end{aligned}$$

18. $\frac{dy}{dx} = y;\ y(0) = 1;\ h = .2$; find $y(1)$.

Using the program for Euler's method in the Graphing Calculator Manual, the following values are obtained:

x_i	y_i	$y(x_i)$	$y_i - y(x_i)$
0	1	1	0
.2	1.2	1.2214028	−.0214028
.4	1.44	1.4918247	−.0518247
.6	1.728	1.8221188	−.0941188
.8	2.0736	2.2255409	−.1519409
1.0	2.48832	2.7182818	−.2299618

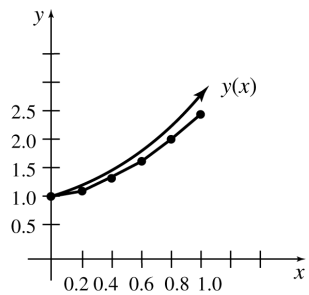
20. $\frac{dy}{dx} = x - xy;\ y(0) = .5,\ h = .2$; find $y(1)$.

Using the program in the Graphing Calculator Manual, the following values are obtained:

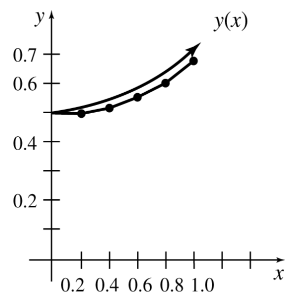
x_i	y_i	$y(x_i)$	$y_i - y(x_i)$
0	.5	.5	0
.2	.5	.5099007	−.0099007
.4	.52	.5384418	−.0184418
.6	.5584	.5823649	−.0239649
.8	.611392	.6369255	−.0255335
1.0	.6735693	.6967347	−.0231654

22. $\dfrac{dy}{dx} = y;\ y(0) = 1$

$y = e^x$ See Exercise 18.



24. $y' = x - xy;\ y(0) = 5$
$y = 1 - .5e^{-x^2/2}$ See Exercise 20.



26. **(a)** $\dfrac{dy}{dt} = ky(150 - y);\ k = .002$

$$\frac{dy}{dt} = .002y(150 - y)$$

$$\frac{dy}{dt} = .3y - .002y^2$$

(b) If $t_0 = 0$ corresponds to 2000, then $t_4 = 4$ corresponds to 2004.

$y_0 = 20;\ g(t, y) = .3y - .002y^2$
$g(t_0, y_0) = .3(20) - .002(20)^2$
$= 5.2$

$t_1 = 1;\ y_1 = 20 + (5.2)(1) = 25.2$
$g(t_1, y_1) = .3(25.2) - .002(25.2)^2$
$= 6.28992$

$t_2 = 2;\ y_2 = 25.2 + (6.28992)(1)$
$= 31.48992$
$g(t_2, y_2) = .3(31.48992)$
$- .002(31.48992)$
$= 7.463745877$

$t_3 = 3;\ y_3 = 31.48992$
$+ (7.463745877)(1)$
$= 38.95366588$

$g(t_3, y_3) = .3(38.95366588)$
$- .002(38.95366588)$
$= 8.651323593$

$t_4 = 4;\ y_4 = 38.95366588$
$+ (8.651323593)$
$= 47.60498947$

About 48 firms are bankrupt in 2004.

28. $\dfrac{dy}{dt} = .02(100 - y^{1/2}) = 2 - .02\sqrt{y}$
$t_0 = 0;\ y_0 = 10;\ h = .5$

$$g(t, y) = 2 - .02\sqrt{y}$$

$g(t_0, y_0) = 2 - .02\sqrt{10} = 1.9368$

$t_1 = .5;\ y_1 = 10 + 1.9368(.5)$
$= 10.9684$

$g(t_1, y_1) = 2 - .02\sqrt{10.9684}$
$= 1.9338$

$t_2 = 1,\ y_2 = 10.9684 + 1.9338(.5)$
$= 11.9353$
$g(t_2, y_2) = 2 - .02\sqrt{11.9353}$
$= 1.9309$

$t_3 = 1.5;\ y_3 = 11.9353 + 1.9309(.5)$
$= 12.9008$
$g(t_3, y_3) = 2 - .02\sqrt{12.9008}$
$= 1.9282$

$t_4 = 2;\ y_4 = 12.9008 + 1.9282(.5)$
$= 13.8649$

$g(t_4, y_4) = 2 - .02\sqrt{13.8649}$
$= 1.9255$

$t_5 = 2.5;\ y_5 = 13.8649 + 1.9255(.5)$
$= 14.8277$
$g(t_5, y_5) = 2 - .02\sqrt{14.8277}$
$= 1.9230$

$t_6 = 3,\ y_6 = 14.8277 + 1.9230(.5)$
$= 15.7892$
$g(t_6, y_6) = 2 - .02\sqrt{15.7892}$
$= 1.9205$

$t_7 = 3.5,\ y_7 = 15,7892 + 1.9205(.5)$
$= 16.7495$
$g(t_7, y_7) = 2 - .02\sqrt{16.7495}$
$= 1.9181$

$t_8 = 4,\ y_8 = 16.7495 + 1.9181(.5)$
$= 17.7086$

$$\begin{aligned} g(t_8, y_8) &= 2 - .02\sqrt{17.7086} \\ &= 1.9158 \end{aligned}$$

$$\begin{aligned} t_9 = 4.5;\ y_9 &= 17.7086 + 1.9158(.5) \\ &= 18.6665 \\ g(t_9, y_9) &= 2 - .02\sqrt{18.6665} \\ &= 1.9136 \\ t_{10} = 5;\ y_{10} &= 18.6665 + 1.9136(.5) \\ &= 19.6233 \end{aligned}$$

There will be about 20 species.

30. $\dfrac{dy}{dt} = -y + .02y^2 + .003y^3$; for $[0, 4]$

$h = 1,\ t_0 = 0,\ y_0 = 15$

$$g(t, y) = -y + .02y^2 + .003y^3$$

$$\begin{aligned} g(t_0, y_0) &= -15 + .02(15)^2 + .003(15)^3 \\ &= -.375 \end{aligned}$$

$$\begin{aligned} t_1 = 1;\ y_1 &= 15 + (-.375)(1) \\ &= 14.625 \\ g(t_1, y_1) &= -14.625 + .02(14.625)^2 \\ &\quad + .003(14.625)^3 \\ &= -.963 \end{aligned}$$

$$\begin{aligned} t_2 = 2;\ y_2 &= 14.625 + (-.963)(1) \\ &= 13.662 \\ g(t_2, y_2) &= -13.662 + .02(13.662)^2 \\ &\quad + .003(13.662)^2 \\ &= -2.279 \end{aligned}$$

$$\begin{aligned} t_3 = 3;\ y_3 &= 13.662 + (-2.279)(1) \\ &= 11.383 \\ g(t_3, y_3) &= -11.383 + .02(11.383)^2 \\ &\quad + .003(11.383)^3 \\ &= -4.367 \end{aligned}$$

$$\begin{aligned} t_4 = 4;\ y_4 &= 11.383 + (-4.367)(1) \\ &= 7.016 \text{ thousand} \end{aligned}$$

There will be about 7000 whales.

32. (a) Using Method 2 described after Example 1 of the text, store $\frac{dy}{dt}$ for the function variable Y_1 with $k = .5$ and $m = 4$. That is, Y_1 should equal $.5(P - P^2)^{1.5}$. Store 5 for H (use keystrokes $5 \to H$), to $-h = -5$ for T ($-5 \to T$), and $p_0 = .1$ for $P(.1 \to P)$. Next enter the keystrokes $T + H \to T$: $P + Y_1H \to P$. Each time the ENTER key is pressed, the subsequent values for t_i will be stored into T and the corresponding values for p_i will appear on the screen. This summarized in the table below.

t_i	p_i
0	.1
5	.1675
10	.297678
15	.536660
20	.846644
25	.963605
30	.980024

(b) By continuing to press the ENTER key, it appears that the values for p_i are approaching 1.

10.4 Applications of Differential Equations

2.
$$\begin{aligned} A &= \frac{-2000 + 2120e^{.06t}}{.06} \\ 20{,}000 &= \frac{-2000 + 2120e^{.06t}}{.06} \\ \frac{80}{53} &= e^{.06t} \\ \ln\left(\frac{80}{53}\right) &= .06t \\ t &= \frac{1}{.06}\ln\left(\frac{80}{53}\right) \approx 6.9 \text{ yr} \end{aligned}$$

4.
$$\begin{aligned} \frac{dA}{dt} &= .07A + D \\ \frac{1}{.07A + D}\,dA &= dt \\ \frac{\ln(.07A + D)}{.07} &= t + C \\ A &= \frac{D}{.07}(-1 + e^{.07t}) \end{aligned}$$

When $t = 3,\ A = 50{,}000$.

$$\begin{aligned} 50{,}000 &= \frac{D}{.07}(-1 + e^{.07(3)}) \\ &= \frac{D}{.07}(-1 + e^{.21}) \\ D &= \frac{50{,}000(.07)}{-1 + e^{.21}} \approx \$14{,}977.87 \end{aligned}$$

6. **(a)**

$$\frac{dy}{dt} = 3y - 2xy = y(3-2x)$$
$$\frac{dx}{dt} = -2x + 3xy = x(-2+3y)$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y(3-2x)}{x(-2+3y)}$$
$$\int \left(\frac{3y-2}{y}\right) dy = \int \left(\frac{3-2x}{x}\right) dx$$
$$\int \left(3 - \frac{2}{y}\right) dy = \int \left(\frac{3}{x} - 2\right) dx$$
$$3y - 2\ln y = 3\ln x - 2x + C$$

When $x = 1$, $y = 2$,

$$6 - 2\ln 2 = 3\ln 1 - 2 + C$$
$$8 - \ln 4 = C.$$

Therefore, $3y - 2\ln y = 3\ln x - 2x + 8 - \ln 4$ is an equation relating x and y in this case.

(b) $0 = 3y - 2xy = -y(-3+2x)$
$0 = -2x + 3xy = x(-2+3y)$

$0 = -y(-3+2x)$ and $0 = x(-2+3y)$
$y = 0$ and $x = 0$ or
$-3 + 2x = 0$ $\quad -2 + 3y = 0$
$x = \frac{3}{2}$ and $y = \frac{2}{3}$

8. **(a)** Let $y =$ the number of individuals infected. The differential equation is

$$\frac{dy}{dt} = a(N-y)y.$$

The solution is Equation 12 in Example 4, which is

$$y = \frac{N}{1 + (N-1)e^{-aNt}}, \text{ where } a = \frac{k}{N}.$$

The number of individuals uninfected at time t is

$$y = N - \frac{N}{1+(N-1)e^{-aNt}}$$
$$= \frac{N + N(N-1)e^{-aNt} - N}{1 + (N-1)e^{-aNt}}$$
$$= \frac{N(N-1)}{N - 1 + e^{aNt}}.$$

Now substitute $N = 5000$ and $a = .00005$.

$$y = \frac{5000(5000-1)}{5000 - 1 + e^{(.00005)(5000)t}}$$
$$= \frac{24{,}995{,}000}{4999 + e^{.25t}}$$

(b) $t = 30$

$$y = \frac{24{,}995{,}000}{4999 + e^{.25(30)}} = 3672$$

(c) $t = 50$

$$\frac{24{,}995{,}000}{4999 + e^{.25(50)}} = 91$$

(d) From Example 4,

$$t_m = \frac{\ln(N-1)}{aN}$$
$$= \frac{\ln(5000-1)}{(.00005)(5000)}$$
$$= 34.$$

The maximum infection rate will occur on the 34th day.

10. **(a)** The differential equation is

$$\frac{dy}{dt} = a(N-y)y.$$

$y_0 = 50;\ y = 300$ when $t = 10,\ N = 10{,}000.$

The solution is Equation 11 in Example 4, which is

$$y = \frac{N}{1 + be^{-kt}},$$

where $b = \frac{N - y_o}{y_o}$ and $k = aN$.
Since $y_0 = 50$ and $N = 10{,}000$,
$b = \frac{10{,}000 - 50}{50} = 199;\ k = 10{,}000\text{a}.$

Therefore,

$$y = \frac{10{,}000}{1 + 199e^{-10{,}000at}}.$$

$y = 300$ when $t = 10$.

$$300 = \frac{10{,}000}{1 + 199e^{-10{,}000(10)a}}$$
$$300 + 300(199)e^{-100{,}000a} = 10{,}000$$
$$e^{-100{,}000a} = \frac{9700}{300(199)}$$
$$= .1624791$$
$$a = \frac{\ln(.1624791)}{-100{,}000}$$
$$= .000018$$

$k = 10{,}000a = 10{,}000(.000018) = .18$
Therefore,

$$y = \frac{10{,}000}{1 + 199e^{-.18t}} \quad \text{or} \quad \frac{10{,}000e^{.18t}}{e^{.18t} + 199}.$$

(b) Half the community is $y = 5000$. Find t for $y = 5000$.

$$\begin{aligned} 5000 &= \frac{10{,}000}{1+199e^{-.18t}} \\ 5000 + 5000(199)e^{-.18t} &= 10{,}000 \\ e^{-.18t} &= \frac{5000}{5000(199)} = .005 \\ t &= \frac{\ln(.005)}{-.18} \\ &= 29.44 \end{aligned}$$

Half the community will be infected in about 29 days.

12. (a) $\dfrac{dy}{dt} = -ay + b(f-y)Y$

$a = 1,\ b = 1,\ f = .5,\ Y = .01;$
$y = .02$ when $t = 0$.

$$\begin{aligned} \frac{dy}{dt} &= -y + 1(.5-y)(.01) \\ &= -1.010y + .005 \end{aligned}$$

$$\int \frac{dy}{-1.010y + .005} = \int dt$$

$$\begin{aligned} \frac{1}{-1.010}\ \ln|-1.010y + .005| &= t + C_2 \\ \ln|-1.010y + .005| &= -1.010t + C_1 \\ |-1.010y + .005| &= e^{-1.010t + C_1} \\ &= e^{C_1}e^{-1.010t} \\ -1.010y + .005 &= Ce^{-1.010t} \\ y &= .005 - .990Ce^{-1.010t} \end{aligned}$$

Since $y = .02$ when $t = 0$,

$$\begin{aligned} .02 &= .005 - .990Ce^0 \\ -.990C &= .015. \end{aligned}$$

Therefore,

$$y = .005 + .015e^{-1.010t}.$$

(b) $\dfrac{dY}{dt} = -AY + B(F-Y)y$

$A = 1,\ B = 1,\ y = .1, F = .03;$
$Y = .01$ when $t = 0$.

$$\begin{aligned} \frac{dY}{dt} &= -Y + 1(.03 - Y)(.1) \\ &= -1.1Y + .003 \\ \frac{dY}{-1.1Y - .003} &= dt \\ -\frac{1}{1.1}\ \ln|-1.1Y + .003| &= t + C_2 \\ \ln|-1.1Y + .003| &= -1.1t + C_1 \\ |-1.1Y + .003| &= e^{-1.1t + C_1} \\ &= e^{C_1}e^{-1.1t} \\ -1.1Y + .003 &= Ce^{-1.1t} \\ Y &= \frac{C}{-1.1}e^{-1.1t} - \frac{.003}{-1.1} \\ &= .909Ce^{-1.1t} + .00273 \end{aligned}$$

Since $Y = .01$ when $t = 0$,

$$\begin{aligned} .01 &= -.909Ce^0 + .00273 \\ -.909C &= .00727. \end{aligned}$$

Therefore,

$$Y = .00727e^{-1.1t} + .00273.$$

14. (a) $\dfrac{dy}{dt} = a(N-y)y$

$y_0 = 5;\ y = 15$ when $t = 3;\ N = 50$

The solution to this differential equation is Equation 11 in Example 4, which is

$y = \dfrac{N}{1+be^{-kt}}$ where $b = \dfrac{N - y_0}{y_0}$ and $k = aN$.

Since $y = 5$ and $N = 50$,

$$\begin{aligned} b &= \frac{50-5}{5} = 9;\ k = 50a. \\ y &= \frac{50}{1+9e^{-50at}}. \end{aligned}$$

$y = 15$ when $t = 3$, so

$$\begin{aligned} 15 &= \frac{50}{1 + 9e^{-50a}} \\ 15 + 135e^{-150a} &= 50 \\ e^{-150a} &= \frac{35}{135} \\ &= \frac{7}{27} \\ -150a &= \ln \frac{7}{27} \\ &= -1.350 \\ -50a &= -.45. \end{aligned}$$

Therefore,

$$y = \frac{50}{1 + 9e^{-.45t}}.$$

(b) When $y = 30$,

$$\begin{aligned} 30 &= \frac{50}{1 + 9e^{-.45t}} \\ 30 + 270e^{-.45t} &= 50 \\ e^{-.45t} &= \frac{20}{270} = \frac{2}{27} \\ t &= -\frac{1}{.45} \ln \frac{2}{27} = 5.78. \end{aligned}$$

In about 6 days, 30 employees have heard the rumor.

16. (a) $\frac{dy}{dt} = kye^{-at}$; $a = .1$; $y = 5$ when $t = 0$; $y = 15$ when $t = 3$.

$$\begin{aligned} \int \frac{dy}{y} &= k \int e^{-.1t}\, dt \\ \ln |y| &= -10ke^{-.1t} + C_1 \\ |y| &= e^{-10ke^{-.1t} + C_1} \\ &= e^{C_1} e^{-10ke^{-.1t}} \\ y &= Ce^{-10ke^{-.1t}} \end{aligned}$$

Since $y = 5$ when $t = 0$,

$$\begin{aligned} 5 &= Ce^{-10k} \\ C &= 5e^{10k}. \end{aligned}$$

Since $y = 15$ when $t = 3$,

$$\begin{aligned} 15 &= Ce^{-10ke^{-.3}} = Ce^{-7.41k} \\ C &= 15e^{7.41k}. \end{aligned}$$

Solve the system

$$\begin{aligned} C &= 5e^{10k} \\ C &= 15e^{7.41k}. \end{aligned}$$

$$\begin{aligned} 5e^{10k} &= 15e^{7.41k} \\ e^{10k} &= 3e^{7.41k} \end{aligned}$$

Take natural logarithms on both sides.

$$\begin{aligned} 10k \ln e &= \ln 3 + 7.41k \ln e \\ 2.59k &= \ln 3 \\ k &= \frac{1}{2.59} \ln 3 = .424 \\ C &= 5e^{10(.424)} = 347 \end{aligned}$$

Therefore,

$$\begin{aligned} y &= 347e^{-10(.424)e^{-.1t}} \\ &= 347e^{-4.24e^{-.1t}}. \end{aligned}$$

(b) If $y = 30$,

$$\begin{aligned} 30 &= 347e^{-4.24e^{-.1t}} \\ e^{-4.24e^{-.1t}} &= \frac{30}{347} = .0865 \\ -4.24e^{-.1t} \ln e &= \ln .0865 \\ e^{-.1t} &= -\frac{1}{4.24} \ln .0865 \\ &= .5773 \\ -.1t \ln e &= \ln .5773 \\ t &= -10 \ln .5773 \\ &= 5.493. \end{aligned}$$

30 employees have heard the rumor in about 5.5 days.

18. Let $y =$ the amount of salt present at time t.

(a)

$$\begin{aligned} \frac{dy}{dt} &= (\text{rate of salt in}) \\ &\quad - (\text{rate of salt out}) \\ \text{rate of salt in} &= (3 \text{ gal/min})\,(2 \text{ lb/gal}) \\ &= 6 \text{ lb/min} \\ \text{rate of salt out} &= \left(\frac{y}{V} \text{ lb/gal}\right)(2 \text{ gal/min}) \\ &= \left(\frac{2y}{V} \text{ lb/min}\right) \\ \frac{dy}{dt} &= 6 - \frac{2y}{V};\ y(0) = 20 \text{ lb} \\ \frac{dV}{dt} &= (\text{rate of liquid in}) \\ &\quad - (\text{rate of liquid out}) \\ &= 3 \text{ gal/min} - 2 \text{ gal/min} \\ &= 1 \text{ gal/min} \\ \frac{dV}{dt} &= 1 \\ V &= t + C_1 \end{aligned}$$

When $t = 0,\ V = 100$. Thus,

$$C_1 = 100.$$
$$V = t + 100.$$

Therefore,

$$\frac{dy}{dt} = 6 - \frac{2y}{t+100}$$
$$\frac{dy}{dt} + \frac{2}{t+100}y = 6.$$

$$\begin{aligned} I(t) &= e^{\int 2\,dt/(t+100)} \\ &= e^{2\ \ln|t+100|} \\ &= (t+100)^2 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt}(t+100)^2 + 2y(t+100) &= 6(t+100)^2 \\ D_t\left[y(t+100)^2\right] &= 6(t+100)^2 \\ y(t+100)^2 &= 6\int (t+100)^2\,dt \\ y(t+100)^2 &= 2(t+100)^3 + C \\ y &= 2(t+100) + \frac{C}{(t+100)^2} \end{aligned}$$

Since $t = 0$ when $y = 20$,

$$\begin{aligned} 20 &= 2(100) + \frac{C}{100^2} \\ C &= -1{,}800{,}000. \\ y &= 2(t+100) - \frac{1{,}800{,}000}{(t+100)^2} \\ &= \frac{2(t+100)^3 -\ 1{,}800{,}000}{(t+100)^2}. \end{aligned}$$

(b) $t = 1$ hr $= 60$ min

$$y = \frac{2(160)^3 -\ 1{,}800{,}000}{(160)^2} = 249.69$$

After 1 hr, about 250 lb of salt are present.

(c) As time increases, salt concentration continues to increase.

20. Let y = the amount of salt present at time t minutes.

(a) $\frac{dy}{dt}$ = (rate of salt in) − (rate of salt out)

rate of salt in $= 0$

$$\begin{aligned} \text{rate of salt out} &= \left(\frac{y}{V}\ \text{lb/gal}\right)(2\ \text{gal/min}) \\ &= \frac{2y}{V}\ \text{lb/min} \\ \frac{dy}{dt} &= -\frac{2y}{V};\ y(0) = 20 \\ \frac{dV}{dt} &= (\text{rate of liquid in}) \\ &\quad - (\text{rate of liquid out}) \\ &= 2\ \text{gal/min} - 2\ \text{gal/min} = 0 \\ \frac{dV}{dt} &= 0 \\ V &= C_1 \end{aligned}$$

When $t = 0,\ V = 100$, so $C_1 = 100$.

Therefore,

$$\begin{aligned} \frac{dy}{dt} &= -\frac{2y}{100} = -.02y \\ \frac{dy}{y} &= -.02\,dt \\ \ln|y| &= -.02t + C_1 \\ |y| &= e^{-.02t+C_1} = e^{C_1}e^{-.02t} \\ &= Ce^{-.02t}. \end{aligned}$$

Since $t = 0$ when $y = 20$,

$$\begin{aligned} 20 &= Ce^0 \\ C &= 20. \\ y &= 20e^{-.02t.} \end{aligned}$$

(b) $t = 1$ hr $= 60$ min

$$y = 20e^{-.02(60)} = 6.024$$

After 1 hr, about 6 lb of salt are present.

(c) As time increases, salt concentration continues to decrease.

22. Let y = amount of the chemical at time t.

(a) $\frac{dy}{dt}$ = (rate of chemical in) − (rate of chemical out)

rate of chemical in
$= (2 \text{ liters/min})(.1 \text{ g/liter})$
$= .2 \text{ g/min}$

rate of chemical out
$= \left(\frac{y}{V} \text{ g/liter}\right)(1 \text{ liter/min})$
$= \frac{y}{V} \text{ g/liter}$

$$\frac{dy}{dt} = .2 - \frac{y}{V};\ y(0) = 5$$

$\frac{dV}{dt}$ = (rate of liquid in) − (rate of liquid out)
$= 2 \text{ liter/min} - 1 \text{ liter/min}$
$= 1 \text{ liter/min}$

$$\frac{dV}{dt} = 1$$
$$V = t + C_1$$

When $t = 0,\ V = 100$, so $C_1 = 100$.

$$V = t + 100$$

Therefore,

$$\frac{dy}{dt} = .2 - \frac{y}{t+100}$$
$$\frac{dy}{dt} + \frac{1}{t+100} \cdot y = .2$$
$$I(t) = e^{\int dt/(t+100)} = e^{\ln|t+100|} = t + 100$$
$$\frac{dy}{dt}(t+100) + y = .2(t+100)$$
$$D_x\,(t+100)y = .2(t+100)$$
$$(t+100)y = \int .2(t+100)\,dt$$
$$(t+100)y = .1(t+100)^2 + C$$
$$y = .1(t+100) + \frac{C}{t+100}$$

$t = 0,\ y = 5$

$$5 = .1(100) + \frac{C}{100}$$
$$500 = 1000 + C$$
$$C = -500$$

Therefore,

$$y = .1(t+100) + \frac{-500}{t+100} = \frac{.1(t+100)^2 - 500}{t+100}.$$

(b) When $t = 30$ min,

$$y = \frac{.1(130)^2 - 500}{130} = 9.154,$$

After 30 min, about 9.2 g of chemical are present.

Chapter 10 Review Exercises

6. $\frac{dy}{dx} + y^2 = xy^2$

Since the equation can be rewritten in the form $\frac{dy}{y^2} = (x-1)dx$, then the equation is separable, but since it cannot be rewritten in the form $\frac{dy}{dx} + P(x)y = Q(x)$, then the equation is not linear.

8. $\frac{dy}{dx} = xy + \ln x$

Since the equation can be rewritten in the form $\frac{dy}{dx} + (-x)y = \ln x$, then the equation is linear. Since the equation cannot be rewritten in the form $g(y)dy = f(x)dx$, then it is not separable.

10. $\frac{x}{y}\frac{dy}{dx} = 1 + x^{3/2}$

Since the equation can be rewritten in the form $\frac{dy}{dx} + \left(-\frac{1+x^{3/2}}{x}\right)y = 0$, then the equation is linear. Since it can be rewritten in the form $\frac{dy}{y} = \frac{1+x^{3/2}}{x}dx$, then it is also separable. Therefore, it is both linear and separable.

12. $\frac{dy}{dx} = x^2 + y^2$

Since the equation cannot be rewritten in either form $\frac{dy}{dx} + P(x)y = Q(x)$ or the form $g(y)dy = f(x)dx$, then the equation is neither linear nor separable.

14. $\frac{dy}{dx} = x^2 + 5x^4$

$$y = \frac{x^3}{3} + x^5 + C$$

16. $\frac{dy}{dx} = \frac{1}{2x+3}$

$$y = \frac{1}{2}\ln|2x+3| + C$$

18. $$\frac{dy}{dx} = \frac{e^x + x}{y - 1}$$
$$(y-1)\,dy = (e^x + x)\,dx$$
$$\frac{y^2}{2} - y = e^x + \frac{x^2}{2} + C$$

20. $$\frac{dy}{dx} = \frac{3-y}{e^x}$$
$$\frac{1}{3-y}\,dy = e^{-x}\,dx$$
$$-\ln|3-y| = -e^{-x} + C$$
$$\ln|3-y| = e^{-x} - C$$
$$3 - y = ke^{e^{-x}}$$
$$y = 3 - ke^{e^{-x}}$$
or $$y = 3 + Me^{e^{-x}}$$

22. $$\frac{dy}{dx} + xy = 4x$$
$$I(x) = e^{\int x\,dx} = e^{x^2/2}$$
$$e^{x^2/2}\,y = \int 4xe^{x^2/2}\,dx$$
$$= 4e^{x^2/2} + C$$
$$y = 4 + Ce^{-x^2/2}$$

24. $$x\,\frac{dy}{dx} + y - e^x = 0$$
$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x}e^x$$
$$I(x) = e^{\int dx/x} = e^{\ln\,x} = x$$
$$xy = \int x \cdot \frac{1}{x}e^x\,dx$$
$$= \int e^x\,dx = e^x + C$$
$$y = \frac{e^x}{x} + \frac{C}{x}$$

26. $\dfrac{dy}{dx} = (3x+2)^2e^y;\ y = 0$ when $x = 0$.

$$e^{-y}\,dy = (3x+2)^2\,dx$$
$$-e^{-y} = \frac{1}{9}(3x+2)^3 + C$$
$$-1 = \frac{1}{9}(2)^3 + C$$
$$C = -\frac{17}{9}$$
$$e^{-y} = \frac{17}{9} - \frac{1}{9}(3x+2)^3$$
$$y = -\ln\left[\frac{17 - (3x+2)^3}{9}\right]$$

Notice that $x < \dfrac{\sqrt[3]{17} - 2}{3}$.

28. $e^x\dfrac{dy}{dx} - e^xy = x^2 - 1;\ y = 42$ when $x = 0$.

$$\frac{dy}{dx} - y = (x^2-1)e^{-x}$$
$$I(x) = e^{\int -1\,dx} = e^{-x}$$
$$e^{-x}y = \int (x^2-1)e^{-x} \cdot e^{-x}\,dx$$
$$= \int (x^2-1)e^{-2x}\,dx$$

Integration by parts:

Let $u = x^2 - 1$ $\quad dv = e^{-2x}\,dx$

$du = 2x\,dx$ $\quad v = -\dfrac{1}{2}e^{-2x}$.

$$e^{-x}y = -\frac{(x^2-1)}{2}e^{-2x} + \int xe^{-2x}\,dx$$

Let $u = x$ $\quad dv = e^{-2x}\,dx$

$du = dx$ $\quad v = -\dfrac{1}{2}e^{-2x}$.

$$e^{-x}y = -\frac{(x^2-1)}{2}e^{-2x} - \frac{x}{2}e^{-2x} + \int \frac{1}{2}e^{-2x}\,dx$$
$$= -\frac{(x^2-1)}{2}e^{-2x} - \frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$$
$$y = -\frac{(x^2-1)}{2}e^{-x} - \frac{x}{2}e^{-x} - \frac{1}{4}e^{-x} + Ce^x$$

$$42 = \frac{1}{2} - 0 - \frac{1}{4} + C$$
$$C = 41.75$$
$$y = -\frac{(x^2-1)}{2}e^{-x} - \frac{x}{2}e^{-x} - \frac{1}{4}e^{-x} + 41.75e^x$$
$$= e^{-x}\left[-\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right] + 41.75e^x$$
$$= \frac{-x^2e^{-x}}{2} - \frac{xe^{-x}}{2} + \frac{e^{-x}}{4} + 41.75e^x$$

30. $\frac{dy}{dx} + x^2y = x^2;\ y = 8$ when $x = 0$.

$$I(x) = e^{\int x^2\,dx} = e^{x^3/3}$$
$$e^{x^3/3}\frac{dy}{dx} + x^2e^{x^3/3}y = x^2e^{x^3/3}$$
$$D_x(e^{x^3/3}y) = x^2e^{x^3/3}$$
$$e^{x^3/3}y = \int x^2e^{x^3/3}\,dx$$
$$= e^{x^3/3} + C$$
$$y = 1 + Ce^{-x^3/3}$$

Since $x = 0$ when $y = 8$,

$$8 = 1 + Ce^0$$
$$C = 7.$$

Therefore,
$$y = 1 + 7e^{-x^3/3}.$$

32. $x\frac{dy}{dx} - 2x^2y + 3x^2 = 0$

$y = 15$ when $x = 0$.

$$\frac{dy}{dx} - 2xy = -3x$$
$$I(x) = e^{\int -2x\,dx} = e^{-x^2}$$
$$e^{-x^2}y - 2xe^{-x^2}y = -3xe^{-x^2}$$
$$D_x\,(e^{-x^2}y) = -3xe^{-x^2}$$
$$e^{-x^2}y = \int -3xe^{-x^2}\,dx$$
$$= \frac{3}{2}e^{-x^2} + C$$
$$y = \frac{3}{2} + Ce^{x^2}$$

Since $x = 0$ when $y = 15$,

$$15 = \frac{3}{2} + Ce^0$$
$$C = \frac{27}{2}.$$

Therefore,

$$y = \frac{3}{2} + \frac{27e^{x^2}}{2}.$$

34. $\frac{dy}{dx} = \frac{x}{x^2-3};\ y = 52$ when $x = 2$.

$$dy = \frac{x}{x^2-3}\,dx$$
$$\int dy = \int \frac{x}{x^2-3}\,dx$$
$$y = \frac{1}{2}\ln\left|x^2-3\right| + C$$

Since $y = 52$ when $x = 2$,

$$52 = \frac{1}{2}\ln\left|(2)^2-3\right| + C$$
$$= \frac{1}{2}\ln 1 + C$$
$$= \frac{1}{2}(0) + C$$
$$C = 52$$

The particular solution is

$$y = \frac{1}{2}\ln\left|x^2-3\right| + 52.$$

36. $x^2\frac{dy}{dx} + 4xy - e^{2x^3} = 0;\ y = e^2$ when $x = 1$.

$$\frac{dy}{dx} + \frac{4}{x}y = \frac{1}{x^2}e^{2x^3}$$
$$I(x) = e^{\int 4/x\,dx} = e^{4\ln x} = x^4$$
$$x^4y = \int x^4\cdot\frac{1}{x^2}e^{2x^3}\,dx$$
$$= \int x^2e^{2x^3}\,dx$$
$$= \frac{1}{6}e^{2x^3} + C$$
$$1\cdot e^2 = \frac{1}{6}e^2 + C$$
$$C = \frac{5}{6}e^2$$
$$y = \frac{e^{2x^3} + 5e^2}{6x^4}$$

38. $\dfrac{dy}{dx} = x + y^{-1};\ y(0) = 1;\ h = .2;$

$$g(x,y) = x + \frac{1}{y}$$

$$x_0 = 0;\ y_0 = 1$$

$$g(x_0,y_0) = 0 + \frac{1}{1} = 1$$

$$x_1 = .2;\ y_1 = 1 + 1(.2) = 1.2$$

$$g(x_1,y_1) = .2 + \frac{1}{1.2} = 1.0333$$

$$x_2 = .4;\ y_2 = 1.2 + 1.0333(.2) = 1.4067$$

$$g(x_2,y_2) = .4 + \frac{1}{1.4067} = 1.1109$$

$$x_3 = .6;\ y_3 = 1.4067 + 1.1109(.2) = 1.6289$$

$$g(x_3,y_3) = .6 + \frac{1}{1.6289} = 1.2139$$

$$x^4 = .8;\ y_4 = 1.6289 + 1.2139(.2) = 1.8717$$

$$g(x_4,y_4) = .8 + \frac{1}{1.8717} = 1.3343$$

$$x_5 = 1.0;\ y_5 = 1.8719 + 1.3343(.2)$$
$$y_5 = 2.13855$$
$$y(1) \approx 2.139$$

40. $\dfrac{dy}{dx} = \dfrac{x}{2} + 4;\ y(0) = 0;\ h = .1,$

$$g(x,y) = \frac{x}{2} + 4$$

$$x_0 = 0;\ y_0 = 0$$

$$g(x_0,y_0) = \frac{0}{2} + 4 = 4$$

$$x_1 = .1;\ y_1 = 0 + 4(.1) = .4$$

$$g(x_1,y_1) = \frac{.1}{2} + 4 = 4.05$$

$$x_2 = .2;\ y_2 = .4 + 4.05(.1) = .805$$

$$g(x_2,y_2) = \frac{.2}{2} + 4 = 4.1$$

$$x_3 = .3;\ y^3 = .805 + 4.1(.1) = 1.215$$

Solving the differential equation gives

$$\frac{dy}{dx} = \frac{x}{2} + 4$$
$$\int dy = \int \left(\frac{x}{2} + 4\right) dx$$
$$y = \frac{x^2}{4} + 4x + C.$$

Since $x = 0$ when $y = 0,\ C = 0.$

$$y = \frac{x^2}{4} + 4x$$
$$y(x_3) = y(.3) = \frac{.3^2}{4} + 4(.3) = 1.223$$
$$y_3 - y(x_3) = 1.215 - 1.223 = -.008$$

44. $A =$ 10,000 when $t = 0;\ r = .05,$ $D = -1000$

(a) $\dfrac{dA}{dt} = .05A - 1000$

(b)
$$\frac{1}{.05A - 1000}\, dA = dt$$
$$\frac{1}{.05} \ln|.05A - 1000| = t + C$$
$$\ln|.05A - 1000| = .05t + k$$
$$\ln|.05(10{,}000) - 1000| = k$$
$$k = \ln|-500| = \ln 500$$
$$\ln|.05A - 1000| = .05t + \ln 500$$
$$|.05A - 1000| = 500e^{.05t}$$

Since $.05A < 1000,$

$$|.05A - 1000| = 1000 - .05A$$
$$1000 - .05A = 500e^{.05t}$$
$$A = \frac{1}{.05}(1000 - 500e^{.05t}) = 10{,}000(2 - e^{.05t}).$$

When $t = 1,$

$A = 10{,}000(2 - e^{.05(1)}) \approx \$9487.29.$

46. $\dfrac{dy}{dx} = .2(125 - y);\ y = 20$ when $x = 0;$ $y \le 125$

$$\frac{1}{125 - y}\, dy = .2\, dx$$
$$-\ln(125 - y) = .2x + C$$
$$-\ln 105 = C$$
$$-\ln(125 - y) = .2x - \ln 105$$
$$\ln(125 - y) = \ln 105 - .2x$$
$$125 - y = 105e^{-.2x}$$
$$y = 125 - 105e^{-.2x}$$

(a) If $x = 10$,

$$\begin{aligned} y &= 125 - 105e^{-.2(10)} \\ &\approx 111 \text{ items.} \end{aligned}$$

(b) For large x, $e^{-2x} \approx 0$, so $y \approx 125$. For example, if $x = 30$, $y = 125 - 105e^{-6} \approx 124.7$ which rounds to 125. Mathematically, however, $y = 125$ requires $e^{-.2x} = 0$ which is impossible.
Although not exactly possible mathematically, for practical purposes, a worker can produce 125 items in a day.

48. $\frac{dy}{dt} = ky;\ k = .1,\ t = 0,\ y = 120$

$$\begin{aligned} \frac{dy}{y} &= k\,dt \\ \ln|y| &= kt + C_1 \\ |y| &= e^{kt+C_1} \\ y &= Me^{kt} \\ y &= Me^{.1t} \\ 120 &= Me^0 \\ M &= 120 \\ y &= 120e^{.1t} \end{aligned}$$

Let $t = 6$ and find y.

$$\begin{aligned} y &= 120e^{.6} \\ &\approx 219 \end{aligned}$$

After 6 weeks, about 219 are present.

50. $\frac{dy}{dt}$ = rate of smoke in

− rate of smoke out

rate of smoke in = 0

$$\text{rate of smoke out} = \left(\frac{y}{V}\right)(1200) = \frac{1200y}{V}$$

$$\frac{dy}{dt} = \frac{-1200y}{V}$$

$$\frac{dV}{dt} = 1200 - 1200 = 0$$

$V(t) = C_1;$ at $t = 0,\ V = 15{,}000$.

$$\begin{aligned} C_1 &= 15{,}000 \\ V(t) &= 15{,}000 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{-1200y}{15{,}000} = -.08y \\ \frac{dy}{y} &= -.08\,dt \\ \ln y &= -.08t + C \\ y &= ke^{-.08t} \end{aligned}$$

At $t = 0,\ y = 20$.

$$\begin{aligned} 20 &= ke^0 \\ k &= 20 \end{aligned}$$

Therefore,

$$y = 20e^{-.08t}.$$

At $y = 5$,

$$\begin{aligned} 5 &= 20e^{-.08t} \\ .25 &= e^{-.08t} \\ t &= \frac{-\ln(.25)}{.08} \\ &= 17.3 \text{ min.} \end{aligned}$$

52. **(a)** The differential equation for y, the number of individuals infected, is

$$\frac{dy}{dt} = a(N - y)y.$$

The solution is equation 12 in Example 4 of Section 9.4, which is

$$y = \frac{N}{1 + (N-1)e^{-aNt}};\ t \text{ is in weeks.}$$

The number of individuals uninfected at time t is

$$\begin{aligned} y &= N - \frac{N}{1 + (N-1)e^{-aNt}} \\ &= \frac{N(N-1)}{N - 1 + e^{aNt}}. \end{aligned}$$

Substitute $N = 700$.

$$\begin{aligned} y &= \frac{700(699)}{699 + e^{700at}} \\ &= \frac{489{,}300}{699 + e^{700at}} \end{aligned}$$

Substitute $t = 6$, $y = 300$.

$$\begin{aligned} 300 &= \frac{489{,}300}{699 + e^{700(6)a}} \\ &= \frac{489{,}300}{699 + e^{4200a}} \\ 209{,}700 + 300e^{4200a} &= 489{,}300 \\ e^{4200a} &= 932 \\ a &= \frac{\ln 932}{4200} = .00163 \\ 700a &= 700(.00163) = 1.140 \end{aligned}$$

Therefore,

$$y = \frac{489{,}300}{699 + e^{1.140t}}.$$

(b) At $t = 7$ weeks,

$$y = \frac{489{,}300}{699 + e^{1.140(7)}}$$
$$\approx 135 \text{ people.}$$

(c) From Example 4,

$$t_m = \frac{\ln(N-1)}{aN}$$
$$= \frac{\ln(700-1)}{(.00163)(700)}$$
$$t_m = 5.7 \text{ wk.}$$

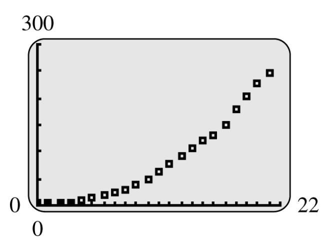
54. $N = \dfrac{\frac{1}{y_1} + \frac{1}{y_3} - \frac{2}{y_2}}{\frac{1}{y_1 y_3} - \frac{1}{y_2^2}}$

(a) Using the formula for N with $y_1 = 5.3$, $y_2 = 23.2$, and $y_3 = 76.0$, $N \approx 185$. So, the upper bound that the U.S. population was approaching during these years was approximately 185 million.

(b) Using the formula for N with $y_1 = 23.2$, $y_2 = 76.0$, and $y_3 = 150.7$, $N \approx 207$. So, the upper bound that the U.S. population was approaching during these years was approximately 207 million.

(c) Using the formula for N with $y_1 = 39.8$, $y_2 = 105.7$, and $y_3 = 203.3, N \approx 326$. So, the upper bound that the U.S. population was approaching during these years was approximately 326 million.

56. (a)



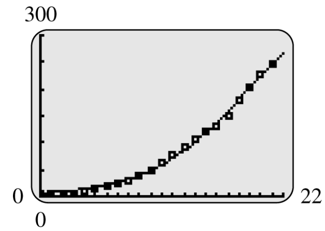
The points suggest the lower portion of a logistic growth curve, so yes, a logistic function seems appropriate.

(b) A calculator with a logistic regression function determines that

$$y = \frac{389}{1 + 54.1e^{-.227x}}$$

best fits the data.

(c)



The model does produce appropriate y-values for the given x-values.

(d) As x gets larger and larger, $e^{-.227x}$ approaches 0, so y approaches

$$\frac{389}{1 + 54.1 \cdot 0} = \frac{389}{1} = 389.$$

Therefore, according to this model, the U.S. population has a limiting size of 389 million.

58. (a) $\dfrac{dy}{dx} = a(N-y)y;\ N = 100;\ y_0 = 4;$
$y = 15$ when $x = 3$.

The solution to the differential equation is equation 11 in Example 4, which is

$$y = \frac{N}{1 + be^{-kx}},$$

where $b = \frac{N - y_0}{y_0}$ and $k = aN$.

$$b = \frac{100-4}{4} = 24;\ k = 100a$$

$$y = \frac{100}{1 + 24e^{-100ax}}$$

Since $y = 15$ when $x = 3$,

$$15 = \frac{100}{1 + 24e^{-100a(3)}}$$
$$15 + 360e^{-300a} = 100$$
$$e^{-300a} = \frac{85}{360}$$
$$-300a = \ln\left(\frac{85}{360}\right)$$
$$-100a = \frac{1}{3}\ln\left(\frac{85}{360}\right) = -.481.$$

Therefore,

$$y = \frac{100}{1 + 24e^{-.481x}}.$$

(b) For $x = 5$ days,

$$y = \frac{100}{1 + 24e^{-.481(5)}}$$
$$y = 31.58$$

In 5 days, about 32 people have heard the rumor.

60. $\frac{dT}{dt} = k(T - T_F)$; $T_F = 300$; $T = 40$ when $t = 0$.

$T = 150$ when $t = 1$.
From Section 10.2, Exercise 27, the solution to the differential equation is

$$T = ce^{kt} + T_0$$

where c is a constant ($-k$ has been replaced by k in this exercise.)
Here, $T = ce^{kt} + 300$.

$$\begin{aligned} 40 &= c + 300 \\ c &= -260 \\ T &= 300 - 260e^{kt} \\ 150 &= 300 - 260e^{k} \\ e^k &= \frac{15}{26} \\ k &= \ln\left(\frac{15}{26}\right) \approx -.55 \\ T &= 300 - 260e^{-.55t} \\ 250 &= 300 - 260e^{-.55t} \\ e^{-.55t} &= \frac{5}{26} \\ t &= -\frac{1}{.55}\ \ln\left(\frac{5}{26}\right) \approx 3 \text{ hr} \end{aligned}$$

Extended Application: Pollution of the Great Lakes

1. $t = \frac{1}{k} \ln\left(\frac{P_L(0)}{P_L(t)}\right)$

(a) $\frac{1}{k} = 2.6$; $\frac{P_L(t)}{P_L(0)} = .4$

$$t = 2.6 \ln\left(\frac{1}{.4}\right) = 2.4 \text{ yr}$$

(b) $\frac{1}{k} = 2.6$; $\frac{P_L(t)}{P_L(0)} = .3$

$$t = 2.6 \ln\left(\frac{1}{.3}\right) = 3.1 \text{ yr}$$

2. $t = \frac{1}{k} \ln\left(\frac{P_L(0)}{P_L(t)}\right)$

(a) $\frac{1}{k} = 30.8$; $\frac{P_L(t)}{P_L(0)} = .4$

$$t = 30.8 \ln\left(\frac{1}{.4}\right) = 28.2 \text{ yr}$$

(b) $\frac{1}{k} = 30.8$; $\frac{P_L(t)}{P_L(0)} = .3$

$$t = 30.8\ \ln\left(\frac{1}{.3}\right) = 37.1 \text{ yr}$$

3. $t = \frac{1}{k} \ln\left(\frac{P_L(0)}{P_L(t)}\right)$

(a) $\frac{1}{k} = 189$; $\frac{P_L(t)}{P_L(0)} = .4$

$$t = 189\ \ln\left(\frac{1}{.4}\right) = 173 \text{ yr}$$

(b) $\frac{1}{k} = 189$; $\frac{P_L(t)}{P_L(0)} = .3$

$$t = 189\ \ln\left(\frac{1}{.3}\right) = 228 \text{ yr}$$

4. (a)

$$\begin{aligned} e^{kt}P_L(t) &= P_L(0) + k\int_0^t P_i(x)e^{kx}dx \\ e^{kt}\frac{dP_L}{dt} + ke^{kt}P_L(t) &= kP_i(t)e^{kt} \\ \frac{dP_L}{dt} + kP_L(t) &= kP_i(t) \end{aligned}$$

(b) Substituting $t = 0$ in both sides of

$$P_L(t) = e^{-kt}\left[P_L(0) + k\int_0^t P_i(x)e^{kx}dx\right]$$

produces $P_L(0)$ on both sides. On the right side, e^{-kt} becomes $e^0 = 1$, and the integral goes to zero (because lower limit = upper limit).

(c) Since $30.8 = \frac{1}{k}, k = \frac{1}{30.8} \approx 0.0325$. This represents $\frac{r}{V} = \frac{\text{flow rate}}{\text{lake volume}}$, which means that every year, the total flow is 3.25% of the lake's volume. That is the percentage replaced. The higher k is, the lower $\frac{1}{k}$ is. So the lake with the biggest turnover is Lake Erie.

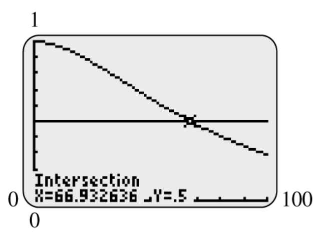
5. Substitute $P_i(x) = P_L(0)e^{-px}$ in $\int_0^t P_i(x)e^{kx}dx$.

$$\int_0^t P_L(0)e^{-px}e^{kx}dx = P_L(0)\int_0^t e^{(k-p)x}dx$$
$$= P_L(0)\left[\frac{1}{k-p}e^{(k-p)x}\right]\Bigg|_0^t$$
$$= \frac{1}{k-p}P_L(0)\left[e^{(k-p)t} - 1\right]$$

6. **(a)** For Lake Michigan, $k \approx 0.0325$. Graph

$$y_1 = \frac{1}{.0125}(.0325e^{-.02x} - .02e^{-.0325x}) \text{ and}$$
$$y_2 = .50,$$

and find the intersection point:



It will take about 67 years to reduce pollution to 50% of its current value. This compares with 21 years assuming pollution-free inflow.

(b) With $p = 0$, the inflow is constant at the same pollution level as the lake. The ratio

$$\frac{P_L(t)}{P_L(0)} = \frac{1}{k-p}(ke^{-pt} - pe^{-kt})$$

becomes

$$\frac{1}{k-0}(ke^0 - 0) = 1,$$

which means that the lake pollution level is constant, as would be expected.

Chapter 11

PROBABILITY AND CALCULUS

11.1 Continuous Probability Models

2. $f(x) = \frac{1}{3}x - \frac{1}{6}$; [3, 4]

Show that condition 1 holds.
Since $3 \le x \le 4$,

$$\begin{aligned} 1 &\le \frac{1}{3}x \le \frac{4}{3} \\ \frac{5}{6} &\le \frac{1}{3}x - \frac{1}{6} \\ &\le \frac{7}{6}. \end{aligned}$$

Hence, $f(x) \ge 0$ on [3, 4].
Show that condition 2 holds.

$$\begin{aligned} \int_3^4 \left(\frac{1}{3}x - \frac{1}{6}\right) dx &= \left(\frac{x^2}{6} - \frac{x}{6}\right)\bigg|_3^4 \\ &= \frac{1}{6}(16 - 4 - 9 + 3) \\ &= 1 \end{aligned}$$

Yes, $f(x)$ is a probability density function.

4. $f(x) = \frac{3}{98}x^2$; [3, 5]

Since $x^2 \ge 0,\ f(x) \ge 0$ on [3, 5].

$$\begin{aligned} \int_3^5 \frac{3}{98}x^2\, dx &= \frac{x^3}{98}\bigg|_3^5 \\ &= \frac{1}{98}(125 - 27) \\ &= 1 \end{aligned}$$

Yes, $f(x)$ is a probability density function.

6. $f(x) = \frac{x^3}{81}$; [0, 3]

$$\begin{aligned} \int_0^3 \frac{x^3}{81}\, dx &= \frac{x^4}{324}\bigg|_0^3 \\ &= \frac{1}{4} \\ &\ne 1 \end{aligned}$$

No, $f(x)$ is not a probability density function.

8. $f(x) = 2x^2$; [−1, 1]

$$\begin{aligned} \int_{-1}^1 2x^2\, dx &= \frac{2}{3}x^3\bigg|_{-1}^1 \\ &= \frac{2}{3}(1 + 1) \\ &= \frac{4}{3} \ne 1 \end{aligned}$$

No, $f(x)$ is not a probability density function.

10. $f(x) = \frac{3}{13}x^2 - \frac{12}{13}x + \frac{45}{52}$; $[0, 4]$

Let $x = 2$. Then $f(x) = f(2) = -\frac{3}{52} < 0$.

So $f(x) < 0$ for at least one x-value in $[0, 4]$.
No, $f(x)$ is not a probability density function.

12. $f(x) = kx^{3/2}$; [4, 9]

$$\begin{aligned} \int_4^9 kx^{3/2}\, dx &= \frac{2k}{5}x^{5/2}\bigg|_4^9 \\ &= \frac{2k}{5}(243 - 32) \\ &= \frac{422k}{5} \end{aligned}$$

If $\frac{422k}{5} = 1,\ k = \frac{5}{422}$.

Notice that $f(x) = \frac{5}{422}x^{3/2} \ge 0$ for all x in [4, 9].

14. $f(x) = kx^2$; [−1, 2]

$$\begin{aligned} \int_{-1}^2 kx^2\, dx &= \frac{k}{3}x^3\bigg|_{-1}^2 \\ &= \frac{k}{3}(8 + 1) \\ &= 3k = 1 \\ k &= \frac{1}{3} \end{aligned}$$

Notice that $f(x) = \frac{1}{3}x^2 \ge 0$ for all x in [−1, 2].

16. $f(x) = kx$; [2, 3]

$$\int_2^3 kx\, dx = \frac{k}{2}x^2\bigg|_2^3 = \frac{k}{2}(9 - 4) = \frac{5k}{2} = 1$$

$$k = \frac{2}{5}$$

Notice that $f(x) = \frac{2}{5}x \ge 0$ for all x in [2, 3].

18. $f(x) = kx^3;\ [2,\ 4]$

$$\begin{aligned}\int_2^4 kx^3\,dx &= \frac{k}{4}x^4\Big|_2^4\\ &= \frac{k}{4}(256-16)\\ &= 60k = 1\\ k &= \frac{1}{60}\end{aligned}$$

Notice that $f(x) = \frac{1}{60}x^3 \geq 0$ for all x in $[2,\ 4]$.

24. $f(x) = e^{-x};\ [0,\ \infty)$

$$\begin{aligned}\int_0^\infty e^{-x}\,dx &= \lim_{b\to\infty}\int_0^b e^{-x}\,dx\\ &= \lim_{b\to\infty}\left(-e^{-x}\Big|_0^b\right)\\ &= \lim_{b\to\infty}\left(1-\frac{1}{e^b}\right) = 1\end{aligned}$$

$f(x) \geq 0$ for all x.
$f(x)$ is a probability density function.

(a) $$\begin{aligned}P(0\leq x\leq 1) &= \int_0^1 e^{-x}\,dx\\ &= -e^{-x}\Big|_0^1\\ &= 1-\frac{1}{e}\approx .6321\end{aligned}$$

(b) $$\begin{aligned}P(1\leq x\leq 2) &= \int_1^2 e^{-x}\,dx\\ &= -e^{-x}\Big|_1^2\\ &= \frac{1}{e}-\frac{1}{e^2}\approx .2325\end{aligned}$$

(c) $$\begin{aligned}P(x\leq 2) &= \int_0^2 e^{-x}\,dx\\ &= -e^{-x}\Big|_0^2\\ &= 1-\frac{1}{e^2}\approx .8647\end{aligned}$$

Notice that

$$P(x\leq 2) = P(0\leq x\leq 1) + P(1\leq x\leq 2).$$

26. $f(x) = \dfrac{20}{(x+20)^2};\ [0,\ \infty)$

$$\begin{aligned}\int_0^\infty \frac{20}{(x+20)^2}\,dx &= \lim_{b\to\infty}\int_0^b 20(x+20)^{-2}\,dx\\ &= \lim_{b\to\infty} -20(x+20)^{-1}\Big|_0^b\\ &= \lim_{b\to\infty}\left(1-\frac{20}{b+20}\right) = 1\end{aligned}$$

$f(x) \geq 0$ for all x.
Therefore $f(x)$ is a probability density function.

(a) $$\begin{aligned}P(0\leq x\leq 1) &= \int_0^1 20(x+20)^{-2}\,dx\\ &= -20(x+20)^{-1}\Big|_0^1\\ &= -20\left(\frac{1}{21}-\frac{1}{20}\right)\\ &= \frac{1}{21}\approx .0476\end{aligned}$$

(b) $$\begin{aligned}P(1\leq x\leq 5) &= \int_1^5 20(x+20)^{-2}\,dx\\ &= -20(x+20)^{-1}\Big|_1^5\\ &= -20\left(\frac{1}{25}-\frac{1}{21}\right)\\ &= \frac{16}{105}\approx .1524\end{aligned}$$

(c) $$\begin{aligned}P(x\geq 5) &= \int_5^\infty 20(x+20)^{-2}\,dx\\ &= \lim_{b\to\infty}\int_5^b 20(x+20)^{-2}\,dx\\ &= \lim_{b\to\infty}\left[-20(x+20)^{-1}\right]\Big|_5^b\\ &= \lim_{b\to\infty}\left[-20\left(\frac{1}{b+20}-\frac{1}{25}\right)\right]\\ &= \frac{4}{5} = .8\end{aligned}$$

Alternatively,

$$\begin{aligned}P(x\geq 5) &= 1-P(0\leq x\leq 5)\\ &= 1-(.048+.152)\\ &= 1-.2 = .8.\end{aligned}$$

28. $f(x) = \begin{cases} \dfrac{20x^4}{9} & \text{if } 0 \le x \le 1 \\ \dfrac{20}{9x^5} & \text{if } x > 1 \end{cases}$

First, note that $f(x) > 0$ for all $x > 0$. Next,

$$\begin{aligned}\int_0^\infty f(x)dx &= \int_0^1 \frac{20x^4}{9}\,dx + \lim_{a\to\infty}\int_1^a \frac{20}{9x^5}\,dx \\ &= \left(\frac{4x^5}{9}\right)\Bigg|_0^1 + \lim_{a\to\infty}\left(-\frac{5}{9x^4}\right)\Bigg|_1^a \\ &= \left(\frac{4}{9} - 0\right) + \left[\lim_{a\to\infty}\left(-\frac{5}{9a^4}\right) - \left(-\frac{5}{9}\right)\right] \\ &= \frac{4}{9} + \frac{5}{9} \\ &= 1.\end{aligned}$$

Therefore, $f(x)$ is a probability density function.

(a) $$\begin{aligned}P(0 \le x \le 1) &= \int_0^1 \frac{20x^4}{9}\,dx \\ &= \left(\frac{4x^5}{9}\right)\Bigg|_0^1 \\ &= \frac{4}{9}\end{aligned}$$

(b) $$\begin{aligned}P(x \ge 1) &= P(x > 1) \\ &= \lim_{a\to\infty}\int_1^a \frac{20}{9x^5}\,dx \\ &= \lim_{a\to\infty}\left(-\frac{5}{9x^4}\right)\Bigg|_1^a \\ &= \lim_{a\to\infty}\left(-\frac{5}{9a^4}\right) - \left(-\frac{5}{9}\right) \\ &= \frac{5}{9}\end{aligned}$$

(c) $$\begin{aligned}P(0 \le x \le 2) &= \int_0^1 \frac{20x^4}{9}\,dx + \int_1^2 \frac{20}{9x^5}\,dx \\ &= \left(\frac{4x^5}{9}\right)\Bigg|_0^1 + \left(-\frac{5}{9x^4}\right)\Bigg|_1^2 \\ &= \left(\frac{4}{9} - 0\right) + \left(-\frac{5}{144} + \frac{5}{9}\right) \\ &= \frac{139}{144}\end{aligned}$$

30. $f(x) = \dfrac{1}{11}\left(1 + \dfrac{3}{\sqrt{x}}\right)$; $[4, 9]$

(a) $$\begin{aligned}P(6 \le x \le 9) &= \int_6^9 \frac{1}{11}(1 + 3x^{-1/2})dx \\ &= \frac{1}{11}(x + 6x^{1/2})\Bigg|_6^9 \\ &= \frac{1}{11}(9 + 18 - 6 - 6\sqrt{6}) \\ &= \frac{1}{11}(21 - 6\sqrt{6}) \approx .5730\end{aligned}$$

(b) $$\begin{aligned}P(4 \le x \le 5) &= \int_4^5 \frac{1}{11}(1 + 3x^{-1/2})dx \\ &= \frac{1}{11}(x + 6x^{1/2})\Bigg|_4^5 \\ &= \frac{1}{11}(5 + 6\sqrt{5} - 4 - 12) \\ &= \frac{1}{11}(6\sqrt{5} - 11) \approx .2197\end{aligned}$$

(c) $$\begin{aligned}P(4 \le x \le 7) &= \int_4^7 \frac{1}{11}(1 + 3x^{-1/2})dx \\ &= \frac{1}{11}(x + 6x^{1/2})\Bigg|_4^7 \\ &= \frac{1}{11}(7 + 6\sqrt{7} - 4 - 12) \\ &= \frac{1}{11}(6\sqrt{7} - 9) \approx .6250\end{aligned}$$

32. $f(x) = \dfrac{1}{(\ln\ 20)x}$; $[1,\ 20]$

(a) $$\begin{aligned}P(1 \le x \le 5) &= \int_1^5 \frac{1}{(\ln\ 20)x}dx \\ &= \frac{1}{\ln\ 20}\ \ln\ x\Bigg|_1^5 \\ &= \frac{\ln\ 5}{\ln\ 20} \approx .5372\end{aligned}$$

(b) $$\begin{aligned}P(10 \le x \le 20) &= \int_{10}^{20} \frac{1}{\ln\ 20x}\,dx \\ &= \frac{1}{\ln\ 20}\ \ln\ x\Bigg|_{10}^{20} \\ &= 1 - \frac{\ln\ 10}{\ln\ 20} \approx .2314\end{aligned}$$

34. $p(x,t) = \dfrac{e^{-x^2/(4Dt)}}{\int_0^L e^{-u^2/(4Dt)}du}$

(a) Let $t = 12,\ L = 6$, and $D = 38.3$.

$$p(x,\ 12) = \frac{e^{-x^2/1838.4}}{\int_0^6 e^{-u^2/1838.4}du}$$

Evaluate this (and other)integrals using the integration feature on a graphing calculator.

$$\begin{aligned} p(x,\ 12) &\approx \frac{1}{5.9611}e^{-x^2/1838.4} \\ P(0 \le x \le 3) &= \int_0^3 p(x,\ 12)dx \\ &\approx \int_0^3 \frac{1}{5.9611}e^{-x^2/1838.4}dx \\ &\approx .5024 \end{aligned}$$

The probability that a flea beetle will be recaptured withing 3 meters of the release point is about .50.

(b)
$$\begin{aligned} P(1 \le x \le 5) &= \int_1^5 p(x,\ 12)dx \\ &\approx \int_1^5 \frac{1}{5.9611}e^{-x^2/1838.4}dx \\ &\approx .6673 \end{aligned}$$

The probability that a flea beetle will be recaptured between 1 and 5 meters of the release point is about .67.

36. $f(x) = \dfrac{8}{7(x-2)^2}$; [3, 10]

(a)
$$\begin{aligned} P(3 \le x \le 4) &= \frac{8}{7}\int_3^4 (x-2)^{-2}\ dx \\ &= -\frac{8}{7}(x-2)^{-1}\Big|_3^4 \\ &= -\frac{8}{7}\left(\frac{1}{2} - 1\right) \\ &= -\frac{8}{7}\left(-\frac{1}{2}\right) \\ &= \frac{4}{7} \approx .5714 \end{aligned}$$

(b)
$$\begin{aligned} P(5 \le x \le 10) &= \frac{8}{7}\int_5^{10} (x-2)^{-2}\ dx \\ &= -\frac{8}{7}(x-2)^{-1}\Big|_5^{10} \\ &= \frac{8}{7}\left(\frac{1}{8} - \frac{1}{3}\right) \\ &= -\frac{8}{7}\left(-\frac{5}{24}\right) \\ &= \frac{5}{21} \\ &\approx .2381 \end{aligned}$$

38. $f(t) = \dfrac{1}{960}e^{-t/960}$

(a)
$$\begin{aligned} P(t < 365) &= \int_0^{365} \frac{1}{960}e^{-t/960}dt \\ &= -e^{-t/960}\Big|_0^{365} \\ &= -e^{-365/960} + 1 \\ &\approx .32 \end{aligned}$$

(b)
$$\begin{aligned} P(t > 960) &= \int_{960}^{\infty} \frac{1}{960}e^{-t/960}dt \\ &= \lim_{b\to\infty} \int_{960}^{b} e^{-t/960}dt \\ &= \lim_{b\to\infty} (-e^{-t/960})\Big|_{960}^{b} \\ &= \lim_{b\to\infty} (-e^{-b/960} + e^{-1}) \\ &= 0 + e^{-1} \\ &\approx .37 \end{aligned}$$

40. $f(x) = \dfrac{.1906}{x^{.5012}}$; $[16, 44]$

(a)
$$\begin{aligned} P(16 \le x \le 25) &= \int_{16}^{25} \frac{.1906}{x^{.5012}}\,dx \\ &= \frac{.1906}{.4988}x^{.4988}\Big|_{16}^{25} \\ &\approx .3821x^{.4988}\Big|_{16}^{25} \\ &= .3821(25^{.4988} - 16^{.4988} \\ &\approx .3798 \end{aligned}$$

(b)
$$\begin{aligned} P(35 \le x \le 44) &= \int_{35}^{44} \frac{.1906}{x^{.5012}}\,dx \\ &\approx .3821(x^{.4988})\Big|_{35}^{44} \\ &= .3821(44^{.4988} - 35^{.4988}) \\ &\approx .2722 \end{aligned}$$

(c) $P(21 \le x \le 30) = \int_{21}^{30} \frac{.1906}{x^{.5012}}\,dx$

$$\approx .3821(x^{.4988})\Big|_{21}^{30}$$
$$= .3821(30^{.4988} - 21^{.4988})$$
$$\approx .3397$$

42. $f(x) = 3x^{-4};\ [1,\ \infty)$

(a) $P(1 \le x \le 2) = \int_1^2 3x^{-4}\,dx$

$$= -x^{-3}\Big|_1^2$$
$$= 1 - \frac{1}{8}$$
$$= \frac{7}{8} = .875$$

(b) $P(3 \le x \le 5) = \int_3^5 3x^{-4}\,dx$

$$= -x^{-3}\Big|_3^5$$
$$= \frac{1}{27} - \frac{1}{125} \approx .029$$

(c) $P(x \ge 3) = \int_3^{\infty} 3x^{-4}\,dx$

$$= \lim_{b\to\infty} \int_3^b 3x^{-4}\,dx$$
$$= \lim_{b\to\infty} (-x^{-3})\Big|_3^b$$
$$= \lim_{b\to\infty} \left(\frac{1}{27} - \frac{1}{b^3}\right)$$
$$= \frac{1}{27} \approx .037$$

11.2 Expected Value and Variance of Continuous Random Variables

2. $f(x) = \frac{1}{10};\ [0,\ 10]$

$$E(x) = \mu = \int_0^{10} x\left(\frac{1}{10}\right)dx$$
$$= \frac{x^2}{20}\Big|_0^{10} = 5$$

$$\text{Var}(x) = \int_0^{10} (x-5)^2\left(\frac{1}{10}\right)dx$$
$$= \frac{1}{10}\cdot\frac{(x-5)^3}{3}\Big|_0^{10}$$
$$= \frac{25}{6} + \frac{25}{6}$$
$$= \frac{25}{3} \approx 8.33$$
$$\sigma \approx \sqrt{\text{Var}(x)} \approx 2.89$$

4. $f(x) = 2(1-x);\ [0,\ 1]$

$$\mu = \int_0^1 2x(1-x)dx = \int_0^1 (2x - 2x^2)dx$$
$$= \left(x^2 - \frac{2x^3}{3}\right)\Big|_0^1 = 1 - \frac{2}{3}$$
$$= \frac{1}{3} \approx .33$$

Use the alternative formula to find

$$\text{Var}(x) = \int_0^1 2x^2(1-x)dx - \left(\frac{1}{3}\right)^2$$
$$= \int_0^1 (2x^2 - 2x^3)dx - \frac{1}{9}$$
$$= \left(\frac{2x^3}{3} - \frac{x^4}{2}\right)\Big|_0^1 - \frac{1}{9}$$
$$= \frac{2}{3} - \frac{1}{2} - \frac{1}{9} = \frac{1}{18} \approx .06.$$
$$\sigma = \sqrt{\text{Var}(x)} \approx .24$$

6. $f(x) = \frac{1}{11}\left(1 + \frac{3}{\sqrt{x}}\right);\ [4,\ 9]$

$$\mu = \int_4^9 \frac{x}{11}\left(1 + \frac{3}{\sqrt{x}}\right)dx$$
$$= \int_4^9 \frac{1}{11}(x + 3x^{1/2})dx$$
$$= \frac{1}{11}\left(\frac{x^2}{2} + 2x^{3/2}\right)\Big|_4^9$$
$$= \frac{1}{11}\left(\frac{81}{2} + 54 - 8 - 16\right)$$
$$= \frac{141}{22} \approx 6.41$$

$$\begin{aligned}\text{Var}(x) &= \int_4^9 \frac{x^2}{11}\left(1+\frac{3}{\sqrt{x}}\right)dx - \mu^2\\ &= \int_4^9 \frac{1}{11}(x^2+3x^{3/2})dx - \mu^2\\ &= \frac{1}{11}\left(\frac{x^3}{3}+\frac{6}{5}x^{5/2}\right)\Big|_4^9 - \mu^2\\ &= \frac{1}{11}\left(243+\frac{1458}{5}-\frac{64}{3}-\frac{192}{5}\right)\\ &\quad -\left(\frac{141}{22}\right)^2\\ &\approx 2.09\end{aligned}$$

$$\sigma \approx \sqrt{\text{Var}(x)} \approx 1.45$$

8. $f(x) = 3x^{-4};\ [1,\ \infty)$

$$\begin{aligned}\mu &= \int_1^\infty x(3x^{-4})dx\\ &= \lim_{b\to\infty}\int_1^b 3x^{-3}\,dx\\ &= \lim_{b\to\infty}\left(-\frac{3x^{-2}}{2}\right)\Big|_1^b\\ &= \lim_{b\to\infty}\left(\frac{3}{2}-\frac{3}{2b^2}\right) = \frac{3}{2} = 1.5\end{aligned}$$

$$\begin{aligned}\text{Var}(x) &= \int_1^\infty x^2(3x^{-4})dx - \left(\frac{3}{2}\right)^2\\ &= \lim_{b\to\infty}\int_1^b 3x^{-2}\,dx - \frac{9}{4}\\ &= \lim_{b\to\infty}(-3x^{-1})\Big|_1^b - \frac{9}{4}\\ &= \lim_{b\to\infty}\left(3-\frac{3}{b}\right)-\frac{9}{4}\\ &= 3-\frac{9}{4} = \frac{3}{4} = .75\end{aligned}$$

$$\sigma = \sqrt{\text{Var}(x)} \approx .87$$

12. $f(x) = \dfrac{x^{-1/3}}{6};\ [0,\ 8]$

(a) $$\begin{aligned}\mu &= \int_0^8 x\left(\frac{x^{-1/3}}{6}\right)dx\\ &= \int_0^8 \frac{x^{2/3}}{6}\,dx\\ &= \frac{x^{5/3}}{10}\Big|_0^8\\ &= \frac{16}{5} = 3.2\end{aligned}$$

(b) $$\begin{aligned}\text{Var}(x) &= \int_0^8 x^2\left(\frac{x^{-1/3}}{6}\right)dx - \left(\frac{16}{5}\right)^2\\ &= \int_0^8 \frac{x^{5/3}}{6}\,dx - \frac{256}{25}\\ &= \frac{x^{8/3}}{16}\Big|_0^8 - \frac{256}{25}\\ &= 16-\frac{256}{25}\\ &= \frac{144}{25} = 5.76\end{aligned}$$

(c) $\sigma = \sqrt{\text{Var}(x)} = 2.4$

(d) $$\begin{aligned}P(x>\mu) &= \int_\mu^8 \frac{x^{-1/3}}{6}\,dx\\ &= \frac{x^{2/3}}{4}\Big|_{3.2}^8\\ &= 1-\frac{(3.2)^{2/3}}{4}\\ &\approx .46\end{aligned}$$

(e) $$\begin{aligned}&P(3.2-2.4<x<3.2+2.4)\\ &= P(.8<x<5.6)\\ &= \int_{.8}^{5.6}\frac{x^{-1/3}}{6}\,dx\\ &= \frac{x^{2/3}}{4}\Big|_{.6}^{5.6}\\ &= \frac{1}{4}[(5.6)^{2/3}-(.8)^{2/3}]\\ &\approx .57\end{aligned}$$

14. $f(x) = \dfrac{3}{2}(1-x^2);\ [0,\ 1]$

(a) $$\begin{aligned}\mu &= \int_0^1 \frac{3}{2}x(1-x^2)dx\\ &= \int_0^1 \frac{3}{2}(x-x^3)dx\\ &= \frac{3}{2}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_0^1\\ &= \frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{3}{8} \approx .38\end{aligned}$$

(b) $$\begin{aligned}\text{Var}(x) &= \int_0^1 \frac{3}{2}x^2(1-x^2)dx - \left(\frac{3}{8}\right)^2\\ &= \int_0^1 \frac{3}{2}(x^2-x^4)dx - \frac{9}{64}\\ &= \frac{3}{2}\left(\frac{x^3}{3}-\frac{x^5}{5}\right)\Big|_0^1 - \frac{9}{64}\\ &= \frac{3}{2}\left(\frac{1}{3}-\frac{1}{5}\right) - \frac{9}{64}\\ &= \frac{1}{5}-\frac{9}{64}\\ &= \frac{19}{320}\\ &\approx .06\end{aligned}$$

(c) $\sigma \approx \sqrt{\text{Var}(x)} \approx .24$

(d) $$\begin{aligned}P(x>\mu) &= \int_\mu^1 \frac{3}{2}(1-x^2)dx\\ &= \frac{3}{2}\left(x-\frac{x^3}{3}\right)\Big|_{3/8}^1\\ &= \frac{3}{2}\left(1-\frac{1}{3}-\frac{3}{8}+\frac{9}{512}\right)\\ &\approx .46\end{aligned}$$

(e) $$\begin{aligned}&P(.375-.24<x<.375+.24)\\ &= P(.135<x<.615)\\ &= \int_{.135}^{.615}\frac{3}{2}(1-x^2)dx\\ &= \frac{3}{2}\left(x-\frac{x^3}{3}\right)\Big|_{.135}^{.615}\\ &= \frac{3}{2}\left[.615-\frac{(.615)^3}{3}-.135+\frac{(.135)^3}{3}\right]\\ &\approx .60\end{aligned}$$

16. $f(x) = \frac{1}{10}$; $[0,\ 10]$

(a) $$\int_0^m \frac{1}{10}\,dx = \frac{x}{10}\Big|_0^m = \frac{m}{10}$$
$$= \frac{1}{2} \text{ when } m = 5.$$

(b) $E(x) = \mu = 5$ (from Exercise 2)
$P(5 \le x \le 5) = P(x=5) = 0$

18. $f(x) = 2(1-x)$; $[0,1]$

(a) $$\begin{aligned}\int_0^m 2(1-x)dx &= (2x-x^2)\Big|_0^m\\ &= 2m-m^2\\ &= \frac{1}{2}\\ 2m^2-4m+1 &= 0\\ m &= \frac{2-\sqrt{2}}{2}\\ &\approx .293\end{aligned}$$

(Reject $m = \frac{2+\sqrt{2}}{2}$, which is not in the interval [0, 1].)

(b) $E(x) = \mu = \frac{1}{3}$ (from Exercise 4)

$$\begin{aligned}&P(.29<x<.33)\\ &= \int_{.29}^{.33} 2(1-x)dx\\ &= (2x-x^2)\Big|_{.29}^{.33}\\ &= 2(.33)-(.33)^2-2(.29)+(.29)^2\\ &\approx .06\end{aligned}$$

20. $f(x) = 3x^{-4}$; $[1,\ \infty)$

(a) $$\begin{aligned}\int_1^m 3x^{-4}\,dx &= -x^{-3}\Big|_1^m\\ &= 1-\frac{1}{m^3} = \frac{1}{2}\\ m^3 &= 2\\ m &= \sqrt[3]{2}\\ &\approx 1.26\end{aligned}$$

(b) $E(x) = \mu = 1.5$ (From Exercise 8)

$$\begin{aligned}&P(1.26<x<1.5)\\ &= \int_{1.26}^{1.5} 3x^{-4}\,dx\\ &= -x^{-3}\Big|_{1.26}^{1.5}\\ &= \frac{1}{(1.26)^3}-\frac{1}{(1.5)^3}\\ &\approx .204\end{aligned}$$

22. $f(x) = \begin{cases} \dfrac{20x^4}{9} & \text{if } 0 \le x \le 1 \\ \dfrac{20}{9x^5} & \text{if } x > 1 \end{cases}$

Expected value:

$$\begin{aligned} E(x) = \mu &= \int_0^\infty x\,f(x)dx \\ &= \int_0^1 x\,\frac{20x^4}{9}\,dx + \lim_{a\to\infty}\int_1^a x\,\frac{20}{9x^5}\,dx \\ &= \int_0^1 \frac{20x^5}{9}\,dx + \lim_{a\to\infty}\int_1^a \frac{20}{9x^4}\,dx \\ &= \left(\frac{20x^6}{54}\right)\Bigg|_0^1 + \lim_{a\to\infty}\left(-\frac{20}{27x^3}\right)\Bigg|_1^a \\ &= \left(\frac{10}{27} - 0\right) + \left[\lim_{a\to\infty}\left(-\frac{20}{27a^3}\right) - \left(-\frac{20}{27}\right)\right] \\ &= \frac{10}{27} + \frac{20}{27} \\ &= \frac{10}{9} \end{aligned}$$

Variance:

$$\begin{aligned} \text{Var}(x) &= \int_0^\infty x^2 f(x)dx - \mu^2 \\ &= \int_0^1 x^2\frac{20x^4}{9}\,dx \\ &\quad + \lim_{a\to\infty}\int_1^a x^2\frac{20}{9x^5}\,dx - \left(\frac{10}{9}\right)^2 \\ &= \int_0^1 \frac{20x^6}{9}\,dx + \lim_{a\to\infty}\int_1^a \frac{20}{9x^3}\,dx - \frac{100}{81} \\ &= \left(\frac{20x^7}{63}\right)\Bigg|_0^1 + \lim_{a\to\infty}\left(-\frac{10}{9x^2}\right)\Bigg|_1^a - \frac{100}{81} \\ &= \left(\frac{20}{63} - 0\right) + \left[\lim_{a\to\infty}\left(-\frac{10}{9a^2}\right) - \left(-\frac{10}{9}\right)\right] - \frac{100}{81} \\ &= \frac{20}{63} + \frac{10}{9} - \frac{100}{81} \\ &= \frac{110}{567} \end{aligned}$$

Standard deviation:

$$\sigma = \sqrt{\text{Var}(x)} \approx .44046$$

24. $f(x) = \dfrac{1}{11}\left(1 + \dfrac{3}{\sqrt{x}}\right)$; $[4,\ 9]$

(a) From Exercise 6, $\mu \approx 6.41$ yr.

(b) $\sigma \approx 1.45$ yr

(c) $P(x > 6.41)$

$$\begin{aligned} &= \int_{6.41}^9 \frac{1}{11}(1 + 3x^{-1/2})dx \\ &= \frac{1}{11}(x + 6x^{1/2})\Bigg|_{6.41}^9 \\ &= \frac{1}{11}[9 + 18 - 6.41 - 6(6.41)^{1/2}] \\ &\approx .49 \end{aligned}$$

26. $f(x) = \dfrac{1}{(\ln\ 20)x}$; $[1,\ 20]$

(a)
$$\begin{aligned} \mu &= \int_1^{20} x \cdot \frac{1}{(\ln\ 20)x}\,dx \\ &= \int_1^{20} \frac{1}{\ln\ 20}\,dx \\ &= \frac{x}{\ln\ 20}\Bigg|_1^{20} \\ &= \frac{19}{\ln\ 20} \approx 6.34 \text{ seconds} \end{aligned}$$

(b)
$$\begin{aligned} \text{Var}(x) &= \int_1^{20} x^2 \cdot \frac{1}{(\ln\ 20)x}\,dx - \mu^2 \\ &= \int_1^{20} \frac{x}{\ln\ 20}\,dx - \mu^2 \\ &= \frac{x^2}{2\ \ln\ 20}\Bigg|_1^{20} - (6.34)^2 \\ &= \frac{399}{2\ \ln\ 20} - (6.34)^2 \\ &\approx 26.40 \\ \sigma &\approx \sqrt{26.40} \\ &\approx 5.14 \text{ sec} \end{aligned}$$

(c)
$$\begin{aligned} &P(6.34 - 5.14 < x < 6.34 + 5.14) \\ &= P(1.2 < x < 11.48) \\ &= \int_{1.2}^{11.48} \frac{1}{(\ln\ 20)x}\,dx \\ &= \frac{\ln\ x}{\ln\ 20}\Bigg|_{1.2}^{11.48} \\ &= \frac{1}{\ln\ 20}(\ln\ 11.48 - \ln\ 1.2) \\ &\approx .75 \end{aligned}$$

28. $f(x) = \dfrac{1}{2\sqrt{x}}$; $[1,\ 4]$

(a)
$$\begin{aligned}\mu &= \int_1^4 x \cdot \frac{1}{2\sqrt{x}}\,dx \\ &= \int_1^4 \frac{x^{1/2}}{2}\,dx \\ &= \frac{x^{3/2}}{3}\Big|_1^4 \\ &= \frac{1}{3}(8-1) \\ &= \frac{7}{3} \approx 2.33 \text{ cm}\end{aligned}$$

(b)
$$\begin{aligned}\text{Var}(x) &= \int_1^4 x^2 \cdot \frac{1}{2\sqrt{x}}\,dx - \left(\frac{7}{3}\right)^2 \\ &= \int_1^4 \frac{x^{3/2}}{2}\,dx - \frac{49}{9} \\ &= \frac{x^{5/2}}{5}\Big|_1^4 - \frac{49}{9} \\ &= \frac{1}{5}(32-1) - \frac{49}{9} \\ &\approx .76 \\ \sigma &= \sqrt{\text{Var}(x)} \\ &\approx .87 \text{ cm}\end{aligned}$$

(c)
$$\begin{aligned}&P(x > 2.33 + 2(.87)) \\ &= P(x > 4.07) \\ &= 0\end{aligned}$$

The probability is 0 since two standard deviations falls out of the given interval $[1,\ 4]$.

30. $f(x) = 1.185 \cdot 10^{-9} x^{4.5222} e^{-.049846x}$

$$E(x) = \int_1^{1000} x\,f(x)\,dx$$

Using the integration function on our calculator.

$$E(x) \approx 110.80$$

The expected size is about 111.

32. $f(x) = \dfrac{5.5 - x}{15}$; $[0,\ 5]$

(a)
$$\begin{aligned}\mu &= \int_0^5 x\left(\frac{5.5-x}{15}\right)dx \\ &= \int_0^5 \left(\frac{5.5}{15}x - \frac{1}{15}x^2\right)dx \\ &= \frac{5.5}{30}x^2 - \frac{1}{45}x^3\Big|_0^5 \\ &= \left(\frac{5.5}{30}\cdot 25 - \frac{1}{45}\cdot 125\right) - 0 \\ &\approx 1.806\end{aligned}$$

(b)
$$\begin{aligned}\text{Var}(x) &= \int_0^5 x^2\left(\frac{5.5-x}{15}\right)dx - \mu^2 \\ &= \int_0^5 \left(\frac{5.5}{15}x^2 - \frac{1}{15}x^3\right)dx - \mu^2 \\ &= \left(\frac{5.5}{45}x^3 - \frac{1}{60}x^4\right)\Big|_0^5 - \mu^2 \\ &= \frac{5.5}{45}\cdot 125 - \frac{1}{60}\cdot 625 - 0 - \mu^2 \\ &\approx 1.60108 \\ \sigma &= \sqrt{\text{Var}(x)} \\ &\approx 1.265\end{aligned}$$

(c)
$$\begin{aligned}&P(x \le \mu - \sigma) \\ &= P(x \le 1.806 - 1.265) \\ &= P(x \le .541) \\ &= \int_0^{.541} \frac{5.5-x}{15}\,dx \\ &= \left(\frac{5.5}{15}x - \frac{1}{30}x^2\right)\Big|_0^{.541} \\ &= \left(\frac{5.5}{15}(.541) - \frac{1}{30}(.541)^2 - 0\right) \\ &\approx .1886\end{aligned}$$

34. $f(x) = \dfrac{.1906}{x^{.5012}}$; or $[16, 44]$

(a) Expected value:

$$\begin{aligned} E(x) = \mu &= \int_{16}^{44} x \frac{.1906}{x^{.5012}}\, dx \\ &= \int_{16}^{44} .1906x^{.4988} dx \\ &= \frac{.1906}{1.4988} x^{1.4988} \Big|_{16}^{44} \\ &= \frac{.1906}{1.4988}(44^{1.4988} - 16^{1.4988}) \\ &\approx 28.8358 \approx 28.8 \text{ years} \end{aligned}$$

(b) Standard deviation:

$$\begin{aligned} \text{Var}(x) &= \int_{16}^{44} x^2 \frac{.1906}{x^{.5012}}\, dx - 28.8358^2 \\ &= \int_{16}^{44} .1906x^{1.4988} dx - 28.8358^2 \\ &= \frac{.1906}{2.4988} x^{2.4988} \Big|_{16}^{44} - 28.8358^2 \\ &= \frac{.1906}{2.4988}(44^{2.4988} - 16^{2.4988}) - 28.8358^2 \\ &\approx 65.75 \end{aligned}$$

$$\begin{aligned} \sigma &= \sqrt{\text{Var}(x)} \\ &= \sqrt{65.75} \\ &\approx 8.1 \end{aligned}$$

(c) $$\begin{aligned} P(16 \le x \le 28.8 - 8.1) &= \int_{16}^{20.7} \frac{.1906}{x^{.5012}}\, dx \\ &= \frac{.1906}{.4988} x^{.4988} \Big|_{16}^{20.7} \\ &= \frac{.1906}{.4988}(20.7^{.4988} - 16^{.4988}) \\ &\approx .2088 \end{aligned}$$

11.3 Special Probability Density Functions

2. $f(x) = 2$ for $[1.25, 1.75]$

This is a uniform distribution.

(a) $\mu = \dfrac{1}{2}(1.75 + 1.25) = \$1.50$

(b) $$\begin{aligned} \sigma &= \frac{1}{\sqrt{12}}(1.75 - 1.25) \\ &= \frac{.5}{\sqrt{12}} \\ &\approx \$.14 \end{aligned}$$

(c) $$\begin{aligned} &P(1.5 < x < 1.5 + .14) \\ &= P(1.5 < x < 1.64) \\ &= \int_{1.5}^{1.64} 2\, dx = 2x \Big|_{1.5}^{1.64} = .28 \end{aligned}$$

4. $f(t) = .05e^{-.05t}$ for $[0, \infty)$

This is an exponential distribution.

(a) $\mu = \dfrac{1}{.05} = 20$ yr

(b) $\sigma = \dfrac{1}{.05} = 20$ yr

(c) $$\begin{aligned} &P(20 < x < 20 + 20) \\ &= P(20 < x < 40) \\ &= \int_{20}^{40} .05e^{-.05t}\, dt \\ &= -e^{-.05t} \Big|_{20}^{40} \\ &= e^{-1} - e^{-2} \\ &\approx .23 \end{aligned}$$

6. $f(x) = .1e^{-.1x}$ for $[0, \infty)$

This is an exponential distribution.

(a) $\mu = \dfrac{1}{.1} = 10\ m$

(b) $\sigma = \dfrac{1}{.1} = 10\ m$

(c) $$\begin{aligned} &P(10 < x < 10 + 10) \\ &= P(10 < x < 20) \\ &= \int_{10}^{20} .1e^{-.1x}\, dx \\ &= -e^{-.1x} \Big|_{10}^{20} \\ &= e^{-1} - e^{-2} \approx .23 \end{aligned}$$

In Exercises 8-14, use the table in the Appendix for areas under the normal curve.

8. $z = 1.68$

Area between mean $z = 0$ and $z = 1.68$ is

$$.9535 - .5000 = .4535.$$

Percent of area = 45.35%

10. Area between $z = -2.13$ and $z = -.04$ is

$$.4840 - .0166 = .4674.$$

Percent of area = 46.74%

12. Since 2% = .02, the z-score that corresponds to the area of .02 to the left of z is -2.05.

14. 22% of the total area to the right of z means $1-.22$ of the total area to the left of z.

$$1 - .22 = .78$$

The closest z-score that corresponds to the area of .78 is .77.

18. For the uniform distribution, $f(x) = \frac{1}{b-a}$ for x in $[a,\ b]$.
If m is the median,

$$P(x \le m) = \frac{1}{2}$$
$$\int_a^m \frac{1}{b-a}\,dx = \frac{1}{2}$$
$$\frac{1}{b-a}\int_a^m dx = \frac{1}{2}$$

Multiply both sides by $b - a$.

$$\int_a^m dx = \frac{1}{2}b - \frac{1}{2}a$$
$$x\Big|_a^m = \frac{1}{2}b - \frac{1}{2}a$$
$$m - a = \frac{1}{2}b - \frac{1}{2}a$$
$$m = \frac{1}{2}b + \frac{1}{2}a$$
$$m = \frac{b+a}{2}$$

20. $f(x) = ae^{-ax}$ for $[0,\ \infty)$ with $a > 0$.

$$\mu = \int_0^\infty xae^{-ax}\,dx$$

We first consider $\int xae^{-ax}\,dx$ using integration by parts.

Let $u = x$ and $dv = ae^{-ax}\,dx$;
$du = dx$ and $v = -e^{-ax}$.

$$\int xae^{-ax}\,dx = -xe^{-ax} - \int(-e^{-ax})dx$$
$$= -xe^{-ax} - \frac{1}{a}e^{-ax} + C$$
$$= -\frac{ax+1}{ae^{ax}} + C$$

$$\mu = \int_0^\infty xae^{-ax}\,dx$$
$$= \lim_{b\to\infty}\int_0^b xae^{-ax}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{ax+1}{ae^{ax}}\right)\Bigg|_0^b$$
$$= \lim_{b\to\infty}\left(-\frac{ab+1}{ae^{ab}} + \frac{1}{a}\right)$$

$$\mu = \frac{1}{a}$$

since e^{ab} grows more rapidly than ab.

$$\text{Var}(x) = \int_0^\infty x^2ae^{-ax}\,dx - \left(\frac{1}{a}\right)^2$$

We first consider $\int x^2ae^{-ax}\,dx$ using integration by parts.

Let $u = x$ and $dv = xae^{-ax}\,dx$

$du = dx$ and $v = -xe^{-ax} - \frac{1}{a}e^{-ax}$

from the previous integration by parts.

$$\int x^2 ae^{-ax}\,dx$$

$$= x\left(-xe^{-ax} - \frac{1}{a}e^{-ax}\right)$$

$$- \int \left(-xe^{-ax} - \frac{1}{a}e^{-ax}\right) dx$$

$$= -x^2e^{-ax} - \frac{1}{a}xe^{-ax} + \int xe^{-ax}\,dx$$

$$+ \frac{1}{a}\int e^{-ax}\,dx$$

$$= -x^2e^{-ax} - \frac{1}{a}xe^{-ax}$$

$$+ \frac{1}{a}\left(-xe^{-ax} - \frac{1}{a}e^{-ax}\right) - \frac{1}{a^2}e^{-ax}$$

$$= -x^2e^{-ax} - \frac{1}{a}xe^{-ax} - \frac{1}{a}xe^{-ax}$$

$$- \frac{1}{a^2}e^{-ax} - \frac{1}{a^2}e^{-ax}$$

$$= -\frac{a^2x^2 + 2ax + 2}{a^2e^{ax}}$$

$$\begin{aligned}\text{Var}(x) &= \int_0^\infty x^2ae^{-ax}\,dx - \frac{1}{a^2}\\ &= \lim_{b\to\infty}\int_0^b x^2ae^{-ax}\,dx - \frac{1}{a^2}\\ &= \lim_{b\to\infty} -\frac{a^2x^2+2ax+2}{a^2e^{ax}}\bigg|_0^b - \frac{1}{a^2}\\ &= \lim_{b\to\infty}\left[-\frac{a^2b^2+2ab+2}{a^2e^{ab}} + \frac{2}{a^2}\right] - \frac{1}{a^2}\\ &= \frac{2}{a^2} - \frac{1}{a^2},\end{aligned}$$

since e^{ab} grows more rapidly than $a^2b^2 + 2ab + 2$.

$$\text{Var}(x) = \frac{1}{a^2}$$

Thus,

$$\sigma = \sqrt{\frac{1}{a^2}} = \frac{1}{a}.$$

22. $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}\left[-2\frac{(x-\mu)}{2\sigma^2}\right]$$

$$f'(x) = \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}e^{(x-\mu)^2/2\sigma^2}}$$

$$f''(x) = \frac{1}{\sigma^3\sqrt{2\pi}}\left(\frac{e^{(x-\mu)^2/2\sigma^2}(-1)+(x-\mu)e^{(x-\mu)^2/2\sigma^2}\frac{(x-\mu)}{\sigma^2}}{e^{(x-\mu)^2/\sigma^2}}\right)$$

$$f''(x) = \frac{-e^{(x-\mu)^2/2\sigma^2}\left(1-\frac{(x-\mu)^2}{\sigma^2}\right)}{\sigma^3\sqrt{2\pi}e^{(x-\mu)^2/\sigma^2}}$$

$$f''(x) = -\frac{1-\frac{(x-\mu)^2}{\sigma^2}}{\sigma^3\sqrt{2\pi}e^{(x-\mu)^2/2\sigma^2}}$$

If $f''(x) = 0$, then $1 - \frac{(x-\mu)^2}{\sigma^2} = 0$

$$\begin{aligned}\frac{(x-\mu)^2}{\sigma^2} &= 1\\ (x-\mu)^2 &= \sigma^2\\ x-\mu &= \pm\sigma\\ x &= \mu\pm\sigma.\end{aligned}$$

If $x < \mu - \sigma$, $f''(x) > 0$.
If $\mu - \sigma < x < \mu + \sigma$, $f''(x) < 0$.
If $x > \mu + \sigma$, $f''(x) > 0$.

Therefore, there are points of inflection when $x = \mu - \sigma$ and when $x = \mu + \sigma$.

24. From Exercise 23(b) and 23(c),

$$\int_0^{50} = .5xe^{-.5x}\,dx \approx 1.99999 \text{ and}$$

$$\int_0^{50} = .5x^2e^{-.5x}\,dx \approx 8.00003$$

From Exercise 23(a), we have the exponential distribution $f(x) = .5e^{-.5x}$ for x in $[0, \infty)$ with $a = .5$.
For this distribution,

$$\mu = \int_0^\infty .5xe^{-5x}\,dx \approx \int_0^{50} .5xe^{.5x}\,dx$$
$$\approx 1.99999$$

$$1.99999 \approx \frac{1}{a} \text{ since } \frac{1}{a} = \frac{1}{.5} = 2.$$

Also, for this distribution,

$$\begin{aligned}\text{Var}(x) &= \int_0^\infty .5x^2e^{-.5x}\,dx - \mu^2\\ &\approx \int_0^{50} .5x^2e^{-.5x}\,dx - (1.99999)^2\\ &\approx 8.00003 - (1.99999)^2\\ &\approx 4.00007.\\ \sigma &\approx \sqrt{4.00007} \approx 2.00002\end{aligned}$$

$2.00002 \approx \frac{1}{a}$ since $\frac{1}{a} = \frac{1}{.5} = 2.$

26. Use $f(x) = abx^{b-1}e^{-ax^b}$ with $a = 4$ and $b = 1.5$.

$$\begin{aligned}f(x) &= (4)(1.5)x^{1.5-1}e^{-4x^{1.5}}\\ &= 6x^{.5}e^{-4x^{1.5}}\end{aligned}$$

(a) $$\begin{aligned}\mu &= \int_0^\infty x(6x^{.5}e^{-4x^{1.5}})dx\\ &= \int_0^\infty 6x^{1.5}e^{-4x^{1.5}}dx\\ \mu &\approx \int_0^3 6x^{1.5}e^{-4x^{1.5}}dx\end{aligned}$$

Using the integration feature on a graphing calculator,

$$\mu \approx .3583.$$

(b) $\sigma^2 \approx \int_0^3 (x - .3583)^2 6x^{.5}e^{-4x^{1.5}}dx$

Using the integration feature on a graphing calculator,

$$\begin{aligned}\sigma^2 &\approx .05916\\ \sigma &\approx .2432.\end{aligned}$$

28. We have an exponential distribution with mean $\mu = 5$.

$$\mu = \frac{1}{a} = 5$$
$$a = .2$$

(a) $f(x) = .2e^{-.2x}$ for $[0,\ \infty)$

(b) $$\begin{aligned}P(2 < x < 6) &= \int_2^6 .2e^{-.2x}dx\\ &= -e^{-.2x}\Big|_2^6\\ &= e^{-.4} - e^{-1.2}\\ &\approx .369\end{aligned}$$

30. We have a normal distribution with $\mu = 32.8$, $\sigma = 1.1$.
Let x = number of ounces of juice.

(a) $$\begin{aligned}&P(x < 32)\\ &= P\left(\frac{x - 32.8}{1.1} < \frac{32 - 32.8}{1.1}\right)\\ &= P(z < -.73)\\ &= .2327\end{aligned}$$

(b) $$\begin{aligned}&P(x > 33)\\ &= P\left(\frac{x - 32.8}{1.1} < \frac{33 - 32.8}{1.1}\right)\\ &= P(z > .18)\\ &= 1 - P(z \le .18)\\ &= 1 - .5714\\ &= .4286\end{aligned}$$

32. We have a normal distribution, with $\mu = 54.40$, $\sigma = 13.50$.

$$P(-a < z < a) = \frac{1}{2}$$
$$P(z < -a) = .25$$

Since the closest value to .25 is .2514, we use $z = -.67$.

$$\begin{aligned}-.67 &< z < .67\\ -.67 &< \frac{x - 54.40}{13.50} < .67\\ 45.36 &< x < 63.45\end{aligned}$$

Therefore, $P(45.36 < x < 63.45) = \frac{1}{2}$, and the middle 50% of the customers spend between \$45.36 and \$63.45.

34. For an exponential distribution, $f(x) = ae^{-ax}$ for $[0,\ \infty)$. Since $a = 2$, $f(x) = 2e^{-2x}$ for $[0,\ 1]$.

(a) The expected proportion is

$$\mu = \frac{1}{2} = .5.$$

(b) $$\begin{aligned}P\left(0 < x < \tfrac{1}{3}\right) &= \int_0^{1/3} 2e^{-2x}dx\\ &= -e^{-2x}\Big|_0^{1/3}\\ &= -\frac{1}{e^{2/3}} + 1\\ &\approx .49\end{aligned}$$

36. $\mu = 3.2$ ft, $\sigma = .2$ ft

If we wish to find the middle 50%, or .50, we want to find the value of z corresponding to a probability of .25 in the standard normal distribution table. This is about .675. Since the desired area is symmetric about the mean, we use $\pm.675$.

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ \pm.675 &= \frac{x-3.2}{.2} \\ \pm.135 &= x - 3.2 \\ x &= 3.2 \pm .135 \\ x &= 3.065,\ 3.335 \end{aligned}$$

The largest height is 3.335 ft, and the smallest height is 3.065 ft.

38. For an exponential distribution, $f(x) = ae^{-ax}$ for $[0, \infty)$.

Since $\mu = \frac{1}{a} = 24.8, a = \frac{1}{24.8}$.

(a) $$\begin{aligned} P(x \geq 25) &= \int_{25}^{\infty} \frac{1}{24.8} e^{-x/24.8} dx \\ &= \lim_{b\to\infty} \int_{25}^{b} \frac{1}{24.8} e^{-x/24.8} dx \\ &= \lim_{b\to\infty} \left(-e^{-x/24.8} \Big|_{25}^{b} \right) \\ &= \lim_{b\to\infty} \left(-e^{-b/24.8} + e^{-25/24.8} \right) \\ &= e^{-25/24.8} \\ &\approx .36 \end{aligned}$$

(b) $$\begin{aligned} P(x < 20) &= \int_{0}^{20} e^{-x/24.8} dx \\ &= -e^{-x/24.8} \Big|_{0}^{20} \\ &= -e^{-20/24.8} + 1 \\ &\approx .55 \end{aligned}$$

40. **(a)** We have a normal distribution with $\mu = 6.9$ and $\sigma = 4.6$.

$$\begin{aligned} P(x > 6) &= P\left(z \geq \frac{6-6.9}{4.6} \right) \\ &= P(z \geq -.19) \\ &\approx .58 \end{aligned}$$

No, this is insufficient evidence.

(b) We have a normal distribution with $\mu = .6$ and $\sigma = .3$.

$$\begin{aligned} P(x \geq 6) &= P\left(z \geq \frac{6-.6}{.3} \right) \\ &= P(z \geq 18) \\ &\approx 0 \end{aligned}$$

Yes, by today's standards there is sufficient evidence to conclude that Andrew Jackson suffered from mercury poisoning.

42. Uniform distribution on $[32,\ 44]$

$$f(x) = \frac{1}{44-32} = \frac{1}{12} \text{ for } [32,\ 44]$$

(a) $\mu = \frac{1}{2}(32 + 44) = 38$ inches

(b) $$\begin{aligned} &P(38 < x < 40) \\ &= \int_{38}^{40} \frac{1}{12}\, dx \\ &= \frac{x}{12} \Big|_{38}^{40} \\ &= \frac{1}{6} \approx .17 \end{aligned}$$

44. For an exponential distribution, $f(x) = ae^{-ax}$ for $[0, \infty)$. Since $a = \frac{1}{609.5}$, $f(x) = \frac{1}{609.5} e^{-x/609.5}$.

(a) The expected number of days is

$\mu = \frac{1}{a} = 609.5$. The standard deviation is

$\sigma = \frac{1}{a} = 609.5$.

(b) $$\begin{aligned} P(x > 365) &= \int_{365}^{\infty} \frac{1}{609.5} e^{-x/609.5} dx \\ &= 1 - \int_{0}^{365} \frac{1}{609.5} e^{-x/609.5} dx \\ &= 1 + \left(e^{-x/609.5} \Big|_{0}^{365} \right) \\ &= 1 + (e^{-365/609.5} - 1) \\ &= e^{-365/609.5} \\ &\approx .55 \end{aligned}$$

Chapter 11 Review Exercises

4. $f(x) = \frac{1}{27}(2x+4);\ [1,\ 4]$

$$\int_1^4 \frac{1}{27}(2x+4)dx = \frac{1}{27}(x^2+4x)\Big|_1^4$$
$$= \frac{1}{27}(32-5)$$
$$= 1$$

Since $1 \le x \le 4$, $f(x) \ge 0$.
Therefore, $f(x)$ is a probability density function.

6. $f(x) = .1;\ [0,\ 10]$

$$\int_0^{10} .1\ dx = .1x\Big|_0^{10}$$
$$= .1(10) - 0 = 1$$

$f(x) \ge 0$ for all x in $[0,\ 10]$.
Therefore, $f(x)$ is a probability density function.

8. $f(x) = k\sqrt{x};\ [1,\ 4]$

$$\int_1^4 k\sqrt{x}\ dx = \int_1^4 kx^{1/2}\ dx$$
$$= \frac{2}{3}kx^{3/2}\Big|_1^4$$
$$= \frac{2}{3}k(8-1)$$
$$= \frac{14}{3}k$$

Since $f(x)$ is a probability density function,

$$\frac{14}{3}k = 1$$
$$k = \frac{3}{14}.$$

10. $f(x) = 1 - \frac{1}{\sqrt{x-1}};\ [2,\ 5]$

(a) $P(3 \le x \le 5)$
$$= \int_3^5 [1-(x-1)^{-1/2}]dx$$
$$= [x - 2(x-1)^{1/2}]\Big|_3^5$$
$$= 5 - 2(2) - 3 + 2\sqrt{2}$$
$$\approx .828$$

(b) $P(2 \le x \le 4)$
$$= \int_2^4 [1-(x-1)^{-1/2}]dx$$
$$= [x - 2(x-1)^{1/2}]\Big|_2^4$$
$$= 4 - 2\sqrt{3} - 2 + 2$$
$$\approx .536$$

(c) $P(3 \le x \le 4)$
$$= \int_3^4 [1-(x-1)^{-1/2}]dx$$
$$= [x - 2(x-1)^{1/2}]\Big|_3^4$$
$$= 4 - 2\sqrt{3} - 3 + 2\sqrt{2}$$
$$\approx .364$$

12. If we consider the probabilities as weights, the expected value or mean of a probability distribution represents the point at which the distribution is balanced.

14. $f(x) = \frac{1}{5};\ [4,\ 9]$

$$E(x) = \mu = \int_4^9 x\left(\frac{1}{5}\right)dx = \frac{x^2}{10}\Big|_4^9$$
$$= \frac{81}{10} - \frac{16}{10} = \frac{65}{10} = 6.5$$
$$\text{Var}(x) = \int_4^9 \frac{1}{5}(x-6.5)^2\ dx$$
$$= \frac{1}{15}(x-6.5)^3\Big|_4^9$$
$$= \frac{1}{15}(2.5^3 + 2.5^3)$$
$$\approx 2.083$$
$$\sigma = \sqrt{\text{Var}(x)} \approx 1.443$$

16. $f(x) = \frac{1}{7}\left(1 + \frac{2}{\sqrt{x}}\right);\ [1,\ 4]$

$$E(x) = \mu = \int_1^4 x\left[\frac{1}{7}(1+2x^{-1/2})\right]dx$$
$$= \frac{1}{7}\int_1^4 (x+2x^{1/2})dx$$
$$= \frac{1}{7}\left(\frac{x^2}{2} + \frac{4}{3}x^{3/2}\right)\Big|_1^4$$
$$= \frac{1}{7}\left(8 + \frac{32}{3} - \frac{1}{2} - \frac{4}{3}\right)$$
$$\approx 2.405$$

$$\begin{aligned}\text{Var}(x) &= \int_1^4 \frac{x^2}{7}(1+2x^{-1/2})dx - (2.405)^2\\ &= \int_1^4 \left(\frac{x^2}{7}+\frac{2}{7}x^{3/2}\right)dx - (2.405)^2\\ &= \left(\frac{x^3}{21}+\frac{4}{35}x^{5/2}\right)\Big|_1^4 - (2.405)^2\\ &\approx \left(\frac{64}{21}+\frac{128}{35}-\frac{1}{21}-\frac{4}{35}\right) - (2.405)^2\\ &\approx .759\end{aligned}$$

$$\sigma = \sqrt{\text{Var}(x)} \approx .871$$

18. $f(x) = 4x - 3x^2;\ [0,\ 1]$

(a) $$\begin{aligned}\mu &= \int_0^1 x(4x-3x^2)dx\\ &= \int_0^1 (4x^2-3x^3)dx\\ &= \left(\frac{4x^3}{3}-\frac{3x^4}{4}\right)\Big|_0^1\\ &= \frac{4}{3}-\frac{3}{4}\\ &= \frac{7}{12}\\ &\approx .583\end{aligned}$$

(b) $$\begin{aligned}\text{Var}(x) &= \int_0^1 x^2(4x-3x^2)dx - \left(\frac{7}{12}\right)^2\\ &= \int_0^1 (4x^3-3x^4)dx - \left(\frac{7}{12}\right)^2\\ &= \left(x^4-\frac{3x^5}{5}\right)\Big|_0^1 - \left(\frac{7}{12}\right)^2\\ &= 1-\frac{3}{5}-\left(\frac{7}{12}\right)^2\\ &\approx .0597\end{aligned}$$

$$\begin{aligned}\sigma &\approx \sqrt{\text{Var}(x)}\\ &\approx .244\end{aligned}$$

(c) $$\begin{aligned}&P\left(0 \le x < \tfrac{7}{12}\right)\\ &= \int_0^{7/12} (4x-3x^2)dx\\ &= (2x^2-x^3)\Big|_0^{7/12}\\ &= 2\left(\frac{7}{12}\right)^2-\left(\frac{7}{12}\right)^3\\ &\approx .482\end{aligned}$$

(d) $$\begin{aligned}&P(\mu-\sigma \le x \le \mu+\sigma)\\ &\approx P(.339 \le x \le .827)\\ &= \int_{.339}^{.827} (4x-3x^2)dx\\ &= (2x^2-x^3)\Big|_{.339}^{.827}\\ &= 2(.827)^2-(.827)^3\\ &\quad -2(.339)^2+(.339)^3\\ &\approx .611\end{aligned}$$

20. $f(x) = \frac{5}{112}(1-x^{-3/2});\ [1,\ 25]$

(a) $$\begin{aligned}\mu &= \int_1^{25} \frac{5x}{112}(1-x^{-3/2})dx\\ &= \frac{5}{112}\int_1^{25} (x-x^{-1/2})dx\\ &= \frac{5}{112}\left(\frac{x^2}{2}-2x^{1/2}\right)\Big|_1^{25}\\ &= \frac{5}{112}\left(\frac{625}{2}-10-\frac{1}{2}+2\right)\\ &\approx 13.6\end{aligned}$$

(b) $$\begin{aligned}\text{Var}(x) &= \int_1^{25} \frac{5x^2}{112}(1-x^{-3/2})dx - (13.6)^2\\ &= \frac{5}{112}\int_1^{25} (x^2-x^{1/2})dx - (13.6)^2\\ &= \frac{5}{112}\left(\frac{x^3}{3}-\frac{2}{3}x^{3/2}\right)\Big|_1^{25} - (13.6)^2\\ &= \frac{5}{112}\left(\frac{25^3}{3}-\frac{250}{3}-\frac{1}{3}+\frac{2}{3}\right) - (13.6)^2\\ &\approx 43.85\end{aligned}$$

$$\sigma \approx \sqrt{\text{Var}(x)} \approx 6.6$$

(c) $$\begin{aligned}&P(\mu-\sigma \le x \le \mu+\sigma)\\ &= P(7 \le x \le 20.2)\\ &= \int_7^{20.2} \frac{5}{112}(1-x^{-3/2})dx\\ &= \frac{5}{112}(x+2x^{-1/2})\Big|_7^{20.2}\\ &= \frac{5}{112}\left(20.2+\frac{2}{\sqrt{20.2}}-7-\frac{2}{\sqrt{7}}\right)\\ &\approx .58\end{aligned}$$

For Exercises 22-28, use the table in the Appendix for the areas under the normal curve.

22. Area to right of $z = 1.53$ is

$$1 - .9370 = 0.63 \text{ or } 6.3\%.$$

24. Area to left of $z = 1.03$ is

$$.8485.$$

Areas to left of $z = -1.47$ is equivalent to area to right of $z = 1.47$:

$$1 - .9292 = .0708.$$

Area between is

$$.8485 - .0708 = .7777 \text{ or } 77.77\%.$$

26. $\sigma = 2.5,\ \mu = 0,\ z = 0 + 2.5$

Area to left of $z = 2.5$ is

$$.9938 \text{ or } 99.38\%.$$

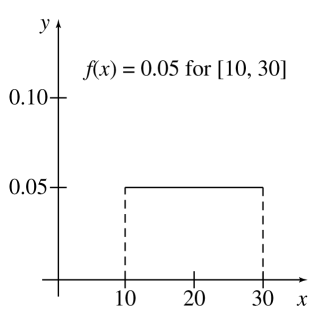
28. We want to find the z-score for 21% of the area under the normal curve to the left of z. We note that $21\% < 50\%$, so z must be negative. The z-score for the value in the table nearest .21 is $z \approx -.81$.

30. $f(x) = .05$ for $[10,\ 30]$

(a) This is a uniform distribution.

(b) The domain of f is [10, 30].
The range of f is $\{.05\}$.

(c)



(d) For a uniform distribution,

$$\mu = \frac{1}{2}(b + a) \text{ and}$$
$$\text{Var}(x) = \frac{b^2 - 2ab + a^2}{12}.$$

Thus,

$$\mu = \frac{1}{2}(30 + 10) = \frac{1}{2}(40) = 20$$

and

$$\text{Var}(x) = \frac{30^2 - 2(10)(30) + 10^2}{12} = \frac{400}{12}.$$

$$\sigma = \sqrt{\frac{400}{12}} \approx 5.77$$

(e) $P(\mu - \sigma \le x \le \mu + \sigma)$
$= P(20 - 5.77 \le x \le 20 + 5.77)$
$= P(14.23 \le x \le 25.77)$
$= \int_{14.23}^{25.77} .05\, dx$
$= .05x\Big|_{14.23}^{25.77}$
$= .05(25.77 - 14.23)$
$\approx .58$

32. $f(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$ for $(-\infty,\ \infty)$

(a) Since the exponent of e in $f(x)$ may be written

$$-x^2 = \frac{-(x - 0)^2}{2\left(\frac{1}{\sqrt{2}}\right)^2},$$

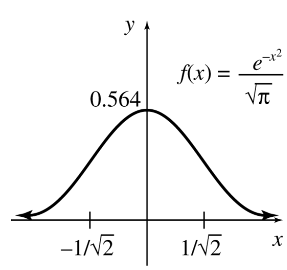
and

$$\frac{1}{\sqrt{\pi}} = \frac{1}{\frac{1}{\sqrt{2}}\sqrt{2\pi}},$$

$f(x)$ is a normal distribution with $\mu = 0$ and $\sigma = \frac{1}{\sqrt{2}}$.

(b) The domain of f is $(-\infty,\ \infty)$.
The range of f is $\left(0, \frac{1}{\sqrt{\pi}}\right)$.

(c)



(d) For this normal distribution, $\mu = 0$ and $\sigma = \frac{1}{\sqrt{2}}$.

(e) $P(\mu - \sigma \le x \le \mu + \sigma)$
$= 2P(0 \le x \le \mu + \sigma)$
$= 2P\left(0 \le x \le \frac{1}{\sqrt{2}}\right)$

If $x = \frac{1}{\sqrt{2}}$, $z = \dfrac{\frac{1}{\sqrt{2}} - 0}{\frac{1}{\sqrt{2}}} = 1.00.$

Thus,

$P(\mu - \sigma \le x \le \mu + \sigma)$
$= 2P(0 \le z \le 1.00)$
$= 2(.3413)$
$\approx .68.$

34. $f(t) = \dfrac{5}{112}(1 - t^{-3/2})$; $[1, 25]$

P(No repairs in years 1-3)
$= P$(First repair needed in years 4-25)

$$= \int_4^{25} \frac{5}{112}(1 - t^{-3/2})dt$$
$$= \frac{5}{112}(t + 2t^{-1/2})\Big|_4^{25}$$
$$= \frac{5}{112}\left[25 + \frac{2}{5} - 4 - 1\right]$$
$$= \frac{51}{56} \approx .911$$

36. $f(x) = \frac{1}{6}e^{-x/6}$ for $[0, \infty)$ is an exponential distribution.

(a) $\mu = \dfrac{1}{\frac{1}{6}} = 6$

(b) $\sigma = \dfrac{1}{\frac{1}{6}} = 6$

(c) $P(x > 6) = \displaystyle\int_6^{\infty} \frac{1}{6}e^{-x/6}\, dx$

$$= \lim_{b\to\infty} \int_6^{b} \frac{1}{6}e^{-x/6}\, dx$$
$$= \lim_{b\to\infty} -e^{-x/6}\Big|_6^{b}$$
$$= \lim_{b\to\infty} \left(e^{-1} - \frac{1}{e^{b/6}}\right)$$
$$= \frac{1}{e} \approx .37$$

38. $f(x) = .01e^{-.01x}$ for $[0, \infty)$ is an exponential distribution.

$P(0 \le x \le 100)$
$$= \int_0^{100} .01e^{-.01x}\, dx$$
$$= -e^{-.01x}\Big|_0^{100}$$
$$= 1 - \frac{1}{e}$$
$$\approx .63$$

40. $f(x) = \dfrac{6}{15{,}925}(x^2 + x)$ for $[20, 25]$

(a) $\mu = \displaystyle\int_{20}^{25} \frac{6}{15{,}925}(x^2 + x)dx$

$$= \frac{6}{15{,}925}\int_{20}^{25}(x^3 + x^2)dx$$
$$= \frac{6}{15{,}925}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]\Bigg|_{20}^{25}$$
$$= \frac{6}{15{,}925} \cdot \left[\frac{(25)^4}{4} + \frac{(25)^3}{3} - \frac{(20)^4}{4} - \frac{(20)^3}{3}\right]$$
$$\approx 22.68°C$$

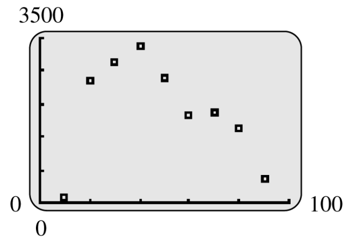
(b) $P(x < \mu)$

$$= \int_{20}^{22.68} \frac{6}{15{,}925}(x^2 + x)dx$$
$$= \frac{6}{15{,}925}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Bigg|_{20}^{22.68}$$
$$= \frac{6}{15{,}925} \cdot \left[\frac{(22.68)^3}{3} + \frac{(22.68)^2}{2} - \frac{(20)^3}{3} - \frac{(20)^2}{2}\right]$$
$$\approx .48$$

42. Normal distribution, $\mu = 2.4$ g, $\sigma = .4$ g, $x =$ tension

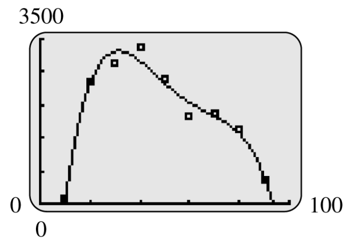
$$P(x < 1.9) = P\left(\frac{x - 2.4}{.4} < \frac{1.9 - 2.4}{.4}\right)$$
$$= P(z < -1.25)$$
$$= .1056$$

44. (a)



polynomial function

(b) $N(x) = -.00114251x^4 + .261752x^3 - 21.6938x^2 + 729.694x - 5266.33$;



The function models the data well.

(c) From part b, $N(x) = -.00114251x^4 + .261752x^3 - 21.6938x^2 + 729.694x - 5266.33$

$$\int_{9.7}^{93.2} N(x)dx \approx 177{,}514$$

So, let $k = \frac{1}{177{,}514}$. Then

$$S(x) = kN(x) = 1.$$

(d) Using the integration feature on the calculator,

$$P(25 \le x \le 35) = \int_{25}^{35} S(x)dx \approx .181$$

$$P(45 \le x \le 65) = \int_{45}^{65} S(x)dx \approx .266$$

$$P(x \ge 55) = \int_{55}^{93.2} S(x)dx \approx .338$$

From the table, the actual probabilities are

$$P(25 \le x \le 35) = \frac{3010}{17{,}559} \approx .171$$

$$P(45 \le x \le 65) = \frac{2647 + 1859}{17{,}559} \approx .257$$

$$P(x \ge 55) = \frac{1859 + 1906 + 1608 + 494}{17{,}559}$$

$$\approx .334$$

(e) $E(x) = \int_{9.7}^{93.2} xS(x)dx \approx 46.4$ years

(f) $$\int_{9.7}^{m} S(x)dx = \frac{1}{177{,}514}\left(-\frac{.00114251}{5}x^5 + \frac{.261752}{4}x^4 - \frac{21.6938}{3}x^3 + \frac{729.694}{2}x^2 - 5266.33x\right)\Bigg|_{9.7}^{m}$$
$$\approx \frac{1}{177{,}514}\left(-\frac{.00114251}{5}m^5 + \frac{.261752}{4}m^4 - \frac{21.6938}{3}m^3 + \frac{729.694}{2}m^2 - 5266.33m\right) - (-.1284)$$
$$= \frac{1}{2} \text{ when } m \approx 43.9 \text{ years,}$$

using the INTERSECT feature of a graphing calculator.

46. Normal distribution, $\mu = 40$, $\sigma = 13$, $x =$ "take"

$$\begin{aligned} P(x > 50) &= P\left(\frac{x-40}{13} > \frac{50-40}{13}\right) \\ &= P(z > .77) \\ &= 1 - P(z \le .77) \\ &= 1 - .7794 \\ &= .2206 \end{aligned}$$

Extended Application: Exponential Waiting Times

1. If the density function is continuous, the probability of any event of the form $(x = c)$ is 0, so it does not matter whether we use strict or inclusive inequalities.

2. We already know that the probability of a gap of 30 minutes or longer is about .05. Over 40 trips, the expected number of waits that are 30 minutes or longer is $40(.05) = 2$.

3. $$P(t > 20) = \int_{20}^{\infty} \frac{1}{10}e^{-t/10}dt = 1 - \int_{0}^{20} \frac{1}{10}e^{-t/10}dt = 1 - \left(-e^{-t/10}\Big|_{0}^{20}\right)$$
$$= 1 - (-e^{-2} + 1) = e^{-2} \approx .135$$

4. For the exponential schedule,

$$P(9 < t < 10) = \int_{9}^{10} \frac{1}{10}e^{-t/10}dt = -e^{-t/10}\Big|_{9}^{10} = -e^{-1} + e^{-9/10} \approx .039$$

For the uniform schedule,

$$P(9 < t < 10) = \int_{9}^{10} \frac{1}{10}dt = \frac{1}{10}t\Big|_{9}^{10} = 1 - \frac{9}{10} = .1$$

5. For the exponential model,

$$P(8 < t < 12) = \int_{8}^{12} \frac{1}{10}e^{-t/10}dt = -e^{-t/10}\Big|_{8}^{12} = -e^{-12/10} + e^{-8/10} \approx .148$$

For the uniform model, *all* interarrival times are 10 minutes, so they all meet the ± 2 criteria.

6. The trick used at the HotBits site is the following: they measure times between successive *pairs* of decay events; if the first wait is shorter than the second, the generator omits a 0, and if the first wait is longer than the second, it omits a 1. For more details, see http://www.fourmilab.ch/hotbits/hw/html.

Chapter 12

SEQUENCES AND SERIES

12.1 Geometric Sequences

2. $a_1 = 4,\ r = 2,\ n = 5$

Since $n = 5$, we must find a_1, a_2, a_3, a_4 and a_5 with $a = a_1 = 4$.

$$\begin{aligned}
a_1 &= 4\\
a_2 &= 4(2)^{2-1} = 4(2)^1 = 4(2) = 8\\
a_3 &= 4(2)^{3-1} = 4(2)^2 = 4(4) = 16\\
a_4 &= 4(2)^{4-1} = 4(2)^3 = 4(8) = 32\\
a_5 &= 4(2)^{5-1} = 4(2)^4 = 4(16) = 64
\end{aligned}$$

The first five terms of this geometric sequence are 4, 8, 16, 32, and 64.

4. $a_1 = \frac{2}{3},\ r = 6,\ n = 3$

Since $n = 3$, we must find a_1, a_2, and a_3 with $a = a_1 = \frac{2}{3}$.

$$\begin{aligned}
a_1 &= \frac{2}{3}\\
a_2 &= \frac{2}{3}(6)^{2-1} = \frac{2}{3}(6)^1 = \frac{2}{3}(6) = 4\\
a_3 &= \frac{2}{3}(6)^{3-1} = \frac{2}{3}(6)^2 = \frac{2}{3}(36) = 24
\end{aligned}$$

The first three terms of this geometric sequence are $\frac{2}{3}$, 4, and 24.

6. $a_2 = 9,\ a_3 = 3,\ n = 4$

Since $n = 4$, we must find a_1, a_2, a_3, and a_4 with $r = \frac{a_3}{a_2} = \frac{3}{9} = \frac{1}{3}$. To find a, use $a_2 = 9,\ r = \frac{1}{3}$, and $n = 2$ in the formula.

$$\begin{aligned}
9 &= a\left(\frac{1}{3}\right)^{2-1}\\
9 &= a\left(\frac{1}{3}\right)^{1}\\
9 &= a\left(\frac{1}{3}\right)\\
27 &= a
\end{aligned}$$

$$\begin{aligned}
a_1 &= 27\\
a_2 &= 27\left(\frac{1}{3}\right)^{2-1} = 27\left(\frac{1}{3}\right)^{1} = 27\left(\frac{1}{3}\right) = 9\\
a_3 &= 27\left(\frac{1}{3}\right)^{3-1} = 27\left(\frac{1}{3}\right)^{2} = 27\left(\frac{1}{9}\right) = 3\\
a_4 &= 27\left(\frac{1}{3}\right)^{4-1} = 27\left(\frac{1}{3}\right)^{3} = 27\left(\frac{1}{27}\right) = 1
\end{aligned}$$

The first four terms of this geometric sequence are 27, 9, 3, and 1.

8. $a_1 = 8,\ r = 4$
Since we want a_5, use $n = 5$ in the formula with $a = a_1 = 8$ and $r = 4$.

$$\begin{aligned}
a_5 &= 8(4)^{5-1} = 8(4)^4\\
&= 8(256) = 2048\\
a_n &= 8(4)^{n-1}
\end{aligned}$$

10. $a_1 = -4,\ r = -2$
Since we want a_5, use $n = 5$ in the formula with $a = a_1 = -4$ and $r = -2$.

$$\begin{aligned}
a_5 &= -4(-2)^{5-1} = -4(-2)^4\\
&= -4(16) = -64\\
a_n &= -4(-2)^{n-1}
\end{aligned}$$

12. $a_3 = 6,\ r = 3$
To find a, use $a_3 = 6$, $r = 3$, and $n = 3$ in the formula.

$$\begin{aligned}
6 &= a(3)^{3-1}\\
6 &= a(3)^2\\
6 &= a(9)\\
\frac{2}{3} &= a
\end{aligned}$$

Since we want a_5, use $n = 5$ in the formula with $a = \frac{2}{3}$ and $r = 3$.

$$\begin{aligned}
a_5 &= \frac{2}{3}(3)^{5-1} = \frac{2}{3}(3)^4 = \frac{2}{3}(81) = 54\\
a_n &= \frac{2}{3}(3)^{n-1}
\end{aligned}$$

14. $a_4 = 81$, $r = -3$
To find a, use $a_4 = 81$, $r = -3$, and $n = 4$ in the formula.

$$\begin{aligned} 81 &= a(-3)^{4-1} \\ 81 &= a(-3)^3 \\ 81 &= a(-27) \\ -3 &= a \end{aligned}$$

Since we want a_5, use $n = 5$ in the formula with $a = -3$ and $r = -3$.

$$a_5 = -3(-3)^{5-1} = -3(-3)^4 = -3(81) = -243$$
$$a_n = -3(-3)^{n-1} = (-3)^{1+(n-1)} = (-3)^n$$

16. 4, 16, 64, 256, ...

$$r = \frac{16}{4} = \frac{64}{16} = \frac{256}{64} = 4$$

Since $r = 4$ and $a = a_1 = 4$, $a_n = 4(4)^{n-1} = 4^n$.

18. −7, −5, −3, −1, 1, 3, ...
Since $\frac{-5}{-7} = \frac{5}{7}$ and $\frac{-3}{-5} = \frac{3}{5}$, the ratio is not constant, so the sequence is not geometric.

20. 6, 8, 10, 12, 14, ...
Since $\frac{8}{6} = \frac{4}{3}$ and $\frac{10}{8} = \frac{5}{4}$, the ratio is not constant, so the sequence is not geometric.

22. $\frac{7}{4}, -\frac{7}{12}, \frac{7}{36}, -\frac{7}{108}, \ldots$

$$r = \frac{-\frac{7}{12}}{\frac{7}{4}} = \frac{\frac{7}{36}}{-\frac{7}{12}} = \frac{-\frac{7}{108}}{\frac{7}{36}} = -\frac{1}{3}$$

Since $r = -\frac{1}{3}$ and $a = a_1 = \frac{7}{4}$, $a_n = \frac{7}{4}\left(-\frac{1}{3}\right)^{n-1}$.

24. 5, 20, 80, 320, ...

Since $a = a_1 = 5$ and $r = \frac{20}{5} = 4$,

$$\begin{aligned} S_5 &= \frac{5\left(4^5 - 1\right)}{4 - 1} \\ &= \frac{5(1024 - 1)}{3} \\ &= 1705 \end{aligned}$$

The sum of the first five terms of this geometric sequence is 1705.

26. $18, -3, \frac{1}{2}, -\frac{1}{12}, \ldots$

Since $a = a_1 = 18$ and $r = \frac{-3}{18} = -\frac{1}{6}$,

$$\begin{aligned} S_5 &= \frac{18\left[\left(-\frac{1}{6}\right)^5 - 1\right]}{\left(-\frac{1}{6}\right) - 1} \\ &= \frac{18\left[-\frac{1}{7776} - 1\right]}{-\frac{7}{6}} \\ &= \frac{1111}{72} \end{aligned}$$

The sum of the first five terms of this geometric sequence is $\frac{1111}{72}$.

28. $a_1 = -4$, $r = 3$
Since $a = a_1 = -4$,

$$\begin{aligned} S_5 &= \frac{-4\left(3^5 - 1\right)}{3 - 1} \\ &= \frac{-4(243 - 1)}{2} \\ &= -484 \end{aligned}$$

The sum of the first five terms of this geometric sequence is −484.

30. $a_1 = -3.772$, $r = -1.553$
Since $a = a_1 = -3.772$,

$$\begin{aligned} S_5 &= \frac{-3.772\left[(-1.553)^5 - 1\right]}{(-1.553) - 1} \\ &\approx \frac{-3.772(-9.0335 - 1)}{-2.553} \\ &\approx -14.8243 \end{aligned}$$

The sum of the first five terms of this geometric sequence is about −14.8243.

32. For $\sum_{i=0}^{3} 2\left(3^i\right)$, use the formula with $n = 4$, $r = 3$, and $a = 2$.

$$S_4 = \frac{2\left(3^4 - 1\right)}{3 - 1} = \frac{2(81 - 1)}{2} = 80$$

Therefore, $\sum_{i=0}^{3} 2\left(3^i\right) = 80$.

34. For $\sum_{i=0}^{4} \frac{3}{2}\left(2^i\right)$, use the formula with $n = 5$, $r = 2$, and $a = \frac{3}{2}$.

$$S_5 = \frac{\frac{3}{2}\left(2^5 - 1\right)}{2-1} = \frac{\frac{3}{2}(32-1)}{1} = \frac{93}{2}$$

Therefore, $\sum_{i=0}^{4} \frac{3}{2}\left(2^i\right) = \frac{93}{2}$.

36. For $\sum_{i=0}^{4} \frac{5}{3}\left(3^i\right)$, use the formula with $n = 5$, $r = 3$, and $a = \frac{5}{3}$.

$$S_5 = \frac{\frac{5}{3}\left(3^5 - 1\right)}{3-1} = \frac{\frac{5}{3}(243-1)}{2} = \frac{605}{3}$$

Therefore, $\sum_{i=0}^{4} \frac{5}{3}\left(3^i\right) = \frac{605}{3}$.

38. For $\sum_{i=0}^{6} 81\left(\frac{2}{3}\right)^i$, use the formula with $n = 7$, $r = \frac{2}{3}$, and $a = 81$.

$$S_7 = \frac{81\left[\left(\frac{2}{3}\right)^7 - 1\right]}{\frac{2}{3}-1} = \frac{81\left(\frac{128}{2187} - 1\right)}{-\frac{1}{3}} = \frac{2059}{9}$$

Therefore, $\sum_{i=0}^{6} 81\left(\frac{2}{3}\right)^i = \frac{2059}{9}$.

40. The yearly incomes produced by the well form a geometric sequence with $r = \frac{1}{2}$ and $a_1 = 4{,}000{,}000$. To determine the total amount produced in 6 years, use the formula to find S_n with $n = 6$, $r = \frac{1}{2}$, and $a = a_1 = 4{,}000{,}000$.

$$\begin{aligned} S_6 &= \frac{4{,}000{,}000\left[\left(\frac{1}{2}\right)^6 - 1\right]}{\frac{1}{2}-1} \\ &= \frac{4{,}000{,}000\left(\frac{1}{64} - 1\right)}{-\frac{1}{2}} \\ &= 7{,}875{,}000 \end{aligned}$$

The total amount of income produced by the well in six years is \$7,875,000.

42. If the machine loses 20% of its value, it maintains 80% of its value. With an initial value of \$100,000, its value at the end of the first year will be 80% of \$100,000, and its value at the end of each subsequent year will be 80% of its value at the end of the previous year. Thus, the end-of-year values form a geometric sequence with $r = .80$. If we let $a_1 = 100{,}000$, then a_2 represents the value of the machine at the end of the first year, a_3 is the value at the end of the second year, and so on. To find the value at the end of the sixth year, we are looking for a_7 in the geometric sequence with $n = 7$, $r = .80$, and $a = a_1 = 100{,}000$.

$$\begin{aligned} a_7 &= 100{,}000\,(.80)^{7-1} \\ &= 100{,}000\,(.80)^6 \\ &= 100{,}000\,(.262144) \\ &= 26{,}214.4 \end{aligned}$$

The value of the machine at the end of the sixth year will be \$26,214.

44. If 10^{15} molecules are present initially, then $\frac{1}{2}\left(10^{15}\right)$ molecules will be present after 3 years. The numbers of molecules present at the end of each 3-year period form a geometric sequence with $r = \frac{1}{2}$ and $a_1 = 10^{15}$. To determine the number of molecules unchanged after 15 years, we want the number of molecules after five 3-year periods. If we let $a_1 = 10^{15}$, then a_2 represents the number of molecules after the first 3-year period, a_3 is the number present after the second 3-year period, and so on. To find the number of molecules after five 3-year periods, we are looking for a_6. Use the formula to find a_n with $n = 6$, $r = \frac{1}{2}$, and $a = a_1 = 10^{15}$.

$$\begin{aligned} a_6 &= 10^{15}\left(\frac{1}{2}\right)^{6-1} \\ &= 10^{15}\left(\frac{1}{2}\right)^{5} \\ &= 10^{15}\left(\frac{1}{32}\right) \\ &= 10^{15}\,(.03125) \\ &= 3.125 \times 10^{13} \end{aligned}$$

After fifteen years, 3.125×10^{13} molecules will be unchanged.

46. **(a)** The thicknesses of the paper form a geometric sequence with $r = 2$ and $a_1 = .008$. If $a_1 = .008$, then a_2 represents the thickness after the first fold, a_3 is the thickness after the second fold, and so on. To determine the thickness after 12 folds, use the formula to find a_n with $n = 13$, $r = 2$, and $a = a_1 = .008$.

$$\begin{aligned} a_{13} &= .008\,(2)^{13-1} \\ &= .008\,(2)^{12} \\ &= .008\,(4096) \\ &= 32.768 \end{aligned}$$

After twelve folds, the final stack of paper is 32.768 inches thick.

(b) $a_{50} = .008\,(2)^{50-1} \approx 4.5036 \times 10^{12}$

The stack would be about 4.5036×10^{12} inches thick, or about 71 million miles.

48. **(a)** In round one, there are 81 players playing in $\frac{81}{3} = 27$ games that produce 27 winners to play in round two. In round two, there will be $\frac{27}{3} = 9$ games that produce 9 winners to play $\frac{9}{3} = 3$ games in round three. These games will determine the 3 winners that will play in the championship in round four. Therefore, the total number of games is the sum $27 + 9 + 3 + 1$, or $3^3 + 3^2 + 3^1 + 3^0$, which is the sum of the geometric sequence $a_1 = 3^3$, $a_2 = 3^2$, $a_3 = 3^1$, $a_4 = 3^0$.

(b) Notice that the ratio for the geometric sequence in part (a) is $r = \frac{3^2}{3^3} = \frac{1}{3}$. The number of players and the number of games decreases by $\frac{1}{3}$ from one round to the next. The total number of games therefore is the sum S_n with $n = 4$, $r = \frac{1}{3}$, and $a = a_1 = 27$, or

$$\begin{aligned} S_5 &= \frac{27\left(\left(\frac{1}{3}\right)^4 - 1\right)}{\frac{1}{3} - 1} \\ &= \frac{\frac{1}{3} - 27}{-\frac{2}{3}} \\ &= -\frac{3}{2}\left(\frac{1-81}{3}\right) \\ &= \frac{-80}{-2} \\ &= 40, \end{aligned}$$

so 40 games are required in all to determine a champion.

(c) When there are $3^4 = 81$ players initially, we know from part (a) that 4 rounds are required to determine a champion. For 3^n players, n rounds are required.

To determine the total number of games in such a tournament, we know that 3^n players will play 3^{n-1} games in the first round, and for each succeeding round, $\frac{1}{3}$ fewer games will be played. Therefore, the number of games played in each round forms a geometric sequence with $a_1 = a = 3^{n-1}$ and $r = \frac{1}{3}$.

The total number of games played is then

$$\begin{aligned} S_n &= \frac{3^{n-1}\left(\left(\frac{1}{3}\right)^n - 1\right)}{\frac{1}{3} - 1} \\ &= \frac{\left(\frac{1}{3} - 3^{n-1}\right)}{-\frac{2}{3}} \\ &= -\frac{3}{2}\left(\frac{1}{3} - 3^{n-1}\right) \\ &= \frac{1}{2}\left(-1 + 3^n\right), \end{aligned}$$

or

$$S_n = \frac{3^n - 1}{2} = 3^{n-1} + \cdots + 3^2 + 3^1 + 1.$$

(d) For a tournament where t^n players are initially present, if the total number of games played is a sum of a geometric sequence as in parts (a) and (c), then it must be that t^n players will play t^{n-1} games in the first round. This means the number of winners and the number of games played in subsequent rounds will decrease by a factor of $\frac{1}{t}$. We also know from parts (a) and (c) that a tournament that begins with t^n players will require n rounds to produce a champion. Therefore, the total number of games played is the sum

$$t^{n-1} + t^{n-2} + \cdots + t^2 + t + 1,$$

which is equal to

$$\begin{aligned} S_n &= \frac{t^{n-1}\left(\left(\frac{1}{t}\right)^n - 1\right)}{\frac{1}{t} - 1} \\ &= \frac{\left(\frac{1}{t} - t^{n-1}\right)}{\frac{1-t}{t}} \\ &= \frac{t}{1-t}\left(\frac{1}{t} - t^{n-1}\right) \\ &= \frac{1-t^n}{1-t} \\ &= \frac{t^n - 1}{t - 1}. \end{aligned}$$

12.2 Annuities: An Application of Sequences

2. $R = 1000$, $i = .06$, $n = 12$

$$S = 1000\left[\frac{(1.06)^{12}-1}{.06}\right]$$

The number in brackets, $s_{\overline{12}|.06}$, is 16.8699412, so that

$$S = 1000\,(16.8699412) \approx 16{,}869.94,$$

or \$16,869.94.

4. $R = 100{,}000$, $i = .08$, $n = 23$

$$S = 100{,}000\left[\frac{(1.08)^{23}-1}{.08}\right]$$

The number in brackets, $s_{\overline{23}|.08}$, is 60.89329557, so that

$$S = 100{,}000\,(60.89329557) \approx 6{,}089{,}329.56,$$

or \$6,089,329.56.

6. $R = 11{,}200$, $i = .08$, $n = 25$

$$S = 11{,}200\left[\frac{(1.08)^{25}-1}{.08}\right]$$

The number in brackets, $s_{\overline{25}|.08}$, is 73.10593995, so that

$$S = 11{,}200\,(73.10593995) \approx 818{,}786.53,$$

or \$818,786.53.

8. Interest of $\frac{12\%}{2} = 6\%$ earned semiannually In 11 years, there are $11 \times 2 = 22$ semiannual periods. Since $s_{\overline{22}|.06} = \left[\frac{(1.06)^{22}-1}{.06}\right] = 43.39229028$, the \$3700 deposits will produce a total of

$$S = 3700\,(43.39229028) \approx 160{,}551.47,$$

or \$160,551.47.

10. Interest of $\frac{16\%}{4} = 4\%$ earned quarterly In 9 years, there are $9 \times 4 = 36$ quarterly periods. Since $s_{\overline{36}|.04} = \left[\frac{(1.04)^{36}-1}{.04}\right] = 77.59831385$, the \$4600 deposits will produce a total of

$$S = 4600\,(77.59831385) \approx 356{,}952.24,$$

or \$356,952.24.

12. This describes an ordinary annuity with $S = 100{,}000$, $i = .08\left(=\frac{16\%}{2}\right)$, and $n = 9 \cdot 2 = 18$ periods.

$$\begin{aligned}100{,}000 &= R \cdot s_{\overline{18}|.08}\\ 100{,}000 &= R\,(37.45024374)\\ R &= 2670.21\end{aligned}$$

The periodic payment should be \$2670.21.

14. $R = 1000$, $i = .08$, and $n = 9$ payments.

$$a_{\overline{9}|.08} = \left[\frac{1-(1.08)^{-9}}{.08}\right] = 6.246887911,$$

so

$$P = 1000\,(6.246887911) \approx 6246.89,$$

or \$6246.89.

16. $R = 890$, $i = .08$, and $n = 16$ payments.

$$a_{\overline{16}|.08} = \left[\frac{1-(1.08)^{-16}}{.08}\right] = 8.851369155,$$

so

$$P = 890\,(8.851369155) \approx 7877.72,$$

or \$7877.72.

18. $R = 10{,}000$, $i = \frac{10\%}{2} = 5\%$, and $n = 15 \cdot 2 = 30$ payments.

$$a_{\overline{30}|.05} = \left[\frac{1-(1.05)^{-30}}{.05}\right] = 15.37245103,$$

so $P = 10{,}000(15.37245103) \approx 153{,}724.51$, or \$153,724.51.

20. $a_{\overline{15}|.04} = \left[\dfrac{1-(1.04)^{-15}}{.04}\right] = 11.11838743$, so

$P = 10{,}000\,(11.11838743) \approx 111{,}183.87$, or \$111,183.87. A lump sum deposit of \$111,183.87 today at 4% compounded annually will yield the same total after 15 years as deposits of \$10,000 at the end of each year for 15 years at 4% compounded annually.

22. $a_{\overline{15}|.06} = \left[\dfrac{1-(1.06)^{-15}}{.06}\right] = 9.712248988$, so

$P = 10{,}000\,(9.712248988) \approx 97{,}122.49$, or \$97,122.49. A lump sum deposit of \$97,122.49 today at 6% compounded annually will yield the same total after 15 years as deposits of \$10,000 at the end of each year for 15 years at 6% compounded annually.

24. \$1000 is the present value of this annuity of R dollars, with 9 periods, and $i = 8\% = .08$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 1000 &= R \cdot a_{\overline{9}|.08} \\ R &= \frac{1000}{a_{\overline{9}|.08}} \\ &= \frac{1000}{6.246887911} \\ &\approx 160.08 \end{aligned}$$

Each payment is \$160.08.

26. \$41,000 is the present value of this annuity of R dollars, with $10 \cdot 2 = 20$ periods, and $i = \frac{12\%}{2} = 6\% = .06$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 41{,}000 &= R \cdot a_{\overline{20}|.06} \\ R &= \frac{41{,}000}{a_{\overline{20}|.06}} \\ &= \frac{41{,}000}{11.46992122} \\ &\approx 3574.57 \end{aligned}$$

Each payment is \$3574.57.

28. \$5500 is the present value of this annuity of R dollars, with 24 periods, and $i = \frac{18\%}{12} = 1.5\% = .015$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 5500 &= R \cdot a_{\overline{24}|.015} \\ R &= \frac{5500}{a_{\overline{24}|.015}} \\ &= \frac{5500}{20.03040537} \\ &\approx 274.58 \end{aligned}$$

Each payment is \$274.58.

30. Pat's payments form an ordinary annuity with $R = 12{,}000$, $n = 9$, and $i = .08$. The amount of this annuity is

$$S = 12{,}000\left[\frac{(1.08)^9 - 1}{.08}\right].$$

The number in brackets, $s_{\overline{9}|.08}$, is 12.48755784, so that

$$S = 12{,}000\,(12.48755784) \approx 149{,}850.69,$$

or \$149,850.69.

32. From Exercise 30, Pat will have 149,850.69 on deposit using the bank that pays 8% interest compounded annually. From Exercise 31, she will have \$137,895.79 on deposit using her brother-in-law's bank which pays only 6% interest compounded annually. If she uses her brother-in-law's bank, she will lose $149{,}850.69 - 137{,}895.79 = 11{,}954.90$, or \$11,954.90 over 9 years.

34. This ordinary annuity will amount to \$10,000 in 8 years at 12% compounded quarterly. Thus, $S = 10{,}000$, $n = 8 \cdot 4 = 32$, and $i = \frac{12\%}{4} = 3\% = .03$, so

$$\begin{aligned} 10{,}000 &= R \cdot s_{\overline{32}|.03} \\ R &= \frac{10{,}000}{s_{\overline{32}|.03}} \\ &= \frac{10{,}000}{52.50275852} \\ &\approx 190.47, \end{aligned}$$

or \$190.47.

36. This ordinary annuity will amount to \$12,000 in 4 years at 16% compounded semiannually. Thus, $S = 12{,}000$, $n = 4 \cdot 2 = 8$, and $i = \frac{16\%}{2} = 8\% = .08$, so

$$12{,}000 = R \cdot s_{\overline{8}|.08}$$
$$R = \frac{12{,}000}{s_{\overline{8}|.08}} = \frac{12{,}000}{10.63662763} \approx 1128.18,$$

or \$1128.18.

38. Interest of $\frac{8\%}{2} = 4\%$ is earned semiannually. In $65 - 40 = 25$ years, there are $25 \cdot 2 = 50$ semiannual periods. Since

$$s_{\overline{50}|.04} = \left[\frac{(1.04)^{50} - 1}{.04}\right] = 152.6670837,$$

the \$1000 semiannual deposits will produce a total of

$$S = 1000\,(152.6670837) \approx 152{,}667.08,$$

or \$152,667.08.

40. Interest of $\frac{12\%}{2} = 6\%$ is earned semiannually. In $65 - 40 = 25$ years, there are $25 \cdot 2 = 50$ semiannual periods. Since

$$s_{\overline{50}|.06} = \left[\frac{(1.06)^{50} - 1}{.06}\right] = 290.3359046,$$

the \$1000 semiannual deposits will produce a total of

$$S = 1000\,(290.3359046) \approx 290{,}335.90,$$

or \$290,335.90.

42. This ordinary annuity will amount to \$40,000 in 7 years at 8% compounded annually. Thus, $S = 40{,}000$, $n = 7$, and $i = .08$, so

$$40{,}000 = R \cdot s_{\overline{7}|.08}$$
$$R = \frac{40{,}000}{s_{\overline{7}|.08}} = \frac{40{,}000}{8.92280336} \approx 4482.90,$$

or \$4482.90.

44. **(a)** The total amount of interest paid is $(4000)\,(.06)\,(5) = \$1200$. This total is divided into $5 \cdot 2 = 10$ equal semiannual interest payments. Since $\frac{1200}{10} = 120$, each semiannual interest payment will be \$120.

(b) This ordinary annuity will amount to \$4000 in 5 years at 8% compounded annually. Thus, $S = 4000$, $n = 5$, and $i = .08$, so

$$4000 = R \cdot s_{\overline{5}|.08}$$
$$R = \frac{4000}{s_{\overline{5}|.08}} = \frac{4000}{5.86660096} \approx 681.83,$$

or \$681.83.

(c)

Payment Number	Amount of Deposit	Interest Earned	Total in Account
1	\$681.83	\$0	\$681.83
2	\$681.83	$(681.83)\,(.08) = \$54.55$	$681.83 + 681.83 + 54.55 = \$1418.21$
3	\$681.83	$(1418.21)\,(.08) = \$113.46$	$1418.21 + 681.83 + 113.46 = \$2213.50$
4	\$681.83	$(2213.50)\,(.08) = \$177.08$	$2213.50 + 681.83 + 177.08 = \$3072.41$
5	\$681.83	$(3072.41)\,(.08) = \$245.79$	$3072.41 + 681.83 + 245.79 = \$4000.03$

So that the final total in the account is \$4000, subtract \$.03 from the last amount of deposit. Thus, line 5 of the table will be:

5	\$681.80	$(3072.41)\,(.08) = \$245.79$	$3072.41 + 681.80 + 245.79 = \$4000.00$

46. We want to find the present value of an annuity of \$50,000 per year for 20 years at 6% compounded annually.

$$a_{\overline{20}|.06} = \left[\frac{1-(1.06)^{-20}}{.06}\right] = 11.46992122, \text{ so}$$
$$P = 50{,}000\,(11.46992122) = 573{,}496.06,$$

or \$573,496.06. A lump sum deposit of \$573,496.06 today at 6% compounded annually will yield the same total after 20 years as deposits of \$50,000 at the end of each year for 20 years at 6% compounded annually.

48. First, we want to find the amount of the annuity with $R = 1000$, $n = 8$, and $i = .06$.

$$S = 1000\left[\frac{(1.06)^8-1}{.06}\right].$$

The number in brackets, $s_{\overline{8}|.06}$, is 9.897467909, so that

$$S = 1000\,(9.897467909) = 9897.47,$$

or \$9897.47.
Next, we want to find the lump sum that must be deposited today at 5% compounded annually for 8 years to provide \$9897.47.
$A = 9897.47$, $i = .05$, and $n = 8$, so

$$9897.47 = P\,(1.05)^8$$
$$P = \frac{9897.47}{(1.05)^8} \approx 6699.00,$$

or \$6699.00.

50. For parts (a) and (b), if \$1 million is divided into 20 equal payments, each payment is \$50,000.

(a) $i = .05, n = 20$

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$
$$= 50{,}000\left[\frac{1-(1+.05)^{-20}}{.05}\right]$$
$$\approx 623{,}110.52$$

The present value is \$623,110.52.

(b) $i = .09, n = 20$

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$
$$= 50{,}000\left[\frac{1-(1+.09)^{-20}}{.09}\right]$$
$$\approx 456{,}427.28$$

The present value is \$456,427.28.

For parts (c) and (d), if \$1 million is divided into 25 equal payments, each payment is \$40,000.

(c) $i = .05, n = 25$

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$
$$= 40{,}000\left[\frac{1-(1+.05)^{-25}}{.05}\right]$$
$$\approx 563{,}757.78$$

The present value is \$563,757.78.

(d) $i = .09, n = 25$

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$
$$= 40{,}000\left[\frac{1-(1+.09)^{-25}}{.09}\right]$$
$$\approx 392{,}903.18$$

The present value is \$392,903.18.

52. The present value, P, is 170,892, $i = \frac{.0811}{12} \approx .0067583333$, and $n = 12 \cdot 30 = 360$.

$$170{,}892 = R \cdot a_{\overline{360}|.0067583333}$$
$$= R\left[\frac{1-(1+.0067583333)^{-360}}{.0067583333}\right]$$
$$= R\left[\frac{1-.0884944586}{.0067583333}\right]$$
$$= R\left[\frac{.9115055414}{.0067583333}\right]$$
$$R \approx 1267.07.$$

Monthly payments of \$1267.07 will be required to amortize the loan.

Use the formula for the unpaid balance of a loan with
$R = 1267.07, i = .0067583333, n = 12 \cdot 30 = 360,$ and $x = 12 \cdot 5 = 60.$

$$y = R\left[\frac{1-(1+i)^{-(n-x)}}{i}\right] = 1267.07\left[\frac{1-(1+.0067583333)^{-300}}{.0067583333}\right] \approx 162{,}628.91$$

The unpaid balance after 5 years is \$162,628.91.

54. The present value, P, is 196,511, $i = \frac{.0757}{12} \approx .0063083333$, and $n = 12 \cdot 25 = 300$.

$$\begin{aligned} 196{,}511 &= R \cdot a_{\overline{300}|.0063083333} = R\left[\frac{1-(1+.0063083333)^{-300}}{.0063083333}\right] \\ &= R\left[\frac{1-.1515930387}{.0063083333}\right] = R\left[\frac{.8484069613}{.0063083333}\right] \\ R &\approx 1461.16. \end{aligned}$$

Monthly payments of \$1461.16 will be required to amortize the loan.

Use the formula for the unpaid balance of a loan with $R = 1461.16, i = .0063083333, n = 12 \cdot 25 = 300,$ and $x = 12 \cdot 5 = 60.$

$$y = R\left[\frac{1-(1+i)^{-(n-x)}}{i}\right] = 1461.16\left[\frac{1-(1+.0063083333)^{-240}}{.0063083333}\right] \approx 180{,}417.11$$

The unpaid balance after 5 years is \$180,417.11.

56. **(a)** \$150,000 is the present value of this annuity of $5 \cdot 2 = 10$ periods with interest of $\frac{6\%}{2} = 3\% = .03$ per year.

$$\begin{aligned} a_{\overline{10}|.03} &= 8.530202837, \text{ so} \\ 150{,}000 &= R\,(8.530202837) \\ R &\approx 17{,}584.58 \end{aligned}$$

Withdrawals of \$17,584.58 at the end of each six-month period are needed.

(b) \$150,000 is the present value of this annuity of $7 \cdot 2 = 14$ periods with interest $\frac{6\%}{2} = 3\% = .03$ per year.

$$\begin{aligned} a_{\overline{14}|.03} &= 11.29607314, \text{ so} \\ 150{,}000 &= R\,(11.29607314) \\ R &\approx 13{,}278.95 \end{aligned}$$

Withdrawals of \$13,278.95 at the end of each six-month period are needed.

58. First, find the amount of each payment. \$72,000 is the present value of an annuity of R dollars, with 9 periods, and $i = \frac{9.5\%}{2} = 4.75\% = .0475$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 72{,}000 &= R \cdot a_{\overline{9}|.0475} \\ R &= \frac{72{,}000}{a_{\overline{9}|.0475}} = \frac{72{,}000}{7.187624181} \approx 10{,}017.22 \end{aligned}$$

Each payment is \$10,017.22.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	\$72,000.00
1	\$10,017.22	72,000(.0475) = \$3420.00	10,017.22 − 3420.00 = \$6597.22	72,000.00 − 6597.22 = \$65,402.78
2	\$10,017.22	65,402.78(.0475) = \$3106.63	10,017.22 − 3106.63 = \$6910.59	65,402.78 − 6910.59 = \$58,492.19
3	\$10,017.22	58,492.19(.0475) = \$2778.38	10,017.22 − 2778.38 = \$7238.84	58,492.19 − 7238.84 = \$51,253.35
4	\$10,017.22	51,253.35(.0475) = \$2434.53	10,017.22 − 2434.53 = \$7582.69	51,253.35 − 7582.69 = \$43,670.66
5	\$10,017.22	43,670.66(.0475) = \$2074.36	10,017.22 − 2074.36 = \$7942.86	43,670.66 − 7942.86 = \$35,727.80
6	\$10,017.22	35,727.80(.0475) = \$1697.07	10,017.22 − 1697.07 = \$8320.15	35,727.80 − 8320.15 = \$27,407.65
7	\$10,017.22	27,407.65(.0475) = \$1301.86	10,017.22 − 1301.86 = \$8715.36	27,407.65 − 8715.36 = \$18,692.29
8	\$10,017.22	18,692.29(.0475) = \$887.88	10,017.22 − 887.88 = \$9129.34	18,692.29 − 9129.34 = \$9562.95
9	\$10,017.22	9562.95(.0475) = \$454.24	10,017.22 − 454.24 = \$9562.98	9562.95 − 9562.98 = −\$.03

So that the final principal is zero, subtract \$.03 from the last payment. Thus, line 9 of the table will be:

9	\$10,017.19	9562.95 (.0475) = \$454.24	10,017.19 − 454.24 = \$9562.95	9562.95 − 9562.95 = \$0

60. Marissa's total cost is $14{,}000 + 7200 = \$21{,}200.00$. After making a down payment of \$1200, she owes a balance of $21200 - 1200 = \$20{,}000$. To amortize this balance, first find the amount of each payment. \$20,000 is the present value of an annuity of R dollars, with $5 \cdot 2 = 10$ periods, and $i = \frac{8\%}{2} = 4\% = .04$ per period.

$$
\begin{aligned}
P &= R \cdot a_{\overline{n}|i} \\
20{,}000 &= R \cdot a_{\overline{10}|.04} \\
R &= \frac{20{,}000}{a_{\overline{10}|.04}} \\
&= \frac{20{,}000}{8.110895779} \\
&\approx 2465.82
\end{aligned}
$$

Each payment is \$2465.82.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	\$20,000.00
1	\$2465.82	$(20{,}000)(.08)\left(\frac{1}{2}\right) = \$800.00$	2465.82 − 800.00 = \$1665.82	20,000.00 − 1665.82 = \$18,334.18
2	\$2465.82	$(18{,}334.18)(.08)\left(\frac{1}{2}\right) = \$733.37$	2465.82 − 733.37 = \$1732.45	18,334.18 − 1732.45 = \$16,601.73
3	\$2465.82	$(16{,}601.73)(.08)\left(\frac{1}{2}\right) = \$664.07$	2465.82 − 664.07 = \$1801.75	16,601.73 − 1801.75 = \$14,799.98
4	\$2465.82	$(14{,}799.98)(.08)\left(\frac{1}{2}\right) = \$592.00$	2465.82 − 592.00 = \$1873.82	14,799.98 − 1873.82 = \$12,926.16
5	\$2465.82	$(12{,}926.16)(.08)\left(\frac{1}{2}\right) = \$517.05$	2465.82 − 517.05 = \$1948.77	12,926.16 − 1948.77 = \$10,977.39
6	\$2465.82	$(10{,}977.39)(.08)\left(\frac{1}{2}\right) = \$439.10$	2465.82 − 439.10 = \$2026.72	10,977.39 − 2026.72 = \$8950.67
7	\$2465.82	$(8950.67)(.08)\left(\frac{1}{2}\right) = \$358.03$	2465.82 − 358.03 = \$2107.79	8950.67 − 2107.79 = \$6842.88
8	\$2465.82	$(6842.88)(.08)\left(\frac{1}{2}\right) = \$273.72$	2465.82 − 273.72 = \$2192.10	6842.88 − 2192.10 = \$4650.78
9	\$2465.82	$(4650.78)(.08)\left(\frac{1}{2}\right) = \$186.03$	2465.82 − 186.03 = \$2279.79	4650.78 − 2279.79 = \$2370.99
10	\$2465.82	$(2370.99)(.08)\left(\frac{1}{2}\right) = \$94.84$	2465.82 − 94.84 = \$2370.98	2370.99 − 2370.98 = \$.01

So that the final principal is zero, add \$.01 to the last payment. Thus, line 10 of the table will be:

10	\$2465.83	$(2370.99)(.08)\left(\frac{1}{2}\right) = \$94.84$	2465.83 − 94.84 = \$2370.99	2370.99 − 2370.99 = \$0

12.3 Taylor Polynomials at 0

2.

Derivative	Value at 0
$f(x) = e^{3x}$	$f(0) = 1$
$f^{(1)}(x) = 3e^{3x}$	$f^{(1)}(0) = 3$
$f^{(2)}(x) = 9e^{3x}$	$f^{(2)}(0) = 9$
$f^{(3)}(x) = 27e^{3x}$	$f^{(3)}(0) = 27$
$f^{(4)}(x) = 81e^{3x}$	$f^{(4)}(0) = 81$

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 + \frac{3}{1!}x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4 = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 \end{aligned}$$

4.

Derivative	Value at 0
$f(x) = e^{-x}$	$f(0) = 1$
$f^{(1)}(x) = -e^{-x}$	$f^{(1)}(0) = -1$
$f^{(2)}(x) = e^{-x}$	$f^{(2)}(0) = 1$
$f^{(3)}(x) = -e^{-x}$	$f^{(3)}(0) = -1$
$f^{(4)}(x) = e^{-x}$	$f^{(4)}(0) = 1$

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 + \frac{-1}{1!}x + \frac{1}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{1}{4!}x^4 = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 \end{aligned}$$

6.

Derivative	Value at 0
$f(x) = \sqrt{x+16} = (x+16)^{1/2}$	$f(0) = 4$
$f^{(1)}(x) = \frac{1}{2}(x+16)^{-1/2} = \frac{1}{2(x+16)^{1/2}}$	$f^{(1)}(0) = \frac{1}{8}$
$f^{(2)}(x) = -\frac{1}{4}(x+16)^{-3/2} = -\frac{1}{4(x+16)^{3/2}}$	$f^{(2)}(0) = -\frac{1}{256}$
$f^{(3)}(x) = \frac{3}{8}(x+16)^{-5/2} = \frac{3}{8(x+16)^{5/2}}$	$f^{(3)}(0) = \frac{3}{8192}$
$f^{(4)}(x) = -\frac{15}{16}(x+16)^{-7/2} = -\frac{15}{16(x+16)^{7/2}}$	$f^{(4)}(0) = -\frac{15}{262{,}144}$

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 = 4 + \frac{\frac{1}{8}}{1!}x + \frac{-\frac{1}{256}}{2!}x^2 + \frac{\frac{3}{8192}}{3!}x^3 + \frac{-\frac{15}{262{,}144}}{4!}x^4 \\ &= = 4 + \frac{1}{8}x - \frac{1}{512}x^2 + \frac{1}{16{,}384}x^3 - \frac{5}{2{,}097{,}152}x^4 \end{aligned}$$

8.

Derivative	Value at 0
$f(x) = \sqrt[3]{x+8} = (x+8)^{1/3}$	$f(0) = 2$
$f^{(1)}(x) = \frac{1}{3}(x+8)^{-2/3} = \frac{1}{3(x+8)^{2/3}}$	$f^{(1)}(0) = \frac{1}{12}$
$f^{(2)}(x) = -\frac{2}{9}(x+8)^{-5/3} = -\frac{2}{9(x+8)^{5/3}}$	$f^{(2)}(0) = -\frac{1}{144}$
$f^{(3)}(x) = \frac{10}{27}(x+8)^{-8/3} = \frac{10}{27(x+8)^{8/3}}$	$f^{(3)}(0) = \frac{5}{3456}$
$f^{(4)}(x) = -\frac{80}{81}(x+8)^{-11/3} = -\frac{80}{81(x+8)^{11/3}}$	$f^{(4)}(0) = -\frac{5}{10{,}368}$

$$\begin{aligned}
P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
&= 2 + \frac{\frac{1}{12}}{1!}x + \frac{-\frac{1}{144}}{2!}x^2 + \frac{\frac{5}{3456}}{3!}x^3 + \frac{-\frac{5}{10{,}368}}{4!}x^4 \\
&= 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \frac{5}{20{,}736}x^3 - \frac{5}{248{,}832}x^4
\end{aligned}$$

10.

Derivative	Value at 0
$f(x) = \sqrt[4]{x+16} = (x+16)^{1/4}$	$f(0) = 2$
$f^{(1)}(x) = \frac{1}{4}(x+16)^{-3/4} = \frac{1}{4(x+16)^{3/4}}$	$f^{(1)}(0) = \frac{1}{32}$
$f^{(2)}(x) = -\frac{3}{16}(x+16)^{-7/4} = -\frac{3}{16(x+16)^{7/4}}$	$f^{(2)}(0) = -\frac{3}{2048}$
$f^{(3)}(x) = \frac{21}{64}(x+16)^{-11/4} = \frac{21}{64(x+16)^{11/4}}$	$f^{(3)}(0) = \frac{21}{131{,}072}$
$f^{(4)}(x) = -\frac{231}{256}(x+16)^{-15/4} = -\frac{231}{256(x+16)^{15/4}}$	$f^{(4)}(0) = -\frac{231}{8{,}388{,}608}$

$$\begin{aligned}
P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
&= 2 + \frac{\frac{1}{32}}{1!}x + \frac{-\frac{3}{2048}}{2!}x^2 + \frac{\frac{21}{131{,}072}}{3!}x^3 + \frac{-\frac{231}{8{,}388{,}608}}{4!}x^4 \\
&= 2 + \frac{1}{32}x - \frac{3}{4096}x^2 + \frac{7}{262{,}144}x^3 - \frac{77}{67{,}108{,}864}x^4
\end{aligned}$$

12.

Derivative	Value at 0
$f(x) = \ln(1+2x)$	$f(0) = 0$
$f^{(1)}(x) = \dfrac{2}{1+2x}$	$f^{(1)}(0) = 2$
$f^{(2)}(x) = -\dfrac{4}{(1+2x)^2}$	$f^{(2)}(0) = -4$
$f^{(3)}(x) = \dfrac{16}{(1+2x)^3}$	$f^{(3)}(0) = 16$
$f^{(4)}(x) = -\dfrac{96}{(1+2x)^4}$	$f^{(4)}(0) = -96$

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 0 + \frac{2}{1!}x + \frac{-4}{2!}x^2 + \frac{16}{3!}x^3 + \frac{-96}{4!}x^4 = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 \end{aligned}$$

14.

Derivative	Value at 0
$f(x) = \ln(1-x^3)$	$f(0) = 0$
$f^{(1)}(x) = \dfrac{-3x^2}{1-x^3} = \dfrac{3x^2}{x^3-1}$	$f^{(1)}(0) = 0$
$f^{(2)}(x) = \dfrac{-3x^4-6x}{(x^3-1)^2}$	$f^{(2)}(0) = 0$
$f^{(3)}(x) = \dfrac{6x^6+42x^3+6}{(x^3-1)^3}$	$f^{(3)}(0) = -6$
$f^{(4)}(x) = \dfrac{-18x^8-288x^5-180x^2}{(x^3-1)^4}$	$f^{(4)}(0) = 0$

$$P_4(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 = 0 + \frac{0}{1!}x + \frac{0}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{0}{4!}x^4 = -x^3$$

16.

Derivative	Value at 0
$f(x) = x^2e^x$	$f(0) = 0$
$f^{(1)}(x) = x^2e^x + 2xe^x$	$f^{(1)}(0) = 0$
$f^{(2)}(x) = x^2e^x + 4xe^x + 2e^x$	$f^{(2)}(0) = 2$
$f^{(3)}(x) = x^2e^x + 6xe^x + 6e^x$	$f^{(3)}(0) = 6$
$f^{(4)}(x) = x^2e^x + 8xe^x + 12e^x$	$f^{(4)}(0) = 12$

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 0 + \frac{0}{1!}x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{12}{4!}x^4 = x^2 + x^3 + \frac{1}{2}x^4 \end{aligned}$$

18.

Derivative	Value at 0
$f(x) = (1+x)^{3/2}$	$f(0) = 1$
$f^{(1)}(x) = \frac{3}{2}(1+x)^{1/2}$	$f^{(1)}(0) = \frac{3}{2}$
$f^{(2)}(x) = \frac{3}{4}(1+x)^{-1/2} = \frac{3}{4(1+x)^{1/2}}$	$f^{(2)}(0) = \frac{3}{4}$
$f^{(3)}(x) = -\frac{3}{8}(1+x)^{-3/2} = -\frac{3}{8(1+x)^{3/2}}$	$f^{(3)}(0) = -\frac{3}{8}$
$f^{(4)}(x) = \frac{9}{16}(1+x)^{-5/2} = \frac{9}{16(1+x)^{5/2}}$	$f^{(4)}(0) = \frac{9}{16}$

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 + \frac{\frac{3}{2}}{1!}x + \frac{\frac{3}{4}}{2!}x^2 + \frac{-\frac{3}{8}}{3!}x^3 + \frac{\frac{9}{16}}{4!}x^4 \\ &= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \frac{3}{128}x^4 \end{aligned}$$

20.

Derivative	Value at 0
$f(x) = \frac{1}{x-1} = (x-1)^{-1}$	$f(0) = -1$
$f^{(1)}(x) = -(x-1)^{-2} = -\frac{1}{(x-1)^2}$	$f^{(1)}(0) = -1$
$f^{(2)}(x) = 2(x-1)^{-3} = \frac{2}{(x-1)^3}$	$f^{(2)}(0) = -2$
$f^{(3)}(x) = -6(x-1)^{-4} = -\frac{6}{(x-1)^4}$	$f^{(3)}(0) = -6$
$f^{(4)}(x) = 24(x-1)^{-5} = \frac{24}{(x-1)^5}$	$f^{(4)}(0) = -24$

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= -1 + \frac{-1}{1!}x + \frac{-2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{-24}{4!}x^4 \\ &= -1 - x - x^2 - x^3 - x^4 \end{aligned}$$

For Exercises 22–34 even, each approximation can be determined using the Taylor polynomials from Exercises 2–14 even, respectively.

22. Using the result of Exercise 2, with $f(x) = e^{3x}$ and $P_4(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$, we can approximate $e^{.03}$ by evaluating $f(.01) = e^{3(.01)} = e^{.03}$. Using $P_4(x)$ from Exercise 2 with $x = .01$ gives

$$\begin{aligned} P_4(.01) &= 1 + 3(.01) + \frac{9}{2}(.01)^2 + \frac{9}{2}(.01)^3 + \frac{27}{8}(.01)^4 \\ &\approx 1.030454534 \end{aligned}$$

To four decimal places, $P_4(.01)$ approximates the value of $e^{.03}$ as 1.0305.

24. Using the result of Exercise 4, with $f(x) = e^{-x}$ and $P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4$, we can approximate $e^{-.04}$ by evaluating $f(.04) = e^{-(.04)} = e^{-.04}$. Using $P_4(x)$ from Exercise 4 with $x = .04$ gives

$$\begin{aligned} P_4(.04) &= 1 - .04 + \frac{1}{2}(.04)^2 - \frac{1}{6}(.04)^3 + \frac{1}{24}(.04)^4 \\ &\approx .96078944 \end{aligned}$$

To four decimal places, $P_4(.04)$ approximates the value of $e^{-.04}$ as .9608.

26. Using the result of Exercise 6, with $f(x) = \sqrt{x+16}$ and $P_4(x) = 4 + \frac{1}{8}x - \frac{1}{512}x^2 + \frac{1}{16{,}384}x^3 - \frac{5}{2{,}097{,}152}x^4$, we can approximate $\sqrt{16.01}$ by evaluating $f(.01) = \sqrt{.01+16} = \sqrt{16.01}$. Using $P_4(x)$ from Exercise 6 with $x = .01$ gives

$$\begin{aligned} P_4(.01) &= 4 + \frac{1}{8}(.01) - \frac{1}{512}(.01)^2 + \frac{1}{16{,}384}(.01)^3 - \frac{5}{2{,}097{,}152}(.01)^4 \\ &\approx 4.001249805 \end{aligned}$$

To four decimal places, $P_4(.01)$ approximates the value of $\sqrt{16.01}$ as 4.0012.

28. Using the result of Exercise 8, with $f(x) = \sqrt[3]{x+8}$ and $P_4(x) = 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \frac{5}{20{,}736}x^3 - \frac{5}{248{,}832}x^4$, we can approximate $\sqrt[3]{7.98}$ by evaluating $f(-.02) = \sqrt[3]{-.02+8} = \sqrt[3]{7.98}$. Using $P_4(x)$ from Exercise 8 with $x = -.02$ gives

$$\begin{aligned} P_4(-.02) &= 2 + \frac{1}{12}(-.02) - \frac{1}{288}(-.02)^2 + \frac{5}{20{,}736}(-.02)^3 - \frac{5}{248{,}832}(-.02)^4 \\ &\approx 1.998331943 \end{aligned}$$

To four decimal places, $P_4(-.02)$ approximates the value of $\sqrt[3]{7.98}$ as 1.9983.

30. Using the result of Exercise 10, with $f(x) = \sqrt[4]{x+16}$ and $P_4(x) = 2 + \frac{1}{32}x - \frac{3}{4096}x^2 + \frac{7}{262{,}144}x^3 - \frac{77}{67{,}108{,}864}x^4$, we can approximate $\sqrt[4]{15.97}$ by calculating $f(-.03) = \sqrt[4]{-.03+16} = \sqrt[4]{15.97}$. Using $P_4(x)$ from Exercise 10 with $x = -.03$ gives

$$\begin{aligned} P_4(-.03) &= 2 + \frac{1}{32}(-.03) - \frac{3}{4096}(-.03)^2 + \frac{7}{262{,}144}(-.03)^3 - \frac{77}{67{,}108{,}864}(-.03)^4 \\ &\approx 1.99906184 \end{aligned}$$

To four decimal places, $P_4(-.03)$ approximates the value of $\sqrt[4]{15.97}$ as 1.9991.

32. Using the result of Exercise 12, with $f(x) = \ln(1+2x)$ and $P_4(x) = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4$, we can approximate $\ln 1.04$ by evaluating $f(.02) = \ln(1 + 2(.02)) = \ln(1 + .04) = \ln 1.04$. Using $P_4(x)$ from Exercise 12 with $x = .02$ gives

$$\begin{aligned} P_4(.02) &= 2(.02) - 2(.02)^2 + \frac{8}{3}(.02)^3 - 4(.02)^4 \\ &\approx .0392206933 \end{aligned}$$

To four decimal places, $P_4(.02)$ approximates the value of $\ln 1.04$ as .0392.

34. Using the result of Exercise 14, with $f(x) = \ln\left(1 - x^3\right)$ and $P_4(x) = -x^3$, we can approximate $\ln .999$ by evaluating $f(.1) = \ln\left(1 - .1^3\right) = \ln .999$. Using $P_4(x)$ from Exercise 14 with $x = .1$ gives

$$\begin{aligned} P_4(.1) &= -(.1)^3 \\ &\approx -.001 \end{aligned}$$

To four decimal places, $P_4(.1)$ approximates the value of $\ln .999$ as $-.0010$.

36. $P_3(x) = f(0) + \dfrac{f^{(1)}(0)}{1!}x + \dfrac{f^{(2)}(0)}{2!}x^2 + \dfrac{f^{(3)}(0)}{3!}x^3 = 3 + \dfrac{6}{1}x + \dfrac{12}{2}x^2 + \dfrac{24}{6}x^3 = 3 + 6x + 6x^2 + 4x^3$

38. **(a)**

Derivative	**Value at 0**
$f(r) = (1+r)^D$	$f(0) = 1$
$f^{(1)}(r) = D(1+r)^{D-1}$	$f^{(1)}(0) = D$

$P_1(r) = 1 + \dfrac{D}{1!}r = 1 + rD$

So the Taylor polynomial of degree 1 for the function $f(r) = (1+r)^D$ is $P_1(r) = 1 + rD$. If r is small, then $(1+r)^D \approx 1 + rD$, and so $V(1+r)^D \approx V(1+rD)$. Therefore, for small r, the two formulas for S give approximately equal values.

(b) Using $S \approx V(1+rD)$ with $V = 1000$, $r = .1$, and $D = 3.2$ gives $S \approx 1000[1 + (.1)(3.2)] = 1320$.
Using $S \approx V(1+r)^D$ with $V = 1000$, $r = .1$, and $D = 3.2$ gives $S \approx 1000(1+.1)^{3.2} \approx 1356.61$.
The two approximations for S are 1320 and 1356.61 which are relatively close.

40.

Derivative	**Value at 0**
$P(x) = \dfrac{20+x^2}{50+x}$	$P(0) = .4$
$P^{(1)}(x) = \dfrac{x^2+100x-20}{(50+x)^2}$	$P^{(1)}(0) = -.008$
$P^{(2)}(x) = \dfrac{(50+x)^2(2x+100) - \left(x^2+100x-20\right)[2(50+x)]}{(50+x)^4}$	$P^{(2)}(0) = .04032$

$P_2(x) = P(0) + \dfrac{P^{(1)}(0)}{1!}x + \dfrac{P^{(2)}(0)}{2!}x^2 = .4 + \dfrac{-.008}{1}x + \dfrac{.04032}{2}x^2 = .4 - .008x + .02016x^2$

$P_2(.3) = .4 - .008(.3) + (.02016)(.3)^2 = .3994144$

The Taylor polynomial $P_2(.3)$ approximates $P(.3)$ as .3994144. The approximate profit when 300 tons of apples are sold is .399 thousand dollars, or \$399.
Finding $P(.3)$ by direct substitution gives:

$$P(.3) = \frac{20+(.3)^2}{50+(.3)} \approx .3994035785$$

Direct substitution also estimates the profit as .399 thousand dollars, or \$399.

42.

Derivative	**Value at 0**
$C(x) = e^{-x/50}$	$C(0) = 1$
$C^{(1)}(x) = -\dfrac{1}{50}e^{-x/50}$	$C^{(1)}(0) = -\dfrac{1}{50}$
$C^{(2)}(x) = \dfrac{1}{2500}e^{-x/50}$	$C^{(2)}(0) = \dfrac{1}{2500}$

$P_2(x) = C(0) + \dfrac{C^{(1)}(0)}{1!}x + \dfrac{C^{(2)}(0)}{2!}x^2 = 1 + \dfrac{-\frac{1}{50}}{1!}x + \dfrac{\frac{1}{2500}}{2!}x^2 = 1 - \dfrac{1}{50}x + \dfrac{1}{5000}x^2$

$P_2(5) = 1 - \dfrac{1}{50}(5) + \dfrac{1}{5000}(5)^2 = .905$

The Taylor polynomial $P_2(5)$ approximates $C(5)$ as .905. The approximate cost is .905 dollar, or about \$.91.

Finding $C(5)$ by direct substitution gives:

$$C(5) = e^{-5/50} \approx .904837418$$

Direct substitution estimates the profit as .905 dollar, or about \$.90.

44.

Derivative	Value at 0
$A(x) = \frac{6x}{1+10x}$	$A(0) = 0$
$A^{(1)}(x) = \frac{6}{(1+10x)^2}$	$A^{(1)}(0) = 6$
$A^{(2)}(x) = -\frac{120}{(1+10x)^3}$	$A^{(2)}(0) = -120$

$$P_2(x) = A(0) + \frac{A^{(1)}(0)}{1!}x + \frac{A^{(2)}(0)}{2!}x^2 = 0 + \frac{6}{1}x + \frac{-120}{2}x^2 = 6x - 60x^2$$

$$P_2(.05) = 6(.05) - 60(.05)^2 = .15$$

The Taylor polynomial $P_2(.05)$ approximates $A(.05)$ as .15. The approximate amount of the drug in the bloodstream after .05 minutes is .15 milliliters.
Finding $A(.05)$ by direct substitution gives:

$$A(.05) = \frac{6(.05)}{1+10(.05)} = .2$$

Direct substitution indicates that .2 milliliter of the drug remain in the bloodsteam after .05 minutes.

12.4 Infinite Series

2. $1 + .9 + .81 + .729 + \ldots$ is a geometric series with $a = a_1 = 1$ and $r = .9$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{1}{1-.9} = \frac{1}{.1} = 10.$$

4. $3 + 9 + 27 + 81 + \ldots$ is a geometric series with $a = a_1 = 3$ and $r = 3$. Since $r > 1$, the series diverges.

6. $81 + 27 + 9 + 3 + 1 + \ldots$ is a geometric series with $a = a_1 = 81$ and $r = \frac{1}{3}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{81}{1-\frac{1}{3}} = \frac{81}{\frac{2}{3}} = \frac{243}{2}.$$

8. $128 + 64 + 32 + \ldots$ is a geometric series with $a = a_1 = 128$ and $r = \frac{1}{2}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{128}{1-\frac{1}{2}} = \frac{128}{\frac{1}{2}} = 256.$$

10. $\frac{4}{5} + \frac{2}{5} + \frac{1}{5} + \ldots$ is a geometric series with $a = a_1 = \frac{4}{5}$ and $r = \frac{1}{2}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{\frac{4}{5}}{1-\frac{1}{2}} = \frac{\frac{4}{5}}{\frac{1}{2}} = \frac{8}{5}.$$

12. $1+\frac{1}{1.01}+\frac{1}{(1.01)^2}+\ldots$ is a geometric series with $a=a_1=1$ and $r=\frac{1}{1.01}$. Since r is in $(-1,1)$, the series converges and has sum

$$\frac{a}{1-r}=\frac{1}{1-\frac{1}{1.01}}=\frac{1}{\frac{.01}{1.01}}=101.$$

14. $e+e^2+e^3+e^4+\ldots$ is a geometric series with $a=a_1=e$ and $r=e$. Since $r>1$, the series diverges.

16.
$$S_1=a_1=\frac{1}{1+1}=\frac{1}{2}$$
$$S_2=a_1+a_2=\frac{1}{1+1}+\frac{1}{2+1}=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$$
$$S_3=a_1+a_2+a_3=\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}$$
$$S_4=a_1+a_2+a_3+a_4=\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}+\frac{1}{4+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{77}{60}$$
$$S_5=a_1+a_2+a_3+a_4+a_5=\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}+\frac{1}{4+1}+\frac{1}{5+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{29}{20}$$

18.
$$S_1=a_1=\frac{1}{3(1)-1}=\frac{1}{2}$$
$$S_2=a_1+a_2=\frac{1}{3(1)-1}+\frac{1}{3(2)-1}=\frac{1}{2}+\frac{1}{5}=\frac{7}{10}$$
$$S_3=a_1+a_2+a_3=\frac{1}{3(1)-1}+\frac{1}{3(2)-1}+\frac{1}{3(3)-1}=\frac{1}{2}+\frac{1}{5}+\frac{1}{8}=\frac{33}{40}$$
$$S_4=a_1+a_2+a_3+a_4=\frac{1}{3(1)-1}+\frac{1}{3(2)-1}+\frac{1}{3(3)-1}+\frac{1}{3(4)-1}=\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}=\frac{403}{440}$$
$$S_5=a_1+a_2+a_3+a_4+a_5=\frac{1}{3(1)-1}+\frac{1}{3(2)-1}+\frac{1}{3(3)-1}+\frac{1}{3(4)-1}+\frac{1}{3(5)-1}$$
$$=\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}=\frac{3041}{3080}$$

20. $S_1 = a_1 = \dfrac{1}{(1+3)\,[2\,(1)+1]} = \dfrac{1}{12}$

$S_2 = a_1 + a_2 = \dfrac{1}{(1+3)\,[2\,(1)+1]} + \dfrac{1}{(2+3)\,[2\,(2)+1]} = \dfrac{1}{12} + \dfrac{1}{25} = \dfrac{37}{300}$

$S_3 = a_1 + a_2 + a_3 = \dfrac{1}{(1+3)\,[2\,(1)+1]} + \dfrac{1}{(2+3)\,[2\,(2)+1]} + \dfrac{1}{(3+3)\,[2\,(3)+1]} = \dfrac{1}{12} + \dfrac{1}{25} + \dfrac{1}{42} = \dfrac{103}{700}$

$S_4 = a_1 + a_2 + a_3 + a_4 = \dfrac{1}{(1+3)\,[2\,(1)+1]} + \dfrac{1}{(2+3)\,[2\,(2)+1]} + \dfrac{1}{(3+3)\,[2\,(3)+1]} + \dfrac{1}{(4+3)\,[2\,(4)+1]}$

$= \dfrac{1}{12} + \dfrac{1}{25} + \dfrac{1}{42} + \dfrac{1}{63} = \dfrac{1027}{6300}$

$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$

$= \dfrac{1}{(1+3)\,[2\,(1)+1]} + \dfrac{1}{(2+3)\,[2\,(2)+1]} + \dfrac{1}{(3+3)\,[2\,(3)+1]} + \dfrac{1}{(4+3)\,[2\,(4)+1]} + \dfrac{1}{(5+3)\,[2\,(5)+1]}$

$= \dfrac{1}{12} + \dfrac{1}{25} + \dfrac{1}{42} + \dfrac{1}{63} + \dfrac{1}{88} = \dfrac{24{,}169}{138{,}600}$

22. (a) Consider the finite sequence

$$C\,(1+r)^{-1},\ C\,(1+r)^{-2},\ C\,(1+5)^{-3},\ \ldots,\ C\,(1+r)^{-n}.$$

This is a finite geometric sequence having first term $a = a_1 = C\,(1+r)^{-1}$ and common ratio $w = (1+r)^{-1}$. Thus,

$$\begin{aligned} P = S_n &= \frac{a\,(w^n - 1)}{w-1} \\ &= \frac{C\,(1+r)^{-1}\left[(1+r)^{-n} - 1\right]}{(1+r)^{-1} - 1} \\ &= C\frac{(1+r)^{-1}\left[(1+r)^{-n} - 1\right]}{(1+r)^{-1}\,[1-(1+r)]} \\ &= C\frac{(1+r)^{-n} - 1}{-r} \\ &= C\frac{(1+r)^{-n} - 1}{-r} \cdot \frac{(1+r)^n}{(1+r)^n} \\ &= C\frac{(1+r)^n - 1}{r\,(1+r)^n}. \end{aligned}$$

(b) The present value over an infinite amount of time is given by the infinite geometric series

$$P = C\,(1+r)^{-1} + C\,(1+r)^{-2} + C\,(1+r)^{-3} + \ldots$$

where $a = a_1 = C\,(1+r)^{-1}$ and the common ratio $w = (1+r)^{-1}$. This series converges to

$$\frac{a}{1-w} = \frac{C\,(1+r)^{-1}}{1-(1+r)^{-1}} = \frac{C\,(1+r)^{-1}}{(1+r)^{-1}\,[(1+r)-1]} = \frac{C}{r}$$

Here, $P = \frac{C}{r}$.

24. The height that the ball returns to after the first bounce is given by $10\left(\frac{3}{4}\right)$, after the second bounce $10\left(\frac{3}{4}\right)^2$, after the third bounce $10\left(\frac{3}{4}\right)^3$, and so on. Thus, the distance that the ball travels before it comes to rest is given by

$$10+2(10)\left(\frac{3}{4}\right)+2(10)\left(\frac{3}{4}\right)^2+2(10)\left(\frac{3}{4}\right)^3+\dots\ .$$

$2(10)\left(\frac{3}{4}\right)+2(10)\left(\frac{3}{4}\right)^2+2(10)\left(\frac{3}{4}\right)^3+\dots$ is an infinite geometric series with $a_1=a=2(10)\left(\frac{3}{4}\right)=15$ and $r=\frac{3}{4}$. This series converges to

$$\frac{a}{1-r}=\frac{15}{1-\frac{3}{4}}=\frac{15}{\frac{1}{4}}=60.$$

So, the total distance traveled by the ball is $10+60=70$ meters.

26. On its second swing, the pendulum bob will swing through an arc $40(.8)$ centimeters long, on its third swing $40(.8)^2$ centimeters long, on its fourth swing $40(.8)^3$ centimeters long, and so on. Thus, the distance it will swing altogether before coming to a complete stop is given by

$$40+40(.8)+40(.8)^2+40(.8)^3+\dots\ .$$

This is an infinite geometric series with $a_1=a=40$ and $r=.8$. This series converges to

$$\frac{a}{1-r}=\frac{40}{1-.8}=\frac{40}{.2}=200.$$

So, the total distance altogether before coming to a complete stop is 200 centimeters.

28. The first triangle has sides 2 meters in length, the second triangle has sides 1 meter in length, the third triangle has sides $\frac{1}{2}$ meter in length, the fourth triangle has sides $\frac{1}{4}$ meter in length, and so on. Thus, the height of the first triangle is $\sqrt{3}$ meters, the height of the second triangle is $\frac{\sqrt{3}}{2}$ meter, the height of the third triangle is $\frac{\sqrt{3}}{4}$ meter, the height of the fourth triangle is $\frac{\sqrt{3}}{8}$ meter, and so on. The total area of the triangles, disregarding the overlaps, is given by

$$\frac{1}{2}(2)\left(\sqrt{3}\right)+\frac{1}{2}(1)\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{4}\right)+\frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{8}\right)+\cdots$$
$$=\sqrt{3}+\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{16}+\frac{\sqrt{3}}{64}+\cdots\ .$$

This is an infinite geometric series with $a=a_1=\sqrt{3}$ and $r=\frac{1}{4}$. This series converges to

$$\frac{a}{1-r}=\frac{\sqrt{3}}{1-\frac{1}{4}}=\frac{\sqrt{3}}{\frac{3}{4}}=\frac{4\sqrt{3}}{3}.$$

The total area of the triangles, disregardng the overlaps, is given by $\frac{4\sqrt{3}}{3}$ square meters.

12.5 Taylor Series

2. This function most nearly matches $\frac{1}{1-x}$. To get 1 in the denominator, instead of 4, divide the numerator and denominator by 4.

$$\frac{-3}{4-x}=\frac{-\frac{3}{4}}{1-\frac{x}{4}}$$

Thus, we can find the Taylor series for $\dfrac{-\frac{3}{4}}{1-\frac{x}{4}}$ by starting with the Taylor series for $\frac{1}{1-x}$, multiplying each term by $-\frac{3}{4}$, and replacing each x with $\frac{x}{4}$.

$$\begin{aligned}\frac{-3}{4-x} &= \frac{-\frac{3}{4}}{1-\frac{x}{4}}\\ &= \left(-\frac{3}{4}\right)\cdot 1 + \left(-\frac{3}{4}\right)\left(\frac{x}{4}\right) + \left(-\frac{3}{4}\right)\left(\frac{x}{4}\right)^2 + \left(-\frac{3}{4}\right)\left(\frac{x}{4}\right)^3 + \cdots + \left(-\frac{3}{4}\right)\left(\frac{x}{4}\right)^n + \cdots\\ &= -\frac{3}{4} - \frac{3x}{16} - \frac{3x^2}{64} - \frac{3x^3}{256} - \cdots - \frac{3x^n}{4^{n+1}} - \cdots\end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $\frac{x}{4}$ gives

$$-1 < \frac{x}{4} < 1 \quad \text{or} \quad -4 < x < 4.$$

The interval of convergence of the new series is $(-4, 4)$.

4. $\dfrac{7x}{1+2x} = x \cdot \dfrac{7}{1-(-2x)}$
Use the Taylor series for $\frac{1}{1-x}$, multiply each term by 7, and replace x with $-2x$. Also, use property (3) with $k = 1$.

$$\begin{aligned}\frac{7x}{1+2x} &= x \cdot \frac{7}{1-(-2x)} \\ &= x \cdot 7 \cdot 1 + x \cdot 7(-2x) + x \cdot 7(-2x)^2 + x \cdot 7(-2x)^3 + \ldots + x \cdot 7(-2x)^n + \ldots \\ &= 7x - 14x^2 + 28x^3 - 56x^4 + \ldots + (-1)^n \cdot 7 \cdot 2^n x^{n+1} + \ldots\end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $-2x$ gives

$$-1 < -2x < 1 \text{ or } \frac{1}{2} > x > -\frac{1}{2}.$$

The interval of convergence of the new series is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

6. $\dfrac{9x^4}{1-x} = x^4 \cdot \dfrac{9}{1-x}$
Use the Taylor series for $\frac{1}{1-x}$, and multiply each term by 9. Also, use property (3) with $k = 4$.

$$\begin{aligned}\frac{9x^4}{1-x} &= x^4 \cdot \frac{9}{1-x} \\ &= x^4 \cdot 9 \cdot 1 + x^4 \cdot 9(x) + x^4 \cdot 9(x)^2 + x^4 \cdot 9(x)^3 + \ldots + x^4 \cdot 9(x)^n + \ldots \\ &= 9x^4 + 9x^5 + 9x^6 + 9x^7 + \ldots + 9x^{n+4} + \ldots\end{aligned}$$

The Taylor series for $\frac{9x^4}{1-x}$ has the same interval of convergence, $(-1, 1)$, as the Taylor series for $\frac{1}{1-x}$.

8. We find the Taylor series of $\ln\left(1 - \frac{x}{2}\right)$ by starting with the Taylor series for $\ln(1+x)$ and replacing each x with $-\frac{x}{2}$.

$$\begin{aligned}\ln\left(1 - \tfrac{x}{2}\right) &= \left(-\frac{x}{2}\right) - \frac{\left(-\frac{x}{2}\right)^2}{2} + \frac{\left(-\frac{x}{2}\right)^3}{3} - \frac{\left(-\frac{x}{2}\right)^4}{4} + \cdots + \frac{(-1)^n\left(-\frac{x}{2}\right)^{n+1}}{n+1} + \cdots \\ &= -\frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{24} - \frac{x^4}{64} - \cdots - \frac{x^{n+1}}{(n+1)\,2^{n+1}} - \cdots\end{aligned}$$

The Taylor series for $\ln(1+x)$ is valid when $-1 < x \le 1$. Replacing x with $-\frac{x}{2}$ gives

$$-1 < -\frac{x}{2} \le 1 \text{ or } 2 > x \ge -2.$$

The interval of convergence of the new series is $[-2, 2)$.

10. We find the Taylor series for e^{-3x^2} by starting with the Taylor series for e^x and replacing each x with $-3x^2$.

$$\begin{aligned}e^{-3x^2} &= 1 + \left(-3x^2\right) + \frac{1}{2!}\left(-3x^2\right)^2 + \frac{1}{3!}\left(-3x^2\right)^3 + \cdots + \frac{1}{n!}\left(-3x^2\right)^n + \cdots \\ &= 1 - 3x^2 + \frac{9}{2}x^4 - \frac{9}{2}x^6 + \cdots + \frac{(-1)^n 3^n x^{2n}}{n!} + \cdots\end{aligned}$$

The Taylor series for e^{-3x^2} has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x.

12. Use the Taylor series for e^x and property (3) with $k = 5$.

$$\begin{aligned} x^5 e^x &= x^5 \cdot 1 + x^5 \cdot x + x^5 \cdot \frac{1}{2!}x^2 + x^5 \cdot \frac{1}{3!}x^3 + \cdots + x^5 \cdot \frac{1}{n!}x^n + \cdots \\ &= x^5 + x^6 + \frac{1}{2}x^7 + \frac{1}{6}x^8 + \cdots + \frac{x^{n+5}}{n!} + \cdots \end{aligned}$$

The Taylor series for $x^5 e^x$ has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x.

14. $\dfrac{6}{3+x^2} = \dfrac{2}{1 - \left(-\dfrac{x^2}{3}\right)}$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by 2, and replace x with $-\frac{x^2}{3}$.

$$\begin{aligned} \frac{6}{3+x^2} &= \frac{2}{1 - \left(-\dfrac{x^2}{3}\right)} \\ &= 2 \cdot 1 + 2\left(-\frac{x^2}{3}\right) + 2\left(-\frac{x^2}{3}\right)^2 + 2\left(-\frac{x^2}{3}\right)^3 + \cdots + 2\left(-\frac{x^2}{3}\right)^n + \cdots \\ &= 2 - \frac{2}{3}x^2 + \frac{2}{9}x^4 - \frac{2}{27}x^6 + \cdots + \frac{(-1)^n \cdot 2 \cdot x^{2n}}{3^n} + \cdots \end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $-\frac{x^2}{3}$ gives

$$-1 < -\frac{x^2}{3} < 1 \text{ or } 3 > x^2 > -3.$$

The inequality is satisfied by any x in the interval $\left(-\sqrt{3}, \sqrt{3}\right)$.

16. $\dfrac{e^x - e^{-x}}{2} = \dfrac{1}{2}e^x - \dfrac{1}{2}e^{-x}$

We find the Taylor series for $\frac{1}{2}e^x$ by starting with the Taylor series for e^x and multiplying each term by $\frac{1}{2}$.

$$\begin{aligned} \frac{1}{2}e^x &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot x + \frac{1}{2} \cdot \frac{1}{2!}x^2 + \frac{1}{2} \cdot \frac{1}{3!}x^3 + \cdots + \frac{1}{2} \cdot \frac{1}{n!}x^n + \cdots \\ &= \frac{1}{2} + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{12}x^3 + \cdots + \frac{1}{2n!}x^n + \cdots \end{aligned}$$

We find the Taylor series for $-\frac{1}{2}e^{-x}$ by starting with the Taylor series for e^x, multiplying each term by $-\frac{1}{2}$, and replacing x with $-x$.

$$\begin{aligned} -\frac{1}{2}e^{-x} &= -\frac{1}{2} \cdot 1 + \left(-\frac{1}{2}\right)(-x) + \left(-\frac{1}{2}\right)\frac{1}{2!}(-x)^2 + \left(-\frac{1}{2}\right)\frac{1}{3!}(-x)^3 + \cdots + \left(-\frac{1}{2}\right)\frac{1}{n!}(-x)^n + \cdots \\ &= -\frac{1}{2} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 + \cdots + \frac{(-1)^{n+1}}{2n!}x^n + \cdots \end{aligned}$$

Use property (1) with $f(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$.

$$\frac{e^x - e^{-x}}{2}$$
$$= \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$
$$= \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{2}\right)x + \left(\frac{1}{4} - \frac{1}{4}\right)x^2 + \left(\frac{1}{12} + \frac{1}{12}\right)x^3 + \cdots + \left(\frac{1}{2n!} + \frac{(-1)^{n+1}}{2n!}\right)x^n + \cdots$$
$$= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots + \frac{1 + (-1)^{n+1}}{2n!}x^n + \cdots$$
$$= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots + \frac{1}{(2n+1)!}x^{2n+1} + \cdots$$

The new series has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x.

18. $\ln\left(1 - 5x^2\right) = \ln\left[1 + \left(-5x^2\right)\right]$
Use the Taylor series for $\ln(1+x)$ and replace x with $-5x^2$.

$$\ln\left(1 - 5x^2\right) = \left(-5x^2\right) - \frac{\left(-5x^2\right)^2}{2} + \frac{\left(-5x^2\right)^3}{3} - \frac{\left(-5x^2\right)^4}{4} + \cdots + \frac{(-1)^n\left(-5x^2\right)^{n+1}}{n+1} + \cdots$$
$$= -5x^2 - \frac{25}{2}x^4 - \frac{125}{3}x^6 - \frac{625}{4}x^8 - \cdots - \frac{5^{n+1}x^{2n+2}}{n+1} - \cdots$$

The Taylor series for $\ln(1+x)$ is valid when $-1 < x \le 1$. Replacing x with $-5x^2$ gives

$$-1 < -5x^2 \le 1 \text{ or } \frac{1}{5} > x^2 \ge -\frac{1}{5}.$$

The inequality is satisfied by any x in the interval $\left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

20. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$
The Taylor series for $\ln(1+x)$ is given as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n x^{n+1}}{n+1} + \cdots$$

We find the Taylor series for $-\ln(1-x)$ by starting with the Taylor series for $\ln(1+x)$ multiplying each term by -1, and replacing x with $-x$.

$$-\ln(1-x) = -1(-x) - (-1)\frac{(-x)^2}{2} + (-1)\frac{(-x)^3}{3} - (-1)\frac{(-x)^4}{4} + \cdots + (-1)\frac{(-1)^n(-x)^{n+1}}{n+1} + \cdots$$
$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^{n+1}}{n+1} + \cdots$$

Use property (1) with $f(x) = \ln(1+x)$ and $g(x) = -\ln(1-x)$.

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$
$$= (1+1)x + \left(-\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{1}{3} + \frac{1}{3}\right)x^3 + \left(-\frac{1}{4} + \frac{1}{4}\right)x^4 + \cdots + \left(\frac{(-1)^n}{n+1} + \frac{1}{n+1}\right)x^{n+1} + \cdots$$
$$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots + \frac{(-1)^n + 1}{n+1}x^{n+1} + \cdots$$
$$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots + \frac{2x^{2n+1}}{2n+1} + \cdots$$

22. Use the Taylor series for e^x and replace x with $-x$ to find the Taylor series for e^{-x}.

$$\begin{aligned} e^{-x} &= 1 + (-x) + \frac{1}{2!}(-x)^2 + \frac{1}{3!}(-x)^3 + \frac{1}{4!}(-x)^4 + \frac{1}{5!}(-x)^5 + \cdots + \frac{1}{n!}(-x)^n + \cdots \\ &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots + \frac{(-1)^n x^n}{n!} + \cdots \\ &= 1 - x + \frac{x^2}{2}\left[1 - \frac{x}{3} + \frac{x^2}{12} - \frac{x^3}{60} + \cdots + \frac{(-1)^n x^{n-2}}{\frac{n!}{2}} + \cdots\right] \\ &= 1 - x + \frac{x^2}{2}\left[1 - \frac{x}{3}\left(1 - \frac{x}{4}\right) - \frac{x^3}{60}\left(1 - \frac{x}{6}\right) - \cdots - \frac{x^{n-2}}{\frac{n!}{2}}\left(1 - \frac{x}{n+1}\right) - \cdots\right] \end{aligned}$$

Compare the series $-\frac{x}{3}\left(1 - \frac{x}{4}\right) - \frac{x^3}{60}\left(1 - \frac{x}{6}\right) - \cdots - \frac{x^{n-2}}{\frac{n!}{2}}\left(1 - \frac{x}{n+1}\right) - \cdots$ with the series $-\frac{x}{3}\left(1 - \frac{x}{4}\right) - \frac{x^3}{48}\left(1 - \frac{x}{4}\right) - \cdots - \frac{x^{n-2}}{3 \cdot 4^{n-3}}\left(1 - \frac{x}{4}\right) - \ldots$. The second series is an infinite geometric series that is less than or equal to the first series term by term. The geometric series has $a = -\frac{x}{3}\left(1 - \frac{x}{4}\right)$ and $r = \frac{x^2}{16}$, and sums to $-\frac{16x - 4x^2}{48 - 3x^2}$. Thus, for $0 < x < 1$,

$$\begin{aligned} 1 - x + \frac{x^2}{2}\left[1 - \frac{16x - 4x^2}{48 - 3x^2}\right] &< 1 - x + \frac{x^2}{2}\left[1 - \frac{x}{3}\left(1 - \frac{x}{4}\right) - \frac{x^3}{60}\left(1 - \frac{x}{6}\right) - \cdots - \frac{x^{n-2}}{\frac{n!}{2}}\left(1 - \frac{x}{n+1}\right) - \cdots\right] \\ &< 1 - x + \frac{x^2}{2}, \end{aligned}$$

or

$$1 - x + \frac{x^2}{2}\left[1 - \frac{16x - 4x^2}{48 - 3x^2}\right] < e^{-x} < 1 - x + \frac{x^2}{2}.$$

For values of x sufficiently close to 0, $\frac{16x - 4x^2}{48 - 3x^2}$ approaches 0, and

$$1 - x + \frac{x^2}{2}\left[1 - \frac{16x - 4x^2}{48 - 3x^2}\right] \approx 1 - x + \frac{x^2}{2}.$$

Thus, $e^{-x} \approx 1 - x + \frac{x^2}{2}$.
For $x < 0$, a similar argument can be made.

24. The area is given by

$$\int_0^{\frac{1}{2}} e^{x^2}\,dx.$$

Find the Taylor series for e^{x^2} by using the Taylor series for e^x and replacing x with x^2.

$$e^{x^2} = 1 + \left(x^2\right) + \frac{1}{2!}\left(x^2\right)^2 + \frac{1}{3!}\left(x^2\right)^3 + \frac{1}{4!}\left(x^2\right)^4 + \cdots + \frac{1}{n!}\left(x^2\right)^n + \cdots$$

Using the first five terms of this series gives

$$\begin{aligned}\int_0^{\frac{1}{2}} e^{x^2}\,dx &\approx \int_0^{\frac{1}{2}} \left[1 + x^2 + \frac{1}{2!}\left(x^2\right)^2 + \frac{1}{3!}\left(x^2\right)^3 + \frac{1}{4!}\left(x^2\right)^4\right] dx \\ &= \int_0^{\frac{1}{2}} \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8\right) dx \\ &= \left(x + \frac{1}{3}x^3 + \frac{1}{10}x^5 + \frac{1}{42}x^7 + \frac{1}{216}x^9\right)\Bigg|_0^{\frac{1}{2}} \\ &= \frac{1}{2} + \frac{1}{24} + \frac{1}{320} + \frac{1}{5376} + \frac{1}{110{,}592} - 0 \\ &\approx .5450\end{aligned}$$

26. The area is given by

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{1-\sqrt{x}}\,dx.$$

Find the Taylor series for $\frac{1}{1-\sqrt{x}}$ by using the Taylor series for $\frac{1}{1-x}$ and replacing x with $\sqrt{x}$.

$$\frac{1}{1-\sqrt{x}} = 1 + \sqrt{x} + \sqrt{x}^2 + \sqrt{x}^3 + \cdots + \sqrt{x}^n + \cdots$$

Using the first five terms of this series gives

$$\begin{aligned}\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{1-\sqrt{x}}\,dx \approx \int_{\frac{1}{4}}^{\frac{1}{2}} &\left(1 + \sqrt{x} + x + x^{3/2} + x^2\right) dx \\ &= \left(x + \frac{2}{3}x^{3/2} + \frac{x^2}{2} + \frac{2}{5}x^{5/2} + \frac{x^3}{3}\right)\Bigg|_{\frac{1}{4}}^{\frac{1}{2}} \\ &= \left(\frac{1}{2} + \frac{2}{3}\left(\frac{1}{2}\right)^{3/2} + \frac{1}{8} + \frac{2}{5}\left(\frac{1}{2}\right)^{5/2} + \frac{1}{24}\right) - \left(\frac{1}{4} + \frac{1}{12} + \frac{1}{32} + \frac{1}{80} + \frac{1}{192}\right) \\ &\approx .5908\end{aligned}$$

28. The area is given by

$$\int_0^{.5} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_0^{.5} e^{-x^2/2} dx.$$

Find the Taylor series for $e^{-x^2/2}$ by using the Taylor series for e^x and replacing x with $-\frac{x^2}{2}$.

$$e^{-x^2/2} = 1 - \frac{1}{2}x^2 + \frac{1}{2!2^2}x^4 - \frac{1}{3!2^3}x^6 + \frac{1}{4!2^4}x^8 + \cdots + \frac{(-1)^n}{n!2^n}x^{2n} + \cdots$$

Using the first five terms of this series gives

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_0^{.5} e^{-x^2/2}dx &\approx \frac{1}{\sqrt{2\pi}}\int_0^{.5}\left(1-\frac{1}{2}x^2+\frac{1}{2!2^2}x^4-\frac{1}{3!2^3}x^6+\frac{1}{4!2^4}x^8\right)dx\\
&= \frac{1}{\sqrt{2\pi}}\int_0^{.5}\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4-\frac{1}{48}x^6+\frac{1}{384}x^8\right)dx\\
&= \frac{1}{\sqrt{2\pi}}\left(x-\frac{1}{6}x^3+\frac{1}{40}x^5-\frac{1}{336}x^7+\frac{1}{3456}x^9\right)\Bigg|_0^{.5}\\
&= \frac{1}{\sqrt{2\pi}}\left(.5-\frac{1}{6}(.5)^3+\frac{1}{40}(.5)^5-\frac{1}{336}(.5)^7+\frac{1}{3456}(.5)^9-0\right)\\
&\approx \frac{1}{\sqrt{2\,(3.1416)}}(.479925)\\
&\approx .1915.
\end{aligned}$$

30. The doubling time n for a quantity that increases at an annual rate r is given by

$$n=\frac{\ln 2}{\ln(1+r)}=\frac{\ln 2}{\ln .0425}\approx 16.47.$$

It will take about 16.47 years, or about 16 years 8 months.

According to the Rule of 70, the doubling time is given by

$$\text{Doubling time} \approx \frac{70}{100r}=\frac{70}{100(.0425)}=16.47.$$

It will take about 16.47 years, or about 16 years 6 months, a difference of 2 months.

32. **(a)** $\displaystyle\sum_{x=0}^{\infty} f(x)=\sum_{x=0}^{\infty}\frac{\lambda^x e^{-\lambda}}{x!}$

$$\begin{aligned}
&= \frac{\lambda^0 e^{-\lambda}}{0!}+\frac{\lambda^1 e^{-\lambda}}{1!}+\frac{\lambda^2 e^{-\lambda}}{2!}+\frac{\lambda^3 e^{-\lambda}}{3!}+\cdots+\frac{\lambda^n e^{-\lambda}}{n!}+\cdots\\
&= e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2 e^{-\lambda}}{2!}+\frac{\lambda^3 e^{-\lambda}}{3!}+\cdots+\frac{\lambda^n e^{-\lambda}}{n!}+\cdots\\
&= e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots+\frac{\lambda^n}{n!}+\cdots\right)\\
&= e^{-\lambda}\cdot e^{\lambda}\\
&= e^0\\
&= 1
\end{aligned}$$

(b)
$$\begin{aligned}\sum_{x=0}^{\infty} xf(x) &= \sum_{x=0}^{\infty} x\frac{\lambda^x e^{-\lambda}}{x!}\\ &= 0\frac{\lambda^0 e^{-\lambda}}{0!} + 1\frac{\lambda^1 e^{-\lambda}}{1!} + 2\frac{\lambda^2 e^{-\lambda}}{2!} + 3\frac{\lambda^3 e^{-\lambda}}{3!} + \cdots + n\frac{\lambda^n e^{-\lambda}}{n!} + \cdots\\ &= 0 + \lambda e^{-\lambda} + \lambda^2 e^{-\lambda} + \frac{\lambda^3 e^{-\lambda}}{2!} + \cdots + \frac{\lambda^n e^{-\lambda}}{(n-1)!} + \cdots\\ &= \lambda e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \cdots + \frac{\lambda^{n-1}}{(n-1)!} + \cdots\right)\\ &= \lambda e^{-\lambda}\cdot e^{\lambda}\\ &= \lambda e^0\\ &= \lambda\end{aligned}$$

(c) Since 7 is the expected value, from part b we have

$$7 = \sum_{x=0}^{\infty} xf(x) = \lambda.$$

For $x < 4$, find

$$\begin{aligned}\sum_{x=0}^{3} f(x) &= \sum_{x=0}^{3} \frac{\lambda^x e^{-\lambda}}{x!}\\ &= \sum_{x=0}^{3} \frac{7^x e^{-7}}{x!}\\ &= \frac{7^0 e^{-7}}{0!} + \frac{7^1 e^{-7}}{1!} + \frac{7^2 e^{-7}}{2!} + \frac{7^3 e^{-7}}{3!}\\ &= e^{-7} + 7e^{-7} + \frac{49}{2}e^{-7} + \frac{343}{6}e^{-7}\\ &\approx .08177\end{aligned}$$

34. (a) The probability, p, of popping a 6 is $p = \frac{1}{6}$.
From Exercise 31b, the expected value is given by

$$\sum_{x=1}^{\infty} xf(x) = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6.$$

(b) For $x \geq 4$, find

$$\begin{aligned}\sum_{x=4}^{\infty} f(x) &= 1 - \sum_{x=1}^{3} f(x)\\ &= 1 - \sum_{x=1}^{3} (1-p)^{x-1} p\\ &= 1 - \sum_{x=1}^{3} \left(1 - \frac{1}{6}\right)^{x-1}\left(\frac{1}{6}\right)\\ &= 1 - \left[\left(1 - \frac{1}{6}\right)^{1-1}\left(\frac{1}{6}\right) + \left(1 - \frac{1}{6}\right)^{2-1}\left(\frac{1}{6}\right) + \left(1 - \frac{1}{6}\right)^{3-1}\left(\frac{1}{6}\right)\right]\\ &\approx 1 - .42\\ &= .58.\end{aligned}$$

12.6 Newton's Method

2. $f(x) = x^2 - 6x + 4$
$f'(x) = 2x - 6$

$f(5) = -1 < 0$ and $f(6) = 4 > 0$ so a solution exists in $(5, 6)$.
Let $c_1 = 5$.

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \\ &= 5 - \frac{-1}{4} = 5.25 \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \\ &= 5.25 - \frac{.0625}{4.5} \\ &= 5.2361 \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \\ &= 5.2361 - \frac{1.4 \cdot 10^{-4}}{4.4722} \\ &= 5.2361 \end{aligned}$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 5.24$.

4. $f(x) = 5x^3 - 2x^2 - 2x - 7$
$f'(x) = 15x^2 - 4x - 2$

$f(1) = -6 < 0$ and $f(2) = 21 > 0$ so a solution exists in $(1, 2)$.
Let $c_1 = 1$.

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{-6}{9} = 1.6667 \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} = 1.6667 - \frac{7.2604}{33.002} = 1.4467 \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} = 1.4467 - \frac{1.06}{23.607} = 1.4018 \\ c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} = 1.4018 - \frac{.0393}{21.868} = 1.4000 \end{aligned}$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = 1.40$.

6. $f(x) = 3x^3 - 14x^2 + 17x - 22$
$f'(x) = 9x^2 - 28x + 17$

$f(3) = -16 < 0$ and $f(4) = 14 > 0$ so a solution exists in $(3, 4)$.
Let $c_1 = 3$.

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} = 3 - \frac{-16}{14} \\ &= 4.1429 \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} = 4.1429 - \frac{21.46}{55.471} \\ &= 3.7560 \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} = 3.7560 - \frac{3.3102}{38.8} \\ &= 3.6707 \\ c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} = 3.6707 - \frac{.14282}{35.487} \\ &= 3.6667 \end{aligned}$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = 3.67$.

8. $f(x) = 2x^4 + 7x^3 + 6x^2 + 7x - 6$
$f'(x) = 8x^3 + 21x^2 + 12x + 7$

$f(-3) = 0$ so $x = -3$ is an exact solution.
$f(0) = -6 < 0$ and $f(1) = 16 > 0$ so a solution exists in $(0, 1)$.

Let $c_1 = 0$.

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} = 0 - \frac{-6}{7} = .8571 \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} = .8571 - \frac{9.8943}{37.749} = .5950 \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} = .5950 - \frac{2.0143}{23.26} = .5084 \\ c_5 &= c_4 - \frac{f(c_4)}{f'(c_4)} = .5084 - \frac{.16308}{19.58} = .5001 \\ c_6 &= c_5 - \frac{f(c_5)}{f'(c_5)} = .5001 - \frac{.00193}{19.254} = .5000 \end{aligned}$$

Subsequent approximations will agree with c_5 and c_6 to the nearest hundredth. Thus, $x = -3$ is a solution in $[-3, -2]$ and $x = .50$ is a solution in $[0, 1]$.

10. $f(x) = x^{1/3} - x^2 + 3$

$f'(x) = \frac{1}{3}x^{-2/3} - 2x$

$f(0) = 3 > 0$ and $f(3) = -4.55775 < 0$ so a solution exists in $(0,3)$.

Let $c_1 = 0$.

$f'(0)$ is undefined, and $c_2 = c_1 = 0$, so let $c_1 = 1$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{3}{-1.667} = 2.7996$$
$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 2.7996 - \frac{-3.428}{-5.431} = 2.1684$$
$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 2.1684 - \frac{-.4076}{-4.138} = 2.0699$$
$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 2.0699 - \frac{-.0101}{-3.935} = 2.0673$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = 2.07$.

12. $f(x) = e^{2x} + 3x - 4$

$f'(x) = 2e^{2x} + 3$

$f(0) = -3 < 0$ and $f(3) = 408.42879 > 0$ so a solution exists in $(0,3)$.

Let $c_1 = 0$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 0 - \frac{-3}{5} = .6$$
$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = .6 - \frac{1.1201}{9.6402} = .4838$$
$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = .4838 - \frac{.08302}{8.2632} = .4738$$
$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = .4738 - \frac{9.1 \cdot 10^{-4}}{8.159} = .4737$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = .47$.

14. $f(x) = x^2e^{-x} + x^2 - 2$

$f'(x) = -x^2e^{-x} + 2xe^{-x} + 2x$

$f(-3) = 187.76983 > 0$ and $f(0) = -2 < 0$ so a solution exists in $(-3,0)$.

Let $c_1 = -3$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = -3 - \frac{187.77}{-307.3} = -2.3890$$
$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = -2.3890 - \frac{65.932}{-119.1} = -1.8354$$
$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = -1.8354 - \frac{22.482}{-47.79} = -1.3650$$
$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = -1.3650 - \frac{7.1591}{-20.72} = -1.0195$$
$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = -1.0195 - \frac{1.9203}{-10.57} = -.8378$$
$$c_7 = c_6 - \frac{f(c_6)}{f'(c_6)} = -.8378 - \frac{.32421}{-7.171} = -.7926$$
$$c_8 = c_7 - \frac{f(c_7)}{f'(c_7)} = -.7926 - \frac{.01602}{-6.475} = -.7901$$

Subsequent approximations will agree with c_7 and c_8 to the nearest hundredth. Thus, $x = -.79$.

16. $f(x) = 2\ln x + x - 3$

$f'(x) = \frac{2}{x} + 1$

$f(1) = -2 < 0$ and $f(4) = 3.7725887 > 0$ so a solution exists in $(1,4)$.

Let $c_1 = 1$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{-2}{3} = 1.6667$$
$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 1.6667 - \frac{-.3116}{2.2} = 1.8083$$
$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 1.8083 - \frac{-.0069}{2.106} = 1.8116$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 1.81$.

18. $\sqrt{3}$ is a solution of $x^2 - 3 = 0$.

$$f(x) = x^2 - 3$$
$$f'(x) = 2x$$

Since $1 < \sqrt{3} < 2$, let $c_1 = 1$.

$$c_2 = 1 - \frac{-2}{2} = 2$$
$$c_3 = 2 - \frac{1}{4} = 1.75$$
$$c_4 = 1.75 - \frac{.0625}{3.5} = 1.732$$
$$c_5 = 1.732 - \frac{-2 \cdot 10^{-4}}{3.464} = 1.732$$

Since $c_4 = c_5 = 1.732$, to the nearest thousandth, $\sqrt{3} = 1.732$.

20. $\sqrt{15}$ is a solution of $x^2 - 15 = 0$.

$$f(x) = x^2 - 15$$
$$f'(x) = 2x$$

Since $3 < \sqrt{15} < 4$, let $c_1 = 3$.

$$c_2 = 3 - \frac{-6}{6} = 4$$
$$c_3 = 4 - \frac{1}{8} = 3.875$$
$$c_4 = 3.875 - \frac{.01563}{7.75} = 3.873$$
$$c_5 = 3.873 - \frac{1.3 \cdot 10^{-4}}{7.746} = 3.873$$

Since $c_4 = c_5 = 3.873$, to the nearest thousandth, $\sqrt{15} = 3.873$.

22. $\sqrt{300}$ is a solution of $x^2 - 300 = 0$.

$$f(x) = x^2 - 300$$
$$f'(x) = 2x$$

Since $17 < \sqrt{300} < 18$, let $c_1 = 17$.

$$c_2 = 17 - \frac{-11}{34} = 17.324$$
$$c_3 = 17.324 - \frac{.12098}{34.648} = 17.321$$
$$c_4 = 17.321 - \frac{.01704}{34.642} = 17.321$$

Since $c_3 = c_4 = 17.321$, to the nearest thousandth, $\sqrt{300} = 17.321$.

24. $\sqrt[3]{15}$ is a solution of $x^3 - 15 = 0$.

$$f(x) = x^3 - 15$$
$$f'(x) = 3x^2$$

Since $2 < \sqrt[3]{15} < 3$, let $c_1 = 2$.

$$c_2 = 2 - \frac{-7}{12} = 2.583$$
$$c_3 = 2.583 - \frac{2.2335}{20.016} = 2.471$$
$$c_4 = 2.471 - \frac{.08753}{18.318} = 2.466$$
$$c_5 = 2.466 - \frac{-.0039}{18.243} = 2.466$$

Since $c_4 = c_5 = 2.466$, to the nearest thousandth, $\sqrt[3]{15} = 2.466$.

26. $\sqrt[3]{121}$ is a solution of $x^3 - 121 = 0$.

$$f(x) = x^3 - 121$$
$$f'(x) = 3x^2$$

Since $4 < \sqrt[3]{121} < 5$, let $c_1 = 4$.

$$c_2 = 4 - \frac{-57}{48} = 5.188$$
$$c_3 = 5.188 - \frac{18.637}{80.746} = 4.957$$
$$c_4 = 4.957 - \frac{.80266}{73.716} = 4.946$$
$$c_5 = 4.946 - \frac{-.0064}{73.389} = 4.946$$

Since $c_4 = c_5 = 4.946$, to the nearest thousandth, $\sqrt[3]{121} = 4.946$.

28. $f(x) = x^3 + 9x^2 - 6x + 4$
$f'(x) = 3x^2 + 18x - 6$

To find critical points, solve $f'(x) = 3x^2 + 18x - 6 = 0$.

$f''(x) = 6x + 18$

$f'(-7) = 15 > 0$ and $f'(-6) = -6 < 0$ so a solution exists in $(-7, -6)$.

Let $c_1 = -7$.

$$c_2 = -7 - \frac{15}{-24} = -6.38$$
$$c_3 = -6.38 - \frac{1.2732}{-20.28} = -6.32$$
$$c_4 = -6.32 - \frac{.0672}{-19.92} = -6.32$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = -6.32$. Since $f''(-6.32) = -19.92 < 0$, the graph has a relative maximum at $x = -6.32$.

$f'(0) = -6 < 0$ and $f'(1) = 15 > 0$ so a solution exists in $(0, 1)$.

Let $c_1 = 0$.

$$c_2 = 0 - \frac{-6}{18} = .33$$
$$c_3 = .33 - \frac{.2667}{19.98} = .32$$
$$c_4 = .32 - \frac{.0672}{19.92} = .32$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = .32$. Since $f''(.32) = 19.92 > 0$, the graph has a relative minimum at $x = .32$.

30. $f(x) = x^4 + 2x^3 - 5x + 2$
$f'(x) = 4x^3 + 6x^2 - 5$

To find critical points, solve $f'(x) = 4x^3 + 6x^2 - 5 = 0$.

$f''(x) = 12x^2 + 12x$

$f'(0) = -5 < 0$ and $f'(1) = 5 > 0$ so a solution exists in $(0, 1)$.

Let $c_1 = 0$.

$$c_2 = 0 - \frac{-5}{0}$$

Since c_2 is undefined, let $c_1 = .5$.

$$c_2 = .5 - \frac{-3}{9} = .83$$
$$c_3 = .83 - \frac{1.4205}{18.227} = .75$$
$$c_4 = .75 - \frac{.0625}{15.75} = .75$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = .75$. Since $f''(.75) = 15.75 > 0$, the graph has a relative minimum at $x = .75$.

32. The process should be used at least until the savings produced is equal to the increased costs incurred, or when $S(x) = C(x)$. Therefore, solve $S(x) - C(x)$.

$$f(x) = S(x) - C(x) = \left(x^3 + 5x^2 + 9\right) - \left(x^2 + 40x + 20\right) = x^3 + 4x^2 - 40x - 11$$
$$f'(x) = 3x^2 + 8x - 40$$

$f'(x) = 3x^2 + 8x - 40$

$f(4) = -43 < 0$ and $f(5) = 14 > 0$ so a solution exists in $(4, 5)$.

Let $c_1 = 4$.

$$c_2 = 4 - \frac{-43}{40} = 5.08$$
$$c_3 = 5.08 - \frac{20.122}{78.059} = 4.82$$
$$c_4 = 4.82 - \frac{1.1098}{68.257} = 4.80$$
$$c_5 = 4.80 - \frac{-.248}{67.52} = 4.80$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = 4.80$. The process should be used for at least 4.80 years.

34.
$$\begin{aligned} i_2 &= .02 - \frac{57\,(.02) - 57\,(.02)\,(1+.02)^{-12} - 600\,(.02)^2}{57\left[-1 + (12)\,(.02)\,(1+.02)^{-12-1} + (1+.02)^{-12}\right]} \\ &= .02 - \frac{.0011177798}{-1.480804049} \\ &= .02075485 \end{aligned}$$

$$\begin{aligned} i_3 &= .02075485 - \frac{57\,(.02075485) - 57\,(.02075485)\,(1+.02075485)^{-12} - 600\,(.02075485)^2}{57\left[-1 + (12)\,(.02075485)\,(1+.02075485)^{-12-1} + (1+.02075485)^{-12}\right]} \\ &= .02075485 - \frac{4.0638916 \cdot 10^{-6}}{-1.583924743} \\ &= .02075742 \end{aligned}$$

12.7 L'Hospital's Rule

2. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 3} \frac{3x^2 + 2x - 11}{2x - 3} = \frac{3\,(3)^2 + 2\,(3) - 11}{2\,(3) - 3} = \frac{22}{3}.$$

By l'Hospital's rule,

$$\lim_{x\to 3} \frac{x^3 + x^2 - 11x - 3}{x^2 - 3x} = \frac{22}{3}.$$

4. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 0} \frac{48x^5 + 12x^3 - 9}{63x^6 - 8x^3 + 3x^2} \text{ which does not exist.}$$

By l'Hospital's rule,

$$\lim_{x\to 0} \frac{8x^6 + 3x^4 - 9x}{9x^7 - 2x^4 + x^3} \text{ does not exist.}$$

6. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 0} \frac{\frac{1}{x+1}}{1} = \frac{1}{0+1} = 1.$$

By l'Hospital's rule,

$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1.$$

8. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 0} \frac{2e^{2x}}{10x - 1} = \frac{2e^{2(0)}}{10\,(0) - 1} = -2.$$

By l'Hospital's rule,

$$\lim_{x\to 0} \frac{e^{2x} - 1}{5x^2 - x} = -2.$$

10. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 0} \frac{e^{-x} - xe^{-x}}{4e^{2x}} = \frac{e^0 - 0 \cdot e^0}{4e^{2(0)}} = \frac{1}{4}.$$

By l'Hospital's rule,

$$\lim_{x\to 0} \frac{xe^{-x}}{2e^{2x} - 2} = \frac{1}{4}.$$

12. $\lim\limits_{x\to 0} e^x = 1$ and l'Hospital's rule does not apply. However

$$\lim_{x\to 0} \frac{e^x}{8x^5 - 3x^4} \text{ does not exist.}$$

14. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 0} \frac{\frac{1}{2(9+x)^{1/2}}}{1} = \frac{1}{2\,(9+0)^{1/2}} = \frac{1}{6}.$$

By l'Hospital's rule,

$$\lim_{x\to 0} \frac{\sqrt{9+x} - 3}{x} = \frac{1}{6}.$$

16. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 9} \frac{\frac{1}{2x^{1/2}}}{1} = \frac{1}{2 \cdot 9^{1/2}} = \frac{1}{6}.$$

By l'Hospital's rule,

$$\lim_{x\to 9} \frac{\sqrt{x} - 3}{x - 9} = \frac{1}{6}.$$

18. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 27} \frac{\frac{1}{3x^{2/3}}}{1} = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}.$$

By l'Hospital's rule,

$$\lim_{x\to 27} \frac{\sqrt[3]{x} - 3}{x - 27} = \frac{1}{27}.$$

20. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 2} \frac{7x^6}{1} = 7 \cdot 2^6 = 448.$$

By l'Hospital's rule,

$$\lim_{x\to 2} \frac{x^7 - 128}{x - 2} = 448.$$

22. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 0} \frac{e^x + 1}{-e^{-x} - 1} = \frac{e^0 + 1}{-e^0 - 1} = \frac{2}{-2} - 1.$$

By l'Hospital's rule,

$$\lim_{x\to 0} \frac{e^x - 1 + x}{e^{-x} - 1 - x} = -1.$$

24. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 5} \frac{\frac{x}{(x^2+11)^{1/2}}}{2x} = \lim_{x\to 5} \frac{1}{2\left(x^2+11\right)^{1/2}} = \frac{1}{2\left(5^2+11\right)^{1/2}} = \frac{1}{12}.$$

By l'Hospital's rule,

$$\lim_{x\to 5} \frac{\sqrt{x^2+11}-6}{x^2-25} = \frac{1}{12}.$$

26. $\lim_{x\to 2} 2x^2 - 10x + 8 = -4$ and l'Hospital's rule does not apply. However, by substitution

$$\lim_{x\to 2} \frac{2-\sqrt{x+2}}{2x^2-10x+18} = \frac{2-\sqrt{2+2}}{2\left(2\right)^2-10\left(2\right)+9} = \frac{0}{-4} = 0$$

28. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 0} \frac{\frac{-1}{2(3-x)^{1/2}} - \frac{1}{2(3+x)^{1/2}}}{1} = \lim_{x\to 0} \left(\frac{-1}{2\left(3-x\right)^{1/2}} - \frac{1}{2\left(3+x\right)^{1/2}} \right) = \frac{-1}{2\cdot 3^{1/2}} - \frac{1}{2\cdot 3^{1/2}} = -\frac{1}{\sqrt{3}} \text{ or } -\frac{\sqrt{3}}{3}.$$

By l'Hospital's rule,

$$\lim_{x\to 0} \frac{\sqrt{3-x}-\sqrt{3+x}}{x} = \frac{1}{\sqrt{3}} \text{ or } -\frac{\sqrt{3}}{3}.$$

30. $\lim_{x\to 1} \sqrt{x^2+5x+9} = \sqrt{15}$ and l'Hospital's rule does not apply. However

$$\lim_{x\to 1} \frac{\sqrt{x^2+5x+9}}{x-1} \text{ does not exist.}$$

32. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x\to 0} \frac{-\ln\left(1-x\right) - \frac{7-x}{1-x}}{-e^{-x}} = \lim_{x\to 0} \frac{-\left(1-x\right)\ln\left(1-x\right) - 7 + x}{-e^{-x}\left(1-x\right)} = \frac{-\left(1-0\right)\ln\left(1-0\right) - 7 + 0}{-e^{0}\left(1-0\right)} = 7$$

By l'Hospital's rule,

$$\lim_{x\to 0} \frac{\left(7-x\right)\ln\left(1-x\right)}{e^{-x}-1} = 7.$$

34. $\lim_{x\to 0} \left(\frac{1}{x^2} + \frac{1}{x^3} \right) = \lim_{x\to 0} \frac{x+1}{x^3}$ which does not exist.

36. $\lim_{x\to 0} \left(\frac{1}{\sqrt[5]{x}} - \frac{1}{x} \right) = \lim_{x\to 0} \frac{x^{4/5}-1}{x}$ which does not exist.

Chapter 12 Review Problems

2.

Derivative	Value at 0
$f(x) = 5e^{2x}$	$f(0) = 5$
$f^{(1)}(x) = 10e^{2x}$	$f^{(1)}(0) = 10$
$f^{(2)}(x) = 20e^{2x}$	$f^{(2)}(0) = 20$
$f^{(3)}(x) = 40e^{2x}$	$f^{(3)}(0) = 40$
$f^{(4)}(x) = 80e^{2x}$	$f^{(4)}(0) = 80$

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 5 + \frac{10}{1!}x + \frac{20}{2}x^2 + \frac{40}{6}x^3 + \frac{80}{24}x^4 = 5 + 10x + 10x^2 + \frac{20}{3}x^3 + \frac{10}{3}x^4 \end{aligned}$$

4.

Derivative	Value at 0
$f(x) = \sqrt[3]{x+27} = (x+27)^{1/3}$	$f(0) = 3$
$f^{(1)}(x) = \frac{1}{3}(x+27)^{-2/3} = \frac{1}{3(x+27)^{2/3}}$	$f^{(1)}(0) = \frac{1}{27}$
$f^{(2)}(x) = -\frac{2}{9}(x+27)^{-5/3} = -\frac{2}{9(x+27)^{5/3}}$	$f^{(2)}(0) = -\frac{2}{2187}$
$f^{(3)}(x) = \frac{10}{27}(x+27)^{-8/3} = \frac{10}{27(x+27)^{8/3}}$	$f^{(3)}(0) = \frac{10}{177{,}147}$
$f^{(4)}(x) = -\frac{80}{81}(x+27)^{-11/3} = -\frac{80}{81(x+27)^{11/3}}$	$f^{(4)}(0) = -\frac{80}{14{,}348{,}907}$

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 3 + \frac{\frac{1}{27}}{1}x + \frac{-\frac{2}{2187}}{2}x^2 + \frac{\frac{10}{177{,}147}}{6}x^3 + \frac{-\frac{80}{14{,}348{,}907}}{24}x^4 \\ &= 3 + \frac{1}{27}x - \frac{1}{2187}x^2 + \frac{5}{531{,}441}x^3 - \frac{10}{43{,}046{,}721}x^4 \end{aligned}$$

6.

Derivative	Value at 0
$f(x) = \ln(3+2x)$	$f(0) = \ln 3$
$f^{(1)}(x) = \frac{2}{3+2x} = 2(3+2x)^{-1}$	$f^{(1)}(0) = \frac{2}{3}$
$f^{(2)}(x) = -4(3+2x)^{-2} = -\frac{4}{(3+2x)^2}$	$f^{(2)}(0) = -\frac{4}{9}$
$f^{(3)}(x) = 16(3+2x)^{-3} = \frac{16}{(3+2x)^3}$	$f^{(3)}(0) = \frac{16}{27}$
$f^{(4)}(x) = -96(3+2x)^{-4} = -\frac{96}{(3+2x)^4}$	$f^{(4)}(0) = -\frac{32}{27}$

$$P_4(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 = \ln 3 + \frac{\frac{2}{3}}{1}x + \frac{-\frac{4}{9}}{2}x^2 + \frac{\frac{16}{27}}{6}x^3 + \frac{-\frac{32}{27}}{24}x^4$$
$$= \ln 3 + \frac{2}{3}x - \frac{2}{9}x^2 + \frac{8}{81}x^3 - \frac{4}{81}x^4$$

8. Using the result of Exercise 1, with $f(x) = e^{2-x}$ and $P_4(x) = e^2 - e^2 x + \frac{e^2}{2}x^2 - \frac{e^2}{6}x^3 + \frac{e^2}{24}x^4$, we can approximate $e^{1.99}$ by evaluating $f(.01) = e^{2-.01} = e^{2-.01} = e^{1.99}$. Using $P_4(x)$ from Exercise 1 with $x = .01$ gives

$$P_4(.01) = e^2 - e^2(.01) + \frac{e^2}{2}(.01)^2 - \frac{e^2}{6}(.01)^3 + \frac{e^2}{24}(.01)^4$$
$$\approx 7.315533762$$

To four decimal places, $P_4(.01)$ approximates the value of $e^{1.99}$ as 7.3155.

10. Using the result of Exercise 3, with $f(x) = \sqrt{x+1}$ and $P_4(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$, we can approximate $\sqrt{.98}$ by evaluating $f(-.02) = \sqrt{-.02+1} = \sqrt{.98}$. Using $P_4(x)$ from Exercise 3 with $x = -.02$ gives

$$P_4(-.02) = 1 + \frac{1}{2}(-.02) - \frac{1}{8}(-.02)^2 + \frac{1}{16}(-.02)^3 - \frac{5}{128}(-.02)^4$$
$$\approx .9899494938.$$

To four decimal places, $P_4(-.02)$ approximates the value of $\sqrt{.98}$ as .9899.

12. Using the result of Exercise 5, with $f(x) = \ln(2-x)$ and $P_4(x) = \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3 - \frac{1}{64}x^4$, we can approximate $\ln 2.03$ by evaluating $f(-.03) = \ln(2-(-.03)) = \ln 2.03$. Using $P_4(x)$ from Exercise 5 with $x = -.03$ gives

$$P_4(-.03) = \ln 2 - \frac{1}{2}(-.03) - \frac{1}{8}(-.03)^2 - \frac{1}{24}(-.03)^3 - \frac{1}{64}(-.03)^4$$
$$\approx .7080386123 \text{ (using } \ln 2 = .69315)$$

To four decimal places, $P_4(-.03)$ approximates the value of $\ln 2.03$ as .7080.

14. Using the result of Exercise 7, with $f(x) = (1+x)^{2/3}$ and $P_4(x) = 1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 - \frac{7}{243}x^4$, we can approximate $(1.01)^{2/3}$ by evaluating $f(.01) = (1+.01)^{2/3} = (1.01)^{2/3}$. Using $P_4(x)$ from Exercise 7 with $x = .01$ gives

$$P_4(.01) = 1 + \frac{2}{3}(.01) - \frac{1}{9}(.01)^2 + \frac{4}{81}(.01)^3 - \frac{7}{243}(.01)^4$$
$$\approx 1.006655605$$

To four decimal places, $P_4(.01)$ approximates the value of $(1.01)^{2/3}$ as 1.0067.

16. $2 + 1.4 + .98 + .686 + \ldots$ is a geometric series with $a = a_1 = 2$ and $r = .7$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{2}{1-.7} = \frac{2}{.3} = \frac{20}{3}.$$

18. $4 + 4.8 + 5.76 + 6.912 + \ldots$ is a geometric series with $a = a_1 = 4$ and $r = 1.2$. Since $r > 1$, the series diverges.

20. $1 + \frac{1}{.99} + \frac{1}{(.99)^2} + \frac{1}{(.99)^3} + \ldots$ is a geometric series with $a = a_1 = 1$ and $r = \frac{1}{.99}$. Since $r > 1$, the series diverges.

22. $S_1 = a_1 = \dfrac{1}{(1+2)(1+3)} = \dfrac{1}{12}$

$S_2 = a_1 + a_2 = \dfrac{1}{(1+2)(1+3)} + \dfrac{1}{(2+2)(2+3)} = \dfrac{1}{12} + \dfrac{1}{20} = \dfrac{2}{15}$

$S_3 = a_1 + a_2 + a_3 = \dfrac{1}{(1+2)(1+3)} + \dfrac{1}{(2+2)(2+3)} + \dfrac{1}{(3+2)(3+3)} = \dfrac{1}{12} + \dfrac{1}{20} + \dfrac{1}{30} = \dfrac{1}{6}$

$S_4 = a_1 + a_2 + a_3 + a_4 = \dfrac{1}{(1+2)(1+3)} + \dfrac{1}{(2+2)(2+3)} + \dfrac{1}{(3+2)(3+3)} + \dfrac{1}{(4+2)(4+3)}$

$= \dfrac{1}{12} + \dfrac{1}{20} + \dfrac{1}{30} + \dfrac{1}{42} = \dfrac{4}{21}$

$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$

$= \dfrac{1}{(1+2)(1+3)} + \dfrac{1}{(2+2)(2+3)} + \dfrac{1}{(3+2)(3+3)} + \dfrac{1}{(4+2)(4+3)} + \dfrac{1}{(5+2)(5+3)}$

$= \dfrac{1}{12} + \dfrac{1}{20} + \dfrac{1}{30} + \dfrac{1}{42} + \dfrac{1}{56} = \dfrac{5}{24}$

24. $\dfrac{2x}{1+3x} = x \cdot \dfrac{2}{1-(-3x)}$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by 2, and replace x with $-3x$. Also, use property (3) with $k = 1$.

$$\frac{2x}{1+3x} = x \cdot \frac{2}{1-(-3x)} = x \cdot 2 \cdot 1 + x \cdot 2(-3x) + x \cdot 2(-3x)^2 + x \cdot 2(-3x)^3 + \cdots + x \cdot 2(-3x)^n + \cdots$$

$$= 2x - 6x^2 + 18x^3 - 54x^4 + \cdots + (-1)^n \cdot 2 \cdot 3^n \cdot x^{n+1} + \cdots$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $-3x$ gives

$$-1 < -3x < 1 \text{ or } \frac{1}{3} > x > -\frac{1}{3}.$$

The interval of convergence of the new series is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

26. $\dfrac{3x^3}{2-x} = x^3 \cdot \dfrac{\frac{3}{2}}{1-\frac{x}{2}}$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by $\frac{3}{2}$, and replace x with $\frac{x}{2}$. Also, use property (3) with $k = 3$.

$$\frac{3x^3}{2-x} = x^3 \cdot \frac{\frac{3}{2}}{1-\frac{x}{2}} = x^3 \cdot \frac{3}{2} \cdot 1 + x^3 \cdot \frac{3}{2}\left(\frac{x}{2}\right) + x^3 \cdot \frac{3}{2}\left(\frac{x}{2}\right)^2 + x^3 \cdot \frac{3}{2}\left(\frac{x}{2}\right)^3 + \cdots + x^3 \cdot \frac{3}{2}\left(\frac{x}{2}\right)^n + \cdots$$

$$= \frac{3x^3}{2} + \frac{3x^4}{4} + \frac{3x^5}{8} + \frac{3x^6}{16} + \cdots + \frac{3x^{n+3}}{2^{n+1}} + \cdots$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $\frac{x}{2}$ gives

$$-1 < \frac{x}{2} < 1 \text{ or } -2 < x < 2.$$

The interval of convergence of the new series is $(-2, 2)$.

28. We find the Taylor series for $\ln\left(1+\frac{1}{3}x\right)$ by starting with the Taylor series for $\ln(1+x)$ and replacing each x with $\frac{1}{3}x$.

$$\begin{aligned}\ln\left(1+\frac{1}{3}x\right) &= \frac{1}{3}x-\frac{\left(\frac{1}{3}x\right)^2}{2}+\frac{\left(\frac{1}{3}x\right)^3}{3}-\frac{\left(\frac{1}{3}x\right)^4}{4}+\cdots \\ &\quad +\frac{(-1)^n\left(\frac{1}{3}x\right)^{n+1}}{n+1}+\cdots \\ &= \frac{1}{3}x-\frac{1}{18}x^2+\frac{1}{81}x^3-\frac{1}{324}x^4+\cdots \\ &\quad +\frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}+\cdots\end{aligned}$$

The Taylor series for $\ln(1+x)$ is valid when $-1<x\le 1$. Replacing x with $\frac{1}{3}x$ gives

$$-1<\frac{1}{3}x\le 1 \quad\text{or}\quad -3<x\le 3.$$

The interval of convergence of the new series is $(-3,3]$.

30. We find the Taylor series for e^{-5x} by starting with the Taylor series for e^x and replacing each x with $-5x$.

$$\begin{aligned}e^{-5x} &= 1+(-5x)+\frac{1}{2!}(-5x)^2+\frac{1}{3!}(-5x)^3+\cdots \\ &\quad +\frac{1}{n!}(-5x)^n+\cdots \\ &= 1-5x+\frac{25}{2}x^2-\frac{125}{6}x^3+\cdots \\ &\quad +\frac{(-1)^n 5^n x^n}{n!}+\cdots\end{aligned}$$

The Taylor series for e^{-5x} has the same interval of convergence, $(-\infty,\infty)$, as the Taylor series for e^x.

32. Use the Taylor series for e^x and replace x with $-x$. Also, use property (3) with $k=6$.

$$\begin{aligned}x^6e^{-x} &= x^6\cdot 1+x^6\cdot(-x)+x^6\cdot\frac{1}{2!}(-x)^2+ \\ &\quad x^6\cdot\frac{1}{3!}(-x)^3+\cdots+x^6\cdot\frac{1}{n!}(-x)^n+\cdots \\ &= x^6-x^7+\frac{x^8}{2}-\frac{x^9}{6}+\cdots+\frac{(-1)^n x^{n+6}}{n!}+\cdots\end{aligned}$$

The Taylor series for x^6e^{-x} has the same interval of convergence, $(-\infty,\infty)$, as the Taylor series for e^x.

34. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x\to 0}\frac{3x^2-8x+6}{3}=\frac{3(0)^2-8(0)+6}{3}=2.$$

By l'Hospital's rule,

$$\lim_{x\to 0}\frac{x^3-4x^2+6x}{3x}=2.$$

36. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x\to 0}\frac{\frac{3}{3x+1}}{1}=\frac{3}{3(0)+1}=3.$$

By l'Hospital's rule,

$$\lim_{x\to 0}\frac{\ln(3x+1)}{x}=3.$$

38. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x\to 0}\frac{\frac{1}{2(5+x)^{1/2}}}{1}=\frac{1}{2(5+0)^{1/2}}=\frac{1}{2\sqrt{5}}\text{ or }\frac{\sqrt{5}}{10}.$$

By l'Hospital's rule,

$$\lim_{x\to 0}\frac{\sqrt{5+x}-\sqrt{5}}{x}=\frac{1}{2\sqrt{5}}\text{ or }\frac{\sqrt{5}}{10}.$$

40. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x\to 16}\frac{\frac{1}{2x^{1/2}}}{1}=\frac{1}{2(16)^{1/2}}=\frac{1}{8}.$$

By l'Hospital's rule,

$$\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16}=\frac{1}{8}.$$

42. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\begin{aligned}
&\lim_{x\to 0}\frac{\frac{1}{2(5+x)^{1/2}}+\frac{1}{2(5-x)^{1/2}}}{2}\\
&=\frac{1}{2}\lim_{x\to 0}\frac{1}{2(5+x)^{1/2}}+\frac{1}{2(5-x)^{1/2}}\\
&=\frac{1}{2}\left[\frac{1}{2(5+0)^{1/2}}+\frac{1}{2(5-0)^{1/2}}\right]\\
&=\frac{1}{2\sqrt{5}}\text{ or }\frac{\sqrt{5}}{10}
\end{aligned}$$

By l'Hospital's rule,

$$\lim_{x\to 0}\frac{\sqrt{5+x}-\sqrt{5-x}}{2x}=\frac{1}{2\sqrt{5}}\text{ or }\frac{\sqrt{5}}{10}.$$

44. $\lim_{x\to 0}\left(\frac{1}{x}+\frac{1}{x^2}\right)=\lim_{x\to 0}\frac{x+1}{x^2}$ which does not exist.

46. $\lim_{x\to 0}\left(\frac{1}{\sqrt[3]{x}}+\frac{1}{\sqrt{x}}\right)=\lim_{x\to 0}\frac{x^{2/3}+x^{1/2}}{x}$

The limit in the numerator is 0, as is the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numrator and denominator gives,

$$\lim_{x\to 0}\frac{\frac{2}{3}x^{-1/3}+\frac{1}{2}x^{-1/2}}{1}=\lim_{x\to 0}\left(\frac{2}{3\sqrt[3]{x}}+\frac{1}{2\sqrt{x}}\right)$$

which does not exist. Therefore, by l'Hospital's rule,

$$\lim_{x\to 0}\left(\frac{1}{\sqrt[3]{x}}+\frac{1}{\sqrt{x}}\right)$$

does not exist.

48. $f(x)=3x^3-4x^2-4x-7$
$f'(x)=9x^2-8x-4$

$f(2)=-7<0$ and $f(3)=26>0$ so a solution exists in $(2,3)$.
Let $c_1=2$.

$$\begin{aligned}
c_2&=c_1-\frac{f(c_1)}{f'(c_1)}=2-\frac{-7}{16}=2.4375\\
c_3&=c_2-\frac{f(c_2)}{f'(c_2)}=2.4375-\frac{2.9309}{29.973}=2.3397\\
c_4&=c_3-\frac{f(c_3)}{f'(c_3)}=2.3397-\frac{.16835}{26.55}=2.3334\\
c_5&=c_4-\frac{f(c_4)}{f'(c_4)}=2.3334-\frac{.00176}{26.336}=2.3333
\end{aligned}$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x=2.33$.

50. $f(x)=x^4+x^3-14x^2-15x-15$
$f'(x)=4x^3+3x^2-28x-15$

$f(3)=-78<0$ and $f(4)=21>0$ so a solution exists in $(3,4)$.
Let $c_1=3$.

$$\begin{aligned}
c_2&=c_1-\frac{f(c_1)}{f'(c_1)}=3-\frac{-78}{36}=5.1667\\
c_3&=c_2-\frac{f(c_2)}{f'(c_2)}=5.1667-\frac{384.31}{472.11}=4.3527\\
c_4&=c_3-\frac{f(c_3)}{f'(c_3)}=4.3527-\frac{95.883}{249.83}=3.9689\\
c_5&=c_4-\frac{f(c_4)}{f'(c_4)}=3.9689-\frac{15.586}{171.202}=3.8779\\
c_6&=c_5-\frac{f(c_5)}{f'(c_5)}=3.8779-\frac{.75897}{154.8}=3.8730\\
c_7&=c_6-\frac{f(c_6)}{f'(c_6)}=3.8730-\frac{.00256}{153.94}=3.8730
\end{aligned}$$

Subsequent approximations will agree with c_6 and c_7 to the nearest hundredth. Thus, $x=3.87$.

52. $\sqrt{39.5}$ is a solution of $x^2-39.5=0$.

$$\begin{aligned}
f(x)&=x^2-39.5\\
f'(x)&=2x
\end{aligned}$$

Since $6<\sqrt{39.5}<7$, let $c_1=6$.

$$\begin{aligned}
c_2&=6-\frac{-3.5}{12}=6.292\\
c_3&=6.292-\frac{.08926}{12.584}=6.285\\
c_4&=6.285-\frac{.00123}{12.57}=6.285
\end{aligned}$$

Since $c_3=c_4=6.285$, to the nearest thousandth, $\sqrt{39.5}=6.285$.

54. $\sqrt[4]{70.9}$ is a solution of $x^4 - 70.9 = 0$.

$$\begin{aligned} f(x) &= x^4 - 70.9 \\ f'(x) &= 4x^3 \end{aligned}$$

Since $2 < \sqrt[4]{70.9} < 3$, let $c_1 = 2$.

$$\begin{aligned} c_2 &= 2 - \frac{-54.9}{32} = 3.716 \\ c_3 &= 3.716 - \frac{119.78}{205.25} = 3.132 \\ c_4 &= 3.132 - \frac{25.325}{122.89} = 2.926 \\ c_5 &= 2.926 - \frac{2.3989}{100.2} = 2.902 \\ c_6 &= 2.902 - \frac{.02341}{97.758} = 2.902 \end{aligned}$$

Since $c_5 = c_6 = 2.902$, to the nearest thousandth, $\sqrt[4]{70.9} = 2.902$.

56. This ordinary annuity will amount to \$5000 in 4 years at 8% compounded semiannually. Thus, $S = 5000$, $n = 4 \cdot 2 = 8$, and $i = \frac{8\%}{2} = 4\% = .04$, so

$$\begin{aligned} 5000 &= R \cdot s_{\overline{8}|.04} \\ R &= \frac{5000}{s_{\overline{8}|.04}} \\ &= \frac{5000}{9.21423} \\ &\approx 542.64 \end{aligned}$$

or \$542.64.

58. The payments form an ordinary annuity with $R = 1526.38$, $n = 5 \cdot 2 = 10$, and $i = \frac{7.6\%}{2} = 3.8\% = .038$. The amount of this annuity is

$$S = 1526.38 \left[\frac{(1.038)^{10} - 1}{.038} . \right]$$

The number in brackets, $s_{\overline{10}|.038}$, is 11.89534558, so that

$$S = 1526.38\,(11.89534558) = 18{,}156.82$$

or \$18,156.82.

60. \$49,275 is the present value of an annuity of R dollars, with 48 periods, and $i = \frac{12.2\%}{12} = 1.01\bar{6}\% = .0101\bar{6}$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 49{,}275 &= R \cdot a_{\overline{48}|.0101\bar{6}} \\ R &= \frac{49{,}275}{a_{\overline{48}|.0101\bar{6}}} \\ &= \frac{49{,}275}{37.83272858} \\ &\approx 1302.44 \end{aligned}$$

Each payment is \$1302.44.

62. The present value, P, is 177,110, $i = \frac{.0845}{12} = .00704\bar{16}$, and $n = 12 \cdot 30 = 360$.

$$\begin{aligned} 177{,}110 &= R \cdot a_{\overline{360}|.0070416} \\ &= R \left[\frac{1 - (1 + .007041\bar{6})^{-360}}{.007041\bar{6}} \right] \\ &= R \left[\frac{1 - .0799689882}{.007041\bar{6}} \right] \\ &= R \left[\frac{.9200310118}{.007041\bar{6}} \right] \\ R &\approx 1355.55 \end{aligned}$$

Monthly payments of \$1355.55 will be required to amortize the loan.

64. The doubling time n for a quantity that increases at an annual rate r is given by

$$n = \frac{\ln 2}{\ln(1 + r)} = \frac{\ln 2}{\ln 1.1} \approx 7.27.$$

It will take about 7.27 years, or about 7 years 3 months.

According to the Rule of 72, the doubling time is given by

$$\text{Doubling time } \approx \frac{72}{100r} = \frac{72}{100(.1)} = 7.2.$$

It will take about 7.2 years, or about 7 years 2 months, a difference of 1 month.

Chapter 13

THE TRIGONOMETRIC FUNCTIONS

13.1 Definitions of the Trigonometric Functions

2. $90° = 90\left(\frac{\pi}{180}\right) = \frac{\pi}{2}$

4. $135° = 135\left(\frac{\pi}{180}\right) = \frac{3\pi}{4}$

6. $300° = 300\left(\frac{\pi}{180}\right) = \frac{5\pi}{3}$

8. $480° = 480\left(\frac{\pi}{180}\right) = \frac{8\pi}{3}$

10. $\frac{2\pi}{3} = \frac{2\pi}{3}\left(\frac{180°}{\pi}\right) = 120°$

12. $-\frac{\pi}{4} = -\frac{\pi}{4}\left(\frac{180°}{\pi}\right) = -45°$

14. $\frac{7\pi}{10} = \frac{7\pi}{10}\left(\frac{180°}{\pi}\right) = 126°$

16. $5\pi = 5\pi\left(\frac{180°}{\pi}\right) = 900°$

18. Let α = the angle with terminal side through $(-12, -5)$. Then $x = -12$, $y = -5$, and

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{144 + 25} \\ &= \sqrt{169} = 13. \end{aligned}$$

$$\sin\alpha = \frac{y}{r} = -\frac{5}{13} \qquad \cot\alpha = \frac{x}{y} = \frac{12}{5}$$

$$\cos\alpha = \frac{x}{r} = -\frac{12}{13} \qquad \sec\alpha = \frac{r}{x} = -\frac{13}{12}$$

$$\tan\alpha = \frac{y}{x} = \frac{5}{12} \qquad \csc\alpha = \frac{r}{y} = -\frac{13}{5}$$

20. Let α = the angle with terminal side through $(-7, 24)$. Then $x = -7$, $y = 24$, and

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{49 + 576} \\ &= \sqrt{625} = 25. \end{aligned}$$

$$\sin\alpha = \frac{y}{r} = \frac{24}{25} \qquad \cot\alpha = \frac{x}{y} = -\frac{7}{24}$$

$$\cos\alpha = \frac{x}{r} = -\frac{7}{25} \qquad \sec\alpha = \frac{r}{x} = -\frac{25}{7}$$

$$\tan\alpha = \frac{y}{x} = -\frac{24}{7} \qquad \csc\alpha = \frac{r}{y} = \frac{25}{24}$$

22. In quadrant II, $x < 0$ and $y > 0$. Furthermore, $r > 0$.

$$\sin\theta = \frac{y}{r} > 0, \text{ so the sign is } +.$$
$$\cos\theta = \frac{x}{r} < 0, \text{ so the sign is } -.$$
$$\tan\theta = \frac{y}{x} < 0, \text{ so the sign is } -.$$
$$\cot\theta = \frac{x}{y} < 0, \text{ so the sign is } -.$$
$$\sec\theta = \frac{r}{x} < 0, \text{ so the sign is } -.$$
$$\csc\theta = \frac{r}{y} > 0, \text{ so the sing is } +.$$

24. In quadrant IV, $x > 0$ and $y < 0$. Also, $r > 0$.

$$\sin\theta = \frac{y}{r} < 0, \text{ so the sign is } -.$$
$$\cos\theta = \frac{x}{r} > 0, \text{ so the sign is } +.$$
$$\tan\theta = \frac{y}{x} < 0, \text{ so the sign is } -.$$
$$\cot\theta = \frac{x}{y} < 0, \text{ so the sign is } -.$$
$$\sec\theta = \frac{r}{x} > 0, \text{ so the sign is } +.$$
$$\csc\theta = \frac{r}{y} < 0, \text{ so the sign is } -.$$

26. When an angle θ of $45°$ is drawn in standard position, $(x, y) = (1, 1)$ is one point on its terminal side. Then

$$r = \sqrt{1 + 1} = \sqrt{2}.$$
$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\sec\theta = \frac{r}{x} = \sqrt{2}$$
$$\csc\theta = \frac{r}{y} = \sqrt{2}$$

28. When an angle θ of $120°$ is drawn in standard position, $(x, y) = (-1, \sqrt{3})$ is one point on its terminal side. Then

$$r = \sqrt{1+3} = 2.$$
$$\cos\theta = \frac{x}{r} = -\frac{1}{2}$$
$$\cot\theta = \frac{x}{y} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$
$$\sec\theta = \frac{r}{x} = -2$$

30. When an angle θ of $150°$ is drawn in standard position, $(x, y) = (-\sqrt{3}, 1)$ is one point on its terminal side. Then

$$r = \sqrt{3+1} = 2.$$
$$\sin\theta = \frac{y}{r} = \frac{1}{2}$$
$$\cot\theta = \frac{x}{y} = -\sqrt{3}$$
$$\sec\theta = \frac{r}{x} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

32. When an angle θ of $240°$ is drawn in standard position, $(x, y) = (-1, -\sqrt{3})$ is one point on its terminal side.

$$\tan\theta = \frac{y}{x} = \sqrt{3}$$
$$\cot\theta = \frac{x}{y} = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$

34. When an angle of $\frac{\pi}{6}$ is drawn in standard position, $(x, y) = (\sqrt{3}, 1)$ is one point on its terminal side. Then

$$r = \sqrt{3+1} = 2.$$
$$\cos\frac{\pi}{6} = \frac{x}{r} = \frac{\sqrt{3}}{2}$$

36. When an angle of $\frac{\pi}{3}$ is drawn in standard position, $(x, y) = (1, \sqrt{3})$ is one point on its terminal side.

$$\cot\frac{\pi}{3} = \frac{x}{y} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

38. When an angle of $\frac{\pi}{2}$ is drawn in standard position, $(x, y) = (0, 1)$ is one point on its terminal side. Then

$$r = \sqrt{0+1} = 1.$$
$$\sin\frac{\pi}{2} = \frac{1}{1} = 1$$

40. When an angle of π is drawn in standard position, $(x, y) = (-1, 0)$ is one point on its terminal side. Then

$$r = \sqrt{1+0} = 1.$$
$$\sec\pi = \frac{r}{x} = -1$$

42. When an angle of $\frac{3\pi}{4}$ is drawn in standard position, $(x, y) = (-1, 1)$ is one point on its terminal side.

$$\tan\frac{3\pi}{4} = \frac{y}{x} = -1$$

44. When an angle of 5π is drawn in standard position, $(x, y) = (-1, 0)$ is one point on its terminal side.
Then

$$r = \sqrt{1+0} = 1.$$
$$\cos 5\pi = \frac{x}{r} = -1$$

46. When an angle of $-\frac{2\pi}{3}$ is drawn in standard position, $(x, y) = (-1, -\sqrt{3})$ is one point on its terminal side.

$$\cot\left(-\frac{2\pi}{3}\right) = \frac{x}{y} = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$

48. When an angle of $-\frac{\pi}{6}$ is drawn in standard position, $(x, y) = (\sqrt{3}, -1)$ is one point on its terminal side.
Then

$$r = \sqrt{3+1} = 2.$$
$$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

50. $\cos 58° = .5299$

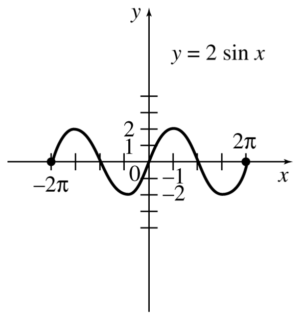
52. $\tan 54° = 1.3764$

54. $\tan 1.0123 = 1.6004$

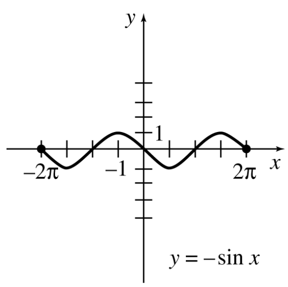
56. $\sin 1.5359 = .9994$

58. $g(t) = 5\sin\left(\frac{\pi}{6}t - 2\right)$ is of the form $g(t) = a\sin(bt + c)$ where $a = 5, b = \frac{\pi}{6}$, and $c = -2$. Thus, $a = 5$ and $T = \frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{6}} = 2\pi \cdot \frac{6}{\pi} = 12$.

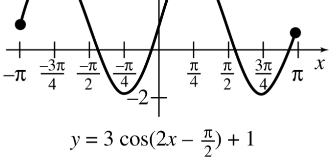
60. The graph of $y = 2\sin x$ is similar to the graph of $y = \sin x$ except that it has twice the amplitude. (That is, its height is twice as great.)



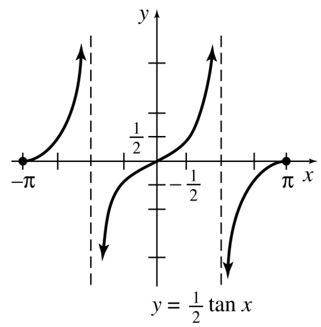
62. The graph of $y = -\sin x$ is similar to the graph of $y = \sin x$ except that it is reflected about the x-axis.



64. $y = 3\cos\left(2x - \frac{\pi}{2}\right) + 1$ has amplitude $a = 3$, period $T = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$, phase shift $\frac{c}{b} = \frac{-\frac{\pi}{2}}{2} = -\frac{\pi}{4}$, and vertical shift $d = 1$. Thus, the graph of $y = 3\cos\left(2x - \frac{\pi}{2}\right) + 1$ is similar to the graph of $f(x) = \cos x$ except that it has 3 times the amplitude, half the period, and is shifted up 1 unit vertically. Also, $y = 3\cos\left(2x - \frac{\pi}{2}\right) + 1$ is shifted $\frac{\pi}{4}$ units to the right relative to the graph of $g(x) = \cos(2x)$.



66. The graph of $y = \frac{1}{2}\tan x$ is similar to the graph of $y = \tan x$ except that the y-values of points on the graph are one-half the y-values of points on the graph of $y = \tan x$.



68. $S(t) = 500 + 500\cos\frac{\pi}{6}t$

(a) November corresponds to $t = 0$. Therefore,

$$\begin{aligned} S(0) &= 500 + 500\cos\left(\frac{\pi}{6}\right)(0) \\ &= 500 + 500\cos 0 \\ &= 1000 \text{ snowblowers.} \end{aligned}$$

(b) January corresponds to $t = 2$. Therefore,

$$\begin{aligned} S(2) &= 500 + 500\cos\left(\frac{\pi}{6}\right)(2) \\ &= 500 + 500\cos\frac{\pi}{3} \\ &= 500 + 500\left(\frac{1}{2}\right) \\ &= 750 \text{ snowblowers.} \end{aligned}$$

(c) February corresponds to $t = 3$. Therefore,

$$\begin{aligned} S(3) &= 500 + 500\cos\left(\frac{\pi}{6}\right)(3) \\ &= 500 + 500\cos\frac{\pi}{2} \\ &= 500 + 500(0) \\ &= 500 \text{ snowblowers.} \end{aligned}$$

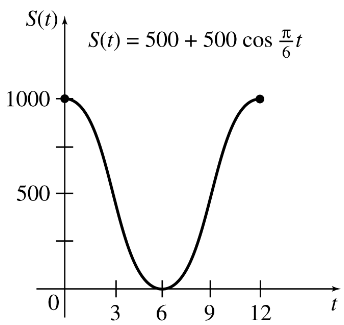
(d) May corresponds to $t = 6$. Therefore,

$$\begin{aligned} S(6) &= 500 + 500\cos\left(\frac{\pi}{6}\right)(6) \\ &= 500 + 500\cos\pi \\ &= 500 + 500(-1) \\ &= 0 \text{ snowblowers.} \end{aligned}$$

(e) August corresponds to $t = 9$.
Therefore,

$$\begin{aligned} S(9) &= 500 + 500\cos\left(\frac{\pi}{6}\right)(9) \\ &= 500 + 500\cos\frac{3\pi}{2} \\ &= 500 + 500(0) \\ &= 500 \text{ snowblowers.} \end{aligned}$$

(f) Use the ordered pairs obtained in parts (a)–(e) to plot the graph.



70. (a) The period is $\dfrac{2\pi}{\left(\frac{\pi}{14.77}\right)} = 29.54$

There is a lunar cycle every 29.54 days.

(b) $y = 100 + 1.8\cos\left(\frac{(x-6)\pi}{14.77}\right)$ reaches a maximum value when $\cos\left(\frac{(x-6)\pi}{14.77}\right) = 1$ which occurs when

$$\begin{aligned} x - 6 &= 0 \\ x &= 6 \end{aligned}$$

Six days from December 8 is December 14.

$$y = 100 + 1.8\cos\left(\frac{(6-6)\pi}{14.77}\right)$$

$$y = 101.8$$

There is a percent increase of 1.8 percent.

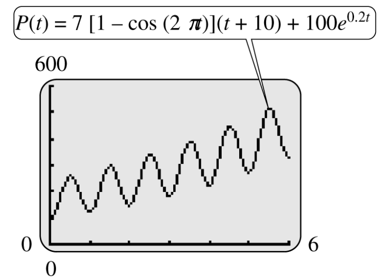
(c) On December 21, $x = 13$.

$$y = 100 + 1.8\cos\left(\frac{(13-6)\pi}{14.77}\right)$$

$$\approx 100.15$$

The formula predicts that the number of consulations was 100.15% of the daily mean.

72.



74. Solving $\dfrac{c_1}{c_2} = \dfrac{\sin\theta_1}{\sin\theta_2}$ for c_2 gives

$$c_2 = \frac{c_1 \sin\theta_2}{\sin\theta_1}.$$

$c_1 = 3 \cdot 10^8$, $\theta_1 = 46°$, and $\theta_2 = 31°$ so

$$\begin{aligned} c_2 &= \frac{3 \cdot 10^8(\sin 31°)}{\sin 46°} \\ &= 214{,}796{,}150 \\ &\approx 2.1 \times 10^8 \text{ m/sec.} \end{aligned}$$

76. On the horizontal scale, one whole period clearly spans four squares, so $4 \cdot 30° = 120°$ is the period.

78. $T(x) = 60 - 30\cos\left(\dfrac{x}{2}\right)$

(a) $x = 0$ represents January, so the maximum afternoon temperature in January is

$$T(0) = 60 - 30\cos 0 = 30°\text{F}.$$

(b) $x = 2$ represents March, so the maximum afternoon temperature in March is

$$T(2) = 60 - 30\cos 1 \approx 44°\text{F}.$$

(c) $x = 9$ represents October, so the maximum afternoon temperature in October is

$$T(9) = 60 - 30\cos\frac{9}{2} \approx 66°\text{F}.$$

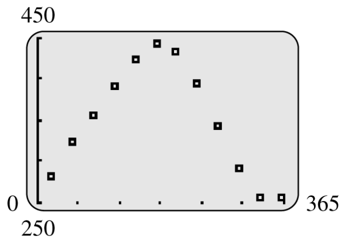
(d) $x = 5$ represents June, so the maximum afternoon temperature in June is

$$T(5) = 60 - 30\cos\frac{5}{2} \approx 84°\text{F}.$$

(e) $x = 7$ represents August, so the maximum afternoon temperature in August is

$$T(7) = 60 - 30\cos\frac{7}{2} \approx 88°\text{F}.$$

80. **(a)**



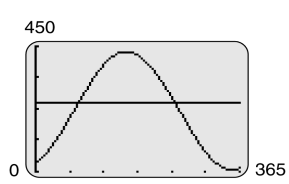
Yes; because of the cyclical nature of the days of the year, it is reasonable to assume that the times of the sunset are periodic.

(b) The function $s(x)$, derived by a $TI-83$ using the sine regression function under the STAT–CALC menu, is given by

$$s(x) = 94.0872\sin(.0166x - 1.2213) + 347.4158.$$

(c)
$$\begin{aligned} s(60) &= 94.0872\sin(.0166(60) - 1.2213) \\ &\quad + 347.4158 \\ &= 326 \text{ minutes} \\ &= 5{:}26 \text{ P.M.} \\ s(120) &= 94.0872\sin(.0166(120) - 1.2213) \\ &\quad + 347.4158 \\ &= 413 \text{ minutes} + 60 \text{ minutes} \\ &\quad \text{(daylight savings)} \\ &= 7{:}53 \text{ P.M.} \\ s(240) &= 94.0872\sin(.0166(240) - 1.2213) \\ &\quad + 347.4158 \\ &= 382 \text{ minutes} + 60 \text{ minutes} \\ &\quad \text{(daylight savings)} \\ &= 7{:}22 \text{ P.M.} \end{aligned}$$

(d) The following graph shows $s(x)$ and $y = 360$ (corresponding to a sunset at 6:00 P.M.). These graphs first intersect on day 82. However because of daylight savings time, to find the second value we find where the graphs of $s(x)$ and $y = 360 - 60 = 300$ intersect. These graphs intersect on day 295. Thus, the sun sets at approximately 6:00 P.M. on the 82$^{\text{nd}}$ and 295$^{\text{th}}$ days of the year.



82. Let h = the height of the building.

$$\tan 37.4° = \frac{h}{48}$$
$$\begin{aligned} h &= 48 \tan 37.4° \\ &\approx 48(.7646) \\ &\approx 36.7 \end{aligned}$$

The height of the building is approximately 36.7 m.

84. Let θ = the average angle with the horizontal.

$$\tan\theta = \frac{26}{5280}$$

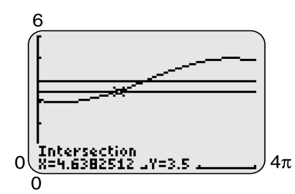
Using the TAN^{-1} key on the calculator,

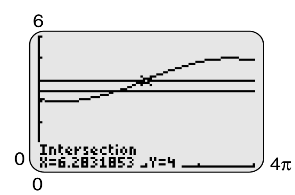
$$\theta = TAN^{-1}\left(\frac{26}{5280}\right) \approx .28°.$$

86. We need to find the values of t for which

$$3.5 \le h(t) \le 4.$$

The following graphs show where $h(t) = \sin\left(\frac{t}{\pi} - 2\right) + 4$ intersects the horizontal lines $y = 3.5$ and $y = 4$.





Thus, $3.5 \le h(t) \le 4$ when t is in the interval $[4.6, 6.3]$, to the nearest tenth.

13.2 Derivatives of Trigonometric Functions

2. $y = -\cos 4x + \cos\left(\frac{\pi}{4}\right)$

$$\frac{dy}{dx} = (\sin 4x) \cdot D_x(4x) + 0 = 4 \sin 4x$$

4. $y = -3\cos(8x^2 + 2)$

$$\begin{aligned} \frac{dy}{dx} &= [3\sin(8x^2+2)] \cdot D_x(8x^2+2) \\ &= 16x \cdot 3\sin(8x^2+2) \\ &= 48x \sin(8x^2+2) \end{aligned}$$

6. $y = -9 \sin^5 x$

$$\frac{dy}{dx} = -45(\sin x)^4 \cdot D_x(\sin x) = -45 \sin^4 x \cos x$$

8. $y = 2 \cot^4 x$

$$\frac{dy}{dx} = 8(\cot x)^3 \cdot D_x(\cot x) = -8 \cot^3 x \csc^2 x$$

10. $y = 6x \cdot \sec 3x$

$$\begin{aligned}\frac{dy}{dx} &= 6x \cdot D_x(\sec 3x) + (\sec 3x)D_x(6x)\\ &= 6x \cdot \sec 3x \tan 3x \cdot D_x(3x) + 6 \sec 3x\\ &= 18x \sec 3x \tan 3x + 6 \sec 3x\end{aligned}$$

12. $y = \dfrac{\tan x}{2x+4}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x+4) \cdot D_x(\tan x) - (\tan x)D_x(2x+4)}{(2x+4)^2}\\ &= \frac{(2x+4)\sec^2 x - 2\tan x}{(2x+4)^2}\end{aligned}$$

14. $y = \cos 4e^{2x}$

$$\frac{dy}{dx} = (-\sin 4e^{2x}) \cdot D_x(4e^{2x}) = -8e^{2x} \sin 4e^{2x}$$

16. $y = -8e^{\tan x}$

$$\frac{dy}{dx} = -8e^{\tan x} \cdot D_x(\tan x) = (-8 \sec^2 x)e^{\tan x}$$

18. $y = \cos(\ln |2x^3|)$

$$\begin{aligned}\frac{dy}{dx} &= [-\sin(\ln |2x^3|)] \cdot D_x(\ln |2x^3|)\\ &= -\sin(\ln |2x^3|) \cdot \frac{D_x(2x^3)}{2x^3}\\ &= -\frac{3}{x} \sin(\ln |2x^3|)\end{aligned}$$

20. $y = \ln |\tan^2 x|$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\tan^2 x} \cdot D_x(\tan^2 x)\\ &= \frac{1}{\tan^2 x} \cdot (2\tan x) \cdot D_x(\tan x)\\ &= \frac{2\sec^2 x}{\tan x}\end{aligned}$$

22. $y = \dfrac{4\cos x}{2 - \cos x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2-\cos x) \cdot D_x(4\cos x) - (4\cos x) \cdot D_x(2-\cos x)}{(2-\cos x)^2}\\ &= \frac{(2-\cos x)(-4\sin x) - 4\cos x \sin x}{(2-\cos x)^2}\\ &= \frac{-8\sin x}{(2-\cos x)^2}\end{aligned}$$

24. $y = \sqrt{\dfrac{\cos 4x}{\cos x}} = \left(\dfrac{\cos 4x}{\cos x}\right)^{1/2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}\left(\frac{\cos 4x}{\cos x}\right)^{-1/2} \cdot D_x\left(\frac{\cos 4x}{\cos x}\right)\\ &= \frac{1}{2}\left(\frac{\cos 4x}{\cos x}\right)^{-1/2}\\ &\quad \cdot \left[\frac{(\cos x) \cdot D_x(\cos 4x) - (\cos 4x) \cdot D_x(\cos x)}{\cos^2 x}\right]\\ &= \frac{1}{2}\left(\frac{\cos x}{\cos 4x}\right)^{1/2} \cdot \left(\frac{-4\cos x \sin 4x + \cos 4x \sin x}{\cos^2 x}\right)\\ &= \frac{-4\cos x \sin 4x + \cos 4x \sin x}{2\cos^{3/2} x \cos^{1/2} 4x}\end{aligned}$$

26. $y = (\sin 3x + \cot(x^3))^8$

$$\begin{aligned}\frac{dy}{dx} &= 8(\sin 3x + \cot(x^3))^7 \cdot D_x(\sin 3x + \cot(x^3))\\ &= 8(\sin 3x + \cot(x^3))^7\\ &\quad \cdot (\cos 3x \cdot D_x(3x) - \csc^2(x^3) \cdot D_x(x^3))\\ &= 8(\sin 3x + \cot(x^3))^7(3\cos 3x - 3x^2 \csc^2(x^3))\end{aligned}$$

28. $y = \sin x; x = \dfrac{\pi}{4}$

Let $f(x) = \sin x$.
Then $f'(x) = \cos x$ and

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

The slope of the tangent line at $x = \frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$.

30. $y = \cos x; x = -\dfrac{\pi}{4}$

Let $f(x) = \cos x$.
Then $f'(x) = -\sin x$ and

$$f'\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

The slope of the tangent line at $x = -\frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$.

32. $y = \cot x; x = \frac{\pi}{2}$

Let $f(x) = \cot x$.
Then $f'(x) = -\csc^2 x$ and

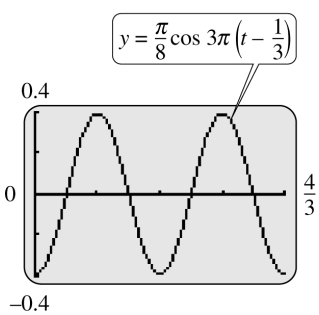
$$f'\left(\frac{\pi}{2}\right) = -\csc^2 \frac{\pi}{2} = -\frac{1}{\sin^2 \frac{\pi}{2}} = -\frac{1}{1} = -1.$$

The slope of the tangent line at $x = \frac{\pi}{2}$ is -1.

34.
$$\begin{aligned} D_x(\sec x) &= D_x\left(\frac{1}{\cos x}\right) \\ &= D_x\left[(\cos x)^{-1}\right] \\ &= -1(\cos x)^{-2}(-\sin x) \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\ &= \sec x \tan x \end{aligned}$$

38. $y = \frac{\pi}{8} \cos 3\pi \left(t - \frac{1}{3}\right)$

(a) The graph should resemble the graph of $y = \cos x$ with the following difference: The maximum and minimum values of y are $\frac{\pi}{8}$ and $-\frac{\pi}{8}$. The period of the graph will be $\frac{2\pi}{3\pi} = \frac{2}{3}$ units. The graph will be shifted horizontal $\frac{1}{3}$ units to the right.



(b) velocity $= \frac{dy}{dt}$

$$\begin{aligned} &= D_t\left[\frac{\pi}{8} \cos 3\pi \left(t - \frac{1}{3}\right)\right] \\ &= \frac{\pi}{8} D_t\left[\cos 3\pi \left(t - \frac{1}{3}\right)\right] \\ &= \frac{\pi}{8}\left[-\sin 3\pi \left(t - \frac{1}{3}\right)\right] D_t\left[3\pi \left(t - \frac{1}{3}\right)\right] \\ &= \frac{\pi}{8}\left[-\sin 3\pi \left(t - \frac{1}{3}\right)\right] \cdot 3\pi \\ &= -\frac{3\pi^2}{8} \sin 3\pi \left(t - \frac{1}{3}\right) \end{aligned}$$

Acceleration $= \frac{d^2y}{dt^2}$

$$\begin{aligned} &= D_t\left[\frac{-3\pi^2 \sin 3\pi \left(t - \frac{1}{3}\right)}{8}\right] \\ &= \frac{-3\pi^2}{8} D_t\left[\sin 3\pi \left(t - \frac{1}{3}\right)\right] \\ &= \frac{-3\pi^2}{8}\left[\cos 3\pi \left(t - \frac{1}{3}\right)\right] D_t\left[3\pi \left(t - \frac{1}{3}\right)\right] \\ &= \frac{-3\pi^2}{8}\left[\cos 3\pi \left(t - \frac{1}{3}\right)\right] \cdot 3\pi \\ &= -\frac{9\pi^3}{8} \cos 3\pi \left(t - \frac{1}{3}\right) \end{aligned}$$

(c) $\frac{d^2y}{dt^2} + 9\pi^2 y$

$$\begin{aligned} &= -\frac{9\pi^3}{8} \cos 3\pi \left(t - \frac{1}{3}\right) \\ &\quad + 9\pi^2\left[\frac{\pi}{8} \cos 3\pi \left(t - \frac{1}{3}\right)\right] \\ &= -\frac{9\pi^3}{8} \cos 3\pi \left(t - \frac{1}{3}\right) \\ &\quad + \frac{9\pi^3}{8} \cos 3\pi \left(t - \frac{1}{3}\right) \\ &= 0 \end{aligned}$$

(d)
$$\begin{aligned} a(1) &= -\frac{9\pi^3}{8} \cos 3\pi \left(t - \frac{1}{3}\right) \\ &= -\frac{9\pi^3}{8} \cos 2\pi \\ &= -\frac{9\pi^3}{8} \cdot 1 \\ &= -\frac{9\pi^3}{8} < 0 \end{aligned}$$

$$\begin{aligned} y(1) &= \frac{\pi}{8} \cos 3\pi \left(t - \frac{1}{3}\right) \\ &= \frac{\pi}{8} \cos 2\pi \\ &= \frac{\pi}{8} \cdot 1 = \frac{\pi}{8} \end{aligned}$$

Therefore, at $t = 1$ second, the force is clockwise and the arm makes an angle of $\frac{\pi}{8}$ radians forward from the vertical. The arm is moving clockwise.

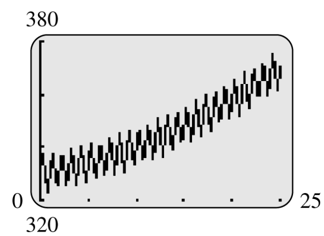
$$a\left(\frac{4}{3}\right) = -\frac{9\pi^3}{8}\cos 3\pi\left(\frac{4}{3}-\frac{1}{3}\right) = -\frac{9\pi^3}{8}\cos(3\pi) = -\frac{9\pi^3}{8}(-1) = \frac{9\pi^3}{8} > 0$$
$$y\left(\frac{4}{3}\right) = \frac{\pi}{8}\cos 3\pi\left(\frac{4}{3}-\frac{1}{3}\right) = \frac{\pi}{8}\cos(3\pi) = \frac{\pi}{8}(-1) = -\frac{\pi}{8}$$

Therefore, at $t = \frac{4}{3}$ seconds, the force is counterclockwise and the arm makes an angle of $-\frac{\pi}{8}$ radians from the vertical. The arm is moving counterclockwise.

$$a\left(\frac{5}{3}\right) = -\frac{9\pi^3}{8}\cos 3\pi\left(\frac{5}{3}-\frac{1}{3}\right) = -\frac{9\pi^3}{8}\cos(4\pi) = -\frac{9\pi^3}{8}\cdot 1 = -\frac{9\pi^3}{8} < 0$$
$$y\left(\frac{5}{3}\right) = \frac{\pi}{8}\cos 3\pi\left(\frac{5}{3}-\frac{1}{3}\right) = \frac{\pi}{8}\cos(4\pi) = \frac{\pi}{8}\cdot 1 = \frac{\pi}{8}$$

Therefore, at $t = \frac{5}{3}$ second, the answer corresponds to $t = 1$ second. So the arm is moving clockwise and makes an angle of $\frac{\pi}{8}$ from the vertical.

40. **(a)** Using the calculator to graph $C(x) = .04x^2 + .6x + 330 + 7.5\sin(2\pi x)$ in a $[0, 25]$ by $[320, 380]$ viewing window, gives the following graph.



(b) $C(25) = .04(25)^2 + .6(25) + 330 + 7.5\sin(2\pi(25)) = 370$ parts per million
$C(35.5) = .04(35.5)^2 + .6(35.5) + 330 + 7.5\sin(2\pi(35.5)) = 401.71$ parts per million
$C(50.2) = .04(50.2)^2 + .6(50.2) + 330 + 7.5\sin(2\pi(50.2)) \approx 468.05$ parts per million

(c) Since $C'(x) = .08x + .6 + 15\pi\cos(2\pi x)$, $C'(50.2)$ is given by

$$C'(50.2) = .08(50.2) + .6 + 15\pi\cos(2\pi(50.2)) \approx 19.18 \text{ parts per million per year.}$$

The level of carbon dioxide will be increasing at the beginning of 2010 at 19.18 parts per million.

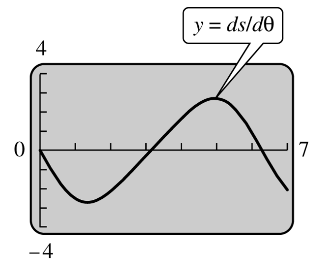
42. $s(\theta) = 2.625\cos\theta + 2.625(15 + \cos^2\theta)^{1/2}$

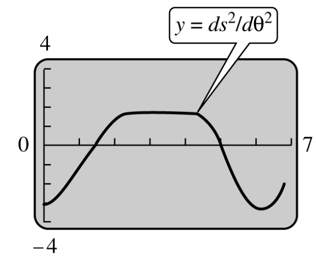
(a) The position of the piston when $\theta = \frac{\pi}{2}$ is given by

$$s\left(\frac{\pi}{2}\right) = 2.625(0) + 2.625(15+0)^{1/2} = 2.625\sqrt{15} \approx 10.17 \text{ in.}$$

(b) $$\frac{ds}{d\theta} = 2.625(-\sin\theta) + 2.625\left(\frac{1}{2}\right)(15+\cos^2\theta)^{-1/2}\cdot 2\cos\theta(-\sin\theta)$$
$$= -2.625\sin\theta - \frac{2.625\sin\theta\cos\theta}{\sqrt{15+\cos^2\theta}}$$
$$= -2.625\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{15+\cos^2\theta}}\right)$$

(c) One way to find the maximum velocity of the piston is to take the second derivative of $s(\theta)$ and then find the values of θ where the second derivative is equal to zero.
Using the second derivative, we can determine which critical points maximize $ds/d\theta$. Because the second derivative is so complicated, we will employ a graphing calculator to find the critical point(s).
The figures below show the calculator-generated graphs of the functions $ds/d\theta$ and $d^2s/d\theta^2$, respectively.





Although there are two critical points, the graph of $ds/d\theta$ shows that only one critical point is associated with the maximum velocity. This occurs when $\theta \approx 5$. The equation solver on the graphing calculator can be used to find a more precise value. The maximum velocity occurs at 4.944 radians.

44. (a) $u(x,t) = T_0 + A_0 e^{-ax}\cos\left(\frac{\pi}{6}t - ax\right)$

$$= 16 + 11e^{-.00706x}\cos\left(\frac{\pi}{6}t - .00706x\right)$$

The amplitude of $\mu(x,t)$ is given by $11e^{-.00706x}$. We need to find where $11e^{-.00706x} \le 1$.

$$\begin{aligned} 11e^{-.00706x} &\le 1 \\ e^{-.00706x} &\le \frac{1}{11} \\ -.00706x &\le \ln\left(\frac{1}{11}\right) \\ x &\le \frac{\ln\left(\frac{1}{11}\right)}{-.00706} \approx 340 \text{ cm} \end{aligned}$$

The amplitude is at most 1°C at a minimum depth of about 340 centimeters.

(b) We wish to find x for which $14 \le u(x,t) \le 18$. Since $-1 \le \cos\left(\frac{\pi}{6}t - .00706x\right) \le 1$, we have

$$\begin{aligned} -11e^{-.00706x} &\le 11e^{-.00706x}\cos\left(\frac{\pi}{6}t - .00706x\right) \\ &\le 11e^{-.00706x} \\ 16 - 11e^{-.00706x} &\le 16 + 11e^{-.00706x}\cdot\cos\left(\frac{\pi}{6}t - .00706x\right) \\ &\le 16 + 11e^{-.00706x} \\ 16 - 11e^{-.00706x} &\le u(x,t) \\ &\le 16 + 11e^{-.00706x} \end{aligned}$$

For $14 \le u(x,t) \le 18$, we need to find where $16 - 11e^{-.00706x} = 14$ and $16 + 11e^{-.00706x} = 18$.

These two conditions are equivalent to

$$\begin{aligned}
11e^{-.00706x} &= 2 \\
e^{-.00706x} &= \frac{2}{11} \\
-.00706x &= \ln\left(\frac{2}{11}\right) \\
x &= \frac{\ln\left(\frac{2}{11}\right)}{-.00706} \approx 242 \text{ cm}
\end{aligned}$$

A minimum depth of about 242 centimeters will keep the wine at a temperature between 14°C and 18°C.

(c) The phase shift will correspond to $\frac{1}{2}$ year or 6 months when

$$\begin{aligned}
\frac{.00706x}{\frac{\pi}{6}} &= 6 \\
.00706x &= \pi \\
x &= \frac{\pi}{.00706} \approx 445 \text{ cm}
\end{aligned}$$

A depth of about 445 centimeters gives a ground temperature prediction of winter when it is summer and vice versa.

(d) $u = T_0 + A_0e^{-ax}\cos(wt - ax)$

$$\begin{aligned}
\frac{du}{dt} &= A_0e^{-ax}[-\sin(wt-ax)](w) = -wA_0e^{-ax}\sin(wt-ax) \\
\frac{du}{dx} &= -aA_0e^{-ax}\cos(wt-ax) + A_0e^{-ax}[-\sin(wt-ax)](-a) \\
&= -aA_0e^{-ax}\cdot[\cos(wt-ax) - \sin(wt-ax)]
\end{aligned}$$

$$\begin{aligned}
\frac{d^2u}{dx^2} &= a^2A_0e^{-ax}[\cos(wt-ax) - \sin(wt-ax)] - aA_0e^{-x}[a\sin(wt-ax) + a\cos(wt-ax)] \\
&= a^2A_0e^{-ax}[\cos(wt-ax) - \sin(wt-ax) - a^2A_0e^{-ax}[\sin(wt-ax) + \cos(wt-ax)] \\
&= -2a^2A_0e^{-ax}\sin(wt-ax)
\end{aligned}$$

Now $a = \sqrt{\frac{w}{2k}}$ implies that

$$\begin{aligned}
a^2 &= \frac{w}{2k} \\
k &= \frac{w}{2a^2}.
\end{aligned}$$

It follows that

$$k\,\frac{d^2u}{dx^2} = \frac{w}{2a^2}[-2a^2A_0e^{-ax}\sin(wt-ax)] = -wA_0e^{-ax}\sin(wt-ax) = \frac{du}{dt}.$$

Thus,

$$\frac{du}{dt} = k\frac{d^2u}{dx^2}.$$

46. **(a)** $s(\theta) = 2.625\cos\theta + 2.625(15+\cos^2\theta)^{1/2}$

$$\frac{ds}{dt} = \frac{ds}{d\theta}\cdot\frac{d\theta}{dt} = \left[-2.625\sin\theta + 1.3125(15+\cos^2\theta)^{-1/2}\cdot D_\theta(15+\cos^2\theta)\cdot\frac{d\theta}{dt}\right]$$

$$= \left[-2.625\sin\theta + \frac{1.3125}{\sqrt{15+\cos^2\theta}}\cdot(-2\sin\theta\cos\theta)\right]\cdot\frac{d\theta}{dt} = -2.625\sin\theta\left(1+\frac{\cos\theta}{\sqrt{15+\cos^2\theta}}\right)\frac{d\theta}{dt}$$

(b) With $\theta = 4.944$ and $\frac{d\theta}{dt} = 505{,}168.1\,\frac{\text{radians}}{\text{hour}}$, we have

$$\frac{ds}{dt} = -2.625\sin(4.944)\left(1+\frac{\cos(4.944)}{\sqrt{15+\cos^2(4.944)}}\right)(505{,}168.1) \approx 1{,}367{,}018.749\,\frac{\text{inches}}{\text{hour}}$$

$$= 1{,}367{,}018.749\,\frac{\text{inches}}{\text{hour}}\times\frac{1\text{ foot}}{12\text{ inches}}\times\frac{1\text{ mile}}{5280\text{ feet}} \approx 21.6\text{ miles per hour}$$

48. **(a)** $\tan\theta = \frac{x}{40}$

Differentiate both sides with respect to time, t.

$$D_t(\tan\theta) = D_1\left(\frac{x}{40}\right)$$

$$\sec^2\theta\cdot\frac{d\theta}{dt} = \frac{1}{40}\cdot\frac{dx}{dt}$$

Since the light rotates twice per minute,

$$\frac{d\theta}{dt} = \frac{2(2\pi\text{ radians})}{1\text{ min}} = 4\pi\text{ radians per minute.}$$

When the light beam and shoreline are at right angles, $\theta = 0$ and $\sec\theta = 1$. Thus,

$$(1)^2(4\pi) = \frac{1}{40}\cdot\frac{dx}{dt}$$

$$\frac{dx}{dt} = 160\pi.$$

The beam is moving along the shoreline at 160π m/min.

(b) When the beam hits the shoreline 40 m from the point on the shoreline closest to the lighthouse, $\theta = \frac{\pi}{4}$ and $\sec\theta = \sqrt{2}$. Thus,

$$(\sqrt{2})^2(4\pi) = \frac{1}{40}\cdot\frac{dx}{dt}$$

$$\frac{dx}{dt} = 320\pi.$$

The beam is moving at 320π m/min.

50. Let x be the length of the ladder and let y be the distance from the wall to the bottom of the ladder. Then

$$\cos\theta = \frac{y+2}{x} \quad \text{and} \quad \cot\theta = \frac{y}{9}$$
$$x\cos\theta = y+2 \quad \text{and} \quad y = 9\cot\theta.$$

Thus, $x\cos\theta = 9\cot\theta + 2$

$$x = \frac{9\cot\theta + 2}{\cos\theta}$$
$$x = 9\csc\theta + 2\sec\theta.$$

This expression gives the length of the ladder as a function of θ. Find the minimum value of this function.

$$\begin{aligned}\frac{dx}{d\theta} &= -9\csc\theta\cot\theta + 2\sec\theta\tan\theta \\ &= -9\left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) + 2\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right) \\ &= \frac{-9\cos\theta}{\sin^2\theta} + \frac{2\sin\theta}{\cos^2\theta}\end{aligned}$$

If $\frac{dx}{d\theta} = 0$, then

$$\begin{aligned}\frac{2\sin\theta}{\cos^2\theta} &= \frac{9\cos\theta}{\sin^2\theta} \\ 2\sin^3\theta &= 9\cos^3\theta \\ \frac{\sin^3\theta}{\cos^3\theta} &= \frac{9}{2} \\ \tan^3\theta &= \frac{9}{2} \\ \tan\theta &= \sqrt[3]{\frac{9}{2}} \\ \theta &\approx 1.02619 \text{ radians.}\end{aligned}$$

If $\theta < 1.02619$, $\frac{dx}{d\theta} < 0$.

If $\theta > 1.02619$, $\frac{dx}{d\theta} > 0$.

Therefore, there is a minimum when $\theta = 1.02619$. If $\theta = 1.02619$,

$$x \approx 14.383.$$

The minimum length of the ladder is approximately 14.38 ft.

13.3 Integrals of Trigonometric Functions

2. $\int \sin 8x\, dx$

Let $u = 8x$, so that $du = 8\, dx$.

$$\begin{aligned}\int \sin 8x\, d &= \frac{1}{8}\int \sin 8x(8\, dx) \\ &= \frac{1}{8}\int \sin u\, du \\ &= -\frac{1}{8}\cos u + C \\ &= -\frac{1}{8}\cos 8x + C\end{aligned}$$

4. $\int (7\sin x - 8\cos x)\, dx$

$$\begin{aligned}&= 7\int \sin x\, dx - 8\int \cos x\, dx \\ &= -7\cos x - 8\sin x + C\end{aligned}$$

6. $\int 2x\cos x^2\, dx$

Let $u = x^2$, so that $du = 2x\, dx$.

$$\begin{aligned}\int 2x\cos x^2\, dx &= \int \cos u\, du \\ &= \sin u + C \\ &= \sin x^2 + C\end{aligned}$$

8. $-\int 2\csc^2 8x\, dx$

Let $u = 8x$, so that $du = 8\, dx$.

$$\begin{aligned}-\int 2\csc^2 8x\, dx &= -\frac{1}{4}\int \csc^2 8x(8\, dx) \\ &= \frac{1}{4}\int(-\csc^2 u)\, du \\ &= \frac{1}{4}\cot u + C \\ &= \frac{1}{4}\cot 8x + C\end{aligned}$$

10. $\int \sin^6 x\cos x\, dx$

Let $u = \sin x$, so that $du = \cos x\, dx$.

$$\begin{aligned}\int \sin^6 x\cos x\, dx &= \int u^6\, du \\ &= \frac{1}{7}u^7 + C \\ &= \frac{1}{7}\sin^7 x + C\end{aligned}$$

12. $\int \frac{\cos x}{\sqrt{\sin x}}\,dx$

Let $u = \sin x$, so that $du = \cos x\,dx$.

$$\begin{aligned}\int \frac{\cos x}{\sqrt{\sin x}}\,dx &= \int \sin^{-1/2} x \cos x\,dx\\ &= \int u^{-1/2}\,du\\ &= 2u^{1/2} + C\\ &= 2\sin^{1/2} x + C\end{aligned}$$

14. $\int \frac{\cos x}{1-\sin x}\,dx$

Let $u = 1 - \sin x$, so that $du = -\cos x\,dx$.

$$\begin{aligned}&\int \frac{\cos x}{1-\sin x}\,dx\\ &= -\int (1-\sin x)^{-1}(-\cos x)\,dx\\ &= -\int u^{-1}\,du\\ &= -\ln|u| + C\\ &= -\ln|1-\sin x| + C\end{aligned}$$

16. $\int (x+2)^4 \sin(x+2)^5\,dx$

Let $u = (x+2)^5$, so that
$du = 5(x+2)^4\,dx$.

$$\begin{aligned}&\int (x+2)^4 \sin(x+2)^5\,dx\\ &= \frac{1}{5}\int \sin(x+2)^5 \cdot 5(x+2)^4\,dx\\ &= \frac{1}{5}\int \sin u\,du\\ &= -\frac{1}{5}\cos u + C\\ &= -\frac{1}{5}\cos(x+2)^5 + C\end{aligned}$$

18. $\int \cot\left(-\frac{3x}{8}\right)dx$

Let $u = -\frac{3x}{8}$, so that $du = -\frac{3}{8}\,dx$.

Then

$$\begin{aligned}\int \cot\left(-\frac{3x}{8}\right)dx &= -\frac{8}{3}\int \cot\left(-\frac{3x}{8}\right)\left(-\frac{3}{8}\,dx\right)\\ &= -\frac{8}{3}\int \cot u\,du\\ &= -\frac{8}{3}\ln|\sin u| + C\\ &= -\frac{8}{3}\ln\left|\sin\left(-\frac{3x}{8}\right)\right| + C.\end{aligned}$$

20. $\int \frac{x}{4}\tan\left(\frac{x}{4}\right)^2 dx$

Let $u = \left(\frac{x}{4}\right)^2$, so that

$$du = 2\left(\frac{x}{4}\right)\left(\frac{1}{4}\,dx\right) = \frac{x}{8}\,dx.$$

$$\begin{aligned}&\int \frac{x}{4}\tan\left(\frac{x}{4}\right)^2 dx\\ &= 2\int \frac{x}{8}\tan\left(\frac{x}{4}\right)^2 dx\\ &= 2\int \tan u\,du\\ &= -2\ln|\cos u| + C\\ &= -2\ln\left|\cos\left(\frac{x}{4}\right)^2\right| + C\end{aligned}$$

22. $\int e^{-x}\tan e^{-x}\,dx$

Let $u = e^{-x}$, so that $du = -e^{-x}\,dx$.

$$\begin{aligned}\int e^{-x}\tan e^{-x}\,dx &= -\int \tan u\,du\\ &= \ln|\cos u| + C\\ &= \ln|\cos e^{-x}| + C\end{aligned}$$

24. $\int x^4 \sec x^5 \tan x^5 dx$

Let $u = x^5$, so that $du = 5x^4 dx$

$$\begin{aligned}\int x^4 \sec x^5 \tan x^5 dx &= \frac{1}{5}\int \sec x^5 \tan x^5 (5x^4 dx)\\ &= \frac{1}{5}\int \sec u \tan u\,du\\ &= \frac{1}{5}\sec u + C\\ &= \frac{1}{5}\sec x^5 + C\end{aligned}$$

26. $\int 9x\sin 2x\,dx$

Let $u = 9x$ and $dv = \sin 2x\,dx$.
Then $du = 9\,dx$ and $v = -\frac{1}{2}\cos 2x$.

$$\begin{aligned}&\int 9x\sin 2x\,dx\\ &= -\frac{9}{2}x\cos 2x - \int\left(-\frac{1}{2}\cos 2x\right)(9\,dx)\\ &= -\frac{9}{2}x\cos 2x + \frac{9}{2}\int \cos 2x\,dx\\ &= -\frac{9}{2}x\cos 2x + \frac{9}{4}\sin 2x + C\end{aligned}$$

28. $\int -11x \cos x \, dx$

Let $u = -11x$ and $dv = \cos x \, dx$.
Then $du = -11 \, dx$ and $v = \sin x$.

$$\int -11x \cos x \, dx$$
$$= -11x \sin x - \int (\sin x)(-11 \, dx)$$
$$= -11x \sin x + 11 \int \sin x \, dx$$
$$= -11x \sin x - 11 \cos x + C$$

30. $\int 10x^2 \sin \frac{1}{2}x \, dx$

Let $u = 10x^2$ and $dv = \sin \frac{1}{2}x \, dx$

Then $du = 20x \, dx$ and $v = -2 \cos \frac{1}{2}x$.

$$\int 10x^2 \sin \frac{1}{2}x \, dx$$
$$= (10x^2)\left(-2 \cos \frac{1}{2}x\right)$$
$$- \int \left(-2 \cos \frac{1}{2}x\right)(20x \, dx)$$
$$= -20x^2 \cos \frac{1}{2}x + 40 \int x \cos \frac{1}{2}x \, dx$$

Let $u = x$ and $dv = \cos \frac{1}{2}x \, dx$.

Then $du = dx$ and $v = 2 \sin \frac{1}{2}x$.

$$\int 10x^2 \sin \frac{1}{2}x \, dx$$
$$= -20x^2 \cos \frac{1}{2}x$$
$$+ 40\left(2x \sin \frac{1}{2}x - \int 2 \sin \frac{1}{2}x \, dx\right)$$
$$= -20x^2 \cos \frac{1}{2}x + 80x \sin \frac{1}{2}x$$
$$+ 160 \cos \frac{1}{2}x + C$$

32. $\int_{-\pi/2}^{0} \cos x \, dx = \sin x \Big|_{-\pi/2}^{0}$

$$= \sin 0 - \sin\left(-\frac{\pi}{2}\right)$$
$$= 0 - (-1)$$
$$= 1$$

34. $\int_{\pi/4}^{\pi/2} \cot x \, dx$

$$= \ln |\sin x| \Big|_{\pi/4}^{\pi/2}$$
$$= \ln \left|\sin \frac{\pi}{2}\right| - \ln \left|\sin \frac{\pi}{4}\right|$$
$$= \ln 1 - \ln \frac{\sqrt{2}}{2}$$
$$= -\ln \frac{\sqrt{2}}{2}$$

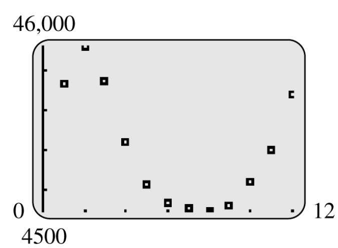
or $\ln \left(\frac{\sqrt{2}}{2}\right)^{-1} = \ln \sqrt{2} = \frac{1}{2} \ln 2$

36. $\int_{\pi/4}^{3\pi/4} \sin x \, dx = -\cos x \Big|_{\pi/4}^{3\pi/4}$

$$= -\cos \frac{3\pi}{4} - \left(-\cos \frac{\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)$$
$$= \sqrt{2}$$

38. Use the fnInt function on the graphing calculator to enter fnInt $(e\hat{}(-x)\cos x, x, 0, b)$, for successively larger values of b, which returns a value of .5 for sufficiently large enough b. Thus, an estimate of $\int_0^\infty e^{-x} \cos x \, dx$ is .5.

40. (a)



The data appears to be periodic, although there is a strong increase in February due to an extended period of cold weather.

(b) The function $C(x)$, derived by a $TI-83$ Plus calculator, is given by

$$C(x) = 20,277.8 \sin(.484742x + 1.02112) + 21,442.2.$$

(c) The estimate is given by

$$\int_0^{12} C(x)dx = \int_0^{12} [20{,}277.8\sin(.484742x + 1.02112) + 21{,}442.2]dx$$
$$= \left[-\frac{20{,}277.8}{.484742}\cos(.484742x + 1.02112) + 21{,}442.2x\right]\Bigg|_0^{12}$$
$$\approx 243{,}603 \text{ million cubic feet.}$$

The actual value is 240,755 million cubic feet.

42. $V(t) = 170\,\sin(120\,\pi t)$

Root mean square $= \sqrt{\dfrac{\int_0^T V^2(t)\,dt}{T}}$

(a) The period is $T = \dfrac{2\pi}{120\pi} = \dfrac{1}{60}$ sec.

(b) $\displaystyle\int_0^T V^2(t)\,dt$

$$= \int_0^{1/60} [170\,\sin(120\,\pi t^2)]^2\,dt$$
$$= 170^2 \int_0^{1/60} \sin^2(120\,\pi t)\,dt$$
$$= 170^2 \int_0^{1/60} \frac{1}{2}[1 - \cos(240\pi t)]\,dt$$
$$= \frac{170^2}{2}\int_0^{1/60} [1 - \cos(240\,\pi t)\,dt$$
$$= \frac{170^2}{2}\left[t - \frac{1}{240\pi}\sin(240\pi t)\right]\Bigg|_0^{1/60}$$
$$= \frac{170^2}{2}\left[\frac{1}{60} - \frac{1}{240\pi}\sin\left(240\pi\frac{1}{60}\right)\right] - \frac{170^2}{2}\cdot 0$$
$$= \frac{170^2}{120}$$

Root mean square $= \sqrt{\dfrac{\frac{170^2}{120}}{\frac{1}{60}}} = \sqrt{\frac{170^2}{2}} \approx 120.21$

Thus, 120 volts is the root mean square value for $V(t)$.

Chapter 13 Review Exercises

4. Exact value for the trigonometric functions can be determined for any integer multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

6. $120° = 120\left(\dfrac{\pi}{180}\right) = \dfrac{2\pi}{3}$

8. $270° = 270\left(\dfrac{\pi}{180}\right) = \dfrac{3\pi}{2}$

10. $420° = 420\left(\dfrac{\pi}{180}\right) = \dfrac{7\pi}{3}$

12. $\dfrac{3\pi}{4} = \dfrac{3\pi}{4}\left(\dfrac{180°}{\pi}\right) = 135°$

14. $\dfrac{7\pi}{15} = \dfrac{7\pi}{15}\left(\dfrac{180°}{\pi}\right) = 84°$

16. $\dfrac{11\pi}{15} = \dfrac{11\pi}{15}\left(\dfrac{180°}{\pi}\right) = 132°$

18. When an angle of 120° is drawn in standard position, $(x, y) = (-1, \sqrt{3})$ is one point on its terminal sides, so

$$\tan 120° = \frac{y}{x} = -\sqrt{3}.$$

20. When an angle of 45° is drawn in standard position, $(x, y) = (1, 1)$ is one point on its terminal side. Then

$$r = \sqrt{1+1} = \sqrt{2},$$

so

$$\sec 45° = \frac{r}{x} = \sqrt{2}.$$

22. When an angle of 300° is drawn in standard position, $(x, y) = (1, -\sqrt{3})$ is one point on its terminal sides, so

$$\cot 300° = \frac{x}{y} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

24. When an angle of $\frac{2\pi}{3}$ is drawn in standard position, $(x, y) = (-1, \sqrt{3})$ is one point on its terminal side. Then

$$r = \sqrt{1+3} = 2,$$

so

$$\cos\frac{2\pi}{3} = \frac{x}{r} = -\frac{1}{2}.$$

26. When an angle of $\frac{7\pi}{3}$ is drawn in standard position, $(x, y) = (1, \sqrt{3})$ is one point on its terminal side. Then

$$r = \sqrt{1+3} = 2,$$

so

$$\csc\frac{7\pi}{3} = \frac{r}{y} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

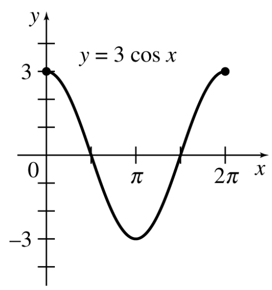
28. $\cos 59° \approx .5150$

30. $\sin(-32°) \approx -.5299$

32. $\cos .3142 \approx .9510$

34. $\tan 1.2915 \approx 3.4868$

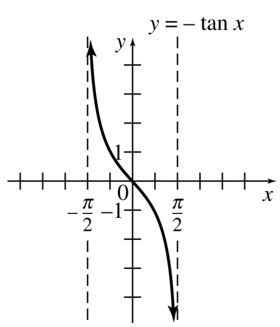
36. The graph of $y = \cos x$ appears in Figure 14 in Section 13.1 in the textbook. To get $y = 3\cos x$, each value of y in $y = \cos x$ must be multiplied by 3. This gives a graph going through $(0, 3)$, $(\pi, -3)$ and $(2\pi, 3)$.



38. The graph of $y = \tan x$ appears in Figure 15 in Section 13.1. The difference between the graph of $y = \tan x$ and $y = -\tan x$ is that the y-values of points on the graph of $y = -\tan x$ are the opposites of the y-values of the corresponding points on the graph of $y = \tan x$.
A sample calculation:

When $x = \frac{\pi}{4}$,

$$y = -\tan\frac{\pi}{4} = -1.$$



40. $y = -4\sin 7x$

$$\frac{dy}{dx} = -4(\cos 7x)\cdot D_x(7x) = -4(\cos 7x)7 = -28\cos 7x$$

42. $y = \tan(4x^2+3)$

$$\frac{dy}{dx} = \sec^2(4x^2+3)\cdot D_x(4x^2+3) = \sec^2(4x^2+3)\cdot(8x) = 8x\sec^2(4x^2+3)$$

44. $y = 3\cos^6 x$

$$\frac{dy}{dx} = 3\,D_x(\cos x)^6 = [3\cdot 6(\cos x)^5]\cdot D_x(\cos x) = 18(\cos x)^5(-\sin x) = -18\sin x\cos^5 x$$

46. $y = \cot(4x^5)$

$$\frac{dy}{dx} = [-\csc^2(4x^5)]\cdot D_x(4x^5) = [-\csc^2(4x^5)]\cdot 20x^4 = -20x^4\csc^2(4x^5)$$

48. $y = x^2\csc x$

$$\frac{dy}{dx} = x^2 D_x(\csc x) + \csc x\,D_x(x^2) = x^2(-\csc x\cot x) + \csc x(2x) = -x^2\csc x\cot x + 2x\csc x$$

50. $y = \dfrac{\sin x - 1}{\sin x + 1}$

$$\frac{dy}{dx} = \frac{(\sin x+1)(\cos x) - (\sin x - 1)(\cos x)}{(\sin x+1)^2} = \frac{\sin x\cos x + \cos x - \sin x\cos x + \cos x}{(\sin x+1)^2} = \frac{2\cos x}{(\sin x+1)^2}$$

52. $y = \dfrac{x-2}{\sec x}$

$$\frac{dy}{dx} = \frac{\sec x\cdot 1 - (x-2)\sec x\tan x}{\sec^2 x} = \frac{\sec x - (x-2)\sec x\tan x}{\sec^2 x} = \frac{\sec x[1-(x-2)\tan x]}{\sec^2 x} = \frac{1-(x-2)\tan x}{\sec x}$$

54. $y = \ln|\cos x|$

$$\frac{dy}{dx} = \frac{D_x(\cos x)}{\cos x} = \frac{-\sin x}{\cos x} = -\tan x$$

56. $\int \sin 2x\,dx$

Let $u = 2x$.
Then $du = 2\,dx$

$\frac{1}{2}\,du = dx.$

$$\begin{aligned}\int \sin 2x\,dx &= \int (\sin u)\left(\frac{1}{2}\,du\right)\\ &= \frac{1}{2}\int \sin u\,du\\ &= \frac{1}{2}(-\cos u) + C\\ &= -\frac{1}{2}\cos 2x + C\end{aligned}$$

58. $\int \tan 9x\,dx$

Let $u = 9x$. Then $\frac{1}{9}\,du = dx$.

$$\begin{aligned}\int \tan 9x\,dx &= \int (\tan u)\left(\frac{1}{9}\,du\right)\\ &= \frac{1}{9}\int \tan u\,du\\ &= \frac{1}{9}(-\ln|\cos u|) + C\\ &= -\frac{1}{9}\ln|\cos 9x| + C\end{aligned}$$

60. $$\begin{aligned}\int 5\sec^2 x\,dx &= 5\int \sec^2 x\,dx\\ &= 5\tan x + C\end{aligned}$$

62. $\int x\sin 3x^2\,dx$

Let $u = 3x^2$. Then $\frac{1}{6}\,du = x\,dx$.

$$\begin{aligned}\int x\sin 3x^2\,dx &= \int \sin u\left(\frac{1}{6}\,du\right)\\ &= \frac{1}{6}\int \sin u\,du\\ &= \frac{1}{6}(-\cos u) + C\\ &= -\frac{1}{6}\cos 3x^2 + C\end{aligned}$$

64. $\int \sqrt{\cos x}\,\sin x\,dx$

Let $u = \cos x$.
Then $du = -\sin x\,dx$
$-du = \sin x\,dx.$

$$\begin{aligned}\int \sqrt{\cos x}\,\sin x\,dx &= \int \sqrt{u}(-du)\\ &= -\int u^{1/2}\,du\\ &= -\frac{2}{3}u^{3/2} + C\\ &= -\frac{2}{3}(\cos x)^{3/2} + C\end{aligned}$$

66. $\int x\tan 11x^2\,dx$

Let $u = 11x^2$. Then $du = 22x\,dx$.

$$\begin{aligned}&\int x\tan 11x^2\,dx\\ &= \frac{1}{22}\int (\tan 11x^2)\cdot(22x\,dx)\\ &= \frac{1}{22}\int \tan u\,du\\ &= \frac{1}{22}(-\ln|\cos u|) + C\\ &= -\frac{1}{22}\ln\left|\cos 11x^2\right| + C\end{aligned}$$

68. $\int (\sin x)^{5/2}\cos x\,dx$

Let $u = \sin x$. Then $du = \cos x\,dx$.

$$\begin{aligned}&\int (\sin x)^{5/2}\cos x\,dx\\ &= \int u^{5/2}\,du\\ &= \frac{u^{5/2+1}}{\frac{5}{2}+1} + C\\ &= \frac{u^{7/2}}{\frac{7}{2}} + C\\ &= \frac{2}{7}u^{7/2} + C\\ &= \frac{2}{7}(\sin x)^{7/2} + C\end{aligned}$$

70. $\int \sec^2 3x \tan 3x \, dx$

Let $u = \tan 3x$. Then $du = 3 \sec^2 3x \, dx$.

$$\begin{aligned}\int \sec^2 3x \tan 3x \, dx &= \frac{1}{3}\int u \, du \\ &= \frac{1}{3} \cdot \frac{u^2}{2} + C \\ &= \frac{1}{6}u^2 + C \\ &= \frac{1}{6}\tan^2 3x + C\end{aligned}$$

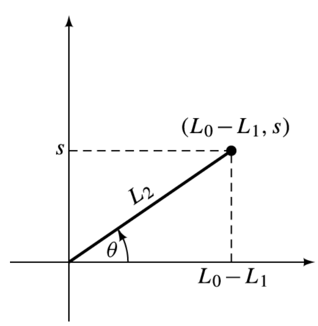
72. $\int_{\pi/2}^{\pi} \sin x \, dx$

$$\begin{aligned}&= -\cos x \Big|_{\pi/2}^{\pi} \\ &= -\cos \pi \left(-\cos \frac{\pi}{2}\right) \\ &= 1 - 0 = 1\end{aligned}$$

74. $\int_0^{2\pi} (5 + 5 \sin x) \, dx$

$$\begin{aligned}&= \int_0^{2\pi} 5 \, dx + \int_0^{2\pi} 5 \sin x \, dx \\ &= 5x \Big|_0^{2\pi} + (-5 \cos x) \Big|_0^{2\pi} \\ &= (10\pi - 0) + [-5 - (-5)] \\ &= 10\pi\end{aligned}$$

76. We draw θ in standard position.



From this diagram, we see that

$$\sin \theta = \frac{s}{L_2}.$$

78. Using the sketch of θ in the solution to Exercise 76 and the definition of cotangent, we see that

$$\cot \theta = \frac{L_0 - L_1}{s}.$$

80. The length of AD is L_1, and the radius of that section of blood vessel is r_1, so the general equation

$$k = \frac{L}{r^4}$$

is similar to

$$R_1 = k\frac{L_1}{{r_1}^4}$$

for that particular segment of the blood vessel.

82. $$\begin{aligned}R &= R_1 + R_2 \\ &= k\frac{L_1}{{r_1}^4} + k\frac{L_2}{{r_2}^4} \\ &= k\left(\frac{L_1}{{r_1}^4} + \frac{L_2}{{r_2}^4}\right)\end{aligned}$$

84. Since k, L_1, L_0, s, r_1, and r_2 are all constants, the only letter left as a variable is θ, so the differentiation indicated in the symbol R' must be differentiation with respect to θ.

$$\begin{aligned}R' &= D_\theta R \\ &= D_\theta\left(k\frac{L_0}{{r_1}^4} - \frac{sk}{{r_1}^4}\cot\theta + \frac{s}{{r_2}^4}\cdot\frac{k}{\sin\theta}\right) \\ &= kD_\theta\left(\frac{L_0}{{r_1}^4} - \frac{s}{{r_1}^4}\cot\theta + \frac{s}{{r_2}^4}\cdot\frac{1}{\sin\theta}\right) \\ &= k\left[D_\theta\left(\frac{L_0}{{r_1}^4}\right) - D_\theta\left(\frac{s}{{r_1}^4}\cot\theta\right) + D_\theta\left(\frac{s}{{r_2}^4}\cdot\frac{1}{\sin\theta}\right)\right] \\ &= k\left[0 - \frac{s}{{r_1}^4}D_\theta(\cot\theta) + \frac{s}{{r_2}^4}D_\theta\left(\frac{1}{\sin\theta}\right)\right] \\ &= k\left[-\frac{s}{{r_1}^4}(-\csc^2\theta) + \frac{s}{{r_2}^4}\left(\frac{-\cos\theta}{\sin^2\theta}\right)\right] \\ &= k\left(\frac{s}{{r_1}^4}\frac{1}{\sin^2\theta} - \frac{s}{{r_2}^4}\cdot\frac{\cos\theta}{\sin^2\theta}\right) \\ &= \frac{ks}{\sin^2\theta}\left(\frac{1}{{r_1}^4} - \frac{\cos\theta}{{r_2}^4}\right) \\ &= \frac{ks\csc^2\theta}{{r_1}^4} - \frac{ks\cos\theta}{{r_2}^4\sin^2\theta}\end{aligned}$$

86. If the left side of the equation in the solution to Exercise 85 is multiplied by $\frac{\sin^2\theta}{s}$, we get

$$\begin{aligned}&\frac{\sin^2\theta}{s}\cdot\frac{ks}{\sin^2\theta}\left(\frac{1}{{r_1}^4} - \frac{\cos\theta}{{r_2}^4}\right) \\ &= k\left(\frac{1}{{r_1}^4} - \frac{\cos\theta}{{r_2}^4}\right)\end{aligned}$$

This gives the equation

$$\frac{k}{{r_1}^4} - \frac{k\cos\theta}{{r_2}^4} = 0.$$

88. If $r_1 = 1$ and $r_2 = \frac{1}{4}$, then

$$\cos\theta = \left(\frac{\frac{1}{4}}{1}\right)^4 = \left(\frac{1}{4}\right)^4 = \frac{1}{256} \approx .0039,$$

from which we get

$$\theta \approx 90^\circ.$$

90.
$$\begin{aligned} s(t) &= A\cos(Bt + C) \\ s'(t) &= -A\sin(Bt + C)\cdot D_t(Bt + C) \\ &= -A\sin(Bt + C)\cdot B \\ &= -AB\sin(Bt + C) \\ s''(t) &= -AB\cos(Bt + C)\cdot D_t(Bt + C) \\ &= -AB\cos(Bt + C)\cdot B \\ &= -B^2A\cos(Bt + C) \\ &= -B^2s(t) \end{aligned}$$

92. (a)
$$\begin{aligned} y &= x\tan\alpha - \frac{16x^2}{V^2}\sec^2\alpha + h \\ &= 39\tan\frac{\pi}{24} - \frac{16(39)^2}{73^2}\sec^2\frac{\pi}{24} + 9 \\ &\approx 9.5 \end{aligned}$$

Yes, the ball will make it over the net since the height of the ball is about 9.5 feet when x is 39 feet.

(b) Entering

$$Y_1 = 39\tan x - \frac{16(39)^2}{44^2}\sec^2 x + 9 \text{ and}$$

$$Y_2 = \frac{44^2\sin x\cos x + 44^2\cos^2 x\sqrt{\tan^2 x + \frac{576}{44^2}\sec^2 x}}{32}$$

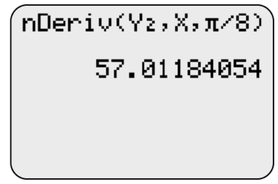
into the graphing calculator and using the table function, indicates that the tennis ball will clear the net and travel between 39 and 60 feet for $.18 \le x \le .41$ or $.18 \le \alpha \le .41$ in radians. In degrees,

$$10.3 \le \alpha \le 23.5.$$

X	Y1	Y2
.17	2.754	44.141
.18	3.1103	44.827
.22	4.5223	47.571
.34	8.6526	55.486
.38	10.002	57.9
.41	11.006	59.602
.42	11.339	60.146

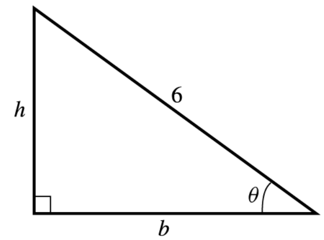
X=.42

(c) Using Y_2 from part (b) and the graphing calculator, we get



Note that $57.1\,\frac{\text{feet}}{\text{radian}} \approx .995\,\frac{\text{feet}}{\text{degree}}$. The distance the tennis ball travels will increase by approximately 1 foot by increasing the angle of the tennis racket by one degree.

94. Refer to the figure below.



Let A be the area of the triangle.

$$A = \frac{1}{2}bh$$

In this triangle,

$$\sin\theta = \frac{h}{6} \quad \text{and} \quad \cos\theta = \frac{b}{6}$$

$$h = 6\sin\theta \quad \text{and} \quad b = 6\cos\theta.$$

Thus,

$$\begin{aligned} A &= \frac{1}{2}(6\cos\theta)(6\sin\theta) \\ &= 18\sin\theta\cos\theta \\ &= 9(2\sin\theta\cos\theta) \\ A &= 9\sin(2\theta) \\ \frac{dA}{d\theta} &= 9\cos(2\theta)\cdot 2 = 18\cos(2\theta). \end{aligned}$$

If $\frac{dA}{d\theta} = 0$,

$$\begin{aligned} 18\cos(2\theta) &= 0 \\ \cos 2\theta &= 0 \\ 2\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{4}. \end{aligned}$$

If $\theta < \frac{\pi}{4}$, $\frac{dA}{d\theta} > 0$.

If $\theta > \frac{\pi}{4}$, $\frac{dA}{d\theta} < 0$.

Thus, A is maximum when $\theta = \frac{\pi}{4}$ or 45°.

Extended Application: The Shortest Time and the Cheapest Path

1. Since both rays are coming from point A, but meet the interface at two different points, they cannot be parallel. If the two rays were parallel, the alternate interior angle of the labeled right angle would also measure $90°$ implying that the two angles θ_1 are equal. As Δx becomes closer to zero the rays become more nearly parallel.

2. The segment labeled Δp is only an approximation to the change in length of the ray from point A. The actual change in length would be found by measuring the length of the original ray from point A to the crossing point along the new ray which would leave the change in length left over. As Δx becomes closer to zero the two rays become more nearly parallel and Δp becomes a better approximation.

3. $\dfrac{\text{Speed of light in air}}{\text{Speed of light in plastic}} = 1.6$

$$\text{Speed of light in plastic} = \frac{\text{Speed of light in air}}{1.6}$$

The percentage of the speed of light in 1.6-index plastic is given by $\frac{1}{1.6} = .625$ or 62.5%.

4. The cheapest route would be where the route goes almost perpendicularly across the swamp, making θ_1 almost zero. Since θ_1 is very close to zero, the road would be built to go perpendicularly across the swamp.

5. If construction over land was actually more expensive than construction over the swamp, we would solve

$$k\,\frac{7-x}{\sqrt{(7-x)^2+5^2}} = \frac{x}{\sqrt{x^2+3^2}}$$

where k represents how many more times expensive building over land is than building over the swamp. Letting $k = 2$ in the above equation, we get

$$2\,\frac{7-x}{\sqrt{(7-x)^2+5^2}} = \frac{x}{\sqrt{x^2+3^2}}.$$

The solver on the calculator gives $x \approx 4.68$ miles. The angle θ_1 has tangent equal to $4.68/3$ which means that θ_1 is about $57.3°$. For the straight line route, θ_1 has tangent equal to $7/8$ which means that θ_1 is about $41.2°$. The angle between the north-south line which passes A and the line from A to the point where the road emerges from the swamp is also θ_1, the larger θ_1 corresponds to the route which lies to the west of the other route. Thus, the route which minimizes cost lies to the west of the straight line route.

6. When you see the sun begin to set, the center of the sun looks to be $\frac{1}{2}(0.53°) = 0.265°$ above the horizon, but is actually $0.265° - 0.57° = -0.305°$ below the horizon. Since the radius of the sun's disk is only $0.265°$, this means the sun is already below the horizon.

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